Phys 152: Fundamentals of Physics II

Unit #2 - Electric Fields & Gauss's Law

Aaron Wirthwein

Department of Physics and Astronomy
University of Southern California

Continuous Charge Distributions

In this unit, we discuss two ways to find the electric field of a continuous charge distribution. One way is to use the principle of superposition, where the summation over point charges will become an integral over a line, surface, or volume.

The second way is to use Gauss's law. Gauss's law is based on the inverse square behavior of the electric force between point charges, and we use it to calculate the electric fields of highly symmetric charge distributions.

Integration is Summation

An integral is a sophisticated way to add an infinite number of infinitesimally small pieces. As a concrete example, consider an object with total charge Q. Divide the object into a large number of small pieces such that

$$Q = \sum_{i=1}^{N} \Delta q_i$$

The integral is the limit $N \to \infty$ such that each $\Delta q_i \to 0$ and

$$Q = \lim_{N \to \infty} \sum_{i=1}^{N} \Delta q_i = \int_{\text{object}} dq.$$

Charge Density

If we divide a charged object into many small pieces, some pieces may have more charge than others. Charge densities are designed to handle situations where the amount of charge in a small region of space can vary throughout an object.

$$dq = \lambda d\ell, \qquad dq = \sigma dA, \qquad dq = \rho dV$$

$$Q = \int_{\text{line}} \lambda d\ell, \quad Q = \int_{\text{surface}} \sigma dA, \quad Q = \int_{\text{volume}} \rho dV$$

Example 1: Nonuniform Line Charge

A straight wire carries a *nonuniform* linear charge density λ that is zero on one end and increases quadratically with distance along the wire. In other words, if the wire lies on the x axis from $x = 0$ to $x = L$ (the total length of the wire), then $\lambda(x) = cx^2$ where c is some constant.

(a) Find the units of c and the total charge Q of the wire.

(b) Determine the total charge of a circular wire in the xy plane with linear charge density $\lambda(\theta) = k \cos \theta$, where θ is an angle above the x axis. The circular wire has radius a and the origin is placed at its center. Around the circle, θ increases from 0 to 2π.

Electric Field of Continuous Charge Distributions

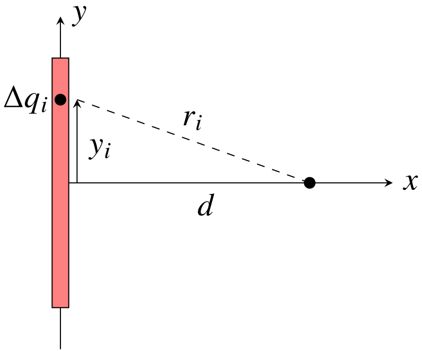
Using the principle of superposition, we can find the electric field of nearly any continuous distribution of charge.

1. Divide the total charge Q into many point-like charges dq. When appropriate, we may also consider small line charges or small surface charges.
2. Find the electric field produced by each dq. For a point charge, we already know the result!
3. Calculate the total electric field by summing up the fields from all infinitesimal contributions. Remember, the electric field is a vector, so it must add like a vector. The summation will become an integral.

The best way to learn is by practicing, so we'll work through many examples. Don't bother to memorizing the results–you'll be tested on your ability to calculate fields in real time.

Example 2: Electric field of a line of charge

Consider a thin, uniformly charged rod of length L and total charge Q. Find the electric field at a perpendicular distance d in the plane that bisects the rod.



Example 3: Electric field of a ring of charge

Consider a thin, uniformly charged ring with radius R and total charge Q. Find the electric field at a point on the axis perpendicular to the plane of the ring and passing through its center. Let the point be a distance d above the center of the ring.

Example 4: Electric field of a disk

Consider a thin, circular disk with radius R and total charge Q spread uniformly over its surface. Find the electric field a distance d above the center of the disk, along an axis perpendicular to the plane of the disk. Hint: use the electric field of a ring of charge and the principle of superposition (slice the disk into rings rather than points).

Example 5: Electric field outside a spherical shell

A thin, spherical shell with radius R carries a uniform charge per unit area σ. Set up, but do not solve, a calculation for the electric field a distance $d > R$ away from the center of the sphere. Hint: use symmetry to simplify the problem and the principle of superposition (slice the spherical shell into rings and set up the integral).

Summary: $\vec{\mathbf{E}}$ of a continuous charge distribution

The total electric field of a continuous charge distribution is

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{\mathbf{r}}$$

where r is the distance from a point charge element dq to a point P and $\hat{\mathbf{r}}$ is a unit vector directed from the element to P.

In a small volume, area, or length element,

$$dq = \rho dV, \qquad dq = \sigma dA, \qquad dq = \lambda d\ell$$

Symmetric Charges

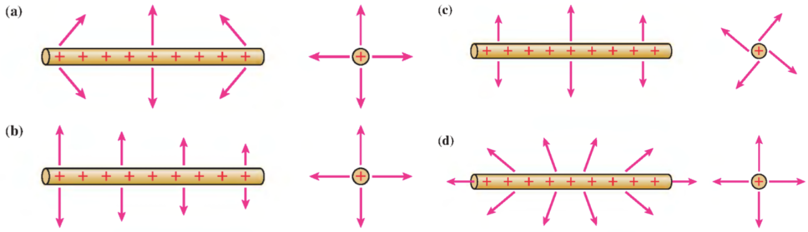
In the examples we've solved, symmetry played an important role in determining the electric fields. We say that a charge distribution is symmetric if there is a group of geometric transformations that don't cause any physical change. Examples of geometric transformations include (i) translations, (ii) rotations, and (iii) reflections.

There are three fundamental symmetries corresponding to one or more of the geometric transformations above: (i) planar, (ii) cylindrical, and (iii) spherical.

The symmetry of the field must match the symmetry of the charge distribution!

Conceptual Question 1

A uniformly charged rod has a finite length. Which field shape or shapes match the symmetry of the rod?



Follow up question: which field shape is the most physically plausible, i.e. consistent with Coulomb's law?

Using Symmetry to Find $\vec{\mathbf{E}}$

In addition to helping us determine the shape of the field, symmetry can also help us determine surfaces over which the electric field strength must be constant. For example, a point charge is spherically symmetric since a rotation around any axis by any amount doesn't change anything about the point charge.

Therefore, the electric field must be spherically symmetric as well, which implies that $\vec{\mathbf{E}}$ can only depend on the radial distance r from the point charge. The electric field must be constant over spherical surfaces centered on the point charge by symmetry arguments alone.

Introduction to Gauss's Law I

Consider a point charge q at the origin. Imagine surrounding the point charge by a spherical surface with radius r. Gauss's law begins with a simple question: if we only know the field on the surface, can we figure out how much charge is inside?

Since the field lines are coming out of the surface, we know that it contains a positive charge, and if we take the product of the field strength and the total surface area of the sphere, we get a quantity proportional to the value of the point charge:

$$E \ A = \frac{q}{4\pi\varepsilon_0 r^2}(4\pi r^2) = \frac{q}{\varepsilon_0}$$

Introduction to Gauss's Law II

Take the spherical surface and divide it into many little patches that are so small that each one is essentially flat and the electric field does not vary significantly over its surface. Each patch has area Δa_i and a unique direction–the outward-pointing normal to its surface $\hat{\mathbf{n}}_i$.

The electric field over each patch points in the same direction as $\hat{\mathbf{n}}_i$ (radially outward). Taking the limit as $N \rightarrow \infty$, the area of each patch approaches zero, and we arrive at a *surface integral*:

$$E\ A = \lim_{N\rightarrow\infty} \sum_{i=1}^{N} \vec{\mathbf{E}} \cdot (\Delta a_i \hat{\mathbf{n}}_i) = \oint \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = \frac{q}{\varepsilon_0}$$

The Concept of Flux

Imagine a wire frame with *area vector* $\Delta\vec{\mathbf{a}}$ dipped into a flowing river characterized by a velocity vector field $\vec{\mathbf{v}}$. The quantity $\vec{\mathbf{v}} \cdot \Delta\vec{\mathbf{a}}$ measures the rate of flow ("fluxus" in Latin) of water through the frame. In other words, it measures the *flux* of water through the frame.

When Michael Faraday introduced the concept of lines of force, he thought of electric fields as something that flowed through space. He viewed electric flux as akin to water or fluid flowing through space, imagining the lines of force radiating from charged objects and permeating a surrounding medium.

Electric Flux

In 1873, James Clerk Maxwell defined the electric flux in a mathematical sense as the integral of an electric field over a surface:

$$\Phi \equiv \int \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}}$$

The following special cases are worth noting:

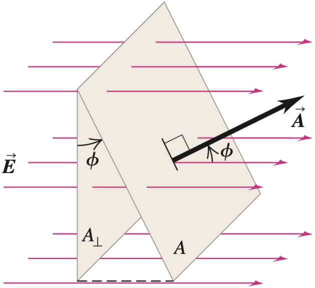
(i) If $\vec{\mathbf{E}}$ is parallel to the surface, $\vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = 0$.

(ii) If $\vec{\mathbf{E}}$ is perpendicular to the surface, $\vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = \pm E\,da$

(iii) Otherwise, at each point on the surface $\vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = E\,da\cos\theta$, where θ is the smallest angle between $\vec{\mathbf{E}}$ and $d\,\vec{\mathbf{a}}$.

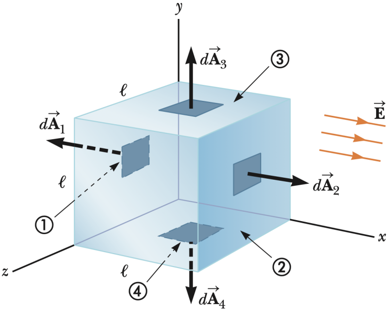
Conceptual Question 2

Which of the surfaces below has greater electric flux, A or $A_\perp$?



Example 6: Flux through a cube

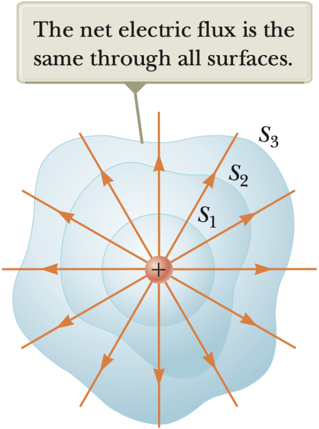
Consider a uniform electric field $\vec{\mathbf{E}}$ oriented in the x direction in empty space. A cube of edge length ℓ is placed in the field, oriented as shown in the figure. Find the net electric flux through the surface of the cube.



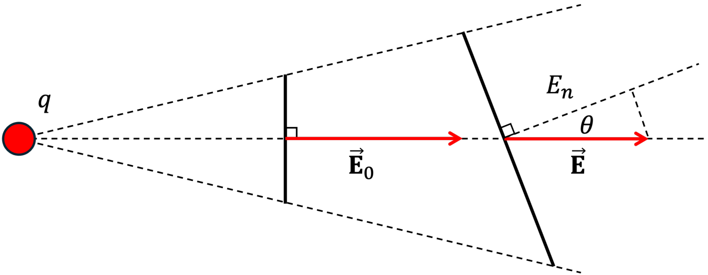
Flux is Independent of Size and Shape

The total flux is independent of both the size and shape of the enclosing surface! Let S_1 be the spherical shape in the figure on the right (we know its flux).

Connect each patch of S_2 with a patch of S_1 intersecting the same radial line. We can show that the flux through each patch of S_2 is the same as that of its corresponding patch on S_1. Hence, the total flux must be the same.



Flux is Independent of Size and Shape

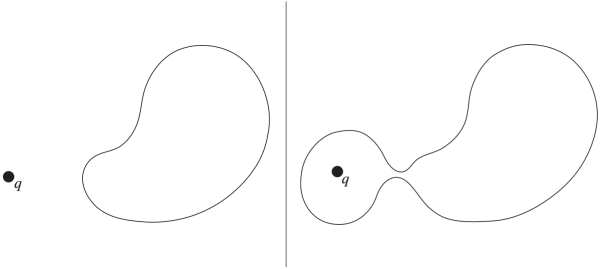


$$\Phi = \vec{\mathbf{E}} \cdot (A\,\hat{\mathbf{n}}) = EA\cos\theta = E_0\left(\frac{r_0^2}{r^2}\right)\left(A_0\frac{r^2}{r_0^2}\frac{1}{\cos\theta}\right)\cos\theta = E_0A_0$$

Area increases with a factor of $(r/r_0)^2$ due to the larger size and $1/\cos\theta$ due to the orientation. The electric field decreases with a factor of $(r_0/r)^2$. Overall, the flux is the same through both surfaces.

Flux Depends Only on Charge Enclosed

The total flux through a closed surface is zero if the point charge lies outside the surface. Hint: show that the flux through the closed surface on the left is zero using the diagram on the right.



Gauss's Law

The flux of $\vec{\mathbf{E}}$ through any closed surface is

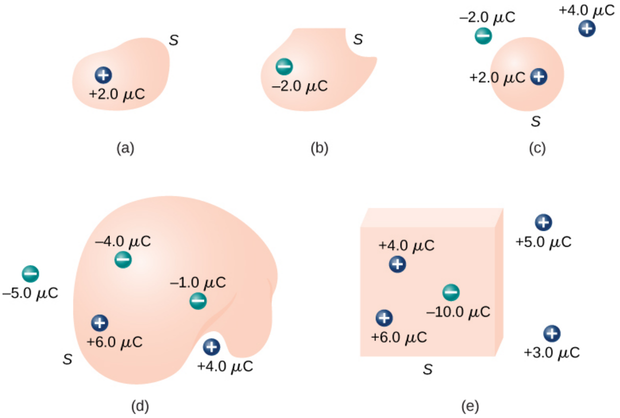
$$\oint \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = \frac{Q_{\mathrm{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \rho dV$$

where Q_{enc} is the total amount of charge enclosed by the surface, and the volume integral on the right-hand side is performed over the volume bounded by the closed surface. Positive flux corresponds to outward-pointing field lines.

The proof of Gauss's law (beyond the scope of this class) follows from (i) the inverse square nature of Coulomb's law and (ii) the principle of superposition. Surprisingly, Gauss's law works for moving charges and thus covers a wider class of electric fields than Coulomb's law (for electrostatic fields only).

Conceptual Question 3

Rank the electric flux from highest to lowest through each of the closed surfaces shown (use units of $\mu\mathrm{C}/\varepsilon_0$).



Example 7: Spherical charges

Use Gauss's law to find the electric field everywhere in space due to the following spherically symmetric charges:

(a) A thin, spherical shell has radius a and uniform surface charge density σ on its surface.

(b) A solid sphere has radius a and uniform volume charge density ρ throughout its interior.

How to Apply Gauss's Law

Gauss's law is always true, but not always useful! It is useful when you can find a Gaussian surface S (a closed surface through which an electric field passes) that satisfies one or more of these properties:

1. $E = |\vec{\mathbf{E}}|$ is constant over S or some portions of S.
2. $\vec{\mathbf{E}}$ is parallel or perpendicular to S or portions of S.

$$\int \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = \pm EA \quad \text{or} \quad \int \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{a}} = 0$$

Example 8: Infinite Lines and Planes

While Coulomb's law is capable of handling infinite distributions, the calculations are nearly effortless with Gauss's law.

(a) Find the electric field of an infinitely long line of positive charge with constant charge per unit length λ.

(b) Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

Discontinuity of $\vec{\mathbf{E}}$

The electric field experiences a discontinuity at singular points in the charge density. For example, the electric field changes abruptly on either side of any surface charge. Refer to the examples we've done (spherical shell and infinite plane), and see if you can determine the discontinuity in terms of σ. Show that $|\Delta E_\perp| = |\sigma|/\varepsilon_0$.

Earnshaw's Theorem I

It is impossible to construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space.

Proof by contradiction. Suppose there is a point P in otherwise empty space where a positively charged particle would be in stable equilibrium. A Gaussian sphere around P would need to have inward electric flux, and by Gauss's law there must be a net negative charge inside the sphere. Since we assumed the region was otherwise empty, this amounts to a contradiction and hence there is no such point P.

Take-home message: classical electrostatics cannot possibly explain the stability of matter! We need quantum mechanics.