Phys 153: Fundamentals of Physics III

Unit #2 – Waves and Wave Optics

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Reading

- Please read the following sections of your textbook carefully and skim the rest:
  - 16.1-16.2, 16.5, 16.9
  - 17.1-17.6
  - 36.1-36.3
  - 37.1-37.3

Traveling Waves

In a linear, non-dispersive medium or field, a traveling wave propagates with a fixed shape at a constant speed.

A 1D traveling wave with speed v is simply any function of the form $\psi(x,t) = \psi(x \pm vt)$

- $\psi(x - vt)$ moves in the $+x$ direction
- $\psi(x + vt)$ moves in the $-x$ direction

The **polarization** of a wave indicates the direction of the disturbance itself—whatever is 'waving.'

- Disturbance is perpendicular to the direction of travel in a transverse wave (wave on string).
- Disturbance is parallel to the direction of travel in a longitudinal wave (sound wave).

The Wave Equation

Any function $\psi(x,t) = \psi(x \pm vt)$ satisfies the 1D wave equation, a **partial differential equation**:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

This means "take the second derivative of $\psi(x,t)$ with respect to x treating t as a constant"

Likewise, this means "take the second derivative of $\psi(x,t)$ with respect to t treating x as a constant"

Example 1: Traveling Wave

Let a 1D traveling wave, such as the vertical displacement of a string, be represented by the function

$$\psi(x,t) = -A(x - vt)^2 + B$$

where $A = 1\ \mathrm{m}^{-1}$, $B = 1\ \mathrm{m}$, and $v = 1\ \mathrm{m}/s$. Also, note that ψ is restricted to positive values.

(a) Plot $\psi(x,t)$ when $t = 0, 3\ s$, and $5\ s$. Show that the wave is indeed moving in the $+x$ direction.

(b) Show that $\psi(x,t)$ satisfies the wave equation (previous slide). It will be more illuminating if you do this part symbolically.

Periodic Waves

An oscillating source can produce waves in a material that repeat themselves after a given spatial distance or duration.

A periodic traveling wave is characterized by its wavelength λ and period T such that

$$\psi(x+\lambda, t+T) = \psi(x,t)$$

- Consider a periodic wave moving to the "right," i.e. $\psi(x,t) = \psi(x - vt)$.

$$\psi(x+\lambda, t+T) = \psi(x - vt + \lambda - vT)$$

$$\Rightarrow \lambda = vT \quad \text{or} \quad v = \lambda f \quad \text{where } f = \frac{1}{T}$$

- f is the frequency of the wave, in Hz = 1/s

Wave Speed Equation

A periodic traveling wave has wavelength and frequency related to the wave speed by

$$v = \lambda f$$

Here's another way to think of it:

$$v = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{\text{wavelength}}{\text{period}}$$

Sinusoidal Waves

If the source exhibits simple harmonic oscillations with a constant frequency, the resulting traveling wave is sinusoidal—it has the shape of a cosine (or sine) graph moving left or right.

$$\psi(x, t) = A\cos(kx \pm \omega t + \phi_0)$$

- $k = 2\pi/\lambda$ is the **wavenumber**
- $\omega = 2\pi/T = 2\pi f$ is the **angular frequency**
- The wave speed is $v = \omega/k = \lambda f$
- ϕ_0 is the **phase constant**, and altogether, $\phi(x, t) = kx \pm \omega t + \phi_0$ is the phase of the wave.
- Lastly, A is the **amplitude** of the wave.

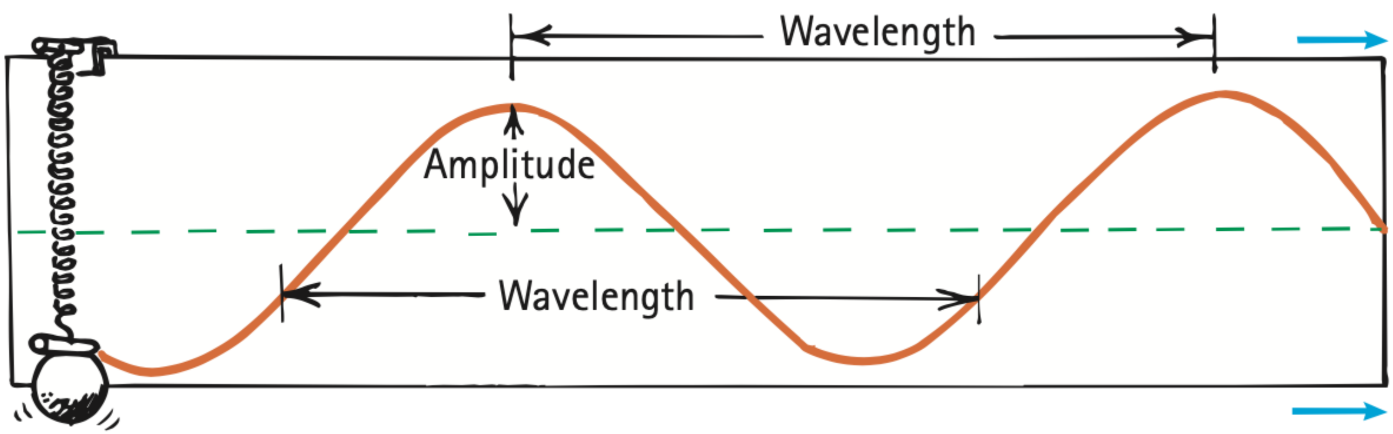
Example 2: Sinusoidal Waves

a) Show that a sinusoidal wave is periodic in space and time, i.e. $\psi(x+\lambda, t+T) = \psi(x,t)$

b) Show that a sinusoidal wave has the same form as any traveling wave; $\psi(x,t) = \psi(x \pm vt)$

c) Show by explicit calculation that a sinusoidal wave satisfies the 1D wave equation.

d) Determine the value of ϕ_0 so that $\psi(0,0) = -A$



Mechanical Waves

The wave speed for a mechanical wave depends properties of the medium itself.

- Proportional to stiffness and inversely proportional to mass density.

The frequency of a periodic mechanical wave is determined by the source. That leaves the wavelength as the only dependent variable:

$$\lambda = \frac{v}{f}$$

Conceptual Question 1

A traveling wave of frequency f is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string. What is the **wave speed** of the second wave?

A. twice that of the first wave

B. half that of the first wave

C. the same as that of the first wave

D. impossible to determine

Conceptual Question 2

A traveling wave of frequency f is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string. What is the **wavelength** of the second wave?

A. twice that of the first wave

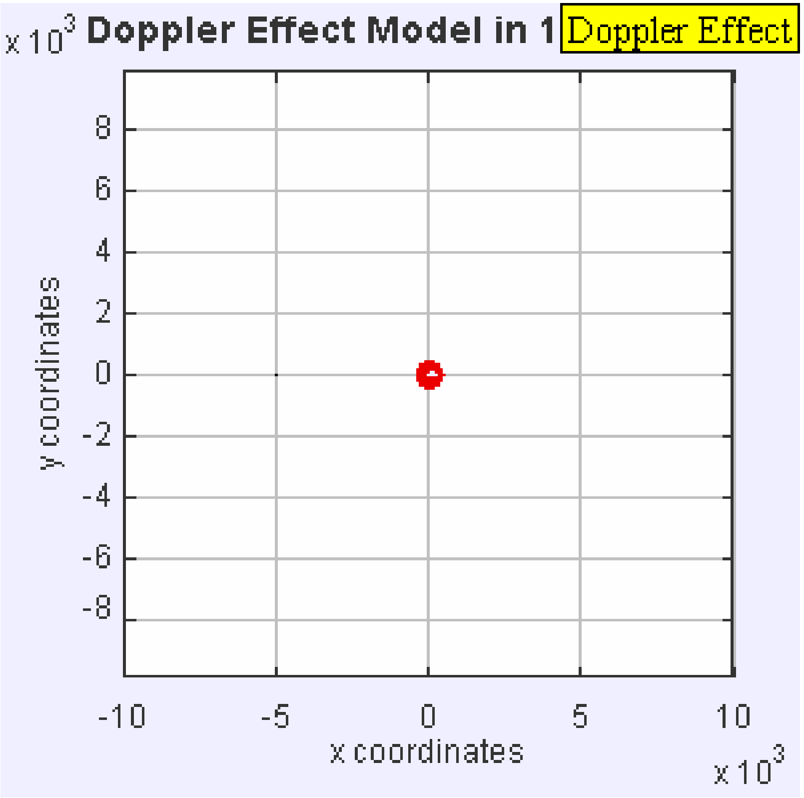
B. half that of the first wave

C. the same as that of the first wave

D. impossible to determine

Doppler Effect

Shift in the frequency of a sound wave resulting from relative motion of either the observer or the source with respect to the medium.



Moving Observer (Qualitative)

- Imagine moving towards a source emitting sound waves with frequency f . The wave crests travel at a fixed speed relative to the air, and as you move towards the source, the wave crests arrive at your location *in shorter time intervals* than they leave the source. Thus, you measure a higher frequency of sound $f' > f$.
- Moving away from the source, the wave crests arrive at your location *in longer time intervals* than they leave the source, so you measure a lower frequency $f' < f$.
  - Imagine moving at the same speed as the sound waves; you would not perceive any variations in the air pressure, so you would hear no sound at all!

Moving Observer (Quantitative)

If the source is stationary, the wavelength of the wave is the same in all directions relative to source and observer.

A moving observer effectively sees a different wave speed and frequency:

$$\lambda = \frac{v}{f} = \frac{v'}{f'} = \frac{v \pm v_O}{f'}$$

$$f' = \left(1 \pm \frac{v_O}{v}\right) f$$

$+$ observer moving directly towards source with speed v_O
$-$ observer moving directly away from source with v_O
v is the speed of sound relative to stationary medium

Moving Source (Qualitative)

- After wave crest leaves the source, its speed is governed by the medium. If the source is moving, the wave crests bunch up in the direction the source is moving and stretch out behind it.
- When the source is moving towards a stationary observer, the wave crests arrive in shorter time intervals, and the observer measures a higher frequency.
  - If source moves towards observer at the same speed as the sound, all the wave crests arrive together!
- When the source is moving away from a stationary observer, the wave crests arrive in longer time intervals, and the observer measures a lower frequency.

Moving Source (Quantitative)

The observed wavelength is shorter in front of a moving source and longer behind it. Along the direction of motion, the distance between crests is $\lambda = (v \pm v_S)T$ and the observed frequency is

$$f' = \frac{v}{\lambda} = \frac{v}{(v \pm v_S)T}$$

$$f' = \frac{f}{(1 \pm v_S/v)}$$

$-$ source moving directly towards observer with speed v_S

$+$ source moving directly away from observer with speed v_S

Pitfall Prevention

Don't bother memorizing all the rules for when to choose + or − in the equations. Instead, use conceptual reasoning!

- When the source and observer approach each other, the frequency should increase.
- When the source and observer recede from each other, the frequency should decrease.

Approximation for a Slow-Moving Source

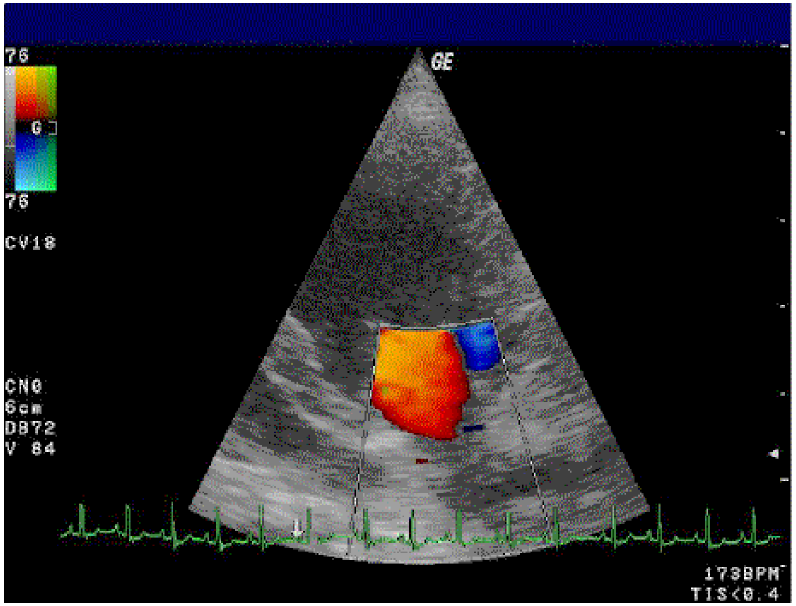
Consider a source moving slowly compared to the speed of sound ($\sim 340\ \mathrm{m}/s$ in air):

$$\frac{1}{1-x} \approx 1 + x \quad (|x| \ll 1)$$

$$f' = \frac{f}{(1 \mp v_S/v)} \approx (1 \pm v_S/v)f$$

Example 3: Doppler Imaging

Cardiologists use Doppler ultrasound to determine the flow speed of blood through the heart. A typical system uses 3.5 MHz ultrasound and can detect a frequency change as small as 0.1 kHz. Assuming the speed of sound in soft tissue is 1540 m/s, what is the smallest flow speed measurable with this device? Give your answer in cm/s.



Echocardiography, animation of a dog’s mitral valve

Linearity of the Wave Equation

The wave equation is a **linear partial differential equation** which means the sum of any two solutions is itself a solution.

The general solution to the 1D wave equation is a linear combination of a right-moving wave and a left-moving wave:

$$\psi(x,t) = \psi_{+x}(x - vt) + \psi_{-x}(x + vt)$$

Right-moving wave

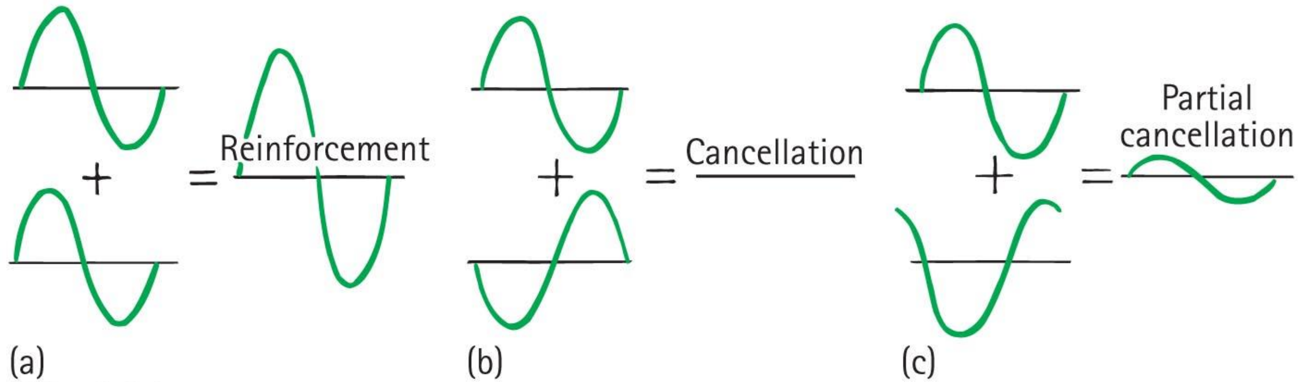
Left-moving wave

Principle of Superposition

When two or more waves are simultaneously present at a single point, the displacement of the medium at that point is the sum of the displacements due to each individual wave.

Wave interference is a consequence of the principle of superposition.

- **Constructive interference** occurs when the individual waves reinforce each other, and the amplitude of the resultant wave is increased.
- **Destructive interference** occurs when the individual waves cancel each other out, and the amplitude of the resultant wave is decreased.

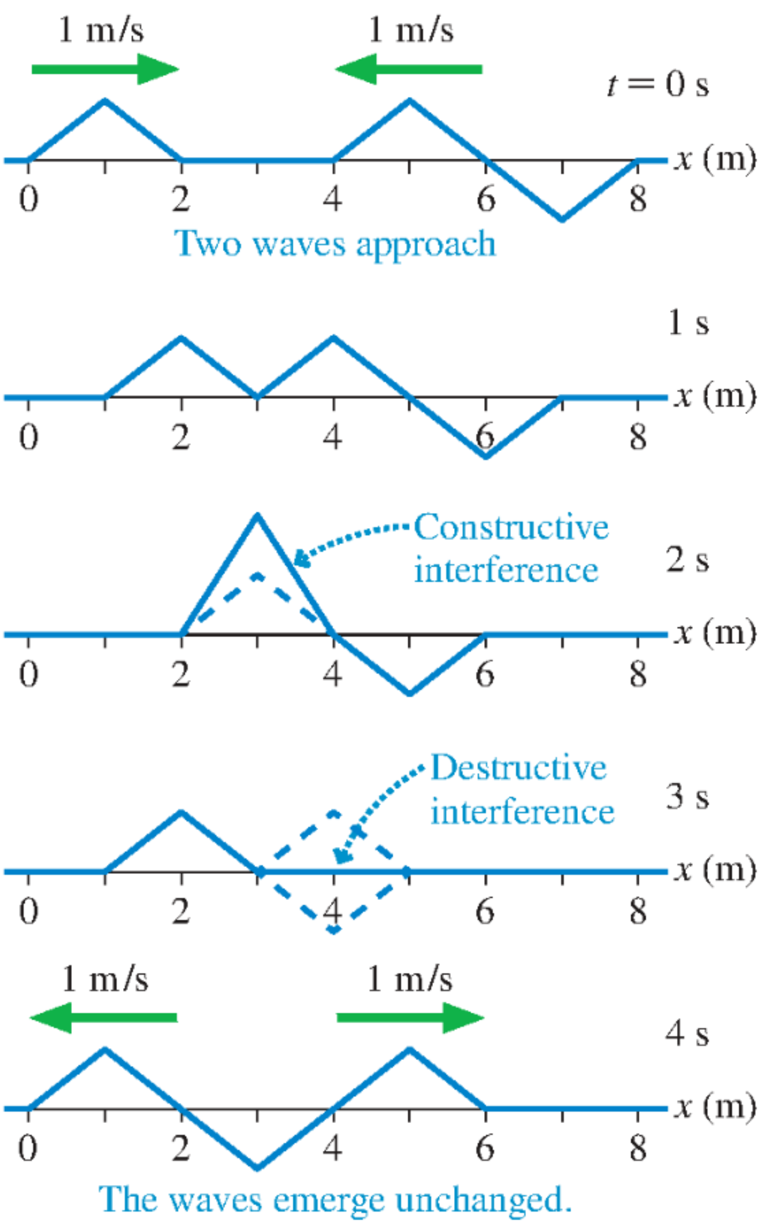


Conceptual Question 3

A small wave pulse and a large wave pulse approach each other on a string; the large pulse is moving to the right. Some time after the pulses have met and passed each other, which of the following statements is correct?

A. The large pulse continues unchanged, moving right.
B. The large pulse continues moving to the right but is now smaller in amplitude.
C. The small pulse is reflected and moves with its original amplitude.
D. The small pulse is reflected and moves with a smaller amplitude.
E. The two pulses combine into a single pulse moving to the right.

Wave Interference in 1D



Standing Waves

A standing wave is produced by two waves of equal amplitude and wavelength traveling in opposite directions.

$$\psi(x,t) = A\cos(kx - \omega t) + A\cos(kx + \omega t)$$

$$= \underbrace{2A\cos(kx)}\cos(\omega t)$$

Each point oscillates with "effective amplitude" given by this term

and with the same frequency as the waves

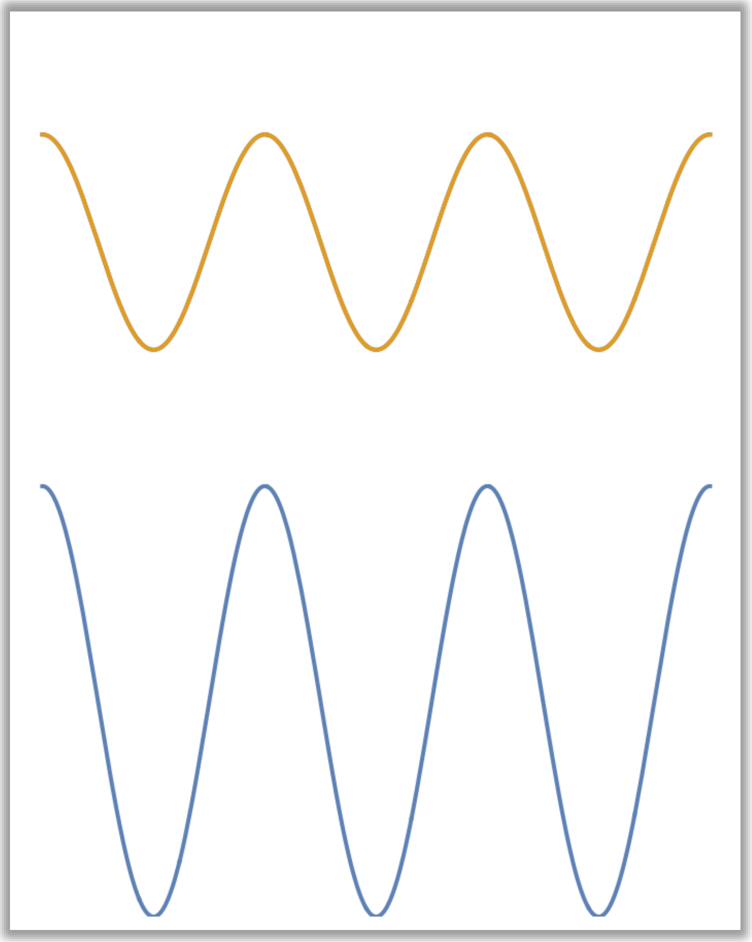
Trig identity: $\cos(a+b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

Standing Wave Visualization

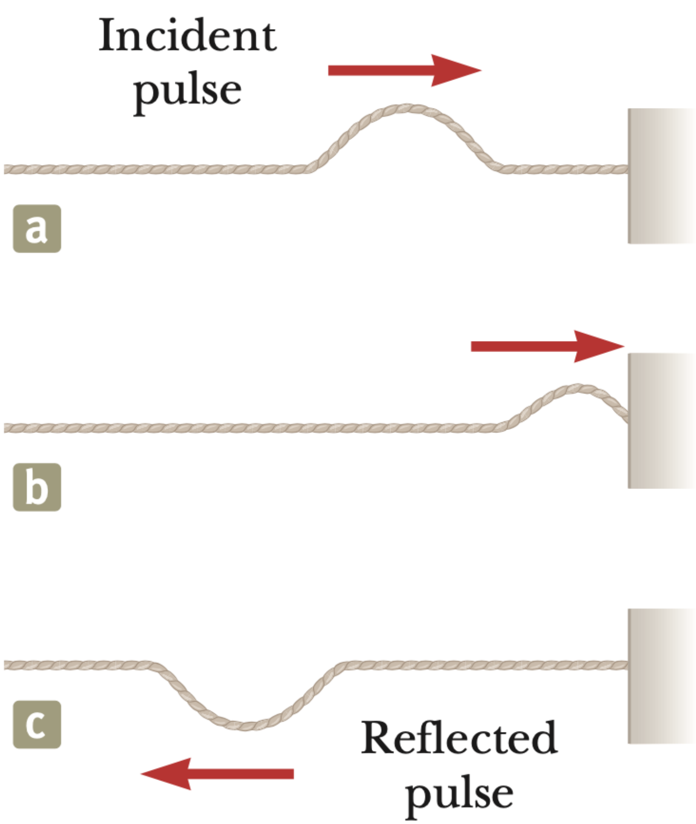
Two sinusoidal traveling waves with the same amplitude and frequency combine to make a standing wave.

Nodes are points where the standing wave has zero effective amplitude—they exhibit no oscillations.

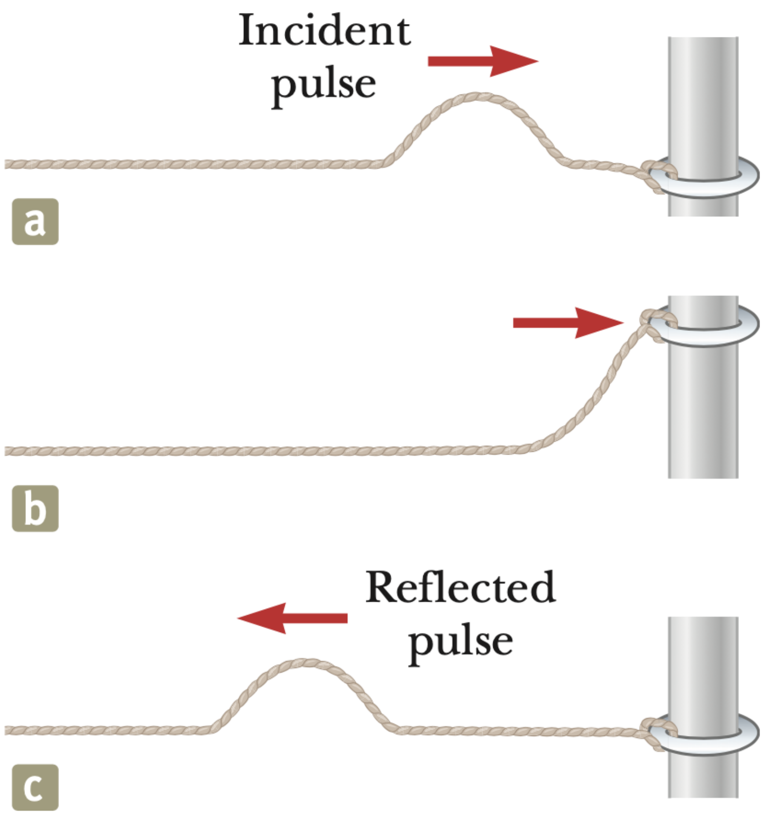
Antinodes are points where the standing wave has maximum effective amplitude.



Boundary Conditions



Fixed End



Free End

Boundary Conditions

Consider sound waves in a column of air such as that inside an organ pipe or a clarinet.

- In a pipe closed at one end, the closed end is a displacement node and pressure antinode.
- In a pipe open at one end, the open end is a displacement antinode and pressure node.

Standing waves on both a string and in a column of air are the result of traveling waves that fit the boundary conditions imposed on them.

String Fixed at Both Ends

Consider a string of length L that carries waves with constant amplitude and frequency.

<u>Boundary conditions</u>: $\psi(0,0) = \psi(L,0) = 0$

$$\psi(x,t) = A\sin(kx)\cos(\omega t) \Rightarrow \psi(0,0) = 0$$

$$\psi(L,0) = A\sin(kL) = 0$$

$\Rightarrow kL = \pi n$ where $n = 1, 2, 3, \ldots$

Quantization is common when waves are subject to boundary conditions

Normal Modes - String Fixed at Both Ends

When a string is fixed at both ends, the wave is limited to a set of **normal modes** with quantized wavelengths and quantized natural frequencies.

$$\lambda_n = \frac{2L}{n} \qquad\qquad f_n = \frac{nv}{2L} = nf_1$$

Tube Closed at One End

Consider a tube of length L open at one end, closed at the other.

<u>Boundary conditions</u>: $\psi(0,0) = A,\ \psi(L,0) = 0$

$$\psi(x,t) = A\cos(kx)\cos(\omega t) \Rightarrow \psi(0,0) = A$$

$$\psi(L,0) = A\cos(kL) = 0$$

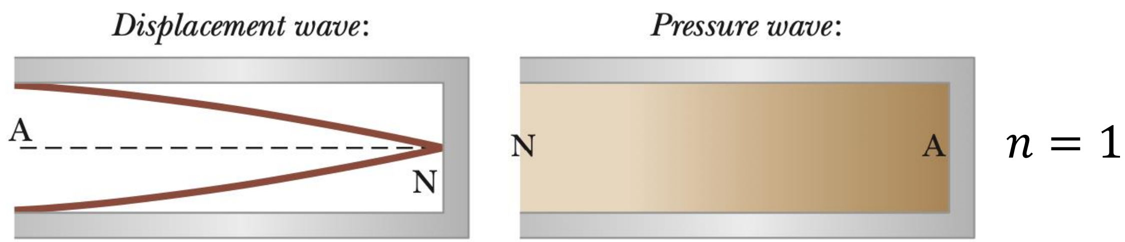
$$\Rightarrow kL = \frac{\pi}{2}(2n-1) \text{ where } n = 1,2,3,\ldots$$

In other words, for
odd multiples of $\pi/2$

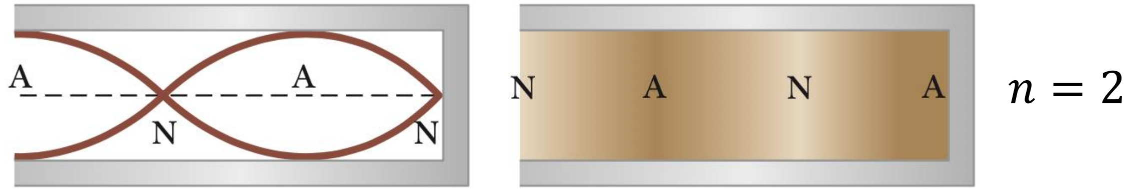
Normal Modes – Closed Tube

In a pipe closed at one end,

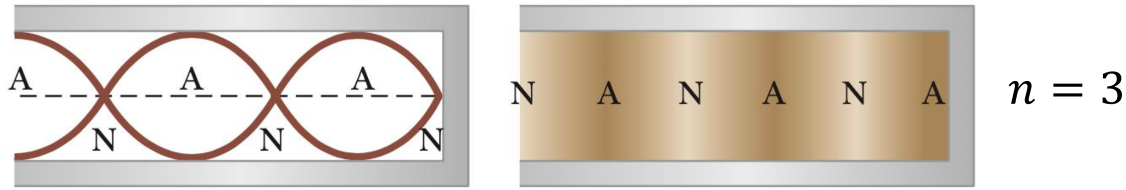
$$\lambda_n = \frac{4L}{(2n-1)} \qquad f_n = \frac{v}{4L}(2n-1) = (2n-1)f_1$$



First harmonic: $\lambda_1 = 4L \qquad f_1 = \frac{v}{4L}$



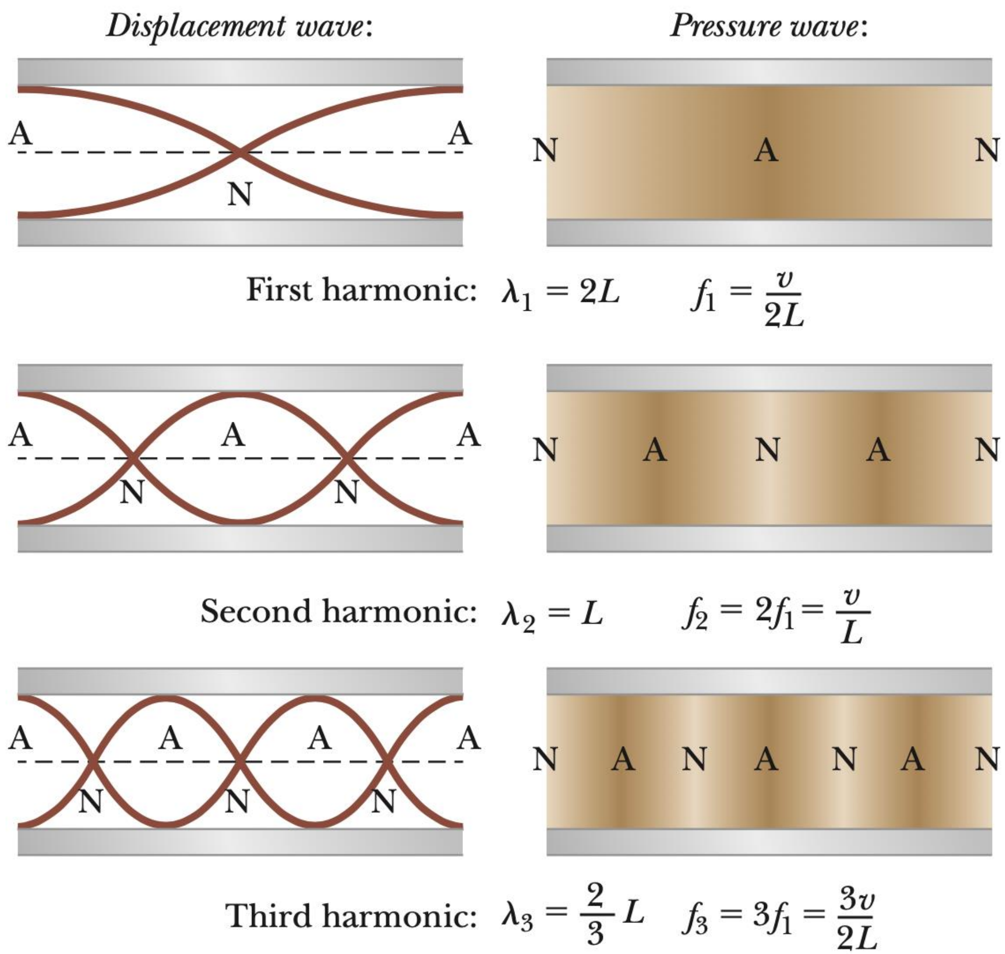
Third harmonic: $\lambda_3 = \frac{4}{3}L \qquad f_3 = 3f_1 = \frac{3v}{4L}$



Fifth harmonic: $\lambda_5 = \frac{4}{5}L \qquad f_5 = 5f_1 = \frac{5v}{4L}$

Normal Modes – Open Tube

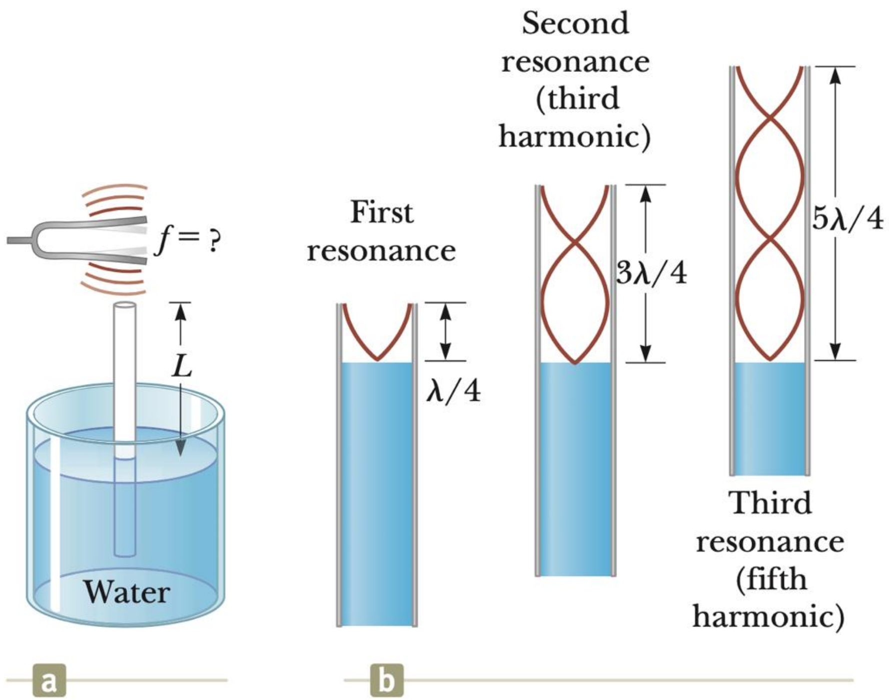
A tube of length L that is open at both ends has the same normal modes as string fixed at both ends.



Example 4: Standing Waves

A vertical pipe open at both ends is partially submerged in water, and a tuning fork vibrating at an unknown frequency is placed near the top of the pipe. The length L of the air column can be adjusted by moving the pipe vertically. The sound waves generated by the fork are reinforced when L corresponds to one of the resonance frequencies of the pipe. For a certain pipe, the smallest value of L for which a peak occurs in the sound intensity is 9.00 cm.

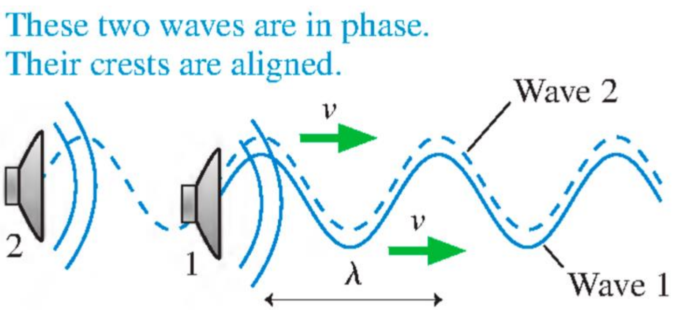
a) What is the frequency of the tuning fork?
b) What are the values of L for the next two resonance conditions?



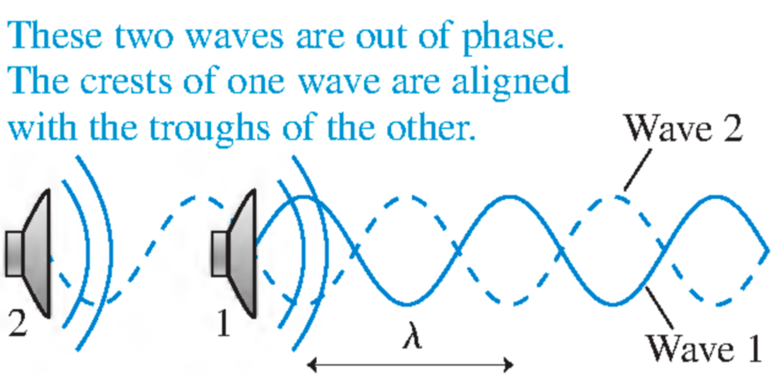
Interference in 1D

Consider two identical sources that produce sinusoidal waves with the same frequency and amplitude that travel to the right.

- **Maximum constructive interference** occurs when the path-length difference is an integer number of wavelengths.

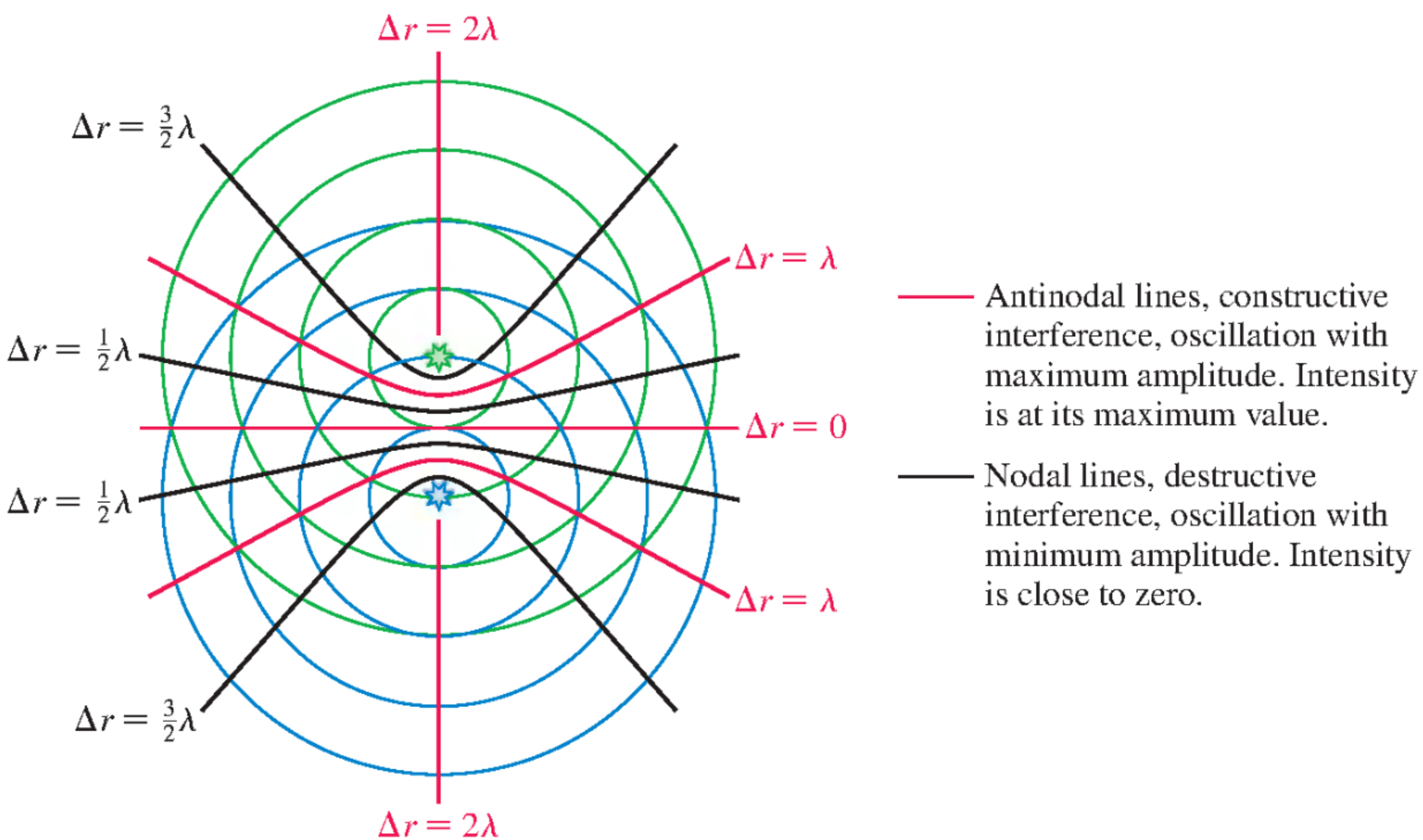


- **Maximum destructive interference** occurs when the path-length difference is a half-integer number of wavelengths.



Interference in 2D

Consider two identical point sources producing circular sinusoidal waves. The conditions for maximum constructive/destructive interference are the same as in 1D with regard to path length.



Example 5: Interference

Two loudspeakers in a plane are 2.0 m apart and in phase with each other. Both emit 700 Hz sound waves into a room where the speed of sound is 341 m/s. A listener stands 5.0 m in front of the loudspeakers and 2.0 m to one side of the center. Is the interference at this point maximum constructive, maximum destructive, or in between? How will the situation differ if the loudspeakers are out of phase with each other?

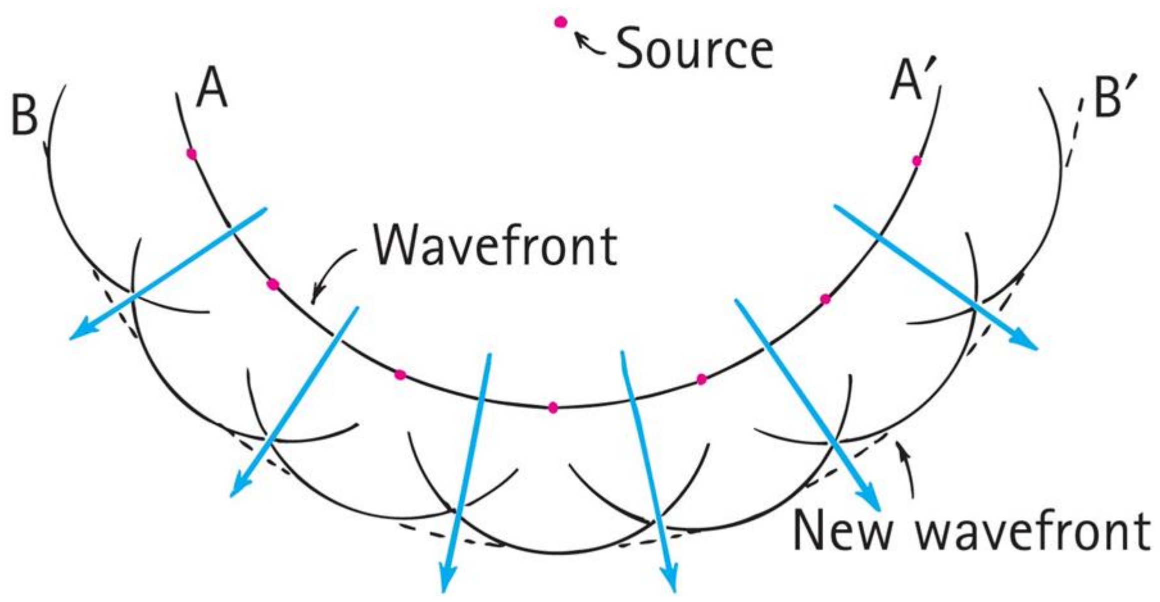
Christiaan Huygens

- Throw a rock in a quiet pool, and waves appear along the surface of the water.
- Huygens proposed that the wavefronts of light waves spreading out from a point source can be regarded as the overlapped crests of tiny secondary waves.
- Wavefronts are made up of tinier wavefronts—this idea is called **Huygens' principle** and like Fermat's principle of least time, it can be derived from Maxwell's equations!

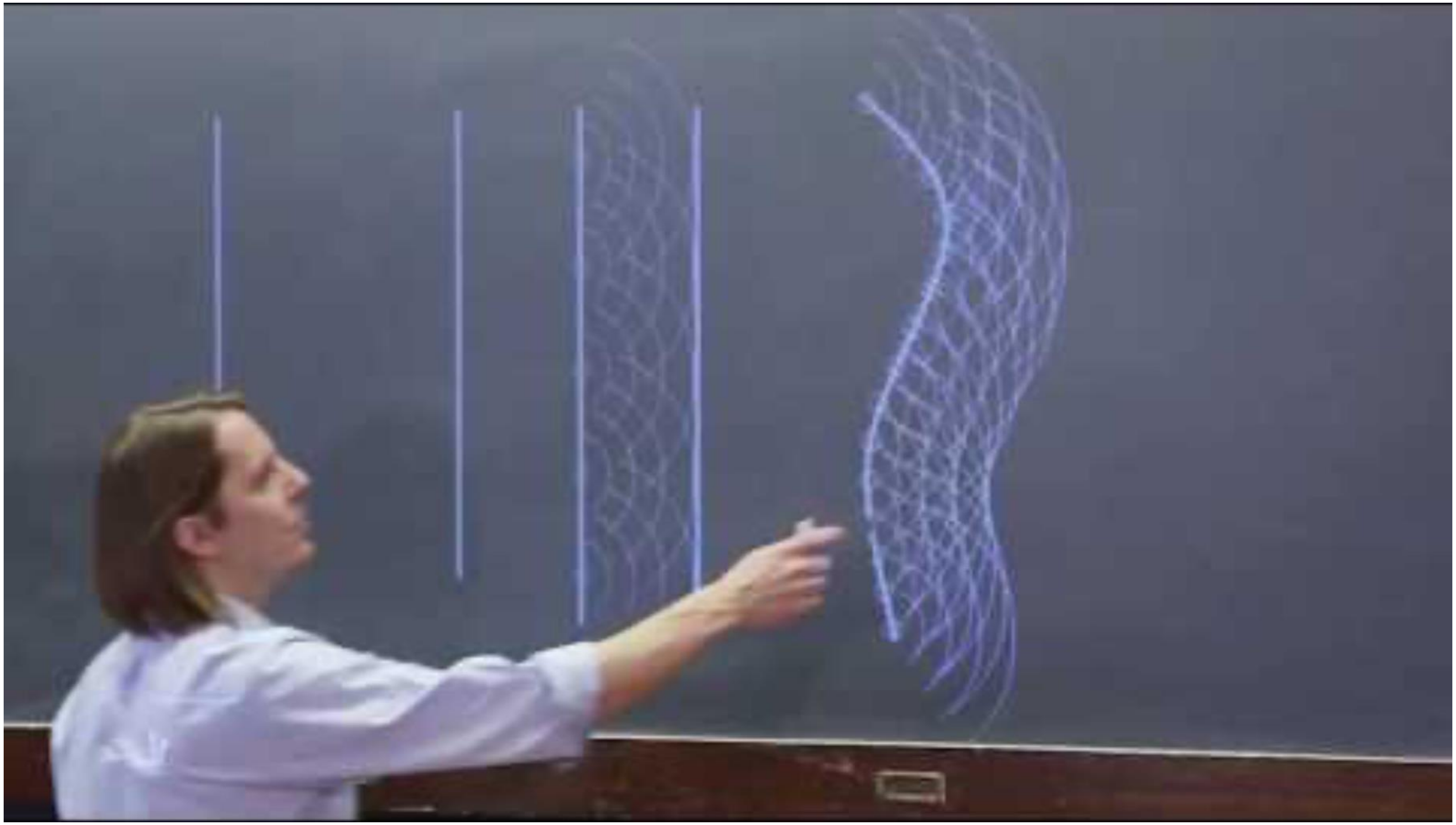
Christiaan Huygens

Huygens’ Principle

- Every point of a wavefront may be considered the source of secondary wavelets that spread out in all directions with a speed equal to the speed of propagation of the waves.

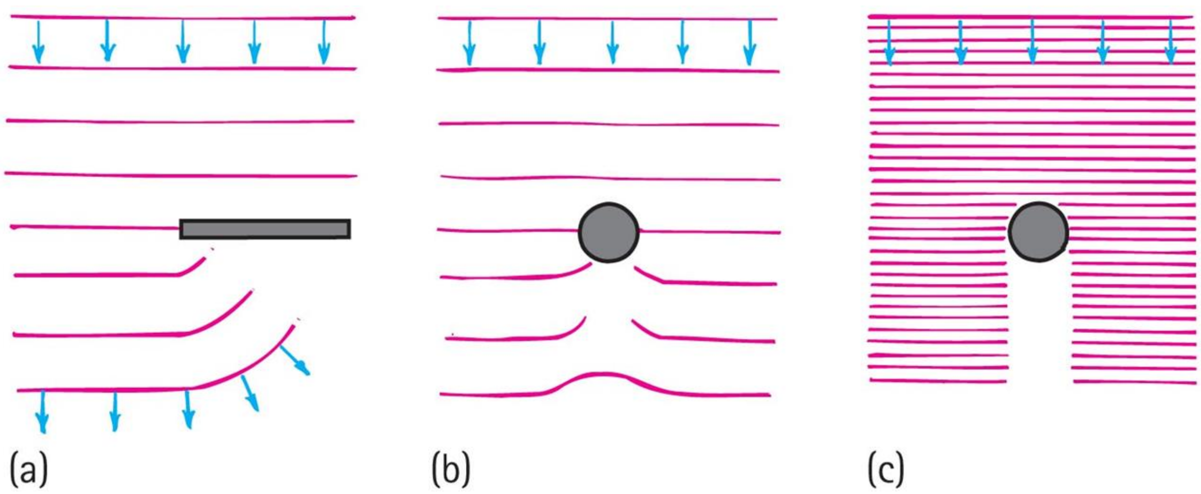


Visualization of Huygens' Principle

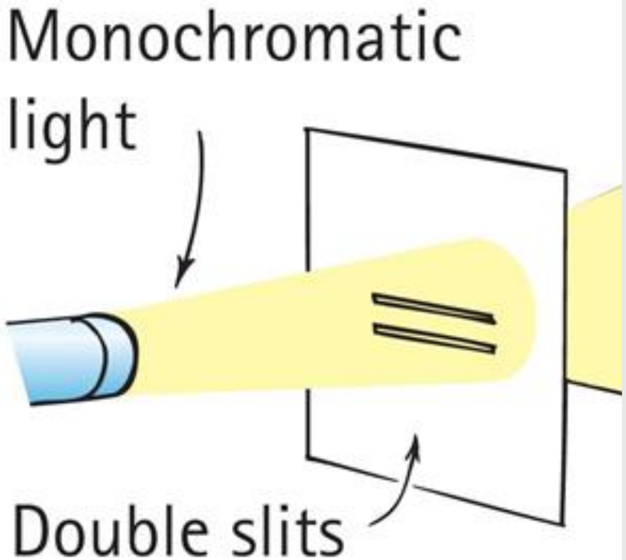


Diffraction

- Bending of waves by means other than reflection and refraction
  - Property of **all** kinds of waves
  - Seen around edges of many shadows
- Amount of diffraction depends on wavelength of the wave compared to the size of the obstruction.
  - We'll learn that resolution is limited by wavelengths!



Conceptual Question 4

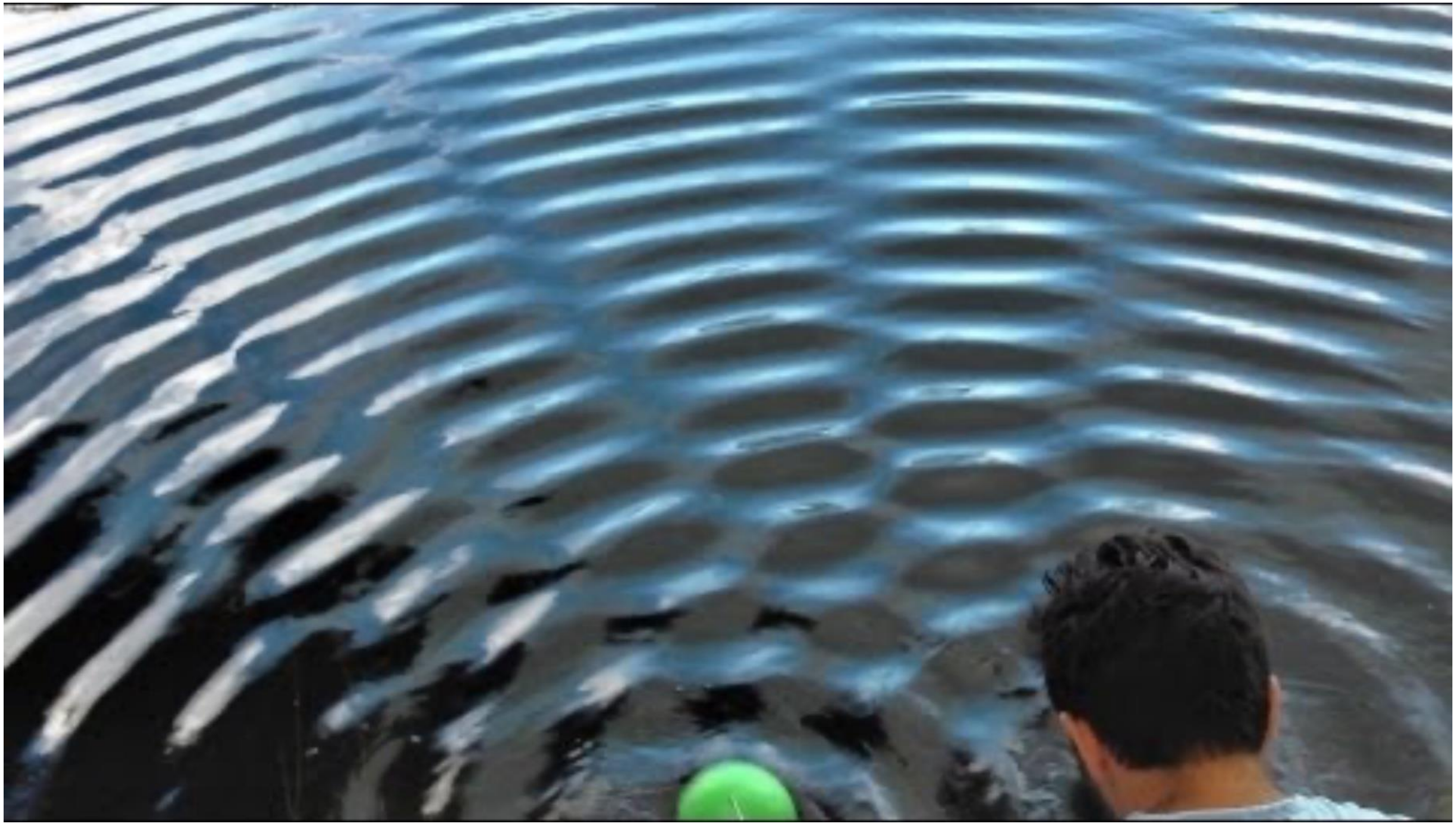


What pattern appears on the screen behind the slits?

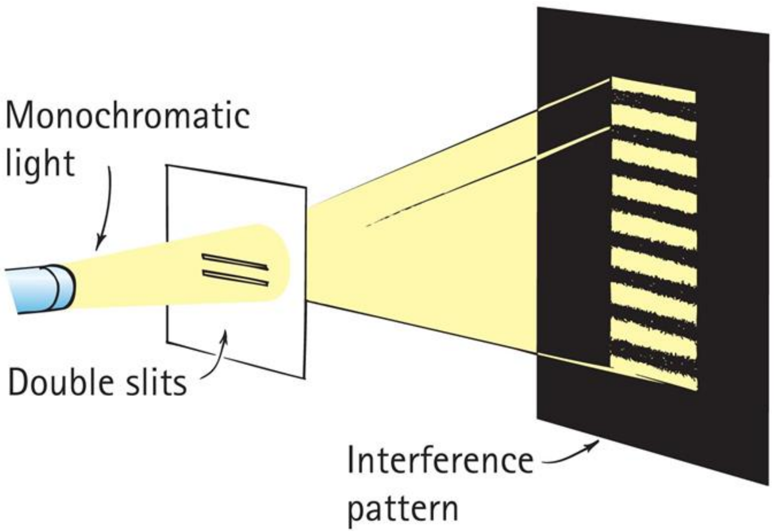
Thomas Young

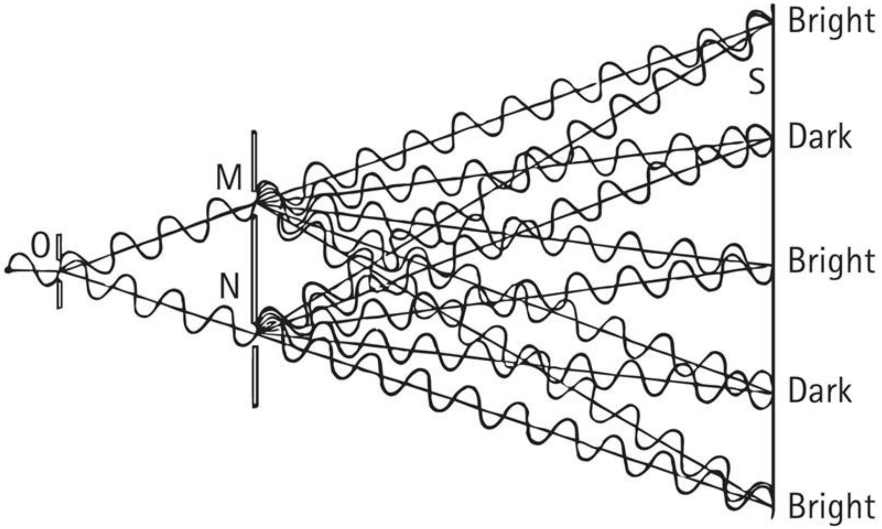
- Acceptance of the wave character of light came in 1801, when the English physicist and physician Thomas Young (1773–1829) demonstrated optical interference with his double slit experiment.
- In Young's experiment, sunlight was passed through a pinhole in a board. The emerging beam then fell on two pinholes in a second board. The light emanating from the two pinholes then fell on a screen where a pattern of bright and dark spots, called fringes, was observed. This pattern can only be explained through interference!

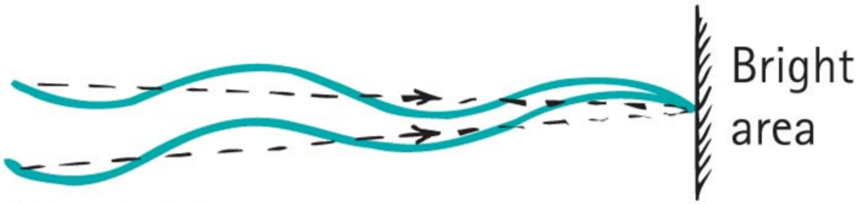
Young’s Double Slit Experiment

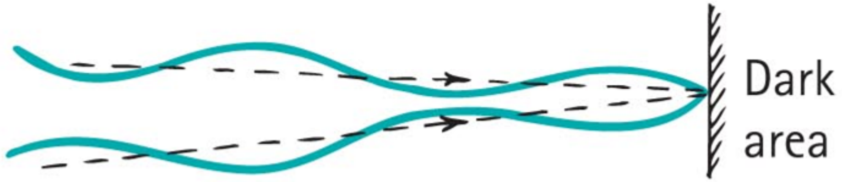


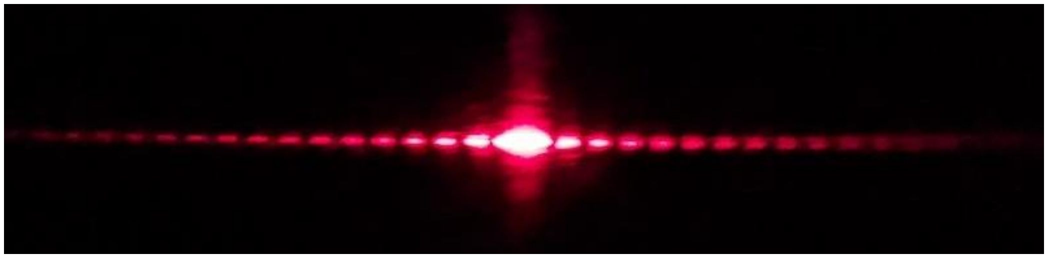
Young's Double Slit Experiment



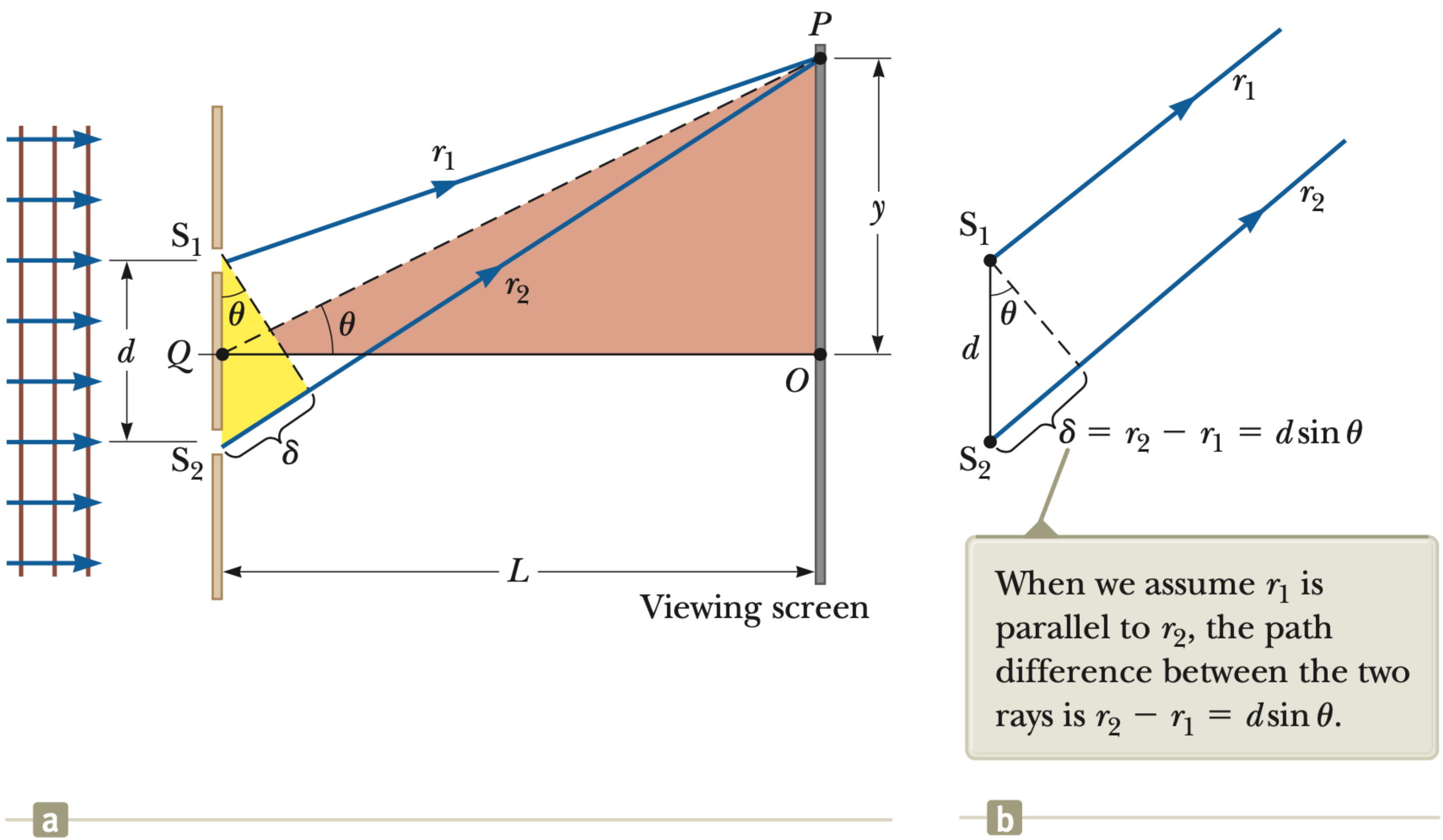








Geometry of the Double Slit Exp.



Angular Position of Fringes

If viewing screen is far away, $\delta = d\sin\theta$ and we can predict the locations of bright and dark fringes in terms of θ.

i. Bright fringes occur when the waves coming from the two slits constructively interfere on the screen:

$$d\sin\theta_{\text{bright}} = m\lambda$$

ii. Dark fringes occur when the waves coming from the two slits destructively interfere on the screen:

$$d\sin\theta_{\text{dark}} = \left(m + \frac{1}{2}\right)\lambda$$

$m = 0, \pm1, \pm2, \ldots$ is the order of the fringe

Linear Positions of the Fringes

For small angles, $y = L\tan\theta \approx L\sin\theta$

i. Bright fringes occur when the waves coming from the two slits constructively interfere on the screen:

$$y_{\text{bright}} = m\frac{\lambda L}{d}$$

ii. Dark fringes occur when the waves coming from the two slits destructively interfere on the screen:

$$y_{\text{dark}} = \left(m + \frac{1}{2}\right)\frac{\lambda L}{d}$$

Conceptual Question 5

Which of the following causes the fringes in a two-slit interference pattern to move farther apart?

A. decreasing the wavelength of the light

B. decreasing the screen distance L

C. decreasing the slit spacing d

D. immersing the entire apparatus in water

Example 6: Two-Slit Interference

Two thin parallel slits that are 0.0116 mm apart are illuminated by a laser beam of wavelength 585 nm.

(a) On a very large distant screen, what is the **total number** of bright fringes, including the central fringe and those on both sides of it?

(b) At what angle, relative to the original direction of the beam, will the fringe that is most distant from the central bright fringe occur?

Mathematics of Interference

The spherical waves emanating from each slit differ only in phase at any point on the viewing screen due to the path length difference. The interference pattern is dominated by this phase difference.

$$E_n = E_0 \sin(kr_n - \omega t + \phi_0) = E_0 \sin(\phi_n)\,, \qquad n = 1, 2$$

$$E_{\text{net}} = E_1 + E_2 = 2E_0 \cos\left(\frac{\phi_2 - \phi_1}{2}\right) \sin\left(\frac{\phi_2 + \phi_1}{2}\right)$$

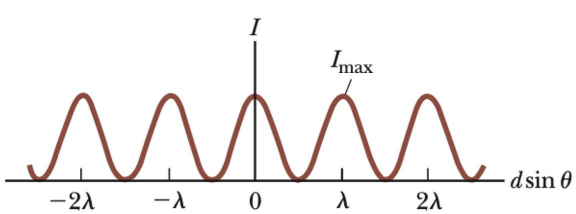
$$= \underbrace{2E_0 \cos(\pi\delta)} \sin\left(kr_{\text{avg}} - \omega t + \phi_0\right)$$

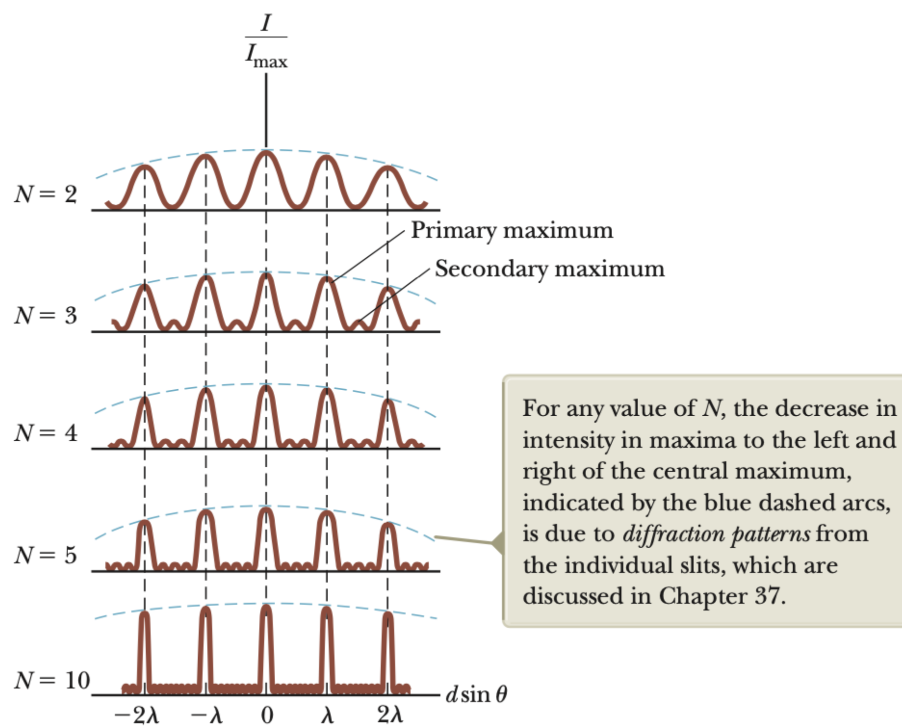
Each point on the screen has an
"effective amplitude" given by this term

Intensity of Double Slit Interference

Intensity of an EM wave is proportional to the square-amplitude of the wave. For $L \gg y$,

$$I = 4I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) = 4I_0 \cos^2\left(\frac{\pi d}{\lambda L} y\right)$$

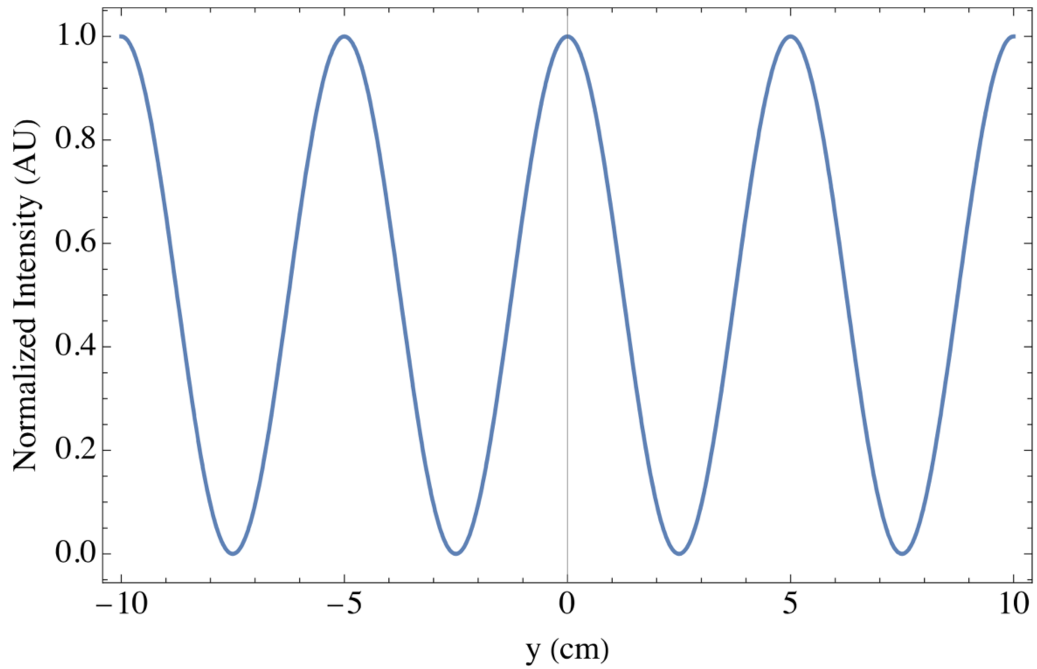




Example 7: Intensity

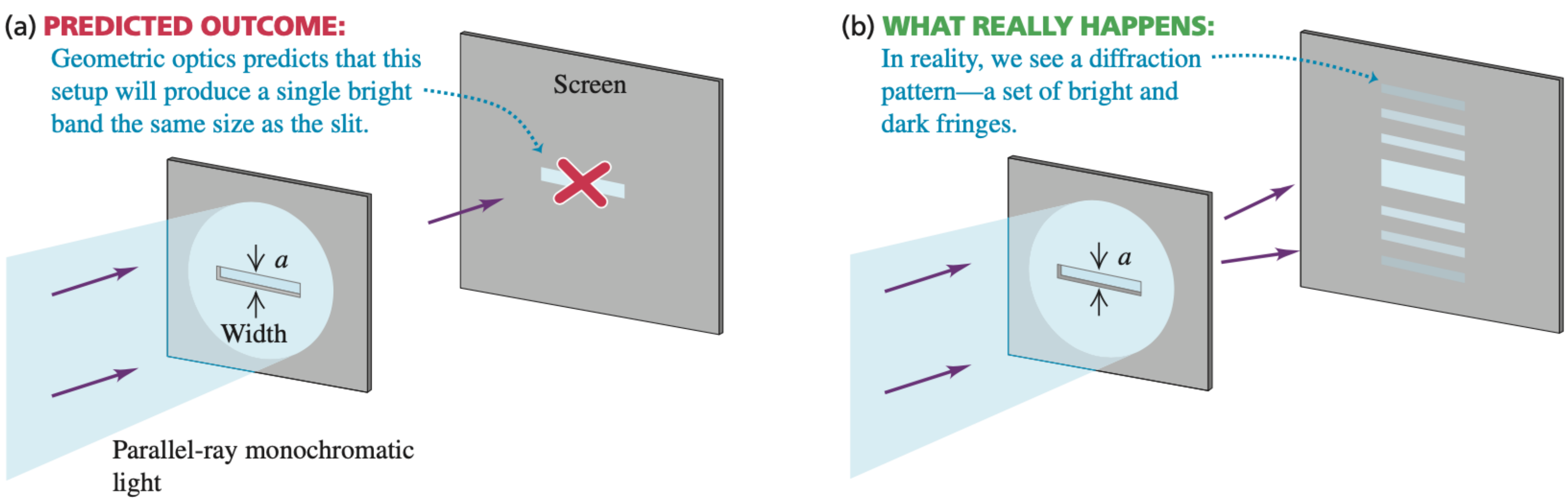
Light of wavelength 500 nm passes through two closely-spaced, narrow slits and illuminates a screen 10 m away with the pattern shown in the figure below.

a) How far apart are the two slits?
b) How would the interference pattern change if the slit spacing were doubled?
c) How would the pattern change if the entire apparatus were immersed in water?

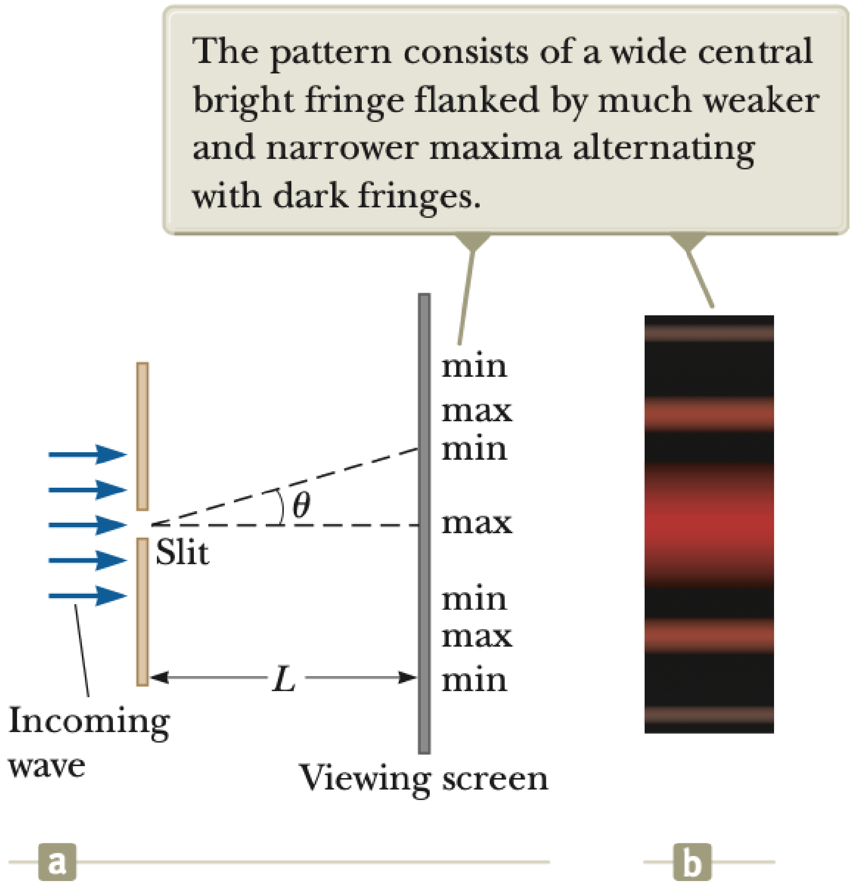


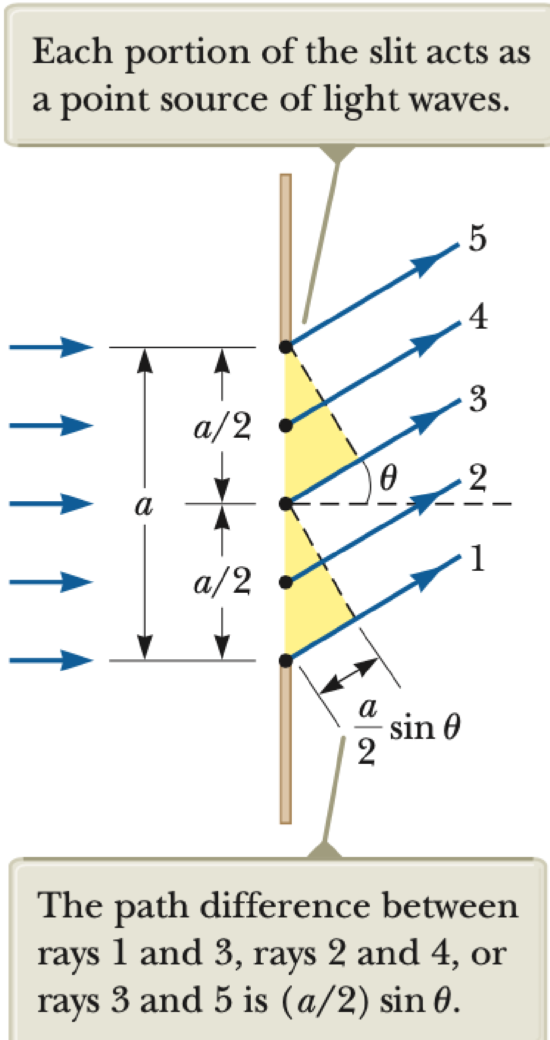
Single Slit Diffraction

Consider light passing through a narrow slit that projects onto a distant screen. According to Huygens's principle, each portion of the slit acts as a source of light waves. Hence, light from one portion of the slit can interfere with light from another portion, and the resultant light intensity on a viewing screen should display an interference pattern!



Single Slit Diffraction





Intensity of Single Slit Diffraction

Analysis of the intensity variation in a diffraction pattern from a single slit of width a shows that the intensity is given by

$$I = I_{\text{max}} \left[\frac{\sin(\pi a \sin\theta/\lambda)}{(\pi a \sin\theta)/\lambda}\right]^2$$

i. I_{max} is the intensity of light at $\theta = 0$, recall $\lim_{x\to 0} \frac{\sin x}{x} = 1$

ii. λ is the wavelength of light used to illuminate the slit

iii. Dark bands will occur whenever $\sin(\pi a \sin\theta/\lambda) = 0$

$$\sin\theta_{\text{dark}} = m\frac{\lambda}{a}, \qquad m = \pm 1, \pm 2, \ldots$$

Conceptual Question 6

Suppose the slit width in a single-slit diffraction experiment is made half as wide. Does the central bright fringe

A. become wider,

B. remain the same, or

C. become narrower?

Conceptual Question 7

Coherent electromagnetic radiation is sent through a slit of width 0.0100 mm. For which of the following wavelengths will there be *no* points in the diffraction pattern where the intensity is zero?

A. Blue light of wavelength 500 nm

B. infrared light of wavelength 10.6 mm

C. microwaves of wavelength 1.00 mm

D. ultraviolet light of wavelength 50.0 nm

Example 8: Single-Slit Diffraction

Red light of wavelength 633 nm from a helium–neon laser passes through a slit 0.350 mm wide. The diffraction pattern is observed on a screen 3.00 m away. Define the width of a bright fringe as the distance between the minima on either side.

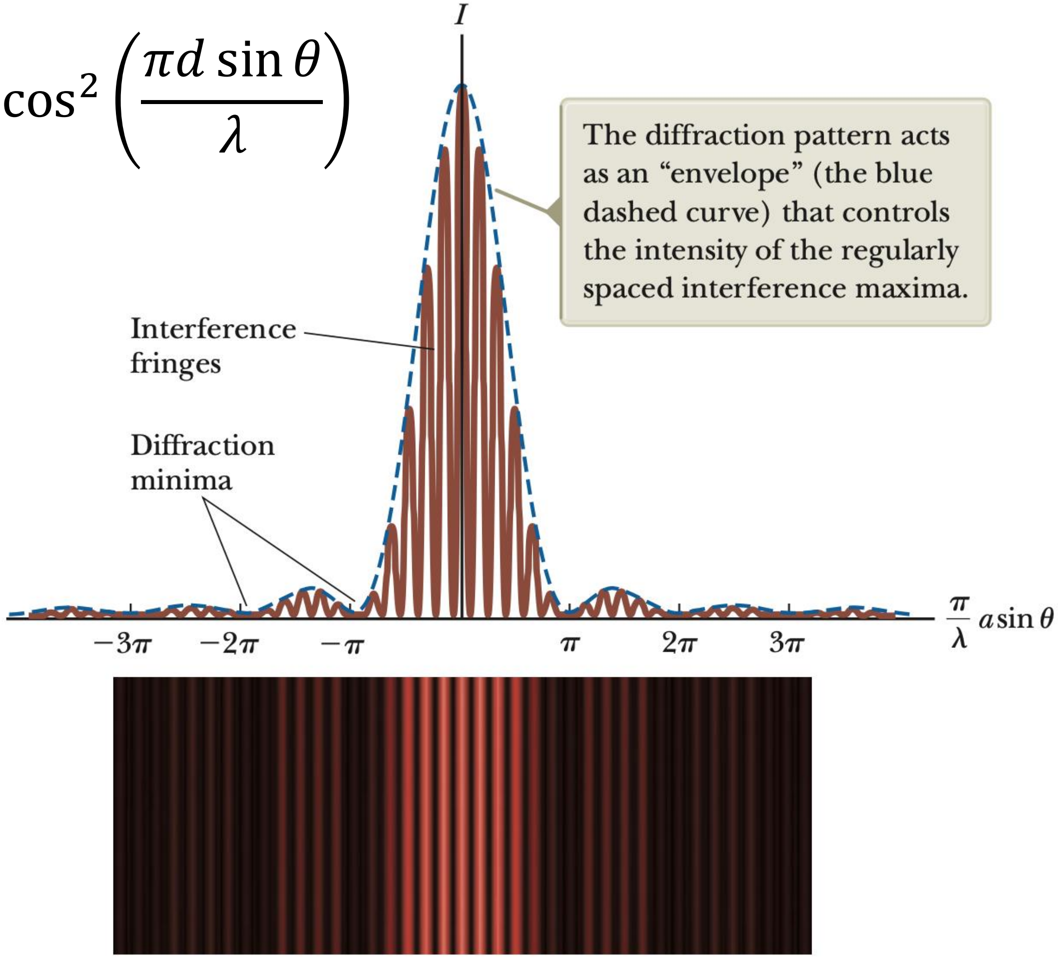
a) What is the width of the central bright fringe?

b) What is the width of the first bright fringe on either side of the central one?

Intensity of Two Slit Diffraction

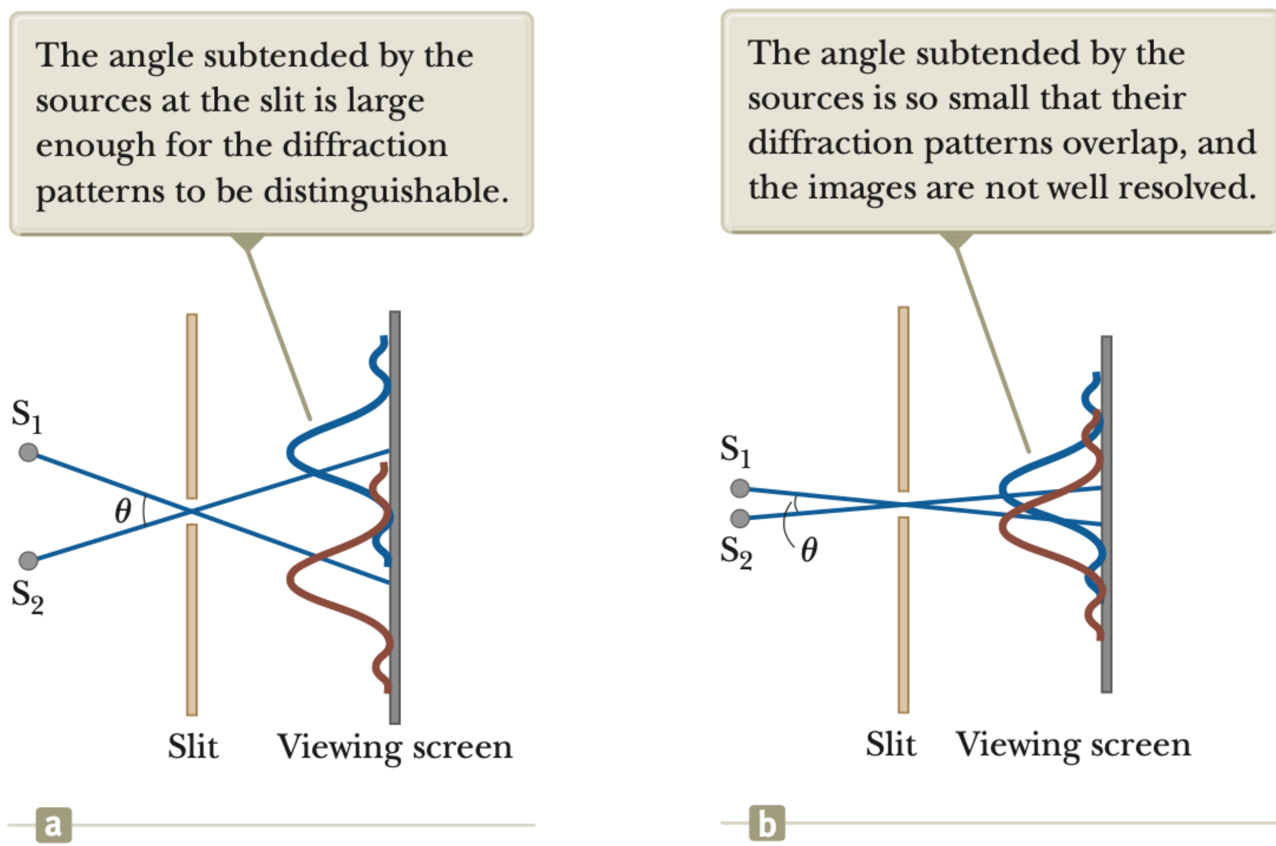
When more than one slit is present, we must consider not only diffraction patterns due to individual slits but also interference due to the waves coming from different slits.

$$I = I_{\text{max}} \left[\frac{\sin(\pi a \sin\theta/\lambda)}{(\pi a \sin\theta)/\lambda}\right]^2 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$



Rayleigh’s Criterion

When the central maximum of one image falls on the first minimum of another image, the images are said to be *just resolved*. This limiting condition of resolution is known as **Rayleigh’s criterion.**



Resolution of Single-Slit and Circular Apertures

The smallest angular separation for which the two images are resolved depends on the aperture:

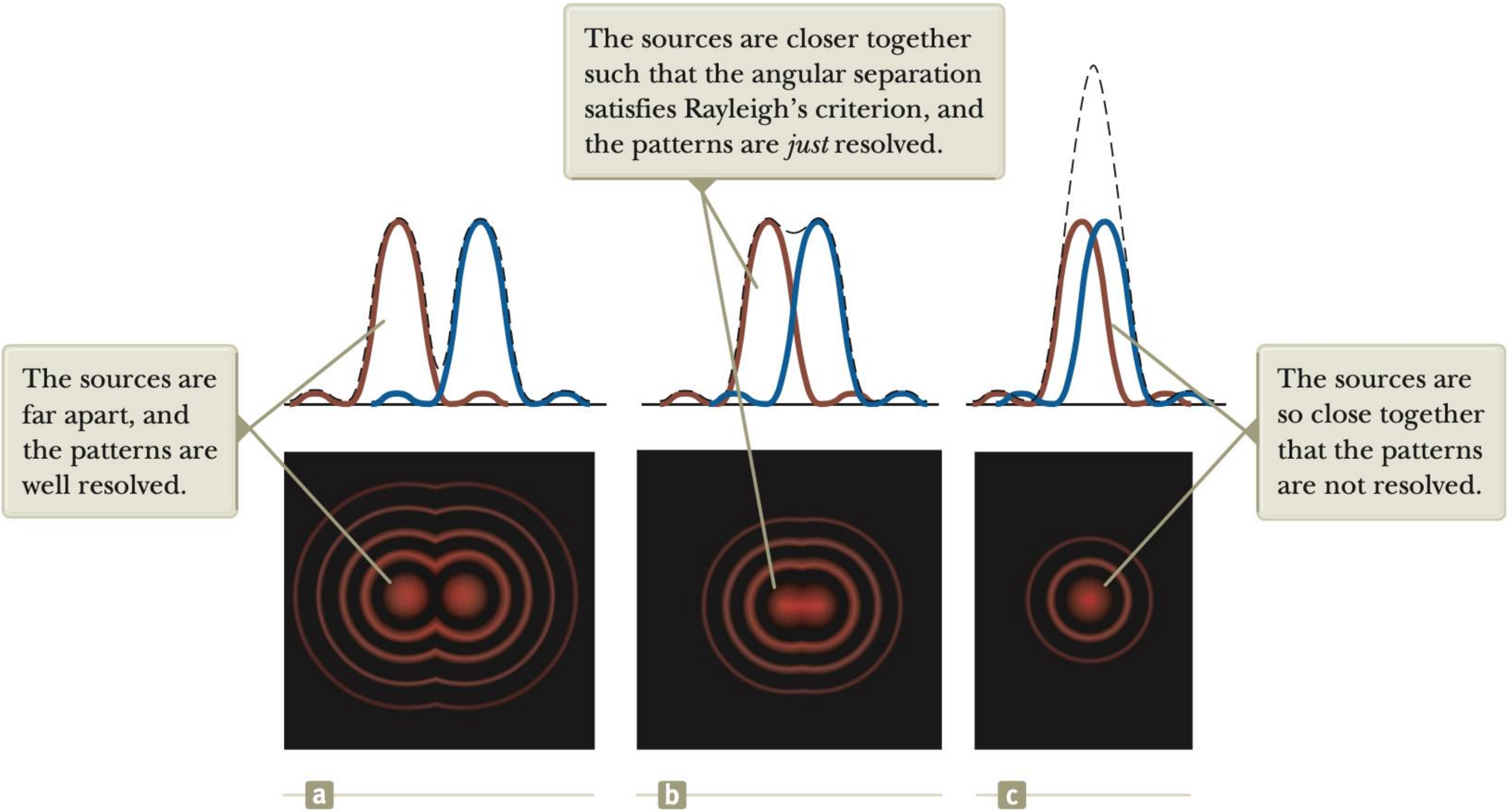
i. Single slit of width a and light with $\lambda \ll a$

$$\theta_{\min} \approx \frac{\lambda}{a}$$

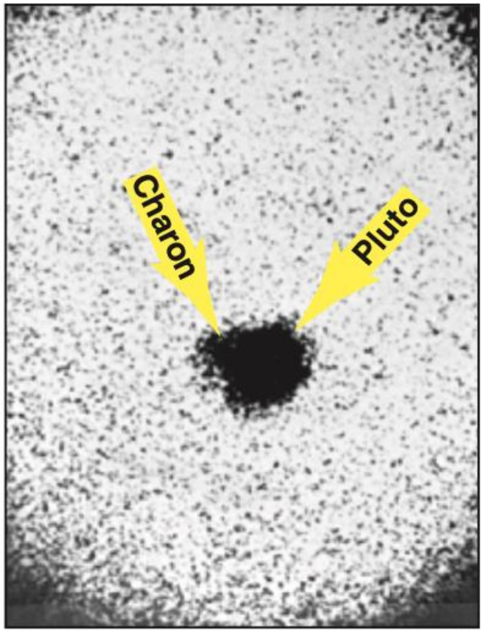
ii. Circular aperture with diameter D ($\lambda \ll D$)

$$\theta_{\min} \approx 1.22\,\frac{\lambda}{D}$$

Circular Aperture

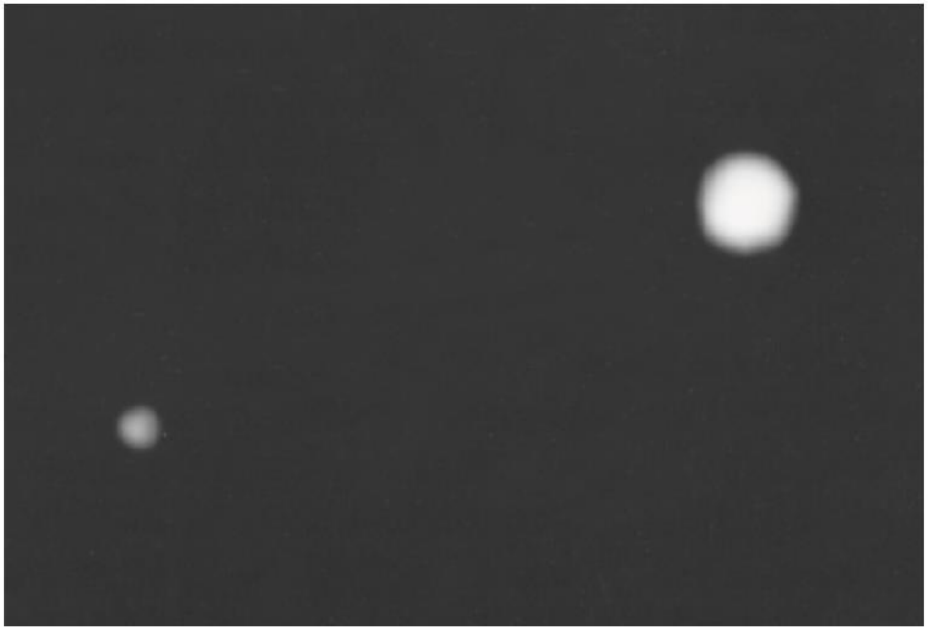


Discovery of Charon



Courtesy U.S. Naval Observatory/James W. Christy

a



Dr. R. Albrecht, ESA/ESO Space Telescope European Coordinating Facility; NASA

b

Conceptual Question 8

Suppose you are observing a binary star with a telescope and are having difficulty resolving the two stars. You decide to use a colored filter to maximize the resolution. (A filter of a given color transmits only that color of light.) What color filter should you choose?

A. blue

B. green

C. yellow

D. red