Independence, mutual independence

MATH/STAT 394: Probability I
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Introduction to Probability
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§ 2.3

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Practice solution

Practice

A firm wants to know if can lower the prescribed 40hr work week and maintain the same productivity level from its employees.

- Each employee is asked to flip a fair coin,
  - If head ($\mathtt{H}$), answer the question "Do you carpool to work?"
  - If tail ($\mathtt{T}$), answer the question "Have you worked less than 40hr weeks in the last month?"
- Out of 8000 responses, 4820 answered "YES" (assuming honesty)
- The company knows that 35% of its employees carpool to work.
- What is the probability that an employee (chosen at random) worked less than a 40hr week in the last month?

Solution

$$\begin{aligned} P(\text{yes}) &= \mathbb{P}(\text{carpool} \mid \mathtt{H})\mathbb{P}(\mathtt{H}) + \mathbb{P}(\text{less work} \mid \mathtt{T})\mathbb{P}(\mathtt{T}) \\ &= \mathbb{P}(\text{carpool})P(\mathtt{H}) + \mathbb{P}(\text{less work})\mathbb{P}(\mathtt{T}) \\ &= 0.35 \times 0.5 + \mathbb{P}(\text{less work}) \times 0.5, \end{aligned}$$

$$\text{Hence, } P(\text{less work}) = \frac{(4820/8000 - 0.35 \times 0.5)}{0.5} = 0.855$$

Recap

Bayes' rule[1]

- $$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c)}$$
- for $B_1, \ldots, B_n$ a partition of Ω
$$\mathbb{P}(B_k \mid A) = \frac{\mathbb{P}(A \mid B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}$$

Independence

- A and B are independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

[1] Provided that the conditional probabilities are well defined, i.e., that the conditioning event has a non-zero prob.

Outline

Independence

Independent random variables

Independence

Example
Consider an urn with 4 red and 7 green balls.

Sample in order two balls and define the events

$$A = \{\text{first ball is red}\} \quad B = \{\text{second ball is green}\}$$

1. If the sampling is with replacement, are A and B independent?
2. If the sampling is without replacement, are A and B independent?

Solution

- (Intuition)
  - If you replace the balls, the sampling restart, and the events should be independent.
  - If you do not replace the ball, the second sampling will be affected by which ball you got first.
- (with replacement) In this case,

  $$\#\Omega = 11^2, \quad \#A = 4 \cdot 11, \quad \#B = 11 \cdot 7 \quad \#A \cap B = 4 \cdot 7$$

  So

  $$\mathbb{P}(A \cap B) = \frac{4 \cdot 7}{11^2} \quad \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{4 \cdot 11}{11^2} \frac{7 \cdot 11}{11^2}$$

  So $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, the events are independent

Independence

Example

Consider an urn with 4 red and 7 green balls.

Sample in order two balls and define the events

$$A = \{\text{first ball is red}\} \quad B = \{\text{second ball is green}\}$$

1. If the sampling is with replacement, are A and B independent?
2. If the sampling is without replacement, are A and B independent?

Solution

- (without replacement) In this case,

  $\#\Omega = 11{\cdot}9, \quad \#A = 4{\cdot}10, \quad \#B = \#A{\cap}B + \#A^c{\cap}B = 4{\cdot}7 + 7{\cdot}6 = 70 \quad \#A{\cap}B = 4{\cdot}7$

  So
  $$\mathbb{P}(A \cap B) = \frac{4 \cdot 7}{11 \cdot 10} = \frac{28}{110} \quad \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{4 \cdot 10}{11 \cdot 10} \frac{70}{11 \cdot 10} = \frac{28}{121}$$
  So $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$, the events are not independent

Independence

Proposition

Suppose A and B are independent. Then

1. *A and B^c are independent*
2. *A^c and B are independent*
3. *A^c and B^c are independent*

Proof

- Again intuitively it makes sense, if A and B are independent not A and B should be independent too. *We still need a rigorous proof.*
- If A and B are independent, then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- Now since $\mathbb{P}(B) = \mathbb{P}(A^c \cap B) + \mathbb{P}(A \cap B)$,

  $\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(B)(1 - \mathbb{P}(A)) = \mathbb{P}(B)\mathbb{P}(A^c)$

  So B and A^c are independent
- The proofs of the other statements follow the same way.

Independence

Exercise

Assume A and B independent and s.t. $\mathbb{P}(A) = 1/3$, $\mathbb{P}(B) = 1/4$.

Find the prob. that exactly one is true.

Solution The event is $AB^c \cup A^cB$. Note that AB^c and A^cB are disjoint. We have

$$\mathbb{P}(AB^c \cup A^cB) = \mathbb{P}(AB^c) + \mathbb{P}(A^cB) = \mathbb{P}(A)\mathbb{P}(B^c) + \mathbb{P}(A^c)\mathbb{P}(B) = \frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{1}{4} = \frac{5}{12}$$

Mutual independence

Definition

Events $A_1, \ldots, A_n$ are ***independent*** *(or* ***mutually independent)*** *if for any collection $A_{i_1}, \ldots, A_{i_k}$ with $2 \leq k \leq n$, $1 \leq i_1 < \ldots < i_k \leq n$,*

$$\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \ldots \mathbb{P}(A_{i_k})$$

Events $A_1, \ldots, A_n$ are ***pairwise independent*** *if for any $i \neq j$,*

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$$

Note:

- Clearly mutual independence $\implies$ pairwise independence
- But **the reverse is false**
- Mutual independence requires that **any** collection of the variables satisfy the factorization.

Mutual independence

Exercise

Flip a fair coin three times. Let:

$$A = \{\text{exactly one tails in the first two flips}\}$$

$$B = \{\text{exactly one tails in the last two flips}\}$$

$$C = \{\text{exaxctly one tails in the first and last flips}\}$$

1. Are A, B, C pairwise independent ?
2. Are A, B, C mutually independent ?

Solution

- We have

$$\begin{aligned} A &= \{(H,T,H),(H,T,T),(T,H,H),(T,H,T)\}, \\ B &= \{(H,T,H),(T,T,H),(H,H,T),(T,H,T)\}, \\ C &= \{(H,T,T),(T,H,H),(T,T,H),(H,H,T)\} \end{aligned}$$

- So

$$\begin{aligned} A \cap B &= \{(H,T,H),(T,H,T)\}, \\ A \cap C &= \{(H,T,T),(T,H,H)\}, \\ B \cap C &= \{(T,T,H),(H,H,T)\} \end{aligned}$$

Mutual independence

Exercise

A $= \{1$ T in flips 1&2 $\}$, B $= \{1$ T in flips 2&3$\}$, C $= \{1$ T in the flips 1&3 $\}$

Are A, B, C pairwise independent ? Are A, B, C mutually independent ?

Solution

- $\#A = \#B = \#C = 4$, and $\#AB = \#AC = \#BC = 2$
- Therefore
$$\mathbb{P}(A \cap B) = \frac{2}{2^3}, \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{4}{2^3} \cdot \frac{4}{2^3}, \text{ so } \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$
same for $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$, $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$,
- Therefore they are pairwise independent
- Yet $A \cap B \cap C = \emptyset$ so
$$\mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) > 0$$
They are not mutually independent.

Mutual Independence

Example
Flip a fair coin three times. Consider

$$G_i = \{\text{the ith flip is tails}\}$$

Are G_1, G_2, G_3 independent, i.e., mutually independent ?

Solution

- $G_1 = \{(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$, so $\mathbb{P}(G_1) = 4/2^3 = 1/2$ same for $\mathbb{P}(G_2), \mathbb{P}(G_3)$.
- $G_1 \cap G_2 = \{(T,T,H),(T,T,T)\}$ so $\mathbb{P}(G_1 \cap G_2) = 2/2^3 = \mathbb{P}(G_1)\mathbb{P}(G_2)$ same for $\mathbb{P}(G_2 \cap G_3) = \mathbb{P}(G_2)\mathbb{P}(G_3)$ and $\mathbb{P}(G_1 \cap G_3) = \mathbb{P}(G_1)\mathbb{P}(G_3)$
- Finally $\mathbb{P}(G_1 \cap G_2 \cap G_3) = 1/2^3 = \mathbb{P}(G_1)\mathbb{P}(G_2)\mathbb{P}(G_3)$
- So they are independent.

Mutual independence

The following prop. simply generalizes the fact that if A and B are independent then their complements are also independent.

Proposition

Assume $A_1, \ldots A_n$ are mutually independent. Then for any collection $A_{i_1}, \ldots, A_{i_k}$ with $2 \leq k \leq n$, $1 \leq i_1 < \ldots < i_k \leq n$,

$$\mathbb{P}(A^*_{i_1} \cap \ldots \cap A^*_{i_k}) = \mathbb{P}(A^*_{i_1}) \ldots \mathbb{P}(A^*_{i_k})$$

with A^*_i is either A_i or A^c_{ix}.

Outline

Independence

Independent random variables

Independent random variables

Recall that on probability space $(\Omega, \mathcal{F}, P)$,

A random variable X is a function from Ω to $\mathbb{R}$.

And a **discrete random variable** X is a function from Ω to a countable (or finite) set of values $\{k_1, k_2, \dots\}$.

Definition

Random variables $X_1, \dots, X_n$ on a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ are independent if

$$\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n) = \mathbb{P}(X_1 \in B_1) \dots \mathbb{P}(X_n \in B_n)$$

for any[2] subsets $B_1, \dots, B_n \subseteq \mathbb{R}$

Note:

This means that the distribution of multiple independent r.v.s can be **factorized** in the distributions of each r.v.

[2] The right definition requires the subsets must be Borel sets i.e. intersections/unions of intervals.

Practice next lecture

Practice

Chose a number uniformly at random on $[0, 1]$. Consider the events

$$A = [0, 1/2] \quad B = [1/3, 2/3], \quad C = [11/24, 17/24]$$

Are A, B, C independent (i.e. mutually independent)?