Phys 152: Fundamentals of Physics II

Unit #8 - Electromagnetic Induction

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Michael Faraday

By the 1830's it was well established that an electric current creates a magnetic field. In 1831, Michael Faraday demonstrated the reverse was also true–a magnetic field could create an electric current under the right circumstances. The mathematical relationship between electricity and magnetic fields is called Faraday's law of induction.

Faraday's law revolutionized our understanding of electromagnetism and led to the development of electrical generators and transformers, making practical electricity generation and transmission possible. As the apocryphal tale goes, Faraday was once asked by an inquiring politician about the usefulness of his discoveries, to which he replied "I know not, but I wager that one day *your government will tax it*."

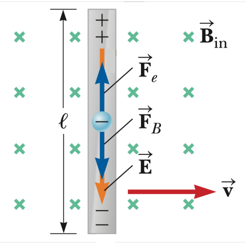
Motional emf with a uniform magnetic field

Motional emf refers to the electromotive force induced in a conductor when it's moved through a magnetic field. The magnetic force causes the ends of the conductor to become oppositely charged, which establishes an electric field.

In steady state, electric and magnetic forces on an electron in the conductor are balanced.

$$\sum F = qE - qvB = 0 \Rightarrow E = vB$$

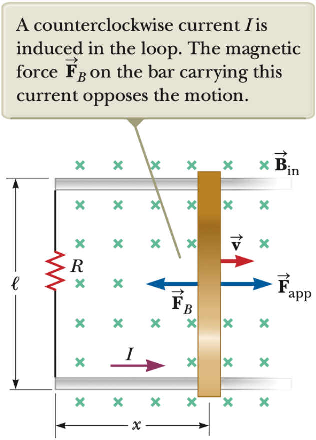
$$\Delta V = E\ell = B\ell v$$



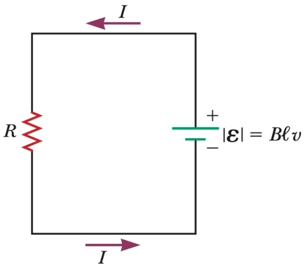
Motional emf in a circuit

When the moving conductor is part of a closed conducting path, it serves as a constant source of emf for an electric circuit. Consider a circuit consisting of a conducting bar of length ℓ, sliding along two fixed, parallel conducting rails. Assume the total resistance of the circuit is R, and a uniform magnetic field $\vec{\mathbf{B}}$ is applied perpendicular to the plane of the circuit.

When the bar is pulled to the right at constant speed, a motional emf will develop in the bar and a current will flow in the counterclockwise direction. In steady state, the applied force will balance the magnetic force on the bar!



$$I = \frac{|\mathcal{E}|}{R} = \frac{B\ell v}{R}$$



Magnetic Flux

Magnetic flux is defined in a similar manner as electric flux:

$$\Phi_B \equiv \int \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{a}}$$

The SI unit of magnetic flux is the weber (Wb); $1\ \mathrm{Wb} = 1\ \mathrm{T} \cdot \mathrm{m}^2$.
In the previous example, the emf produced in the circuit is directly related to the rate of change of magnetic flux passing through the area spanned by the circuit:

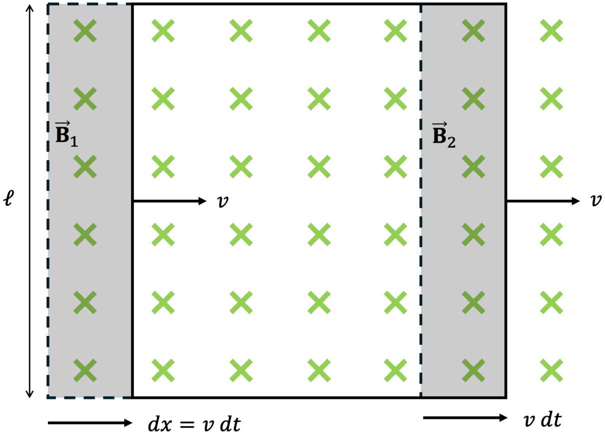
$$|\mathcal{E}| = B\ell v = \left|\frac{d}{dt}(B\ell x)\right| = \left|\frac{d\Phi_B}{dt}\right|$$

Motional emf with a nonuniform magnetic field

Consider a conducting loop moving with speed v in the x direction. Let its position at some instant t be such that the magnetic field strength is B_1 at the left side of the loop and B_2 along the right side. We can demonstrate that even in the case of a nonuniform magnetic field, there is a relationship between the induced emf and the change in magnetic flux. *This is the key to understanding Faraday's law.*

$$|\mathcal{E}| = B_2\ell v - B_1\ell v = (B_2 - B_1)\ell v$$

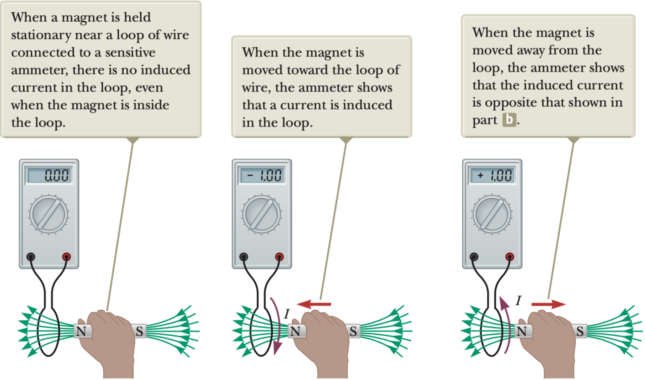
$$d\Phi_B = +B_2\ell v\ dt - B_1\ell v\ dt \Rightarrow \left|\frac{d\Phi_B}{dt}\right| = (B_2 - B_1)\ell v = |\mathcal{E}|$$


$\vec{\mathbf{B}}_1$
$\vec{\mathbf{B}}_2$
ℓ
v
v
$dx = v\,dt$
$v\,dt$

Induction with a moving bar magnet

We expect the results to be independent of whether the conducting loop or the source of the nonuniform magnetic field is moving (motion is relative), and we demonstrate this with a simple experiment.

Consider a stationary loop of conducting wire connected to an ammeter. Indeed, a current will be induced in the loop when a bar magnet moves towards or away from it. No current is induced when the bar magnet is stationary. Interestingly, the induced current flows in one direction when the magnet moves toward the loop and in the opposite direction when the magnet moves away.

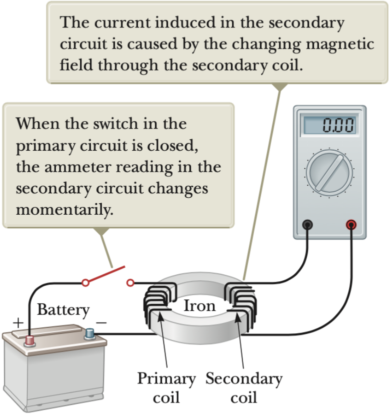

When a magnet is held stationary near a loop of wire connected to a sensitive ammeter, there is no induced current in the loop, even when the magnet is inside the loop.
0.00
N
S
When the magnet is moved toward the loop of wire, the ammeter shows that a current is induced in the loop.
− 1.00
I
N
S
When the magnet is moved away from the loop, the ammeter shows that the induced current is opposite that shown in part b.
+ 1.00
I
N
S

As an aside, we typically explain the case of a moving bar magnet with Faraday's law (see below) whereas the case of a moving conductor is explained by motional emf (the Lorenz force law). Albert Einstein found this unsatisfactory: Why should the physical explanation implicitly depend on the state of motion of the observer? Einstein sought a more fundamental explanation where the laws of physics, including electromagnetic phenomena, are the same for all observers, regardless of their relative motion. This led him to develop the special theory of relativity in 1905.

Faraday's Discovery

From our discussion thus far, induction can be seen as a natural consequence of the force on a charge moving in a magnetic field. From the magnetic force law, it follows that the induced emf is equal in magnitude to the rate of change in the magnetic flux.

Faraday used an apparatus consisting of two closely-spaced conducting coils to demonstrate that a changing current in one coil (the primary) induces a current in the other (the secondary). Although the experiment works well if the coils are separated by air, the effects can be enhanced if both coils are wrapped around an iron ring. This experiment cannot be easily explained in terms of the magnetic force law–we've discovered another fundamental law of nature!


The current induced in the secondary circuit is caused by the changing magnetic field through the secondary coil.
When the switch in the primary circuit is closed, the ammeter reading in the secondary circuit changes momentarily.
0.00
Battery
+
−
Iron
Primary coil
Secondary coil

Faraday's Law of Induction

Almost all of the experiments considered so far have one thing in common; an emf is induced in a loop when the magnetic flux through the loop changes with time. Mathematically,

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)}$$

If a coil consists of N loops with the same area, and Φ_B is the magnetic flux through one loop,

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

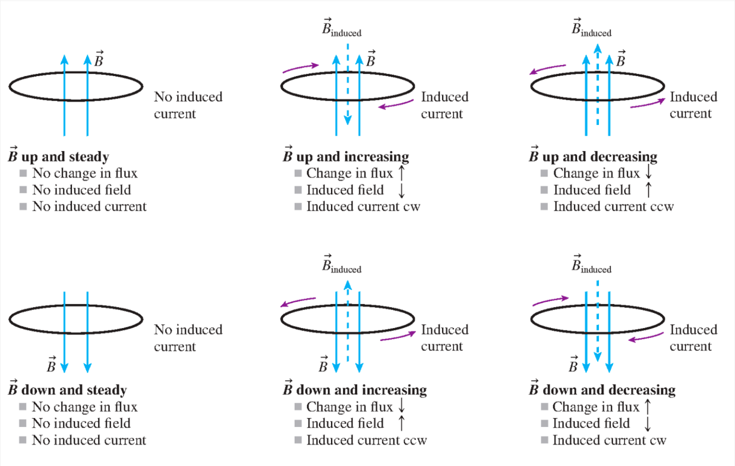
Lenz's Law

The minus sign in Faraday's law tells us which way an induced current will flow, and it is best understood in the language of vector calculus. For now, it will be easier to use Lenz's law:

The induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop.

Electromagnetic induction is "inertial." A conducting loop likes to maintain a constant flux. If you try to change the flux, the loop responds by producing a current that opposes the change.

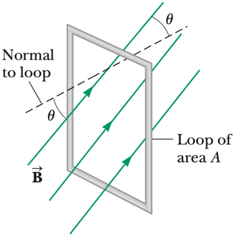
Using Lenz's Law



Induction in a Conducting Loop

Suppose a loop enclosing an area A lies in a uniform magnetic field $\vec{\mathbf{B}}$. The magnetic flux through the loop is equal to $BA\cos\theta$, where θ is the angle between the magnetic field and the normal to the loop; hence, the induced emf is

$$\mathcal{E} = -\frac{d}{dt}(BA\cos\theta)$$



Conceptual Question 1

A circular loop of wire is held in a uniform magnetic field, with the plane of the loop perpendicular to the field lines. Which of the following will not cause a current to be induced in the loop?

(a) crushing the loop

(b) rotating the loop about an axis perpendicular to the field lines

(c) keeping the orientation of the loop fixed and moving it along the field lines

(d) pulling the loop out of the field

Conceptual Question 2

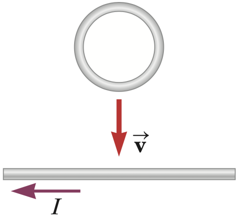
The figure shows a circular loop of wire falling toward a wire carrying a current to the left. What is the direction of the induced current in the loop of wire?

(a) clockwise

(b) counterclockwise

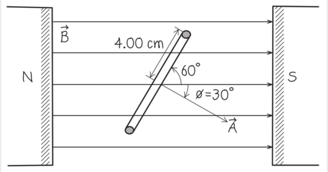
(c) zero

(d) impossible to determine



Example 1: Faraday's law

A 500-loop circular coil with radius 4.00 cm is placed between the poles of a large electromagnet. The magnetic field is uniform and makes an angle of 60.0° with the plane of the coil. If the magnetic field magnitude decreases at a rate of 0.200 T/s, what is the magnitude of induced emf and which way will the induced current flow?



Generators and Motors

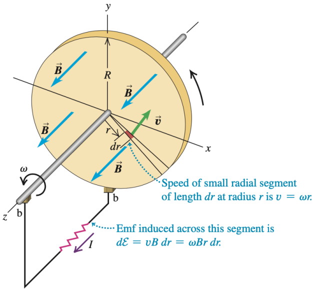
An electric generator is a device that converts mechanical or chemical energy into electrical energy. The simplest mechanical generators consist of conducting coils rotated by some external means in a magnetic field. As the coil rotates, the magnetic flux through the area enclosed by the loop changes with time, and this change induces an emf and a current in the loop according to Faraday's law.

An electric motor is a device that converts electrical energy into mechanical work. A motor is essentially a generator operating in reverse. Instead of generating a current by rotating a coil, a current is supplied to the coil by a battery, and the torque acting on the current-carrying coil causes it to rotate.

Example 2: Faraday disk dynamo

The first electromagnetic generator, the Faraday disk, was invented in 1831 and operates on the principle of motional emf.

A conducting disk with radius R lies in the xy-plane and rotates with constant angular velocity ω about the z-axis. The disk is in a uniform $\vec{\mathbf{B}}$ field in the z-direction.



Find the induced emf between the center and the rim of the disk.

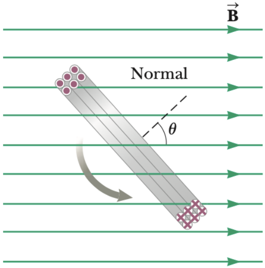
Induced current in a rotating coil

Suppose a coil with N turns, with the same area A, rotates in a magnetic field with a constant angular speed ω. If $\theta = \omega t$ is the angle between the magnetic field and the normal to the plane of the coil (as a function of time), the induced emf in the coil is

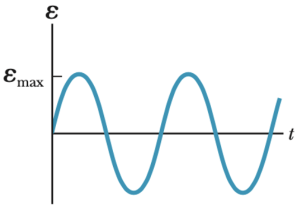
$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(NBA\cos\omega t) = NBA\omega\sin\omega t$$

$$\mathcal{E}_{\max} = NBA\omega$$

A generator that makes use of a rotating coil naturally produces alternating current, but it can be made to produce DC.



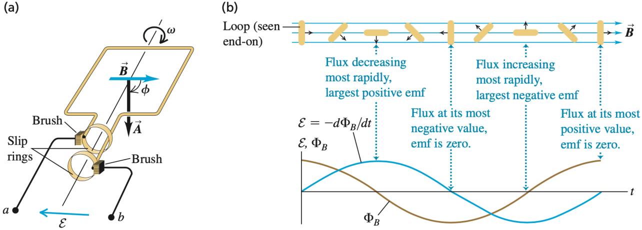
$\vec{B}$
Normal
θ



$\mathcal{E}$
$\mathcal{E}_{max}$
t

Alternating-Current Generator

A rotating coil can become a source of emf in an external circuit by using two slip rings that rotate with the coil. The rings slide against stationary contacts, called brushes, connected to output terminals.



Conceptual Question 3

In an AC generator, a coil with N turns of wire spins in a magnetic field. Of the following choices, which does not cause an increase in the emf generated in the coil?

(a) replacing the coil wire with one of lower resistance

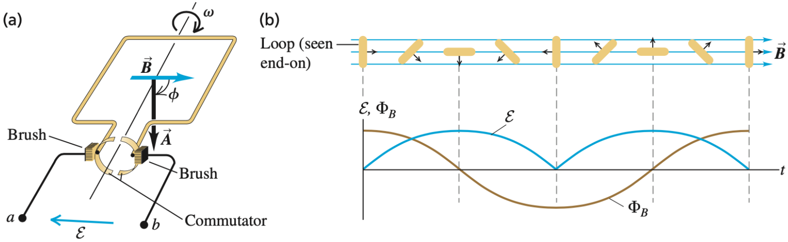
(b) spinning the coil faster

(c) increasing the magnetic field

(d) increasing the number of turns of wire on the coil

Direct-Current Generator

A direct-current generator produces an emf that always has the same sign. An arrangement of split rings, called a commutator, reverses the connections to the external circuit at angular positions at which the emf reverses. Commercial DC generators have a large number of coils and commutator segments, smoothing out the bumps in the emf so that the terminal voltage is not only one-directional but also practically constant. The same device can function as a DC motor when run in reverse–when connected to an external source of constant voltage.



(a)
ω
$\vec{B}$
ϕ
$\vec{A}$
Brush
Brush
Commutator
a
b
ℰ
(b)
Loop (seen end-on)
$\vec{B}$
ℰ, Φ_B
ℰ
Φ_B
t

Example 3: Back emf in a DC motor

A motor's back emf is the emf induced by the changing magnetic flux through its rotating coil. Consistent with Lenz's law, the back emf opposes the applied voltage and effectively limits the current supplied to the motor. This isn't necessarily a bad thing–if a very heavy load jams the motor so that it cannot rotate, the lack of a back emf can lead to dangerously high current in the motor's wire.

Consider a motor with a square, 500-turn coil, 10.0 cm on a side. If the magnetic field has magnitude 0.200 T, at what rotation speed is the average back emf of the motor equal to 112 V?

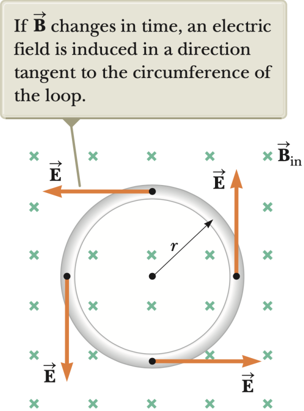
General Form of Faraday's Law

An emf implies that work is being done on conduction electrons. The source of the work is an electric field induced in the wires.

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

(i) Emf measures the circulation of the electric field around a closed loop through which a changing magnetic flux passes.

(ii) Point the thumb of your right hand in the direction of the *change in magnetic flux* and your fingers will curl in the *opposite direction* of the induced electric field. OR use Lenz's law and simply replace "induced current" with "induced electric field."


If $\vec{\mathbf{B}}$ changes in time, an electric field is induced in a direction tangent to the circumference of the loop.
$\vec{\mathbf{B}}_{\text{in}}$
$\vec{\mathbf{E}}$
$\vec{\mathbf{E}}$
r
$\vec{\mathbf{E}}$
$\vec{\mathbf{E}}$

Pitfall Prevention

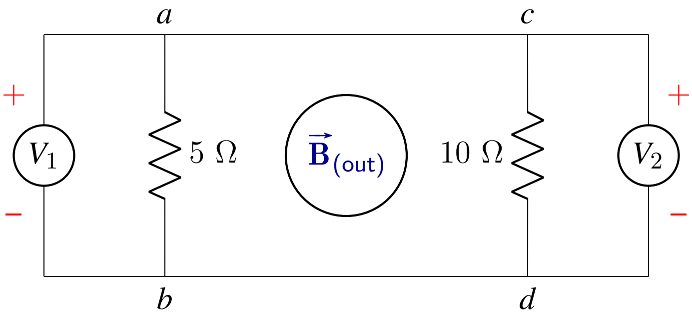
Faraday's law implies an induced electric field is a nonconservative field. We cannot associate an electric potential function with a nonconservative electric field. We reserve the term potential difference for electrostatic fields. The term voltage difference applies to any electric field. Voltmeters measure voltage difference:

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

When there is changing magnetic flux, the line integral (and hence the voltage reading between two points) will depend on the path!

Example 4

A solenoid is so long that its external magnetic field is negligible. Its cross-sectional area is $20\ \mathrm{cm}^2$, and the field inside is increasing at the rate of 100 gauss/s. Two identical voltmeters with high internal resistance are connected to points on a loop that encloses the solenoid and two resistors. What will each voltmeter read?



Self-Induction

Self-induction is the tendency of a current-carrying coil to resist any change in current. When the current i is changing, there will be a corresponding change in the magnetic flux through the circuit, and consequently an electromotive force $\mathcal{E}_L$ is induced. The induction law holds, whatever the source of the flux:

$$\mathcal{E}_L = -N\frac{d\Phi_B}{dt}$$

The magnetic flux through the circuit depends on the magnetic field created by the moving charges. From the Biot-Savart law, it follows that $B \propto i$, and hence $\Phi_B \propto i$.

Self-Inductance

Since $\Phi_B \propto i$, we define a quantity called self-inductance as

$$L \equiv \frac{N\Phi_B}{i} \quad \Rightarrow \quad \mathcal{E}_L = -L\frac{di}{dt}$$

Self-inductance is measured in Henries (H) and only depends on the physical characteristics of the circuit.

$$1\ \mathrm{H} = 1\ \mathrm{V} \cdot \mathrm{s}/\mathrm{A}$$

Example 5

Consider a uniformly wound solenoid having N turns and length ℓ. Assume ℓ is much longer than the radius of the windings and the core of the solenoid is air.

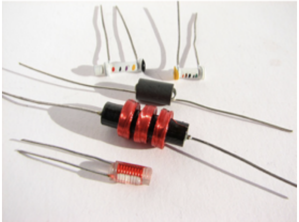
(a) Find the inductance of the solenoid.

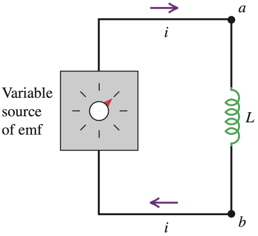
(b) Calculate the inductance of the solenoid if it contains 300 turns, its length is $25.0 \mathrm{~cm}$, and its cross-sectional area is $4.00 \mathrm{~cm}^2$.

(c) Calculate the self-induced emf in the solenoid if the current it carries decreases at the rate of 50.0 A/s.

Inductors

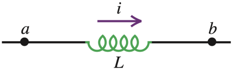
An inductor is a circuit element with relatively large self-inductance (and negligible internal resistance). They are usually tightly wound coils with many loops. Inductors can store energy in a magnetic field and they are important circuit elements when current is changing.





Voltage Difference Across an Inductor

The voltage difference across an inductor is given by Faraday's law. For current flowing from point a to point b through an inductor,



$$V_b - V_a = -L\frac{di}{dt}$$

(i) If the current is increasing, induced emf opposes source emf.

(ii) If current is decreasing, induced emf supports source emf.

(iii) If the current it steady, there is no induced emf.

The next three slides attempt to explain the equation above but may be skipped for brevity. TLDR: it's not as obvious as it seems but Faraday is always right.

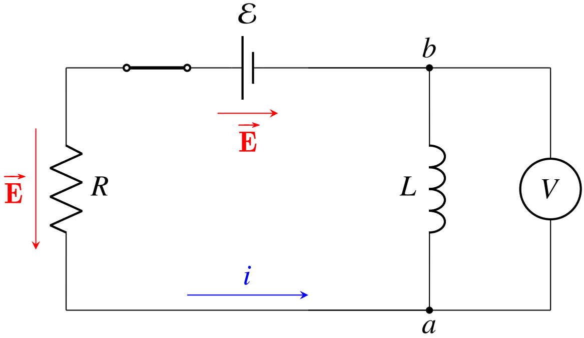
Voltage Difference Across an Inductor*

Imagine taking the line integral of $\vec{\mathbf{E}}$ around a circuit consisting of an emf source, an inductor, and a resistor. Faraday's law gives

$$\oint \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\mathcal{E} + iR = -N\frac{d\Phi_B}{dt} = -L\frac{di}{dt}$$

Rearranging, we end up with an expression that we would get using Kirchhoff's loop rule *as long as we respect the convention for the voltage difference across the inductor:*

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$



ℰ
b
E
R
L
V
E
i
a

Voltage Difference Across an Inductor*

It is often easier to analyze circuits with Kirchhoff's loop rule than Faraday's law, so we'll stick to the traditional approach even though it requires an ad-hoc convention for the voltage difference across an inductor. *Technically, there is no voltage drop across an ideal inductor.* You may protest and say "but when I place a voltmeter across an inductor, I measure a voltage drop of $L\ di/dt$." Of course, but that is not because of the (negligible) electric field inside the inductor; rather, it is because of a time-varying magnetic flux through the voltmeter circuit consisting of the inductor, the voltmeter leads, and the large internal resistance in the voltmeter. A typical voltmeter only provides the voltage drop across its own internal resistance.

Voltage Difference Across an Inductor*

You may also wonder whether there can truly be zero electric field inside an inductor, even in the ideal case. We just learned that changing magnetic fields create electric fields, so surely there's an electric field inside the inductor! Alas, our model of electric current relies on Ohm's law, and in a current-carrying wire with zero resistance, the electric field must be zero.

The key to resolving this apparent paradox is to remember that electric fields can be created by changing magnetic fields and by static charge distributions. It must be the case that charges rearrange themselves to create an electric field inside the inductor that exactly cancels the electric field induced by the changing magnetic flux.

Conceptual Question 4

A coil with zero resistance has its ends labeled a and b. Using a voltmeter, you discover $V_b - V_a = -3.5\ \mathrm{V}$. Which of the following could be consistent with this situation? Select all that apply.

(a) The current is constant and is directed from a to b.

(b) The current is constant and is directed from b to a.

(c) The current is increasing and is directed from a to b.

(d) The current is decreasing and is directed from a to b.

(e) The current is increasing and is directed from b to a.

(f) The current is decreasing and is directed from b to a.

Energy of an Inductor

As the current through an inductor increases, the magnetic field grows, and the *induced electric field does work on moving charges*. The work done is the negative of the change in the inductor's energy.

$$i\mathcal{E}_L = -Li\frac{di}{dt} = -\frac{dU_B}{dt}$$

When $i = 0$, there is no magnetic field, so we set $U_B(i = 0) = 0$.

$$\Rightarrow U_B = \int_0^t Li'\frac{di'}{dt'}dt' = \int_0^{i(t)} Li'di' = \frac{1}{2}Li^2$$

Energy Density of a Magnetic Field

A long solenoid maintains a uniform magnetic field in its interior. The energy stored in the solenoid's magnetic field is

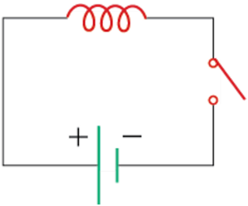
$$U_B = \frac{1}{2}Li^2 = \frac{1}{2}\mu_0 n^2 V \left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}V$$

The energy density (energy per unit volume) is

$$u_B = \frac{U_B}{V} = \frac{B^2}{2\mu_0}$$

Inductors in Circuits

When a switch is initially closed, an inductor will at first resist the current, acting like an open switch. No energy is stored. When the switch has been closed a long time and the current has settled down, the inductor has zero voltage across it, and acts like a straight piece of wire. The inductor has stored energy.

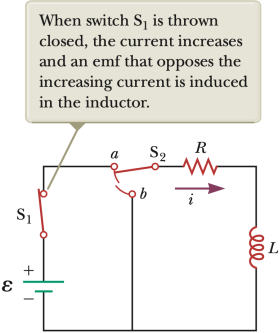


Current Growth in RL Circuits

Apply Kirchhoff's loop rule to the series combination of battery, resistor, and inductor. At $t = 0$, $i = 0$.

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$

$$i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right), \quad \tau = \frac{L}{R}$$

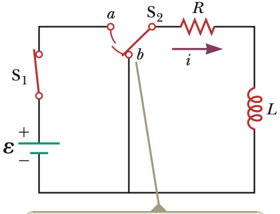


Current Decay in RL Circuits

Apply Kirchhoff's loop rule to the series
combination of resistor and inductor. This time, $i = \mathcal{E}/R$ when $t = 0$.

$$-iR - L\frac{di}{dt} = 0$$

$$i(t) = \frac{\mathcal{E}}{R}e^{-t/\tau}, \quad \tau = \frac{L}{R}$$



When the switch S_2 is thrown to position *b*, the battery is no longer part of the circuit and the current decreases.

Example 6

Consider an RL circuit connected to a battery with the current having reached a steady-state value. When a switch is thrown to remove the battery, the current in the circuit decays exponentially with time. Show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.