Phys 153: Fundamentals of Physics III

Unit #3 – Relativistic Mechanics

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Reading

Please read the following sections of your textbook carefully and skim the rest:

- Chapter 38 – just read all of it ☺

This is the point in the semester where we begin heavily supplementing the textbook, so please pay careful attention in lectures.

Since relativity can be hard to digest, we will first go through a series of results from a conceptual standpoint and save the equations for later.

Introduction to Relativity

Relativity is about how observations made by different observers in different frames of reference are related to one another.

In some sense, it should be obvious that certain aspects of motion depend on the frame of reference—they are **frame dependent**.

What's not so obvious is that some quantities and relationships appear to be **frame independent**. Everyone will agree on them regardless of how they are moving (to some extent).

Conceptual Question 1

An object is dropped from an airplane flying at constant speed in a straight line at a constant altitude. If there is no air resistance, the falling object will (as seen from the ground)

(A) Lag behind the airplane, but still move forward.

(B) Lag behind the airplane, and fall straight down.

(C) Lag behind the airplane, and move backward.

(D) Remain directly below the airplane.

Principle of Relativity

We postulate that the laws of physics are the same in all inertial frames of reference, in every direction of space (isotropy of space), and at all locations and times (homogeneity).

Classical Mechanics	Relativistic Mechanics
Additional assumption: time is a universal parameter of change, simultaneity is absolute	Experimental fact*: the speed of light in vacuum is constant in all inertial frames, independent of the motion of the source or observer

These statements are incompatible!

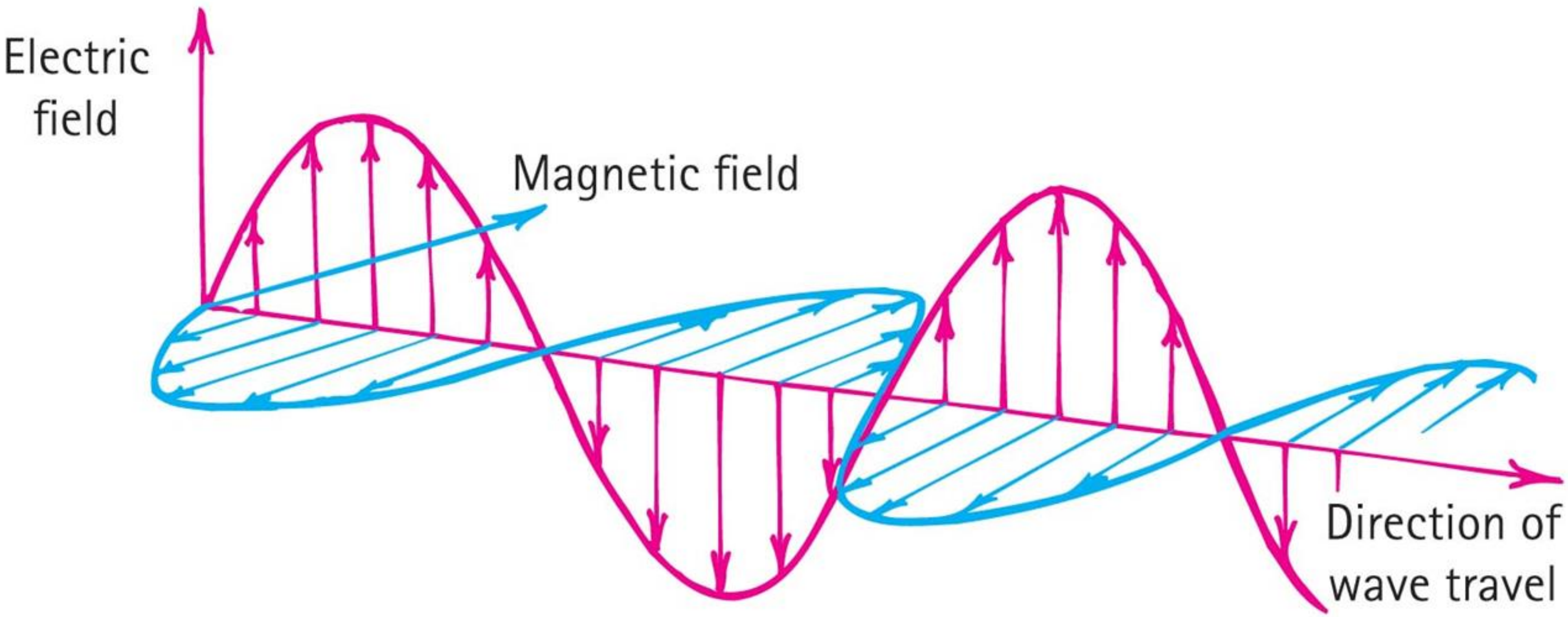
Inertial Frames of Reference

An inertial frame of reference is defined to be one in which Newton's first law is valid. A free particle moves with constant speed in a straight line:

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = 0 \qquad (1)$$

- From one inertial frame, we can obtain others by performing transformations on the coordinates
  - Rotations
  - Translations
  - Boosts
- All transformations must be **linear** to preserve Eq. (1)

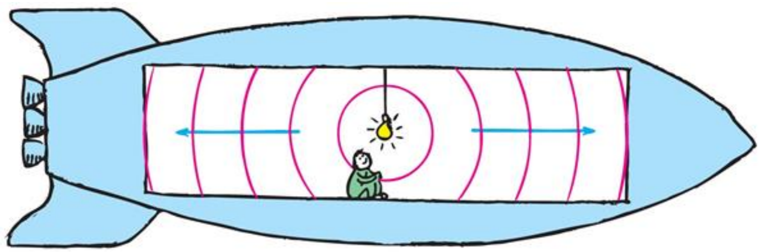
Can you surf an EM wave?

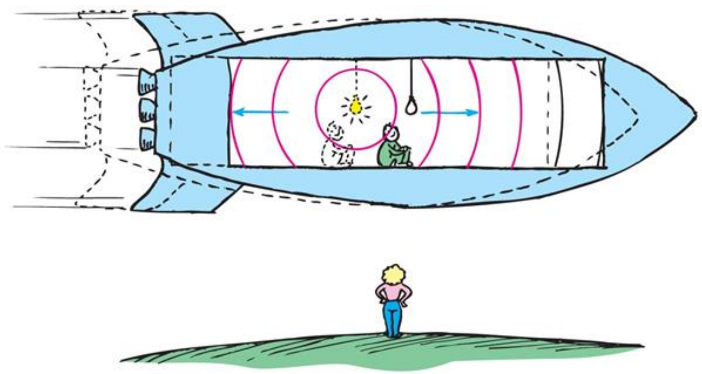


Changing magnetic field

Changing electric field

Relativity of Simultaneity





- From the point of view of the observer who travels with the compartment, light from the source travels equal distances to both ends of the compartment and therefore strikes both ends simultaneously.
- Because of the ship’s motion, light that strikes the back of the compartment doesn’t have as far to go and strikes sooner than light that strikes the front of the compartment.

Conceptual Question 2

Is the non-simultaneity of hearing thunder after seeing lightning similar to relativistic non-simultaneity?

A. Yes, it is exactly the same phenomenon.

B. No, it is a completely different phenomenon.

C. It depends upon how loud the thunder is.

D. It depends upon how far the thunder is.

Conceptual Question 3

Suppose that an observer standing on a planet sees a pair of lightning bolts simultaneously strike the front and rear ends of the compartment in a high-speed rocket ship. Will the lightning strikes be simultaneous to an observer in the middle of the compartment in the rocket ship?

A. Yes, they will be simultaneous.

B. No, they will be nonsimultaneous.

C. It depends upon how fast the ship is moving.

D. It depends upon how long the ship is.

Relativity of Simultaneity

Spacetime

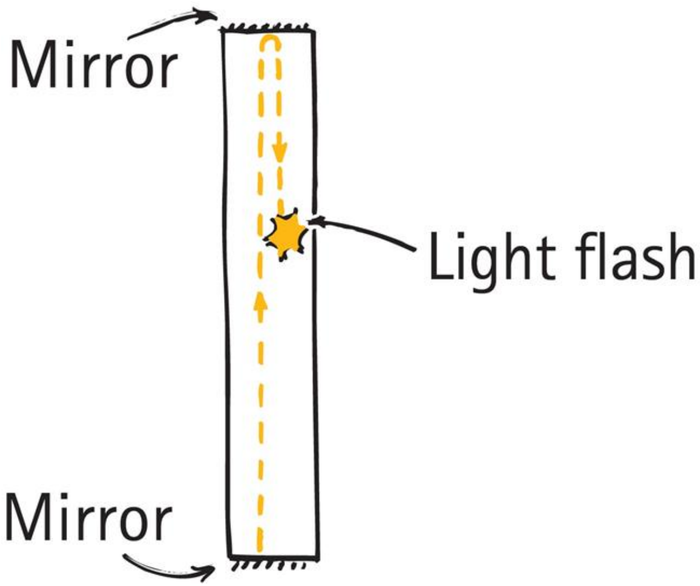
One observer's measurements of distance and time differ from similar measurements by another observer in a different reference frame.

Both measure the same ratio of the distance to the time separating events linked by a light beam: a greater measured distance in space corresponds with a greater measured time interval.

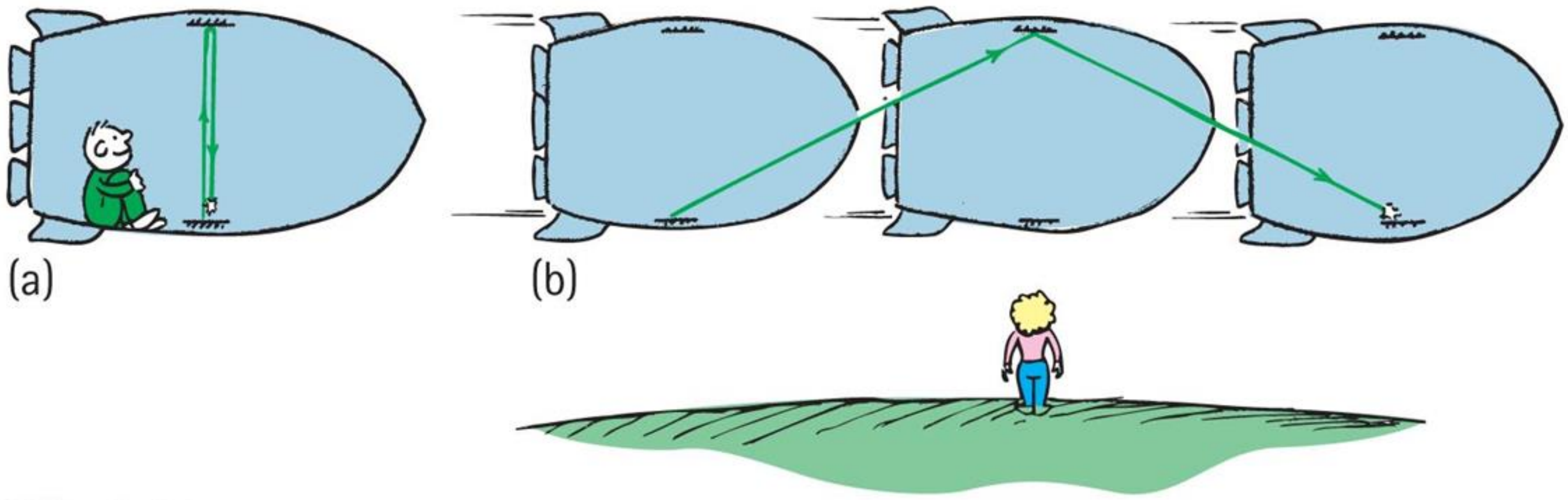
$$c = \frac{\text{distance}}{\text{time}} = \frac{\textbf{distance}}{\textbf{time}}$$

Lightclock

Imagine that we are somehow able to observe a flash of light bouncing up and down between a pair of parallel mirrors. If the distance between the mirrors is fixed, then the arrangement constitutes a **light clock**, because the back-and-forth trips of the flash take equal time intervals.



Time Dilation

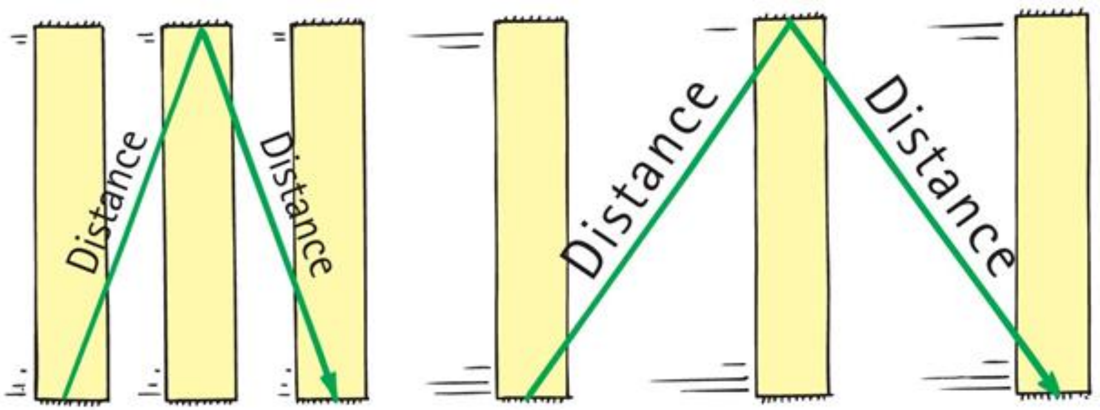


(a) An observer moving with the spaceship observes the light flash moving vertically between the mirrors of the light clock.
(b) An observer who sees the moving ship pass by observes the flash moving along a diagonal path.

Time Dilation

Observers agree that light traveled at speed c between two ticks of the clock, but they disagree about how far the light went, so they must disagree about how much time passed.

In fact, the observer who sees the clock moving will conclude that more time passed between two consecutive ticks of the clock.



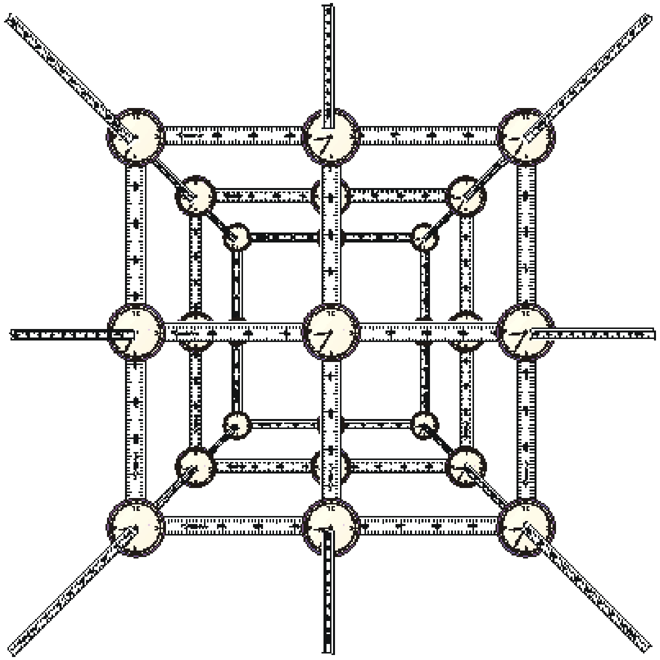
$$\frac{\text{Distance}}{\text{Time}} = \frac{\text{Distance}}{\text{Time}} = c$$

Proper Time vs. Coordinate Time

Proper time is the time between two events that occur at the same location, as measured by a clock that is present at both events. It's like the time between when you wake up and go to sleep *as measured by the watch you carry*.

Coordinate time is the time measured between two events according to a system of *stationary synchronized clocks*.

- A proper time interval is always a coordinate time interval, but not all coordinate time intervals are proper time intervals.
- Events may occur at different locations or the same location.



Conceptual Question 4

If you were moving in a spaceship at a high speed relative to Earth, would you notice a difference in your pulse rate or the pulse rate of people on Earth?

A. Yes, you would notice a difference in both pulse rates.
B. You would notice a difference in your pulse rate, but not the pulse rate of people on Earth.
C. You would notice a difference in the pulse rate of people on Earth, but not in your own pulse rate.
D. You would not notice a difference in either pulse rate.

Conceptual Question 5

Will observers A and B agree on measurements of time if observer A moves at half the speed of light relative to observer B?

A. Yes, they would agree completely.

B. No, they would disagree completely.

C. They would agree half of the time and disagree the other half of the time.

D. None of the above is correct.

Conceptual Question 6

Will observers A and B agree on measurements of time if both A and B move together at half the speed of light relative to Earth?

A. Yes, they would agree completely.

B. No, they would disagree completely.

C. They would agree half of the time and disagree the other half of the time.

D. None of the above is correct.

Conceptual Question 7

Does time dilation mean that time really passes more slowly in moving systems or only that it seems to pass more slowly?

A. Time really passes more slowly in moving systems.

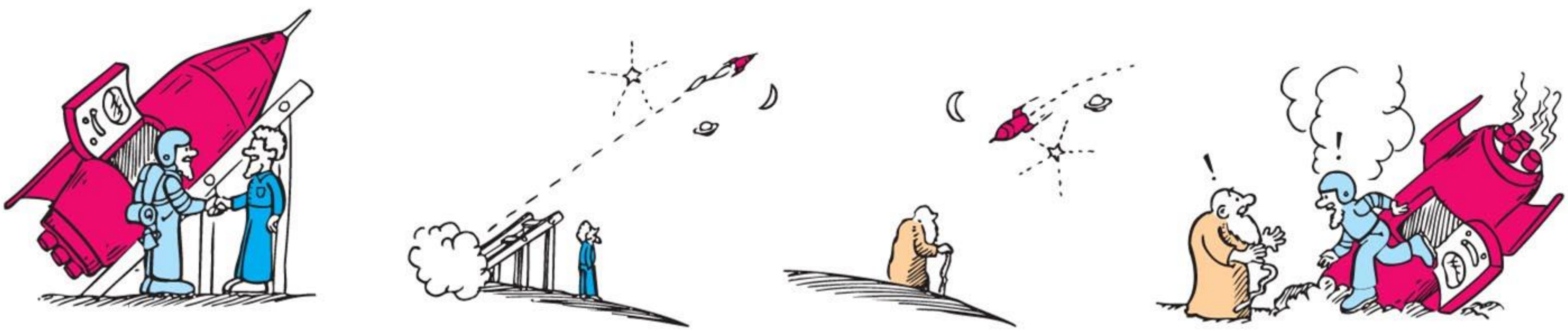
B. Time only seems to pass more slowly in moving systems.

C. It depends upon how fast the system is moving.

D. It depends upon the direction in which the system is moving.

The Twin Paradox

- Identical twins, one an astronaut who takes a high-speed round-trip journey while the other stays home on Earth:
  - When the traveling twin returns, he's younger than the stay-at-home twin! How much younger depends on the relative speed.
- Since motion is relative, why doesn't the effect work equally well the other way around? Why wouldn't the traveling twin return to find his stay-at-home twin younger than himself?



Physics can be non-intuitive but never contradictory!

Where did the time go?

Conceptual Question 8

The ship sends equally spaced flashes every 6 minutes while approaching the receiver at constant speed. How will these flashes be spaced when they encounter the receiver?

A. They will be equally spaced 6 minutes apart.
B. They will be equally spaced less than 6 minutes apart.
C. They will be equally spaced more than 6 minutes apart.
D. They will not be equally spaced.

Conceptual Question 9

Since motion is relative, can’t we say as well that the spaceship is at rest and the Earth moves, in which case the twin on the spaceship ages more?

A. Yes.

B. No.

C. It depends on how fast the ship is moving.

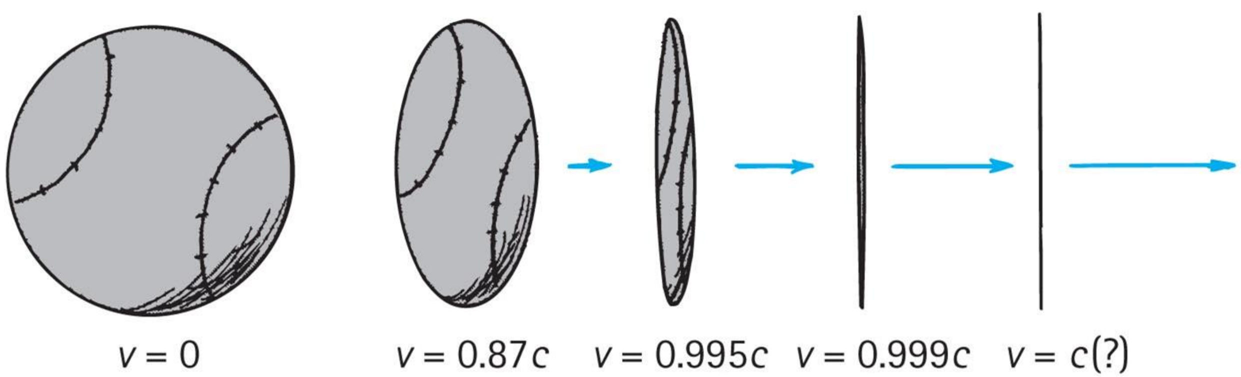
D. It depends upon the direction in which the ship is moving.

Length Contraction

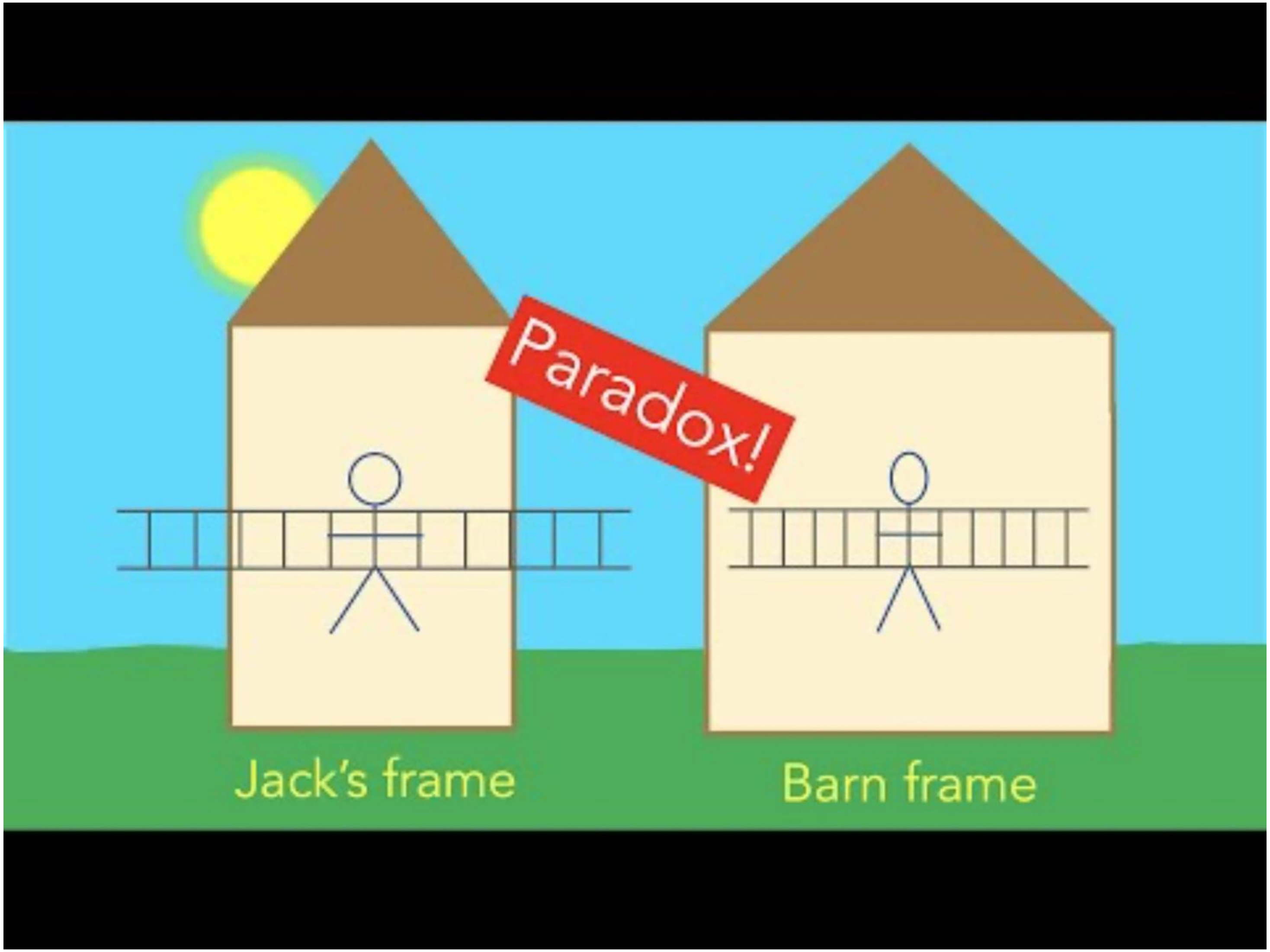
From the rod's perspective, the person moves to the right with relative speed v, and from the person's perspective, the rod moves to the left with the same relative speed v. Since the person measures a shorter time interval between the passing of both ends of the rod, they measure the rod to have *shorter length* compared to when it's at rest.

Length Contraction

- A moving object's length is measured to be shorter than its **proper length**, which is the length as measured in the object's own rest frame.
- Length contraction takes place only in the direction of travel. An object traveling horizontally experiences horizontal contraction but **not** vertical contraction (due to isotropy).



Einstein’s Ladder Paradox



Standard Orientation

Time to roll up our sleeves. Before we get started, we adopt a few conventions that make life easier:

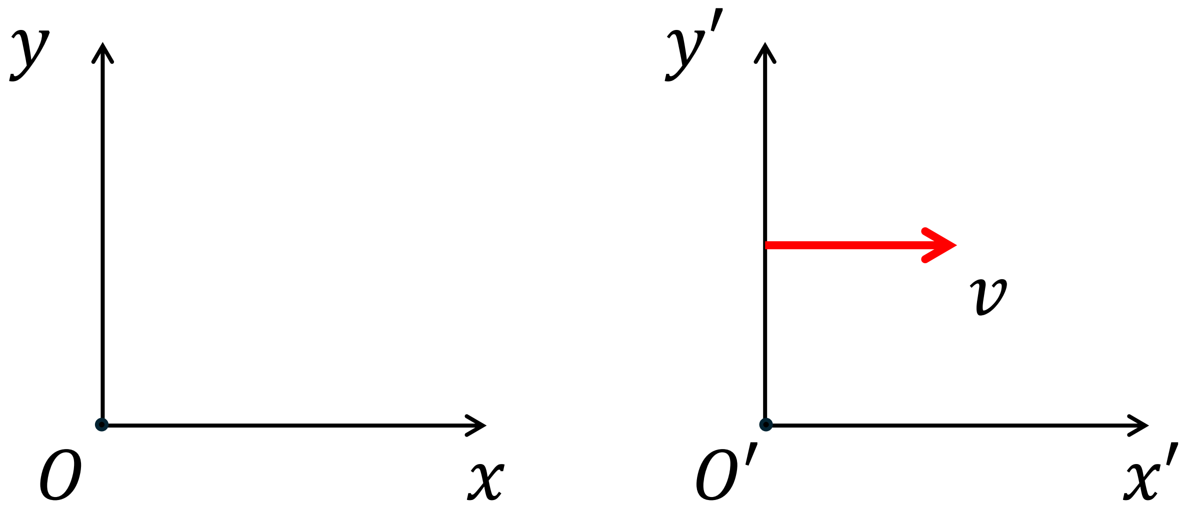
Home Frame: inertial reference frame S with origin O and spacetime coordinates (ct, x, y, z)

Other Frame: inertial frame S' with origin O' and spacetime coordinates (ct', x', y', z')

- The Other Frame moves with constant relative speed v along the x axis of the Home Frame.
- The spatial axes are aligned in both frames, e.g. the x' axis in S' points the same way as the x axis in S.
- The origins coincide at $t = t' = 0$.

We will call this the “standard orientation.”

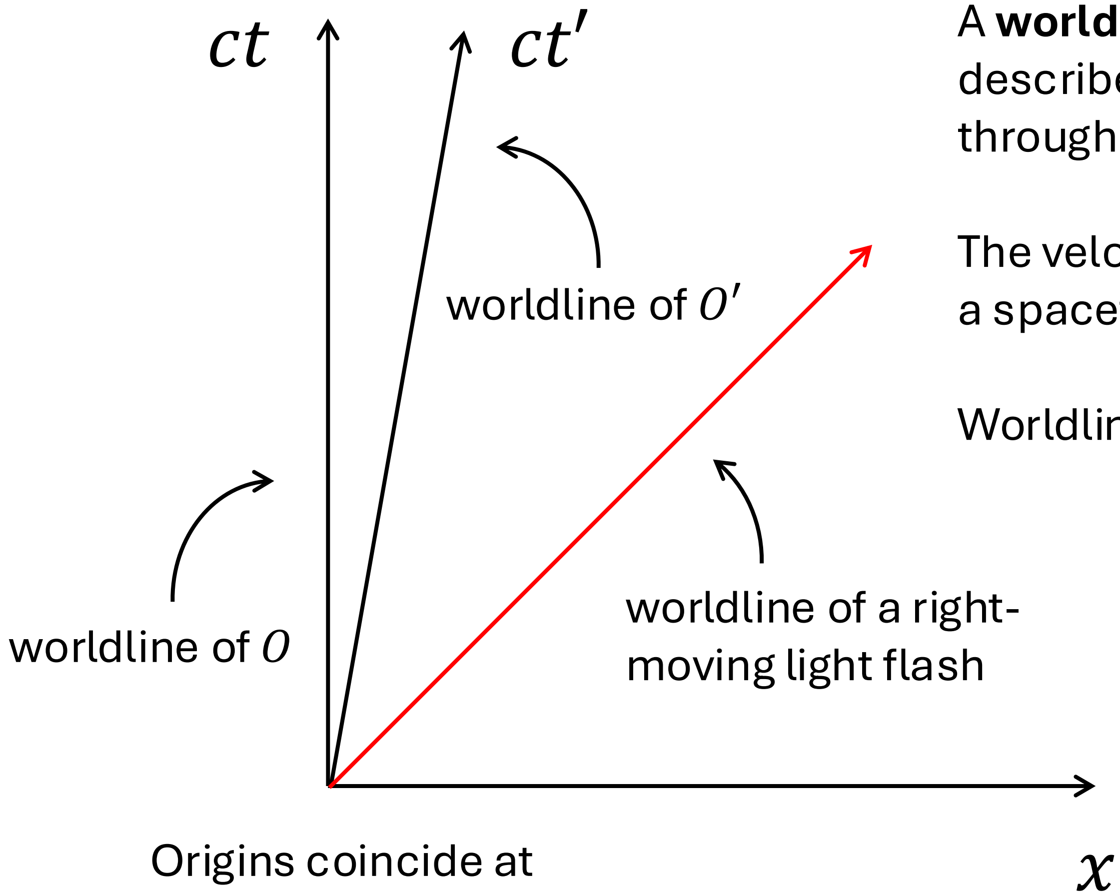
Standard Orientation



Home Frame **Other Frame**

The Other Frame moves at constant speed v relative to the Home Frame

Spacetime Diagram



Origins coincide at
$t = t' = 0$

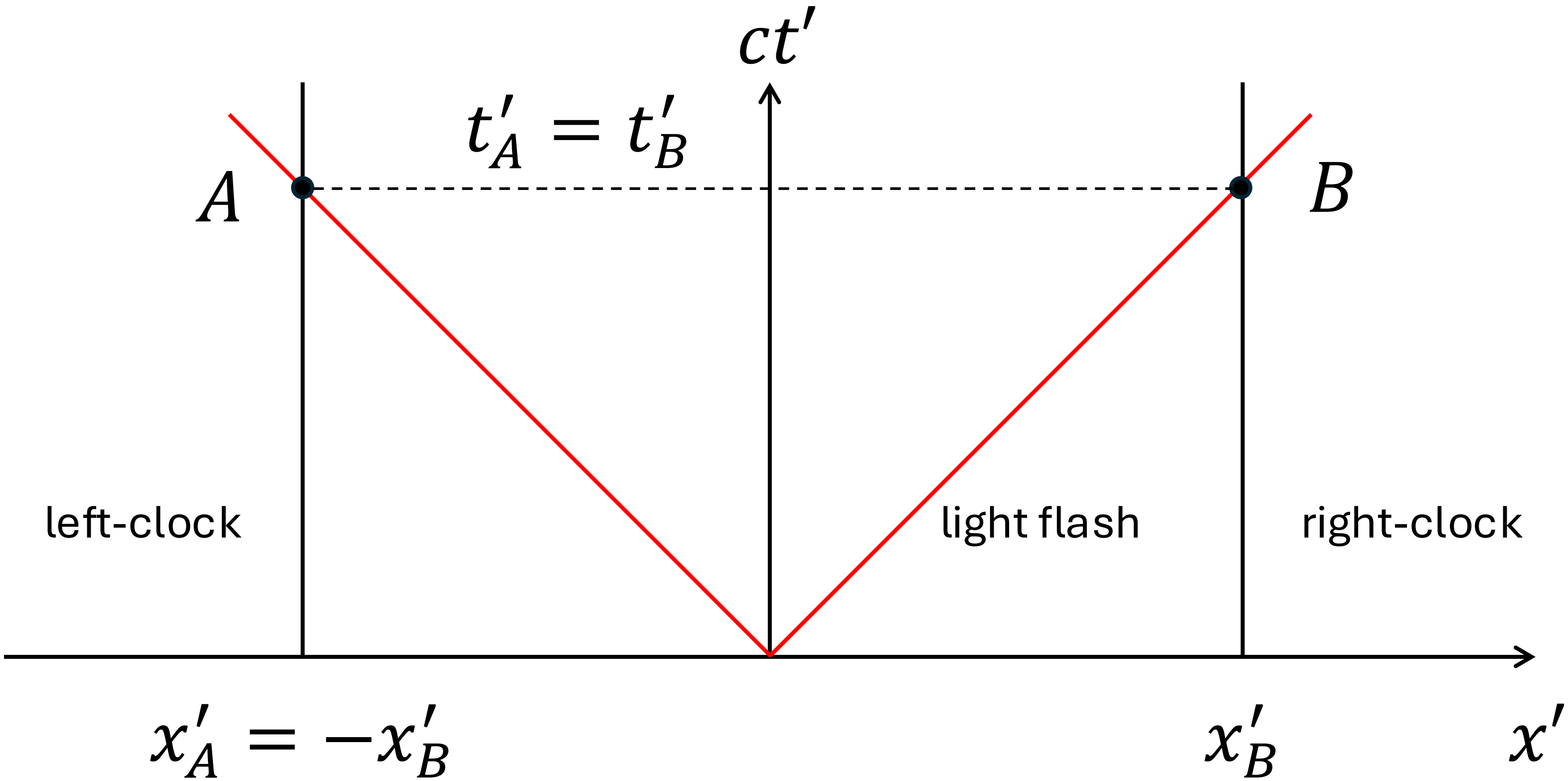
A **worldline** is a series of events that describes the trajectory of a particle through spacetime.

The velocity of a particle is 1/slope on a spacetime diagram.

Worldline of light flash has slope ± 1

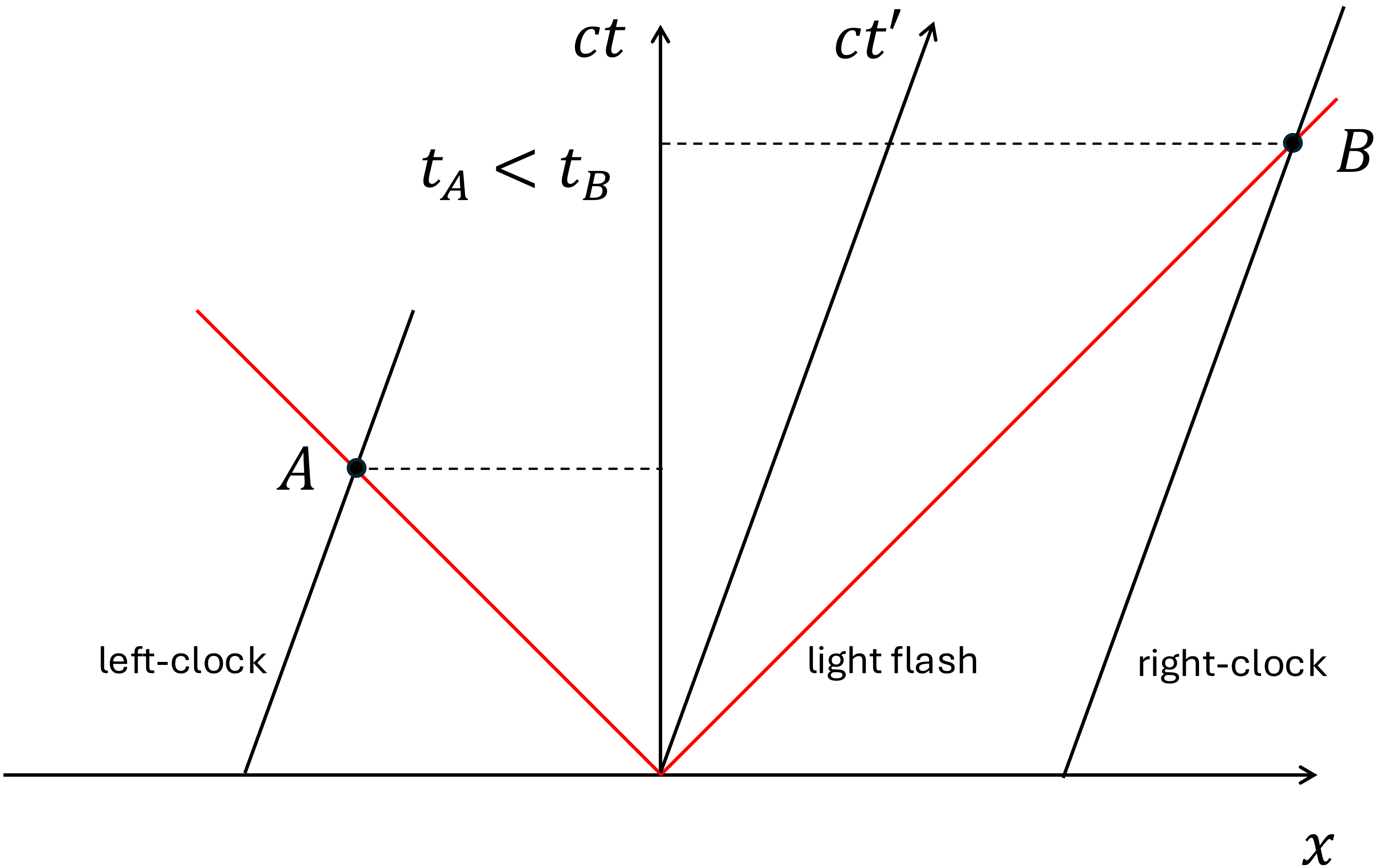
Relativity of Simultaneity

Two events that are simultaneous in the Other Frame…



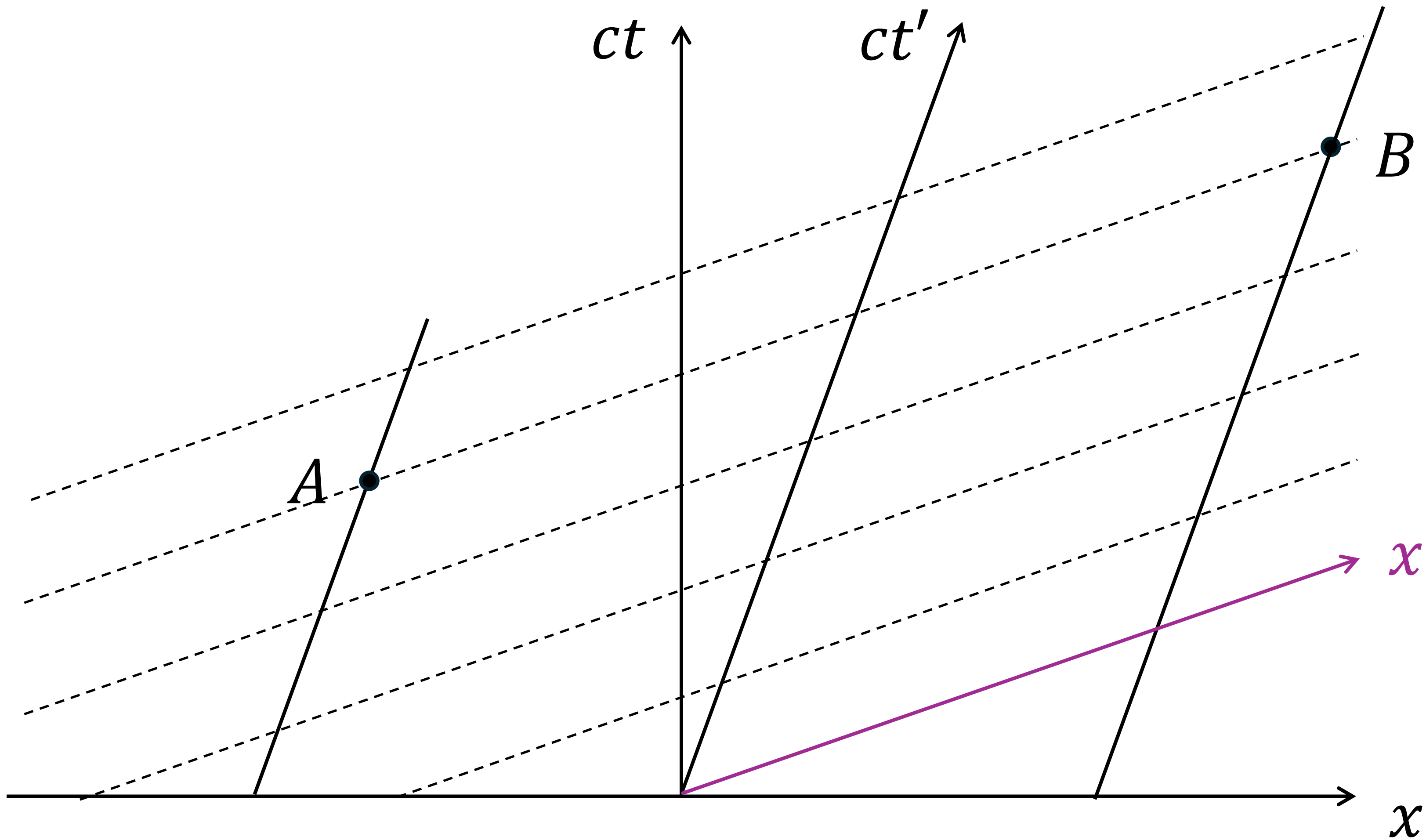
Relativity of Simultaneity

...are not simultaneous in the Home Frame



Lines of Simultaneity

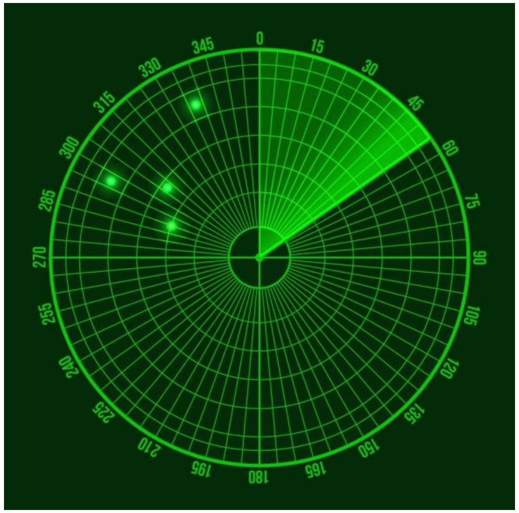
The lines of simultaneity connect all events that have the same time coordinate in a given frame of reference.



Remember, A and B are simultaneous in S'. That hasn't changed!

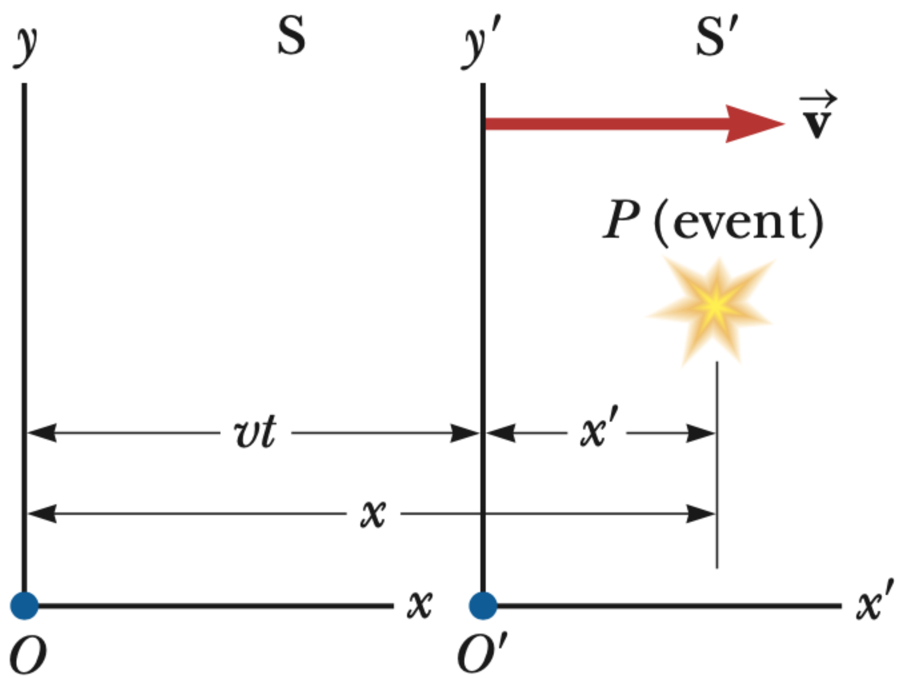
Example 1: Radar

Construct a spacetime diagram for the following scenario. A "master clock" sits at the origin of coordinates and periodically emits flashes of light in the $\pm x$ directions. At time t_1, the clock sends out a light flash to reflect off a target (call this event A). The reflected light travels back along the x axis to the master clock, which registers the reflected flash at time t_2. Use the diagram and basic kinematics to determine the spacetime coordinates of A.



Galilean Transformation

In classical mechanics, where simultaneity is absolute, it makes sense to talk about the position of an object in different frames of reference *at the same time*. This allows us to derive a set of coordinate transformations called the **Galilean transformation.**



$$t' = t$$

$$x' = x - vt$$

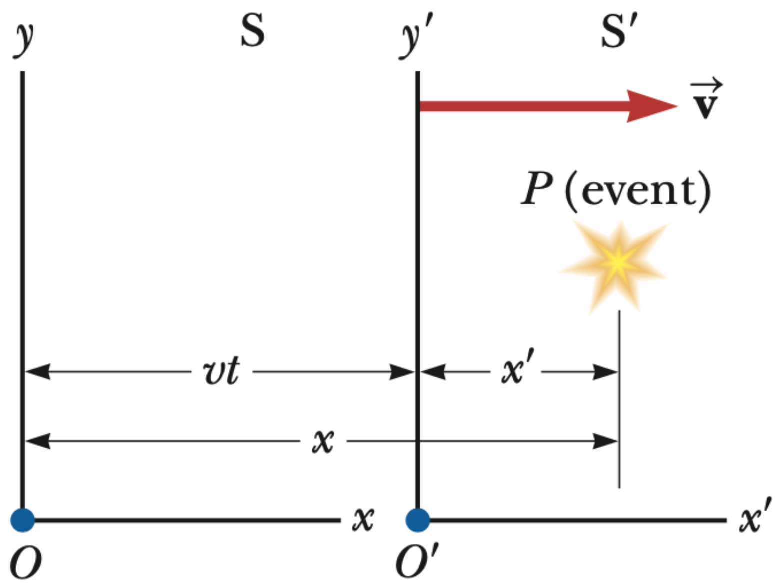
$$y' = y$$

$$z' = z$$

Good approximation when $v \ll c$

Lorentz Transformation

The **Lorentz transformation** is the *correct set* of equations that relate the spacetime coordinates of S and S' for all relative speeds in the range $0 < v < c$.



$$t' = \gamma(t - vx/c^2)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

Lorentz factor: $$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Matrix Notation

It is much easier to remember the Lorentz transformation in matrix notation. Ignoring the y and z coordinates,

$$x' = \gamma(x - vt)$$

$$ct' = \gamma(ct - vx/c)$$

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \underbrace{\gamma\begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix}}_{\textbf{Lorentz transformation matrix}}\begin{pmatrix} x \\ ct \end{pmatrix}$$

Inverse Transformation

If S' moves right with speed v relative to S, then S moves to the *left* with speed v according to S'.

The **inverse transformation** from S' to S is therefore the Lorentz transformation from S to S' with $v \to -v$.

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \gamma \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

If you don't buy it, you're welcome to invert the matrix yourself.

Note that $\gamma(-v) = \gamma(v)$

Alternate Notation

Let $\beta = v/c$ to simplify the notation. Always include a factor of c in front of the time coordinate!

Lorentz transformation

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Inverse transformation

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \gamma \begin{pmatrix} 1 & +\beta \\ +\beta & 1 \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

Example 2: Time Dilation

The Other Frame measures a time interval Δt_0 between two events that occur at the origin O', i.e. $\Delta x' = 0$. What are the corresponding coordinate intervals in S?

$$\begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix}$$

Inverse Lorentz transformation for coordinate intervals

Time Dilation

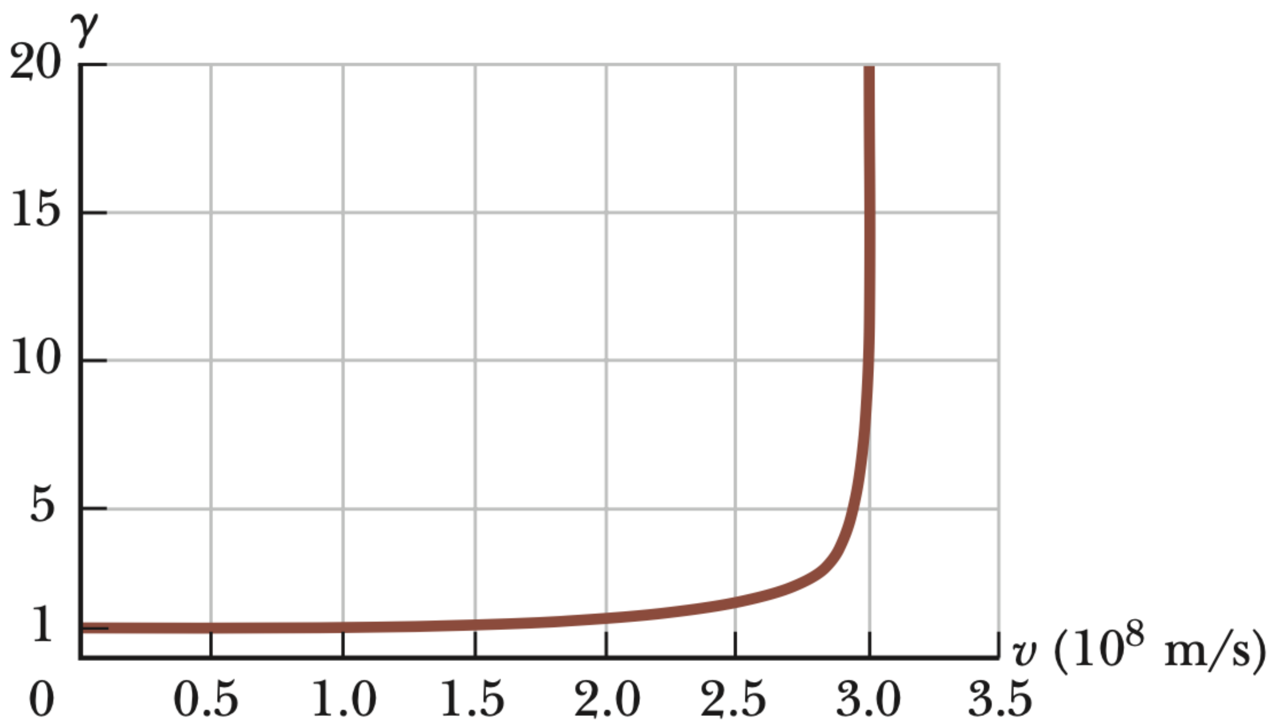
$$\boxed{\Delta t = \gamma \Delta t_0 = \frac{\Delta t_0}{\sqrt{1-\beta^2}}}$$

- The **coordinate time** between two events measured in any inertial reference frame is greater than or equal to any **proper time** measured between the events.
- This equation also applies to situations where the speed of the clock measuring Δt_0 is constant, even if the velocity is changing (as in uniform circular motion for example).

Lorentz Factor

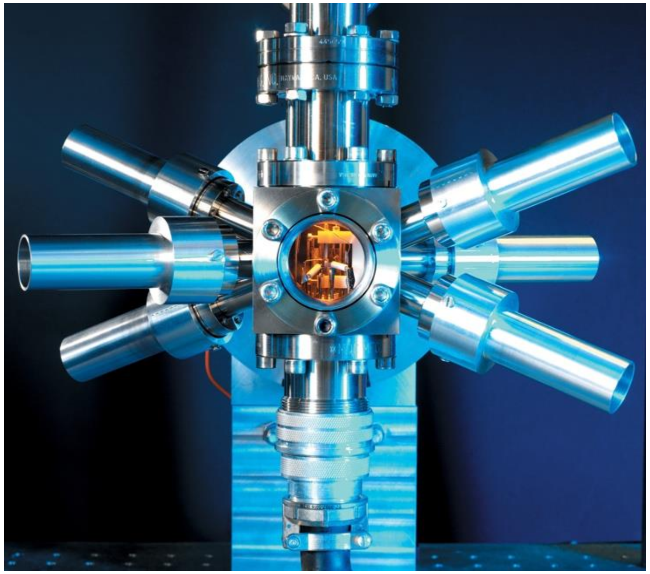
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-v^2/c^2}}$$

The Lorentz factor differs significantly from a value of 1 only for very high speeds



Example 3: Atomic Clocks

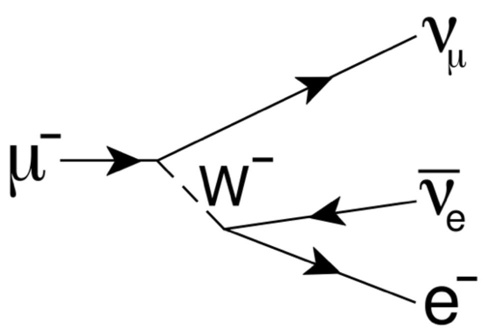
Ignoring gravitational effects and the rotation of the Earth, estimate the amount by which two atomic clocks will disagree if one clock on an airplane goes all the way around the equator at a constant speed of 300 m/s and another clock sits at rest on the Earth's surface.



Example 4: Muon Decay

The half-life of an unstable particle is the amount of time it takes for one half of a large sample of identical particles to decay, on average. The half-life of a muon at rest is $1.52\ \mu s$, meaning that after $1.52\ \mu s$, a collection of N muons will decay until $N/2$ muons are remaining.

Two detectors are vertically aligned and placed 2 km apart. The detectors register only the muons traveling at $0.995c$ vertically downward in the Earth's rest frame. If N muons are detected at the top detector, what fraction are detected at the bottom one? Calculate two values, one using the Galilean transformation and another using the Lorentz transformation.



Length Contraction

Consider a stick of length L_0 at rest in the Other Frame. To measure the length L of the stick in the Home Frame, we must record the positions of each end of the stick *simultaneously* in S.

Event A has coordinates (t, x_A) and B has coordinates (t, x_B) such that $L = \Delta x = x_B - x_A$

$$\begin{pmatrix} L_0 \\ c\Delta t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} L \\ 0 \end{pmatrix}$$

$$\boxed{L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \beta^2}}$$

Example 5: Galilean Transformations

Why don't we observe time dilation or length contraction in everyday life?

The simple answer is that we typically transform quantities between reference frames moving with relative speeds much slower than light.

Show that the Lorentz transformation reduces to the Galilean transformation in the limit $v \ll c$.

Relativistic Doppler Effect

$$f_{\text{obs}} = \sqrt{\frac{1-\beta}{1+\beta}}\, f_{\text{src}}$$

Source and receiver moving directly **away from each other**:

- observed frequency decreases
- observed wavelength increases

RED-SHIFT

(receding)

$$f_{\text{obs}} = \sqrt{\frac{1+\beta}{1-\beta}}\, f_{\text{src}}$$

Source and receiver moving directly **toward each other**:

- observed frequency increases
- observed wavelength decreases

BLUE-SHIFT

(bound for)

$$f_{\text{obs}} \approx (1 \pm \beta) f_{\text{src}} \qquad \text{when } \beta \ll 1 \text{ or } v \ll c$$

Conceptual Question 10

A star that is visible from Earth is moving **away** from us at a very high speed. Compared to the color of the light emitted by the star in its rest frame, observers on Earth will say

A. the star appears more red

B. the star appears more blue

C. the color of the star is the same

D. it depends how far away the star is

Conceptual Question 11

A star that is visible from Earth is moving **toward** us at a very high speed. Compared to the color of the light emitted by the star in its rest frame, observers on Earth will say

A. the star appears more red

B. the star appears more blue

C. the color of the star is the same

D. it depends how far away the star is

Example 6: Doppler Effect

Light from excited atoms in a certain quasar is received by observers on earth. The wavelength of a certain spectral line of this light is measured by those observers to be 1.12 times longer than it would be if the atoms were at rest in the laboratory (i.e., the light has been *red-shifted* by about 12%).

What is the quasar's speed relative to earth (assuming it is moving directly away from earth)?

Spacetime Interval

Consider two events E_1 and E_2 <u>connected by a light flash</u>. Observers in S and S' will generally disagree on the distance and time interval between events, but they will both agree the distance between them is the same distance traveled by light in the time elapsed:

$$(c\Delta t)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$(c\Delta t')^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

$$(c\Delta t)^2 - \Delta x^2 = (c\Delta t')^2 - \Delta x'^2$$

S and S' agree on this quantity, at least for two events separated by a light pulse

Spacetime Interval

Now consider *any two events in spacetime*. We define the **spacetime interval** such that

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The spacetime interval is invariant under Lorentz transformations, in a way that is analogous to the invariance of length under rotations.

Invariance means $\Delta s^2 = \Delta s'^2$ for any two inertial frames in relative motion. The equation for Δs^2 is often called "the metric equation."

The Metric Equation

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The metric equation is a generalization of the concept of length applied to spacetime.

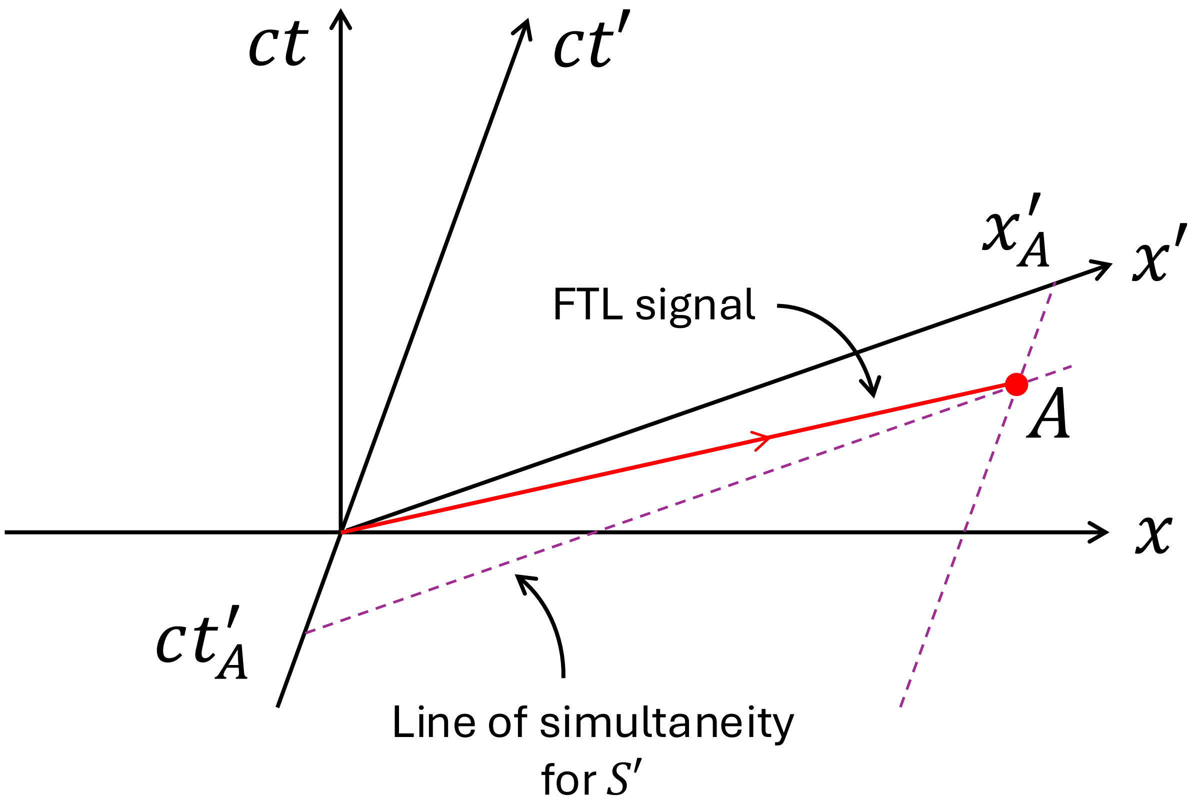
- When $\Delta s^2 > 0$, there exists a frame of reference where $\Delta s/\mathrm{c}$ is a proper time interval $(\Delta r = 0)$.
- When $\Delta s^2 < 0$, there exists a frame of reference where $\sqrt{-\Delta s^2}$ is a proper length $(\Delta t = 0)$.

Cosmic Speed Limit

"Nothing can go faster than light." There are different ways to understand this result, but for one, it must be true to preserve causality—the idea that a causal event must precede its effect. If faster-than-light travel were possible, effects could precede their causes, and that's absurd!

The speed of light is the "cosmic speed limit" in the sense that no physical object capable of exerting causal influence can travel faster than c.

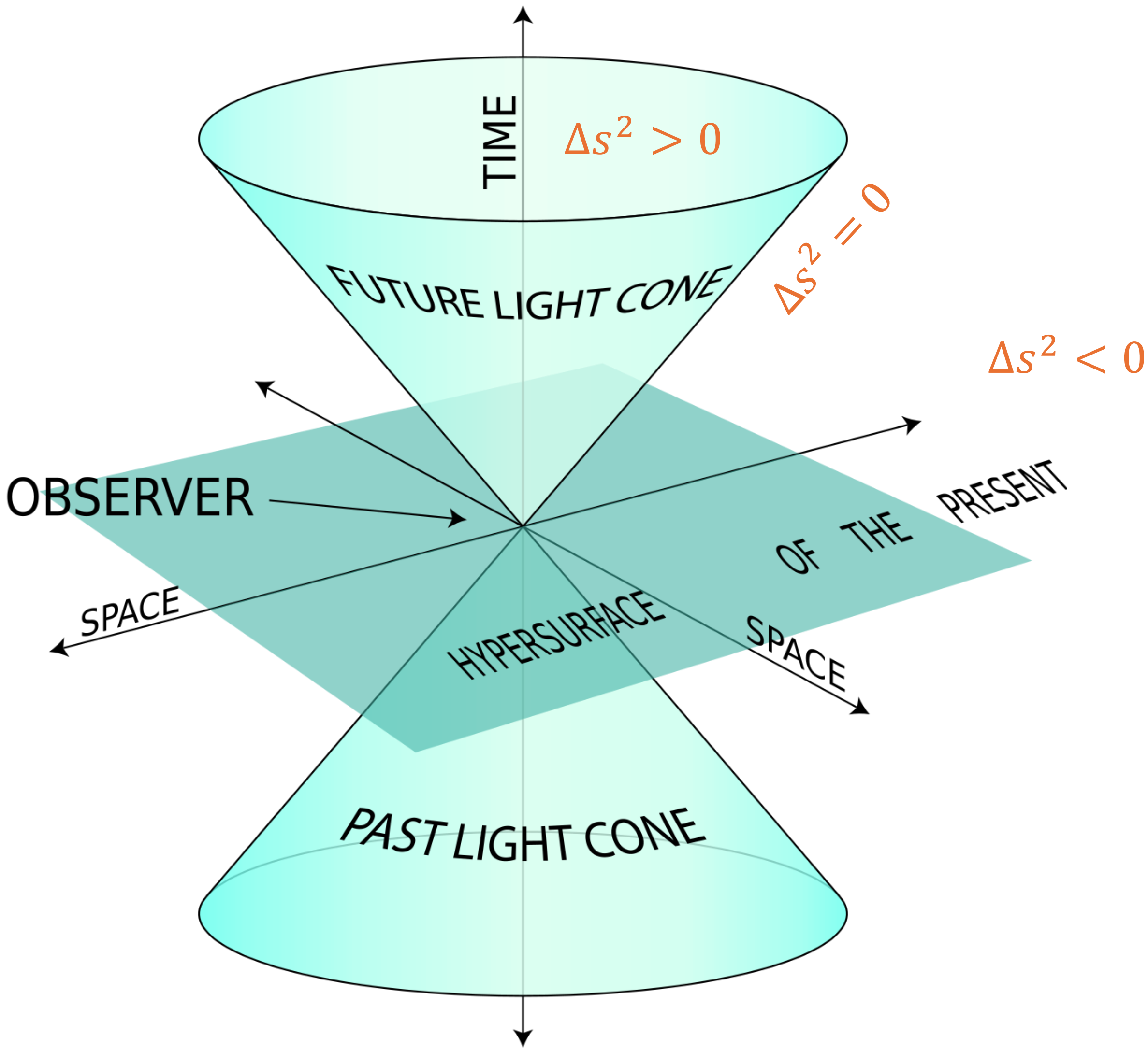
Proof by Contradiction



In the Home Frame, the cause is at the origin,
and its hypothetical effect is event A.


186,000 miles per second. It's not just a good idea…
It's the law!

Light Cone


TIME
$\Delta s^2 > 0$
$\Delta s^2 = 0$
$\Delta s^2 < 0$
FUTURE LIGHT CONE
OBSERVER
PRESENT
OF THE
HYPERSURFACE
SPACE
SPACE
PAST LIGHT CONE

Example 7: Causality

A meteor strikes the moon (event A), causing a large and vivid explosion. Exactly 0.47 s later (as measured in an inertial reference frame attached to the earth) the radio telescope receiving signals from the moon goes on the fritz. Could these events be causally related? (Hint: The distance between the earth and the moon is roughly 384,000 km.)

Velocity Addition

Particle moves with x-velocity u'_x in S'. The velocity of the same particle in S can be found with the inverse transformation:

$$u_x \equiv \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)}$$

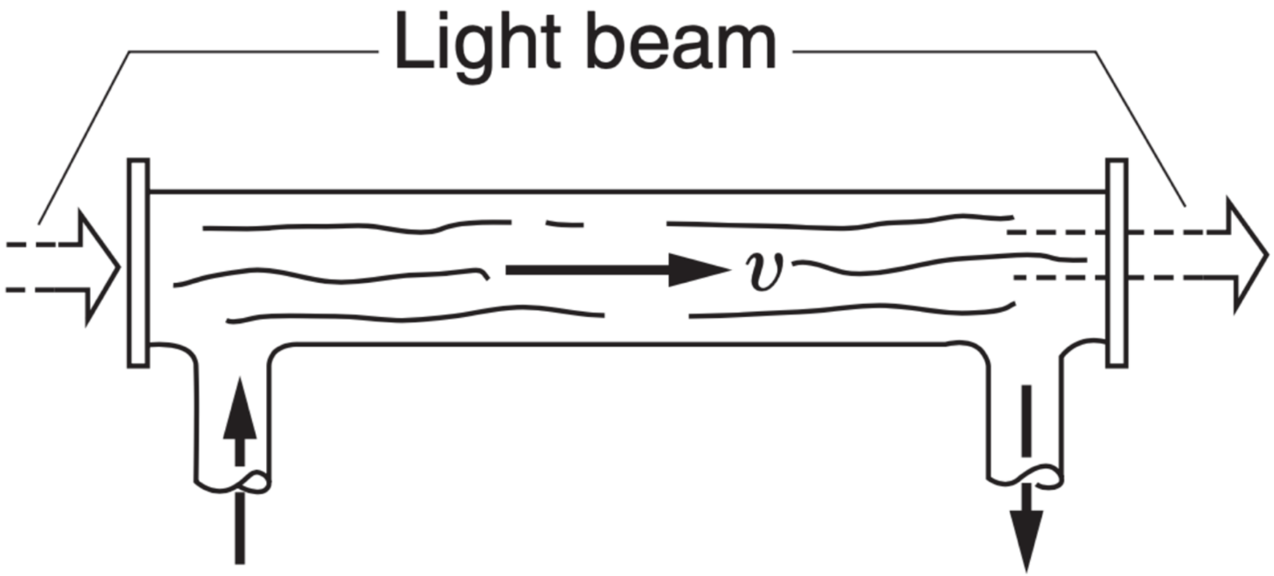
$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}$$

$$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c\text{^}2\ )}$$

$$u_z = \frac{u'_z}{\gamma(1 + vu'_x/c\text{^}2\ )}$$

Example 8: Velocity Addition

Consider a tube filled with water. If the water is at rest, the speed of light in the water with respect to the laboratory is $u = c/n$. What is the speed of light when the water is flowing with speed v?



Four Velocity

It's annoying that ordinary velocity doesn't transform the same way coordinates do. On the other hand, we'll find that a new quantity called **four velocity** does.

Consider an infinitesimal spacetime interval between two successive events in a massive particle's worldline:

$$0 < ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Proper time interval

$$c^2 = c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2$$

Components of the
four velocity

Four Velocity

The **four velocity** of a particle is

$$\mathbf{U} \equiv \gamma(u)(u_x, u_y, u_z, c)$$

$$\frac{dt}{d\tau} = \gamma(u) \qquad\qquad \frac{dx}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau} = \gamma(u) u_x$$

$$\begin{pmatrix} U'_x \\ U'_t \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} U_x \\ U_t \end{pmatrix}$$

The **four velocity** components transform like spacetime coordinates!

Four Momentum

Ordinary Newtonian momentum does not obey a conservation law that is valid in all inertial frames!

A particle of mass m has a **four momentum** $\mathbf{P} = m\mathbf{U}$ which is conserved in the absence of forces, and similarly, the total four momentum of an isolated system of particles is conserved. The conservation of four momentum is a frame-independent law of physics!

The spatial components of $\mathbf{P}$ are related to Newtonian momentum. A particle's **relativistic momentum** is

$$\vec{\mathbf{p}} = \gamma\, m\, \vec{\mathbf{u}}$$

Conceptual Question 12

When an object of mass m is pushed to relativistic speed u, its momentum is

A. greater than mu.

B. smaller than mu.

C. equal to mu.

Relativistic Energy

What about the time component of the four momentum, the quantity γmc? Let's consider the limiting case $u \ll c$ for a moment and see what we get:

$$\gamma mc = mc\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx mc + \frac{1}{c}\left(\frac{1}{2}mu^2\right) + \cdots$$

Define **relativistic energy** as

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

Kinetic Energy & Rest Energy

The equation for relativistic energy implies that even when the particle isn't moving, it has rest energy mc^2 (finally!)

Define **relativistic kinetic energy** as

$$K = E - mc^2 = (\gamma - 1)mc^2$$

When $u \ll c$,

$$K \approx \frac{1}{2}mu^2$$

Pitfall Prevention

The $\gamma(v)$ that appears in the Lorentz transformation depends on the relative speed between frames S and S', whereas the $\gamma(u)$ in the relativistic energy and momentum equations depends on the speed of the particle.

Know your gamma's!

Helpful Trick

We use the conservation principle to find energy and momentum, and once we have those, we can find the velocity components. Energy and momentum are highly nonlinear expressions in the velocity components, but we can use the following trick to find them:

$$\frac{p_x c}{E} = \frac{\gamma m u_x c}{\gamma m c^2} = \frac{u_x}{c}$$

$$u_x = \frac{p_x c^2}{E} \qquad u_y = \frac{p_y c^2}{E} \qquad u_z = \frac{p_z c^2}{E}$$

Energy-Momentum Relation

Returning to our introduction to four velocity, after some algebraic manipulations, it follows that

$$c^2 = \frac{E^2}{m^2c^2} - \frac{p^2}{m^2}$$

[we can trace this back to the metric equation]

$$E^2 = (pc)^2 + (mc^2)^2$$

This is the **energy-momentum relation**, where p is the magnitude of relativistic momentum.

Example 9: Non-relativistic Limit

(a) Use the energy-momentum relation to show that $pc = \sqrt{K(K + 2mc^2)}$, where K is the relativistic kinetic energy.

(b) Show that for a massive particle moving slowly compared to the speed of light, K/mc^2 is a very small quantity (second order in v/c).

(c) When $K \ll mc^2$, show that $p = \sqrt{2mK}$, consistent with Newtonian physics.

Consequences for Dynamics

- The postulates of special relativity call for revisions to core concepts of Newtonian physics.
  - Force, momentum, and energy are all redefined so that **if energy and momentum are conserved in one inertial frame of reference, they are conserved in all inertial frames of reference.**
  - Just as space and time make up a single concept of spacetime, energy and momentum make up a single quantity composed of *relativistic energy* and *relativistic momentum (four momentum)*.
  - The mass of a single particle is a relativistic invariant, but the total mass of a system of particles is not. The total mass depends on your frame of reference!

Example 10: Four Momentum

Somewhere in deep space, an indestructible crystal with mass $m_1 = 12$ kg is moving in the $+x$ direction with $u_{1x} = +4c/5$ in some inertial frame. The crystal strikes another crystal of mass $m_2 = 28$ kg at rest. The crystals elastically bounce off each other and the less massive one moves with velocity $u_{3x} = -5c/13$.

What is the x-velocity of the larger crystal after the collision? Show that the collision does not conserve Newtonian momentum.

Example 11: Mass Conservation?

Two balls of putty, each of mass m, move in opposite directions toward each other with speeds of $0.95c$. The balls stick together, forming a single motionless ball of putty at rest.

What is the mass of this ball of putty? (Express your answer as a multiple of m.)