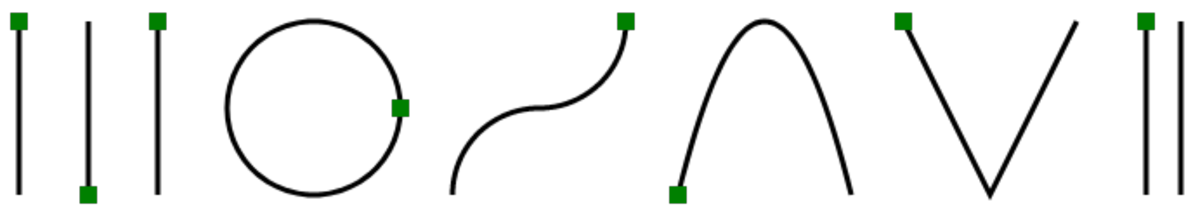
Workbook 5 Questions – Sample Answers

CS559

March 9, 2019

1 Question 1



Given the animation, this question should be done by visual inspection: a curve has arc-length parameterization iff the pen moves at a constant speed.

- The curves named line1, line2, circ, twoQuarterCircles have arc-length parameterization.
- For the curves named line3, twoLines, parabola, the pen moves at varying speed, therefore they do not have arc-length parameterization.
- For the curve name disconnect, the pen jumps, therefore it does not have arc-length parameterization.

Mathematically, a curve f has arc-length parameterization iff

$$\left\|f'\left(u\right)\right\| \text{ is constant for all } u.$$

- For line1,

$$\begin{aligned}\left\|f'\left(u\right)\right\| &= \left\|\begin{bmatrix}0\\100\end{bmatrix}\right\| \\ &= \sqrt{0^2+100^2} \\ &= 100 \text{ independent of } u.\end{aligned}$$

- For line2,

$$\left\|f'\left(u\right)\right\| = \left\|\begin{bmatrix}0\\-100\end{bmatrix}\right\|$$

arc-length parameterization

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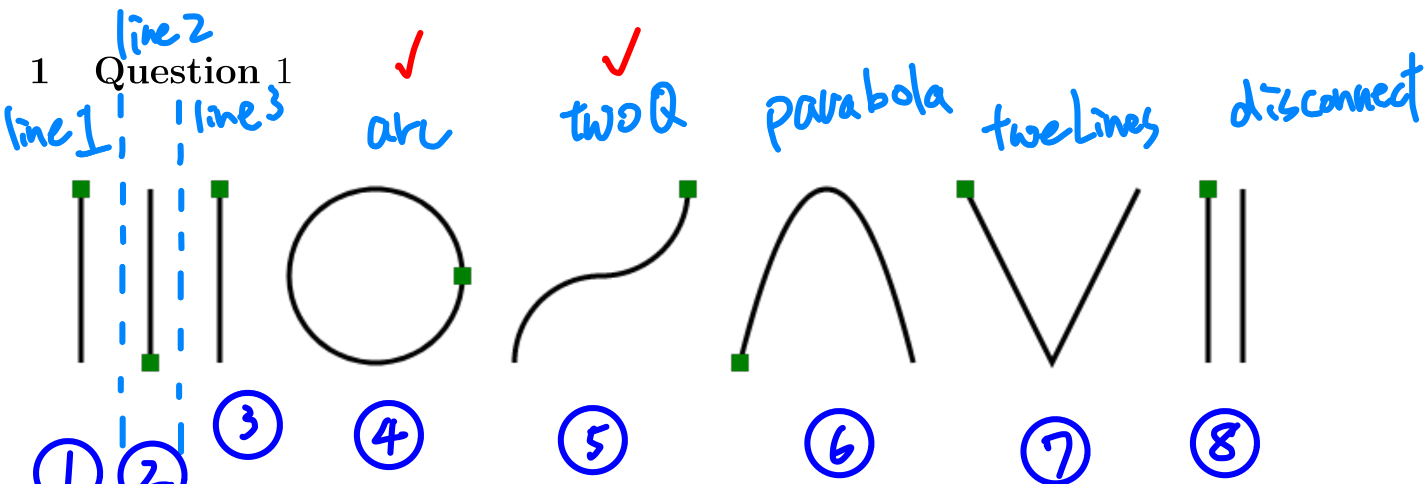


Given the animation, this question should be done by visual inspection: a curve has arc-length parameterization iff the pen moves at a constant speed.

- The curves named line1, line2, circ, twoQuarterCircles have arc-length parameterization.
- For the curves named line3, twoLines, parabola, the pen moves at varying speed, therefore they do not have arc-length parameterization.
- For the curve name disconnect, the pen jumps, therefore it does not have arc-length parameterization.

Mathematically, a curve f has arc-length parameterization iff

$$\|f'(u)\| \text{ is constant for all } u.$$

(x, y, x', y')

- For line1,

$$\begin{aligned}\|f'(u)\| &= \left\|\begin{bmatrix}0\\100\end{bmatrix}\right\| \\ &= \sqrt{0^2+100^2} \\ &= 100 \text{ independent of } u.\end{aligned}$$

```
function line1(u) {
    return [0,100*u,0,100];
}
```

- For line2,

$$\|f'(u)\| = \left\|\begin{bmatrix}0\\-100\end{bmatrix}\right\|$$

```
function line2(u) {
    return [0,100*(1-u),0,-100];
}
```

$$
\begin{aligned}
&= \sqrt{0^2 + (-100)^2} \\
&= 100 \text{ independent of } u.
\end{aligned}
$$

- For line3,

```
function line3(u) {
    return [0,100*u*u,0,200*u];
}
```

$$
\begin{aligned}
\left\| f'(u) \right\| &= \left\| \begin{bmatrix} 0 \\ 200u \end{bmatrix} \right\| \\
&= \sqrt{0^2 + (200u)^2} \\
&= 200u, \text{ depends on } u.
\end{aligned}
$$

- For circ,

```
function circ(u) {
    let ur = Math.PI*2*u;
    return [50+50*Math.cos(ur),50+50*Math.sin(ur),
            -50*Math.sin(ur),50*Math.cos(ur)];
}
```

$$
\begin{aligned}
\left\| f'(u) \right\| &= \left\| \begin{bmatrix} -50\sin(2\pi u) \\ 50\cos(2\pi u) \end{bmatrix} \right\| \\
&= \sqrt{(-50\sin(2\pi u))^2 + (50\cos(2\pi u))^2} \\
&= \sqrt{2500\left(\sin(2\pi u)^2 + \cos(2\pi u)^2\right)} \\
&= 50 \text{ independent of } u.
\end{aligned}
$$

- For twoQuarterCircles

```
function twoQuarterCircles(u) {
   let pu = Math.PI * u;
   if (u<0.5) return [50+50*Math.cos(pu),50*Math.sin(pu),
       -50*Math.sin(pu),50*Math.cos(pu)];
   else return [50+50*Math.cos(pu),100-50*Math.sin(pu),
       -50*Math.sin(pu),-50*Math.cos(pu)];
 }
```

If $u < \frac{1}{2}$,

$$
\begin{aligned}
\left\| f'(u) \right\| &= \left\| \begin{bmatrix} -50\sin(\pi u) \\ 50\cos(\pi u) \end{bmatrix} \right\| \\
&= \sqrt{(-50\sin(\pi u))^2 + (50\cos(\pi u))^2} \\
&= \sqrt{2500\left(\sin(\pi u)^2 + \cos(\pi u)^2\right)} \\
&= 50.
\end{aligned}
$$

If $u > \frac{1}{2}$,

$$
\begin{aligned}
\left\| f'(u) \right\| &= \left\| \begin{bmatrix} -50\sin(\pi u) \\ -50\cos(\pi u) \end{bmatrix} \right\| \\
&= \sqrt{(-50\sin(\pi u))^2 + (-50\cos(\pi u))^2} \\
&= \sqrt{2500\left(\sin(\pi u)^2 + \cos(\pi u)^2\right)} \\
&= 50.
\end{aligned}
$$

If $u = \frac{1}{2}$,

$$\left\| f'\left(\frac{1}{2}\right) \right\| = \left\| \begin{bmatrix} -50 \\ 0 \end{bmatrix} \right\| \\ = \sqrt{(-50)^2 + 0^2} \\ = 50.$$

Therefore, $\|f'(u)\|$ is independent of u.

- For twoLines,

```
function twoLines(u) {
    if (u<0.5) return [100*u,200*u,100,200];
    else return [100*u,100-200*(u-0.5),100,-200];
}
```

At $u = \frac{1}{2}$, the derivative from the left is $\begin{bmatrix} 100 \\ 200 \end{bmatrix}$ and the derivative from the right is $\begin{bmatrix} 100 \\ -200 \end{bmatrix}$. Therefore, the derivative does not exist, so the curve is not arc-length parameterized.

- For disconnect,

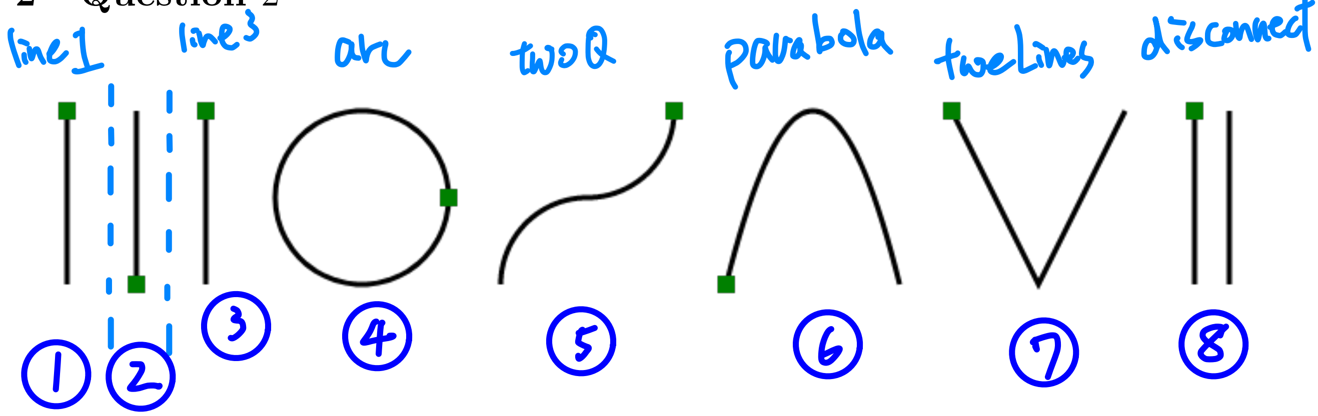
At $u = \frac{1}{2}$, the function is not continuous. Therefore, the derivative does not exist, so the curve is not arc-length paramerized.

- For parabola,

```
function parabola(u) {
    // Begin Example Solution
    return [100*u, 400*(u-0.5) * (u-0.5),100,800*(u-0.5)];
    // End Example Solution
}
```

$$\|f'(u)\| = \left\| \begin{bmatrix} 100 \\ 800\left(u - \frac{1}{2}\right) \end{bmatrix} \right\| \\ = \sqrt{100^2 + \left(800\left(u - \frac{1}{2}\right)\right)^2} \text{ depends on } u.$$

2 Question 2



This question can also be solved by visual inspection: a curve is $C^{(1)}$ if its derivative is continuous (and does not change abruptly anywhere).

- For the curves named line1, line2, line3, circ, twoQuarterCircles and parabola, the derivatives changes smoothly.
- For the curve named twoLines, the derivative changes from $\begin{bmatrix} 100 \\ 200 \end{bmatrix}$ to $\begin{bmatrix} 100 \\ -200 \end{bmatrix}$ abruptly at $u = \frac{1}{2}$.

```
function disconnect(u) {
    if (u<0.5) return [0,200*u,0,200];
    else return [20,100-200*(u-0.5),0,-200];
}
```

- For the curve named disconnect, it is not $C^{(0)}$ (not continuous), therefore not $C^{(2)}$.

Mathematically, a curve f is $C^{(1)}$ if f' is continuous.

- For line1,

$$f'(u) = \begin{bmatrix} 0 \\ 100 \end{bmatrix} \text{ is continuous .}$$

```
function line1(u) {
    return [0,100*u,0,100];
}
```

- For line2,

$$f'(u) = \begin{bmatrix} 0 \\ -100 \end{bmatrix} \text{ is continuous .}$$

```
function line2(u) {
    return [0,100*(1-u),0,-100];
}
```

- For line3,

$$f'(u) = \begin{bmatrix} 0 \\ 200u \end{bmatrix} \text{ is continuous .}$$

```
function line3(u) {
    return [0,100*u*u,0,200*u];
}
```

- For circ,

$$f'(u) = \begin{bmatrix} -50\sin(2\pi u) \\ 50\cos(2\pi u) \end{bmatrix} \text{ is continuous .}$$

```
function circ(u) {
    let ur = Math.PI*2*u;
    return [50+50*Math.cos(ur),50+50*Math.sin(ur),
            -50*Math.sin(ur),50*Math.cos(ur)];
}
```

- For twoQuarterCircles,

If $u < \dfrac{1}{2}$,

$$f'(u) = \begin{bmatrix} -50\sin(\pi u) \\ 50\cos(\pi u) \end{bmatrix} \text{ is continuous .}$$

```
function twoQuarterCircles(u) {
    let pu = Math.PI * u;
    if (u<0.5) return [50+50*Math.cos(pu),50*Math.sin(pu),
        -50*Math.sin(pu),50*Math.cos(pu)];
    else return [50+50*Math.cos(pu),100-50*Math.sin(pu),
        -50*Math.sin(pu),-50*Math.cos(pu)];
 }
```

If $u > \dfrac{1}{2}$,

$$f'(u) = \begin{bmatrix} -50\sin(\pi u) \\ -50\cos(\pi u) \end{bmatrix} \text{ is continuous .}$$

At $u = \dfrac{1}{2}$,

$$f'\left(\frac{1}{2}\right) = \begin{bmatrix} -50 \\ 0, \end{bmatrix}$$

substitute $u = \dfrac{1}{2}$ into the derivative from the left $\left(u < \dfrac{1}{2}\right)$,

right = mid point

$$\lim_{u \to \left(\frac{1}{2}\right)^+} f'(u) = \begin{bmatrix} -50 \\ 0 \end{bmatrix},$$

substitute $u = \frac{1}{2}$ into the derivative from the right $\left(u > \frac{1}{2}\right)$,

Left = mid point

$$\lim_{u \to \left(\frac{1}{2}\right)^-} f'(u) = \begin{bmatrix} -50 \\ 0 \end{bmatrix}.$$

Therefore, $f'(u)$ is continuous at $u = \frac{1}{2}$ too.

```
function twoLines(u) {
    if (u<0.5) return [100*u,200*u,100,200];
    else return [100*u,100-200*(u-0.5),100,-200];
}
```

- For twoLines,

At $u = \frac{1}{2}$, the derivative from the left is $\begin{bmatrix} 100 \\ 200 \end{bmatrix}$ and the derivative from the right is $\begin{bmatrix} 100 \\ -200 \end{bmatrix}$. Therefore, the derivative is not continuous, the curve is not $C^{(1)}$.

- For disconnect,

The curve is not $C^{(0)}$. Therefore, it is not $C^{(1)}$.

```
function disconnect(u) {
    if (u<0.5) return [0,200*u,0,200];
    else return [20,100-200*(u-0.5),0,-200];
}
```

- For parabola,

$$f'(u) = \begin{bmatrix} 100 \\ 800\left(u - \frac{1}{2}\right) \end{bmatrix} \text{ is continuous .}$$

```
function parabola(u) {
    // Begin Example Solution
    return [100*u, 400*(u-0.5) * (u-0.5),100,800*(u-0.5)];
    // End Example Solution
}
```

3 Question 3

A curve f is $C^{(2)}$ if f'' is continuous.

- For line1,

```
function line1(u) {
    return [0,100*u,0,100];
}
```

$$f''(u) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is continuous .}$$

- For line2,

```
function line2(u) {
    return [0,100*(1-u),0,-100];
}
```

$$f''(u) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is continuous .}$$

- For line3,

$$f''(u) = \begin{bmatrix} 0 \\ 200 \end{bmatrix} \text{ is continuous .}$$

```
function line3(u) {
    return [0,100*u*u,0,200*u];
}
```

- For circ,

$$f''(u) = \begin{bmatrix} 100\pi \cos(2\pi u) \\ 100\pi \sin(2\pi u) \end{bmatrix} \text{ is continuous .}$$

```
function circ(u) {
    let ur = Math.PI*2*u;
    return [50+50*Math.cos(ur),50+50*Math.sin(ur),
            -50*Math.sin(ur),50*Math.cos(ur)];
}
```

5

```
function twoQuarterCircles(u) {
    let pu = Math.PI * u;
    if (u<0.5) return [50+50*Math.cos(pu),50*Math.sin(pu),
        -50*Math.sin(pu),50*Math.cos(pu)];
    else return [50+50*Math.cos(pu),100-50*Math.sin(pu),
        -50*Math.sin(pu),-50*Math.cos(pu)];
}
```

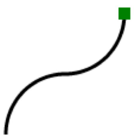
- For twoQuarterCircles,

If $u < \dfrac{1}{2}$,

+

$$f''(u) = \begin{bmatrix} 50\pi\cos(\pi u) \\ 50\pi\sin(\pi u) \end{bmatrix} \text{ is continuous .}$$

twoQ



5

If $u > \dfrac{1}{2}$,

−

$$f''(u) = \begin{bmatrix} 50\pi\cos(\pi u) \\ -50\pi\sin(\pi u) \end{bmatrix} \text{ is continuous .}$$

right ≠ left

At $u = \dfrac{1}{2}$,

$$f''\left(\frac{1}{2}\right) = \begin{bmatrix} 0 \\ -50\pi \end{bmatrix},$$

substitute $u = \dfrac{1}{2}$ into the derivative from the left $\left(u < \dfrac{1}{2}\right)$,

$$\lim_{u \to \left(\frac{1}{2}\right)^{+}} f''(u) = \begin{bmatrix} 0 \\ 50\pi \end{bmatrix}.$$

It is not the same as $f''\left(\dfrac{1}{2}\right)$, which means the derivative is not continuous, the curve is not $C^{(1)}$.

- For twoLines,

The curve is not $C^{(1)}$. Therefore, it is not $C^{(2)}$.

- For disconnect,

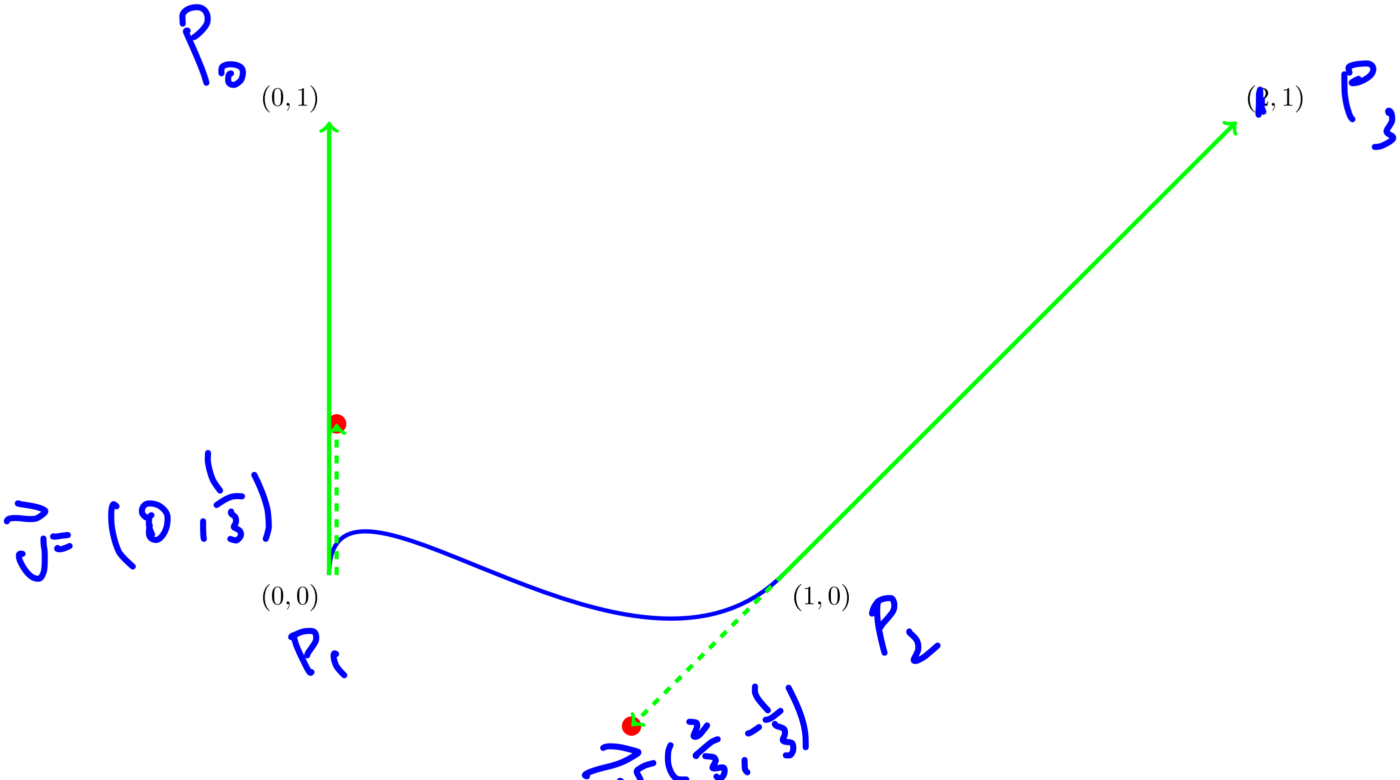
The curve is not $C^{(1)}$. Therefore, it is not $C^{(2)}$.

- For parabola,

$$f''(u) = \begin{bmatrix} 0 \\ 800 \end{bmatrix} \text{ is continuous .}$$

4 Question 4

The curve looks like following blue curve,

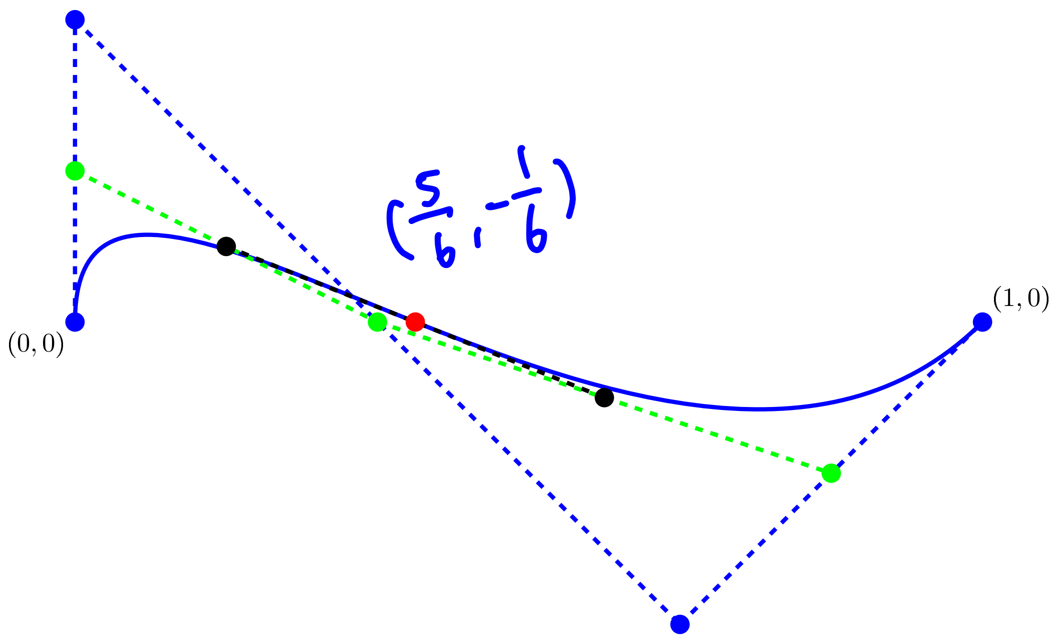


First convert the curve into Bezier form with the following control points,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where the second and third control points are found as $\frac{1}{3}$ and - $\frac{1}{3}$ of the derivatives at the two endpoints (red points on the diagram),

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix},$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}.$$



Then try to split the curve into two halves using De Casteljau. In the diagram, first find the green dots as the midpoints between the consecutive blue control points.
The midpoint between the first two control points is,

$$\frac{1}{2}\left(\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0\\1/3\end{bmatrix}\right)=\begin{bmatrix}0\\1/6\end{bmatrix},$$

the midpoint between the next two control points is,

$$\frac{1}{2}\left(\begin{bmatrix}0\\1/3\end{bmatrix}+\begin{bmatrix}2/3\\-1/3\end{bmatrix}\right)=\begin{bmatrix}1/3\\0\end{bmatrix},$$

and the midpoint between the last two control points is,

$$\frac{1}{2}\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}2/3\\-1/3\end{bmatrix}\right)=\begin{bmatrix}5/6\\-1/6\end{bmatrix}.$$

Repeat this process, and find the black points as the midpoints between the consecutive green points computed previously.
The midpoint between the first two midpoints is,

$$\frac{1}{2}\left(\begin{bmatrix}0\\1/6\end{bmatrix}+\begin{bmatrix}1/3\\0\end{bmatrix}\right)=\begin{bmatrix}1/6\\1/12\end{bmatrix},$$

and the midpoint between the last two midpoints is,

$$\frac{1}{2}\left(\begin{bmatrix}1/3\\0\end{bmatrix}+\begin{bmatrix}5/6\\-1/6\end{bmatrix}\right)=\begin{bmatrix}7/12\\-1/12\end{bmatrix}.$$

Finally, the red midpoint between the two black points is the point of split, which is also where $u = \dfrac{1}{2}$. The midpoint between the above two midpoints is,

$$\frac{1}{2}\left(\begin{bmatrix}1/6\\1/12\end{bmatrix}+\begin{bmatrix}7/12\\-1/12\end{bmatrix}\right)=\begin{bmatrix}3/8\\0\end{bmatrix}.$$

Therefore, the point where $u = \dfrac{1}{2}$ is $\begin{bmatrix}3/8\\0\end{bmatrix}$. To verify that this is correct, use the basis functions formula,

$$f(u) = \left(1-3u^2+2u^3\right)p_0 + \left(u-2u^2+u^3\right)p_0' + \left(3u^2-2u^3\right)p_1 + \left(-u^2+u^3\right)p_1'.$$

Here,

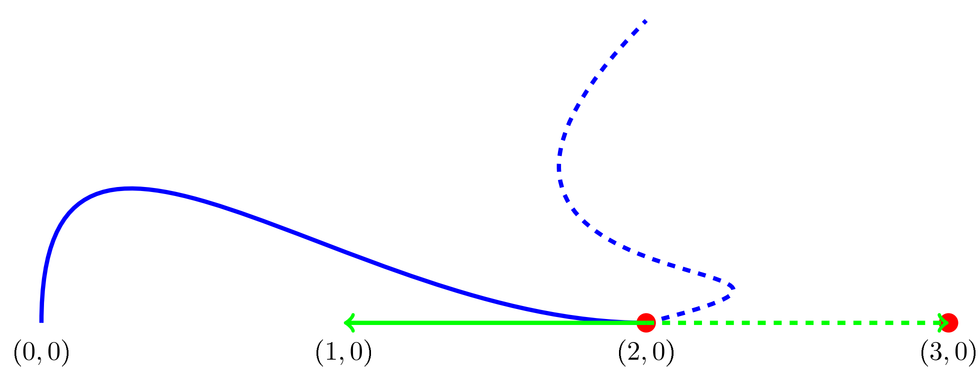
$$\begin{aligned} f\left(u=\frac{1}{2}\right) &= \frac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{8}\begin{bmatrix}0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}-\frac{1}{8}\begin{bmatrix}1\\1\end{bmatrix}\\ &= \begin{bmatrix}3/8\\0\end{bmatrix}. \end{aligned}$$

5 Question 5

The tangent vector at $u = 1$ is equal to p_1',

$$\begin{bmatrix}1\\1\end{bmatrix}.$$

6 Question 6



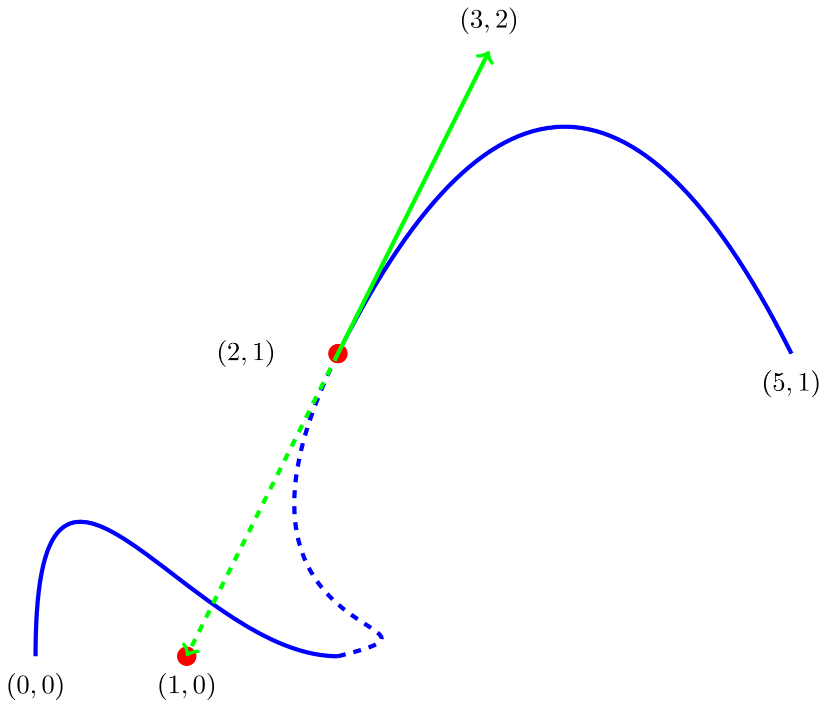
For the combined curve to be $C^{(0)}$, the first control point of the second curve must be equal to the last control point of the first curve,

$$\begin{bmatrix}2\\0\end{bmatrix}.$$

For the combined curve to be $C^{(1)}$, the second control point must be the same distance away and opposite direction from $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ compared to the second-to-last control point of the first curve,

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

7 Question 7



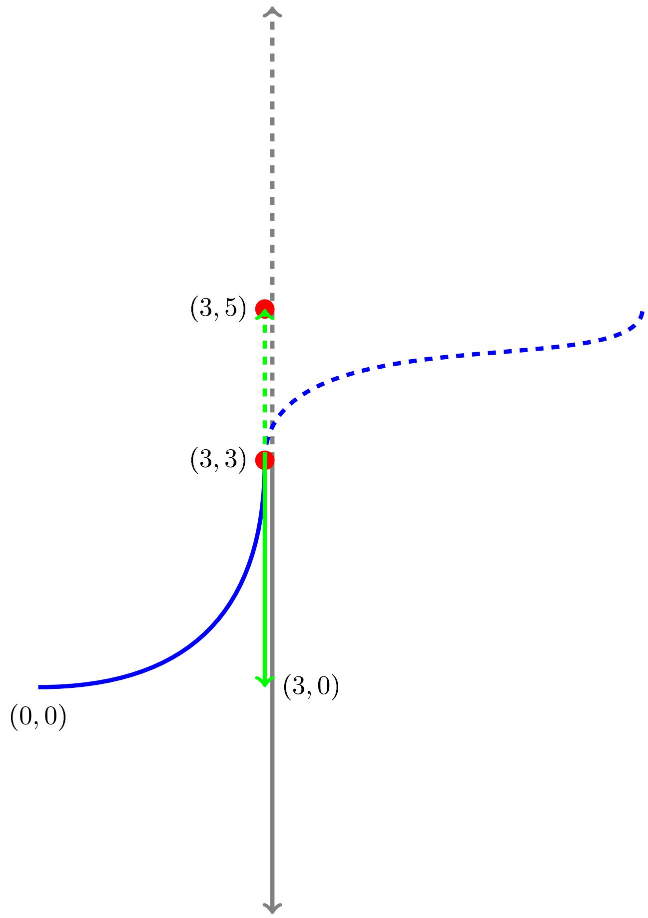
For the combined curve to be $C^{(0)}$, the last control point of the second curve must be equal to the first control point of the third curve,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

For the combined curve to be $C^{(1)}$, the third control point must be the same distance away and opposite direction from $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ compared to the second control point of the third curve,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

8 Question 8



For the combined curve to be $C^{(0)}$, the first control point of the second curve must be equal to the last control point of the first curve,

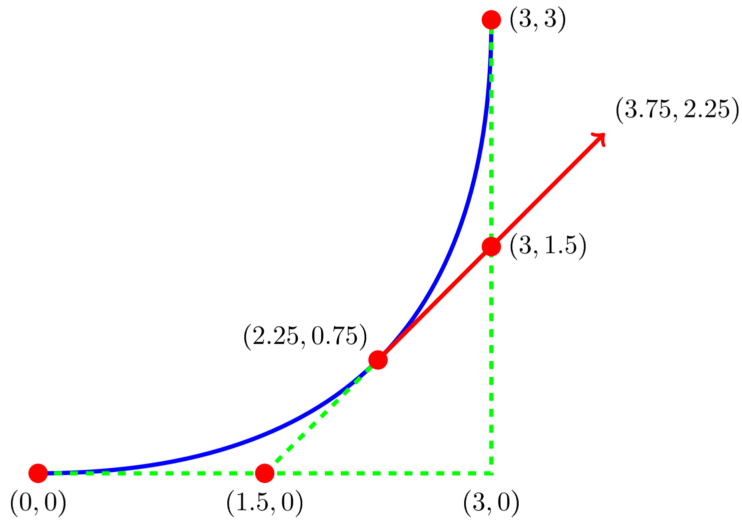
$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

For the combined curve to be $C^{(1)}$, the second control point must be $\frac{2}{3}$ (see below for explaination) the distance away and opposite direction from $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ compared to the second-to-last control point of the first curve,

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} - \frac{2}{3}\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

To see where the $\frac{2}{3}$ comes from, note that the vector between the last two control points of a quadratic (degree 2) Bezier curve is - $\frac{1}{2}$ of the derivative at the endpoint, and the vector between the first two control points of a cubic (degree 3) Bezier curve is $\frac{1}{3}$ of the derivative at the endpoint. The derivative at $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is diagrammed as the dotted gray line: the green dotted line connecting the first two control points of the cubic curve is $\frac{1}{3}$ of it and the green solid line connecting the last two control points of the quadratic curve is $\frac{1}{2}$ of its negative.

9 Question 9



To split the Bezier curve using De Casteljau, find the midpoint between first two control points,

$$\frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix},$$

and the midpoint between last two control points,

$$\frac{1}{2}\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3/2 \end{bmatrix}.$$

Then the point at the split is the midpoint between the above two midpoints,

$$\frac{1}{2}\left(\begin{bmatrix} 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 3/2 \end{bmatrix}\right) = \begin{bmatrix} 9/4 \\ 3/4 \end{bmatrix}.$$

Therefore, the curve is split into two quadratic Bezier curves with the control points, for the first piece,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/4 \\ 3/4 \end{bmatrix},$$

and for the seconde piece,

$$\begin{bmatrix} 9/4 \\ 3/4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The derivative at $u = \dfrac{1}{2}$ is the derivative at $\begin{bmatrix} 9/4 \\ 3/4 \end{bmatrix}$, which is twice the vector between first two control points of the second piece,

$$2\left(\begin{bmatrix} 3 \\ 3/2 \end{bmatrix} - \begin{bmatrix} 9/4 \\ 3/4 \end{bmatrix}\right) = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}.$$