UNIVERSITY of WASHINGTON

Counting, Set Operations

MATH/STAT 394: Probability I
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Introduction to Probability
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Appendices C, B

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Recap

- From n distinct items, number of ways to draw k of them

	without replacement (repeats not allowed)	with replacement (repeats allowed)
counting tuples (order matters)	$(n)_k = \frac{n!}{(n-k)!}$	n^k
counting subsets (order doesn't matter)	$\binom{n}{k} = \frac{n!}{(n-k)!k!}$	—

Outline

Practice solution

Practice

Among 4 married couples, we want to select a group of 3 people that is not allowed to contain a married couple. How many choices are there?

Solution Number of choices if the group can contain married couple(s):

$$N_1 = \binom{8}{3} = \frac{8!}{3! \times 5!} = 56$$

Number of choices if the group contain at least one married couple(s)?
Then it can only contain one couple.

$$N_2 = \binom{4}{1} \times \binom{6}{1} = 24$$

The number of choices that the group does not have a couple:

$$N_1 - N_2 = 32$$

Alternatively: there are $8 \times 6 \times 4$ ways of permuting 3 people where no married couple is contained. However, the order plays a role in this calculation, which we do not want. Therefore, there are $\frac{8 \times 6 \times 4}{3!} = 32$ number of choices that the group does not have a couple.

Properties of choose numbers

Choose number

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 0 \le k \le n.$$

Reduction property.

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$$

- Consider counting subsets $E \subseteq \{1, 2, 3, 4, 5\}$ of size 2. There are two possibilities:
  1. $5 \in E$. Then $E \setminus \{5\}$ is a one-element subset of $\{1, 2, 3, 4\}$; there are $\binom{4}{1}$ such subsets.
  2. $5 \notin E$. Then E is a two-element subset of $\{1, 2, 3, 4\}$. There are $\binom{4}{2}$ such subsets.

  Hence, adding the two cases, $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$.

Yang Hui's / Pascal's triangle

$n=0$:					1					
$n=1$:				1		1				
$n=2$:			1		2		1			
$n=3$:		1		3		3		1		
$n=4$:	1		4		6		4		1	

- Each element is the sum of two elements "on the shoulders" — the reduction formula.
- Starting from the edges, fill them with ones: $\binom{n}{0} = \binom{n}{n} = 1$.
- Then we fill the inside from top to bottom using this rule.

Yang Hui's / Pascal's triangle

$$
\begin{array}{lccccccccc}
n = 0: & & & & & \binom{0}{0} & & & & \\
n = 1: & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
n = 2: & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
n = 3: & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
n = 4: & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

- Each element is the sum of two elements "on the shoulders" — the reduction formula.
- Starting from the edges, fill them with ones: $\binom{n}{0} = \binom{n}{n} = 1$.
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Binomial theorem

Note

$$(x+y)^2 = x^2 + 2xy + y^2,$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

- Coefficients are taken from corresponding lines in Yang Hui's triangle!
- Why? Since $(x+y)^3 = (x+y)(x+y)(x+y)$, the coefficient before x^2y is just the number of ways of getting one y (and hence two x's), i.e., $\binom{3}{2}$. Hence,

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3.$$

In general,

$$\boxed{(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n}y^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.}$$

Power set

Example

How many subsets are there of the set $\{1, 2, \ldots, n\}$?

- For each $0 \leq k \leq n$, there are $\binom{n}{k}$ different subsets of size k. Then

$$\#(\text{subsets}) = \sum_{k=0}^{n} \binom{n}{k}$$

- Use the Binomial Theorem to simplify

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

- <u>Any other way to solve this question?</u>
- For each set, an element either belong to that set or does not (2 choices for each element, 2^n choices for all subsets)

Definition

For a set A, the power set of A is the set of its subsets and is often denoted 2^A.

Outline

Multinomial coefficients

Example
Putting 10 balls into 3 baskets: 5 into red basket, 2 into blue and 3 into yellow. How many combinations? POLL

Solution

$$\binom{10}{5}\binom{5}{2}\binom{3}{3} = \frac{10!}{5!(10-5)!} \cdot \frac{5!}{3!(5-3)!} = \frac{10!}{5!3!2!}$$

Definition (Multinomial Coefficient)

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, \ldots, n_r$, where $n_1 + n_2 + \cdots + n_r = n$.

Number of possible divisions is

$$\boxed{\binom{n}{n_1, n_2, \ldots, n_r} := \frac{n!}{n_1! n_2! \cdots n_r!}}$$

- Multinomial Coefficient decomposes as

$$\boxed{\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}}$$

- When $r = 2$, becomes binomial coefficient (choose function)

$$\boxed{\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2},}$$

  where $n_1 + n_2 = n$.

- **Multinomial Theorem**

$$\boxed{(a_1 + a_2 + \cdots + a_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}}$$

  The Binomial theorem is a special case when $r = 2$.

Outline

Combinations

Multinomial coefficients

Set Theory

Set theory

Definition

Let Ω be a set, that is a collection of elements or points

- *A is a subset of Ω, denoted $A \subseteq \Omega$, if it is a set composed of elements of Ω,*
- *Given an element ω of Ω and a subset A of Ω, either*
    - *ω belongs to A, denoted $\omega \in A$*
    - *ω does not belong to A, denoted $\omega \notin A$.*

Example

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, examples of subsets are

$$A = \{2, 4, 6\}, \quad B = \{1, 2, 3, 4\}, \quad C = \{1\}$$

There are alternative ways to specify a set, namely we can equivalently write

$$A = \{\omega \in \Omega, \omega \text{ is even}\}, \quad B = \{\omega \in \Omega : 1 \leq \omega \leq 4\}$$

Which statements are true? POLL

$$2 \in A, \qquad 5 \notin A, \qquad C \subseteq B$$

Set Theory

Note

- Subsets of the form $\{\omega\}$ are called singletons
- Note that $\omega \in \{\omega\}$ but ω is not the same as $\{\omega\}$

Definition

For A, B two subsets of Ω, either

- $A \subseteq B$: *A is a subset of B*: $\forall \omega \in A,\ \omega \in B$
- $A \nsubseteq B$: *A is not a subset of B*: $\exists \omega \in A, \omega \notin B$

Moreover, if $A \subseteq B$ and $B \subseteq A$, then $A = B$

Note

- Ω is a subset of Ω,
- the empty set denoted $\emptyset$ is a subset of Ω
- for any subset A of Ω, $\emptyset \subseteq A \subseteq \Omega$

Set operations

Definition

Let A, B be two subsets of a set Ω.

1. ***Intersection***, $A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}$
2. ***Union***, $A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}$
3. **Complement** *of A*, $A^c = \{\omega \in \Omega : \omega \notin A\}$
4. ***Set difference*** $A \setminus B := \{\omega \in \Omega : \omega \in A \text{ and } \omega \notin B\}$

Note

- $A^c = \Omega \setminus A$
- $A \setminus B = A \cap B^c$

Definition

Two sets A, B are disjoint if $A \cap B = \emptyset$.

Definition

The cardinality of a finite set A is the numbers of elements in the set and is denoted $\#A$.

Venn diagrams

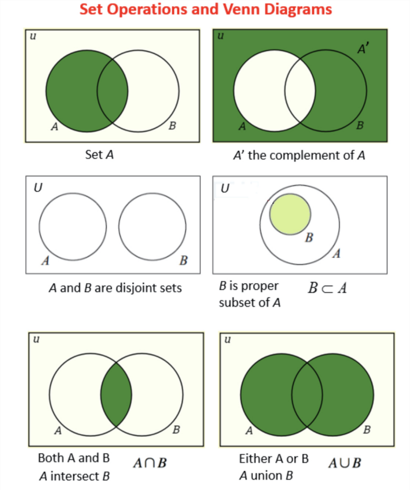

Figure from https://www.onlinemathlearning.com/venn-diagrams.html

Some rules

1. Commutative laws

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

2. Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. Distributive laws

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

DeMorgan's Laws

Lemma
For two subsets A, B of a set Ω,

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

- Outline of proof (to the first equation).
  1. Left $\subseteq$ Right: For any $x \in (A \cup B)^c$, then $x \in A^c \cap B^c$.
  2. Right $\subseteq$ Left: For any $x \in A^c \cap B^c$, then $x \in (A \cup B)^c$.

- DeMorgan's laws can be generalized to n events $A_1, \ldots, A_n$:

$$\boxed{\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c, \quad \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.}$$

Infinite sets, Infinite unions

Note

- The set Ω can be infinite, e.g. $\Omega = \mathbb{N}$
- We distinguish **countable sets** (s.t. that these sets can be mapped to a subset of $\mathbb{N}$ with every mapping different than the others), often called *discrete sets*
- and **uncountable sets**, for example $\mathbb{R}$, often called *continuous sets* (such sets cannot be mapped to $\mathbb{N}$ with a mapping that is distinct for all elements)

Definition

Given a sequence of subsets A_i of Ω we define

$$\bigcup_{i=1}^{+\infty} A_i = \{\omega \in \Omega : \text{ s.t. } \omega \in A_i \text{ for at least one index } i \in \{1, 2, \ldots\}\}$$

$$\bigcap_{i=1}^{+\infty} A_i = \{\omega \in \Omega : \text{ s.t. } \omega \in A_i \text{ for all indexes } i \in \{1, 2, \ldots\}\}$$

Practice for next lecture

Practice

Given a standard deck of 52 cards, how many poker hands (set of 5 cards) are designated three of a kind? That is, exactly 3 of the same value/number and two other cards that have different values from each other and from the trio. (Feel free to look up poker hands on wikipedia if that helps understand the definition)