WEEK 1

Counting w/ Product & Sums

Cartesian Product:

$A = \{ \ldots \}, \quad B = \{ \ldots \}$

$$A \times B = \{ \underbrace{(a_1, b_1), (a_2, b_2) \ldots}_{\text{All possible pairs b/w } A \,\&\, B} \}$$

$$|A \times B| = |A| \cdot |B|$$

Unions:

$$A \cup B = \{ x : x \in A \lor x \in B \}$$

If disjoint: $|A \cup B| = |A| + |B|$

Intersection: $A \cap B = \{ x : x \in A \land x \in B \}$

Subsets + Lists

Subset: a set of elements also in another set

Q: Given a set S, find # of subset

↳ $S = \{ a, b, c, d \ldots \}$

In subset / Not in subset $\Rightarrow$ Every elem. has 2 choice.

$\therefore$ $|\text{subsets of } S| = 2^n$

List: orders of elements.

Subset: $\{a, b, c\} = \{c, b, a\}$

List: $[a, b, c] \neq [c, b, a]$

Q: # of lists for n elem set:

$$\underbrace{\underline{n} \cdot \underline{n-1} \ \underline{n-2}}_{\text{Common approaches to find \# of arrangements.}} \ldots = n!$$

Partial List

Partial list = k-length list

$$\# \text{ of partial lists} = \underline{n} \times \underline{n-1} \times \underline{n-2} \ldots \underline{n-k+1}$$

k slots

$$= \frac{n!}{(n-k)!}$$ $\Rightarrow$ $k > n \Rightarrow 0$ partial lists

Assumption: partial lists are distinct

Count Subsets

$$\# \text{ of k-length subsets} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Considerations:

1. $k > n$: 0
2. $k = 1 \Rightarrow \binom{n}{k} = n$
3. $k = 0 \Rightarrow \binom{n}{0} = 1$
4. $k = n \Rightarrow \binom{n}{n} = 1$

Combinatorial Proof Techniques: Double Counting

Idea: show equality by counting in 2 diff ways

↳ often helpful when trying to prove a formula

Ex://

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

Trick: understand what each term means in terms of # of lists, subsets / partial lists.

① LHS:

Create subset of length 1, or 2, or 3, ...

Create all possible subsets of set of length n!

② RHS:

\# of create all subsets of set of n elements.

∵ Counting same thing ⇒ ☒

Can only use double counting if counting from same set

↳ Ex: // $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ cannot be proven by double counting b/c LHS is from set of n elem, RHS is from set of $n-1$ elem.

Combinatorial Proof Techniques: Bijections.

Idea: show how 1 elem on LHS ⟶ 1 elem on right hand side.

↳ $|LHS| = |RHS|$

How to show bijection:

① Create function: $f: A \longrightarrow B$

(2a) Injection / surjective method:

Show:

① $f(a) = f(a') \Rightarrow a = a'$ (injection)

② $\forall b \in B, \exists a \; f(a) = b$ (all elem. in range has domain elem.)

(2b) Inverse:

Show that $g(\cdot)$ exists s.t.

$$\left.\begin{aligned} &1.\ \forall a \in A: \ g(f(a)) = a \\ &\qquad\text{OR} \\ &2.\ \forall b \in B: \ f(g(b)) = b \end{aligned}\right\} \text{Not super mathy}$$

③ Well formed:

$\forall a \in A \quad f(a) \in B$

Ex: // Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

① Draw:
↳ Inspiration for bijection

② Define bijection:

A = k elem subsets of n elem.

B = k elem subset of n-1 elem OR k-1 subsets of n-1 elem.

$f: A \to B$

$$f(x) = \begin{cases} x & \text{if } n \notin x \\ x \setminus \{n\} & \text{if } n \in x \end{cases}$$

some subset in A

③ Show well formed:

Explanation.

④ Inverse method:

$g: B \to A$

$$g(Y) = \begin{cases} Y & \text{if } |Y| = k \\ Y \cup \{n\} & \text{if } |Y| = k-1 \end{cases}$$

Expl. of whether g is inverse of A

⑤ Claim:

This is a bijection $\Rightarrow |A| = |B|$

Complement Rule:

$$\binom{n}{k} = \binom{n}{n-k}$$

$\Rightarrow$ Idea of not choosing is equiv. to choosing is a good bijection example.

Proof:

#1: Double counting:

LHS: choosing k objects from n objects

RHS: to choose k objects from n objects, I can choose n-k objects to NOT include in subset $\Rightarrow$ equiv.

$\therefore$ Double counts = $|LHS| = |RHS|$

#2: Bijection:

Let A = set of all k-elem subsets of n items.

B = " $(n-k)$ "

$f: A \to B$: set of n elem.

$$f(x) = S - x$$

$g: B \to A$:

$$g(y) = S - x$$

Proving inverse:

$$g(f(x)) = S - (S - x)$$
$$= S - S + x$$
$$= x \quad \blacksquare$$

Multiset

Choosing t types, each with $n_1, n_2 \ldots n_t$ # of items s.t.

$n_1 + n_2 + n_3 + \ldots + n_t = n$

Representation: $(n_1, n_2, \ldots n_t)$

Another way: # of combos of t types of items s.t. always n items in total

Formula:

$$\text{\# of multisets of } t \text{ type from } n \text{ elements} = \binom{n+t-1}{t-1}$$

WEEK 2

Power Series

$$P(x) = 1 - 2x + 3x^2 + \ldots \Rightarrow \text{polynomial rep.}$$

$$= \sum_{n \geq 0} (-1)^n (n+1) x^n \Rightarrow \text{Sigma rep.}$$

Be able to:

1. Convert polynomial ⟺ sigma
2. Reindex sigma sums
↳ Common! Biggest suggestion: write out 1st few terms to reindex

Common power series:

① Geometric series:

$$P(x) = 1 + x + x^2 + \dots$$
$$= \frac{1}{1-x}$$
$$= \sum_{n \geq 0} x^n$$

② Binomial series / binomial theorem:

$$P(x) = ((1+x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Proof: induction on n, keep k constant

↳ Useful: if LHS or RHS doesn't look like set ⟹ use induction

↳ Useful: if induction required; keep 1 variable constant

③ Negative binomial series

$$P(x) = (1-x)^{-t} = \sum_{n \geq 0} \binom{n+t-1}{t-1} x^n$$

Generating series: function that describes some process, w/ coef. of each x^i telling # of objects of size i

$$\Phi(x) = 3x + 4x^2 + 5x^3 \dots + 264x^{15} \dots$$

At $n = 15$ ⟹ 264 items!

Beauty: generalize counting problem ⟹ only look @ coefficients.

Be able to take polynomial ⟷ sequence

Ex:// $\Phi(x) = \frac{1}{2-x^2}$ → Find the generating series.

① Put this into form of 3 known generating series.

$$\Phi(x) = \frac{1}{2}\left(\frac{1}{1-\frac{x^2}{2}}\right)$$

$$= \frac{1}{2} \sum_{n \geq 0} \left(\frac{x^2}{2}\right)^n$$

$$= 2^{-1} \sum_{n \geq 0} 2^{-n} x^{2n}$$

$$= \sum_{n \geq 0} 2^{-n-1} x^{2n}$$

Ex://

$$\frac{1}{1 - \frac{x}{1-x}}$$

① Simplify:

$$\frac{1-x}{1-x-x} = \frac{1-x}{1-2x}$$

② Split up & solve part that looks like known series.

$$\frac{1-x}{1-2x} = (1-x) \cdot \frac{1}{1-2x}$$

$$= (1-x) \cdot \sum_{n \geq 0} (2x)^n$$

$$= \sum_{n \geq 0} (2x)^n - x \sum_{n \geq 0} (2x)^n$$

③ Combine via reindexing?

$$\frac{1-x}{1-2x} = \sum_{n \geq 0} (2x)^n - \sum_{n \geq 0} 2^n x^{n+1}$$

Can combine if x raised to same power ⇒ reindex s.t. pow is same

$$\sum_{n \geq 0} (2x)^n - \sum_{n \geq 1} 2^{n-1} x^n$$

Make the same summation

$$1 + \sum_{n \geq 1} (2x)^n - \sum_{n \geq 1} 2^{n-1} x^n$$

$$\left\{ \begin{array}{l} 2^n x^n - 2^{n-1} x^n \\ 2^{n-1} x^n (2-1) \\ 2^{n-1} x^n \end{array} \right.$$

Combine:

$$1 + \sum_{n \geq 1} 2^{n-1} x^n$$

Ex: $\left(\frac{x}{1+2x}\right)^5$

① Simplify & make it look like common functn.

$$\left(\frac{x}{1+2x}\right)^5 = x^5\left(\frac{1}{1+2x}\right)^5$$

$$= x^5 \underbrace{(1+2x)^{-5}}_{NBT}$$

② Apply NBT

$$\left(\frac{x}{1+2x}\right)^5 = x^5 \sum_{n\geq 0} \binom{n+5-1}{5-1} (-2x)^n$$

$$= x^5 \sum_{n\geq 0} \binom{n+4}{4} (-2)^n x^n$$

$$= \sum_{n\geq 0} \binom{n+4}{4} (-2)^n x^{n+5}$$

$$= \sum_{n\geq 5} \binom{n-1}{4} (-2)^{n-5} x^n$$

Coefficient Extraction

3 rules:

① $[x^k](aF(x) + bG(x)) = a[x^k]F(x) + b[x^k]G(x)$

↳ Linearity of coefficients.

② $[x^k](x^\ell F(x)) = [x^{k-\ell}]F(x)$

↳ Shifting of coefficients

③ $[x^k](F(x)G(x)) = \sum_{\ell=0}^{k} ([x^\ell]F(x))([x^{k-\ell}]G(x))$

↳ Product of coefficients.

Ex: $[x^4](1+2x)(1-3x)^{-3}$

$$[x^4](1+2x)(1-3x)^{-3} = [x^4](1+2x)\sum_{n\ge0}\binom{n+2}{2}(3x)^n$$

$$= [x^4]\left[\sum_{n\ge0}\binom{n+2}{2}(3x)^n + 2x\sum_{n\ge0}\binom{n+2}{2}(3x)^n\right]$$

$$= [x^4]\sum_{n\ge0}\binom{n+2}{2}(3x)^n + 2[x^3]\sum_{n\ge0}\binom{n+2}{2}(3x)^n$$

$$= 3^4\binom{6}{2} + 2\cdot 3^3\cdot\binom{5}{2}$$

Usual process for extraction

① Split up into individual series s.t. it is a common series.

↳ If cannot & product → Rule 3

② Create series

③ Find coefficients.

Generating Series for Sets

We want a series that describes a set

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

weight

# of items in set that has weight.

For some set S and weight function $w(\sigma)$, $\sigma \in S$

Description:

$$\Phi_S(x) = \sum_{\sigma\in S} x^{w(\sigma)}$$

$$= \sum_{n\ge0} \left|\{\sigma : \sigma\in S, w(\sigma)=n\}\right| x^n$$

Coefficient:

Combinatorics!!

$$[x^n]\,\Phi_S(x) = \left|\{\sigma : \sigma\in S, w(\sigma)=n\}\right|$$

Ex:// $S = \{\text{multisets on } t \text{ types}\}$. $w(\sigma) = |\sigma|$

Find a generating series that describes S

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

$$= \sum_{n \geq 0} |\#\text{ of multisets w/ } n \text{ elements}|\, x^n$$

$$= \sum_{n \geq 0} \binom{n+t-1}{t-1} x^n$$

Ex:// $S = \{\text{subsets of } \{1 \ldots t\}\}$. $w(\sigma) = |\sigma|$

Find a gen. series for S

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

$$= \sum_{n \geq 0} \|\text{subsets of } (1 \ldots t) \text{ of length } n\|\, x^n$$

$$= \sum_{n \geq 0} \binom{t}{n} x^n$$

Binomial theorem

$$= (1+x)^n$$

WEEK 3

Sum and Product Lemmas

Sum lemma: $\Phi_{S_1 \cup S_2}(x) = \Phi_{S_1}(x) + \Phi_{S_2}(x)$

All have same weight functions.

Product lemma:

$$\Phi_{S_1 \times S_2}(x) = \Phi_{S_1}(x) \cdot \Phi_{S_2}(x)$$

$w(\sigma_1, \sigma_2) = w_1(\sigma_1) + w_2(\sigma_2)$ $\qquad w_1(\sigma)$ $\qquad w_2(\sigma)$

Infinite sum lemma: $S = S_1 \cup S_2 \ldots\ldots$

$$\Phi_S(x) = \sum_{n \geq 0} \Phi_{S_n}(x)$$ $\Rightarrow$ Gen. series of gen. series.

Ex:// Let $S = \{1, 3, 5, \ldots\}$. How many pairs $(n_1, n_2) \in S$ s.t. $n_1 + n_2 = 100$.

① Create a generating series for base set

$$S = \{1, 3, 5, \ldots\} \quad) \; w(\sigma) = \sigma$$

$$= x + x^3 + x^5 + \ldots$$

$$= x(1 + x^2 + x^4 + \ldots)$$

$$= x \sum_{n \geq 0} x^{2n}$$

$$= x \cdot \frac{1}{1-x^2}$$

$$= \frac{x}{1-x^2}$$

② Create generating series for pairs:

$$\underset{\substack{\uparrow \\ n_1}}{\Phi_S(x) = \frac{x}{1-x^2}} \cdot \underset{\substack{\uparrow \\ n_2}}{\frac{x}{1-x^2}} = \left(\frac{x}{1-x^2}\right)^2$$

③ Find 100^{th} coefficient:

Reasoning: If we want $n_1 + n_2 = 100$, we want series where

$100 = w(\sigma_{n_1}, \sigma_{n_2}) = w_1(\sigma_{n_1}) + w_2(\sigma_{n_2}) \Rightarrow$ PRODUCT LEMMA.

$$[x^{100}]\Phi_S(x) = [x^{100}]\left(\frac{x}{1-x^2}\right)^2$$

$$= [x^{100}]\, x^2 \cdot \frac{1}{(1-x^2)^2}$$

$$= [x^{98}] \sum_{n \geq 0} \binom{n+1}{1} (x^2)^n$$

$$n = 49 \Rightarrow \text{Coef.: } \binom{49+1}{1} = \binom{50}{1} = 50$$

String Lemma

Let's say we have set $S = \{\ldots\}$

$S^* = \{$ all possible tuple of elem. in S $\}$

Ex:// $S = \{0, 1\}$

$S^* = \{\emptyset, (0), (1), (0,0), (0,1), \ldots\}$

Special weight function: $w(\sigma) = \sigma$

String lemma: $\Phi_{S^*}(x)$ on $w(\cdot)$, then:

$$\Phi_{S^*}(x) = \frac{1}{1 - \Phi_S(x)}$$

Compositions

Composition: tuple of > 0 ints $(n_1, n_2 \ldots)$

parts part

- Weight: $n_1 + n_2 + \ldots$
- Alt. idea: $\{1, 2, 3, \ldots\}^*$ and $w((n_1, \ldots n_t)) = n_1 + \ldots + n_t$

Basic composition:

$$\Phi_{\{\text{compositions}\}}(x) = \Phi_{\overset{n_1}{\mathbb{Z}_1}}(x) + \Phi_{\overset{n_2}{\mathbb{Z}_2}}(x) + \ldots$$

$$= \Phi_{\mathbb{Z}^*}(x)$$

$$= \frac{1}{1 - \Phi_{\mathbb{Z}>0}(x)} \implies \Phi_{\mathbb{Z}>0}(x) = 1 + 2 + \ldots$$

$$= \sum_{n \geq 1} x^n$$

$$= \sum_{n \geq 0} x^{n+1}$$

$$= x \sum_{n \geq 0} x^n$$

$$= \frac{1}{1 - x \sum_{n \geq 0} x^n}$$

$$= \frac{1}{1 - \frac{x}{1-x}}$$

$$= \frac{1-x}{1-2x}$$

Ex: # of compositions of 239

$$[x^{239}] \frac{1-x}{1-2x} = [x^{239}] (1-x) \sum_{n \geq 0} (2x)^n$$

?

?

$$= 2^{238}$$

Ex: # of compositions of 239 with odd parts.

① Example: $(1, 211, 5, \ldots)$

② Series for 1 part:

1 part $\in \{1, 3, 5, \ldots\}$

$$\Phi_A(x) = x + x^3 + x^5 + \ldots$$
$$= \frac{x}{1-x^2}$$

③ Sum/product/infinite/string lemma for 1 composition.

$$\Phi_S(x) = \underbrace{\left(\frac{x}{1-x^2}\right)}_{n_1} \cdot \underbrace{\left(\frac{x}{1-x^2}\right)}_{n_2} \cdot \boxed{\ldots} \Rightarrow \text{String lemma}$$

$$= \frac{1}{1 - \frac{x}{1-x^2}}$$

$$= \frac{1-x^2}{1-x-x^2}$$

④ Find $[x^{234}]$

To find x^{234} ⇒ put into series.

$$\frac{1-x^2}{1-x-x^2} = \sum_{n\geq 0} a_n x^n$$

$$\frac{1-x^2}{1-x-x^2} = a_0 + a_1 x^1 + a_2 x^2 + \ldots$$

$$1 - x^2 = (a_0 + a_1 x + a_2 x^2 + \ldots)(1 - x - x^2)$$

$$1 - x^2 = (a_0 + a_1 x + \ldots) - (a_0 x + a_1 x^2 + \ldots) - (a_0 x^2 + a_1 x^3 + \ldots)$$

$$= a_0 + \underbrace{(a_1 - a_0)}_{0} x + (a_2 - a_1 - a_0) x^2 + \underbrace{(a_3 - a_2 - a_1)}_{0} x^3 \ldots$$

Linear eqn by equating coeff.

$a_0 = 1$ — ①

$a_1 - a_0 = 0 \Rightarrow a_1 = 1$ — ②

$a_2 - a_1 - a_0 = -1$

$a_n - a_{n-1} - a_{n-2} = 0$

$a_n = a_{n-1} + a_{n-2}$

$$a_2 - 2 = -1$$
$$a_2 = 1$$

Recursion:

$$a_n \begin{cases} a_0 = 1 \\ a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

Tip: If particular next $\Phi_S(x) \Rightarrow$ build recurrence.

① Find:

$$\Phi_S(x) = a_0 + c_1 x + c_2 x^2 + \ldots$$

② Multiply denom

③ Group into common x

④ Coefficient equating on LHS & RHS to find base case.

⑤ For all higher powers of x w/ no coef. on LHS $\Rightarrow$ find recusion.

If changing content of part $\Rightarrow$ change step ① accordingly

If changing # of parts $\Rightarrow$ change step ② (infinite sum / string!)

WEEK 4

Binary Strings + Regex

Binary: 0, 1 . . . .

- Empty: ε

Ex:// Find all binary strings of length:

Think of this exactly as you think about compositions

$(\underset{q}{0}, 1, 0 \ldots \ldots)$

① # of choices in part

$\{0, 1\} \Rightarrow 2x \Rightarrow$ Gen. series.

② G.S. for entire composition.

Infinite # of parts $\Rightarrow$ string lemma.

$$\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A}$$

$$= \frac{1}{1-2x}$$

Concatenation: $\sigma_1 \sigma_2 \ldots$

- $(110)(011) \Rightarrow 110011$
- Sets of binary strings:
  - ↳ Concat. product: $S_1 S_2 = \{\sigma_1 \sigma_2 : \sigma_1 \in S_1, \sigma_2 \in S_2\}$ All possible concats.
  - ↳ Concat. power: $S_1^{(k)} = \underbrace{S_1 S_1 \ldots S_1}_{k \text{ times.}}$

Regular expressions: lang. of binary string creation.

- $\varepsilon, 0, 1$
- $R_1^*, R_1 \cup R_2, R_2 R_1$ are all regex if R_1, R_2 are regexs.

$R_1^* = R_1 R_1 \ldots \ldots, \varepsilon, R_1 R_2, R_1, R_1 \ldots$

$R_1 \cup R_2 =$ "or"

$R_1 R_2 \Rightarrow$ concat.

Ambiguity:

Regex is ambiguous $\Rightarrow$ $\exists$ 2≥ ways of creating 1 string from regex

Ex:// $(1 \cup 10)(1 \cup 01)$. Show that this is ambg.

We need to show that $\exists$ string that is created multiple times.

Generate couple of strings.

$101 \to 1, 01$
$101 \to 10, 1$
} Ambiguous.

How to prove unambiguity:

Recursive defn. of unambiguity:

- $\varepsilon, 0, 1 \Rightarrow$ unambiguous.
- R, S are unambiguous.

$R \cup S$ is unambig's $\Leftrightarrow R \cap S = \emptyset$

RS is unambig $\Leftrightarrow \alpha$ is a bijection between $A \times B \to AB$

R^* is unambig $\Leftrightarrow \alpha$ is a bijection from Cart. product $\to$ concat. product.

α: tuple of bits $\to$ concat. of bits

$\alpha((0, 1, 1)) = 011$

Trick: decompose string $\Rightarrow$ show that parts are unambiguous by

a) Showing well defined

b) Can't get same string via "math arg."

c) Can get all binary strings.

Ex:// Show $0^*(10^*)^*$ is unambiguous.

① Show general string:

$$0 \ldots 0 \; (10 \ldots 0)(10 \ldots 0), \ldots (10 \ldots 0)$$

② All binary strings can be created. "Math arg"

Always have n leading zeros, m ones and p 0s between each 1.
Unique definition of binary string b/c n, m, p are unique to string.
$\therefore$ Unambiguous.

③ Recursively for each part

$$0^* \cap (10^*)^* = \emptyset$$

$0^*(10^*)^* \Rightarrow$ can't get a string that has same concat.

Block decomposition: unambiguous method to show all strings:

$$0^*(11^*00^*)^*1^*$$
or $$1^*(00^*11^*)^*0^*$$
} Produces all binary strings.

- Use decomposition if:
  - # of 0s / 1s
  - Positions
  - Relations between $0 \leftrightarrow 1$

} Substitute 0, 1 blocks to match.

Ex:// Regex for all binary strings w/ even # of 0s.

$$(00)^*(11^*00(00)^*)1^*$$

Ex:// No 111 as substring:

$$(\varepsilon \cup 1 \cup 11)(00^*(1 \cup 11))^*0^*$$

No way to create 111

Note that we need ε if remove 0

String + Gen. Series.

Convert unambiguous regex $\longrightarrow$ generating series (mapping length to $[n]$)

Allow us to find # of strings of some length that follows prop.

① Convert property into regex (unambiguous)

② Convert regex into generating series.

- $\varepsilon \Rightarrow 1$
- $0, 1 \Rightarrow x$
- R, S are regexes:

$$R \cup S: \quad f(x) + g(x)$$
$$RS: \quad f(x)\, g(x)$$
$$R^* = \frac{1}{1-f(x)}$$

③ Find coefficient.

Ex:// # of strings of length 20 where each block is length 1/2.

① Regex:

$$0^* (11^* 00^*) 1^*$$

$$(\varepsilon \cup 0 \cup 00) \; ((1 \cup 11)(0 \cup 00))^* \; (\varepsilon \cup 1 \cup 11)$$

② Convert into gen. series:

$$\varepsilon \cup 0 \cup 00 \Rightarrow 1 + x + x^2 \quad \Longrightarrow \quad \varepsilon \cup 1 \cup 11$$

$$((1 \cup 11)(0 \cup 00)) = (x + x^2)(x + x^2) = (x + x^2)^2$$

$$((1 \cup 11)(0 \cup 00))^* = \frac{1}{1 - (x + x^2)^2}$$

$$\Phi_R(x) = (1 + x + x^2)^2 \cdot \frac{1}{1 - (x + x^2)^2}$$

③ Find coef:

$$[x^{20}] \, \Phi_R(x)$$

WEEK 5

Recursive Binary strings

① How to construct a recursive decomp.

Define regex S in terms of itself.

$$S = S \cup 01$$

$$S = 0S^* 1$$

Unambiguous: only produce 1 string once.

↳ Playing around. ⇒ strategy to prove unambiguity.

Strategy: get a set of string ⇒ regex recursively?

↳ Piece strings together.

↳ Look at # of 0s, 1s ⇒

↳ Look at anomalies (ε ...)

② Generating series:

Convert your regex into $\Phi_S(x)$ ⇒ solve for $\Phi_S(x)$ w/ alg.

Ex:// $S = \varepsilon \cup 1 \cup 0S0$

① Replace regex w/ gen. series:

$$\Phi_S(x) = 1 + x + x^2 \Phi_S(x)$$

② Solve:

$$(1-x^2)\Phi_S(x) = 1 + x$$

$$\Phi_S(x) = \frac{1+x}{1-x^2}$$

How do you write recursion for strings that don't have a substring?

Ex:// Find gen. series w/ no 1111

① Define another set such that it only has 1 occurrence of the substring at the very end.

T = {set of all strings w/ 1 occurrence of 1111 at end}

② Define 2 equations

1. S → something in T
2. S (substring) → possible combos in relation to T

000111

① $\varepsilon \cup S(0 \cup 1) = S \cup T$

② $S(1111) = T \cup T(1) \cup T(11) \cup T(111)$

...00001111

...0001111 1

$\underbrace{\text{...0001111}}_{T}$

$\underbrace{\text{0011 1}}_{T}\text{111}$

③ Solve 'simultaneous eqn' w/ gen. series.

① $1 + \Phi_S(x) \cdot x^2 = \Phi_S(x) + \Phi_T(x)$

② $x^4 \Phi_S(x) = \Phi_T(x) + x\Phi_T(x) + x^2\Phi_T(x) + x^3\Phi_T(x)$

Extracting Coefficients.

Problem: rational expressions are ugly $\Rightarrow$ can't be NB, Geometric, Binomial.

Solution: Partial Fraction Decomp.

Ex://

$$\Phi_S(x) = \frac{1+x}{1-5x+6x^2} = \frac{1+x}{(1-2x)(1-3x)}$$

$$= \frac{-3}{1-2x} + \frac{4}{1-3x}$$

$$\therefore [x^n]\,\Phi_S(x) = [x^n]\frac{-3}{1-2x} + [x^n]\frac{4}{1-3x} \Rightarrow \text{We can do!}$$

Partial Fraction Decomp (PFD).

Basic: denomi is factored into distinct linear factors.

$$\frac{P(x)}{Q(x)} = \frac{A}{(1-\lambda_1)} + \frac{B}{(1-\lambda_2)} + \cdots + \frac{C}{(1-\lambda_n)}$$

$$P(x) = \frac{A}{1-\lambda_1}Q(x) + \cdots$$

Equate coefficients.

To do this quicker, use cover-up method

Int: denom. is factored into linear factors that are not distinct (eg. $(x+1)^2(x-2)$)

Exact same thing, but then add another frac. for each repeated factor.

$$\frac{-2+6x+2x^2}{(1-2x)^2(1-5x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-5x}$$

Recurrence Relations

We have recurrence $\Rightarrow$ want a formula that is deterministic

Soln: find rational expr $\Rightarrow$ use PFD $\Rightarrow$ $[x^n]$ will give us formula.

Ex://

$$a_n = \begin{cases} 2 & \text{if } n=0 \\ 7 & \text{if } n=1 \\ a_{n-1}+2a_{n-2} & \text{if } n \geq 2. \end{cases}$$

Find formula:

① Create homogenous linear recurrence:

Recurrence w/ RHS = 0

$\therefore\ a_n - a_{n-1} + 2a_{n-2} = 0\ ,\ n \geq 2$

② Find rel. of each term in linear recurrence w/ $[x^n]A(x)$

$a_n \Rightarrow [x^n]A(x)$

$a_{n-1} \Rightarrow [x^{n-1}]A(x) = [x^n](x A(x)) \longrightarrow [x^k]\, x^\ell B(x) = [x^{k-\ell}]B(x)$

$2a_{n-2} \Rightarrow 2[x^{n-2}]A(x) = 2x^2[x^n]A(x)$

$[x^n]A(x) - x[x^n]A(x) + 2x^2[x^n]A(x) = 0 \qquad n \geq 2$

$[x^n]A(x)\ (1 - x + 2x^2) = 0 \qquad n \geq 2$

$[x^n]\,P(x) = 0 \quad \forall n \geq 2$

↳ $P(x)$ has degree of ≤ 2 !!

③ Find RHS polynomial:

$$A(x)\,(1-x+2x^2) = c_1 + c_2 x$$

$$A(x) = \frac{c_1 + c_2 x}{1-x+2x^2}$$

$$= \frac{c_1 + c_2 x}{(1+x)(1-2x)}$$

$$= \frac{D_1}{1+x} + \frac{D_2}{1-2x}$$

$$[x^n]A(x) = D_1(-1)^n + D_2(2)^n$$

④ Plug in initial conditions:

$\vdots$

$$a_n = 3 \cdot 2^n - (-1)^n$$

TIME CONSUMING

Theorem: $c_1, \dots, c_k, \lambda_1 \dots \lambda_s \in \mathbb{C}$. Each λ_i is distinct.

Also: $1 + c_1 x + c_2 x^2 + \dots + c_k x^k = (1+\lambda_1)^{d_1} (1+\lambda_2)^{d_2} \dots (1+\lambda_s)^{d_s}$

If we have a recurrence: $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = 0$

then:

$$a_n = p_1(n) \lambda_1^n + p_2(n) + \lambda_2^n + \dots + p_s(n) \cdot \lambda_s^n$$

↳ Each $p_i < d_i$

Ex://

$$a_n = \begin{cases} 0 & , n = 0 \\ -5 & , n = 1 \\ -1 & , n = 2 \\ 3a_{n-2} - 2a_{n-3} & , n \geq 3 \end{cases}$$

Formula:

① Linear homogeneous recurrence:

$$a_n - 3a_{n-2} + 2a_{n-3} = 0 \quad \forall n \geq 3$$

② Characteristic polynomial:

Coef. of each term ⇒ coef. of x^n

$$1 - 3x^2 + 2x^3 = 0$$

③ Factor:

$$1 - 3x^2 + 2x^3 = (1-x)^2(1+2x)$$

$\therefore \lambda_1 = 1, \quad \lambda_2 = -2$

④ Formula:

$$a_n = P_1(x)(1)^n + P_2(x)(-2)^n$$

$\deg(P_1(x)) < 2 \Rightarrow P_1(x) = An + B$

$\deg(P_2(x)) < 1 \Rightarrow P_2(x) = C$

$$\therefore a_n = (An + B)(1)^n + C(-2)^n$$

⑤ Sub in initial conditions + solve for A, B, C

$\vdots$

$$a_n = -2n - 1 + (-2)^n$$

Recurrence $\rightarrow$ want a rational expr.

Ex://

$$g_n = \begin{cases} 2, & n=0 \\ 5, & n=1 \\ 6, & n=2 \\ 3g_{n-2} + 2g_{n-3}, & n \geq 3 \end{cases}$$

Find the gen. series.

① Write out general formula:

$$\Phi_G(x) = \sum_{n=0}^{\infty} g_n x^n$$

② Sub in values for g_n via recurrence:

$$\Phi_G(x) = g_0 x^0 + g_1 x^1 + g_2 x^2 + \sum_{n\geq 3} (3g_{n-2} + 2g_{n-3}) x^n$$

$$\Phi_G(x) = 2 + 5x + 6x^2 + \sum_{n\geq 3} (3g_{n-2} + 2g_{n-3}) x^n$$

③ Solve for $\Phi_G = \sum_{n\geq 0} g_n x^n \Rightarrow$ transform recursive part into this form!!

$$\sum_{n\geq 3} (3g_{n-2} + 2g_{n-3}) x^n = 3\sum_{n\geq 3} g_{n-2} x^n + 2\sum_{n\geq 3} g_{n-3} x^n$$

i) Make everything into g_n, not $g_{f(n)}$, reindex sum

$$3\sum_{n\geq 3} g_{n-2} x^n = 3\sum_{n\geq 1} g_n x^{n+2} = 3x^2 \sum_{n\geq 1} g_n x^n$$

Write out 1st few terms

$$g_1 x^3 + g_2 x^4 + g_3 x^5$$

$$2\sum_{n\geq 3} g_{n-3} x^n = 2\sum_{n\geq 0} g_n x^{n+3} = 2x^3 \sum_{n\geq 0} g_n x^n$$

$$g_0 x^3 + g_1 x^4 + g_2 x^5$$

$$\therefore \sum_{n\geq 3} (3g_{n-2} + 2g_{n-3}) x^n = 3x^2 \sum_{n\geq 1} g_n x^n + 2x^3 \sum_{n\geq 0} g_n x^n$$

ii) All sums start at zero: add/subtract terms

$$\sum_{n\geq 3} (3g_{n-2} + 2g_{n-3}) x^n = 3x^2 \sum_{n\geq 1} g_n x^n + 2x^3 \sum_{n\geq 0} g_n x^n$$

$$= 3x^2\left(\sum_{n\geq 0} s_n x^n - s_0 x_0\right) + 2x^3 \sum_{n\geq 0} s_n x^n$$
$$= 3x^2\left(\sum_{n\geq 0} s_n x^n - 2\right) + 2x^3 \sum_{n\geq 0} s_n x^n$$
$$= 3x^2 \sum_{n\geq 0} s_n x^n - 6x^2 + 2x^3 \sum_{n\geq 0} s_n x^n$$

$$\sum_{n\geq 1} s_n x^n = s_1 x^1 + s_2 x^2 + \cdots$$
$$= s_0 x_0 - s_0 x_0 + s_1 x_1 \cdots$$
$$= (s_0 x_0 + s_1 x_1 \cdots) - s_0 x_0$$
$$= \sum_{n\geq 0} s_n x^n - s_0 x_0$$

(iii) Substitute $\Phi_6(x) = \sum_{n\geq 0} s_n x^n$

$$\therefore \quad \Phi_6(x) = 2 + 5x + 6x^2 + 3x^2\,\Phi_6(x) - 6x^2 + 2x^3 \sum_{n\geq 0} s_n x^n$$
$$= 2 + 5x + 3x^2\,\Phi_6(x) + 2x^3\,\Phi_6(x)$$
$$\Phi_6(x)\left(1 - 3x^2 + 2x^3\right) = 2 + 5x$$
$$\Phi_6(x) = \frac{2+5x}{1 - 3x^2 + 2x^3}$$

WEEK 6

Introduction

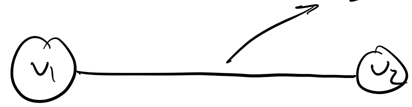
Graphs

Edges: $E(G)$ ⇓ $\{v_1v_2, v_3v_4, \ldots\}$

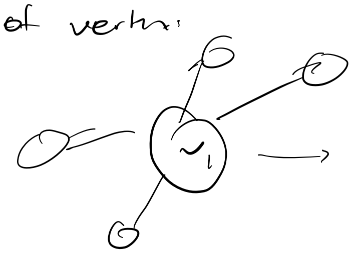
$v_1v_2 = v_2v_1$

Vertices: $V(G)$ ↳ Finite, non-empty

If edge: v_1 is adj to v_2

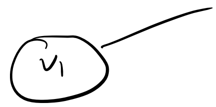


Neighbours of vertex:



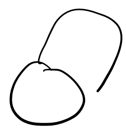
→ All nodes adjacent

Incident: edge incident to vertex: ⇒ one of endpoint of edge is on vertex.

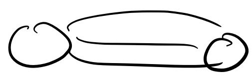


Studying simple graphs:

1. Edges unordered:
2. Loops not allowed:

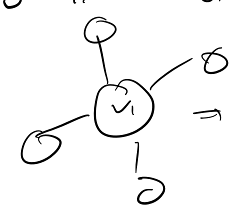


3. Multigraphs not allowed:



Degrees

$\deg(v_1)$ = # of edges incident to v_1



$\Rightarrow \deg(v_1) = 4$

Handshaking Lemma:

$$\sum_{v \in V(G)} \deg(v) = 2 \cdot |E(G)|$$

→ Use Lemma if looking at $|E(G)|$, or sum of degrees.

→ 1 edge = 2 degrees
(1 degree on 1 endpoint, another on another endpoint)

Corollary: # of vertices w/ odd degrees is even.

Sum of degrees is even ⇒ odd degree # is even.

Avg. degree: $\dfrac{\text{Sum of degrees}}{\#\text{ of vertices}} = \dfrac{2 \cdot |E(G)|}{|V(G)|}$

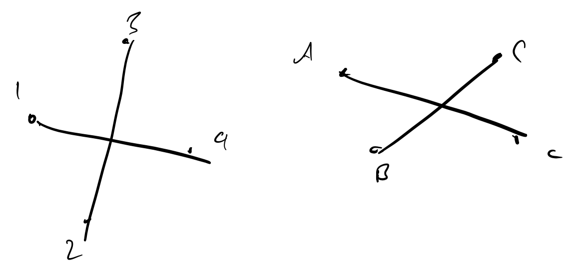
Isomorphism

Q: 2 graphs are the same or diff?

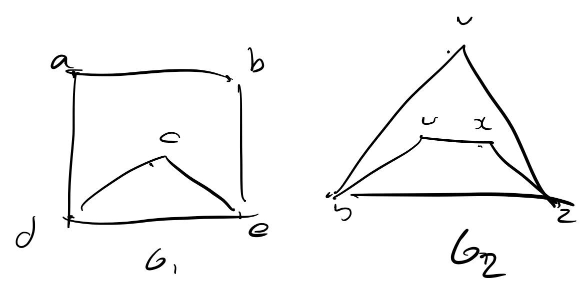
To show 2 graphs are the same, you need to do the following:

① Create a bijection between $V(G_1) \to V(G_2)$.

② Show that $\forall\ uv \in E(G_1) \iff f(u)f(v) \in E(G_2)$
$\forall\ uv \in E(G_2) \iff f^{-1}(u)f^{-1}(v) \in E(G_1)$



Bijection of vertices & edges

Ex:// 

Show isomorphic

① Define isomorphism

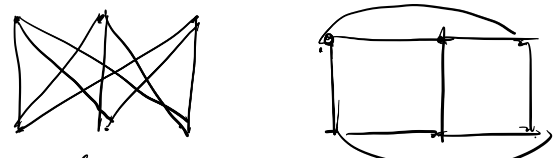
$f: V(G_1) \to V(G_2)$

$f(a) = w$ $\quad f(c) = v$ $\quad f(e) = z$
$f(b) = x$ $\quad f(d) = y$

② Bijection of edges:

$ab \Rightarrow f(a)f(b) = wx$ ✓ $\qquad vy \to$ ✓
$ad \Rightarrow f(a)f(d) = wy$ ✓ $\qquad zz -$ ✓
$dc \Rightarrow yv$ ✓ $\quad \ldots \quad wz \to$ ✓

Ex://



G_1 G_2

Show that it's not isomorphic

Key tricks: structural property

- \# of mutually adjacent ↳ Collection of nodes that are neighbors of each other.
- \# of degrees.
- \# of vertices
- \# of edges

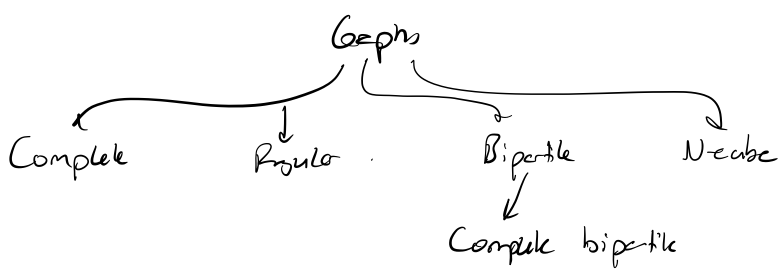
G_2 has 2 groups of 3 mutually adj. nodes $\Rightarrow$ does not exist G_1

$\therefore$ $G_1 \neq G_2$

Isomorphic class: collection of isomorphic graphs of particular structural property.

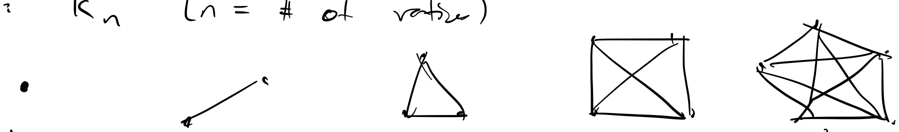
Isomorphism is an equivalence rel.

Graph Classification.



Complete: every pair of vertices has an edge.

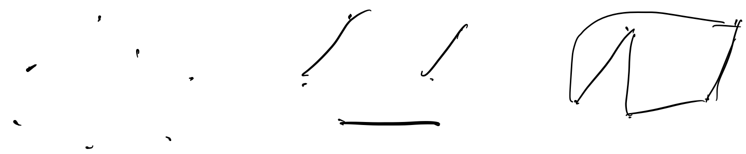
- Notation: K_n (n = # of vertices)



K_1 K_2 K_3 K_4 K_5

- $|E(G_n)| = \binom{n}{2}$ $\Rightarrow$ # of pairs in n vertices.

Regular graphs: k-regular ⇒ all vertices have deg k.



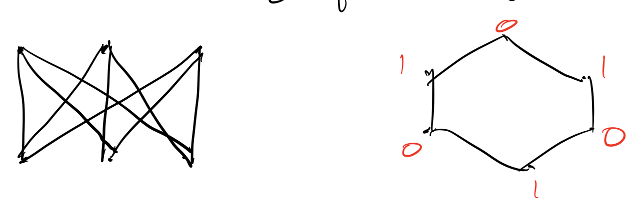
0 - regular   1 - regular   2 - regular graph

- # of edges?
  ↳ HL: $\sum_{v \in V(G)} \deg(v) = 2 \cdot |E(G)|$

$$nk = 2 \cdot |E(G)|$$

$$|E(G)| = \frac{nk}{2}$$

Bipartite graph: split graph into 2 groups, all edges connect between groups (no edges w/in groups)



- Visually identifying a bipartite graph:
  1. Stick 1 node in 1 group
  2. All neighbors in other group
  3. Alternate w/in graph.

- Smallest bi-partite graph is k_3
- Show something is/is not bipartite:

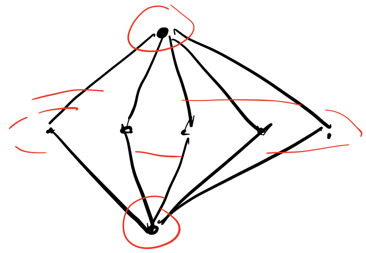
  Not bipartite: contradiction ⇒ edge w/in a group

  Bipartite: structural property that would connect 2 groups $\forall e \in E(G)$

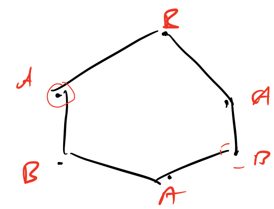
Subtype: complete bipartite graph.

- $K_{M,N}$: Group A = # of nodes ⇒ m; Group B = # of nodes ⇒ n } All vertices in a $K_{m,n}$ graph will connect <u>all</u> m vertices w/ <u>all</u> n vertices!

$K_{2,5}$



Not a complete bipartite:



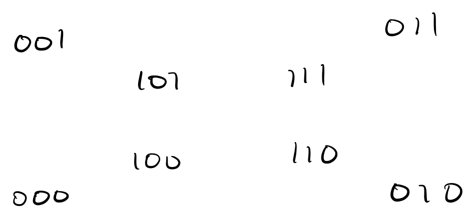
⇒ A node in A is not connected w/ ALL nodes in B.

\# of edges = mn (# of ways to pair node in A w/ B)

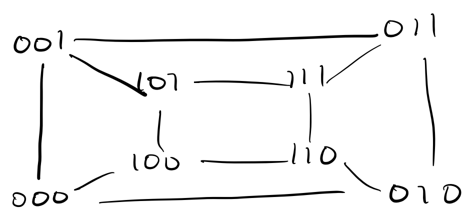
N-cube:

① All binary strings of length n ($n=?$): 010, 111, 110, ...

② Vertex = string:



③ Edges: between nodes w/ exactly 1 difference in character.



⇒ n-cube ($n=3$)

Statistics:

\# of vertices?

\# of binary strings of length n = 2^n

Regular graph?

n-regular graph: n neighbors by changing 1 character once among n characters.

\# of edges?

$$\sum \deg = 2 \cdot |E(G)|$$

$$|E(G)| = \frac{\sum \deg}{2} = \frac{2^n \cdot n}{2} = n \cdot 2^{n-1}$$

Bipartite graph: Yes?

↳ Proving a bipartite graph

① Divide vertices into 2 groups w/ diff properties.

A = vertices w/ odd 0, B = vertices w/ even 0

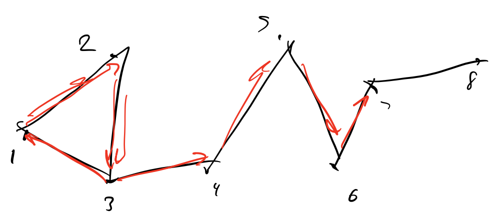
② For any arb. edge: always maps from $A \to B$.

Edge between 2 nodes ⇒ nodes w/ diff parity of 0s.

WEEK 7

Walks & Paths

Walk: seq. of vertices + edges.



Walk from $v_1 \to v_7$

$v_1, v_1v_2, v_2, v_2v_3, v_3 \ldots$

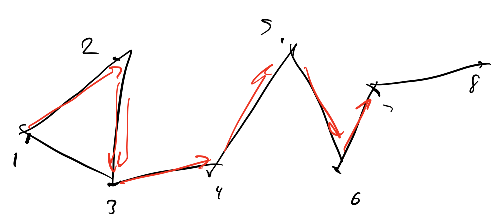
↓

$v_1, v_2, v_3, v_1, v_2, v_3, v_4 \ldots$

Length of walk: # of edges of edge walk = k

Special type ⇒ closed walk:

Path:



⇒ A vertex is visited 1x

↳ No repeated edges, vertices.

Theorem: uv walk ⇒ uv path.

Proof: Induction on # of repeated vertices (k) in walk.

Base case: $k=0$ ⇒ walk = path

Ind. hyp: Assume that uv walk w/ k repeated vertices $\Rightarrow$ u,v path exists.

Ind. step: Show that uv walk w/ $k+1$ repeated vertices $\Rightarrow$ u,v path.

Assume that walk $u, v_1, v_2 \dots, v_i, v_{i+1} \longrightarrow v_j, v_j \longrightarrow v$

repeated vertex.

Remove repeated vertex

$\hookrightarrow$ $u \longrightarrow v_i$ has $\leq k$ repeated vertices $\Rightarrow$ path exists

$v_j \longrightarrow v$ ''

$\therefore$ $u-v$ path exists.

Proof: Longest/shortest path proofs.

Suppose $v_0 \dots v_k$ is a walk that is shortest in length.

If no repeated vertices $\Rightarrow$ path ✓

If repeated vertex: $\exists$ i, j s.t. $i < j$, $v_i = v_j$

$\therefore$ $v_0 \dots v_i \dots v_{j+1} \longrightarrow v_k$ is also a path to v_k $\Rightarrow$ shorter walk! Contradiction

$\therefore$ u-v path exists.

Theorem: $u-v$ path $\wedge$ $v-w$ path $\rightarrow$ $u-w$ path.

Subgraphs & Cycles

Recursively decompose graphs $\rightarrow$ subgraphs

Theorem: H is a subgraph $\Rightarrow$ $V(H) \subseteq V(G)$ $\wedge$ $E(H) \subseteq E(G)$ w/ same endpoints.

Spanning subgraph: $V(H) = V(G)$

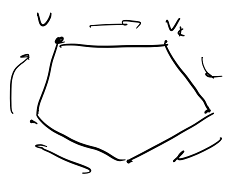
Proper subgraph: $H \neq G$

Notation:

- Subgraph w/out edge: $H - e$ $\rightarrow$ e is $\subseteq E(H)$
- Subgraph w/ edge: $H + e$

Cycles:

Subgraph w/ distinct edges + vertices s.t. $v_1 \dots v_k$ has edges $v_1v_2, v_2v_3 \dots, v_kv$

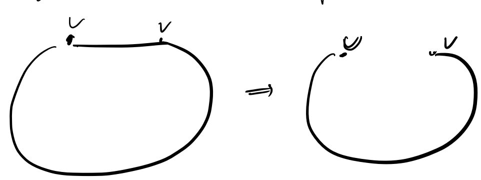


Length: # of edges $\Rightarrow k$

Cycle has minimum of 3

Cycle = closed walk w vertices in seq.

<u>Theorem</u>: Edge uv in cycle C. $C - uv \Rightarrow$ path between u and v.



<u>Theorem</u>: P is a u-v path $\Rightarrow P + uv$ is a cycle

<u>Theorem</u>: Cycle is 2-regular.

<u>Theorem</u>: $\forall v \in V(G), \deg(v) \geq 2 \Rightarrow G$ has a cycle

Proof: Longest path arg.

$v_0 \dots v_k$ is longest path between v_0 & v_k

End vertex has 2 edges $\rightarrow$ 1 edge in path, w/b other edge?

A. Edge is not in path

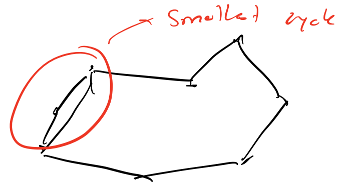
Larger path! $\Rightarrow$ Contradiction.

Edge must be in path. Call this node x.

$\therefore$ Cycle: $v_0 \dots v_i, x, v_0$

Property of graph: girth

- Length of smallest cycle in graph.



$\therefore$ Girth = 3

Hamilton cycle: spanning cycle ($V(C) = V(G)$)

Connectedness

Connected: $\forall\, v,u \in V(G) \Rightarrow \exists$ path between u,v

↳ # of paths: # of pairs of vertices $\Rightarrow \binom{|V(G)|}{2}$

Theorem If $\forall\, u,v \in V(G)$ have a path $\Rightarrow$ connected.

↳ If any node has $|V(G)|-1$ paths from it, we're good.

B/c of transitivity of paths.

Proof: Assume that $\forall\, u,v \in V(G)$ have a path

WLOG, pick vertex x. To construct path between any 2 vertices in graph: path $u-x$, path $x-v \Rightarrow u-v$.

Repeat for all pairs $\Rightarrow$ path between all pairs $\Rightarrow$ connected.

Ex:// Show that n-cube is connected.

① Fix a node

$V_0 : 0000 \ldots 0 \Rightarrow n$ 0's

② Show that we can get to any arb. vertex from fixed.

x = Vertex w/ k 1's. To go from $v_0 \to x$, process:

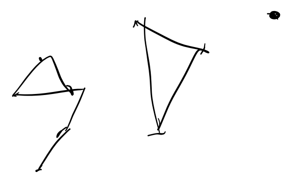
V_j has j 1's. V_{j+1} has $j+1$ 1's & differ in 1 position. $\therefore$ V_j & V_{j+1} has an edge.

$\therefore$ Path: $v_0, v_1, v_2 \ldots v_k = x$. We have a $v_0 - x$ path.

$\therefore$ By theorem, we know it's connected

Components & Cuts

Component: maximally connected graph

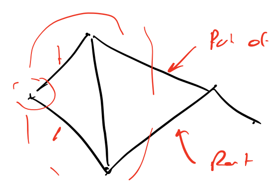


$\Rightarrow$ 3 components

Theorem: Graph is connected $\Rightarrow$ 1 component

Theorem: No edge from 1 component to another

Cuts: $X \subseteq V(G)$. Cut induced by $X \Rightarrow$ all edges w/ 1 endpoint in X



Theorem: Graph is disconnected $\Rightarrow$ empty cut induced by comp.

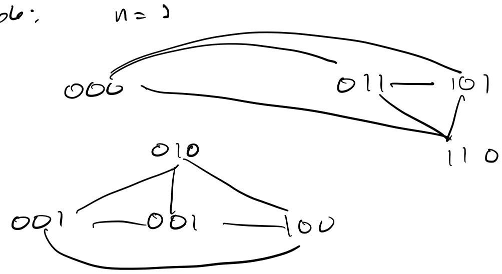
Theorem: A graph is disconnected $\Longleftrightarrow$ $\exists$ a non-empty $X \subseteq V(G)$ s.t. cut induced by X is empty

$\Rightarrow$: Graph is disconnected. At least 2 comp. H_1, H_2. Assume for contr. that there is an edge connecting $v_1 \in V(H_1)$, $v_2 \in V(H_2)$. $\therefore H_1, H_2$ are not maximally connected. Contr.

$\Leftarrow$: Assume path $\overset{G_1}{v_0} \ldots \overset{G_2}{v_k}$. $u = v_0$, $v_k = v$. Every vertex in path in X or not in X. Assume that one vertex in path is not in X. Let's say $\exists j \in j$ is maximal $v_j \in X$. $\therefore v_{j+1}$ not in X. Edge in cut induced by $v_j \Rightarrow$ contr. No u, v path exists $\Rightarrow$ disconnected.

Ex:// G_n is a graph of binary strings of length n. 2 strings adj. iff differ in 2 posns. Show that G_n is disconnected.

① Example: $n = 3$



② Property of distinct components

$A =$ vertices w/ even # of 1's

$B =$ vertices w/ odd # 1s

③ Show that no cut exists. between 2

Arbitrary string in A w/ k 1's. If we change 2 positions, 4 ways:

$k+2$	$0 \to 1,\ 0 \to 1$	Parity of 1's remain same.
k	$1 \to 0,\ 0 \to 1$	
$k-2$	$1 \to 0,\ 1 \to 0$	
k	$0 \to 1,\ 1 \to 0$	

∴ No edge in $A \to$ to B, parity is different.

∴ $\exists\ X \subseteq V(G)$, non-empty & proper w/ empty cut

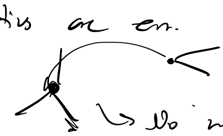
∴ Disconnection

Eulerian Circuit

Closed walk $\Rightarrow$ every edge used once.

Properties:

1. Connected: could have isolated nodes, assume that isn't
2. Degree of all vertices are even.



↳ No way to get out

Theorem: G is connected & degree is even $\forall v \in V(G) \Rightarrow$ Eulerian circuit exists.

Proof: Induction on # of edges m

Base case: $m = 0$

↳ No edges $\Rightarrow$ Eulerian circuit

Ind. hyp: Connected graph w/ even degrees + $< m$ edges $\Rightarrow$ Eulerian circuit

Ind. step: G is connected graph w/ m edges & all even degrees.

B/c even edges $\Rightarrow$ cycle in G.

If cycle contains all edges $\rightarrow$ Eulerian circuit

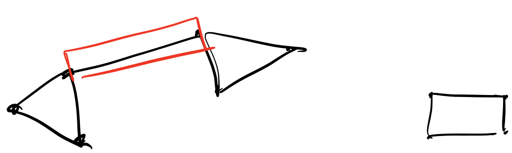
If not:

WEEK 8

Bridges

Bridge: edge that if removed $\Rightarrow$ increases # of comp.

Ex://



Bridge lemma: $e = uv$ is a bridge in graph G in component $H \Leftrightarrow H - e$ has exactly 2 components.

- Implications:
  1. $G - e$ has 1 more component than G
  2. u and v are in diff. components.
- Proof idea:
  a) Show that there is exactly 2 components.

     By contr., assume another component J that is formed after $H - e$. Non-empty subset of H w/ an empty cut. If we add back e,

J is not changed $\Rightarrow$ still has an empty cut $\Rightarrow$ H was disconnected. Contradiction!

b) Show that u, v in opposite components.

By cont. assume u, v in same component J. e is not in empty cut of J. $\therefore$ Add back e $\Rightarrow$ J still has empty cut $\Rightarrow$ H is disconnected $\Rightarrow$ cont.

IDEA: Try to prove by contradiction by adding & removing edges

Theorem: Edge e is a bridge in $G \Leftrightarrow e$ not in any cycles in G

Proof idea:

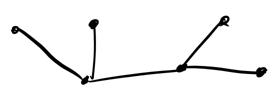
$\Rightarrow$: Contradiction proof. If e was in cycle, then $G-e$ in same component b/c u, v path exists. Cont!

$\Leftarrow$: Contrapositive proof (e is not a bridge $\Rightarrow$ e in cycle).

Always a path between u, v in component $H-e$. $H+e \Rightarrow$ cycle.

Trees (Part 1)

Tree Defn: connected graph w/ no cycles



Forest Defn: multiple trees (doesn't have to be connected)

Theorem (Bridge-Tree Theorem): Every edge in tree is a bridge.

↳ Property (Minimally connected): Removing edge in tree $\Rightarrow$ disconnected

Theorem (Edge-Vertex Rel. In Trees)

$$T \text{ is a tree} \Rightarrow |E(T)| = |V(T)| - 1$$

Proof: Induction on # of edges m.

Base case: $m = 0 \Rightarrow 1$ vertex. $\therefore |E(T)| = |V(T)| - 1$

Ind. hyp: Assume claim holds for any tree w/ less than m edges.

Ind step: T is a tree w/ $m \geq 1$ edges.

Remove edge $\rightarrow T - e$ is smaller components. These smaller comp. are also trees $\Rightarrow$ connected graph w/ no cycles. T_1, T_2 are smaller trees. Ind. hyp. holds

$$\begin{aligned}
\therefore |E(T)| &= |E(T_1)| + |E(T_2)| + 1 \\
&= |V(T_1)| - 1 + |V(T_2)| - 1 + 1 \\
&= |V(T_1)| + |V(T_2)| - 1 \\
&= |V(T)| - 1 \quad \boxtimes
\end{aligned}$$

Theorem (Forest - Edge Theorem)

F is a forest w/ n vertices & k components.

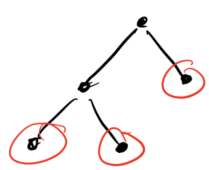
$$|E(F)| = n - k$$

Proof:

$$\begin{aligned}
|E(F)| &= \sum_{i=1}^{k} E(i) \\
&= \sum_{1}^{k} V(i) - 1 \\
&= V(k) + \sum_{i=1}^{k} (-1) \\
&= V(k) - k \\
&= n - k
\end{aligned}$$

Trees (Part 2)

Structure trees:



$\rightarrow$ Leaves: vertex of degree 1

Theorem (Vertex $\leftrightarrow$ leaf thm):

All trees w/ at least 2 vertices have at least 2 leaves.

Proof:

Idea: If talking about leaves ⇒ think paths in trees.

Longest path proof:

Assume a path P from $v_1 \dots v_k$. Path will have at least 2 vertices

This means that we have 2 endpoints.

Analyze endpoints:

v_1: v_1 cannot have another neighbor:

a) If it had another neighbor not in P ⇒ P no longer longest path

b) If " in P ⇒ cycle! Cannot exist in tree

v_1 has a degree of 1!

Same applies to v_k

∴ Have at least 2 leaves in tree w/ ≥ 2 vertices.

Theorem (Number of leaves)

$$\#\text{ of leaves} = 2 + \sum_{i \geq 3} (i-2) n_i$$

n_i = # of vertices w/ degree i

Proof.

$$|V(T)| = n_1 + n_2 + n_3 + \dots$$

⇒ Summing all vertices w/ all degrees.

From prev. theorem:

$$|E(T)| = |V(T)| - 1$$
$$= -1 + n_1 + n_2 + n_3 + \dots$$

From handshaking lemma:

$$\sum_{v \in V(G)} \deg(v) = 2 \cdot |E(G)|$$

$$2|E(T)| = \sum \deg(v)$$

$$2|E(T)| = \sum_{i \geq 1} i n_i$$

$$2(-1 + n_1 + n_2 \dots) = \sum_{i \geq 1} i n_i$$

$$-2 + 2n_1 + 2n_2 + \dots = \sum_{i \geq 1} i n_i$$

n_1 = # of leaves ⇒ solve for n_1

$$2n_1 = 2 - (2n_2 + 2n_3 + \dots) + n_1 + \sum_{i \geq 2} i n_i$$

$(= -2n_2 - 2n_3 - 2n_4 + \dots)$

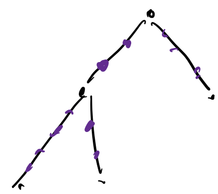
$+ (2n_2 + 3n_3 + 4n_4 + \dots)$

$(3-2)n_3 + (4-2)n_4 + \dots$

$$n_1 = 2 + \sum_{i \geq 3} (i-2) n_i$$

Implications:

- Vertices of deg 2 have no impact ⇐ can always put deg(2) vertex in edge & won't change # of leav.



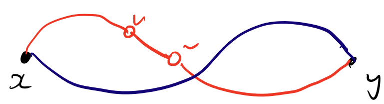
- Higher # of high degree vertices ⇒ more leaves.

Theorem (Unique Path in Tree)

Always a unique path between 2 nodes in tree.

Proof Idea:

By cont, assume 2 distinct paths between x and y ⇒ Edge uv in 1 path but not in other path



Remove edge uv. However, x and y are in same component ⇒ uv is not a bridge. We know that all edges are bridges, contradiction.

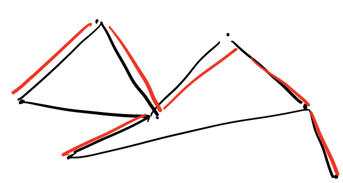
Theorem (Bipartite Tree Theorem):

A tree is bipartite.

WEEK 9

Spanning Tree:

Defn: subgraph that is a tree that uses all vertizes.



Theorem (Connected Spanning Tree):

G is connected $\Leftrightarrow$ has a spanning tree.

Proof idea:

$\Leftarrow$: Tree $\Rightarrow$ connected. Path b/w 2 vertices in tree $\Rightarrow$ Graph is connected.

$\Rightarrow$: Induction on # of cycles in graph. k

Claim: If G has k cycles $\Rightarrow$ has spanning tree.

Base case: $k=0 \Rightarrow G$ is a tree, G is a spanning tree.

Ind. hyp: Graph w $< k$ cycles has a spanning tree.

Ind. step: Graph w/ k cycles. Let e be an edge in a cycle. Then $G-e$ would have less than k cycles $\Rightarrow$ $G-e$ has spanning tree by ind. hyp

Since $V(G) = V(G-e)$ (no vertex removed), and $E(G-e) \subseteq E(G) \Rightarrow$ Spanning tree in G.

Theorem (Connected + $|V(G)|$ + $|E(G)|$)

G is connected $\wedge$ n vertices & $n-1$ edges $\Rightarrow$ G is a tree.

Proof: G has a spanning tree $\Rightarrow E[T] = n-1$. $|E(G)| = |E(T)| \Rightarrow$ $G = T \Rightarrow G$ is a tree.

How do you know if a graph G is a tree? 2 of 3 conditions:

1. G is connected
2. G has no cycles
3. G has $n-1$ edges.

Proof technique: changing in trees

1) Adding edges in spanning tree $\Rightarrow$ cycle
2) Remove edge in spanning tree but it's not a bridge $\Rightarrow$ still a tree

Collorcries:

1. T is a spanning tree e not in T ⇒ T+e has 1 cycle. -

   ↳ If e' is in cycle ⇒ T+e-e' is back at spanning tree of G

2. T is a span. tree ∧ e ∈ E(T) ⇒ T-e is 2 components.

   ↳ Reconnect to form spanning tree if e' is in cut induced by either component in G.

   ↳ T-e+e' is a spanning tree

Bipartite Characterization

Theorem: G is bipartite ⟺ no odd cycles

Proof:

⇒: Assume G is bipartite w/ groups A and B. Have a cycle

$v_1, v_2, \ldots, v_k, v_1$

WLOG, $v_1 \in A$. Then $v_2 \in B$ ...

From obs./induction: $v_i \in A \iff i$ is odd

Then: $v_1 \in A$, then $v_k \in B \Rightarrow B$ is even

Only even length cycle.

⇐: Prove contrapositive:

G is not bipartite ⇒ odd cycle.

If G is not bipartite, some component H that is not bipartite.

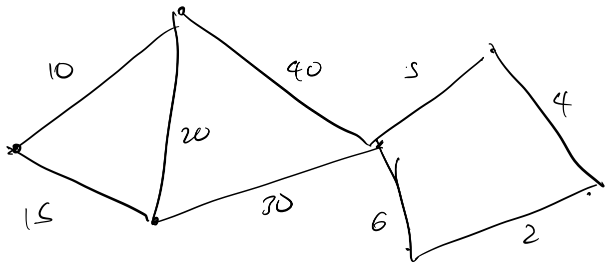
H has a spanning tree T, which is bipartite.

∃ e ∈ E(H) s.t. join 2 vertices in A or B. Now, for T, assume $u, v \in A$. Then, path length between u, v is even.

T + e ⇒ odd length ~~cycle.~~

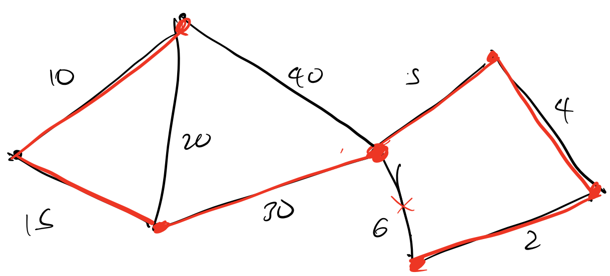
Minimum Spanning Tree

Defn: graph has weights on each edge. MST is spanning tree w/ min. amount of weight on edges.



Prim's Algorithm:

1. Choose arbitrary vertex v. Create Tree T w/ just v
2. While T is not spanning tree:
   a) Get cut induced by V(T)
   b) Choose e = uv w/ smallest weight s.t. u is in cut or v ∈ V(T)
   c) Add e and v in T



Total weight: 10 + 15 + 30 + 5 + 4 + 2 = 66

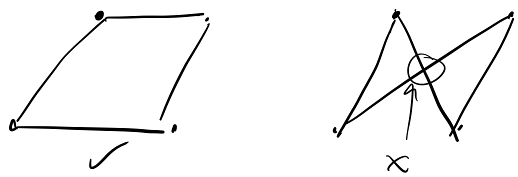
WEEK 10

Planar Graphs

Planar graph: has a planar embedding

↳ Planar embedding: drawing of graph s.t. vertices at diff. endpoints & edges only cross at common vertex endpoints.

Ex://

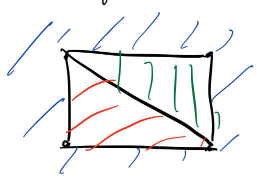


<u>Theorem (Components & Planarity)</u>

Graph is planar $\iff$ components are planar

Face: Connected region in embedding (not sep. by edges)

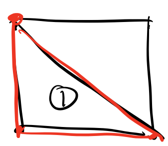
- Intuition: MS paint drop $\Rightarrow$ colors a face
- Ex://



$\Rightarrow$ 3 faces

Face boundaries: Subgraph that touches face

Ex://

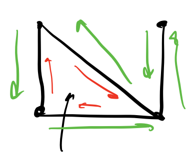


Adjacent faces $\iff$ common edge in boundaries

Boundary walk: walk around perimeter of face (walking on boundary)

Degree of face: length of walk

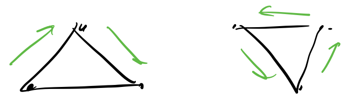
Ex://



$\Rightarrow$ Degree: 5

Degree = 3

Ex://



Degree of outer face = 6

Note: if graph is disconnected, get sum of degrees for each boundary.

<u>Theorem: Faceshaking lemma</u>

$$\sum_{f \in F} \deg(F) = 2|E(G)|$$

- Proof: each edge contributes 1 degree to inner face, another to another face / same face (faces are same on both sides)

- Corollary: If edge has same face $\Leftrightarrow$ edge is a bridge
  - Proof: If $e \notin$ bridge $\Rightarrow$ e is part of cycle $\Rightarrow$ ≥ 2 faces. Contr.

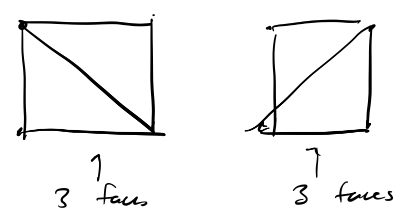
Theorem: Jordan Curve Theorem

Every planar embedding of cycle $\Rightarrow$ plane with 2 faces

Euler's Formula

All planar embeddings have same # of faces

Ex://



Euler's formula: G is connected planar graph w/ n vertices, m edges, s faces.

$$n - m + s = 2$$

- Proof: fix n, induct on m.

Base case: Min # of edges in $m = n-1$

Since G is connected & $n-1$ edges $\Rightarrow$ G is a tree.

Now, since each edge in bridge, there is only 1 face.

$\therefore \quad n - (n-1) + (1) = n - n + 1 + 1$
$= 2$ ✓

Ind. hyp: All graphs w/ n vertices & $m-1$ edges follow Euler's formula.

Ind. step: G w/ n vertices, m edges, s faces.

Take e that is not a bridge. Remove e from G. Since e is not a bridge $\Rightarrow$ e is in cycle, So G is still connected. $\therefore$ By ind. hyp, this graph $G-e$ still follows Euler formula.

If e is not a bridge $\Rightarrow$ merged 2 faces to create $G-e$. Then:

$$n - (\underset{\text{removed edge}}{m-1}) + (\underset{\text{merged}}{s-1}) = 2$$

$$n - m + s = 2 \quad \blacksquare$$

<u>Extension</u>: if G has c components $\Rightarrow$ $n - m + s = 1 + c$

Platonic Solids

Polyhedron: put a planar graph on sphere $\Longrightarrow$ cut to fit edges.

Platonic graph: every vertex & face of planar embedding of planar graph has same degree (≥ 3)

Suppose we have a platonic graph w/ $d_v \geq 3$, $d_f \geq 3$ w/ n vertices, m edges and s faces.

↳ What is relationship b/w d_v & d_f ?

① By Handshaking lemma: $n d_v = 2m$
$$d_v = \frac{2m}{n}$$

② By faceshaking lemma: $s d_f = 2m$
$$d_f = \frac{2m}{s}$$

③ Euler's formula: $n - m + s = 2$

④ Substitution into formula:

$$\frac{2m}{d_v} - m + \frac{2m}{d_f} = 2$$

Trivially:

$$m(2d_f - d_v d_f + 2d_v) = 2 d_v d_f > 0$$

$$\Downarrow$$

$$(d_v - 2)(d_f - 2) < 4 \Longrightarrow$$ Only 5 possible (d_v, d_f) pairs.

With a platonic graph ⇒ can make polyhedrons.

1 graph / platonic solid.

Nonplanar Graphs

Showing drawing of graph that is nonplanar is not sufficient

Lemma (Planar - Edge Check)

G w/ n vertices & m edges. G is a planar embedding w/ every face has degree at least $d \geq 3 \Rightarrow m \leq \frac{d(n-2)}{d-2}$

Proof:

Case 1: G is connected.

From faceshaking Lemma:

$$\sum_{f \in F} \deg(f) = 2m$$

$$\sum_{f \in F} d \leq 2m$$

$$sd \leq 2m$$

From Euler's Formula:

$$n - m + s = 2$$

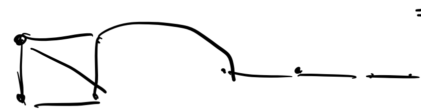
$$s = 2 - n + m$$

$$\therefore \quad d(2 - n + m) \leq 2m$$

$$m \leq \frac{d(n-2)}{d-2}$$

Case 2: G is not connected.

Create connected planar graph by adding edges in outer face to join components.



⇒ Degree of faces does not decrease,

m ↑ ⇒ inequality holds.

↳ not long.) ⇒ ||

Use this theory if we know minimum face degree & # of edges ⇒ require an embedding.

<u>Lemma (Cycle planarity)</u>

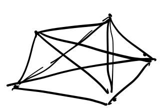
G contains a cycle ⇒ in any planar embedding of G, every face boundary contains cycle.

<u>Theorem (Nonplanarity w/ vertices & edges):</u>

G is planar graph w/ $n \geq 3$ vertices & m edges ⇒ $m \leq 3n-6$

Ex:// Show K_5 is not planar

# of edges: 10 edges.   # of vertices: 5



If planar:

5 vertices ⇒ $m \leq 3(5)-6 = 9$

$m = 10$ ⇒ non planar

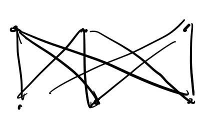
Cannot use to prove planarity

<u>Theorem (Bipartite Inequality for Planar Graphs)</u>

G is a bipartite planar graph ⇒ $m \leq 2n-4$

Ex:// Show $K_{3,3}$ is not planar.

# of vertices: 6
# of edges: 9 edges ⇒ $m \leq 2(6)-4$
≤ 8



WEEK 11

Kuratowski's Theorem

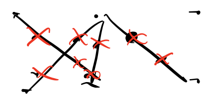
To show non planar

Show too many edges

Graph 'looks like' K_5 or $K_{3,3}$

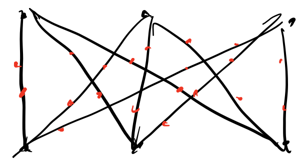
$\Downarrow$

Remove degree 2 vertices



Edge subdivision: replace each edge w a path of length ≥ 1 (adding deg 2 vertices)

Ex://



Theorem (edge subdivision $\Leftrightarrow$ planarity)

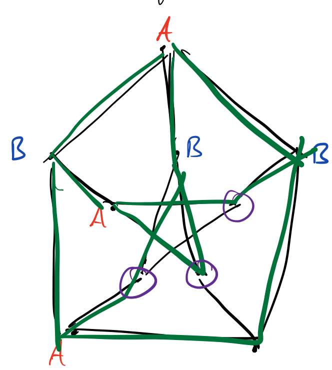
Graph is planar $\Leftrightarrow$ any edge subdivision of graph is planar

$\therefore$ If graph has edge subdiv. of K_5 or $K_{3,3}$ $\Rightarrow$ non planar

Kuratowski's Theorem

Graph is planar $\Leftrightarrow$ no edge subdiv. of K_5 or $K_{3,3}$

Ex://



Show non planar.

Either count edges / Kuratowski's theorem.

1. Count edges & vertices

   Vertices: 10

   Edges: 13 $\Rightarrow$ Inequality check fails

2. Kuratowski's Theorem

   Try $K_{3,3}$ $\Leftarrow$ K_5 needs set of interconnected 5 ver

CANNOT REUSE EDGES TO CREATE EDGE SUBDIV.

$\therefore K_{3,3} \Rightarrow$ nonplan.

Check:

1. No edges reused
2. All vertices $\neq$ part of $K_{3,3}$ / K_5 have degree 2

Coloring

k-coloring: assign each vertex 1/k colors w/ no adj. vertices w/ same color.

- Formally: function: Colors $\longrightarrow V(G)$ s.t. $uv \in E(G)$, $f(u) \neq f($
- No need to use all k cols. if k colorable
  - $\therefore$ k-colorable graph $\Rightarrow$ k+1 colorable

Q: Minimum # of colors to color graph?

Graph classes that are colorable

1. Bipartite graph $\Longrightarrow$ 2 colorable
2. Complete graph $\Longrightarrow$ n colorable

Theorem (5-deg theorem)

Every planar graph has degree of vertex at most 5.

Proof:

By way of contr., assume that all n vertices have $\deg(v) \geq 6$

By Handshaking lemma: $|E(G)| \geq \frac{6n}{2} = 3n$

Contr. $\Rightarrow$ Every planar graph has $|E(G)| \leq 3n - 6$

6-Color Theorem

Every planar graph is 6 colorable.

Proof:

Induction on # of vertices.

Base case: $n = 1 \Rightarrow 1$ vertex is trivially 6-colorable

Ind. hyp: a graph with $< n$ vertices is 6-colorable.

Ind. step: G is a graph w/ n vertices. We want to define
$f: V(G) \to C$

Choose random vertex $v \Rightarrow$ remove v & incident edges.

By ind. hyp $\Rightarrow$ G' is 6-colorable $\Rightarrow f'(G') : V(G') \to C$

To define f: ✓ Same colorings

$\forall w \in V(G),\ w \neq v \Rightarrow f(w) = f'(w)$

For v:

$\mathcal{U} = \{ f(w) : vw \in E(G) \}$. Since v has degree at most $5 \Rightarrow |\mathcal{U}| \leq 5$. Since 6 colors available $\Rightarrow$ use missing color on $v \Rightarrow f(v) = C - \mathcal{U}$

Verify f is coloring:

Show that neighbors are diff. colored

For edges x, y, $x \neq v \wedge y \neq v \Rightarrow$ Already have coloring.

For edge v, w, $f(w) \in \mathcal{U}$, $f(v) \notin \mathcal{U}$ so $f(v) \neq f(w)$

5-Color Theorem

Every planar graph is 5-colorable.

Proof:

Trick: Contract vertices into 1

Induction on n (# of vertices)

Base case:

$\left.\begin{array}{l} n = 1 \\ n = 2 \end{array}\right\}$ Trivial

Ind. hyp

Assume any graph $< n$ vertices is 5-colorable

Ind. step:

G is a planar graph w/ n vertices. Let v be a vertex w/ deg at most deg 5.

If $\deg(v) \le 4 \Rightarrow$ use 6-color theorem proof method to do same for 5-color

$\deg(v) = 5$:

v must have 2 neighbors x, y s.t. x is not adj. to y.

↳ Proof. Assume opposite $\Rightarrow K_5 \Rightarrow$ Not planar $\Rightarrow$ Contra.

Contract x, v, y $\Rightarrow$ new vertex v^*

↓

G has $\underline{\underline{n-2}}$ vertices $\Rightarrow$ 5 colorable

Define coloring:

$\forall w \in V(G) \wedge w \notin \{x, v, y\} \Rightarrow f(w) = f'(w)$

For $v, x, y \Rightarrow f(x) = f(y)$ b/c not adj.

Let $\mathcal{U} = \{f(w) : vw \in E(G)\}$. Since $f(x) = f(y) \Rightarrow |\mathcal{U}| \le 4$.

Since $|C| \le 5 \Rightarrow \exists$ a color for v!

Verify: exercise.

4 - Color Theorem

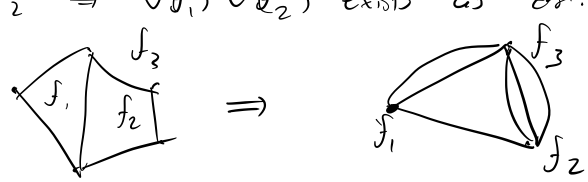
Every planar graph is 4 colorable

Planar Duals

Inversion of planar embedding –

↳ Define vertex for each face. $\forall$ edges in graph w/ faces $f_1, f_2 \Rightarrow v(f_1)\, v(f_2)$ exists as edge.

Ex://



Not always a simple graph

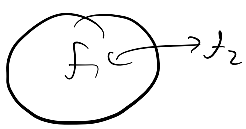
Theorem (Planar Dual is Planar)

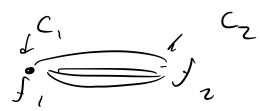
Dual graph is planar

Coloring faces = coloring planar dual.

Notes

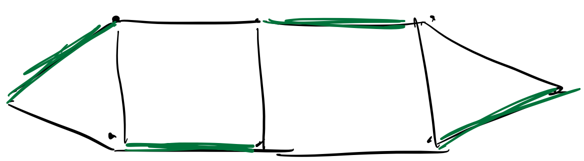
1. Planar dual of planar dual of G = G
2. A face of degree k $\rightarrow$ vertex of degree k
3. Dual of platonic $\rightarrow$ platonic
4. Closed curve on plane $\rightarrow$ 2 colorable.





Matchings

Defn: set of edges w/ no common endpoints



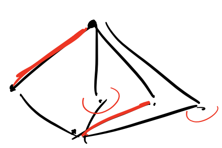
Q: Largest matching possible in graph?

Saturated vertex: vertex incident w/ edge in matching.

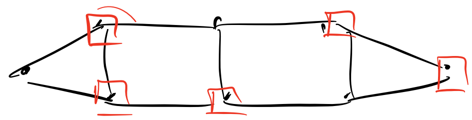
Perfect matching: saturates all vertices.

↳ Not every graph has perfect matching

Ex.// $K_{4,2}$



Cover: set of vertices in G s.t. each edge has 1 endpoint in set.



Q: Minimal cover in graph?

Theorem (Matching $\Leftrightarrow$ Cover)

$$\underset{\substack{\uparrow \\ \text{matching}}}{|M|} \leq \underset{\substack{\uparrow \\ \text{cover}}}{|C|}$$

Proof: $\forall\, e = uv \in M$. One of u/v is in cover.

Since each edge is disjoint in M (no shared endpoints), at least $|M|$ distinct vertices.

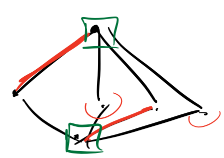
$$\therefore \quad |M| \leq |C|$$

Lemma (Maximum Matching $\Leftrightarrow$ Minimum Cover)

If M is maximal matching, C is cover $\Rightarrow$ $\underset{\substack{\uparrow \\ \text{maximal}}}{|M|} = \underset{\substack{\uparrow \\ \text{minimal}}}{|C|}$

To show maximal matching, provide cover of same size.

Ex://



Since $|M| = |C| \Rightarrow M$ is maximal

Note that this isn't only key

↳ Possible to have graphs w/ |maximal matching| < |cover|

↳ Not used to disprove maximal matching.

Konig's Theorem

If G is bipartite $\Rightarrow$ |maximal matching| = |cover|

Augmenting Paths

To show maximal matching

$|M| = |C|$

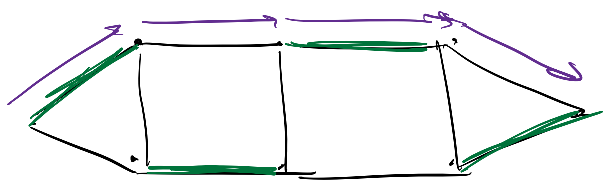
↳ If $|M| \neq |C|$ and G is not bipartite $\Rightarrow$ can't prove that M is/is not maximal

No augmenting paths

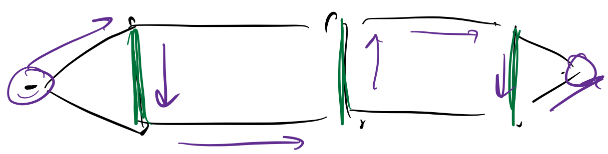
$\Downarrow$

Solid

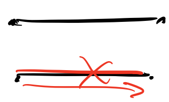
Alternating path: path w/ edges alternating in Matching.



Augmenting path: alternating path that starts & ends in distinct unsaturated vertices.



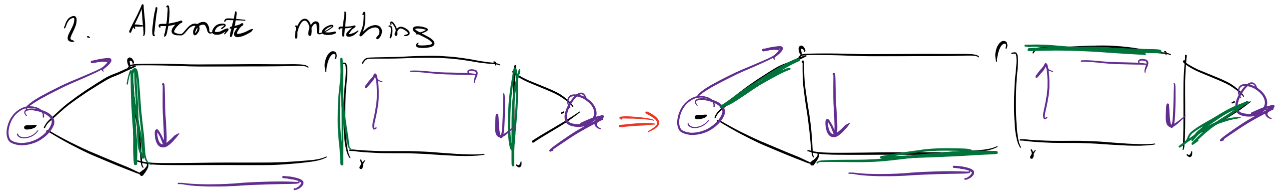
- Trivially: single edge w/ 2 unsaturated vertices $\Rightarrow$ augmenting.



- Augmenting paths have odd length $\Rightarrow$ start & end w/ non-matching edges.

To create a bigger matching from old matching:

1. Create augmenting path
2. Alternate matching



Lemma (Augmenting Path ⇒ Non-Maximal Matching)

If M has aug. path $\Rightarrow$ M is not maximal

WEEK 12

Bipartite Matching Algo

Goal: create a maximal matching in a bipartite graph (A, B)

Algo: we have $G(A, B)$ and matching M

1. Initialize sets:

   X_0: unsaturated vertices in A

   $X = X_0$

   $Y = \emptyset$

2. Find all neighbors of X that is not in Y

   Cases:

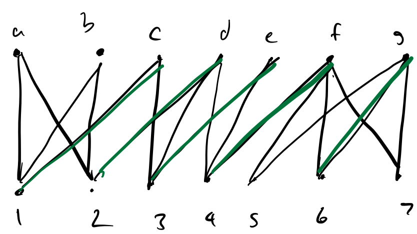
   ① If vertex is unsaturated $\Rightarrow$ found a augmenting path
   Extend matching: flip matching in path. Repeat 1.

   ② If vertex is saturated $\Rightarrow$ put it in Y (all in B)
   Put matching neighbors in X (all part of A)

   ③ If no vertex added in Y $\Rightarrow$ maximal matching.

   Maximal matching w/ cover $Y \cup (A \setminus X)$

Ex://



Find maximal matching from initial matching.

① Initialize sets:

$X_0 = \{a, b\}$ $\quad Y = \emptyset$

$X = \{a, b\}$

② Find all neighbors of x not in Y

$Nei(x) = \{1, 2\}$

Both 1, 2 are saturated $\Rightarrow$ put in Y

$Y = \{1, 2\}$

$X = \{a, b, c, d\}$

③ Find all neighbors in X not in Y

$Nei(x) = \{3, 4\}$) sat.

$Y = \{1, 2, 3, 4\}$

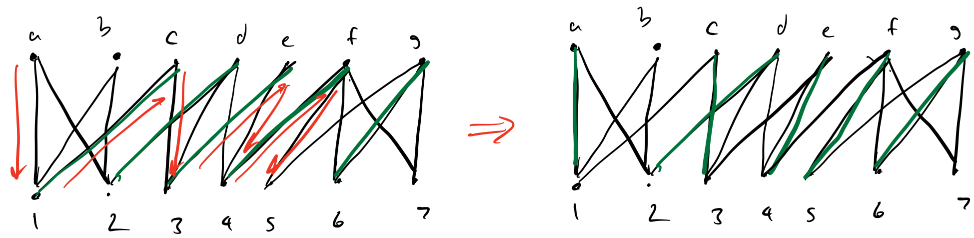
$X = \{a, b, c, d, e, f\}$

④ Find all nei. of x not in Y

$Nei(x) = \{5, 6, 7\}$.

Vertex 5 is unsaturated $\Rightarrow$ augmenting path. (b/c started as unsat.)

Flip matching:



⑤ Restart:

$X_0 = \{b\}$ $Y = \{\emptyset\}$

$X = \{b\}$

⑥ Algo restart:

1. $Nei(x) \Rightarrow \{1, 2\}$

   $Y = \{1, 2\}$, $X = \{b, a, d\}$

2. $Nei(x) \Rightarrow \{3, 4\}$

   $Y = \{1, 2, 3, 4\}$, $X = \{b, a, d, c, e\}$

3. $Nei(x) \Rightarrow \{\}$ (all in Y)

   ↳ Maximal matching: Cover: $Y \cup (A \setminus X)$

   $= \{1, 2, 3, 4\} \cup \{f, g\}$

   $= \{1, 2, 3, 4, f, g\}$ = Matching!

$\therefore$ Maximal matching found.

Meaning behind X_0, X, Y

1. X_0: set of all unsat. vertices
2. X: vertices in A reachable to X_0 via alt. path
3. Y: 〃 B 〃

Konig's Theorem

Proof:

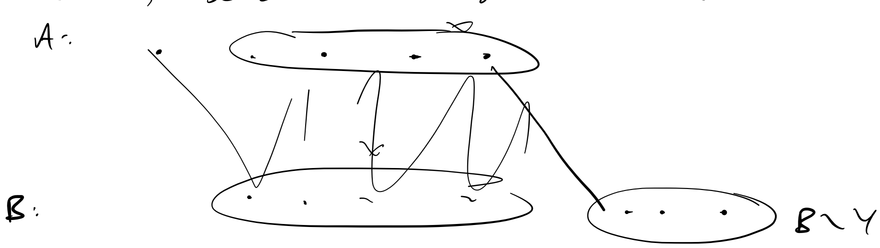
At end of algo $\Rightarrow$ X_0, X, Y and a matching M

Claim #1: No edge between X and $B \setminus Y$

X: set of all $v \in A$ s.t. you can reach v from X_0 via alt. path.

Y: 〃 $v \in B$ 〃

For contr., assume that $\exists$ edge between X and $B \setminus Y$



Then I could have extended alt. path b/c neighbor of X not in Y!

If algo terminated $\Rightarrow$ no more edges between X and $B \setminus Y$

Concl. from Claim #1:

Every edge either has a endpoint in Y or $A \setminus X$

Why?

4 groups: X, Y, $A \setminus X$ & $B \setminus Y$

$X \longrightarrow Y$ ✓

$X \longrightarrow B \setminus Y$ ✗ (from claim)

$X \longrightarrow A \setminus X$ ✗ (bipartite)

$Y \longrightarrow X$ ✓

$Y \longrightarrow B \setminus Y$ ✗ (bipartite)

$Y \longrightarrow A \setminus X$ ✓ (not matching neighbor)

Every edge has endpoint in Y or $A \setminus X$

$\Downarrow$

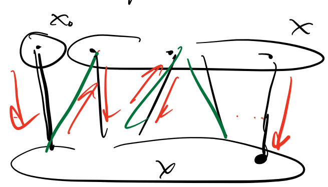
Cover: $Y \cup (A \setminus X)$

Claim #2

Every vertex in Y is saturated.

Proof: Contr. $\Rightarrow$ assume there was a vertex $\in Y$ not saturated

$\therefore$ Augmenting path could have been found from $X_0 \to Y$



We know algo terminated $\Rightarrow$ no more augmenting paths! Contr.

By algo: if vertex in Y saturated, other endpoint of matching edge in X

Claim #3

Every vertex in $A \setminus X$ is saturated.

Why? $\underline{\underline{X_0}} \Rightarrow$ contains all unsat. vertices in A (so $A \setminus X$ is saturated)
$X_0 \subseteq X$

Implication:

$A \setminus X$ have edges in either Y or $B \setminus Y$

However, matching cannot come from Y! Why? Y is saturated by an edge from X from algo, not by $A \setminus X$.

$\therefore$ All matchings that saturate $A \setminus X$ is from $B \setminus Y$

Concl.

Matching in Y is DISJOINT from $A \setminus X$!!

$$\therefore \quad |M| = |Y| + |A \setminus X|$$
$$= |Y \cup (A \setminus X)|$$

Maximal matching & minimal cover.

Theorem (Lower bound of bipartite matchings)

Bipartite graph w/ m edges & max degree d has matching at least $\frac{m}{d}$.

Proof: C is cover of G. If at most d edges incident w/ vertex, then # of vertices $\geq \frac{m}{d}$. Thus $|C| \geq \frac{m}{d}$. From cover $\Leftrightarrow$ matching theorem:

$$|M| \geq |C| \geq \frac{m}{d}$$

$$|M| \geq \frac{m}{d}$$

How to justify $\geq$ vs. $>$?

↳ Maximal matching = minimal cover by König's Theorem!

↳ Possible that $|M| = m/d$!

Hall's Theorem

Q: Conditions to saturate all vertices in 1 bipartition?

Hall's Theorem

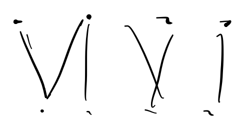
Neighbors of D

G has matching that saturates $A \Leftrightarrow \forall D \subseteq A$, $|N(D)| \geq |D|$

Proof:

$\Rightarrow$ Assume G has matching that saturates A.

For any $D \subseteq A$, every matching that connects D comes from $N(D)$!



$\Rightarrow$ If matching for every vertex, it must come from distinct neighbors! O.w. not matching!

$\therefore |N(D)| \geq |D|$ (cannot be less, o.w. sharing vertices!)

$\Leftarrow$ Prove contrapositive:

G does not have saturating matching for $A \Rightarrow \exists D \subseteq A$, $|N(D)| < |D|$

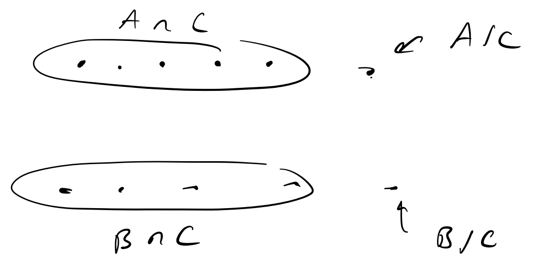
Assume G does " . Let M be the maximum matching. We know that $|M| < |A|$ (vertex in A not connected w/ matching).

By König's Theorem: $|M| = |$minimal cover$|$ C

Consider: $A \cap C$, $A \setminus C$, $B \cap C$, $B \setminus C$

Obj: Show one subsets has neighbors excl. in other subset

+ other subset is less than itself.



B/C cover ⇒ no edge between A/C or B/C

↳ Why? Cover must have all vertices w/ endpoints of all edges. O.w. both endpoints in A/C and B/C then C is not cover.

All neighbors of A/C is in $B \cap C \Rightarrow N(A \setminus C) \subseteq B \cap C$

Recall: $C = (A \cap C) \cup (B \cap C)$

$A = (A \cap C) \cup (A - C)$

Also:

$$|C| < |A|$$

$$|(A \cap C) \cup (B \cap C)| < |(A \cap C) \cup (A - C)|$$

$$|B \cap C| < |A - C|$$

Since $N(A - C) \subseteq B \cap C$, $|N(A \setminus C)| < |A \setminus C|$ ■

Applications of Hall's Theorem

Theorem (Perfect matching in regular bipartite)

G is a k-reg. bipartite $G \Rightarrow$ perfect matching.

Proof: Since k-reg. $|A| = |B|$. We need to find matching that saturates A.

Let $D \subseteq A$. $\sum_{v \in D} \deg(v) \leq \sum_{v \in N(D)} \deg(v)$

b/c regular:

$$k \cdot |D| \leq k\, |N(D)|$$

$$|D| \leq |N(D)|$$

Good trick!!

$\therefore$ By Hall's Theorem, exists a matching that satisfies A!

This a perfect matching!