Physics 438A – Lecture #4
Rotations & Angular Momentum

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1 Rotations in 3D Space

Rotations play a central role in quantum mechanics and serve as a gentle introduction to the *group-theoretic framework* that underlies modern physics. Group theory provides the mathematical language for describing symmetries in a precise way. Informally, a group is a set of transformations that can be combined and inverted, capturing the structure of symmetry operations. By studying the group of rotations, we will uncover an underlying algebraic structure that allows us to derive quantum states systematically, understand their behavior under rotations, and describe phenomena such as spin.

We begin by studying the group of *proper rotations*, denoted SO(3). Given a unit vector $\hat{\mathbf{u}} = (u_x, u_y, u_z)$, a proper rotation R is a linear map from $\hat{\mathbf{u}}$ to another unit vector

$$\hat{\mathbf{u}}' = R\hat{\mathbf{u}}$$

and it preserves the orientation of a right-handed triplet. Namely,

$$R\hat{\mathbf{z}} \cdot (R\hat{\mathbf{x}} \times R\hat{\mathbf{y}}) = \det(R) \det \begin{bmatrix} \hat{\mathbf{z}} & \hat{\mathbf{x}} & \hat{\mathbf{y}} \end{bmatrix} = \det(R) = 1$$

Since R is a linear map between unit vectors, it must be orthogonal:

$$\hat{\mathbf{u}}' \cdot \hat{\mathbf{u}}' = (R\hat{\mathbf{u}}) \cdot (R\hat{\mathbf{u}}) = \hat{\mathbf{u}}^T R^T R \hat{\mathbf{u}} = 1 \quad \Rightarrow \quad R^T R = \mathbb{1}$$

Hence, the elements of SO(3) are represented by matrices with the following properties:

$$R^T R = R R^T = \mathbb{1} \quad \text{and} \quad \det(R) = 1 \tag{1.1}$$

1.1 Rotations About a Fixed Axis

Consider a rotation about some axis $\hat{\mathbf{n}}$ by an angle θ in which the sense is determined by the right hand rule. Denote this rotation by $R(\hat{\mathbf{n}}, \theta)$. Rotations about the same axis commute

and the angles add under matrix multiplication:

$$R(\hat{\mathbf{n}}, \theta_1)R(\hat{\mathbf{n}}, \theta_2) = R(\hat{\mathbf{n}}, \theta_2)R(\hat{\mathbf{n}}, \theta_1) = R(\hat{\mathbf{n}}, \theta_1 + \theta_2)$$

Consecutive rotations about different axes do not commute; the group of rotations is called "non-abelian" for this reason. The rotations about the three coordinate axes are

$$R(\hat{\mathbf{x}}, \theta) = R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{1.2}$$

$$R(\hat{\mathbf{y}}, \theta) = R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{1.3}$$

$$R(\hat{\mathbf{z}}, \theta) = R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4}$$

and you can work out for yourself simple examples where $R_z(\theta_1)R_x(\theta_2) \neq R_z(\theta_2)R_x(\theta_1)$ for instance. I'll probably ask you to perform rotations on your phone in class...

One can show that any proper rotation can be represented as $R(\hat{\mathbf{n}}, \theta)$, for some axis $\hat{\mathbf{n}}$ and some angle θ. Thus, there is no loss of generality in writing a proper rotation in this form. We will call this the *axis-angle parameterization* of the rotations.

1.2 Infinitesimal Rotations

Consider an infinitesimal rotation of the vector $\mathbf{u}$ about the axis $\hat{\mathbf{n}}$ by a small angle $\delta\theta$ (see Figure 1.1 for a geometrical argument):

$$R(\hat{\mathbf{n}}, \delta\theta)\mathbf{u} = \mathbf{u} + \delta\theta\hat{\mathbf{n}} \times \mathbf{u} + \mathcal{O}(\delta\theta^2) \tag{1.5}$$

The cross product can be written as a matrix equation using the following identity:

$$\mathbf{a} \times \mathbf{u} = A\mathbf{u} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \tag{1.6}$$

The matrix A is antisymmetric ($A^T = -A$) and admits the following decomposition:

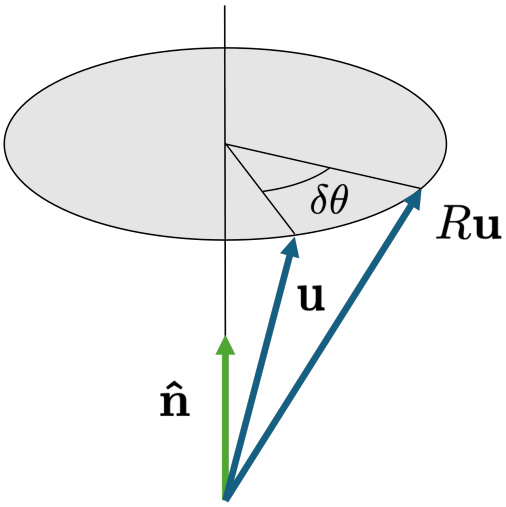

Figure 1.1: Infinitesimal rotation of the vector $\mathbf{u}$ about the axis specified by $\hat{\mathbf{n}}$. The component of $\mathbf{u}$ parallel to $\hat{\mathbf{n}}$ remains constant while the perpendicular component rotates in a positive sense (given by the right hand rule) around the axis defined by $\hat{\mathbf{n}}$.

$$A = \mathbf{a} \cdot \boldsymbol{\mathcal{J}} = a_x \mathcal{J}_x + a_y \mathcal{J}_y + a_z \mathcal{J}_z \tag{1.7}$$

where $\boldsymbol{\mathcal{J}} = (\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z)$ is a "vector" of the matrices

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.8}$$

The matrices $\mathcal{J}_x, \mathcal{J}_y$ and $\mathcal{J}_z$ encode the geometric structure of rotations (and ultimately the structure of 3D Euclidean space). The non-commutativity of rotations is fully captured by the commutators, which you can verify by direct calculation:

$$[\mathcal{J}_x, \mathcal{J}_y] = \mathcal{J}_z, \quad [\mathcal{J}_z, \mathcal{J}_x] = \mathcal{J}_y, \quad [\mathcal{J}_y, \mathcal{J}_z] = \mathcal{J}_x \tag{1.9}$$

Any infinitesimal rotation can be written as

$$R(\hat{\mathbf{n}}, \delta\theta) = 1 + \delta\theta \hat{\mathbf{n}} \cdot \boldsymbol{\mathcal{J}} \tag{1.10}$$

and it follows that any finite rotation can then be written as the matrix exponential

$$R(\hat{\mathbf{n}}, \theta) = \exp\left(\theta \hat{\mathbf{n}} \cdot \boldsymbol{\mathcal{J}}\right) \tag{1.11}$$

It turns out that knowledge of the commutation relations in Eq. (1.9) is enough to compute the multiplication of any two finite rotations in terms of axis-angle parameters.

2 Rotations in Quantum Mechanics

Our goal for this section is to find operators that act on quantum states and represent rotations at the classical level–rotations in 3D Euclidean space. We saw in Lecture #3 there was a sense in which a spin-1/2 system precesses in a magnetic field. Specifically, the expectation value of the spin vector rotates over time around the direction of the magnetic field. In the context of spin precession, a unitary operator evolves the state over time in direct correspondence with a rotating classical vector $\mathbf{S}$. We postulate for each proper rotation matrix R in 3D Euclidean space, there corresponds a unitary operator $U(R)$ acting on state vectors. The specific form of $U(R)$ depends on the system, but the essential properties of quantum rotations follow from the properties of classical rotations.

2.1 Angular Momentum

Recall that Eq. (1.11) for a classical rotation depends on the product $\theta\hat{\mathbf{n}}$. Define $\boldsymbol{\theta} = \theta\hat{\mathbf{n}}$ such that for an infinitesimal rotation

$$R(\boldsymbol{\theta}) = \mathbb{1} + \boldsymbol{\theta} \cdot \boldsymbol{\mathcal{J}} + \cdots \tag{2.1}$$

which implies

$$\mathcal{J}_x = \left.\frac{\partial R(\boldsymbol{\theta})}{\partial \theta_x}\right|_{\boldsymbol{\theta}=0}, \quad \mathcal{J}_y = \left.\frac{\partial R(\boldsymbol{\theta})}{\partial \theta_y}\right|_{\boldsymbol{\theta}=0}, \quad \mathcal{J}_z = \left.\frac{\partial R(\boldsymbol{\theta})}{\partial \theta_z}\right|_{\boldsymbol{\theta}=0} \tag{2.2}$$

The unitary operator that corresponds to $R(\boldsymbol{\theta})$ is ultimately a function of $\boldsymbol{\theta}$, and for an infinitesimal rotation, we can approximate $U(\boldsymbol{\theta})$ by the leading terms of its Taylor series

$$U(\boldsymbol{\theta}) = \mathbb{1} - \frac{i}{\hbar}\boldsymbol{\theta} \cdot \mathbf{J} + \cdots \tag{2.3}$$

where (by analogy with the classical case)

$$J_x = i\hbar \left.\frac{\partial U(\boldsymbol{\theta})}{\partial \theta_x}\right|_{\boldsymbol{\theta}=0}, \quad J_y = i\hbar \left.\frac{\partial U(\boldsymbol{\theta})}{\partial \theta_y}\right|_{\boldsymbol{\theta}=0}, \quad J_z = i\hbar \left.\frac{\partial U(\boldsymbol{\theta})}{\partial \theta_z}\right|_{\boldsymbol{\theta}=0} \tag{2.4}$$

Formally, components of $\mathbf{J}$ are the *generators of rotation* in quantum mechanics. When a classical system is invariant under rotation about some axis, the angular momentum about

that axis is conserved. Likewise, when the Hamiltonian for a quantum system commutes with the generator of rotation about some axis, the expectation value of the generator is conserved. We define the **angular momentum** of a quantum system as the vector operator $\mathbf{J} = (J_x, J_y, J_z)$. The factor $\hbar$ ensures (i) $\mathbf{J}$ has the units of angular momentum and (ii) $\mathbf{J}$ corresponds to classical angular momentum in the appropriate limit. The factor of i ensures the generators are self-adjoint, i.e. $J_x^\dagger = J_x$, etc. Any finite rotation corresponds to the unitary operator

$$U(\boldsymbol{\theta}) = \exp\left(-\frac{i}{\hbar}\boldsymbol{\theta}\cdot\mathbf{J}\right) = \exp\left(-\frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\mathbf{J}\right) \tag{2.5}$$

Consider a unitary transformation $U(\hat{\mathbf{y}}, \delta\theta)$ corresponding to a classical rotation about the y axis by a small angle. On a quantum system in the state $|\psi\rangle$,

$$|\psi'\rangle = U|\psi\rangle$$

The expectation values of $\mathbf{J}$ in the states $|\psi\rangle$ and $|\psi'\rangle$ should be related by the classical rotation $R(\hat{\mathbf{y}}, \delta\theta)$. In other words,

$$\langle\psi'|\mathbf{J}|\psi'\rangle = \langle\psi|U^\dagger\mathbf{J}U|\psi\rangle = R\langle\psi|\mathbf{J}|\psi\rangle \tag{2.6}$$

This must hold for all $|\psi\rangle$, so it follows that

$$U^\dagger(\hat{\mathbf{y}}, \delta\theta)\mathbf{J}U(\hat{\mathbf{y}}, \delta\theta) = R(\hat{\mathbf{y}}, \delta\theta)\mathbf{J} \tag{2.7}$$

In fact, Eq. (2.7) should hold for finite rotations around any axis and for any *vector observable*, i.e. one that corresponds to an ordinary vector in 3D Euclidean space. It might be easier to understand Eq. (2.7) component-wise, where the left-hand side is a product of operators and the right-hand side is the operator resulting from the rotation of $\mathbf{J} = (J_x, J_y, J_z)$,

$$U^\dagger J_i U = \sum_{j=1}^{3} R_{ij} J_j$$

We can also write Eq. (2.7) in matrix form as long as we keep in mind that each component of $\mathbf{J}$ is itself an operator on the vector space of state vectors:

$$\begin{pmatrix} U^\dagger J_x U \\ U^\dagger J_y U \\ U^\dagger J_z U \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}$$

Taylor expand each operator to leading order in $\delta\theta$ and collect terms:

$$\left(1+\frac{i}{\hbar}\delta\theta J_y\right)\mathbf{J}\left(1-\frac{i}{\hbar}\delta\theta J_y\right)=\mathbf{J}+\delta\theta\hat{\mathbf{y}}\times\mathbf{J}$$

$$\mathbf{J}-\frac{i}{\hbar}\delta\theta\begin{pmatrix}[J_x,J_y]\\0\\[J_z,J_y]\end{pmatrix}=\mathbf{J}+\delta\theta\begin{pmatrix}J_z\\0\\-J_x\end{pmatrix}$$

From this, we immediately get two of the three commutators, and it follows under cyclic permutation of coordinates $x\to y\to z\to x$ that

$$[J_x,J_y]=i\hbar J_z,\quad [J_z,J_x]=i\hbar J_y,\quad [J_y,J_z]=i\hbar J_x \tag{2.8}$$

We have achieved something rather remarkable; starting only from the geometric properties of rotations in ordinary three-dimensional space and the requirement that quantum expectation values transform in the same way as classical vectors under these rotations, we have deduced–without ever referring to any specific physical system–that the generators of rotations in quantum mechanics must satisfy the commutation relations in Eq. (2.8).

Example 2.1: Rotations in Quantum Mechanics

Consider a spin-1/2 system. Verify Eq. (2.7) for rotation around the z axis by an angle φ by explicit computation. Use the matrix representations in the S_z basis.

Solution: According to Eq. (2.7),

$$U^\dagger(\hat{\mathbf{z}},\varphi)\mathbf{S}U(\hat{\mathbf{z}},\varphi)=R(\hat{\mathbf{z}},\varphi)\mathbf{S}$$

$$\begin{pmatrix}\exp(i\varphi S_z/\hbar)S_x\exp(-i\varphi S_z/\hbar)\\ \exp(i\varphi S_z/\hbar)S_y\exp(-i\varphi S_z/\hbar)\\ \exp(i\varphi S_z/\hbar)S_z\exp(-i\varphi S_z/\hbar)\end{pmatrix}=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}S_x\\S_y\\S_z\end{pmatrix}$$

Since S_z commutes with any analytic function of itself, the z component is trivially satisfied. Let's check the x component by explicit calculation. On the left-hand side,

$$\exp(i\varphi S_z/\hbar)S_x\exp(-i\varphi S_z/\hbar)=\begin{pmatrix}e^{i\varphi/2} & 0\\ 0 & e^{-i\varphi/2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\varphi/2} & 0\\ 0 & e^{i\varphi/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}$$

And on the right-hand side,

$$S_x \cos\varphi - S_y \sin\varphi = \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\varphi \\ \cos\varphi & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sin\varphi \\ i\sin\varphi & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}$$

As for the y component, work out the calculations for yourself and verify that both sides equal

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix}$$

It's good to see that the expression works when we're given the observables for a physical system, but ultimately Eq. (2.7) is a foundational relationship that connects unitary rotation operators to their corresponding rotations in 3D Euclidean space. In the next section, we'll see how the observables corresponding to the components of **S** in a spin-1/2 system (and others) naturally arise from the quantum theory of rotations.

3 Representations of Angular Momentum

We will now explore the consequences of the angular momentum commutation relations for the construction of angular momentum eigenstates. Consider the operator

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{3.1}$$

and note that $[J^2, J_x] = [J^2, J_y] = [J^2, J_z] = 0$. Since J^2 and **J** commute, we can construct simultaneous eigenstates of J^2 and any one of the components of **J**. However, since these components do not commute with each other, we cannot find simultaneous eigenstates of more than one component of **J**. By convention we look for simultaneous eigenstates of J^2 and J_z. Every basis vector is labeled by two numbers (the eigenvalues)

$$J^2|\alpha,\beta\rangle = \hbar^2\alpha|\alpha,\beta\rangle \tag{3.2}$$

$$J_z|\alpha,\beta\rangle = \hbar\beta|\alpha,\beta\rangle \tag{3.3}$$

We'll assume without loss of generality that the eigenstates are orthonormal. Note that

$$J^2 - J_z^2 = J_x^2 + J_y^2 \geq 0$$

since J_x and J_y are self-adjoint. It follows that

$$\alpha \geq \beta^2$$

Since $0 \leq \alpha < \infty$, β is bounded above and below for a given value of α. In other words,

$$-\sqrt{\alpha} \leq \beta \leq \sqrt{\alpha}$$

Therefore, β has an upper bound $\beta_{\max}$ and a lower bound $\beta_{\min}$.

Whenever eigenvalues are bounded, it is useful to search for *raising* and *lowering* operators that can be used to construct eigenvectors by repeated application. The only generators that can map an eigenstate to a different vector are J_x and J_y, and after careful consideration, we define the following raising and lowering operators (ladder operators)

$$J_+ = J_x + iJ_y, \quad J_- = J_x - iJ_y \tag{3.4}$$

I realize this is the least motivated step in the whole process, but it really is the key that unlocks the eigenstates. This process of considering a complex linear combination, instead of the original generators, is called a *complexification*, and it's a common technique. The commutation relations for the ladder operators are

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [J_+, J_-] = 2\hbar J_z \tag{3.5}$$

Notice that complexification breaks the cyclical symmetry of the original commutation relations. Not only that, but we can use the commutation relations to justify the claim that $J_\pm$ are the raising and lowering operators. For starters, $J_\pm$ does not alter the eigenvalue of J^2 since $[J^2, J_\pm] = 0$.

$$J^2 \left(J_\pm|\alpha, \beta\rangle\right) = J_\pm J^2|\alpha, \beta\rangle = \hbar^2\alpha \left(J_\pm|\alpha, \beta\rangle\right)$$

From the commutator of L_z and $L_\pm$, it follows that

$$\begin{aligned} J_z\left(J_\pm|\alpha,\beta\rangle\right) &= J_\pm J_z|\alpha,\beta\rangle + [J_z, J_\pm]|\alpha,\beta\rangle \\ &= \hbar\beta J_\pm|\alpha,\beta\rangle \pm \hbar J_\pm|\alpha,\beta\rangle \\ &= \hbar(\beta\pm 1)\left(J_\pm|\alpha,\beta\rangle\right) \end{aligned}$$

Hence, $J_\pm|\alpha,\beta\rangle$ is proportional to $|\alpha,\beta\pm 1\rangle$. We'll find the constant of proportionality later by requiring that $|\alpha,\beta\rangle$ is an orthonormal set. Since β is bounded above and below,

$$J_+|\alpha,\beta_{\max}\rangle = 0 \quad \text{and} \quad J_-|\alpha,\beta_{\min}\rangle = 0 \tag{3.6}$$

Furthermore,

$$\begin{aligned} 0 = J_-J_+|\alpha,\beta_{\max}\rangle &= \left(J_x^2 + J_y^2 - \hbar J_z\right)|\alpha,\beta_{\max}\rangle \\ &= \left(J^2 - J_z^2 - \hbar J_z\right)|\alpha,\beta_{\max}\rangle \\ &= \left(\hbar^2\alpha - \hbar^2\beta_{\max}^2 - \hbar^2\beta_{\max}\right)|\alpha,\beta_{\max}\rangle \\ \alpha &= \beta_{\max}\left(\beta_{\max}+1\right) \end{aligned}$$

Similarly, using $J_-|\alpha,\beta_{\min}\rangle = 0$ and $J_+J_- = J^2 - J_z^2 - \hbar J_z$, we find

$$\alpha = \beta_{\min}\left(\beta_{\min}-1\right)$$

We now introduce the standard notation $j = \beta_{\max}$ and $\beta = m$ and relabel the eigenstates

$$J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \quad \text{and} \quad J_z|j,m\rangle = \hbar m|j,m\rangle$$

where

$$-j \le m \le j \tag{3.7}$$

Starting from the vector $|j,j\rangle$, we obtain the vector $|j,-j\rangle$ by repeated application of J_-. After each application, the eigenvalue is lowered by one, so the difference between the highest and lowest eigenvalue must be an integer n. It follows that

$$j - (-j) = 2j = n \quad \rightarrow \quad j = \frac{n}{2} \tag{3.8}$$

Hence, the only values of j allowed are non-negative integers or half-integers:

$$j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \ldots$$

When we say that j belongs to the indicated set of values (integers and half-integers), we mean that the commutation relations alone tell us that j can take on only these values. They do not tell us which of these values actually occur in a specific application.

Let's look at the eigenvalues for the first few representations:

$$\begin{array}{lll} j = 0 & j(j+1) = 0 & m = 0 \\ j = 1/2 & j(j+1) = 3/4 & m = -1/2, 1/2 \\ j = 1 & j(j+1) = 2 & m = -1, 0, 1 \\ \ldots & & \end{array}$$

The dimension of the vector space is $2j+1$ for a given representation, and we can construct the eigenvectors of J_z using the ladder operators (the raising the lowering operators). Recall

$$J_\pm |j, m\rangle = C_\pm |j, m \pm 1\rangle$$

for constants $C_\pm$ that depend on j and m. The norm squared of $J_+|j, m\rangle$ is

$$\begin{aligned} \|J_+|j,m\rangle\|^2 = |C_+|^2 &= \langle j, m| J_+^\dagger J_+ |j, m\rangle \\ &= \langle j, m| J_- J_+ |j, m\rangle \\ &= \langle j, m| \left(J^2 - J_z^2 - \hbar J_z\right) |j, m\rangle \\ &= \hbar^2 \left(j(j+1) - m^2 - m\right) \end{aligned}$$

A similar calculation gives $|C_-|^2$. There is ambiguity in the overall phase factor, so we choose

$$C_\pm = \hbar\sqrt{j(j+1) - m(m \pm 1)} \tag{3.9}$$

Given an eigenstate $|j, m\rangle$, a ladder operator will produce a higher or lower eigenstate:

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle \tag{3.10}$$

3.1 Spin 1/2 and SU(2)

The smallest non-trivial value of j is 1/2. The generator J_z has eigenvalues $\hbar/2$ and $-\hbar/2$, and because J_z is conventionally chosen to be diagonal,

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the corresponding eigenvectors are

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.11}$$

As for the ladder operators,

$$J_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle \quad \text{and} \quad J_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

and the corresponding matrix representations are

$$J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.12}$$

From $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$, we can invert the expressions to find

$$J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

These are the spin-1/2 angular momentum operators we inferred from experiments earlier in the course! Keep in mind there is no guarantee that any physical system will align with this mathematical description. As it turns out, if you combine quantum mechanics, special relativity, and rotational invariance, *spin is unavoidable*. You would learn more about this in a course on quantum field theory.

There's a topic I've been putting off for a while, but now seems like a good time to discuss the group SU(2), the group of special unitary 2×2 complex matrices. In the same way that complex numbers can be used to represent 2D rotations (rotation around a single, fixed axis), there is a way to use 2×2 special unitary matrices to represent rotations in 3D

Euclidean space. I'd like to show you how that works.

The unitary rotation operators in a spin-1/2 system belong to SU(2). They are 2×2 complex, unitary matrices, and they are "special" because $\det U = 1$. In the axis-angle parameterization, any unitary rotation operator in a spin-1/2 system can be written as

$$U(\hat{\mathbf{n}}, \theta) = \exp\left(-\frac{i}{\hbar}\theta\hat{\mathbf{n}} \cdot \mathbf{S}\right) = \exp\left(-\frac{i}{2}\theta\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}\right)$$

where we use $\mathbf{S}$ to denote angular momentum, and $\boldsymbol{\sigma}$ denotes the **Pauli matrices**:

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \quad \text{where} \quad S_x = \frac{\hbar}{2}\sigma_x, \quad \text{etc.}$$

Any 2×2 Hermitian matrix can be written as a linear combination of the identity operator and the Pauli matrices. To see why, consider an arbitrary 2×2 Hermitian matrix

$$A = \begin{pmatrix} a_0 + a_z & a_x + ia_y \\ a_x - ia_y & a_0 - a_z \end{pmatrix}$$

where c and the components of $\mathbf{a} = (a_x, a_y, a_z)$ are real numbers. It follows that

$$A = a_0\mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma}$$

This result suggests that any 2×2 Hermitian matrix encodes four parameters, and three of them correspond to a vector in 3D Euclidean space.

Just as in the Heisenberg picture, we regard state vectors to be stationary while the operators evolve unitarily over time, we can do the same thing for rotations. Start by considering the expectation value of A under unitary rotation of the state $|\psi\rangle$:

$$\langle\psi'|A|\psi'\rangle = \langle\psi|U^\dagger AU|\psi\rangle$$

The unitary transformation of A is then

$$A' = U^\dagger AU$$

which has the same structure as a time-evolving Heisenberg operator. It follows that

$$A' = U^\dagger(a_0\mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma})U = a_0\mathbb{1} + U^\dagger(\mathbf{a} \cdot \boldsymbol{\sigma})U$$

Using Eq. (2.7), we can expand the second term as

$$U^\dagger(\mathbf{a}\cdot\boldsymbol{\sigma})U = \sum_i a_i U^\dagger \sigma_i U = \sum_i a_i \sum_j R_{ij}\sigma_j$$

Note that $R_{ji} = R^T_{ji}$ such that

$$U^\dagger(\mathbf{a}\cdot\boldsymbol{\sigma})U = \sum_j \left(\sum_i R^T_{ji} a_i\right)\sigma_j = \left(R^T\mathbf{a}\right)\cdot\boldsymbol{\sigma}$$

Altogether, we find

$$A' = U^\dagger A U = a_0\mathbb{1} + \left(R^T\mathbf{a}\right)\cdot\boldsymbol{\sigma}$$

In the *Schrödinger picture of rotations*, the state vectors unitarily rotate in correspondence with the classical rotation R of a Bloch vector. In the *Heisenberg picture of rotations*, state vectors are fixed and the operators are unitarily rotating in the opposite sense since $R^T = R^{-1}$. This is certainly an interesting result, but let's revisit that second term. Let $R^T \to R$ and note that $U(R^T) = U^\dagger(R)$. Hence,

$$U^\dagger(R)(\mathbf{a}\cdot\boldsymbol{\sigma})U(R) = (R\mathbf{a})\cdot\boldsymbol{\sigma}$$

This result, known as the adjoint formula for rotations, constitutes a mapping from elements of SU(2) to elements of SO(3). Notice the mapping is two-to-one because both $U(R)$ and $-U(R)$ map to the same rotation matrix R.

3.2 Spin 1

In the case $j = 1$,

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}, \quad J_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \tag{3.13}$$

and the basis vectors are

$$|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{3.14}$$

Example 3.1: Stern-Gerlach experiment with spin-1 particles

Consider a Stern-Gerlach experiment with a neutral beam of spin-1 particles. What fraction of particles with $S_y = \hbar$ will be found to have $S_z = \hbar$ with the appropriate arrangement of analyzers?

Solution: The quantitative answer to this equation is the square of an inner product:

$$|\langle 1,1|1,1\rangle_y|^2$$

where $|1,1\rangle$ is the eigenvector of S_z with eigenvalue $\hbar$ and $|1,1\rangle_y$ is the eigenvector of S_y with eigenvalue $\hbar$. Find $|1,1\rangle_y$ in the basis of S_z. The eigenvalue equation is

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \hbar\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

From which it follows that

$$-ib = \sqrt{2}a, \quad i(a-c) = \sqrt{2}b, \quad ib = \sqrt{2}c$$

Hence, $b = i\sqrt{2}a$ and $c = -a$. After normalizing the state,

$$|1,1\rangle_y = \frac{1}{2}\begin{pmatrix}1\\ i\sqrt{2}\\ 1\end{pmatrix}$$

and finally,

$$|\langle 1,1|1,1\rangle_y|^2 = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

Furthermore, 1/2 of the particles will have $S_z = 0$ and 1/4 of them will have $S_z = -\hbar$.

Your turn. What fraction of particles with $S_y = 0$ with be found to have $S_z = \hbar, 0$ and $-\hbar$ with the appropriate arrangement of analyzers?