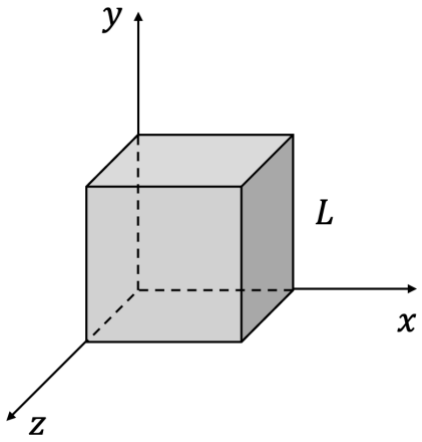
Blackbody Radiation

Physics 153 – Spring 2026

1 Cavity Blackbody Radiation

By definition, a blackbody is an object that emits radiation in a manner that is independent of the body's material composition or shape. Let's assume the blackbody is a cavity in the shape of a cube with side length L and perfectly conductive walls. We use a Cartesian coordinate system with the origin at one corner of the cube and the x, y, and z axes along the edges:



The electric field must be zero at each of the six conductive walls of the cavity. This constitutes boundary conditions that are similar to those for a string fixed at two ends or a tube closed at both ends. This cavity can therefore support *electromagnetic standing waves* with definite frequency f and wavelength λ that can be related to the geometry of the cube.

For a moment, consider two sinusoidal electromagnetic waves moving along the x axis with their electric fields polarized along the y axis. One wave moves to the right while the other moves to the left. Only at certain frequencies will the two waves cancel at the boundaries:

$$E = E_0 \sin(k_x x - \omega t) + E_0 \sin(k_x x + \omega t) = 2E_0 \cos \omega t \sin k_x x$$

- $E = 0$ at $x = 0$ is already satisfied (I cheated when I used sine functions to begin with).
- $E = 0$ at $x = L$ is a little trickier, but we've seen this before in a different context. This requires $\sin k_x L = 0$, which is possible if and only if $k_x L = \pi n_x$ where n_x is a positive integer. Since $k_x = 2\pi/\lambda_x$, we find the familiar result $\lambda_x = 2L/n_x$.

We can easily extend this discussion to three dimensions, where the standing wave propagates along some arbitrary direction inside the cube. Let $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k}$ be the **wave vector** for the electromagnetic wave, where each component satisfies

$$k_x = \frac{2\pi}{\lambda_x}\,, \qquad k_y = \frac{2\pi}{\lambda_y}\,, \qquad k_z = \frac{2\pi}{\lambda_z}$$

The wave vector indicates the direction of propagation. For a standing wave, it's the direction along which two waves travel in opposite directions to produce a standing wave pattern. Just as in the 1D case, only certain wave vector components will satisfy the boundary conditions:

$$k_x L = \pi n_x\,, \qquad k_y L = \pi n_y\,, \qquad k_z L = \pi n_z$$

The magnitude of the wave vector (also called the wave number) is related to the wavelength of the wave by $k = \left|\vec{k}\right| = 2\pi/\lambda$. Like any vector, $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$, so we find

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

Furthermore, since a monochromatic electromagnetic wave satisfies $c = \lambda f$,

$$f = \frac{c}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

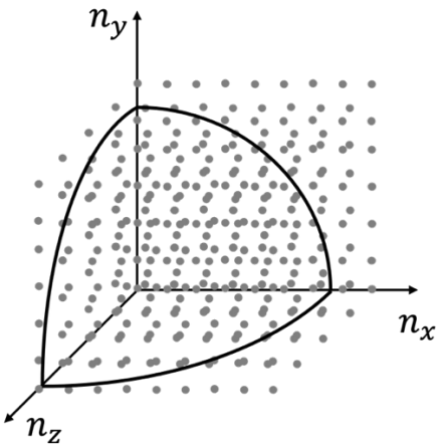
This is the frequency of an electromagnetic standing wave inside the cavity.

2 Number of Modes Per Frequency

Let N be the number of standing waves (called modes) in the cavity with frequency less than or equal to f. Each combination of $n_x, n_y,$ and n_z that satisfies the equation for f in the previous section represents a unique standing wave with frequency f. In a typical cavity, $n_x, n_y,$ and n_z are large numbers. For instance, consider EM waves of visible light with $f \approx 10^{14}$ Hz inside a cube with a side length of 1 m:

$$\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{2Lf}{c} = \frac{2(1\text{ m})(10^{14}\text{ Hz})}{3\times 10^8\,\frac{\text{m}}{\text{s}}} \approx 10^6$$

We can therefore use a trick to solve for the number of modes. Imagine a lattice of points each corresponding to a unique combination of $n_x, n_y,$ and n_z for standing waves with frequency less than or equal to f. The difference between any two nearby points is so small compared to their maximum values, the discrete lattice becomes effectively continuous. The trick to calculate the number of lattice points is to instead compute the volume contained within a sphere of radius



$$R = \frac{2Lf}{c}$$

Since the values of $n_x, n_y,$ and n_z are positive, we only need the volume of one-eighth of the sphere in "n-space." Hence,

$$\#\text{ of lattice points} = \frac{1}{8}\left(\frac{4}{3}\pi R^3\right) = \frac{4}{3}\frac{\pi L^3 f^3}{c^3}$$

Technically, the number of standing waves is twice the number of lattice points because for each lattice point, there are two independent polarization states for an electromagnetic wave. For example, for a circularly polarized wave, there are two ways the electric field can rotate—clockwise or counterclockwise around the direction of propagation. Thus,

$$N = 2 \times (\#\text{ of lattice points}) = \frac{8}{3}\frac{\pi L^3 f^3}{c^3}$$

It will be useful to know how many modes exist within a small frequency interval df around f. We can determine this by taking the derivative of N with respect to f:

$$\frac{dN}{df} = \frac{8\pi L^3}{c^3} f^2$$

Thus, for a small frequency interval df around frequency f, the number of modes is given by

$$dN = \frac{8\pi L^3}{c^3} f^2\, df$$

3 Rayleigh-Jeans Law

The **spectral energy density** is the average energy per unit volume per unit frequency of the radiation within the cavity. Denoted $u_f(f,T)$, the spectral energy density is

$$u_f(f,T) = \frac{\#\text{ of modes}}{\text{frequency}} \times \frac{\text{average energy per mode}}{\text{cavity volume}}$$

We can use concepts from statistical mechanics to derive the average energy per mode; assuming the energy of the standing waves is continuous, the average energy per mode is $k_B T$ where k_B is the Boltzmann constant and T is the temperature of the cavity in thermal equilibrium with its radiation. We'll discuss how the average energy per mode is derived in a moment. For now, we can use the result to derive Rayleigh-Jeans law for the spectral energy density:

$$u_f(f,T) = \frac{8\pi L^3}{c^3}\frac{k_B T}{L^3} = \frac{8\pi f^2}{c^3}\, k_B T$$

4 Statistical Mechanics

A system is in thermal equilibrium with its surroundings when its temperature is well-defined and constant. The system's energy fluctuates as it exchanges energy with its surroundings, but there is no net energy flow in or out. Although the system may be able to take on many different energies, not all energies are equally likely. Lower energies occur more often than higher ones, and the likelihood of finding the system with energy E decreases exponentially as E increases. This exponential weighting is described by the *Boltzmann factor*, $e^{-E/(k_B T)}$, where k_B is Boltzmann's constant. Physically, the Boltzmann factor reflects the fact that when the system has higher energy, the surrounding environment has fewer microscopic configurations available, making such situations less probable.

4.1 Continuous Energy Exchange

When the energy of the system is continuous, probabilities are described using a probability density function $\rho(E)$, where $\rho(E)\,dE$ gives the probability that the energy lies between E and $E + dE$. In thermal equilibrium, this probability density is proportional to the Boltzmann factor and is written as $\rho(E) = \frac{1}{Z} e^{-E/(k_B T)}$. The constant Z, called the *partition function*, ensures that the total probability is equal to one. It is determined by imposing the normalization condition $\int_0^\infty \rho(E)\,dE = 1$, which leads to $Z = \int_0^\infty e^{-E/(k_B T)}\,dE$.

The average, or mean, energy of the system is defined as a probability-weighted average over all possible energies. For continuous energies, this is given by

$$\langle E \rangle = \int_0^\infty E\,\rho(E)\,dE = \frac{1}{Z}\int_0^\infty E\, e^{-E/k_B T}\,dE$$

Evaluating the integrals shows that the partition function equals $k_B T$ and the numerator equals $(k_B T)^2$, so the average energy is $\langle E \rangle = k_B T$. This result highlights the role of temperature in thermal physics; at equilibrium, the temperature sets the characteristic energy scale of the system, with higher temperatures corresponding to broader energy distributions and larger average energies. Technically, this result assumes a constant "density of states," meaning each interval dE contains the same number of microscopic states, regardless of the total energy of the macroscopic system. Many systems do not have a constant density of states. For example, as the energy of an ideal gas grows, there are more states per unit of energy, so the density of states grows with energy.

4.2 Discrete Energy Exchange & Planck's Distribution

Max Planck was an expert in statistical mechanics and had privileged access to precise spectra of blackbodies from experimental studies. He was able to derive a formula for the spectral density by replacing the integral in the calculation above with a summation. The only way to justify the calculation was to consider discrete energy exchanges by an amount $\Delta E = hf$. This implies the energy is quantized as $E = nhf$ where n is an integer. In the case of a discrete energy exchange, the integral becomes a summation:

$$\langle E \rangle = \frac{1}{Z} \sum_{all\ states} E\, e^{-\frac{E}{k_B T}} = \frac{1}{Z} \sum_{n=0}^{\infty} nhf\, e^{-\frac{nhf}{k_B T}}$$

where the partition function integral is also replaced by a summation:

$$Z = \sum_{all\ states} e^{-\frac{E}{k_B T}} = \sum_{n=0}^{\infty} e^{-\frac{nhf}{k_B T}}$$

Let $x = hf/k_B T$. The average energy can then be written as

$$\langle E \rangle = \frac{hf}{Z} \sum_{n=0}^{\infty} n\, e^{-nx}, \text{where } Z = \sum_{n=0}^{\infty} e^{-nx}$$

This calculation can be done "by hand" various ways, but the coolest way is to use a trick! Notice that taking the derivative of the partition function with respect to x gives

$$\frac{\partial Z}{\partial x} = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} e^{-nx} = -\sum_{n=0}^{\infty} n\, e^{-nx}$$

Furthermore, consider the derivative of the natural logarithm of the partition function:

$$\frac{\partial}{\partial x}\ln Z = \frac{1}{Z}\frac{\partial Z}{\partial x} = -\frac{1}{Z}\sum_{n=0}^{\infty} n\, e^{-nx}$$

If we simply multiply this quantity by $-hf$, we recover the average energy! In other words,

$$\langle E\rangle = -hf\frac{\partial}{\partial x}\ln Z$$

Once we evaluate the partition function, we can use this expression to find the average energy. The partition function above is in fact a *geometric series*. Let's write a few terms:

$$Z = \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty}(e^{-x})^n = 1 + (e^{-x}) + (e^{-x})^2 + \cdots$$

Since $e^{-x} < 1$ for all $x > 0$, each term gets smaller and smaller, so it contributes less and less to the sum. One can prove that Z is finite even though the series has infinitely many terms. Assuming Z is finite, there's a slick way to evaluate the series. Let $r = e^{-x}$, multiply the partition function by r and write out the first few terms:

$$rZ = r + r^2 + r^3 + \cdots = Z - 1$$

Solving for the partition function, we find

$$Z = \frac{1}{1-r} = \frac{1}{1-e^{-x}}$$

Substitute this result into the expression for the average energy:

$$\langle E\rangle = -hf\frac{\partial}{\partial x}\ln Z = -hf\frac{\partial}{\partial x}\ln(1-e^{-x})^{-1} = hf\frac{\partial}{\partial x}\ln(1-e^{-x})$$

After using the chain rule and simplifying,

$$\langle E\rangle = \frac{hf}{e^x - 1} = \frac{hf}{e^{hf/k_BT} - 1}$$

This represents the average energy per mode, so the spectral energy density is

$$u_f(f,T) = \frac{8\pi f^2}{c^3}\,\frac{hf}{e^{hf/k_BT} - 1} = \frac{8\pi h}{c^3}\,\frac{f^3}{e^{hf/k_BT} - 1}$$

This is Planck's distribution, and it exactly matches the experimentally determined spectrum of blackbody radiation as long as $h = 6.626 \times 10^{-34}\ \mathrm{J\cdot s}$.