Axioms of Probability

MATH/STAT 394: Probability I
Summer 2021 A Term

Introduction to Probability
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§1.1

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Logistics

- There is no freely available copy of the text
- Synchronous class time: Lecture + reviewing difficult problems.
- Scheduling office hours, please pick times on: https://www.when2meet.com/?12178993-7Kr1Q
- I'll try to make some extra credit problems on HWs, one with coding and one without
- Participation points: in-class polls, activity on Piazza
- Homework writeup expectations: Demonstrate you understand how/why you've arrived at your solution

Arrangements, Permutations, and Combinations

- Permutation means the arrangement/ordering of some things.
- Arrangement means an ordered selection of things.
- Combination means a selection of things (order invariant).

Example

Suppose we want to make a 3-digit number using the digits in $\Omega = \{1, 2, 3, 4\}$. To form this number we will make an arrangement using elements from Ω. We might make the number 231, a particular arrangement of the subset $A = \{1, 2, 3\} \subset \Omega$. A different permutation of A generates a different number, and a different arrangement, eg 321.

On the other hand, consider selecting 7 players out of a team of 20 ultimate frisbee players to start a game. The order of the selected players does not matter since the starting line will be the same. In order to form a different starting line, you would need to change out at least one player.

Practice solution

Practice

Given a classical deck of 52 cards, how many poker hands (set of 5 cards) are in the category three of a kind? (that is, no better than three of a kind, look on wikipedia if you do not know the possible categories of a poker hand)

Solution Two possible ways
(i) build an unordered hand
(ii) build an ordered hand and divide by 5! to remove the overcounting,
First way

1. Consider $\Omega = \{\{x_1, x_2, x_3, x_4, x_5\}$ with x_i a card from the deck$\}$.
2. Consider building all possible hands with 3 of a kind denote this subset A
3. to build A
   3.1 Choose the rank of the three of a kind (13 choices)
   3.2 Pick 3 cards out of the 4 possible suits for this rank ($\binom{4}{3}$ choices)
   3.3 Choose 2 other ranks for the two remaining cards ($\binom{12}{2}$ choices)
   3.4 Choose one of the suit for both of these cards (4 choices for each so 4^2 for both)
4. in total $\#A = 13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54912$

Practice solution

Practice

Given a classical deck of 52 cards, how many poker hands (set of 5 cards) are in the category three of a kind? (that is, no better than three of a kind, look on wikipedia if you do not know the possible categories of a poker hand)

Solution Now second way

1. Consider $\widetilde{\Omega} = \{(x_1, x_2, x_3, x_4, x_5)$ with x_i a card from the deck, all different$\}$.
2. Consider building all possible ordered hands with 3 of a kind denote this subset $\widetilde{A}$
3. to build $\widetilde{A}$
   3.1 First choose three slots for the three of a kind $\binom{5}{3}$ for this choice
   3.2 Assign 1 card to one of these slots (52 choices)
   3.3 Assign a second card to one of these slots with the same rank (3 choices)
   3.4 Assign a third card to the last slot with the same rank (2 choices)
   3.5 Assign one card different than the three others (48 choices)
   3.6 Assign one card different to all other ones (44 choices)
4. in total $\#\widetilde{A} = \binom{5}{3} \cdot 52 \cdot 3 \cdot 2 \cdot 48 \cdot 44$
5. divide by 5! to only consider the possible **sets**, i.e. the solution is $\#\widetilde{A}/5! = 54912$

Outline

Sample space

Definition

A sample space Ω is the set of all possible outcomes of an experiment.

Example

3 coin tosses	S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
One die roll	S = {1,2,3,4,5,6}
Sum of two rolls	S = {2,3,...,11,12}
Seconds waiting for bus	S = $[0, \infty)$
Number of attempts to win a lottery	S = $\{1, \ldots, +\infty\}$

Examples of sample spaces

Exercise

- Playing five rounds of Russian roulette
  Is this the same as coin flipping?

  Solution $\Omega = \{D, LD, LLD, LLLD, LLLLD, LLLLL\}$.

- Sequencing three nucleotides (each nulceotide is A, C, G or T)
  $\Omega = \{AAA, CCC, GGG, TTT, AAC, AAT, AAG, ...\}$
  What is the cardinality of the space?

  Solution $\#\Omega = |\Omega| = 4^3 = 64$.

Event

Definition

An event E is any subset of the sample space Ω.

Example

2 heads out of three flips	$E = \{\text{HHT, HTH, THH}\}$
Even number on a roll of a die	$E = \{2,4,6\}$
<2 minutes when waiting for the bus	$E = [0, 120)$

Collection of events

Collection of events

In the following, we consider probabilities associated to a collection of events, denoted $\mathcal{F}$.

- If Ω is *discrete* (i.e. countable) we will simply consider

$$\mathcal{F} = 2^{\Omega}$$

  the power set (the set of all subsets of Ω)
- If Ω is *continous* (i.e. uncountable) like $\mathbb{R}$, the power set of Ω is too complex

  We restrict ourselves to events that are intersections or unions of intervals $[a, +\infty)$
- (Formally the collection of events of interest must be a σ-algebra)
- In this lecture we consider finite sets, next lecture we'll consider infinite sets

Note

- In all cases, $\emptyset \in \mathcal{F}, \quad \Omega \in \mathcal{F}$.

Set operations interpretations

Let A, B be two events.

Definition

1. ***Intersection*** $A \cap B$*: both* A *and* B *occur*
2. ***Union*** $A \cup B$*: at least one of* A *or* B *occur*
3. *The* **complement** *of* A, A^c*:* A *does not occur*
4. $A \subset B$*: occurrence of* A *implies occurrence of* B*.*
5. *Set difference* $A \setminus B := A \cap (B^c)$*:* A *occurs,* B *does not.*

Definition

Two events A *and* B *are* **mutually exclusive** *or* **disjoint** *if they have no outcomes in common, i.e.* $A \cap B = \emptyset$*.*

Example

Singletons of distinct elements are always disjoint. So for a die, rolling a 6 ($A = \{6\}$) and rolling a 4 ($B = \{4\}$) are mutualy disjoint events.

DeMorgan's Laws

Lemma

For two subsets A, B of a set Ω,

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

- Outline of proof (to the first equation).
  1. Left $\subseteq$ Right: For any $x \in (A \cup B)^c$, then $x \in A^c \cap B^c$.
  2. Right $\subseteq$ Left: For any $x \in A^c \cap B^c$, then $x \in (A \cup B)^c$.

- DeMorgan's laws can be generalized to n events $A_1, \ldots, A_n$:

$$\boxed{\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c, \quad \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.}$$

- Or an infinite sequence $A_1, A_2, \ldots$

$$\boxed{\left(\bigcup_{i=1}^{+\infty} A_i\right)^c = \bigcap_{i=1}^{+\infty} A_i^c, \quad \left(\bigcap_{i=1}^{+\infty} A_i\right)^c = \bigcup_{i=1}^{+\infty} A_i^c.}$$

Set practice

Exercise

Let A be the event that a person is male, B that the person is under 30, and C that the person speaks French. Describe in symbols (feel free to use a Venn diagram to help):

- A male at least 30 years old
- A female under 30 who speaks French
- A male who either is under 30 or who speaks French

Solution

- $A \cap B^c$
- $A^c \cap B \cap C$
- $A \cap (B \cup C)$

Practice for next lecture

Practice

You roll 2 dice, what is the cardinality of the event:
$E =$ "the sum larger than or equal to 10"? By cardinality I mean the number of possible outcomes that result in this event.