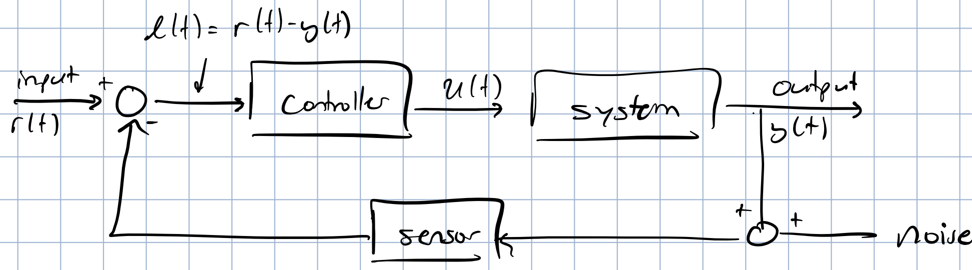
Mathematical Models of Systems

Feedback control model



Often modelled in form of diff. eq.

State space models

State vector x: minimum # of variables to define sys. (not params!)
↳ $|x|$ = order of diff. eq.

Using $x(t_0)$, $u(t_0)$ $\Rightarrow$ $x(t_1)$, $y(t_1)$ $(t_0 < t_1)$

Model:

$$\begin{cases} \dot{x} = \underbrace{\boxed{Ax + Bu}}_{} \; f(x,u) & \text{dynamics} \\ y = \underbrace{\boxed{Cx + Du}}_{} \; h(x,u) & \text{output} \end{cases}$$

$\Rightarrow$ We can find out $x(t_1)$, $y(t_1)$ by solving & updating Δt!

Restriction: if $\dot{x}$ happens to be non-linear (e^x, $\sin(x)$...), we can't write A or B.

Soln: linearize:

① Find state vector $\bar{x}$ & input $\bar{u}$ s.t. $f(\bar{x}, \bar{u}) = 0$ (equilibrium configs)
↳ no change

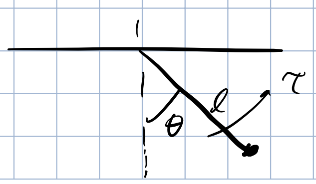
② $$\underbrace{A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}}}_{\text{Calculate}}, \quad \underbrace{C = \left.\frac{\partial h}{\partial x}\right|_{\bar{x},\bar{u}}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{\bar{x},\bar{u}}}_{\text{By inspection}}$$

③ State space model:

$$\begin{cases} \overbrace{\delta \dot{x}}^{\dot{\bar{x}} - f(\bar{x},\bar{u})} = A \overbrace{\delta x}^{x - \bar{x}} + B \overbrace{\delta u}^{u - \bar{u}} \\ \delta y = C \delta x + D \delta u \end{cases}$$

Note:
$\delta \dot{x} = (x - \bar{x})'$
$= \dot{x}$

Ex://



Q: find state space model & linearize around equilibrium configs.

(1) Eqn:

$$\tau = m\ell^2\ddot{\theta} + mg\ell\sin\theta$$

(2) State:

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \text{guess! No unique soln.}$$

(2) Define input & output

$$u(t) = \tau \quad , \quad y(t) = \theta = x_1$$

(3) Find dynamics eqn:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m\ell^2}u - \frac{g}{\ell}\sin(x_1) \end{bmatrix} \leftarrow f(x,u)$$

solve from eqn in term of (u, x)!

(4) Find output eqn:

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(5) Find equilibrium:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{1}{m\ell^2}u - \frac{g}{\ell}\sin(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \dot{x}_1 = \bar{x}_2 = \dot{\theta} = 0$$

$$\dot{x}_2 = \frac{1}{m\ell}\bar{u} = g\sin x_1 = 0$$

If $g\sin x_1 = 0 \rightarrow x_1 \in k\pi, \ k \in \mathbb{Z}$
& $\bar{u} = 0$

$$\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \quad \bar{u} = 0$$

(6) Linearize:

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\bar{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\right|_{\bar{x},\bar{u}} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

⑦ Linearized state space:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Laplace Transform

$$\left. u(t) \longrightarrow \boxed{\text{Sys}} \longrightarrow y(t) \right\} \quad \text{Obj: } y(t) = f(u(t))$$

Note: doesn't replace state space model

General procedure:

① $u(t) \xrightarrow{\mathcal{L}} U(s)$

② Dynamics Laplace

③ Acquire transfer func: $T(s) = \dfrac{Y(s)}{U(s)}$

④ If $u(t)$ known $\longrightarrow y(t) = \mathcal{L}^{-1}\{T(s) \cdot U(s)\}$

Properties:

① Linearity: $f_1 + f_2 = f_3 \longrightarrow \mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_3\}$

② Translation in t: $\mathcal{L}\{f(t-\tau)\} = e^{-\tau s} \cdot \mathcal{L}\{f(t)\}$

③ Exponential scaling: $\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$

④ Convolution: $\mathcal{L}\{f_1 * f_2\} = F_1(s)\, F_2(s)$

⑤ Initial value: $f(0^+) = \lim_{s \to \infty} sF(s)$

⑥ Final value: $f(\infty) = \lim_{s \to 0} sF(s)$

} Poles of $F(s)$ must have neg. real comp.

Proper vs. strictly proper:

Proper: $\lim_{s \to \infty} G(s) \in \mathbb{C}$

Strictly proper: $\lim_{s \to \infty} G(s) = 0$

Poles: roots of denom $\left(\lim_{s \to p} |G(s)| = \infty \right)$ } $G(s)$ is transfer

Zeros: " numerator $\left(\lim_{s \to z} |G(s)| = 0 \right)$

Zero-pole-gain rep.

$$\underset{\text{transfer}}{G(s)} = \underset{\text{gain}}{k} \cdot \frac{(s - z_1) \cdots (s - z_n)}{(s - p_1) \cdots (s - p_n)} \begin{array}{l} \} \to \text{zeros} \\ \} \to \text{pole} \end{array}$$

Ex:// Multidimensional Laplace:

$$\begin{cases} \dot{x} = Ax + Bu & - A \\ y = Cx + Du & - B \end{cases} \quad \text{(assume zero init. cond)}$$

① Take Laplace of both functions

A: $sX(s) = AX(s) + BU(s)$

$$(sI - A)\,X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1} B\,U(s)$$

B: $Y(s) = CX(s) + DU(s)$

② Sub A → B:

$$Y(s) = C(sI - A)^{-1} B\,U(s) + D\,U(s)$$

$$= \underbrace{\left[C(sI - A)^{-1} B + D \right]}_{\text{Transfer fnc. matrix } G(s)} U(s)$$

③ Extract func:

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & & \vdots \\ & & G_{pm}(s) \end{bmatrix}$$

$G_{ij}(s)$ is transfer func b/w $U_j(s)$ & $Y_i(s)$

Q: Transfer func. for all input on 1 output?

$$Y_i(s) = \sum_{j=1}^{m} G_{ij}(s)\, U_j(s)$$

Zero-state & zero-input

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

① Zero-input : assume $u(t) = 0$

<u>1D</u>: $\dot{x} = ax$

Guess for best soln : $x = e^{a(t-t_0)} x_0$

Assume $x(0) = x_0 \Rightarrow t_0 = 0 \Rightarrow x(t) = e^{at} x_0$

Check :

$$\dot{x} = (e^{at} x_0)' = a \underbrace{e^{at} x_0}_{x(t)} \checkmark \quad \} \text{ Dynamics}$$

$$x(0) = a x_0 \checkmark \quad \} \text{ Initial cond.}$$

<u>Multidimensional</u>

From 1D $\rightarrow$ $x(t) = \underbrace{e^{At}}_{e^{\text{matrix}}\ ?!} \cdot x_0$

Defining e^{At} :

By Taylor.

1d: $e^{at} = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \dots$

multi-d: $e^{At} = 1 + At + \dots = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

↳ Use this if finite ($A^n = 0 \ \forall n > n_0$)

↳ Use this if more complicated A

<u>Theorem</u> : unique soln to

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

is $x = \underbrace{e^{At}}_{\text{state transition matrix}} \cdot x_0 \Rightarrow$ zero-input soln

② Zero-state soln :

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du, \end{cases} \quad x(t_0) = 0$$

Guess : $x(t) = \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau$?

<u>Derivation from 1D</u>:

$\dot{x} = ax + bu$

$\downarrow \mathcal{L}$

$X(s) = \frac{b}{s-a} U(s)$

$\downarrow \mathcal{L}^{-1}$

$x(t) = be^{at} * u(t)$

For any system:

$$x(t) = \underbrace{e^{A(t-t_0)} x_0}_{\text{zero input}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{zero state}} \quad \Big\} \begin{array}{l}\text{particular} \\ + \\ \text{homogeneous} \\ \text{sln.}\end{array}$$

Either use this formula or Laplace to calculate!

LINEAR SYSTEM THEORY

Stability

System

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

① Stability

$$\underbrace{\forall x(t_0) = x_0 \in \mathbb{R}^m}_{\text{for all init. cond.}} \rightarrow \underbrace{\text{zero input}}_{u(t)=0} \text{ response is bounded}$$

$$\therefore \lim_{t \to \infty} \underbrace{e^{A(t-t_0)} x_0}_{\text{zero input respon}} \leq |B|$$

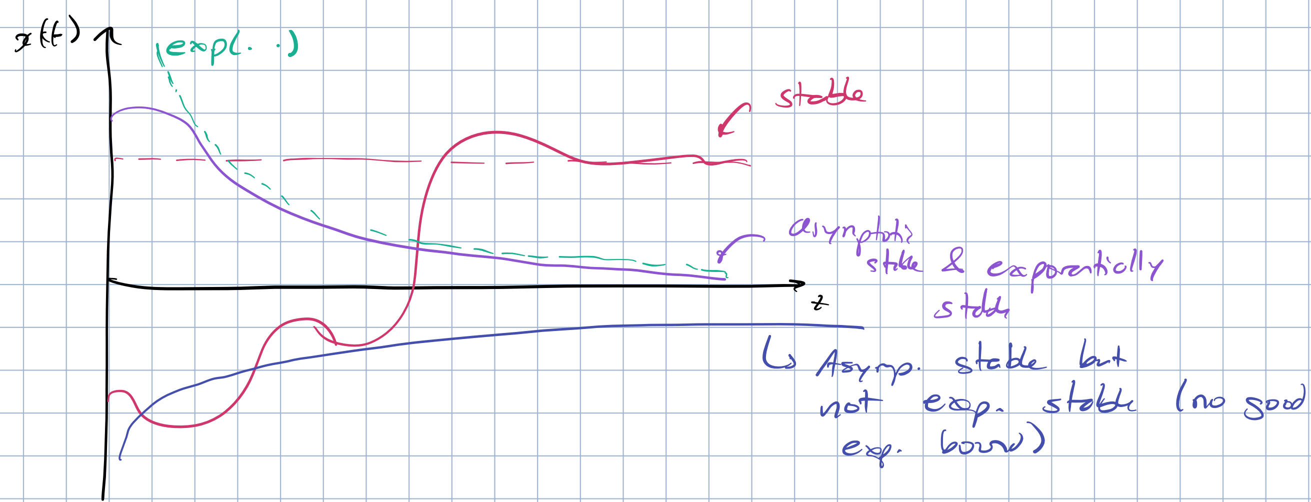
② Asymptotic stability:

Stable & $\lim_{t \to \infty} x(t) = 0 \quad \forall$ init. cond.

③ Exponential stability:

Asymptotically stable & $\exists c, \lambda > 0 \rightarrow \|x(t)\| \leq c e^{\lambda t} \|x_0\|, \forall t \geq 0$

↳ Rate of approach is like exponential



* Note: system $\underline{\text{linear}}$ & asym. stable $\rightarrow$ exponentially stable

$\hookrightarrow$ Why? Linear system $\Rightarrow e^{\cdots}$ $\leftarrow$ this is exponential

④ BIBO stable

$\exists B_1, B_2 \; \forall t. \; \underbrace{|u(t)| \le B_1}_{\text{Bounded input}} \;\&\; \underbrace{|y(t)| \le B_2}_{\text{Bounded output}} \Rightarrow$ BIBO stable

* Note: if asymp stable $\rightarrow$ BIBO stable (b/c eigenvalues of A are superset of G(s) poles)

Q: How to figure if system is stable?

① State space stability theorem.

Thm: $\dot{x} = Ax$ is asym. stable $\Leftrightarrow$ eigenvalues of A have neg. real part

Ex://

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\sigma}{m} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u$$

. Is system stable?

① Eigenvalues of A

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda + \frac{\sigma}{m} \end{bmatrix} \right|$$

$$= \lambda^2 + \frac{\sigma}{m}\lambda + \frac{k}{m}$$

Set to 0 & solve for λ:

$$\lambda^2 + \frac{\sigma}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = \underbrace{-\frac{\sigma}{2m}}_{\text{neg. real part}} \pm \sqrt{\underbrace{\frac{\sigma^2}{4m^2} - \frac{k}{m}}}$$

even if real, smaller than $\frac{\sigma}{2m}$

① Case analysis

If $m, k > 0$:

1) $\sigma > 0 \Rightarrow$ neg. real part $\Rightarrow$ stable

2) $\sigma < 0 \Rightarrow \neg$ 〃 $\Rightarrow$ not stable

② BIBO stable

$\underbrace{\text{Rational \& proper}}_{\text{If not, can't conclude}}$ $G(s)$ is BIBO stable $\Leftrightarrow$ all poles have neg. real part.

Representation of transfer functions

$$G(s) = \frac{k(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \Leftarrow \begin{array}{l} z_i : \text{real zero} \\ p_i : \text{real pole} \end{array}$$

$$= \frac{k\prod_i (s-z_i)\overbrace{\prod_i (s^2+2\xi\alpha_{n_i}s+\alpha_{n_i}^2)}^{\text{complex conjugate}}}{s^{\gamma}\prod_i (s-p_i)\prod_i (s^2+2\xi_i\omega_{n_i}s+\omega_{n_i}^2)} \Rightarrow$$

ξ, ξ: damping constant

$\alpha_{n_i}, \omega_{n_i}$: natural freq.

γ: # of singularities at origin

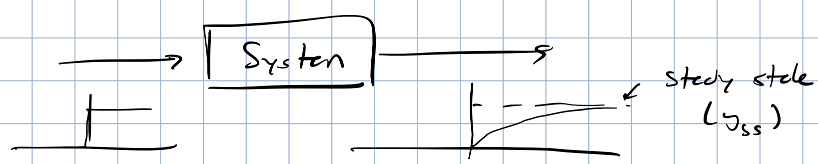
Useful $\Big\{$ $$= \frac{\mu\prod_i (1+T_i s)\prod_i \left(\frac{s^2}{\alpha_{n_i}^2}+\frac{2\xi_i}{\alpha_{n_i}}s+1\right)}{s^{\gamma}\prod_i (1+\tau_i s)\prod_i \left(\frac{s^2}{\omega_{n_i}^2}+\frac{2\xi_i}{\omega_{n_i}}s+1\right)} \Rightarrow$$

μ: steady state gain

T_i, τ_i: time constants

Performance Metrics

① Steady state gain



Mathematically: $y = \lim_{t\to\infty} y(t)$

If sys. is BIBO stable $\Rightarrow$ neg. real parts of denom $\Rightarrow$ FVT!

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\cdot \underbrace{U(s)}_{\text{unit step}} = G(0)$$

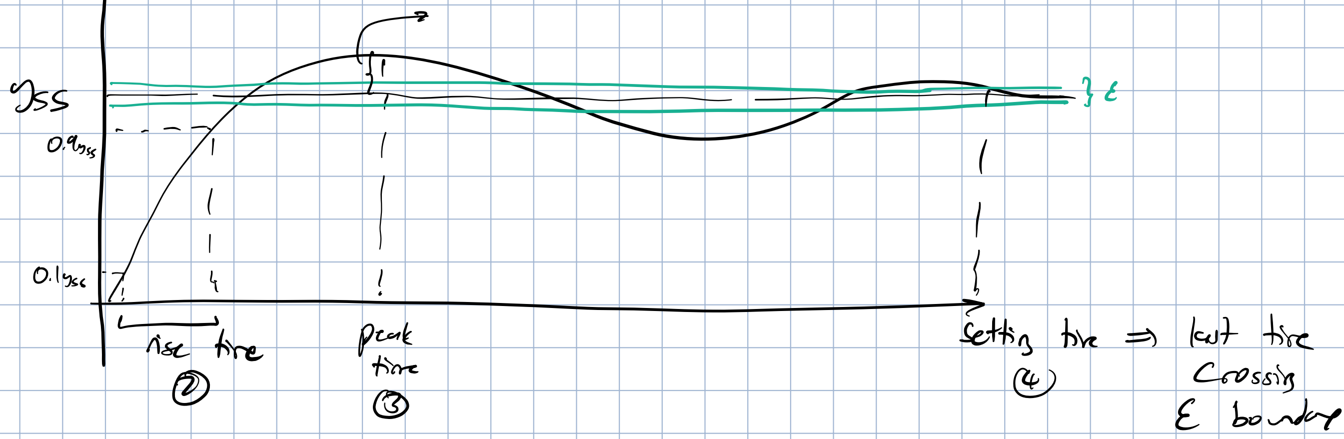
If:

$\gamma > 0 \Rightarrow$ not BIBO stable

$\gamma < 0 \Rightarrow G(0) = 0$

$\gamma = 0 \Rightarrow G(0) = \mu$

(5) Overshoot: $\frac{y_{max} - y_{ss}}{y_{ss}} \times 100$

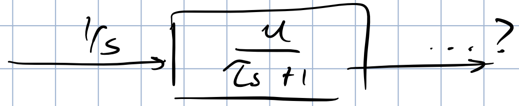


First Order System

Form: $\dot{y} + a_0 y = b_0 u$

$\mathcal{L}_s$ $\quad G(s) = \frac{b_0}{s + a_0} = \frac{\overbrace{b_0/a_0}^{\mu}}{\underbrace{\frac{1}{a_0}}_{\tau} s + 1}$

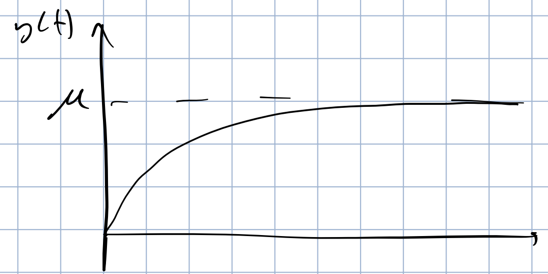
Step response:



$$Y(s) = \frac{1}{s} \cdot \frac{\mu}{1 + \tau s}$$

$$y(t) = \mu H(t) - \mu e^{-t/\tau}$$

Graph:



Performance metrics:

(1) Steady state: $G(0) = \mu$

(2) Rise time:

i) $y(t_2) = 0.9\mu \Rightarrow t_2 = -\tau \ln(0.1)$

ii) $y(t_1) = 0.1\mu \Rightarrow t_1 = -\tau \ln(0.9)$

$\Big\}$ Rise time $= t_2 - t_1$
$= -\tau(\ln(0.1) - \ln(0.9))$
$\approx -2\tau$

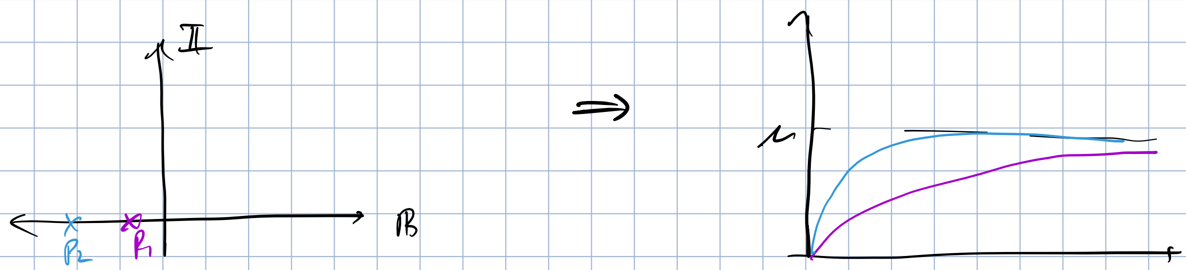
③ Overshoot: $0 \Rightarrow$ never exceed μ

④ Settling time:

$$y(t^{5\%}) = \mu(1 - e^{t^{5\%}/\tau}) = 0.95\mu$$
$$\vdots$$
$$\approx 3\tau$$

* $2\% \to 4\tau_1$ $1\% \to 5\tau$

Effect of changing poles:



Obs: as poles real neg $\to -\infty \Rightarrow$ approach step unit step

Second Order System

Case ①: Real & distinct poles

$$G(s) = \frac{\mu}{(1+\tau_1 s)(1+\tau_2 s)}$$

Step response:

$$y(t) = \mu\left(1 - \frac{\tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}\right)$$

If $\tau_1 >> \tau_2$:

$$y(t) = \mu\left(1 - \underbrace{\cancel{\frac{\tau_1}{\tau_1 - \tau_2}}}_{\tau_1}^{1} e^{-t/\tau_1} + \cancel{\frac{\tau_2}{\tau_2 - \tau_1}}^{0} e^{-t/\tau_2}\right)$$

↳ Becomes like a 1^{st} order system

Aside:

If zero $\Rightarrow$ this will not change steady state

Case ②: Real & coincident poles:

$$G(s) = \frac{\mu}{(1+\tau s)^2}$$

Step response:

$$y(t) = \mu\left(1 - e^{-t/\tau} - \frac{t}{\tau} e^{-t/\tau}\right)$$

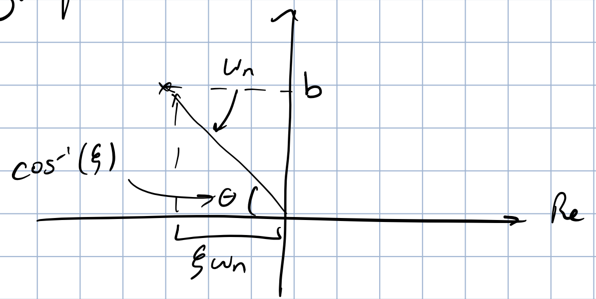
Case ③: Complex poles:

$$G(s) = \frac{k}{(s-s_1)(s-s_2)}$$

Assume:
$s_1 = a + jb$
$s_2 = a - jb$

$$= \frac{k}{s^2 - \underbrace{2\xi\omega_n}_{-a} s + \underbrace{\omega_n^2}_{a^2+b^2}}$$

Pole graph:



In another rep:

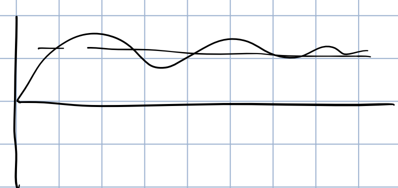
$$G(s) = \frac{\mu \cdot \omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2}$$

Step response:

$$y(t) = \mu\left(1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t + \cos^{-1}(\xi)\right)\right)$$

Analysis based off ξ:

1) $0 < \xi < 1$: system oscillates. Underdamped



← Asymp. stable

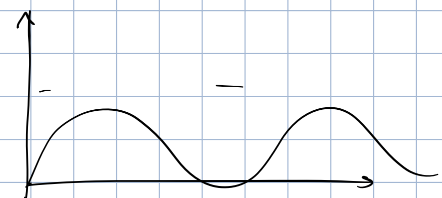
* If input at ω_n freq ⇒ resonance

2) $\xi = 1$: system does not oscillate. Critically damped

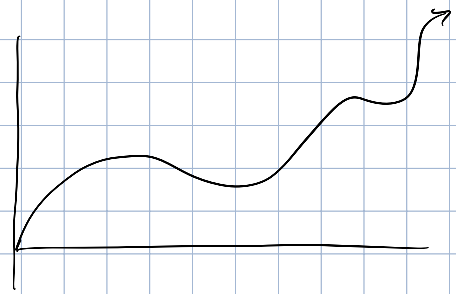
almost same

3) $\xi > 1$: overdamped

4) $\xi = 0$: undamped oscillations



5) $\xi < 0$: unstable



Aside: how to do analysis on test

① Derive pole formula

$$s = -\zeta w_n \pm w_n j \sqrt{1-\xi^2}$$

② Do case analysis on ξ

Performance metrics:

① Steady state gain : μ

② % overshoot = $100\, e^{\frac{-\zeta \pi}{\sqrt{1-\zeta^2}}}$

③ Settling time at ε % $= -\frac{1}{\zeta w_n} \ln(0.01\varepsilon)$

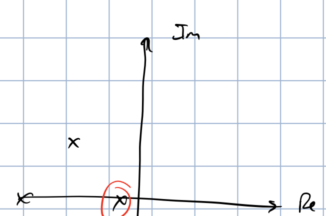
Higher order systems

All systems can be approx. by either 1^{st} order or 2^{nd} order system.

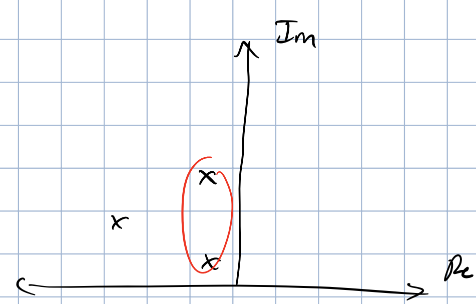
How to do approx?

1. Transfer function
2. Pole graph
3. Dominant pole analysis
   i) Identify which pole/s are near Im axis
   ii) Dominant poles

Ex://



4^{th} order sys $\longrightarrow$ 1^{st} order sys.



3^{rd} order sys $\rightarrow$ 2^{nd} order sys.

System Identification

Q:

$$u(t) \longrightarrow \boxed{\;?\;} \longrightarrow y(t)$$

what is system?

Methods:

① E2E method

Sensor inputs $\longrightarrow$ control inputs (hacky method)

② ML

Determine system by ML

③ Diff. eq method

Assume system:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = b_m u^{(m)} + \dots + b_0 \quad \Rightarrow \text{ } n \text{ \& } m \text{ are params.}$$

Goal: if we know output & input, $\Rightarrow$ find $a_0 \to a_n$, $b_0 \longrightarrow b_m$

i) Rearrange system

$$y^{(n)} = b_0 u + \dots + b_m u^{(m)} - a_0 y - \dots - a_{n-1} y^{(n-1)}$$

$$= \underbrace{\begin{bmatrix} u & \dot{u} & \cdots & y & \cdots & y^{(n-1)} \end{bmatrix}}_{d} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \\ a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\theta}$$

$$\therefore \; \underset{\text{known}}{Y} = \underset{\text{known}}{d^T} \, \underset{\text{solve}}{\theta}$$

ii) Collect data over multiple times

$$\begin{bmatrix} y^{(n)}(t_0) \\ \vdots \\ y^{(n)}(t_n) \end{bmatrix} = \begin{bmatrix} \longleftarrow & d^T(t_0) & \longrightarrow \\ & \vdots & \\ \longleftarrow & d^T(t_n) & \longrightarrow \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix}$$

iii) Linear regression

$$\text{Obj: } \hat{\theta} = \underset{\theta}{\operatorname{argmin}} \; \| Y - D^T \theta \|_2$$

$$\vdots$$

$$\hat{\theta} = (D^T D)^{-1} D^T Y$$

Analysis of Feedback Control Systems

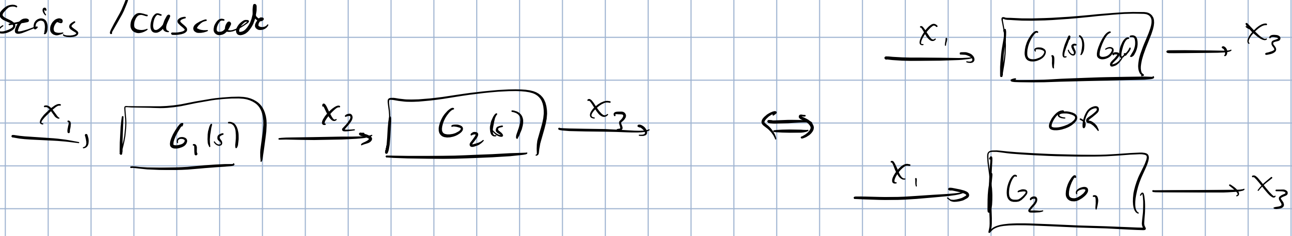
Block Diagrams

Components:

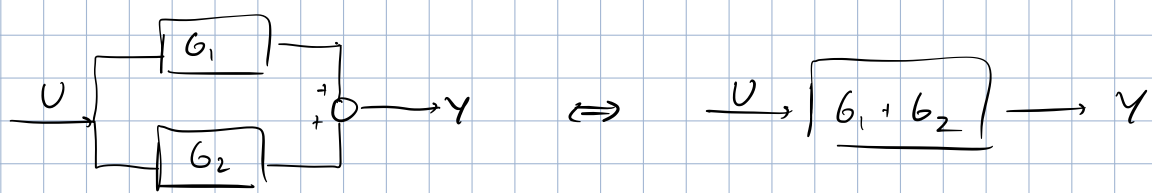
1. Block: transfer function of particular signal
2. Adder/subtractors: adds/subtracts freq. resp.

Block diagram transformations:

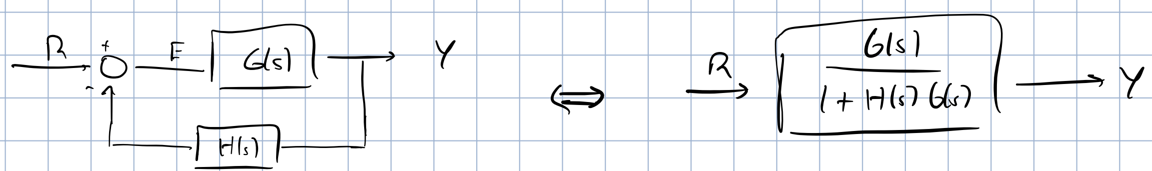
1. Series / cascade



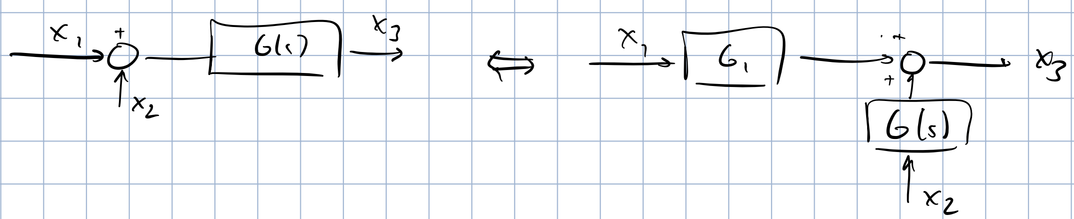
2. Parallel



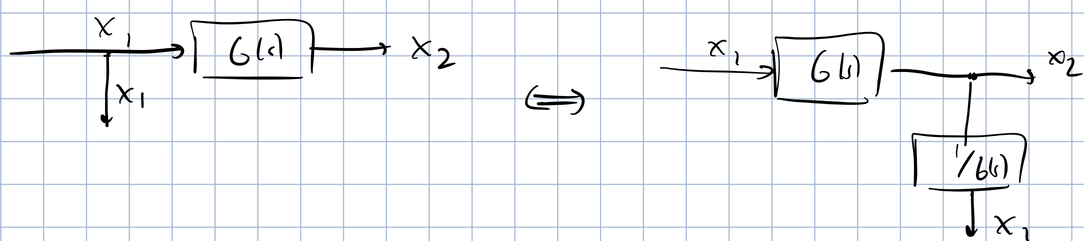
3. Feedback



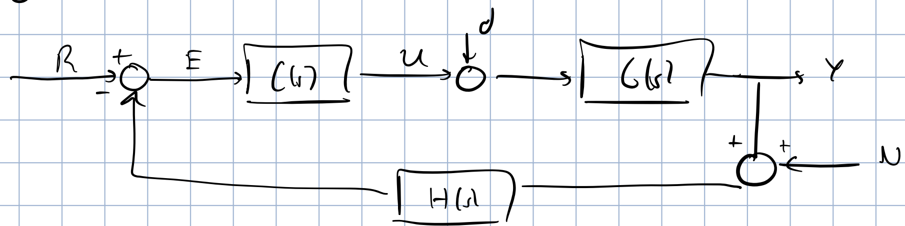
4. Moving summing point



5. Moving seperation point



Ex: Negative feedback control system



Goal:

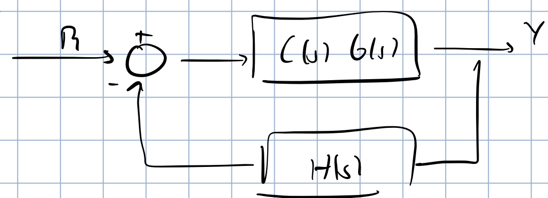
$$\begin{bmatrix} Y \\ U \\ E \end{bmatrix} = \begin{bmatrix} & 3\times 3 & \end{bmatrix} \begin{bmatrix} R \\ D \\ N \end{bmatrix}$$

Steps:

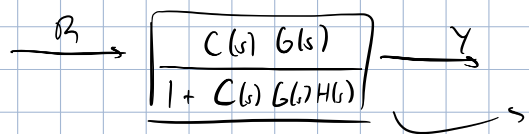
Consider each pair b/w input & output. Disregard other inputs

$R \to Y$

1.



2.



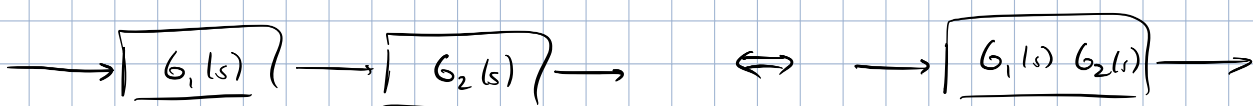
Poles closer to Im axis

Note: if poles of system are faster than others, can approximate by steady state gain!

Stability of Interconnected Systems

Q: if individual sys. are stable, will combo be stable?

1. Series interconnections.

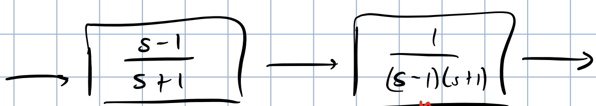


If G_1 & G_2 are indiv. stable & no zero-pole cancel $\Rightarrow$ $G_1 G_2$ is stable

↳ Reason: poles of $G_1 G_2$ are union of G_1 & G_2

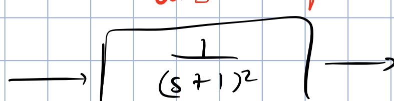
If G_1 or G_2 are not stable $\Rightarrow$ $G_1 G_2$ is not stable

Ex:



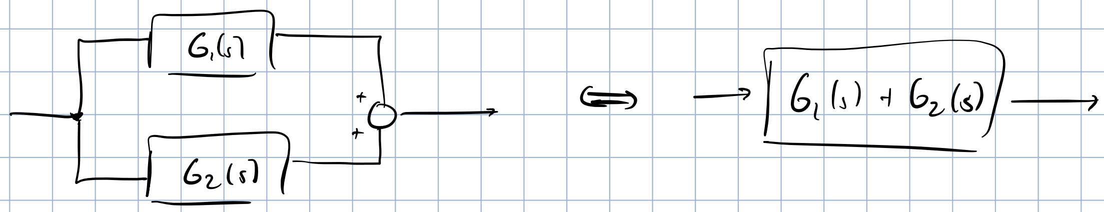
unstable pole

Final system:



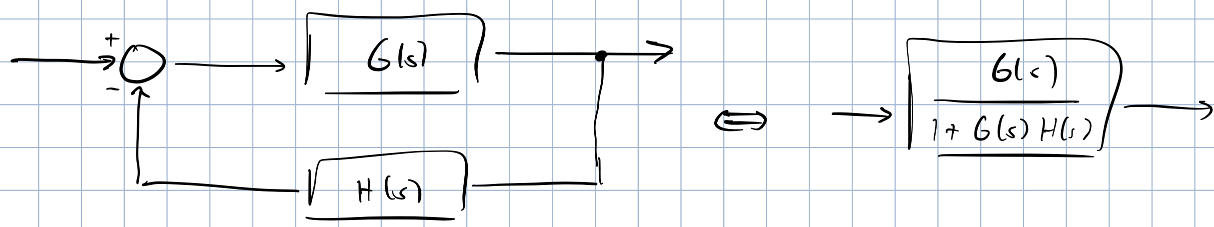
∴ Even though final TF is stable, the sys. is actually unstable.

② Parallel Interconnections



Same result as series interconnections

③ Feedback



Poles of sys. : $1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = 0$

Stability depends on $G(s)$ & $H(s)$

Closed-loop stability & Routh-Hurwitz Criterion

Hurwitz polynomial: $\pi(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$
s.t. all roots have −ve real parts

If charac. eqn. of T.F is Hurwitz $\longleftrightarrow$ stable

Q: Can we determine if smth. is Hurwitz w/out finding roots?

A: Yup! Routh-Hurwitz criterion

Routh-Hurwitz criterion:

① Build table

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}
s^{n-2}				
⋮				
⋮				
s^1				
s^0				

} Filled from poly.
If unequal $\Rightarrow$ put 0s at end.

② Calculate rows s^{n-2} onwards.

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}
s^{n-2}	①	②		
⋮				
s^1				
s^0				

① : $-\dfrac{1}{\text{val on top}} \left|\text{matrix above}\right| = -\dfrac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3}\end{vmatrix}$

$$= \frac{-(a_n a_{n-3} - a_{n-1}a_{n-2})}{a_{n-1}}$$

② : $-\dfrac{1}{\text{val on top}} \left|\text{matrix above}\right| = \dfrac{-1}{a_{n-3}}\begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-3} & a_{n-5}\end{vmatrix}$

③ Analyze Routh array using Routh-Hurwitz theorem.

Routh array: 1st column of values in table

RH theorem:

i) $\pi(s)$ is Hurwitz $\Leftrightarrow$ all Routh array elem. have same sign & non-zero

ii) Routh array has no zeros:

a) # of sign changes & changes to 0 = # of unstable roots

b) No roots on imag. axis.

Necessitate that if polynomial doesn't have consec. power → unstable.

↳ Ex:// $G(s) = \dfrac{1}{s^4+1}$ → unstable b/c no s^3, s^2, s

Ex:// $\pi(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$

① Build table

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s	45/7	0	0
s^0	10	0	0

$s_{2,1} = -\dfrac{1}{1}\begin{vmatrix} 2 & 3 \\ 1 & 5\end{vmatrix} = -(10-3) = -7$

$s_{2,2} = -\dfrac{1}{5}\begin{vmatrix} 3 & 10 \\ 5 & 0\end{vmatrix} = 10$

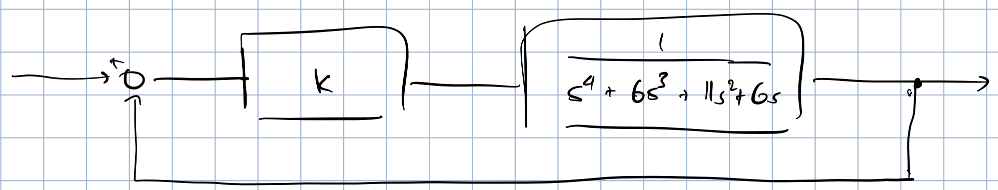
$s_{3,1} = -\dfrac{1}{-7}\begin{vmatrix} 1 & 5 \\ -7 & 10\end{vmatrix} = \dfrac{1}{7}\left|10+35\right| = 45/7$

$s_{4,1} = \dfrac{-7}{45}\begin{vmatrix} -7 & 10 \\ 45/7 & 0\end{vmatrix} = 10$

② RH theorem:

Sign change ⇒ unstable polynomial w/ 2 unstable roots

Ex://



Choose $K>0$ s.t. system is stable

① Find TF

$$\frac{KG(s)}{1+KG(s)}$$

② Find characteristic polynomial:

$$1+KG(s)=\frac{s^4+6s^3+11s^2+6s+K}{s^4+6s^3+11s^2+6s}$$ ← actual denom. of TF

$$\therefore \pi(s) = s^4+6s^3+11s^2+6s+K$$

③ RH table

s^4	1	11	K
s^3	6	6	0
s^2	10	K	0
s	$6-\frac{3}{5}k$	0	0
s^0	K		

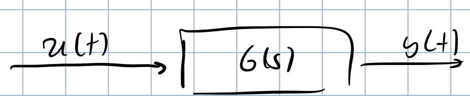
④ RH criteria:

$K>0 \Rightarrow$ no sign change

$6-\frac{3}{5}k>0 \Rightarrow K<10 \Rightarrow$ no sign change

$$\therefore \boxed{0<k<10}$$

Frequency response



Q: What happens in $u(t)=a\sin(\omega t)$?

A: Fundamental theory of freq. respon.

Fundamental theory of freq. response

If $u(t) = a\sin(\omega t)$, then $y_{ss}(t) = a\underbrace{|G(j\omega)|}_{\text{amplification}}\sin(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase shift}})$

Ex://

$$\begin{cases} \dot{x} = -10x + u \\ y = x \end{cases}, \quad u(t) = 2\cos(3t + \pi/2)$$

Find $y_{ss}(t)$:

① TF

$$G(s) = C(sI - A)^{-1}B + D$$
$$= \frac{1}{s+10}$$

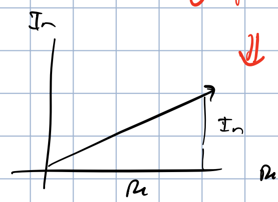
② Evaluate $G(j\omega)$

$$G(j\omega) = \frac{1}{3j + 10} = \frac{10 - 3j}{109}$$

$$|G(j\omega)| = \frac{1}{109}|10 - 3j|$$
$$= \frac{1}{109}\sqrt{10^2 + 9}$$
$$= \frac{\sqrt{109}}{109} \approx 0.1$$

Use graph of complex etc

⇓



$$\angle G(j\omega) = \tan^{-1}\left(\frac{-3}{10}\right) = -0.29 \text{ rad}$$

③ Fundamental theorem

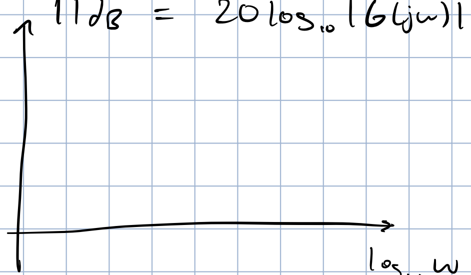
$$y_{ss}(t) = 2 \cdot 0.1\cos(3t + \pi/2 - 0.29)$$

← Initial type of sinusoid & phase shift don't matter

If $u(t) = \sum a_i\sin(\omega_i t) \Rightarrow y_{ss}(t) = \underbrace{\sum a_i|G(j\omega_i)|\sin(\omega_i t + \angle G(j\omega_i))}_{\text{superposition}}$

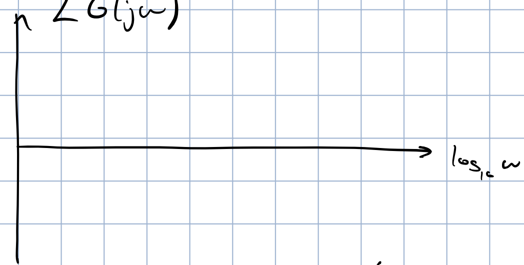
Bode Plots

$|\,|_{dB} = 20\log_{10}|G(j\omega)|$



Magnitude

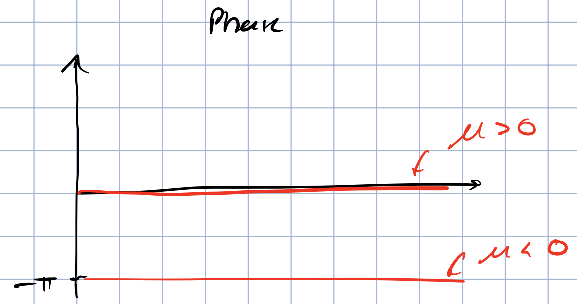
$\angle G(j\omega)$



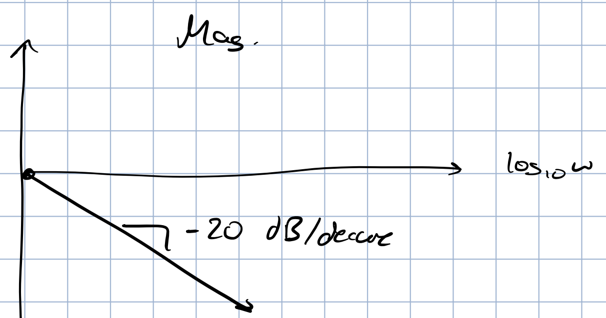
Phase plot (TIP: Re-Im plot)

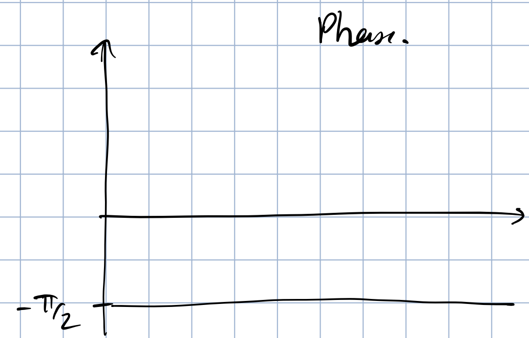
Inclusion of logs $\Rightarrow$ we can plot complex transfer func. as <u>superposition</u> of multiple simpler plots

① $G(s) = \mu$



② $G(s) = 1/s$

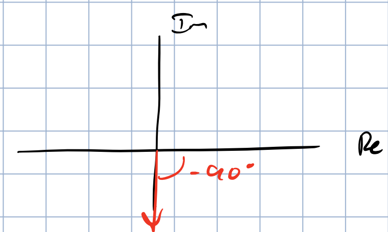




Derivation:

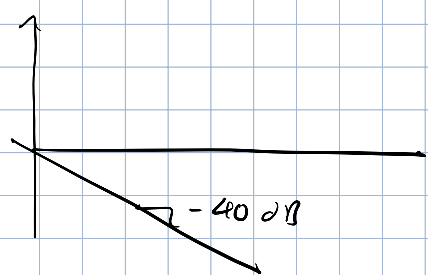
$$|G(j\omega)|_{dB} = 20\log_{10}\left|\frac{1}{j\omega}\right|$$
$$= -20\log_{10}|j\omega|$$
$$= -20\log_{10}\omega$$

Mag $\rightarrow$ drop;



$$\frac{1}{j\omega} = -\frac{j}{\omega} = -\frac{1}{\omega}j$$

③ $G(s) = 1/s^2$



Derivation:

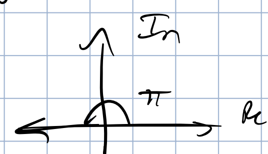
$$|G(j\omega)|_{dB} = 20\log_{10}\left|\left(\frac{1}{j\omega}\right)^2\right|$$
$$= -40\log_{10}|j\omega|$$
$$= -40\log_{10}\omega$$

Derivation:

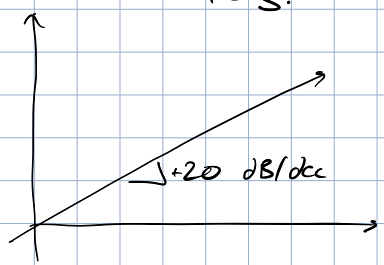
① Simplify $G(j\omega)$

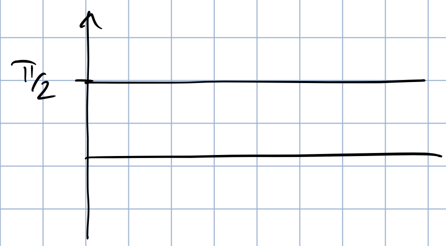
$$\left(\frac{1}{j\omega}\right)^2 = \frac{1}{-\omega}$$

② Re - Im



④ $G(s) = s$

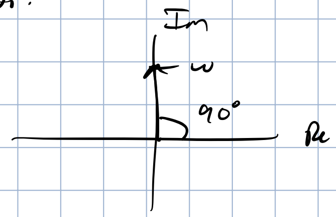




Derivation:

$$|G(j\omega)|_{dB} = 20\log_{10}|j\omega|$$
$$= 20\log_{10}\omega$$

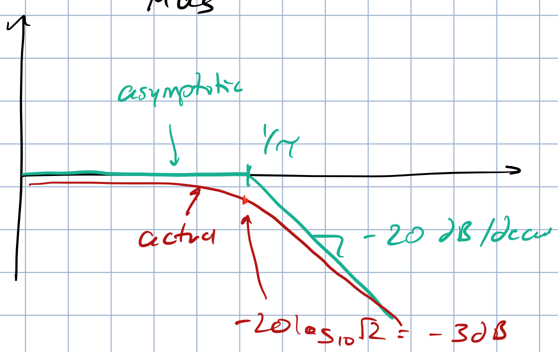
Derivation:

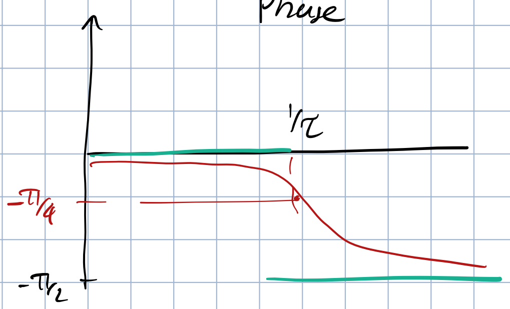


Note: zero increase rise times for transient & high amp. at high freq.

integrators ($1/s$): good for steady state & tracking.

⑤ $G(s) = \dfrac{1}{1+\tau s}$





Derivation:

$$|G(j\omega)|_{dB} = 20\log_{10}\left|\frac{1}{1+\tau s}\right|$$
$$= -20\log_{10}|1+\tau s|$$
$$= \begin{cases} -20\log 1 = 0 & \text{if } \omega \ll |1/\tau| \rightarrow 1 \text{ dominates} \\ -20\log_{10}|\omega\tau| & \text{if } \omega \gg |1/\tau| \rightarrow \tau s \text{ dominates.} \end{cases}$$

$$\angle G(j\omega) = \angle \frac{1}{1+\tau j\omega}$$
$$= \begin{cases} 0° & \omega \ll 1/\tau \rightarrow 1 \text{ dominates} \\ -90° & \omega \gg 1/\tau \rightarrow 1/\tau j\omega \text{ dominates} \end{cases}$$

Meaning:

① At low freq. $|G(j\omega)|_{dB} = 1 \rightarrow |G(j\omega)| = 1 \rightarrow$ tracks input

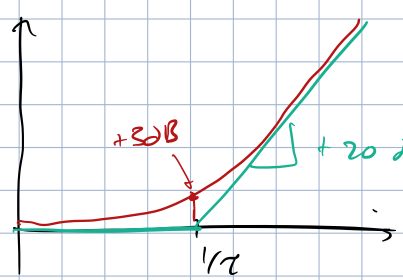
② At high freq. $|G(j\omega)|_{dB} \rightarrow -\infty \rightarrow |G(j\omega)| = 0 \rightarrow$ filter out high frequencies

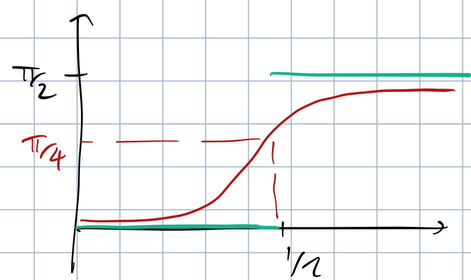
$\therefore$ $\boxed{\text{Low pass filter}}$

⑥ $G(s) = 1 + \tau s$

Flip all plots:

Phase

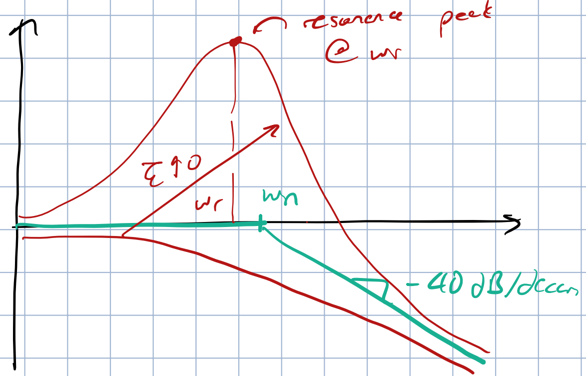




⑦ $G(s) = \dfrac{1}{1 + \dfrac{2\zeta}{\omega_n} s + \dfrac{s^2}{\omega_n}}$

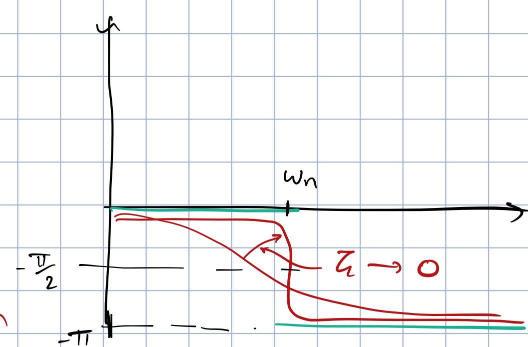
Mag. plot

Phase plot



$\zeta < 1/\sqrt{2}$
$\downarrow$
multiple osc. @ ss.

$\zeta > 1/\sqrt{2}$
$\downarrow$
One oscillation around ss.



Peaks approach ω_n as $\zeta \rightarrow 0$

$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$

$G(j\omega_r) = \dfrac{1}{2|\zeta|\sqrt{1-\zeta^2}}$

If $\zeta \rightarrow 0$, mag. $\uparrow\uparrow$

Derivation:

$$|G(j\omega)|_{dB} = -20 \log \left| 1 - \frac{\omega^2}{\omega_n^2} + j \frac{2\zeta}{\omega_n} \omega \right|$$

$$= \begin{cases} 0, & \omega \ll \omega_n \rightarrow 1 \text{ dominates} \\ -40 \log \left| \frac{\omega}{\omega_n} \right|, & \omega \gg \omega_n \rightarrow \frac{\omega^2}{\omega_n^2} \text{ dominates} \end{cases}$$

$$\angle G(j\omega) = \begin{cases} 0^\circ, & \omega \ll \omega_n \\ -180^\circ, & \omega \gg \omega_n \end{cases}$$

⑧ $G(s) = 1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n}$

Flip all plots of ⑦

If complex T.F.:

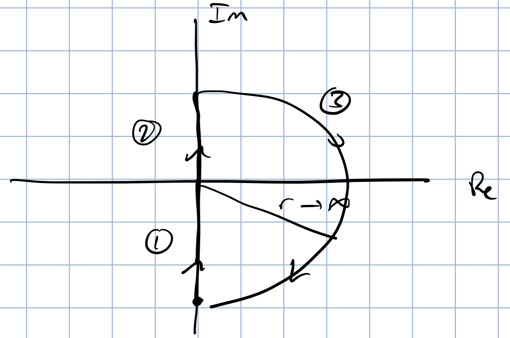
① Break into any of 8 components

② Draw ①

③ Superposition all plots

Nyquist Plots

Nyquist contour

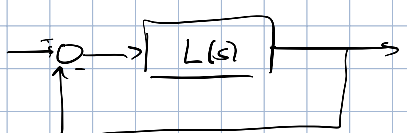


Nyquist plot: image w/ contour → T.F.

Nyquist criterion:

Let $L(s)$ be T.F. p = # of poles of $L(s)$ w real +ve roots. N = # of loops Nyquist plot of $L(s)$ has around ± 1

Then:



is asymp. stable $\iff$ $N = P$

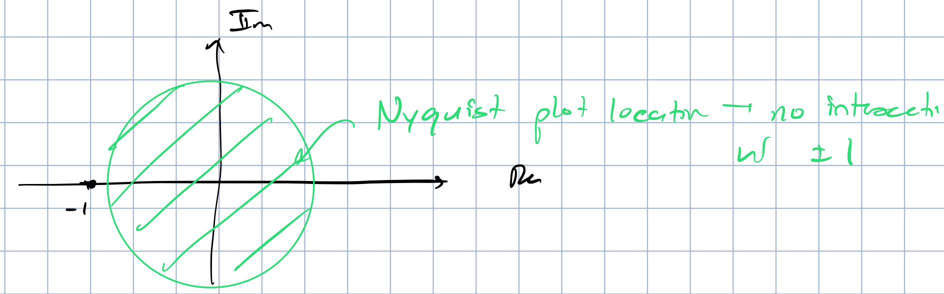
$\therefore N - P$ = # of unstable poles of $L(s)$

If plot goes through ± 1 → system is not A.S., could be stable

Corollaries: If $G(s)$ is A.S.

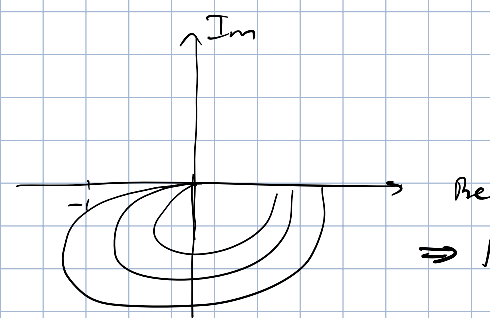
1) $\forall \omega \,.\; |G(j\omega)| < 1 \Rightarrow G(s)$ is A.S.

↳ Reasoning:



2) $\forall \omega \,.\; \angle G(j\omega) \in [0, \pi) \Rightarrow G(s)$ is A.S.

↳ Reasoning:



⇒ No intersections w/ the real axis.

Margins: indicator of how much slack we have until unstable.

Gain margin:

$\omega_\pi : \angle G(j\omega_\pi) = -180°$

margin: $K_m = \dfrac{1}{|G(j\omega_\pi)|}$

Phase margin:

$\omega_c : |G(j\omega_c)| = 1$

margin $= \varphi_m = -180° - \angle G(j\omega_c)$

Bode stability criterion: If $G(s)$ has no unstable poles & $|G(j\omega)|_{dB}$ crosses 0_{dB} once

$\therefore$ A.S. $\Leftrightarrow K_m > 1$ & $\varphi_m > 0°$

Reading margins on Bode plot:

Gain margin:

① Find ω_π: cross $-180°$

② On mag. plot: take diff. b/w 0 & $|G(j\omega_\pi)|_{dB}$

$0 - |G(j\omega_\pi)|_{dB}$

Phase margin:

① Find ω_c: freq. at which mag. crosses x axis

② Go to phase plot & find diff. b/w $-\pi$ & $\angle G(j\omega_c)$

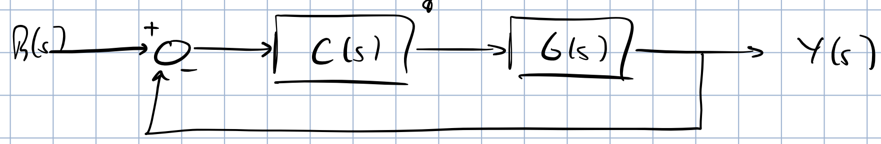
$\angle G(j\omega_c) - (-\pi)$

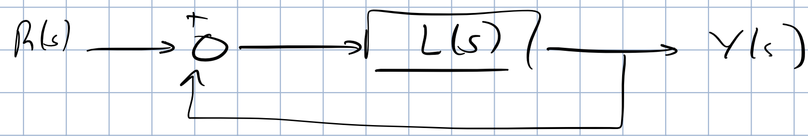
Special note: $\zeta \approx \dfrac{\varphi_m}{100}$

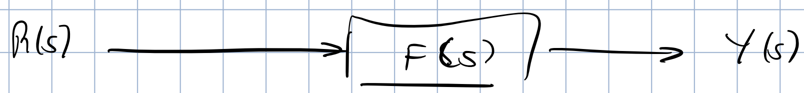
CONTROLLER SYNTHESIS

Obj.:

set this for certain output



≡ 

≡ 

System metrics

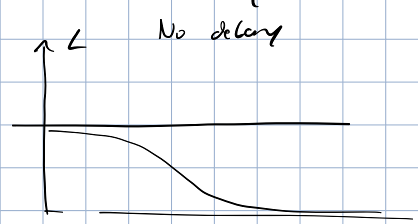
① Stability

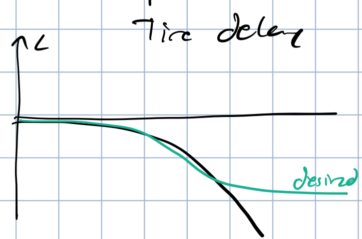
Result: $K_{m_{dB}} > 0 \wedge \varphi_m > 0$ of $L(s)$ assuming no cancellations

② Robust stability

Result: high K_m & φ_m → further from instability

* Note: Time delay of τ creates $-\omega_c\tau$ phase delay





③ Static performance

Result: high μ or $p > 0$ ⟹ low steady state error

Reason:

$$L(s) = C(s)\,G(s) = \frac{\mu}{s^p} \cdot \underbrace{\frac{\Pi_i(\ldots)\,\Pi_i(\ldots)}{\Pi_i(\ldots)\,\Pi_i(\ldots)}}_{\text{Poles \& zeros}}$$

At steady state:

$$L(s) \approx \frac{\mu}{s^p}$$

$\therefore \quad F(s) = \dfrac{\mu}{s^{\rho} + \mu} \xrightarrow{\text{at steady state}} \begin{cases} \dfrac{\mu}{1+\mu}, & \rho = 0 \\ 1, & \rho > 0 \\ 0, & \rho < 0 \end{cases}$

If input has unit step $\left(\frac{r_0}{s}\right)$:

a) $\rho = 0$

$$e_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{\mu}{1+\mu} = \frac{\mu}{1+\mu}$$

high μ, $e_\infty \to 0$

b) $\rho > 0$

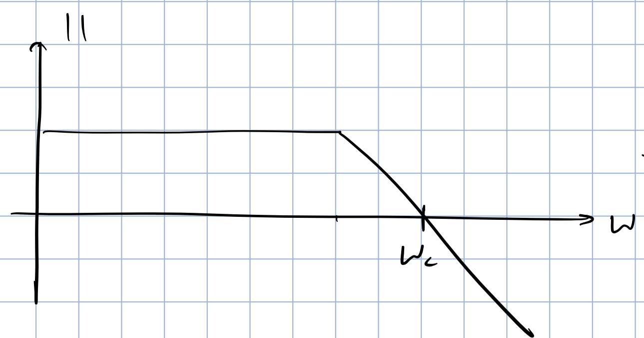
$$e_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} F(s) = r_0$$

+ve ρ leads to constant error.

④ Dynamic performance

Result: High w_c for $F(s)$ → can track high-freq. input signals

Reason:



⇒ If w_c is higher, $||_{dB} \to 0$ later

If $||_{dB} \to 0$, can no longer track

Increasing w_c → keep track of reference for longer.

⑤ Disturbance rejection

Result: set minimum w_c to ensure disturbances rejected

⑥ Noise attenuation

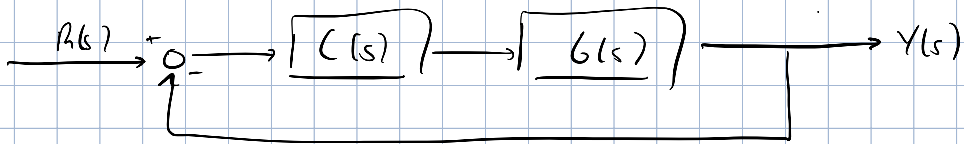
Result: set max w_c to stop high frequencies

⑦ Realized infinity of $C(s)$

Result: slope of $|L(j\omega)| \leq |G(j\omega)|$ ($C(s)$ should be proper)

Controller design

Ex://



$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}$$

Requirements:

a) $|e_\infty| \leq 0.1$

b) $\omega_c \geq 0.2$

c) $\varphi_m \geq 60°$

} Design controller to satisfy these if $r(t) = H(t)$

① Tackle $|e_\infty|$

Put initial guess: $C(s) = \frac{\mu}{s^\nu}$ (no other poles/zeros yet)

If $\nu = 0$:

$$|e_\infty| = \lim_{t\to\infty}(y(t) - r(t)) = \frac{\mu}{1+\mu} - 1 \leq 0.1$$

If $\nu = 1$:

$$|e_\infty| = \lim_{t\to\infty} y(t) - 1 = 1 - 1 = 0$$

Both work. We'll go w/ $\nu = 0$ for now.

Quantify error:

$$|e_\infty| = \lim_{t\to\infty} |y(t) - r(t)|$$

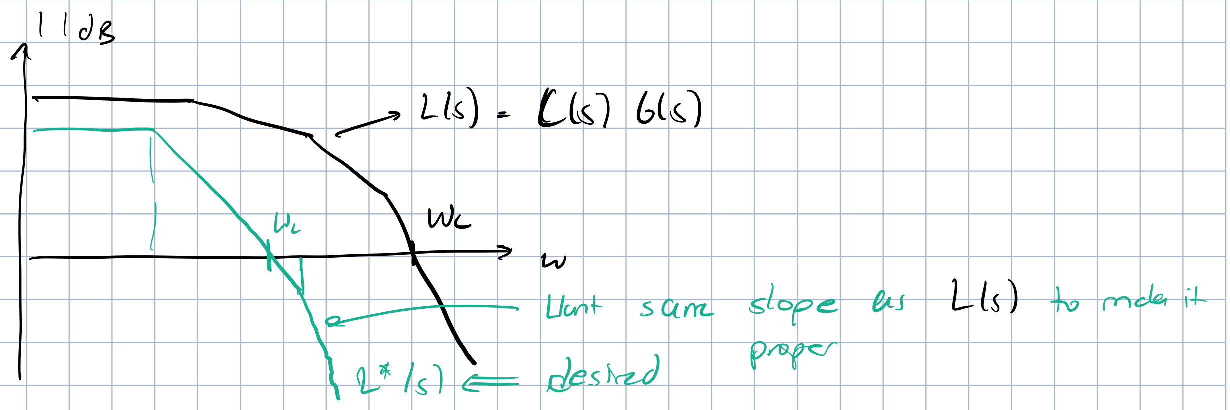
$$= \lim_{s\to 0} s\, C(s)\, G(s)\, \cancel{R(s)} - 1$$

$$= \frac{10\mu}{1+10\mu} - 1 \leq 0.1$$

$$\frac{10\mu}{1+10\mu} \geq 0.9$$

$$\boxed{\mu \geq 0.9}$$

② Bode plot : create desired Bode plot



We want ω_c of $L^*(s)$ to be ≥ 0.2.

① Set ω_c of $L^*(s) = 0.3$

② Make pole to straddle ω_c of $0.3 \rightarrow$ 1 pole at 0.03, another at 3

③ Make 2^{nd} pole squeezed to get -60 dB slope @ end

$$\therefore \quad L^*(s) = \frac{10\mu}{\left(1+\frac{s}{0.03}\right)\left(1+\frac{s}{3}\right)^2}$$

④ Solve for controller w/ desired $L^*(s)$

$$L^*(s) = C(s)\,G(s)$$

$$C(s) = \frac{L^*(s)}{G(s)} = \frac{\cancel{10\mu}\ 1}{\left(1+\frac{s}{0.03}\right)\left(1+\frac{s}{3}\right)^2} \times \frac{(1+10s)(1+5s)(1+s)}{\cancel{10}}$$

$$= \frac{(1+10s)(1+5s)(1+s)}{\left(1+\frac{s}{0.03}\right)\left(1+\frac{s}{3}\right)^2}$$

This also solves φ_m constraint

Q: Why not $\frac{\mu}{s}$?

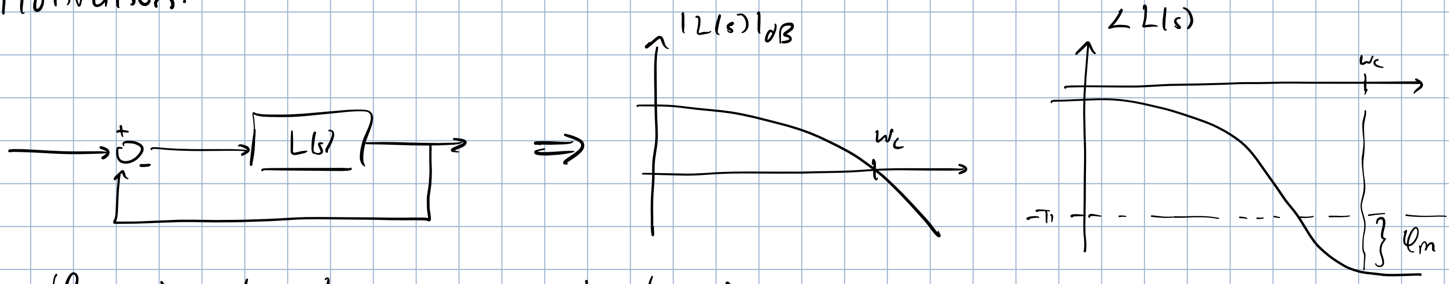
Integrators tradeoff:

Pros: $|e_\infty| = 0$

Cons: slow response (cancel slow poles), margins can be neg. $\rightarrow$ unstable sometimes

Lead compensator

Motivation:

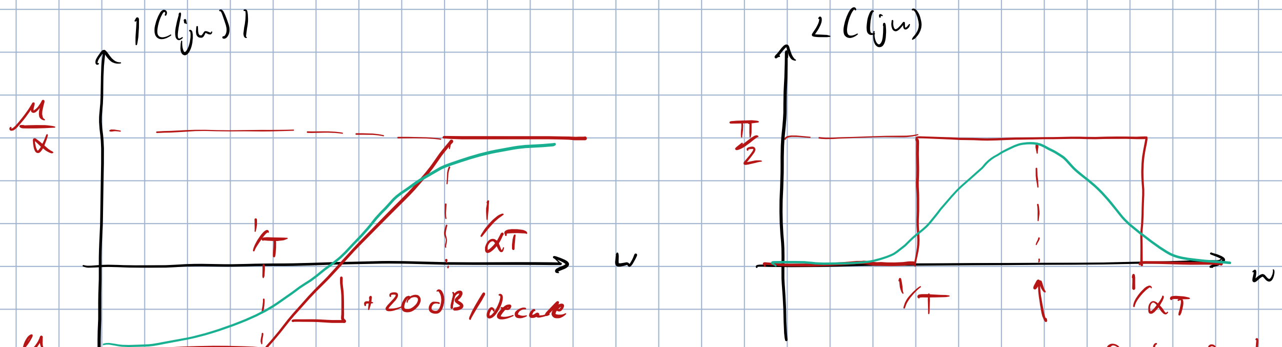


φ_m is too low, we want to increase it

Introduce following controller:

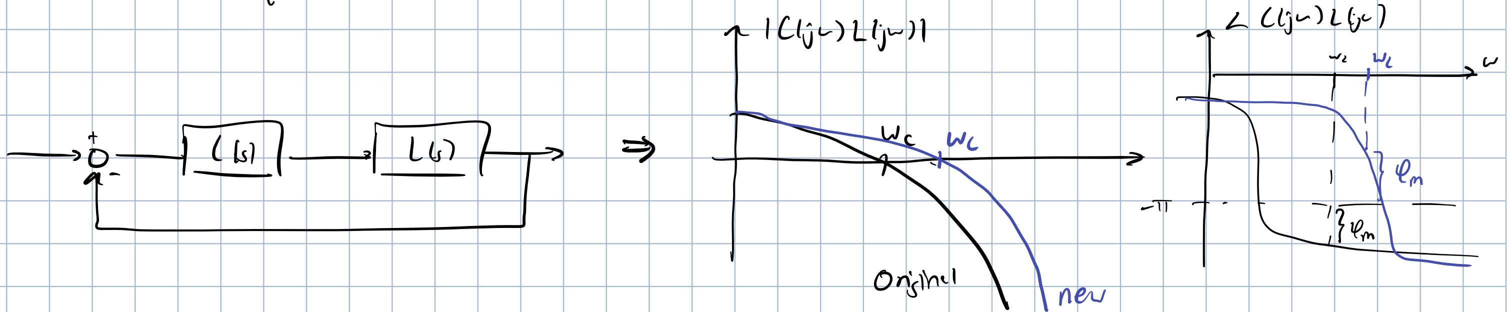
$$C(s) = \mu \frac{1 + Ts}{1 + \alpha T s} \quad , \quad T > 0, \ \mu > 0, \ \underline{\underline{0 < \alpha < 1}}$$

Bode plot of $C(s)$



Geom mean of $1/T$ & $\frac{1}{\alpha T}$ = $\frac{1}{T\sqrt{\alpha}}$

To increase phase of $L(s)$, set $1/T$ & $\frac{1}{\alpha T}$ to straddle w_c



Con: Bode plot changes & $w_c \uparrow$ $\rightarrow$ system might respond to higher freq. more

Best practices:

① Keep $1/\alpha T$ & $\frac{1}{T}$ far away to get good phase shift

↳ Recommendation: $\alpha \approx 0.1 \Rightarrow$ phase plot increases by $\sim 55°$

Farther poles $\longrightarrow$ $C(s)$ real Bode $\longrightarrow$ $C(s)$ asymptotic Bode

Note: $\alpha \uparrow \rightarrow \omega_c \uparrow$ (faster response, but more noise)

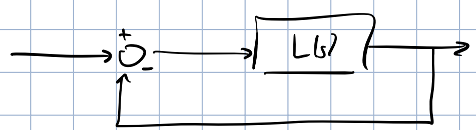
② $\frac{\mu}{\alpha}$ is gain @ high freq. $\rightarrow$ increase ω_c

Q: What if $\alpha \rightarrow 0$

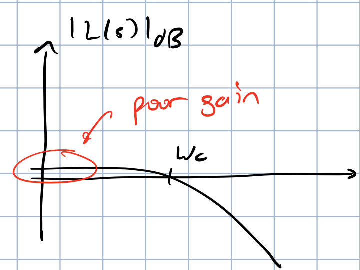
$\therefore$ $C(s) = \underbrace{\mu}_{\text{proportional}} + \underbrace{\mu T s}_{\text{derivative}}$ } PD controller. Improper $\rightarrow$ not realizable

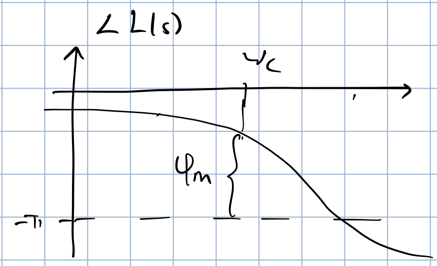
Lag compensator

Motivation:



$\Rightarrow$



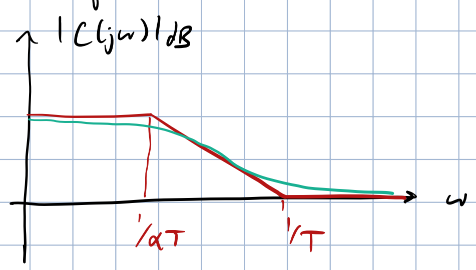


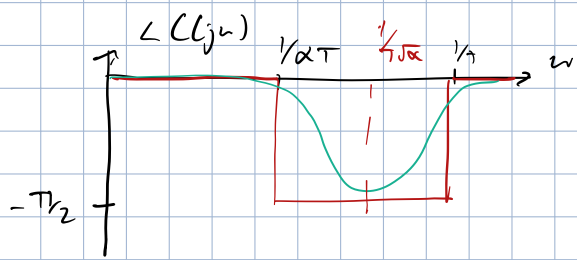
Goal: improve static performance (steady state vel. should be high for err $\downarrow$)

Introduce following controller:

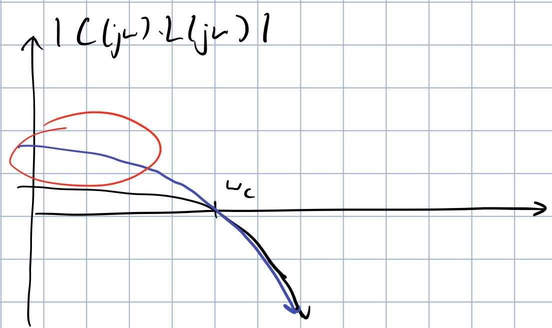
$$C(s) = \mu \frac{1 + Ts}{1 + \alpha T s}, \quad T > 0, \mu > 0, \underline{\underline{\alpha > 1}}$$

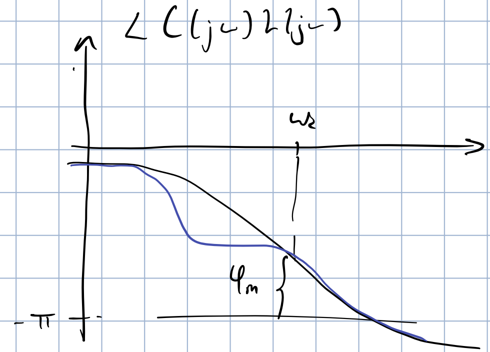
Bode plot





System:





Con: if you put phase lag close to $\omega_c \longrightarrow$ could have $\varphi_m < 0$

↳ Prevent: $1/\alpha T < 1/T \ll \omega_c$

Pro:

① Improves static perf.

② Improve disturbance rejection:

↳ Reason:

$$\frac{Y(s)}{D(s)} = \frac{G(s)}{1 + C(s)G(s)}$$

At low freq.: high $C(s)$ gain $\to$ denom $\uparrow$ $\to$ T.F. $\to 0$

At high freq.: denom $\uparrow$ $\to$ T.F. $\to 0$

Notes:

① If $\alpha = 10 \longrightarrow -55°$ phase shift at $\omega = \frac{1}{T\sqrt{\alpha}}$

② $1/T < \frac{\omega_c}{10} \rightarrow$ prevents $\varphi_m \downarrow$

③ $\mu = \alpha$: $\lim_{\omega \to \infty} |C(j\omega)| = 1$

④ $\mu < 1$: lag compensator can improve φ_m ($\omega_c \downarrow$, so static perf $\downarrow$)

Q: What happens if $\alpha \to \infty$

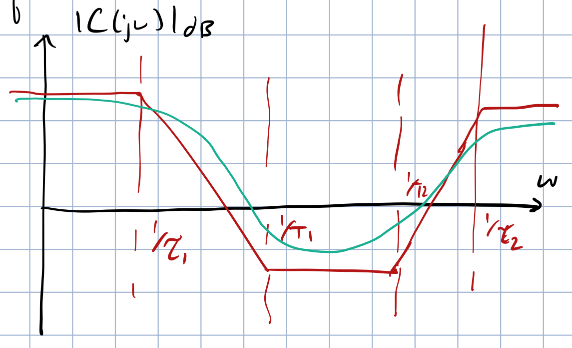
$$C(s) \approx \underbrace{\frac{\mu}{s}}_{\text{Integral}} + \underbrace{\mu T}_{\text{Proportional}} \quad \Big\} \ \underline{\text{PI controller}}$$

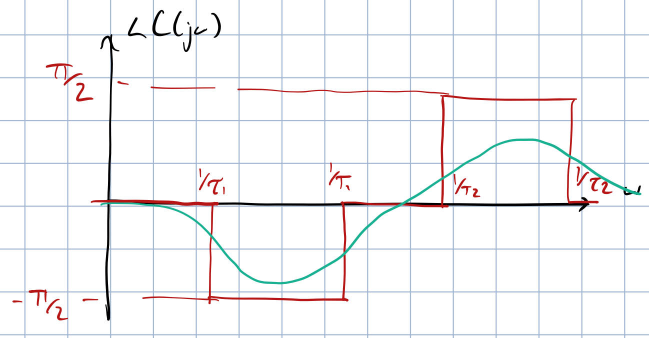
∴ Really good gain @ low freq., no e_∞, slow

Lead-lag compensator

$$C(s) = \frac{\mu(1 + T_1 s)(1 + T_2 s)}{(1 + \tau_1 s)(1 + \tau_2 s)} , \quad \left|\frac{1}{\tau_1}\right| < \left|\frac{1}{T_1}\right| < \left|\frac{1}{T_2}\right| < \left|\frac{1}{\tau_2}\right|$$

Bode plot:





Benefits:

① At low $\omega \to 0$, gain is high $\longrightarrow$ static perf. & disturbance rejection $\uparrow$

② At high ω, gain is high $\longrightarrow$ $\omega_c \uparrow$ $\longrightarrow$ bandwidth $\uparrow$

③ At second spike of $\angle$ plot $\longrightarrow$ improve φ_m $\left(\frac{1}{T_2} < \omega_c < \frac{1}{\tau_2} \right)$

Lead-lag compensator is physical realization of PID controller

PID controller

$e(t) \longrightarrow \boxed{C_{PID}} \longrightarrow u(t) \quad \Rightarrow$

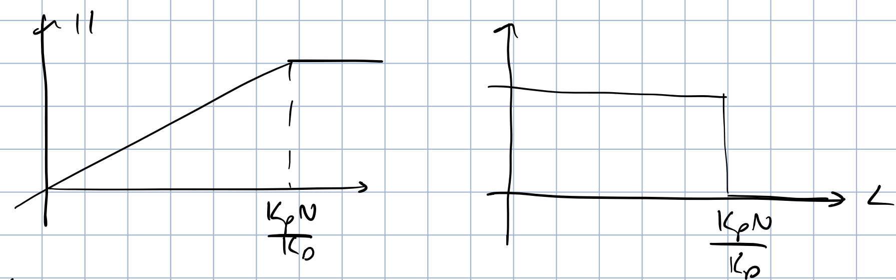
Time: $u(t) = K_p e(t) + K_I \int_0^t e(\tau)\,d\tau + K_D \frac{de(t)}{dt}$

Freq: $\frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_D s$

Problem: differentiator $\rightarrow$ improper, cannot realize

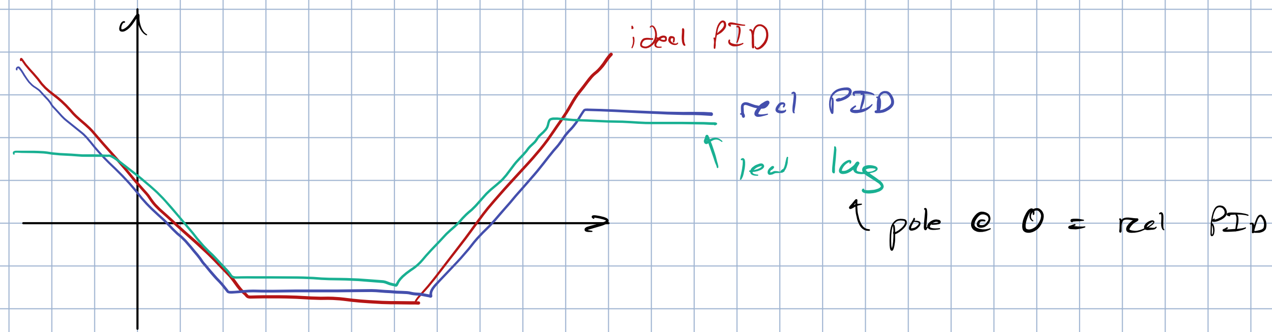
Soln: modify differentiator to "roll off" at far away pole

$$P_D(s) = \frac{K_D s}{1 + \frac{K_D}{K_p N} s} \Rightarrow$$



Need to make $N \gg \frac{K_D}{K_p}$ s.t. rolloff happens at irrelevant freq.

Modified PID results in following mag. Bode plot



Practical considerations:

① Differentiator can cause massive impulses if discontinuous signal differentiated.

↳ Eg: // at $t=0$ of $r(t) = u_{-1}(t)$. $\frac{dr}{dt}\Big|_{t=0}$ is undef.

Soln: Use $e(t)$ to hook into P_D rather than $r(t)$

② Integration windup

If plant reaches max value, integrator will dominate & windup. Takes time for proportional controller to start countering huge windup of P_I

Soln: turn integrator off when sys. reaches limit

③ Tuning K_P, K_I, K_D

A: Manual

	O.S. %	T_s (Ex.)	e_∞
K_P ↑	↑	=	↓
K_I ↑	↑	↑	0
K_D ↑	↓	↓	=

Based off table, change K_P, K_I, K_D

B: Auto-tune via Ziegler-Nichols method

Root Locus

Motivation: analyze unstable systems → Bode & Nyquist cannot handle this

Idea: change parameters of system from $0 \to \infty$ ⇒ poles of C.L.S. change

Setup:

① System

② Eqn. for poles:

Poles: $1 + \mu G(s) = 0$ ⇒ $G(s) = \dfrac{N_G(s)}{D_G(s)}$

$$D_G(s) + \mu N_G(s) = 0$$

$$\mu = -\frac{D_G(s)}{N_G(s)}$$

$$|\mu| = \left|\frac{D_G(s)}{N_G(s)}\right|$$

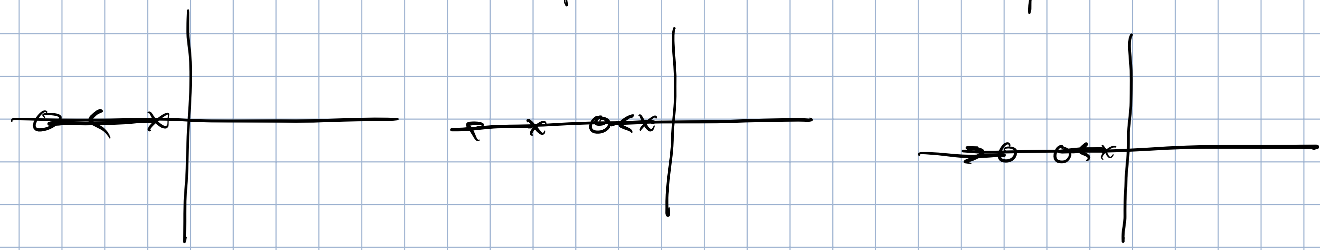
$$\angle\mu = \begin{cases} (2k+1)\pi, & \mu > 0 \\ 2k\pi, & \mu < 0 \end{cases}$$

How to plot? Rules:

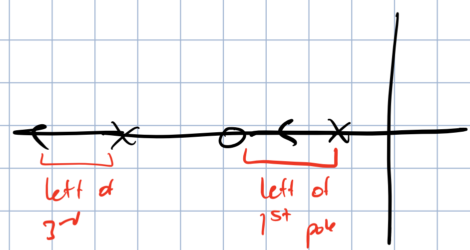
1. n loci, $n = \max(\deg(D_G(s)), \deg(N_G(s)))$
2. $\mu: 0 \to \infty \Rightarrow$ poles of C.L.S go from poles of $G(s)$ $\to$ zeros of $G(s)$

   Pole zero pair.

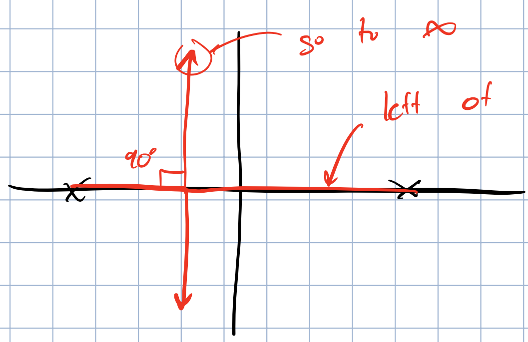
   poles = zeros   poles > zeros   poles < zeros



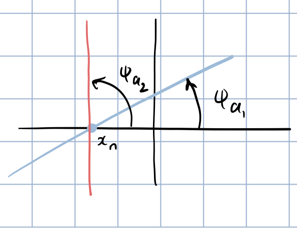
3. Loci of complex poles are symmetric about $\mathbb{R}e$
4. No loci path can cross over itself
5. Portion of $\mathbb{R}e$ axis to left of odd # of open loop poles & zeros are part of loci



6. Lines leave & enter $\mathbb{R}e$ axis @ $90°$
7. Not enough pole-zero pairs? Extra lines go/come from ∞



8. Asymptotes for lines $\to \infty$



$$\Rightarrow x_n = \frac{1}{n-m}\left(\sum p_i - \sum z_i\right)$$

n = # of poles, m = # of zeros

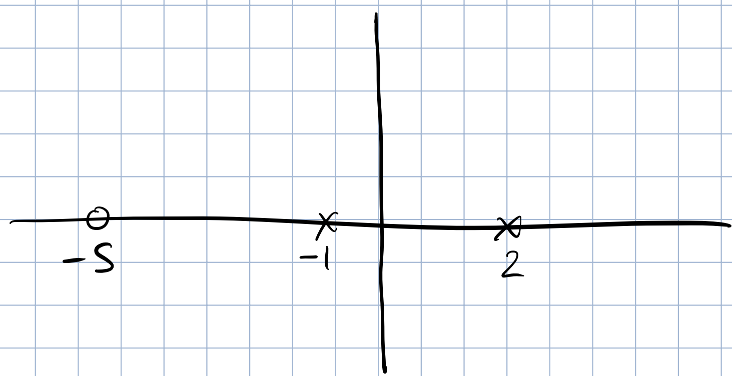
$$\varphi_{a_k} = \frac{2k+1}{n-m}\pi \qquad (k \in [0, n-m-1])$$

⑨ If at least 2 lim → ∞, sum of all roots is constant

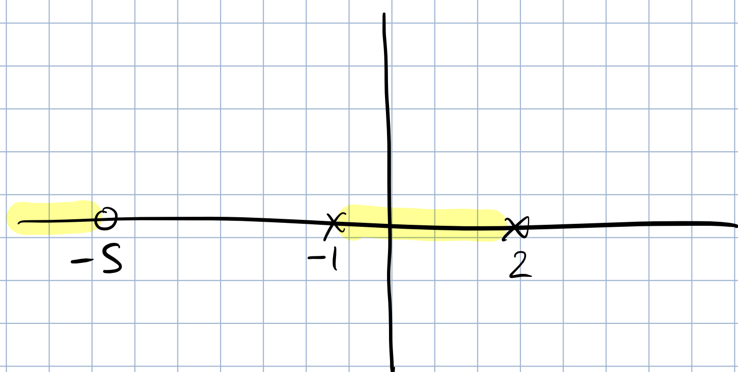
⑩ $k: 0 \to -\infty$, reverse rules & add 180° to asymptotic angle.

Ex:// What is root locus if zero at -5, roots at -1 & 2.

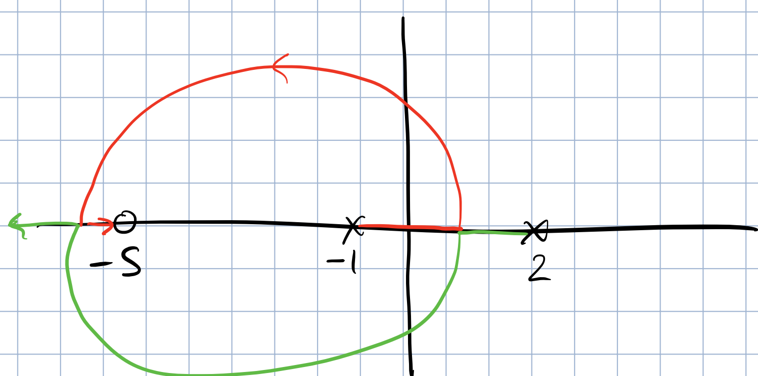
① Draw root locus w/ zero & poles



② Color the axis that should be included



③ Draw & circle pairs



Ex:// Find values of μ s.t. $\frac{1}{s-1}$ is stable.

$$|\mu| = \frac{|D_G(s)|}{|N_G(s)|} = \frac{|s-1|}{|1|}$$

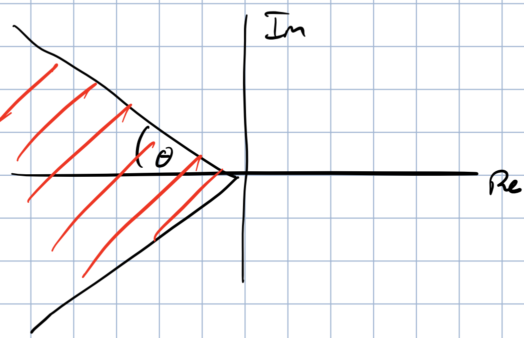
$$= |s-1|$$

If stable, $s < 0$ $\therefore$ $|\mu| = 1 - s \Rightarrow \mu > 1$

Use root locus for specifications:

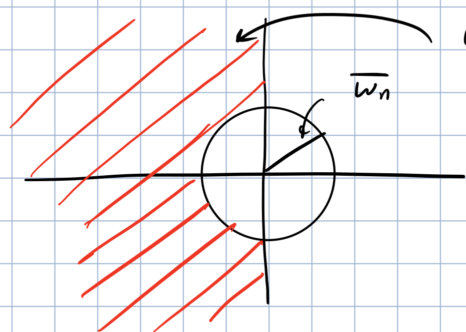
A Damping ratio $(\zeta \geq \bar{\zeta}) \Rightarrow$ O.S.%, T_p, oscillation freq, settling time

Choose gain w/ all poles in red



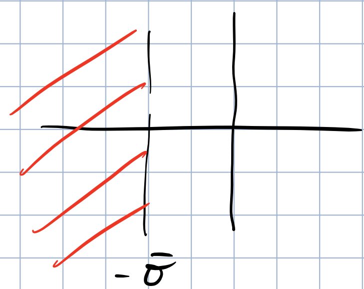
$\theta = \cos^{-1}(\bar{\zeta})$

B Natural freq $(\omega_n \geq \bar{\omega}_n) \Rightarrow T_p$, oscillation freq., settling time w/



$\omega_n \geq \bar{\omega}_n \Rightarrow$ choose gain s.t. poles in this region

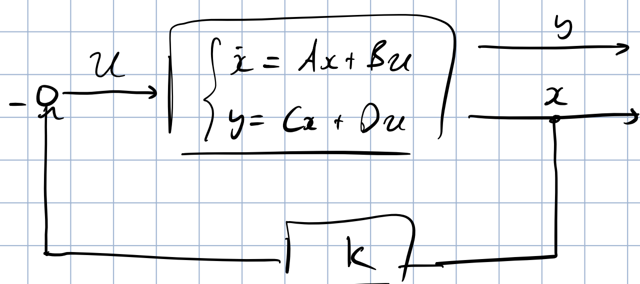
C Real component $\sigma = \zeta\omega_n \geq \bar{\sigma}$ (controls settling time)



State space controllers / pole placement

Problem w/ T.F.: can't use state for feedback

Goal:



Steps to design controller:

① Create controllable canonical form ($G(s) \rightarrow$ s.s. model)

$$G(s) = \frac{b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

$$\Downarrow$$

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & 0 & & & 0 \\ 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} x \end{cases}$$

No D matrix b/c strictly proper

② Construct u

$$u = -kx \quad \Rightarrow \quad k^T \in \mathbb{R}^n \quad (n = \# \text{ of states})$$

③ Solve for $\dot{x}$ using u

$$\therefore \dot{x} = Ax + Bu$$
$$= Ax + B(-kx)$$
$$= (A - Bk)x$$

④ Find eigenvalues for stability

$$|\lambda I - (A - Bk)| = 0$$

$\Downarrow$

characteristic poly.

⑤ Choose n stable values for eigenvalues & equate to char. polynomial

$$P_c(\lambda, k) = (\lambda - \lambda_1^*)(\lambda - \lambda_2^*) \dots (\lambda - \lambda_n^*)$$

⑥ Solve for k!

Ex://

$$G(s) = \frac{1}{s(s-1)} = \frac{1}{s^2 - s}$$

① Controllable canonical

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

② Solve for $\dot{x}$:

$$\dot{x} = (A - Bk)x$$
$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) x$$
$$= \begin{bmatrix} 0 & 1 \\ -k_1 & 1-k_2 \end{bmatrix} x$$

③ Char. poly:

$$|\lambda I - (A - Bk)| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k_1 & 1-k_2 \end{bmatrix} \right|$$
$$= \left| \begin{bmatrix} \lambda & -1 \\ k_1 & \lambda + k_2 - 1 \end{bmatrix} \right|$$
$$= \lambda^2 + (k_2 - 1)\lambda + k_1$$

④ Choose stable eigenvalues & solve for c:

$$\lambda^2 + (k_2 - 1)\lambda + k_1 = (\lambda + 1)(\lambda + 7)$$
$$\lambda^2 + (k_2 - 1)\lambda + k_1 = \lambda^2 + 8\lambda + 7$$
$$\therefore \quad k_1 = 7, \quad k_2 = 9$$

Theorem: If A & B of canonical form + n complex #s $\lambda_1^*, \dots, \lambda_n^*$, then $\exists\, k^T \in \mathbb{R}^n$ s.t. eigen values of $A - Bk$ are $\lambda_1^*, \dots, \lambda_k^*$

Drawbacks:

① Need access to state, requires sensors
↳ Can overcome by estimating state