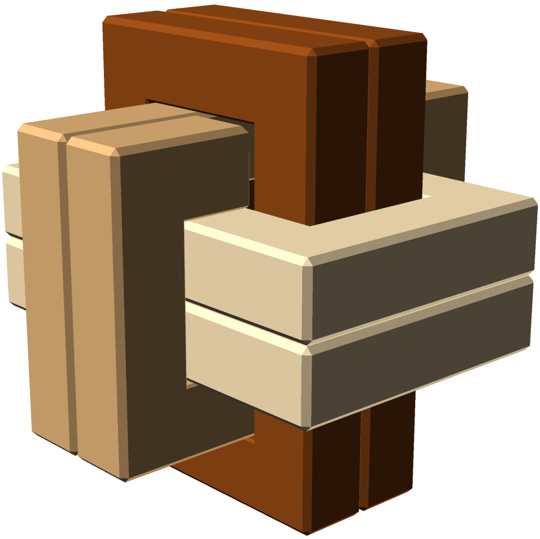
Board Burr Bundle



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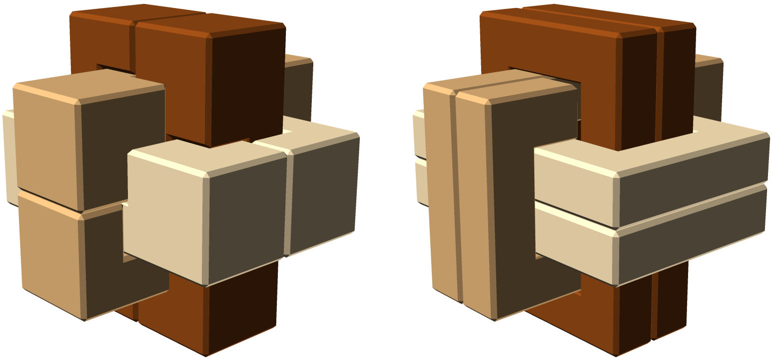


Figure 1. Ordinary six-piece burr (left) and six board burr (right).

Overview

Mechanical puzzles with six interlocking pieces have been a source of fascination for generations of puzzle enthusiasts. The familiar *six-piece burr* consists of six stick-shaped pieces with dimensions $6 \times 2 \times 2$ (see Figure 1, left). It is surely the oldest and best-known six-piece interlocking configuration: references date to at least 1733, and sets of pieces are frequently made and widely available.

The relatively neglected *six board burr* configuration (Figure 1, right) has identical assembled shape, but with the $6 \times 2 \times 2$ pieces replaced by flat $6 \times 4 \times 1$ "boards". It has a far shorter history, with the first documented examples appearing in the late twentieth century. In the early 2000s, an exhaustive computer analysis was carried out by Bill Cutler and Frans de Vreugd; some of their work is summarized in the *Analysis of Six Board Burrs* section of this pamphlet, below.

Although board burrs appear superficially similar to their better-known cousins, they provide a very different puzzle solving experience. The typical board burr puzzle admits many assemblies with a high degree of interlock, offering a substantial and fascinating challenge. Their unique structure has captivated designers and puzzle solvers alike and led to a growing repertoire of examples and variations.

The *Board Burr Bundle* is a set of eighteen pieces, organized into three color groups, that combine to form 55 distinct board burr challenges. Each challenge consists of two pieces of each color, which are paired in the solution as shown in Figure 1. The color constraint reduces the complexity of the solution space, yielding a nice selection of challenges of (mostly) manageable difficulty. Despite being drawn from the same limited set of pieces, the puzzle solutions exhibit surprisingly rich variation, both in difficulty and in solution contour. For a more extreme challenge, there is an additional set of puzzles that can be made if colors are disregarded.

The Challenges

In each of the challenges below, the goal is to assemble a particular set of six pieces, two from each color group, into the shape of Figure 1, with pieces of like color paired along each axis (the *color constraint*). For example, challenge #1 uses pieces 488 and 896 from the first color group, 256 and 496 from the second, and 1016 and 1024 from the third group. Every challenge has a unique solution: the pieces can be assembled into the color-constrained shape in one way only.

Each challenge is given with its *level* and *assembly count*, which together can be interpreted as an approximate measure of the puzzle's difficulty. The level indicates how many moves are necessary to disassemble the first three puzzle pieces: For example, 3.3.2 means that 3 moves are necessary to remove the first piece, an additional 3 moves for the second, and a further 2 moves for the third.

The assembly count gives the total number of possible assemblies. An assembly count of 40 means that the pieces *in theory* could fit together in 40 distinct ways. Since the challenges all have unique solutions, however, only one of those 40 can actually be assembled. The other 39 are *false assemblies*—theoretically achievable if one could "teleport" the pieces into place, but physically unattainable, since the pieces will block each other from moving into position. (All assembly counts assume that color groups are respected as in Figure 1. If the color constraint is ignored, then assembly counts are usually much higher.)

#	Piece Selection			Level	Assemblies
The puzzles in the first group have low levels and relatively low assembly counts, making them good challenges to start with.					
1	(488,896)	(256,496)	(1016,1024)	3.4.1	2
2	(488,960)	(256,496)	(1016,1024)	3.5.1	4

3	(256,511) (1016,1024) (384,1012)	3.3.2	6
4	(511,896) (256,496) (1016,1024)	4.3.2	8
5	(256,511) (512,1016) (384,1024)	3.4.1	12
6	(488,960) (256,1016) (1016,1024)	3.5.1	12
7	(511,896) (256,1016) (1012,1016)	4.4.3	12

The challenge increases somewhat with more false assemblies.			
8	(508,896) (256,1016) (1012,1016)	3.4.3	28
9	(256,511) (512,512) (1012,1024)	3.2.2	40
10	(488,508) (512,512) (384,1024)	3.4.1	48
11	(256,960) (512,512) (384,1016)	3.4.1	48
12	(508,896) (256,512) (1012,1024)	3.4.3	52
13	(256,508) (512,1024) (1012,1016)	3.4.3	56
14	(508,511) (1016,1024) (384,1016)	3.5.4	63

The puzzles in this group have higher levels for the first or second piece. Puzzle #26 has the highest (first-piece) level of any six-piece board burr in three-color configuration.			
15	(488,960) (256,1016) (384,1016)	3.8.1	8
16	(488,508) (256,1024) (384,1016)	3.8.7	8
17	(256,896) (496,1016) (1012,1024)	3.7.1	20
18	(256,960) (496,1016) (1012,1016)	4.6.2	22
19	(511,896) (256,1016) (1024,1936)	3.11.2	26
20	(511,896) (256,512) (1024,1936)	8.3.2	24
21	(256,896) (496,1024) (1024,1936)	2.8.1	28
22	(508,896) (256,1024) (384,1016)	3.7.7	28
23	(511,960) (256,1024) (1012,1016)	4.6.5	30
24	(896,960) (256,1016) (384,1016)	3.7.1	40

25	(508,511)	(256,1024)	(1024,1936)	3.10.3	46
26	(511,960)	(256,512)	(1024,1936)	11.3.2	46
27	(896,960)	(256,512)	(1012,1024)	6.2.2	60
28	(256,508)	(512,1024)	(384,1016)	3.6.3	64
29	(256,511)	(512,1024)	(1024,1936)	8.3.1	72

The puzzles in this group have large numbers of assemblies, but still with unique solutions.					
30	(896,960)	(256,1016)	(480,1016)	3.3.2	86
31	(508,960)	(256,1024)	(1012,1016)	3.6.5	88
32	(488,960)	(512,1024)	(1016,1936)	4.5.3	92
33	(511,960)	(496,1024)	(384,1024)	4.5.3	112
34	(511,960)	(1016,1024)	(384,1016)	2.4.3	126
35	(511,960)	(496,1024)	(1012,1024)	5.5.3	127
36	(511,960)	(1016,1024)	(1016,1936)	4.4.4	143
37	(896,960)	(496,1024)	(1012,1016)	4.6.4	151
38	(508,896)	(496,512)	(1012,1024)	2.4.1	184
39	(896,960)	(496,1016)	(1016,1024)	3.4.4	192
40	(508,511)	(512,1016)	(1012,1024)	3.5.3	195
41	(508,960)	(1016,1024)	(1016,1936)	3.4.6	208
42	(508,896)	(512,1016)	(384,1024)	3.6.1	216
43	(896,960)	(496,512)	(1012,1024)	4.1.4	240
44	(511,960)	(1016,1024)	(480,1016)	5.5.4	265
45	(508,960)	(1016,1024)	(1012,1016)	2.3.3	276
46	(508,960)	(1016,1024)	(480,1012)	3.4.4	415
47	(256,508)	(512,512)	(480,1024)	3.4.1	448
48	(511,960)	(512,1024)	(480,1016)	5.5.3	516
49	(508,960)	(496,512)	(1012,1024)	2.4.3	524

With high intermediate levels and many false assemblies, the puzzles in this group present a serious challenge. Puzzle #55 has the highest combined level of any six-piece board burr in three-color configuration.					
50	(488,960)	(496,512)	(480,1016)	3.12.2	63
51	(896,960)	(496,512)	(480,1016)	3.11.2	324
52	(508,960)	(512,1016)	(480,1016)	3.5.8	840
53	(508,960)	(512,512)	(480,1016)	3.11.2	928
54	(508,960)	(512,1024)	(480,1012)	3.7.4	952
55	(508,511)	(512,512)	(480,1024)	4.19.1	576

Additional Puzzles

If one allows any six pieces to be combined without regard for color, then hundreds more puzzles with unique solutions can be constructed. Here is a sampling. In each of these puzzles, ignore colors completely—pieces of like color will not necessarily line up in the solution! Several of the puzzles on this list have thousands of false assemblies.

#	Piece Selection						Level	Assemblies
56	256	256	488	496	1024	1024	3.3.1	5
57	256	256	488	512	1016	1024	3.4.3	12
58	256	488	511	1012	1016	1024	3.5.1	20
59	256	384	488	1012	1016	1024	3.8.1	38
60	256	511	1012	1024	1024	1936	3.10.3	168
61	256	256	480	508	1024	1024	3.7.10	432
62	256	488	508	512	960	1016	3.11.6	445
63	256	480	511	512	512	1016	4.10.10	672
64	480	488	512	896	960	1016	3.12.2	1748
65	256	480	896	960	1012	1024	7.2.2	1480
66	384	480	508	511	960	1024	3.3.3	5144
67	480	511	512	512	896	960	5.8.3	5584
68	384	480	496	512	960	1024	5.5.2	6584

Analysis of Six Board Burrs

Six board burrs bear a superficial resemblance to their better-known cousins, the ordinary six-piece burrs. They have the same assembled shape, and their assemblies have the same internal structure. The similarities mostly end there, however: The board burr configuration interlocks more tightly, so that the vast majority of assemblies (more than 99.99%!) cannot be taken apart.

The following comparison illustrates the differences. In this table, an assembly is "solvable" if it can be completely taken apart. A "puzzle" is a particular selection of six pieces that can form at least one solvable assembly.

	Six-Piece Burr		Six Board Burr	
Assemblies	35,657,131,235		34,443,659,727	
Solvable Assemblies	5,748,127,225	(16.1%)	1,620,359	(0.005%)
Puzzles	1,549,741,261		211,638	
Unique Sol'n Puzzles	741,320,226		62,666	

Solvable board burr puzzles are evidently a rarefied breed, and they are also rather difficult. High-level board burrs are especially rare. The highest level for any six-piece board burr assembly is 13; the highest level for any *puzzle* (with or without the three-color scheme) is 11.[1] A level 11 example is given by Puzzle #26 in this set. Remarkably, there are several level 11 puzzles with solutions that are unique, even without any color scheme.

A peculiar feature of board burrs is that the number of moves to remove the second, or even third, piece is occasionally strikingly high, even when the first-move level is low. Two closely related assemblies have level 8.19.2.1.2, requiring 32 total moves to disassemble, the highest possible. Puzzle #55 comes close, with level 4.19.1.3.4, or 31 total moves, and it has the advantage of a unique solution using the three-color scheme.

Solvable board burr assemblies—and particularly the higher-level ones—also tend to be mostly hollow (since the denser ones have little freedom of movement and usually can't be taken apart), so that the pieces fit together in many orientations, leading to many false assemblies. The highest overall for a board burr is a whopping 29,568 assemblies with just 7 solutions, and there are several puzzles with 18,432 assemblies and a unique solution! An example of an ultra-high assembly count that can be made with

[1] Every set of pieces that can form a level-13 assembly can also form another assembly with lower level. This is true even when the three-color scheme is used; but other, more restrictive, color schemes can enforce the level 13 solution. See, for example, Cutler and de Vreugd's *Chocolate Dip Burr*.

the *Board Burr Bundle* is {480,496,508,512,960,1024}, which has 15,312 assemblies (disregarding colors) and 3 solutions. (Since it has multiple solutions, it does not appear in the above list of challenges.)

In designing the *Board Burr Bundle*, the three-color scheme was used in order to reduce the solution space, making the puzzle difficulty more manageable. The particular set of pieces used in the set can make a remarkable 113 puzzles with unique solution. If one eliminates "duplicates"—puzzles that can be obtained from another in the set by simply adding or removing voxels to the solution assembly—then the number of puzzles reduces to 70. The 55 challenges were curated from that initial list of 70 by further removing puzzles that have identical level and/or are very similar in solution texture.

The Numbering Scheme

The pieces are identified using Jürg von Känel's numbering scheme, originally introduced in the analysis of ordinary six-piece burrs. Every board burr piece can be obtained by removing various holes from a solid $6 \times 4 \times 1$ block. Of the 24 voxels in the $6 \times 4 \times 1$ block, 12 must always be present in order to realize the desired assembled shape. The other 12 are each assigned a power of 2 as in the following diagram:

		1024	2048		
	16	32	64	128	
	1	2	4	8	
		256	512		

The calculation is as follows: start with 1 (the Känel number of a solid $6 \times 4 \times 1$ piece) and add up all the numbers corresponding to voxels that are removed. For example, piece 488 is given by

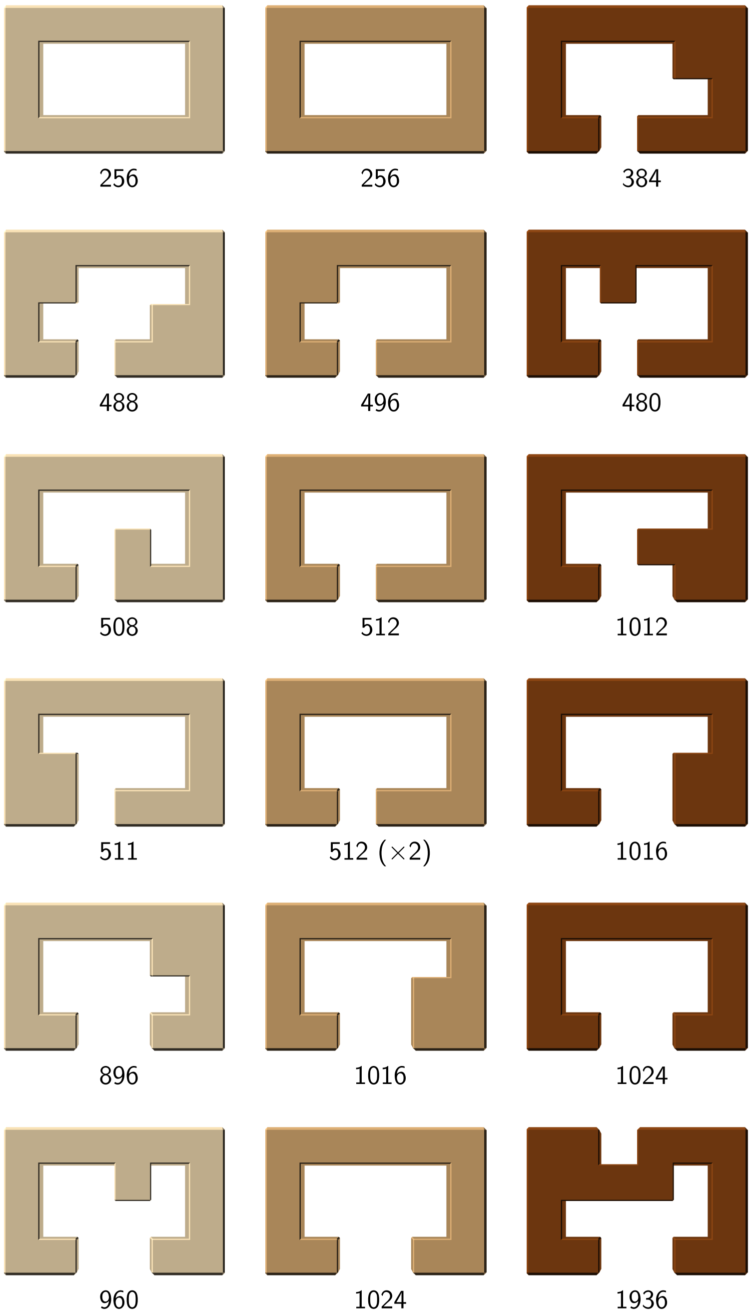
$$1 + 1 + 2 + 4 + 32 + 64 + 128 + 256 = 488.$$

In general, each piece can be rotated into several different orientations, and each distinct orientation gives a different value for this calculation. The Känel number of a physical piece is the *smallest* value across all of that piece's orientations.

Out of the 4,096 possible ways of removing voxels from a $6 \times 4 \times 1$ board, 1,871 would give disconnected shapes, which cannot be realized in a physical puzzle. This leaves 2,225 valid piece orientations. Those 2,225 "oriented pieces" reduce to 581 distinct physical pieces (and hence, 581 distinct Känel numbers).

Most of those 581 pieces are useless for the purpose of constructing puzzles. 133 pieces do not participate in any assemblies at all. (To visualize why, consider piece 1, the solid $6 \times 4 \times 1$ block. If it appears anywhere in the assembly, then it severs the entire puzzle in two, preventing any pieces from passing crosswise through it.) Of the remaining 448, just 113 participate in at least one solvable puzzle.

For high-level puzzles, the situation is even more constrained. A previous exhaustive analysis by Cutler and de Vreugd determined that just 31 distinct pieces collectively make "all puzzles having a first level of 8 or higher, a second level of 13 or higher, and/or a total number of moves for the first and second level of at least 16 moves." (This is perhaps a less striking observation when one considers that there are only 3,045 assemblies that meet the Cutler–de Vreugd criterion, out of a total of 34 billion!) The pieces in the *Board Burr Bundle* are drawn from the "Cutler–de Vreugd 31," with the sole exception of number 488, which is included to introduce more variety and allow for a larger number of easier puzzles.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