Physics 438A – Lecture #10
Quantum Foundations

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1 Introduction

Quantum foundations is the part of physics that asks not just *how to calculate* with quantum mechanics, but *what the theory is really saying about the world*—about reality, causality, probability, space, and time. It probes whether quantum mechanics offers a complete description of physical systems, or whether deeper principles or structures might lie beneath its remarkably successful formalism. Quantum foundations sits at the intersection of physics, mathematics, and philosophy, and over the last few decades it has become an increasingly sharp, experimentally-driven science.

Furthermore, it has repeatedly generated *new science and technology* rather than remaining a purely interpretive enterprise. John Bell's analysis of locality, originally aimed at clarifying the implications of entanglement, led to device-independent quantum cryptography and certified randomness. The 2022 Nobel prize in physics was awarded jointly to Alain Aspect, John F. Clauser, and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science." Efforts to understand measurement and classical emergence through *decoherence*, a concept invented by David Bohm of Bohmian mechanics or pilot-wave theory, underpin modern noise models and error correction. David Deutsch's many-worlds–inspired question about computation in a quantum universe launched quantum computing itself. The history of quantum foundations shows that probing the *conceptual structure* of a theory can reshape both our picture of the world and our technological reach.

We begin in Sec. 2 by introducing the Einstein–Podolsky–Rosen (EPR) argument in Bohm's spin formulation, reviewing the addition of two spin-1/2 systems and deriving the singlet state. This leads naturally to the notion of *entanglement*. In Sec. 3 we turn to Bell's theorem: we exploit the rotational invariance of the singlet state to compute the quantum spin correlation function, and then contrast this prediction with what would follow from local hidden-variable models using the "instruction set" picture. The resulting contradiction highlights the tension between locality and hidden variables and sets the stage for a discussion of modern Bell tests and the no-communication theorem.

2 Einstein-Padolsky-Rosen (EPR)

In 1935, Albert Einstein, Boris Podolsky, and Nathan Rosen published a paper [1] presenting a thought experiment designed to expose the shortcomings of quantum mechanics. They considered two particles with *perfectly correlated positions and momenta* and claimed that, if measurements on one cannot instantaneously affect the other, then the distant particle must possess definite values of both quantities, implying that quantum mechanics is incomplete. By incomplete, they mean to reject the idea that the state vector (or wave function) is a complete description of the system. There should exist additional variables (later on, these were called hidden variables) that specify the state of the system.

David Bohm reformulated EPR using two spin-1/2 particles, making the argument cleaner and experimentally accessible[1]. In the EPRB thought experiment, an unstable particle with spin 0 decays into two spin-1/2 particles, which by conservation of angular momentum must have opposite spin components and by conservation of linear momentum must travel in opposite directions. For example, a neutral pi meson decays into an electron and a positron: $\pi^0 \to e^- + e^+$. Observers A and B are on opposite sides of the decaying particle and each has a Stern-Gerlach apparatus to measure the spin component of the particle headed in its direction. Whenever one observer measures spin up along a given direction, then the other observer measures spin down along that same direction.

2.1 Adding Two Spin-1/2 Particles

Before it decays, the neutral pion is in the spin angular momentum state $|s=0, m=0\rangle$. Since angular momentum is conserved in all fundamental interactions, this must be the spin angular momentum state of the composite system comprised of two spin-1/2 particles. The first postulate of quantum mechanics states that all systems have an associated state-vector space, and the goal of this section is to explore the combined spin state of two spin-1/2 particles. The particles have spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$. We start with the *uncoupled basis*, which simultaneously diagonalizes the commuting operators S_{1z} and S_{2z}. Conceptually, it is possible for observer A or B to measure either spin up or spin down in their apparatus, and the possible measurement outcomes can be enumerated as

$$\{(\uparrow,\uparrow), (\uparrow,\downarrow), (\downarrow,\uparrow), (\downarrow,\downarrow)\}$$

where the first item in each tuple indicates the spin measured by A and the second by B.

[1]Technically, in order to use an S-G apparatus, the particles must be neutral. Otherwise the Lorentz force screws up the measurement. This is why modern Bell tests use entangled photons.

The composite system must allow you to describe all joint measurement outcomes, allow independent operations on each part, and reduce to the usual description when the systems don't interact. The mathematical structure that does exactly this is the *tensor product.*

The **tensor product** of two vector spaces is itself a vector space whose elements are linear combinations of product states denoted $|\psi\rangle \otimes |\phi\rangle$ satisfying bilinearity:

$$\Big(a|\psi_1\rangle + b|\psi_2\rangle\Big) \otimes |\phi\rangle = a|\psi_1\rangle \otimes |\phi\rangle + b|\psi_2\rangle \otimes |\phi\rangle$$
$$|\psi\rangle \otimes \Big(c|\phi_1\rangle + d|\phi_2\rangle\Big) = c|\psi\rangle \otimes |\phi_1\rangle + d|\psi\rangle \otimes |\phi_2\rangle$$

The inner product in a tensor product space is defined multiplicatively as

$$\langle\psi_1 \otimes \phi_1|\psi_2 \otimes \phi_2\rangle = \langle\psi_1|\psi_2\rangle\langle\phi_1|\phi_2\rangle$$

Lastly, for linear operators A on the first space and B on the second, the tensor product $A \otimes B$ is the unique linear operator satisfying

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A|\psi\rangle) \otimes (B|\phi\rangle)$$

Note that there exist vectors in the tensor product space that cannot be written as product states, and likewise, there are linear operators which may not be expressed as a tensor product of two linear operators acting on each space as above.

In the case of two spin-1/2 particles, the product states of the uncoupled basis are

$$|{+}{+}\rangle = |{+}z\rangle_1 \otimes |{+}z\rangle_2$$
$$|{+}{-}\rangle = |{+}z\rangle_1 \otimes |{-}z\rangle_2$$
$$|{-}{+}\rangle = |{-}z\rangle_1 \otimes |{+}z\rangle_2$$
$$|{-}{-}\rangle = |{-}z\rangle_1 \otimes |{-}z\rangle_2$$

and the operators S_{1z} and S_{2z} can be written as

$$S_{1z} = S_z \otimes \mathbb{1} \quad \text{and} \quad S_{2z} = \mathbb{1} \otimes S_z$$

Let's verify the product states of the uncoupled basis really are eigenstates of the spin operators above. From the properties of a tensor product space, we find

$$S_{1z}|{+}{+}\rangle = (S_z|{+}z\rangle) \otimes |{+}z\rangle = \frac{\hbar}{2}|{+}z\rangle \otimes |{+}z\rangle = \frac{\hbar}{2}|{+}{+}\rangle$$

By similar calculations,

$$S_{1z}|{+}{\pm}\rangle = \frac{\hbar}{2}|{+}{\pm}\rangle,\ S_{1z}|{-}{\pm}\rangle = -\frac{\hbar}{2}|{-}{\pm}\rangle,\ S_{2z}|{\pm}{+}\rangle = \frac{\hbar}{2}|{+}{\pm}\rangle,\ S_{2z}|{\pm}{-}\rangle = -\frac{\hbar}{2}|{\pm}{-}\rangle,$$

Hence, the uncoupled basis simultaneously diagonalizes the spin operators, and it's not too surprising such a basis exists given $[S_{1z}, S_{2z}] = 0$, as mentioned earlier.

Whether our system consists of a single particle or many, the theory of rotations we developed in Lecture #4 can be applied to any system residing in 3D Euclidean space. On the tensor product space, we postulate a unitary transformation corresponding to local rotations of each system about the same axis by the same angle:

$$U_{\text{tot}}(\boldsymbol{\theta}) = U_1(\boldsymbol{\theta}) \otimes U_2(\boldsymbol{\theta})$$

Expanding for small $\boldsymbol{\theta}$ (you know the drill),

$$U_{\text{tot}}(\boldsymbol{\theta}) = \left(\mathbb{1} - \frac{i}{\hbar}\boldsymbol{\theta}\cdot\mathbf{S}_1 + \cdots\right) \otimes \left(\mathbb{1} - \frac{i}{\hbar}\boldsymbol{\theta}\cdot\mathbf{S}_2 + \cdots\right)$$

$$= \mathbb{1}\otimes\mathbb{1} - \frac{i}{\hbar}\boldsymbol{\theta}\cdot\left(\mathbf{S}_1\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{S}_2\right) + O(\theta^2)$$

The first term is the identity on the tensor product space, and the second term contains the generator of rotations in the tensor product space. We identify the **total spin angular momentum** as the tensor product operator

$$\mathbf{S} = \mathbf{S}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{S}_2$$

often abbreviated as $\mathbf{S}_1 + \mathbf{S}_2$. It's exactly what you would have guessed, perhaps naively.

We now seek simultaneous eigenstates, the *coupled basis*, of $\mathbf{S}^2$ and S_z. Since $\mathbf{S}$ is an angular momentum, its components will satisfy the usual commutation relations, and we can simply make use of our results from Lecture #4. Notice the largest value of m, related to the eigenvalue of S_z ($\hbar m$), is given by $1/2 + 1/2 = 1$. There is only one product state with this value of m, and that's $|{+}{+}\rangle$. Hence,

$$|s = 1, m = 1\rangle = |1, 1\rangle = |{+}{+}\rangle$$

We can generate the remaining states with $s = 1$ by applying the lowering operator. Starting with the state $|1,1\rangle$,

$$S_-|1,1\rangle = (S_{1-} + S_{2-})|{+}{+}\rangle = \hbar|{-}{+}\rangle + \hbar|{+}{-}\rangle$$

and on the other hand, with $s = 1$ and $m = 1$,

$$S_-|1,1\rangle = \hbar\sqrt{2}|1,0\rangle$$

Hence,

$$|s = 1, m = 0\rangle = |1,0\rangle = \frac{1}{\sqrt{2}}\Big(|{+}{-}\rangle + |{-}{+}\rangle\Big)$$

Applying S_- once more, we find

$$|s = 1, m = -1\rangle = |1,-1\rangle = |{-}{-}\rangle$$

So far, we've found a "triplet" of states with $s = 1$. If we look at the possible values of m in the uncoupled basis, we only have three of the four values ($m = +1, 0, -1$, just once each), so there must be a single state with $m = 0$ remaining. This state necessarily has a total spin value of 0. Physically, there can be different spin configurations that give the same total z-projection. Since the S_z operator is two-fold degenerate in the uncoupled basis, it must be two-fold degenerate in the coupled basis. A change of basis will only map the vectors $|{+}{-}\rangle$ and $|{-}{+}\rangle$ into linear combinations of themselves. The remaining spin state of the coupled basis must have $m = 0$ and be orthogonal to $|0,0\rangle$. The so-called "singlet" state is

$$|0,0\rangle = \frac{1}{\sqrt{2}}\Big(|{+}{-}\rangle - |{-}{+}\rangle\Big)$$

In summary, we've derived the simultaneous eigenstates of $\mathbf{S}^2$ and S_z:

$$|1,1\rangle = |{+}{+}\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\Big(|{+}{-}\rangle + |{-}{+}\rangle\Big)$$

$$|1,-1\rangle = |{-}{-}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\Big(|{+}{-}\rangle - |{-}{+}\rangle\Big)$$

We'll complete the topic of angular momentum addition in Physics 438B. In this lecture, we really only care about the singlet state. This is the combined spin state of the electron and the position after the neutral pion decays.

2.2 Elements of Reality

In the original EPR paper, the authors conjectured: "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity." Suppose observer A measures S_{1z} of the electron with a suitable S-G apparatus and finds $+\hbar/2$. The measurement collapses the state to $|+-\rangle$, allowing observer A to predict with certainty that observer B will measure $-\hbar/2$ for the positron's spin along z. This perfect anti-correlation, occurring even when the measurements are spacelike separated, appears to involve "spooky action at a distance." How could the positron "know" the electron's outcome? EPR assumed that such spooky action is impossible, so they argued that the distant particle must already know what measurement result it would reveal, implying there is an element of reality not encoded in the quantum state. Therefore, they concluded that quantum mechanics is incomplete[1].

One important consequence of the EPR argument, and the work it inspired, was to force the physics community to take the concept of *entanglement* seriously. Shortly after the EPR paper was published, Schrödinger coined the term entanglement (Verschränkung) which he claimed was *the* characteristic feature of quantum mechanics (whatever that means). "The best possible knowledge of a whole does not include the best possible knowledge of its parts." Notice that the singlet state *cannot* be written as a product state. In other words, there exist no vectors $|\psi\rangle$ or $|\phi\rangle$ such that

$$\frac{1}{\sqrt{2}}\Big(|+-\rangle - |-+\rangle\Big) = |\psi\rangle \otimes |\phi\rangle \qquad \text{(impossible)}$$

According to quantum mechanics, the electron and positron are therefore *entangled.* The composite system is in a perfectly well-defined state, but neither particle can be assigned its own independent pure state. In Physics 438B, we'll learn how to assign "mixed states" to the electron and positron in situations like this.

[1] There are interesting things to say about incompatible observables, but this is the crux of the argument.

3 Bell's Theorem

An important result of special relativity is the concept of *causal locality*: causal influences should not be able to travel faster than light. In 1964, John Bell wrote a paper "On the Einstein-Padolsky-Rosen Paper" [2] where he confronts us with a logical fork . We must either accept causal nonlocality or assert both the incompleteness of quantum theory and the existence of a causally local way for measurement results to be "predetermined." The conclusion of Bell's paper was that all roads lead to nonlocality:

> "In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant."

For now we'll put aside the question of whether Bell's treatment is sufficiently general or necessarily implies that quantum theory is causally nonlocal. Bell himself realized that the 1964 argument left room for confusion and had technical limitations. Over the next two decades he wrote several more papers to clarify, generalize, and strengthen the result. A more conservative conclusion of his 1964 paper is to rule out measurement deterministic hidden-variables theories obeying causally local dynamics.

3.1 Rotational Invariance

The singlet state, denoted henceforth as $|\Psi_-\rangle$, is *rotationally invariant* in the sense that

$$\Big(U(R)\otimes U(R)\Big)|\Psi_-\rangle = |\Psi_-\rangle$$

for proper rotation R by any angle θ about any axis. We can therefore choose a rotation R that maps $\hat{\mathbf{z}} \to \hat{\mathbf{n}}$ such that

$$U(R)|{+z}\rangle = |{+n}\rangle \quad \text{and} \quad U(R)|{-z}\rangle = |{-n}\rangle$$

The singlet state is now expressible in terms of the S_n eigenstates as

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\Big(|{+-}\rangle_n - |{-+}\rangle_n\Big)$$

where, for example, $|+-\rangle_n = |+n\rangle \otimes |-n\rangle$. Since the singlet has the same form in every spin basis, the measurement results for spin along the same direction are perfectly anti-correlated for all directions of physical space.

It is easy to show that $|\Psi_-\rangle$ is rotationally invariant. Consider a finite rotation by $\boldsymbol{\theta}$,

$$\Big(U(R) \otimes U(R)\Big)|\Psi_-\rangle = |\Psi_-\rangle = \exp\left(-\frac{i}{\hbar}\boldsymbol{\theta} \cdot (\mathbf{S}_1 + \mathbf{S}_2)\right)|\Psi_-\rangle$$

Since $|\Psi_-\rangle$ is a spin state with $s = 0$, $\mathbf{S}^2|\Psi_-\rangle = 0$, and it follows that $\mathbf{S}|\Psi_-\rangle = 0$. The singlet state $|\Psi_-\rangle$ is also an eigenstate of $\mathbf{S}$ with eigenvalue 0. Hence,

$$\exp\left(-\frac{i}{\hbar}\boldsymbol{\theta} \cdot (\mathbf{S}_1 + \mathbf{S}_2)\right)|\Psi_-\rangle = \exp(0)|\Psi_-\rangle = |\Psi_-\rangle$$

Observers A and B can measure spin along the directions $\mathbf{a}$ and $\mathbf{b}$ respectively, where both $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. When $\mathbf{a} = \mathbf{b}$, the spins will be anti-correlated no matter which direction the vectors point in physical space. In general, for some nonzero angle between $\mathbf{a}$ and $\mathbf{b}$, the spin measurements will not be anti-correlated, but they will still display correlations that can be inferred from a statistical distribution of measurement outcomes. Intuitively, we expect the correlations to only depend on the angle between $\mathbf{a}$ and $\mathbf{b}$ due to the rotational invariance of the singlet state.

3.2 Spin Correlation Function

Observers A and B measure a spin component along directions $\mathbf{a}$ and $\mathbf{b}$, getting outcomes $\pm\hbar/2$, or ± 1 after dividing by $\hbar/2$. This is equivalent to A and B measuring the observables $\boldsymbol{\sigma} \cdot \mathbf{a}$ and $\boldsymbol{\sigma} \cdot \mathbf{b}$ respectively, where $\boldsymbol{\sigma}$ is the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Since the measurement outcomes for each observer are random,

$$\langle \boldsymbol{\sigma} \cdot \mathbf{a} \rangle = 0, \quad \text{and} \quad \langle \boldsymbol{\sigma} \cdot \mathbf{b} \rangle = 0,$$

for the singlet state. The *local statistics* don't tell us anything interesting. We quantify the correlations between the subsystems using the **spin correlation function** defined as

$$E(\mathbf{a}, \mathbf{b}) \equiv \langle (\boldsymbol{\sigma} \cdot \mathbf{a}) \otimes (\boldsymbol{\sigma} \cdot \mathbf{b}) \rangle$$

When $\mathbf{b} = \mathbf{a}$, we can use rotational invariance of the singlet state to find

$$E(\mathbf{a}, \mathbf{a}) = \langle \Psi_- | (\boldsymbol{\sigma} \cdot \mathbf{a}) \otimes (\boldsymbol{\sigma} \cdot \mathbf{a}) | \Psi_- \rangle$$

$$= \frac{1}{\sqrt{2}}\Big(\langle +-|_a - \langle -+|_a\Big)\Big((\boldsymbol{\sigma}\cdot\mathbf{a})\otimes(\boldsymbol{\sigma}\cdot\mathbf{a})\Big)\frac{1}{\sqrt{2}}\Big(|+-\rangle_a - |-+\rangle_a\Big)$$

where

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\otimes(\boldsymbol{\sigma}\cdot\mathbf{a})|+-\rangle_a = -|+-\rangle_a$$

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\otimes(\boldsymbol{\sigma}\cdot\mathbf{a})|-+\rangle_a = -|-+\rangle_a$$

so

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\otimes(\boldsymbol{\sigma}\cdot\mathbf{a})\Big(|+-\rangle_a - |-+\rangle_a\Big) = -|+-\rangle_a + |-+\rangle_a$$

The product states are orthonormal, so it follows that

$$E(\mathbf{a},\mathbf{a}) = \frac{1}{2}(-1-1) = -1$$

We could have guessed this result. The expectation value corresponds to the average of many measurement results, and since the results are anti-correlated, when observer A measures $+1$, B measures -1. Similarly, when A measures -1, B measures $+1$. The product of their measurement outcomes is always -1. Nonetheless, it's good to see how the tensor product works even when the answer is obvious.

There's a very nice way to calculate the correlation function for arbitrary $\mathbf{a}$ and $\mathbf{b}$ that relies on rotational invariance of $|\Psi_-\rangle$. Begin by writing

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\otimes(\boldsymbol{\sigma}\cdot\mathbf{b}) = \sum_{i,j} a_i b_j\ \sigma_i\otimes\sigma_j$$

where i and $j = 1,2,3$ corresponds to x, y, z respectively. The correlation function is

$$E(\mathbf{a},\mathbf{b}) = \sum_{i,j} a_i b_j C_{ij}$$

where $C_{ij} = \langle\Psi_-|\sigma_i\otimes\sigma_j|\Psi_-\rangle$ are the matrix elements to be determined. When $i = j$,

$$C_{ii} = \langle\Psi_-|\sigma_i\otimes\sigma_i|\Psi_-\rangle = -1$$

since we can write $|\Psi_-\rangle$ in the basis of x, y, or z and simply re-use our result from earlier. When $i \neq j$, we can write $\langle\Psi_-|$ in the basis of the ith component and $|\Psi_-\rangle$ in the basis of the jth component, and it becomes clear that

$$C_{ij} = \langle \Psi_- | \sigma_i \otimes \sigma_j | \Psi_- \rangle = 0, \quad i \neq j$$

by orthogonality of the product states. Thus, $C_{ij} = -\delta_{ij}$ and we find

$$E(\mathbf{a}, \mathbf{b}) = -\sum_{i,j} a_i b_j \delta_{ij} = -\mathbf{a} \cdot \mathbf{b}$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then we have

$$E(\mathbf{a}, \mathbf{b}) = -\cos\theta$$

Clearly, when $\theta = 0$, the "measurement settings" are aligned and $E(\mathbf{a}, \mathbf{b}) = -1$. When $\theta = \pi/2$, the two axes are orthogonal to each other, and we don't predict any correlations in the data. For example, if A measures spin up along the x direction, they will have nothing to say about what B will measure along y.

Having obtained the spin correlation function, it is worth pausing to ask what this quantity means operationally. In an actual experiment, A and B do not directly measure a product of outcomes; they record long lists of individual outcomes, and from these lists they count how often their results agree and how often they disagree. Let the outcomes henceforth be denoted $A = \pm 1$ for observer A and $B = \pm 1$ for observer B. Let the disagreement rate r be the probability that $A \neq B$. The agreement rate, given by $1 - r$, is the probability that $A = B$. Because the product AB equals -1 when the outcomes disagree and $+1$ when they agree, the correlation function can be written as

$$E(\mathbf{a}, \mathbf{b}) = (-1)(r) + (+1)(1 - r) = 1 - 2r$$

and the disagreement rate is given by

$$r = \frac{1 - E(\mathbf{a}, \mathbf{b})}{2}$$

Substituting the prediction of quantum mechanics, the disagreement rate becomes

$$r = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2}$$

This reformulation makes two important points clear. First, it connects the abstract correlator to directly observable frequencies in an experiment. "Bell tests" are ultimately about how often two distant observers agree or disagree. Second, it highlights the striking content of the singlet correlations. When the analyzers are aligned, the outcomes always disagree; when they are orthogonal, agreement and disagreement are equally likely; and when they are opposite, the outcomes always agree.

3.3 Instruction Sets

In this section, we present an argument that arrives at the same conclusion as Bell's 1964 paper, but from a different perspective. This argument is similar to the one presented in your textbook; it relies on the "instruction set" conception of hidden variables. In this approach, a local hidden variable theory assigns to each particle a set of instructions specifying in advance the outcomes of measurements along any two directions **a** and **b**. This is arguably a strong assumption for a hidden-variables theory, but it is likely close to what EPR had in mind. If the particles do carry a set of instructions, then no influence traveling faster than light between the two observers would be needed to explain the correlations.

The argument is one by contradiction. Suppose the particles carry a set of instructions denoted λ such that outcome A depends only on λ and $\mathbf{a}$ (not on $\mathbf{b}$); likewise, B depends only on λ and $\mathbf{b}$ (not on $\mathbf{a}$). Furthermore, assume the outcome statistics will reproduce the predictions of quantum mechanics; the disagreement rate must be given by $\cos^2(\theta/2)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{a}$ and $\mathbf{b}$ are co-planar unit vectors, we can specify the measurement settings by angles θ_a and θ_b above some reference line, e.g. a common x-axis. Then we can write $A = A(\theta_a)$ and $B = B(\theta_b)$. We will now outline a series of concrete thought experiments that will arrive at a contradiction:

1. Perfectly aligned measurements: $\theta_a = \theta_b = 0$

- $\theta = \theta_a - \theta_b = 0$
- $r(\theta) = r(0) = 1$, 100% disagreement
- $A(0) = -B(0)$ for 100% of outcomes
- Note that $A(\theta_a) = -B(\theta_b)$ for 100% of outcomes whenever $\theta_a = \theta_b$.

2. Let $\theta_a = +60°$ and $\theta_b = 0$

- $\theta = \theta_a - \theta_b = 60°$
- $r(\theta) = r(60°) = 0.75$, 75% disagreement
- $A(+60°) = -B(0)$ for 75% of outcomes
- Since $B(0) = -A(0)$, $A(+60°) = A(0)$ for 75% of outcomes

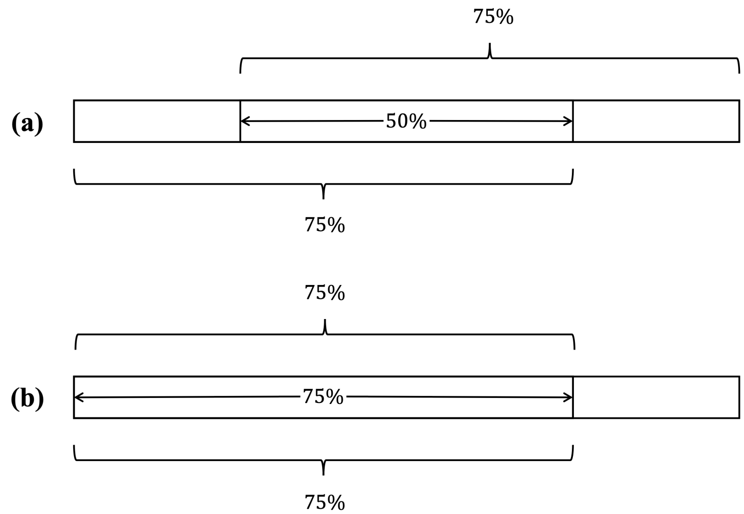


Figure 3.1: Illustrations of two overlapping sets. (a) Minimum overlap of two sets of size 75% is 50%. (b) Maximum overlap between the two sets is 75%.

3. Let $\theta_a = 0$ and $\theta_b = -60°$.

- $\theta = \theta_b - \theta_a = 60°$
- $r(\theta) = r(60°) = 0.75$, 75% disagreement
- $A(0) = -B(-60°)$ for 75% of outcomes
- Since $B(-60°) = -A(-60°)$, $A(0) = A(-60°)$ for 75% of outcomes

4. Combining the results of experiments 2 and 3,

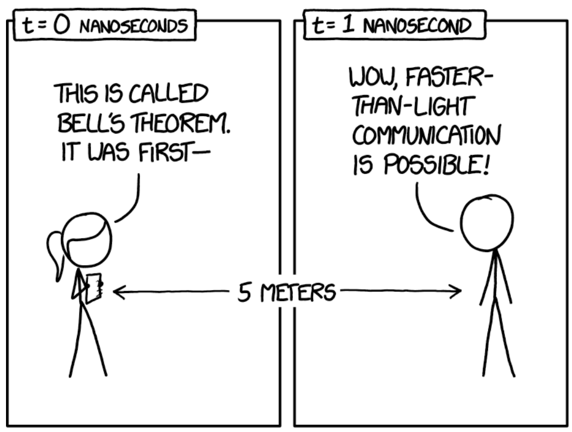
- if $A(+60°) = A(0)$ for 75% of outcomes and $A(-60°) = A(0)$ for 75% of outcomes,
- then for *at least* 50% of outcomes, $A(+60°) = A(-60°)$. [This follows because two sets of size 75% must overlap in at least 50% of outcomes; see Figure 3.1.]
- Since $A(-60°) = -B(-60°)$, $A(+60°) = -B(-60°)$ for at least 50% of outcomes.
- The disagreement rate must be $\geq$ 50% between $A(+60°)$ and $B(-60°)$. This is an example of a *Bell inequality*.

5. However, if $\theta_a = +60°$ and $\theta_b = -60°$,

- then $\theta = \theta_a - \theta_b = 120°$,
- and $r(120°) = 25\%$, which is lower than 50%. The Bell inequality is violated.

Conclusion: either the particles do not carry a set of instructions that predetermine the measurement outcomes, or the outcomes depend on distant measurement settings[1]. Bell's theorem shows that quantum phenomena cannot be explained by any theory that combines hidden variables with locality. One may interpret this as evidence for nonlocal causal influences, but a more moderate view is that quantum phenomena exhibit *nonlocal correlations* that may or may not result from signals or causes propagating through space.

Whether or not the correlations can be explained by some "spooky action at a distance," the **no-communication theorem** (or no-signaling theorem) states that entanglement cannot be used to transmit information faster than light. Even though entangled particles exhibit strong correlations, no local operation on one particle can influence the measurement-outcome statistics of the other in a way that could be used for communication. [By local operations, we mean local unitary operations or local measurements.]



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[2] J. S. Bell. "On the Einstein Podolsky Rosen paradox". In: *Physics Physique Fizika* 1 (3 Nov. 1964), pp. 195–200. DOI: 10.1103/PhysicsPhysiqueFizika.1.195. URL: https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195.

[1] Another option is that quantum mechanics makes the wrong predictions, but we find that real-world experiments agree with the predictions of quantum mechanics.

The tables below will be explained in class.

$A(+60°)$	$A(0)$	$B(0)$	$B(-60°)$
$+1$	$+1$	-1	-1
$+1$	-1	$+1$	$+1$
$+1$	$+1$	-1	-1
-1	$+1$	-1	$+1$
-1	-1	$+1$	$+1$
-1	-1	$+1$	$+1$
$+1$	$+1$	-1	$+1$
-1	-1	$+1$	$+1$

(θ_a, θ_b)	r_{bell}	r_{quantum}
$(0, 0)$	1	1
$(+60°, 0)$	0.75	0.75
$(0, -60°)$	0.75	0.75
$(+60°, -60°)$	0.75	0.25

$A(+60°)$	$A(0)$	$B(0)$	$B(-60°)$
$+1$	$+1$	-1	$+1$
$+1$	-1	$+1$	$+1$
$+1$	$+1$	-1	$+1$
$+1$	$+1$	-1	-1
$+1$	-1	$+1$	$+1$
-1	-1	$+1$	$+1$
$+1$	$+1$	-1	-1
-1	-1	$+1$	$+1$

(θ_a, θ_b)	r_{bell}	r_{quantum}
$(0, 0)$	1	1
$(+60°, 0)$	0.75	0.75
$(0, -60°)$	0.75	0.75
$(+60°, -60°)$	0.50	0.25