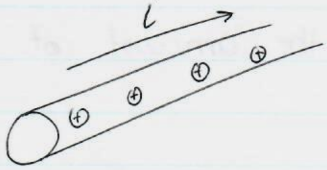
ECE 140 FEYNMAN

CURRENT



$\Rightarrow$ Current direction is based on $\oplus$ charge movement

Mathematically:

$$\underset{\text{current (A)}}{i} = \frac{dq}{dt} \quad \begin{matrix} \leftarrow \text{charge (C)} \\ \leftarrow \text{time (s)} \end{matrix}$$

- To get q: integration

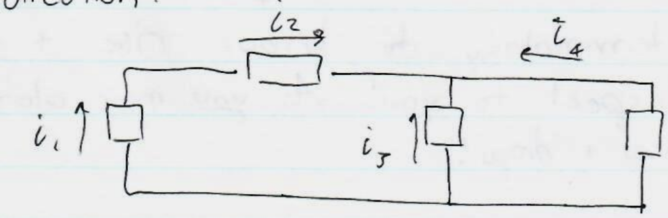
$$dq = i\,dt$$

$$\therefore \quad q = q(t_0) + \int_{t_0}^{t_1} i(\tau)\,d\tau$$

To get all-time current, assume $t_0 = -\infty$ and $q(t_0) = 0$

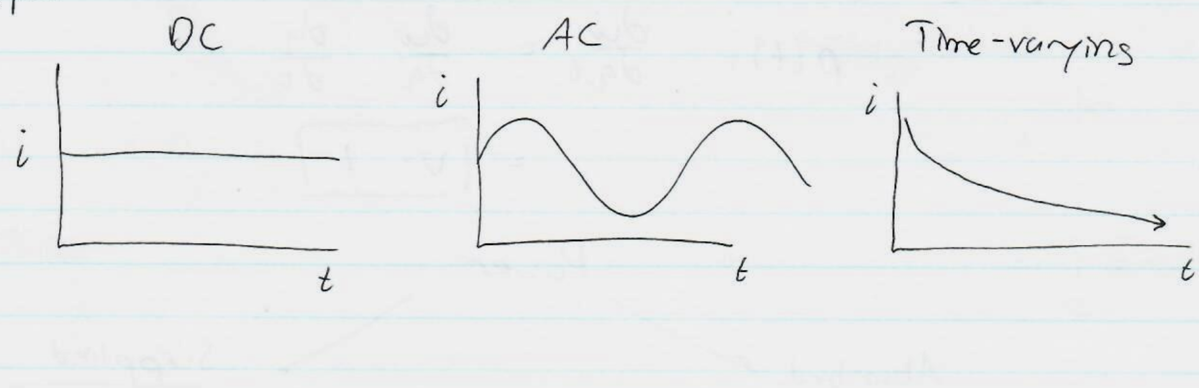
$$\therefore \quad q = \int_{-\infty}^{t_1} i(\tau)\,d\tau$$

Reference direction:



1. Choose current direction arbitrarily
2. If current val. is negative, reverse direction

Types:



VOLTAGE

Think of voltage as the amount of work a single charge can do.

Mathematically:

$$V = \frac{dw(t)}{dq}$$

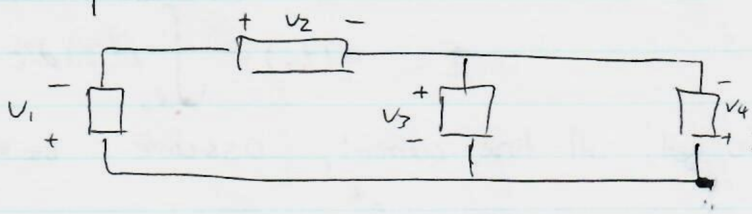
$dw(t)$ ← work (J)

dq → charge (C)

voltage ($J/C = V$)

Much like current, find work via integration.

Reference polarities:



1. Put +, − to indicate high + lo volt respect.
2. Arbitrary (negation rule like current

- Can also use an arrow (− to +)

Quick terminology to know: rise + drop

↳ Respect to you! As you move along circuit, you have rises + drops!

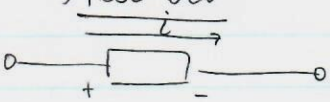
POWER

$$p(t) = \frac{dw}{dq\,t} = \frac{dw}{dq} \cdot \frac{dq}{dt}$$

$$= \boxed{v \cdot i}$$

Power

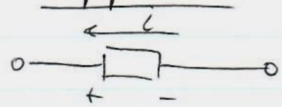
Absorbed



⇒ Passive sign

$P > 0$

Supplied



$P > 0$

Negative: switch!

Energy conservation:

$$\sum \text{supplied pow} = \sum \text{absorbed pow}$$

To find energy from power: simple integration.

Circuit Basics

Elements

Active: supply energy (ability)

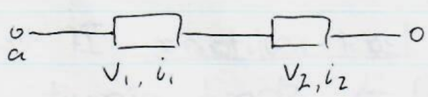
- Independent (voltage/current source)
- Dependent

Passive: do not generate electricity

↳ resistors, capacitors, inductors

Connections

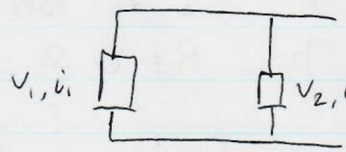
Serial:



$V_1 \neq V_2 \Rightarrow$ Valid + correct

$i_1 \neq i_2 \Rightarrow$ Invalid

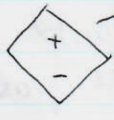
Parallel:



$V_1 \neq V_2 \Rightarrow$ ~~≢~~ Invalid

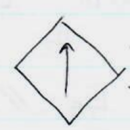
$i_1 \neq i_2 \Rightarrow$ Valid

Dependent Sources



→ $V = \alpha V_x$

→ $V = \alpha i_x$



→ $i = \alpha V_x$

→ $i = \alpha i_x$

Resistors

↳ Resistance: $R \geq 0, \Omega$

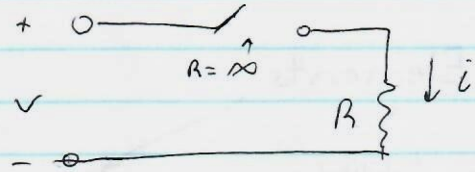
Ohm's Law = $\boxed{V = I \cdot R}$ (change into dif. forms: $\frac{V^2}{R} = P$) using power rule)

Conductance:

$$G = \frac{1}{R} \quad (\Omega^{-1} = \text{Siemens (S)})$$

Know how to express V, I, R, P is terms of G.

OPEN + SHORT CIRCUITS



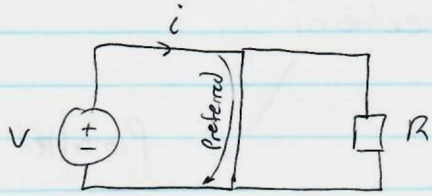
Open circuit

$\Rightarrow$ $V = IR$

$I = \frac{V}{R}$ + let $R \to \infty$

$\therefore \boxed{\lim_{R \to \infty}} I = 0$

Open circuit is when $I = 0$.



Short circuit

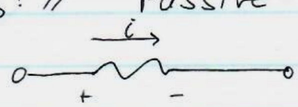
$I = \frac{V}{R} \leftarrow 0!$

e^- like to take path of least resistance. If the path has $R = 0\ \Omega \Rightarrow \boxed{V = 0} \Rightarrow$ short circuit.

BASIC CIRCUIT SOLVING

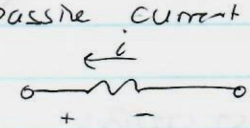
- Recipie:

1. Assign unknown voltages, current, resistance.
   ↳ Use serial + parallel laws to simplify problem.
2. Resistor: use Ohm's Law.
3. Solve the system. w/ polarities in mind.
   ↳ Eg: // Passive vs not passive current



$\boxed{V = IR}$

vs.



$\boxed{V = -iR}$

* Passive sign convention + Ohm's Law are extremely important. Follow this as dogma

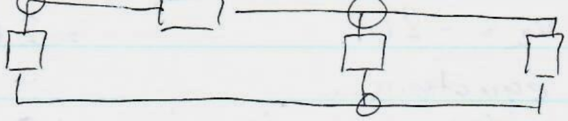
KIRCHOFF LAWS

1. Kirchoff's Current Law (KCL):

   The sum of current at a node is 0.

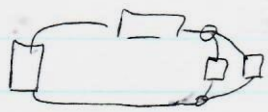
   $$\sum i_{in} = \sum i_{out}$$ (assign polarity to currents) ONLY @ node

   Node: 2/more element connected at 1 point

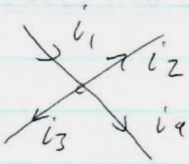


   Trick to finding nodes: take empty wires + try to remove as much as possib.

   Ex://



   Ex://



   $\Rightarrow$ KCL: $i_1 = i_2 + i_3 + i_4$

   To solve circuit using only KCL:

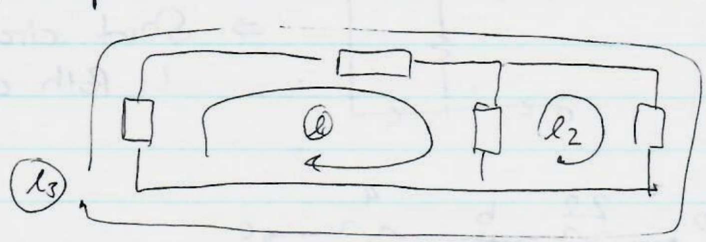
   # of eq. = # of nodes − 1

2. Kirchoff's Voltage Law (KVL):

   The algebraic sum of voltages along a loop is 0

   $$\sum V_{drops} = \sum V_{rises}$$

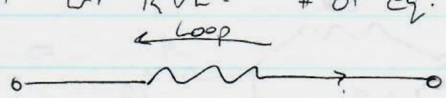
   Loop: path where no element encountered more than once.



   Loop → mesh: loop w/ no inner loop (l_1, l_2 are meshes)

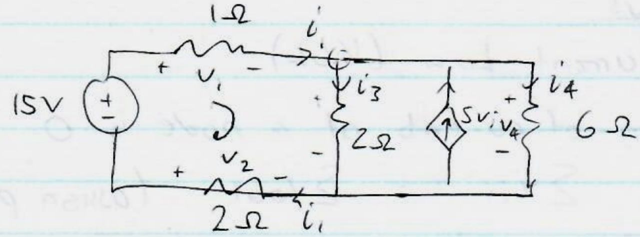
   Solving circuit w/ KVL: # of eq. = # of meshes.

Note*:



← Consider as voltage rise, no passive sign needed as i going + → −

- Ex: //



Find $i_1, i_3, i_4, v_1, v_2, v_4$

1. Ohm's law equations.

   ①: $v_1 = i_1$ ③: $v_4 = 6 i_4$

   ②: $v_2 = -2 i_1$ $= 2 i_3 \Leftarrow$ 2 Ω resistor

2. Kirchoff equations:

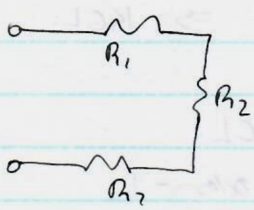
   KCL: $5 i_1 + i_1 = i_3 + i_4$ - ④

   KVL $15 + v_2 = v_1 + v_4$ - ⑤

   Solve.

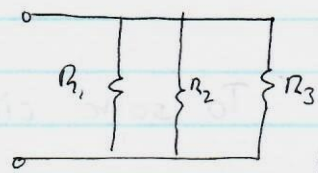
Resistor Equivalence

Series:



$$R_{eq} = R_1 + R_2 + R_3$$

Parallel:



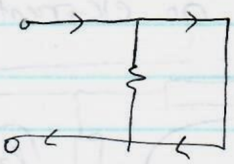
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

↳ $G_{eq} = G_1 + G_2 + G_3$

- Common formula for 2 resistor parallel circuit.

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

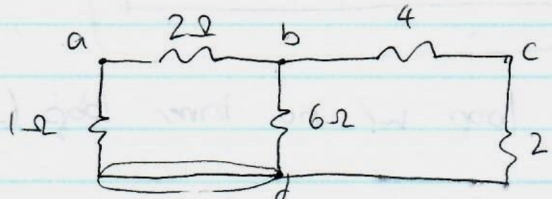
- Resistor short circuits:



⇒ Short circuit

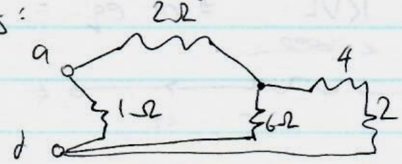
↳ Path of no resistance.

- Ex: //



Find R_{eq} between a and d

Redrawing:



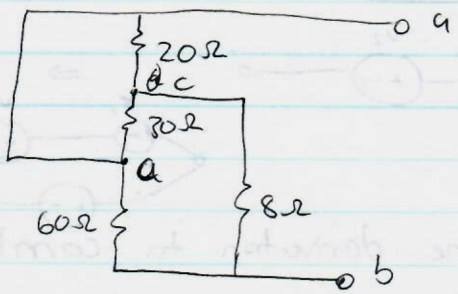
⇒ 4-2 ⇒ series

↳ parallel w 6

↳ series with 2 Ω

Parallel ~~Series~~ w 1 Ω

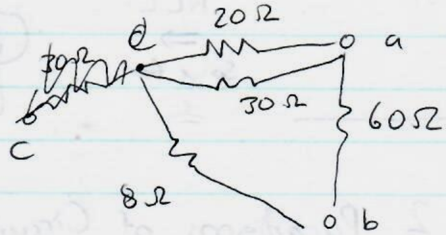
- Ex: //



Find R_{eq} between a and b.

Is this a short circuit? No ⇒ every path has resistance.

1. Redraw diagram:



⇐ Key!!
- ↳ Identify nodes + connect.
- ↳ Minimize empty wire.

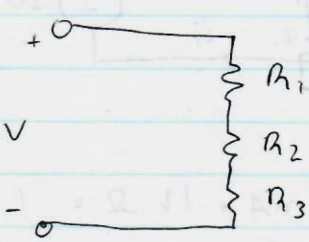
2. Calculate:

$$R_{eq} = \frac{20 \times 30}{20 + 30} = 12\ \Omega$$

8Ω in series: $20\ \Omega$

$$R_T = \frac{20 \times 60}{20 + 60} = \frac{1200}{80} = \boxed{15\ \Omega}$$

CURRENT + VOLTAGE DIVISION

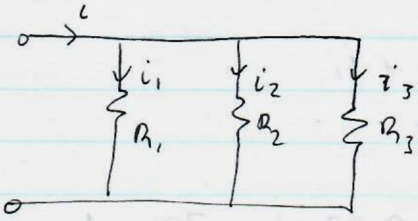


$$V_{R_1} = \frac{R_1}{R_1 + R_2 + R_3} V$$

$$V_{R_2} = \frac{R_2}{R_1 + R_2 + R_3} V$$

$\vdots$

→ Series circuit!



$$i_1 = \frac{G_1}{G_1 + G_2 + G_3} i$$

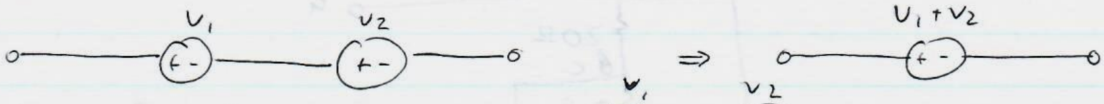
$$i_2 = \frac{G_2}{G_1 + G_2 + G_3} i$$

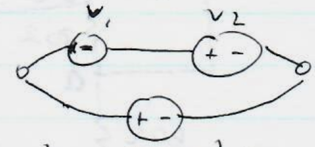
$\vdots$

Parallel circuit!

Use: when you know all resistor values.

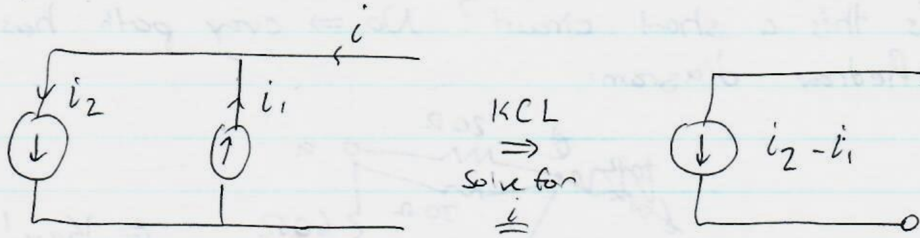
VOLTAGE + CURRENT SOURCE EQUIVALENCE





← KVL derivation

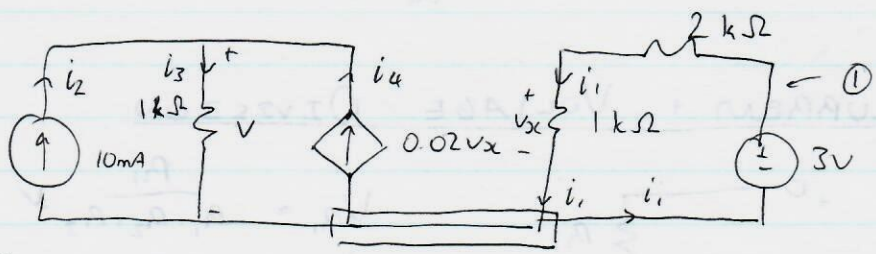
Use the same derivation to combine sources in differing polarities.



2 Paradigms of Circuit

KCL, KVL + Ohm's Law	Circuit Reduction
◦ Requires special attention to passive sign convention	◦ Requires resistor + source equivalence + Ohm's Law.

- Ex: //



Find V:

①: Analyze circuit branch ①

$R_{T①} = 3\,k\Omega$ $\qquad v_x = 1mA \times 1k\,\Omega = 1\,V$

$i_① = 1\,mA$

②: KCL on squared node:

$$i_1 + i_3 = i_2 + i_4 + i_1$$

$$i_3 = i_2 + i_4$$

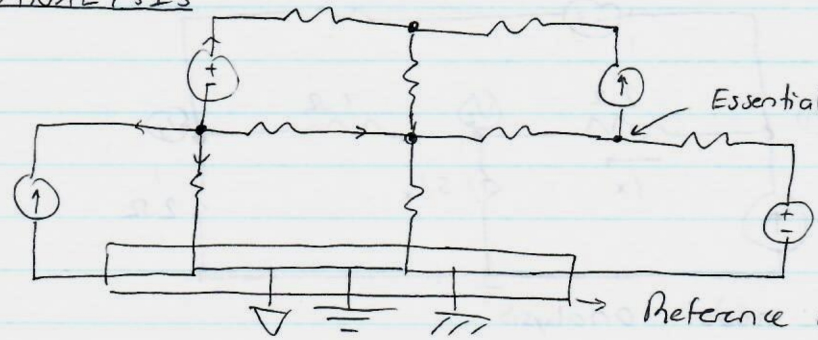
③: Solve for i_3 + V:

$$i_3 = 10\,mA + 0.02 \times 1 = 30\,mA$$

$$V = i_3 \cdot R_3$$

$$= \boxed{30\ V}$$

Nodal Analysis

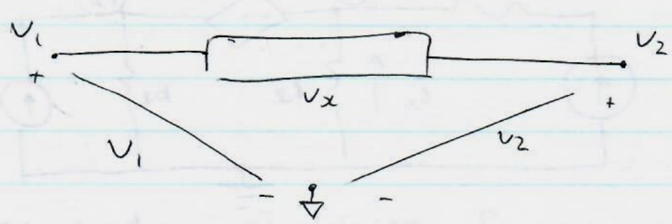


Each essential node has node voltage (voltage drop to ref. node)

General recipie:

1. Define essential + reference nodes + node voltages
2. Assign currents to branches (optional) + make them leave node
3. Perform KCL on all essential nodes.
4. Express currents in terms of node voltages + resistors (Ohm's Law)
   ↳ Passive sign convention: don't worry about it.
5. Solve.

Step 4: 3-point rule/ voltage source



Performing KVL: $V_x = V_1 - V_2$

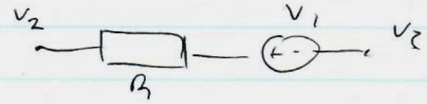
$$i_x = \frac{V_1 - V_2}{R}$$

Just take voltage difference across branch + divide by resistor.

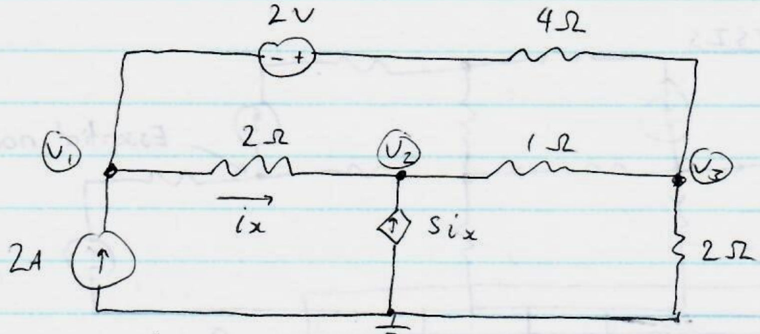
If voltage source in branch:

① Source is between node + reference.
   Define voltage at node = voltage source.

② Source is between 2 nodes:
   Account for it in 3 point rule



$$\Rightarrow i_x = \frac{V_2 - V_3 - V_1}{R}$$



Apply nodal analysis:

At node ①:

$$\frac{v_1 + 2 - v_3}{4} + \overset{i_x}{\frac{v_1 - v_2}{2}} - 2 = 0 \quad -(1)$$

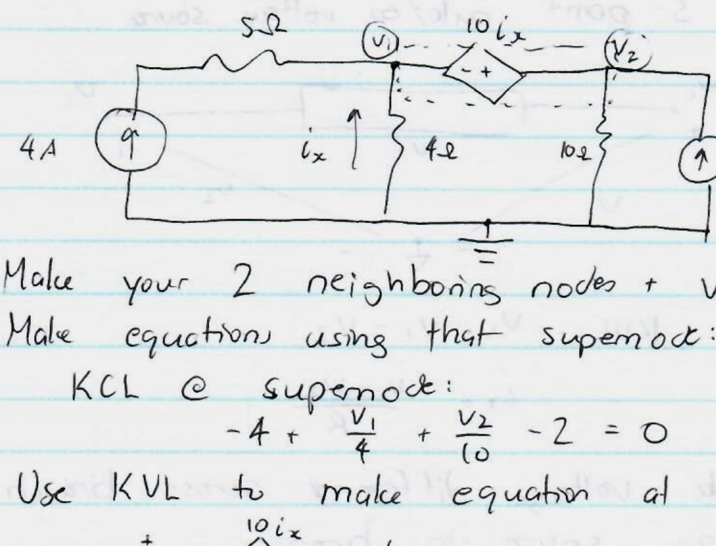
At node ②:

$$\frac{v_2 - v_1}{2} - 5\left(\frac{v_1 - v_2}{2}\right) + \frac{v_2 - v_3}{1} = 0 \quad -(2)$$

At node ③:

$$\frac{v_3 - v_1 - 2}{4} + \frac{v_3 - v_2}{1} + \frac{v_3}{2} = 0 \quad -(3)$$

Supernode: only a voltage source between two non-reference nodes.



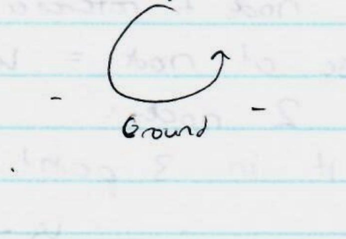
Can be defined anywhere.

1. Make your 2 neighboring nodes + voltage source as 1 node (supernode)
2. Make equations using that supernode:

   KCL @ supernode:

   $$-4 + \frac{v_1}{4} + \frac{v_2}{10} - 2 = 0$$

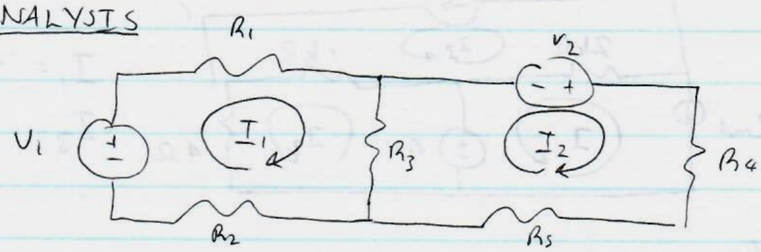
3. Use KVL to make equation at supernode:



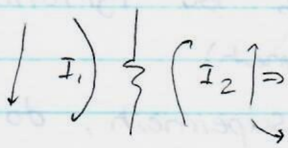
$$\Rightarrow \quad \boxed{V_1 - V_2 + 10i_x = 0}$$

4. Solve.

Mesh Analysis



1. Define a mesh current: imaginary current that flows around a mesh
2. Write branch currents in terms of mesh current
   ↳ What if an element is shared between 2 meshes:
   Current: Cur of 1 loop − Cur of other



$\Rightarrow$ $i = I_1 - I_2$

$i = I_2 - I_1$

3. Perform KVL in each mesh in direction of mesh current
   - Assume every resistor is drop
   - Sources:

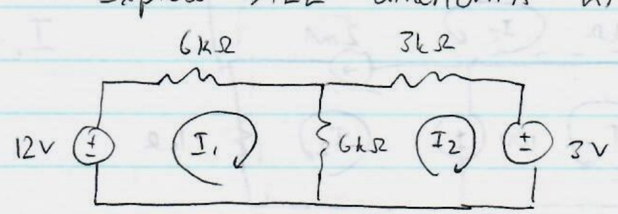
     Voltage source: contributes to KVL

     Current source:

     ↳ Only in 1 mesh: mesh current = current sour.

     ↳ Shared: supermesh
4. Use Ohm's Law + mesh currents to solve for mesh currents.
   ↳ Express ALL unknowns w/ mesh currents

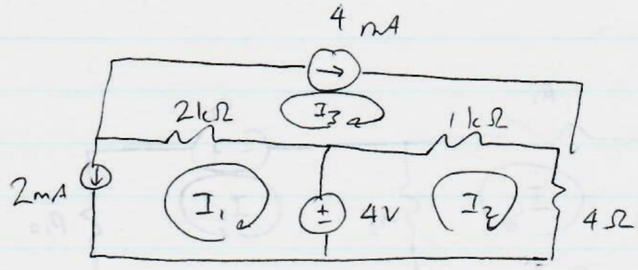
- Ex: //



①: $-12 + 6I_1 + 6(I_1 - I_2) = 0$

②: $6(I_2 - I_1) + 3(I_2) + 3 = 0$

- Ex: //



$I_1 = -2\ mA$

$I_3 = 4\ mA$

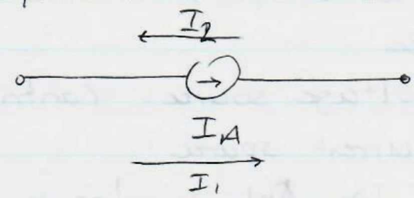
①: $2(I_1 - I_3) + 4 = 0$

②: $-4 + (I_2 - I_3) + 4I_2 = 0$ ⇒ This one is needed! I_2 is unknown

③: $2(I_3 - I_1) + (I_3 - I_2) = 0$

Supermeshes: current source shared between meshes

1. Create meshes by ignoring branch w/ shared current (supermesh)
2. Within the supermesh, do KVL w/ original mesh currents.
3. Create a equation for shared current source.

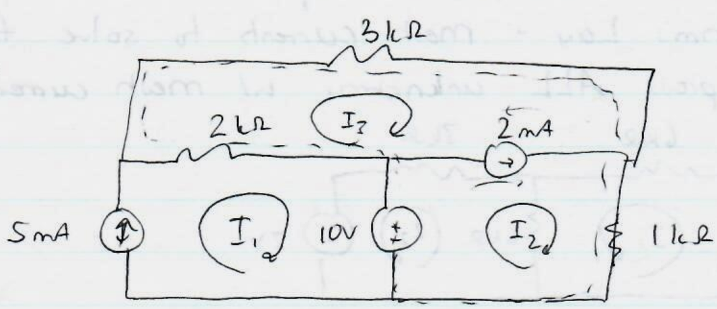


I_A is going in I_1 dir.

$\therefore\ I_1 > I_2$

$\therefore\ I_A = I_1 - I_2$

- Ex: //



$I_1 = 5mA$

①: $-10 + 2(I_3 - I_1) + 3I_3 + I_2 = 0$

②: $I_2 - I_3 = 2$

Nodal vs. Mesh

Nodal:
- Parallel circuits
- Current sources
- Less nodes & meshes

Mesh:
- Series circuits
- Voltage sources
- Less meshes & nodes

LINEARITY

Linearity
↙ ↘
Superposition — Homogeneity.

①: Superposition:

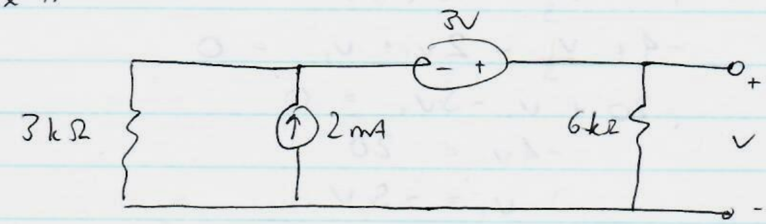
Current/voltage across an element = sum of the individual contribution of each indep. source.

1. Turn off voltage/current:

   0 V ⇒ short circuit. 0 A ⇒ open circuit

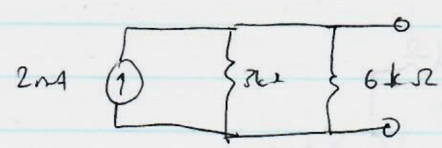
2. Calculate desired output
3. Add them up.

Ex: //



⇐ Find v.

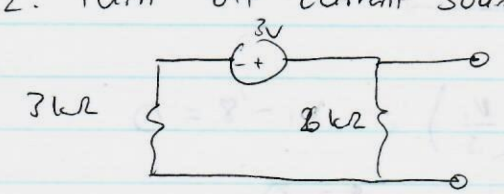
1. Turn off voltage.



$R_T = \frac{6 \times 3}{6+3} = 2\,\Omega$

$\therefore V_T = I_T \times R_T = \cancel{4A}\ \boxed{4V}$

2. Turn off current source.



Perform voltage division:

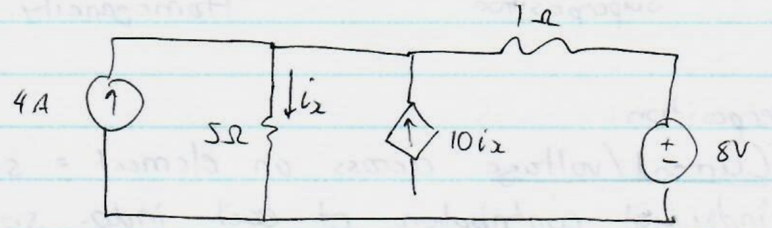
$V_{6\Omega} = 3 \cdot \frac{6}{9} = \boxed{2\ V}$

3. Total:

$V_T = V_{2mA} + V_{3V} = 4 + 2 = \underline{6V}$

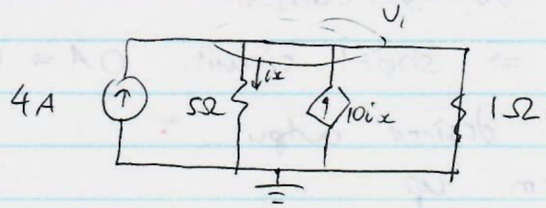
Dealing w/ dependent sources: stick to rules (turning off indep. sources) + use other technique.

Ex: 11



Find i_x.

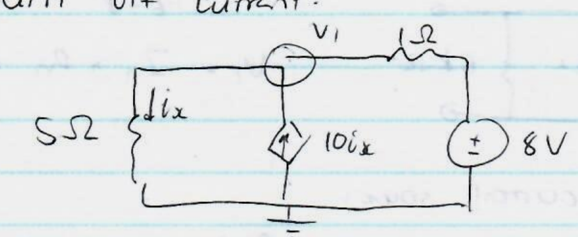
1. Turn off voltage circuit:



Nodal analysis:

$$-4 + \frac{v_1}{5} - 10\left(\frac{v_1}{5}\right) + v_1 = 0$$
$$-4 + \frac{v_1}{5} - 2v_1 + v_1 = 0$$
$$-20 + v_1 - 5v_1 = 0$$
$$-4v_1 = 20$$
$$v_1 = -5\ V$$
$$\therefore \quad i_x = -1A$$

2. Turn off current:

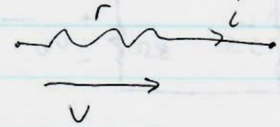


Nodal analysis:

$$\frac{v_1}{5} - 10\left(\frac{v_1}{5}\right) + v_1 - 8 = 0$$
$$\frac{v_1}{5} - 2v_1 + v_1 - 8 = 0$$
$$v_1 - 5v_1 - 40 = 0$$
$$v_1 = -10\ V$$
$$i_x = -2\ A$$

3. Total: $i_x = i_x' + i_x'' = -1 - 2 = \boxed{-3A}$

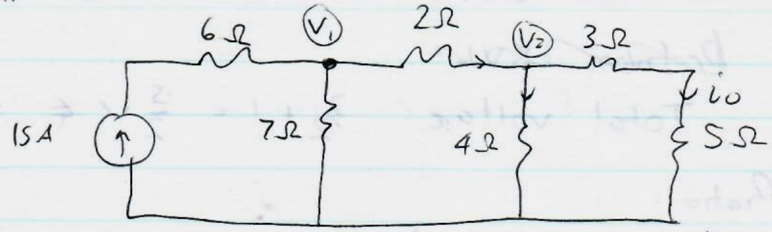
② Homogeniety:



} 2x V ⇒ 2x i ⇒ Ratios constant

Useful: find ratios between known + unknown parts of circuit + solve.

Ex://



Find i_o.

①: Assume unknown value ⇒ $i_o = 1A$ (usually put 1)

$V_2 = 8\Omega \cdot i_o = 8V$

Current across $4\Omega = \frac{8}{4} = 2A$

∴ By KCL, current across 2Ω: 3A

②: Reverse engineer to known value.

$\frac{V_1 - V_2}{2} = 3$ ⇐ By nodal

$V_1 = 6 + 8 = 14V$

Current across 7Ω: $\frac{V_1}{7} = \frac{14V}{7\Omega} = 2A$

By KCL: $I_s = 2A + 3A = 5A$.

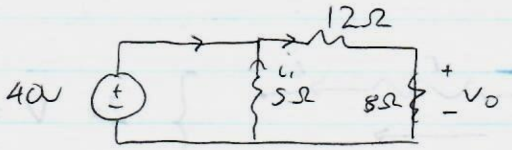
③ Ratio:

$\frac{I_s}{I_o} = \frac{15}{i_o}$

$\frac{5}{1} = \frac{15}{i_o}$

$\boxed{i_o = 3A}$

Ex://



Find v_o via homogeneity:

①: Assumptions: $v_o = 1\,V$

If $v_o = 1\,V \Rightarrow I = \frac{V}{R} = \frac{1}{8}\,A$

Voltage across 12 Ω: $V = \frac{1}{8} \times 12 = \frac{3}{2}\,V$

~~Perform KVL:~~

Total voltage: $\frac{3}{2} + 1 = \frac{5}{2}\,V \Leftarrow$ <u>voltage source</u>

② Ratio:

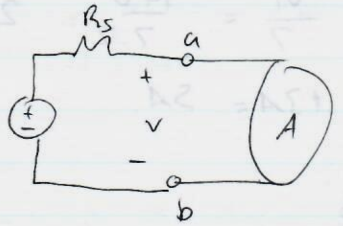
$$\frac{V_s}{V_o} = \frac{40}{x}$$

$$\frac{5}{2} = \frac{40}{x}$$

$$x = \frac{\overset{8}{\cancel{40}} \times 2}{\cancel{5}} = \boxed{16\,V}$$

<u>SOURCE TRANSFORMATIONS</u>

Convert between voltage sources ↔ current sources.
ONLY in following combos:



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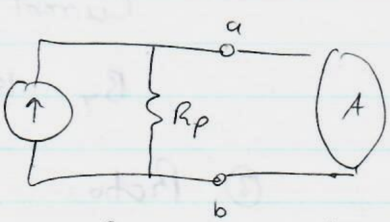


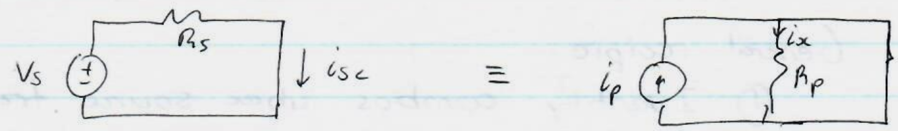
Series voltage

Parallel current

Circuit gets same input regardless!!

To do this, we will use a combination of short + open circuits ⇒ GOOD PROOF TECHNIQUE.

①: Analyzing short circuit current:



$i_{sc} = \frac{V_s}{R_s}$

By KCL: $i_p = i_x + i_{sc}$

By shorting: $V_{sc} = 0$!

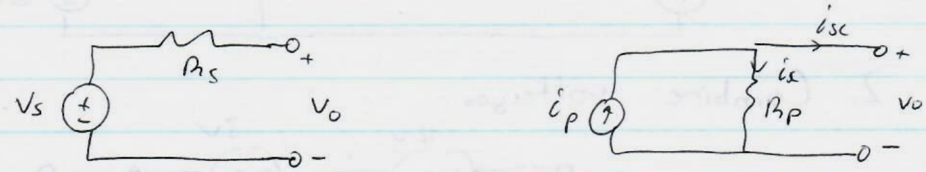
$\therefore i_x = V_{sc}/R_p = 0$

$\therefore i_p = i_{sc}$

Equating:

$$\frac{V_s}{R_s} = i_p$$

②: Analyze open circuit voltage:



$i_{sc} = 0 \Rightarrow i_x = i_p$

$V_o = V_s$ b/c $i_o = 0 \Rightarrow R_s$ has no voltage

$\therefore V_p = R_p \cdot i_p$

$\therefore V_o = R_p \cdot i_p$

Equate: $V_s = R_p \cdot i_p$

③: Equate short + open:

$V_s = R_s \cdot i_p$ from short.

$$V_{s\,short} = V_{s\,open}$$

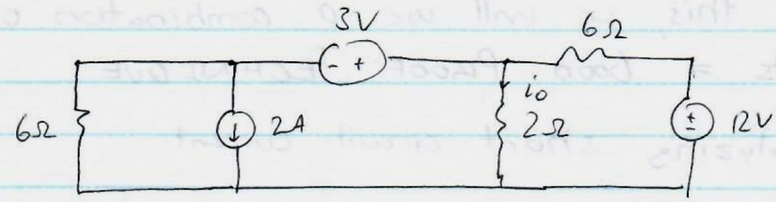
$$R_s \cdot i_p = R_p \cdot i_p$$

$$\therefore \boxed{R_s = R_p}$$

$$\therefore \boxed{V_s = R_{s/p} \cdot i_p}$$

Use this for source transform.

- Ex: //



Find i_o using source transform.

A: General recipie:

① Identify combos where source transform can be done

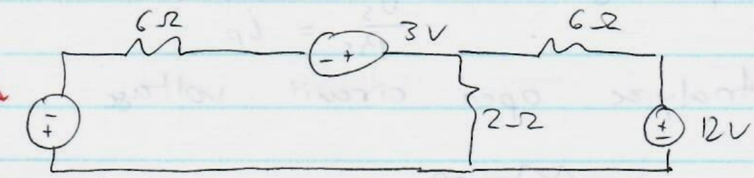
②: Transform s.t. sources can be combine + redraw
↳ Volt ⇒ series
↳ current ⇒ parallel.

⇈ Follow this heuristic when determing what to transform

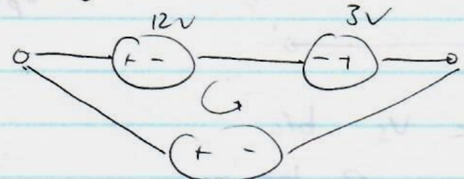
1. Convert 6Ω – 2A combo to V:

$$V_s = 6 \cdot 2 = 12\,V$$

Direction: (↑) = (±)



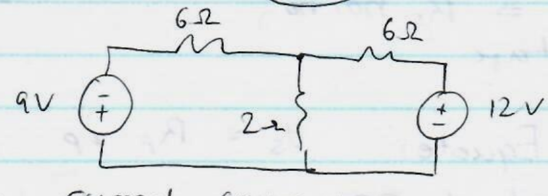
2. Combine voltages:



By KVL:

$-12 + v + 3 = 0$

$v = 9$



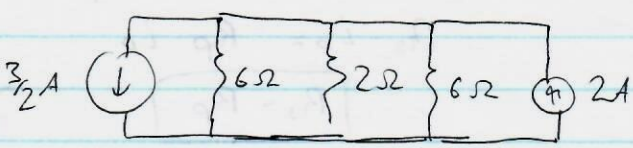
3. Creating current sources:

9V – 6Ω:

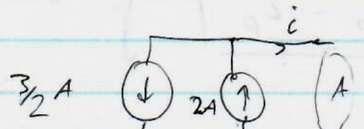
$i_p = \frac{9}{6} = \frac{3}{2}\,A$

12V – 6Ω:

$i_p = \frac{12}{6} = 2A$



4. Combining current:



$\cancel{i = 2}$

$i + \frac{3}{2}A = 2$

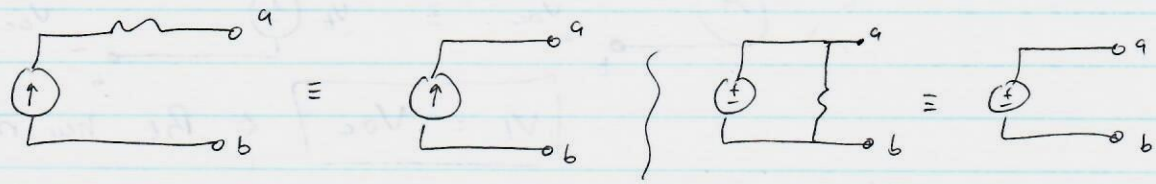
$i = \frac{1}{2}A$

5. Current division

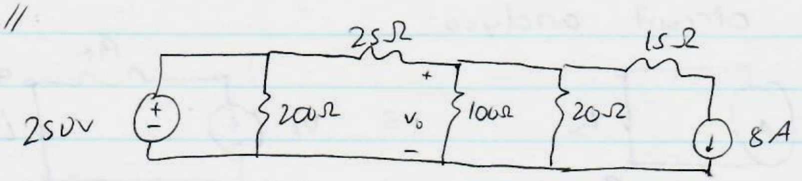
$$i_o = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} \cdot \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{6}{5} \cdot \frac{1}{2} = \boxed{0.3\,A}$$

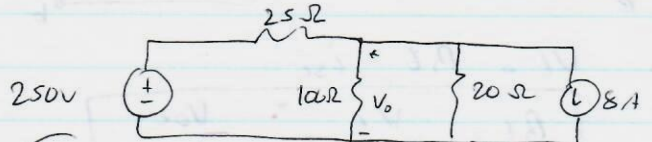
- If voltage + resistor in parallel or current + resistor in series, ignore the resistor. ⇒ no impact on rest of circuit.

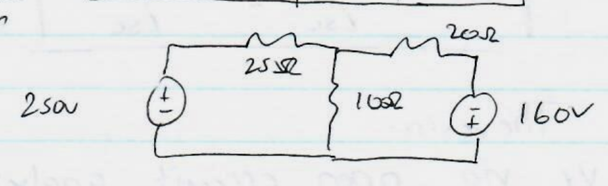


∘ Ex://

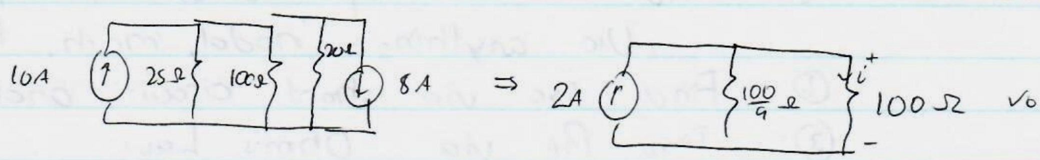


Find V_o





Does not hurt!
↳ Voltage sources not in series! Cannot combine



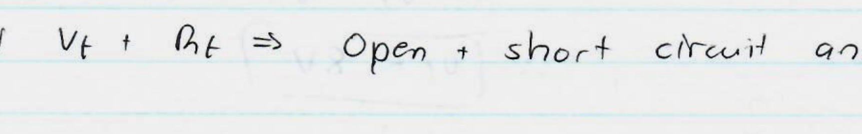
Current division:

$$i = 2 \cdot \frac{100/9}{\frac{1000}{9}} = 0.2\ A$$

$$V_o = 0.2 \times 100 = \boxed{20\ V}$$

- Dependent source: treat it same

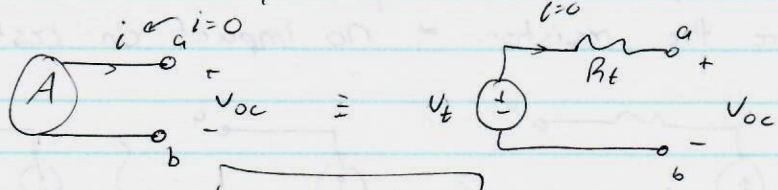
THEVENIN EQUIVALENT CIRCUIT



To B, A is just a voltage src + resistor

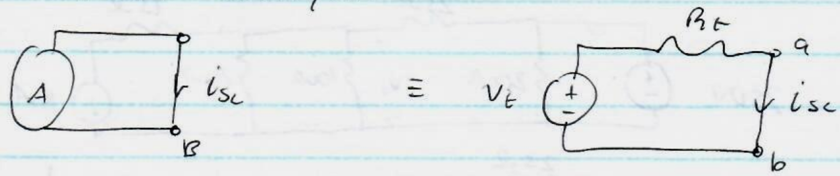
To find V_t + R_t ⇒ Open + short circuit analysis.

Open circuit analysis:



$\therefore \boxed{V_t = V_{oc}}$ ⇐ R_t has no impact on voltage.

Short circuit analysis:



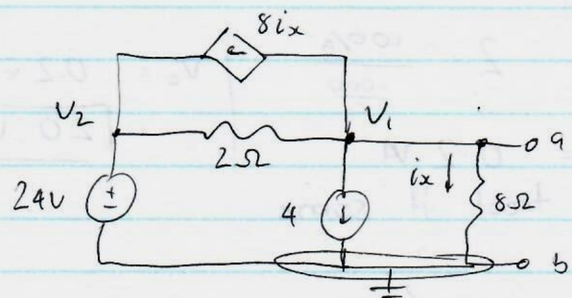
$$V_t = R_t \cdot i_{sc}$$

$$\boxed{R_t = \frac{V_t}{i_{sc}} = \frac{V_{oc}}{i_{sc}}}$$

Steps to solve Thevenin:

1. Find V_t via open circuit analysis:
   Use anything: nodal, mesh, KVL, KCL...
2. Find i_{sc} via short circuit analysis:
3. Find R_t via Ohm's Law:

Ex: //



Find Thevenin equivalent between a and b.

1: Find V_t via open circuit analysis:

Find via nodal:

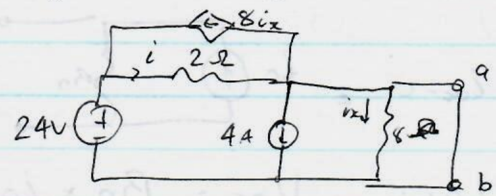
$V_2 = 24$, $V_{oc} = V_t = V_1$

$$\therefore \frac{V_1}{8} + \frac{V_1 - 24}{2} + 4 + V_1 = 0$$

$$V_1 = 8V$$

$$\therefore \boxed{V_t = 8V}$$

②: Finding i_{sc} using short circuit analysis:



From node 1 before:

KCL:

$$4 + 8i_x + i_x + i_{sc} = i$$

However: $V_{sc} = 0 \Rightarrow i_x = \frac{V}{R} = \frac{0}{8} = 0$

$$\therefore i_{sc} + 4 = i$$

$$i_{sc} + 4 = \frac{24}{2} \Rightarrow \boxed{2\Omega \parallel \text{battery}}$$ → Excellent trick!

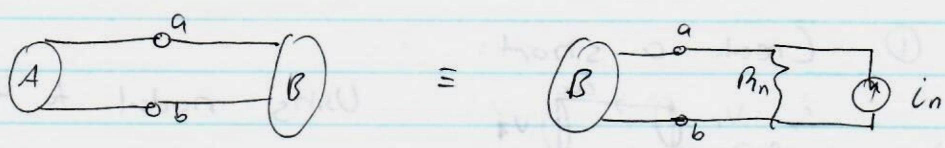
$$\therefore i_{sc} = 8A$$

③: R_T: $\therefore R_T = \frac{V_{oc}}{i_{sc}} = 1\Omega$

Tips:

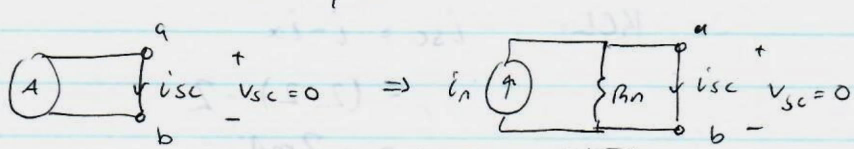
1. Always find V_t first ⇒ easy
2. i_{sc}: Use properties ($sc \Rightarrow V_{sc} = 0$) to eliminate variables.
3. Always note down changed nodes! Shorts cause nodes to change! ⇒ Redraw if stuck.

NORTON EQUIVALENT



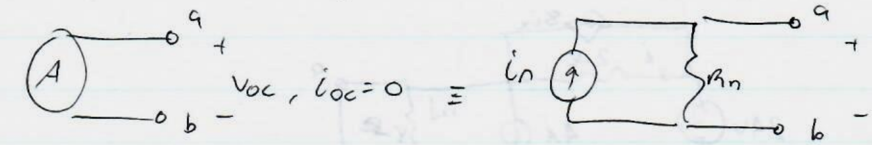
Again, use short + open circuit analysis.

~~Open~~ Short circuit analysis:



R_n has no effect: $\boxed{i_n = i_{sc}}$

Open circuit analysis:



By Ohm's Law: $V_{oc} = R_n \times i_n$

$= R_n \times i_{sc}$

$$\boxed{R_n = \frac{V_{oc}}{i_{sc}}}$$

Comparison between Thevenin + Norton:

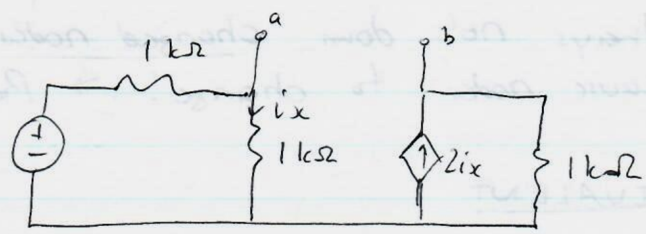
$V_t = V_{oc}$ $\qquad$ $i_n = i_{sc}$ $\Rightarrow$ Easy way to remember: Choose non-zero corresponding property.

$R_t = \frac{V_{oc}}{i_{sc}} \longleftrightarrow R_n = \frac{V_{oc}}{i_{sc}}$

Steps to solve Norton:

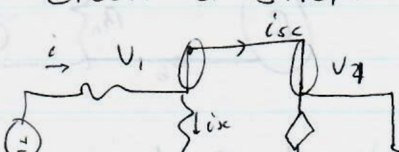
1: Short circuit → i_n
2: Open circuit → V_{oc}
3: R_n → Ohm's Law.

Ex://



Find Norton equiv. between a and b.

1: Create a short:



Using nodal to find i_{sc}:

$\frac{V_1 - 2}{1} + V_1 - 2V_1 + V_1 = 0$

$\therefore V_1 = 2V$

$\therefore$ KCL: $i_{sc} = i - i_x$

$= (2-2) - 2$

$= -2mA$

$\therefore \boxed{i_n = -2mA}$

② Create open: find V_{oc}:

Nodal analysis:

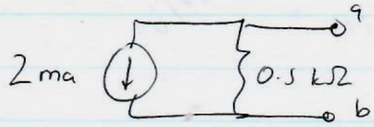
$$V_1 - 2 + V_1 = 0 \qquad -2V_1 + V_2 = 0$$

$$\therefore V_1 = 1V \qquad V_2 = 2V$$

$$\therefore V_{oc} = -1V \quad \text{(defined from } a \rightarrow b\text{)}$$

③ Find R_T:

$$R_T = \frac{V_{oc}}{i_{sc}} = \frac{-1V}{-2mA} = 0.5\ k\Omega$$



Tips:

①: Always do short first → much easier

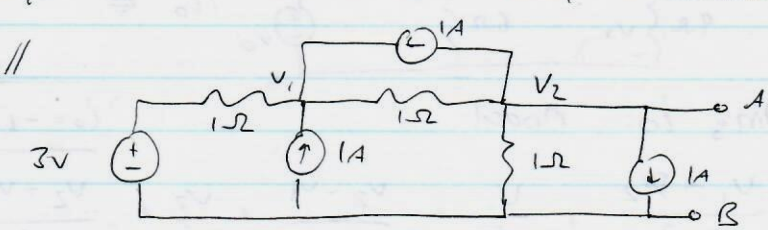
↳ Be careful w/ nodes!

②: Pay attention to direction!

FINDING R_T / R_N TRICKS

①: Equivalent resistance: only if all sources are indep.

Ex://



Find Thevenin equivalent.

1. Open circuit analysis:

Nodal:

①: $$\frac{V_1 - 3}{1} - 1 - 1 + \frac{V_1 - V_2}{1} = 0$$

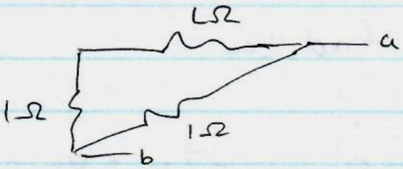
②: $$1 + \frac{V_2 - V_1}{1} + \frac{V_2}{1} + 1 = 0$$

$$V_{oc} = V_2 = \tfrac{1}{3} V$$

2. Short circuit analysis via i_{sc} / equivalent resistance: ⇒ R_T

#1: Turn off all sources

#2: Combine resistors to find R_T

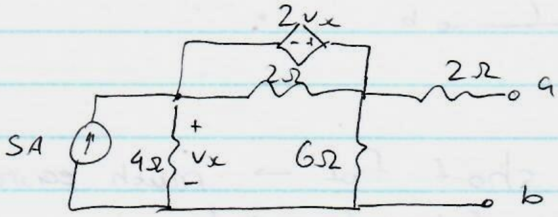


$\Rightarrow \boxed{R_T = 2/3\ \Omega}$

② Test sources: if dependent source in circuit

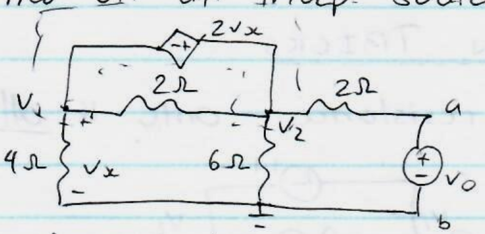
#1: Find v_{oc} / i_{sc} based on prev. method.
#2: Turn off all indep. sources
#3: Insert a voltage/current source: v_o / i_o
#4: Find current/voltage across element: i_o/v_o $\Rightarrow$ In terms of #3
#5: $R_{T/N} = v_o / i_o$

Ex://



Find Thevenin equivalent.

Finding R_T:
Turned off all indep. sources + inserted voltage source v_o:



TIP: WRITE DOWN DIRECTIONS!

Finding i_o: nodal:

$i_o = -i_3$

$$\frac{v_1 - v_2}{2} + \frac{v_1}{4} + \frac{v_2 - v_1}{2} + \frac{v_2}{6} + \frac{v_2 - v_o}{2} = 0 \quad - ①$$

$$v_1 + 2(v_1) = v_2$$

$$3v_1 = v_2 \quad - ②$$

①: $$3v_1 + 2v_2 + 6v_2 - 6v_o = 0$$

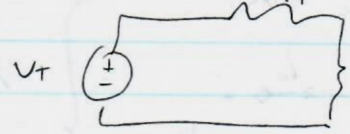
$$9v_2 = 6v_o$$

$$v_2 = 2/3\ v_o$$

$$i_o = -\left(\frac{v_2 - v_o}{2}\right) = -\left(-\frac{v_o}{6}\right) = \frac{v_o}{6}$$

$$\therefore R_{TH} = \frac{v_o}{i_o} = \boxed{6\ \Omega}$$

MAXIMUM POWER TRANSFER



$\Rightarrow$ What should R_L be s.t. power across load is max.

Derivation:

$$P = IV = I^2 R_L = \left(\frac{V_t}{R_t + R_L}\right)^2 \cdot R_L$$

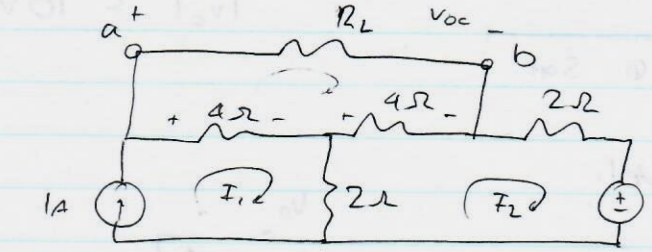
Maximize P w/ respect to R_L.

Solving: ①: $R_{L\,max} = R_t$

②: $P_{L\,max} = \dfrac{V_t^2}{4R_t}$

To solve max power transfer problem: find Thevenin equivalent w/ respect to R_L:

Ex://



Find R_L s.t. it absorbs max pow.

#1: Find Thevenin equivalent: mesh analysis

$I_1 = 1A$

①: $2(I_2 - 1) + 4I_2 + 2I_2 + 4 = 0$

$I_2 = -\frac{1}{4}A$

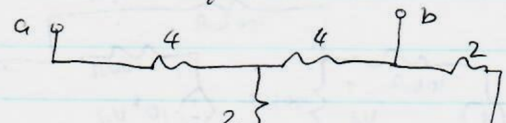
Find $V_{OC} \Rightarrow$ KVL

$V_{OC} = 4I_1 + 4I_2$

$= 4 - 1$

$= 3\ V$

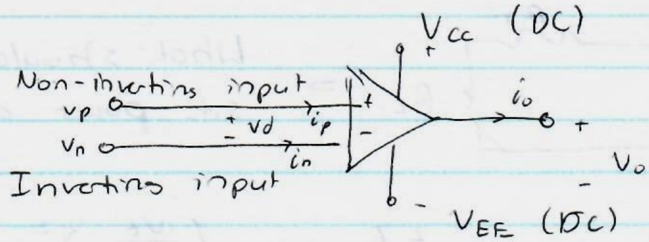
Find R_T via equivalent resistors:



$\Rightarrow R_T = 6\,\Omega$

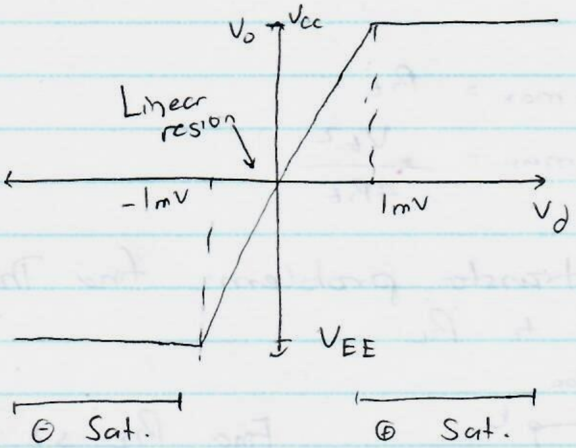
$\therefore\ R_L = \boxed{6\,\Omega}$ $\quad P_L = \dfrac{V_t^2}{4R_T} = \dfrac{9}{4 \cdot 6} = \boxed{\frac{3}{8}\,W}$

OPERATIONAL AMPLIFIERS



Ability:
- Addition
- Subtraction
- Amp.
- Integ.
- Diff.

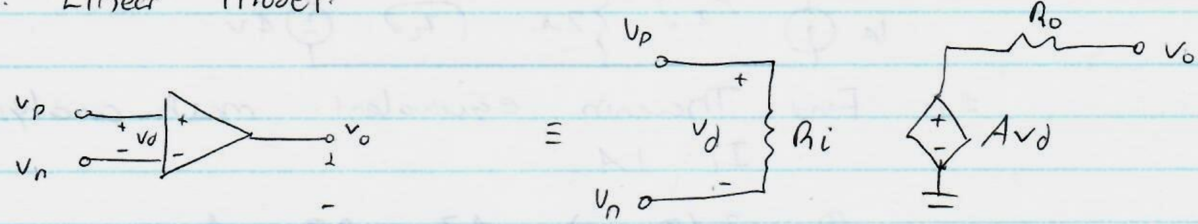
- Characteristics:



Slope: A
↳ Open-circuit voltage gain (10^4 - 10^8)

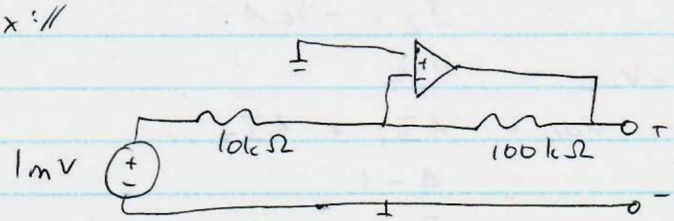
Analysis in linear region.
$|v_d| \leq 1mV$
$|v_o| \leq 10V$

- 2 models:
  i. Linear model:



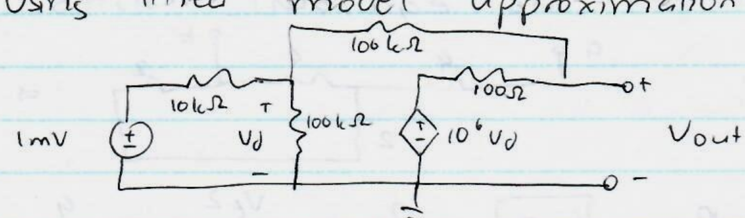
To solve q., simply replace op-amp w/ circuit.

- Ex://



$R_i = 100\ k\Omega$
$R_o = 100\ \Omega$
$A = 100{,}000$
Find v_d + v_o.

Using linear model approximation:



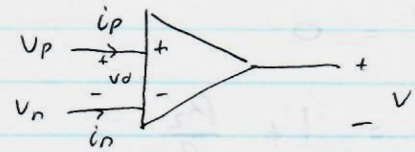
Using nodal:

$$\frac{V_1 - 10^{-3}}{10 \times 10^3} + \frac{V_1}{100 \times 10^3} + \frac{V_1 - V_0}{100 \times 10^3} = 0 \quad - ①$$

Works really well ⇒ if Nodal + op-amp

$$\frac{V_0 - V_1}{100 \times 10^3} + \frac{V_0 - 10^6 V_1}{100} = 0 \quad - ②$$

2. Ideal model:

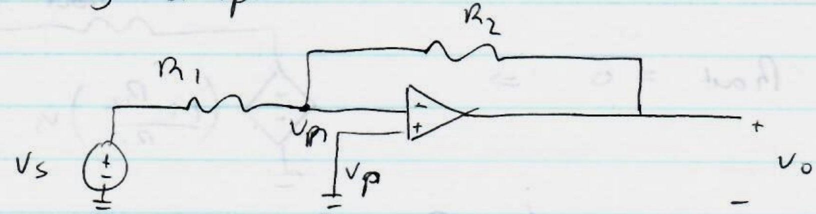


①: $V_p = V_n$ (roughly)

②: $i_p = i_n = 0$.

Extremely versatile. Use nodal w/ ideal model by identifying V_p + V_n.

- Inverting amplifier:



Analysing via ideal model:

① Identify V_p + V_n

$V_p = 0 \Rightarrow V_n = 0$.

②: KCL on nodes (only on input):

$$\frac{V_n - V_s}{R_1} + \frac{V_n - V_0}{R_2} + i_n = 0$$

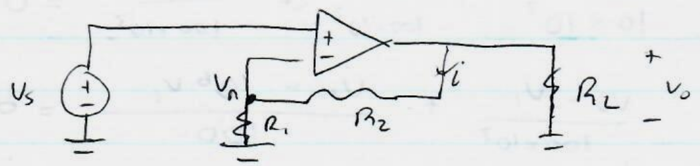
$$-\frac{V_s}{R_1} - \frac{V_0}{R_2} = 0$$

$$\therefore \boxed{V_0 = -\frac{R_2}{R_1} V_s}$$

G_V (closed circuit gain)

° $R_{in} = R_1$, $R_{out} = 0$ (resistance seen by V_s + V_0 respectively)

- Non-inverting amplifier:



$$V_p = V_n = V_s.$$

KCL at v_n:

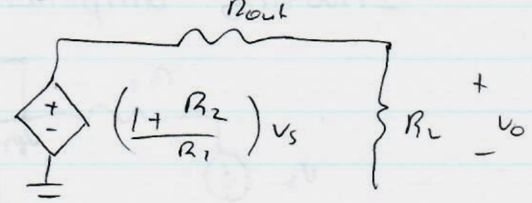
$$\frac{V_n}{R_1} + \frac{V_n - V_o}{R_2} = 0$$

$$\therefore \quad G_v = \frac{V_o}{V_s} = 1 + \frac{R_2}{R_1}$$

- Input resistance:

$$R_{in} = \frac{V_s}{i_s} \Rightarrow \lim_{i_s \to 0} R_{in} = \infty$$

$R_{out} = 0 \Rightarrow$



Since $V_o = \left(1 + R_2/R_1\right) V_s \Rightarrow R_{out} = 0$

- Output current:

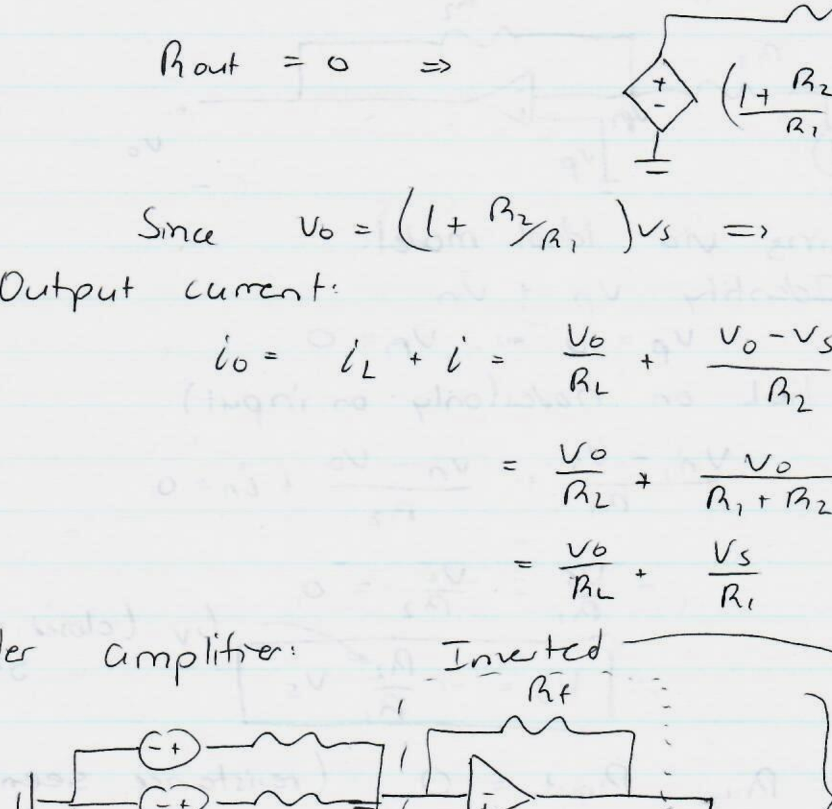
$$i_o = i_L + i = \frac{V_o}{R_L} + \frac{V_o - V_s}{R_2}$$

$$= \frac{V_o}{R_L} + \frac{V_o}{R_1 + R_2}$$

$$= \frac{V_o}{R_L} + \frac{V_s}{R_1}$$

R_1 is in series w/ R_2.

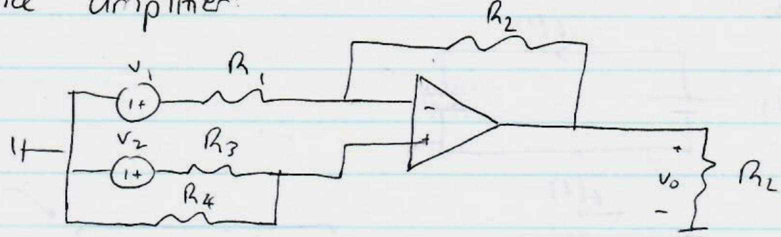
- Adder amplifier: Inverted



$$V_o = -\left(\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2 + \frac{R_f}{R_3} v_3\right)$$

NOTE: Terminal from output → ⊖. Want to prevent ⊕ saturation

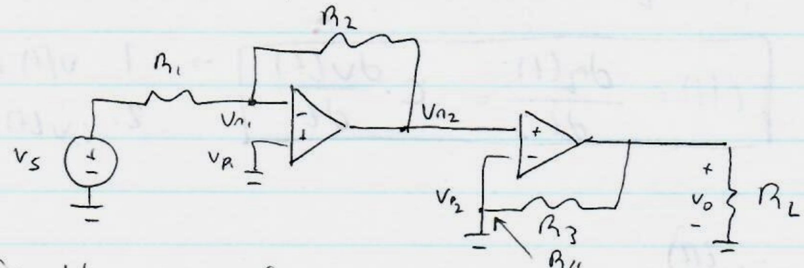
- Difference amplifier:



$$V_o = \frac{R_2}{R_1}\left[\left(\frac{R_1/R_2 + 1}{R_3/R_4 + 1}\right) v_2 - v_1\right]$$

For $\frac{R_1}{R_2} = \frac{R_3}{R_4} \Rightarrow V_o = R_2/R_1 \, (v_2 - v_1)$

- Ex: // Chained amplifier



①: $v_n + v_p$ for each amp:

②: Apply KCL @ input:

$$\frac{v_{n_1} - v_s}{R_1} + \frac{v_{n_1} - v_{n_2}}{R_2} = 0 \quad , \quad v_{p_1} = v_{n_1} = 0$$

$$v_{n_2} = -\frac{R_2}{R_1} v_s \Rightarrow \text{Inverting amplifier}$$

③: Apply KCL @ 2nd input:

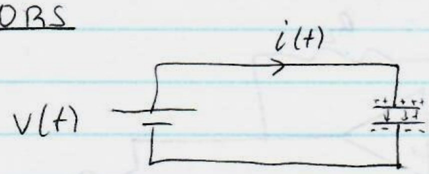
$$\frac{v_{p_2}}{R_4} + \frac{v_{p_2} - v_o}{R_3} = 0 \qquad v_{p_2} = v_{n_2} = -\frac{R_2}{R_1} v_s$$

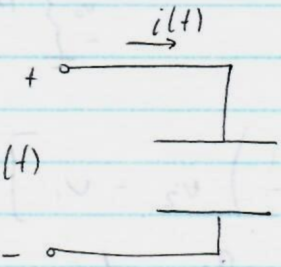
$$v_o = \left(1 + \frac{R_3}{R_4}\right)\left(-\frac{R_2}{R_1}\right) v_s$$

We multiplied effect on inverting + non-inverting amplifier.

Identify individual op amps → find individual output voltages which are input voltages of next op amp → continue forward.

CAPACITORS





$$\boxed{C = \frac{q(t)}{v(t)}}$$

Measured in Farads

(C is property of capacitor)

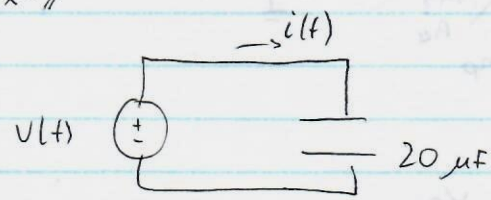
To put this in terms of current:

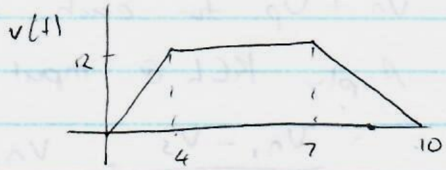
$$q(t) = C \cdot v(t)$$

$$\boxed{i(t) = \frac{dq(t)}{dt} = C \cdot \frac{dv(t)}{dt}}$$

→ 1. v(t) is constant → $i = 0$
2. v(t) is continuous

- Ex: //





Find $i(t)$.

From $0 \le t \le 4$:

$$i(t) = C \cdot \frac{dv(t)}{dt} = 20 \times 10^{-6} \cdot 3 = 60 \text{ µA}$$

From $4 \le t \le 7$:

$$i(t) = C \cdot \frac{dv(t)}{dt} = 0 \text{ A}$$

From $7 \le t \le 10$:

$$i(t) = C \cdot \frac{dv(t)}{dt} = 20 \times 10^{-6} \cdot -4 = -80 \times 10^{-6} \text{ A}$$

- Finding voltage from capacitors:

$$i(t) = C \cdot \frac{dv(t)}{dt}$$

$$dv(t) = \frac{1}{C} \cdot i(t)\,dt$$

$$\int dv(t) = \int \frac{1}{C} \cdot i(t)\,dt$$

$$v(t) - v(t_0) = \frac{1}{C} \int_{t_0}^{t} i(\tau)\,d\tau$$

$$\boxed{\therefore v(t) = V(t_0) + \frac{1}{C} \int_{t_0}^{t} i(\tau)\,d\tau}$$

- Power + energy:

$$P(t) = v(t) \cdot i(t)$$
$$= v(t) \cdot C \cdot \frac{dv}{dt}$$
$$\frac{dw(t)}{dt} = v(t) \cdot C \cdot \frac{dv}{dt}$$

$$dw(t) = C v(t)\, dv$$

$$\int_{t_1}^{t_2} dw(t) = \int_{v(t_1)}^{v(t_2)} C v(t)\, dv$$

$$w(t_2) - w(t_1) = C \cdot \frac{v^2(t_2) - v^2(t_1)}{2}$$

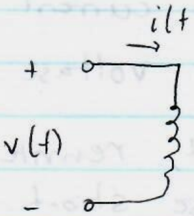
Take $t_1 = -\infty$, $v(t_1) = 0$

$$\therefore \boxed{\Delta W = \frac{C \cdot v^2(t)}{2}} \Rightarrow$$ Energy stored in capacitor electric field

Be able from graphs

↳ Recognize that ^{2} on $v(t) \Rightarrow$ line $\rightarrow$ quadratics.

<u>INDUCTORS</u>



$$\Rightarrow v(t) = L \cdot \frac{di(t)}{dt}$$

$L \Rightarrow$ inductance $\left(H = \frac{V \cdot s}{A}\right)$

Formulas for inductors v. similar to capacitors:

$v \leftrightarrow i \qquad C \leftrightarrow L$

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\, d\tau \quad \text{- ①}$$

$$W = \frac{1}{2} L \left(i^2(t_2) - i^2(t_1)\right)$$

- Current follows passive sign convention + <u>must be continuous</u>

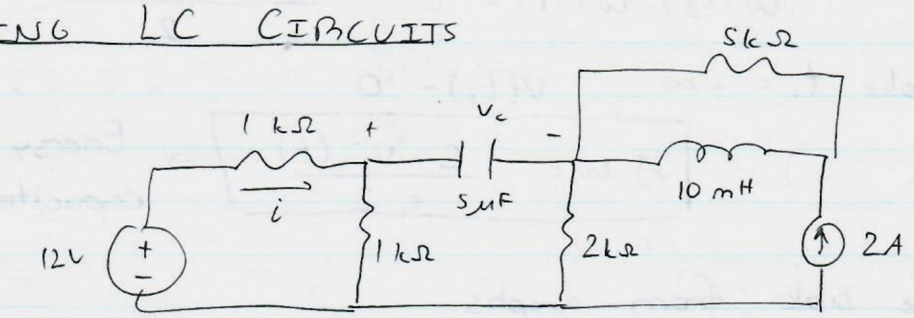
① : Set i_0 = prev i (limits one-sided) ⇐ continuity requirement.

② : Use integrals on next stage to get formula

③ : Repeat

	Capacitors	Inductors
$v(t)$	Continuous	Stepwise
$i(t)$	Stepwise	Continuous

SOLVING LC CIRCUITS

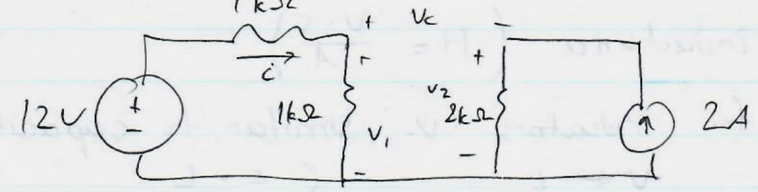


Note that all currents + voltages are constant!

↳ For capacitor: no current $\left(\frac{dv}{dt} = 0\right)$

↳ For inductor: no voltage $\left(\frac{di}{dt} = 0\right)$

① If sources are constant, rewrite w/ capacitors = <u>open circuit</u>, inductors = ~~close~~ <u>short circuits</u>.

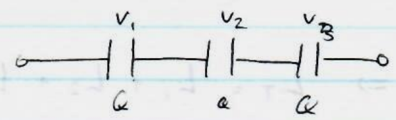


② : Solve for current/voltage running through:

$\therefore i = 6\ mA$

$V_c = V_1 - V_2 = 2V$

EQUIVALENT CAPACITORS

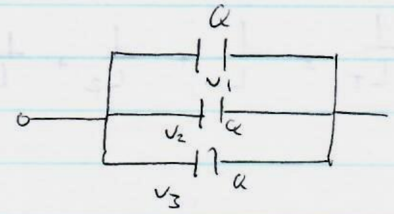


$V_T = V_1 + V_2 + V_3$

$\frac{Q_T}{C_T} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3}$

$\boxed{\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$

Q is same: induction



$Q_T = Q_1 + Q_2 + Q_3$

$C_T V_T = C_1 V_1 + C_2 V_2 + C_3 V_3$

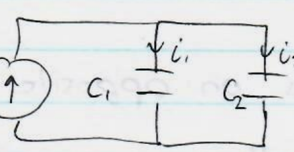
$\boxed{C_T = C_1 + C_2 + C_3}$

V is same: parallel laws

Tips for solving capacitor proofs:

1. Recognize Q is same in all capacitors in series.
2. Recognize Q is summed in parallel.
3. Try to put variables in terms of each other via KVL, KCL, etc.

Ex:// Show $i_1 = \frac{C_1}{C_1 + C_2} i_s$ and $i_2 = \frac{C_2}{C_1 + C_2} i_s$



Don't mess w/ equivalent!

①: $i_s = i_1 + i_2 \Rightarrow Q_s = Q_1 + Q_2$

②: $V_1 = V_2$

From ② + using capacitance formula:

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

Using ①:

$$\frac{Q_1}{C_1} = \frac{Q_s}{C_2} - \frac{Q_1}{C_2}$$

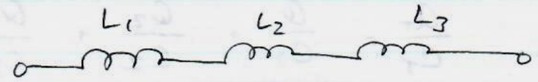
$$Q_1 \left(\frac{1}{C_1} + \frac{1}{C_2} \right) = \frac{1}{C_2} Q_s$$

$$Q_1 = \frac{1}{\cancel{C_2}} \cdot \frac{C_1 \cancel{C_2}}{C_1 + C_2} Q_s$$

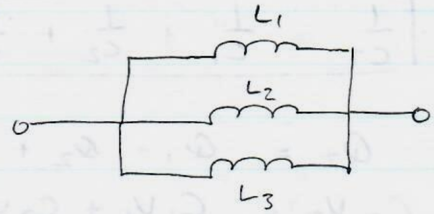
$$I_1 = \frac{C_1}{C_1 + C_2} Q_s$$

Equivalent Inductors

Mimic resistors!



$\Rightarrow \quad L_T = L_1 + L_2 + L_3$

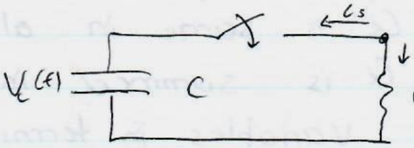


$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Transient Analysis

Constant current + voltage $\longrightarrow$ time-based V, I

①: Discharge of RC circuit



Assume $V_c(0) = V_i$

We want to find $V_c(t)$ for $t \geq 0$:

$$i_s + i_R = 0$$

← Want to look at dissipitation, so need i_R!

$$C \cdot \frac{dv_c(t)}{dt} + \frac{V_c(t)}{R} = 0$$

To solve DE, put differentials on opposite sides + integrate.

$$\frac{dV_c(t)}{V_c(t)} = -\frac{dt}{RC}$$

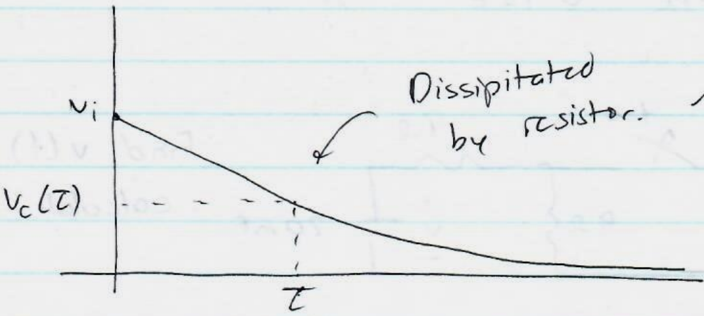
$$\int_{v(0)}^{v(t)} \frac{dV_c(t)}{V_c(t)} = \int_0^t -\frac{dt}{RC}$$

$$\ln |v_c(t)| \Big|_{V_i}^{v(t)} = -\frac{t}{RC}$$

$$\ln\left(\frac{v(t)}{v_i}\right) = -\frac{t}{RC}$$

$$\boxed{v(t) = V_i \, e^{-t/RC}}$$

Time constant: $\tau = RC \Rightarrow V(t) = V_i e^{-t/\tau}$



After 5τ, we assume

$V_c(5\tau) = 0$

$\tau \downarrow \rightarrow$ reach steady state faster

$$i_R = \frac{V(t)}{R} = \frac{V_i}{R} e^{-t/\tau}$$

$$\text{Power} = V(t) \cdot i(t) = \frac{V_i^2}{R} e^{-2t/\tau}$$

Energy: $$E(t) = \int_0^t p(\lambda)\, d\lambda = \int_0^{t} \frac{V_i^2}{R} e^{-2\lambda/\tau}\, d\lambda$$

$$= \left[-\frac{\tau V_i^2}{2R} e^{-2\lambda/\tau} \right]_0^t$$

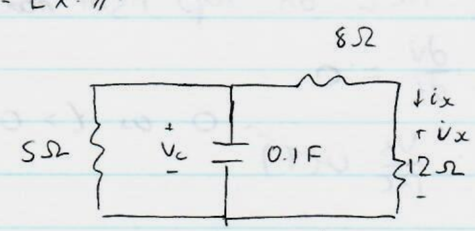
$$= \tfrac{1}{2} C V_i^2 \left(1 - e^{-2t/\tau}\right)$$

W/ energy: $t \to \infty$, $E(t) = \tfrac{1}{2} C V_i^2 \Rightarrow$ Energy dissipated by resistor = initial energy

Tips:

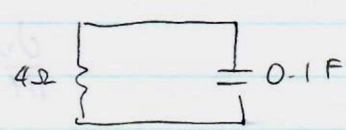
1. Find $V(0)$ ASAP + τ
2. $R = R_{TH}$
3. If $v(t)$ and $i(t)$ given, use it w/ τ to find $R + C$. $R = \frac{v(t)}{i(t)}$
   $E = \tfrac{1}{2} C (v(t))^2$!

- Ex: //:



Find V_c, V_x, i_x at $t > 0$ if $v_c(0) = 15$ V.

①: Finding τ by doing R_{TH}



$\Rightarrow \tau = RC$

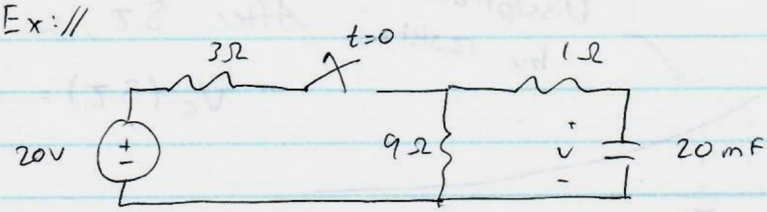
$= 0.4$

②: V_c: $V_c(t) = 15 e^{-t/0.4}$

③: $v_x = V_T \cdot \frac{12}{12+8} = 9e^{-2.5t}$ V (voltage division on series)

④: $i_x = V_x / 12 = 0.75e^{-2.5t}$ A

- Ex://



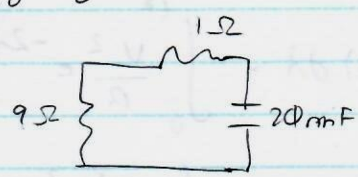
Find $v(t)$ for $t \geq 0$ and calculate initial energy.

Note that V_i is simply found via normal techniques. When switch is opened, indep. source has no impact.

①: Find V_i:

$$V_i = \frac{9}{3+9} \cdot 20 = 15\,V$$ ⇐ Capacitor acts as open!

②: Find τ:



$$\Rightarrow \tau = 10 \cdot 20 \times 10^{-3} = 0.2\ s$$

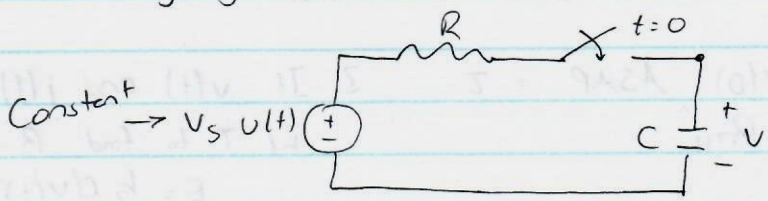
③: $v(t)$:

$$v(t) = 15e^{-t/0.2}\ V$$

④: Initial energy:

$$E(t) = \tfrac{1}{2} C v(0)^2 = \tfrac{1}{2}(20\times10^{-3})(15) = 2.25\,J$$

②: Charging an RC circuit



Assume $v(0) = V_0$

Finding $v(t)$ by doing KCL on top right node:

$$\frac{v - V_s u(t)}{R} + C\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}\cancel{u(t)}$$ — 0 as $t > 0$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Doing integration:

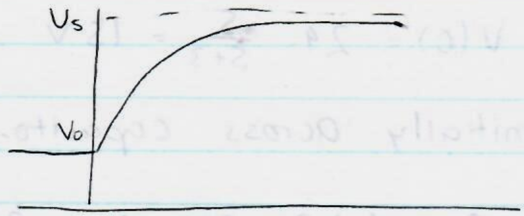
$$\ln(v - V_s)\Big|_{v(0)}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln\left(\frac{v - V_s}{V_0 - V_s}\right) = -\frac{t}{RC}$$

Complete response. → $\boxed{v(t) = V_s + (V_0 - V_s)\,e^{-t/\tau}}$ ⇒ $\tau = RC$

Uncharged cap. initial

$$v(t) = \begin{cases} 0 \;/\; V_0 & t = 0 \\ V_s(1-e)^{-t/\tau} \;/\; V_s + (V_0 - V_s)\,e^{-t/\tau} & t > 0 \end{cases}$$



⇒ Graph of step response / complete response.

Current.

$$i(t) = C \cdot \frac{dv}{dt} = C \cdot \left(\frac{V_s e^{-t/\tau}}{\tau}\right) = \frac{C}{\tau} V_s e^{-t/\tau}$$

OR: $$i(t) = \frac{V_s e^{-t/\tau}}{R}$$

Easier way of analyzing step response:

①: Complete response = natural response (stored) + forced response (indep. source)

$$\therefore \quad V = V_n + V_f$$

$$V_n = V_0 e^{-t/\tau} \quad \text{(from discharge)}$$

$$V_f = V_s (1 - e)^{-t/\tau} \quad \text{(from charging)}$$

②: Complete response = transient response + steady-state response.

$$V = V_t + V_{ss}$$

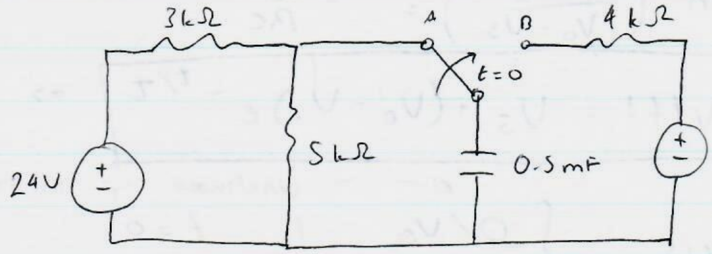
$$V_t = (V_0 - V_s)\,e^{-t/\tau}, \quad V_{ss} = V_s$$

Generally: $\boxed{v(t) = v(\infty) + (v(0) - v(\infty))\,e^{-t/\tau}}$

If time delay:

$$V(t) = V(\infty) + [v(t_0) - V(\infty)]\, e^{-\frac{t-t_0}{\tau}}$$

- Ex: //



Determine $v(t)$ for $t = 1, 4$ s.

①: $v(0)$:

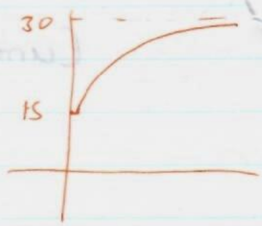
Voltage division across 5kΩ:

$$V_{5k\Omega} = V(0) = 24 \cdot \frac{5}{5+3} = 15\ V$$

∴ 15 V initially across capacitor.

②: Switch moves:

$$\tau = R \cdot C = 4\,k\Omega \cdot 0.5\,mF = 2\ s$$

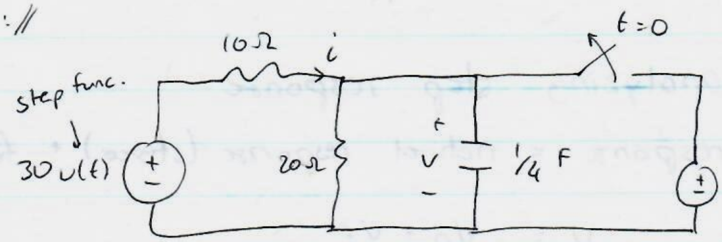


$$V(\infty) = 30\ V$$

$$\therefore\ v(t) = 30 + (15 - 30)\, e^{-t/2}$$

$$\therefore\ v(1) = 30 - 15\, e^{-1/2} = 20.9V, \quad v(4) = 27.97\ V$$

- Ex: //



Find i + v for all time.

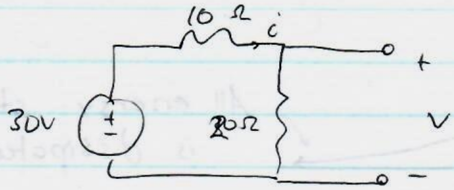
$t < 0$:

$$30\ u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 30 & \text{for } t \geq 0 \end{cases}$$

All components in parallel:

$$\therefore\ V_0 = 10\ V, \quad i = -\frac{V}{R} = -1\ A$$

$\underline{t > 0:}$



$\therefore \quad \tau = R_{TH} \cdot C = \cancel{30 \cdot \tfrac{1}{4}} F \quad 10 // 20 \cdot \tfrac{1}{4} F = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3} s$

$v(\infty) = \cancel{30V} : 30 \cdot \frac{20}{30} = 20V$

$\therefore \quad v(t) = \overset{20}{\cancel{30}} + (10 - \overset{20}{\cancel{30}}) e^{-3t/5}$

Common mistake! Need voltage for capacitor NOT source.

$i = \frac{V}{20} + C \frac{dv}{dt} = (1 + e^{-0.6t}) A$

↑ current on 20 Ω res.

$\therefore \quad i(t) = \begin{cases} -1 & t < 0 \\ 1 + e^{-0.6t} & t \geq 0 \end{cases}, \quad v(t) = \begin{cases} 10 \\ 20 - 10e^{-0.6t} \end{cases}$

How to solve RC problems:

① : Analyze $t < 0$ for $v(t)$, $i(t)$

↳ Keep in mind that $v(t)$ must be continuous + $i(t)$ ~~ca~~ can be stepwise.

② : Analyze $t > 0$

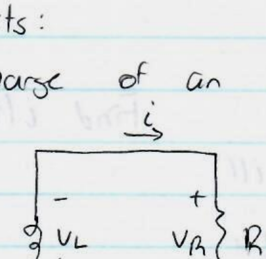
↳ Find $v(0)$, $v(\infty)$, R_{TH} (using tricks)

↳ Plug into formula.

Redraw circuits! Esp. if switches.

RL circuits:

① Discharge of an RL circuit



KVL along circuit:

$V_L + V_R = 0$

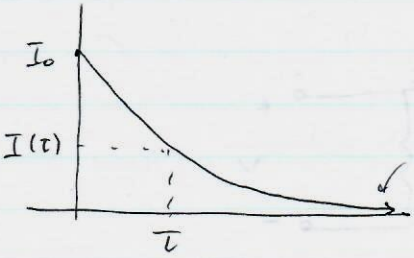
$L \frac{di}{dt} + \overset{iR}{\cancel{V_R}} = 0$

$\int_0^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$

$I(t) = I_0 e^{-\frac{Rt}{L}}$

$= \boxed{I_0 e^{-t/\tau}}$

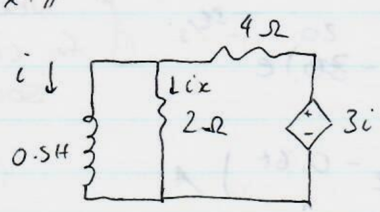
$\boxed{\tau = \frac{L}{R}}$



All energy stored in inductor is dissipated by resistor

RL circuit problems: find $i(t)$ first. $\Rightarrow \tau + I_0$

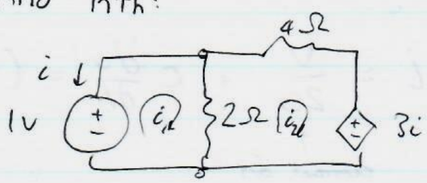
- Ex://



Assume $i(0) = 10A$. Find $i(t)$ and $i_x(t)$

Want to find τ:

Find R_{Th}:



Use KVL:

$$\text{①: } 1 - 2(i_1 - i_2) = 0$$
$$\text{②: } -2(i_2 - i_1) + 4(i_2) + 3i_1 = 0$$

$i_1 = -3A$
$\hookrightarrow i = 3A$

Mesh analysis for R_{Th} works well.

$$\therefore R_{Th} = \frac{V_0}{i_1} = \frac{1}{3}\ \Omega$$

$$\therefore \tau = \frac{L}{R} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$$

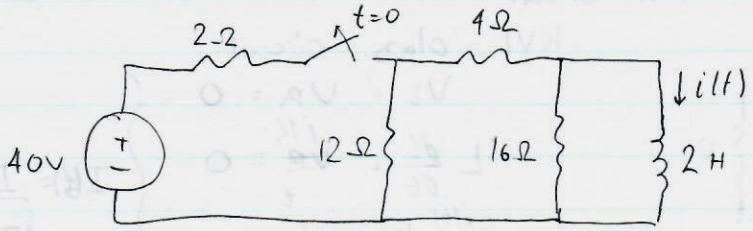
$$\therefore i(t) = 10e^{-\frac{2}{3}t}$$

To find i_x: $\cancel{i_x = \frac{3(10e^{-2/3 t})}{2} = 15}$

Voltage along inductor $= L \cdot \frac{di}{dt} = -\frac{10}{3}e^{-2/3 t}$ ← Find volltage first.

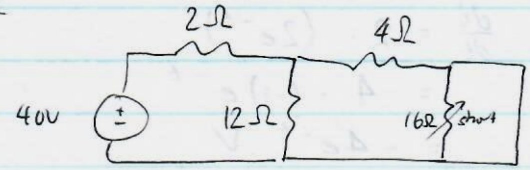
$$i_x = \frac{V}{2} = -\frac{5}{3}e^{-2/3\, t}\ A$$

- Ex://



Find $i(t)$ for $t > 0$

$t < 0$:



We want to find $i(0) = i(0^-)$
↳ Find i of 4Ω:

$12\,\Omega \,//\, 4\,\Omega \Rightarrow R_{eq} = 3\,\Omega$

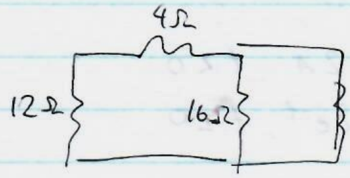
$\therefore \quad I_T = 40/5 = 8A$

Voltage division: $V_{2\Omega} = 40 \cdot \frac{2}{5} = 16\,V$

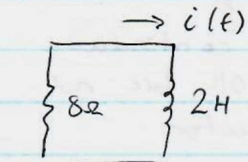
$\therefore$ 24 V across $4\,\Omega$:

$\therefore \quad i(0) = 24/4 = 6\,A \Rightarrow i(0)!$

$t > 0$:



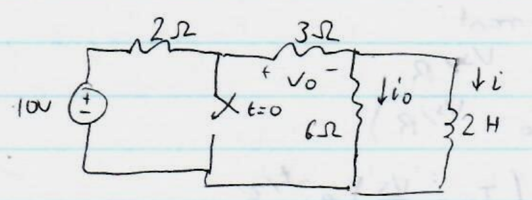
$R_{Th} \Rightarrow$



$\therefore \quad \tau = L/R = 2/8 = 1/4\,s$

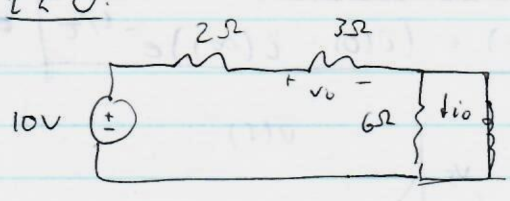
$\therefore \quad i(t) = 6e^{-4t}\,A$

- Ex://



Find i_o, v_o, i for all time.

$t < 0$:

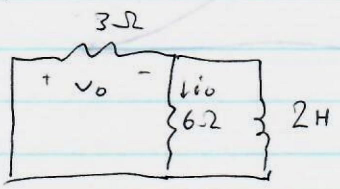


$V_o = 10 \cdot 3/5 = 6\,V$

$i(0) = i(0^-) = 10/5 = 2\,A$

$i_o = 0$

$t > 0$:



$3\,\Omega \,//\, 6\,\Omega \Rightarrow R_{Th} = 2\,\Omega$

↳ $\tau = L/R = 1\,s$

$\therefore \quad i(t) = 2e^{-t}\,A$

$\boxed{i_o = 2e^{-t} \times 1/3 = 2/3\, e^{-t}}$ ⇒ Not proper b/c no idea which direction is in. Insted, find V from inductor + apply to problem.

Trying another option:

$$V_L = L \cdot \frac{di}{dt} = 2 \cdot (2e^{-t})'$$
$$= 4 \cdot (-1) e^{-t}$$
$$= -4e^{-t} \text{ V}$$

$$\therefore \quad v_o = -V_L = 4e^{-t}$$
$$i_o = V_L / 6 = -\tfrac{2}{3} e^{-t}$$

$\therefore$ Conclusion:

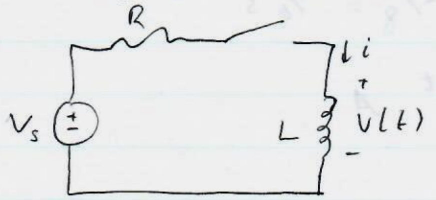
$$i_o(t) = \begin{cases} 0 \text{ A}, & t < 0 \\ -\tfrac{2}{3}e^{-t}, & t \geq 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6 \text{ V}, & t < 0 \\ 4e^{-t} \text{ V}, & t \geq 0 \end{cases}$$

Not continuous, but ok b/c not inductor

Not continuous.

$$i(t) = \begin{cases} 2 \text{ A}, & t < 0 \\ 2e^{-t}, & t \geq 0 \end{cases}$$

② : Charging an RL circuit



$$i = i_t + i_{ss}$$
$$= Ae^{-t/\tau} + V_s/R$$

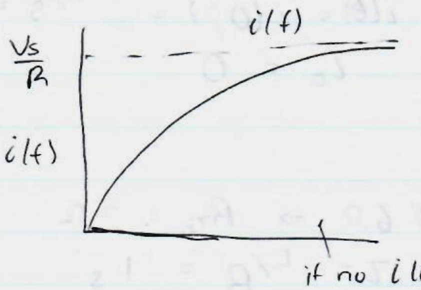
After long time, inductor = short

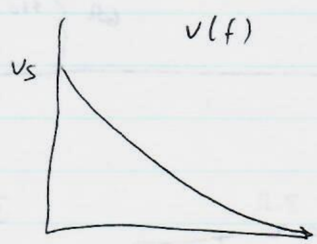
Let i_o = initial current:

$$I_o = A + V_s/R$$
$$\therefore A = (I_o - V_s/R)$$

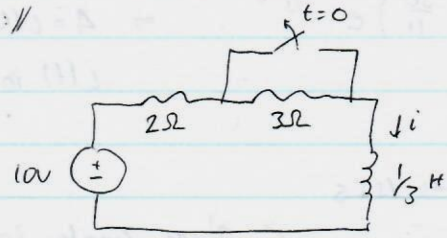
$$\therefore \quad i(t) = \frac{V_s}{R} + \left(I_o - \frac{V_s}{R}\right) e^{-t/\tau}$$

$$\boxed{i(t) = i(\infty) + (i(0) - i(\infty)) e^{-t/\tau}} \leftarrow \text{Same as RC!}$$



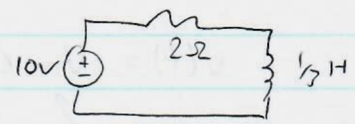


- Ex:



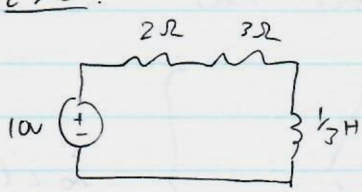
Find $i(t)$ for all time.

$t<0$:



$\Rightarrow \; i(0) = i(0^-) = \frac{10}{2} = 5A$

$t>0$:



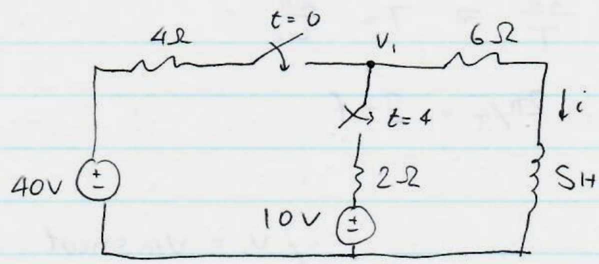
$$i(\infty) = \frac{10}{5} = 2\,A$$

$$\tau = L/R_{TH} = \tfrac{1}{3} \times \tfrac{1}{5} = \tfrac{1}{15}\,s$$

$$\therefore \; i(t) = 2 + (5-2)e^{-15t} \; A$$

$$= 2 + 3e^{-15t} \; A$$

- Ex:

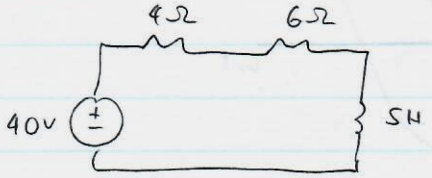


Find $i(t)$ for all time.

$t<0$:

No closed switches $\Rightarrow \; i(0) = 0$

$0<t<4$:



$$i(\infty) = \frac{40}{10} = 4\,A$$

$$\tau = 5/10 = 0.5\,s$$

$$\therefore \; i(t) = 4 + (-4)e^{-2t}$$

$$= 4(1 - e^{-2t})$$

$t>4$:

$$R_{TH} = 4\Omega \,//\, 2\Omega + 6\Omega = 22/3\,\Omega \quad \rightarrow \quad \tau = L/R_{TH} = \frac{15}{22}\,s$$

Find $i(\infty)$:

$$\frac{V_1 - 40}{4} + \frac{V_1 - 10}{2} + \frac{V_1}{6} = 0 \quad \Rightarrow \quad V_1 = \frac{180}{11}\,V$$

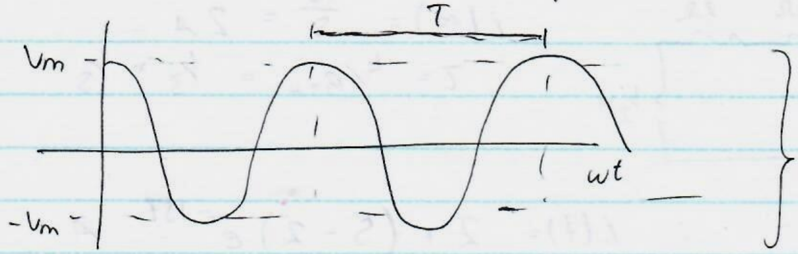
$$\therefore \; i(\infty) = V_1/6 = \frac{30}{11}\,A$$

$$i(t) = \frac{30}{11} + \left(4 - \frac{30}{11}\right) e^{-\frac{22}{15}t} \quad \Rightarrow \quad 4 = i(0) \text{ from } i(t) \text{ in } t < 4.$$

STEADY-STATE SINUSOIDAL ANALYSIS

Sinusoid: $v(t) = V_m \cos(\omega t + \theta)$ ← phase angle in rad (shift

V_m = amplitude, ω = angular frequency (rad/sec)



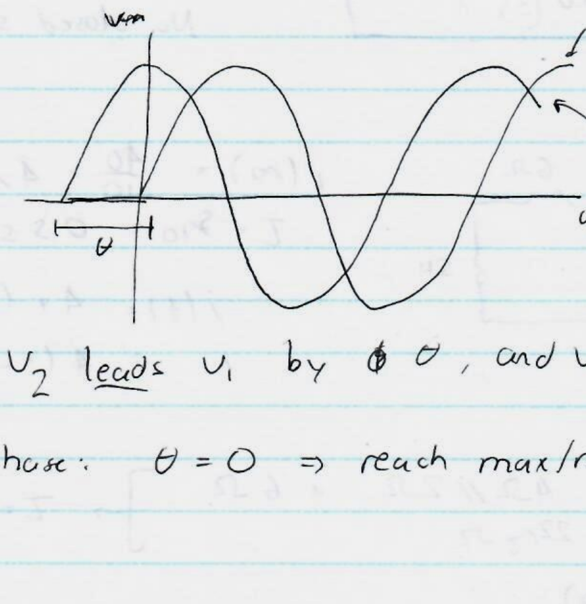
$-\infty < t < \infty$

Period: how many sec. for one rotation?

$$\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega}$$

Frequency: $1/T \rightarrow \omega = 2\pi/T = 2\pi f$

Phase:



$v_1 = V_m \sin \omega t$

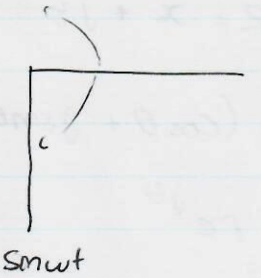
$v_2 = V_m \sin(\omega t + \theta)$

v_2 leads v_1 by θ, and v_1 lags v_2 by θ

In-phase: $\theta = 0 \Rightarrow$ reach max/min at same time.

Convert sin $\leftrightarrow$ cos + combine trig functions:

Graphical formula:



$\Rightarrow$ $\cos(\omega t \mp 90°) = \pm\sin(\omega t)$

$\cos(\omega t \pm 180°) = -\cos(\omega t)$

$\sin(\omega t \pm 180°) = -\sin(\omega t)$

$\sin(\omega t + 90°) = \cos(\omega t)$

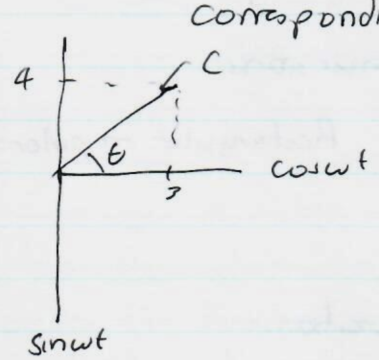
Combine: 117 knowledge:

$$A\cos\omega t + B\sin\omega t = \underset{\text{cos/sin}}{C}\cos(\omega t - \theta)$$

$$C = \sqrt{A^2 + B^2} \qquad \theta = \tan^{-1}(B/A)$$

Ex.// $3\cos(\omega t) - 4\sin\omega t = - \;..\; \cos$

①: Draw graphical: make points on graph corresponding to function.



②: Find C and θ:

$$C = \sqrt{4^2 + 3^2} = 5$$

$$\theta = \tan^{-1}(-4/3) = -53.1°$$

$$\therefore\; 3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1°)$$

<u>PHASORS</u>

Complex # that represents sinusoid:

- Complex # review:

$$j = \sqrt{-1} \Rightarrow j^2 = -1$$

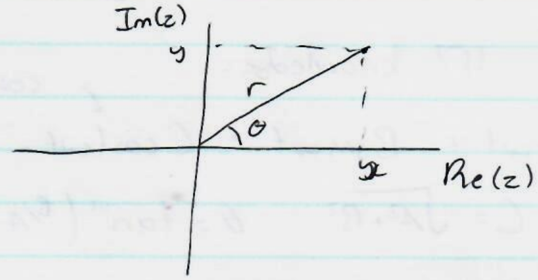
Complex numbers: $Z = x + jy$

Real ↗ x, Imaginary ↖ jy

Rectangular form: $Z = x + jy$

Polar form: $Z = r(\cos\theta + j\sin\theta) = r\angle\theta$

Exponential: $Z = re^{j\theta}$



Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Conversions:

Rectangular → polar: $x = r\cos\theta, \quad y = r\sin\theta$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)$$

Operations:

Addition + subtraction: Use rectangular coordinates:

$$Z_1 \pm Z_2 = (x_1 \pm jy_1) + (x_2 \pm jy_2)$$
$$= (x_1 \pm x_2) + (y_1 \pm y_2)j$$

Multiplication / division: use polar:

$$Z_1 Z_2 = r_1 r_2 \angle \theta_1 + \theta_2$$

$$Z_1/Z_2 = r_1/r_2 \angle \theta_1 - \theta_2$$

Conjugate: $z^* = x - yj$ (division in rectangular).

$v(t) = V_m \cos(\omega t + \theta)$] → Time domain rep.

↳ Phasor = $\boxed{V_m e^{j\theta}} = V_m \angle\theta$ ⇒ Complex number.

↳ Phasor - domain representation

Ex:// Chart for phasors: Based on $Re(z)$

$$v(t) = V_m \cos(\omega t + \theta) \Leftrightarrow V_m e^{j\theta} = V_m \angle\theta$$

$$V_m \sin(\omega t + \theta) \Leftrightarrow V_m \angle\theta - 90°$$

$$I_m \cos(\omega t + \theta) \Leftrightarrow I_m \angle\theta$$

$$I_m \sin(\omega t + \theta) \Leftrightarrow I_m \angle\theta - 90°$$

Differentiation / integration:

$$\boxed{\underbrace{\frac{dv}{dt}}_{\text{Time domain}} = \underbrace{j\omega V}_{\text{Phasor}}}$$ (multiply phasor by $j\omega$ for dif.)

$$\boxed{\underbrace{\int v\,dt}_{\text{time}} = \underbrace{\frac{V}{j\omega}}_{\text{phasor}}}$$

- Ex:// Convert sinusoids to phasors:

a) $i = 6\cos(50t - 40°)$

$$i_m = 6 \Rightarrow \theta = -40° \Rightarrow \tilde{i} = 6 \angle -40°$$

b) $v = -4\sin(30t + 50°)$

Real but defined in terms of sin. Real → cos!

①: Cos conversion:

$$v = -4\sin(30t + 50°) = 4\cos(30t + 140°)$$

$$\therefore \tilde{V} = 4 \angle 140°$$

- Ex:// Find sinusoids described by phasors:

  a) $\tilde{I} = -3 + 4j$ A

  Converting to polar:

  $$-3 + 4j = 5(\cos 126.87° + j\sin 126.87°)$$

  $$\therefore \quad \tilde{I} = 5\angle 126.87°$$

  $\therefore$ Sinusoid:

  $$I = 5\cos(\omega t + 126.87°) \text{ A}$$

  b) $\hat{V} = j8e^{-j20°}$ V

  2 complex # multiplied together!

  1. $j \Rightarrow 1\angle 90°$
  2. $8e^{-j20°} \Rightarrow 8\angle 20°$

  $\therefore$ Multiply:

  $$j8e^{-j20} = (1\angle 90)(8\angle 20°)$$
  $$= 8\angle 90-20$$
  $$= 8\angle 70°$$

  We cannot find ω from phasor.

  $$\therefore \quad v(t) = 8\cos(\omega t + 70°) \text{ V}$$

- Adding currents / voltage waveforms is easy using phasors
  1. Convert to phasor
  2. Convert to rectangular form + add
  3. Convert back to sinusoid.
- Integrodifferential equations: remember integration + differentiation for phasors.
  - Ex:// Determine $i(t)$ if:

  $$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75°)$$

  We know from RHS that $\omega = 2$

  $$\therefore \quad 4\tilde{I} + 8\cdot\frac{\tilde{I}}{2j} - 3\tilde{I}\cdot 2j = \cancel{\phi}\ 50\angle 75°$$

  $$\tilde{I}\left(4 + \frac{4}{j} - 6j\right) = 50\angle 75°$$

  ↳ Solve $\tilde{I}$ using complex # division.

Phasors + Circuit Elements

Resistor: Ohm's Law holds:

$$\boxed{\tilde{V} = R\tilde{I}}$$

Inductor:

In a nomal circuit: $v = L \cdot \frac{di}{dt}$

In an AC circuit:

$$\boxed{\tilde{V} = L \cdot j\omega \tilde{I}}$$ ⇒ Phase of $\tilde{V}$ is 90° out of phase w/ $\tilde{I}$ (current lags by 90°)

Capacitor:

In a nomal circuit: $i = C \cdot \frac{dv}{dt}$

In an AC circuit:

$$\boxed{\tilde{I} = C \cdot j\omega \tilde{V}}$$ ⇒ Phase of $\tilde{I}$ is $\underbrace{90° + \phi}$ out of phase. or (current leads by 90°)

- Ex:// $v = 12\cos(60t + 45°)$ → 0.1-H inductor. Find current.

① Convert to phasor:

$$\tilde{V} = 12\angle 45°$$

②: Use the inductor formula:

$$\tilde{V} = L \cdot j\omega \tilde{I}$$

$$\tilde{I} = \frac{12\angle 45°}{j\omega L} = \frac{12\angle 45°}{j \cdot 60 \cdot 0.1} \xrightarrow{\text{Conversion!}} = \frac{12\angle 45°}{6\angle 90°}$$

$$= 2\angle -45°$$

$$\therefore \quad I(t) = 2\cos(60t - 45°)$$

IMPEDANCE + ADMITTANCE

$$\tilde{V} = R\tilde{I} \qquad \tilde{V} = j\omega L\tilde{I} \qquad \tilde{V} = \frac{\tilde{I}}{j\omega C}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\tilde{V}}{\tilde{I}} = R \qquad \frac{\tilde{V}}{\tilde{I}} = j\omega L \qquad \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$$

Impedance: $Z = \frac{\tilde{V}}{\cancel{R}} = \frac{\tilde{V}}{\tilde{I}} \Rightarrow$ Opposition to flow of sinusoidal current.

Impact of frequency on impedance.

$\omega \to 0$ (DC circuit):

$$Z_L \to 0 \;, \quad Z_C \to \infty \quad \left(\begin{array}{l} L\text{: short circuit} \\ C\text{: open circuit} \end{array}\right)$$

$\omega \to \infty$:

$$Z_L \to \infty \;, \quad Z_C \to 0 \quad \left(\begin{array}{l} L\text{: open circuit} \\ C\text{: short circuit} \end{array}\right)$$

Alternative rep. of impedance:

B/c impedance is a complex #:

$$Z = R \pm jX$$

$R = Re(z) =$ resistance $\qquad X = Im(z) =$ reactance.

↳ $X > 0$: inductor, $\tilde{I}$ lags $\tilde{V}$

$X < 0$: cap., $\tilde{I}$ leads $\tilde{V}$

All measured in ohms.

$$Z = |z| \angle \theta$$

$$|z| = \sqrt{R^2 + X^2} \;, \quad \theta = \tan^{-1}\left(\frac{X}{R}\right)$$

→ Remember polar graph

Admittance:

$$Y = \frac{1}{Z} = \frac{\tilde{I}}{\tilde{V}}$$

$$= G \pm jB$$

Conductance: $G = \frac{R}{R^2 + X^2}$ $\qquad$ Susceptance: $B = -\frac{X}{R^2 + X^2}$

PHASOR KCL + KVL

KCL: At a particular node

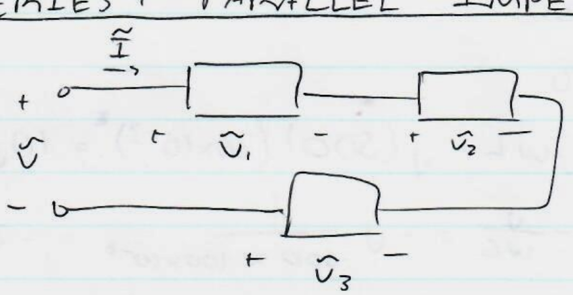
$$\sum_n \tilde{I}_n = 0$$

KVL: In a loop:

$$\sum_n \hat{V}_n = 0$$

No change!

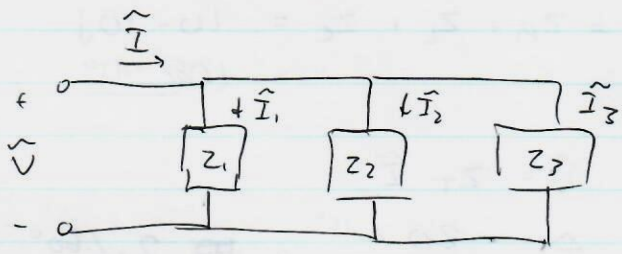
SERIES + PARALLEL IMPEDANCES



$$\tilde{V} = Z_1 \tilde{I} + Z_2 \tilde{I} + Z_3 \tilde{I}$$

$$\frac{\hat{V}}{\hat{I}} = Z_1 + Z_2 + Z_3$$

$$\boxed{Z_T = Z_1 + Z_2 + Z_3}$$



$$\tilde{I} = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$$

$$\frac{\hat{V}}{Z_T} = \frac{\tilde{V}}{Z_1} + \frac{\tilde{V}}{Z_2} + \frac{\tilde{V}}{Z_3}$$

$$\therefore \boxed{\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}}$$

Act exactly like resistors.

All techniques that we learned: nodal, mesh ... all apply

CIRCUIT ANALYSIS VIA PHASORS + IMPEDANCE

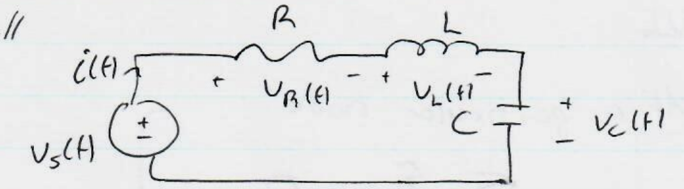
1: Convert circuit from time-domain → phasor.

- All I, V → $\tilde{I}$, $\tilde{V}$
- All elements should have impedances.

2: Solve circuit in phasor domain

3: Convert into time domain

- Ex://



$R = 10\ \Omega$

$L = 20\ mH$

$C = 100\ \mu F$

$V_s(t) = 20\cos(500t + 15°)$

①: Find $i(t)$, $v_R(t)$, $v_L(t)$, $v_C(t)$:

1) Conert $V_s(t) \rightarrow$ phasor:

$$V_s(t) = 20\angle 15°$$

2) Find impedances:

$$Z_R = 10$$

$$>0 \rightarrow Z_L = j\omega L = j(500)(20\times10^{-3}) = 10j$$

$$<0 \rightarrow Z_C = -\frac{j}{\omega C} = -j\frac{1}{500\times100\times10^{-6}} = -20j$$

$$\therefore Z_T = Z_R + Z_L + Z_C = 10 - 10j$$
$$= 10\sqrt{2}\angle -45°$$

3) $i(t)$:

$$\tilde{V} = Z_T\, \hat{I}$$

$$\hat{I} = \frac{20\angle 15°}{10\sqrt{2}\angle -45°} = \cancel{10}\ 2/\sqrt{2}\angle 60°$$
$$= \sqrt{2}\cos(500t + 60°)\ A$$

4) $V_R(t)$:

$$\tilde{V}_R = Z_R \cdot \hat{I}$$
$$= 10 \cdot \sqrt{2}\angle 60° = 10\sqrt{2}\cos(500t + 60°)$$

5) $V_L(t)$:

$$\tilde{V}_L = Z_L \cdot \hat{I}$$
$$= 10\angle 90° \cdot \sqrt{2}\angle 60°$$
$$= 10\sqrt{2}\cos(500t + 150°)$$

6) $V_C(t)$:

$$\tilde{V}_C = Z_c \cdot \hat{I}$$
$$= 20\angle -90° \cdot \sqrt{2}\angle 60°$$
$$= 20\sqrt{2}\cos(500t - 30°)$$

- Series resonance:

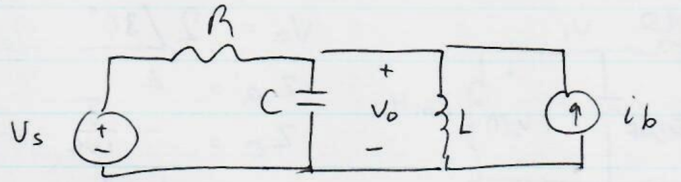
$$Z_{eq} = R + Z_L + Z_C = Re$$

$$\therefore \quad Z_L + Z_C = 0$$

$$\boxed{\omega L = \frac{1}{\omega C}}$$

This leads to $\tilde{V}_L + \tilde{V}_C = 0$

- Ex://



$v_s = 2\cos(1000t + 30°)$

$i_b = \cos(1000t - 45°)$

$R = 4\Omega, \; C = 250\,\mu F, \; L = 2mH$

Find $v_o(t)$:

Perform nodal analysis!

①: $v_s \Rightarrow 2\angle 30°, \quad i_b = 1\angle -45°$

$$Z_R = 4, \quad Z_C = -\frac{j}{\omega C} = -4j, \quad Z_L = j\omega L = 2j$$

②: Nodal:

$$\frac{\tilde{V}_o - \hat{V}_s}{Z_R} + \frac{\hat{V}_o}{Z_C} + \frac{\hat{V}_o}{Z_L} - \tilde{I}_b = 0$$

$$\frac{\hat{V}_o - \hat{V}_s}{4} + \frac{\hat{V}_o}{(-4j)} + \frac{\hat{V}_o}{2j} - \hat{I}_b = 0$$

$$j(\tilde{V}_o - \hat{V}_s) + \hat{V}_o + 2\tilde{V}_o - 4j\tilde{I}_b = 0$$

Only plus in V + I phasor @ end.

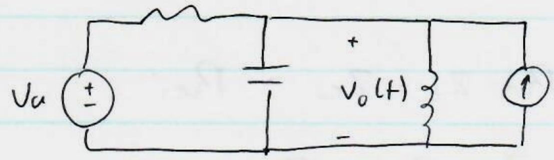
$$(1+j)v_o = j\hat{V}_s + 4j\hat{I}_b$$

$$= (1\angle 90°)(2\angle 30°) + (4\angle 90°)(1\angle -45°)$$

$$= 2\angle 120° + 4\angle 45°$$

$$v_o = \frac{2\angle 120° + 4\angle 45°}{\sqrt{2}\angle 45°}$$

- Ex: 11



$V_a = 2\cos(1000t + 30°)$

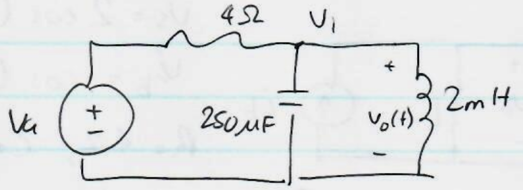
$i_b = \cos(2000t - 45°)$

$R = 4\,\Omega$, $C = 250\,\mu F$, $L = 2mH$

Find $v_o(t)$

Since there are different angular frequencies $\Rightarrow$ superposition.

①: Just for V_a:



$V_a = 2\angle 30°$

$Z_R = 4$

$Z_C = -\frac{j}{\omega C} = -4j$

$Z_L = j\omega L = 2j$

$\therefore \frac{\tilde{V}_1 - \tilde{V}_a}{4} + \frac{\tilde{V}_1}{Z_C} + \frac{\tilde{V}_1}{Z_L} = 0$

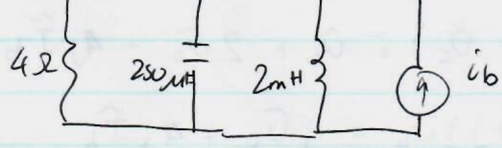
$\frac{\tilde{V}_1 - 2\angle 30°}{4} + \frac{\tilde{V}_1 j}{4} - \frac{\tilde{V}_1 j}{2} = 0$

$\tilde{V}_1 - 2\angle 30° + \tilde{V}_1 j - 2\tilde{V}_1 j = 0$

$V_1(1-j) = 2\angle 30°$

$V_1 = \frac{2\angle 30°}{\sqrt{2}\angle -45°} = \frac{2}{\sqrt{2}}\angle 75° = \sqrt{2}\angle 75°$

②: Just for i_b:



$i_b = 1\angle -45°$

$\frac{\tilde{V}_2}{4} + \frac{V_2 j}{4} - \frac{V_2 j}{2} - 1\angle 45° = 0$

$\tilde{V}_2 = \frac{4}{\sqrt{2}}\cos(2000t - 90°)$

$\therefore V_o(t) = \tilde{V}_1 + \tilde{V}_2 = \sqrt{2}\cos(1000t + 75°)$

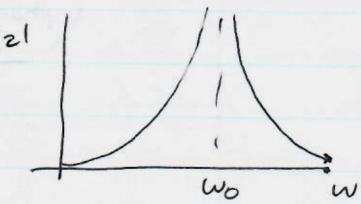
$+ \frac{4}{\sqrt{2}}\cos(2000t - 90°)$

- Resonance:

$$Z = j\omega L \,//\, \frac{1}{j\omega C} \Rightarrow \text{Can be used as a resonance circuit.}$$

$$= \frac{j\omega L \cdot \frac{1}{j\omega C}}{j\omega L - j\frac{1}{\omega C}} = \frac{L/C}{j(\omega L - 1/\omega C)}$$

$$\therefore |Z| = \frac{L/C}{|\omega L - 1/\omega C|} \Rightarrow$$



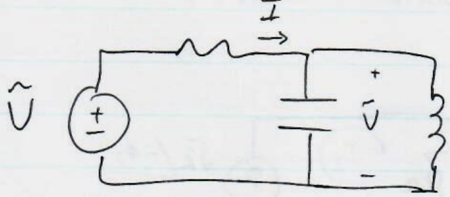
$\omega \to 0 \Rightarrow |Z| \to 0$, $\omega \to \infty, \Rightarrow |Z| \to 0$

If $|Z| \to \infty$ iff $|\omega L - 1/\omega C| \to 0$

Recall: $\omega_0 L = \dfrac{1}{\omega_0 C}$

$$\boxed{\omega_0 = \pm \frac{1}{\sqrt{LC}}} \rightarrow \text{Resonance frequency}$$

At ω_0:



$\tilde{I} = 0$ (infinite impedance)

$\therefore \tilde{I}_C + \tilde{I}_L = 0$

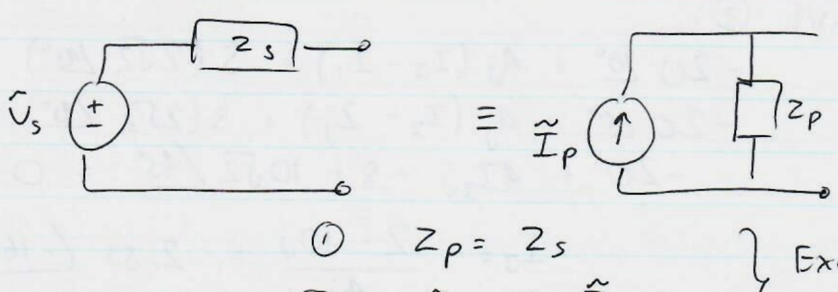
Recall: C + L have the same voltage $\tilde{V}$:

$$\tilde{I}_C = \frac{\tilde{V}}{R_C} = \frac{\hat{V}}{1/j\omega_0 C} = j\sqrt{\frac{C}{L}}$$

$$\tilde{I}_L = \frac{\tilde{V}}{Z_L} = \frac{\tilde{V}}{j\omega L} = -j\sqrt{\frac{C}{L}}$$

$\Bigg\}\ \tilde{V}_0 = \tilde{V}$ (max.)

- Source transformation

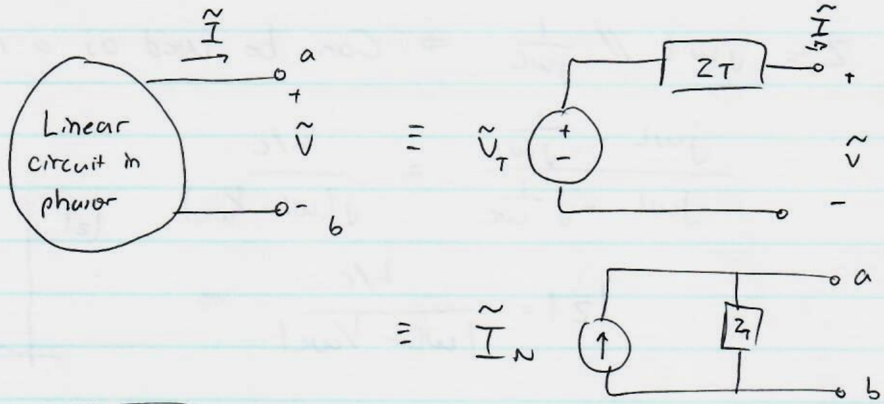


① $Z_p = Z_s$

②: $\tilde{V}_s = Z_p \tilde{I}_p$

$\Big\}$ Exactly like DC source transform.

THEVENIN + NORTON IN PHASOR DOMAIN



⇒ Thevenin equiv.

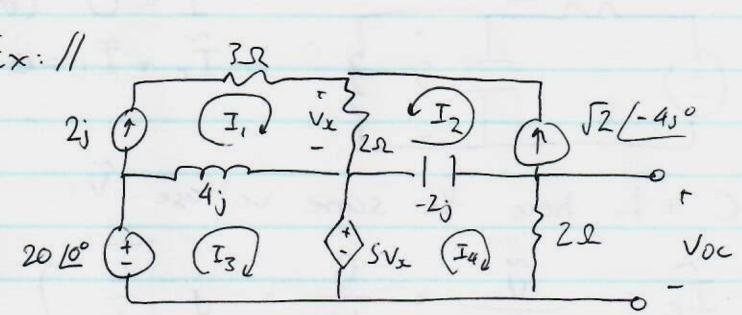
⇒ Norton equiv.

$$\boxed{\tilde{V}_T = \tilde{V}_{oc}} \qquad \boxed{\tilde{I}_N = \tilde{I}_{sc}} \qquad Z_T = Z_n = \boxed{\frac{\tilde{V}_{oc}}{\tilde{I}_{sc}}}$$

$$\boxed{Z_t = Z_n = Z_{eq}} \qquad = \boxed{\frac{\tilde{V}_s}{\tilde{I}_s}} \Leftarrow \text{Ext. source.}$$

Same tips in DC apply in AC (eg. set indep. sources to 0 to calculate Z_{Th})

Ex: //



Find Thevenin equivalent between a and b.

$I_1 = 2j, \quad I_2 = \sqrt{2}\angle{-45^\circ} \quad \Rightarrow \quad V_x = 2(I_1 + I_2)$

$$= 2(2j + 1 - j)$$
$$= 2 - 2j$$
$$= 2\sqrt{2}\angle 45^\circ$$

KVL ③:

$$-20\angle 0^\circ + 4j(I_3 - I_1) + 5(2\sqrt{2}\angle 45^\circ) = 0$$
$$-20\angle 0^\circ + 4j(I_3 - 2j) + 5(2\sqrt{2}\angle 45^\circ) = 0$$
$$-20 + 4I_3 j + 8 + 10\sqrt{2}\angle 45^\circ = 0$$
$$I_3 = \frac{2 - 10j}{4j} = 2.55\angle{-168.7^\circ}$$

KVL ④:

$$-5V_x - 2j(I_4 - I_2) + 2I_4 = 0$$

$$\therefore I_4 = 6j$$

$$V_{oc} = V_t = 2I_4 = \boxed{12j}$$

Recognized that only 1 part in prev. circuit chg.

Finding $\tilde{I}_{sc}$ ⇒ Changes KVL 4 ⇒ short circuiting 2Ω resistor.

$$-5V_x - 2j(I_4 - I_2) = 0$$

$$I_4 = \frac{12}{\sqrt{2}} \angle 135° = I_{sc}$$

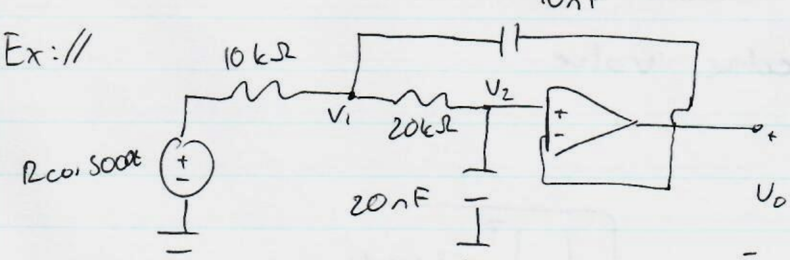
$$Z_T = \frac{\tilde{V}_{oc}}{\tilde{I}_{sc}} = \frac{12 \angle 90°}{12/\sqrt{2} \angle 135°} = \sqrt{2} \angle -45° = \boxed{1 - j}$$

OP-AMPS IN PHASOR DOMAIN

Pretty much same analysis as DC Op Amp

1. Look at $\tilde{V}_p$ + $\tilde{V}_n$ ⇒ possible relation to $\tilde{V}_o$
2. Construct nodal equations
3. Look for common op-amp (inverting, non-inverting, adding, difference, voltage follower)

Ex://



Find v_o:

①: Convert to phasor domain:

$$V_s = 12\cos(5000t) = 12 \angle 0°$$

$$C_1 = 10nF \Rightarrow -20j \quad , \quad C_2 = 20nF \Rightarrow -10j$$

7: Find relations:

$$V_n = V_o \Rightarrow V_p = V_2 = V_o$$

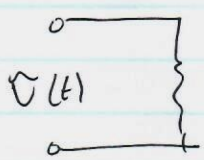
8: Nodal:

1) $$\frac{V_1 - V_s}{10} + \frac{V_1 - V_o}{-20j} + \frac{V_1 - V_o}{20}$$

2) $$\frac{V_o - V_1}{10} + \frac{V_o}{-10j} = 0$$

$$\therefore \quad V_o = 4 \angle -90^\circ = 4 \sin(5000t)$$

Root - Mean Squared Value (RMS):



$$\Rightarrow \quad P_{absorbed} = \frac{v^2(t)}{R}$$

$$\boxed{P_{avg} = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}\,dt} \Rightarrow !!!$$

$$P_{avg} = \frac{1}{R}\Bigg(\underbrace{\sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}}_{V\ RMS}\Bigg)^2$$

$$\therefore \quad \boxed{P_{avg} = \frac{V_{rms}^2}{R}} \Rightarrow \boxed{V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}}$$

RMS ⇒ effective value

Similarly:

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

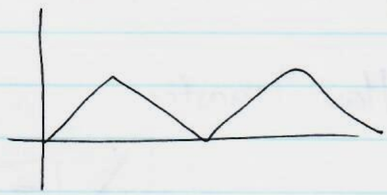
$$\therefore \quad \boxed{P_{av} = R \cdot I_{RMS}^2}$$

RMS FOR SINUSOIDS

$$\boxed{V_{rms} = \frac{V_m}{\sqrt{2}}} , \quad \boxed{I_{rms} = \frac{I_m}{\sqrt{2}}}$$

For a function that is non-sinusoidal but periodic:

$$V_{rms} = \frac{V_m}{\sqrt{3}} , \quad I_{rms} = \frac{I_m}{\sqrt{3}} \Rightarrow \text{Triangular!}$$



→ Triangular function.

How do you find RMS for sums of functions?

Ex:// $v(t) = V_{m_1} \cos(\omega_1 t + \theta_1) + V_{m_2} \cos(\omega_2 t + \theta_2)$

1) If $\omega_1 \neq \omega_2$

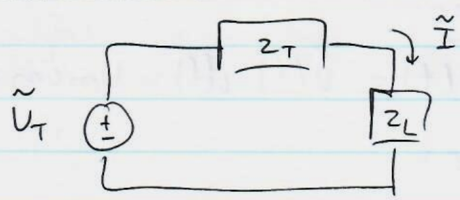
$$\therefore \quad V_{rms} = \sqrt{V_{rms_1}^2 + V_{rms_2}^2}$$

2) If $\omega_1 = \omega_2$

$$V_{rms} = \sqrt{V_{rms_1}^2 + V_{rms_2}^2 + V_{m_1} V_{m_2} \cos(\theta_1 - \theta_2)}$$

If DC $\Rightarrow$ $V_{rms} = V_{DC}$ (no $\div \sqrt{2}$)

MAX POWER TRANSFER



($\tilde{U}_T + \tilde{Z}_T \Rightarrow$ Thevenin)

$Z_T = R_t + jX_T$

$Z_L = R_L + jX_L$

Max avg power transfer happens when:

$$R_L = R_t \quad \wedge \quad \boxed{X_L = -X_T}$$

↳ Z_T is inductive $\Rightarrow$ Z_L is capacitive

Z_T is capacitive $\Rightarrow$ Z_T is inductive

P_{max} of L: $\quad P_{max} = \dfrac{|\hat{V}_t|^2}{8 R_t}$

If $Z_L = R_L$ (real circuit element)

$\Rightarrow \quad R_L = |Z_t| = \sqrt{R_t^2 + X_t^2}$

$\Rightarrow \quad P_{max} = \frac{1}{2} |\hat{I}|^2 R_L = I_{rms}^2 \cdot R_L$

Max Transfer

$\boxed{Z_L \in \mathbb{C}}$

$Z_L = R_t - X_T j$

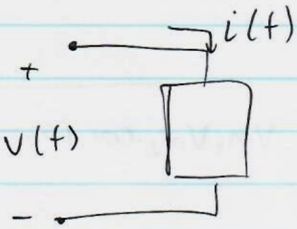
$P_{max} = \dfrac{|\hat{V}_T|^2}{8 R_T}$

$\boxed{Z_L \in \mathbb{R}}$

$R_L = \sqrt{R_t^2 + X_t^2}$

$P_{max} = I_{rms}^2 \cdot R_L$

DIFFERENT POWER TYPES



$v(t) = V_m \cos(\omega t + \theta_v)$

$i(t) = I_m \cos(\omega t + \theta_i)$

Subtract θ_i from both equations

$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$

$i(t) = I_m \cos(\omega t)$

Instantaneous power:

$$p(t) = v(t) \cdot i(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos(\omega t)$$

Active (real) power:

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) \quad \text{(watts)}$$

↳ electric energy → non-electric energy

Reactive power

$$Q = V_{rms}\, i_{rms}\, \sin(\theta_v - \theta_i) \qquad (\overset{\text{volts}}{V}-\overset{\text{ampere}}{A}-\overset{\text{reactive}}{R})$$

↳ Not transformed to non-electric energy, transfered between elements.

Linking equations:

$$\boxed{p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t)}$$

Also $P_{avg} = P$ (active power)

Power factor: $pf = \cos(\theta_v - \theta_i)$ $\xrightarrow{\text{lag}}$ $\tilde{I}$ lags $\tilde{V}$ / $Q > 0$

$\xrightarrow{\text{lead}}$ $\tilde{I}$ leads $\tilde{V}$ / $Q < 0$

$$\therefore \quad P = V_{rms} I_{rms} \cdot pf$$
$$= \frac{1}{2} V_m I_m (pf)$$

POWER IN A RESISTOR

$\theta_v \leftrightarrow \theta_i \Rightarrow$ equal to each other.

↳ $pf = \cos(\theta_v - \theta_i) = \cos(0) = 1$

$$\therefore \quad P = \frac{1}{2} V_m I_m = \frac{1}{2}\frac{V_m^2}{R} = \frac{1}{2} I_m^2 R$$

All normal formulas work.

$$Q = V_{rms}\, i_{rms}\, \sin(\underbrace{\theta_v - \theta_i}_{0}) = 0$$

POWER IN AN INDUCTOR

$$\theta_v - \theta_i = 90° \Rightarrow pf = \cos(90°) = 0$$

$$\therefore \quad P = 0$$

$$Q = \frac{1}{2} V_m I_m = \frac{1}{2}(\omega L I_m) I_m = \frac{1}{2}\omega L I_m^2 = \boxed{\omega L I_{rms}^2}$$

$$\text{OR} \quad Q = \frac{1}{2}\left(\frac{V_m}{\omega L}\right) V_m = \frac{1}{2}\frac{V_m^2}{\omega L} = \boxed{\frac{V_{rms}^2}{\omega L}}$$

No energy stored ⇒ store in one half of cycle + then release.

POWER IN A CAPACITOR

$$\theta_v - \theta_i = -90^\circ \Rightarrow \begin{array}{l} \cos(\theta_v - \theta_i) = 0 \\ \sin(\quad " \quad) = -1 \end{array}$$

$\therefore$ $P = 0$

$Q = -\frac{1}{2} V_m I_m = -V_{rms} I_{rms}$ (always negative!)

Expanding:

In a capacitor: $I = \omega C V_m$

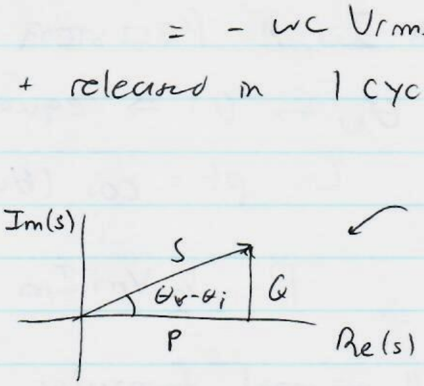
$$\therefore \quad Q = -\frac{1}{2}\frac{I_m^2}{\omega C} = -\frac{I_{rms}^2}{\omega C}$$

$$\Rightarrow Q = -\frac{1}{2} V_m(\omega C V_m) = -\frac{1}{2}\omega C V_m^2$$
$$= -\omega C V_{rms}^2$$

No P absorbed ⇒ stored + released in 1 cycle.

COMPLEX POWER

$S = P + jQ$ ⇒
↳ [VA]



Power triangle

Notice:

$$\frac{Q}{P} = \frac{\sin(\theta_v - \theta_i)}{\cos(\theta_v - \theta_i)} = \tan(\theta_v - \theta_i)$$

$$\therefore \boxed{\theta_v - \theta_i = \tan^{-1}(Q/P) = \cos^{-1}(pf)}$$

$$\boxed{P_{app} = |S| = \sqrt{P^2 + Q^2}} \rightarrow [VA]$$

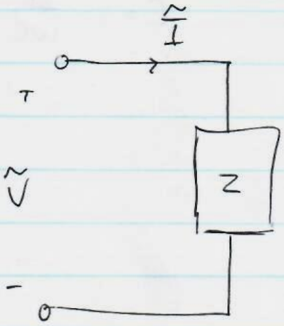
Conjugate of I!: $I^* = I_m \angle -\theta_i$

$$\boxed{S = \frac{1}{2}\tilde{V}\tilde{I}^* = \tilde{V}_{rms} \cdot \tilde{I}_{rms}} \rightarrow |S| = V_{rms} \cdot I_{rms}$$

↳ Remember: $\tilde{V}_{rms} = V_{rms} \angle \theta_v \Rightarrow \hat{V}_{rms} = \frac{\hat{V}}{\sqrt{2}}$

$\tilde{I}_{rms} = I_{rms} \angle \theta_i \Rightarrow \hat{I}_{rms} = \frac{\hat{I}}{\sqrt{2}}$

- Complex power in impedance:



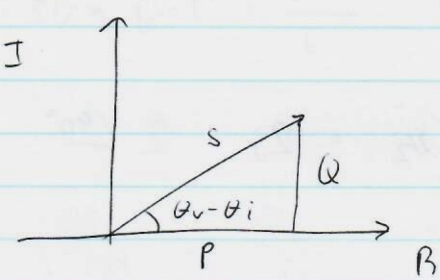
$$\left.\begin{aligned} \tilde{V} &= V_m \angle \theta_v \\ \tilde{I} &= I_m \angle \theta_i \\ \hat{V} &= Z \cdot \tilde{I} \end{aligned}\right\}$$

$$\begin{aligned} S &= \frac{1}{2}\tilde{V} \cdot \tilde{I}^* \\ &= \frac{1}{2} Z \tilde{I} \cdot \tilde{I}^* \\ &= \frac{1}{2} Z \cdot I_m \angle \theta_i \cdot I_m \angle -\theta_i \\ &= \frac{1}{2} I_m^2 \cdot Z \\ &= I_{rms}^2 Z \Rightarrow Z = R + jX \\ &= \underbrace{R I_{rms}^2}_{P} + \underbrace{jX I_{rms}^2}_{Q} \end{aligned}$$

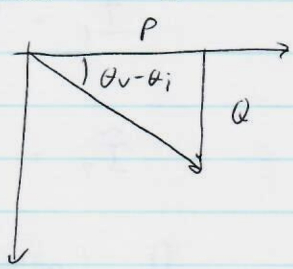
$\therefore \quad P = R I_{rms}^2$

$Q = X I_{rms}^2$ → $\omega L I_{rms}^2 \Rightarrow (+)$ if inductive Z
→ $-\frac{1}{\omega C} I_{rms}^2 \Rightarrow (-)$ if capacitive Z

Inductive Z:



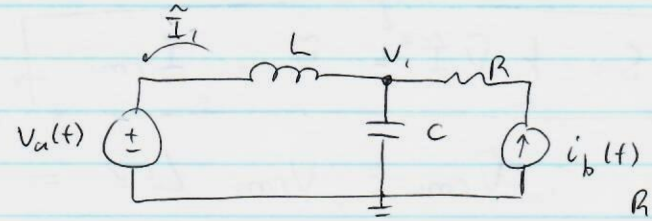
Capacitive Z



$\therefore \quad |S| = \sqrt{P^2 + Q^2}$

$$\tan(\theta_v - \theta_i) = \frac{Q}{P} = \frac{X}{R}$$

- Ex: //



$V_a(t) = \cos(100t - 90°)$
$i_b(t) = \sqrt{2}\cos(100t + 45°)$
$R = 1\,\Omega$ $L = 1/100\ H$, $C = 1/50\ F$

Lookins at power of each element:

①: Convert to phasor domain:

$V_a(t) = 1\angle{-90°} = -j$ $R = 1\,\Omega$ $C = -\frac{1}{wC}j = -j/2$

$i_b(t) = \sqrt{2}\angle 45° = 1+j$ $L = j$

②: Using nodal analysis to find $\tilde{V}_1$

$$\frac{\tilde{V}_1 - \tilde{V}_a}{Z_L} + \frac{\tilde{V}_1}{Z_C} = \tilde{I}_b$$

$$\frac{\tilde{V}_1 + j}{j} + \frac{\tilde{V}_1}{-j/2} = \tilde{I}_b$$

$$-j\tilde{V}_1 + 1 + 2j\hat{V}_1 = 1+j$$

$$\therefore V_1 = 1 \Rightarrow 1\angle 0°$$

③: Compute current in each branch + voltage

$$\hat{I}_1 = \frac{\tilde{V}_1 - \hat{V}_a}{j} = \frac{1+j}{j} = 1-j = \sqrt{2}\angle{-45°}$$

$$\tilde{I}_2 = \frac{\tilde{V}_1}{-j/2} = 1/{-j/2} = 2j = 2\angle 90°$$

$\hat{V}_2$ (voltage across R):

$$\frac{\tilde{V}_2 - \tilde{V}_1}{R} = \tilde{I}_b(t)$$

$$\tilde{V}_2 - 1 = 1+j$$

$$\hat{V}_2 = 2+j = \sqrt{5}\angle 26.57°$$

④: Computing power:

1. $\tilde{V}_a(t)$:

$$P_a = V_{rms} \, i_{rms} \cos(\theta_v - \theta_i)$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\cos(-90° - (-45°))$$
$$= 1/\sqrt{2} \cos(-45°)$$
$$= 1/\sqrt{2} \cdot 1/\sqrt{2}$$
$$= \boxed{1/2 \text{ W}}$$

$$Q_a = V_{rms} \cdot i_{rms} \cdot \sin(\theta_v - \theta_i)$$
$$= 1/\sqrt{2} \sin(-45°)$$ ⇐ Use power triangles if unsure.
$$= \boxed{-1/2 \text{ VAR}}$$

2. $i_b(t)$:

$$P_a = V_{rms} \cdot i_{rms} \cos(\theta_v - \theta_i)$$
$$= \left(\frac{\sqrt{5}}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\cos(26.57° - 45°)$$
$$= \boxed{3/2 \text{ W}}$$
$$Q_a = \boxed{-1/2 \text{ VAR}}$$

3. For R:

Good trick! No need to calculate $\tilde{V}_R$

$$P_R = V_{rms} I_{rms} = R I_{rms}^2 = 1 \cdot \left(\frac{\sqrt{2}}{\sqrt{2}}\right)^2 = \boxed{1 \text{ W}}$$
$$Q_R = 0$$

4. For L:

$$P_L = 0$$

$\tilde{V}_L = \tilde{V}_1 - \tilde{V}_a$

$$Q_L = V_{rms} \cdot i_{rms} = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \boxed{1 \text{ VAR}}$$

Could have also used: $Q = X I_{rms}^2 = \omega L I_{rms}^2$

5. For C:

$$P_C = 0$$

$$Q_C = V_{rms} \cdot I_{rms} \cdot \sin(\theta_v - \theta_i)$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)\sin(-90^\circ)$$

$$= \boxed{-1 \text{ VAR}}$$

Recogniz.

$$Q = X I_{rms}^2 = -\frac{1}{\omega C} I_{rms}^2$$

Note:

$$\underline{\sum P = 0}, \quad \underline{\sum Q = 0}$$