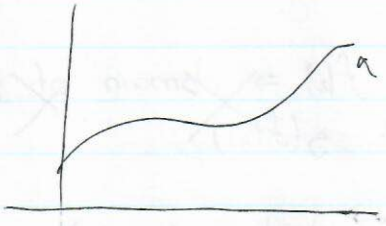
MATH 117

PART 1

INVERSES



$y = f(x)$

$x = f^{-1}(y)$ ⇐ Switching input and output

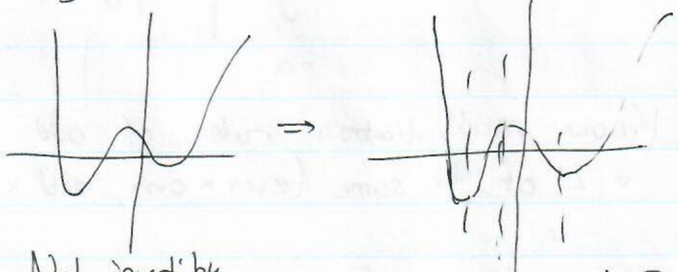
- To find inverse:

1. Solve for indep. variable

$$y = e^{x^2} + 1$$
$$y - 1 = e^{x^2}$$
$$\ln(y-1) = x^2$$
$$x = \pm\sqrt{\ln(y-1)}$$
$$\therefore f^{-1}(y) = \pm\sqrt{\ln(y-1)}$$

Can switch variable if needed

2. Domain:
   - Domain of inverse = ~~dom~~ range of original
   - Domain of original = range of inverse.
   - Restricting domain:



Not invertible

⇐ Intervals are good!

1. Note intervals where you can invert
2. Note impact on y (range) ⇐ BE CAREFUL

- Ex:

$$f(x) = \begin{cases} 1-x, & x \in (-\infty, -1) \\ \frac{2}{\pi}\cos^{-1}, & x \in [-1, 1) \\ -\sqrt{x-1}, & x \in [1, \infty) \end{cases}$$

$\Rightarrow f^{-1}(y) = 1-x \quad (x \geq 2)$

$\Rightarrow f^{-1}(y) = \cos\left(\frac{\pi}{2}x\right) \quad (0 < x \leq 2)$

$\Rightarrow f^{-1}(y) = x^2+1 \quad (0 < x \ldots)$

Found using restrictions on range of original

$$\therefore f^{-1}(x) = \begin{cases} x^2+1, & x < 0 \\ \cos\left(\frac{\pi}{2}x\right), & 0 \leq x < 2 \\ 1-x, & x \geq 2 \end{cases}$$

Composition

$$(g \circ f)(x) = g(f(x))$$

Domain + range:

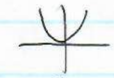
~~1. Find range of $f(x) \Rightarrow$ domain of $g(x)$~~

~~2. Find range of $g(f(x))$~~

* $g(x)$ may be more restrictive! *
↳ Choose more restrictive

1. Find domain restriction of $g(x) \Rightarrow$ range restriction on $f(x)$
2. Find domain of $f(x)$ which satisfies range restriction

Symmetry

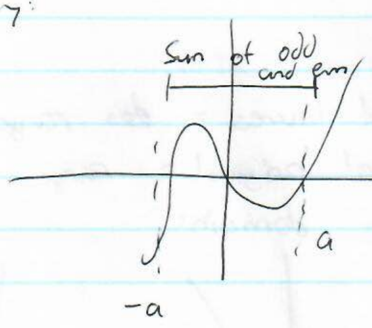
Even: $f(-x) = f(x) \Rightarrow$ Symmetric about y 

Odd: $f(-x) = -f(x) \Rightarrow$ Symmetric about x: 

- Check:

$$f(-x) \Rightarrow \text{output } (-f(x) \text{ or } f(x)?)$$

- Cool property:



- Know multiplication rules of odd and even functions
  - 2 of the same (even × even, odd × odd) ⇒ even. Otherwise: odd

Absolute Value Function

- Ex: $\left| \frac{x}{x+1} \right| < 1 - |x|$

1. Manipulate into a simpler form:

$$|x| < (1-|x|)|x+1|$$

2. Identify points st where defn change.

$$x = 0, -1$$

3. Consider cases: $x < -1$, $-1 \le x < 0$, $x > 0$

① : $x < -1$:

$$-x < -(1+x)(x+1)$$

$$x > x^2 + 2x + 1$$

$$x^2 + x + 1 < 0 \Rightarrow \text{No solution.}$$

② : $-1 \leq x < 0$:

$$-x < (1+x)(x+1)$$

$$x^2 + 3x + 1 > 0 \Rightarrow \text{Solve}: \quad x > \frac{\sqrt{5} - 3}{2}, \quad x < \frac{-\sqrt{5}+3}{2}$$

③ : $x > 0$:

$$x < (1-x)(x+1)$$

$$x^2 + x - 1 < 0$$

$$\frac{-1-\sqrt{5}}{2} < x < \frac{-1+\sqrt{5}}{2}$$

4. Combine solution w/ case domain restriction:
   - ① ✓
   - ②: Only $\frac{\sqrt{5}-3}{2} \leq x < 0$ works
   - ③: Only $0 \leq x < \frac{\sqrt{5}-1}{2}$
5. Final solution:

$$x \in \left(\frac{\sqrt{5}-3}{2}, \frac{\sqrt{5}-1}{2}\right)$$

HEAVISIDE

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

1. Multiply function by $H(x)$ to "turn on" function after x value.
2. Shift $x-a$ to turn on a unit right
3. Negate functions!

$$f(t) = \begin{cases} t & t < 0 \\ 1-t^2 & t \in [0,2] \\ 1 & t \in (2,4 \\ 0 & t \geq 4 \end{cases}$$

$$f(t) = t + (1-t^2 \underline{-t})H(t) + (\underline{1-1+t^2})H(t-2) + \underline{(-1)}H(t-4)$$

PARTIAL FRACTION DECOMPOSITION

1. Convert to proper form via long division
2. Factor your denominator.
3. Make general solution:

1) $$\frac{- \; - \; -}{(x-a)(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2} \Rightarrow \text{Repeat } x^n \text{ factors } n \text{ times.}$$

2) $$\frac{- \; - \; - \; -}{(x-a)(x^2-bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2-bx+c}$$

4. Multiply and solve for A and B

↳ Cover-up: assume x value to cancel term OR Equate powers of x.

- Sketching:

$$f(x) = p(x) + \frac{A}{x-a} + \frac{B}{x-b}$$

$x \to \pm\infty$: $p(x)$ dominates

$x \to a, b$: Asymptotic behavior.

↳ Look at left + right behavior.

- Ex://

$$\frac{1}{(x-2)(x^2+2x+5)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+5}$$

Multiply denom:

$$1 = A(x^2+2x+5) + \cancel{B(x-2)(x^2+2x+5)}\ (Bx+c)(x-2)$$

Let $x = 2$:

$$1 = A(4+4+5) + 0$$

$$A = 1/13$$

Finding B by equating x^2:

$$0 = A + B$$

$$0 = 1/13 + B$$

$$\therefore B = -1/13$$

Finding C by equating constant term:

$$1 = -2C + 5A$$

$$\therefore C = \frac{1}{2}(5A - 1)$$

$$= \frac{1}{2}\left(\frac{5}{13} - \frac{13}{13}\right)$$

$$= -4/13$$

$$\therefore \frac{1}{(x-2)(x^2+2x+5)} = \frac{1}{13}\left(\frac{1}{x-2} - \frac{(x+4)}{x^2+2x+5}\right)$$

Hyperbolic Curves

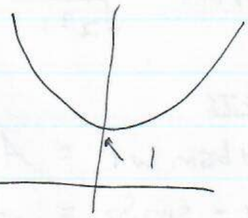
$$e^x = \tfrac{1}{2}e^x + \tfrac{1}{2}e^x$$

$$= \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x} + \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}$$

$$= \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd}}$$

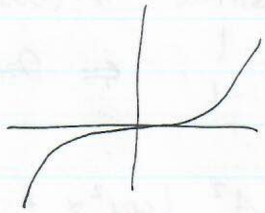
- $\cosh(x)$:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



- $\sinh(x)$:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$



- Identities: share same trig identities except when 2 sines multiplied → (−)
- Solving:

$$\sinh(x) = \tfrac{1}{2}$$
$$\frac{e^x - e^{-x}}{2} = \tfrac{1}{2}$$
$$e^x - e^{-x} = 2(\tfrac{1}{2})$$
$$\ln(e^x - e^{-x}) = \ln 1$$

Correct way
1. Find $\sinh^{-1}(x)$
2. Plug in values.
   ↳ 1 soln if sinh, 2 if cosh

Inverse Trig

Restrict domain:

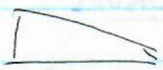
$$\sin^{-1}(x) \Rightarrow x \in [-\pi/2, \pi/2]$$
$$\cos^{-1}(x) \Rightarrow x \in [0, \pi]$$
$$\tan^{-1}(x) \Rightarrow x \in [\pi/2, \pi/2]$$

Numbers: plug + chug:

$$\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

- Remember domain restriction when looking @ multiple solution.

"In terms of x"

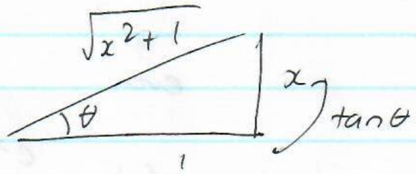
Use $\theta =$ ratio + 

- Express $\sin(\tan^{-1} x)$ in terms of x.

Let $y = \sin(\tan^{-1} x)$

Let $\theta = \tan^{-1} x$

$\therefore \quad x = \tan\theta$



$\therefore \sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

AMPLITUDES + SHIFTS

$a\cos\omega t + b\sin\omega t = A\sin(\omega t + \alpha) \,/\, A\cos(\omega t + \alpha)$

- Ex:// Put $\cos 5x - \sin 5x = A\cos(\omega t + \alpha)$

$\cos 5x - \sin 5x = A\cos(5t + \alpha)$

$\cos 5x - \sin 5x = A(\cos 5t \cos\alpha) + -A(\sin 5t \sin\alpha)$

① $A\cos\alpha = 1$

② $-A\sin\alpha = -1$

$\Leftarrow$ Derived eqns from equating coefficients.

Find A:

$①^2 + ②^2 \qquad A^2(\cos^2\alpha + \sin^2\alpha) = 2$

$A = \pm\sqrt{2} \Rightarrow (+)$ for amplitude.

① / ② for tan:

$\tan\alpha = 1$

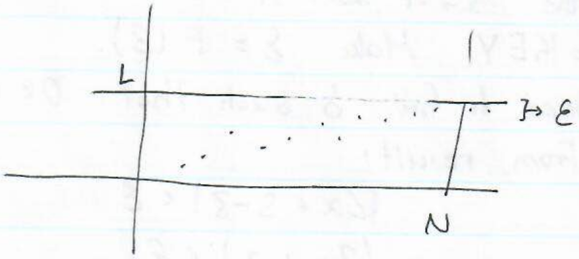
$\alpha = \pm \pi/4 \Rightarrow \tan^{-1}(x) \in [-\pi/2, \pi/2)$

Check! If ① and ②'s signs for trig ratio don't match to angle, add π to it.

$\therefore \cos 5x - \sin 5x = \sqrt{2}\cos(5x + \pi/4)$

PART 2

- Sequences:
  - Limit: $\lim_{n\to\infty} a_n = L$ means:
    For any ε (>0), there exist a number N s.t.
    $$n > N \Rightarrow |a_n - L| < \varepsilon$$



  - Proofs: must find N (usu. in terms of ε) such that above defn is true.
    - Ex:// Prove $\frac{2n+1}{n-3} \to 2$ as $n \to \infty$
      1: Write down what I need to prove.
      $$n > N \Rightarrow \left|\frac{2n+1}{n-3} - 2\right| < \varepsilon \quad (\forall \varepsilon)$$
      2: Work w/ conclusion to find N (in terms of ε).
      $$\left|\frac{2n+1}{n-3} - 2\right| < \varepsilon$$
      Sequence: always +
      $$\Leftrightarrow \frac{2n+1}{n-3} < \varepsilon + 2$$
      $$\Leftrightarrow \frac{2n+1-2n+6}{n-3} < \varepsilon$$
      $$\Leftrightarrow \frac{7}{n-3} < \varepsilon$$
      $$\Leftrightarrow \frac{7}{\varepsilon} < n-3$$
      $$\Leftrightarrow n > \frac{7}{\varepsilon} + 3$$
      → N! Equality satisfied when $n > \frac{7}{\varepsilon} + 3$
      3: Write down proof.
      $\forall \varepsilon$ ($\varepsilon > 0$), if $n > \frac{7}{\varepsilon} + 3$, $\left|\frac{2n+1}{n-3} - 2\right| < \varepsilon$
      - Retrace back to original.
  - Different definitions:
    $\lim_{n\to -\infty}$, $\lim_{n\to\infty} = \infty$, $\lim_{n\to\infty} = -\infty$
    $n < N$ $\quad a_n > M$ $\quad a_n < M$

- Functions:

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon, \exists \delta \text{ s.t. } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

I make x close enough to a (δ error) such that $f(x)$ is close enough to L for all ε

- Ex:// Prove $\lim_{x \to -1} (2x+5) = 3$
  - KEY: Make $\delta = F(\varepsilon)$

  Need to find δ such that $0 < |x+1| < \delta \Rightarrow |2x+5-3| < \varepsilon$

  From result:

$$|2x+5-3| < \varepsilon$$
$$|2x+2| < \varepsilon$$
$$2|x+1| < \varepsilon$$
$$|x+1| < \frac{\varepsilon}{2} \Leftarrow \text{We have } \delta$$

  Proof: If $0 < |x+1| < \frac{\varepsilon}{2}$, then $|2x+5-3| < \varepsilon \ \forall \varepsilon$

  ↳ Expand on conclusion

- Calculating limits.
  - Indeterminate forms: $0/0, \frac{\pm\infty}{\pm\infty}, \infty - \infty, 0 \cdot \infty, 0^0, 1^\infty, 0^0$
  - Squeeze theorem: v. useful if functions have bounds.

$$\lim_{n \to \infty} \frac{(-1)^n + 2\sin n}{\sqrt{n^2+5}}$$

$$-1 \le \sin n \le 1$$
$$-2 \le \sin n \le 2$$
$$-3 \le (-1)^n + \sin n \le 3$$
$$-\frac{3}{\sqrt{n^2+5}} \le \frac{(-1)^n + \sin n}{\sqrt{n^2+5}} \le \frac{3}{\sqrt{n^2+5}}$$

1. Take bounded function
2. Build up to current w/ limits
3. Take limit

$$\lim_{n \to \infty} \frac{-3}{\sqrt{n^2+5}} \le \lim_{n \to \infty} \dots \le \lim_{n \to \infty} \frac{3}{\sqrt{n^2+5}}$$

$$0 \le \_\_\_\_\_ \le 0$$

$$\therefore \lim_{n \to \infty} \frac{(-1)^n + 2\sin n}{\sqrt{n^2+5}} = 0$$

  - Strategies:
    - If continuous: $\lim_{n \to a} f(x) = f(a)$
    - if $n \to \pm\infty$: divide by n/biggest growth

      ↳ If $n \to -\infty$, put $\ominus$ if goins into root

- Rationalization: <u>sqrt</u>
- For $\lim_{x \to a}$: check $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ (if not, DNE)

- Factoring
- Fundamental trig limit: $\cancel{\lim_{n \to \infty}}$ $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$

- Different limit definitions:

① : $x < N \Rightarrow f(x) < M$
↳ $\lim_{x \to -\infty} f(x) = -\infty$

④ : ~~dian~~ $x < N \Rightarrow |f(x) - L| < \varepsilon$
↳ $\lim_{x \to -\infty} f(x) = L$

② : $0 \leq |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
↳ $\lim_{x \to a} f(x) = L$

⑤ : $x > N \Rightarrow |f(x) - L| < \varepsilon$
$\lim_{x \to \infty} f(x) = L$

③ : $0 < |x-a| < \delta \Rightarrow f(x) > M$
$\lim_{x \to a} f(x) = \infty$

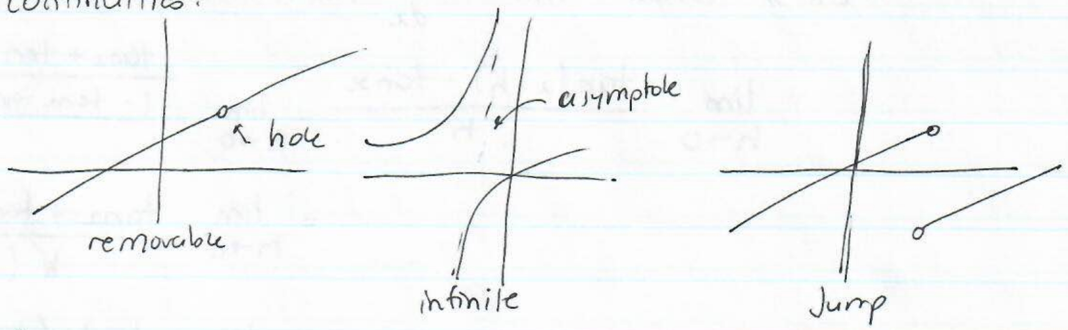
⑥ : $x > N \Rightarrow f(x) < M$
$\lim_{x \to \infty} f(x) = -\infty$

- Continuity:

$$\lim_{x \to a} f(x) = f(a)$$

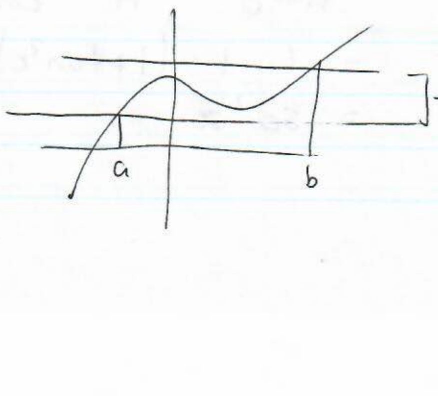
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

- Use in piecewise function continuity evaluation.
- Discontinuities:



- Intermediate Value Theorem:

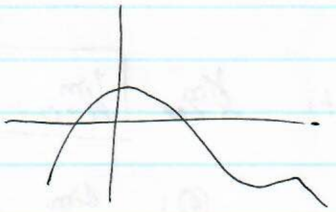
If $f(x)$ is continuous from $[a, b]$, then ~~f(x) must exist~~ $f(x) = c$ where $a < c < b$.



→ All values must be outputted from $f(a) \to f(b)$

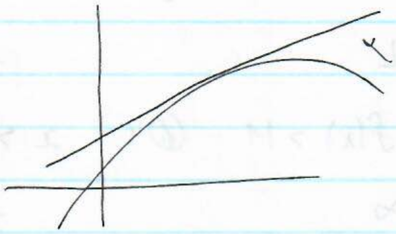
* Note: closed interval! *

- Used to prove certain value of a function exists
- Root finding:



1. Make rough sketch
2. Choose values that are on opposite signs
3. Narrow domain

PART 3



Slope of tangent line?
↳ Derivative.

- Definitions of derivative:

1: $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2: $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Be able to prove derivatives via definition.

- Ex: // Show that $\frac{d}{dx} \tan x = \sec^2 x$

$$\lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h} = \lim_{h \to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h}$$

$$= \lim_{h \to 0} \frac{\tan x + \tan h - \tan(x)(1 - \tan x \tan h)}{h(1 - \tan x \tan h)}$$

$$= \lim_{h \to 0} \frac{\tanh(1 + \tan^2 x)}{h(1 - \tan x \tanh)}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{1}{\cos h} \cdot \frac{1 + \tan^2 x}{1 - \tan x \tanh}$$

$$= 1 \cdot 1 \cdot (1 + \tan^2 x) \leftarrow (\tanh \to 0 \text{ as } h \to 0$$

$$= \sec^2 x$$

- Diffrentiation:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\left.\begin{aligned} \frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\cos^{-1}x &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2} \end{aligned}\right\}$$

Important: $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$$\left.\begin{aligned} \frac{d}{dx}(\sinh x) &= \cosh x \\ \frac{d}{dx}(\cosh x) &= \sinh x \end{aligned}\right\}$$

Can be proven via definition of $\sinh x, \cosh x$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

- Implicit differentiation: cannot isolate for y
  1. Take derivative w/ respect to $x, t, \ldots$ for whole expression
  2. Isolate for $\frac{dy}{dx}$ or rate in common.
- Inverse differentiation:

1. Rewrite to remove inv
2. Implicit differentiation

1. Find coordinates on original + inverse. (a, b) on original, (b, a) inverse
2. $\left.(f^{-1}(x))'\right|_{x=b} = \frac{1}{f'(a)}$

- Ex: Derive derivative for $\tan^{-1} x$:

Let $y = \tan^{-1} x$. Thus: $x = \tan y$

Use implicit differentiation:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = \sec^2 y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

- Logarithmic differentiation: used when you have really nasty expression.
  1. Take logarithm of both sides.
  2. Implicitly differentiate + use chain rule.

- Ex:// Find $y'(1)$ if $y = x^3 e^{-x} \ln(x^2+1)$

Take logarithm:

$$\ln y = \ln(x^3 e^{-x} \ln(x^2+1))$$

$$\ln y = \ln x^3 - x\ln e + \ln(\ln(x^2+1))$$

$$\ln y = \ln x^3 - x + \ln(\ln(x^2+1))$$

Take derivative:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - 1 + \frac{1}{\ln(x^2+1)} \cdot \frac{1}{x^2+1} \cdot 2x$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} - 1 + \frac{1}{\ln(x^2+1)} \cdot \frac{1}{x^2+1} \cdot 2x\right)$$

$$= (x^3 e^{-x} \ln(x^2+1))\left(\frac{3}{x} - 1 + \frac{1}{\ln(x^2+1)} \cdot \frac{1}{x^2+1} \cdot 2x\right)$$

Evaluate at $x = 1$

$$\left.\frac{dy}{dx}\right|_{x=1} = \left(\frac{1}{e} \cdot \ln 2\right)\left(3 - 1 + \frac{1}{\ln 2} \cdot \frac{1}{2} \cdot 2\right)$$

$$= \frac{\ln 2}{e}\left(2 + \frac{1}{\ln 2}\right)$$

$$= \frac{2\ln 2}{e} + \frac{1}{e}$$

- Derivative theorems:
  1. Differentiability $\Rightarrow$ continuity
  2. Mean Value Theorem:

     If f is differentiable $\in [a,b]$, $\exists\, c \in [a,b]$ where

     $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

     - Often used in proofs.
  3. L'Hopital's

     If $\lim_{x \to \square} \frac{f(x)}{g(x)} = \frac{0}{0} = \frac{\infty}{\infty}$, then:

     $$\lim_{x \to \square} \frac{f(x)}{g(x)} = \lim_{x \to \square} \frac{f'(x)}{g'(x)}.$$

     1. Try using other limit techniques first
     2. Manipulate indeterminate forms into this form.
        1. $a^{\infty} \Rightarrow$ use log to put into correct form.

           $$\lim_{x \to 0} (1+x)^{1/x}$$

           Let $y = (1+x)^{1/x}$

           $$\ln y = \frac{\ln(1+x)}{x}$$

           $$\therefore \lim_{x \to 0} (\ln y) = \lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{1+x} = 1$$

           $$\therefore \lim_{x \to 0} (\ln y) = 1$$

           $\downarrow$ $\boxed{\text{Remove logarithm}}$

           $$\lim_{x \to 0} y = e$$
        2. $0 \cdot \infty \Rightarrow$ manipulate into better form:

           $$\lim_{x \to 0} x e^{1/x} = \lim_{x \to 0} \frac{e^{1/x}}{1/x}$$

           $$= \lim_{x \to 0} \frac{e^{1/x} \ln x}{\ln x}$$

           $$= \lim_{x \to 0} e^{1/x}$$

           $$= \infty$$

- Applications:
  1. Related rates:
     1. Make equation that ~~des~~ inter-relates information (Pythagoras, trig, volumes)
        - Make in terms of 1 variable (similar triangles, etc...)
     2. Implicitly differentiate
        - Careful: rate signs
     3. Solve for rate.
  2. Differentials:

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} \quad (\Delta x \text{ is really small})$$

- Estimation:
  1. Find function + take derivative.
  2. Find derivative at closest point $(x=a) \Rightarrow$ Known
  3. Use following equation:

$$\Delta f = \Delta x \cdot \frac{df}{dx}$$

- Ex: // Use differentials to find $24^{1/4}$.

Function: $\sqrt[4]{x}$

Derivative: $\frac{1}{4} x^{-3/4}$

$$\therefore \quad \Delta f = \Delta x \cdot \frac{1}{4} x^{-3/4}$$

$$\left.\begin{aligned} \Delta f &= 8 \cdot \frac{1}{4} \cdot \left(\frac{1}{x^{1/4}}\right)^{3/4} \\ &= 8 \cdot \frac{1}{4} \cdot \left(\frac{1}{16^{1/4}}\right)^{3} \end{aligned}\right\} \text{Evaluating at } x = 16 \text{ (known value)}$$

$$= 8 \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\right)^3$$

$$= 1/4$$

$$\therefore \quad \Delta f = f_f - f_i$$

$$1/4 = f_f - \cancel{1/4}\ 4$$

$$f_f = 9/4$$

- Error rates / related rates:

Error rate of x: $\frac{dx}{x}$

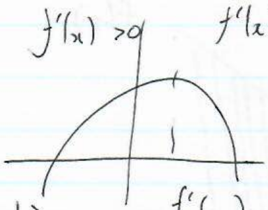
Manipulate equations to show error rate relations

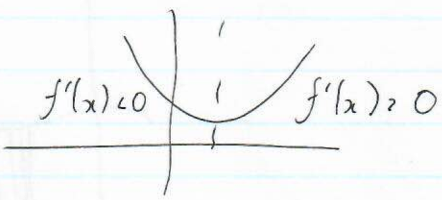
1. Take derivative.
2. Divide derivative expr. by formula given.

Look @ Quiz #5.

3. Curve sketching:

1. $f(x)$:
   1. $\lim_{x \to \infty}$, $\lim_{x \to -\infty}$, $\lim_{x \to a}$ (a is an asymptote)
      ↳ Use tricks!
   2. Points (x int, y int)
2. $f'(x)$:
   1. Critical points
      $f'(x) = 0$ or undefined] → max / min



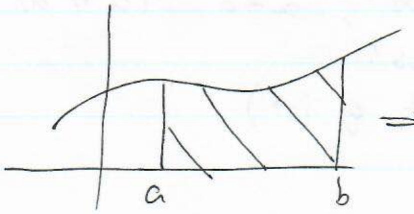


   2. $\lim_{x \to a/\pm\infty} f'(x)$
3. $f''(x)$:
   1. Verify max + min
      Max: $f'(x) = 0 \wedge f''(x) < 0$
      Min: $f'(x) = 0 \wedge f''(x) > 0$
   2. Inflection
      ↳ Undefined points!

∘ Ex: // Sketch $y = x - 2\ln(x^2+1)$.

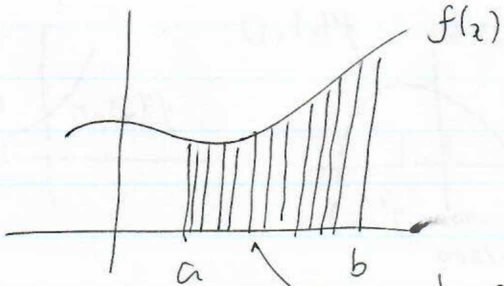
1. $f(x)$:
   $x = 0 \Rightarrow y = 0 - 2\ln(1) = 0$
   $\lim_{x \to \infty} f(x) =$
   $\lim_{x \to -\infty} f(x) = -\infty$
2. $f'(x)$:
   $f'(x) = 1 - \frac{2}{x^2+1} \cdot 2x$
   Critical points:
   $f'(x) = 0 \Rightarrow x = 2 \pm \sqrt{3}$ → $f(2+\sqrt{3}) = \ldots$ → $f(2-\sqrt{3}) = \ldots$
   $\lim_{x \to \infty} f'(x) = 1$
   ↳ Function goes to infinity.
3. $f''(x)$:
   $y'' = \frac{4x^2 - 4}{(x^2+1)^2}$
   Inflection: $x = \pm 1$ | Max + Min analysis.

Pt. 4: Integral Calculus.



$\Rightarrow \displaystyle\int_a^b f(x)dx.$

Approximate:



$\Delta x = \dfrac{b-a}{n}$, $x_i = x_a + i\Delta x$

height: $f(x_i^*)$

$$\int_a^b f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{b-a}{n} i\right)}_{\text{Height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}$$

→ Riemann Sum
↳ Use for proofs.

Summation properties.

$$\sum_{i=1}^{n} 1 = n \quad , \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad , \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Prove: $\displaystyle\int_1^2 x^2 = \frac{7}{3}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{1}{n} i\right)\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{1}{n} i\right)^2 \left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2}{n} i + \frac{i^2}{n^2}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n} + \frac{2}{n^2} i + \frac{i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{2n^2+4n}{2n^2} + \frac{2n^2+3n+1}{6n^2}\right)$$

$$= \frac{7}{3}$$

- Few integrals you should know:

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C \qquad \int \sec x \, dx = \ln|\sec x + \tan x|$$

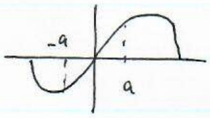
$$\int \csc x \, dx = -\ln|\csc x - \cot x| + C \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

- Symmetry:

If function is odd:

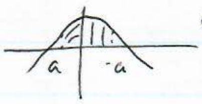
$$\int_{-a}^{a} f(x) \, dx = 0$$



⇐ Areas cancel.

If function is even:

$$\int_{-a}^{a} f(x) \, dx = 2\int_{0}^{a} f(x) \, dx$$



⇐ Same area.

- Substitution: replace function w/ u.
  1. Find functions whose derivative is present in integrand.
  2. Replace function w/ u
  3. Calculated du + substitute (eliminate variables)
  4. Definite integrals: convert bounds + calculate new integral.
  5. Substitute variable back.

- Ex://

1. $$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \qquad u = \cos x, \quad du = -\sin x \, dx, \quad dx = \frac{du}{-\sin x}$$

$$= -\int \frac{\sin x}{u} \frac{du}{\sin x}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

2. $$\int x^3 \sqrt{1+x^2} \, dx = \frac{1}{2}\int x^2 \sqrt{u} \, du \qquad u = 1+x^2, \quad du = 2x \, dx$$

$$= \frac{1}{2}\int (u-1)\sqrt{u} \, du$$

$\vdots$

3.

$$\int \frac{dx}{1+e^x} \to \int \frac{1}{1+e^x}\,dx = \int \frac{1}{u} \cdot \frac{du}{u-1} \Rightarrow \text{Solve via PFD}$$

$$\to \int \frac{e^{-x}}{1+e^{-x}}\,dx = -\int \frac{1}{u}\,du = -\ln|1+e^{-x}| + C$$

$$\to \int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int \left(1 - \frac{e^x}{1+e^x}\right)dx = x - \ln|1+e^x| + C$$

4.

$$\int \sin^7 x \cos^8 x = \int \sin^6 x \cos^8 x \cdot \sin x\,dx$$

$$= \int (1-\cos^2 x)^3 \cos^8 x \cdot \sin x\,dx \qquad u = \cos x,\quad du = -\sin x\,dx$$

$$= -\int (1-u^2)^3 u^8\,du$$

$\vdots$

$$\int \sin^6 x \sin^4 x = \int \left(\frac{1-\cos 2x}{2}\right)^3 \left(\frac{1+\cos 2x}{2}\right)^2 dx \ldots$$

- Substitution tricks:
  - Turn everything (function + derivatives) in terms of u
  - Experiment
  - Trig:
    - $\int \sin^n \cos^m dx$:
      - $n + m$ are diff. ~~po~~ odd + even: remove 1 ratio from odd → substitute
      - $n + m$ are even: double angle.
  - Fractions: convert top to look like denominator
    - ↳ Split up fractions!

- Taking derivative of integral:

Let $F(x)$ be antiderivative of $f(x)$.

$$\frac{d}{dx}\int_a^{g(x)} F(x)\,dx = f(g(x)) \cdot g'(x)$$

- Integration By Parts:

$$(uv)' = u'v + uv'$$

$$uv = \int u'v + \int uv'$$

$$\int uv' = uv - \int u'v$$

1. Identify u and dv in question. (could be whole factors)
   - ILATE (order of derivatives) $\Rightarrow u$
   - $dv = dx$ (esp. if product of inverse / log)

   } Simpler
2. Go through integration
   - Substitution
   - More IBP $\Rightarrow$ brings integral to same side.

- Ex://

1. $$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$
$$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right)$$
$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$
$$\int e^x \cos x\,dx = \frac{e^x(\sin x + \cos x)}{2} + C$$

2. $$\int e^{\sqrt{x}}\,dx = 2\int e^u \sqrt{x}\,dx$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

Note substitution ux.

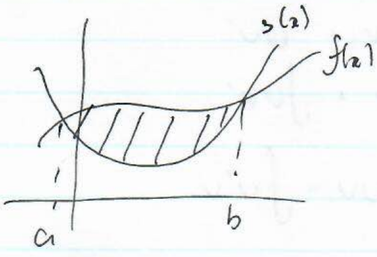
$$= 2\int e^u u\,dx$$
$$= 2\left(e^u u - \int e^u du\right)$$
$$= 2e^{\sqrt{x}}\sqrt{x} - 2e^{\sqrt{x}} + C$$
$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$

Key: Try the strategy + be willing to retry.

- Application:
  - I: Direct application: summing up (what does dx + $f(x)$ mean)
  - II: Mean value of a function:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

II: Area between curves:



1. Find intersection points.
2. Note which function is greater than in the interval.
3. $\int_a^b (f(x) - g(x))\,dx$ = area

- Note: sometimes easier by integrating on y.
- Use symmetry

- Trigonometric Substitution: power of quadratic:

$\sqrt{1+x^2} \Rightarrow x = \tan\theta$
$\sqrt{1-x^2} \Rightarrow x = \sin\theta$
$\sqrt{x^2-1} \Rightarrow x = \sec\theta$

Do $x = a$ ___ if $\sqrt{a^2 + \dots}$
↳ Factor out a.

1. Identify quadratics in powers.
2. Perform substitution (simple $x = \dots$ or complete square)
3. Solve (don't forget regular integrals: $\int \sec^2 d\theta$, $\int -\csc\cot$)
   - IBP: divide ratios into common identities + remainder
     $\int \cot^3 x\,dx = \int \cot^2 x \cdot \cot x\,dx$ → Bad example.
   - If you get (ratios)^2: use trig identities (double θ)
   - Complete square
   - If $\int \frac{1}{\sqrt{1-u^2}}$ or $\int \frac{1}{1+u^2} \Rightarrow \sin^{-1}(x)$ and $\tan^{-1} x$
   - Convert to common identities.

$$\int \tan^3 x\,dx = \int \tan^2 x \cdot \tan x\,dx$$
$$= \int \tan x\,dx - \int \sec^2 x \tan x\,dx$$
$$= \int \tan x\,dx - \left(\tan^2 x - \int \tan x \sec^2 x\,dx\right)$$

4. Convert $\theta \to x$ via triangles.

- Integration of Rational Functions.
  1. Try using substitution

$$\int \frac{1}{(x-4)^2} = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{x-4} + C$$

  2. Divide fraction using PFD.
  3. Integrate:

$$\frac{1}{x-a} \Rightarrow \ln|x-a|$$

$$\frac{1}{x^2+a^2} \Rightarrow \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \Leftarrow$$ Use complete the square + substitution if unable to split up.

$$\frac{1}{\sqrt{1-x^2}} \Rightarrow \sin^{-1}x + C$$

$$\frac{1}{(x^2+\ldots)^2} \Rightarrow$$ Other methods (substitution, trig)

• Ex://

1. $$\int \frac{1}{x^2-3x+2}\,dx = \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx = \ln|x-2| + \ln|x-1| + C$$

2. $$\int \frac{1}{x^2-2x+5}\,dx = \int \frac{dx}{(x-1)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

3. $$\int \frac{x+2}{x^2+x+1}\,dx = \int \frac{x+\frac{1}{2}+\frac{3}{2}}{x^2+x+1}\,dx$$ ← Split up to get to substitionable fraction

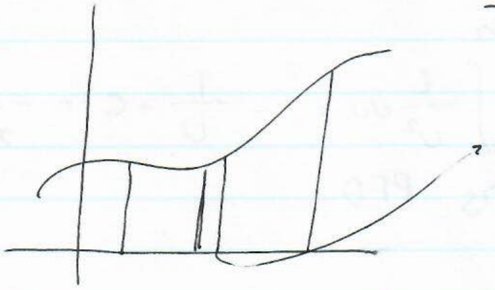
$u = x^2+x+1$
$du = (2x+1)\,dx$
$du = 2(x+\frac{1}{2})\,dx$

$$= \int \frac{x+\frac{1}{2}}{x^2+x+1}\,dx + \frac{3}{2}\int \frac{1}{x^2+x+1}$$

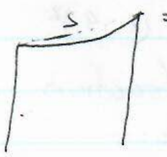
$$= \frac{1}{2}\int \frac{1}{u}\,du + \frac{3}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\left(\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)\right) + C$$

- Length of Curves:



To find length of curve:



$\Rightarrow$ Approximated via line!

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$\therefore$ Summing up over interval:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

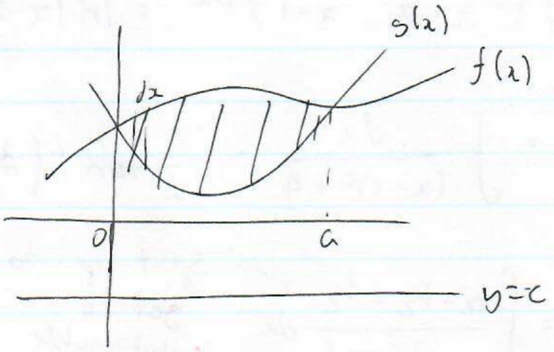
In respect to y:

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

3 cases:
1. Hyperbolic (identities!)
2. Parabola (trig substitution)
3. $f(x) = \frac{x}{a} + \frac{a}{x}$

- Volumes:

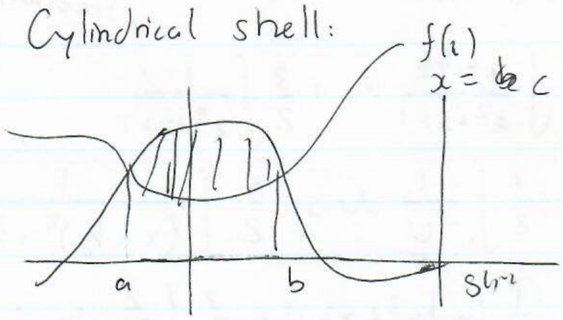
1. Washer method:



$$V = \pi \int_0^a \left(f^2(x) - (g(x))^2\right) dx$$

$$= \pi \int_0^a \left[(f(x) + c)^2 - (g(x) + c)^2\right] dx$$

2. Cylindrical shell:



$$V = 2\pi \int_a^b \underset{\text{radius}}{(c - x)}\ \overset{\text{Height}}{(f(x) - g(x))}\, dx$$

Be able to do this w/ respect to the y axis.

IMPROPER INTEGRALS

Usage: discontinuities, asymptotes, infinity

Let a be the limit which you cannot evaluate.

$$\int_b^a f(x)\,dx = \lim_{t \to a} \int_b^t f(x)\,dx$$

$$= \quad \vdots$$

$$= \text{Substitute } t + \text{take limit.}$$

Use combo of substitution + IBP + trig.

Discontinuity:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

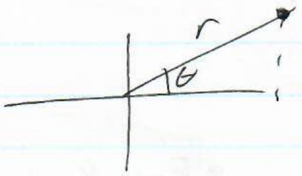
$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

Note:

$$\int_0^\infty f(x) = \int_0^a f(x)\,dx + \int_a^\infty f(x)\,dx \Rightarrow \text{Do not eval. directly.}$$

POLAR COORDINATES

$$x = r\sin\theta$$
$$y = r\cos\theta$$



Be careful w/ angles + use sign methodology

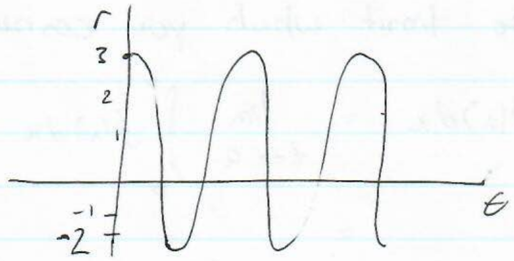
If $r < 0 \Rightarrow$ flip r.

Convert between $(x,y) \leftrightarrow (r,\theta)$

↳ $(x,y) \rightarrow (r,\theta)$: substitute $x = r\cos\theta$, $y = r\sin\theta$.

↳ $(r,\theta) \rightarrow (x,y)$: Create r^2 ($r^2 = x^2+y^2$) and group into $r\cos\theta$, $r\sin\theta$ to convert.

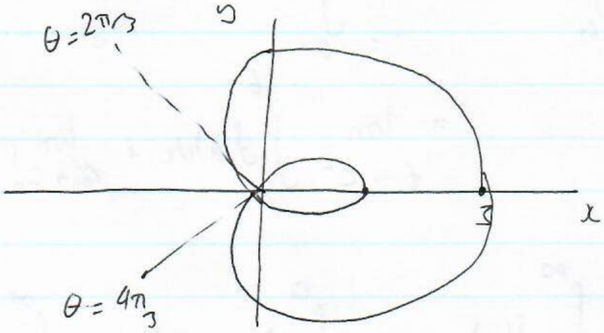
- Sketching polar graphs:
  - Sketch ~~y~~ $r = 2\cos\theta + 1$ on Cartesian plan.
    1. Write graph on r, θ plane.



Find ~~x-a~~ θ intercepts:

$$\therefore 0 = 2\cos\theta + 1$$
$$\theta = 2\pi/3,\ 4\pi/3.$$

2. On x, y plane: rotate through $0 - 2\pi$ and use graph to change radii.

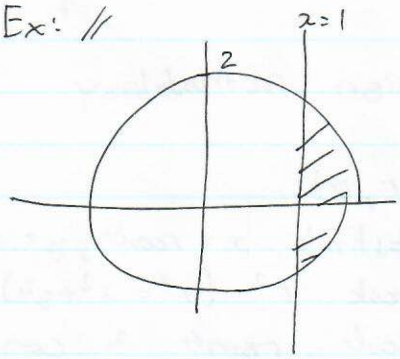


Also look at key points: $\theta = 0, \pi, \pi/2, \frac{3\pi}{2}$

- Areas:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}(f(\theta))^2 \, d\theta$$

- Ex: //



1. Convert to polar:
   Circle: $r = 2$
   Line: $x = 1 \Rightarrow r\cos\theta = 1 \Rightarrow r = \sec\theta$
2. ~~Integrate~~ Intersections:
   $\sec\theta = 2 \Rightarrow \theta = \pm\frac{\pi}{3}$
3. Integral:

$$A = 2\int_0^{\pi/3} (4 - \sec^2\theta)\, d\theta$$

Be careful w/ angles! Do not want to cross area.