Differential Calculus

SOC 512 & CSSS 505

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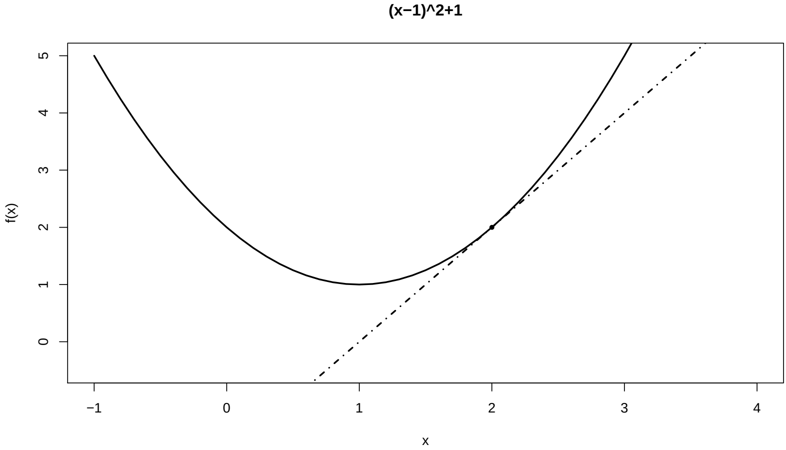
January 21, 2021

Outline

- Differentiation of functions
  - Defining the derivative
  - Basic differentiation rules
  - Second, third, etc... derivatives
- Critical points of functions
  - What is a critical point?
  - Maximum, minimum, and using the second derivative to tell the difference
- Taylor Series

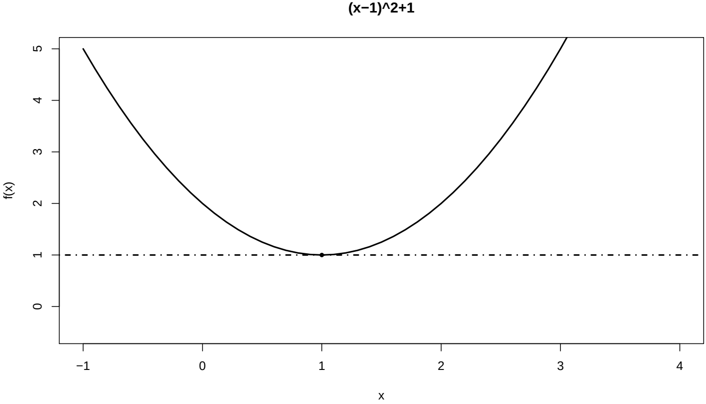
What is the derivative?

The derivative can be thought of as the slope of the line tangent to $f(x)$ at the point x. (Skims the curve, touching only at the point x).



What is the derivative?

In statistics we are often interested in derivatives to help us find the values that maximize (or minimize) functions. We will be particularly interested in the values x such that the slope of the derivative is zero.



Finding the Derivative

The derivative of a function $f(x)$ is the instantaneous rate at which the function is changing at x. Why would be interested in finding the derivative of a function?

- growth rate of a population relative to change in time
- change in distance relative to a change in time
- marginal revenue - change in amount of money from item sales relative to change in demand for the items

Think of it as finding the 'slope' of a function at specific point. To find the average rate of change over an interval $[a, b]$, we look at the change in $f(x)$ over the length of the interval.

$$\frac{f(b) - f(a)}{b - a}$$

Finding the Derivative

So if we want to find the rate of change at a value x, we find the average rate of change over a very small interval (usually of length δ).

$$\frac{f(x+\delta)-f(x)}{\delta}$$

We look at what happens when δ becomes very very small, i.e. when the interval essentially just becomes the point x.

The derivative of $f(x)$ at x is then:

$$lim_{\delta \to 0} \frac{f(x+\delta)-f(x)}{\delta}$$

It is denoted by $\frac{d}{dx}f(x)$ or $f'(x)$.

Differentiation with Limits

Given an $f(x)$, how do we find the derivative $f'(x)$?

$$f'(x) = lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

to start, let's write out the algebra and then take the limit.
Example: $f(x) = mx + b$

$$\begin{aligned}
lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} &= lim_{\delta \to 0} \frac{m(x+\delta) + b - (mx+b)}{\delta} \\
&= lim_{\delta \to 0} \frac{mx - mx + m\delta + b - b}{\delta} \\
&= lim_{\delta \to 0} \frac{m\delta}{\delta} \\
&= lim_{\delta \to 0} m = m
\end{aligned}$$

Differentiation with Limits

Example: $f(x) = ax^2$

$$\begin{aligned}
lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} &= lim_{\delta \to 0} \frac{a(x+\delta)^2 - ax^2}{\delta} \\
&= lim_{\delta \to 0} \frac{a(x^2 + 2x\delta + \delta^2) - ax^2}{\delta} \\
&= lim_{\delta \to 0} \frac{2ax\delta + a\delta^2}{\delta} \\
&= lim_{\delta \to 0} 2ax + a\delta \\
&= 2ax
\end{aligned}$$

Differentiation Rules

Derivative of a constant:

$$f(x) = a; \;\; f'(x) = 0$$

Derivative of a power:

$$f(x) = ax^n; \;\; f'(x) = n \cdot a \cdot x^{n-1}$$

Derivative of an exponential:

$$f(x) = e^x; \;\; f'(x) = e^x$$

Examples:

- $f(x) = x^4; f'(x) = 4x^3$
- $f(x) = 3x^7; f'(x) = 21x^6$
- $f(x) = 3e^x; f'(x) = 3e^x$

Differentiation rules

logs and trigonometric functions

Derivative of an Logarithmic Function:

$$f(x) = log(x); \;\; f'(x) = 1/x$$

Derivative of a Trigonometric Functions:

$$f(x) = sin(x); \;\; f'(x) = cos(x) \quad \& \quad f(x) = cos(x); \;\; f'(x) = -sin(x)$$

Derivative of a Sum of Functions:

$$f(x) = g(x) + h(x); \;\; f'(x) = g'(x) + h'(x)$$

Examples:

- $f(x) = 2log(x); f'(x) = 2/x$
- $f(x) = sin(x) + e^x; f'(x) = cos(x) + e^x$
- $f(x) = 3x^2 + 4x; f'(x) = 6x + 4$

Derivative of the product of two functions:

$$f(x) = g(x) \cdot h(x); \;\; f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$$

Examples:

- $f(x) = x^2 \cdot e^x; f'(x) = 2x \cdot e^x + x^2 \cdot e^x$
- $f(x) = 3x \cdot log(x); f'(x) = 3 \cdot log(x) + 3x \cdot 1/x = 3log(x) + 3$
- $f(x) = 3x \cdot log(x) + 2x; f'(x) = 3log(x) + 3 + 2 = 3log(x) + 5$

Derivative of the division of two functions:

$$f(x) = \frac{g(x)}{h(x)}; \;\; f'(x) = \frac{g'(x) \cdot h(x) - h'(x) \cdot g(x)}{h(x)^2}$$

Examples:

- $f(x) = \frac{x^2}{e^x}; f'(x) = \frac{2x \cdot e^x - x^2 \cdot e^x}{e^{2x}} = \frac{2x - x^2}{e^x}$
- $f(x) = \frac{3x}{log(x)}; f'(x) = \frac{3 \cdot log(x) - 3x \cdot 1/x}{log(x)^2} = \frac{3log(x) - 3}{log(x)^2}$

Differentiation rules

Chain Rule

Derivative of a function within a function:

$$f(x) = g(h(x)); \;\; f'(x) = g'(h(x)) \cdot h'(x)$$

Examples:

- $f(x) = e^{3x}; g(h) = e^h, h(x) = 3x$
  $g'(h) = e^h, h'(x) = 3 \Rightarrow f'(x) = g'(3x)h'(x) = 3e^{3x}$
- $f(x) = log(1-x); g(h) = log(h), h(x) = 1-x$
  $g'(h) = 1/h, h'(x) = -1 \Rightarrow f'(x) = g'(1-x)h'(x) = \frac{-1}{(1-x)}$
- $f(x) = (2x+2)^2; g(h) = h^2, h(x) = 2x+2$
  $g'(h) = 2h, h'(x) = 2 \Rightarrow f'(x) = g'(2x+2)h'(x) =$
  $2(2x+2) \cdot 2 = 4(2x+2)$

Derivative of the division of two functions:

$$f(x) = \frac{g(x)}{h(x)} = g(x) \cdot h^{-1}(x)$$

$$f'(x) = g'(x) \cdot h^{-1}(x) + (-1) \cdot h^{-2}(x) \cdot h'(x) \cdot g(x)$$

Example:

- $f(x) = \frac{x^2}{e^x} = x^2 \cdot e^{-x}$;

$$\begin{aligned} f'(x) &= 2x \cdot e^{-x} + (-1) \cdot e^{-2x} \cdot e^x \cdot x^2 \\ &= \frac{2x}{e^x} - \frac{e^x \cdot x^2}{e^{2x}} \\ &= \frac{2x}{e^x} - \frac{x^2}{e^x} \\ &= \frac{2x - x^2}{e^x} \end{aligned}$$

We can combine many rules:

$$f(x) = 3x(2x+1)^4$$

will require the product rule and the chain rule, where $g(x) = 3x, k(x) = 2x + 1$, and $h(k) = k^4$.

$$\begin{aligned} f'(x) &= g'(x) \cdot h(k) + g(x) \cdot h'(k) \cdot k'(x) \\ &= 3 \cdot (2x+1)^4 + 3x \cdot 4(2x+1)^3 \cdot 2 \\ &= 3(2x+1)^4 + 24x(2x+1)^3 \end{aligned}$$

Second & Third Derivatives

We can find the second derivative by taking the derivative of the derivative. The third derivative is found by taking the derivative of the second derivative and so on.

The second derivative is the rate of change of the first derivative and can be written as $f''(x)$ or $\frac{d^2}{dx^2}f(x)$.
Example:

$$\begin{aligned} f(x) &= log(4x) \\ f'(x) &= \frac{1}{4x} \cdot 4 = 1/x = x^{-1} \\ f''(x) &= (-1) \cdot x^{-2} = -x^{-2} \\ f'''(x) &= (-2) \cdot -1x^{-3} = 2x^{-3} \end{aligned}$$

Differentiation rules

distance, velocity, acceleration

Let's take d=distance, v=velocity, a=acceleration. You may remember from physics, the distance travel after time t

$$d(t) = \frac{a}{2}t^2$$

The velocity at any time t is the instantaneous rate of change of the distance, $v(t) = d'(t)$:

$$v(t) = 2 \cdot \frac{a}{2}t = at$$

The acceleration at any time t is the instantaneous rate of change of the velocity, $a(t) = v'(t) = d''(t)$:

$$a(t) = a$$

Critical Values

Critical value occurs when the behavior of a function changes.

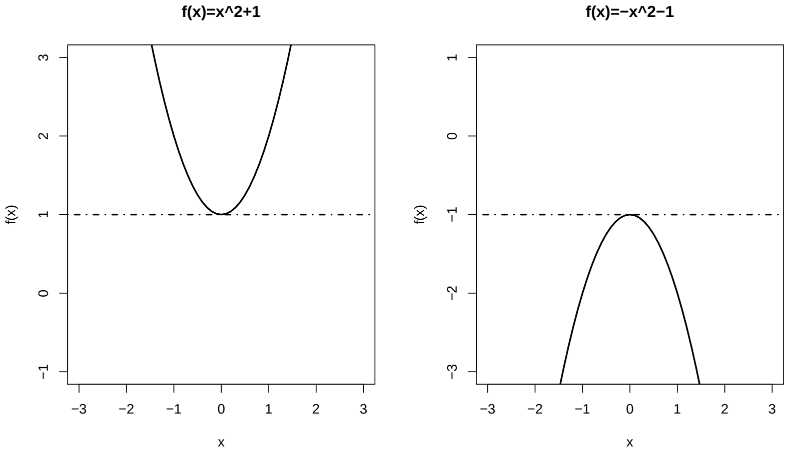
A *maximum* occurs when a function stops increasing and starts to decrease.

A *minimum* occurs when a function stops decreasing and starts increasing.

If we are modeling a population or a behavior, knowing when the maximum or minimum occurs is very useful. In statistics, finding the maximum helps us find values of interest (Maximum Likelihood Estimates).

Mathematically, critical points are defined as the points where the derivative is zero. As a function passes through a critical point, the derivative goes from positive to negative (maximum) or negative to positive (minimum).

Critical Values



Critical Values

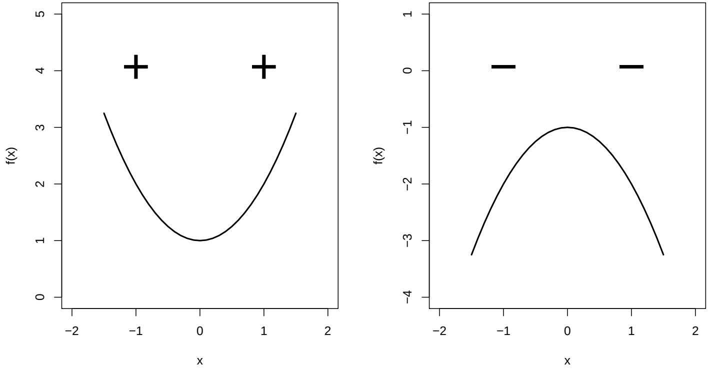
We can us the first derivative to find the critical point by setting it equal to zero and then solving for x, the root. The goal is to find x such that $f'(x) = 0$.

However, as seen on the previous slide, the derivative is zero for maximums **and** minimums. How do we tell the difference?

We use the second derivative.

Critical Values

For the max, the derivative decreases from positive to negative, so the second derivative will be negative. For the min, the derivative increases from negative to positive, so the second derivative will be positive.

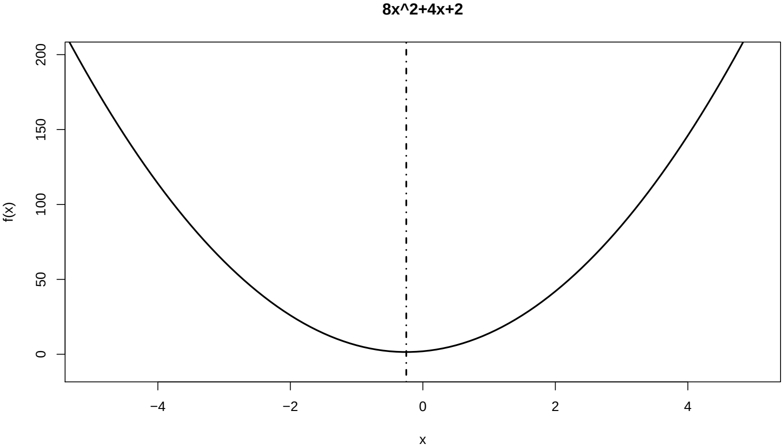


$$\begin{aligned} f(x) &= 8x^2 + 4x + 2 \\ f'(x) &= 16x + 4 \\ 0 &= 16x + 4 \Rightarrow 16x = -4 \Rightarrow x = \frac{-1}{4} \\ f''(x) &= 16 \end{aligned}$$

The cricital value is at $x = -1/4$ and the second derivative is positive, so it is a minimum.

Critical Values

Examples



Critical Values

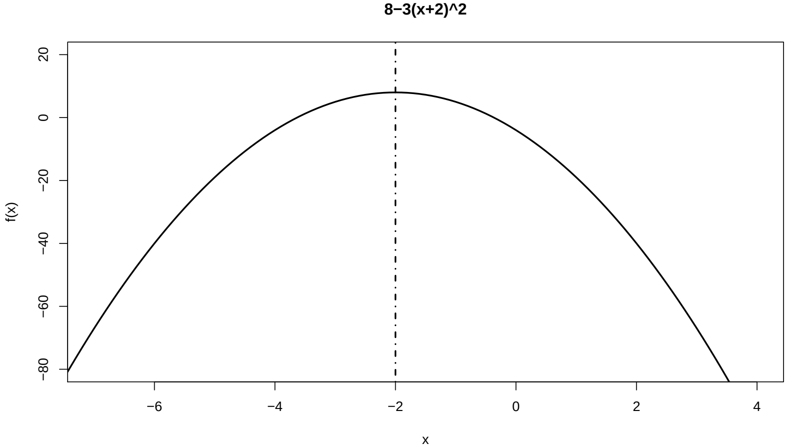
Examples

$$
\begin{aligned}
f(x) &= 8 - 3(x+2)^2 \\
f'(x) &= -6(x+2) \\
0 &= -6x - 12 \Rightarrow -6x = 12 \Rightarrow x = -2 \\
f''(x) &= -6
\end{aligned}
$$

The cricital value is at $x = -2$ and the second derivative is negative, so it is a maximum.

Critical Values

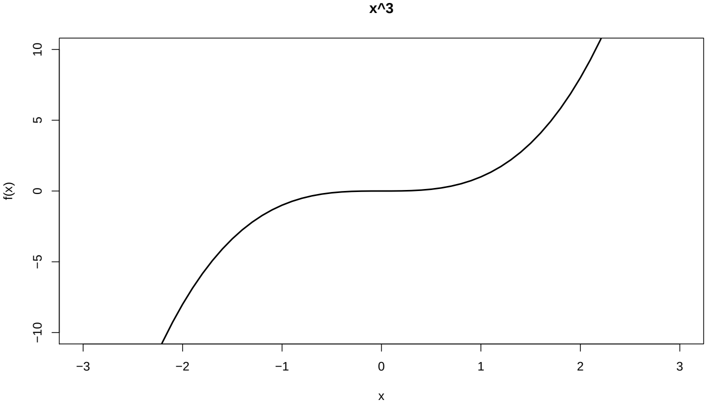
Examples



Critical Values

Saddle Points

If $f''(x) = 0$, then you have a *saddle point*. This is a critical point where the overall behavior of your function does not change.
For example: $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$. At $x = 0$, we have $f'(x) = f''(x) = 0$.

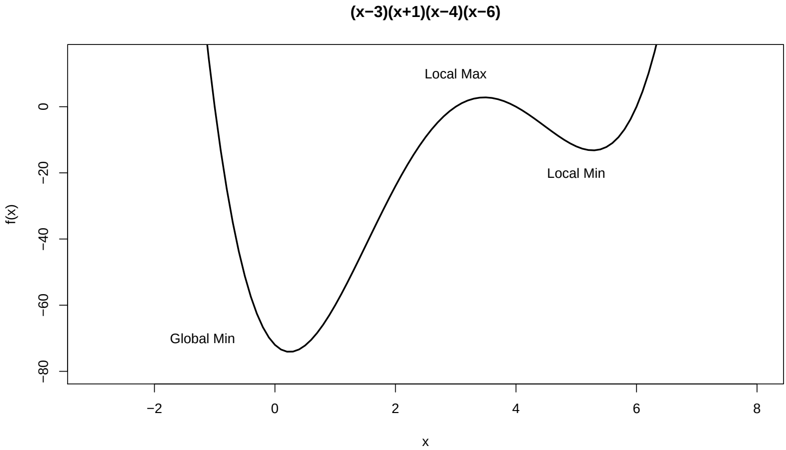


Some functions have more than one maximum or minimum.

We call the largest maximum or the lowest minimum the *global* critical point. All others are referred to as *local* critical points. When looking, ideally we want to find the global maximum or minimum.

Critical Values

Global vs. Local



Taylor Series

We can write any function as a sums of its derivatives. This trick is often used when we want to find an approximation for $f(x)$ around a given value a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

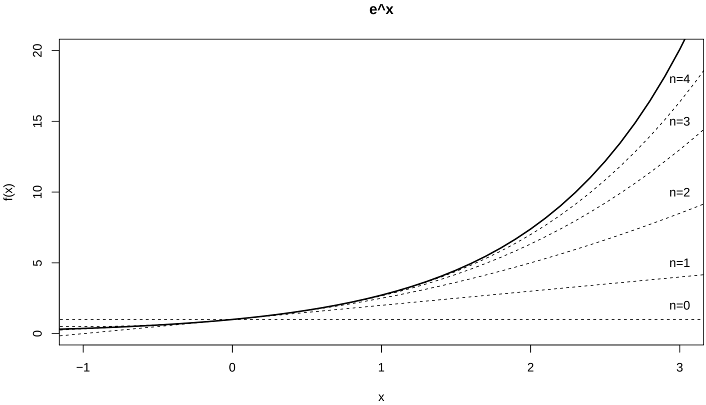
We can't write out the entire sum, so we write out a few terms and then put the rest into a bounded reminder. This is often used if we want to find values near a.

Taylor Series

Example

If we expand $f(x) = e^x$ around the value $a = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{e^0}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



The End

Questions?