CSSS 505 / SOC 512

Algebra, Functions, & Limits

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Week 1

Outline

- Math notation
- Order of operations
- Rules of exponents, logarithms
- Equation of a line
- Functions, domain, range, examples
- Function transformations
- Continuous and piecewise functions
- Limits

Notation

Real Numbers

- Any number that falls on the continuous line. Often represented by a, b, c, d
- Examples: 2, 3.234, 1/7, $\sqrt{5}$, π
- The set of real numbers is denoted by $\mathbb{R}$. Then $a \in \mathbb{R}$ means a is in the set of real numbers.

Integers

- Any whole number. Often represented by i, j, k, l
- Examples: ...,-3,-2,-1,0,1,2,3,...

Variables

- Can take on different values
- Often represented by x, y, z

Notation

Functions

- Often represented by f, g, h
- Examples: $f(x) = x^2 + 3$, $g(y) = 6y^2 - 2y$, $h(z) = z^3$

Summations

- Often represented by $\sum$ and summed over some integer
- Example:
$$\sum_{i=1}^{3}(i+1)^2 = (1+1)^2 + (2+1)^2 + (3+1)^2 = 2^2 + 3^2 + 4^2 = 29$$

Products

- Often represented by $\prod$ and multiplied over some integer
- Example: $\prod_{k=1}^{3}(y_k + 1)^2 = (y_1 + 1)^2 \times (y_2 + 1)^2 \times (y_3 + 1)^2$

Fractions

Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

All numbers can be written as fractions. Examples:
$\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4} = \frac{1}{2}, \frac{8}{1}(=8)$.

Multiplication: Multiply the numerators; multiply the denominators. Examples: $\frac{1}{2} \times \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$

Division: Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction. Examples: $\frac{1/2}{7/8} = \frac{1}{2} \times \frac{8}{7} = \frac{1 \cdot 8}{2 \cdot 7} = \frac{8}{14}$.

Simplify: $\frac{8}{14} = \frac{2 \cdot 4}{2 \cdot 7} = \frac{2}{2} \times \frac{4}{7} = 1 \times \frac{4}{7} = \frac{4}{7}$

Fractions

Adding & Subtracting

Adding and subtracting requires that fractions must have the same denominator. If not, we need to find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add (or subtract) the two numerators.

Examples: $\frac{1}{7} + \frac{4}{7} = \frac{5}{7}$

$\frac{1}{3} + \frac{1}{4} = \frac{1}{3} \times \frac{4}{4} + \frac{1}{4} \times \frac{3}{3} = \frac{1 \cdot 4}{3 \cdot 4} + \frac{1 \cdot 3}{4 \cdot 3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

$\frac{17}{20} - \frac{3}{4} = \frac{17}{20} \times \frac{1}{1} - \frac{3}{4} \times \frac{5}{5} = \frac{17 \cdot 1}{20 \cdot 1} - \frac{3 \cdot 5}{4 \cdot 5} = \frac{17}{20} - \frac{15}{20} = \frac{2}{20} = \frac{1}{10}$

Exponents

a^n is 'a to the power of n'. a is multiplied by itself n times. Often a is called the base, n the exponent. Examples:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

$$6^4 = 6 \cdot 6 \cdot 6 \cdot 6 = 1296$$

Exponents do not have to be whole numbers. They can be fractions or negative.
Examples:

$$4^{1/2} = \sqrt{4} = 2$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Common Rules

- $a^1 = a$
- $a^k \cdot a^l = a^{k+l}$
- $(a^k)^l = a^{kl}$
- $(ab)^k = a^k \cdot b^k$
- $\left(\frac{a}{b}\right)^k = \left(\frac{a^k}{b^k}\right)$
- $a^{-k} = \frac{1}{a^k}$
- $\frac{a^k}{a^l} = a^{k-l}$
- $a^{1/2} = \sqrt{a}$
- $a^{1/k} = \sqrt[k]{a}$
- $a^0 = 1$

Logarithms

A logarithm is the power (x) required to raise a base (c) to a given number (a).

$$\log_c(a) = x \Rightarrow c^x = a$$

Examples:

- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $4^6 = 4096 \Rightarrow \log_4(4096) = 6$
- $9^{1/2} = 3 \Rightarrow \log_9(3) = \frac{1}{2}$

Logarithms

The three most common bases are 2, 10, and $e \approx 2.718$, the natural logarithm. e is often called Euler's number after Leonhard Euler.

Examples:

- $10^2 = 100 \Rightarrow \log_{10}(100) = 2$
- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $e^2 = 7.3891... \Rightarrow \log(7.3891) = 2$

The natural logarithm ($\log_e$) is the most common; used to model exponential growth (populations, etc). If no base is specified, i.e. $\log(a)$, most often the base is e. Sometimes written as $\ln(a)$.

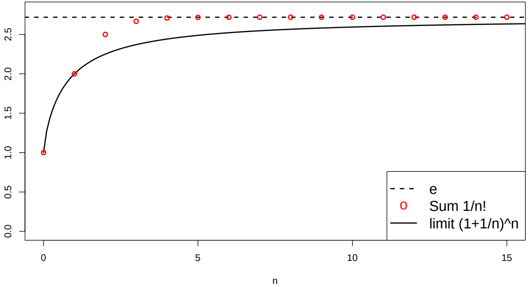
Logarithms

What is e?

The number e is a famous irrational number. The first few digits are $e = 2.718282...$

Two ways to express e:

- $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
- $\sum_{n=0}^{\infty}\frac{1}{n!}$



Logarithms

Rules

$$\log_c(a \cdot b) = \log_c(a) + \log_c(b)$$

$$\begin{aligned}
x &= \log_c(a \cdot b) \iff c^x = a \cdot b \\
& \text{let } x_1 + x_2 = x \\
\Rightarrow & \quad c^{x_1 + x_2} = a \cdot b \\
\Rightarrow & \quad c^{x_1} \cdot c^{x_2} = a \cdot b \Rightarrow c^{x_1} = a; c^{x_2} = b \\
\Rightarrow & \quad x_1 = \log_c(a); x_2 = \log_c(b) \\
\Rightarrow & \quad x = x_1 + x_2 \Rightarrow \log_c(a \cdot b) = \log_c(a) + \log_c(b)
\end{aligned}$$

Logarithms

Rules

$$\log_c(a^n) = n \cdot \log_c(a)$$

For $n = 2$:

$$\begin{aligned}
x &= \log_c(a^2) \iff c^x = a^2 \\
& \text{let } x_1 + x_2 = x \\
&\Rightarrow c^{x_1+x_2} = a \cdot a \\
&\Rightarrow c^{x_1} \cdot c^{x_2} = a \cdot a \Rightarrow c^{x_1} = a; c^{x_2} = a \\
&\Rightarrow x_1 = \log_c(a); x_2 = \log_c(a) \\
&\Rightarrow x = x_1 + x_2 \Rightarrow \log_c(a^2) = \log_c(a) + \log_c(a) = 2 \cdot \log_c(a)
\end{aligned}$$

Logarithms

Rules

$$\log_c\left(\tfrac{a}{b}\right) = \log_c(a) - \log_c(b)$$

$$\begin{aligned}
x &= \log_c\left(\frac{a}{b}\right) \iff c^x = \frac{a}{b} \\
&\quad \text{let } x_1 + x_2 = x \\
\Rightarrow &\quad c^{x_1+x_2} = \frac{a}{b} \\
\Rightarrow &\quad c^{x_1} \cdot c^{x_2} = \frac{a}{b} \Rightarrow c^{x_1} = a; c^{x_2} = \frac{1}{b} = b^{-1} \\
\Rightarrow &\quad x_1 = \log_c(a); x_2 = (-1) \cdot \log_c(b) \\
\Rightarrow &\quad x = x_1 + x_2 \Rightarrow \log_c\left(\frac{a}{b}\right) = \log_c(a) - \log_c(b)
\end{aligned}$$

Logarithms
Examples

- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$
- $\log_{10}\left(\frac{1000}{10}\right) = \log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$
- $\log_4(6^4) = 4 \cdot \log_4(6)$
- $\log(x^3) = 3 \cdot \log(x)$

Order of Operations

Please **E**xcuse **M**y **D**ear **A**unt **S**ally

- **P**arentheses
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction

Order of Operations

Examples

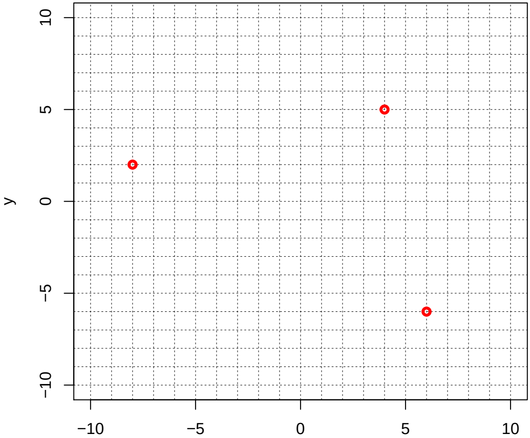
When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

- $\left((1+2)^3\right)^2 = (3^3)^2 = 27^2 = 729$
- $4^3 \cdot 3^2 - 10 + 27/3 = 64 \cdot 9 - 10 + 9 = 576 - 10 + 9 = 575$
- $(x+x)^2 - 2x + 3 = (2x)^2 - 2x + 3 = 4x^2 - 2x + 3$

Cartesian Coordinates

Given pairs of points: $(x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n)$ The plot has two axes: x (horizontal) and y (vertical).

Examples: (-8,2),(4,5),(6,-6)



Equation of a Line

Linear Equations

If we have two pairs of points $(x_1, y_1), (x_2, y_2)$, we can find a line between the two points.

A common equation for a line is: $y = mx + b$ where m is the *slope* and b is the *y-intercept*.

Another common form (often used in the regression setting) is: $y = \beta_0 + \beta_1 x$, where β_0 is the *y-intercept* and β_1 is the *slope*.

Equation of a Line

Linear Equations

The slope is a measure of the steepness of a line. A line with a slope 5 is steeper than a line with a with slope 2. The slope is the ratio of the difference in the two y-values to the difference in the two x-values. Commonly referred to as *rise* over *run*.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The intercept is the value of y when $x = 0$. This is the vertical height where the line crosses the y-axis.

Once you have the slope, you can find the intercept by plugging in one point and the slope into the equation and then solving for the intercept.

$$b = y_1 - m \cdot x_1$$

Equation of a Line

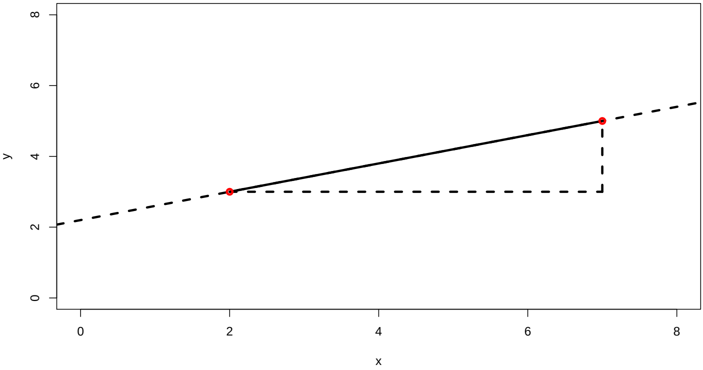
Linear Equations Example

Given the points $(2,3),(7,5)$:

Slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{7-2} = \frac{2}{5}$

Intercept: $b = y_1 - mx_1 = 3 - \frac{2}{5} \cdot 2 = 3 - \frac{4}{5} = 11/5$

Equation of the line: $y = \frac{2}{5}x + \frac{11}{5}$



Functions and their Limits

A *function* is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

The *domain* of a function is the set of all possible values that you can plug into the function. The *range* is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- Domain: all real numbers $\mathbb{R}$
- Range: zero and all positive real numbers, $f(x) \geq 0$

Functions and their Limits

Examples continued

$f(x) = \sqrt{x}$

- Domain: zero and all positive real numbers, $x \geq 0$
- Range: zero and all positive real numbers, $x \geq 0$

$f(x) = 1/x$

- Domain: all real numbers except zero
- Range: all real numbers except zero

Solving Linear Equations

Often we would like to find the *root* of a linear equation. This is the value of x that maps $f(x)$ to 0 (where the line crosses the x-axis).

$$f(x) = mx + b$$

To find the root we need to solve

$$\begin{aligned} 0 &= mx + b \\ -b &= mx \\ \frac{-b}{m} &= x \end{aligned}$$

The value $-b/m$ is the root of $f(x) = mx + b$.

Solving Linear Equations

Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?
Let x be hours and $f(x)$ total price.

$$f(x) = 7 + 11x$$

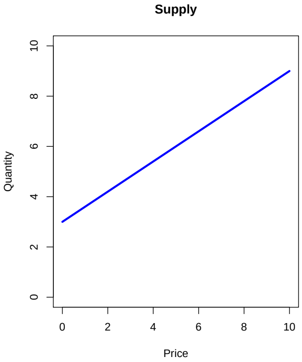
How long can you bowl?

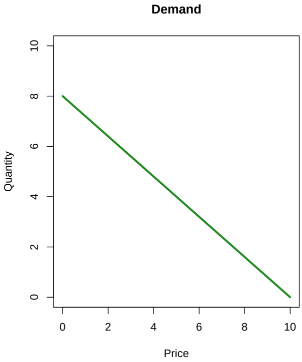
$$\begin{aligned} 40 &= 11x + 7 \\ 40 - 7 &= 11x \\ 33 &= 11x \\ 33/11 &= 3 = x \end{aligned}$$

Solving Systems of Linear Equations

We often are interested in finding the values of x and y where two lines cross. This is called solving the system of linear equations. A common example is supply and demand curves.

Supply Curve: As the price of oil increases, producers are willing to provide more to the market. **Demand Curve**: As price increases, consumers will demand less oil.





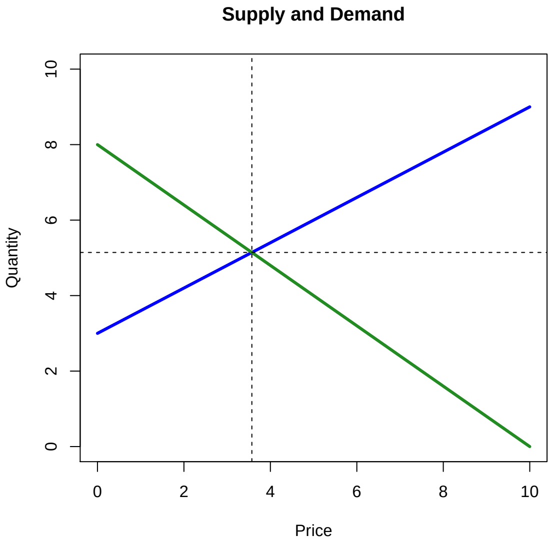
Solving Systems of Linear Equations

There are many ways to approach solving linear equations. We are interested in finding the point (x, y) that falls on both lines. If Supply is $y = 3 + 0.6x$ and Demand is $y = 8 - 0.8x$ we could take the following approach:

$$\begin{aligned} 3 + 0.6x &= 8 - 0.8x \\ 3 - 3 + 0.6x + 0.8x &= 8 - 3 - 0.8x + 0.8x \\ 1.4x &= 5 \\ x &= 5/1.4 = 3.571429 \end{aligned}$$

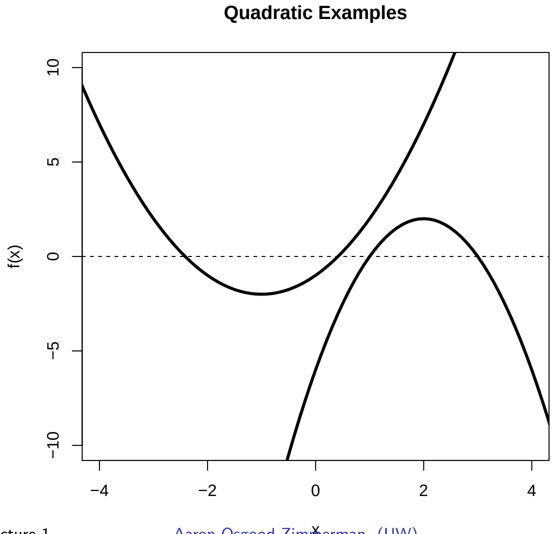
The y-value is found using either equation:
$y = 3 + 0.6(3.571429) = 5.142857$

Solving Systems of Linear Equations



Quadratic Equations

A *quadratic* function has the form $f(x) = ax^2 + bx + c$. The quadratic function is associated with the parabola.



Quadratic Equations

Examples

For any quadratic equation $f(x) = ax^2 + bx + c$, we find the root(s) (values of x such that $f(x) = 0$) by using the 'quadratic equation':

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note: Quadratics may have only one root (both roots are the same) or no real root.

Many quadratic equations can be factored into a more simple form. For example:

$$2x^2 - 6x - 8 = (x-4)(2x+2)$$

To see that they are equivalent we can FOIL.

- **F**irst: $x \cdot 2x = 2x^2$
- **O**uter: $x \cdot 2 = 2x$
- **I**nner: $-4 \cdot 2x = -8x$
- **L**ast: $-4 \cdot 2 = -8$

Thus, $(x-4)(2x+2) = 2x^2 + 2x - 8x - 8 = 2x^2 - 6x - 8$

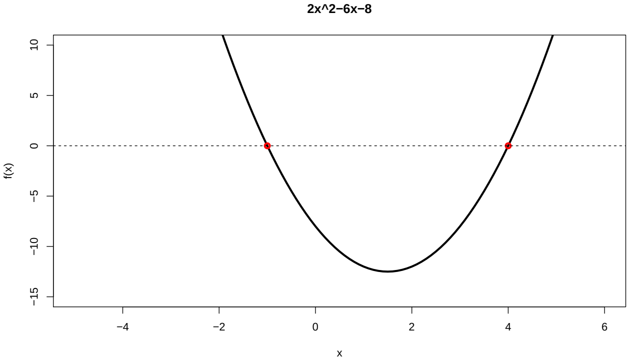
Quadratic Equations

Factoring and FOIL

When your quadratic has been factored you can find the roots by solving each term for zero. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

has roots when $x - 4 = 0$ and $2x + 2 = 0$. Thus, the roots are found at $x = -1, 4$.



Quadratic Equations

Factoring and FOIL

Hunting for the FOIL factors can be tricky. Here are some hints:

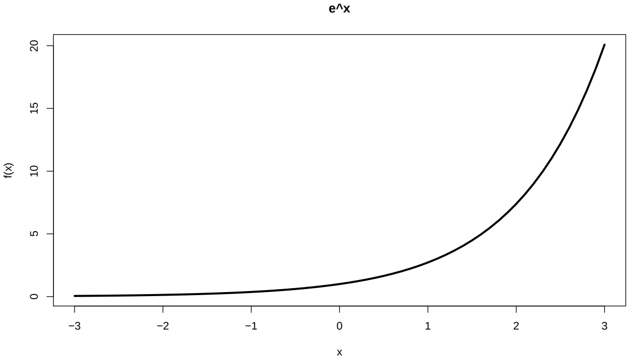
- If $b^2 - 4ac$ is positive, there will be two roots.
- If $b^2 - 4ac$ is zero, there will be one root.
- If $b^2 - 4ac$ is negative, there will be no real roots.
- If $b^2 - 4ac$ is a whole number, a fraction, a squared number, then it can be factored into something simple, if not use the quadratic formula.

Examples:

- $2x^2 + 4x - 16 \Rightarrow 4^2 - 4 \cdot 2 \cdot (-16) = 144$; 2 roots; factors
- $3x^2 - 2x + 9 \Rightarrow (-2)^2 - 4 \cdot 3 \cdot 9 = -104$; no real roots

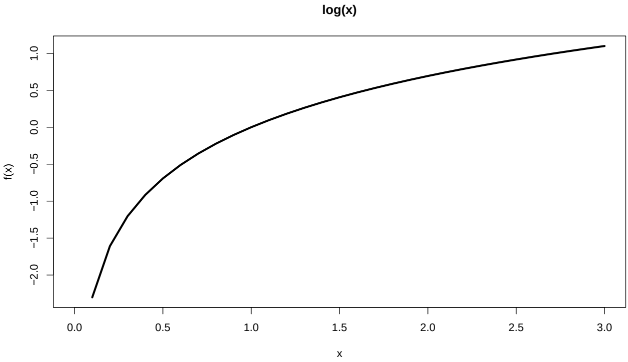
Exponential Functions

Exponential Functions are of the form $f(x) = ae^{bx}$. Often used as a model for population increase where $f(x)$ is the population at time x.



Logarithmic Functions

Logarithmic Functions, $f(x) = c + d \cdot log(x)$, can be used to find the time $f(x)$ necessary to reach a certain population x. It can be thought of as an 'inverse' of the exponential function.



Note: $c = -1/b \cdot log(a)$ and $d = 1/b$ from the previous exponential model.

Continuous & Piecewise Functions

A *continuous* function behaves without break or interruption. If you can follow the ENTIRE graph of a function with your pencil without picking it up, the function is continuous. Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

A *piecewise* functioncan either have 'jumps' in it or can be made up of different functions for different parts of the domain (possible x-values). Example:

- Absolute Value $f(x) = |x|$ can be written as $f(x) = x, x \geq 0$ and $f(x) = -x, x < 0$

Limits

Often we are interested in what a function does as it approaches a certain value. This behavior is called the *limit*.

The limit of $f(x)$ as x approaches a is L:

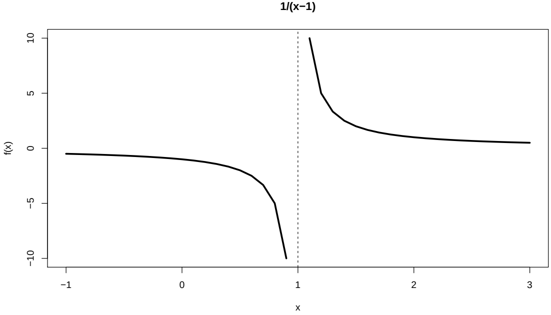
$$lim_{x \to a} f(x) = L$$

It may be that a is not in the domain of $f(x)$ but we can still find the limit by seeing what value $f(x)$ is approaching as x gets very close to a. Examples:

- $lim_{x \to 3} x^2 = 9$ (3 is in the domain)
- $lim_{x \to \infty} (1 + 1/x)^x = e$

Limits

Often limits are different depending on the direction from which you approach a. The limit 'from above' is approaching from the right ($x \downarrow a$) and the limit 'from below' ($x \uparrow a$)is approaching from the left.



If $f(x) = \frac{1}{x-1}$ we have $lim_{x \downarrow 1} \frac{1}{x-1} = \infty$ and $lim_{x \uparrow 1} \frac{1}{x-1} = -\infty$

The End

Please bring questions to class or office hours