# Random Variables
# A first look

MATH/STAT 394: Probability I
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Introduction to Probability
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§ 1.5

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## Logistics

- Office Hours:
    - Tuesday 8:30-9:30am PST
    - Thursday 4-5pm PST

# Practice solution

**Practice**

Randomly pair 4 keys {a, b, c, d} with 3 locks {a, b, c}.
What is $\mathbb{P}(\text{at least one match})$?

**Solution** Let $A$ denote the event that lock $a$ and key $a$ matches. Similarly, $B, C$.

- $\mathbb{P}(A) = \frac{3\times 2}{4\times 3\times 2}$
- $\mathbb{P}(A \cap B) = \frac{2}{4\times 3\times 2}$
- $\mathbb{P}(A \cap B \cap C) = \frac{1}{4\times 3\times 2}$
- $\mathbb{P}(\text{at least one match}) = \mathbb{P}(A \cup B \cup C) =$

$$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$
$$= \frac{6}{24} \times 3 - \frac{2}{24} \times 3 + \frac{1}{24} = \frac{13}{24}.$$

# Outline

Random Variables

# Random variables

**Definition**
Given a sample space $\Omega$, *a random variable (r.v.) $X$ on $\Omega$*[1] is a function from $\Omega$ to $\mathbb{R}$.

Key concepts to get comfortable with:

- A random variable is not a variable but a function
- r.v. are denoted with capital letters (unlike functions in, eg, calc)
- The value of a r.v. at a sample point $\omega$ is $X(\omega)$

**Example**
Flip a coin $n$ times, define $X$ as the number of heads you got
$X : \Omega \to \{0, \ldots, n\}$ with $\Omega = \{(a_1, \ldots, a_n) : a_i \in \{H, T\}\}$

[1] See the *ASV* §1.6 for a rigorous complete definition

## Random variable

**Example**
Flip three coins. Denote

- $X_h$ the outcome of the $h^{th}$ coin,
- $S$ the sum

Let's express them as functions of the sample space

**Solution**

- The sample space is made of outcomes $\omega = (i, j, k)$ with $i$ the first flip, $j$ the second flip, and $k$ the third.
- We have $X_1(\omega) = X_1(i, j, k) = i$, $X_2(\omega) = j$, $X_3(\omega) = k$ for $\omega = (i, j, k)$
- We have $S(\omega) = X_1(\omega) + X_2(\omega) + X_3(\omega)$.

# Random Variables

## Notations

- Recalling that events are subsets of $\Omega$, we will denote

$$\{X = x\} = \{\omega \in \Omega : X(\omega) = x\}$$

- In the prior example, the probability than the sum is 2 is written

$$\begin{aligned}\mathbb{P}(S = 2) &= \mathbb{P}(\omega \in \Omega : X(\omega) = 2) \\ &= \mathbb{P}((i, j, k) \in \Omega : i, j, k \in \{H, T\}, X(i, j, k) = 2)\end{aligned}$$

- Note: inside events a comma is short for "and" which represents $\cap$
- In probability, we usually only bother with the essential features:

only writing the event $\{S = 2\}$, or the probability $\mathbb{P}(S = 2)$

# Random Variables

## Notations

- Similarly we denote

$$\{a \leq X \leq b\} = \{\omega \in \Omega : a \leq X(\omega) \leq b\}$$

  So for example the prob. that sum is larger than or equal to 2 is written

$$\mathbb{P}(S \geq 2)$$

- **Idea:** we can measure probabilities in $\Omega$ by expressing them as the set of elements such that the r.v. satisfies some equality/inequality (namely the reciprocal image)

## Random variables

**Example**
A fair die is rolled

- If the roll is 1,2 or 3 the player loses 1$
- if the roll is 4 the player wins 1$
- if the rol is 5 or 6 the player wins 3$

Express the gain of the player as a r.v., give its probability of gain

**Solution**

- Denote $W$ the gain of the player, we have

$$W(1) = W(2) = W(3) = -1, \quad W(4) = 1, \quad W(5,6) = 3$$

- The probabilities are then

$$\mathbb{P}(W = -1) = \mathbb{P}(\{1,2,3\}) = 1/2, \quad \mathbb{P}(W = 1) = 1/6, \quad \mathbb{P}(W = 3) = 1/3$$

## Random variables

**Exercise**

Pick a point uniformly at random on $[0,1]$. Let $Y$ be twice the point.

1. What is the prob. that $Y$ is less than some $a \in [0,2]$?
2. Same for $a < 0$ and $a > 2$

**Solution**

1.

$$\begin{aligned}\{Y \leq a\} &= \{\omega \in [0,1] : Y(\omega) \leq a\} \\ &= \{\omega \in [0,1] : 2\omega \leq a\} \\ &= \{\omega \in [0,1] : \omega \leq a/2\} = [0, a/2]\end{aligned}$$

   Hence $\mathbb{P}(Y \leq a) = a/2$

2. For $a < 0$, $\{Y \leq a\} = \emptyset$ so $\mathbb{P}(Y \leq a) = 0$, and for $a > 2$, $\{Y \leq a\} = \Omega$ so $\mathbb{P}(Y \leq a) = 1$.

## Random variables

**Definition**
A r.v. is degenerate if there exists some value $b$ such that $\mathbb{P}(X = b) = 1$

**Example**

- A constant function $X(\omega) = b$ is degenerate
- Not all degenerate r.v. are constant functions

## Probability distribution

### Notations

Generally for $A \subseteq \mathbb{R}$, and $X$ a r.v. on $\Omega$, the *reciprocal image* of $A$ by $X$ is

$$\{X \in A\} := X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\}$$

This defines the events of interest

### Definition

The **probability distribution**[2] of a r.v. $X$ on $\Omega$ is the function

$$\mathbb{P}_X : B \to \mathbb{P}(X \in B)$$

for $B$ a subset in $\mathbb{R}$

[2] See *ASV* §1.6 for a more rigorous treatment

## Discrete random Variables

**Definition**
A r.v. $X$ is a **discrete r.v.** if there exists a finite or countable set $K = \{k_1, k_2, \ldots\}$ of real numbers such that

$$\sum_{i \in K} \mathbb{P}(X = k_i) = 1$$

**Example**

- If the range of $X$ (i.e. the set of values that $X$ can take) is finite or countable, then $X$ is a discrete r.v.
- Example the roll of a die, the tosses of a coin, ...
- Counter example: picking a point uniformly at random on $[0,1]$, then as we've seen $\mathbb{P}(X = a) = 0$ for all $a \in [0,1]$ so it is not discrete.

# Probability mass function

**Definition**
The **probability mass function** (p.m.f) of a discrete r.v. is the function

$$p_X(k) = \mathbb{P}(X = k)$$

for all possible values $k$ of $X$.

## Probability mass function

**Example**
For a fair die,

| k | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p_X$ | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |

For the sum of two dice

| S | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_S$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |

# Bernoulli Random variable

**Definition**

*A r.v. $X$ has a Bernoulli dist. with param. $p \in [0,1]$ if it takes its values in $\{0,1\}$ and*

$$\mathbb{P}(X=1) = p \quad \mathbb{P}(X=0) = 1-p$$

*We denote it $X \sim \mathrm{Ber}(p)$*

**Example**

This is typically the toss of a coin where you say 1 corresponds to tail and the prob. to have a tail is $p$.

## Bernoulli Random variable

**Example**

Define the **indicator function** of a subset $A \subset \Omega$ as

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

The indicator function of $A$ is a Bernoulli random variable such that

$$\mathbb{P}(1_A = 1) = \mathbb{P}(A) \quad \mathbb{P}(1_A = 0) = 1 - \mathbb{P}(A)$$

The Bernoulli r.v. is one of the primary building blocks for other discrete r.v. we will cover

**Exercise**

Two fair dice are rolled. We are interested in counting how many of the rolls were 2 or less. Write down the PMF for this r.v.

**Solution**

- We can consider each roll to be binary with prob of 1/3 of being 2 or less
- The possible outcomes are $\{SS, SF, FS, FF\}$
- Which map to $2, 1, 1, 0$
- We can then calculate the probabilities to be:

| k | 0 | 1 | 2 |
|---|---|---|---|
| $p_X$ | $(4/6)^2$ | $2 * (2/6 * 4/6)$ | $(2/6)^2$ |

## Practice for next lecture

**Practice**

Flip a fair coin 5 times. For each heads you pay me 1$ and for each tails I pay you 1$. Denote by $X$ my net wining.

- What are the possible values for $X$?
- What is $\mathbb{P}(X = 3)$?