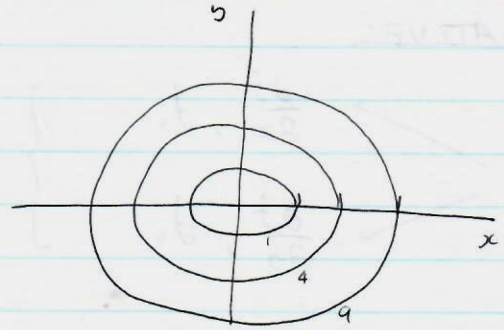
MATH 119 FEYNMAN

CONTOUR PLOTS

$f(x) = x^2 + y^2$

1. Choose a certain output (0, 1, 2 . . .)
2. Graph onto 2D plot
3. Repeat values:

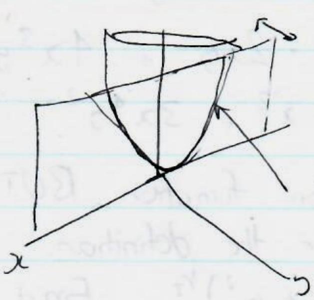


⇒ Stacking multiple graphs! Visualize taking plot + moving up + down.

Tips + tricks:

1. Steepness: closer the lines, steeper the graph (didn't need to move much to get to next plot)
2. Cross-sections: $z-x$, $z-y$

   Set x/y at a specific value. This is creating a plane w/ a cross-section of graph



cross-section at $y=0$!

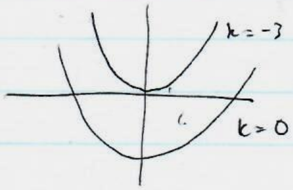
3. Level curve graphing: let output be some constant, then manipulate into graphable format.

- Ex: // $z = x^2 + y - 3$.

Contour plot:

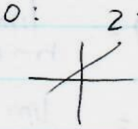
$x^2 + y - 3 = k$

$y = x^2 - (k+3)$

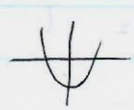


Cross-sections:

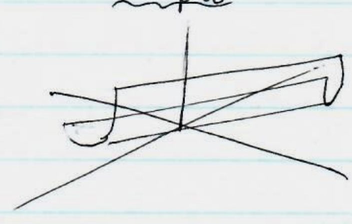
$x=0$: $z = y - 3$

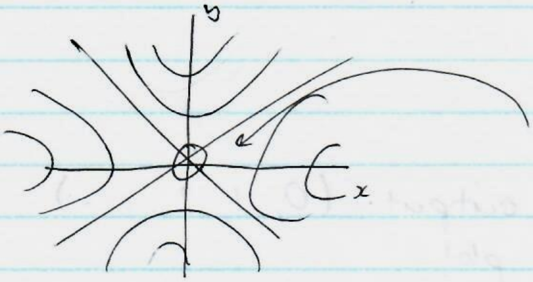


$y=0$: $z = x^2 - 3$



3D plot:





Saddle point: graph changing from increasing ↔ decreasing.

PARTIAL DERIVATIVES

$$f(x,y) \to \frac{\partial f}{\partial x},\ f_x \quad ; \quad \frac{\partial f}{\partial y},\ f_y$$

} Treat other variables as constants.

Definition:

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,\ b) - f(a,\ b)}{h} \to \text{Moving in } + \ x \text{ axis}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a,\ b+h) - f(a,\ b)}{h} \to \text{ '' in } + \ y \text{ axis.}$$

Ex://

$$f(x,y) = \sin(x^2 y) + x^4 y^3$$

$$f_x = \cos(x^2 y)\cdot 2xy + 4x^3 y^3$$

$$f_y = \cos(x^2 y)\cdot x^2 + 3x^4 y^2$$

Sometimes, point exists on function BUT you create discontinuity by taking derivative. Use the definition.

- Ex:// $f(x,y) = (x^3+y^3)^{1/3}$ Find $f_y(0,0)$.

$$f_y = \tfrac{1}{3}(x^3+y^3)^{-2/3}\cdot 3y^2$$

$$f_y(0,0) = 0/0 \Rightarrow \lim f_y \to (0,0) \text{ is } \underline{\text{HARD}}$$

At definition:

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,\ 0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{(h^3)^{1/3}}{h}$$

$$= 1$$

- Higher order derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{YY} \quad \Bigg\} \quad \underset{\text{forwards}}{f_{xy}} = \frac{\partial}{\partial y}\underset{\text{backwards}}{\left(\frac{\partial f}{\partial x}\right)} = \frac{\partial^2 f}{\partial y \partial x}$$

Clairaut's Theorem:

f_x, f_y, f_{xy} exist around (a,b) + f_{xy} is cts. at (a,b),

then $f_{xy} = f_{yx}$

- Simplify problems that have complex higher order.

Ex:// $f(x,y) = \tan y - xye^{y^2}$. Find f_{yyxxx}

Naive: $f_y \to f_y \to f_x \ldots$

Clairut Theorem: Remove terms as quickly ASAP (linear terms ... exp., trig)

constants

$f_x = ye^{y^2}$

$f_{xx} = 0 \Rightarrow f_{yyxxx} = 0$

TANGENT PLANES + DIFFERENTIALS

Tangent plane at (a,b): $z = f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b)$

1. Evaluate $f(a,b)$
2. Find f_x, f_y $\Rightarrow$ evaluate at (a,b)
3. Combine.

Applications → Approximation, Differentials.

Approximation:

1. Find your function that you are to approximate.

   Ex:// $\frac{4.02 \cdot 3.96^2}{\sqrt{3.96}} \Rightarrow \frac{xy^2}{\sqrt{y}}$ (Look at if # are same + operations on #)

2. Create tangent plane at point P

   $P \Rightarrow$ easy to compute + close to approx.

3. Plug in point to original numbers $\Rightarrow$ approximation.

Ex:// Estimate $\frac{4.02}{\sqrt{3.96}}$.

1. Function: $f(x,y) = \frac{x}{\sqrt{y}}$

2. Tangent plane:

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{y}}, \quad \frac{\partial f}{\partial y} = -\frac{x}{2y\sqrt{y}}$$

At $(4,4)$:

$$f(x,y) = 2 + \tfrac{1}{2}(x-4) - \tfrac{1}{4}(y-4)$$

3. Plug in:

$$f(4.02, 3.96) = 2 + \tfrac{1}{2}(4.02-4) - \tfrac{1}{4}(3.96-4)$$
$$= 2.02$$

Differentials:

$$\Delta f = f_x(x-a) + f_y(y-b)$$
$$\downarrow \quad \to 0$$
$$df = f_x \cdot dx + f_y \cdot dy$$

1. Relationships between rate of change:

Ex: Cylinder w/ $h=5$, $r=1$. Which dimension change has more of an impact on volume:

$$V = \pi r^2 h$$

$$dV = \left.\frac{\partial f}{\partial r}\right|_{h=5,\, r=1} dr + \left.\frac{\partial f}{\partial h}\right|_{h=5,\, r=1} dh$$

$$= 10\pi\, dr + \pi\, dh$$

$\therefore$ dr has $10\times$ impact vs. dh.

2. % change: % change $= \boxed{\frac{dx}{x}}$

- Absolute change: $\left|\frac{dx}{x}\right| \Rightarrow$ Triangle ineq.

Ex:// $T = 2\pi\sqrt{L/g}$ Absolute error of $L = 0.5\%$, $g = 0.1\%$.

Find % error of T.

$$\left|\frac{dL}{L}\right| \le 0.005, \quad \left|\frac{dg}{g}\right| \le 0.001$$

Find $\left|\frac{dT}{T}\right|$

1. Create differentials:

$$dT = \frac{\partial T}{\partial L} \cdot dL + \frac{\partial T}{\partial g} \cdot dg$$

$$dT = \frac{\pi}{\sqrt{Lg}} \cdot dL + - \frac{\pi\sqrt{L}}{g\sqrt{g}} dg$$

2. Put it in terms of $\frac{dT}{T}$

$$\frac{dT}{T} = \left(\frac{\pi}{\sqrt{Lg}} \cdot \frac{\sqrt{g}}{\sqrt{L} \cdot 2\pi}\right) dL - \left(\frac{\pi\sqrt{L}}{g\sqrt{g}} \cdot \frac{\sqrt{g}}{2\pi\sqrt{L}}\right) dg$$

$$\frac{dT}{T} = \frac{1}{2}\frac{dL}{L} - \frac{1}{2}\frac{dg}{g}$$

3. Δ Ineq:

$$\left|\frac{dT}{T}\right| \leq \frac{1}{2}\left|\frac{dL}{L}\right| + \frac{1}{2}\left|\frac{dg}{g}\right|$$

$$\leq 0.003$$

VECTORS + PARAMETERIZATION

$$(x, y) = \begin{cases} x = f_1(t) \\ y = f_2(t) \end{cases} \Rightarrow$$ Using another variable (param) to define (x, y)

Simple example: Circle:

$(x, y) \Rightarrow x = \cos\theta$, $y = \sin\theta$ (θ is param)

Power of parameterization: can parameterize any path

$$\vec{r}(t) = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix} \begin{array}{l} \rightarrow x \text{ component} \\ \rightarrow y \text{ component} \end{array}$$

Deriv

$$\vec{r}\,'(t) = \begin{bmatrix} g'(t) \\ h'(t) \end{bmatrix}$$

Magnitude: $\|\vec{r}(t)\| = \sqrt{g^2(t) + h^2(t)}$

Parameterize a curve:

1. Draw the graph

2. If we have a circle/ellipse:

$$\vec{r}(t) = \begin{bmatrix} \alpha \cos\omega t \\ \beta \cos\omega t \end{bmatrix} \Rightarrow \text{General form.}$$

Determine α and ω; check with $x^2+y^2=1$...

α: $\underline{\text{Max / min}}$, $\quad \omega = \frac{2\pi}{\text{period}}$ ⇐ Changes speed of motion, not graph.

Length of 1 cycle.

3. If simple function / not multivariable:

Set one variable as parameter + substitute.

Ex: $y = x^2 \Rightarrow y = t^2,\ x = t$

4. Line segments.

Param. line segment $(2,3) \to (4,7)$

a) Direction vector:

$$\vec{r}\,'(t) = \begin{bmatrix} 4-2 \\ 7-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

b) Find antiderivative:

$$\vec{r}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

c) Plug in $t=0 \Rightarrow$ determine $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ from init

$$\therefore \vec{r}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore \vec{r}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

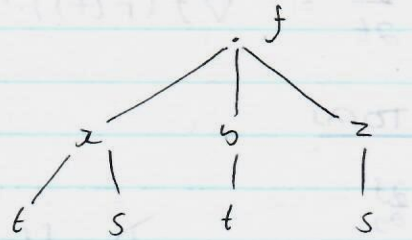
5. General trick: find equation that graph satisfies.

- Ex: Hyperbola: $x = \cosh(t),\ y = \sinh(t)$

CHAIN RULE

1. Tree diagram: variables relations:

$$f(x, y, z) = f(x(t, s), y(t), z(s))$$



2. Create partial derivatives at 2^{nd} level

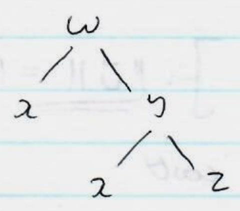
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \_\_\_ + \frac{\partial f}{\partial y} \cdot \_\_\_ + \frac{\partial f}{\partial z} \cdot \_\_\_$$

3. Create derivatives for 3^{rd} level... Trick: if one-to-one, use simple derivatives.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

4. Evaluate:
   - If you still have combo of 2^{nd} level variables ⇒ make it a function of 3^{rd} level (definition of function)
   - Evaluate 3^{rd} level derivatives at point.

Ex:// $w = x^3 + 2y$, $y = x \sin z$. Find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial z}$



1. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x}$

base var

$= 3x^2 + 2 \cdot \sin z$

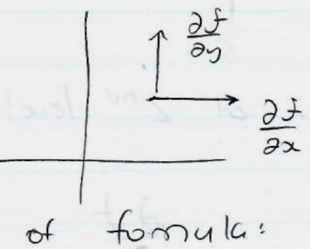
2. $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial z} = 2 \cdot x \cos z$

GRADIENT VECTOR

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Chain rule: $\dfrac{\partial f}{\partial t} = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

DIRECTIONAL VECTORS



$$D_{\vec{u}}\, f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(\ldots)}{h}$$

Derivation of formula:

$$g(h) = f(a + hu_1, b + hu_2)$$

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

$$= D_{\vec{u}}\, f(a, b)$$

Finding $g'(h)$:

$$\frac{dg}{dh} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial h}$$

$\leftarrow g(x, y)$ where $x = a + hu_1$, $y = b + hu_2$

$$= \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2$$

$$= \nabla f \cdot \vec{u}$$

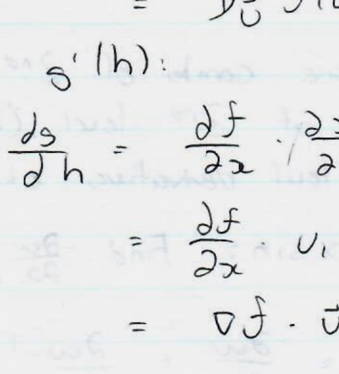
$$\therefore \; D_{\vec{u}}\, f(a, b) = \nabla f \cdot \vec{u} \quad \Big] \; \underline{\underline{\|\vec{u}\| = 1}}$$

$$= \|\nabla f\| \cdot \|\vec{u}\| \cos\theta$$

Maximum is when $\cos\theta = 1 \Rightarrow \theta = 0$

$\therefore$ ∇f is the <u>maximum</u> derivative!

$\hookrightarrow$ $\nabla f \perp$ contour.



Ex:// $f(x,y) = x^2 + y^2$. Find slope at $(1,-1)$ in direction of $\vec{v} = \langle 3, 4 \rangle$

$$D_{\vec{u}} f(1,-1) = \nabla f \cdot \vec{u}$$

$\swarrow$ Normalized

$$= \langle 2, -2 \rangle \cdot \tfrac{1}{5} \langle 3, 4 \rangle$$

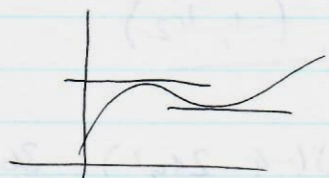
$$= -2/5$$

Asking directional deriv. q's:

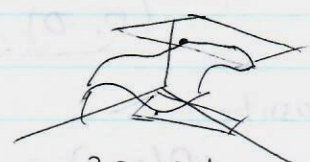
1. Vector: convert to normalized $\Rightarrow$ calc.
2. Point: make own vector
3. Angle: $D_{\vec{u}} f(a,b) = \| \nabla f \| \cdot \cos\theta$

UNCONSTRAINED OPTIMIZATION

We need to find points where tangent plane is horizontal



1D calc



2D calc

1. Find critical points:

$$\nabla f = 0$$

$\therefore$ $f_x = 0$, $f_y = 0$ AT THE SAME TIME

2. Classify extrema:

$D(x,y)$: $>0 \rightarrow$ Extrema; $<0 \rightarrow$ Saddle Point; $=0 \rightarrow$ Undetermined

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$$ $\leftarrow$ Evaluate partial @ critical pt

3. Local min/max: 2^{nd} derivative test.

$f_{xx} / f_{yy} > 0 \Rightarrow$ minimum

$f_{xx} / f_{yy} < 0 \Rightarrow$ maximum.

What happens if $D = 0$:

- Look at derivatives/second derivatives.
- Graphs (contour plot, cross-sections)

-Ex:// Find max + min of $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2y$

1. Find critical points:

$$f_x = 3x^2 + 6xy = 0 \Rightarrow 3x(x+2y) = 0 \Rightarrow x=0,\ x=-2y$$
$$f_y = -4y - 8y^3 + 3x^2 = 0$$

If $x = 0$:

$$\left.\begin{aligned} f_y: \quad -4y - 8y^3 &= 0 \\ 2y^3 + y &= 0 \\ y(2y^2+1) &= 0 \\ \therefore y &= 0 \end{aligned}\right\} (0,0)$$

If $x = -2y$:

$$f_y = -4y - 8y^3 + 12y^2 = 0$$
$$y = \tfrac{1}{2}, 1, 0$$
$$\therefore \underline{(0,0),\ (-2,1),\ (-1,\tfrac{1}{2})}$$

2. Classify:

$$D(x,y) = (6x+6y)(-4-24y^2) - 36x^2$$

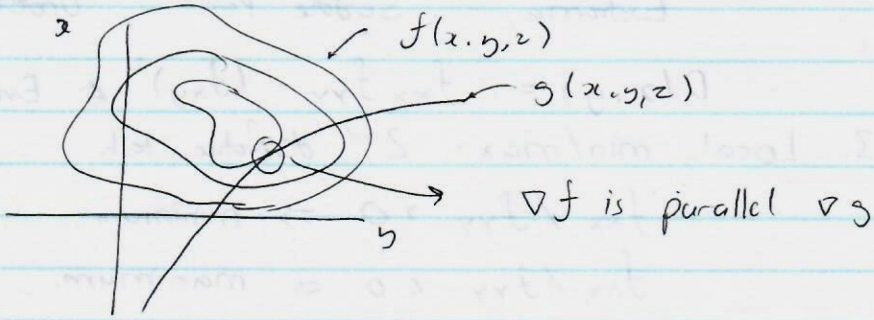
$D(0,0) = 0 \Rightarrow$ Cross-section @ $y=0 \Rightarrow f(x,y) = x^3$. No extremum

$D(-2,1) = 24 \Rightarrow$ Local max b/c $f_{xx} < 0$

$D(-1,\tfrac{1}{2}) = -6 \Rightarrow$ Saddle point.

CONSTRAINED OPTIMIZATION

$f(x,y,z)$ subjected to constrainted $g(x,y,z)$



$\therefore$ 2 equations to consider:

①: $\nabla f = \lambda \nabla g \Leftarrow \boxed{\nabla g = k} \rightarrow$ Rewrite into this form.

②: $\nabla g = 0 \Rightarrow$ On a critical point of f

We need to solve ① and ② to find critical points.
↳ Solve for $\lambda \Rightarrow x$ and y.
Check if critical points are local max/min/ saddle point.

Ex: // Rectangular open box w/ $V = 4m^3$. Find dimensions s.t. it requires least amount of material.

1. Functions:

$$f(x,y,z) = 2xz + 2yz + xy \Rightarrow \text{Optimize this}$$
$$g(x,y,z) = xyz = 4$$

2. Equations:

$$\left.\begin{aligned}
&①: \quad 2z + y = \lambda yz \Rightarrow \lambda = \frac{2}{y} + \frac{1}{z} \\
&②: \quad 2z + x = \lambda xz \Rightarrow \lambda = \frac{2}{x} + \frac{1}{z} \\
&③: \quad 2x + 2y = \lambda xy \Rightarrow \lambda = \frac{2}{y} + \frac{1}{x} \\
&④: \quad xyz = 4.
\end{aligned}\right\} \begin{aligned}&\nabla g = 0 \text{ would} \\ &\text{have violated} \\ &\text{conditions.}\end{aligned}$$

$$\therefore \lambda = \frac{2}{y} + \frac{1}{z} = \frac{2}{x} + \frac{1}{z} = \frac{2}{y} + \frac{R}{x}$$

$$\therefore x = y, \quad x = 2z$$

$$④: \quad x \cdot x \cdot \frac{x}{2} = 4$$
$$x^3 = 8$$
$$\boxed{x = 2 \Rightarrow y = 2, \ z = 1}$$

MULTIVARIABLE INTEGRATION

Simple antidifferentiating: $f_{xy} = f_{yx}$

General recipie: we are given $f_x(x,y)$, $f_y(x,y)$. To find $f(x)$:

①: Choose f_x / f_y + antidifferentiat w/ respect to x/y

②: Make the constant a function of the other variable.

③: Differentiate w/ respect to other variable + equate.

④: Solve for constant.

- Ex: // $f_x = 1 + y\cos(xy)$, $f_y = -2y + x\cos(xy)$.

① Antidifferentiate f_x:

$$f(x,y) = \int (1 + y\cos(xy))\,dx = x + \sin(xy) + g(y)$$

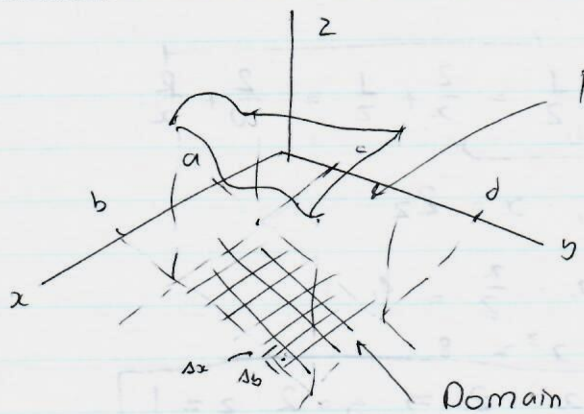
② Take partial w/ respect to y:

$$f(x,y) = x + \sin(xy) + g(y)$$
$$f_y = x\cos(xy) + g'(y)$$

③: Equate + solve:

$$x\cos(xy) + g'(y) = -2y + x\cos(xy)$$
$$g'(y) = -2y$$
$$g(y) = -y^2 + C$$
$$\therefore f(x,y) = x\cos(xy) - y^2 + C + x$$

DOUBLE INTEGRALS

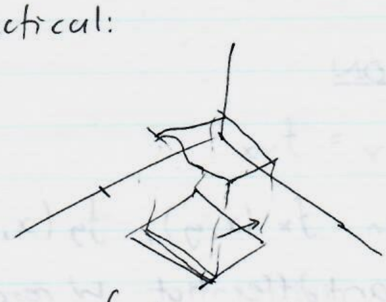


Find volume under surface for defined domain.

$$V = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_i^*)\,\Delta y\,\Delta x$$

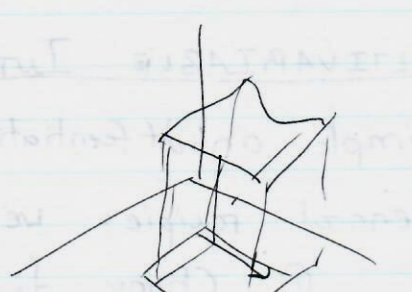
↳ Adding up rectangular bars.

Practical:



Fix x, find area of slice + move down x

Every x: run from all values permissible for y.



Fix y, find area of slice + move down y

Every y: run all permissible x

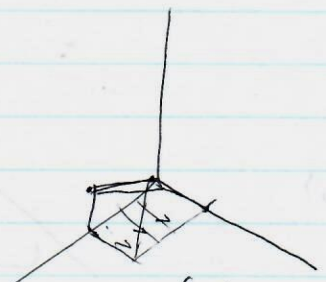
Take all permissible x

$$V = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

For certain value of x, running all values of y

- Evaluation: evaluate inner integral → evaluate outer.
  - Factoring: if variables are independent, can do dx + dy indep.
- Ex:// Find volume under $z = xy^2 + x$ in domain $x \in [0,2]$, $y \in [0,1]$

  ①: Domain + graph:



  ②: Integral

for x ← Limits for y

$$V = \int_0^2 \int_0^1 (xy^2 + x)\,dy\,dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} + xy\right]_0^1 dx$$

Antidifferentiate w/ y

$$= \int_0^2 \left(\frac{x}{3} + x\right) dx$$

Into x

$$= \left[\frac{x^2}{6} + \frac{x^2}{\cancel{3}2}\right]_0^2$$

$$= \frac{2}{3} + \frac{4}{\cancel{3}}2$$

$$= 8/3$$

Evaluated.

Alternatively: factoring ⇒ $z = xy^2 + x = x(y^2+1)$

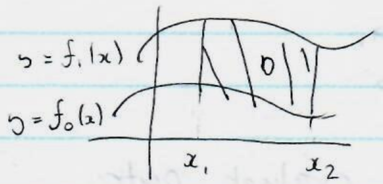
$$V = \int_0^2\int_0^1 x(y^2+1)\,dy\,dx = \int_0^2 x\,dx \int_0^1 (y^2+1)\,dy = \frac{x^2}{2}\Big|_0^2 \cdot \frac{y^3}{3} + y\Big|_0^1$$

split.

$$= \underline{\underline{8/3}}$$

Double Integrals: Non-Rectangular Domain

Type I:



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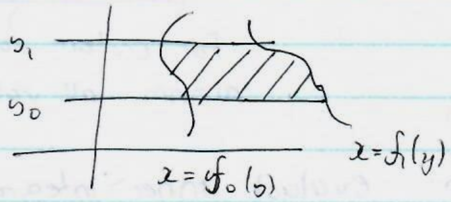
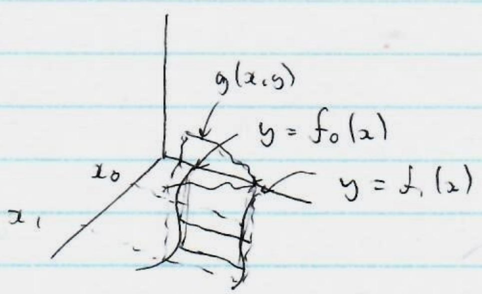


Type III:

Either type I or type II.

Using sheet anology:

Type I:



Creating sheets: For every x, y ranges from $f_0(x) \to f_1(x)$

$$V = \int_{x_0}^{x_1} \int_{f_0(x)}^{f_1(x)} g(x,y)\, dy\, dx$$

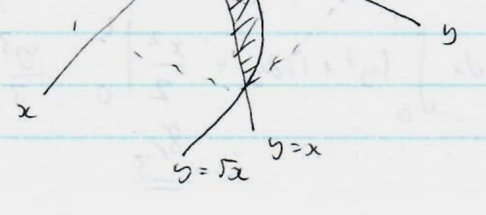
Type II:

Creating sheets: for every y-value, x ranges from $f_0(y) \to f_1(y)$

$$V = \int_{y_0}^{y_1} \int_{f_0(y)}^{f(y)} g(x,y)\, dx\, dy$$

- Ex: // Find volume of solid bounded by $z = xy$ ~~+x-y~~ and on sides by $y = x$ and $y = \sqrt{x}$.

①: Domain:



②: Integral:

T_1: $$V = \int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$$

T_2: $$V = \int_0^1 \int_{y^2}^{y} xy\, dx\, dy$$

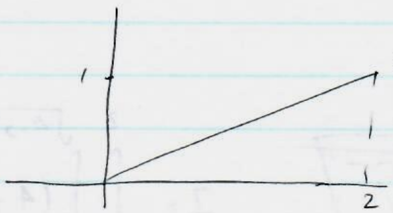
← Had to convert to function of y

- Use T1 ↔ T2 conversion to simplify integrals:
  - Ex://

    Not ideal: e^{x^2} cannot be integrated w/ x

    $$\int_0^1 \int_{2y}^{2} e^{x^2}\,dx\,dy \Rightarrow \text{Convert to T1 integral.}$$

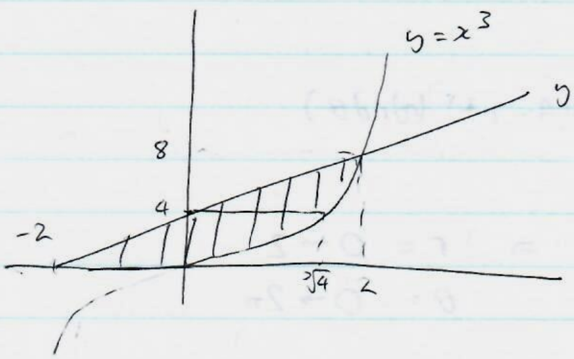
    ①: Domain:



②: Integral:

$$V = \int_0^2 \int_0^{x/2} e^{x^2}\,dy\,dx$$

- Divide complex areas into combo of T1 and T2 integrals.
  - Ex:// $I = \iint_D 3x^2\,dA$ if D is:



This integral can be broken into 2 parts:

①: $-2 \le x \le 0$

$$I = \int_{-2}^{0} \int_0^{2x+4} 3x^2\,dy\,dx$$

②: $0 \le x \le 2$:

T1

$$I = \int_0^2 \int_{x^3}^{2x+4} 3x^2\,dy\,dx$$

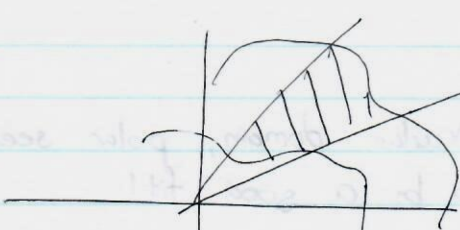
OR

T2

$$\int_0^{\cancel{8}\,4} \int_0^{\cancel{x^2}\,\sqrt[3]{y}} 3x^2\,dx\,dy + \int_4^8 \int_{\cancel{2x+4}\,\frac{y-4}{2}}^{\sqrt[3]{y}} 3x^2\,dx\,dy$$

- Note how we divided into different parts.

MULTIVARIABLE INTEGRATION W/ POLAR



Somewhat circular ⇒ polar

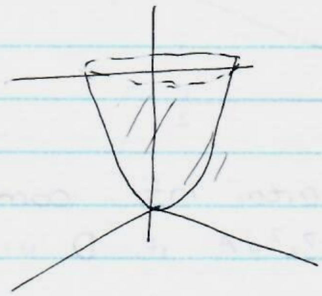
$$V = \iint f(x,y)\,dy\,dx = \iint f(r,\theta)\,(r\,dr\,d\theta)$$

General recipie:

1. Draw your original domain
2. Convert each part, including function, into polar
3. Determine bounds
4. Integrate.

- Ex:// Find volume of bowl with bottom $z = x^2 + y^2$ and top of $z = 4$.

1: Domain:



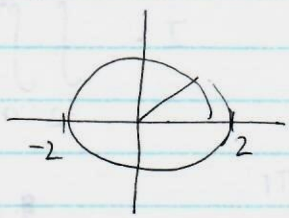
$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 + y^2)\, dy\, dx$$

Note: domain based on max bounds!

2: Polar:

$$I = \int \int (4 - r^2)(r) r\, d\theta)$$

3: Bounds:



$\Rightarrow$ $r = 0 \to 2$

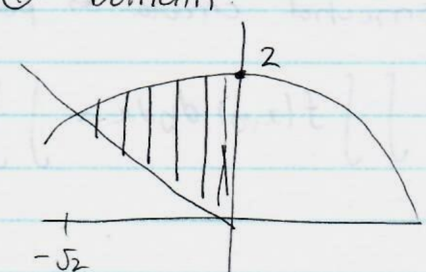
$\theta = 0 \to 2\pi$

4: Integral

$$V = \int_0^{2\pi} \int_0^{2} (4r - r^3)\, dr\, d\theta$$

- Ex:// Evaluate $\displaystyle\int_{-\sqrt{2}}^{0} \int_{-x}^{\sqrt{4-x^2}} x^2\, dy\, dx$

1: Domain:



$\Rightarrow$ Circular domain, polar seems to be a good fit!

② : Polar conversion:

$$V = \int\int r^2\cos^2\theta \, (r\,dr\,d\theta)$$

If we are going from y-axis to $y=-x$, we are going from $\pi/2 \to \frac{3\pi}{4}$

③ : Bounds:

$r = 0 \to 2$, θ: $\pi/2 \to 3\pi/4$.

④ : Integral:

$$V = \int_{\pi/2}^{3\pi/4}\int_0^2 r^3\cos^2\theta \, dr\,d\theta =$$

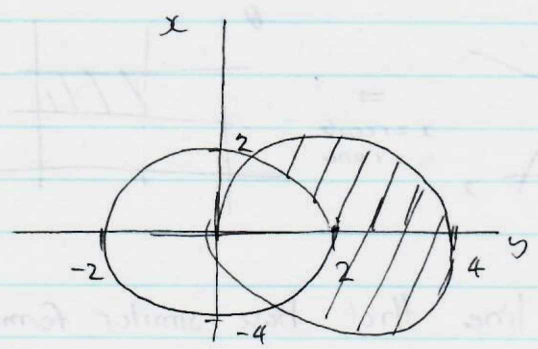
$$= \int_{\pi/2}^{3\pi/4}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \int_0^2 r^3 dr$$

$$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{3\pi/4} \cdot \frac{r^4}{4}\Big|_0^2$$

$$= \frac{\pi}{2} - 1$$

- Ex:// Find V of region below $z = 1/x$ above xy plane, inside cylinder $(x-2)^2+y^2=4$ but outside $x^2+y^2=4$.

① : Domain:



② : Convert to polar:

$$V = \int\int \frac{1}{r\cos\theta}(r\,dr\,d\theta)$$

③: Bounds:

r: $2 \rightarrow$ some function of θ

Finding function of θ:

$$(x-2)^2 + y^2 = \cancel{4}\ 4^2$$

$$(r\cos\theta - 2)^2 + r^2\sin\theta = \cancel{4}\ 4^2$$

$$r^2\cos^2\theta + -4r\cos\theta + 4 + r^2\sin\theta = \cancel{4}\ 4$$

$$\cancel{-4r\cos\theta = r^2} \qquad r^2 - 4r\cos\theta = 0$$

$$\cancel{r = 4\cos\theta} \qquad r(r - 4\cos\theta) = 0$$

$$\therefore r = 4\cos\theta$$

θ: Intersection points

$$4\cos\theta = 2$$

$$\cos\theta = 1/2$$

$$\theta = \pi/3,\ -\pi/3.$$

④: Integral:

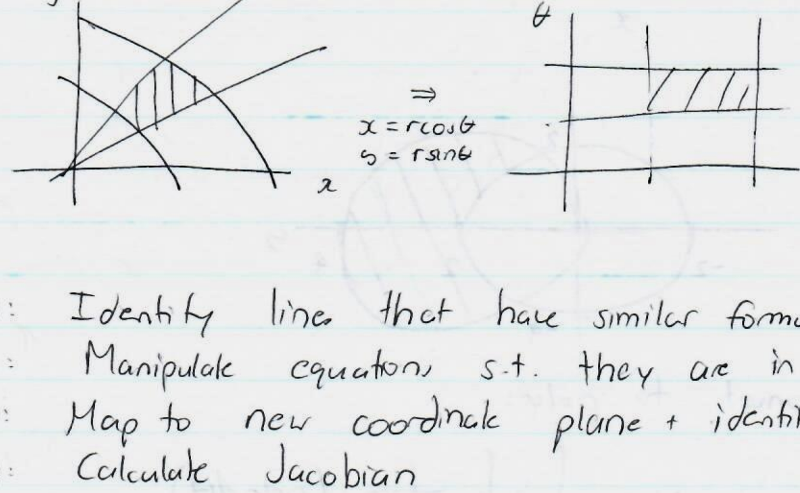
$$V = \int_{-\pi/3}^{\pi/3}\int_{2}^{4\cos\theta} \frac{1}{r\cos\theta}\,(r\,dr\,d\theta)$$

Biggest tip: follow the θ and $r \Rightarrow$ look @ ranges.

CHANGE OF VARIABLES

Nasty integral $\rightarrow$ different coordinate plane.

Ex: //



Recipie:

① : Identify lines that have similar formula.
② : Manipulate equations s.t. they are in terms of constant
③ : Map to new coordinate plane + identify bounds.
④ : Calculate Jacobian
⑤ : Evaluate integral.

- Jacobian:

$$x = f_1(u, v)$$
$$y = f_2(u, v)$$

Jacobian $\Rightarrow$ $\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

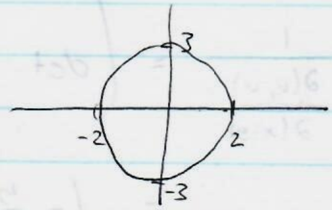
$$\therefore \quad \iint g(u,v) \cdot \frac{\partial(x,y)}{\partial(u,v)} \cdot du\,dv$$

Sometimes, not easy to get x and y as functions of u, v. $\therefore$ Trick:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$$

- Ex: $I = \iint_D x^2\,dA$ where D is the interior of $9x^2 + 4y^2 = 36$

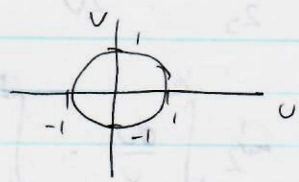
①: Domain:



This is really annoying to integrate.

②: Change of variables: $\boxed{\text{Unit circle!}}$

$$u = \frac{x}{2}, \quad v = \frac{y}{3} \Rightarrow \begin{array}{l} x = 2u \\ y = 3v \end{array} \Rightarrow Jac = \det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

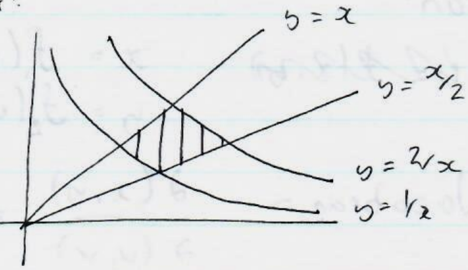


$$\Rightarrow I = \iint_{D_{xy}} x^2\,dx\,dy = \iint_{D_{uv}} 4u^2\,du\,dv \cdot 6$$
$$= 24 \int_{D_{uv}} u^2\,du\,dv.$$

③: Change of variables: $\boxed{\text{polar:}}$

$$I = 24 \int_0^{2\pi} \int_0^1 r^2 \cos^2\theta\,dr\,d\theta = \boxed{6\pi}$$

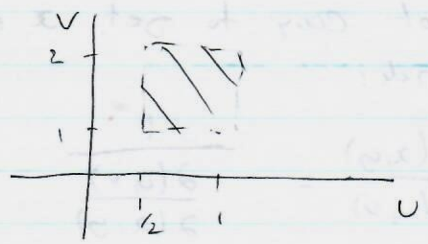
- Ex: // $\iint_R e^{xy}\, dA$ if R is:



①: Create mappings:

$$U = \frac{y}{x} \Rightarrow \tfrac{1}{2} \le U \le 1$$

$$V = xy \Rightarrow 1 \le V \le \cancel{\tfrac{1}{2}}\ 2$$



②: Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \left(\det\begin{bmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{bmatrix}\right)^{-1}$$

$$= \left(-\frac{y}{x} - \frac{y}{x}\right)^{-1}$$

$$= \left(-\frac{2y}{x}\right)^{-1}$$

$$= -\frac{x}{2y} = -\frac{1}{2u}$$ ⇐ Convert Jacobian in terms of new coord.

③: $$I = \int_1^2 \int_{1/2}^1 e^v \left|-\frac{1}{2u}\right| du\, dv = \frac{1}{2}\int_1^{1/2} \frac{du}{u} \cdot \int_{1}^{2} e^v\, dv$$

$$= \boxed{\frac{1}{2}\ln 2 \cdot (e^2 - e^{1/2})}$$

AVERAGE VALUES

$$\bar{f}(x,y) = \frac{1}{Area(D)} \cdot \iint_D f(x,y)\, dA$$

$$\bar{f}(x,y,z) = \frac{1}{Volume(D)} \cdot \iiint_D f(x,y,z)\, dA$$

- Ex:// Find avg. distance from origin of all points within unit circle.

$$\bar{f}(r,\theta) = \frac{1}{\pi \cdot (1)^2} \cdot \int_0^{2\pi}\int_0^1 r\,(r\,dr\,d\theta)$$

$$= \frac{1}{\pi} \cdot 2\pi \cdot \tfrac{1}{3}$$

$$= \tfrac{2}{3}$$

TRIPLE INTEGRALS

Domain: 3D region.

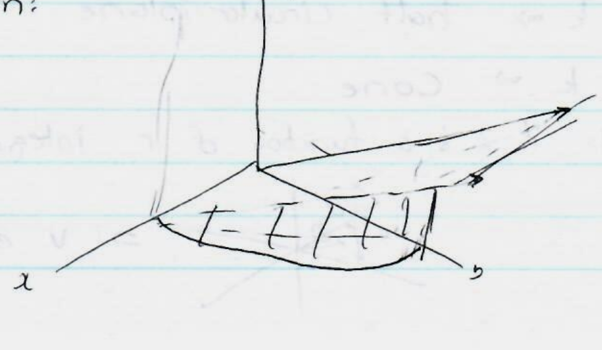
Splitting into rectangular $(dx\,dy\,dz)$ + calculate w/ product of 3-var function.

2 tricks:

1. Change up order of integration (6 dif. ways!)
2. Think about where variable exits to create integral.

- Ex:// $\iiint_R xz\, dV$ where R is region below $z=y$ and cylinder $x^2+y^2=1$

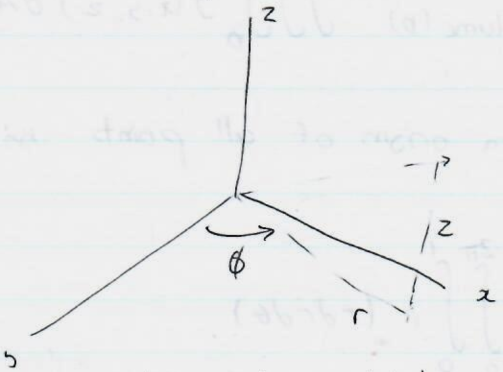
(1): Domain:



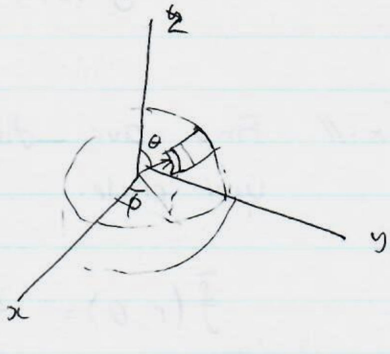
$$I = \int_0^1 \int_z^1 \int_0^{\sqrt{1-y^2}} xz \, dx\, dy\, dz = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} xz \, dx\, dz\, dy$$

- 2 coordinate systems in 3D:

Cylindrical:



Spherical



$dV = dz \cdot r\,d\phi\,dr$

$dV = r\,dr\,d\theta \cdot r\sin\theta\,d\phi$
$= r^2 \sin\theta \, dr\, d\theta\, d\phi$

∘ Anything remotely alike ⇒ convert

- Last example:

Convert to cylindrical system:

$$I = \int_0^1 \int_0^{r\sin\phi} \int_0^{\pi/2} (r\cos\phi) z \, (r\,d\phi\,dr\,dz)$$

$$= 1/30$$

- Conversions:

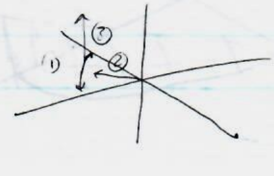
Cartesian	Cylindrical	Spherical
x	$r\cos\phi$	$r\sin\theta\cos\phi$
y	$r\sin\phi$	$r\sin\theta\sin\phi$
z	z	$r\cos\theta$

$r = k \Rightarrow$ sphere

$\phi = k \Rightarrow$ half circular plane

$\theta = k \Rightarrow$ cone

- For cylindrical: if z is a function of r, integrate <u>z first</u>



⇒ v. easy!

PARTIAL DIFFERENTIAL EQUATIONS

$f_x - 3f_y = x \Rightarrow$ Hard to find f b/c partials related to each other.

Thus, we introduce a new variable + express f_x, f_y in terms of new variable, Hopefully, this makes it easier to integrate.

Ex:// PDE of $3z_x + z_y = 1$.

We introduce change of variables: $u = x - 3y$, $v = y$.

Find $f(x, y)$.

①: Express f_x and f_y in terms of f_u and f_v

Chain rule:

$$
\begin{tikzcd}
 & f \arrow[dl] \arrow[dr] & \\
u \arrow[d] \arrow[dr] & & v \arrow[d] \\
x & y & y
\end{tikzcd}
$$

$$f_x = f_u u_x + f_v v_x = f_u$$

$$f_y = f_u u_y + f_v v_y = -3f_u + f_v$$

②: Substitute into PDE:

$$3(f_u) + (-3f_u + f_v) = 1$$

$$f_v = 1$$

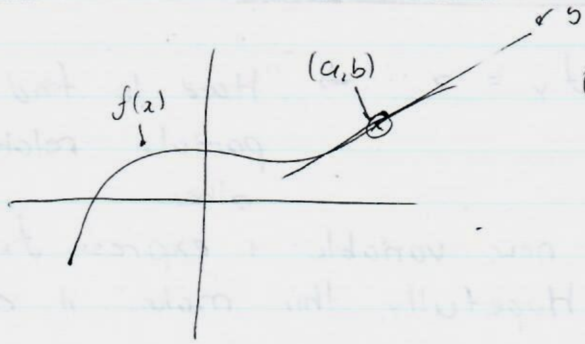
③: Antidifferentiate w/ respect to v:

$f(u, v) = v + g(u)$ ⇐ From simple antidifferentiation.

④: Substitute variables back:

$f(x, y) = y + g(x - 3y)$ ⇐ Any function of $x - 3y$ works.

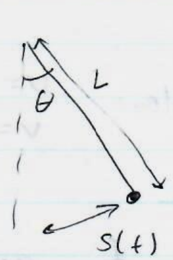
LINEAR APPROXIMATIONS



Use tangent line to find values near construction.

$$y = b + f'(a,b)(x-a)$$

- Ex: pendulum:



We know from Newton's laws that

$$m \cdot \frac{d^2 s}{dt^2} = -mg\sin\theta$$

Problem: way too many variables!

①: Replace variables w/ more important ones (anything in derivative (both $s(t)$

We know that: $s = L\theta$

$$\therefore \frac{d^2 s}{dt^2} = L \cdot \frac{d^2\theta}{dt^2}$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

②: Replace functions w/ reasonable approximation.

At $\theta \ll \Rightarrow \sin\theta \sim \theta$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

③: Solve DE.

ROOT FINDING

①: Bisection method w/ IVT

②: Newton's method

③: Fixed point iteration

① Bisection method:

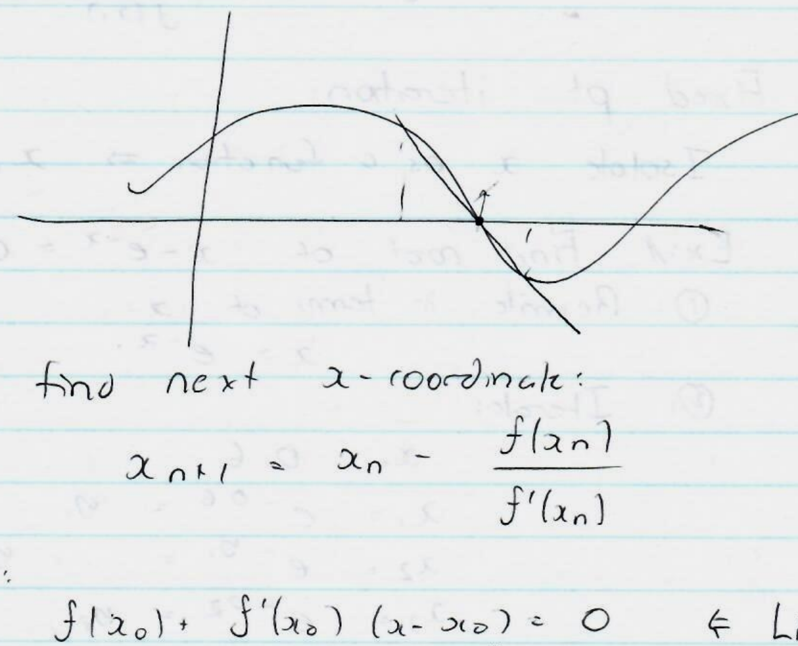
IVT: if $f(x)$ is continuous and $f(a) < 0$ and $f(b) > 0$, then root exists between $[a, b]$

Do binary search within interval

② Newton's Method:

1. Find a reasonable interval using bisection method
   ↳ Root exist within $[3, 4]$.
2. Construct a linear approximation w/ pt. that is close to root.
   ↳ Ex: $f(3) = 0.05$, $f(4) = 1$

   ∴ I will use $x = 3$ as my linear approximation starter.
3. Find when $L(x) = 0$ ⇒ use the x coordinate to make another approximation + repeat.



To find next x-coordinate:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Derivation:

$$f(x_0) + f'(x_0)(x - x_0) = 0 \quad \Leftarrow \text{Linear approximation}$$

$$\therefore x - x_0 = -f(x_0)/f'(x_0)$$

$$x = x_1 = x_0 - f(x_0)/f'(x_0)$$

Beaware the followins

①: $f'(x)$ does not exist / not cts at root $\Rightarrow$ failure

②: $f'(x) = 0$ at pt. $\Rightarrow$ converge slowly

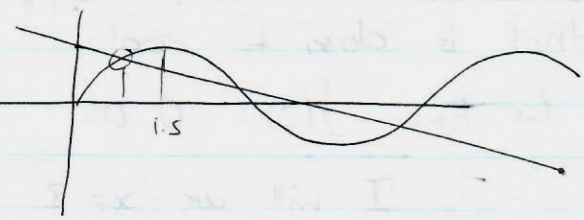
$f''(x) \to \infty$

③: Make sure guess is extremely good!

↳ Can completely miss root!

Ex:// Find root of $\sin x + x - 1 = 0$

①: Reasonable guess (graph is faster):



Estimate: $x_0 = 0.5$

②: Iteration:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.51\ldots$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.51\ldots$$

③: Fixed pt. iteration:

Isolate x as a function $\Rightarrow$ $x_{n+1} = g(x_n)$

Ex:// Find root of $x - e^{-x} = 0$

①: Rewrite in terms of x:

$$x = e^{-x}$$

②: Iterate:

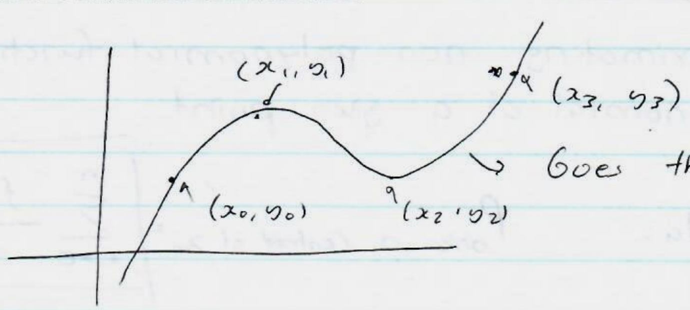
$$\left.\begin{aligned} x_0 &= 0.6 \\ x_1 &= e^{-0.6} = y_1 \\ x_2 &= e^{-y_1} = \ldots y_2 \\ x_3 &= e^{-y_2} = y_3 \\ &\vdots \end{aligned}\right\}$$

Simplistic!

⇑

$|f'(x)| < 1$ for convergence.

Polynomial Interpolation



→ Goes through all pts.

- To do this → Newton's Forward differece formula

$$y = y_0 + \frac{x - x_0}{h} \Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\, h^2} \Delta^2 y_0 + \dots$$

$$\dots \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n+1})}{n!\, h^n} \Delta^n y_0$$

- Ex:// Create a polynomial s.t. it goes through $(2, 4), (2.2, 5)$ $(2.4, 4), (2.6, 2)$.

①: Make sure points are equidistant

$$h = \Delta x = 0.2$$

②: Triangular table of changes.

y	Δy	$\Delta^2 y$	$\Delta^3 y$
4			
	1		
5		-2	
	-1		1
4		-1	
	-2		
2			

⇒ Only interested in 1^{st} changes!

③: Create polynomial via formula:

$$y = 4 + \frac{(x-2)}{0.2}(1) + \frac{(x-2)(x-2.2)}{0.2^2 \cdot 2!}(-2) + \frac{(x-2)(x-2.2)(x-2.4)}{0.2^3 \cdot 3!}(1)$$

Iterate through points → look at last point, $(x - pt)$ in num.,
→ $h^{\text{iteration}}$ · iteration! in denom → multiply change by iteration

TAYLOR POLYNOMIAL

Approximating non-polynomial function as a polynomial at a given point.

Formula: $P_{\text{order } n,\ \text{centred at } x_0} = \boxed{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}$

Maclaurin: $x_0 = 0$.

Ex:// Construct 4^{th} order Taylor polynomial for $\sqrt{x}$ at $x=4$.

①: Derivative + evaluation:

$$f(4) = 2$$
$$f'(x) = \tfrac{1}{2}x^{-1/2} \Rightarrow f'(4) = 1/4$$
$$f''(x) = -\tfrac{1}{4}x^{-3/2} \Rightarrow f''(4) = -1/32$$
$$f^{(3)}(x) = \tfrac{3}{8}\cdot\tfrac{1}{x^{5/2}} \Rightarrow f^{(3)}(4) = 3/256$$
$$f^{(4)}(x) = -\tfrac{15}{16}\cdot\tfrac{1}{x^{7/2}} \Rightarrow f^{(4)}(4) = -15/2048$$

②: Use formula:

Do not forget factorials ⇒ $P_{4,4} = 2 + \tfrac{1}{4}(x-4) - \left(\tfrac{1}{32}\cdot\tfrac{1}{2}\right)(x-4)^2 + \ldots$

Note: Edge case of $f^{(k)}(x_0) = 0$ is important ⇒ n^{th} + $(n^{th}+1)$ order polynomials may look the same

$\therefore$ n^{th} order poly. ⇒ n derivatives to include.

1) $\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k$

5) $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

2) $\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k$

3): $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

4): $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

Identify their common ratios / functions in problems

TAYLOR'S INEQUALITY

$$|\text{error}| \leq \frac{k \cdot (x - x_0)^{n+1}}{(n+1)!} \Rightarrow n^{th}\text{order polynomial.}$$

Taking the next term in series + finding its max value.

k: next derivative's max value within interval.
↳ If we have non-whole number ⇒ round up to nearest whole #.

- Ex.// Estimate $\cos(1/2)$ using 4^{th}-order Maclaurin + give upper bound on error.

1: Taylor polynomial:

$$P_{4,0} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

2: Next term:

$$f^{(5)} = -\sin x \Rightarrow \text{How large can this be in } (0, 1/2)$$

Largest value: $f^{(5)} \Rightarrow x = 1/2 \Rightarrow |\sin(1/2)| \leq 1$

$\therefore k = 1$

3: Error formula:

$$|\text{error}| \leq \frac{1 \cdot (1/2 - 0)^5}{5!} = 1/3840$$

Note: We could have done even better:

Since $P_{4,0} = P_{5,0} \Rightarrow$ error bound on $P_{5,0}$
(6^{th} order for error)

Approximation of Integrals

$$\int_0^{0.5} e^{t^2}\,dt \Rightarrow e^{t^2} \text{ has no antiderivative!}$$

Approximate $\Rightarrow$ Taylor polynomials.

①: Construct an n-order polynomial in integrand.

I will use 4^{th}-order Maclaurin centred at 0 (integral near zero)

Note: e^{t^2} is e^u when $u = t^2$.

$$\therefore \quad e^u \approx 1 + u + \frac{u^2}{2}$$
$$e^{t^2} \approx 1 + t^2 + \frac{t^4}{2}$$

Particularly useful still! 2^{nd} order $e^u \to 4^{th}$ order e^{t^2}!

②: Integrate!

$$\int_0^{0.5} e^{t^2}\,dt \approx \int_0^{0.5} \left(1 + t^2 + \frac{t^4}{2}\right) dt = \ldots$$

③: Error analysis:

Find max error for next term:

$$|\text{error}| \leq k \cdot \frac{x^3}{3!}$$

k: max of e^u from $u \in [0, 1/4]$

How? $t \in [0, 0.5] \Rightarrow$ Integral

$t^2 \in [0, 1/4]$

$u \in [0, 1/4]$

Analysing e^u, not e^{t^2}. Simpler, but need to change bound

$$\therefore \quad k = \lceil e^{1/4} \rceil = 2$$

$$\therefore |R_2(u)| \leq \frac{2u^3}{6} \leq \frac{u^3}{3} \Rightarrow \boxed{R_4(t^2) \leq \frac{t^6}{3}}$$

$u = t^2$

Now integrate error term:

$$\int_0^{x} e^{t^2}\,dt = \int_0^{x} P_{2,0}(t^2)\,dt + \int_0^{x} R_2(t^2)\,dt$$

$$\therefore \left| \int_0^{x} R_2(t^2)\,dt \right| \le \int_0^{x} |R_2(t^2)|\,dt$$

$$\le \int_0^{x} \frac{t^6}{3}\,dt$$

$$\le \frac{x^7}{21} \Rightarrow x \in [-1/2, 1/2]$$

Now, use definite limits.

Summary:

1: Construct a Taylor polynomial for integrand
   a) Manual construction
   b) Look @ common functions (e^x, $\sin x$ ...)

2: Error analysis
   1. Find remainder upper bound
      ↳ Use simpler polynomial in 1, b)
        NOT complex one in 1, a)
   2. Integrate.

3: Integrate whole thing.

TAYLOR SERIES

Q: If we add infinite terms to Taylor polynomial, do we get our function?

A: Depends on error behaviour as $n \to \infty$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$$

only true if $\lim_{n\to\infty} R_n(x) = 0$

If that is true, then we have a Taylor series:

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \Leftarrow \text{Taylor Series is a limit.}$$

- Ex:// Find Taylor series for $\tan^{-1}(x)$.

① Create a Taylor polynomial of degree n:

Given that we have arbitrary $n \to$ pre-derived formula.

↳ Since $\tan^{-1}(x)$ is not pre-derived formula, consider derivative:

$$(\tan^{-1}(x))' = \frac{1}{1+x^2}$$

Only func similar: $\frac{1}{1-x} \Rightarrow u = -x^2$

$$\therefore \frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots \pm R_n$$

Find error:

Interval: $x \in [-a, a] \Rightarrow u \in [-a^2, 0]$.

$$f(u) = \frac{1}{1-u} \Rightarrow f'(u) = -\frac{\cancel{2}}{(1-u)^2} \dots$$

$$f^{(n)}(u) = \frac{n!}{(1-u)^{n+\cancel{2}1}}$$

Requires multiple derivatives + pattern match.

Upper bound:

$$f^{(n+1)}(u) = \frac{(n+1)!}{(1-u)^{n+2}}$$

At $u=0 \Rightarrow f^{(n+1)}(u) \le (n+1)! \Leftarrow$ Upper bound at $u=0$

$$\therefore R_n(u) = \frac{k\,u^{n+1}}{(n+1)!} \le \frac{(n+1)!\,u^{n+1}}{(n+1)!} = u^{n+1}$$

$$\therefore \frac{1}{1-u} = 1 + u + u^2 + \dots + u^n \pm u^{n+1}$$

$$\frac{1}{1+x^2} = 1 + -x^2 + x^4 - x^6 + \dots \pm |(-x^2)|^{n+1} \quad x^{2n+2}$$

Abs. val ⇒ ⊖ removed. $(x^2)^{n+1} = \cancel{x^{2n+2}}$

Integrating:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} \pm \frac{x^{2n+3}}{2n+3}$$

②: Take limit of error term:

$$\lim_{n \to \infty} \frac{x^{2n+3}}{2n+3} \to 0 \quad \text{if} \quad |x| \le 1$$

↳ Proof using power series shown later.

Since $R_n \to 0$ if $a \le 1$:

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad \text{in interval } [-1, 1].$$

- Ex:// Find Taylor series for $\sin x$:

①: Find Taylor polynomial:

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_n$$

②: Error term:

$$R_{2n+1} \Rightarrow |\text{error}| \le \frac{k\, x^{2n+2}}{(2n+2)!}$$

We know that upper bound of any derivative of $\sin x$ is 1:

$$\therefore |R_{2n+1}| \le \frac{x^{2n+2}}{(2n+2)!}$$

③: Take limit:

$$\lim_{n \to \infty} |R_{2n+1}| \to 0$$ since factorial grows faster than exponential.

$$\therefore \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

INFINITE SERIES CONVERGENCE

When is $f(x) = \sum_{k=0}^{\infty} \ldots$?

Instead of using Taylor error, we are using infinite series properties.

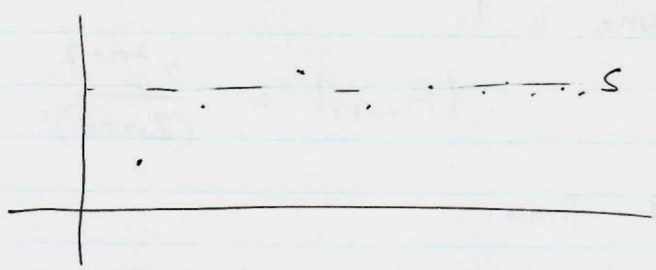
$$\sum_{k=0}^{\infty} a_k = \left.\lim_{n \to \infty} \sum_{k=0}^{n} a_k \right] \rightarrow \text{Adding infinite terms!}$$

$$\{a_0, a_1, a_2, a_3, a_4 \ldots \ldots a_n\}$$

Partial sums:

$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = S_1 + a_2$$
$$\vdots$$
$$S_n = S_{n-1} + a_n$$

If S converges $\left(\lim_{n\to\infty} S_n = s\right)$, then $\sum_{k=0}^{\infty} a_k$ converges to the sum of s.



Tips:

1. Sequence vs. series: sequence ⇒ individual terms, series ⇒ sum of terms.
2. Two different sequences: $\{a_n\}$, $\{S_n\}$

3. Reindexing:

$$\sum_{k=q}^{\infty} a_k = \sum_{k=0}^{\infty} a_{(k+q)}$$ } Encourage writing down first few terms to understand $a, r \ldots$

Convergence tests:

①: Look @ first few terms

Particularly useful for <u>telescoping series</u>

Ex:// Find sum of $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

1. Divide common term into fractions via PFD

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Indication of tel. series.

2. Write out first few terms:

$$\left\{ \frac{1}{2}\left(1-\frac{1}{3}\right), \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right), \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right) \ldots \right\}$$

Notice that terms are cancelling out

∴ General formula:

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2} - \frac{1}{(2N+1)2}$$

3. Take $N \to \infty$:

$$N \to \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

②: Convergence test

1. Geometric series:

$\sum_{k=0}^{\infty} ar^k$ (multiply r every single time).

$$\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$$ if $|r| < 1$, otherwise it diverges

Way to analyze geometric series:

1. Find 1st term + write out sequence
2. Find r from sequence $\Rightarrow$ determine convergence
3. Sum formula:

Tip: If series is a sum $\rightarrow$ decompose it into smaller geometric series.

Ex:// Determine if following series converges + determine sum.

$$\sum_{n=0}^{\infty} \frac{11(-2)^n - 7^n}{8^n} \Rightarrow$$ Note: looks like a geometric series b/c $\boxed{\text{ratio}^n}$

$$\sum_{n=0}^{\infty} \frac{11(-2)^n - 7^n}{8^n} = \underbrace{\sum_{n=0}^{\infty} \frac{11(-2)^n}{8^n}}_{1} - \underbrace{\sum_{n=0}^{\infty} \frac{7^n}{8^n}}_{2}$$

1: $$\sum_{n=0}^{\infty} \frac{11(-2)^n}{8^n} \Rightarrow \frac{11}{1+1/4} = \frac{44}{5}$$

2: $$\sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n \Rightarrow \frac{1}{1-7/8} = 8$$

$$\therefore \sum_{n=0}^{\infty} \frac{11(-2)^n - 7^n}{8^n} = 1 - 2 = \frac{44}{5} - 8 = \boxed{4/5}$$

We can set limits of convergence!

Ex:// $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow$$ What values of x does this converge to?

Geometric series w/ initial term of 1 + $r = x$

$$r < |1|$$

$$-1 < x < 1 \Rightarrow \text{Convergence limits.}$$

2. Divergence Test / N^{th} - term Test.

$$\lim_{k \to \infty} a_k \neq 0 \Rightarrow \sum a_k \text{ diverges!}$$

Ex://

$$\sum_{n=0}^{\infty} 2^n \text{ diverges! } \{1, 2, 4, \ldots\}$$

Not converging to particular value b/c not <u>decaying</u>!

Key: Converse is not true! If $\lim_{k \to \infty} a_k = 0$, this tells us <u>nothing</u> about convergence.

↳ Use other techniques

<u>Harmonic series</u>: $\sum \frac{1}{k}$ diverges although $\lim_{k \to \infty} 1/k = 0$

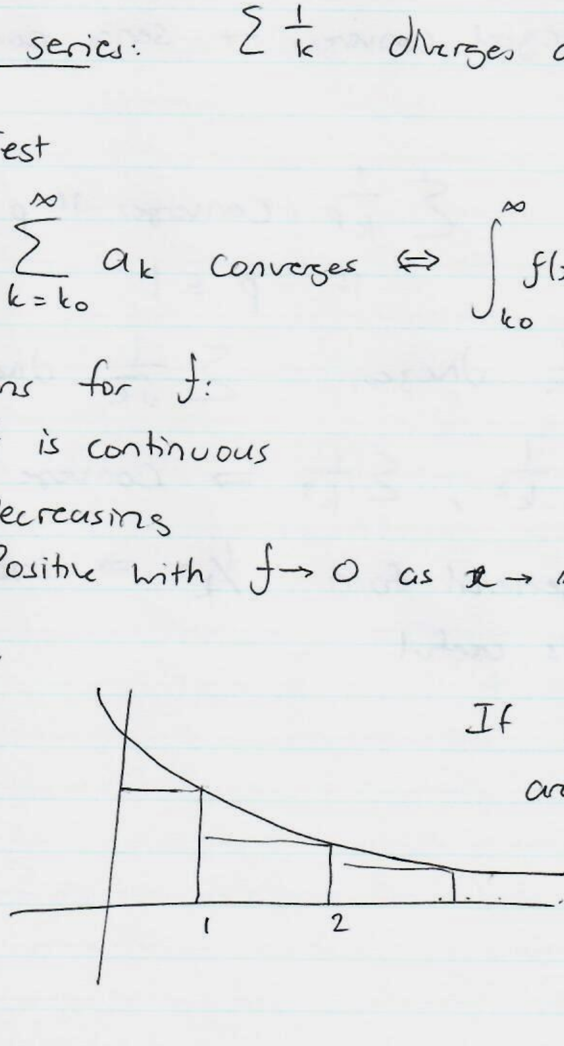
3. Integral Test

$$\sum_{k=k_0}^{\infty} a_k \text{ converges} \Leftrightarrow \int_{k_0}^{\infty} \overset{a_k}{f(x)}\, dx \text{ converges.}$$

Limitations for f:

1. f is continuous
2. Decreasing
3. Positive with $f \to 0$ as $x \to \infty$

Intuition:



If $f(x)$ is finite, then area must converge

area ⇒ series!

Useful if series looking particularly nasty.

Ex:// Determine if $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

Consider

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^2}$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^2}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2}\right)$$

$$= 1/\ln 2$$ $\Rightarrow$ This is NOT guaranteed to be the sum.

B/c integral converges $\rightarrow$ series converges too!

4. P-series:

$\sum \frac{1}{k^p}$ converges if $p > 1$ + diverges if $p \leq 1$

$\therefore$ $\sum \frac{1}{k}$ diverges, $\sum \frac{1}{\sqrt{k}}$ diverges

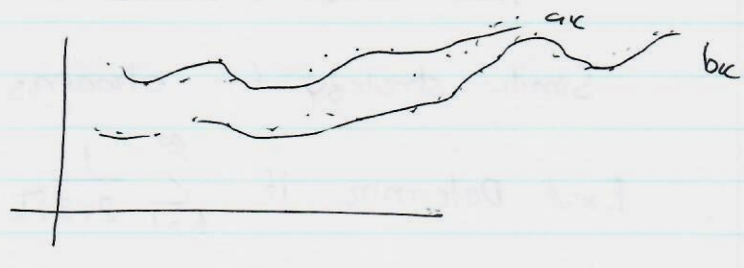
$\sum \frac{1}{k^2}$, $\sum \frac{1}{k^3}$ $\Rightarrow$ converge

If in general form $1/k^n$ $\Rightarrow$ indication that p-series is useful.

5. Comparison test:

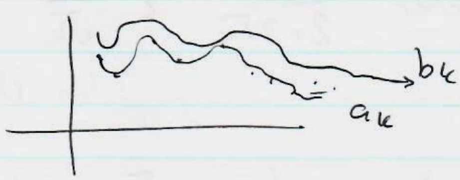
1) $\sum a_k$ and $\sum b_k$ where $\forall a_k$, $a_k > b_k$.

If $\sum b_k$ diverges, then $\sum a_k$ also diverges.



2) If $\sum a_k$ and $\sum b_k$ where $\forall a_k$, $a_k < b_k$.

If $\sum b_k$ converges, then $\sum a_k$ also converge.



$\sum b_k$ is usually geometric / p-series.

Find geometric / p-series functions that look really similar!

Ex: Determine if $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ converges

Recognize that $\ln k / k$ looks like harmonic series!

Also note: $\frac{\ln k}{k} > \frac{1}{k}$ if $k \geq 3$

Replace functions / compare w/ constants.

Since $1/k$ diverges and $\frac{\ln k}{k} > \frac{1}{k}$, series diverges.

Ex: Determine if $\sum_{k=1}^{\infty} \frac{1}{k^2+2}$ converges.

$\frac{1}{k^2+2} < \frac{1}{k^2}$. Since $1/k^2$ converges,

then $\frac{1}{k^2+2}$ also converges.

6. Limit Comparison Test

If $\lim_{k\to\infty} \frac{a_k}{b_k} = L \quad (L \neq 0) \Rightarrow \sum a_k$ and $\sum b_k$

have same behaviour!

Similar strategy for choosing b_k!

Ex:// Determine if $\sum_{k=1}^{\infty} \frac{1}{2+3\sqrt{k}}$ converges.

① : Look through past tests. Looks similar to $1/\sqrt{k}$.

So comparison test could be used.

However $\frac{1}{2+3\sqrt{k}} < \frac{1}{\sqrt{k}}$, so doesn't work.

② : LCT: ↳ Usually fails CT b/c not > / < known series.

$$\lim_{k\to\infty} \frac{1}{2+3\sqrt{k}} \cdot \sqrt{k} = 1/3$$

Since $1/3 \neq 0$, $\frac{1}{\sqrt{k}}$ and $\frac{1}{2+3\sqrt{k}}$ should have same behavior → diverges.

7. Alternating Series Test

Consider $\sum_{k=0}^{\infty} (-1)^k a_k$. If $\lim_{k\to\infty} a_k = 0 \wedge \{a_k\}$ is decreasing ⇒ series converges.

Tip: Use AST if $(-1)^k$ spotted.

Ex:// $\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{\ln k}$ converges.

1. $\lim_{k\to\infty} a_k = 0$ 2. $\{a_k\}$ decreases

2. Alternating $\therefore$ Converges!

Just b/c $\lim_{k \to \infty} a_k = 0$ does not mean $\{a_k\}$ is decreasing

↳ Consider $\{1, -1/2^2, 1/3, -1/4^2 \dots\}$ Increase!

Alternating Series Estimation Theorem:

If $\sum (-1)^k a_k$ converging + n^{th} partial sum S_n.

$\therefore$ $|\text{error}| \leq a_{n+1}$ (error < first term omitted)

Ex:// $\sum_{k=3}^{\infty} (-1)^k \frac{8}{k^4 \ln k}$

1) Is this converging:

1. Alternating, 2. $\lim_{k \to \infty} \frac{8}{k^4 \ln k} = 0$, 3. $\{a_n\}$ is decreasing

$\therefore$ Converging

2) Error:

Let's say we use 4 terms to estimate sum:

$|\text{error}| \leq \frac{8}{8^4 \ln 8}$

Sum of infinite series can change if order changes b/c partial sums have changed.

<u>Absolute vs. Conditional Convergence</u>

Absolutely convergent - $\sum a_k$ converges $\Rightarrow$ $\sum |a_k|$ converges.

Ex:// $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2}$ converges + $\sum_{k=0}^{\infty} \frac{1}{k^2}$ converges

$\therefore$ $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2}$ absolutely converges.

This simplifies problems w/ $(-1)^k$!

↳ If abs (a_k) converges $\rightarrow$ a_k converges!

Conditionally convergent:

$\sum a_k$ converges but $\sum |a_k|$ diverge.

↳ Ex:// $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges but $\sum \frac{1}{k}$ diverges.

$\therefore$ Conditionally convergent.

Changing order of terms has <u>NO</u> impact on absolutely converging series.

8. Ratio Test

Suppose $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L$

$L < 1 \Rightarrow \sum a_k$ is absolutely convergent

$L > 1 \Rightarrow \sum a_k$ is divergent

$L = 1 \Rightarrow$ test fails (use other tests)

Ex:// $\sum (-1)^k \frac{2^k}{k!}$

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{(-1)^{k+1} 2^{k+1}}{(k+1)!} \times \frac{k!}{(-1)^k 2^k} \right|$$

$$= \lim_{k \to \infty} \left| \frac{(-1)^1 \cdot 2}{k+1} \right|$$

$$= \lim_{k \to \infty} \frac{2}{k+1}$$

$= 0 \rightarrow$ ~~diverging!~~ absolutely converging!

Ex: 11 $\sum_{k=1}^{\infty} \frac{3^k}{k^2 2^k}$

$$\lim_{k\to\infty}\left|\frac{3^{k+1}}{(k+1)^2 2^{k+1}} \times \frac{k^2 2^k}{3^k}\right| = \lim_{k\to\infty}\left|\frac{3k^2}{2(k^2+2k+1)}\right|$$

$$= \lim_{k\to\infty} \frac{3}{2(1+2/k+1/k^2)}$$

$$= 3/2 > 1 \Rightarrow \text{Diverging!}$$

General Framework:

1: Write out first few terms (esp. telescoping!)

2: Use limit test + determine abs. conv. / divergence/failure

3: Failure $\Rightarrow$ use any other tests.

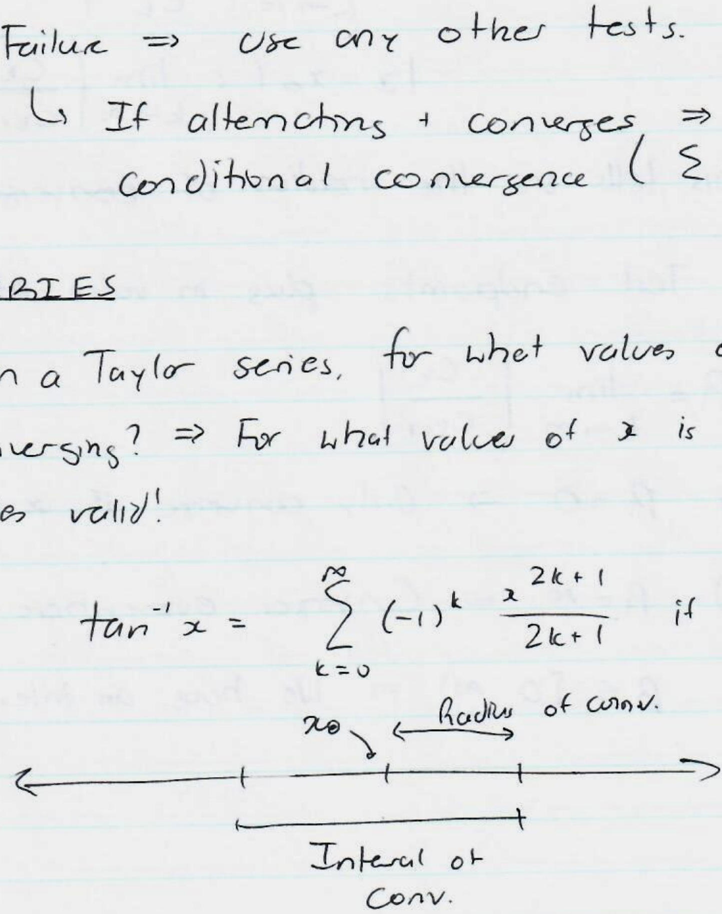
↳ If alternating + converges $\Rightarrow$ determine if conditional convergence ($\sum |a_k|$)

POWER SERIES

Q: Given a Taylor series, for what values of x are we converging? $\Rightarrow$ For what value of x is the taylor series valid!

Interval of convergence.

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad \text{if } x \in [-1, 1]$$



Power series: $\sum_{k=0}^{\infty} c_k (x-x_0)^k$ ⇐ V. similar to Taylor series.

To find out if series converges + for what values of x:

①: Apply limit test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{c_{k+1}(x-x_0)^{k+1}}{c_k (x-x_0)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| \cdot |x-x_0|$$

②: Apply conditions:

$$L < 1 \Rightarrow \text{absolute conv.}$$

$$|x-x_0| \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| < 1$$

$$|x-x_0| < \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right|$$ ⇐ DO NOT MESS W ABS VALUE!

This tells us the radius of convergence.

③: Test endpoints. plus in value at endpoints + test

Let $R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right|$

1) $R = 0 \Rightarrow$ Only converge if $x = x_0$

2) $R = \infty \Rightarrow$ Converges everywhere

3) $R \in [0, \infty) \Rightarrow$ We have an interval → $[x_0 - R, x_0 + R]$

- Examples: //

1) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ What values of x does this converge?

① : Apply Limit test

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right|$$

$$= \lim_{k\to\infty} |x| \cdot \frac{1}{k+1}$$

② : Apply condition:

$$|x| \cdot \lim_{k\to\infty} \frac{1}{k+1} < 1$$

$$|x| < \lim_{k\to\infty} k+1$$

Since $\lim_{k\to\infty} k+1 \to \infty \Rightarrow$ Converges for all x.

2) $\sum_{k=0}^{\infty} k!\, x^k$

① Apply limit test:

$$\lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} |k \cdot x|$$

② : Apply condition:

$$\lim_{k\to\infty} k \cdot |x| < 1$$

$$|x| < \lim_{k\to\infty} \frac{1}{k} \to 0!$$

$\therefore$ Converges only if $x = 0$

3) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-3)^k}{k \cdot 4^k}$

①: Apply Limit Test

$$\lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{(k+1)\,4^{k+1}} \cdot \frac{k \cdot 4^k}{(x-3)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x-3}{4} \cdot \frac{k}{k+1} \right|$$

②: Apply condition:

$$\lim_{k\to\infty} \tfrac{1}{4}\,|x-3| \cdot \frac{k}{k+1} < 1$$

$$\tfrac{1}{4}\,|x-3| < 1$$

$$|x-3| < 4$$

$$\therefore\ R = 4 \Rightarrow x \in (-1, 7)$$

② Test endpoints:

$x = 1$

$$\therefore \sum_{k=1}^{\infty} \frac{(1-3)^k}{k \cdot 4^k} = \sum_{k=1}^{\infty} \frac{(-4)^k}{k \cdot 4^k} = \sum_{k=1}^{\infty} (-1)^k \cdot \frac{1}{k}$$

$\therefore$ Converges!

$x = 7$:

$$\sum_{k=1}^{\infty} \frac{(7-3)^k}{k \cdot 4^k} = \sum_{k=1}^{\infty} 1/k \Rightarrow \text{Diverges.}$$

$\therefore$ Interval of convergence: $x \in \cancel{[-1, 7]}$

$[-1, 7)$

$\sum c_k (x - x_0)^k$ w/ radius of convergence of R

- Integrale
- Differentiale
- Multiply by constant
- Add to another series w/ $R_{new} \geq R$

} will also have radius of convergence of R!

Ex:// Find radius of convergence for $\frac{1}{(1-x)^2}$

Notice that $\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ if } |x| \leq 1! \Rightarrow \boxed{\text{Geometric series.}}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$$

Diff. term by term

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$\therefore \frac{1}{(1-x)^2} \text{ converges if } |x| \leq 1$$

Even though R remains same, interval may have changed (esp endpoints)

Tip: Geometric series should be a classic building block: Looks similar to sum formula

Ex:// Maclaurin series for $\frac{x}{3+2x}$

$$\frac{x}{3+2x} = x\left(\frac{1}{3+2x}\right) = \frac{x}{3}\left(\frac{1}{1 + \frac{2}{3}x}\right) = \frac{x}{3}\left(\frac{1}{1-(-\frac{2}{3}x)}\right)$$

$$\therefore \frac{x}{3}\sum_{k=0}^{\infty}\left(-\tfrac{2}{3}x\right)^k = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{2^k x^{k+1}}{3^{k+1}}}$$

Radius of convergence: $\left|-\frac{2}{3}x\right| < 1 \Rightarrow |x| < \frac{3}{2}$

$$\therefore R = \tfrac{3}{2}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x|<1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \forall x$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad \forall x$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad \forall x$$

m could be <0!

$$\Rightarrow (1+x)^m = 1 + mx + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + \dots + (m-n+1)\frac{x^n}{n!} + \dots$$

$\hookrightarrow R = 1$

Ex:// Find Radius of convergence for $\frac{1}{\sqrt{2-x}}$

$$\frac{1}{\sqrt{2-x}} = \frac{1}{\sqrt{2}}\left(1+\left(-x/2\right)\right)^{-1/2} \Rightarrow \text{Binomial expansion!}$$

From above: $R<1 \Rightarrow \left|-x/2\right| < 1$

$$|x| < 2$$

BIG-O ORDER SYMBOL

$f(x) = O(g(x))$ as $x \to x_0$ if

$\exists$ a constant A s.t. $|f(x)| \leq A|g(x)|$ on interval around x_0.

Common in computer science.

- Ex://

$$x^3 = O(x^2) \text{ as } x \to 0$$
$$= O(x) \text{ as } x \to 0$$

$$x^3 \neq O(x^4) \text{ as } x \to 0$$
$$x^3 = O(x^4) \text{ as } x \to \infty$$

→ V. important.

$$\sin x = O(x) \text{ as } x \to 0$$

- Ex://

$\frac{x^2}{8+x^3} \in [x: -1, 1]$. What is order of this function?

$$\left|\frac{x^2}{8+x^3}\right| = \left|\frac{1}{8+x^3}\right| |x|^2 \leq \frac{1}{7}|x^2|$$

We have a constant $(1/7)$ s.t. $x^2 \geq f(x)$.

$$\therefore \frac{x^2}{8+x^3} = O(x^2) \text{ as } x \to 0 \text{ in } [-1, 1]$$

Very similar error calculation for K in Taylor's Ineq.

- Connection to Taylor's Inequality:

$$|R_n(x)| \leq \frac{k|x-x_0|^{n+1}}{(n+1)!}$$

$$|\text{error}| \leq O((x-x_0)^{n+1}) \text{ as } x \to x_0!$$

Why is this useful ⇒ no need to calculate k
+ easy to manipulate series.

- Ex://

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x + O(x^2), \quad \sin(x) = x + O(x^3) \quad (x \to 0)$$

$$\sqrt{1+x} + \sin x = \left(1 + \tfrac{1}{2}x + O(x^2)\right) + x + O(x^3)$$
$$= 1 + \tfrac{3}{2}x + O(x^2) + O(x^3)$$
$$= 1 + \tfrac{3}{2}x + O(x^2)$$

$x^3 \in O(x^2)$!

- Ex:// Find Maclaurin series for $\sinh x$

①: Look back @ definition

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

② : Use individual Maclaurin $\Rightarrow$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^5) \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots + O(x^5)\right)\right.$$

③ combine.

$$= x + \frac{x^3}{3!} + O(x^5) \quad \text{as } x \to 0$$

- Ex:// Substitution:

$$\sqrt{1+x} = 1 + x/2 + O(x^2) \quad \text{as } x \to 0$$

$$\therefore \sqrt{1+u^4} = 1 + \frac{u^4}{2} + O(x^8) \quad \text{as } x \to 0$$

- Ex:// Multiplication

$$e^{x} \sin x = \left[1 + x + \frac{x^2}{2!} + O(x^3)\right]\left[x + O(x^2)\right]$$

$$= x + x^2 + \frac{x^3}{2!} + x\,O(x^3) + O(x^3) + x\,O(x^3) + \frac{x^2}{2!}\,O(x^3) + \left[O(x^3)\right]^2$$

Combine all functions + order terms into 1 if possible.

$$= x + x^2 + \frac{x^3}{2!} + O(x^3)$$

$$= x + x^2 + O(x^3)$$

As $x \to 0$

- Rules:
  1. $k\,O(x^n) = O(x^n)$
  2. $O(x^m) + O(x^n) = O(x^{\min(m,n)})$
  3. $O(x^m) \cdot O(x^n) = O(x^{m+n})$
  4. $[O(x^m)]^n = O(x^{mn})$
  5. $\frac{O(x^m)}{x^n} = O(x^{m-n})$

Use sparingly.

- Extremely easy to evaluate limits:

Ex://
$$\lim_{x \to 0} \frac{x\cos x - \sin x}{x^3} = \lim_{x \to 0} \frac{x\left[1 - \frac{x^2}{2!} + O(x^4)\right] - \left[x - \frac{x^3}{3!} + O(x^5)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{x - \frac{x^3}{2!} + O(x^5) - x + \frac{x^3}{3!} + O(x^5)}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{3}x^3 + O(x^5)}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{3} + O(x^2)}{1}$$

$$= \boxed{-\frac{1}{3}}$$

1: Convert into small-order Maclaurin

2: Simplify

3: Use O-rules to get to simplified O(...)

4: Evaluate limit.