Classical random variables
Conditional Independence

MATH/STAT 394: Probability I
Summer 2021 A Term

Introduction to Probability
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§ 2.4, 2.5

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Recap

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- "i.i.d." stands for **independent and identically distributed**

Outline

Bernoulli random variable

Reminder:

Definition

A r.v. X has a ***Bernoulli*** *dist. with param. $p \in [0,1]$ if it takes its values in $\{0,1\}$ and*

$$\mathbb{P}(X=1)=p \quad \mathbb{P}(X=0)=1-p$$

We denote it $X \sim \mathrm{Ber}(p)$

Binomial random variable

Many random variables *arise from repeated trials*.

Definition

*A r.v. X has a **Binomial** distribution with parameters $n \in \mathbb{N}$, $n > 0$, and $p \in [0, 1]$, if the possible values of X are $\{0, \dots, n\}$ and*

$$p_X(k) = \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n.$$

We denote it $X \sim \mathrm{Bin}(n, p)$.

Alternative definition

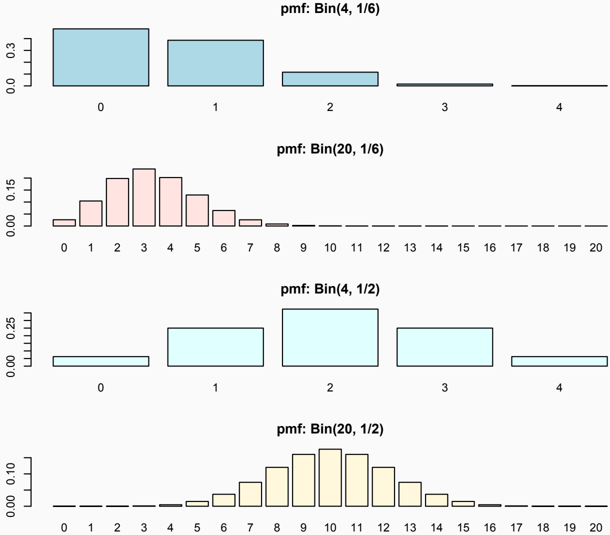
$X \sim \mathrm{Bin}(n, p)$ if and only if $X = Y_1 + \ldots + Y_n$ for $Y_i \overset{i.i.d.}{\sim} \mathrm{Ber}(p)$

where $Y_i \overset{i.i.d.}{\sim} \mathrm{Ber}(p)$ means that the Y_i are independent and identically distributed with a dist. $\mathrm{Ber}(p)$.

Notes:

- For $n = 1$, we retrieve the Bernoulli dist.

Binomial random variable



Geometric random variable

Example
Take a coin whose probability of $\mathtt{H}$ is p. Toss until the the first $\mathtt{H}$.

Recall that this is an experiment defined on

$$\Omega_\infty = \{\text{all infinite sequences of } \{\mathtt{H}, \mathtt{T}\}\}.$$

Let

$$Y = \text{the total number of tosses.}$$

What is the p.m.f. of Y?

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Solution

- Already seen:

$$\mathbb{P}(Y = k) = \mathbb{P}(X_1 = \cdots = X_{k-1} = 0, X_k = 1) = (1-p)^{k-1}p,$$

$$\boxed{p_Y(k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \ldots.}$$

- Exercise: show that $\sum_{k=1}^{+\infty} p_Y(k) = 1$.
  (Hint: this is the sum of a geometric sequence)

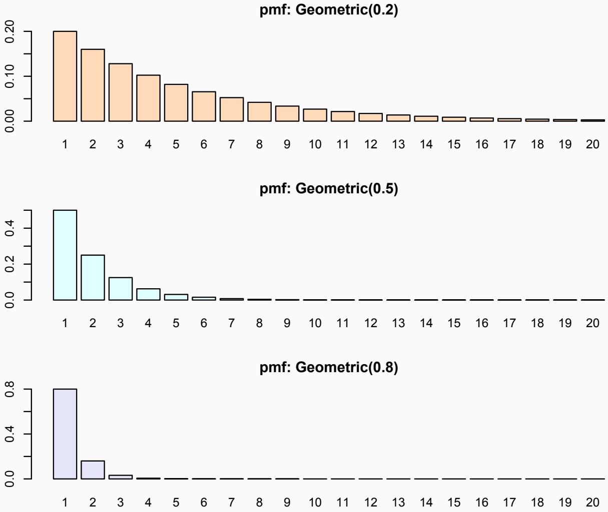
Geometric random variable

Definition

A r.v. X has a ***Geometric*** *dist. with param. $p \in [0,1]$ if it takes its values in $\{1, 2, \ldots\}$ and*

$$p_X(k) = \mathbb{P}(X = k) = (1-p)^{k-1} p$$

We denote it $X \sim \mathrm{Geom}(p)$.


pmf: Geometric(0.2)
0.00 0.10 0.20
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
pmf: Geometric(0.5)
0.0 0.2 0.4
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
pmf: Geometric(0.8)
0.0 0.4 0.8
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

credit: R. Guo

Hypergeometric distribution

Example
An urn contains N balls: N_A labelled as `A`, $N - N_A$ labelled as `B`.
Draw n ($n \leq N$) **without replacement**, and let

$$X = \text{the number of } \texttt{A} \text{ balls.}$$

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Solution

- Denote k a possible value for X. Note that $0 \leq k \leq \min(N, N_A)$.
- Moreover one cannot take more than $N - N_A$ B balls from the urn, so $0 \leq n - k \leq N - N_A$, that is $n - (N - N_A) \leq k$. Overall the possible values of k are

$$\max(0, n - (N - N_A)) \leq k \leq \min(N_A, n).$$

- Now following the definition we have for a possible k,

$$\boxed{p_X(k) = P(X = k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}, \quad k = 0, 1, \ldots, n,}$$

Hypergeometric distribution

Definition

*A r.v. X has a **Hypergeometric** dist. with param. n, N_A, N if it takes values k such that*

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We denote it $X \sim \mathrm{Hypergeom}(N, N_A, n)$.

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Note:

The range of possible values of k is directly given in the p.m.f. since, for k an integer, $P_X(k) = 0$ if k does not satisfy $\max(0, n-(N-N_A)) \leq k \leq \min(N_A, n)$.

Sampling n times with/without replacement

- *Sampling without replacement*: $X \sim \text{Hypergeom}(N, N_A, n)$

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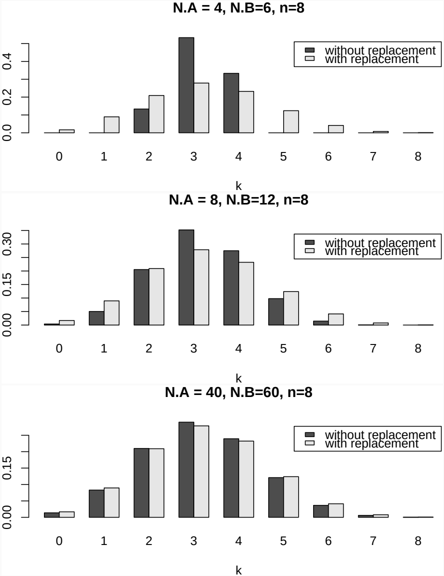
- *Sampling with replacement*: $Y \sim \text{Bin}(N_A/N, n)$

$$\boxed{p_Y(k) = \binom{n}{k}\left(\frac{N_A}{N}\right)^k\left(1 - \frac{N_A}{N}\right)^{n-k}, \quad k = 0, 1, \ldots, n.}$$


N.A = 4, N.B=6, n=8
without replacement
with replacement
0.0
0.2
0.4
0 1 2 3 4 5 6 7 8
k
N.A = 8, N.B=12, n=8
without replacement
with replacement
0.00
0.15
0.30
0 1 2 3 4 5 6 7 8
k
N.A = 40, N.B=60, n=8
without replacement
with replacement
0.00
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0 1 2 3 4 5 6 7 8
k

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$$\boxed{p_Y(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}}$$

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- bombs during the Blitz

Identifying distributions

Exercise

For each of the following examples, select the distribution of the r.v. of interest

1. Draw 4 cards from a deck, $X =$ the number of hearts
2. Observe the weather in Seattle for 7 days. $Y =$ number of times it rains (unique rain events with a break since the last one).
3. Take the bus to school each day for 30 days. $X =$ number of times the bus is late.
4. Survey 100 people and ask which candidate they will vote for, among 4 candidates. $X =$ the number of votes for each candidate.
5. You're stuck in some bad traffic at a stop light. let $X =$ number of light cycles before you get through
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1. Hypergeometric(52, 4, 4)
2. Pois(7 $*$ avg. number of rain events per day) or Pois(avg. number of rain events per week)
3. Binomial(30, p(bus is late))
4. Multinomial (haven't yet learned) note: it takes on vector values
5. ???
6. Geometric(p(socialist candidate gets a vote))

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Definition

Let $B \subseteq \Omega$ s.t. $\mathbb{P}(B) > 0$, events A_1, A_2 are ***conditionally independent given*** *B if*

$$\mathbb{P}(A_1 \cap A_2 \mid B) = \mathbb{P}(A_1 \mid B)\mathbb{P}(A_2 \mid B)$$

Conditional Independence

Exercise

Suppose 90% of coins in the circulation are fair and 10% are biased with $\mathbb{P}(\mathtt{T}) = \frac{3}{5}$. I have a random coin and flip it twice.

Denote $A_1 = \{\text{1st flip is tail}\}$ and $A_2 = \{\text{2nd flip is tail}\}$.

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- Denote $F = \{\text{the coin is fair}\}$, $B = \{\text{the coin is biased}\}$
- For **a given coin** the events are identically distributed, i.e.,

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- Then by the law of total prob., for $i = 1$ or 2,

$$\mathbb{P}(A_i) = \mathbb{P}(A_i \mid F)\mathbb{P}(F) + \mathbb{P}(A_i \mid B)\mathbb{P}(B) = \frac{1}{2} \cdot \frac{9}{10} + \frac{3}{5} \cdot \frac{1}{10} = \frac{51}{100}$$

Conditional Independence

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Solution (continued)

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- Now assume that for **a given coin**, the two events are conditionally independent (natural assumption), i.e.,

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$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1 \cap A_2 \mid F)\mathbb{P}(F) + \mathbb{P}(A_1 \cap A_2 \mid B)\mathbb{P}(P) \\
&= \mathbb{P}(A_1 \mid F)\mathbb{P}(A_2 \mid F)\mathbb{P}(F) + \mathbb{P}(A_1 \mid B)\mathbb{P}(A_2 \mid B)\mathbb{P}(B) \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{9}{10} + \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{1}{10} = \frac{261}{1000}
\end{aligned}$$

Conditional Independence

Solution (continued)

90% fair coins, 10% are biased with $\mathbb{P}(\mathtt{T}) = \frac{3}{5}$. I have a random coin that is flipped 2x

Denote $A_1 = \{\text{1st flip is tail}\}$ and $A_2 = \{\text{2nd flip is tail}\}$. Are A_1, A_2 independent?

- Now assume that for **a given coin**, the two events are conditionally independent (natural assumption), i.e.,

$$\mathbb{P}(A_1 \cap A_2 \mid F) = \mathbb{P}(A_1 \mid F)\mathbb{P}(A_2 \mid F) \quad \mathbb{P}(A_1 \cap A_2 \mid B) = \mathbb{P}(A_1 \mid B)\mathbb{P}(A_2 \mid B)$$

- Then by the law of total prob.

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- Then $\mathbb{P}(A_1 \cap A_2) = \frac{261}{1000} \neq \left(\frac{51}{100}\right)^2 = \mathbb{P}(A_1)\mathbb{P}(A_2)$, the two events **are not independent**

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- Then $\mathbb{P}(A_1 \cap A_2) = \frac{261}{1000} \neq \left(\frac{51}{100}\right)^2 = \mathbb{P}(A_1)\mathbb{P}(A_2)$, the two events **are not independent**
- Why? The first flip gives us some information about the coin, which influences the prob. of getting a tail a second time
- Think for example that $A_1 = \{\text{first 100 flips are tail}\}$ and $A_2 = \{\text{101th flip is tail}\}$, clearly if A_1 is true, the coin has more chances to be biased and so the prob. of A_2 is influenced by this information.

Conditional Independence

- Conditional independence, tells us that given some information B, another event A_2 is no longer relevant

Lemma
If A_1 and A_2 are conditionally independent given B then

$$\mathbb{P}(A_2 \mid A_1, B) = \mathbb{P}(A_2 \mid B)$$

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Lemma
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Proof

$$\begin{aligned}\mathbb{P}(A_2 \mid A_1, B) := \mathbb{P}(A_2 \mid A_1 \cap B) &= \frac{\mathbb{P}(A_2 \cap A_1 \cap B)}{\mathbb{P}(A_1 \cap B)} \\ &= \frac{\mathbb{P}(A_2 \cap A_1 \mid B)}{\mathbb{P}(A_1 \mid B)} = \frac{\mathbb{P}(A_2 \mid B)\mathbb{P}(A_1 \mid B)}{\mathbb{P}(A_1 \mid B)} = \mathbb{P}(A_2 \mid B)\end{aligned}$$

Conditional Independence

Example

Every day I walk a random number of kilometers. Let X_n the distance that I walked after n days. Are the events $\{X_1 = 10\}$ and $\{X_3 = 20\}$ conditionally independent given $\{X_2 = 15\}$?

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Solution This is just an intuitive example, we won't dive into this kind of problems during the course

- Yes, we naturally have that if we know X_2, X_1 is not relevant, namely

$$\mathbb{P}(X_3 = 20 \mid X_2 = 15, X_1 = 10) = \mathbb{P}(X_3 = 15 \mid X_2 = 15)$$

Note:

- This is an example of a Markov chain, a sequence of events such that the future is independent of the past given the present.
- This is a very common model that can be used for example to predict the weather.

Practice next lecture

Practice

At a lottery, there are 10 out of 100 tickets that have prizes.

1. Consider picking 5 tickets with replacement, what is the prob. that you get exactly 2 prizes? (Namely you pick a ticket, look if you win or not and repeat that 5 times)
2. Consider picking 5 tickets without replacement, what is the prob. that you get exactly 2 prizes?

Practice

Roll a fair die twice, define

$$A = \{\text{first die is a 2 or a 3}\}, B = \{\text{4 appears at least once}\}$$

- Are A, B independent?
- Are A, B conditionally independent given that

$$C = \{\text{the sum of the dice is a 6}\}?$$