Inclusion Exclusion Problems examples

MATH/STAT 394: Probability I
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Introduction to Probability
D. Anderson, T.Seppäläinen, B. Valkó

§ 1.4

Aaron Osgood-Zimmerman

Department of Statistics

Practice solution

Practice

John and Lucy take turns rolling two dice.

If John rolls a pair larger than or equal to 10 he wins and the game stops.

If Lucy rolls a pair strictly smaller than 10, she wins and the game stops.

They keep rolling in turn until one of them wins. Suppose John rolls first.

What is the probability that Lucy wins?

Solution

- Define the events

$$B = \{\text{Lucy wins}\} \quad \text{and} \quad B_k = \{\text{Lucy wins on her kth roll}\}.$$

  Then $B = \cup_{k=1}^{+\infty} B_k$, the B_k are mutually exclusive so $\mathbb{P}(B) = \sum_{k=1}^{+\infty} \mathbb{P}(B_k)$.
- Lucy wins on her kth roll with prob. $1 - 1/6 = 5/6$ and John fails on his kth roll with prob. $5/6$
- We have

$$\mathbb{P}(B_k) = \frac{(5 \cdot 1)^{k-1} 5 \cdot 5}{(6 \cdot 6)^k} = (5/36)^{k-1}(25/36)$$

- And so

$$\mathbb{P}(B) = \sum_{k=1}^{+\infty} \mathbb{P}(B_k) = \frac{25}{36} \frac{1}{1 - 5/36} = \frac{25}{31} \approx 0.81$$

Outline

Birthday problem

Exercise

In a room we have k people, each of which can have any day as his/her birthday (equally likely). What is the chance that there are two in the room who have the same birthday?

Suppose Charlie is in the room — what is the chance that there is someone else with the same birthday as Charlie?

In each case, how large does k need to be before the probability is larger than 50%?

Solution Let the event be A, and $A^c = \{\text{no two have the same birthday}\}$.

$$\#\Omega = n^k, \quad \#A^c = (365)_k.$$

Then, by equally likely outcomes,

$$P(A^c) = \frac{\#A^c}{\#\Omega} = \frac{(365)_k}{n^k} = \frac{365(365-1)(365-2)\ldots(365-(k-1))}{365^k}$$
$$= \frac{\prod_{i=0}^{k-1}(365-i)}{365^k}$$

Birthday problem

Exercise

Suppose Charlie is in the room — what is the chance that there is someone else with the same birthday as Charlie?

Solution Let the event be B and
$B^c = \{$every of the $(k-1)$ people has a birthday different from Charlie's$\}$.

$$\#\Omega = 365^{k-1}, \quad \#B^c = (365-1)^{k-1},$$

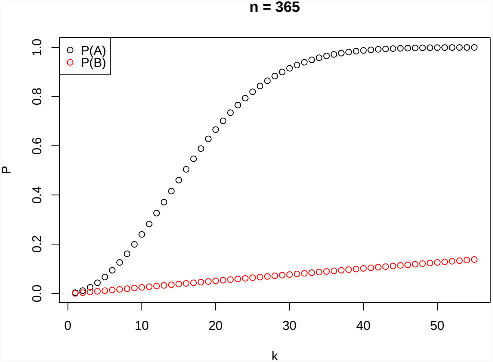
and

$$P(B^c) = \frac{\#B^c}{\#\Omega} = \frac{(365-1)^{k-1}}{365^{k-1}} = (1 - 1/365)^{k-1}.$$

If you plug in values of k, you'll find the first part surpasses 50% probability at $k = 23$ and the second part at $k = 253$

Birthday problem

$A = \{$there are two in the room who have the same birthday$\}$,
$B = \{$someone else with the same birthday as Charlie$\}$.



Some intuition

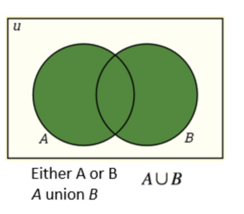
- For any particular pair of people, there is a probability of 1/365 that they share a birthday.
- Thus, if there are approximately 365 pairs of people, there should be a good probability of success.
- For a group of size k, there are $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs.
- So we could expect that k should loosely satisfy $\frac{k(k-1)}{2} \times \frac{1}{365} = .5$
- which gives a k of around 19.6

Outline

Inclusion-exclusion

Consider two events, A and B, which can intersect.

$$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).}$$



Proof Let $C_1 = A \setminus B$, $C_2 = A \cap B$, $C_3 = B \setminus A$. By dfn, C_1, C_2, C_3 are disjoint. Then

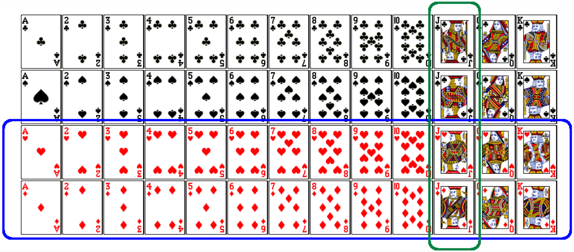
$$\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(C_1 \cup C_2 \cup C_3) \\
&= \mathbb{P}(C_1) + \mathbb{P}(C_2) + \mathbb{P}(C_3), \\
\mathbb{P}(A) &= \mathbb{P}(C_1 \cup C_2) = \mathbb{P}(C_1) + \mathbb{P}(C_2), \\
\mathbb{P}(B) &= \mathbb{P}(C_2 \cup C_3) = \mathbb{P}(C_2) + \mathbb{P}(C_3), \\
\mathbb{P}(A \cap B) &= \mathbb{P}(C_2),
\end{aligned}$$

and the formula follows.

Figure from https://www.onlinemathlearning.com/venn-diagrams.html

Example

From a shuffled full deck, $\mathbb{P}(\text{drawing a jack or a red card}) = ?$



Solution

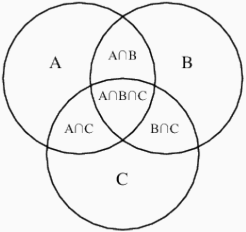
$$\begin{aligned}\mathbb{P}(\text{jack or red}) &= \mathbb{P}(\text{jack}) + \mathbb{P}(\text{red}) - \mathbb{P}(\text{red jack}) \\ &= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}\end{aligned}$$

Figure from http://www.milefoot.com/math/discrete/counting/cardfreq.htm.

Inclusion-exclusion

- For three events A, B, C,

$$\boxed{\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ &\quad - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).\end{aligned}}$$



Exercise

There are 20 people: 10 Norwegians, 7 Swedes and 3 Finns. Randomly choose a committee of 5 people. Assuming outcomes are equally likely,

$$\mathbb{P}(\text{at least one nation is not represented in the committee}) = ?$$

Solution Since ordering does not matter,

$$\Omega = \{\text{5-subsets of } 1, \dots, 20\}, \quad \#\Omega = \binom{20}{5}.$$

Let $F = \{\text{Finns are not represented}\}$, $S = \{\text{Swedes are not represented}\}$ and $N = \{\text{Norwegians are not represented}\}$. We look for $\mathbb{P}(F \cup S \cup N)$.

- What is the probability of F? If Finns are not represented, then we choose from $10 + 7 = 17$ people. Since outcomes are equally likely,

$$\mathbb{P}(F) = \frac{\#F}{\#\Omega} = \frac{\binom{17}{5}}{\binom{20}{5}}.$$

Similarly, $\mathbb{P}(S) = \frac{\binom{13}{5}}{\binom{20}{5}}$ and $\mathbb{P}(N) = \frac{\binom{10}{5}}{\binom{20}{5}}$.

Exercise

There are 20 people: 10 Norwegians, 7 Swedes and 3 Finns. Randomly choose a committee of 5 people. Assuming outcomes are equally likely,

$$\mathbb{P}(\text{at least one nation is not represented in the committee}) = ?$$

- Also,

$$\mathbb{P}(S \cap F) = \frac{\binom{10}{5}}{\binom{20}{5}},$$

because $S \cap F$ means that we choose only from Norwegians, and there are 10 of them.

Similarly, we can compute $\mathbb{P}(F \cap N) = \frac{\binom{7}{5}}{\binom{20}{5}}$, and $\mathbb{P}(S \cap N) = 0$.

- Finally, $\mathbb{P}(S \cap N \cap F) = 0$. Apply the inclusion-exclusion formula:

$$\mathbb{P}(F \cup S \cup N) = \frac{\binom{13}{5}}{\binom{20}{5}} + \frac{\binom{17}{5}}{\binom{20}{5}} + \frac{\binom{10}{5}}{\binom{20}{5}} - 0 - \frac{\binom{10}{5}}{\binom{20}{5}} - \frac{\binom{7}{5}}{\binom{20}{5}} + 0.$$

Inclusion-exclusion

General formula

The general formula for n events is

$$\begin{aligned}\mathbb{P}(A_1 \cup \ldots \cup A_n) = \mathbb{P}(A_1) &+ \ldots + \mathbb{P}(A_n) \\ &-\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \ldots - \mathbb{P}(A_{n-1} \cap A_n) \\ &+\mathbb{P}(A_1 \cap A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_4) + \ldots + \mathbb{P}(A_{n-2} \cap A_{n-1} \cap A_n) \\ &- \ldots + (-1)^{n-1}\mathbb{P}(A_1 \cap \ldots \cap A_n).\end{aligned}$$

We can also write it as

$$\begin{aligned}\mathbb{P}(A_1 \cup \ldots \cup A_n) &= \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \ldots \\ &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \ldots < i_k \le n} \mathbb{P}(A_{i_1} \cap \ldots A_{i_k})\end{aligned}$$

Inclusion Exclusion

Example

n people arrive at a show and leave their hats in the cloakroom. The hats are mixed completely at random and each person gets a random hat when leaving.

1. What is the prob. that no one gets their own hat?
2. How does this prob. behave as $n \to +\infty$

- There $n!$ possible assignment of the hats (number of possible permutations)
- Define $A_i = \{\text{person i gets his/her own hat}\}$ for $1 \leq i \leq n$
- The prob. of interest is

$$\mathbb{P}(\cap_{i=1}^{n} A_i^c) = 1 - \mathbb{P}(\cup_{i=1}^{n} A_i)$$

- Now we want to use the inclusion-exclusion formula, we have

$$\mathbb{P}(A_{i_1} \cap \ldots A_{i_k}) = \mathbb{P}(\text{individuals } i_1, \ldots, i_k \text{ get their own hats}) = \frac{(n-k)!}{n!}$$

  Why? If we assign k hats correctly, there are $(n-k)!$ ways to distribute the remaining hats
- Thus, since there are $\binom{n}{k}$ terms in the sum (choose k indexes among $\{1, \ldots n\}$):

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathbb{P}(A_{i_1} \cap \ldots A_{i_k}) = \binom{n}{k} \frac{(n-k)!}{n!} = \frac{1}{k!}$$

Inclusion Exclusion

Example

n people arrive at a show and leave their hats in the cloakroom. The hats are mixed completely at random and each person gets a random hat when leaving.

1. What is the prob. that no one gets his/her own hat?
2. How does this prob. behave as $n \to +\infty$

- Therefore we get

$$\mathbb{P}(\cup_{i=1}^{n} A_i) = 1 - \frac{1}{2!} + \frac{1}{3!} + \ldots + (-1)^{n+1}\frac{1}{n!}$$

- So

$$\mathbb{P}(\cap_{i=1}^{n} A_i^c) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + (-1)^n \frac{1}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

- Recall that

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{n} \frac{x^k}{k!}$$

- So for $n \to +\infty$,

$$\lim_{n \to +\infty} \mathbb{P}(\cap_{i=1}^{n} A_i^c) = e^{-1}$$

Recap

Probability space $(\Omega, \mathcal{F}, P)$. Three axioms of probability P

1. $0 \leq P(A) \leq 1$, $A \in \mathcal{F}$
2. $P(\Omega) = 1$
3. $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, if $A_i \cap A_j = \emptyset$ for $i \neq j$

Properties

- $P(A^c) = 1 - P(A)$
- $P(B \cap A^c) = P(B) - P(A)$, if $A \subseteq B$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) =$
  $P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Practice next lecture

Practice

Randomly pair 4 keys {a, b, c, d} with 3 locks {a, b, c}.
What is $\mathbb{P}$(at least one match)?