Infinitely many outcomes
Probability calculus

MATH/STAT 394: Probability I
Summer 2021 A Term

Introduction to Probability
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§ 1.3-1.4

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Logistics

- Participation updates and expectation
    - 11 days of lecture with polls (this excludes day 1, and the final review day). Participating in each day will contribute +10% of participation
    - Asking or answering content related questions (on lectures, homeworks, or other practice exercises) will contribute +5% of your participation grade
    - cannot exceed 100%
- The syllabus has been updated with this (and I've removed the breakout groups every 4th lecture)
- Please fill out the office hours poll by 5pm today. I'll announce new office hours on Monday. So far, Tue/Thu 8:30-9:30 is the most available.

Practice solution

Practice

Rodney packs 4 shirts for a trip at random. The closet contains 10 shirts: 5 striped, 3 plaid, 2 solid colored ones.

Prob. that he chose 2 striped and 2 plaid?

Solution

1. (unordered sample space) let's say 1, 2, 3, 4 , 5 are striped, 6, 7, 8, are plaid, 9, 10, are solid colored. Then

$$\Omega = \{\{x_1, x_2, x_3, x_4\} : x_i \in \{1, \dots 10\}\} \quad \#\Omega = \binom{10}{4} = 210$$

   We seek $A = \{\{x_1, x_2, x_3, x_4\} : x_1, x_2 \in \{1, \dots, 5\}, x_3, x_4 \in \{6, 7, 8\}\}$
   We have $\#A = \binom{5}{2}\binom{3}{2} = 30$ so $\mathbb{P}(A) = \frac{30}{210} = 1/7$

2. (ordered sample space) Now consider

   $\widetilde{\Omega} = \{(x_1, x_2, x_3, x_4), x_i \in \{1, \dots, 10\}, x_1, x_2, x_3, x_4 \text{ distinct}\}, \quad \#\widetilde{\Omega} = 10{\cdot}9{\cdot}8{\cdot}7 = 5040$

   Denote $\widetilde{A}$ the event of interest. The elements of $\widetilde{A}$ can be found by

   2.1 choosing two positions for the plaid shirts out of 4 ($\binom{4}{2}$)
   2.2 choosing the first plaid (3 choices), choosing the second plaid (2 choices)
   2.3 choosing the first stripped (5 choices), choosing the second striped shirt (4 choices)

   So $\#\widetilde{A} = \binom{4}{2} \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 720$ and $\mathbb{P}(\widetilde{A}) = \frac{720}{5040} = 1/7$.

Outline

Infinitely many outcomes

Motivation

- Until now we looked at finite sets (e.g. roll of a die, sampling from an urn...)
- What about infinite countable sample space (e.g. number of attempts to win a lottery) ?
- What about continuous sample space (e.g. time waiting for a bus) ?

Finite Additivity

Example (Urn with replacement)

You sample from an urn with 3 red balls and 1 blue ball and you're interested in counting how many draws it takes before the blue ball is selected.

What is the space Ω of possible outcomes for this experiment?

The number of draws required can be any positive integer and we can even imagine the scenario where the blue ball never comes up. So, in addition to the positive numbers, we also include ∞ in our sample space:

$$\Omega = \{\infty, 1, 2, \ldots\}$$

What is the probability of each event in the sample space?

The outcome is k exactly when the first $k-1$ draws are red, followed by a blue draw. The number of ways this occurs is $3^{k-1} \times 1$ and the number of ways we could draw k balls is 4^k. So,

$$P(k^{th} \text{ sample is first blue}) = \frac{3^{k-1} \times 1}{4^k}$$

Finite Additivity cont'd

Example (Urn with replacement cont'd)

$$P(k^{th} \text{ sample is first blue}) = \frac{3^{k-1} \times 1}{4^k}$$

what about $P(\infty)$?

From the axioms of probability, we must have:

$$1 = P(\Omega) = P(\{\infty, 1, 2, \ldots\}) = P(\infty) + \sum_{k=1}^{\infty} P(k), \text{ and}$$

Filling in what we know:

$$\sum_{k=1}^{\infty} P(k) = \sum_{k=1}^{\infty} \frac{3^{k-1} \times 1}{4^k} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{3^k}{4^k}$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} (\frac{3}{4})^k = \frac{1}{3} \times (\frac{1}{1 - 3/4} - 1) = 1$$

Finite Additivity cont'd

Example (Urn with replacement cont'd)

So,

$$P(\Omega) = 1 = P(\infty) + P(\{1, 2, \ldots\}) \rightarrow XSP(\infty) = 0$$

This is actually quite a powerful result that happens to agree with our intuition: the probability of never drawing a blue ball is 0. That outcome is simply not possible!

$$P(k^{th} \text{ sample is first blue}) = \frac{3^{k-1} \times 1}{4^k}$$

defines the *Geometric distribution* with success parameter $1/4$

The number of draws, on average, that it would take to get the first blue ball is finite (in this case it is 4).

An example where the average number of trials isn't finite is keeping track of a running sum during coin flips where you subtract 1 for each tail and add 1 for each head. The average time that it will take to get to the sum of $+1$ is infinite! More on this later.

Continuous sample spaces

Example (Uniform distribution)

- Consider picking a point X uniformly at random from $[0,1]$
- Uniformly at random means that X is equally likely to lie anywhere in $[0,1]$
- Since we can select anything in $[0,1]$, $\Omega = [0,1]$
- We might want to ask the question, what is the prob. that X lies in a smaller interval $[a,b] \subset [0,1]$?
- Since all locations are equally likely, a reasonable choice is to take the proportion of $[a,b]$ in $[0,1]$

$$\mathbb{P}(X \text{ lies in the interval } [a,b]) = \frac{b-a}{1-0} \quad \text{for } 0 \leq a \leq b \leq 1$$

- This equation defines the *Continuous Uniform distribution* on $[0,1]$

Probabilities of continuous sample spaces

Exercise

What is the probability that:

- X is equal to a particular value $c \in [0, 1]$?
- X is equal to a particular value $c > 1$?
- X falls in the interval $[c, d]$ where $0 < c < 1 < d$?
- X falls in the interval $(a, b]$ where $(a, b] \subset [0, 1]$?

Solution

- 0
- 0
- $1 - c$
- $b - a$

Continuous sample spaces

Example (Darts)
Consider a dartboard the shape of a disk with radius 9 inches

The bullseye is a disk of radius 1/2 in the middle of the board

1. What is the sample space?
2. What is the prob. that a randomly thrown dart hits the bullseye, assuming the dart hits the board uniformly at random?

Solution

1. $\Omega = \{(x, y) : x^2 + y^2 \leq 9^2\}$
2. The probability is the **proportion** of the surface of the bullseye w.r.t. the board
   Let A be the event that the dart hits the bullseye

$$A = \{(x, y) : x^2 + y^2 \leq 1/2^2\}$$

   Therefore

$$\mathbb{P}(A) = \frac{\text{Area of the bullseye}}{\text{Area of the board}} = \frac{\pi(1/2)^2}{\pi 9^2} = \frac{1}{18^2} \approx 0.003$$

Discrete versus continuous sample spaces

Discrete (countable) spaces

For discrete spaces (finite or countably infinite), the probability can simply be defined by giving a prob. to each element

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\})$$

Continuous (uncountable) spaces

- For continuous intervals we cannot assign a prob. to each point
- Consider $c = \mathbb{P}(\{x\}) > 0$ for $x \in [0, 1]$ and assume the points are equally likely,
- then for A made of k different points we would get $\mathbb{P}(A) = kc$ and for k large we would get $\mathbb{P}(A) > 1$ which is not possible.
- So we get that in this case

$$\mathbb{P}(\{x\}) = 0 \quad \text{for each } x \in [0, 1]$$

Consequences of continuous sample spaces

- Because probabilities of particular outcomes of continuous sample spaces must all be 0, the definition of probabilities of events on an uncountable space must be based on something other than individual points.
- We have now seed examples on how to approach this problem using length and area to model uniformly distributed points. Later, we will build up tools to handle cases where the random point is not uniformly selected.
- This issue is also closely tied with the additivity probability axiom. Note that the axiom requires additivity only for a sequence of pairwise disjoint events, and not for an uncountable collection of events so the axiom is not violated.
- For the interested reader, I suggest you take a look at §1.6 for a deeper explanation of measuring events of continuous sample spaces

Outline

Finite additivity

Reminder

For $A_1, \ldots, A_n$ **mutually exclusive**, i.e. $A_i \cap A_j = \emptyset$.

$$\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Also valid for an infinite sequence.

$$\mathbb{P}(\cup_{i=1}^{+\infty} A_i) = \sum_{i=1}^{+\infty} \mathbb{P}(A_i)$$

Finite additivity example

Exercise

An urn contains 30 red, 20 green and 10 yellow balls. Draw 2 without replacement.

What is the prob. that the sample contains exactly one red or exactly one yellow? (so it can contain exactly one red, exactly one yellow or both)

Solution

$$\mathbb{P}(\text{exactly 1 r or exactly 1 y}) = \mathbb{P}(\text{r \& g}) + \mathbb{P}(\text{y \& g}) + \mathbb{P}(\text{r \& y})$$

- $\mathbb{P}(\text{red and green}) = \frac{30 \cdot 20}{\binom{60}{2}}$
- $\mathbb{P}(\text{yellow and green}) = \frac{10 \cdot 20}{\binom{60}{2}}$
- $\mathbb{P}(\text{red and yellow}) = \frac{30 \cdot 10}{\binom{60}{2}}$

Events and Complements

Fact
For any event A,

$$\mathbb{P}(A) + \mathbb{P}(A^c) = 1$$

Proof Because A and A^c are disjoint with union Ω

Take-away

Sometimes the probability of the complement is easier to compute than the event itself.

Events and complements

Exercise

Roll a fair die 4 times. What is the prob. that some number appears more than once?

Solution

- Let $A =$ "some number appear more than once"
- Then $A^c =$ "all 4 rolls are different"
- So $\mathbb{P}(A^c) = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5}{18}$ and $\mathbb{P}(A) = \frac{13}{18}$.

Events and complements

Fact
For any two events A, B,

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$$

Proof B can be split into $A \cap B$ and $A^c \cap B$ which are disjoint

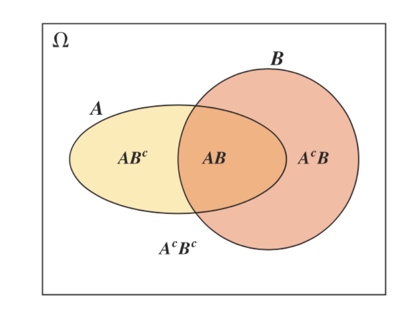

Figure from Introduction to probability, D. Anderson, T. Seppäläinen, B. Valkò

Monotonicity of probability

Fact
If $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Proof

- If $A \subseteq B$ then $B = A \cup (A^c \cap B)$
- Hence $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) \geq \mathbb{P}(A)$

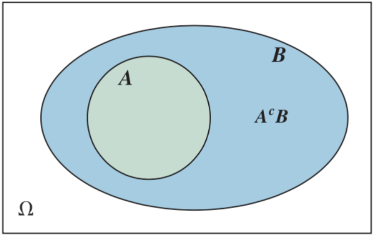

Figure from Introduction to probability, D. Anderson, T. Seppäläinen, B. Valkò

Practice next lecture

Practice

John and Lucy take turns rolling two dice.

If John rolls a pair larger than or equal to 10 he wins and the game stops.

If Lucy rolls a pair strictly smaller than 10, she wins and the game stops.

They keep rolling in turn until one of them wins. Suppose John rolls first.

What is the probability that Lucy wins?