Phys 152: Fundamentals of Physics II

Unit #7 - Magnetostatic Fields

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Introduction to Biot-Savart

In 1820, Jean-Baptiste Biot and Félix Savart discovered that the intensity of the magnetic field set up by a current flowing through a wire is proportional to the current and inversely proportional to the distance from the wire.

Let's revisit Ampère's discovery in a quantitative setting. The force per unit length between two parallel current-carrying wires is found to be proportional to the product of the currents in each wire and inversely proportional to the distance between them. We can use the magnetic force law to infer the strength of the magnetic field of a current-carrying wire.

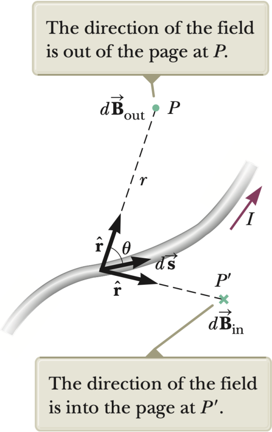
$$\frac{F_1}{L} \propto \frac{I_1 I_2}{d}, \quad F_1 = I_1 L B_2 \quad \Rightarrow \quad B_2 \propto \frac{I_2}{d}$$

Biot-Savart Law

The Biot-Savart law gives the total magnetic field at any point in space due to a steady current distribution. Historically, it was developed to reproduce the experimental observations of Biot and Savart, but it applies to *any steady current distribution*.

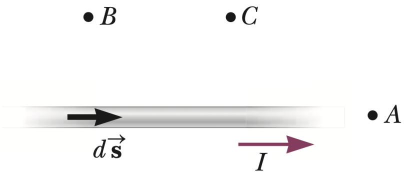
$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int I \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

where the constant $\mu_0 \approx 4\pi \times 10^{-7}\ \mathrm{T} \cdot \mathrm{m/A}$ is called the permeability of free space. It is important to understand that the Biot-Savart law gives the magnetic field *produced by the current distribution*.


The direction of the field is out of the page at P.
$d\vec{\mathbf{B}}_{\text{out}}$
P
r
I
$\hat{\mathbf{r}}$
θ
$d\vec{\mathbf{s}}$
$\hat{\mathbf{r}}$
P'
$d\vec{\mathbf{B}}_{\text{in}}$
The direction of the field is into the page at P'.

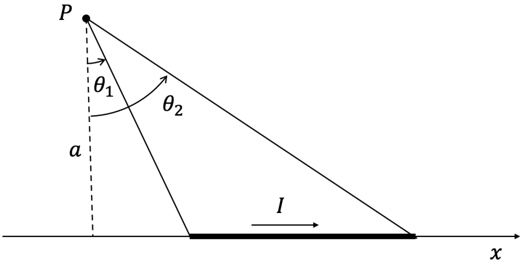
Conceptual Question 1

Consider the magnetic field due to the current in the wire shown in the figure. Rank the points A, B, and C in terms of magnitude of the magnetic field that is due to the current in just the length element $d\vec{\mathbf{s}}$ shown from greatest to least.



Example 1

Consider a thin, straight wire of finite length carrying a constant current I and placed along the x axis. Determine the magnitude and direction of the magnetic field at point P due to this current.

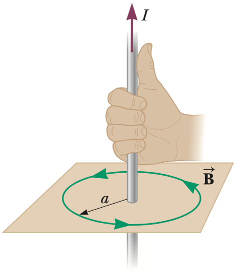


Magnetic Field of a Long Straight Wire

The magnetic field strength of a long straight wire is

$$B = \frac{\mu_0 I}{2\pi a}$$

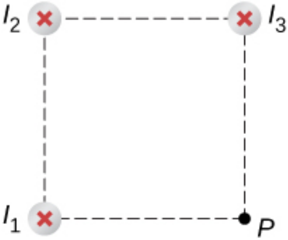
where the field circulates around the wire in a direction given by a right-hand rule.



The Biot-Savart law predicts the findings of Biot and Savart (go figure), but it works for other current distributions as well.

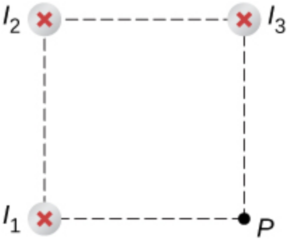
Example 2

Three wires sit at the corners of a square, all carrying currents of 2 amps into the page as shown in the figure. Calculate the magnitude of the magnetic field at the other corner of the square, point P, if the length of each side of the square is 1 cm.



Example 2 cont.

Keeping the currents the same in wires 1 and 3, what should the current be in wire 2 to counteract the magnetic fields from wires 1 and 3 so that there is no net magnetic field at point P?

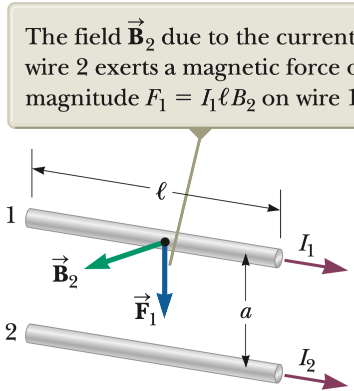


Magnetic Force Between Parallel Conducting Wires

We can now verify Ampère's discovery on theoretical grounds using the magnetic force law and the Biot-Savart law. Consider two long, straight, parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same direction. The force on the wire carrying I_1 is towards the wire carrying I_2 (attractive) and has magnitude

$$F_1 = I_1 \ell B_2 = I_1 \ell \left(\frac{\mu_0 I_2}{2\pi a} \right) \rightarrow \frac{F_1}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

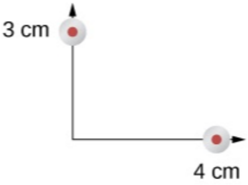
By the RHR, if the two currents point in opposite directions, the force will be repulsive, and by Newton's third law, the force on 1 is equal and opposite to the force on 2.



The field $\vec{\mathbf{B}}_2$ due to the current in wire 2 exerts a magnetic force of magnitude $F_1 = I_1 \ell B_2$ on wire 1.
ℓ
1
I_1
$\vec{\mathbf{B}}_2$
$\vec{\mathbf{F}}_1$
a
2
I_2

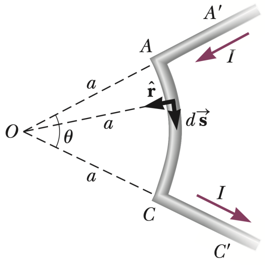
Example 3

Two wires, both carrying current out of the page, have a current of magnitude 5.0 mA. The first wire is located at (0.0 cm, 3.0 cm) while the other wire is located at (4.0 cm, 0.0 cm). What is the magnetic force per unit length of the first wire on the second and the second wire on the first?



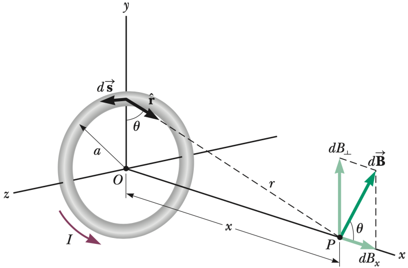
Example 4

Calculate the magnetic field at point O for the current-carrying wire segment shown in the figure. The wire consists of two straight portions and a circular arc of radius a, which subtends an angle θ.



Example 5

A circular loop of wire with radius a carries current I. Find the magnetic field of the current loop at distance x on its central axis.

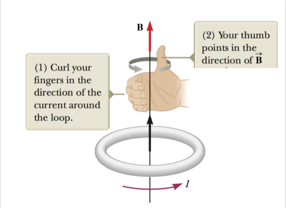


Magnetic Field of a Current Loop

At the very center of a coil with N circular loops carrying current I, the magnetic field strength is

$$B = \frac{\mu_0 N I}{2a}$$

where the direction is given by a right-hand rule explained in the figure.



Revisiting the Magnetic Dipole

Consider $x \gg a$ in the result from Example 5:

$$B_x \approx \frac{\mu_0 I a^2}{2x^3} = \frac{\mu_0 \mu}{2\pi x^3},$$

where $\mu = IA = I(\pi a^2)$ is the magnetic dipole moment of the current loop. The result for the magnetic field is similar in form to the expression for the electric field due to an electric dipole!

Ampère's Law

The line integral of $\vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}}$ around any closed path (the circulation of $\vec{\mathbf{B}}$) equals $\mu_0 I$, where I is the total steady current passing through any surface bounded by the closed path:

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{through}} = \mu_0 \int \vec{\mathbf{J}} \cdot d\,\vec{\mathbf{a}}$$

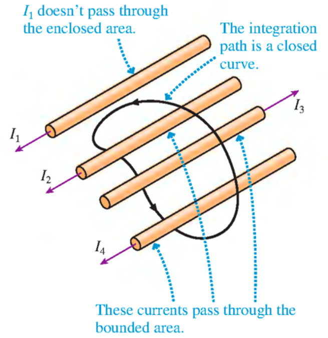
(i) The integration path of Ampère's law is an abstract mathematical curve called an amperian loop. It does not have to match any physical surface or boundary.

(ii) Ampère's law is useful in situations where the current distribution exhibits a high degree of symmetry.

Sign Convention for Ampère's Law

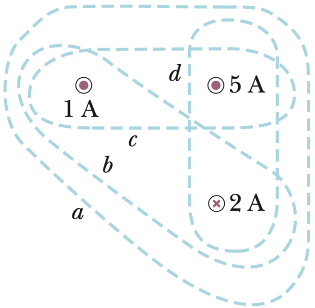
If you curl the fingers of your right hand around the closed path in the direction of integration, any current passing through the bounded area in the direction of your outstretched thumb is "positive current." Any current in the opposite direction is "negative current."

$$I_{\text{through}} = I_2 - I_3 + I_4$$



Conceptual Question 2

Rank the magnitudes of $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ for the closed paths a through d in the figure below from greatest to least.



Using Ampère's Law

When using Ampère's law to find the magnetic field of a symmetric current distribution, use the Biot-Savart law to make symmetry arguments and choose an amperian loop for which one or more of the following hold:

(i) $\vec{\mathbf{B}}$ is constant along portions of the integration path,

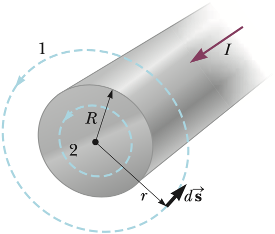
(ii) $\vec{\mathbf{B}}$ is tangent to the integration path so $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \pm B ds$

(iii) $\vec{\mathbf{B}}$ is perpendicular to the integration path so $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0$.

These choices will simplify the line integral, making it easier to determine the magnetic field.

Example 6

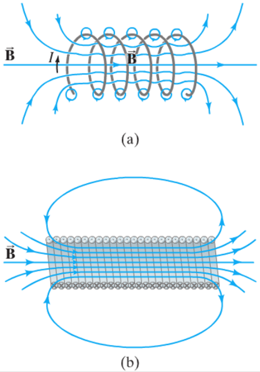
A long, straight conducting wire of radius R carries a steady current I that is uniformly distributed through the cross section of the wire. Calculate the magnetic field a distance r from the center of the wire in the regions $r \geq R$ and $r < R$. Ignore any magnetization effects.



Solenoids

A solenoid is a helical coil of wire with the same current I passing through each loop in the coil. Solenoids may have hundreds or thousands of coils, often called turns, sometimes wrapped in several layers. They are used in practice to generate uniform magnetic fields in their interior.

An ideal solenoid is a tightly-wound, infinitely long solenoid with n turns per unit length. The magnetic field inside an ideal solenoid is constant (by translational symmetry), and outside the solenoid the magnetic field strength is zero.



Example 7

Real solenoids can be approximated by ideal solenoids in the limit that the length is much larger than the radius. Use Ampere's law to find the field inside a long solenoid if it has n turns per unit length and carries current I.

Ans: $B = \mu_0 n I$ (direction given by right-hand rule)

Conceptual Question 3

Consider a solenoid that is very long compared with its radius. Of the following choices, what is the most effective way to increase the magnetic field in the interior of the solenoid?

(a) double its length, keeping the number of turns per unit length constant

(b) reduce its radius by half, keeping the number of turns per unit length constant

(c) overwrap the entire solenoid with an additional layer of current-carrying wire

Example 8

Long, straight conductors with square cross sections and each carrying current I are laid side by side to form an infinite current sheet. The conductors lie in the xy-plane, are parallel to the y-axis, and carry current in the $+y$ direction. There are n conductors per unit length measured along the x-axis. (a) What are the magnitude and direction of the magnetic field a distance a below the current sheet? (b) What are the magnitude and direction of the magnetic field a distance a above the current sheet?