Phys 152: Fundamentals of Physics II

Unit #3 - Electric Potential & Conductors

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Electrostatic Force is Conservative

Consider the work done by the electrostatic force on a test charge q_0 as it moves from point a to point b in the electric field of a stationary source charge q at the origin. Let r be a radial coordinate.

$$W_e = \int_a^b \vec{\mathbf{F}}_e \cdot d\vec{\mathbf{r}} = \int_a^b \frac{qq_0}{4\pi\varepsilon_0}\frac{1}{r^2}dr = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

Since W depends only on the endpoints and not on the path taken, the electrostatic force is conservative! $\vec{\mathbf{F}}_e \cdot d\vec{\mathbf{r}}$ simplifies to Fdr because the electrostatic force of a point charge only has a radial component.

Electrostatic Potential Energy

By superposition, the force of any distribution of charges will be conservative. As a test charge moves from point a to point b in an electrostatic field, the change in potential energy is

$$\Delta U_e = -\int_a^b \vec{\mathbf{F}}_e \cdot d\vec{\mathbf{r}} = -q_0 \int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$$

where $\Delta U_e = U_e(\vec{\mathbf{r}}_b) - U_e(\vec{\mathbf{r}}_a)$. The negative sign means the potential energy decreases when the force does positive work.

Conceptual Question 1

A point charge is interacting with an electrostatic field. Which of the following statements are true? Select all that apply.

A. The system loses potential energy as a positive charge moves in the direction of the field.

B. The system gains potential energy as a negative charge moves in the direction of the field.

C. The system loses potential energy as a positive charge moves in the direction opposite to the field.

D. The system loses potential energy as a negative charge moves in a direction opposite to the field.

Electric Potential Difference

The line integral of $\vec{\mathbf{E}}$ is path-independent, so we can define the electric potential difference as

$$\Delta V_{a\to b} \equiv -\int_a^b \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{r}} = \frac{\Delta U_e}{q_0} \qquad [V] = \text{volts (V)} = \mathrm{J/C}$$

where $\Delta V_{a\to b} = V_b - V_a$. The potential difference is the potential energy difference per unit charge. In other words, electric potential difference is the negative of the work done by the electrostatic field per unit charge.

V is arbitrary up to an overall constant (we can always change the reference point of the electric potential).

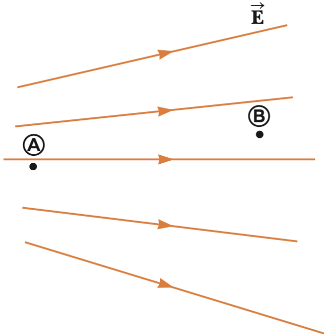
Conceptual Question 2

How would you describe the potential difference $\Delta V = V_B - V_A$?

A. It is positive.

B. It is negative.

C. It is zero.



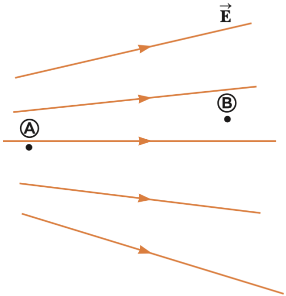
Conceptual Question 3

A negative charge is placed at A and then moved to B. How would you describe the change in potential energy for this process?

A. It is positive.

B. It is negative.

C. It is zero.



Voltage vs Potential Difference

You may be familiar with the term "voltage." Voltage is defined as the negative of the work per unit charge done by any electric field. For electrostatic fields, voltage is the same as potential difference, so we'll use the terms interchangeably. Voltage may depend on the path taken by the charges, but you'll have to wait until we get to Faraday's law (stay tuned).

Voltmeters are devices that measure voltage. They will always measure the voltage between two points, and only in electrostatic situations will they measure the potential difference.

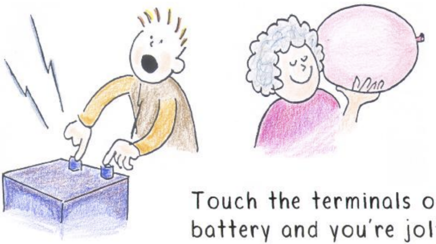
Typical Voltages

Source	Voltage
Thundercloud to ground	10^8 V
High-voltage power line	10^5 – 10^6 V
Household outlet	10^2 V
Across cell membranes	10^{-1} V

Example 1: Work Done by an Electric Field

How much work is done by a battery or some other source of potential difference to move 6.02×10^{23} electrons (Avogadro's number of electrons) all at once from a point where the electric potential is +$9.00\ \mathrm{V}$ to a point where the electric potential is $-5.00\ \mathrm{V}$? The potential in each case is measured (using a voltmeter) relative to a common reference point where $V = 0$.

Conceptual Question 4



Touch the terminals of a 100-volt battery and you're jolted. Touch a 10,000-volt rubber balloon and you feel nothing. Why?

Uniform Electric Field

Consider the potential difference between two points in a uniform electric field:

$$\Delta V = -\int_a^b \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{r}} = -E\Delta r \cos\theta = -Ed$$

where $d = \Delta r \cos\theta$ is the component of the separation between the two points along the direction of the electric field. If a point charge q moves between the two points,

$$\Delta U_e = q\Delta V = -qEd$$

Example 2: Parallel Plates

A potential difference of $120\ \mathrm{V}$ is established between two parallel conducting plates with a separation of $0.45\ \mathrm{mm}$.

(a) What is the magnitude of the electric field between the plates?

(b) A proton is released at rest from the positive plate. What is its speed just before it impacts the negative plate?

(c) An electron is released at rest from the negative plate. What is its speed just before it impacts the positive plate?

$$m_e = 9.11 \times 10^{-31}\ \mathrm{kg} \qquad m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$$

The Electron-Volt

The electron-volt is a unit of energy equal to the kinetic energy gained by a fundamental charge (electron or proton) accelerated by a potential difference of one volt.

$$1 \text{ eV} = q\Delta V = (1.602 \times 10^{-19} \text{ C})(1 \text{ V})$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

Electric Field from the Potential Difference

Consider the potential difference between two points so close that the field does not vary significantly from point to point.

$$\Delta V \approx -\vec{\mathbf{E}} \cdot \Delta \vec{\mathbf{r}} = -E \Delta r \cos\theta$$

When $\Delta \vec{\mathbf{r}}$ is in the same direction as the electric field,

$$E \approx -\frac{\Delta V}{\Delta r} \qquad [E] = \frac{V}{m}$$

The electric field measures the rate of change with position of the electric potential in the direction of its *maximal decrease*.

Gradient of a Scalar Function

The gradient of a (well-behaved) scalar function F is a vector field denoted ∇F whose value at any point gives the direction and rate of fastest increase in F with respect to position.

$$dF = (\nabla F) \cdot d\vec{\mathbf{r}}$$

In Cartesian coordinates,

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}}$$

Gradient of the Electric Potential

The electrostatic field is minus the gradient of the potential:

$$\vec{\mathbf{E}} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

From just one scalar function we can derive three vector components. The electrostatic field is clearly a special vector!

Equipotential Surfaces

An equipotential surface is any surface consisting of a continuous distribution of points having the same electric potential ($\Delta V = 0$). No work is required to move a charge along an equipotential surface. The electric field is everywhere perpendicular to equipotential surfaces.

$$W_{\text{ext}} = q\Delta V = 0$$

$$\Delta V = -\vec{\mathbf{E}} \cdot \Delta \vec{\mathbf{r}} = 0$$

Electric Potential of a Point Charge

Consider the potential difference between two points surrounding a point charge q at the origin.

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

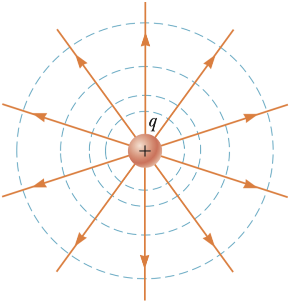
By convention, let $V_a = 0$ when $r_a \to \infty$. Then,

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = k_e\frac{q}{r} \qquad \text{(potential of a point charge)}$$

Equipotential Surfaces for a Point Charge

The equipotential surfaces are spheres with constant radii and the electric field is everywhere (locally) perpendicular to the equipotential surfaces. Also,

$$E_r = -\frac{dV}{dr} = k_e \frac{q}{r^2}$$



Electric Potential of Many Point Charges

By the principle of superposition, the electric potential at any point due to a collection of point charges is

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i} = k_e\sum_{i=1}^{N}\frac{q_i}{r_i}$$

Caution: The electric potential is a <u>scalar</u> function of position. Do not perform vector addition! V does not have components. This makes it a bit easier to do calculations.

Electric Potential Energy of Two Point Charges

Bring a point charge q_2 from infinity to a distance r_{12} away from a point charge q_1. The system's change in potential energy is

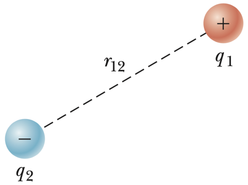
$$U_e = q_2 V_1(r_{12}) = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}} = k_e \frac{q_1 q_2}{r_{12}}$$

We interpret this as the amount of work required to assemble the configuration. The zero of this potential energy corresponds to the situation where two charges are infinitely far apart.

Conceptual Question 5

In the figure, take q_2 to be a negative source charge and q_1 to be a second charge whose sign can be changed. When q_1 is changed from positive to negative, what happens to the potential energy of the two-charge system?

A. It increases.
B. It decreases.
C. It remains the same.



Total Electrostatic Energy

Total electrostatic energy is the total amount of work required to assemble a configuration of charges. For a system of point charges, the total electrostatic energy is the sum of the potential energies of interaction for each pair of charges.

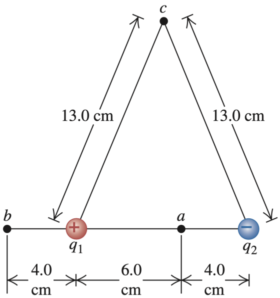
$$W = k_e \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{q_i q_j}{r_{ij}}$$

$$W = k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \qquad (N = 3)$$

Example 3: Electric Dipole

Two point charges $q_1 = +12\ \mathrm{nC}$ and $q_2 = -12\ \mathrm{nC}$ are placed $10.0\ \mathrm{cm}$ apart. Compute the electric potentials at points a, b, and c.

Find the change in electrostatic energy when a $5.00\text{-}\mathrm{nC}$ point charge is brought to points a, b, and c.



Conceptual Question 6

Three identical particles with charge $+q$ sit on an equilateral triangle with sides of length a. What would be the final kinetic energy of the top charge if you released all three from rest?

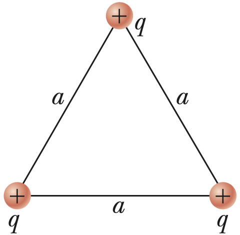
A. $k_e q^2/a$

B. $2k_e q^2/a$

C. $2k_e q^2/3a$

D. $3k_e q^2/a$

E. None of the above.



Electric Potential of a Continuous Charge Distribution

We can use the principle of superposition to find the electric potential of a localized continuous charge distribution by adding the contributions from infinitely many point-like sources:

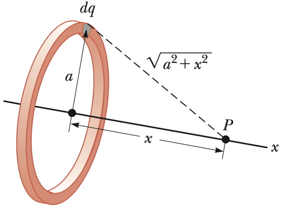
$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r} = k_e \int \frac{dq}{r} \qquad (V = 0 \text{ at } \infty)$$

If the distribution has sufficient symmetry, we could also use Gauss's law to find $\vec{\mathbf{E}}$ and then integrate to find the potential difference.

Example 4: Uniformly Charged Ring

(a) Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius a and total charge Q.

(b) Find an expression for the magnitude of the electric field at P.



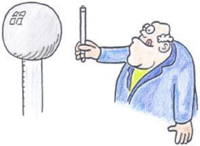
Example 5: Spherical Shell

Determine the electric potential at a distance r from the center of a charged spherical shell with radius a and total charge Q spread uniformly over its surface. Hint: use Gauss's law and compute

$$\Delta V = V(r) - V(\infty) = -\int_{\infty}^{r} \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$$

Set $V(\infty) = 0$ since this is a localized distribution.

Conceptual Question 7



The lamp will not glow when it is held with both ends equidistant from the charged Van de Graaff generator. But when one end is closer to the dome than the other end, a current is established and it glows. Why?

Earnshaw's Theorem II

The electric potential can have no local maxima or minima in a charge-free region.

Proof. We already know from the "first Earnshaw's theorem" that a sphere surrounding a point P in empty space cannot have net inward or outward electric flux. Since the electric field points in the direction of maximal decrease in the potential, inward flux corresponds to a local minimum and outward flux to a local maximum. Since there can be no inward or outward flux, there can be no local max or min.

Wigner's Argument for Charge Conservation

Suppose we create a charge Q at a location $\vec{\mathbf{r}}_1$ by doing some amount of work W. Since $V(\vec{\mathbf{r}}_1)$ is only defined up to an overall arbitrary constant, W should not depend on $V(\vec{\mathbf{r}}_1)$. Move the charge Q to a new location $\vec{\mathbf{r}}_2$ by doing work against an electric field equal to $\Delta U = Q\,[V(\vec{\mathbf{r}}_2) - V(\vec{\mathbf{r}}_1)]$. Now destroy the charge and recover the work W that it took to create it. In the end, the energy of the universe has changed by an amount ΔU, which contradicts the assumption that energy is conserved. Hence, charge must be conserved.

(1949, Eugene Wigner)

Conductors in Electrostatic Equilibrium

When there is no net motion of charge within a conductor, the conductor is in electrostatic equilibrium. A conductor in electrostatic equilibrium has the following properties:

1. $\vec{\mathbf{E}} = 0$ everywhere inside a conductor.
2. An isolated conductor's charge resides on its surface.
3. $\vec{\mathbf{E}}$ at a point just outside a conductor is perpendicular to the surface and has magnitude $|\sigma|/\varepsilon_0$ where σ is the surface charge density at that point.
4. The surface charge density is greatest at locations where the radius of curvature is smallest (sharp points).

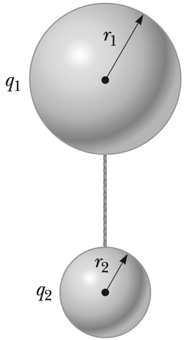
Thomson's Theorem

The electrostatic energy of a body with fixed shape and size is minimized when its charge distributes itself to make the electrostatic potential constant throughout the body.

The proof is beyond the scope of our class, but the essential idea is that conductors have the lowest electrostatic energy possible given their fixed size and shape. Even in the presence of other charges (external to the conductor or within any cavities), the charges inside a conductor will arrange themselves (rather quickly) to reach a constant potential (and zero electric field) everywhere inside.

Example 6

Two spherical conductors of radii r_1 and r_2 are separated by a distance much greater than the radius of either sphere. The spheres are connected by a long conducting wire as shown (not to scale). The charges on the spheres in equilibrium are q_1 and q_2 respectively, and they are uniformly charged. Find the ratio of the magnitudes of the electric fields at the surfaces of the spheres.



A Cavity Within a Conductor

A cavity surrounded by conducting walls is free of an electric field as long as no charges are inside the cavity. The field in the cavity is zero *even if an electric field exists outside the conductor*.

The absence of electric field inside a conducting enclosure is the basis for electrical shielding with Faraday cages. In practice, the enclosure does not have to be perfectly sealed–there can be small holes in a metallic screen and the field inside will be reduced significantly.

Example 7: Conducting spherical shell

A point charge q is located at an arbitrary position inside a neutral conducting spherical shell. Explain why the electric field outside the shell is the same as the spherically symmetric field due to a charge q located at the center of the shell. [The shape of the inner surface is irrelevant. Hence, the external field is radial even if the cavity takes an odd shape.]