Counting

MATH/STAT 394: Probability I
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Introduction to Probability
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Appendix C

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Outline

n-tuple

Definition

An n-tuple is an ***ordered*** *set of n-elements taken from a set.*

Example

Colors on a striped flag from top to bottom

- e.g., (blue, white, red) (Columbia)
- ordered because (blue, white, red) $\neq$ (red, blue, white) (Russia)

Lemma

For 2-tuple (a, b). Suppose $a \in A$ with n_1 elements, and $b \in B$ with n_2 elements, then there are

$$n_1 \times n_2$$

possibilities for the tuple (a, b).

Exercise

A team of one boy and one girl is to be made form a group of 5 girls and 2 boys. How many different teams are there?

Solution

$$\begin{matrix} G_1B_1 & G_2B_1 & G_3B_1 & G_4B_1 & G_5B_1 \\ G_1B_2 & G_2B_2 & G_3B_2 & G_4B_2 & G_5B_2 \end{matrix} \qquad 5 \times 2 = 10$$

The generalized rule of counting

Lemma

Suppose an experiment consists r different outcomes, with the i-th outcome having n_i possibilities, then together there are

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

possibilities for the experiment.

Exercise

How many different license plates? POLL

____	____	____	____	____	____
letter	letter	letter	number	number	number

Solution

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

Outline

Permutations

Definition

A **permutation** *of* $\{1, \ldots, k\}$ *is a k-tuple such that numbers* ***cannot repeat****.*

Example

- How many 3-tuples made of the letters a, b, c?
- 3^3 as seen before
- How many 3-tuples made of the letters a, b, c *with no repetitions*? POLL

$$\underline{\quad\quad} \quad \underline{\quad\quad} \quad \underline{\quad\quad}$$
$$3 \times 2 \times 1 = 3! = 6$$

- Each of these arrangements is a permutation.
- The order matters!
- Number of permutations of n different objects

$$\boxed{n! = n \times (n-1) \times \cdots 2 \times 1}$$

- We define $0! = 1$.

Arrangements

Example

How many 3-tuples without repetition are there, made of the letters a, b, c, d, e, f, g?

Solution

$$7 \times 6 \times 5 = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7!}{(7-3)!} = (7)_3.$$

Definition

*If there are k slots for $1, 2, \ldots, n$, then the number of arrangements is **the number of k-tuples that can be selected from $\{1, 2, \ldots, n\}$ without repeating elements** and **is given by***

$$\boxed{(n)_k := n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.}$$

Outline

Sets

Definition

*A set is an **unordered** collection of **different** elements*

Example

- Colors on a painting $\{yellow, red, purple, green\}$
- Unordered because the painting does not have any order
- Different because we will not count the color twice

Note

- We use $(\cdot, \cdot, \cdot)$ for tuples
- We use $\{\cdot, \cdot, \cdot\}$ for sets

Combinations

Example

How many subsets of 3 elements are there, made of letters a, b, c, d, e, f, g?

- For each subset of size 3, we counted $3! = 6$ permutations (i.e. 6 different orderings).
- We counted $(7)_3$ possible 3-tuples with different elements (arrangements)
- Therefore, we divide the number of arrangements by 6:

$$\frac{(7)_3}{3!} = \frac{7!}{4!3!} = \frac{210}{6} = 35.$$

Combinations: order doesn't matter!

When order matters, there are $k!$ different orderings of the k items selected.

- If we have n items and want to select k of them,

$$\#(\text{combinations}) = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

- Define the **choose number**

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!\,k!}, \quad 0 \leq k \leq n.}$$

- The number $\binom{n}{k}$ is pronounced as n choose k,
  it is the number of ways to pick k objects from a set of n distinct objects.
- $\binom{n}{1} = n$ and $\binom{n}{n} = \binom{n}{0} = 1$.

Exercise

How many handshakes take place between a group of 6 people if everyone need to shake hands with everyone else?
Same question: how many combinations of 2 numbers among $\{1, 2, 3, 4, 5, 6\}$ are there?

Solution

$$\begin{array}{c} \{1,2\}\ \{1,3\}\ \{1,4\}\ \{1,5\}\ \{1,6\} \\ \{2,3\}\ \{2,4\}\ \{2,5\}\ \{2,6\} \\ \{3,4\}\ \{3,5\}\ \{3,6\} \\ \{4,5\}\ \{4,6\} \\ \{5,6\} \end{array}$$

$$\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \times 5}{2 \times 1} = 15$$

Exercise

5 women and 4 men take an exam. We rank them from top to bottom, according to their performance. There are no ties. POLL

1. How many possible rankings?
2. What if we rank men and women separately?
3. As in (ii), but Julie has the third place in women's rankings.

Solution

(i) A ranking in just another name for permutation of nine people. The answer is $\boxed{9!}$

(ii) There are 5! permutations for women and 4! permutations for men. Since any ranking for women can be 'tupled' with any ranking of men, by the counting principle, the total number is $\boxed{5!4!}$.

(iii) We exclude Julie from consideration, because her place is already reserved. There are four women remaining, so the number of permutations is 4!. For men, it is also 4!. The answer is $\boxed{4!^2}$

Properties of choose numbers

Choose number

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 0 \leq k \leq n.$$

Symmetry

$$\boxed{\binom{n}{k} = \binom{n}{n-k}.}$$

- For every subset of $\{1, 2, \ldots, 8\}$ of 2 elements there is a subset of 6 elements: its complement.
- For example,

$$\{3, 5\} \quad \leftrightarrow \quad \{1, 2, 4, 6, 7, 8\}.$$

- This is a one-to-one correspondence. So there are equally many subsets of two elements and subsets of six elements. Hence, $\binom{8}{2} = \binom{8}{6}$.

Practice

Among 4 married couples, we want to select a group of 3 people that is not allowed to contain a married couple. How many choices are there? POLL