

Integral Calculus

SOC 512 & CSSS 505

written by Laina Mercer & Jessica Godwin

Aaron Osgood-Zimmerman

Department of Statistics
University of Washington

January 28, 2021

Outline

- Motivation for Integrals
- Rules of Integration
- Lots of Examples

Motivation for Integrals in Statistics

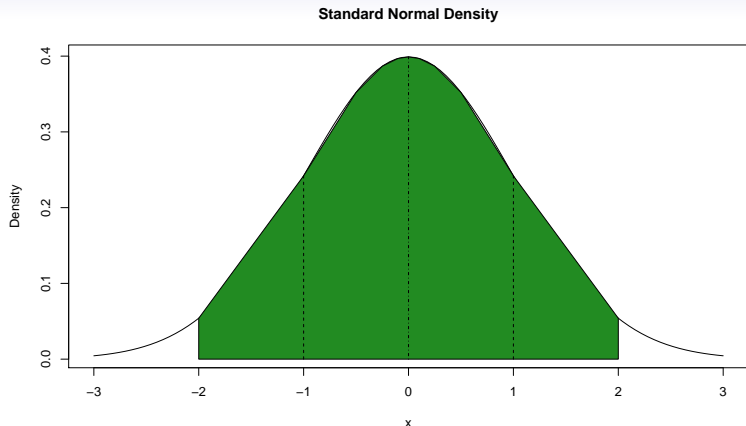


Figure: Standard Normal Density ($N(0,1)$). Approximately 68% of the probability lies within 1 standard deviation and 95% within 2 standard deviations. The area under the whole curve (from $-\infty$ to ∞) is 1.

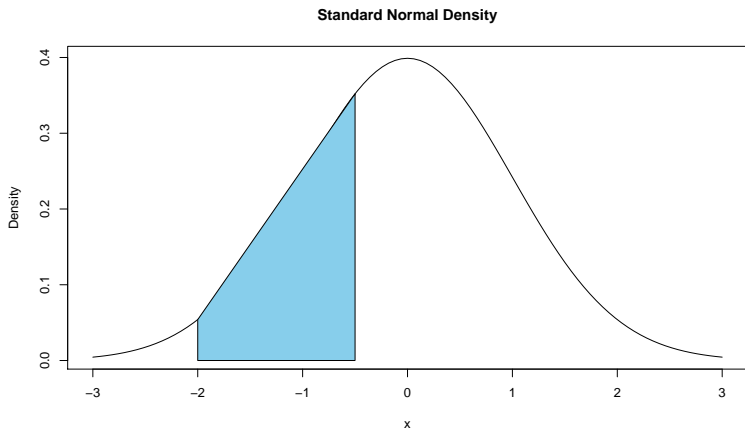
Motivation for Integrals in Statistics

Integral calculus...

- is a tool for computing areas under curves.
- can be used to compute percentile rankings.
- is also used to compute probabilities of events.
- will be needed to compute the expected values and variance of probability distributions.
- is heavily used in statistical theory!

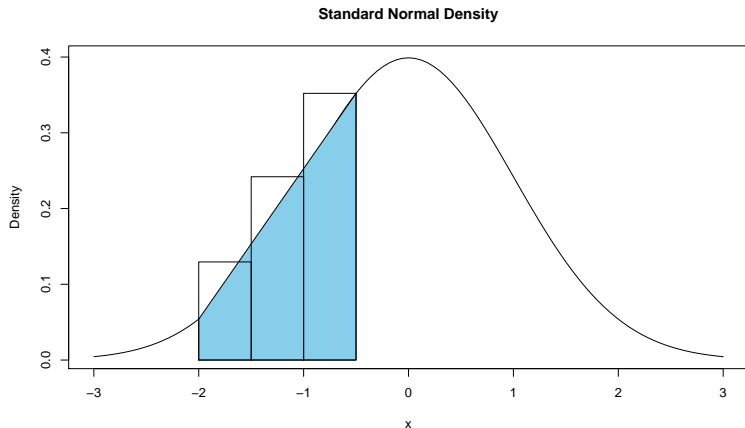
Motivation for Integrals in Statistics

What if we wanted to find the area under the curve from -2 to -0.5 ?



Motivation for Integrals in Statistics

We could approximate with rectangles or trapezoids. Narrower rectangles would give better approximations.



Differentiation Example

distance, velocity, acceleration

Let's take d =distance, v =velocity, a =acceleration. You may remember from physics, the distance travel after time t

$$d(t) = \frac{a}{2}t^2$$

The velocity at any time t is the instantaneous rate of change of the distance, $v(t) = d'(t)$:

$$v(t) = 2 \cdot \frac{a}{2}t = at$$

The acceleration at any time t is the instantaneous rate of change of the velocity, $a(t) = v'(t) = d''(t)$:

$$a(t) = a$$

Distance

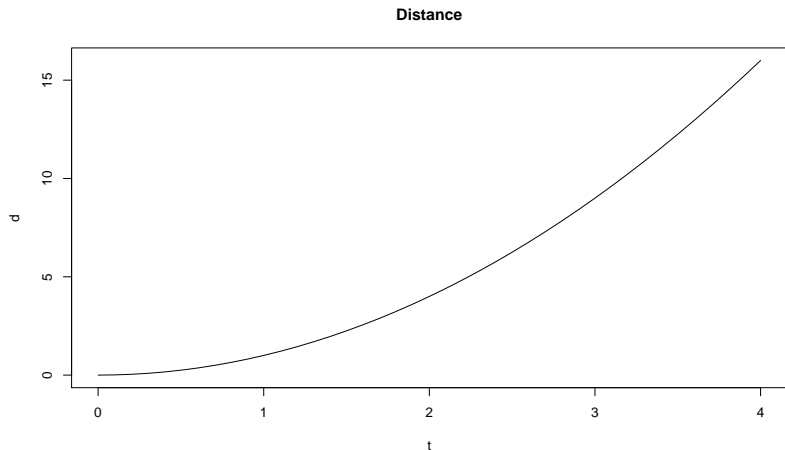


Figure: Distance over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Velocity

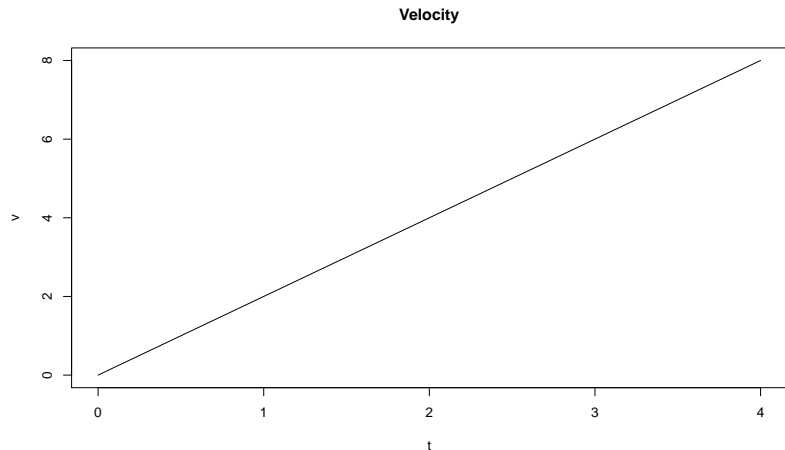


Figure: Velocity over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Acceleration

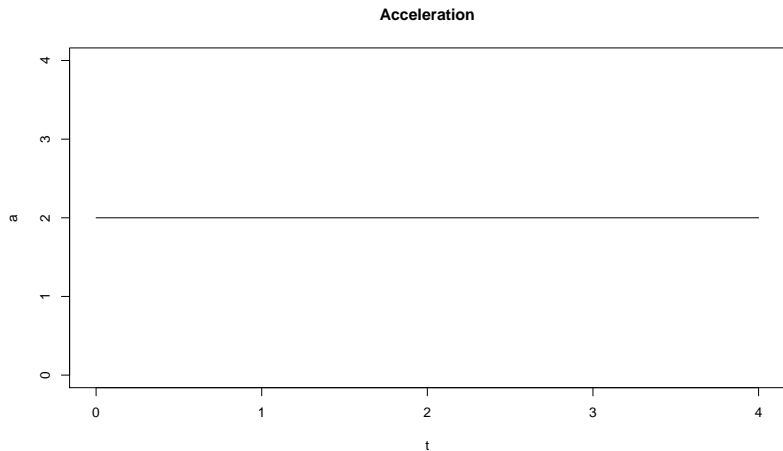


Figure: Acceleration over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

What is the velocity at $t=3$ when $a=2$?

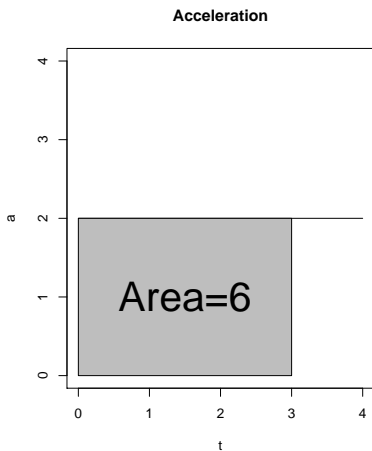
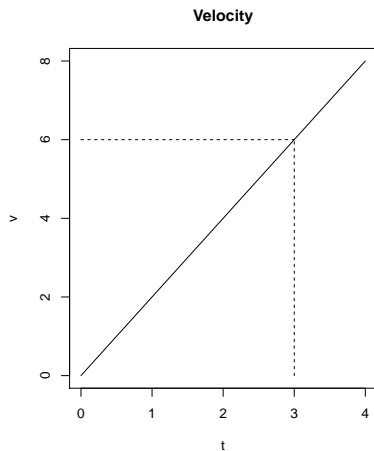
We know that $v(t) = 2t$, so clearly

$$v(3) = 2 \cdot 3 = 6.$$

However we can also find the velocity, by looking at the area under the acceleration curve from $t = 0$ to $t = 3$. This would just be the area of a rectangle (base \times height),

$$(3 - 0) \cdot 2 = 3 \cdot 2 = 6.$$

What is the velocity at $t=3$ when $a=2$?



What is the distance at $t=3$ when $a=2$?

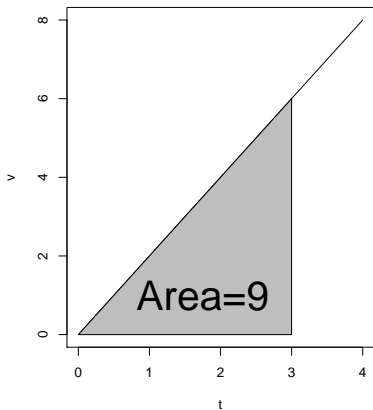
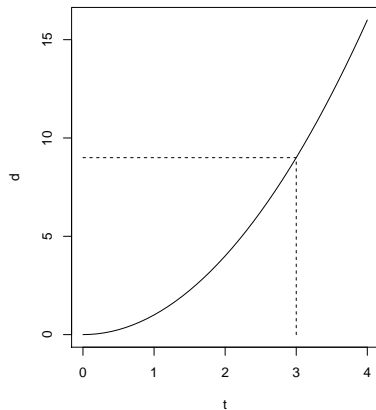
We know that $d(t) = 2/2t^2 = t^2$, so clearly

$$d(3) = 3^2 = 9.$$

However we can also find the distance, by looking at the area under the velocity curve from $t = 0$ to $t = 3$. This would just be the area of a triangle ($1/2 \times \text{base} \times \text{height}$),

$$1/2 \cdot (3 - 0) \cdot 6 = 3/2 \cdot 6 = 18/2 = 9.$$

What is the distance at $t=3$ when $a=2$?



Integration

The area under a curve is written:

$$\int_a^b f(x)dx$$

This formula is called the *definite integral* of $f(x)$ from a to b .

Here a and b are our endpoints of interest. You can think of the integral as the 'opposite' of the derivative.

Integration

More specifically,

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

$F(x)$ is called the *indefinite integral* of $f(x)$. The important relationships between derivatives and integrals are:

$$F'(x) = f(x) \quad \& \quad \int f(x)dx = F(x)$$

What is an integral?

You can think of integrating as looking at a derivative and trying to find the original function.

- $\int 3dx$. What function has a derivate equal to 3? $3x$.
- $\int 2xdx$. What function has a derivate equal to $2x$? x^2 .
- $\int e^x dx$. What function has a derivate equal to e^x ? e^x .

In practice, you don't have to search for the right function. We have handy shortcuts (rules).

Integration Rules

Integrating a Constant

$$\int c dx = cx$$

Examples:

- $\int 1 dx = x$
- $\int 6 dx = 6x$
- $\int y dx = yx$

Integration Rules

Integrating a Power of x

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

Examples:

- $\int x dx = \frac{1}{2} x^2$
- $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1} x^{-1} = -\frac{1}{x}$

Integration Rules

Integrating an Exponential and Logarithmic Functions

Exponential:

$$\int e^x dx = e^x$$

(Natural) Logarithm:

$$\int \frac{1}{x} dx = \log(x)$$

Integration Rules

Basic Trigonometric Functions

Remember, $\frac{d}{dx} \cos(x) = -\sin(x)$, thus

$$\int \sin(x) dx = -\cos(x)$$

and $\frac{d}{dx} \sin(x) = \cos(x)$, thus

$$\int \cos(x) dx = \sin(x).$$

Integration Rules

Multiple of a Function

$$\int af(x)dx = a \cdot \int f(x)dx = aF(x)$$

Examples:

- $\int 4x^2 dx = 4 \int x^2 dx = 4 \left(\frac{1}{3}x^3 \right) = \frac{4}{3}x^3$
- $\int \frac{3}{x^2} dx = 3 \int \frac{1}{x^2} dx = 3 \int x^{-2} dx = \frac{3}{-1}x^{-1} = -\frac{3}{x}$
- $\int \mu y dy = \mu \int y dy = \mu \left(\frac{1}{2}y^2 \right) = \frac{\mu}{2}y^2$

Integration Rules

Sums of Functions

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx = F(x) + G(x)$$

Examples:

- $\int 4x + 3x^2 dx = \int 4x dx + \int 3x^2 dx = 4 \int x dx + 3 \int x^2 dx = 4 \cdot \frac{1}{2}x^2 + 3 \cdot \frac{1}{3}x^3 = 2x^2 + x^3$
- $\int e^x - \frac{2}{x} dx = \int e^x dx - 2 \int \frac{1}{x} dx = e^x - 2\log(x)$

Integration Rules

u -substitution

Sometimes the function we are integrating is similar to a simpler function with an easy derivative.

For example, $\int \frac{1}{1-x} dx$ is similar to $\int \frac{1}{x} dx$ which we know is $\log(x)$. Similar to the chain rule, we can think about functions within functions.

Let's set $u = 1 - x$. If we differentiate the left with respect to u and the right with respect to x we have $du = -1dx$. Solving for dx we have $dx = -1du$. Now we can substitute these values into our original integral.

$$\int \frac{1}{1-x} dx = \int \frac{1}{u} \cdot (-1) du = -1 \int \frac{1}{u} du$$

Integration Rules

u -substitution continued

Now let's take the integral with respect to u :

$$\int \frac{1}{1-x} dx = -1 \int \frac{1}{u} du = -\log(u)$$

Then we can plug in the value for $u = 1 - x$:

$$\int \frac{1}{1-x} dx = -1 \int \frac{1}{u} du = -\log(u) = -\log(1-x)$$

Integration Rules

u -substitution continued

Example:

$$\int (2x + 4)^3 dx$$

We can take $u = 2x + 4$. Then $du = 2dx$ or $\frac{1}{2}du = dx$.

When we make the substitutions in our integral we have:

$$\int (2x + 4)^3 dx = \int u^3 \cdot \frac{1}{2} du = \frac{1}{2} \int u^3 du$$

Now we have an integral we can easily compute

$$\frac{1}{2} \int u^3 du = \frac{1}{2} \cdot \frac{1}{4} u^4 = \frac{1}{8} u^4$$

and then we just need to substitute back in for the functions of x .

$$\int (2x + 4)^3 dx = \frac{1}{2} \int u^3 du = \frac{1}{8} u^4 = \frac{1}{8} (2x + 4)^4$$

Finding Definite Integrals

Often we will be interested in knowing the exact area under the curve $f(x)$, not just the function $F(x)$.

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Examples:

- $\int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$
- $\int_0^{\infty} e^{-x} dx = -e^{-x}|_0^{\infty} = -e^{-\infty} - (-e^0) = -\frac{1}{e^{\infty}} + e^0 = 1$
- $\int_2^8 \frac{1}{x} dx = \log(x)|_2^8 = \log(8) - \log(2) = \log\left(\frac{8}{2}\right) = \log(4)$

Integration Example

distance, velocity, acceleration

Back to our original example, with $a = 2$. The velocity at any time $t = 3$ is the definite integral of the acceleration,

$$v(3) = \int_0^3 a(t) dt:$$

$$v(3) = \int_0^3 2 dt = 2t \Big|_0^3 = 2 \cdot 3 - 2 \cdot 0 = (3 - 0) \cdot 2 = 6$$

Similarly, the distance at any time $t = 3$ is the definite integral of the velocity, $d(3) = \int_0^3 v(t) dt$:

$$d(3) = \int_0^3 v(t) dt = \int_0^3 2t dt = t^2 \Big|_0^3 = 3^2 - 0^2 = 9$$

Example

$$\int_0^3 e^{x/3} dx$$

We could take $u = x/3$. Then $du = 1/3 dx$ and $3du = dx$.

When we substitute in for u and dx it is important to note that we must also substitute in for our limits of integration. The lower value $u = 0/3 = 0$ and the upper value would be $u = 3/3 = 1$.

$$\int_0^3 e^{x/3} dx = \int_0^1 e^u \cdot 3du = 3 \int_0^1 e^u du = 3e^u \Big|_0^1 = 3(e^1 - e^0) = 3(e - 1)$$

The End

Questions?