UNIVERSITY of WASHINGTON

Birthday Problem

MATH/STAT 394: Probability I
Summer 2021 A Term

Introduction to Probability
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§ 1.4

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Outline

Birthday Problem

Birthday problem

Exercise

In a room we have k people, each of which can have any of the n (e.g., $n = 365$) days as his/her birthday (equally likely). What is the chance that there are two in the room who have the same birthday?

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Solution

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Solution Let the event be A, and $A^c = \{\text{no two have the same birthday}\}$.

$$\#\Omega = n^k, \quad \#A^c = (n)_k.$$

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Then, by equally likely outcomes,

$$P(A^c) = \frac{\#A^c}{\#\Omega} = \frac{(n)_k}{n^k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}$$
$$= 1(1-1/n)(1-2/n)\dots(1-(k-1)/n).$$

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$$\begin{aligned} P(A^c) = \frac{\#A^c}{\#\Omega} = \frac{(n)_k}{n^k} &= \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k} \\ &= 1(1-1/n)(1-2/n)\dots(1-(k-1)/n). \end{aligned}$$

Suppose n is big compared to k. Using $1 - x \approx e^{-x}$ for $x \approx 0$,

$$P(A^c) \approx \exp\left(-(1+2+\cdots+k-1)/n\right) = \exp(-k(k-1)/2n) \approx \exp(-k^2/2n),$$

and hence $P(A) \approx 1 - \exp(-k^2/2n)$.

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$B^c = \{$every of the $(k-1)$ people has a birthday different from Charlie's$\}$.

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$$P(B^c) = \frac{\#B^c}{\#\Omega} = \frac{(n-1)^{k-1}}{n^{k-1}} = (1-1/n)^{k-1}.$$

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Again using $e^{-x} \approx 1 - x$, we have

$$P(B) = 1 - P(B^c) \approx 1 - \exp(-(k-1)/n).$$

Birthday problem

$A = \{\text{there are two in the room who have the same birthday}\}$,
$B = \{\text{someone else with the same birthday as Charlie}\}$.