Physics 438A – Lecture #2
Postulates of Quantum Mechanics

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Brief Historical Digression

We'll zoom out from the Stern-Gerlach experiment and talk about the postulates of quantum mechanics more generally (but still within the context of finite dimensional vector spaces). We'll superficially revise the postulates later when we need to work with *wave functions* and infinite dimensional vector spaces. Much of the intuition we gain from the finite-dimensional case will carry over to infinite dimensional spaces anyways. Let's begin with a brief historical digression–not because the historical details matter but because quantum theory is abstract and it can be hard to imagine where these principles came from. You may find quantum theory to be more palatable if you treat it like a game with simple but arbitrary rules.

The rules were first formulated in 1925 by Heisenberg, Born, and Jordan. Heisenberg in particular took a radical step by abandoning particle trajectories and formulating a theory only in terms of measurable quantities (observables). He found that the relationships between observables could be expressed using arrays of numbers (matrices), leading to a new algebraic, non-visual formulation of quantum mechanics called matrix mechanics. Heisenberg's work is significant because it marked the departure from classical intuition and set the stage for the more abstract nature of quantum mechanics.

Schrödinger developed wave mechanics in 1926 as an alternative formulation of quantum mechanics, offering a more intuitive and visualizable approach compared to Heisenberg's matrix mechanics. Inspired by Louis de Broglie's wave-particle hypothesis, Schrödinger proposed that particles, such as electrons, could be described by wave functions evolving in space and time according to a differential equation (the Schrödinger equation). Unfortunately, as elucidated by Max Born, wave functions are ultimately used to compute probabilities, and they do not exist in three-dimensional physical space like ordinary waves, so their interpretation is not as intuitive as Schrödinger initially hoped.

In 1928, Dirac merged Schrödinger's wave mechanics and Heisenberg's matrix mechanics into a single, abstract framework (and also introduced the bra-ket notation)[1]. Finally, in 1932, John von Neumann provided mathematical rigor, turning quantum mechanics into a well-defined axiomatic system and inventing a large portion of a field known as *functional*

[1]It was Schrödinger who demonstrated equivalence between matrix mechanics and wave mechanics.

analysis (beyond the scope of our class). The postulates of quantum mechanics dictate how to treat a quantum mechanical system mathematically and how to interpret the mathematics to learn about the physical system in question. These postulates cannot be proven, but they have been successfully tested by many experiments, so we accept them as an accurate way to describe quantum mechanical systems. Quantum mechanics has five basic rules, and I've come up with a silly mnemonic to help you remember them:

Small **O**bjects **B**reak **C**lassical **T**heories

1 Kinematical Postulates

Postulate 1

The **state** of a quantum system is represented by a normalized vector $|\psi\rangle$ belonging to a complex vector space with an inner product.

The quantum state vector is a normalized vector (ket) and it contains complex components corresponding to all possible measurement configurations. Kets can be multiplied by complex numbers and added together. Any linear combination of kets is also a ket, but this does not mean that the system "exists in multiple states at once" (this is a very common misconception). Two states are said to be *physically equivalent* if they differ only by a multiplicative phase factor, i.e. $e^{i\alpha}$ where α is a real-valued constant.

Postulate 2

Every measurable physical quantity, called an **observable**, is described by a self-adjoint linear operator. The result of measuring an observable is always an eigenvalue of the corresponding operator, and the operator's eigenvectors form a basis for the vector space.

Recall that a self-adjoint operator has the property that it is equal to its adjoint, i.e. $A^\dagger = A$. While postulate #2 states unambiguously (and without proof) that observables are represented by self-adjoint linear operators, we gain confidence and intuition by examining the plausibility of this postulate. We need to establish a few properties of self-adjoint operators before we can remark on their ability to represent observables.

(a) The expectation value of a self-adjoint operator $A = A^\dagger$ is a real number for all vectors.

$$\langle A\rangle^* = \langle\psi|A|\psi\rangle^* = \langle\psi|A^\dagger|\psi\rangle = \langle A\rangle$$

(b) The eigenvalues of a self-adjoint operator $A = A^\dagger$ are real. Let $|a_n\rangle$ be a normalized eigenvector of A, i.e. $A|a_n\rangle = a_n|a_n\rangle$ and $\langle a_n|a_n\rangle = 1$. From (a), it follows immediately that $a_n = \langle a_n|A|a_n\rangle$ is real.

Interpretation of (a) and (b): All measurement outcomes in quantum mechanics will be real numbers. We don't have to worry about computing a complex value for position or spin just because we're working with vectors over complex numbers.

(c) Eigenvectors corresponding to distinct eigenvalues are orthogonal. Let $|a_1\rangle$ and $|a_2\rangle$ be two normalized eigenvectors of A such that $A|a_1\rangle = a_1|a_1\rangle$ and $A|a_2\rangle = a_2|a_2\rangle$. It follows that

$$\langle a_2|A|a_1\rangle = a_1\langle a_2|a_1\rangle \text{ and } \langle a_1|A|a_2\rangle = a_2\langle a_1|a_2\rangle$$

Since A is self-adjoint, $\langle a_1|A|a_2\rangle^* = \langle a_2|A|a_1\rangle$, and since the eigenvalues are real,

$$a_1\langle a_2|a_1\rangle = a_2^*\langle a_1|a_2\rangle^* = a_2\langle a_2|a_1\rangle$$

$$(a_2 - a_1)\langle a_2|a_1\rangle = 0$$

If the eigenvalues are distinct, namely $a_2 \neq a_1$, then $\langle a_2|a_1\rangle = 0$.

Interpretation of (c): Two measurement outcomes that are distinct should correspond to the system being in two different states. Otherwise the states would not be well-defined. The measurement outcomes are mutually exclusive, i.e. if one measurement outcome occurs, a different outcome certainly does not occur. Note that if two states share an eigenvalue, their inner product may not necessarily be zero (the eigenvalues may be *degenerate*).

In finite-dimensional vector spaces, the eigenvectors of a self-adjoint operator provably span the vector space. Under precise conditions, the spectral theorem from functional analysis states that self-adjoint linear operators can be diagonalized with a complete set of "generalized eigenstates," even in infinite dimensions. In this course, we'll simply take completeness as an axiom.

Spectral theorem (finite dimensional case)

Let A be a self-adjoint linear operator on a finite-dimensional complex vector space. The following statements are equivalent:

(i) There exists an orthonormal basis of the vector space consisting of eigenvectors of A.

(ii) A is diagonalizable by a unitary transformation $A = UDU^\dagger$ where U is is a unitary matrix whose columns are orthonormal eigenvectors of A and D is a real diagonal matrix whose entries are the eigenvalues of A.

The **spectral decomposition** of a self-adjoint operator is a weighted sum of projectors given by

$$A = \sum_n a_n A_n = \sum_n a_n |a_n\rangle\langle a_n| \tag{1.1}$$

where $\{a_n\}$ are the eigenvalues (the weights) and $\{|a_n\rangle\}$ is an orthonormal set of eigenvectors. We will accept the spectral theorem without proof in this class, although you are welcome to consult supplementary resources.

Postulate 3

The probability of obtaining an eigenvalue a_n in the measurement of an observable A on a quantum system in the state $|\psi\rangle$ is given by the **Born rule**:

$$p(a_n) = \langle\psi|P_n|\psi\rangle \tag{1.2}$$

where P_n is the projection operator of A corresponding to the eigenvalue a_n.

It is possible for an observable to have multiple orthogonal eigenstates corresponding to the same eigenvalue. In this case, we say the eigenvalue is degenerate. When measuring the observable, there is no way to distinguish between the eigenstates corresponding to a degenerate eigenvalue.

(i) When the eigenvalue a_n is non-degenerate, then $P_n = |a_n\rangle\langle a_n|$ for a unique normalized vector $|a_n\rangle$ and the probability becomes

$$p(a_n) = |\langle a_n|\psi\rangle|^2 \tag{1.3}$$

This can also be viewed as the probability that the state collapses to $|a_n\rangle$, even when the associated eigenvalue is degenerate.

(ii) When the eigenvalue a_n is degenerate, there are multiple eigenvectors with the same eigenvalue, and the projection operator is a sum over projectors

$$P_n = \sum_{i=1}^{k} |a_i\rangle\langle a_i| \tag{1.4}$$

where k is the degeneracy of the eigenvalue and the probability becomes

$$p(a_n) = \sum_{i=1}^{k} |\langle a_i|\psi\rangle|^2 \tag{1.5}$$

From postulate #2, we can express any vector from the vector space in terms of an observable's eigenvectors, which we can assume are orthonormal without loss of generality:

$$|\psi\rangle = \sum_i \langle a_i|\psi\rangle|a_i\rangle$$

$$\langle\psi|\psi\rangle = \sum_i |\langle a_i|\psi\rangle|^2 = 1$$

The quantities $|\langle a_i|\psi\rangle|^2 \geq 0$ sum to 1, so it seems plausible they define a discrete probability distribution over the measurement outcomes in a finite dimensional setting.

1.1 Proof of the Uncertainty Principle

Recall the **uncertainty principle** states that for observables A and B,

$$\Delta A \Delta B \geq \tfrac{1}{2}\big|\langle [A, B]\rangle\big| \tag{1.6}$$

The uncertainty principle alone does not mean that any single measurement is necessarily imprecise. In fact, each time you perform a single measurement of an observable, you may get an infinitely precise value. Instead, the uncertainty principle is a statistical statement about repeated measurements of different observables on identically prepared quantum states. It is possible to derive the uncertainty principle from the postulates of quantum mechanics.

We begin by proving the **Schwarz inequality**

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \tag{1.7}$$

which is analogous to $|\mathbf{a}|^2|\mathbf{b}|^2 \geq |\mathbf{a}\cdot\mathbf{b}|^2$ for real Euclidean space. For kets $|\alpha\rangle, |\beta\rangle$,

$$(\langle\alpha| + \lambda^*\langle\beta|) \cdot (|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

since the inner product of a ket with itself is nonnegative. The relation above holds for any

complex number λ, even when

$$\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} \quad \text{where } |\beta\rangle \neq 0$$

Expanding the inner product and multiplying the relation by $\langle\beta|\beta\rangle$, we find

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle - |\langle\alpha|\beta\rangle|^2 \geq 0$$

Consider $|\alpha\rangle = \tilde{A}|\psi\rangle$ and $|\beta\rangle = \tilde{B}|\psi\rangle$ for an arbitrary ket $|\psi\rangle$ where $\tilde{A} = A - \langle A\rangle$ and $\tilde{B} = B - \langle B\rangle$. Since $\tilde{A}$ and $\tilde{B}$ are Hermitian,

$$(\Delta A)^2(\Delta B)^2 \geq |\langle\tilde{A}\tilde{B}\rangle|^2$$

where $(\Delta A)^2 = \langle\tilde{A}^2\rangle$ and $(\Delta A)^2 = \langle\tilde{B}^2\rangle$ are the uncertainties in A and B respectively. Note

$$\tilde{A}\tilde{B} = \tfrac{1}{2}[\tilde{A}, \tilde{B}] + \tfrac{1}{2}\{\tilde{A}, \tilde{B}\}$$

where $\{\tilde{A}, \tilde{B}\} = \tilde{A}\tilde{B} + \tilde{B}\tilde{A}$ denotes the anti-commutator. The anti-commutator of two Hermitian operators is obviously Hermitian, but the commutator is anti-Hermitian:

$$[\tilde{A}, \tilde{B}]^\dagger = [A, B]^\dagger = (AB - BA)^\dagger = -[A, B]$$

Taking the expectation value, we find

$$\langle\tilde{A}\tilde{B}\rangle = \tfrac{1}{2}\langle[\tilde{A}, \tilde{B}]\rangle + \tfrac{1}{2}\langle\{\tilde{A}, \tilde{B}\}\rangle$$

where $\langle[\tilde{A}, \tilde{B}]\rangle$ is pure imaginary and $\langle\{\tilde{A}, \tilde{B}\}\rangle$ is pure real. Hence,

$$|\langle\tilde{A}\tilde{B}\rangle|^2 = \tfrac{1}{4}|\langle[\tilde{A}, \tilde{B}]\rangle|^2 + \tfrac{1}{4}|\langle\{\tilde{A}, \tilde{B}\}\rangle|^2 \geq \tfrac{1}{4}|\langle[\tilde{A}, \tilde{B}]\rangle|^2 = \tfrac{1}{4}|\langle[A, B]\rangle|^2$$

By omitting the anti-commutator term, we make the resulting inequality even stronger. The uncertainty relation immediately follows:

$$\Delta A \Delta B \geq \tfrac{1}{2}|\langle[A, B]\rangle|$$

1.2 Minimum Uncertainty States

It is natural to wonder, given two incompatible observables A and B such that $[A, B] \neq 0$, under what conditions a state $|\psi\rangle$ would be considered a **minimum uncertainty state**

in the sense that Heisenberg's uncertainty relation is saturated at the lower bound. Well, the **Schwartz inequality** becomes an equality when the two vectors are co-linear, so for a minimum uncertainty state,

$$\tilde{A}|\psi_{\min}\rangle = \lambda\tilde{B}|\psi_{\min}\rangle$$

or

$$\big(A - \langle A\rangle\big)|\psi_{\min}\rangle = \lambda\big(B - \langle B\rangle\big)|\psi_{\min}\rangle$$

where λ is a complex number.

2 Dynamical Postulates

Postulate 4

If a measurement of an observable A on a quantum system in the state $|\psi\rangle$ yields the result a_n, the quantum system irreversibly **collapses** to a new state

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}} \tag{2.1}$$

where P_n is the projection operator of A corresponding to the eigenvalue a_n. The new state is the normalized projection of $|\psi\rangle$ onto the ket (or kets) corresponding to the result of the measurement.

Allow me to emphasize just how *annoying* this postulate is in light of postulate #3. Not only can you not predict with 100% certainty which outcome you will get, but the state irreversibly changes depending on the outcome!

(i) When the eigenvalue a_n is non-degenerate, then the state vector collapses to the single normalized ket $|a_n\rangle$ that corresponds to the eigenvalue a_n.

(ii) When the eigenvalue is degenerate, then the state collapses to the eigensubspace that corresponds to the eigenvalue a_n. It is a common misconception that the state will collapse to an equal superposition of eigenstates with the same eigenvalue obtained in the measurement, but it depends on the state of the system right before measurement.

We know how to describe a quantum system at any given time, and we know how to model the effect of a measurement, but we haven't yet discussed what quantum systems are doing "when nobody's watching."

Postulate 5

The **time evolution** of a closed quantum system is described by a unitary transformation. That is, the state of the system at time t_0 is related to the state of the system at time t by a unitary operator U which depends on t and t_0 such that

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{2.2}$$

For continuous evolution, the state vector obeys a differential equation called the Schrödinger equation given by

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{2.3}$$

where the Hamiltonian $H(t)$ is the observable associated with the total energy of the system and $\hbar$ is the reduced Planck's constant (a universal constant).

Julian Schwinger once remarked, when lecturing on quantum mechanics, "for fundamental properties we will borrow only names from classical physics." You may recognize the terms "Hamiltonian" and "energy" in the fifth postulate, but I would kindly ask you to remember that we are describing a theory for which classical mechanics is merely an approximation. It may be the case that we have to abandon certain ideas attached to classical energy in order to fully capture the behavior of quantum systems.

We'll cover postulate #5 in more detail in the next section, but for now I'd like to discuss unitary transformations and the role they play in time evolution. The fact that one state vector is related to another by a linear operator is not too surprising. A unitary operator has the additional property that its left or right inverse is equal to its corresponding adjoint operator. In other words, if U is a unitary operator, then $U^\dagger U = UU^\dagger = \mathbb{1}$.

Claim: If U is a linear operator, and $|\psi\rangle$ is a normalized vector, then $U|\psi\rangle$ is normalized if and only if U is a unitary operator.

Proof: If U is unitary, then the norm squared of $U|\psi\rangle$ is $\langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \langle\psi|\psi\rangle = 1$. On the other hand, if $U|\psi\rangle$ is normalized, then

$$\langle\psi|U^\dagger U|\psi\rangle = 1 = \langle\psi|\mathbb{1}|\psi\rangle$$

It might be tempting to immediately conclude that $U^\dagger U = \mathbb{1}$ just because $|\psi\rangle$ is arbitrary, but there's no obvious sense of cancellation that applies here. Rearranging the expression,

we find

$$\langle\psi|\left(U^\dagger U - \mathbb{1}\right)|\psi\rangle = \langle\psi|A|\psi\rangle = 0$$

where $A = U^\dagger U - \mathbb{1}$. Again, it might be tempting to immediately conclude that $A = 0$ and $U^\dagger U = \mathbb{1}$, but this isn't obvious, and in fact, it simply isn't true for an arbitrary linear operator A! In case you don't buy it, here's a simple counterexample; consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any real vector $|\psi\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$, we see that $\langle\psi|A|\psi\rangle = xy - yx = 0$ even though $A \neq 0$.

Here's where it gets a bit creative–if you can think of a better proof, please let me know! Consider the linear combination $|\phi + \alpha\psi\rangle = |\phi\rangle + \alpha|\psi\rangle$ where α is an arbitrary scalar. Since this is guaranteed to be a ket in the vector space,

$$\begin{aligned} 0 &= \langle\psi + \alpha\phi|A|\psi + \alpha\phi\rangle \\ &= \langle\psi|A|\psi\rangle + \alpha^*\langle\phi|A|\psi\rangle + \alpha\langle\psi|A|\phi\rangle + \alpha^*\alpha\langle\phi|A|\phi\rangle \\ &= \alpha^*\langle\phi|A|\psi\rangle + \alpha\langle\psi|A|\phi\rangle \end{aligned}$$

Let $c = \langle\psi|A|\phi\rangle$. Since A is Hermitian, $c^* = \langle\phi|A|\psi\rangle$ and it follows that

$$\alpha c + (\alpha c)^* = 0$$

This expression must hold for all complex numbers α. When $\alpha = 1$, $c + c^* = 0$, and we conclude the real part of c must vanish. On the other hand, when $\alpha = i$, $c - c^* = 0$, which means the imaginary part of c must vanish. If both the real and imaginary parts are zero, then $c = \langle\psi|A|\phi\rangle = 0$. In particular, for all $|\psi\rangle = A|\phi\rangle$, we find that $\langle\psi|\psi\rangle = 0$, and it follows from the definition of the inner product that $|\psi\rangle = A|\phi\rangle = 0$. By definition of the zero operator, $A = 0$, and we finally conclude that $U^\dagger U = \mathbb{1}$.

The point of this exercise was to demonstrate that unitary operators are the only operators that map normalized kets to normalized kets, and since a normalized ket contains all the information of a quantum system, a unitary operator is "information conserving" in a very informal sense. It is crucial for the statistical interpretation that unitary operators preserve the norm of the quantum state vector as it evolves over time in a closed system.

3 Generator of Time Translations

In the "Schrödinger picture" of quantum mechanics, the state vector evolves over time between measurements. The law governing time evolution is the unitary transformation

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{3.1}$$

where $U(t, t) = \mathbb{1}$ for any time t. It's easy to demonstrate that the time evolution operators obey a composition rule. Given three times t_0, t_1, and t_2, the state can evolve unitarily from $t_0 \to t_1$ and then again from $t_1 \to t_2$. The composition rule is

$$U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0) \tag{3.2}$$

where the operator $U(t_2, t_1)$ can be understood as $U(t_2, t_0)U^\dagger(t_1, t_0)$ in case there happens to be something "special" about t_0. Assuming time is a continuous parameter, the composition rule implies that time evolution divides into infinitesimal unitary transformations. We imagine the state vector is evolving smoothly just like a particle's trajectory in classical mechanics.

If $U(t', t)$ is differentiable, we may consider an infinitesimal time evolution from time t to time $t' = t + \epsilon$ where ϵ is small. Expanding $U(t + \epsilon, t)$ to first order in ϵ,

$$U(t + \epsilon, t) = U(t, t) + \epsilon \left.\frac{\partial U(t', t)}{\partial t'}\right|_{t'=t} + O(\epsilon^2)$$

or

$$U(t + \epsilon, t) = \mathbb{1} - i\epsilon\Omega(t) + O(\epsilon^2) \tag{3.3}$$

where we've defined a new linear operator $\Omega(t)$ with units of $[\text{time}]^{-1}$ as

$$\Omega(t) = i\left.\frac{\partial U(t', t)}{\partial t'}\right|_{t'=t} \tag{3.4}$$

In defining this operator, we've included the factor of i to ensure $\Omega^\dagger(t) = \Omega(t)$, which follows from unitarity of $U(t', t)$. The adjoint is given by

$$U^\dagger(t + \epsilon, t) = \mathbb{1} + i\epsilon\Omega^\dagger(t)$$

and it follows that

$$U^\dagger(t+\epsilon,t)U(t+\epsilon,t) = \Big(\mathbb{1} + i\epsilon\Omega^\dagger(t)\Big)\Big(\mathbb{1} - i\epsilon\Omega(t)\Big)$$
$$= \mathbb{1} + i\epsilon\Big(\Omega^\dagger(t) - \Omega(t)\Big) + O(\epsilon^2)$$

The linear term must vanish, so $\Omega^\dagger(t) = \Omega(t)$, and we've shown that $\Omega(t)$ is Hermitian. Presumably, it corresponds to some physical quantity that we can measure. Formally, $\Omega(t)$ is the *generator of time translations* in quantum mechanics, because it gives the small correction needed to advance a state from time t to $t+\epsilon$. Evolution over a finite time interval can be regarded as the result of a large number of small displacements in time, so knowing $\Omega(t)$ allows us to evolve the system over any time interval.

3.1 The Schrödinger Equation

We can obtain a differential equation for $U(t,t_0)$ by differentiating with respect to the final time t and expanding the definition of the derivative:

$$\frac{\partial U(t,t_0)}{\partial t} = \lim_{\epsilon\to 0}\frac{U(t+\epsilon,t_0) - U(t,t_0)}{\epsilon} \tag{3.5}$$

Using the composition rule, $U(t+\epsilon,t_0) = U(t+\epsilon,t)U(t,t_0)$, and we have

$$\frac{\partial U(t,t_0)}{\partial t} = \left[\lim_{\epsilon\to 0}\frac{U(t+\epsilon,t) - \mathbb{1}}{\epsilon}\right] U(t,t_0) \tag{3.6}$$

where the term in square brackets is $-i\Omega(t)$. A differential equation for $U(t,t_0)$ is then

$$i\frac{\partial U(t,t_0)}{\partial t} = \Omega(t)U(t,t_0) \tag{3.7}$$

We can also find a differential equation for the quantum state by differentiating Eq. (3.1) with respect to the final time t and substituting Eq. (3.7)

$$\frac{d}{dt}|\psi(t)\rangle = \frac{\partial U(t,t_0)}{\partial t}|\psi(t_0)\rangle = -i\Omega(t)U(t,t_0)|\psi(t_0)\rangle$$

or

$$i\frac{d}{dt}|\psi(t)\rangle = \Omega(t)|\psi(t)\rangle \tag{3.8}$$

So far, there has been no mention of energies or Hamiltonians. In other words, up until this point, there has been very little (recognizable) physics. A straightforward application of the

product rule reveals an important identity concerning the expectation value of $\Omega(t)$:

$$\frac{d}{dt}\langle\psi(t)|\Omega(t)|\psi(t)\rangle = \left[\frac{d}{dt}\langle\psi(t)|\right]\underbrace{\Omega(t)|\psi(t)\rangle}_{i\frac{d}{dt}|\psi(t)\rangle} + \underbrace{\langle\psi(t)|\Omega(t)}_{-i\frac{d}{dt}\langle\psi(t)|}\left[\frac{d}{dt}|\psi(t)\rangle\right] + \left\langle\frac{\partial\Omega(t)}{\partial t}\right\rangle$$

$$\therefore\ \frac{d}{dt}\langle\Omega(t)\rangle = \left\langle\frac{\partial\Omega(t)}{\partial t}\right\rangle \tag{3.9}$$

Hence, the expectation value of $\Omega(t)$ is conserved when the operator itself does not depend on time. In classical mechanics, the Hamiltonian is (i) used to take the state of the system at one time and produce the state of the system at an infinitesimal time later, and (ii) conserved when it does not depend explicitly on time. The analogy could be taken further, but ultimately, we simply define the quantum mechanical Hamiltonian as

$$H(t) = \hbar\Omega(t) = i\hbar\frac{\partial U(t',t)}{\partial t'}\bigg|_{t'=t} \tag{3.10}$$

where $\hbar$ is a universal constant with units of (energy) $\times$ (time) so that $H(t)$ has units of energy, and the precise numerical value of $\hbar$ must be obtained from experiments. We then arrive at the **Schrödinger equation**, a first-order differential equation for the state vector, as a consequence of smooth unitary time evolution:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{3.11}$$

The differential equation for $U(t,t_0)$ can be written in terms of $H(t)$ as

$$i\hbar\frac{\partial U(t,t_0)}{\partial t} = H(t)U(t,t_0) \tag{3.12}$$

The Hamiltonian differs from the generator of time translations by an overall multiplicative constant, so we might as well regard the Hamiltonian itself as the generator. When $H(t)$ doesn't depend explicitly on time, its expectation value is guaranteed to be conserved for any time-dependent state vector. This gives us confidence that the Hamiltonian corresponds to the energy observable. In fact, by the postulates of quantum mechanics, the eigenvalues of the Hamiltonian define the possible energies of the system.

3.2 Time-Translation Invariance

In classical mechanics, you learn that the Hamiltonian is conserved when the laws governing the evolution of a system are time-translation invariant. There is a similar and even more

elegant example of that same idea in quantum mechanics. We say that a quantum system is time-translation invariant when $U(t, t_0)$ depends only on the time difference $t - t_0$ such that

$$U(t, t_0) = U(t - t_0) \tag{3.13}$$

The composition law for a time-translation invariant system becomes

$$U(t_2 - t_0) = U(t_2 - t_1)U(t_1 - t_0) \tag{3.14}$$

Suppose $t_1 = t_0 + s_1$ and $t_2 = t_1 + s_2$, then

$$U(s_1 + s_2) = U(s_2)U(s_1) = U(s_1)U(s_2) \tag{3.15}$$

If we assume that $U(0) = \mathbb{1}$, then $U(t)$ applied to $|\psi(0)\rangle$ gives the state of the system at any later time. We now show that time-translation invariance implies the quantum mechanical Hamiltonian is time-independent, and hence conserved. The time derivative of $U(t)$ is

$$\begin{aligned}
\frac{\partial U(t)}{\partial t} &= \lim_{\delta t \to 0} \frac{U(t + \delta t) - U(t)}{\delta t} \\
&= \lim_{\delta t \to 0} \frac{U(\delta t)U(t) - U(t)}{\delta t} \\
&= \left[\lim_{\delta t \to 0} \frac{U(0 + \delta t) - U(0)}{\delta t} \right] U(t)
\end{aligned}$$

Hence,

$$H(t) = i\hbar \frac{\partial U(t)}{\partial t} U^\dagger(t) = i\hbar \frac{\partial U(t)}{\partial t}\bigg|_{t=0}$$

The expression on the right hand side is the derivative of U with respect to t evaluated at $t = 0$. In other words, it is independent of time. Thus, $H(t)$ must be independent of time, i.e. $H(t) = H$. Additionally, the operator $U(t)$ itself obeys the differential equation

$$i\hbar \frac{\partial U(t)}{\partial t} = HU(t) \tag{3.16}$$

In many situations, the Hamiltonian will be supplied by the model, and when the Hamiltonian is independent of time, we know the evolution will be time-translation invariant, and we can easily solve for the unitary time operators by invoking Stone's theorem. We can understand the result intuitively by taking a finite interval t and dividing it many times into

N intervals of length δt such that $t = N\delta t$. Then,

$$U(t) = U(N\delta t) = \big[U(\delta t)\big]^N \approx \left[1 + \frac{\partial U(t)}{\partial t}\bigg|_{t=0} \delta t\right]^N$$

$$U(t) = \lim_{N\to\infty} \left[1 + \frac{\partial U(t)}{\partial t}\bigg|_{t=0} \delta t\right]^N$$

$$= \lim_{N\to\infty} \left[1 - \frac{iHt/\hbar}{N}\right]^N$$

Recall that, as a function over real numbers, the exponential e^x can be written as

$$e^x = \lim_{N\to\infty} \left(1 + \frac{x}{n}\right)^n$$

It is therefore tempting to write the solution for $U(t)$ as

$$U(t) = e^{-iHt/\hbar} \tag{3.17}$$

and this actually turns out to be correct, but our "derivation" only works for a limited class of operators and the exponential of a self-adjoint operator is properly understood in the context of functional analysis, and in particular, the spectral theorem.

In a finite-dimensional vector space, the exponential of an operator A can be defined via the Taylor series as

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{3.18}$$

If A is self-adjoint and finite dimensional, it has a spectral decomposition $UDU^\dagger$ such that D is diagonal and U is a unitary matrix consisting of an orthonormal set of eigenvectors of A. Hence,

$$e^A = \sum_{n=0}^{\infty} \frac{(UDU^\dagger)^n}{n!} = Ue^D U^\dagger \tag{3.19}$$

Since D is diagonal, e^D can be represented by a diagonal matrix consisting of entries like e^λ where λ is an eigenvalue of A. In fact, many functions (those with convergent series expansions) applied to self-adjoint linear operators may be written as

$$f(A) = Uf(D)U^\dagger. \tag{3.20}$$

In terms of the spectral decomposition,

$$f(A) = \sum_n f(a_n)|a_n\rangle\langle a_n| \tag{3.21}$$

and furthermore, $f(A)|a_n\rangle = f(a_n)|a_n\rangle$ where we are using an abuse of notation where f can act on the space of operators and real numbers. For instance, $\cos(A)|a_n\rangle = \cos(a_n)|a_n\rangle$.

Example 3.1: Spectral Theorem

Use the spectral theorem to compute $e^{i\omega S_x t/\hbar}$ in the basis of S_z where ω is a constant with units of frequency and t is a time parameter.

Solution: Start by writing S_x in terms of its eigenvalues and eigenprojectors:

$$S_x = \frac{\hbar}{2}|+x\rangle\langle+x| - \frac{\hbar}{2}|-x\rangle\langle-x|$$

The exponential is then

$$e^{i\omega S_x t/\hbar} = e^{i\omega t/2}|+x\rangle\langle+x| + e^{-i\omega t/2}|-x\rangle\langle-x|$$

In the standard S_z basis,

$$e^{i\omega S_x t/\hbar} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} e^{i\omega t/2} + \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} e^{-i\omega t/2}$$

$$= \begin{pmatrix}\cos(\omega t/2) & i\sin(\omega t/2)\\ i\sin(\omega t/2) & \cos(\omega t/2)\end{pmatrix}$$

4 Interpretations

We have laid out the basic postulates of quantum mechanics: states as vectors in a complex vector space, observables as operators, unitary time evolution, measurement with probabilistic outcomes, and the projection postulate (state collapse). Together, these rules form a remarkably successful framework for predicting the results of experiments. From atomic spectra to semiconductors to lasers to magnetic resonance, quantum mechanics works extraordinarily well *for all practical purposes*.

For the majority of this course, we'll adopt an *instrumentalist* view of quantum mechanics. For the applications we're interested in, it will be clear when a measurement occurs and there

will be little-to-know ambiguity in the formalism. We will simply use the postulates without worrying about the underlying physical reality they may or may not be describing.

In this section, and later in the course if we have time, we'll address *realism*, the attitude that a successful physical theory is not just a tool for predicting observations, but a description of a world that exists (independently of us). There have been many attempts to understand the physical reality implied by quantum mechanics, and each has a different take on the issues of probability, measurement, and state collapse, to name a few. We'll briefly address each attempt in the context of a thought experiment devised by Eugene Wigner.

4.1 Wigner's Friend

The primary purpose of the Wigner's friend thought experiment is to convince you that the measurement problem is indeed a problem! Wigner asks his friend to perform a Stern-Gerlach experiment inside a large container that can be perfectly isolated from the environment. The initial spin state of the atoms is $|+x\rangle$, a superposition of $|\pm z\rangle$, and this state is known to both Wigner and his friend. Suppose Wigner's friend performs a measurement of the z-component of spin angular momentum. The friend measures and records a definite outcome, for example "spin up," in a notebook. From the friend's point of view, the story is straightforward: before the measurement, the system was in a superposition; after the measurement, it has a definite value. The state vector has collapsed.

Now consider Wigner, outside the sealed lab. He treats everything inside—the atom, the apparatus, the friend, and the notebook—as one big *isolated quantum system*. Quantum mechanics tells him to assign a state that evolves unitarily. For Wigner, the experiment puts the lab into a superposition of two outcomes, meaning no definite outcome has occurred. Both descriptions are allowed by the quantum postulates, depending on where we draw the line between "system" and "observer." But they cannot both be true if collapse is supposed to be an objective physical process. Did the state collapse when the friend looked? Or only later, when Wigner opens the lab? Or never?

Suppose you aren't convinced of the ambiguity. Wigner may have no knowledge of the outcome, but Wigner's friend certainly obtains one. What if Wigner's friend was actually a highly trained capuchin monkey? A very smart dog? What if Wigner's friend was a tardigrade? Can tardigrades collapse quantum states? The magnetic field alone doesn't collapse the state; in the next lecture we'll learn that a spin state evolves *unitarily* inside a magnetic field. When does the collapse actually occur at the level of atoms and interactions between atoms (or subatomic particles)? In the words of John Bell, "...does not any analysis of measurement require concepts more fundamental than measurement?"

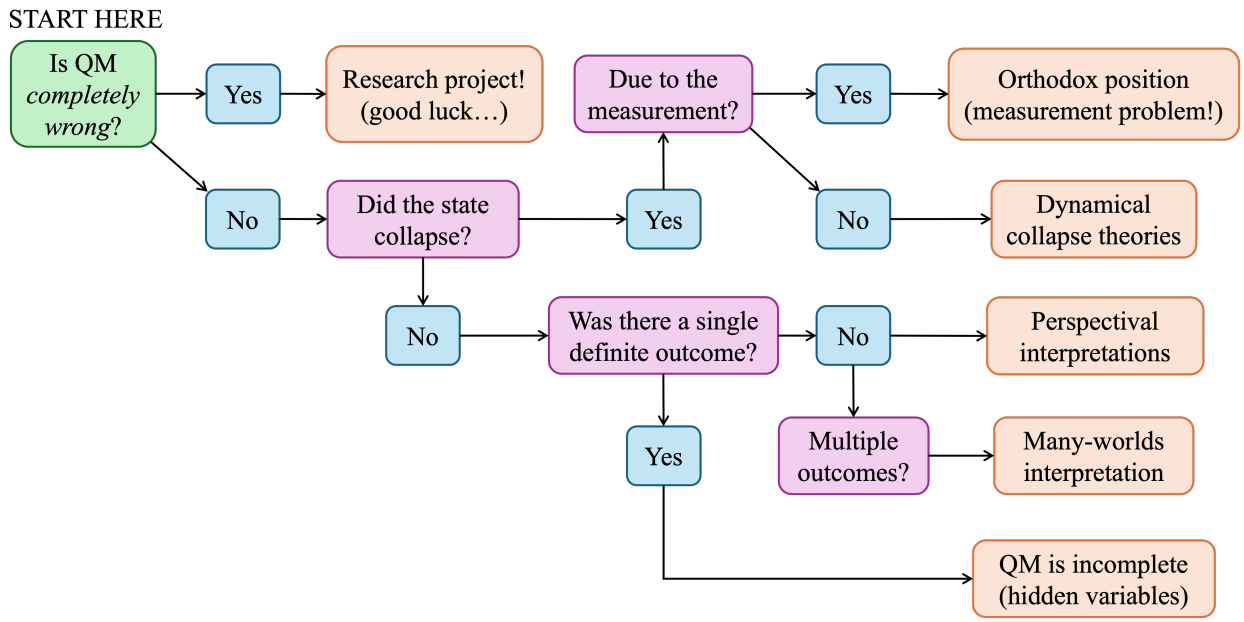

Figure 4.1: The "Wigner's friend flow chart" maps out various interpretations of quantum mechanics (credit: Dr. Jacob Barandes, Havard).

4.2 The Flow Chart

We'll briefly discuss various attempts to better understand, solve, or subvert the measurement problem by navigating the "Wigner's friend flow chart" in Figure 4.1. Almost everyone ends up somewhere on the chart, even if you think quantum mechanics is completely wrong and not just missing some subtle details about nature. Assuming you agree that quantum mechanics is mostly correct at some level, you likely fall into one of several camps:

1. The **orthodox position** remains largely agnostic about physical reality. The postulates are a set of tools, so "shut up and calculate." The measurement problem is not a *practical problem*, but it might be worthy of study. This position is often conflated with the **Copenhagen interpretation**, which crucially involves a distinction between the "classical world" and the "quantum world." Read Heisenberg's book, *Philosophy of Physics*, if you're interested in learning more.

2. **Dynamical collapse theories**, like Ghirardi–Rimini–Weber (GRW), assert that state vector collapse is an objective physical process and try to describe how it works.

3. In a **perspectival or relational interpretation** of quantum mechanics, the properties of a system, including its quantum state, are not absolute, but always defined relative to another system with which it interacts. Look up Carlo Rovelli's *relational quantum mechanics* if you want to learn more.

4. In the **many-worlds interpretation** (by Everett), the state vector is taken to be a real, physical object that always evolves according to the Schrödinger equation and never collapses. What we call a measurement is just an ordinary interaction that *entangles* the system with the apparatus and observer, causing the total state to branch into non-interacting components in which each possible outcome is realized. In each branch, the observer experiences a single definite result, even though all outcomes exist in the global state vector (or wave function). The apparent randomness of measurement reflects which branch an observer finds themselves in, not any fundamental indeterminism, so the theory is fully deterministic at the level of the universal state.

5. In a **hidden-variable theory**, the quantum state does not give a *complete description* of an individual physical system. Instead, there exist additional, unobserved parameters called *hidden variables* that determine the properties and outcomes of measurements. In deterministic hidden-variable theories (e.g., pilot wave theory), the apparent randomness of quantum mechanics reflects our ignorance of the hidden variables like in classical statistical mechanics. In stochastic hidden-variable theories, indeterminism is a fundamental feature of the rules by which the hidden variables change over time. Whether deterministic or indeterministic, the system always possesses at least some well-defined properties, and measurements are physical interactions that can either reveal those properties or create new, emergent properties. Contrary to what you may have read or heard, Bell's theorem does not rule out hidden variables.

Some extreme positions suggest the state vector never collapses and no definite outcome occurs; these are largely self-undermining and rarely discussed seriously.