Phys 152: Fundamentals of Physics II

Unit #9 - Alternating Current

Aaron Wirthwein

Department of Physics and Astronomy
University of Southern California

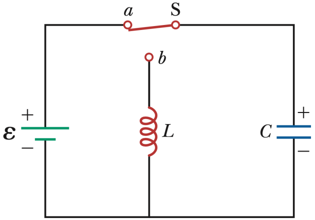
Alternating Current

Alternating current (AC) is the flow of electric charge that periodically reverses direction. In some oscillatory electrical systems, AC can occur naturally, and in many others it is produced by an alternating source, such as a conducting coil rotating in a magnetic field.

An AC circuit contains an AC source that varies periodically in time, usually sinusoidally. We will eventually study the behavior of AC circuits consisting of resistors, capacitors, and inductors. While resistors continuously dissipate energy from the circuit, capacitors and inductors are able to store and release electrical energy on characteristic time scales, making them important components of AC circuits for many practical applications.

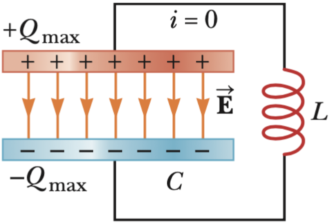
Oscillations in LC Circuits

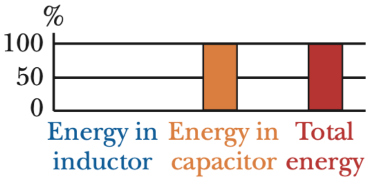
Capacitors store energy in their electric fields and inductors store energy in their magnetic fields. A circuit containing both an inductor and a capacitor can oscillate current and charge without a source of emf by shifting the energy between the electric and magnetic fields.



Energy Transfer in an LC Circuit (1/5)

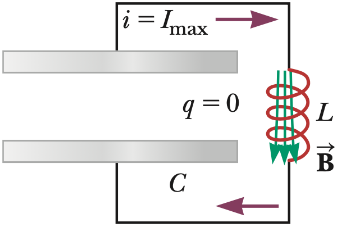
Before we get to the equations, let's run through the steps of one full oscillation and describe them in terms of energy transfers.

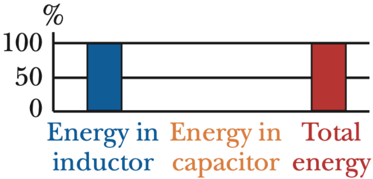




Energy Transfer in an LC Circuit (2/5)

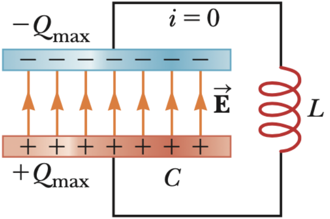
As the capacitor discharges, energy is transferred from the capacitor to the inductor until the maximum current is reached.

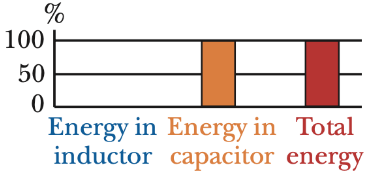




Energy Transfer in an LC Circuit (3/5)

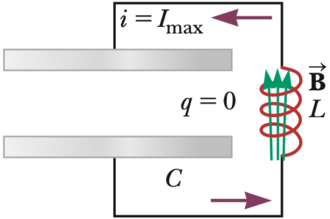
As charges accumulate on the capacitor with reversed polarity, current decreases and energy transfers from inductor back to capacitor.

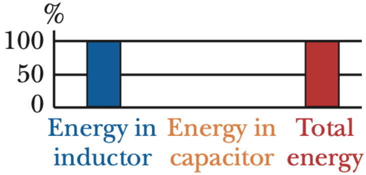




Energy Transfer in an LC Circuit (4/5)

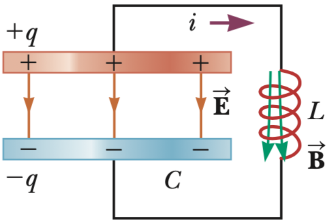
Capacitor discharges back through inductor with current flowing in the opposite direction.

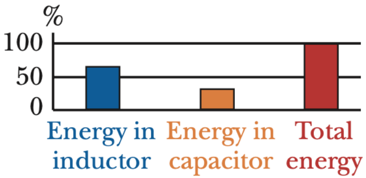




Energy Transfer in an LC Circuit (5/5)

As charges accumulate on the capacitor with original polarity, energy is being transferred from the inductor to the capacitor. The total energy remains constant throughout the entire process.





Oscillation of Charge in LC Circuits

We could use Kirchhoff's loop rule to analyze the LC circuit, but instead we will use the *principle of energy conservation.* At any moment, $U_E + U_B =$ constant $\Rightarrow \frac{dU_E}{dt} + \frac{dU_B}{dt} = 0$. Hence,

$$\frac{d}{dt}\left(\frac{q^2}{2C}\right) + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) = 0 \Rightarrow \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

By conservation of charge, $i = \frac{dq}{dt}$, and it follows that

$$\frac{d^2q}{dt^2} = -\frac{1}{LC}q \quad \text{(simple harmonic oscillations)}$$

Oscillation of Charge in LC Circuits

The charge on the capacitor oscillates sinusoidally with frequency f and period T. The solution for the charge as a function of time is

$$q(t) = Q_{\max} \cos(\omega t + \phi)$$

where ω is the *angular frequency* of oscillation,

$$\omega = \sqrt{\frac{1}{LC}} = 2\pi f = \frac{2\pi}{T}$$

and ϕ is the *phase constant* determined by the initial conditions.

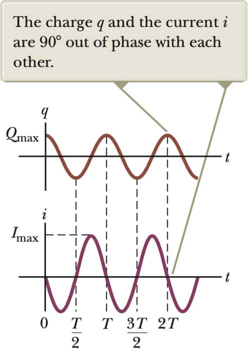
Oscillation of Current in LC Circuits

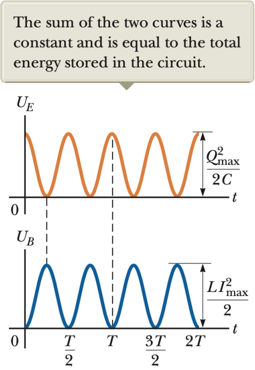
The current in the circuit also varies sinusoidally. Differentiating charge with respect to time,

$$i = \frac{dq}{dt} = -\omega Q_{\max} \sin(\omega t + \phi) = -I_{\max} \sin(\omega t + \phi)$$

Exercise: show that $I_{\max} = \omega Q_{\max}$ using energy considerations and explain in what sense "the current is 90° out of phase with the charge." It may help to look at graphs of both charge and current over time in the LC circuit.

Graphs of Oscillations in LC Circuits





Conceptual Question 1

At an instant of time during the oscillations of an LC circuit, the current is at its maximum value. At this instant, what happens to the voltage across the capacitor? Select all that apply.

(a) It is different from that across the inductor.

(b) It is zero.

(c) It has its maximum value.

(d) It is impossible to determine.

Conceptual Question 2

Now consider an instant when the current is momentarily zero. Describe the magnitude of the voltage across the capacitor at this instant. Select all that apply.

(a) It is different from that across the inductor.

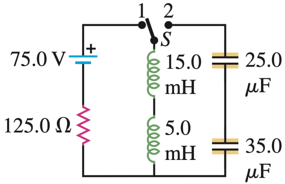
(b) It is zero.

(c) It has its maximum value.

(d) It is impossible to determine.

Example 1: An LC Circuit

In the circuit shown, neither the battery nor the inductors have any appreciable resistance, the capacitors are initially uncharged, and the switch S has been in position 1 for a very long time. (a) What is the current in the circuit? (b) The switch is now suddenly flipped to position 2. Find the maximum charge that each capacitor will receive, and how much time after the switch is flipped will it take to acquire this charge.



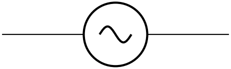
Applications of LC Circuits

LC circuits have a wide range of applications due to their ability to resonate. They are used in radios to select specific frequencies from incoming signals. The resonance frequency of the LC circuit is adjusted to match the desired radio station frequency. LC circuits serve as the basis for oscillators that generate periodic signals, such as sine or square waves. These are used in clocks, signal generators, and communication systems.

LC circuits are used in electronic filters (low-pass, high-pass, band-pass, and band-stop) to block or allow signals of certain frequencies. These are essential in audio processing, telecommunications, and data transmission.

Alternating Source

The following symbol is used to denote a *sinusoidal voltage source*:



Let the voltage across the terminals be given by $\Delta v = \Delta V_{\max} \sin \omega t$. For a typical house in the United States, $\Delta V_{\max} = 170$ V and $f = \omega/2\pi = 60$ Hz. In Europe, residential AC circuits typically operate with $\Delta V_{\max} = 325$ V and $f = 50$ Hz.

Simply put, Europe prioritizes efficient power transmission through while the US prioritizes safety with a lower voltage.

Root-Mean-Square Voltage

For reasons we will discuss soon, alternating voltage is best described in terms of rms voltage rather than maximum (peak) voltage:

$$\Delta V_{\rm rms} = \frac{\Delta V_{\rm max}}{\sqrt{2}} \quad \text{(sinusoidal signal)}$$

The "rms" stands for *root-mean-square* which means to (i) square the signal, (ii) take a time average over one period, (iii) and finally, take the square root.

In the US, $\Delta V_{\rm rms} = 120$ V, and in Europe, $\Delta V_{\rm rms} = 230$ V.

For any time-varying, periodic signal $x(t)$,

$$X_{\mathrm{rms}} \equiv \sqrt{\frac{1}{T}\int_0^T x^2\ dt.}$$

See if you can obtain the result above for a sinusoidal signal. Depending on how you do the calculation, you may need to solve the following integral:

$$\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt$$

Resistor in an AC Circuit

Consider a series circuit consisting of an AC source and a resistor. By Kirchhoff's loop rule and Ohm's law applied to the resistor,

$$\Delta v_R = \Delta v = \Delta V_{\max} \sin \omega t,$$

$$i = \frac{\Delta v_R}{R} = \frac{\Delta V_{\max}}{R} \sin \omega t = I_{\max} \sin \omega t$$

The resistor voltage and current are both in phase with the source emf. There would be a phase difference if $i \propto \sin(\omega t + \phi)$ for some nonzero value of ϕ, as we'll see in the next few examples.

Average Power

In an AC circuit with a resistor, the average current may be zero but the average power delivered to the resistor is certainly nonzero. Since $P = i^2 R$, and $i(t)$ is a sinusoidal signal,

$$P_{\text{avg}} = \frac{1}{T}\int_0^T i^2 R \, dt = \frac{1}{2} I_{\text{max}}^2 R = I_{\text{rms}}^2 R$$

where I_{rms} is the rms current. Hence, rms values provide a meaningful way to compare AC to DC in terms of average power delivery:

$$P_{\text{avg}} = I_{\text{rms}}^2 R = \frac{\Delta V_{\text{rms}}^2}{R} = I_{\text{rms}} \Delta V_{\text{rms}}$$

Inductor in an AC Circuit

Now consider an inductor in an AC circuit. By Kirchhoff's loop rule (modified to be consistent with Faraday's law),

$$\Delta v_L = \Delta v \Rightarrow L\frac{di}{dt} = \Delta V_{\max}\sin\omega t$$

Integrate to find the current, and set the constant of integration to zero since there is no steady current in the circuit:

$$i = -\frac{\Delta V_{\max}}{\omega L}\cos\omega t = \frac{\Delta V_{\max}}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right)$$

The current lags the source emf with a 90° phase difference.

Inductive Reactance

The maximum current through an inductor in an AC circuit is

$$I_{\max} = \frac{\Delta V_{\max}}{\omega L} = \frac{\Delta V_{\max}}{X_L}$$

The quantity $X_L \equiv \omega L$, measured in ohms, is the inductive reactance. Inductors are used in some circuit applications, such as power supplies and radio-interference filters, to block high frequencies while permitting lower frequencies or DC to pass through. A circuit device that uses an inductor for this purpose is called a *low-pass filter*.

Capacitor in an AC Circuit

Consider a capacitor in an AC circuit. By Kirchhoff's loop rule, the definition of capacitance, and charge conservation,

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(C\Delta V_{\max}\sin\omega t\right) = \omega C\Delta V_{\max}\cos\omega t$$

Using trigonometric properties and rearranging the expression,

$$i = \frac{\Delta V_{\max}}{1/\omega C}\sin\left(\omega t + \frac{\pi}{2}\right)$$

The current leads the source emf with a 90° phase difference.

Capacitive Reactance

The maximum current through a capacitor in an AC circuit is

$$I_{\mathrm{max}} = \frac{\Delta V_{\mathrm{max}}}{1/\omega C} = \frac{\Delta V_{\mathrm{max}}}{X_C}$$

The quantity $X_C \equiv 1/\omega C$ is the capacitive reactance (measured in ohms). Capacitors tend to pass high-frequency current and block low-frequency currents and DC (just the opposite of inductors). A device that preferentially passes signals of high frequency is called a *high-pass filter*.

Pitfall Prevention

Resistance R relates the instantaneous voltage to the instantaneous current. Reactances X_L and X_C only relate the amplitudes of voltage and current (not the instantaneous values).

$$i(t) = \frac{\Delta v(t)}{R}, \quad I_{\max} = \frac{\Delta V_{\max}}{X_L}, \quad \text{or} \quad I_{\max} = \frac{\Delta V_{\max}}{X_C}$$

$$X_L \equiv \omega L \quad \text{and} \quad X_C \equiv 1/\omega C$$

$$[R] = [X_L] = [X_C] = \Omega$$

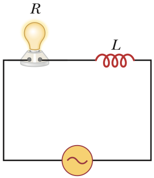
Conceptual Question 3

Consider the AC circuit in the figure. The frequency of the AC source is adjusted while its voltage amplitude is held constant. When does the lightbulb glow the brightest?

(a) It glows brightest at high frequencies.

(b) It glows brightest at low frequencies.

(c) The brightness is the same at all frequencies.



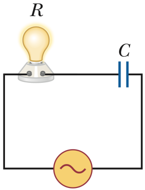
Conceptual Question 4

Consider the AC circuit in the figure. The frequency of the AC source is adjusted while its voltage amplitude is held constant. When does the lightbulb glow the brightest?

(a) It glows brightest at high frequencies.

(b) It glows brightest at low frequencies.

(c) The brightness is the same at all frequencies.



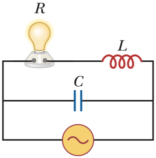
Conceptual Question 5

Consider the AC circuit in the figure. The frequency of the AC source is adjusted while its voltage amplitude is held constant. When does the lightbulb glow the brightest?

(a) It glows brightest at high frequencies.

(b) It glows brightest at low frequencies.

(c) The brightness is the same at all frequencies.



Series RLC Circuit

In an series RLC (AC) circuit, the current must be the same through all components (by conservation of charge). By Kirchhoff's rule,

$$\Delta v - \Delta v_R - \Delta v_L - \Delta v_C = 0$$

Let $i(t) = I_{\max} \sin \omega t$ and $\Delta v = \Delta V_{\max} \sin(\omega t + \phi)$ where ϕ is some phase difference that must be determined. In a steady-state solution, we can write down the voltages across each component:

$$\begin{aligned}
\Delta v_R &= I_{\max} R \sin \omega t && \text{(in phase)} \\
\Delta v_L &= I_{\max} X_L \cos \omega t && \text{(leads current by } 90^\circ\text{)} \\
\Delta v_C &= -I_{\max} X_C \cos \omega t && \text{(lags current by } 90^\circ\text{)}
\end{aligned}$$

From Kirchhoff's rule,

$$\Delta v = I_{\max}\left[R\sin\omega t + (X_L - X_C)\cos\omega t\right]$$

Using properties of trigonometric functions,

$$\begin{aligned}\Delta v &= \Delta V_{\max}\sin(\omega t + \phi)\\ &= \Delta V_{\max}\left[\cos\phi\sin\omega t + \sin\phi\cos\omega t\right]\end{aligned}$$

In order for both expressions to be consistent, it must be the case that

$$\Delta V_{\max}\cos\phi = I_{\max}R \quad \text{and} \quad \Delta V_{\max}\sin\phi = I_{\max}(X_L - X_C).$$

We can solve these equations simultaneously for ϕ and $I_{\max}$.

Phase Angle and Impedance

The phase angle is the phase difference between Δv and i given by

$$\phi = \arctan\left[\frac{X_L - X_C}{R}\right]$$

The amplitudes of voltage and current are related by a quantity called the impedance Z which depends on the resistance and reactances:

$$\Delta V_{\max}^2 = \left[R^2 + (X_L - X_C)^2\right] I_{\max}^2$$

$$I_{\max} = \frac{\Delta V_{\max}}{Z}, \quad \text{where} \quad Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

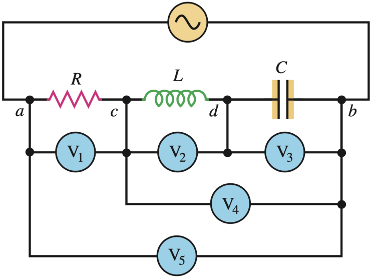
Example 2: RLC Series Circuit

Five ideal voltmeters, calibrated to read rms values, are connected as shown in the figure below. Let $R = 200\ \Omega$, $L = 0.400\ \mathrm{H}$, and $C = 6.00\ \mu\mathrm{F}$, and $\Delta V = 30.0\ \mathrm{V}$ (reading on V_5).

Find the reading of each voltmeter when

(a) $\omega = 200\ \mathrm{rad/s}$

(b) $\omega = 1000\ \mathrm{rad/s}$



Conceptual Question 6

A sinusoidal voltage source is connected in series to a resistor, an inductor, and a capacitor. Compared to the average power when only the resistor is present, the average power of the series RLC circuit is...

(a) always larger

(b) always smaller

(c) exactly the same

(d) none of the above

Resonance in a Series RLC Circuit

The average power in a series RLC circuit depends on the driving frequency of the sinusoidal voltage source:

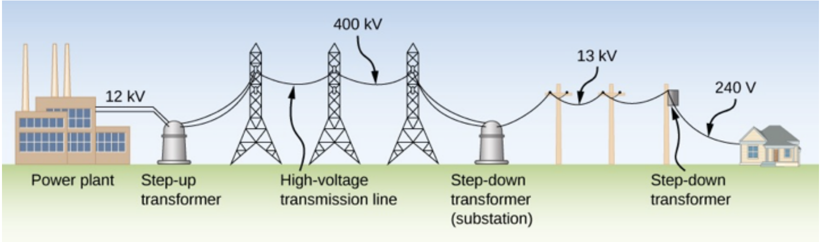
$$P_{\mathrm{avg}} = \frac{(\Delta V_{\mathrm{rms}})^2 R}{R^2 + (X_L - X_C)^2}$$

Near resonance, an oscillating system will exhibit maximum response when driven at its natural frequency. The average power in a series RLC circuit is maximized when $X_L = X_C$, which occurs when

$$\omega = \omega_0 \equiv \sqrt{\frac{1}{LC}} \quad \text{(resonance frequency)}$$

Power Transmission

The primary advantage of AC is that it is easier to step voltage levels up and down. For long-distance power transmission it is desirable to use as high a voltage and as small a current as possible; this reduces i^2R losses in the transmission lines. Present-day transmission lines operate at rms voltages of the order of 500 kV. The necessary voltage conversions are accomplished by using transformers.



Transformers

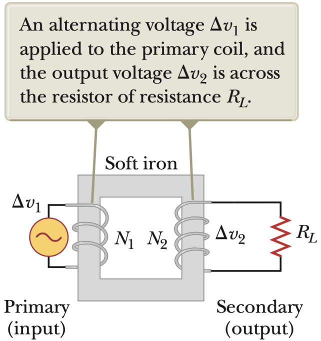
A transformer usually consists of two coils of wire wound around a core of iron. Assuming the magnetic field lines remain in the iron core, the flux through each turn of the primary equals the flux through each turn of the secondary. By Faraday's law,

$$\Delta v_1 = -N_1 \frac{d\Phi_B}{dt}, \quad \Delta v_2 = -N_2 \frac{d\Phi_B}{dt} \quad \Rightarrow \quad \Delta v_2 = \frac{N_2}{N_1} \Delta v_1$$

The voltage conversion depends on the number of loops in the coils.

(i) In a step-up transformer, $N_2 > N_1$.

(ii) In a step-down transformer, $N_2 < N_1$.



An alternating voltage Δv_1 is applied to the primary coil, and the output voltage Δv_2 is across the resistor of resistance R_L.
Soft iron
Δv_1
N_1
N_2
Δv_2
R_L
Primary (input)
Secondary (output)

Transformers

In an ideal transformer, the power supplied to the primary coil is equal to the power in the secondary:

$$I_1 \Delta V_1 = I_2 \Delta V_2 \quad \Rightarrow \quad I_2 = \frac{N_1}{N_2} I_1 \quad \text{(using rms values)}$$

In a step-up transformer, the voltage steps up and the current steps down (limiting losses in a transmission line). In a step-down transformer, the voltage steps down and the current steps up.

Typical transformers, while not 100% efficient, do have pretty high power efficiencies from 90% to 99%.

Circuit diagram for a transformer: