Axioms of probability
Sampling from an urn

MATH/STAT 394: Probability I
Summer 2021 A Term

Introduction to Probability
D. Anderson, T.Seppäläinen, B. Valkó
§ 1.1 - 1.2

Aaron Osgood-Zimmerman

Department of Statistics

UNIVERSITY of WASHINGTON

Experiments, Outcomes, Events, and Sample Spaces

- An experiment is a stochastic process which can be observed but for which the result is uncertain.
- An outcome is one possible result from an experiment
- The sample space consists of all possible results from an experiment
- An event is a collection of outcomes from a sample space

Practice solution

Practice

You roll 2 dice, what is the cardinality of the event:
$E =$ "the sum larger than or equal to 10"? By cardinality I mean the number of possible outcomes that result in this event.

Solution The event is

$$A = \{(4,6),(5,5)(5,6),(6,6),(6,5),(6,4)\}$$

Outline

What is probability?

Definition (Informal)

Given an experiment, a sample space Ω, and all possible events $\mathcal{F}$[1] to define a probability measure is to associate each event $A \in \mathcal{F}$ with a number $\mathbb{P}(A)$ between 0 and 1, called the probability of the event A.

There is no agreement in how probabilities should be *interpreted*.

- Frequentist interpretation: The probability of event A is the proportion of times (frequency) that A occurs in an infinite sequence (or very long run) of separate tries of the experiment.

$$\mathbb{P}(A) = \lim_{n \to \infty} \frac{\#\text{ times A happens}}{n}.$$

- Bayesian interpretation: The probability of event A reflects a (usually subjective) belief on the likelihood of A.

That being said, probability as a *mathematical object*, is well-defined.

[1] If Ω is discrete, $\mathcal{F} = 2^{\Omega}$, otherwise see next lecture for a rigorous definition

Axioms of probability

Definition

Let Ω be the sample space, and $\mathcal{F}$ be the set of all possible events[2]*.*
The ***probability measure*** *(also called* ***probability distribution*** *or simply* ***probabillity****) is a function from $\mathcal{F}$ into the real numbers such that*

1. $0 \leq \mathbb{P}(A) \leq 1$ *for any event A*
2. $\mathbb{P}(\Omega) = 1$
3. *For $A_1, A_2, \ldots$ any sequence of (pairwise) disjoint events ($A_i \cap A_j = \emptyset$ for $i \neq j$),*

$$\boxed{\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).} \qquad \text{(Countable additivity)}$$

The triplet $(\underbrace{\Omega}_{\text{Sample space}}, \underbrace{\mathcal{F}}_{\text{Set of events}}, \underbrace{\mathbb{P}}_{\text{Probability measure}})$ is called a ***probability space***.

[2] If Ω is discrete, $\mathcal{F} = 2^{\Omega}$, otherwise see next lecture for a rigorous definition

Axioms of probability

Immediate consequences

- **Finite additivity**: For disjoint $A_1, A_2, \ldots, A_m \in \mathcal{F}$,

$$\boxed{\mathbb{P}(\cup_{i=1}^m A_i) = \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_m).}$$

- For a finite set $\Omega = \{\omega_1, \ldots, \omega_N\}$,

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\{\omega_1\}) + \ldots + \mathbb{P}(\{\omega_N\})$$

i.e. the probabilities of the singletons must sum up to one

Equally likely outcomes

Definition
Suppose Ω is finite and consists of N elements

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

A probability $\mathbb{P}$ is called equally likely if $\mathbb{P}(\omega_1) = \cdots = \mathbb{P}(\omega_N) = \frac{1}{N}$.

Clearly, $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$ and for any event $E \in \mathcal{F}$,

$$\boxed{\mathbb{P}(E) = \frac{\#E}{N}.}$$

Probability space examples

Example

Toss a fair coin twice. $\Omega = \{HH, TT, HT, TH\}$.
For $A = \{HT, TT\}$, $\mathbb{P}(A) = 2/4 = 1/2$.

Example

An unfair die would be a die such that the probabilities of each outcome is modified such as

$$\mathbb{P}(\{1\}) = 1/6, \quad \mathbb{P}(\{2\}) = 2/6, \quad \mathbb{P}(\{3\}) = 1/6,$$
$$\mathbb{P}(\{4\}) = 1/6, \quad \mathbb{P}(\{5\}) = 0, \quad \mathbb{P}(\{6\}) = 1/6$$

One can define any unfair die, yet the probabilities must always sum up to 1!

Probability space examples

Exercise

- Consider a fair die, such that the probability of rolling each face is equally probable. What is the probability of having an even number?
- Consider two fair dice. What it is the probability that the sum of the rolled faces is 8?

Solution

- The event of interest is$A = \{2,4,6\}$ and the prob. is $\mathbb{P}(A) = \mathbb{P}(\{2,4,6\}) = \mathbb{P}(\{2\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{6\}) = 1/2$.
- The sample space here is

$$\Omega = \{(i,j) : i,j \in \{1,\dots,6\}\}$$

Here we have tuples (ordered pairs). The dice are fair so all outcomes in Ω are equally probable, i.e., $\mathbb{P}(\{(i,j)\}) = 1/36$. The event of interest is

$$D = \{\text{the sum of the two dice is 8}\} = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

Therefore $\mathbb{P}(D) = 5/36$. *Why do we have ordered pairs?*

Probability space examples

Exercise

- Consider flipping a fair coin three times. What is the prob. to have at least two heads in a row?
- You roll 2 fair dice, what is the probability to have a sum larger or equal than 10?

Solution

- Rather than using $\{H, T\}$ we will use $(0, 1)$. In general it is much simpler to use numerical values. The sample space here is

$$\Omega = \{(i, j, k) : i, j, k \in \{0, 1\}\}$$

  The coin is fair so all outcomes in Ω are equally probable, i.e., $\mathbb{P}(\{(i, j, k)\}) = 1/2^3$. The event of interest is

$$D = \{\text{at least two heads in a row}\} = \{(0, 0, 1), (1, 0, 0), (0, 0, 0)\}$$

  Therefore $\mathbb{P}(D) = 3/8$.

- The event is

$$A = \{(4, 6), (5, 5), (5, 6), (6, 6), (6, 5), (6, 4)\}$$

  All events are equally probable, there are 6^2 possible events.
  So the Prob(A) is $6/36 = 1/6$.

Outline

Ordered sampling with replacement

Definition (Ordered sampling **with** replacement)

Consider an urn with $N \geq 2$ balls labeled as $\{1, 2, \ldots, N\}$. Pick one ball without looking, note its label, and put it back. Repeat this k times. The outcome is a k-tuple $(a_1, a_2, \ldots, a_k)$.

$$\Omega = \{\textit{all } k\textit{-tuples of } 1, \ldots, N\}, \quad \#\Omega = N^k.$$

Ordered sampling with replacement

Example

Suppose our urn contains 5 balls labeled 1, 2, 3, 4, 5. Sample 3 balls with replacement and produce an ordered list of the numbers drawn.
The sample space is

$$\Omega = \{1,2,3,4,5\}^3 = \{(s_1, s_2, s_3) : \text{ each } s_i \in \{1,2,3,4,5\}\}, \qquad \#\Omega = 5^3 = 125$$

We have for example

$$\mathbb{P}(\text{the sample is } (2,1,5)) = \mathbb{P}(\text{the sample is } (2,2,3)) = 1/125$$

Example

Repeated flips of a coin or rolls of a die are also sampling with replacements from the set $\{H, T\}$ or$\{1,2,3,4,5,6\}$.

Ordered sampling without replacement

Definition (Ordered sampling without replacement)

Pick one ball without looking, note its label, but don't put it back. Repeat this k times. The outcome is a k-tuple without repeats, i.e.,

$$\Omega = \{\textit{all } k\textit{-arrangements of } 1, \ldots, N\}, \quad \#\Omega = (N)_k.$$

Ordered sampling without replacement

Example

Consider again the urn with 5 balls labeled 1, 2, 3, 4, 5. Sample 3 balls without replacement and produce an ordered list of the numbers drawn.

$$\Omega = \{(s_1, s_2, s_3) : \text{ each } s_i \in \{1,2,3,4,5\} \text{ and } s_1, s_2, s_3 \text{ are all distinct}\},$$

We have 5 chocies for the first, 4 choices for the second, 3 choices for the third so $\#\Omega = 5 \cdot 4 \cdot 3 = (5)_3$ and

$$\mathbb{P}(\text{the sample is } (2,1,5)) = \frac{1}{5 \cdot 4 \cdot 3} = \frac{1}{60} \qquad \mathbb{P}(\text{the sample is } 2,2,3) = 0$$

Unordered sampling without replacement

Definition

Unordered sampling **without** replacement Pick one ball without looking, note its label, but *don't* put it back. Repeat this k times. This time do not consider the order of the balls you drew but only their labels (which ball comes first does not matter).

Namely, consider the outcome to be a k-subset

$$\Omega = \{\text{all } k\text{-subset of } 1, \ldots, N\}, \quad \#\Omega = \binom{N}{k}.$$

Unordered sampling without replacement

Example

Same as before, an urn, 5 balls labeled 1, 2, 3, 4, 5. Sample 3 balls without replacement an produce a set of 3 balls (unordered)

$$\Omega = \{\omega : \omega \text{ is a subset of size 3 from } \{1,2,3,4,5\}\}$$

$$\mathbb{P}(\text{the sample is } \{1,2,5\}) = \frac{1}{\binom{5}{3}} = \frac{2!3!}{5!} = \frac{1}{10}$$

Sampling practice

Exercise

Class of 24 children. All picking is done uniformly at random.

1. Each day one child leads the class to lunch.
   What is the prob. that Alex is chosen on Monday and Wednesday and Julie is chosen on Tuesday?
2. One student chosen to be president, one vice-president, one treasurer.
   They cannot hold more than one position
   Prob. that Mary is president, Cory vice-president, Matt treasurer?
3. Same but now what is the prob. that Ben is either president or vice-president?
4. A team of 3 students is chosen at random.
   What is the prob. that the team consists in Shane, Heather and Laura?
5. A team of 3 is chosen. What is the prob. that Mary is on the team?

Sampling illustration

Class of 24 children. All picking is done uniformly at random. **Solution**

1. $\mathbb{P}((\text{Alex, Julie, Alex})) = 24^{-3}$
2. $\mathbb{P}((\text{Mary pres, Cory vice pres, Matt treas})) = \frac{1}{24 \cdot 23 \cdot 22}$
3. $\mathbb{P}(\text{Ben pres}) = \frac{1 \cdot 23 \cdot 22}{24 \cdot 23 \cdot 22}$, $\mathbb{P}(\text{Ben vice pres}) = \frac{23 \cdot 1 \cdot 22}{24 \cdot 23 \cdot 22}$, $\mathbb{P}((\text{Ben pres or vice pres}) = \mathbb{P}(\text{Ben pres}) + \mathbb{P}(\text{Ben vice pres})$
4. $\mathbb{P}(\{Shane, Heather, Laura\}) = \frac{1}{\binom{24}{3}} = \frac{1}{2024}$
5. $\binom{23}{2}$ teams include Mary so $\mathbb{P}(\text{Mary is in the team}) = \frac{\binom{23}{2}}{\binom{24}{3}}$

Practice for next lecture

Practice

Rodney packs 4 shirts for a trip at random. The closet contains 10 shirts: 5 striped, 3 plaid, 2 solid colored ones.
What is the probability that he chose 2 striped and 2 plaid?