ECE 106 FEYNMAN

PHYSICS INTEGRATION

We need to derive moment of inertia, area, volume using solely integration.

1. Define differential element:

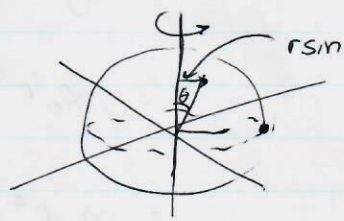
Area: $dA = dx\,dy$ (~~polar~~ Cart.), $dA = r\,d\phi\,dr$ (polar), $dA = r\,d\theta \cdot r\sin\theta\,d\phi$ (spherical)

Volume: $dV = dx\,dy\,dz$, $dV = dz \cdot dr \cdot r\,d\phi$, $dV = r^2 \sin\theta\,d\theta\,d\phi\,dr$

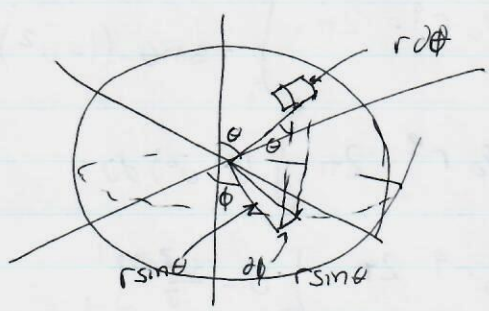
Moment of inertia:

↳ $dI = r^2\,dm \Rightarrow r$: distance from axis!

OR: $dI = (r\sin\theta)^2 dm \Rightarrow$ Sphere!



General trick: graph it + look at element:



$dV = (\text{in} + \text{out}) \cdot (\text{length w/ } z) \cdot (\text{length w/ } z)$

$= dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$

2. Know which coordinate system you are on: Cartesian, polar, spherical, cylindrical?
   - Circular → polar. Sphere → spherical . . .
3. Range of each differential in differential element.

   In spherical: $dr: 0 \to R$ (R is radius)

   $d\phi: 0 \to 2\pi$

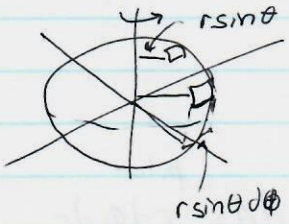
   $d\theta: 0 \to \pi$

*4. For moment of inertia: use $dm = \rho_0\,dV/dA$ + $M = \rho_0 \cdot$ actual volume.
   - $dm \to$ integration
   - $M \to$ convert final expression into terms of M.

5. Integrate → use proper integration techniques w/ double integrals.

- Ex: Find the moment of inertia of a spherical shell w/ radius R.

1. Differential element on spherical coordinate system.



$$dI = (r\sin\theta)^2\, dm \Rightarrow r\sin\theta \text{ is distance from axis.}$$

$$dm = \rho_0\, dA$$

$$= \rho_0 \cdot r\,d\theta \cdot r\sin\theta\, d\phi$$

$$\therefore\ dI = r^4 \sin^3\theta\, \rho_0\, d\theta\, d\phi$$

2. Range: $d\theta: 0 \to \pi$. $d\phi: 0 \to 2\pi$.

3. Integrate:

$$I = \int dI$$

$$= \rho_0 r^4 \int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\theta\, d\phi$$

$$= \rho_0 r^4 \cdot 2\pi \cdot \int_0^{\pi} \sin\theta\,(1-\cos^2\theta)\, d\theta \qquad \rightarrow\ u = \cos\theta,\ du = -\sin\theta\, d\theta$$

$$= \rho_0 r^4 \cdot 2\pi \cdot \int_1^{-1} -\sin\theta\,(1-u^2) \cdot \frac{du}{\sin\theta}$$

$$= \rho_0 r^4 \cdot 2\pi \int_{-1}^{1} (1-u^2)\, du$$

$$= \rho_0 r^4 \cdot 2\pi \left[u - \frac{u^3}{3}\right]_{-1}^{1}$$

$$= \rho_0 r^4 \cdot 2\pi \left(\left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right)\right)$$

$$= \frac{8\pi}{3}\rho_0 r^4$$

4. Convert in terms of mass (total):

$$M = \rho_0 A$$

$$= \rho_0\, 4\pi r^2$$

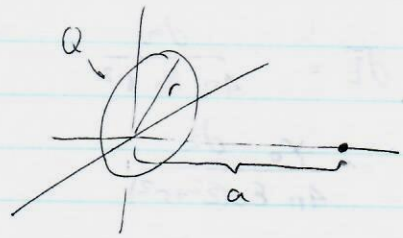
$$I = 4\pi r^2 \rho_0 \cdot \tfrac{2}{3} r^2$$

$$= \boxed{\tfrac{2}{3} M r^2}$$

ELECTRIC FIELD

$$\vec{E} = \frac{dq}{4\pi\varepsilon_0 r^2} \Rightarrow r \text{ is the distance of point to charge.}$$

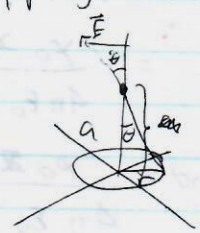
- Ex: //



Find $\vec{E}$ at point from ring.

1. Create differential element:

Flipping coordinate system to cylindrical



$$d\vec{E}_x = \frac{dq}{4\pi\varepsilon_0} \cdot \frac{1}{r^2+a^2} \sin\theta$$

Hard to compute w/ no component

$= 0$ (by symmetry) → Very useful!

$$d\vec{E}_y = \frac{dq}{4\pi\varepsilon_0} \cdot \frac{1}{r^2+a^2} \cos\theta$$

$$= \frac{a\,dq}{4\pi\varepsilon_0 (r^2+a^2)^{3/2}}$$

$$dq = \frac{Q}{2\pi r}(r\,d\phi) = \frac{Q}{2\pi}\,d\phi \Rightarrow \text{Charge density} = \frac{\text{Charge}}{\text{length}}$$

$$\therefore d\vec{E}_y = \frac{aQ\,d\phi}{8\pi^2\varepsilon_0 (r^2+a^2)^{3/2}}$$

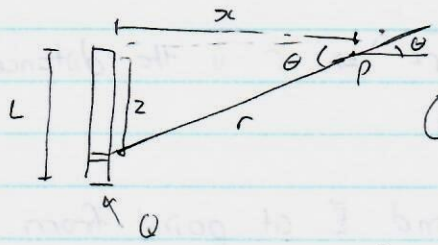
2. Integrate:

$$\vec{E}_y = \frac{aQ}{8\pi^2\varepsilon_0 (r^2+a^2)^{3/2}} \int_0^{2\pi} d\phi$$

$$= \frac{aQ}{4\pi\varepsilon_0 (r^2+a^2)^{3/2}}$$

→ Be comfortable with limit of integration.
↳ Visually think about how dq is changing

∘ Ex: //



What is $\vec{E}$ at point p

① Create differential element:

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 r^2}$$

$$d\vec{E} = \frac{\rho_0\, dz}{4\pi\varepsilon_0 (\sqrt{z^2+x^2})^2} = \frac{\rho_0\, dz}{4\pi\varepsilon_0 (z^2+r^2)}$$

①.a Dividing into components:

$$d\vec{E}_x = \frac{\rho_0\, dz}{4\pi\varepsilon_0 (z^2+r^2)} \cdot \cos\theta = \frac{\rho_0\, dz}{4\pi\varepsilon_0 (z^2+r^2)} \cdot \frac{x}{\sqrt{z^2+r^2}}$$

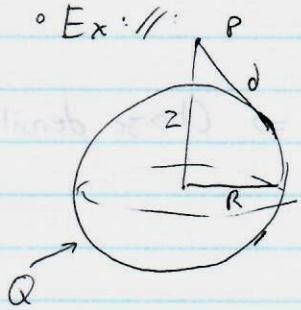
$$= \frac{\rho_0\, x\, dz}{4\pi\varepsilon_0 (z^2+r^2)^{3/2}}$$

$$d\vec{E}_y = \frac{\rho_0\, dz}{4\pi\varepsilon_0 (z^2+r^2)} \cdot \sin\theta = \frac{\rho_0\, z\, dz}{4\pi\varepsilon_0 (z^2+r^2)^{3/2}}$$

②. Integrate:

$$E_x = \frac{\rho_0 x}{4\pi\varepsilon_0} \int_0^L \frac{dz}{(z^2+r^2)^{3/2}}, \quad E_y = \frac{\rho_0}{4\pi\varepsilon_0} \int_0^L \frac{z\, dz}{(z^2+r^2)^{3/2}}$$

∘ Ex: //



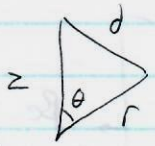
Find $\vec{E}$ at P:

① Differential element:

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 d^2}$$ ⇐ In the eqn, r is distance of dif. elem from point.

$$dq = \frac{Q}{4\pi r^2} dA = \frac{Q r^2 \sin\theta\, d\theta\, d\phi}{4\pi r^2} = \frac{Q \sin\theta\, d\theta\, d\phi}{4\pi}$$

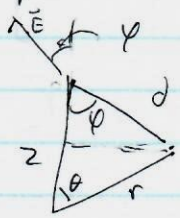
$$\sigma = \frac{Q}{4\pi}$$



$$\Rightarrow d^2 = z^2 + r^2 - 2zr\cos\theta.$$

↳ Cosine Law! Note how I used spherical coordinates.

①.a Components:



Finding cos φ and sin φ

$$\cos\varphi = \frac{z - r\cos\theta}{\sqrt{z^2+r^2-2zr\cos\theta}}$$ ⇐ d!

$$\sin\varphi = \frac{r\sin\theta}{\sqrt{z^2+r^2-2zr\cos\theta}}$$

(1.b) Combine:

By symmetry, $\vec{E}_x = \vec{E}_y = 0$.

$$d\vec{E}_z = \frac{\sigma \sin\theta \, d\theta \, d\phi}{4\pi\varepsilon_0 (z^2+r^2-2zr\cos\theta)} \cdot \overset{\cos\varphi}{\frac{z - r\cos\theta}{\sqrt{z^2+r^2-2zr\cos\theta}}}$$

(2) Integrate:

$$\vec{E}_z = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\sin\theta \, d\theta \, (z - r\cos\theta)}{(z^2+r^2-2zr\cos\theta)^{3/2}}$$

Substitution: $u = \overset{\cos}{\cancel{\sin}}\theta \Rightarrow du = -\sin\theta \, d\theta$

$$\vec{E}_z = \frac{\sigma}{4\pi\varepsilon_0} \cdot 2\pi \cdot \int_{-1}^{1} \frac{z - ru}{(z^2+r^2-2zru)^{3/2}}$$

Integral given.

$$= \frac{\sigma}{2\varepsilon_0} \left[\frac{zu - r}{z^2\sqrt{z^2+r^2-2zru}} \right]_{-1}^{1}$$

$$= \frac{\sigma}{2z^2\varepsilon_0} \left(\frac{z-r}{\sqrt{(z-r)^2}} + \frac{z+r}{\sqrt{(z+r)^2}} \right)$$

If $z < r$

$$\vec{E}_z = 0$$

$z > R$

$$\vec{E}_z = \frac{\sigma}{2\varepsilon_0 z^2} \cdot 2$$

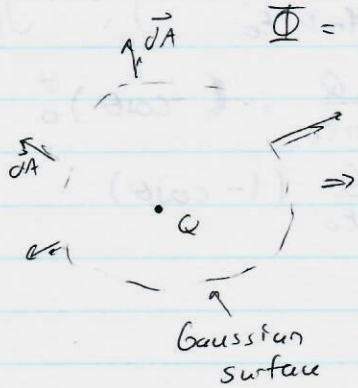
$$= \frac{\sigma}{\varepsilon_0 z^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 z^2}$$

GAUSS' LAW

area vector of G.S

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$



$$\Rightarrow \Phi = \frac{Q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

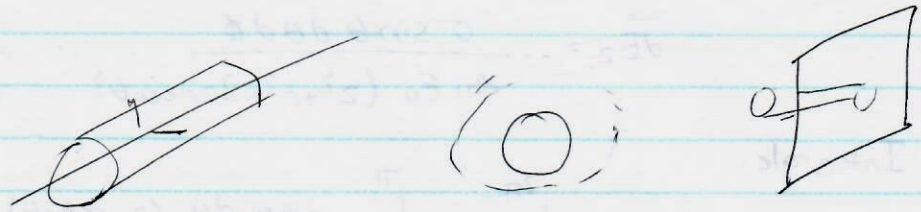
$$\Phi = \oint \vec{E} \cdot d\vec{A} = \oint |E||dA| \overset{\theta = 0}{\cos\theta}$$

$$= \vec{E} \cdot \oint dA = \vec{E} \cdot 4\pi r^2$$

$\vec{E} \cdot dA$ is constant!

$$\rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

1. Choose a Gaussian surface that reflects symmetry of field lines:



Must pay attention if there is a uniform distribution.

$\Rightarrow$ G.S. inside sphere:

$\hookrightarrow$ $\Phi = 0$, but $\vec{E} \neq 0$. No symmetry

$\Phi = \oint \vec{E} \cdot d\vec{A}$ $\Leftarrow$ Cannot separate integral.

$\Phi = \oint |E|\,|dA| \cos\theta$ $\nwarrow$ Not constr

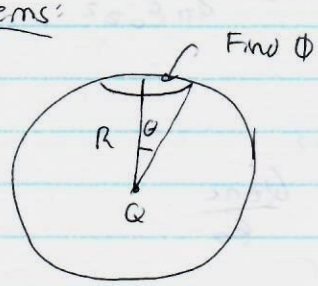
2. Sanity check: draw field lines.
3. To find Φ: either do $\Phi = \frac{Q_{enc}}{\varepsilon_0}$, $\Phi = \oint \vec{E} \cdot d\vec{A}$
4. To find $\vec{E}$:

$$\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E} \cdot d\vec{A} \Rightarrow$$ $\vec{E}$ only found via Gauss' Law if $\vec{E} \cdot dA$ is constant.

Pay special attention to <u>what</u> you are integrating.

<u>Problems:</u>

1.



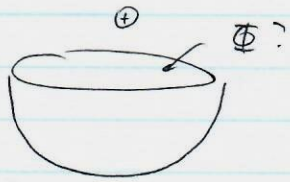
$$\Phi = \oint \vec{E} \cdot dA$$

$$= \frac{Q}{4\pi R^2 \varepsilon_0} \cdot R^2 \int_0^{\theta} \sin\theta \int_0^{2\pi} d\varphi$$ $\vec{E}$ is constant in patch

$$= \frac{Q}{4\pi\varepsilon_0} \cdot (-\cos\theta)\Big|_0^{\theta} \cdot 2\pi$$

$$= \frac{Q}{2\varepsilon_0}(1 - \cos\theta)$$

2.

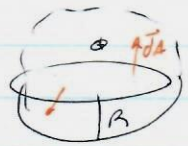


$\Phi_{hemisphere} = 0 \Leftarrow \frac{Q_{enc}}{\varepsilon_0} = \frac{0}{\varepsilon_0} = 0$

$\boxed{\Phi_{flat} + \Phi_{curved} = 0}$ ⇒ Important!

$\Phi_{flat} = -\Phi_{curved}$ $\quad \Phi_{AB} = \Phi_A + \Phi_B$

Full Gaussian sphere: ⇒ Capture symmetry of point charge



$\Phi_{sphere} = \frac{Q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$

$\Phi_{curved\ part} = \frac{\Phi_{sphere}}{2} = \frac{Q}{2\varepsilon_0}$

$\therefore \Phi_{flat} = -\frac{Q}{2\varepsilon_0}$ ⇒ Neg. flux:

ELECTROSTATIC POTENTIAL

Defining work in electricity context:

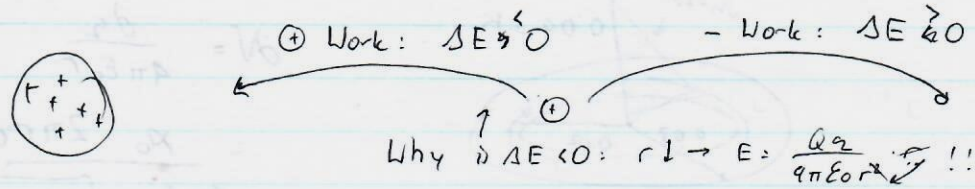
$$W = -\Delta E \Rightarrow \text{Think of gravity:}$$

$$F \cdot d = -\Delta E$$

Distance vector.

$$\Delta E = -\int_a^b \vec{F} \cdot d\vec{s}$$

B/c work is done by field-generating object:

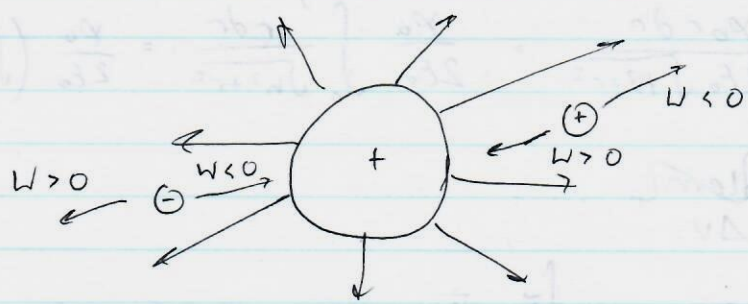


To calculate work done, solve integral.

$$W_{A \to B} = -\int_A^B \vec{F} \cdot d\vec{s} = -\int_A^B \frac{Qq}{4\pi\varepsilon_0 r^2}\, dr$$

(Analyze: $A < B$, $A > B$)

Think about in terms of field:

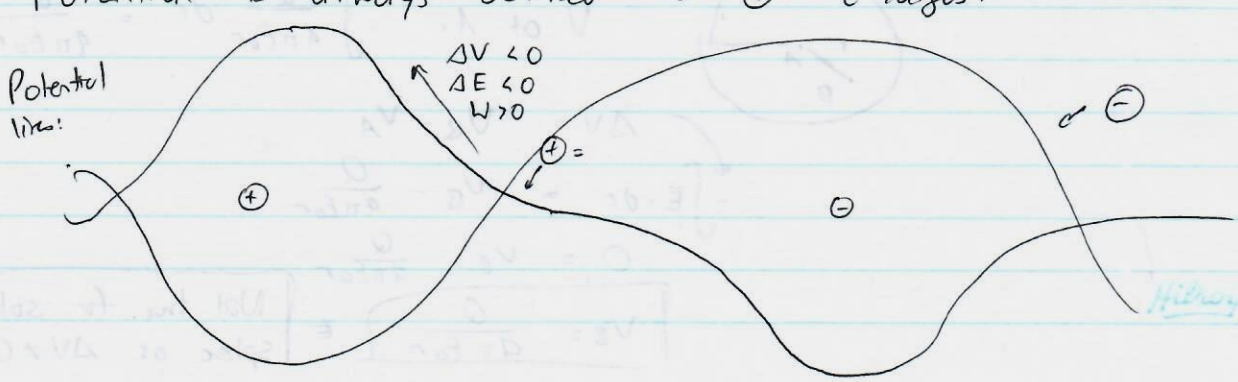


⇒ Analysis of ⊖ charge: Use the integral + formula!

Now, we define potential:

$$\Delta V = \frac{\Delta E}{q} = -\int_A^B \frac{\vec{F} \cdot d\vec{s}}{q} = \boxed{-\int_A^B \vec{E} \cdot d\vec{s}}$$

Potential is always defined for ⊕ charges.



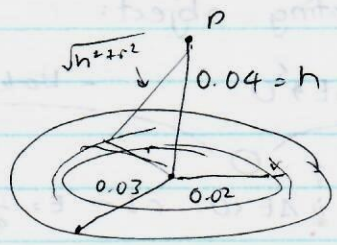
- Solving problems of calculating potential:

① Calculate potential at certain point:

Continuous $\Rightarrow$ $\boxed{dV = \frac{dQ}{4\pi\varepsilon_0 r}}$ $\Leftarrow$ Derivation: $\boxed{\vec{E} = \frac{Qq}{4\pi\varepsilon_0 r} \div q}$

Discrete $\Rightarrow$ Add up potentials (superposition)

Continuous problem:



$\sigma = \rho_0$. V at P.

1. Find dV:

$$dV = \frac{dq}{4\pi\varepsilon_0 r}$$ $\Leftarrow$ Differentiate little rings of radius r

$$= \frac{\rho_0 \cdot 2\pi r\,dr}{4\pi\varepsilon_0 \cdot \sqrt{h^2+r^2}}$$ ← Circumference × charge.

$$= \frac{\rho_0\, r\, dr}{4\pi\varepsilon_0 \sqrt{h^2+r^2}}$$

2. Integrate:

$$V = \int_a^b \frac{\rho_0\, r\, dr}{2\varepsilon_0 \sqrt{h^2+r^2}} = \frac{\rho_0}{2\varepsilon_0}\int_a^b \frac{r\,dr}{\sqrt{h^2+r^2}} = \frac{\rho_0}{2\varepsilon_0}\left(\sqrt{h^2+b^2} - \sqrt{h^2-a^2}\right)$$

~~Discrete problem:~~

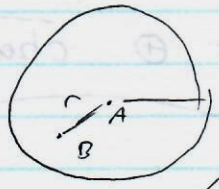
② Calculate ΔV:

$$\Delta V = -\int \vec{E}\cdot d\vec{L}$$

Tip: Calculate $\vec{E}$ via Gauss' Law

- Ex:// Prove $\Delta V = 0$ inside charged spherical shell (i.e V = constant)

By Gauss' Law: $\vec{E} = 0$ inside



V at A: $\int \frac{Q}{4\pi\varepsilon_0 r}\,dr = \frac{Q}{4\pi\varepsilon_0 r}$ } dr is constant!

$$\Delta V = V_B - V_A$$

$$-\int \vec{E}\cdot dr = V_B - \frac{Q}{4\pi\varepsilon_0 r}$$

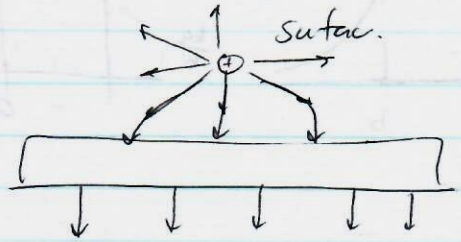
$$0 = V_B - \frac{Q}{4\pi\varepsilon_0 r}$$

$$\boxed{V_B = \frac{Q}{4\pi\varepsilon_0 r}}$$ $\Leftarrow$ Not true for solid charged sphere as $\Delta V \neq 0$!

Conductors

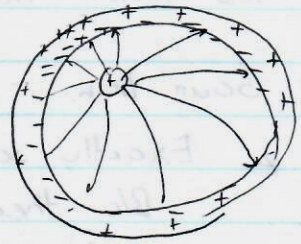
① : $\vec{E} = 0$ in a conductor

② : $\Delta V = 0 \Rightarrow$ conductor equipotential surface.

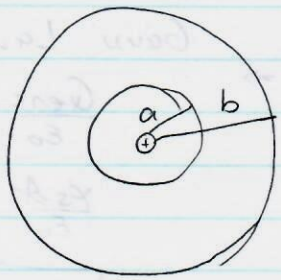


③ : $\vec{E} \perp$ surface of an equipotential

④ : Polarization:



- Ex: //



Find V + $\vec{E}$ at all points in object:

① : $r > b$:

By Gauss' Law: $\vec{E} = \frac{kq}{r^2}$

$\therefore V = E = \frac{kq}{r}$

as $r \to b$: $\vec{E} = \frac{kq}{b^2}$, $V = \frac{kq}{b}$

② : $a < r < b$

$\vec{E} = 0$ b/c conductor

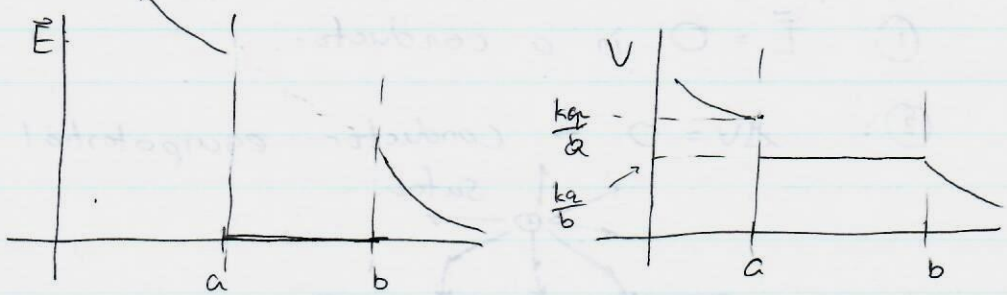
$\vec{V} = \frac{kq}{b}$.

③ : $r < a$:

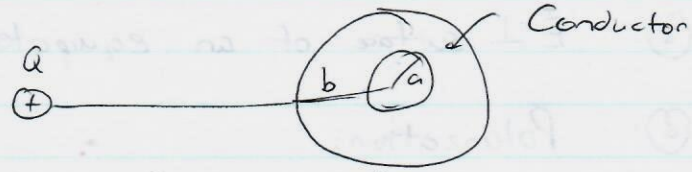
$\vec{E} = \frac{kq}{r^2}$ (Gauss' Law)

$V = \frac{kq}{r} \Rightarrow$ At a: $\boxed{\frac{kq}{a}}$

Graph:



- Ex: //



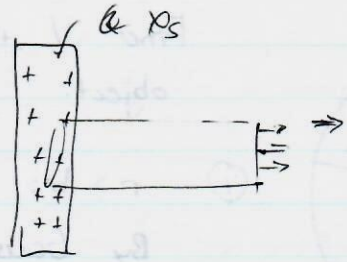
What is potential + field inside hole:

$\vec{E}_{hole} = 0 \Rightarrow$ Gauss' Law:

$$\boxed{V = \frac{kq}{b}}$$ ⇐ Exactly same case as $r > b$ B/c there is not $\vec{E}$ in sphere, V is same throughout

b is distance from centre to charge!

- Ex: // Plate potentials:

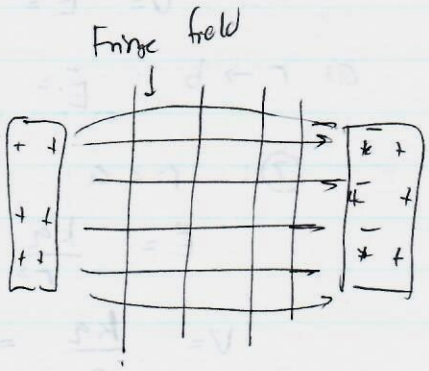


Gauss' Law:

$$\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E} \cdot dA$$

$$\frac{\sigma_s A}{\varepsilon_0} = EA$$ (not 2EA b/c flat surface has no $\vec{E}$)

$$\therefore \boxed{E = \frac{\sigma_s}{\varepsilon_0}}$$

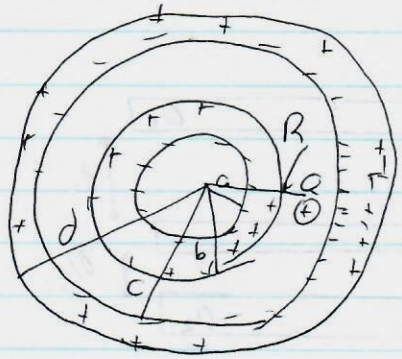


⇒ ΔV is easy to calculate b/c $\vec{E}$ is not changing.

To find potential @ a point:

$\boxed{\text{Superposition}}$

- Ex: //



Find potential at the <u>centre</u>.

① Look @ charge distribution.

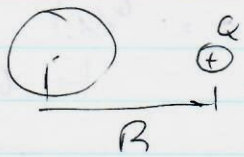
↳ Equal

Gauss' Law:

↳ Unequal:

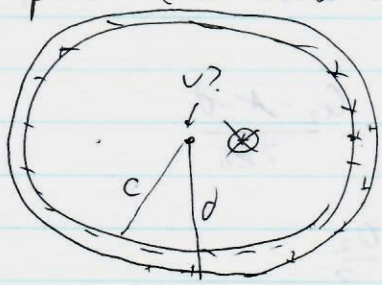
Superposition.

#1 problem:



V_{centre}: $\frac{kQ}{R}$

#2: problem (no charge b/c accounted for in #1 problem)



$$V = \underbrace{\frac{kQ}{d}}_{\text{Potential from outer surface}} - \underbrace{\frac{kQ}{c}}_{\text{Potential from inner surface. BUT } \ominus \text{ b/c induced charge is neg.}}$$

Add it up: $V_T = V_1 + V_2$

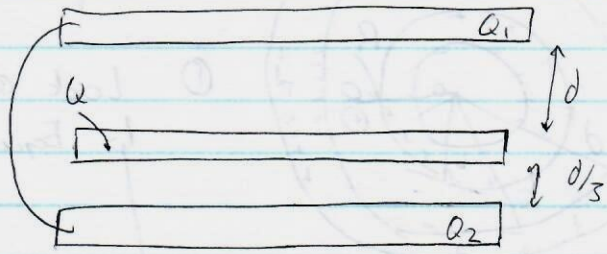
$$= \boxed{\frac{kQ}{R} + \frac{kQ}{c} - \frac{kQ}{d}}$$

Conductors touching ⇒ <u>EQUIPOTENTIAL</u>

$$\frac{kQ_1}{r_1} = \frac{kQ_2}{r_2}$$

$$Q_1 \cdot r_2 = Q_2 \cdot r_1$$

- Ex: //



How much charge goes to top of middle plate?

Q_1: charge induced on top plate.
Q_2: " bottom plate

①: Analyze field + potential from 2 dif. plates.

$$\vec{E} = \cancel{\frac{Q}{\varepsilon_0}} \quad \frac{\sigma}{\varepsilon_0} \qquad (\sigma = Q/A)$$

↳ Q: charge on top of plate.

$$\Delta V_{top} = \vec{E} \cdot d = \frac{\sigma \cdot d}{\varepsilon_0}$$

$$\Delta V_{top} = \vec{E} \cdot d/3 = \frac{\sigma \cdot d}{3\varepsilon_0}$$

$$\therefore \boxed{\begin{aligned} Q_1 &= Q_{top} \\ Q_2 &= Q_{bottom} \end{aligned}}$$

②: Equate:

$$\frac{Q_1 \cdot \cancel{A \cdot d}}{\cancel{\varepsilon_0}} = \frac{Q_2 \cdot \cancel{A \cdot d}}{3\cancel{\varepsilon_0}}$$

$$Q_1 = \frac{Q_2}{3}$$

③: Total charge:

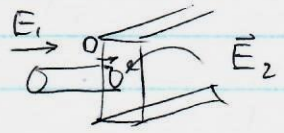
$$Q = Q_1 + Q_2$$
$$= 4Q_1$$

$$\therefore \text{Top: } \frac{Q}{4}, \quad \text{Bottom: } \frac{3Q}{4}$$

DIELECTRICS + BOUNDARY CONDITIONS

Problems where media is insulator → $\vec{E}$ affected.

$$D = \varepsilon \vec{E} \Rightarrow \text{Electric flux density}$$
$$= \varepsilon_0 \varepsilon_r \vec{E} \Rightarrow \varepsilon_r \text{ dependent on material}$$



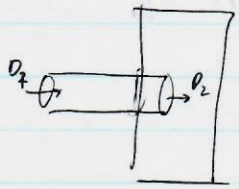
To find relationship between $\vec{E}$ in different media, need to find relationship between D ← boundary conditions.

①: Tangential boundary conditions:

$$\boxed{\vec{E}_{t_1} = \vec{E}_{t_2}}$$

②: Normal boundary condition:

Gauss' Law for dielectrics.



$$\Rightarrow \quad \oint \vec{D} \cdot d\vec{A} = Q_{enc}^{free}$$

$$A(D_{2_n} - D_{1_n}) = Q_{enc}^{free}$$

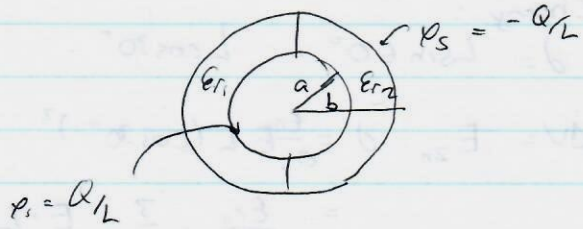
$$\boxed{D_{2_n} - D_{1_n} = \rho_s}$$

Problem-solving steps:

①: Use boundary conditions to find relations between E_t and E_n in regions.

②: Solve.

- Ex:// Cylinder w/ following configuration



Find $\vec{E}$ between a and b:

①: Use Gaussian Law w/ surface (cylindrical) between a and b:

$$\oint \vec{D} \cdot d\vec{A} = QL$$

Divided due to different dielectrics

$$\oint_{left} \vec{D} \cdot d\vec{A} + \oint_{right} \vec{D} \cdot d\vec{A} = QL$$

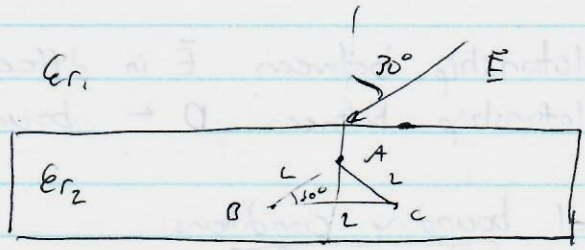
Note: $A = \frac{2\pi r L}{2}$

$$A \varepsilon_1 \vec{E} \pi R L + A \varepsilon_{r_1} \vec{E} \pi R L = QL$$

$$\vec{E} = \frac{Q}{(\varepsilon_1 + \varepsilon_2)\pi R L}$$

Hilroy

- Ex: //



1. Find $\Delta V_{B \to C}$

① : Find E_t in medium.

↳ Boundary conditions: $E_{t_2} = E_{t_1} = E \sin 30°$

② : ΔV:

$$\Delta V = E \Delta d = E \sin 30° \cdot L$$

2. Find $\Delta V_{a \to \text{middle of } B+C}$:

① : Find E_n in medium:

↳ Boundary conditions:

$$\oint \vec{D} \cdot dA = Q_{enc}$$

$$A (D_2 - D_1) = 0$$

$$D_2 = D_1$$

$$E_{2n} \varepsilon_{r_2} = E_{1n} \varepsilon_{r_1}$$

$$= E \cos 30 \cdot \varepsilon_{r_1}$$

$$\therefore E_{2n} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} E \cos 30$$

② : V change:

$$d = L \sin 60° = L \cos 30°$$

$$\therefore \Delta V = E_{2n} \cdot d = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} E L (\cos 30°)^2$$

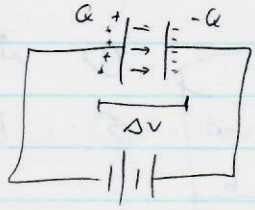
$$= \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \cdot \frac{3}{4} \cdot E \cdot L$$

3. Find $\Delta V_{B \to A}$:

$$\therefore \Delta V = \frac{\Delta V_{B-C}}{2} + \Delta V_{a \to \text{middle}}$$

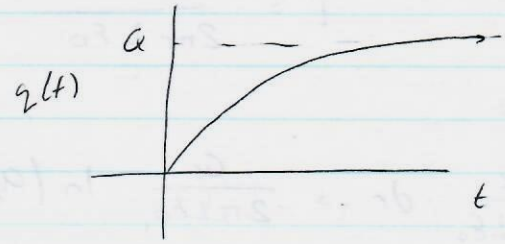
$$= \frac{EL}{2} \cdot \frac{1}{2} + \frac{3}{4} \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \cdot E \cdot L$$

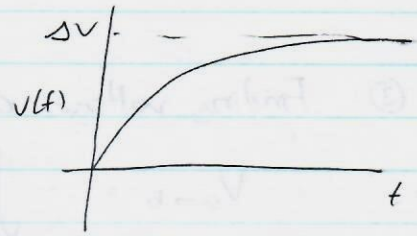
CAPACITANCE



$$\boxed{C = \frac{Q}{V}}$$

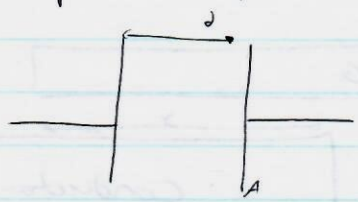
Non-equilibrium graphs:





If we remove battery, capacitor acts as a battery!

Parallel plate capacitor:



① Assume Q + $-Q$ deposited on capacitor

② Find $\vec{E}$:

$$\vec{E} = \frac{\rho_s}{\epsilon_0 \epsilon_r} = \frac{Q}{A\epsilon_0\epsilon_r}$$

③ Find V:

$$V = \vec{E} \cdot d = \frac{Qd}{A\epsilon_0\epsilon_r}$$

④ Put into formula:

Only if PPC

$$\boxed{C = \frac{Q}{V} = \frac{A\epsilon_0\epsilon_r}{d}}$$

Putting in material w/ ϵ_r $\Rightarrow$ $C_{final} = \epsilon_r \cdot C_{initial}$

Know relationships between C, Q, V + C, A, d

↳ If w/ voltage source: V is constant
If disconnected: Q is constant

Series vs. parallel:

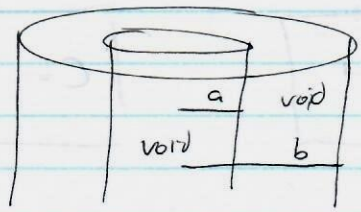
Parallel:

$$C_T = C_1 + C_2 + \dots$$

Series:

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \dots$$

-Ex:// Find C between a and b:



①: Assume Q + -Q are deposited

②: $\vec{E}$ between a and b: Gauss' Law

$$\vec{E} \cdot 2\pi r L = \frac{Q}{\varepsilon_0}$$

$$\vec{E} = \frac{Q}{2\pi r L \varepsilon_0}$$

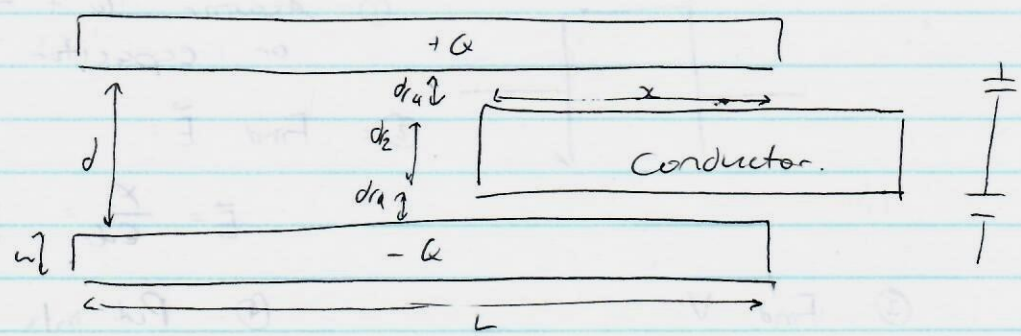
③: Finding voltage change:

$$V_{a\to b} = -\int_a^b \frac{Q}{2\pi r L \varepsilon_0} \cdot dr = \frac{Q}{2\pi L \varepsilon_0} \cdot \ln(a/b)$$

④: Put into equation:

$$C = \frac{Q}{V} = \frac{2\pi L \varepsilon_0}{\ln(a/b)}$$

-Ex://



Find capacitance of system:

$$C_{left} = \frac{A\varepsilon_0}{d} = \frac{(L-x)w\varepsilon_0}{d}$$

$$C_{right}: \quad C_{free} = \frac{4xw\varepsilon_0}{d} \quad (2x) \Big\} \text{ In ~~para~~ series}$$

$$\therefore \frac{1}{C_{right}} = \frac{1}{C_{free}} + \frac{1}{C_{free}} = \frac{2xw\varepsilon_0}{d}$$

$$C_{total} = C_{left} + C_{right} \text{ (parallel)}$$

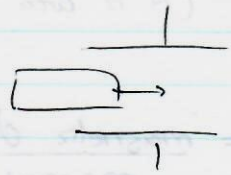
$$= \frac{(L+x)w\varepsilon_0}{d}$$

- Energy of a capacitor:

$$U = \int_{q_1}^{q_2} V\,dq = \int_{q_1}^{q_2} \frac{q}{C}\,dq = \frac{Q^2}{2C}$$

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

- Ex: //



How much work does it take to put in electric:

$$W = U_{final} - U_{initial}$$
$$= \frac{1}{2}C_f V_f^2 - \frac{1}{2}C_i V_i^2$$
$$= \frac{1}{2}(\varepsilon_r C_0)V^2 - \frac{1}{2}C_0 V^2$$
$$= \frac{1}{2}C_0 V^2(\varepsilon_r - 1)$$

- Force to put in:

$$F = -\frac{dU}{dx} \Rightarrow$$ 1. Create potential as $f(x)$
2. Differentiate.

MAGNETISM

$\vec{B}$: magnetic field (magnetic flux density)

$$F_{\vec{B}} = q(\vec{v} \times \vec{B}) = qvB\sin\theta \quad (\theta: \text{angle between } \vec{v} + \vec{B}) \quad [T]$$

①: Cross product: $\vec{F} \perp \vec{v}$, $\vec{F} \perp \vec{B}$

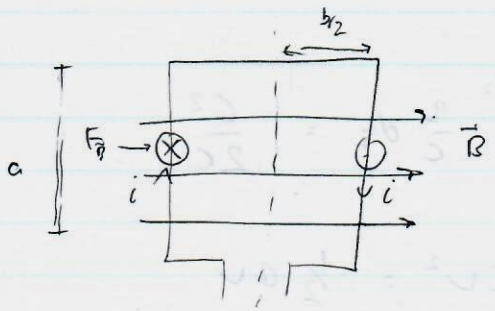
Right hand rule: palm = force, fingers = field, thumb in direction of velocity

②: $F_{\vec{B}} \propto \vec{v}$

Circular motion!

$$qvB = \frac{mv^2}{r}$$

$$\boxed{r = \frac{mv}{qB}}$$



$\therefore$ $F_{\vec{B}}$ is exerting torque on loop.

$F_{\vec{B}} = I \cdot L \cdot \vec{B}$ $\Rightarrow$ Wires

$$\tau = \frac{b}{2} \cdot I a \vec{B} + \frac{b}{2} \cdot I a \vec{B}$$
$$= I a b \vec{B}$$
$$= I A \vec{B} \qquad (A \text{ is area of loop})$$

$$\tau = I \vec{A} \times \vec{B}$$
$$= \vec{\mu} \times \vec{B}$$

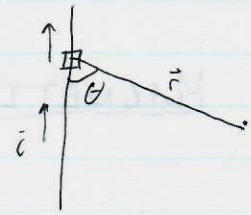
$\vec{\mu} = I\vec{A} \Leftarrow$ <u>magnetic dipole moment</u>

Torque + force decrease as loop moves.
↳ $\sin\theta \downarrow$ as $\vec{A}$ moves!

- Biot - Savart Law:

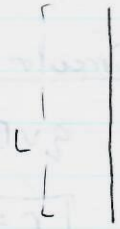
$$d\vec{B} = \frac{\mu_0 I \, d\vec{l} \times \hat{r}}{4\pi r^2}$$

- $d\vec{l}$: points in direction of current
- $\hat{r}$: unit vector of $\vec{r}$: $\hat{r} = \frac{\vec{r}}{|r|}$
- $\therefore |d\vec{l} \times \hat{r}| = |dl| \, |\hat{r}| \sin\theta = dl \sin\theta \Rightarrow$



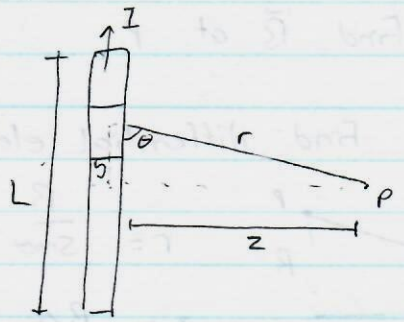
To find magnetic field direction given current: right hand
Thumb is direction of current. Curl fingers → field

Tip: b/c angles are involved, find an origin to use symmetric integrals.



$$\Rightarrow \int_0^L d\vec{B} \quad \text{vs.} \quad \int_{-L/2}^{L/2} d\vec{B} \checkmark$$

- Ex: //



What is $\vec{B}$ at P.

① : Differential element:

$$d\vec{B} = \frac{\mu_0 I \cdot dl \times \hat{r}}{4\pi r^2}$$

Simplify:

$$d\vec{B} = \frac{\mu_0 I}{4\pi r^2} |d\vec{s}| \cancel{|\hat{r}|}^{1} \sin\theta$$

$$= \frac{\mu_0 I}{4\pi r^2} d\vec{s} \cdot \frac{z}{r}$$

$$= \frac{\mu_0 I z}{4\pi r^3} d\vec{s}$$

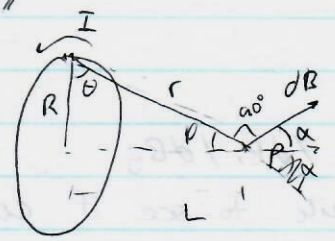
② : Define:

$$r = \sqrt{b^2 + z^2} \Rightarrow d\vec{B} = \frac{\mu_0 I z}{4\pi (y^2+z^2)^{3/2}} d\vec{y}$$

③ : Integrate : symmetric integrals

$$\vec{B} = \int_{-L/2}^{L/2} \frac{\mu_0 I z}{4\pi (y^2+z^2)^{3/2}} dy$$

- Ex: //



Find $\vec{B}$ at point P.

① Differential element:

$$d\vec{B}_z = \left(\frac{\mu_0 I}{4\pi r^2} d\vec{s} \times \hat{r}\right) \cancel{\cos\alpha} \cos\alpha$$

Angle between dB + dB_z

$90° + \varphi + \alpha = 180°$!

Need to find dz, ~~at~~ otherwise it is really here

$$= \left(\frac{\mu_0 I}{4\pi r^2} |d\vec{s}| \cancel{|\hat{r}|}^{1} \sin\theta\right) \cancel{\sin\theta} \sin\varphi$$

$$= \frac{\mu_0 I}{4\pi r^2} (R\, d\phi) \cdot \frac{R}{r}$$

$$= \frac{\mu_0 I R^2}{4\pi (R^2+L^2)^{3/2}} d\phi$$

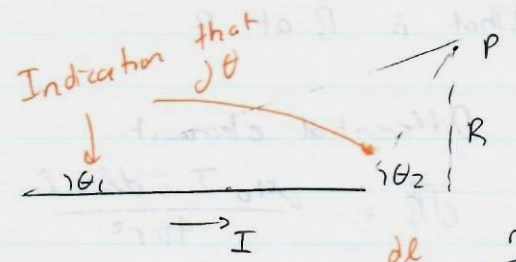
Imagine plane w/ $\hat{r}$ + P. $d\vec{s} \perp$ plane, so $\sin\theta = 1$

② : Integrate:

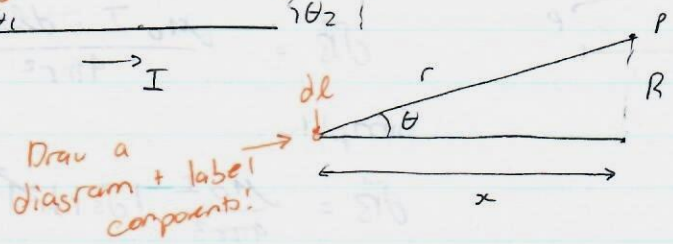
$$\vec{B} = \int_0^{2\pi} \frac{\mu_0 I R^2}{4\pi (R^2+L^2)^{3/2}} = \frac{\mu_0 I R^2}{2(R^2+L^2)^{3/2}}$$

- Ex: //

Find $\vec{B}$ at P.



① Find differential element:



$$r = \frac{R}{\sin\theta} = \underline{R\csc\theta}$$

$$x = R/\tan\theta$$

$$dx = \underline{-R\csc^2\theta\, d\theta}$$

Relation.

$$d\vec{B} = \frac{\mu_0 I\, d\ell \times \hat{r}}{4\pi r^2} = \frac{\mu_0 I\,|(-R\csc^2\theta)|\,|\hat{r}|\sin\theta\, d\theta}{4\pi\,(R\csc\theta)^2}$$

$$= \frac{\mu_0 I \cdot R^2\csc^2\theta\,\sin\theta\, d\theta}{4\pi\, R^2\csc^2\theta}$$

$$= \frac{\mu_0 I \sin\theta}{4\pi R}\, d\theta$$

②: Integrate:

$$\therefore \vec{B} = \int_{line} d\vec{B} = \int_{\theta_1}^{\theta_2} \frac{\mu_0 I \sin\theta}{4\pi R}\, d\theta$$

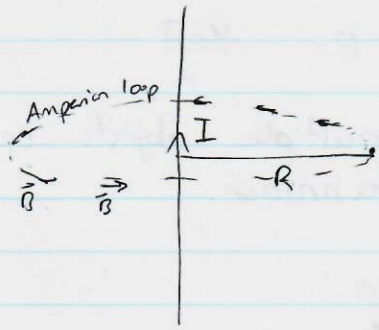
$$= \frac{\mu_0 I}{4\pi R}(\cos\theta_2 - \cos\theta_1)$$

- Tips:
  - Symmetric integrals
  - Sometimes need to do $dB_z / dB_x / dB_y$
    - ↳ Use right-hand curl rule to see if component in x/y/z axis is cancelling ⇒ component B.S.
  - Use given variables for integration (ex: prev. problem)
  - Use derivatives to find relations (ex: prev. problem)
  - DO NOT FORGET ABS. VALUE

AMPERE'S LAW

Ampere's Law simplifies Biot-Savart

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \Leftarrow \underline{\text{Area}}.$$



$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\oint |\vec{B}||dl|\cos 0^\circ = \mu_0 I_{enc}$$

$$B \cdot 2\pi R = \mu_0 I$$

I_{enc} is I running through cross-section

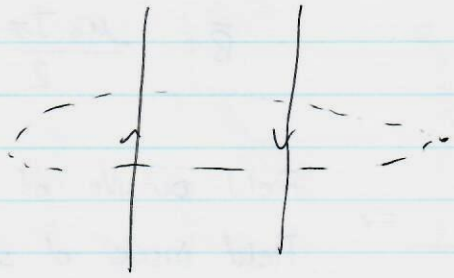
$$B = \frac{\mu_0 I}{2\pi R}$$

Ampere's Law ← Uniformity: field value remains same along loop

Ampere's Law ← Symmetry: field distribution same along loop

$$\oint \vec{B} \cdot d\vec{l} = \oint \underbrace{|B|}_{\text{uniform}} |dl| \underbrace{\cos\theta}_{\text{symmetric} \Rightarrow \text{constant!}}$$

Can be evaluated every where, but easier if uniform + symmetric.



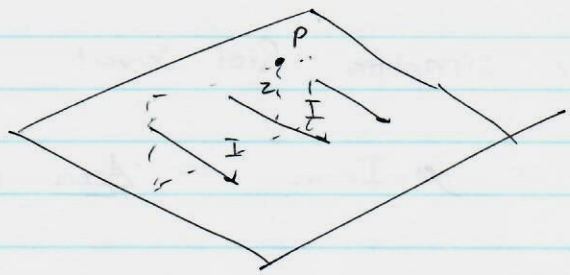
Using Amperian loop:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\oint \vec{B} \cdot dl = 0$$

However, $\vec{B} \neq 0 \Leftarrow$ field not uniform or symmetric!

Better way: superposition!



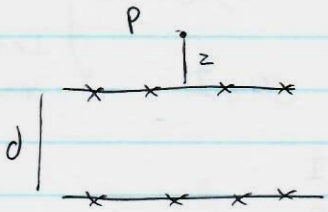
Finding $\vec{B}$ at P:

$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$

$$2BL = \mu_0 \cancel{I} JL$$

$$B = \frac{\mu_0 J}{2}$$

- Superposition: you have holes/ multiple objects contributing to field, use this technique.



Find $\vec{B}$ at P:

①: $\oint \vec{B} \cdot dl = \mu_0 I_{enc}$

$2Bl = \mu_0 \cancel{I} Jl$

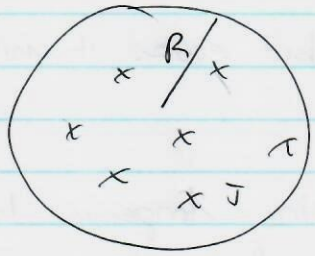
$\therefore B = \frac{\mu_0 J}{2}$

②: ⋮

$B = \frac{\mu_0 J}{2}$

$\therefore$ Total $\vec{B} = \vec{B}_T = \underline{\mu_0 J}$

- $I_{enc} = f(R, \theta, \ldots) \Rightarrow$ integration to find I_{enc}:
  - ↳ Keep in mind that variables from contour integral + ~~the~~ I_{enc} could be the same!



Find $\vec{B}$ at $r < R$:

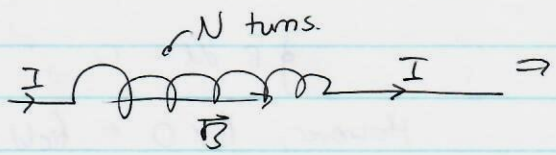
$$\vec{B} \cdot 2\pi r = \mu_0 J \pi r^2$$

$$\therefore \vec{B} = \frac{\mu_0 J r}{2}$$

Find $\vec{B}$ at $r > R$:

$$\vec{B} \cdot 2\pi r = \mu_0 J \pi R^2$$

$$\vec{B} = \frac{\mu_0 J R^2}{2r}$$



$\Rightarrow$ Field outside of solenoid = 0.

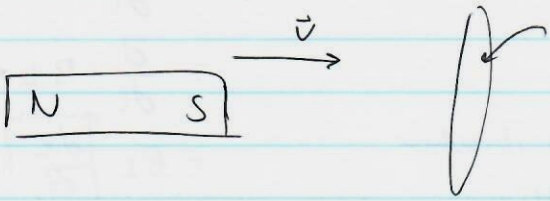
Field inside of solenoid: $\vec{B} = \mu_0 N I$

Direction by solenoid right hand rule!

INDUCTION

Changing magnetic flux $\Rightarrow$ current, $\mathcal{E}$

Flux: $\Phi_B = \vec{B} \cdot \vec{A} = |B||A| \cos\theta$



$B \downarrow$ as S pole gets closer

$\therefore \Delta\Phi \neq 0 \Rightarrow$ induced $\mathcal{E}$.

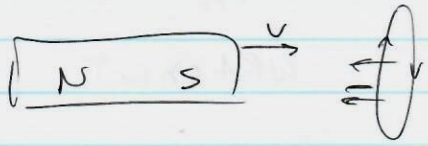
Focusing on EMF:

① : Magnitude:

$$\mathcal{E}_{ind} = \frac{d}{dt}\Phi \quad \Rightarrow \underline{\text{Faraday's Law.}}$$

② : Direction:

Induced current in direction s.t. flux change is opposed:



We want to oppose the increase in $\vec{B}$ going left $\Rightarrow$ $\vec{B}$ to go right

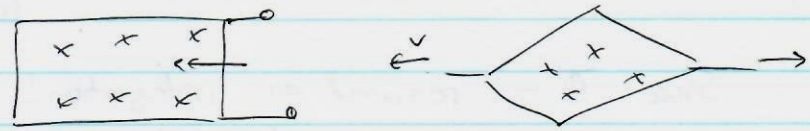
$\hookrightarrow$ Current must go in CW

<u>Lenz' Law</u>

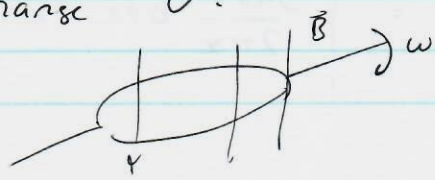
<u>How to change flux:</u>

① : Change $\vec{B}$: alternating field + alt-current

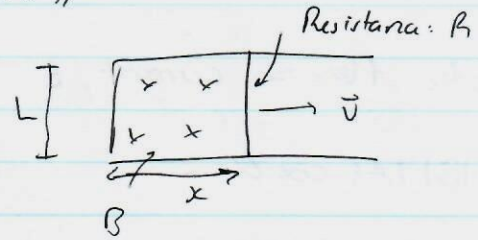
② : Change A :



③ : Change θ :



- Ex: //



Find $\mathcal{E}_{ind}$:

$$\underline{\mathcal{E}_{ind}} = \frac{d\Phi}{dt}$$

$$= \frac{d}{dt}\vec{B}\cdot\vec{A}$$

$$= \frac{d}{dt} BLx$$

$$= BL\boxed{\frac{dx}{dt}}$$

$$= BLv \;!$$

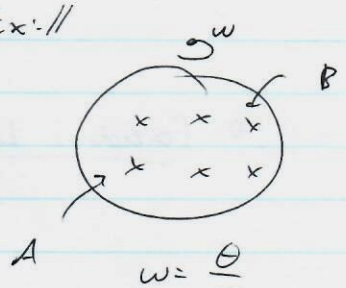
Look @ which var change w time.

Direction:

Current in CCW direction

↳ Oppose ↓ flux

- Ex: //



$$\omega = \frac{\theta}{T}$$

$$\theta = \omega T$$

Find $\mathcal{E}_{ind}$:

$$\underline{\mathcal{E}_{ind}} = \frac{d}{dt}\Phi$$

$$= \frac{d}{dt}\vec{B}\cdot\vec{A}$$

$$= \frac{d}{dt} BA\cos\theta$$

$$= BA\left(\frac{d}{dt}\cos\omega T\right)$$

$$= \omega BA\sin\omega T$$

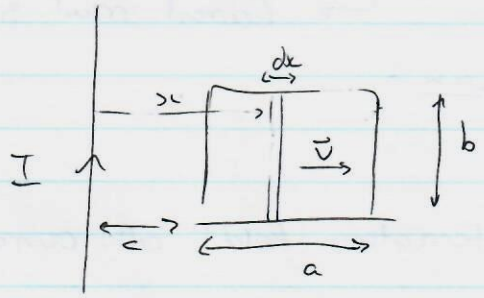
Direction:

Should go in CCW direction.

$$\underline{I_{ind}} = \frac{\mathcal{E}}{R}$$

$$= \frac{\omega BA\sin\omega T}{R}$$

- Ex: //



a) Find Φ:

$$\Phi_T = B_T \cdot A_T$$

Since B is changing, employ Biot Savart or Ampere's Law

Using Ampere's Law:

$$B\cdot 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi R}$$

Since Φ_T required ⇒ integration!

$$\Phi = \int B\,dA = \int_c^{c+a} \frac{\mu_0 I}{2\pi x}\, b\,dx = \frac{\mu_0 I b}{2\pi}\ln\left(\frac{c+a}{c}\right)$$

b) Find $\mathcal{E}_{ind}$:

$$\mathcal{E}_{ind} = \frac{d}{dt}\Phi_T$$

$$= \frac{d}{dt}\frac{\mu_0 I b}{2\pi}\ln\left(\frac{c+a}{c}\right)$$

$$= \frac{\mu_0 I b}{2\pi}\frac{d}{dc}\ln\left(\frac{c+a}{c}\right)\cdot\frac{dc}{dt} \leftarrow v$$

Moving field problems:

①: Expression of field via Ampere / Biot Savart

②: Find $\Phi_{total} \Rightarrow$ integration of $\boxed{d\Phi = B\cdot dA}$

③: Take derivative of Φ_{total}.

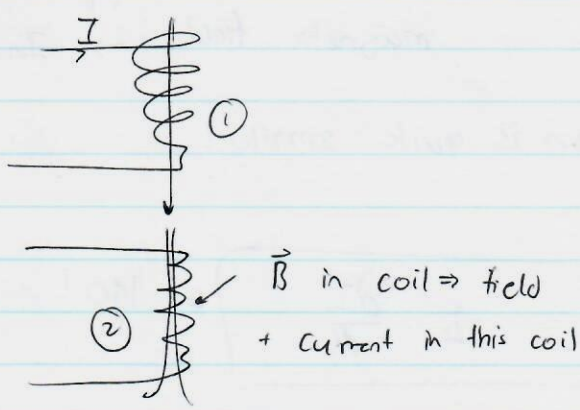
Note: Changing flux $\Rightarrow$ non-conservative field!

$$\oint \vec{E}_{nc}\cdot ds = \frac{d\Phi}{dt} \quad \Leftarrow \quad \mathcal{E} = \oint \vec{E}\cdot ds$$

Direction of I_{ind}

Solenoids: find flux of 1 loop $\Rightarrow$ multiply by N (# of loops)

MUTUAL + SELF - INDUCTANCE



Concept of mutual inductance:

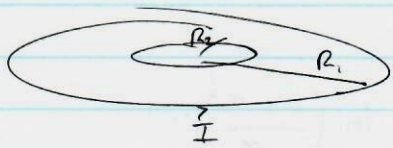
1. Find ~~flux~~ $\vec{B}$ from one object (①)
2. Find flux ~~of~~ in ② from field of ①

$$\Phi_{21} = B_1 \cdot A_2$$

3. Mutual inductance constant

$$m = \frac{\mathcal{E}_{ind}}{dI/dt} = \frac{\Phi_{21}}{I_1} \quad [H]$$

- Ex: //



Find mutual inductance between 2 coils.

1. Field of R_1 loop:

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi r} \Rightarrow \text{At center: } \frac{\mu_0 I}{2\pi R_1}$$

Assume that $\vec{B}_1$ is field in inner loop (not changing b/c $R_2 \ll R_1$)

2. Finding flux:

$$\Phi_{21} = \vec{B}_1 \cdot A_2 = \frac{\mu_0 I}{2\pi R_1} \cdot \pi R_2^2 = \frac{\mu_0 I R_2^2}{2R_1}$$

3. Mutual inductance:

$$M = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 R_2^2}{2R_1}$$

- Self inductance:

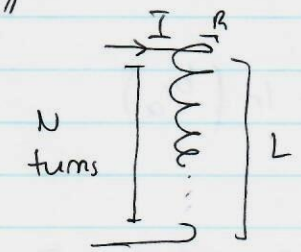
$\Rightarrow$ Each loop has own magnetic field induced s.t. it opposes other loops magnetic field. $\Rightarrow$ Inductance: L

Process of calculation is quite similar!

Important note:

$$\boxed{\mathcal{E}_{ind} = -L \cdot \frac{dI}{dt}} \Leftarrow \text{140!}$$

- Ex: //



Find self inductance of solenoid.

1. Field:

$$\vec{B} = \frac{\mu_0 N I}{L} = \mu_0 n I$$

2. Φ in loops:

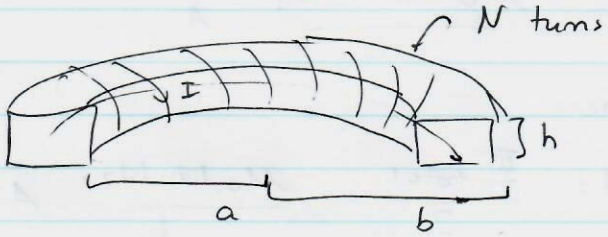
$$\Phi_{1\,loop} = \vec{B} \cdot A_{loop} = \mu_0 n I \cdot \pi R^2$$

$$\Phi_{total} = N \cdot \Phi_{1\,loop} = \frac{\mu_0 N^2 I \pi R^2}{L} = \mu_0 n^2 I \pi R^2 L$$

3. Self inductance:

$$L = \frac{\Phi_{total}}{I} = \boxed{\mu_0 n^2 \pi R^2 L}$$

- Ex: //



Find self-inductance.

1. Find $\vec{B}$:

By Ampere's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\vec{B} \cdot 2\pi r = \mu_0 N I$$

$$\vec{B} = \frac{\mu_0 N I}{2\pi r}$$

2. Find Φ:

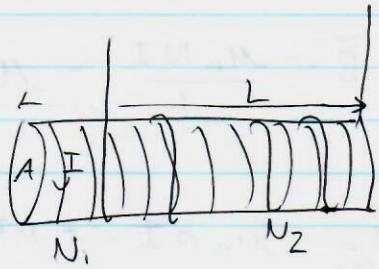
$$\Phi_{1\,loop} = \int_a^b B\,dA = \int_a^b \frac{\mu_0 N I}{2\pi r} h\,dr = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$\therefore \Phi_{total} = N \cdot \Phi_{1\,loop} = \frac{\mu_0 N^2 I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

3. Self-inductance:

$$L = \frac{\Phi_{total}}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(b/a\right)$$

- Ex: //



Mutual inductance?

1. Find field:

$$\vec{B} = \frac{\mu_0 N_1 I_1}{L}$$

2. Find flux:

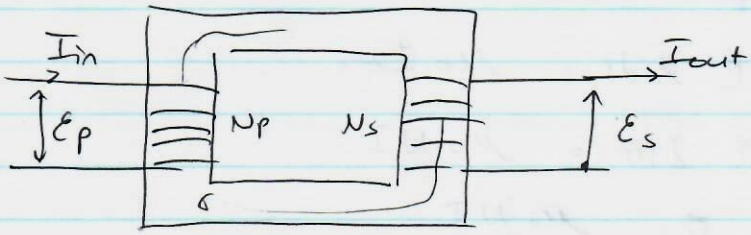
$$\Phi_{loop} = \vec{B} \cdot A = \frac{\mu_0 N_1 I_1}{L} \cdot A$$

$$\Phi_{total} = \frac{\mu_0 N_1 N_2 I_1}{L} \cdot A$$

3. Mutual inductance:

$$M = \frac{\Phi_{total}}{I} = \frac{\mu_0 N_1 N_2}{L} \cdot A$$

- Transformers:



From before:

$$\mathcal{E}_p = N_p \cdot \frac{d\Phi}{dt}$$

$$\mathcal{E}_s = N_s \cdot \frac{d\Phi}{dt}$$

B/c same toroid $\Rightarrow$ same $\vec{B}$ $\Rightarrow$ same Φ/t

$$\therefore \boxed{\frac{\mathcal{E}_p}{N_p} = \frac{\mathcal{E}_s}{N_s}}$$