Lorentz Transformations

Physics 153 – Spring 2026

1 Coordinate Transformations

In special relativity, we postulate that the laws of physics take the same form in all inertial frames, in every direction of space (isotropy), and at all locations and times (homogeneity). Consider two inertial frames, S and S' in uniform relative motion. Assuming the standard orientation, the origins of both frames coincide at $t = t' = 0$ and the frame S' moves with velocity $\vec{v} = v\,\hat{\imath}$ relative to S. Equivalently, the frame S moves with velocity $-\vec{v} = -v\,\hat{\imath}$ with respect to S'. This is because the x axis in each frame points in the same direction in physical space, and if S sees S' moving "forward," then S' must see S moving "backward."

We'd like to derive a rule for transforming the coordinates from one frame to the other. In the process, we'll discover the Lorentz transformations naturally emerge along with a universal constant c that must be determined experimentally. It happens to be finite and coincides with the speed of light, but that's not necessary for this discussion. We will not assume that there exists a maximum speed limit, or that the speed of light is constant in all inertial frames. Nonetheless, we can derive the Lorentz transformations from the principle of relativity and by carefully defining what we mean by coordinate transformations from one inertial frame to another.

A coordinate transformation takes the coordinates of frame S given by (x, y, z, t) and maps them to the appropriate coordinates of S' given by (x', y', z', t'). There should exist a "do nothing" transformation that maps the coordinates from one frame to themselves. We call this the identity transformation. Coordinate transformation can be done successively, say from $S \to S'$ and then again from $S' \to S''$. Each transformation should have an inverse that "undoes" the transformation to arrive at the original coordinates. A transformation followed by its inverse is the same as doing nothing, which is the identity transformation.

2 Linear Transformations

Since S and S' are both inertial frames of reference, the coordinate transformation must be a linear map. A linear transformation is the only transformation that will preserve constant-velocity motion. Remember, in an inertial frame of reference, a free particle must move with constant velocity. This constitutes a law of physics, and it must be preserved in all inertial frames. Since the relative velocity is along the x (and x') axis, nothing happens to the perpendicular directions so we can omit them for brevity (there's a better argument invoking isotropy in case you're interested). Hence, the most general coordinate transformation from S to S' is

$$x' = \alpha(v)\,x + \beta(v)\,t$$
$$t' = \delta(v)\,x + \gamma(v)\,t$$

where $\alpha(v), \beta(v), \gamma(v)$, and $\delta(v)$ are all functions that depend only on the relative velocity. These functions do not depend on x or t due to homogeneity.

It will be slightly easier to work with the transformation in the form of a matrix equation:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

3 Motion of the Origins

Consider the motion of the origin of S' relative to S. In the S' coordinates, the origin of S' has coordinates $(0, t')$, while in S, the origin of S' has coordinates $(x = vt, t)$. Hence,

$$\begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} vt \\ t \end{pmatrix}$$

Assuming $t > 0$, it follows that $\beta = -\alpha v$. Analogously, considering the motion of the origin of S relative to S', the transformation maps $(0, t) \mapsto (x' = -vt', t')$. Thus,

$$\begin{pmatrix} -vt' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix}$$

From the second equation, $t' = \gamma t$, and from the first, $-vt' = -v\gamma t = \beta t$, or $\beta = -v\gamma$. Combing our results so far, $\alpha = \gamma$ and the transformation matrix has already simplified to

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

4 Reflection of Spatial Coordinates

Consider a reflection of the spatial coordinates such that $x \to -x$ and $x' \to -x'$. Since space is isotropic, the transformation rule should take the same form but with $v \to -v$. Hence,

$$\begin{pmatrix} -x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma(-v) & v\gamma(-v) \\ \delta(-v) & \gamma(-v) \end{pmatrix} \begin{pmatrix} -x \\ t \end{pmatrix}$$

or,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma(-v) & -v\gamma(-v) \\ -\delta(-v) & \gamma(-v) \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Comparing this to the original transformation rule, we've learn that $\gamma(-v) = \gamma(v)$, so $\gamma(v)$ is an even function and $\delta(-v) = -\delta(v)$, so $\delta(v)$ is an odd function of the 1D relative velocity.

5 Inverse Transformation and Identity

There are two ways to go from S' to S. The first is to apply the inverse matrix to K' coordinates:

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\gamma^2 + v\gamma\delta} \begin{pmatrix} \gamma(v) & v\gamma(v) \\ -\delta(v) & \gamma(v) \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

The second is to use the transformation rule from S' to S directly with the understanding that S moves with velocity $-v\hat{\imath}$ relative to S'. This results in the matrix equation

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma(-v) & v\gamma(-v) \\ \delta(-v) & \gamma(-v) \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

The two transformations must match each other (each transformation is unique), so evidently,

$$\gamma^2 + v\gamma\delta = 1$$

This constitutes a constraint relating γ and δ, so in principle, we could write γ in terms of δ and our transformation matrix now depends on just one unknown function.

The transformation must reduce to the identity in the case $v = 0$. Hence, $\gamma(0) = 1$ & $\delta(0) = 0$.

6 Closure

We now invoke the requirement that successive transformations should constitute a single transformation. More precisely, consider transforming S to S' and from S' to a third frame S'' moving with velocity v' relative to S'. The composition of transformations is

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} \gamma(v') & -v'\gamma(v') \\ \delta(v') & \gamma(v') \end{pmatrix} \begin{pmatrix} \gamma(v) & -v\gamma(v) \\ \delta(v) & \gamma(v) \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

The resulting transformation matrix must have the same entries on the diagonal, so

$$\gamma(v')\gamma(v) - v'\gamma(v')\delta(v) = -\,v\gamma(v)\delta(v') + \gamma(v')\gamma(v)$$

$$\frac{\delta(v)}{v\gamma(v)} = \frac{\delta(v')}{v'\gamma(v')}$$

For nonzero v, this combination of functions must be a *universal constant*, one and the same for all inertial frames. Define this constant as

$$\frac{\delta(v)}{v\gamma(v)} = \kappa$$

where $[\kappa] = 1/(\text{speed})^2$. From $\gamma^2 + v\gamma\delta = 1$, it follows that

$$\gamma = \frac{1}{\sqrt{1 + \kappa v^2}}$$

where we chose the positive root since $\gamma(0) = 1$, not -1. From the closure requirement, we can also derive the velocity addition rule, but I won't do that here.

7 Asymmetry of Time and Space

Contrary to what you may have heard, time and space are not on "equal footing" in special relativity. If $\kappa > 0$, then there would be transformations (with $\kappa v^2 \gg 1$) which transform time into a spatial coordinate and vice versa. We exclude this on physical grounds, because time can only run in one direction (chosen to be positive by convention). Hence, $\kappa < 0$ and we define a new constant c such that $\kappa = -1/c^2$. We call c the invariant speed because it has units of speed and takes the same value in all inertial frames of reference.

8 Lorentz Transformation

Given that c has the units of speed and takes the same value in all inertial frames, multiply the time coordinates by c such that the transformation equation becomes

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \frac{1}{\sqrt{1 - v^2/c^2}} \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Only experiment or some empirically adequate law of physics can ultimately answer whether c is finite or infinite. In principle, any experiment between inertial frames would be sufficient.

Define $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$ so the transformation can be written as

$$\boxed{\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}}$$