Expectation, Median, Quantiles

MATH/STAT 394: Probability I
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Introduction to Probability
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§ 3.3

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Practice solution

Practice

The time (in min.) to wait for my bus can be modeled as an exponential r.v.

I know that the average time I need to wait for the bus is 10 min.

What is the prob. that I would wait more than 15min?

Hint:

1. Compute $\mathbb{E}[X]$ for $X \sim \text{Exp}(\lambda)$.
2. Deduce how you could know λ if you only knew the expected time.

Solution

- $$\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx = \int_{0}^{+\infty} x\lambda e^{-\lambda x}dx$$
  $$= \left[-xe^{-\lambda x}\right]_0^{+\infty} - \int_0^{+\infty} -e^{-\lambda x}dx = 0 + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{+\infty} = \frac{1}{\lambda}$$

  Take-away: If you know that a r.v. is exponential and you don't know the parameter λ, you can get it from the value of the mean (a.k.a. the expectation)

Practice solution

Practice

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Hint:

1. Compute $\mathbb{E}[X]$ for $X \sim \text{Exp}(\lambda)$.
2. Deduce how you could know λ if you only knew the expected time.

Solution (continued)

- We know that $X \sim \text{Exp}(\lambda)$ with lambda a priori unknown and $\mathbb{E}[X] = 10$
- Since $\mathbb{E}[X] = \frac{1}{\lambda}$ so we deduce that $\lambda = 1/10$
- Therefore we can compute

$$\mathbb{P}(X \geq 15) = e^{-\lambda \cdot 15} = e^{-15/10} \approx 0.22$$

Outline

Infinite expectations

Exercise

Consider the following gamble. You flip a fair coin.

- If it is heads, you get 2 dollars and the game ends,
- If it is tails, the prize is doubled and you flip the coin again

Continue like that: every tails double the prize and at the first heads you get the prize.

What is your expected gain?

Solution

- Denote $Y \sim \text{Geom}(1/2)$ the number of flips before getting a heads and X your gain
- Then
$$\mathbb{E}[X] = \sum_{k=1}^{+\infty} 2^k \mathbb{P}(Y = k) = \sum_{k=1}^{+\infty} 1 = +\infty$$
- So even if the game surely ends (because $\mathbb{P}(Y \leq +\infty) = 1$), your expected gain is infinite.
- Called the St Petersburg Paradox first considered by N. Bernoulli

Undefined expectation

Exercise

John and Lucy flip a fair coin until seeing a tail.

If the number k of flips is odd, Lucy pays 2^k dollars to John, if k is even, John pays 2^k dollars to Lucy.

Denote by X the gain of John. What is $\mathbb{E}[X]$?

Solution

- Let $Y \sim \text{Geom}(1/2)$, s.t. $\mathbb{P}(Y = k) = 2^{-k}$
- We have that $\mathbb{P}(X = 2^k) = \mathbb{P}(Y = k) = 2^{-k}$ for k odd, $\mathbb{P}(X = -2^k) = \mathbb{P}(Y = k) = 2^{-k}$ for k even
- Therefore if we want to sum up the average of the first n gains we get

$$\sum_{k=1}^{n}(-1)^k 2^k \mathbb{P}(Y = k) = \sum_{k=1}^{n}(-1)^k = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

- Therefore $\mathbb{E}[X] = \lim_{n\to+\infty} \sum_{k=1}^{n}(-1)^k 2^k \mathbb{P}(Y = k)$ is not defined.

Recap

Discrete r.v.

- $\mathbb{E}[X] = \sum_{k \in \mathcal{X}} kp(k)$
- If $X \sim \text{Ber}(p)$, $\mathbb{E}[X] = p$
- If $X \sim \text{Geom}(p)$, $\mathbb{E}[X] = \frac{1}{p}$
- If $X \sim \text{Bin}(n, p)$, $\mathbb{E}[X] = np$

Continuous r.v.

- $\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx$
- If $X \sim \text{Unif}([a, b])$, $\mathbb{E}[X] = \frac{a+b}{2}$
- If $X \sim \text{Exp}(\lambda)$, $\mathbb{E}[X] = \frac{1}{\lambda}$

Finite, infinite, undefined expectations

- An expectation can be finite, infinite (positive or negative) or undefined

Outline

Expectation of a function of a r.v.

Motivation

- Given a r.v. X, we can always define a new r.v. as $Y = g(X)$ through a given function g
- We may then want to know:
    - The p.d.f./p.m.f. of Y (this will be seen later)
    - The expectation of Y

Expectation of a function of a r.v.

Example
Roll a fair die and define the gain of a player as

$$W = \begin{cases} -1 & \text{if the roll is 1, 2 or 3} \\ 1 & \text{if the roll is 4} \\ 3 & \text{if the roll is 5 or 6} \end{cases}$$

What is $\mathbb{E}[W]$?
Solution

- Define X the face of the die and

$$g : \begin{cases} \{1, \dots, 6\} & \to \mathbb{R} \\ k & \to \begin{cases} -1 & \text{if } k \in \{1,2,3\} \\ 1 & \text{if } k = 4 \\ 3 & \text{if } k \in \{5,6\} \end{cases} \end{cases}$$

- Then $W = g(X)$ and

$$\begin{aligned}\mathbb{E}[W] &= -1 \cdot (\mathbb{P}(X{=}1) + \mathbb{P}(X{=}2) + \mathbb{P}(X{=}3)) + 1 \cdot \mathbb{P}(X{=}4) + 3 \cdot (\mathbb{P}(X{=}5) + \mathbb{P}(X{=}6)) \\ &= g(1)\mathbb{P}(X{=}1) + g(2)\mathbb{P}(X{=}2) + g(3)\mathbb{P}(X{=}3) + g(4)\mathbb{P}(X{=}4) + g(5)\mathbb{P}(X{=}5) + g(6)\mathbb{P}(X{=}6) \\ &= -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} = \frac{2}{3}\end{aligned}$$

Expectation of a function of a r.v.

Theorem
Let X be a r.v. that takes values in $\mathcal{X}$ and $g : \mathcal{X} \to \mathbb{R}$ be some function.

$$\mathbb{E}[g(X)] = \sum_{k \in \mathcal{X}} g(k)p(k) \qquad \text{if } X \text{ is discrete with p.m.f. } p$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx \qquad \text{if } X \text{ is continuous with p.d.f. } f$$

Proof For the discrete case,

$$\begin{aligned}
\mathbb{E}[g(X)] = \sum_y y\mathbb{P}(g(X) = y) &= \sum_y y \sum_{k:g(k)=y} \mathbb{P}(X = k) \\
&= \sum_y \sum_{k:g(k)=y} y\mathbb{P}(X = k) \\
&= \sum_y \sum_{k:g(k)=y} g(k)\mathbb{P}(X = k) \\
&= \sum_k g(k)\mathbb{P}(X = k)
\end{aligned}$$

For the continuous case, we take this result as granted

Expectation of a function of a r.v.

Example
A stick of length 1 is broken at a random location. Let X be the length of the longer piece.

What is $\mathbb{E}(X)$?

Solution

- Let U denote the random location where the stick is broken. Then $U \sim \text{Unif}[0,1]$.
- Since X is the bigger broken piece, we have $X = g(U)$, where

$$g(u) = \max(u, 1-u) = \begin{cases} 1-u, & \text{if } u < 1/2 \\ u, & \text{if } u \geq 1/2. \end{cases}$$

- Thus

$$\mathbb{E}[X] = \mathbb{E}[g(U)] = \int_0^{1/2} (1-u)du + \int_{1/2}^1 u du = \frac{3}{4}.$$

Outline

Infinite and undefined expectations

Expectation of a function of a r.v

Properties of the expectation

Linearity of the expectation

Proofs

Properties of the expectation

Motivation

- The expectation is a useful characterization to compute
- Yet it may be cumbersome to do the computations each time (e.g. binomial)
- Can we use some properties to simplify the computations?

Outline

Infinite and undefined expectations

Expectation of a function of a r.v

Properties of the expectation

Linearity of the expectation

Proofs

Linearity of the Expectation

As an application of the expectation of a function of a r.v. we have the following result

Theorem
Let X be a r.v. and $a, b \in \mathbb{R}$, then

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Proof (For the continuous case, the discrete case can be treated similarly)

$$\mathbb{E}[aX + b] = \int_{-\infty}^{+\infty} (ax + b)f(x)dx = a\int_{-\infty}^{+\infty} xf(x)dx + b\int_{-\infty}^{+\infty} f(x)dx = a\mathbb{E}[X] + b$$

Linearity of the expectation

More generally we have this very useful result

Theorem
Let X, Y be r.v. on the same sample space, then

$$\boxed{\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]}$$

Proof **Proof is detailed in STAT395, you can have a look in the supp. slides for the discrete case**

Corollary
Let $X_1, \ldots X_n$ be n r.v. defined on the prob. sample space and $g_1, \ldots g_n$ be n functions, we have that

$$\mathbb{E}[g_1(X_1)+\ldots+g_n(X_n)]=\mathbb{E}[g_1(X_1)]+\ldots+\mathbb{E}[g_n(X_n)]$$

Linearity of the expectation

Exercise

Using the previous theorem, compute the expectation of $X \sim \text{Bin}(n, p)$

Solution

- $X \sim \text{Bin}(n, p)$ can be described as the sum of n $Y_i \overset{i.i.d.}{\sim} \text{Ber}(p)$ i.e. $X = Y_1 + \ldots + Y_n$.
- Therefore using that $\mathbb{E}[Y_i] = p$ for any i,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = np$$

Practice next lecture

Practice

The annual maximum one-day rainfall can be modeled by a r.v. X with p.d.f.

$$f(x) = \begin{cases} \frac{2}{\pi}\frac{1}{x^2+1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

What is the expectation of X?

Outline

Linearity of the expectation*

Theorem
Let X, Y be r.v. on the same prob. space, then

$$\boxed{\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]}$$

Proof For the discrete case, summing over all the possible values of X and Y denoted resp. k, ℓ,

$$\begin{aligned}\mathbb{E}[X+Y] &= \sum_{k,\ell}(k+\ell)\mathbb{P}(X=k, Y=\ell) \\ &= \sum_k k \sum_\ell \mathbb{P}(X=k, Y=\ell) + \sum_\ell \ell \sum_k \mathbb{P}(X=k, Y=\ell)\end{aligned}$$

Now by def. the events $A_\ell = \{Y=\ell\}$ form a partition of the sample space Ω. Therefore

$$\sum_\ell \mathbb{P}(X=k, Y=\ell) = \sum_\ell \mathbb{P}(\{X=k\} \cap A_\ell) = \mathbb{P}(X=k) \qquad \text{(by the law of total prob.)}$$

So we get that

$$\mathbb{E}[X+Y] = \sum_k k\mathbb{P}(X=k) + \sum_\ell \ell\mathbb{P}(Y=\ell) = \mathbb{E}[X] + \mathbb{E}[Y]$$