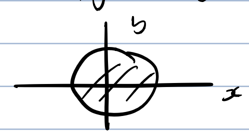
Unit 1: Scalar Functions

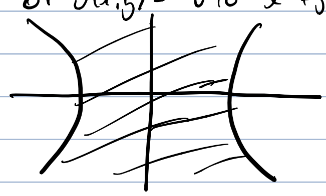
- Basic vocab:
  - $f: A \to B$ associates each $a \in A$ to a unique $f(a) \in B$
    - $f(a)$ is the image of a under f
    - A is the domain, B is the codomain
    - The subset of B consisting of all $f(a)$ is the range of f $(R(f))$
- Scalar function: domain is $\mathbb{R}^n$ and range is subset of $\mathbb{R}$
  - Ex:// $f(x,y) = e^x + e^y$ ✓
    $P(L,K) = bL^\alpha K^{1-\alpha}$ ✓ $\Rightarrow \mathbb{R}^n \to \mathbb{R}$
    $g(x,y) = (\sin(x), \cos(y))$ ✗ $\Rightarrow \mathbb{R}^n \to \mathbb{R}^n$
    $V(r,h) = \pi r^2 h$ ✓
  - Evaluation: simply plug in values.
- Domain and range of scalar functions: examine NPV
  - Ex:// Find domain and range of $f(x,y) = \sqrt{xy}$
    Domain: $xy \geq 0$ so $\Rightarrow x \geq 0, y \geq 0$ or $x \leq 0, y \leq 0$
    Range: $f(x,y) \geq 0$
  - Ex:// Find domain and range of $g(x,y) = \dfrac{x^2 - y^2}{|x| + |y|}$

    Domain: x and y cannot simultaneously be $(0,0)$
    $\therefore$ Domain: $\mathbb{R}^2 - \{(0,0)\}$
    Range: Use variables + simple combos.
    ↳ Consider $(a, 0)$ where $a \in \mathbb{R}$
    $\therefore g(a,0) = \dfrac{a^2 - 0}{|a| + 0} = |a| \Rightarrow$ any positive val.
    Consider $(0, b)$ where $b \in \mathbb{R}$
    $g(0,b) = \dfrac{0^2 - b^2}{0 + |b|} = -|b| \Rightarrow$ any negative val.

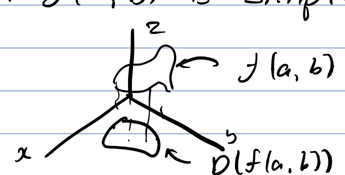
    $\therefore R(f) = \mathbb{R}$
  - Ex:// Find the domain and range of $f(x,y) = \ln(1 - x^2 - y^2)$
    Domain: $1 - x^2 - y^2 > 0$
    $-x^2 - y^2 > -1 \Rightarrow$
    $x^2 + y^2 < 1$



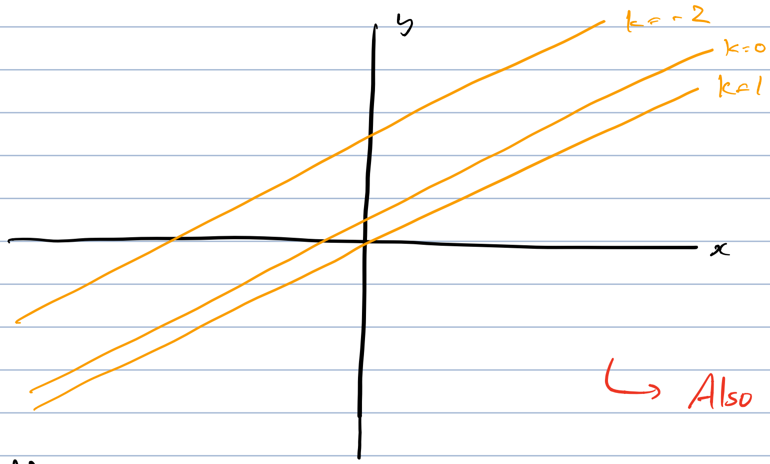
    Range: $f(x,y) \leq \ln(1)$
    $f(x,y) \leq 0$
  - Ex:// Find the domain and range of $f(x,y) = \sqrt{16 - x^2 + y^2}$
    Domain: $16 - x^2 + y^2 \geq 0$
    $x^2 - y^2 \leq 16$



    Range: Consider $(a, 0)$, $(0, b)$
    $(a,0)$:
    $f(a,0) = \sqrt{16 - a^2} \Rightarrow$ goes from $0 \to 4$
    $f(0,b) = \sqrt{16 + b^2} \Rightarrow$ goes from $4 \to \infty$ $\Big\} \geq 0$
- Geometric interpretation of $f(x,y)$:
  - The graph of $f(a,b)$ is simply:



- Hard to draw 3D graph so we use level curves defined as:
  $f(x,y) = k \quad (k \in R(f),\ k \text{ is constant})$
  - Ex:// Find the level curves of $f(x,y) = 2x - 3y + 1$



$k=0$:
$2x - 3y + 1 = 0$
$y = \frac{2}{3}x + \frac{1}{3}$

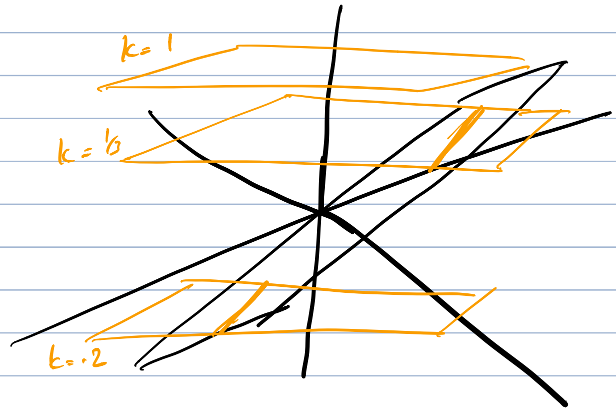
$k=1$:
$y = \frac{2}{3}x$

$k=-2$:
$y = \frac{2}{3}x + 1$

↳ Also called contour/topographic maps

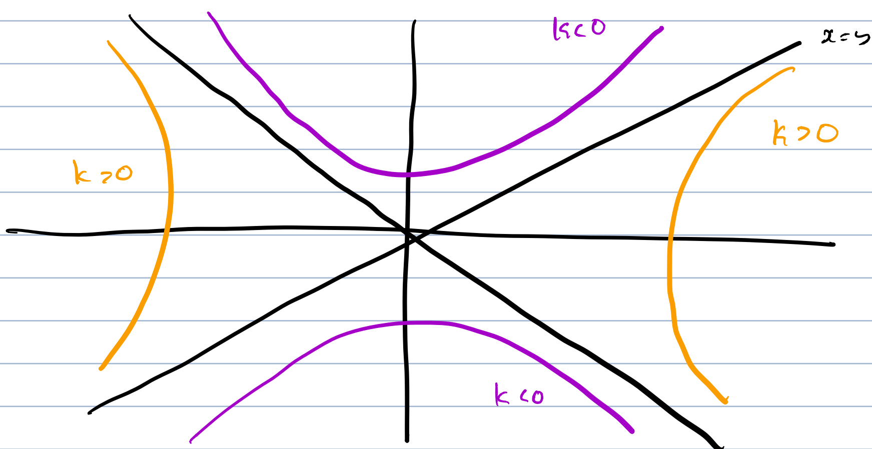
Note that $2x - 3y + 1 = k \Rightarrow 2x - 3y + (1-k) = 0$ so will give family of parallel lines.

  - Think of level curves as horizontal intersection



To visualize: move level curves to different heights!

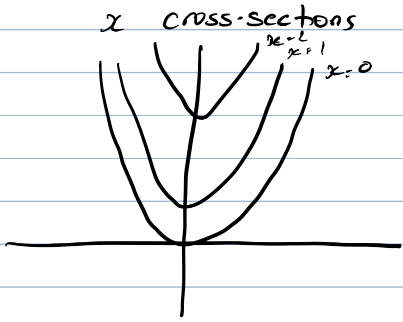
  - Ex:// Sketch level curves of $f(x,y) = x^2 - y^2$



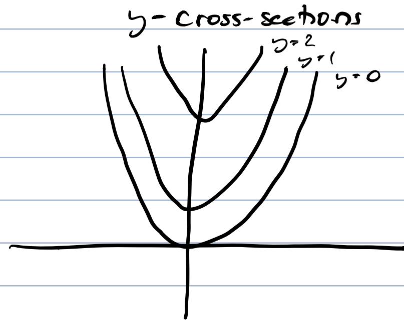
Visualize as a <u>saddle surface</u> or Pringle chip

- We can also use vertical surfaces like $x=k$ or $y=k$ (cross-sections)
  - Ex:// Create x and y cross-sections of $f(x,y) = x^2 + y^2$

x cross-sections



y-cross-sections



- For n inputs, we can always use level curves.
- Level set: $\{\vec{x} \in \mathbb{R} \mid f(\vec{x}) = k \text{ for } k \in R(f)\}$

Unit 2: Limits

Definition of a Limit

- Recall definition of single-variable limit.

  $\forall \epsilon > 0$, $\exists$ a $\delta > 0$ s.t.

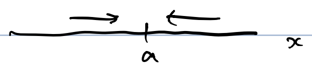
$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

  which defines:

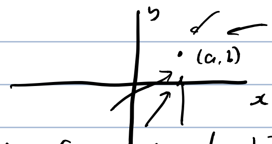
$$\lim_{x \to a} f(x) = L$$

- Defining two-variable limit for $\lim_{(x,y) \to (a,b)} f(x,y) = L$
  - Key difference: multiple ways of approaching (a,b)

Single var:

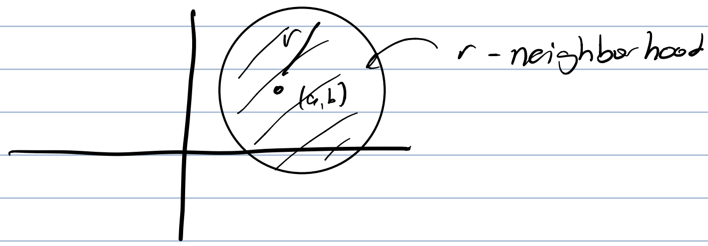


Multivariable



  - r-neighborhood: r-neighborhood of point $(a,b) \in \mathbb{R}^2$ is the set:

$$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 \mid \underbrace{\|(x,y) - (a,b)\|}_{\text{Euclidean distance}} < r,\ r \in \mathbb{R}\}$$



  - Formal definition:

    Assume $f(x,y) \in N_r(a,b)$. If $\forall \epsilon > 0$, there exists a $\delta > 0$ s.t.

$$0 < \|(x,y) - (a,b)\| < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

    then

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

  - Visually: taking a circle centred at point (a,b) and radius δ and function outputs values $L - \epsilon < f(x,y) < L + \epsilon$ in this restricted domain

Limit Theorems

- If $\lim_{(x,y) \to (a,b)} f(x,y)$ and $\lim_{(x,y) \to (a,b)} g(x,y)$ exists then:

a)
$$\lim_{(x,y) \to (a,b)} [f(x,y) + g(x,y)] = \lim_{(x,y) \to (a,b)} f(x,y) + \lim_{(x,y) \to (a,b)} g(x,y)$$
Limit of sums = sum of limits

b)
$$\lim_{(x,y) \to (a,b)} [f(x,y)\, g(x,y)] = \left[\lim_{(x,y) \to (a,b)} f(x,y)\right]\left[\lim_{(x,y) \to (a,b)} g(x,y)\right]$$
Limit of products = product of limits

c)
$$\lim_{(x,y) \to (a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y) \to (a,b)} f(x,y)}{\lim_{(x,y) \to (a,b)} g(x,y)}$$
given $\lim_{(x,y) \to (a,b)} g(x,y) \neq 0$
Limit of division = division of limits

- Proof of a):

  Call $\lim_{(x,y) \to (a,b)} f(x,y) = L_1$ and $\lim_{(x,y) \to (a,b)} g(x,y) = L_2$. Since both exist, by def.,
  $\exists \delta > 0$ s.t.

$$0 < \|(x,y) - (a,b)\| < \delta \Rightarrow |f(x,y) - L_1| < \frac{\epsilon}{2} \wedge |g(x,y) - L_2| < \frac{\epsilon}{2}$$

  Considering Original:

$0 < \|(x,y)-(a,b)\| < \delta \Rightarrow |f(x,y)+g(x,y)-L_1-L_2|$

$$= |(f(x,y)-L_1)+(g(x,y)-L_2)|$$
$$\le |f(x,y)-L_1| + |f(x,y)-L_2| \quad \text{by } \Delta \text{ ineq.}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$< \varepsilon \quad \square$$

Ex:// Find $\lim_{(x,y)\to(1,1)} \frac{x^2-xy+y^2}{x^2+y^2}$

$$\lim_{(x,y)\to(1,1)} \frac{x^2-xy+y^2}{x^2+y^2} = \frac{\lim_{(x,y)\to(1,1)} x^2-xy+y^2}{\lim_{(x,y)\to(1,1)} x^2+y^2}$$
$$= \frac{1-1+1}{1+1}$$
$$= 1/2$$

- If $\lim_{(x,y)\to(a,b)}$ exists, then it is unique.
  - Proof: prove by contradiction.

    Assume there exists 2 different limits L_1 and L_2.

$$|L_1-L_2| = \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} [f(x,y)-f(x,y)] = \underline{\underline{0}}$$

    This means L_1 and L_2 are the same, so it must be unique.

Proving Limit DNE

- In single-variable calc, we showed the left-hand and right-hand limit do not equal.
- Similarly, in multivariable limits, we approach point along any smooth curve and show that it does not equal

Ex:// $f(x,y) = \frac{xy}{x^2+y^2}$. Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

Let's go along $y=0$ and $y=x$

1) $y=0$:

$$f(x,0) = 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,0) = 0$$

2) $y=x$

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \Rightarrow \lim_{(x,y)\to(0,0)} f(x,x) = \frac{1}{2}$$

Since $f(x,y)$ approaches diff. values as $(x,y)\to(0,0)$ along different paths, limit DNE.

- We can introduce an arbitrary m constant for smooth curves. If limit depends on m, then DNE

Ex:// Prove that $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ DNE

Along $y=mx$

Note that x is controlling var

$$\lim_{(x,y)\to(0,0)} f(x,mx) = \lim_{x\to 0} \frac{\sin(mx^2)}{x^2(1+m^2)}$$
$$= \lim_{x\to 0} \frac{\cos(mx^2)\cdot 2mx}{2x(1+m^2)}$$
$$= \frac{m}{1+m^2}$$

By L'Hôpital

Ex:// Let $f(x,y) = \frac{|x|}{|x|+y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x,mx) = 1$ for all $m \in \mathbb{R}$, but $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$$\lim_{(x,y)\to(0,0)} f(x,mx) = \lim_{x\to 0} \frac{|x|}{|x|+m^2x^2}$$

As $x \to 0^+$

$$\lim_{x\to 0^+} \frac{x}{x+m^2x^2} = \lim_{x\to 0^+} \frac{1}{1+2m^2x} = 1$$

$$\lim_{x\to 0^-} \frac{-x}{-x+m^2x^2} = \lim_{x\to 0^-} \frac{-1}{-1+2mx} = 1$$

$\rightarrow \lim_{(x,y)\to(0,0)} f(x,mx) = 1$

Consider $x=0$

$$\lim_{(x,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Since we approached $(0,0)$ on 2 paths and limits do not agree, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

Ex:// Let $f(x,y) = \frac{x^2y}{x^4+y^2}$. Show that $\lim_{(x,y)\to(0,0)}$ DNE

Consider $y=0$

$$\lim_{x\to 0} f(x) = \frac{0}{x^4} = 0$$

Consider $y = x^2$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$\therefore$ DNE

Ex:// Prove that $\lim_{(x,y)\to(1,0)} \frac{(x-1)(y+1)}{|x-1|+y}$

Consider $y=0$

$$\therefore \lim_{x\to 1} = \frac{x-1}{|x-1|}$$

$$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$$

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = -1$$

$\therefore$ Limit DNE on x-axis.

Proving a Limit Exists

- Use the squeeze theorem to prove limits.

If there exists a function $B(x,y)$ such that

$$|f(x,y) - L| \le B(x,y) \text{ for all } (x,y) \ne (a,b)$$

in some neighborhood of (a,b) and $\lim_{(x,y)\to(a,b)} B(x,y) = 0$, then $\lim_{(x,y)\to(a,b)} f(x,y) = 0$

- Basically says that if we bound a function by something that goes to 0, then $f \to L$.

- Proof: Let $\epsilon > 0$. Since $\lim_{(x,y)\to(a,b)} B(x,y) = 0$, by the def. of a limit, $\exists\, \delta > 0$ s.t.

$$0 < \|(x,y) - (a,b)\| < \delta \Rightarrow |B(x,y) - 0| < \epsilon$$

Hence, if $0 < \|(x,y) - (a,b)\| < \delta$, then:

$$|f(x,y) - L| \le B(x,y) < \epsilon$$

$$\therefore \lim_{(x,y)\to(a,b)} f(x,y) = L$$

- Ex:// Prove that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$.

1) Find $B(x,y)$ s.t. $B(x,y) > |f(x,y) - L|$

$$|f(x,y) - L| = \left| \frac{x^2y}{x^2+y^2} - 0 \right| = \frac{x^2|y|}{x^2+y^2}$$

Since $y^2 \geq 0$, then $x^2 \leq x^2 + y^2$

$$\therefore B(x,y) = \frac{(x^2+y^2)|y|}{x^2+y^2} = |y|$$

2) By inspection, $\lim_{(x,y)\to(0,0)} |y| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$

- Ex:// Prove that

$$\frac{|2x^2 - y^2|}{|x| + |y|} \leq 2|x| + |y|$$

Try to manipulate numerator such that $|x| + |y|$ cancels.

$$\begin{aligned} |2x^2 - y^2| &= |2x^2 + (-y^2)| \\ &\leq 2|x|^2 + |-y^2| \\ &\leq 2|x|^2 + |y|^2 \end{aligned}$$ By Δ ineq.

Since $|x| \leq |x| + |y|$ and $|y| \leq |x| + |y|$ $\Rightarrow$ Clever step! $|a| \leq |a| + |b|$

↳ Use if $|a|^2$

$$\begin{aligned} 2|x|^2 + |y|^2 &\leq 2|x|(|x|+|y|) + |y|(|x|+|y|) \\ &\leq (2|x| + |y|)(|x| + |y|) \end{aligned}$$

$$\therefore \frac{|2x^2 - y^2|}{|x| + |y|} \leq \frac{(2|x|+|y|)(|x|+|y|)}{(|x|+|y|)} \leq 2|x| + |y|$$

- Ex:// Determine if $\lim_{(x,y)\to(0,0)} \frac{x^2 - |x| - |y|}{|x| + |y|}$ exists.

1) Linear lines: $y = mx$

$$\lim_{x\to 0} \frac{x^2 - |x| - |m||x|}{|x| + |m||x|} = \lim_{x\to 0} \frac{|x|(|x| - 1 - |m|)}{|x|(1+|m|)} = \lim_{x\to 0} \frac{|x| - (1+|m|)}{1+|m|} = -1$$

2) Prove via squeeze theorem:

$$\begin{aligned} \left| \frac{x^2 - |x| - |y|}{|x| + |y|} - (-1) \right| &= \left| \frac{x^2 - |x| - |y| + |x| + |y|}{|x| + |y|} \right| \\ &= \left| \frac{x^2}{|x| + |y|} \right| \\ &= \frac{x^2}{|x| + |y|} \end{aligned}$$

$$\therefore \frac{x^2}{|x|+|y|} = \frac{|x||x|}{|x|+|y|} \leq \frac{|x|(|x|+|y|)}{|x|+|y|} \leq |x|$$

Since $\lim_{x\to 0} |x| = 0$, limit is -1.

Inequalities and Absolute Values

- Use the following when dealing w/ multivariable limits
  - Trichotomy: either $a < b$, $a = b$, or $a > b$
  - Transistivity: $a < b \wedge b < c \Rightarrow a < c$
  - Additivity: If $a < b$, then $\forall c$, $a + c < b + c$
  - Multiplication: If $a < b$ and $c < 0$, then $bc < ac$
  - Multiplication inverse: If $ab > 0$ with $a < b$, then $\frac{1}{b} < \frac{1}{a}$
- Absolute values:
  - $|a| = \sqrt{a^2}$
  - $|a| < b$ if $-b < a < b$
  - $|a + b| \leq |a| + |b|$
  - If $c > 0$, then $a < a + c$

- Cosine inequality $2|x||y| \le x^2+y^2$

- Ex:// Prove

$$\frac{|x^3-y^3|}{x^2+y^2} \le |x|+|y|$$

$$\frac{|x^3-y^3|}{x^2+y^2} \le \frac{|x^3|+|-y^3|}{x^2+y^2}$$

$$\le \frac{|x^3|+|y^3|}{x^2+y^2}$$

$$\le |x^2||x|+|y^2||y|$$

$|x| \le |x|+|y|$

$|y| \le |x|+|y|$

$$\le \frac{(|x^2|+|y^2|)(|x|+|y|)}{x^2+y^2}$$

$$\le \frac{\cancel{(x^2+y^2)}(|x|+|y|)}{\cancel{x^2+y^2}}$$

$$\le |x|+|y|$$

Final Limits Lesson

- Ex:// $\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x}$

1) Direct plug: does not work
2) Factor + cancel:

$$\lim_{(x,y)\to(1,-2)} = \lim_{\to} \frac{y(y+2x)}{y+2x} = \lim \; y = -2$$

- Ex:// $\lim_{(x,y)\to(0,0)} \frac{x^4-y^5}{x^4+y^4}$

2) $x=0,\ y=0$

$x=0$:

$$\lim_{y\to 0} \frac{-y^5}{y^4} = -y = 0 \quad \Bigg\} \quad y=0 \quad \lim_{x\to 0} \frac{x^4}{x^4} = 1$$

Unit 3: Continuity

- From single-variable function:

$f(x)$ is continuous at $x=a$ iff

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

- For multivariable functions:

$f(x,y)$ is continuous at (a,b) iff

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- Ex:// Let $f(x,y)$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that its continuous at $(0,0)$

From before, we know

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$$

$\therefore$ Continuous as $f(0,0)=0$ as well

- Ex:// Prove that $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$

Examining $\lim_{(x,y)\to(0,0)} f(x,y)$

Consider $y = x \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 1/2 \neq 0$ ∴ not continuous

- Ex:// Consider f defined by

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2} \quad (x,y) \neq (0,0)$$

Is it possible for f to be defined at $(0,0)$ s.t. continuous at $(0,0)$?
Limit does not exist, so not possible.

- Ex:// Let f be defined as

$$f(x,y) = \begin{cases} \frac{xy}{|x|+|y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Determine if continuous at $(0,0)$.

Consider path $y = x$

$$f(x) = \frac{x^2}{|x|+|x|} = \frac{x^2}{2|x|} = \frac{|x||x|}{2|x|} = \frac{|x|}{2} \to 0$$

Proving limit:

$$\left|\frac{xy}{|x|+|y|}\right| = \frac{|x||y|}{||x|+|y||} \le \frac{(|x|+|y|)|x|}{|x|+|y|} \le |x|$$

Since $|x| \to 0$ as $x \to 0$, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

∴ Continuous?

Continuity Theorems

- Take the following as continuous: constants, power function (x^n, y^n), logs, exp, trig, inv. trig, abs $|u|$
- Function operations:
  1. $(f+g)(x,y) = f(x,y) + g(x,y)$
  2. $(fg)(x,y) = f(x,y)g(x,y)$
  3. $(f/g)(x,y) = f(x,y) \div g(x,y)$ if $g(x,y) \neq 0$
  4. $(g \circ f)(x,y) = g(f(x,y))$ for all $(x,y) \in D(f)$ for which $f(x,y) \in D(g)$
     ↳ Make sure inner domain ⊂ outer domain
     ↳ Also make sure that you are being careful w/ scalar vs. vector outputs!
- Theorem 1: f and g are continuous at $(a,b) \Rightarrow f+g$ and fg are continuous at a,b
- Theorem 2: f and g are continuous at (a,b) and $g(a,b) \neq 0 \Rightarrow f/g$ 〃
- Theorem 3: $f(x,y)$ continuous at (a,b) and $g(t)$ continuous at $f(a,b) \Rightarrow g \circ f$ continuous at (a,b)
- Ex:// Prove $h(x,y) = \sin(6x^2y + 3xy^2)$ is continuous for all $(x,y) \in \mathbb{R}^2$

  Applying theorem 1 and rules about constant and power function shows that

  $$f(x,y) = 6x^2y + 3xy^2$$

  is continuous for all $(x,y) \in \mathbb{R}^2$

  Applying theorem 3 implies $h(x,y)$ also continuous for all $(x,y) \in \mathbb{R}^2$
- Remember: if function has separate definitions at a particular point, you need to use limit definitions to prove continuity at that point

Unit 4: Linear Approximations & Partial Derivatives

Partial Derivatives

- A scalar function $f(x,y)$ can be differentiated in 2 ways:
  1. Treat y as constant and differentiate w/ respect to x to obtain $\frac{\partial f}{\partial x}$
  2. Treat x as constant and differentiate w/ respect to y to obtain $\partial f/\partial y$

  Partial derivatives
- Formal definition:

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

- Notation:

$$D_1 = \partial f/\partial x = f_x \quad , \quad D_2 = \partial f/\partial y = f_y$$

  - Operator notation: $D_i \Rightarrow$ differentiate w/ respect to i^{th} variable (in position), keep all else constant
- Use standard diff. rules. If those don't work, use limit definition
- Ex:// Consider $f(x,y) = xe^{kxy}$ where k is a constant.

  Determine f_x and f_y:

$$\frac{\partial f}{\partial x} = x'e^{kxy} + x\left(e^{kxy}\right)' \qquad \frac{\partial f}{\partial y} = xe^{kxy}\cdot xk$$

$$= e^{kxy} + x\left(e^{kxy}\right)(ky) \qquad = kx^2e^{kxy}$$

$$= e^{kxy}(1 + kxy)$$

- Ex:// Consider $f(x,y) = \sin(xy^2)$

$$\therefore f_x = \cos(xy^2)\cdot y^2$$

$$f_y = \cos(xy^2)\cdot 2xy$$

- Ex:// Determine if $\frac{\partial f}{\partial x}(0,0)$ exists for $f(x,y) = (x^3+y^3)^{1/3}$

  1. Determine $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = \tfrac{1}{3}(x^3+y^3)^{-2/3}\cdot 3x^2$$

$$= \frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$$

  2. $(0,0)$ is not defined since $x^3+y^3 \neq 0$
  3. Use partial derivative definition

$$\lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0}\frac{((x+h)^3+y^3)^{1/3} - (x^3+y^3)^{1/3}}{h}$$

  At $(0,0)$:

$$\lim_{h\to 0}\frac{(h^3+0^3)^{1/3} - 0^{1/3}}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$$\therefore \frac{\partial f}{\partial x}(0,0) \text{ exists} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1$$

- Same definition of differentiability holds: limit must exist!
- Ex:// Let

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0. \end{cases}$$

  Calculate $f_x(0,0)$ and $f_y(0,0)$.

$$f_x = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

- Differentiability $\not\Rightarrow$ continuity!
- Ex:// Let $f(x,y) = (x^3+y^3)^{1/3}$. Find $f_x(a,-a)$

$$\lim_{h\to 0} \frac{f(a+h,-a) - f(a,-a)}{h} = \lim_{h\to 0} \frac{((a+h)^3 - a^3)^{1/3} - 0}{h}$$

$$= \lim_{h\to 0} \frac{(a^3 + 3a^2h + 3ah^2 - a^3)^{1/3}}{h}$$

L'Hopital

$$= \lim_{h\to 0} \frac{(3a^2h + 3ah^2 + h^3)^{1/3}}{h}$$

$$= \lim_{h\to 0} \tfrac{1}{3}(3a^2h + 3ah^2 + h^3)^{-2/3} \cdot (3a^2 + 6ah + 3h^2)$$

$$= +\infty$$

DNE

- Ex:// $f(x,y) = |x(y-1)|$

Let $x(y-1) \geq 0$:
$f(x,y) = xy - x$
$f_x(x,y) = y - 1$

Let $x(y-1) < 0$
$f(x,y) = -xy + x$
$f_x(x,y) = -y + 1$

$(0,0) \Rightarrow$ DNE
$(0,1) \Rightarrow 0$

$$\lim_{h\to 0} \frac{f(h,1) - f(0,1)}{h} = \lim_{h\to 0} \frac{(h-h) - 0}{h} = 0$$

- Generalizing to larger functions: simply keep all other variables as constants.

Higher Order Partial Derivatives

- Since partial derivatives are scalar functions, you can take the partial derivative of a partial derivative.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = D_1^2 f \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\cdot\left(\frac{\partial f}{\partial x}\right) = f_{xy} = D_2 D_1 f$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = D_1 D_2 f \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\cdot\left(\frac{\partial f}{\partial y}\right) = f_{yy} = D_2^2 f$$

note order!

- Clairaut's theorem: If f_{xy} and f_{yx} are defined in neighborhood of (a,b) and are continuous at (a,b): then $f_{xy}(a,b) = f_{yx}(a,b)$.
- Ex:// For $f(x,y) = x^y$ for $x > 0$, is $f_{xy} = f_{yx}$?

$f_x = yx^{y-1}$
$f_{yx} = x^{y-1} + y\frac{\partial}{\partial y}(x^{y-1})$
$= x^{y-1} + y(x^{y-1}\cdot \ln x)$
$= x^{y-1}(1 + y\ln x)$

$f_y = x^y \ln x$
$f_{xy} = (x^y)'\ln x + x^y(\ln x)'$
$= yx^{y-1}\ln x + x^y \tfrac{1}{x}$
$= yx^{y-1}\ln x + x^{y-1}$
$= x^{y-1}(1 + y\ln x)$

- Clairaut's theorem extends for higher order partial derivatives
- If the k^{th}-partial order derivatives are continuous of some function f, we say:

$f \in C^k$ ("f in class C^k")

- Important to check before applying Clairaut

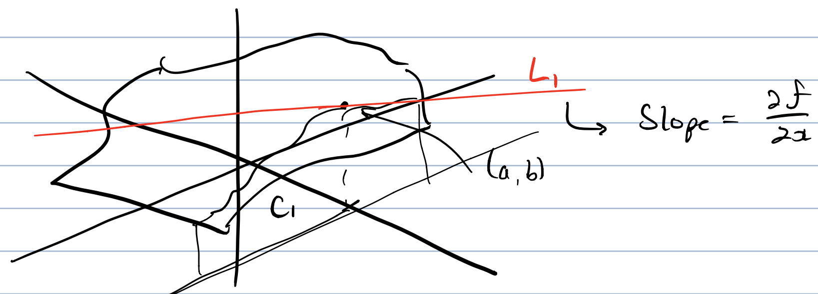
The Tangent Plane

- Think of the tangent plane as the plane which best approximates surface at a point
- Geometric interpretation of partial derivative

  Let $z = f(x,y)$. Let C_1 be the cross-section of surface at $y = b$ and C_2 be the cross-section of $x = a$.

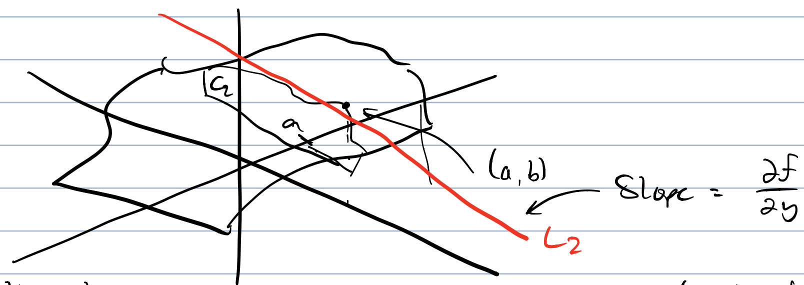
  1. C_1: $z = f(x, b)$.

     $\therefore \frac{\partial f}{\partial x}(a,b)$ = slope of tangent line L_1 of C_1 at point $(a, b, f(a,b))$



  2. C_2 : $z = f(a, y)$

     $\therefore \frac{\partial f}{\partial y}(a,b)$ = slope of tangent line L_2 of C_2 at point $(a, b, f(a,b))$



- We now define the tangent plane to be surface at $(a, b, f(a,b))$ to be plane that has $L_1 + L_2$
  - Note that any plane in form: $z = f(a,b) + m(x-a) + n(y-b)$
  - C_1: $(x, b) \Rightarrow z = f(a,b) + m(x-a)$. $\therefore m = \partial f / \partial x$ at (a,b)
  - Similarly, $n = \partial f / \partial y$ at (a,b).
  - Tangent plane at $(a, b, f(a,b))$:

$$z = f(a,b) + \frac{\partial f}{\partial x}_{(a,b)}(x-a) + \frac{\partial f}{\partial y}_{(a,b)}(y-b)$$

$\Rightarrow$ Assumes function is differentiable

- Ex:// Find tangent plane at $(3, -4, 5)$ for $f(x,y) = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} f(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} f(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\partial f}{\partial x}(3,-4) = \frac{3}{5}, \quad \frac{\partial f}{\partial y}(3,-4) = \frac{-4}{5}$$

$$\therefore z = 5 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4)$$

Linear Approximation for $z = f(x,y)$

- Recall from single variable calculus: tangent line $L_a(x) = f(a) + f'(a)(x-a)$ used to approximate x
  - $L_a(x)$: linearization. Linear approximation: $f(x) \approx L_a(x)$
- Linearization in 2 variable case:

$$L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$\Rightarrow$ Approximating (x,y) with tangent plane defined by (a,b)

- Linear approximation: $f(x,y) \approx L_{(a,b)}(x,y)$

- Ex:// Approximate $\sqrt{\sin(\frac{1}{10}) + \tan(\frac{3}{4})}$

Generally: $f(x,y) = \sqrt{\sin(x) + \tan(y)}$. Constructing tangent plane at $(0, \pi/4)$

$\frac{\partial f}{\partial x} = \frac{1}{2}(\sin(x) + \tan(y))^{-1/2} \times \cos(x)$ $\qquad$ $\frac{\partial f}{\partial y} = \frac{1}{2}(\sin(x) + \tan(y))^{-1/2} \cdot \sec^2(y)$

$\frac{\partial f}{\partial x}(0, \pi/4) = \frac{1}{\sqrt{0 + \tan(\pi/4)}} = \frac{1}{2}$ $\qquad$ $\frac{\partial f}{\partial y}(0, \pi/4) = \frac{1}{2} \cdot \frac{1}{\sqrt{0+1}} \cdot \left(\frac{1}{\cos \pi/4}\right)^2$

$= \frac{1}{2} \cdot \left(\frac{2}{\sqrt{2}}\right)^2$

$= 1$

$\therefore z = 1 + \frac{x}{2} + y - \pi/4 \implies f(\frac{1}{10}, \frac{3}{4}) \approx 1.015$

- Increment form: Define in change instead

$$\Delta f(x,y) \approx \Delta L_{(a,b)}(x,y) = \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y$$

- Ex:// Isocles triangle has base 4 m, equal angles $\pi/4$. Base increased by 0.16 m and angle decreased by 0.16 rad, estimate change in area.

① Define function: we need to define area function f affected by base x and angle θ

$\therefore f(x,\theta) = \frac{1}{2}xh = \frac{1}{2}x\left(\frac{x}{2}\tan\theta\right) = \frac{1}{4}x^2\tan\theta$

② Define increment tangent plan:

$\frac{\partial f}{\partial x} = \frac{\tan\theta}{2}x \Rightarrow \frac{\partial f}{\partial x}(4, \frac{\pi}{4}) = \frac{1}{2} \times 4 = 2$

$\frac{\partial f}{\partial \theta} = \frac{x^2}{4} \times \sec^2\theta \Rightarrow \frac{\partial f}{\partial \theta}(4, \frac{\pi}{4}) = \frac{16}{4} \times \left(\frac{2}{\sqrt{2}}\right)^2 = 4 \times 2 = 8$

$\therefore \Delta f = \Delta L_{(a,b)}(x,\theta) = 2\Delta x + 8\Delta\theta$

③ Plug in changes:

$\Delta f \approx 2(0.16) + 8(-0.1) \approx -0.48$

Area decreases by $0.48\ m^2$

Linear Approximation in Higher Dimensions

- We define the linearization of $f(x,y,z)$ at $\vec{a} = (a,b,c)$ by

$$L_a(x,y,z) = f(\vec{a}) + f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)$$

- We can also write this in terms of the dot product of two vectors:

$$L_a(x,y,z) = f(\vec{a}) + \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \rightarrow \nabla f$$

- Gradient: vector of all possible first-order partial derivatives at some particular point $\vec{a}$
- We can now define linearization + linear approximation compactly.
  - Linearization: $L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$
  - Linear approximation: $f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$

- Ex:// Consider $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Find ∇f and linear approximation for f at $\vec{a} = (1,2,-2)$

$$\nabla f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Evaluate at $\vec{a}$:

$$\nabla f(\vec{a}) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

Create linear approximation:

$$L_{\vec a}(\vec x) = f(\vec a) + \nabla f(\vec a)\cdot(x-1, y-2, z+2)$$
$$= 3 + \left(\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right)\cdot(x-1, y-2, z+2)$$
$$= 3 + \tfrac{1}{3}(x-1) + \tfrac{2}{3}(y-2) - \tfrac{2}{3}(z+2)$$

- Ex: Consider $f(x,y,z) = xyz$. Find $4.99 \times 7.01 \times 9.99$

$$\nabla f(x,y,z) = (yz, xz, xy)$$
$$\nabla f(5,7,10) = (70, 50, 35)$$

$\Rightarrow \vec x$: actual vector you want to approximate
$\Rightarrow \vec a$: vector to construct approximation (should be simple!)

$$\therefore L_{\vec a}(\vec x) = 350 + (70, 50, 35)\cdot(x-5, y-7, z-10)$$
$$= 350 + 70(x-5) + 50(y-7) + 35(z-10)$$
$$= 350 + 70(4.99-5) + 50(7.01-7) + 35(9.99-10)$$
$$= 350 - 0.7 + 0.5 - 0.35$$
$$= 349.45$$

- Vector notation generalizes quite well!

Unit 5: Differentiable Functions

Definition of Differentiability

- One variable definition of differentiability doesn't work too well: possible to have derivative at a non cont. pt
- 1 variable differentiability:
  - Let us define $R_{1,a} = g(x) - L_a(x)$ to be error for 1st order approx.
  - Then:

    Th. 1: If $g'(a)$ exists, then $\lim_{x\to a}\left|\frac{R_{1,a}(x)}{|x-a|}\right| = 0$
  - Proof:

$$\frac{|R_{1,a}(x)|}{|x-a|} = \frac{|g(x) - \overbrace{g(a) - g'(a)(x-a)}^{L_a(x)}|}{|x-a|} = \left|\frac{g(x)-g(a)}{|x-a|} - g'(a)\right| \Rightarrow 0 \text{ as } x\to a$$

  - This tells us that the tangent line at $(a, g(a))$ with slope $g'(a)$ is the best straight line approx. at pt.
- 2 variable differentiability
  - $R_{1,(a,b)} = f(x,y) - L_{(a,b)}(x,y)$
  - A function is differentiable at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

  - If a function $f(x,y)$ satisfies

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - f(a,b) - c(x-a) - d(y-b)|}{\|(x,y)-(a,b)\|} = 0$$

    then $f_x(x,y) = c$, $f_y(x,y) = d$
    - Proof: Since the limit is 0 along any path, approach on $y=b$

$$0 = \lim_{x\to a}\frac{|f(x,b) - f(a,b) - c(x-a)|}{|x-a|}$$
$$0 = \lim_{x\to a}\left|\frac{f(x,b)-f(a,b)}{x-a} - c\right|$$
$$0 = f_x(a,b) - c$$
$$\therefore c = f_x(a,b)$$

  - In other words, differentiable functions have a "good" linear approximation as defined above
  - To prove differentiability: f_x, f_y exist and ratio of error + displacement tend to 0
- Ex: Show $f(x,y) = x^2+y^2$ is differentiable at $(1,0)$

$f_x = 2x \Rightarrow f_x(1,0) = 2$ $\qquad f(1,0) = 1$

$f_y = 2y \Rightarrow f_y(1,0) = 0$

$\therefore L_{(1,2)}(x,y) = 1 + 2(x-1)$

$$\lim_{(x,y)\to(1,0)} \frac{|x^2+y^2-1-2x+2|}{\|(x,y)-(1,0)\|} = \lim_{(x,y)\to(1,0)} \frac{|(x-1)^2+y^2|}{\|(x,y)-(1,0)\|} = \lim_{(x,y)\to(1,0)} \frac{(x-1)^2+y^2}{\sqrt{(x-1)^2+y^2}}$$

$$= \lim_{(x,y)\to(1,0)} \sqrt{(x-1)^2+y^2}$$

$= 0$ by continuity theorems

$\therefore$ Differentiable at $(1,0)$

- Ex:// Determine if $f(x,y) = \sqrt{|xy|}$ is differentiable at $(0,0)$
  1. Construct approximation

$f(0,0) = 0$

$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

$f_y(0,0) = 0$

$\therefore z = 0$

  2. Construct error: $R_{1,(0,0)}(x,y) = \sqrt{|xy|}$
  3. Limit

$$\lim_{(x,y)\to(0,0)} = \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$$

Along $y=x$

$$\lim_{x\to 0} \frac{\sqrt{|x|^2}}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}|x|} = 1/\sqrt{2} \neq 0$$ $\therefore$ Not differentiable

- Ex:// Determine if $f(x,y) = \begin{cases} x^2y/x^2+y^2 & \text{if } (x,y)\neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$ is differentiable at $(0,0)$
  1. Construct approximation

$f_x(0,0) = \lim_{h\to 0} \frac{f(x+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

$f_y(0,0) = 0$

$f(0,0) = 0$

$\therefore L_{(0,0)}(x,y) = 0$

  2. Construct error: $\frac{x^2y}{x^2+y^2}$
  3. Limit:

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^2y}{x^2+y^2}}{\sqrt{x^2+y^2}} \Rightarrow$$ Approach at $y=x$:

$$\lim_{x\to 0} \frac{x^3}{2x^2\sqrt{2x^2}} = \lim_{x\to 0} \frac{x^3}{(2x^2)^{3/2}}$$

$$= \lim_{x\to 0} 1/2^{3/2}$$

$\neq 0 \Rightarrow$ Not differentiable

- If $f(x,y)$ is differentiable at (a,b), then it is continuous at (a,b)

Continuous Partial Derivatives and Differentiability

- Recall Mean Value Theorem: if $f(x)$ is continuous on interval $[x_1, x_2]$ and is differentiable, then there exists a $x_0 \in (x_1, x_2)$ s.t. $f(x_2) - f(x_1) = f'(0)(x_2 - x_1)$
- If f_x and f_y are continuous at (a,b) then $f(x,y)$ is differentiable at (a,b)
- Ex:// Determine at which points $f(x,y) = (x^2+y^2)^{2/3}$ is differentiable.
  1. Find partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{4x}{3(x^2+y^2)^{1/3}} \quad \text{for } (x,y)\neq(0,0)$$

Same for $\partial f/\partial y$

  2. Conclude: since they are continuous everywhere except $(0,0)$, we can conclude it is differentiable

everywhere except $(0,0)$, which needs further investigation.

③ Further investigation using definition.

Need to determine if $f(x,y)$ is differentiable at $(0,0)$.

Ⓐ Constructing approximation.

$f(0,0) = 0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{((0+h)^2)^{2/3} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{(0+h)^{4/3} - 0}{h} = 0$$

Ⓑ Error: $R_{(0,0)}(x,y) = (x^2+y^2)^{2/3}$

Ⓒ Limit:

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^{2/3}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} (x^2+y^2)^{1/6} = 0 \Rightarrow \underline{\text{Differentiable}}$$

- Strategy: use the partial derivatives to show function is differentiable. Use the error definition for hard points.
- This also applies to checking if f_x is differentiable using f_{xx}, f_{xy}, f_{yy}
- These theorems are generalized to n-dimensional functions

Linear Approximation Revisited

- The error of $f(x,y)$ is said to be: $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$
- Linear approximation: $L_{(a,b)}(x,y) = f(a,b) + \nabla f(a,b) \cdot (x-a, y-b)$
- Rearranging error formula:

$$f(x,y) = f(a,b) + \nabla f(a,b)\cdot(x-a, y-b) + R_{1,(a,b)}(x,y)$$

- Now, we can say that the linear approximation is good if we can neglect the error term
  - We can neglect if we know $f(x,y)$ is differentiable at (a,b) since $\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}|}{\|(x,y)-(a,b)\|} = 0$
- Ex:// Is the following a valid approximation?

$$(xy)^{1/3} \approx 2 + 1/3(x-2) + 1/6(y-4)$$

① Construct linear approximation

Let $f(x,y) = (xy)^{1/3}$. Thus:

$$\nabla f = \left(\tfrac{1}{3}x^{-2/3}y^{1/3}, \tfrac{1}{3}x^{1/3}y^{-2/3}\right)$$

At $\nabla f(2,4) = (1/3, 1/6)$

Thus:

$$(xy)^{1/3} = 2 + 1/3(x-2) + 1/6(y-4) + R_{1,(2,4)}(x,y)$$

② Determine if differentiable at point of interest

$f(x,y)$ has continuous partial derivatives at $(2,4)$, so

$$\lim_{(x,y)\to(2,4)} \frac{|R_{1,(2,4)}(x,y)|}{\|(x,y)-(2,4)\|} = 0$$

Thus, we can neglect $R_{1,(2,4)}(x,y)$. So original approximation is valid

Putting it Together

- Ex:// Prove that $f(x,y) = x(|y|-1)$ is differentiable at $(0,0)$

① Find partials

$$\frac{\partial f}{\partial x}(0,0) = -1 \qquad \frac{\partial f}{\partial y} = (x|y| - x)' = x\cdot\frac{y}{|y|}$$

$$\therefore \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{0}{h} = 0$$

② Linear approximation:

$$L_{(0,0)}(x,y) = -x$$

③ Error:

$$R_{1,(0,0)}(x,y) = x|y| - x + x = x|y|$$

④ Determining if differentiable

$$\lim_{(x,y)\to(0,0)} = \frac{x|y|}{\sqrt{x^2+y^2}} \Rightarrow$$ We want to prove differentiability so this must be 0!

Proving:

$$\left|\frac{|x||y|}{\sqrt{x^2+y^2}}\right| \le \frac{|x||y|}{\sqrt{x^2+y^2}} \le \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

Since $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$, above limit is $0 \Rightarrow$ differentiable.

Unit 6: Chain Rule

Basic Chain Rule in 2 Dimensions

- Recall from single-variable calculus $\Rightarrow$ $f(t(x))' = f'(t(x)) \cdot t'(x)$
- If $f(x,y) = f(x(t), y(t))$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt} \Rightarrow$$ Note the difference in ∂ (derivative of multivariable func) and d (derivative of single-variable)

  - More precise:
    Let $G(t) = f(x(t), y(t))$ and let $a = x(t_0)$ and $b = y(t_0)$. If f is differentiable at (a,b) and $x'(t_0)$ and $y'(t_0)$ exist, then $G'(t_0)$ exists and
    $$G'(t_0) = f_x(a,b)\,x'(t_0) + f_y(a,b)\,y'(t_0)$$

- Ex:// Use the Chain Rule to find df/dt for $f(x,y) = xy^3 - x^3y$ with $x(t) = t^2+1$ and $y(t) = t^2-1$ at $t_0 = 1$

  ① Find a and b

  $a = x(t_0) = 2$, $b = y(t_0) = 0$

  ② Find $x'(t_0)$ and $y'(t_0)$

  $x'(t) = 2t$ $\quad y'(t) = 2t$

  $x'(t_0) = 2$ $\quad y'(t_0) = 2$

  ③ Find $G'(t)$:

  $$G'(t_0) = f_x(a,b)\cdot x'(t_0) + f_y(a,b)\cdot y'(t_0)$$
  $$= f_x(2,0)\cdot 2 + f_y(2,0)\cdot y'(t_0)$$

  ④ Find $f_x(a,b)$ and $f_y(a,b)$

  $f_x = y^3 - 3x^2y \Rightarrow f_x(a,b) = 0$

  $f_y = 3xy^2 - x^3 \Rightarrow f_y(a,b) = -8$

  ⑤ Plug in:

  $G'(1) = -8\times 2 = -16$

- Ex:// $f(x,y) = (xy)^{1/3}$, $x(t) = t$, $y(t) = t^2$. $G(t) = f(x(t), y(t))$. Find $G'(0)$

  ① Finding (a,b):

  $a = x(0) = 0$, $b = y(0) = 0$

  ② Finding $x'(t)$, $y'(t)$

  $x'(t) = 1$, $y'(t) = 2t$

  $\therefore x'(0) = 1$, $y'(0) = 0$

  ③ Finding $f_x(0,0)$, $f_y(0,0)$:

  $f_x = \frac{1}{3}(xy)^{-2/3}\cdot y$ $\quad f_y = \frac{1}{3}(xy)^{-2/3}\cdot x$

  $f_x(0,0) = 0$ $\quad f_y(0,0) = 0$

  ④ Plugging in:

  $$\frac{\partial G}{\partial t} = \frac{\partial G}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial G}{\partial y}\cdot\frac{dy}{dt} = 0$$

  OR

  ①: Make $G(t)$:

  $G = (xy)^{1/3}$

  $= (t\cdot t^2)^{1/3}$

  $= t$

  $\therefore G'(t) = 1$

  $G'(0) = 1$

  Discrepance! $G(x(t), y(t))$ is not differentiable at $(0,0)$.

- Ex:// $T(x,y) = 10e^{-0.1(x^2+y^2)}$. $x(t) = 2\cos t$. $y(t) = 4\sin t$. Find rate of change of T at $t = 3\pi/4$

  ① Point:

  $x(3\pi/4) = 2\cos 3\pi/4 = 2\cdot -\frac{\sqrt{2}}{2} = -\sqrt{2}$

  $y(3\pi/4) = 4\sin 3\pi/4 = 4\cdot \sqrt{2}/2 = 2\sqrt{2}$

  ② Derivatives:

$x'(t) = -2\sin t \Rightarrow x'(3\pi/4) = -2\sin\frac{3\pi}{4} = -2\times\sqrt{2}/2 = -\sqrt{2}$

$y'(t) = 4\cos t \Rightarrow y'(3\pi/4) = 4\cos\frac{3\pi}{4} = -4\times\sqrt{2}/2 = -2\sqrt{2}$

$T_x = 10e^{-0.1(x^2+y^2)} \cdot -0.1\times 2x = -2xe^{-0.1(x^2+y^2)}$

$T_y = -2ye^{-0.1(x^2+y^2)}$

$T_x(-\sqrt{2}, 2\sqrt{2}) = \frac{2\sqrt{2}}{e}$, $T_y(-\sqrt{2}, 2\sqrt{2}) = -\frac{4\sqrt{2}}{e}$

③ Chain rule:

$$\frac{dT}{dt}\left(\frac{3\pi}{4}\right) = \frac{2\sqrt{2}}{e}\cdot -\sqrt{2} + \frac{-4\sqrt{2}}{e}\cdot -2\sqrt{2}$$
$$= -4/e + \frac{16}{e} = \boxed{\frac{12}{e}}$$

- Ex:// $g(t) = f(t^2+3, e^t)$. $\nabla f(3,1) = (-2,5)$. Find $g'(0)$. What condition on f will guarantee validity?
  f must be differentiable at $(3,1)$.

$$\frac{dg}{dt} = -2\cdot 0 + 5\cdot 1 = 5$$

- Ex:// $g(t) = f(x,y)$ where $x(t) = \cos t$, $y(t) = \sin(t)$. Calculate $g'(\pi/3)$ if $\nabla f(1/2, \sqrt{3}/2) = (\sqrt{3}, 4)$

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt}$$
$$= \sqrt{3}\cdot(-\sin\pi/3) + 4(\cos\pi/3)$$
$$= -\sqrt{3}\cdot\sqrt{3}/2 + 4\times 1/2$$
$$= -3/2 + 2$$

- Rewriting in vector format:

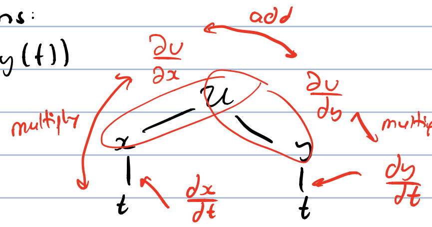
$$\frac{dT}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \nabla f\cdot\frac{d\vec{x}}{dt}$$

  - Holds generally:

$$\frac{dT}{dt} = T_x\, x'(t) + T_y\cdot y'(t) + T_z\cdot z'(t)$$

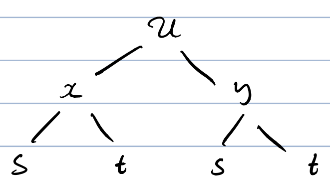
Extensions of Chain Rule

- Dependence diagrams:
  - Ex:// a) $u = f(x(t), y(t))$



$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

b) $u = f(x(s,t), y(s,t))$



u is given func. of x + y and diff wrt x holding y fixed

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

⇒ u is composite function of s and t, diff. wrt to s + hold t

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

- Algorithm:
  1. Identify all variables in dependence tree
  2. Take all possible paths from differentiated variable to differentiating variable
  3. For each link in path, multiply derivatives ⇒ be careful if partial / not
  4. Add products from 3

- Ex:// $z = f(x,y) = (x-y)^4$ where $x = st^4$ and $y = s^4t$. Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$

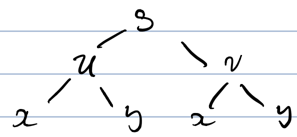
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= 4(x-y)^3\cdot t^4 + 4(x-y)^3(-1)(4s^3t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 4(x-y)^3(4st^3) + 4(x-y)^3(-1)\cdot s^4$$

- Ex:// Calculate $\frac{\partial g}{\partial x}(1,1)$ where $g(x,y) = f(2xy,\ x^2-y^2)$ and f is a differentiable function with $\nabla f(2,0) = (2,3)$

Note that f is a function of $u(x,y) = 2xy$ and $v(x,y) = x^2 - y^2$)

① Dependence tree:



② Construct partial derivatives

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}(u(x,y), v(x,y)) \cdot \frac{\partial u}{\partial x}(x,y) + \frac{\partial g}{\partial v}(u(x,y), v(x,y)) \cdot \frac{\partial v}{\partial x}(x,y)$$

$$= \frac{\partial g}{\partial u}(u(x,y), v(x,y)) \cdot (2y) + \frac{\partial g}{\partial v}(u(x,y), v(x,y)) \cdot (2x)$$

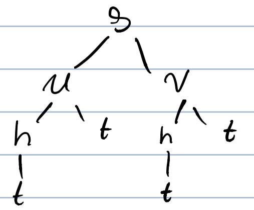
③ Substitute $(x,y) = (1,1)$ and $\nabla f(2,0) = (2,3)$

$$\frac{\partial g}{\partial x}(1,1) = 2\ \frac{\partial g}{\partial u}(2,0) + 2\frac{\partial g}{\partial v}(2,0)$$

$$= 2 \cdot 2 + 2 \cdot 3$$

$$= 10$$

- Ex:// $g(t) = f(h(t)+t,\ h(t)-t)$. Chain rule?



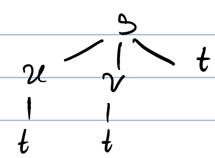
$$\Rightarrow \frac{\partial g}{\partial t} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial h} \cdot \frac{\partial h}{\partial t} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial g}{\partial v} \cdot \frac{\partial}{\partial h} \cdot \frac{\partial h}{\partial t}$$

OR

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial u}(h(t)+t,\ h(t)-t) \cdot (h'(t)+1) + \frac{\partial g}{\partial v}(h(t)+t,\ h(t)-t)(h'(t)-1)$$

- Ex:// $f(3,2) = 5$ and $\nabla f(3,2) = (4,-1)$. Let $g(t) = t^2 f(2t+1,\ 3t^3-t)$. Find $g'(1)$.
f is a function of two variables u and v where $u(t) = 2t+1$ and $v = 3t^3 - t$

① Dependence tree:



② Create partial derivative:

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial u} \cdot \frac{du}{dt} + \frac{\partial g}{\partial v} \cdot \frac{dv}{dt}$$

③ Calculate partial derivatives:

$$\frac{\partial g}{\partial u} = t^2 f_u(u,v), \quad \frac{\partial g}{\partial v} = t^2 f_v(u,v), \quad \frac{\partial g}{\partial t} = 2t\, f(u,v)$$

$$\frac{du}{dt} = 2, \quad \frac{dv}{dt} = 9t^2 - 1$$

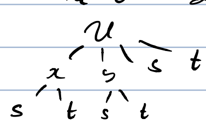
④ Solve:

$$g'(1) = 2f(3,2) + 2\frac{\partial f}{\partial u}(3,2) + 8\ \frac{\partial f}{\partial v}(3,2)$$

$$= 10 + 2 \cdot 4 + 8 \cdot (-1)$$

$$= 10$$

- Ex:// Write chain rule of $\frac{\partial u}{\partial s}$ for $u(s,t) = f(x(s,t),\ y(s,t),\ s,\ t)$



$$\Rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial s} + \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Chain Rule for Second Partial Derivatives

- Ex:// $z = f(x)$ where x is twice differentiable and $x = e^u$. Show: $z''(u) = x^2 f''(x) + x f'(x)$

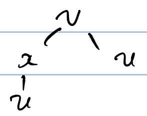
① Find first order:

$$z'(u) = f'(x)x'(u) = f'(x) \cdot e^u$$

② Use dependence tree for second order.

Let $v(x,u) = f'(x) \cdot e^u$. Thus $z'(u) = v(e^u, u)$

$\therefore$



③ Construct partial derivative eqn

$$\frac{\partial v}{\partial u} = \frac{\partial v}{\partial x} \cdot \frac{dx}{du} + \frac{dv}{du}$$

④ Substitute:

$$z''(u) = f''(x)e^u \cdot e^u + f'(x)e^u \quad x = e^u$$

$$= x^2 f''(x) + x f'(x)$$

- Ex:// $g(u,v) = f(u^2 - v^2, 2uv)$. Find $(g_u)^2 + (g_v)^2$ and $g_{uu} + g_{vv}$

Let $x(u,v) = u^2 - v^2$ and $y(u,v) = 2uv$

① First order chain rule

$$g_u = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = 2u f_x + 2v f_y$$

$$g_v = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = -2v f_x + 2u f_y$$

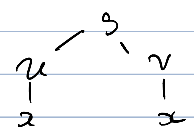
② Second order partials via product rule

$$g_{uu} = 2f_x + 2u[2u f_{xx} + 2v f_{xy}] + 0 + 2v[2u f_{yx} + 2v f_{yy}]$$

- Ex:// Let $g(u,v) \in C^2$ and $f(x) = g(x, 2x)$. Find a, b, c s.t.

$$f''(x) = a g_{uu} + b g_{uv} + c g_{vv}$$

$\therefore u(x) = x, \quad v(x) = 2x$



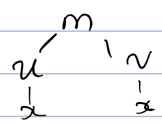
① First order g_x

$$g'(x) = g_u \cdot u'(x) + g_v v'(x)$$

$$= g_u(u,v) + 2 g_v(u,v)$$

② Second order:

Let $m(u,v) = g_u(u,v) + 2 g_v(u,v)$



$\therefore$

$$\frac{\partial m}{\partial x} = \frac{\partial m}{\partial u} \cdot \frac{du}{dx} + \frac{\partial m}{\partial v} \cdot \frac{dv}{dx}$$

$$g''(x) = g_{uu} + 2g_{vu} + (g_{vu} + 2g_{vv}) \cdot 2$$

$$= g_{uu} + 2g_{vu} + 2g_{vu} + 4g_{vv}$$

$$= g_{uu} + 4g_{vu} + 4g_{vv}$$

UNIT 7: DIRECTIONAL DERIVATIVES & GRADIENT VECTOR

Directional Derivatives

- We want to define a derivative in the direction of unit vector $\vec{u}$
- Directional derivative of $f(x,y)$ at (a,b) in direction of $\vec{u} = (u_1, u_2)$ is:

$$D_{\vec{u}} f(a,b) = \frac{d}{ds} f(a+su_1, b+su_2)\Big|_{s=0}$$

- Ex:// Find the directional derivative of $f(x,y) = x^2 - y^2$ at $(1,2)$ in direction of $\vec{u} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$

$$D_{\vec{u}}(1,2) = \frac{d}{ds} f\left(1+\frac{s}{\sqrt{5}}, 2+\frac{2s}{\sqrt{5}}\right)\Big|_{s=0}$$
$$= \frac{d}{ds}\left[\left(1+\frac{s}{\sqrt{5}}\right)^2 - \left(2+\frac{2s}{\sqrt{5}}\right)^2\right]\Big|_{s=0}$$
$$= \left[\frac{2}{\sqrt{5}}\left(1+\frac{s}{\sqrt{5}}\right) - \frac{4}{\sqrt{5}}\left(2+\frac{2s}{\sqrt{5}}\right)\right]\Big|_{s=0}$$
$$= -6/\sqrt{5}$$

Calculate derivative, then evaluate $s=0$.

- Directional derivative theorem: $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$ (dot product)
- Ex:// $f(x,y) = 2x^3 + 4xy^2 + y$ at $(-1,1)$ in direction $\vec{u} = (1,1)$
  1. Find $\nabla f(a,b)$:
     $f_x = 6x^2 + 4y^2 \Rightarrow f_x(-1,1) = 10$
     $f_y = 8xy + 1 \Rightarrow f_y(-1,1) = -7$
  2. Convert to unit vector
     $$\vec{u}^* = \frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
  3. Dot product
     $$D_{\vec{u}^*}(-1,1) = \frac{10}{\sqrt{2}} - \frac{7}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

- If $f(x,y)$ not differentiable at (a,b), you must use directional derivative definition
- If $\vec{u} = (1,0)$, then we get f_x. If $\vec{u} = (0,1)$, we get f_y
- Geometrically, the directional derivative is the rate of change of the function w/ respect to distance from (a,b) along in direction $\vec{u}$

Gradient Vector in 2 Dimensions

- If $f(x,y)$ is differentiable at (a,b) and $\nabla f(a,b) \neq 0$, then $\max(D_{\vec{u}}(a,b))$ occurs when $\vec{u}$ in direction of $\nabla f(a,b)$
  - Gradient vector is maximum directional derivative.
  - Proof:

$$D_{\vec{u}}(a,b) = \nabla f(a,b) \cdot \vec{u}$$
$$= \|\nabla f(a,b)\| \, \|\vec{u}\| \cos\theta$$
$$= \|\nabla f(a,b)\| \cos\theta \Rightarrow \text{max at } \theta = 0$$
, or if $\vec{u}$ in same direction as ∇f

- Ex:// Find largest rate of change of $f(x,y) = \sqrt{x^2+2y^2}$ at $(1,2)$ and direction.
  1. Find gradient:
     $$\nabla f(x,y) = \left(\tfrac{1}{2}(x^2+2y^2)^{-1/2} \cdot 2x, \ \tfrac{1}{2}(x^2+2y^2)^{-1/2} \cdot 4y\right)$$
     $$\nabla f(1,2) = (1/3, 4/3)$$
  2. Max:
     $$\text{Max} = \|\nabla f(1,2)\| = \|(1/3, 4/3)\| = \sqrt{\tfrac{1}{9}+\tfrac{16}{9}} = \frac{\sqrt{17}}{3}$$
  3. Direction:
     $$\vec{u} = \nabla f(1,2) = (1/3, 4/3)$$

- Greatest rate of change theorem applies in any dimension
- Ex:// $f(x,y,z) = z^3 e^{x^2+y^2-2x}$. Find greatest rate of change + direction at $(1,1,1)$
  1. Find gradient:
     $f_x = z^3 e^{x^2+y^2-2x} \cdot (2x-2) \Rightarrow f_x(1,1,1) = 0$
     $f_y = z^3 e^{x^2+y^2-2x} \cdot (2y) \Rightarrow f_y(1,1,1) = 2$

$f_z = 3z^2 e^{x^2+y^2-2x} \Rightarrow f_z(1,1,1) = 3$

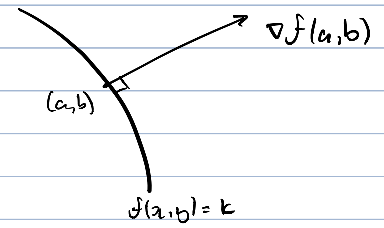
② Max:

$\|\nabla f(1,1,1)\| = \sqrt{0^2+2^2+3^2} = \sqrt{13}$

③ Direction:

$\vec{u} = \nabla f(1,1,1) = (0, 2, 3)$

- Orthogonality theorem: $f(x,y) \in C^1$ in (a,b) neighborhood $\wedge$ $\nabla f(a,b) \neq (0,0) \Rightarrow \nabla f(a,b) \perp$ level curve k



- This means that the greatest rate of change should be in direction orthogonal to level curve at the point (a,b)
- Ex:// Prove that $f(x,y) = \frac{y}{x^2}$ $(x \neq 0)$ and $g(x,y) = x^2 + 2y^2$ intersect orthogonally.

  We need to show that

  $\nabla f \cdot \nabla g = 0$ $\Rightarrow$ wherever they intersect, it should be orthogonal

  $\nabla f = \left(-\frac{2y}{x^3}, \frac{1}{x^2}\right)$, $\nabla g = (2x, 4y)$

  $\therefore \nabla f \cdot \nabla g = -\frac{2y}{x^3} \cdot 2x + \frac{1}{x^2} \cdot 4y$
  $= -\frac{4y}{x^2} + \frac{4y}{x^2}$
  $= 0$

Gradient Vector in 3 Dimensions

- Visualizing $f(x,y,z)$ is hard, but instead think about level surfaces $f(x,y,z) = k$ $\in \mathbb{R}^3$
- Orthogonality theorem: $\nabla f(a,b,c)$ is orthogonal to level surface $f(x,y,z) = k$ through (a,b,c)
- We can use this to define tangent plane at any surface in $\mathbb{R}^3$.

  Let $\vec{x} \in \mathbb{R}^3$ and is on tangent plane to surface $f(x,y,z) = k$ at (a,b,c). Since $\nabla f(a,b,c)$ is orthogonal

  $\nabla f(a,b,c) \cdot (\vec{x} - \vec{a}) = 0$

  $\Rightarrow f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + \ldots = 0$

- Ex:// Find equation of tangent plane to surface $z^3 e^{x^2+y^2-2x} = 1$ at $(1,1,1)$

  $\therefore \nabla f(1,1,1) \cdot (x-1, y-1, z-1) = 0$

  Previously: $\nabla f(1,1,1) = (0,2,3)$

  $\therefore 2y - 2 + 3z - 3 = 0$

  $2y + 3z - 5 = 0$

- Ex:// Find equation of tangent plane to $x^2 + 2y^2 + 3z^2 = 12$ at $(1, 1, \sqrt{3})$

  ① Find gradient

  $f(x,y,z) = x^2 + 2y^2 + 3z^2$

  $\nabla f(x,y,z) = (2x, 4y, 6z)$

  $\nabla f(1,1,\sqrt{3}) = (2, 4, 6\sqrt{3})$

  ② Constructing plane:

  $\nabla f(1,1,\sqrt{3}) \cdot (\vec{x} - (1,1,\sqrt{3})) = 0$

  $2(x-1) + 4(y-1) + 6\sqrt{3}(z - \sqrt{3}) = 0$

  $2x - 2 + 4y - 4 + 6\sqrt{3}z - 18 = 0$

  $2x + 4y + 6\sqrt{3}z = 24$

UNIT 8: TAYLOR POLYNOMIALS & TAYLOR'S THEOREM

Taylor Polynomial of Degree 2

- In single variable calculus, Taylor polynomial of deg. 2 at $x=a$ is

$$P_2(a) = f(a) + f'(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

  - This is just: $P_2(a) = L_a(x) + \frac{1}{2} f''(a)(x-a)^2$
  - Coefficient of $\frac{1}{2}$: from constraint that $P_{2,a}''(a) = f''(a)$

- 2 variable case:
  - Taylor polynomial of deg. 2: $P_{2,(a,b)}(x,y)$

$$\therefore P_{2,(a,b)}(x,y) = L_{(a,b)}(x,y) + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$$

Finding A, B, C:

$$\frac{\partial^2 P_2(x,y)}{\partial x^2} = 2A \qquad \frac{\partial^2 P_2(x,y)}{\partial y^2} = 2C \qquad \frac{\partial^2 P_2(x,y)}{\partial x \partial y} = B$$

Using our constraint that second derivatives of P_2 and f should be same:

$$\left.\begin{matrix} 2A = \frac{\partial^2 f}{\partial x^2}, & 2C = \frac{\partial^2 f}{\partial y^2}, & B = \frac{\partial^2 f}{\partial x \partial y} \end{matrix}\right\} \to \text{evaluated at } (a,b)$$

Combining all together:

$$P_{2,(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)\underset{(x-a)}{}(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

  - $f(x,y) \approx P_2(x,y)$ w/ better accuracy than linear approximation

- Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix} \Rightarrow \text{2D version of gradient}$$

  - Useful for finding coefficients.
  - Ex:// Calculate Hessian matrix of $f(x,y) = \frac{1}{2}y^2 + x - \frac{1}{3}x^3$

$$f_x = 1 - x^2 \to f_{xx} = -2x, \quad f_{xy} = 0$$
$$f_y = y \to f_{yy} = 1, \quad f_{yx} = 0$$

$$Hf(x,y) = \begin{bmatrix} -2x & 0 \\ 0 & 1 \end{bmatrix}$$

- Ex:// Use a Taylor polynomial to 2nd degree to approximate $\sqrt{(0.95)^3 + (1.98)^3}$

① Define function + approx. point

$$f(x,y) = \sqrt{x^3 + y^3} \quad \text{at } (a,b) = (1,2)$$

② Finding Hessian matrix and gradient at (1,2)

$$Hf(1,2) = \begin{bmatrix} 11/12 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \qquad \nabla f(1,2) = \left(\frac{1}{2}, 2\right)$$

③ Create Taylor equation:

$$P_{2(1,2)}(x,y) = 3 + \frac{1}{2}(x-1) + 2(y-2) + \frac{1}{2}\left[\frac{11}{12}(x-2)^2 - \frac{2}{3}(x-1)(y-2) + \frac{2}{3}(y-2)^2\right]$$

④ Plug in actual points

$$\therefore P_{2,(1,2)}(0.95, 1.98) = \ldots$$

Taylor Formula w/ Second Degree Remainder

- Recall in single variable calc:

If $f''(x)$ exists on $[a, x]$, then $\exists\, c \cdot a < c < x$ such that

$$f(x) = f(a) + f'(a)(x-a) + R_{1,a}(x)$$

where $R_{1,a}(x) = \frac{1}{2} f''(c)(x-a)^2$ $\Rightarrow$ error!

- We don't know c but we can estimate the upper bound

$$|f(x)| \leq B$$

$$\therefore \quad |f(x) - L_a(x)| = |R_{1,a}(x)|$$
$$\leq \frac{1}{2} B (x-a)^2$$

- Taylor's Theorem for 2 variables: If $f(x,y) \in C^2$ in neighborhood $N(a,b)$, then for all $(x,y) \in N(a,b)$, there exists a point (c,d) on the line between (a,b) and (x,y) s.t.

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$

where

$$R_{1,(a,b)}(x,y) = \frac{1}{2}\left[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\right]$$

- Again, (c,d) exists. So we need to use same error bound analysis

- Ex// Let $f(x,y) = \sqrt{1+x+2y}$. Find the linear approx. near $(0,0)$ and show that if $x \geq 0, y \geq 0$, then

$$|R_{1,(0,0)}(x,y)| \leq \frac{3}{4}(x^2+y^2)$$

① Finding linearization:

$$\nabla f(x,y) = \left(\frac{1}{2}(1+x+2y)^{-1/2},\ (1+x+2y)^{-1/2}\right)$$
$$\nabla f(0,0) = \left(\frac{1}{2},\ 1\right)$$
$$f(0,0) = 1$$
$$\therefore \quad L_{(0,0)}(x,y) = 1 + \frac{1}{2}x + y$$

② Finding Hessian matrix

$$f_{xx} = -\frac{1}{4}(1+x+2y)^{-3/2}$$
$$f_{xy} = f_{yx} = -\frac{1}{2}(1+x+2y)^{-3/2}$$
$$f_{yy} = -\frac{1}{2}(1+x+2y)^{-3/2} \cdot 2$$

$$\therefore \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -1/4 & -1/2 \\ -1/2 & -1 \end{bmatrix} \Rightarrow \text{At } (0,0)$$

③ Error Function:

$$R_{1,(0,0)}(x,y) = \frac{1}{2}\left[f_{xx}(c,d)(x-0)^2 + f_{xy}(c,d)(x-0)(y-0) + f_{yy}(c,d)(y-0)^2\right]$$

④ Upper bound:

$$|R_{1,(0,0)}(x,y)| = \frac{1}{2}\left(|f_{xx}(c,d)|x^2 + 2|f_{xy}(c,d)||x||y| + |f_{yy}(c,d)|y^2\right)$$
$$\leq \frac{1}{2}\cdot\left(\frac{1}{4}x^2 + 2\cdot\frac{1}{2}|x||y| + y^2\right)$$
$$\leq \frac{1}{2}\cdot\left(\frac{1}{4}x^2 + \frac{1}{2}(x^2+y^2) + y^2\right) \quad 2|x||y| \leq x^2+y^2$$
$$\leq \frac{3}{8}x^2 + \frac{3}{4}y^2$$
$$\leq \frac{3}{4}x^2 + \frac{3}{4}y^2$$
$$\leq \frac{3}{4}(x^2+y^2)$$

- If $f(x,y) \in C^2$ in some neighborhood $N(a,b)$, then there exists a positive constant M s.t.

$$|R_{1,(a,b)}(x,y)| \leq M\,\|(x,y)-(a,b)\|^2 \quad \text{for all } (x,y) \in N(a,b)$$

Generalizations of Taylor Polynomial

- Multi-index: if $f \in C^k$ is a function of n variables, k^{th}-order partial derivative is

$$\partial^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$$

Where α is a multi-index $\Rightarrow$ $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ like $\alpha_i \in \mathbb{N}$.

- Sum of α is k $\Rightarrow$ order of α
- Ex:// $f(x,y,z) = x^3 y^4 z^5$ and $\alpha = (2,1,3)$.

$$\therefore \quad \partial^\alpha = \left(\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial}{\partial y}\right)^1 \left(\frac{\partial}{\partial z}\right)^3 f$$

① $$\frac{\partial^3 f}{\partial z^3} = 5 \cdot 4 \cdot 3\, x^3 y^4 z^2 = 60 x^3 y^4 z^2$$

② $$\frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial z^3}\right) = 240 x^3 y^3 z^2$$

③ $$\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial z^3}\right)\right) = 240 \cdot 3 \cdot 2\, x y^3 z^2$$
$$= 1440\, x y^3 z^2$$

- We can also use it like so:
$\vec{x} = (x_1, x_2, \dots, x_n)$. $\vec{a} = (a_1, a_2, \dots, a_n)$. $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$

$$\therefore \quad (\vec{x} - \vec{a})^\alpha = (x_1 - a_1)^{\alpha_1} (x_2 - a_2)^{\alpha_2} \cdots (x_n - a_n)^{\alpha_n}$$

- Ex:// Let $\vec{x} = (x_1, x_2, x_3)$, $\vec{a} = (2, -1, 0)$, $\alpha = (2, 4, 1)$
$$\therefore (\vec{x} - \vec{a})^\alpha = (x_1 - 2)^2 (x_2 + 1)^4 (x_3)$$

- k^{th} degree polynomial of $f(x,y)$

$$P_{k(a,b)} = \sum_{|\alpha| \le k} \partial^\alpha f(a,b) \frac{[(x,y) - (a,b)]^\alpha}{\alpha!}$$

- Summing over $|\alpha| \le k$ mean summing all partials whose order sum to k/less
- Ex:// What is $P_{2(a,b)}(x,y)$?

We have
$$P_{2,(a,b)}(x,y) = \sum_{\alpha \le 2} \partial^\alpha f(a,b) \frac{[(x,y) - (a,b)]^\alpha}{\alpha!}$$

Since we have two variables $\Rightarrow$ $\alpha = (\alpha_1, \alpha_2)$. We need to find all combos such that $\alpha_1 + \alpha_2 \le k$ and $\alpha_1, \alpha_2 \in \mathbb{N}$.

① $\alpha = (0,0)$: $f(a,b)$
② $\alpha = (1,0)$: $f_x(a,b)\,(x-a)^1 (y-b)^0 = f_x(a,b)(x-a)$
③ $\alpha = (0,1)$: $f_y(a,b)\,(x-a)^0 (y-b)^1 = f_y(a,b)(y-b)$
④ $\alpha = (2,0)$ = $f_{xx}(a,b)\,(x-a)^2$ w/ $\alpha! = 2$
⑤ $\alpha = (1,1)$ = $f_{xy}(a,b)(x-a)(y-b)$ w/ $\alpha = 1$
⑥ $\alpha = (0,2)$ = $f_{yy}(a,b)(y-b)^2$ w/ $\alpha! = 2$

All together:

$$P_{2,(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tfrac{1}{2} f_{xx}(a,b)(x-a)^2$$
$$+ f_{xy}(a,b)(x-a)(y-b) + \tfrac{1}{2} f_{yy}(a,b)(y-b)^2$$

- All previous theories will work
① : Taylor's theorem but now error is of order k
② : Good approximation
$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_{k,(a,b)}(x,y)|}{\|(x,y) - (a,b)\|^k} = 0$$
③ : Corollary on error term max bound

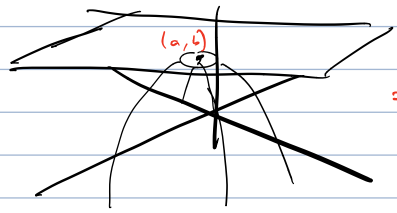
- For n variables:

$$P_{k,\vec{a}}(\vec{x}) = \sum_{|\alpha| \le k} \partial^\alpha f(\vec{a}) \frac{(\vec{x}-\vec{a})^\alpha}{\alpha!}$$

Unit 9: Critical Points

Local Extrema + Critical Points

- Critical value of f: $f'(c) = 0$ or undefined.
  - Remember: even though a point might be a critical point, it is not always a local extremum
- Local maximum: $f(x,y) \le f(a,b)$ for some $N(a,b)$
- Local minimum: $f(x,y) \ge f(a,b)$ for some $N(a,b)$
- Graphically:



$\Rightarrow$ Horizontal tangent plane at (a,b)!

$$\therefore \quad \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

- Theorem: if each partial derivative is 0 or DNE at (a,b), the point is a local extrema
- Critical points: all partials are either 0/DNE at this point
  - Being a critical point does not guarantee that you are at an extremum
- Ex:// Find critical points of $g(x,y) = -x^2 - y^2$
  1. Find partials:
     $$g_x = -2x, \qquad g_y = -2y$$
  2. Points where both g_x & g_y = 0/DNE
     $\therefore$ Only $(0,0) \Rightarrow g_x = g_y = 0$
  3. Determine if actually max, min or neither through graphs.
     $g(x,y) = -(x^2+y^2) \Rightarrow$ upside down paraboloid at $(0,0)$
     $\therefore$ Local max
- Ex:// Find critical points of $h(x,y) = x^2 - y^2$.
  1. Find partials
     $h_x = 2x \qquad h_y = -2y$
  2. Critical points:
     $(0,0) \Rightarrow h_x = h_y = 0$
  3. Max, min or saddle?
     Inspecting level curves.
     $h(x,0) = x^2$
     $h(0,y) = -y^2$ $\Rightarrow$ Saddle point at $(0,0)$
- Ex:// Find critical points of $f(x,y) = x^2y + 3xy^2 + xy$
  1. Partials
     $f_x = 2xy + 3y^2 + y$
     $f_y = x^2 + 6xy + x$
  2. Critical points
     Factor out $\longrightarrow$ cases
     $f_x = y(2x + 3y + 1) = 0$ - (A)
     $f_y = x(x + 6y + 1) = 0$ - (B)

     From A:

Case 1: $y=0$

$\therefore f_y = x(x+1)=0$

$\therefore x=0,-1$ } Two pts: $(0,0)$, $(-1,0)$

Case 2: $2x+3y+1=0$

$\therefore y = \frac{-2x-1}{3}$

$f_y = x(x+2(-2x-1)+1)=0$

$= x(-3x-1)=0$

$\therefore x = 0, -1/3$ } Two pts: $(0,0)$, $(-1/3, -1/9)$

Second Derivative Test

- Single variable:
  - $f''(c)>0 \Rightarrow x=a$ is a min if $x=a$ is a critical point
  - $f''(c)<0 \Rightarrow x=a$ is a max if ''
  - $f''(c)=0 \Rightarrow$ conclude nothing
- Quadratic forms:

$$Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2 \Rightarrow \text{quadratic form on } \mathbb{R}^2$$

where a_{11}, a_{12}, a_{22} are constants

  - Matrix notation

$$Q(u,v) = [u \ v]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}$$

  - 4 main quadratic classes: determined by symmetric constants matrix
    1. Positive definite: $\forall u,v \cdot (u,v)\neq(0,0) \wedge Q(u,v)>0$
    2. Negative definite: '' $Q(u,v)<0$
    3. Indefinite: $\exists u,v,w,z \cdot Q(v,v)>0 \wedge Q(w,z)<0$
    4. Semidefinite: not in other 3 classes
       a) Positive semi definite: $\forall u,v \cdot Q(v,v)\geq 0$
       b) Neg. '' : '' $\cdot Q(u,v)\leq 0$
  - Ex:// Classify $\begin{bmatrix} 1 & 3 \\ 3 & 2\end{bmatrix}$

$$Q(u,v) = u^2+6uv+2v^2$$
$$= (u^2+6uv+9v^2)-7v^2 \quad \text{complete square}$$
$$= (u+3v)^2-7v^2$$

Indefinite!

Good checks! $\rightarrow$ $Q(u,0)=u^2>0$ for $u\neq 0$!
$Q(-3v,v) = -7v^2<0$ for $v\neq 0$!

  - Ex:// Classify $A=\begin{bmatrix} 2 & -1 \\ -1 & 1\end{bmatrix}$

$$Q(u,v) = 2u^2-2uv+v^2$$
$$= 2(u^2-uv+1/4v^2)+3/4v^2$$
$$= \underbrace{2(u-1/2v)^2}_{>0}+\underbrace{3/4v^2}_{>0}$$

$\therefore$ Positive definite

  - Trick:
    1. Positive definite: $\det(A)>0 \wedge a_{11}>0$
    2. Negative definite: $\det(A)>0 \wedge a_{11}<0$
    3. Indefinite if $\det(A)<0$
    4. Semidefinite if $\det(A)=0$
- Second derivative test for 2 variables:

Suppose $f(x,y)\in C^2$ in some $N(a,b)$ and:

$f_x(a,b) = 0 = f_y(a,b) \Rightarrow$ critical point

Then:

1. If $Hf(a,b)$ is positive definite $\Rightarrow (a,b)$ is local minimum
2. If $Hf(a,b)$ is negative definite $\Rightarrow (a,b)$ is local max
3. If $Hf(a,b)$ is indefinite $\Rightarrow (a,b)$ is saddle point
4. If $Hf(a,b)$ is semidefinite $\Rightarrow$ inconclusive

- Ex:// Find + classify critical points of $f(x,y) = x^3 - 4x^2 + 4x - 4xy^2$

  ① Find critical points

  $f_x = 3x^2 - 8x + 4 - 4y^2$ · Ⓐ

  $f_y = -8xy$ · Ⓑ

  From B:

  Case 1: $x = 0$

  $f_x = 4 - 4y^2 = 0 \Rightarrow y = \pm 1$

  Case 2: $y = 0$

  $f_x = 3x^2 - 8x + 4 = 0 \Rightarrow x = 2, 2/3$

  $\therefore$ Critical points: $(0,1), (0,-1), (2,0), (2/3, 0)$

  ② Create Hessian matrix

  $f_{xx}(x,y) = 6x - 8$

  $f_{xy}(x,y) = -8y$

  $f_{yy}(x,y) = -8x$

  At $(2/3, 0)$: $\begin{bmatrix} -4 & 0 \\ 0 & -16/3 \end{bmatrix}$ At $(0,1)$: $\begin{bmatrix} -8 & -8 \\ -8 & 0 \end{bmatrix}$ At $(2,0)$: $\begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}$ At $(0,-1)$: $\begin{bmatrix} -8 & 8 \\ 8 & 0 \end{bmatrix}$

  ③ Classify according to Hessian matrices

  $(2/3, 0) \Rightarrow$ Negative definite $\Rightarrow$ local maximum

  $\{(0,1), (0,-1), (2,0)\} \Rightarrow$ indefinite $\Rightarrow$ saddle points

- If Hessian matrix is semidefinite, consider $f(x,y) - f(a,b)$ + sign
  - Ex:// Show that $(0,0)$ is a degenerate critical point of $f(x,y) = 2(x-y)^2 - x^4 - y^4 + 3$ + classify.

    ① Degenerate proof

    $\nabla f(0,0) = (0,0) \wedge Hf(0,0) = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

    Quadratic form of Hessian matrix: $Q(u,v) = 4u^2 - 8uv + 4v^2 = 4(u-v)^2 \geq 0$

    This is semidefinite, thus $(0,0)$ is degenerate

    ② Classification:

    $f(x,y) - f(0,0) = 2(x-y)^2 - x^4 - y^4$

    Try to substitute variables s.t. elements are cancelled:

    Consider:

    $f(x,x) - f(0,0) = -2x^4 < 0$ for all $x \neq 0$

    Consider

    $f(x,0) - f(0,0) = 2x^2 - x^4 = x^2(2 - x^2) > 0$ for all x s.t. $0 < x < 2$

    Since (x,y) is in $N(0,0)$ and have both pos. + neg. signs, $\Rightarrow (0,0)$ is saddle point

Convex Functions

- Convex function: f is convex if $f''(x) \geq 0$ for all x
- Strictly convex: $f''(x) > 0$
- Properties of convex 1 variable functions: If $f \in C^2$ and is strictly convex:

  1. $f(x) > L_a(x)$

  Proof:

  From Taylor's Theorem: $f(x) = L_a(x) + \frac{f''(c)}{2}(x-a)^2$ where $a \leq c \leq x$

Since $f''(c) > 0$, $f(x) > L_c(x)$

Graphically:

$\Rightarrow$ $f(x)$ above tangent line

2. For $a < b$,

$$f(x) < f(c) + \frac{f(b) - f(c)}{b-a}(x-a) \quad \text{for } x \in (a,b)$$

Proof:

Let $g(x) = f(x) - \left[f(c) + \frac{f(b)-f(c)}{b-a}(x-a) \right]$

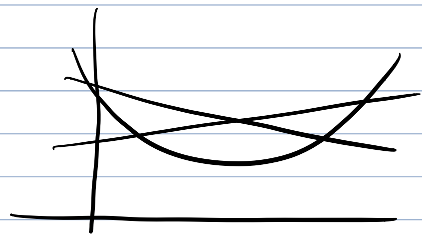
$g(c) = g(b) = 0$ and $g''(x) = f''(x) > 0$. We need to show $g(x) < 0$ for some $x \in (a,b)$

By MVT: $\frac{f(b)-f(c)}{b-a} = f'(c)$ for some $c \in (c,b)$

Note: $g'(x) = f'(x) - \frac{f(b)-f(c)}{b-a} = f'(x) - f'(c)$. Thus $g'(c) = 0$.

Since $g''(x) > 0$, $g'(x)$ is strictly increasing. Thus, $g'(x) < 0$ on $[a,c)$ and $g'(x) > 0$ on $(c,b]$
This implies $g(x)$ is decreasing on $[a,c]$. Since $g(c) = 0$, then $g(x) < 0$ on $(a,c]$.
QED.

Graphically:



$\Rightarrow$ $f(x)$ below secant line

- For 2 variables:
  - Convex: $Hf(x,y)$ is positive semidefinite
  - Strictly convex: $Hf(x,y)$ is positive definite $\Rightarrow$ $f_{xx} > 0 \wedge f_{xx}f_{yy} - f_{xy}^2 > 0$ for all (x,y)
  - Properties: If $f(x,y)$ is strictly convex
    1. $f(x,y) > L_{(a,b)}f(x,y)$ for all $(x,y) \neq (a,b)$ $\Rightarrow$ Always above tangent plane
    2. $f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)) < f(a_1, a_2) + t[f(b_1, b_2) - f(a_1, a_2)]$ for $0 < t < 1$, $(a_1, a_2) \neq (b_1, b_2)$

       ↳ Cross-section lies below secant line
  - Critical point properties
    1. If $f(x,y)$ is convex $\Rightarrow$ every critical point satisfies $f(x,y) \geq f(c,d)$ for all $(x,y) \neq (c,d)$
    2. " strictly convex $\Rightarrow$ critical point (c,d) satisfies $f(x,y) > f(c,d)$ for all $(x,y) \neq (c,d)$ + no other critical points

    □ Proof of 2:

    If (c,d) is a critical point, then $f_x = f_y = 0$. Thus, $L_{(c,d)}(x,y) = f(c,d)$.
    Since $f(x,y) > L_{(a,b)}(x,y)$ for all $(x,y) \neq (a,b)$, then $f(x,y) > f(c,d)$ for all $(x,y) \neq (c,d)$.
    To show uniqueness: Assume second critical point (c_1, d_1) s.t. $(c_1, d_1) \neq (c, d)$
    Similarly: $f(x,y) > f(c_1, d_1)$
    However, this implies $f(c_1, d_1) > f(c,d)$ and $f(c,d) > f(c_1, d_1)$. Contradiction □

Unit 10: Optimization

Extreme Value Theorem

- Extreme value theorem for one-variable calculus:

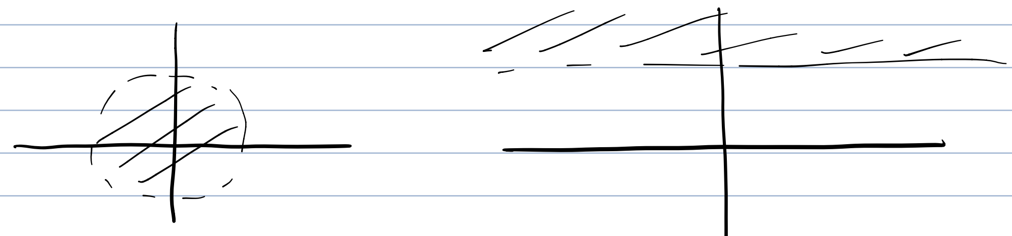
  If $f(x)$ is continuous on a finite, closed interval I, there exists $c_1, c_2 \in I$ s.t.

  $$f(c_1) \leq f(x) \leq f(c_2) \quad \forall x \in I$$

  - Every continuous function has an absolute maximum + minimum
- Absolute max + min for 2-variable functions

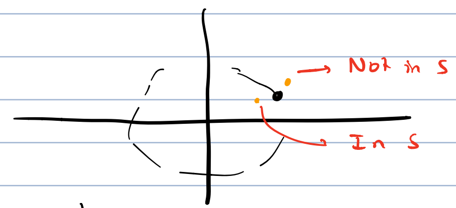
- Absolute max: $(a,b) \in S \subseteq \mathbb{R}^2$ is abs. max of $f(x,y)$ if
$$f(x,y) \le f(a,b) \quad \forall (x,y) \in S$$
- Absolute min: $(a,b) \in S \subseteq \mathbb{R}^2$ is abs. min of $f(x,y)$ if
$$f(a,b) \le f(x,y) \quad \forall (x,y) \in S$$

- Extending "finite closed interval" to $\mathbb{R}^2$:
  - Bounded set ($S \subset \mathbb{R}^2$): contained in some neighborhood of the origin

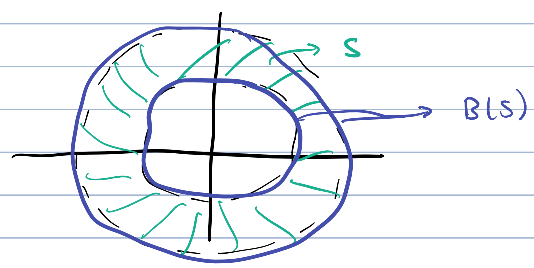


Bounded   Unbounded

  - Bounded sets have finite distance from origin
  - Boundary point: given $S \subset \mathbb{R}^2$, (a,b) is a boundary point $\Leftrightarrow$ neighborhood of (a,b) contains one point in S and one point not in S



  - Boundary: set of all boundary points
  - Closed set ($S \subseteq \mathbb{R}^2$): S contains all boundary points
    - Ex:// Consider $S = \{(x,y) \in \mathbb{R}^2 \mid 1 < \|(x,y)\| \le 2\}$.



Boundary of S: $B(S) = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| = 1 \lor \|(x,y)\| = 2\}$
Lower boundary of $\|(x,y)\| = 1$ is not included in boundary! Look at def. of S!
$\therefore$ Not closed
    - Closed set = closed interval

- Extreme Value Theorem for 2-Variable Functions.
If $f(x,y)$ is continuous on a closed + bounded set $S \subset \mathbb{R}^2$, then there exists points (a,b) and $(c,d) \in S$ such that
$$f(a,b) \le f(x,y) \le f(c,d) \quad \forall (x,y) \in S$$
  - You can have abs. max / min even if EVT does not apply
    - Ex:// $S = \{(x,y) \in \mathbb{R}^2 \mid x > -1, y \in \mathbb{R}\}$
$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \ne 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$
Not continuous function but $f(x,y)$ has max of 1 and min of 0.

Algorithm for Extreme Value

- In single-var calc, we just needed to compare points at critical pts + endpoints to determine max + min
- Similarly, we only need to compare critical pts + boundary points
- Algorithm:
  1. Check if given set is closed and bounded

2. Check if $f(x,y)$ is continuous on set
3. Find all critical points of f contained in S
4. Evaluate f at each point
5. Find maximum + minimum values of f on the boundary $B(s)$
6. Determine max + min

- Ex:// Find maximum value of $f(x,y) = xy$ on $S = \{(x,y) \mid x^2+y^2 \le 1\}$

① Set is closed + bounded + $f(x,y)$ is continuous

② Find critical point:

$$\nabla f(x,y) = (y, x)$$

$\therefore$ Critical point $= (0,0)$

$$f(0,0) = 0$$

③ Find maximum on boundary.

Parameterizing boundary lead to

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

Value of f on boundary

$$g(t) = f(\cos t, \sin t) = \tfrac{1}{2}\sin 2t$$

Maximize:

$$g'(t) = \cos 2t$$

Critical points:

$$2t = \pi/2,\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2}$$

$$t = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$$

Evaluating at critical pts:

$$g(\tfrac{\pi}{4}) = \tfrac{1}{2},\quad g\left(\tfrac{3\pi}{4}\right) = -\tfrac{1}{2},\quad g\left(\tfrac{5\pi}{4}\right) = \tfrac{1}{2},\quad g\left(\tfrac{7\pi}{4}\right) = -\tfrac{1}{2}$$

Also $g(0) = 0$, $g(2\pi) = 0$.

④ Maximum: $\frac{1}{2}$ at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

- Ex:// Find max + min of $f(x,y) = xy - 2x - y + 2$ on triangular region S with vertices $(0,0)$, $(2,0)$, $(0,3)$

① Checks: set is closed, bounded + $f(x,y)$ is continuous

② Critical points:

$$\nabla f(x,y) = (y-2, x-1)$$

$\therefore$ Critical point is $(1,2)$ but outside of S

③ Boundary

Case 1: $x=0$, $0 \le y \le 3$

By inspection, max at $y=0$ (2) and min at $y=3$ (-1)

Case 2: $y=0$, $0 \le x \le 2$

By inspection, max at $x=0$ (2) and min at $x=2$ (-2)

Case 3: $y = 3 - \frac{3}{2}x$, $0 \le x \le 2$

$g(x) = f(x, 3 - \frac{3}{2}x) = -\frac{3}{2}x^2 + \frac{5}{2}x - 1$ } COMMON FOR BOUNDARIES

Critical points: $g'(x) = -3x + \frac{5}{2} = 0$

$\therefore$ $(5/6, 7/4)$ is critical point w/ $f(5/6, 7/4) = 1/24$

④ Compare min + max

Max: 2 at $(0,0)$

Min: -2 at $(2,0)$

Optimization w/ Constraints

- Goal: find max/min of function $f(x,y)$ subject to constraint $g(x,y) = k$

- Geometrically, this is the same problem as finding critical points on boundaries! $g(x,y)$ is a level curve.

- Method of Lagrange Multipliers:
  - Critical point on constraint curve are exactly at
    $$\nabla f(a,b) = \lambda g(a,b)$$ ⇒ Gradient of function + constraint in same direction
  - Algorithm: find max/min at
    1. $\nabla f(a,b) = \lambda g(a,b) \wedge g(a,b) = k$
    2. $\nabla g(a,b) = (0,0) \wedge g(a,b) = k$
    3. (a,b) is endpoint of curve $g(x,y) = k$
  - Ex:// Find max of $6x+4y-7$ on $3x^2+y^2=28$

    ① Check first condition:
    $$f(x) = 6x+4y-7, \quad g(x,y) = 3x^2+y^2 = 28$$
    $$\nabla f(x,y) = (6, 4)$$
    $$\nabla g(x,y) = (6x, 2y)$$
    ∴ Need to find points (a,b) s.t.:
    $$(6,4) = \lambda(6x, 2y)$$
    ∴ 3 equations:
    $$6 = 6\lambda x \quad - 1$$
    $$4 = 2\lambda y \quad - 2$$
    $$3x^2+y^2 = 28$$
    Solving:
    1: $\lambda = \frac{1}{x}$ $(x \neq 0)$
    2: $y = 2x$
    3: $3x^2 + 4x^2 = 28$
    $$x = \pm 2$$
    $$y = \pm 4 \Rightarrow (2,4), (-2,-4)$$

    ② Check second condition:
    $$\nabla g(a,b) = 0 \wedge g(a,b) = 28$$
    Not possible since $(0,0)$ satisfies $\nabla g(a,b) = 0$ but not constraint

    ③ End points: No endpoints.

    ∴ Max of 21 at $(2,4)$ ⇒ Plug into $f(x,y)$
  - Ex:// Find max + min of $f(x,y) = y$ on curve $y^2+x^4-x^3=0$

    ① First condition:
    $$\nabla f(x,y) = (0, 1) \qquad \nabla g(x,y) = (4x^3-3x^2, 2y)$$
    Equations:
    A: $\lambda(4x^3-3x^2) = 0$
    B: $\lambda(2y) = 1$
    C: $y^2+x^4-x^3=0$

    From A:
    $$\lambda x^2(4x-3) = 0$$
    $$\therefore x = 0, 3/4$$
    $x=0 \Rightarrow$ C ensures $y=0$ but violates B
    $x = 3/4 \Rightarrow y = \pm\frac{3\sqrt{3}}{16}$
    $$\therefore \left(\tfrac{3}{4}, \tfrac{3\sqrt{3}}{16}\right), \left(\tfrac{3}{4}, -\tfrac{3\sqrt{3}}{16}\right)$$

    ② Second condition:
    A: $4x^3-3x^2=0$
    B: $2y = 0$
    C: $y^2+x^4-x^3=0$

A: $x = 0, \ 3/4$ B: $y = 0$

$(0, 0)$ works,

③ Third condition: no endpoints.

④ Evaluate:

Max of $\frac{3\sqrt{3}}{16}$ at $(3/4, \frac{3\sqrt{3}}{16})$

Min of $-\frac{3\sqrt{3}}{16}$ at $(3/4, -\frac{3\sqrt{3}}{16})$

- Ex:// R is region bounded by $x = \sqrt{1-y^2}$. Find max + min of $f(x,y) = x^2 - \frac{1}{2}x + y^2$ in R

① Critical points:

$$\nabla f(x,y) = (2x - \tfrac{1}{2}, 2y) = (0,0)$$
$$\therefore x = 1/4, y = 0$$

② Boundary:

Two boundaries, y axis + semi circle

On y-axis: $x = 0, -1 \le y \le 1$

$f(0,y) = y^2 \Rightarrow$ Max at $(0,1)$, min at $(0,-1)$

On semi-circle: $g(x,y) = x^2 + y^2 = 1 \quad (x \ge 0)$

① $\nabla f(a,b) = \lambda g(a,b) \wedge g(a,b) = 1$

$\nabla f(x,y) = (2x - \frac{1}{2}, 2y), \quad \nabla g(x,y) = (2x, 2y)$

$\therefore$ A: $2x - \frac{1}{2} = 2\lambda x$

B: $2y = 2\lambda y$

C: $x^2 + y^2 = 1$

From B: $\lambda = 1$ or $y = 0$

If $y = 0$:

$x = 1 \Rightarrow (1,0)$ is a point

If $\lambda = 1$: $\Rightarrow$ No points.

② $\nabla g(x,y) = (0,0) \wedge g(x,y) = 1$

$$\left.\begin{aligned} A&: 2x = 0 \\ B&: 2y = 0 \\ C&: x^2+y^2 = 1 \end{aligned}\right\} \text{No soln.}$$

③ Check end points:

$(0,1), \quad (0,-1)$

All critical points + evaluations:

$\boxed{f(1/4, 0) = -1/16}, \ \boxed{f(0,1) = 1}, \ f(1,0) = 1/2, \ \boxed{f(0,-1) = 1}$

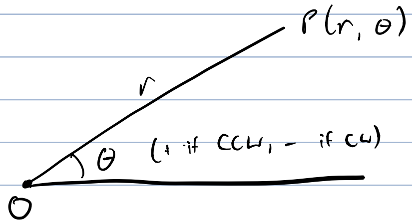
- Generalization:

Let $\vec{a}$ be critical points. Then:

1. $\nabla f(\vec{a}) = \lambda \nabla g(\vec{a}) \wedge g(\vec{a}) = k$
2. $g(\vec{a}) = 0 \wedge g(\vec{a}) = k$
3. Boundary points

Unit 11: Coordinate Systems

Polar Coordinates



- Relationship to Cartesian coordinates:

$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$

$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$

- Ex:// Convert the following to Cartesian

a) $(2, -\pi/3)$

$x = 2\cos(-\pi/3) = 1$

$y = 2\sin(-\pi/3) = -\sqrt{3}$

b) $(1, 3\pi/4)$

$x = \cos(3\pi/4) = -\sqrt{2}/2$

$y = \sin(3\pi/4) = \sqrt{2}/2$

- Ex:// Convert $(1,1)$ to polar:

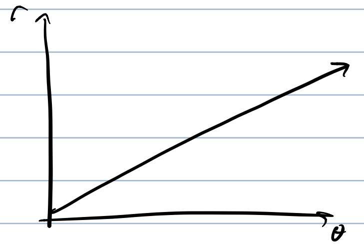
$r = \sqrt{2}$

$\theta = \tan^{-1}(1) = \pi/4 + 2\pi k$ ⇒ We know it has to be in 1st quadrant

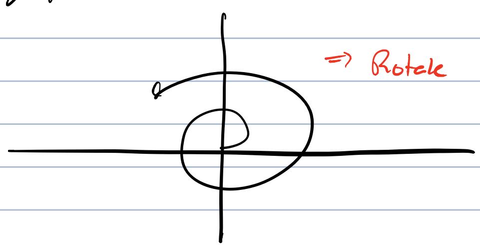
$\therefore (\sqrt{2}, \pi/4 + 2\pi k) \quad k \in \mathbb{Z}$

- Graphing polar functions
  - Best method: Draw r-θ graph → convert to x-y graph by considering all θ + r
  - Ex:// Graph $r = \frac{1}{2}\theta$

① : r-θ graph



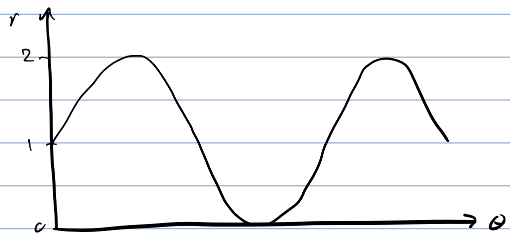
② x-y graph



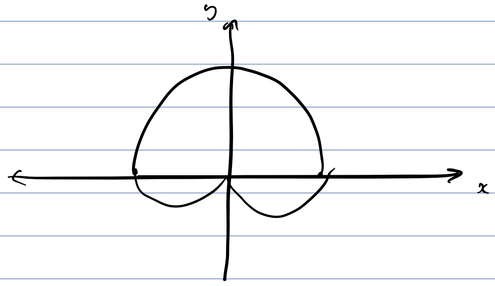
⇒ Rotate across all θ + plot r accordingly

- Ex:// Sketch $r = 1 + \sin\theta$

① r-θ graph



② x-y graph



- We can also convert polar equations into Cartesian equations using formulas.
  - Ex:// Graph $r = \cos\theta$ on Cartesian plane

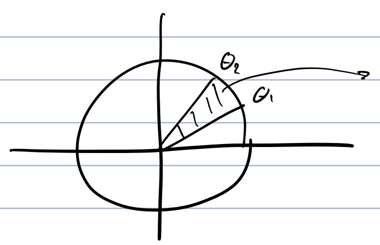
$r = \cos\theta$

$r^2 = r\cos\theta$

$x^2 + y^2 = x$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

- Area of sector:



$A = \frac{1}{2} r^2 (\theta_2 - \theta_1)$

Generalized:

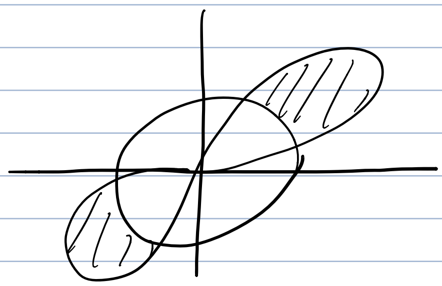
$$A = \int \frac{1}{2} f(\theta)^2 \, d\theta$$

- Area between curves in polar coordinate.
  - Algorithm:
    1. Find points of intersection
    2. Graph curves + split into integrable regions
    3. Integrate
  - Ex:// Find area inside $r = 2\sin(2\theta)$ but outside $r = 1$.

① Intersection points:

$$1 = 2\sin(2\theta)$$
$$2\theta = \pi/6, \ 5\pi/6$$
$$\theta = \pi/12, \ 5\pi/12$$

② Graphing:



$\Rightarrow$ Since symmetric, we can find area in Q1 and double.

③ Integrating:

Wrong: $\frac{1}{2}(2 \cdot \sin(2\theta))^2 - \frac{1}{2}(1)^2$

$$2 \int_{\pi/12}^{5\pi/12} \frac{1}{2} (2\sin(2\theta) - 1)^2 \, d\theta = \int_{\pi/12}^{5\pi/12} (4\sin^2(2\theta) - 4\sin(2\theta) + 1) \, d\theta$$

$$= \int_{\pi/12}^{5\pi/12} 4 \cdot \frac{1-\cos(4\theta)}{2} d\theta - 4 \int_{\pi/12}^{5\pi/12} \sin 2\theta + \frac{\pi}{3}$$

$$= 2 \left[\theta - \frac{\sin(4\theta)}{4} \right]_{\pi/12}^{5\pi/12} - 4 \left[\frac{-\cos(2\theta)}{2} \right]_{\pi/12}^{5\pi/12} + \frac{\pi}{3}$$

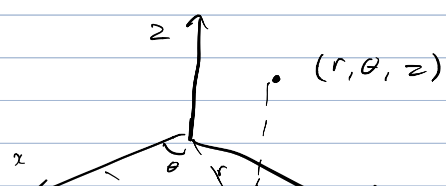
$$= 2 \left(\frac{5\pi}{12} - \frac{1}{8} - \frac{\pi}{12} + \frac{1}{8} \right) - 4 \left[\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \right] + \frac{\pi}{3}$$

$$= \frac{2\pi}{3} - 2\sqrt{3} + \frac{\pi}{3}$$

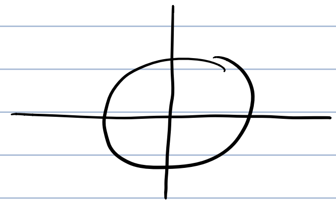
$$= \pi - 2\sqrt{3}$$

Cylindrical Coordinates

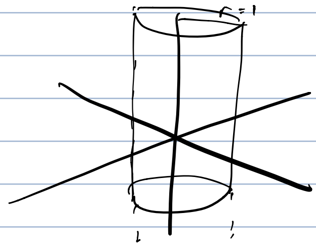
- Polar with z coordinate:



- Same graphing as polar
  - Ex:// Graph $r=1$
    1. Draw x-y graph cross-section



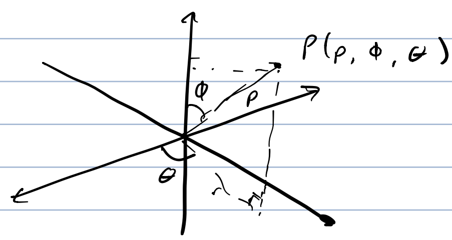
    2. Extend across z axis



  - To convert to x-y-z system, simply use polar formulas.
  - Ex:// Convert $z = r^2 \cos\theta$ to cartesian

$$z = r(r\cos\theta)$$
$$= x\sqrt{x^2+y^2}$$

Spherical Coordinates



- ϕ is the tilt from the z axis s.t. $0 \le \phi \le \pi$
- Equations:

$x = r\cos\theta$ $\qquad$ $z = \rho\cos\phi$

$= \rho\sin\phi\cos\theta$ $\qquad$ $\rho = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta$ $\qquad$ $\tan\theta = y/x$

$= \rho\sin\phi\sin\theta$ $\qquad$ $\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

- Ex:// Convert spherical coordinates to Cartesian

a) $(1, \pi/4, \pi/4)$

$x = 1\sin\pi/4 \cos\pi/4$ $\qquad$ $z = 1\cdot\cos\pi/4$

$= 1/2$ $\qquad$ $= \sqrt{2}/2$

$y = 1\sin\pi/4 \sin\pi/4$

$= 1/2$

$\therefore (1/2, 1/2, \sqrt{2}/2)$

b) $(1, \pi/4, \frac{5\pi}{4})$

$$\left.\begin{aligned} x &= \sin\pi/4 \cos 5\pi/4 = -1/2 \\ y &= \sin\pi/4 \sin 5\pi/4 = -1/2 \\ z &= \cos\pi/4 = \sqrt{2}/2 \end{aligned}\right\} \quad (-1/2, -1/2, \sqrt{2}/2)$$

- Ex:// Convert Cartesian to spherical
  a) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right)$

$$\rho = \sqrt{\frac{1}{2} + \frac{1}{2} + 3} = 2$$
$$\theta = \tan^{-1}(1) = \pi/4$$
$$\phi = \cos^{-1}\left(\sqrt{3}/2\right) = \pi/6$$
$$\left(2, \frac{\pi}{6}, \frac{\pi}{4}\right)$$

  b) $(-1, -1, -1)$

$$\rho = \sqrt{3}$$
$$\phi = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$
$$\theta = 5\pi/4$$ $\Rightarrow$ Look at x, y coordinate! In Q3 $\Rightarrow \theta = \frac{5\pi}{4}$

- Graphs:
  - Be able to convert spherical functions w/ Cartesian functions and v.v.
  - Let k be some constant:

$\rho = k \Rightarrow$ sphere of ~~radius~~ k
$\phi = k \Rightarrow$ cone w/ tilt of k

  - Ex:// Convert $x^2+y^2+z^2 = 2x$ to spherical coordinates given $-\pi/2 \le \theta \le \pi/2$

$$x^2+y^2+z^2 = 2x$$
$$\rho^2 = 2\rho \sin\phi\cos\theta$$
$$\rho = 2\sin\phi\cos\theta$$

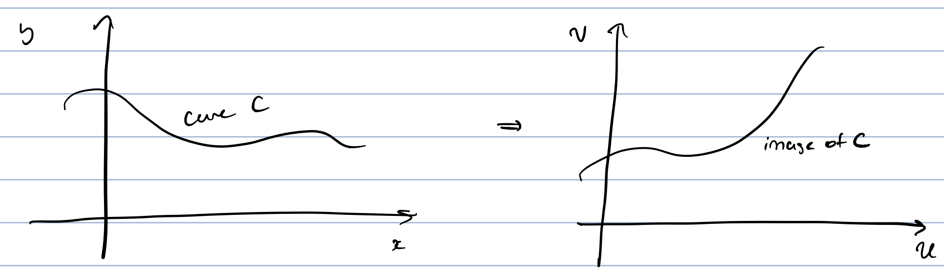
Unit 12: Mappings

Geometry of Mappings

- Vector-valued functions: $\mathbb{R}^n \to \mathbb{R}^m$
  - Ex:// $(x, y) = F(t) = (f(t), g(t)) \Rightarrow$ parametric
    F maps from $\mathbb{R}$ to $\mathbb{R}^2$
- Mapping: function that maps subset of $\mathbb{R}^n$ to subset of $\mathbb{R}^m$
- Consider:

$$\left.\begin{aligned} u &= f(x,y) \\ v &= g(x,y) \end{aligned}\right\}$$ component functions

$\therefore \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2_{(u,v)}$



  - Image is the resultant mapping from domain
- Ex:// Consider $(u, v) = F(x, y) = \left(\frac{1}{2}(x+y), \frac{1}{2}(-x+y)\right)$
  a) Find images of lines $x = k$ and $y = \ell$ under F
     Solve for x and y in terms of u and v

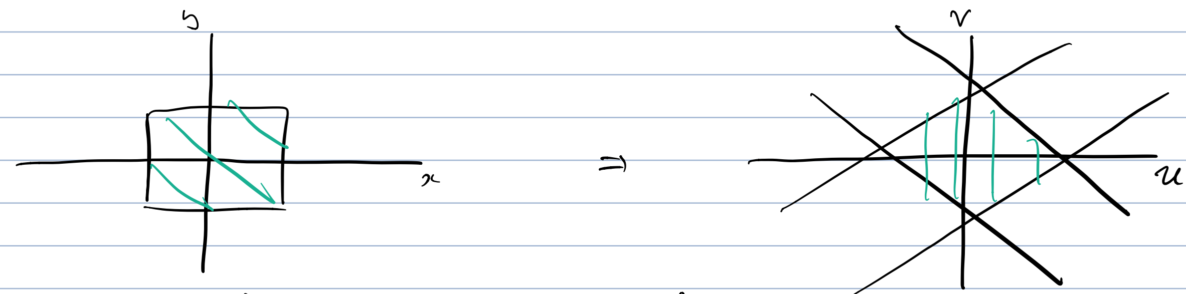
$$u = \tfrac{1}{2}(x+y) \qquad x = u - v$$
$$v = -\tfrac{1}{2}(-x+y) \Rightarrow \qquad y = u + v$$

$\therefore x = k \Rightarrow u - v = k$
$y = \ell \Rightarrow u + v = \ell$

  b) Find image of square $S = \{(x, y) \mid |x| \le 1, |y| \le 1\}$
     Essentially need to find mapping of $x = \pm 1, y = \pm 1$

$\therefore\ u - v = \pm 1, \quad v + V = \pm 1$

Graphically:



- Ex:// Find image of $D = \{(x,y) \mid -1 \le x \le 3,\ 0 \le y \le 2\}$ under mapping $(u,v) = T(x,y) = (x^2 - y^2, 2xy)$.

We need to find image of boundaries. Cannot solve for x and y easily so substitute:

1. $x = -1$ from $y = 0 \le y \le 2$

$$u = 1 - y^2$$
$$v = -y$$

Eliminating y:

$$u = 1 - v^2$$

Constraint of $0 \le y \le 2$ maps to $-2 \le v \le 0$

2. $x = 3,\ 0 \le y \le 2$

$$u = 9 - \tfrac{1}{4} v^2$$
$$v = 3y$$

Since $0 \le \frac{v}{3} \le 2 \Rightarrow 0 \le v \le 6$

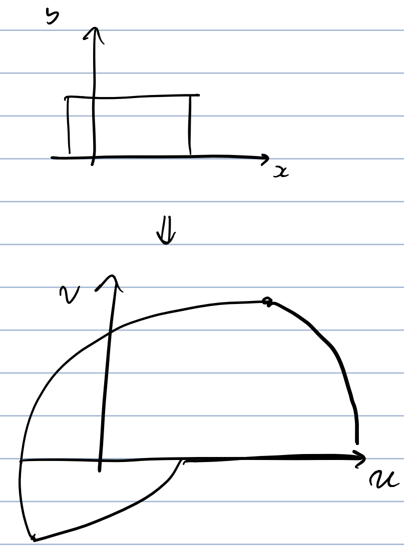
3. $y = 2,\ -1 \le x \le 3,$

$$u = \tfrac{1}{4} v^2 - 4$$
$$v = 2x \Rightarrow -2 \le v \le 6$$

4. $y = 0,\ -1 \le x \le 3$

$$v = 0$$
$$u = x^2$$

Graphically



- Ex:// Image of rectangle $R = \{(r,\theta) \mid 1 \le r \le 2,\ \pi/4 \le \theta \le 3\pi/4\}$ under Cartesian mappings.

Finding image of boundary lines.

1. $r = 1,\ \pi/4 \le \theta \le 3\pi/4$

$\therefore$ Unit circle from $\pi/4 \le \theta \le 3\pi/4$

2. $r = 2,\ \pi/4 \le \theta \le 3\pi/4$.

$\therefore$ Circle w/ radius $\sqrt{2}$ from $\pi/4 \le \theta \le 3\pi/4$

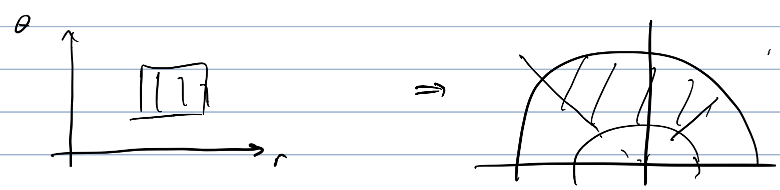
3. $\theta = \pi/4,\ 1 \le r \le 2$:

$$x = r\cos \pi/4 = \frac{1}{\sqrt{2}} r, \quad y = \frac{1}{\sqrt{2}} r$$
$$\therefore y = x$$

Since $r = \sqrt{2}\, x \Rightarrow 1 \le \sqrt{2}\, x \le 2 \Rightarrow \frac{1}{\sqrt{2}} \le x \le 2$

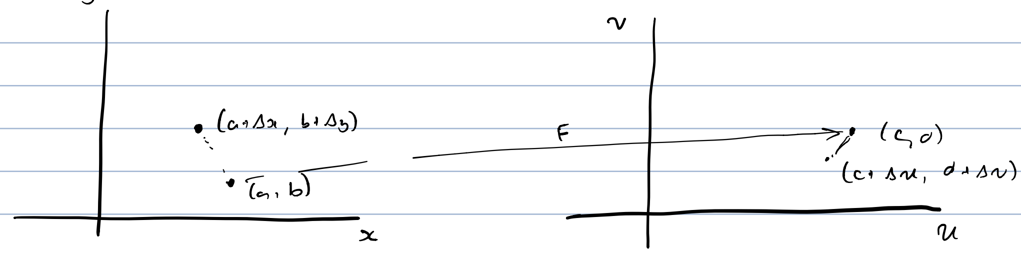
4. $\theta = 2\pi/4,\ 1 \le r \le 2$:

$$x = -1/\sqrt{2}\, r, \quad y = 1/\sqrt{2}\, r$$
$$\therefore y = -x$$
$$-\sqrt{2} \le x \le -1/\sqrt{2}$$



Linear Approximation of a Mappings

- Goal: approximate $(c + \Delta u, d + \Delta v)$ of point $(a + \Delta x, b + \Delta y)$ given $c = f(a,b),\ d = g(a,b)$



- We can say

$$\Delta u = \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y$$

$$\Delta v = \frac{\partial g}{\partial x}(a,b)\,\Delta x + \frac{\partial g}{\partial y}(a,b)\,\Delta y$$

- In matrix form:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{bmatrix}}_{\text{derivative matrix}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

- Ex:// Find derivative matrix of $(u,v) = F(x,y) = (x^2 \sin y,\ y^2 \cos x)$

$$DF(x,y) = \begin{bmatrix} 2x \sin y & x^2 \cos y \\ -y^2 \sin x & 2y \cos x \end{bmatrix}$$

- Increment form of linear approximation for mappings:

$$\Delta \vec{u} = DF(x,y)\,\Delta \vec{x} \quad \text{where } \Delta\vec{u} = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \text{ and } \Delta \vec{x} = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

- Linear approximation:

$$F(x,y) \approx F(a,b) + DF(a,b)\,\Delta\vec{x}$$

- Derivative matrix is mapping of $\Delta\vec{x}$ to $\Delta\vec{u}$
- Ex:// Consider $(u,v) = F(x,y) = \left(-x + \sqrt{x^2+y^2},\ x + \sqrt{x^2+y^2}\right)$. Find approximation of $(3.02, 3.99)$ under F.
  1. Find derivative matrix.

$$DF(x,y) = \begin{bmatrix} -1 + \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ 1 + \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

  2. Approximate using reference point

$$DF(3,4) = \begin{bmatrix} -2/5 & 4/5 \\ 8/5 & 4/5 \end{bmatrix}$$

  3. Displacement vector:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \simeq DF(3,4) \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -2/5 & 4/5 \\ 8/5 & 4/5 \end{bmatrix} \begin{bmatrix} 0.02 \\ -0.01 \end{bmatrix} = \begin{bmatrix} -0.016 \\ 0.024 \end{bmatrix}$$

  4. Approximate:

$$F(3.02, 3.99) \approx (2,8) + (-0.016,\ 0.024) = (1.984,\ 8.024)$$

Plug in reference point into mapping

Composite Mappings + Chain Rule

- Consider successive mappings:

$$F: \begin{cases} p = p(u,v) \\ q = q(u,v) \end{cases} \qquad G: \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

- Composite mapping:

$$F \circ G: \begin{cases} p = p(u(x,y),\ v(x,y)) \\ q = q(u(x,y),\ v(x,y)) \end{cases}$$

- Derivative matrix of composite:

$$D(F \circ G)(x,y) = DF\overbrace{(u,v)}^{G(x,y)}\,DG(x,y)$$

- Ex:// Consider following mappings:

$$(u,v) = G(x,y) = (xy,\ x+y)$$
$$(p,q) = F(u,v) = (u-v,\ u^2)$$

a) Find composite mappings $F \circ G$

$$(p,q) = F(G(x,y)) = (xy-(x+y),\ (xy)^2)$$

b) Find derivative matrices:

$$DG(x,y) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix} \quad DF(u,v) = \begin{bmatrix} 1 & -1 \\ 2u & 0 \end{bmatrix} \quad D(F \circ G) = \begin{bmatrix} y-1 & x-1 \\ 2xy^2 & 2x^2y \end{bmatrix}$$

Finding $D(F \circ G)(x,y)$ using matrix product

$$D(F \circ G) = DF(u,v)\, DG(x,y)$$
$$= \begin{bmatrix} 1 & -1 \\ 2u & 0 \end{bmatrix} \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} y-1 & x-1 \\ 2uy & 2ux \end{bmatrix}$$
$u = xy$ from $G(x,y)$
$$= \begin{bmatrix} y-1 & x-1 \\ 2xy^2 & 2x^2y \end{bmatrix}$$

UNIT 13: JACOBIANS + INVERSE MAPPINGS

Inverse Mapping Theorem

- We want to find the condition that will guarantee $(u,v) = F(x,y)$ has an inverse
  - If F maps from D_{xy} to D_{uv}, then F^{-1} exist if $(x,y) = F^{-1}(u,v) \Leftrightarrow (u,v) = F(x,y)$
  - This also means:
$$(F^{-1} \circ F)(x,y) = (x,y)$$
$$(F \circ F^{-1})(u,v) = (u,v)$$
  - Invertible function is like one-to-one function.
    Function is one-to-one if $F(a,b) = F(c,d) \Rightarrow (a,b) = (c,d)$
  - In fact: if F is one-to-one on D_{xy}, then F is invertibe on D_{xy}
    - In single variable $\Rightarrow f'(x) \neq 0$ to be one-to-one
  - Inverse of derivative matrix
    F maps from $D_{xy} \to D_{uv}$. If F has continuous partial derivatives at $\vec{x} \in D_{xy}$ and there exists inverse mapping F^{-1} which has 〃 at $\vec{u} \in D_{uv}$
    then: $DF^{-1}(u)\, DF(\vec{x}) = I$ $\Rightarrow$ If F is invertible, so is DF
  - Ex:// Consider $(u,v) = F(x,y) = (y+x^2,\ x)$. Solve for F^{-1}, find DF and DF^{-1} and verify that $DF^{-1}(u,v)$ is matrix inverse of $DF(x,y)$.

① Inverse of F: solve for x, y
$$u = y + x^2$$
$$v = x$$
$$\therefore x = v \Rightarrow y = u - v^2$$
$$\therefore (x,y) = F^{-1}(u,v) = (v,\ u-v^2)$$

② Derivative matrices:
$$DF = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \quad DF^{-1} \begin{bmatrix} 0 & 1 \\ 1 & -2v \end{bmatrix}$$

③ Verify:
$$DF\, DF^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2v \end{bmatrix} \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2x-2v & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$v = x$ sub.

- Jacobian:

Let mapping be $(u,v) = F(x,y) = (u(x,y), v(x,y))$. Jacobian denoted $\frac{\partial(u,v)}{\partial(x,y)}$ is

$$\frac{\partial(u,v)}{\partial(x,y)} = \det[DF(x,y)] = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

- Ex:// Calculate Jacobian of:

$$(x,y) = F(r,\theta) = (r\cos\theta, r\sin\theta)$$

① DF:

$$DF = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

② Jacobian

$$\det(DF) = r\cos^2\theta + r\sin^2\theta = r$$

- We know from linear algebra that matrix invertible if determinant is non-zero
- Corollary: F invertible $\Rightarrow$ Jacobian is non zero (converse NOT true)
- Corollary: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$ if F^{-1} exists

- Inverse mapping theorem:
  If mapping $(u,v) = F(x,y)$ has continuous partial derivatives in a neighborhood of (a,b) and Jacobian of F is non-zero, then F^{-1} exists in neighborhood provided F is one-to-one as well
- Ex:// Consider $(u,v) = F(x,y) = (xy - x^2, x+y)$. Show that F has an inverse in neighborhood of $(1,-2)$

① DF

$$DF = \begin{bmatrix} y-2x & x \\ 1 & 1 \end{bmatrix}$$

② Jacobian

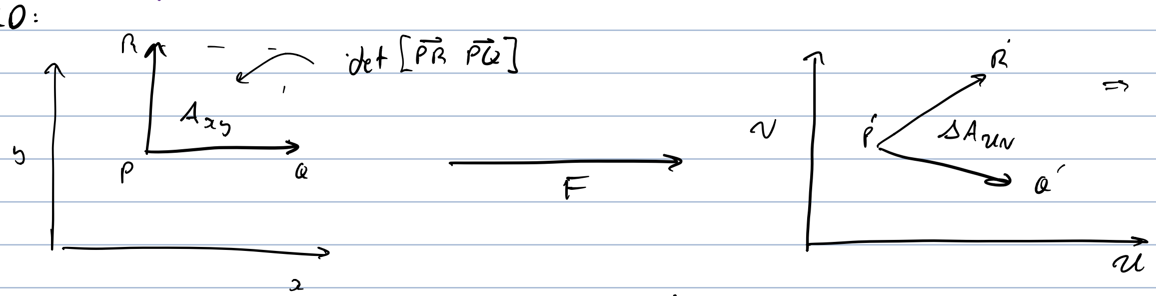
$$\det(DF) = y - 3x$$

③ Evaluate at point

$$\frac{\partial(u,v)}{\partial(x,y)}(1,-2) = 1 > 0 \Rightarrow \text{since continuous derivatives, neighborhood has inverse}$$

Geometric Intrepretation of the Jacobian

- In 2D:



$\Rightarrow$ $P'R' \approx DF \cdot \begin{bmatrix} 0 \\ \Delta y \end{bmatrix}$

$P'Q' \approx DF \cdot \begin{bmatrix} \Delta x \\ 0 \end{bmatrix}$

$\Downarrow$

$\Delta A_{uv} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \Delta A_{xy}$

- Like a scaling factor of areas defined by vectors mapped by F
- Approximation unless linear mapping, where it is exact
- Ex:// Calculate area of rectangle $\Delta x \Delta y$ at $(3,4)$ under $F \Rightarrow (u,v) = F(x,y) = (-x + \sqrt{x^2+y^2},\ x + \sqrt{x^2+y^2})$

① Find DF

$$DF(3,4) = \begin{bmatrix} -2/5 & 4/5 \\ 8/5 & 4/5 \end{bmatrix}$$

② Get Jacobian

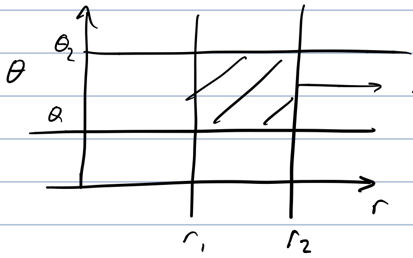
$$\frac{\partial(u,v)}{\partial(x,y)} = \det(DF(3,4)) = -8/5$$

④ Approximate area:

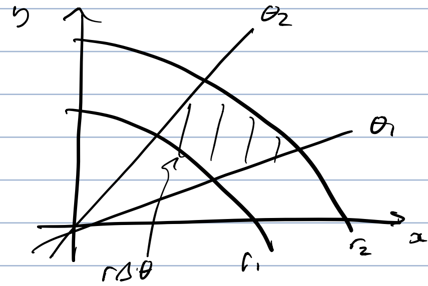
$$\Delta A_{uv} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \Delta A_{xy} = 8/5\ \Delta x \Delta y$$

- Ex:// $(x,y) = F(r,\theta) = (r\cos\theta, r\sin\theta)$. Find image of rectangle in $r\theta$ plane + verify Jacobian mag.

① Use ECE 106 tricks to find area in r-θ plane



$\Rightarrow$



Area in $r-\theta$: $\Delta r \Delta \theta$

Area in $x-y$: $r\Delta\theta \cdot \Delta r = r A_{r\theta}$

② Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)} = r \Rightarrow \text{Scale of } r$$

- In 3D:

$$\Delta V_{uvw} = \left| \frac{\partial(u,v,w)}{\partial(x,y,z)} \right| \Delta V_{xyz}$$

- You can generalize Jacobian + inverse mapping to n dimensions

Constructing Mappings

- Ex:// Find mapping F to tranform parallelogram D_{xy} with $(0,0)$, $(2,1)$, $(3,4)$ and $(1,3)$ into unit square $0 \le u \le 1$ ^ $0 \le v \le 1$ on (u,v) plane.
  ① Find equations of boundary line on x-y plane + u,v

$$\left.\begin{array}{l} 2y - x = 0 \\ 2y - x = 5 \\ 3x - y = 0 \\ 3x - y = 5 \end{array}\right\} \xrightarrow{\ F\ } \left\{\begin{array}{ll} u = 0 & v = 0 \\ u = 1 & v = 1 \end{array}\right.$$

② Construct pairs $\rightarrow F$

$$\begin{array}{l} 2y - x = 0 \\ 2y - x = 5 \end{array} + \begin{array}{l} u = 0 \\ u = 1 \end{array} \Rightarrow u = \frac{2y-x}{5}$$

$$\begin{array}{l} 3x - y = 0 \\ 3x - y = 5 \end{array} + \begin{array}{l} v = 0 \\ v = 1 \end{array} = v = \frac{3x-y}{5}$$

$$F\left(\frac{2y-x}{5}, \frac{3x-y}{5}\right) \Rightarrow \text{Not unique.}$$

Calculate Jacobian:

$$(u,v) = F(x,y) = \left(\frac{2y-x}{5}, \frac{3x-y}{5}\right)$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = -1/5$$

$$\therefore A_{uv} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| A_{xy}$$

$$A_{uv} = \frac{1}{5} A_{xy} \Rightarrow A_{xy} = 5 A_{uv} = 5$$

- Basically need to see how x-y equations can turn into u-v equations.
- Ex:// Find invertible mapping to transform $xy = 1$, $xy = 3$, $x^2 - y^2 = 2$, $x^2 - y^2 = 4$ into square.
  Let $u = xy$, $v = x^2 - y^2$

$$\therefore (u,v) = F(x,y) = (xy, x^2 - y^2)$$

Proving that it is invertible.

Cannot solve for inverse but can use Inverse Mapping Theorem

$$\det DF(x,y) = \det \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix} = -2y^2 - 2x^2$$

Since this is non-zero on region D_{xy} + F has continuous partial deriv., F is invertible in neighborhood

for every point in D_{xy} by Inverse Mapping Theorem
- Just because invertible in neighborhood of each point $\not\Rightarrow$ invertible in entire region

Unit 14: Double Integrals

Definition of Double Integrals

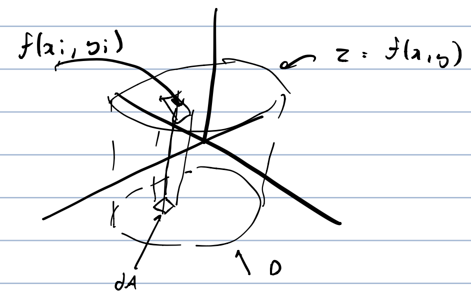
- Reimann sum for volumes:

$$\sum_{i=1}^{n} f(x_i, y_i)\, \Delta x_i\, \Delta y_i$$

- If all Reimann sums for a partition approach the same value, then the function is integrable
- If integrable:

$$\iint_D f(x,y)\, dA = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

- All you really need to determine is whether the function is continuous in the region D
  - If f is piece-wise continuous, continuous on D, but not on boundary, its still integrable
- Interpretations:
  - Area: $f(x,y) = 1$ for all $(x,y) \in D$. Thus, double integral calculates area
  - Volume: If $f(x,y) \geq 0 \Rightarrow$ calculating volum



  - Mass: If $f(x_i, y_i)$ is area density, then double integral can represent mass of entire object
  - Probability:

$$P((X,Y) \in D) = \iint f(x,y)\, dA$$

  - Average value:

$$f_{av} = \frac{1}{A(D)} \iint_D f(x,y)\, dA$$

- Properties:
  1. Linearity: $D \subset \mathbb{R}^2$ is closed + bounded, f and g are both integrable, then for any constant c

$$\iint_D (f+g)\, dA = \iint_D f\, dA + \iint_D g\, dA$$

$$\iint_D cf\, dA = c \iint_D f\, dA$$

  2. Inequality: $\forall (x,y) \in D,\ f(x,y) \leq g(x,y)$

$$\iint_D f(x,y)\, dA \leq \iint_D g(x,y)\, dA$$

  3. Absolute value:

$$\left| \iint_D f\, dA \right| \leq \iint |f|\, dA$$

  4. Decomposition: If $D_1 + D_2 = D$ via some smooth curve C

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$

Iterated Integrals

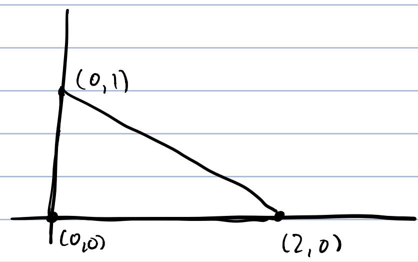
- Idea: write double integrals as succession of single integrals

$$\iint_D f(x,y)\,dA = \int_{x_l}^{x_u} \left(\int_{y_l(x)}^{y_u(x)} f(x,y)\,dy \right) dx$$

- Inner integral must be partial integration ⇒ only integrating 1 variable

- Ex:// Find $\iint_D xy\,dA$ where D is triangular region from (0,0), (2,0) and (0,1)

① Sketch



② Describe via inequalities of variables.

$$\left.\begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le -\frac{1}{2}x + 1 \end{array}\right\} \text{Description of whole set } D$$

1 variable should be bounded by constants, other variable bounded by function of other var

③ Set up integral

$$\iint_D xy\,dA = \int_0^2 \int_0^{1-\frac{1}{2}x} xy\,dy\,dx = \int_0^2 \left[\frac{xy^2}{2}\right]_0^{1-\frac{1}{2}x} dx$$

$$= \frac{1}{2}\int_0^2 x\left(1-\tfrac{1}{2}x\right)^2 dx$$

$$= \frac{1}{6}$$

- Ex:// Doing same problem but opposite constraints

$$D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le -2y+2\}$$

$$\therefore \iint_D xy\,dA = \int_0^1 \int_0^{-2y+2} xy\,dx\,dy = 2\int_0^1 y(1-y)^2\,dy = \frac{1}{6}$$

- Ex:// Let D be bounded by $y=0$, $x=1$ and $y=x$. Find $\iint_D e^{x^2}dA$

Note that integrating with respect to x first is not a good idea! Be strategic about order of integration

$$\iint_D e^{x^2}dA = \int_0^1\int_0^x e^{x^2}dy\,dx = \int_0^1 xe^{x^2}dx = \frac{1}{2}(e-1)$$

- Ex:// Find volume of first octant bounded by cylinder $y^2+z^2=16$ and planes $3y-2x=0$, $x=0$, $z=0$
Solid can be described as:

$$0 \le z \le \sqrt{16-y^2} \quad \text{and } (x,y) \in D$$

Defining D:

We want to integrate with respect to x first. Integrate y with constants for ease

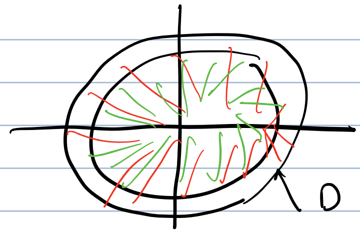
$\therefore\ D = \{(x,y) \mid 0 \le x \le \frac{3}{2}y,\ 0 \le y \le 4\}$ → We know this from volume of solid!

Integral:

$$\iint_D \sqrt{16-y^2}\, dA = \int_0^4\int_0^{3y/2} \sqrt{16-y^2}\, dx\, dy = \int_0^4 \frac{3y}{2}\sqrt{16-y^2}\, dy = -\frac{1}{2}(16-y^2)^{3/2}\Big|_0^4$$

$$= 32$$

- Use decomposed integrals for complex shapes that cannot be described via inequalities.



$$\Rightarrow \iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA - \iint_{D_2} f(x,y)\, dA$$

Change of Variable Theorem

- If integrating shape is too difficult, construct mapping $(u,v) = F(x,y)$
- This requires us to replace dA from $dx\,dy$ to $du\,dv$
  - This conversion is the Jacobian!
- Theorem:

$$\iint_{D_{xy}} G(x,y)\, dx\, dy = \iint_{D_{uv}} G(f(u,v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

- Ex:// Evaluate $\iint_{D_{xy}} (x+y)\, dA$ where D_{xy} is set bounded by parallelogram with vertices $(0,0)$, $(2,1)$, $(1,3)$ and $(3,4)$

① Mapping

$$(u,v) = F(x,y) = \left(\tfrac{1}{5}(2y-x),\ \tfrac{1}{5}(3x-y)\right) \Rightarrow \text{maps to unit square}$$

② Jacobian

inverse mapping!

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{1}{5} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -5$$

③ Convert integral:

$$\iint_{D_{xy}} (x+y)\, dx\, dy = \int_0^1\int_0^1 (4u+3v)\,|-5|\, du\, dv = \frac{35}{2}$$

- Ex:// Use mapping $(u,v) = F(x,y) = (x+y, -x+y)$ to evaluate $\int_0^{\pi}\int_0^{\pi-y} (x+y)\cos(x-y)\, dx\, dy$

① Finding new region:

$x=0,\ 0 \le y \le \pi \Rightarrow v = u,\ 0 \le u \le \pi$

$y=0,\ 0 \le x \le \pi \Rightarrow v = -u,\ 0 \le u \le \pi$

$x+y=\pi,\ 0 \le x \le \pi \Rightarrow u = \pi,\ u-v = 2x \Rightarrow -\pi \le v \le \pi$

② Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

③ Integral conversion + evaluate

$$\int_0^{\pi}\int_0^{\pi-y} (x+y)\cos(x-y)\, dx\, dy = \int_0^{\pi}\int_{-u}^{u} u\cos(-v)\cdot \tfrac{1}{2}\cdot dv\, du$$

- Cartesian to polar: $dxdy = r\,dr\,d\theta$
  - Ex:// Evaluate $\iint_{D_{xy}} \frac{x}{x^2+y^2}\,dA$ where D_{xy} is half disc $(x-1)^2+y^2 \leq 1$, $x \geq 1$

① Convert to cartesian:

$$x = 1 \Rightarrow r\cos\theta = 1$$
$$r = \sec\theta$$
$$x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos\theta$$
$$r = 2\cos\theta$$

Intersection:

$$\frac{1}{\cos\theta} = 2\cos\theta$$
$$\frac{1}{2} = \cos^2\theta$$
$$\theta = \pm \pi/4$$

② Convert integral

$$\iint_{D_{xy}} \frac{x}{x^2+y^2}\,dx\,dy = \iint \frac{r\cos\theta}{r^2}\,r\,dr\,d\theta = \iint \cos\theta\,dr\,d\theta$$

$$\therefore \int_{-\pi/4}^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \cos\theta\,dr\,d\theta = 1$$

- If integral cannot be written on set D, use change of variables or decomposition
- If integral can be written but stuck: change order of integration, rewrite D, use change of vars