MATH 115

COMPLEX NUMBERS

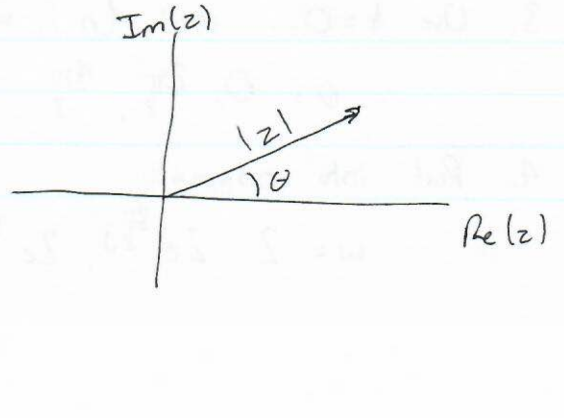
$$\mathbb{C} = \{x + yj,\ x, y \in \mathbb{R}\}$$

$$j^2 = -1$$

- Operations:
  - Add/subtract: add up $Re(z)$ and $Im(z)$
  - Multiply + divide: multiply " "
    - ↳ Division: $\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \boxed{\frac{z\bar{w}}{w^2}}$
- Conjugate: $z = x + yj$, $\bar{z} = x - yj$
  - Purely real: $z = \bar{z}$ Purely imaginary: $z = -\bar{z}$. $\overline{z+w} = \bar{z} + \bar{w}$

    $\overline{zw} = \bar{z} \cdot \bar{w}$
- Modulus: $|z| = \sqrt{x^2 + y^2}$
  - $z\bar{z} = |z|^2$. $\underline{|z + w| \leq |z| + |w|}$
- Common question:
  - Ex:// Find solutions for $z^2 = \bar{z}$
    1. Write z in standard form:
       $$z = x + yj$$
       $$z^2 = (x^2 + 2xyj - y^2)$$
       $$\bar{z} = x - yj$$
    2. Equate Re and Im to form 2 equation.
       ①: $x = x^2 - y^2$
       ②: $-y = 2xy$
    3. Consider cases of 0 and not zero
       Case #1: $y = 0$
       ①: $x = x^2 \Rightarrow x = \pm 1, y = 0$ $\}$ $\boxed{z = \pm 1}$
       Case #2: $y \neq 0$
       ②: $-1 = 2x \Rightarrow \underline{x = -1/2}$,
       $-1/2 = 1/4 - y^2$
       $y^2 = 3/4 \Rightarrow \underline{y = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}}$ $\}$ $\boxed{z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j}$

- Representation:



- Polar form:

$$z = r(\cos\theta + j\sin\theta)$$

- Conversion from standard to polar
  1. Find r ($r = |z|$)
  2. Find θ:

$$x = r\cos\theta \qquad y = r\sin\theta$$

- Multiplying/dividing:

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right)$$

- Complex exponential:

$$e^{j\theta + 2\pi k} = \cos\theta + j\sin\theta \Leftarrow \underline{\text{Infinite}} \ (\theta + 2\pi k)$$

- Ex: Put $1+j$ into complex exponential
  1. Convert into polar form:

$$r = \sqrt{2}, \ \theta = \pi/4$$

$$\therefore \ 1+j = \sqrt{2}(\cos\pi/4 + j\sin\tfrac{\pi}{4})$$

  2. Convert to exponential:

$$1+j = \sqrt{2}e^{\pi/4 j}$$

- De Moivre's Theorem:
  - Power: $(re^{j\theta})^n = r^n e^{jn\theta} = r^n(\cos n\theta + j\sin n\theta)$
  - Used to prove other properties (like addition + subtraction)
- N^{th} Roots:
  - Ex:// Find cube roots of 8

$$\cancel{a^8} \ w^3 = 8$$

$$r^3 e^{3j\theta} = 8$$

  1. Put number into complex exponential.

$$8 = 8e^{j(0 + 2\pi k)}$$

  2. Equate radius and power:

$$\therefore \ ①: \ r^3 = 8 \Rightarrow r = 2$$

$$②: \ \theta = \frac{2\pi k + 0}{3}$$

  3. Use $k = 0 \ldots n-1$ (n is # of roots)

$$\therefore \ \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

  4. Put into original:

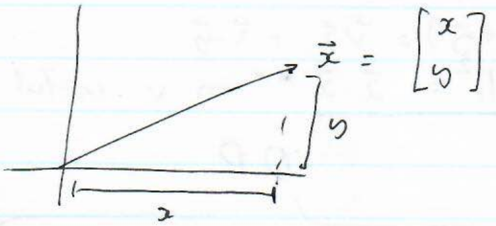
$$w = 2, \ 2e^{\frac{2\pi}{3}j}, \ 2e^{\frac{4\pi}{3}j} \Rightarrow \boxed{2, -1+\sqrt{3}i, -1-\sqrt{3}i}$$

VECTORS

$$\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$\vec{x} = \vec{y}$$

$$x_i = y_i \qquad (i = 1 \ldots n)$$



- Addition:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$$

- Scalar multiplication:

$$a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$$

- Behaves very similarly to real numbers.
- $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$
- Norm: magnitude:

$$\left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\| = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2} \qquad \Leftarrow \quad \|\vec{x}\| = \sqrt{\ldots}$$

- $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$
- $\|t\vec{v}\| = |t| \cdot \|\vec{v}\|$
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ $\Rightarrow$ Equal if collinear
- Distance:

$$\vec{d} = \vec{x} - \vec{y}$$

$$\|\vec{d}\| = \|\vec{x} - \vec{y}\|$$

↳ Similar to distance eqn.

- Dot product:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = (x_1 y_1) + (x_2 y_2) + \ldots + (x_n y_n) \in \mathbb{R}$$

- Properties:
  1) $\vec{x} \cdot \vec{v} = \vec{v} \cdot \vec{x}$
  2) $t(\vec{x} \cdot \vec{v}) = t\vec{x} \cdot t\vec{v}$
  3) $\vec{v}(\vec{x} + \vec{y}) = \vec{v} \cdot \vec{x} + \vec{v} \cdot \vec{y}$
  4) $\underline{\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}}$ ** $\Rightarrow$ v. useful

DP

$x_1 y_1 + x_2 y_2 + \ldots x_n y_n$

$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$

$\Downarrow$

Obtuse, acute, orthogonal ($\vec{x} \cdot \vec{y} = \vec{0}$)

parallel: $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\|$

- Cauchy-Schwarz:

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \times \|\vec{y}\|$$

- Unit vector: $\|\vec{v}\| = 1$

$$e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \rightarrow 1 \text{ is in } n^{th} \text{ component}$$

- To make vectors into unit vectors

$$\frac{\vec{v}}{\|\vec{v}\|} = 1 \Rightarrow \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{38}} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

  - Can use this to make vectors of specific length (multiply by scalar)

- Complex vectors:

$$\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \middle| z \in \mathbb{C} \right\}$$

$$\vec{z} = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix}}_{Re(\vec{z})} + j \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{Im(z)}$$

- Inner product:

$$\langle \vec{z}, \vec{w} \rangle = \sum_{i=1}^{n} \bar{z}_i w_i$$

  - Ex:

$$\vec{z} = \begin{bmatrix} 1 \\ j \end{bmatrix}, \vec{w} = \begin{bmatrix} j+2 \\ 2j \end{bmatrix}, \langle z, w \rangle = j + 2 - j(2j) = j + 4$$

- Complex norm:

$$||\vec{z}|| = \sqrt{\langle \vec{z}, \vec{z} \rangle} = \sqrt{\bar{\vec{z}} \cdot \vec{z}} = \sqrt{||\vec{z}||^2} = ||\vec{z}||$$

  - Norm is square root of inner product of itself.

- Cross product:

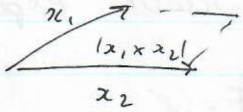
$$\vec{x} \times \vec{y} \in \mathbb{R}^3$$

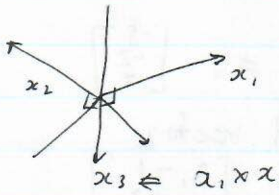
$$\begin{bmatrix} x_2 y_3 - x_3 y_2 \\ -(x_1 y_3 - x_3 y_1) \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

Magnitude

$$||\vec{x} \times \vec{y}|| = ||\vec{x}|| \cdot ||\vec{y}|| \sin\theta$$

↳ Area of parallelogram ($2 \times \Delta_{triangle}$)





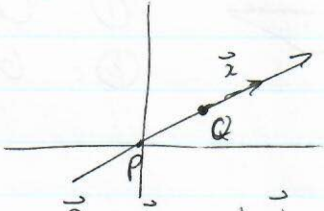
$x_3 \Leftarrow x_1 \times x_2 \Rightarrow$ Orthogonal

  - Properties:
    1) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u}) \Rightarrow$ Order matters
    2) $\vec{u} \times \vec{u} = \vec{0}$
    3) $t(\vec{u} \times \vec{v}) = t\vec{u} \times \vec{v}$
    4) $\vec{u} \times (\vec{v} \pm \vec{w}) = (\vec{u} \times \vec{v}) \pm (\vec{u} \times \vec{w})$

LINES

direction vector

$$\vec{y} = t\vec{x} + C \leftarrow \text{Point}$$



$$\vec{P} - \vec{Q} = k\vec{d}$$

Parametric:

$p_1 - q_1 = k d_1$

$p_2 - q_2 = k d_2$

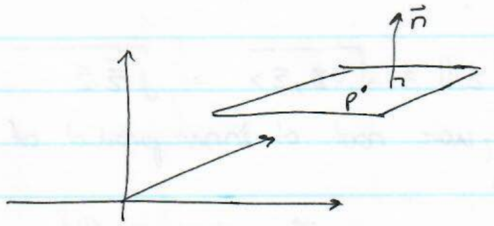
Vector:

$\vec{PQ} = k\vec{d}$

$\vec{p} = k\vec{d} + \vec{Q}$

1. Find vector between 2 point $\Rightarrow$ direction vector
2. Put arbitrary point

PLANES



$\longrightarrow$ Scalar:

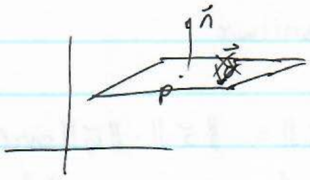
$x_1 + x_2 + x_3 = d$

vector:

$\vec{x} + \vec{s} = \vec{z}$

Parametric

①: Point + direction



1. Define an arbitrary vector from an arbitrary point:

$$\vec{PX} = \vec{P} - \vec{X} = \begin{bmatrix} p_1 - x_1 \\ p_2 - x_2 \\ - \end{bmatrix}$$

2. $\vec{PX} \cdot \vec{n} = \vec{0}$ (from orthogonality)

Scalar: expand dot product:

$$n_1(x_1 - p_1) + n_2(x_2 - p_2) = \vec{0}$$

- Ex://

$P_0(1,3,2)$ and $\vec{n} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$.

1) Defining general vector

$$\vec{P_0 x} = \begin{bmatrix} x_1 - 1 \\ x_2 - 3 \\ x_3 - 2 \end{bmatrix}$$

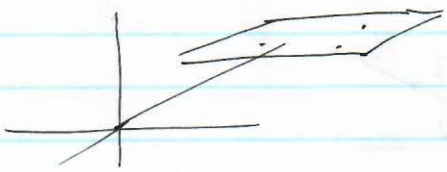
2) Orthogonality:

$$\begin{bmatrix} x_1 - 1 \\ x_2 - 3 \\ x_3 - 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \vec{0}$$

$$5(x_1 - 1) - 2(x_2 - 3) + 3(x_3 - 2) = 0$$

$$5x_1 - 2x_2 + 3x_3 = 5$$

②: 3 points:



①: Create 2 vectors (non-parallel)

②: General form:

$$\vec{x} = s\vec{v}_1 + t\vec{v}_2 + \vec{p}$$

Scalar form:

①: Create 2 vectors (non-parallel)

②: Create cross product to find $\vec{n}$

③: Use point + direction steps.

Parametric:

Split it into components.

③: 1 point + 1 line:

①: Find $\vec{d}$ of L

②: Create vector from p to L (non-parallel)

③: Vector format:

$$\vec{x} = s\vec{v}_1 + t\vec{v}_2 + \vec{p}$$

- Ex:// Consider plane with 2 lines:

$$\begin{bmatrix}1\\2\\1\end{bmatrix} + t\begin{bmatrix}2\\1\\3\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\2\end{bmatrix} + s\begin{bmatrix}-4\\-2\\6\end{bmatrix} \quad (s, t \in \mathbb{R})$$

Find vector and scalar equation.

Note: vectors given are parallel ⇒ can't be used.

$$\vec{v}_1 = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}-1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\1\\-1\end{bmatrix} \Leftarrow \text{New vector from points}$$

$$\therefore \vec{x} = \begin{bmatrix}2\\1\\3\end{bmatrix}t + \begin{bmatrix}2\\1\\-1\end{bmatrix}s + \begin{bmatrix}1\\2\\1\end{bmatrix} \Rightarrow \text{Vector eqn.}$$

Define $\vec{n}$:

$$\vec{n} = \begin{bmatrix}2\\1\\3\end{bmatrix} \times \begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}4\\-8\\0\end{bmatrix}$$

Define arbitrary ~~point~~ vector: $\begin{bmatrix}x_1-1\\x_2-2\\x_3-1\end{bmatrix}$ (point from 1st line)

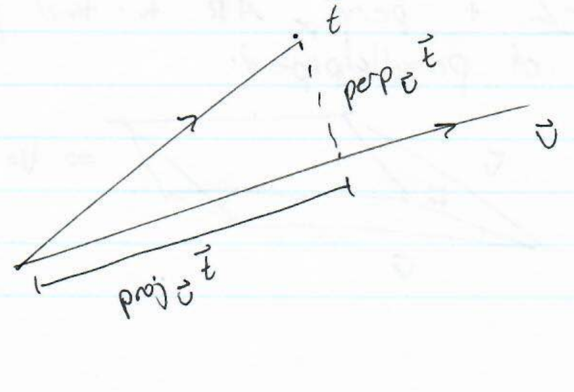
$$\therefore 4(x_1-1) - 8(x_2-2) + 0(x_3-1) = 0$$

$$4x_1 - 8x_2 = 12 \Rightarrow \text{Scalar eqn.}$$

- Scalar to vector:
  - ①: Find 3 noncollinear points on plane (plug in arbitrary points)
  - ②: Construct 2 vectors (should be non-parallel)
  - ③: Combine into vector equation:

$$\vec{x} = t\vec{u} + s\vec{v} + r_0$$

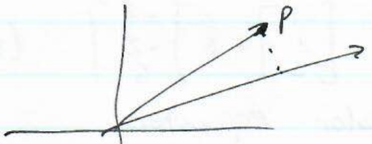
<u>Projections</u>:



$$\vec{t} = proj_{\vec{u}} \vec{t} + \cancel{proj}\ perp_{\vec{u}} \vec{t}$$

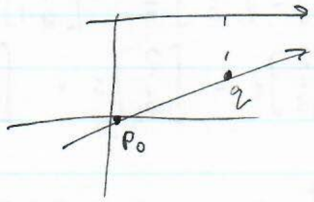
$$proj_{\vec{u}} \vec{t} = \left(\frac{\vec{u} \cdot \vec{t}}{\|\vec{u}\|^2} \right) \vec{u} \qquad perp_{\vec{u}} \vec{t} = \vec{t} - proj_{\vec{u}} \vec{t}$$

- Question examples:
  1. Find shortest distance between 2 ~~vector~~ points and line



$\Rightarrow$ Calculate $|\cancel{pro}\ perp_{\vec{u}} \vec{t}|$ by constructing a vector from point to line

  2. Find closest point on $\vec{u}$ ~~from~~ to $\vec{t}$.



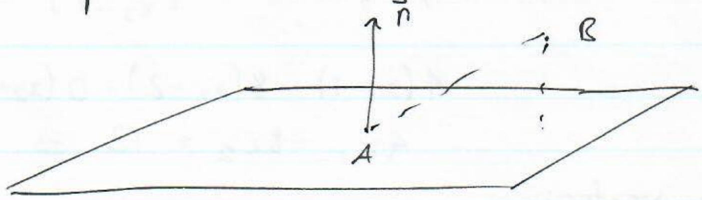
1. Find shortest distance
2. Add distance to original point on line $\vec{u}$

$$\vec{q} = P_0 + proj_{\vec{u}} \vec{t}$$

OR

$$q = P_0 + \cancel{perp_{\vec{u}}} \left(\vec{t} - perp_{\vec{u}} \vec{t} \right)$$ vector from P_0 to q now

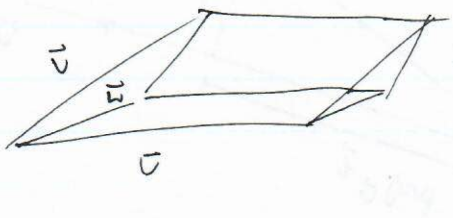
- Projecting on plane:



Common: find distance from B to plane.
1. Construct vector from A to B
2. Project AB to normal
3. Find point on plane underneath by $\vec{B} - proj_{\vec{n}} \vec{AB}$

To project point on plane:
$\underline{1-2}$ + $perp_{\vec{n}} AB$ to find point.

- Volume of parallelpiped:



$\Rightarrow V = |u \cdot (u \times w)|$

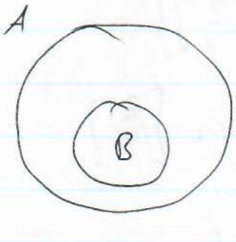
SETS

$A = \{1, 2, ☺\}$ Special set: $\emptyset = \{\} \Rightarrow$ empty set.

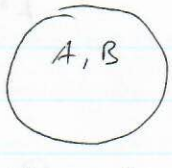
Notation:

$\{x \text{ form} \mid \text{restrictions}\}$.

Subsets + Equality



~~$A \subseteq B$~~
$B \subseteq A$
$B \subsetneq A$



$A \subseteq B \wedge B \subseteq A$

- Problem: show $A = B$
  1. Show $A \subseteq B$ (decomposing A to look like B)
  2. Show $B \subseteq A$ (" B to look like A)

Ex://

$$S = \left\{ a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}2\\3\end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

$$T = \left\{ e\begin{bmatrix}1\\2\end{bmatrix} + f\begin{bmatrix}1\\1\end{bmatrix} \middle| e, f, \in \mathbb{R} \right\}$$

Show $S = T$.

$S \subseteq T$:

Let $\vec{x} \in S$. $\vec{x} = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}2\\3\end{bmatrix}$

3 vectors $\Rightarrow$ 2 vectors (split 1 vector into other 2)

$$\vec{x} = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} + c\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= (a+c)\begin{bmatrix}1\\2\end{bmatrix} + (b+c)\begin{bmatrix}1\\1\end{bmatrix}$$

$\therefore \vec{x} \in T \Rightarrow S \subseteq T$

$T \subseteq S$:

$\vec{y} \in T \therefore \vec{y} = e\begin{bmatrix}1\\2\end{bmatrix} + f\begin{bmatrix}1\\1\end{bmatrix}$ Common trick! Only works if coefficient can be zero

$$= e\begin{bmatrix}1\\2\end{bmatrix} + f\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}2\\3\end{bmatrix} \in S$$

$\therefore \vec{y} \in S \Rightarrow T \subseteq S \Rightarrow T = S$

Spanning Sets

$$\vec{x} \in \text{span}\{v_1 \ldots v_k\} \Rightarrow \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ \vdots \\ v_1 \end{bmatrix} + \ldots + c_k \begin{bmatrix} v_k \\ \vdots \\ v_k \end{bmatrix}$$

1. Show $\vec{x} \in$ span.

   Show that $\vec{x}$ can be made of spanning set.

   ↳ Ex: $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

   $$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

   $$\left.\begin{aligned} ①&: 4c_1 + 2c_2 = 2 \\ ②&: 5c_1 + 2c_2 = 3 \end{aligned}\right\} \text{Solve.}$$

2. Reducing span

   Show 1 vector can be represented as a combo of other.

   $$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

   * If $v_k \in$ big set, $v_k \in$ small $\Leftrightarrow$ span (big) = span (small) *

   $$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ! \Rightarrow \text{Solve for } c_1 \text{ and } c_2$$

   $$\therefore \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

   - Important to demonstrate set equality.

3. Show span$\{\ldots\} = \{\ldots\}$

   Show general vector that belongs in other set can be represented in terms of span.

   Ex.// Show $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3$

   Let $\vec{x} \in \mathbb{R}^3 \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

   $$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

   $$\left.\begin{aligned} ①&: x_1 = c_1 + c_2 + c_3 \\ ②&: x_2 = c_2 + 2c_3 \\ ③&: x_3 = c_3 \end{aligned}\right\} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}(x_1 - x_2 + x_3) + \ldots$$

   ↳ I can choose any $\vec{x}$ and still represent via spanning vectors.

LINEAR DEPENDENCE / INDEPENDENCE

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$$

If all $c_i = 0$ independent

If one or more not 0 dependent.

- Ex:// $B = \left\{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$

1. Write vector equation:

$$c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + c_3\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

2. Parametric equations:

① : $c_1 + c_2 + 2c_3 = 0$

② : $c_2 + c_3 = 0$

③ $-c_1 + c_2 = 0$

3. Solve:

③ $c_2 = c_1$

② : $c_2 = -c_3$

① : $c_2 + c_2 - 2c_2 = 0$

Infinite solutions → not linear independent

↳ We can write vectors in terms of one another

- Subset of a lin. indep. set is also lin. indep.

SUBSPACES

1. All subspaces must have $\vec{0}$
2. All subspaces are closed under vector addition.
3. All subspaces are closed under scalar multiplication

- Ex:// Show S is a subspace where

$$S = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| \begin{matrix} x_1 + x_2 = 0 \\ x_1 + x_3 = 0 \end{matrix} \right\}$$

1. Zero vector:

$x_1 = x_2 = x_3 = 0 \Rightarrow$ satisfies condition.

2. Vector addition:

$$x_1 = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \; y_1 = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}, \; x_1 + y_1 = \begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}$$

$$x_1+y_1+x_2+y_2 = (x_1+x_2)+(y_1+y_2) \quad x, y \in S$$
$$= 0 + 0$$
$$= 0$$
$$x_1+y_1+x_3+y_3 = (x_1+x_3)+(y_1+y_3)$$
$$= 0 + 0$$
$$= 0$$

3. Scalar multiplication.

$$\vec{x} \in S, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad k\vec{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}$$
$$kx_1 + kx_2 = k(x_1+x_2) = 0$$
$$kx_1 + kx_3 = k(x_1+x_3) = 0$$

All three parts of tests were passed. ∴ S is a subspace.

BASES

Minimal # of lin. indep. vectors that can span space.

1. Linearly independent
2. Span the whole subspace

Basic strategy:

1. Show linear independence.
2. Spanning:
   Show some general vector $\vec{x} \in S$, can be decomposed into bases.

- Ex:// $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$ is a bases for $\mathbb{R}^2$

1. Spans $\mathbb{R}^2$.

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

① : $c_1 = x_1$

② : $-c_2 + c_1 = x_2$

$\hookrightarrow c_2 = x_1 - x_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (x_1 - x_2) \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

2. Lin indep:

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Standard bases: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ldots$

- Finding basis of subspace.
  - Manipulate subspace definition to form vectors that span some general vector.
  - Ex:// Find basis of

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \begin{matrix} x_1 + x_2 = 0 \\ x_1 + 7x_3 = 0 \end{matrix} \right\}$$

1. Find a general vector and use definition for components.

$$\vec{x} \in S: \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ -\frac{x_1}{7} \end{bmatrix} =$$

2. Take out free variable:

$$\begin{bmatrix} x_1 \\ -x_1 \\ -\frac{x_1}{7} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -1/7 \end{bmatrix} \rightarrow \text{Basis:}$$

↳ Lin. indep + span

  - Ex:// Find basis of

$$\left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

$$\vec{x} = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

ORTHOGONAL + ORTHONORMLAL SETS

$$\{ v_1, v_2 \ldots v_k \}$$

$$\downarrow$$

Orthogonal:

$$v_i \cdot v_j = \vec{0}$$

$$\downarrow$$

Orthonormal:

$$v_i \cdot v_j = \vec{0} \wedge |v_i| = 1$$ ⇐ Lin. indep + unit vector

- Writing vectors as projections of orthogonal / orthonormal sets.
  - Ex:// $\vec{x}$ as proj of $\{v_1 \ldots v_k\}$ ⇒ orthogonal.

$$\vec{x} = \text{proj}_{\vec{v}_1} \vec{x} + \text{proj}_{\vec{v}_2} \vec{x} + \ldots + \text{proj}_{\vec{v}_k} \vec{x}$$

$$= \frac{v_1 \cdot \vec{x}}{\|v_1\|^2} \vec{v}_1 + \frac{v_2 \cdot \vec{x}}{\|v_2\|^2} \vec{v}_2 + \ldots + \frac{v_k \cdot \vec{x}}{\|v_k\|^2} \cdot \vec{v}_k$$

- Orthonomal: $\| v_i \| = 1$!

$\therefore \vec{x} = (v_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 + \dots + (\vec{v}_k \cdot \vec{x})\vec{v}_k$

- Construct orthonomals sets:
  1. Get orthogonal vectors: look carefully at dot product
  2. Normalize (divide by magniture).

- Ex:// Construct orthonomal basis for $\mathbb{R}^2$ given $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

  1. Get orthogonal vector:

$$\therefore v_1 \cdot v_2 = 0$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0$$

  2. Normalize:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \overset{\frac{1}{\|v_1\|}}{\frac{1}{\sqrt{5}}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \overset{\frac{1}{\|v_2\|}}{\frac{1}{\sqrt{5}}} \times \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

  3. Basis:

$$\mathbb{R}^2 \text{ basis: } \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

SYSTEMS OF EQUATIONS + MATRICES

$$m \text{ rows} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{n \text{ cols}} \Rightarrow M_{m \times n}$$

Use matrix for linear equations:

- Ex://

① : $x_1 - x_2 - x_3 = b_1$

② : $3x_1 - 6x_2 - 5x_3 = b_2$

③ : $2x_1 - x_2 + x_3 = b_3$

$$\underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -5 \\ 2 & -1 & 1 \end{bmatrix}}_{\text{Coefficient matrix}} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

RREF:

$$\begin{bmatrix} 1 & 0 & 0 & & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. All rows have a leading 1. (pivot)
2. All entries below/above pivot $= 0$

To reduce matrix to RREF form:

$$\left[\begin{array}{cccc|c} | & | & | & | & \vec{b} \end{array}\right] \rightarrow \text{row operations (subtracting rows, multip)}$$

- Key: do row op one by one.

Steps to solve matrix problems:

① Represent problem in a matrix (augmented matrix)

$$A\vec{x} = \vec{b}$$

Augmented:

$$[A \mid b] = \vec{x}$$

② Reduce to row-echelon form or RREF.

③ Put this back into equations. Using columns as variabl.

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & 3 & 0 & 4 & 6 \end{array}\right] \iff \begin{array}{l} ①: x_1 - 3x_2 + x_4 = 1 \\ ②: 2x_2 + x_3 = 5 \\ ③: 3x_3 + 4x_4 = 6 \end{array}$$

④. Separate variables into leading vars / free variables.
↳ Leading vars: determined (cols w/ pivot)
↳ Free vars: not determined (cols w/ no pivot)

⑤: Write leading variables in terms of free variables + parametrize them ($x_2 \rightarrow t$, $x_3 \rightarrow s$, $s, t \in \mathbb{R}$)

⑥: Take general solution vector $\vec{x}$ and break into bases:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 + 5x_2 + 1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Bases: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

↳ Bases of soln space.

- Complex matrices: same but be aware that $\times j$ is a <u>valid</u> row op

RANK

# of leading vars in REF $\Rightarrow$ rank(A)

# of free vars in REF $\Rightarrow$ nullity of A.

$$\therefore\ n = \text{rank}(A) + \text{nullity}(A)$$

How do we know if a system will give a solution.

$$\text{rank}(A) \Leftrightarrow \text{rank}([A|\vec{b}])$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

↳ consistent! ↳ Inconsistent $(\text{rank}(A) \neq \text{rank}([A|\vec{b}]))$

$\therefore$ Extension: if row $[0 \ldots 0 \mid b]$ $(b \neq 0)$, no solution!

$[A|\vec{b}]$

- $\text{rank}([A|\vec{b}]) = \text{rank}(A)$ → There is Solution
  - $\text{rank}(A) = n$ → One Unique
    - ↳ $\text{nullity}(A) = 0$
  - $\text{rank}(A) \neq n$ → Infinite
    - ↳ $\text{nullity}(A) \neq 0$
- $\text{rank}([A|\vec{b}]) \neq \text{rank}(A)$ → No Solution

$\Rightarrow$ Reason: $n = \text{rank}(A) + \text{nullity}(A)$

- Ex: // Consider system:

$$x_1 + x_2 + \alpha x_3 = 3$$
$$2x_1 + x_2 + x_3 = \beta$$
$$-x_1 + x_2 + 2x_3 = 1$$

Find out values of β, α such that:

a) No solution

b) Unique solution

c) Infinite solutions.

1. Make matrix + RREF:

$$\left[\begin{array}{ccc|c} 1 & 1 & \alpha & 3 \\ 2 & 1 & 1 & \beta \\ -1 & 1 & 2 & 1 \end{array}\right] \sim \begin{array}{c} \\ 2R_1 - R_2 \\ R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 3 \\ 0 & 1 & 2\alpha - 1 & 6-\beta \\ 0 & 2 & \alpha + 2 & 4 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \sim \\ 2R_2 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\alpha + 1 & -3 + \beta \\ 0 & 1 & 2\alpha - 1 & 6 - \beta \\ 0 & 0 & 3\alpha - 4 & 8 - 2\beta \end{array}\right]$$

2. Make conditions:

a) No solution: $\text{rank}([A|\bar{b}]) \neq \text{rank}(A)$. or $[0 \text{ — } 0 \,|\, \bar{b}]$ $(b \neq 0)$

$\therefore \quad 3\alpha - 4 = 0 \quad \wedge \quad 8 - 2\beta \neq 0$

$\alpha = 4/3 \quad \wedge \quad \beta \neq 4$

b) Unique solution: $3\alpha - 4 \neq 0$

$\therefore \alpha \neq 4/3$

c) Infinite solutions:

$[0 \text{ — } 0 \,|\, 0]$

$\therefore \quad 3\alpha - 4 = 0 \quad \wedge \quad 8 - 2\beta = 0$

$\alpha = 4/3 \quad \wedge \quad \beta = 4 \Rightarrow \text{rank}(A|\bar{b}) = \text{rank}(A)$

but $\text{nullity}(A) \neq 0$.

- Know difference between under/over determined system

APPLICATIONS

1. Chemical reactions:

Represent compound as vector of elements + solve for coefficients.

◦ Ex:// $H_3PO_4 + Mg(OH)_2 \longleftrightarrow Mg_3(PO_4)_2 + H_2O$

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \longrightarrow c_3 \begin{bmatrix} 0 \\ 2 \\ 8 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$ $\Rightarrow$ Put into matrix + solve.

2. Linear Programming.

1. Define variables (want to maximize) } → UNITS
2. Write down constraints.
3. Write down optimization function
4. Put constraints as equalities → matrix → solve for vars.
5. Check constraints + put into optimization

3. Network.

Flow into node = Flow out of node.

1. Look at inflow + outflow for each node
2. Construct equations for each node
3. Solve.

- Optionally: put constraints (esp. if free variables) (rate ≥ 0)

4. Electrical networks:

1. $\Sigma v\downarrow = \Sigma v\uparrow \Rightarrow$ Loop rule
2. $I_{in} = I_{out} \Rightarrow$ Node rule

1. Loop rule:
   1. Assume current ~~dir~~ direction
   2. Traverse through loop in arbitrary direction.
   3. If traversing against current $\Rightarrow$ voltage gain
      Else $\Rightarrow$ voltage drop.

   Make sure to always have same # of eqns as vars.

<u>DECREASING SPANNING SETS</u>

$$B = \{v_1 \dots v_k\}$$

Want to find subset of B that is basis + spans B.

1. Conduct LI test:
$$\begin{bmatrix} v_1 & \text{—} & \text{—} & v_k & \vec{0} \\ | & | & | & | & \downarrow \end{bmatrix}$$
2. Leading variables $\Rightarrow$ candidates. Check L.I.
3. Span of candidates = B.
   ↳ Matrix too:
   Say $v_1 + v_3$ are LI:
   $v_2 = a v_1 + b v_3$
   $v_4 = x v_1 + y v_3$

- To find coefficients of combinations:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 3^{rd} \text{ column: } v_3 = 2v_1 + 5v_2$$

↳ Coefficients correspond to previous pivot column coefficients.

DIMENSION THEORY

Dimension: # of bases of a subspace.

$$A\ \{v_1 \cdots v_k\},\ B = \{v_1 \cdots v_l\}$$

If A is a set of bases, is B also a set of bases?

1. If $l < k$: not bases b/c not spanning whole subspace.
2. If $l > k$: not bases b/c not lin. indep.
3. If $l = k$: it is bases iff vectors in B are L.I.

Problem solving strat:

1. Find bases for your subspace
2. Note your dimension.
3. If $|B| \neq$ dimension $\rightarrow$ exit. Else: check. L.I.

FUNDAMENTAL SUBSPACES

Let A be a matrix:

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

Col (A) = linear combo of lin. indep. columns in A.
Row (A) = " " rows in A.

Problem Solving Strat:

1. RREF
2. Solve for $A\vec{x} = \vec{0}$ system.
   ↳ Get general solution $\Rightarrow$ bases.
3. Col: look at lin. indep columns $\Rightarrow$ span{lin. indep.}
4. Row: " rows $\Rightarrow$ span{original rows/new rows}

Dimensions:

$$\dim(\text{Row}) = \dim(\text{Col}) = \text{rank}(A)$$

$$\dim(\text{Null}) + \text{rank}(A) = n \quad (\text{num of cols})$$

MATRIX ALGEBRA

Special:

$\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \leftarrow$ Square

numRows = numCols

$m = n$

$\begin{bmatrix} \ddots & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddots \end{bmatrix}$

Diagonal

I

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A = IA$

Diagonal w/ 1's

Triangular:

$\begin{bmatrix} \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots \\ 0 & 0 & \ddots \end{bmatrix}$

upper

$\begin{bmatrix} \ddots & 0 & 0 \\ \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots \end{bmatrix}$

lower

Addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} d & e \\ f & g \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+f & d+g \end{bmatrix}$$

Scalar multiplication:

$$k \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ak & ck \\ bk & dk \end{bmatrix}$$

$\Big\}$ Matrix is same size.

Vector multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} a \\ c \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

↳ Represent systems! Spanning sets!

Ex://:

$$x_1 + 3x_2 - 2x_3 = 7$$
$$-x_1 + -4x_2 - 3x_3 = 8$$

$$\begin{bmatrix} 1 & 3 & -2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

To solve:

$$\begin{bmatrix} 7 \\ 8 \end{bmatrix} \in \underline{\text{span}} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right\}$$

MATRIX MULTIPLICATION

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} & \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} f \\ h \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$[\vec{A}_1 \dots \vec{A}_n][\vec{b}_1 \dots \vec{b}_n] = [[\vec{A}_1 \dots \vec{A}_n]\vec{b}_1 \quad \dots \quad [\vec{A}_1 \dots \vec{A}_n]\vec{b}_n]$$

Careful:

1. Order matters: $AB \neq BA$ not always.
2. Size matters:

→ All other properties of $\mathbb{R}$ hold.

$$A \times B = C$$

$$\underbrace{(m \times \underbrace{n)\,(n}_{\text{match}} \times l)}_{\text{dimensions}} = m \times l$$

Easy: Let $C = AB$.

$$C_{ij} = \vec{A}_{i\,\text{row}} \cdot \vec{B}_{j\,\text{column}}$$

TRANSPOSE MATRIX

$$(A_{ij})^T = A_{ji}$$

Visually + take time

Properties:

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T \Rightarrow \text{Addition distribution is unique to transpose!}$$

$$(AB)^T = B^T A^T$$

CONJUGATE TRANSPOSE + HERMITIAN

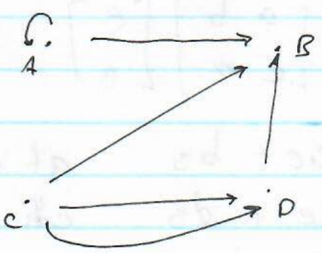
Conj. Transpose: $A^* = \bar{A}^T$

Hermitian: $A^* = A$

Properties:

1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$
3. $(cA)^* = \bar{c}\,A^*$
4. $(AB)^* = B^* A^*$

Adjacency Matrices



1. Adjacency matrix:
   A_{ij} : # of paths from node i to node j
2. Trying to find # of ways in k paths: A^k
3. Look at entry ij to find # of ways from node i to node j

Tips:

1. Use the row + column dot product trick.
2. Know the meaning of:
   "no more": calculates paths below the limit + equal
   Specify route: calculate routes from $A \to B$ and ~~B→A~~ D−C separately ⇒ multiply
   "at least once": Calculate # of ways to not travel to node (adjacency matrix new), subtract from original adj matrix.

Markov Chains

Probability vector: sum of components = 1
↳ Steady state: $P\vec{s} = \vec{s} \Rightarrow \cancel{P}\vec{s} - \vec{s} = 0$
$(P - I)\vec{s} = 0$

1. Update rules ⇒ matrix.
2. Iterate however many times to find state at certain gen.
3. Steady state:
   $[P - I \mid \vec{0}\,] \Rightarrow$ solve.
   ↳ General vector. Find free variable s.t. the sum of the ~~stek~~ vector = 1. ⇒ steady state.

- Regular: all entries are positive for some power.
  Matrix is regular ⇔ steady state vector that is unique.

INVERSES

$$AB = I = BA$$
$$A \text{ is invertible}, \quad B = A^{-1}$$
→ Proofs

$\boxed{\text{Defined for square matrices}}$

① How to find inverse:

$$[A \mid I] \Rightarrow \text{RREF } A \Rightarrow [I \mid B]$$
$$\therefore B = A^{-1}$$

② Usage?

$$A\vec{x} = C$$

Since $B = A^{-1}$:

$$BA^{-1}\vec{x} = BC$$
$$\vec{x} = BC$$

③ How do you know if A is invertible.

1) $\text{RREF}(A) = I$
2) $\text{rank}(A) = n \Rightarrow$ columns are bases.
3) $\forall b \in \mathbb{R}^n, \exists! \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$
4) A^T is invertible
5) Rows of A span $\mathbb{R}^n$ b/c lin. indep.
6) $\text{Null}(A) = \{\vec{0}\}$
   ↳ Only $\vec{x} = \vec{0}$ can give $A\vec{x} = \vec{0}$

④ Properties:

$$(tA)^{-1} = \frac{1}{t} A^{-1}$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^k)^{-1} = (A^{-1})^k$$
$$(A^T)^{-1} = (A^{-1})^T$$
$$(A^{-1})^{-1} = A$$

Note that
①: Order matters
②: Addition distribution NOT Defined

- Ex:// Find A^{-1} given: $(A^T + I)^{-1} = (BA^{-1})^T$ ← Cannot distribute!

$$A^T + I = ((BA^{-1})^T)^{-1}$$
$$(A^T + I)^T = (BA^{-1})^{-1}$$
} Applied T and -1 on both sides

$$A + I = AB^{-1}$$ ← Distribute

$$I = AB^{-1} - A$$
$$I = A(B^{-1} - I)$$
} Manipulated to definition.

Since $AA^{-1} = I$

$$\boxed{A^{-1} = B^{-1} - I}$$

LINEAR TRANSFORMATION

$$\vec{V}_1 \xrightarrow[L(x)]{} \vec{V}_2$$

$L(x)$ is a linear transformation when:

1. $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$
2. $L(s\vec{x}) = s\,L(\vec{x})$

- Proving $L(x)$ is linear:
  1. Take arbitrary $s, t \in \mathbb{R}$, $\vec{x}, \vec{y} \in \mathbb{R}^n$
  2. Prove:

     $$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

     by
     1. In $L(s\vec{x} + t\vec{y})$: combine vectors.
     2. Apply transformation
     3. Decompose.

$L(x)$: $\begin{bmatrix} & \\ & \end{bmatrix} \Rightarrow$ matrix $L(\vec{x}) = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix}\vec{x}\end{bmatrix} = \vec{y}$

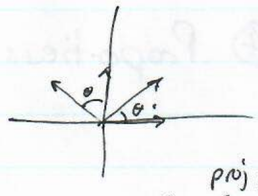
↳ Standard matrix:

$$[L] = [L(\vec{e}_1) \dots L(\vec{e}_n)]$$

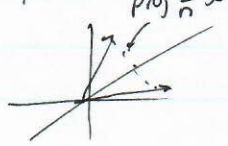
↳ Any basis.

Common transformations:

Rotation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



Reflection: $\text{refl}_{\vec{n}}\,\vec{x} = \vec{x} - 2\,\text{proj}_{\vec{n}}\,\vec{x}$



1. Find $\vec{n}$
2. Perform on each basis.

Stretch: $\begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}$

Contraction: $\begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$

Shear: $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Multiple Linear Transformations:

$$T(L(x)) = [T][L] \Leftarrow \text{matrix for composition.}$$

If

$$[T \circ L] = \mathbb{I} = [L \circ T]$$

$\therefore$ T is invertible and $T^{-1} = L$

$\therefore$ L can reverse transformation. $\Leftarrow$ Solve via inverse also.

Applying linear transformation:

Let $[L] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$. Find $\vec{x}$ such that $L(\vec{x}) = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

Solve $[L \mid \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}]$: $\Leftarrow$ KEY

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 3 & 7 \end{array}\right] \underset{R_3 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right] \underset{R_3 - R_2}{\overset{R_1 - 2R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 3x_3 \\ 3 + -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R})$$

* Don't try to find inverse if you don't know that there is a <u>unique</u> vector mapping to answer.

<u>KERNEL + RANGE</u>

Kernel: $\{\vec{x} \in \mathbb{R}^n : L(\vec{x}) = \vec{0}\} \Rightarrow$ null space of L

Range: $\{\vec{y} \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, L(x) = y\} \Rightarrow$ col. space of L.

Kernel problems:

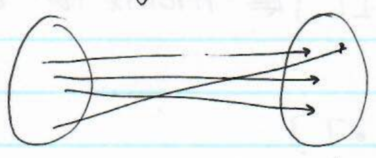
1. Determine if $\vec{x} \in Ker(L)$: $L(\vec{x}) = \vec{0}$?
2. Determine the basis of the kernel: Solve $[L(\vec{x}) \mid \vec{0}]$
   - Let $\vec{x} \in Ker(L) \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow [L][\vec{x}] = \vec{0}$

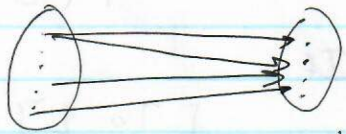
Range problems.

1. Determine if $\vec{x} \in Range(L)$: See if system $[L \mid \vec{x}]$ is consistent!
2. Basis of range: basis for col. space.

INJECTIVE (1-to-1) + SURJECTIVE (ONTO)

Injective: Surjective





$L(x_1) = L(x_2) \Leftrightarrow x_1 = x_2$ $\qquad$ $\forall y \in \mathbb{R}^m, \exists x \in \mathbb{R}^n$ s.t. $L(\vec{x}) = \vec{y}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

Injective: $\operatorname{Ker}(L) = \{\vec{0}\}$ $\qquad$ Surjective: $\operatorname{Range}(L) = \mathbb{F}^m$ (codomain)

$\Leftrightarrow \operatorname{rank}(L) = n$ (# of col) $\qquad$ $\Leftrightarrow \operatorname{rank}(L) = m$ (# of rows)

Proofs: use the definitions.

Bijective: both injective + surjective $\Rightarrow \operatorname{rank}(L) = n = m$

$\Rightarrow$ Invertible

- Ex:// Let $[L] : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$[L] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

a) Find all $\vec{x} \in \mathbb{R}^3$ s.t. $L(\vec{x}) = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

I.e: determine if $\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \in \operatorname{Range}(L)$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 3 & 7 \end{array}\right] R_1 - R_3 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & \cancel{7}\,-3 \end{array}\right] R_3 + R_2 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 2R_2 \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \quad \vec{x} \in \mathbb{R}^3 : \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

①: $x_1 - 3x_3 = -2$

②: $x_2 = 3 - 2x_3$

$$\therefore \quad \vec{x} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

b) Find basis for Range (L). Is it surjective?

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} \Rightarrow$ Since $2 \neq 3 \Rightarrow$ not surject.

c) Find basis for Kernel (L). Is it one to one?

Basis: $\left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \right\} \Rightarrow$ Since $1 \neq 3 \Rightarrow$ not injective.

DETERMINANTS + ADJUCATE MATRICES

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc$$

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A \cdot \text{adj}(A) = \det(A) \cdot I$$

$$A \cdot \boxed{\frac{\text{adj}(A)}{\det(A)}} = I$$

$$\therefore \quad A^{-1} = \frac{\text{adj}(A)}{\det(A)} \Rightarrow \det(A) \neq 0.$$

To check if a matrix has an inverse: make sure $\det(\_\_\_) \neq 0$.

Computing $\det(A)$: Basic

1. Take 1 row/column w/ most # of 0s
2. Iterate through row + column:

Multiply: $(-1)^{i+j} \cdot a \cdot \det(A_{ij}) \Rightarrow A_{ij}$: matrix w/ $i+j$ row + col deleted.

$$\det \begin{vmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{vmatrix} = 1 \cdot \det\left(\begin{bmatrix} -5 & 6 \\ 8 & 9 \end{bmatrix}\right) - 2\det\left(\begin{bmatrix} 4 & 6 \\ -7 & 9 \end{bmatrix}\right) + -3 \begin{vmatrix} 4 & -5 \\ -7 & 8 \end{vmatrix}$$

Computing $\det(A)$: Cofactor expression.

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

$$\therefore \det(A) = A_{ij} \cdot C_{ij} \cdot \ldots$$ (do it along row/column w/ most number of zeros)

Computing $\text{adj}(A)$:

$$\text{adj}(A)_{ij} = [C_{ij}]^T$$

Be careful w/:

1. negatives
2. row + column switching.

If adj (A) and A is known:

$$\det(A) = (1^{st} \text{ row of } A) \cdot (1^{st} \text{ column of } adj(A))$$

Elementary Row + Column Operations

Simplify A → compute det → change to A.

1. Row/column in A that is zero: det (A).
2. Swapping rows/columns: det (new matrix) = - det (A).
3. Adding multiples of 1 row to a ~~colum~~ row: det (new) = det (A)
   - Note: only adding multiples of 1 Col/Row - Don't do scaling of both row + column (ex: $R_1 + 2R_2$ ✓ $2R_1 + 3R_2$ ×)
4. If scaling row/colum: det (new) = k det (A).
5. $\det(kA) = k^n \det(A)$.

Problem solving strat:

1. Perform row + col operations s.t. you get rows + column of zeros.

   OR

2. Perform operation s.t. matrix is a diagonal matrix.

det (diagonal) = product of diagonal entries.

Tips:

1. Do not scale rows/cols when adding.
2. Do operations w/ factoring/scaling separately to get correct det (A) effect.

Ex: $\begin{vmatrix} 3 & 6 \\ 6 & 11 \end{vmatrix} \Rightarrow \underset{B}{\begin{vmatrix} 3 & 6 \\ 6 & 11 \end{vmatrix}} \xrightarrow{1/3\, C_1} \underset{C}{\begin{vmatrix} 1 & 6 \\ 2 & 11 \end{vmatrix}}$

$\frac{1}{3} \det(B) = \det(C)$

$\det(B) = 3\det(C) \rightarrow$ plug in

Determinant Properties

$$\left.\begin{aligned} \det(AB) &= \det(A)\det(B) \\ \det(A^k) &= (\det(A))^k \\ \det(A^T) &= \det(A) \end{aligned}\right\} \rightarrow \det(A+B) \text{ is not defined.}$$

Tip: convert adj (A) in terms of det (A) + A^{-1} for algebraic manipulation calculations.

- Ex:// Let $\det(A) = 32$. Find $\det\left(\frac{1}{16} B(\operatorname{adj}(A)\right)$. $\det B = 2$. ($B \in M_{3\times 3}$)

$$\therefore A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

$$\therefore A^{-1}\det(A) = \operatorname{adj}(A).$$

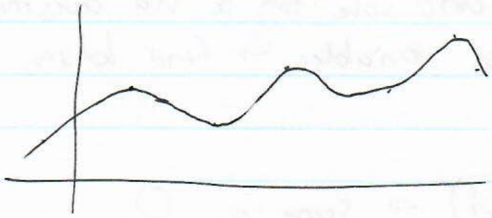
$$\begin{aligned}\therefore \det\left(\tfrac{1}{16} B \operatorname{adj}(A)\right) &= \det\left(\tfrac{1}{16} B \cdot A^{-1}\det(A)\right) \\ &= \frac{1}{16^3} \cdot \det(B) \cdot \frac{1}{\det(A)} \cdot (\det(A))^3 \quad \text{KEY!} \\ &= \left(\tfrac{1}{16}\right)^3 \cdot (32)^3 \cdot \frac{2}{32} \\ &= 8 \cdot \frac{2}{32} \\ &= \tfrac{1}{2}\end{aligned}$$

- Geometric interpretation:

$\rightarrow A = \det\left(\left[\bar{u} \ \ \bar{v}\right]\right)$

Parallelipiped $\rightarrow V = \det\left(\left[\bar{u} . \bar{v} \ \ \bar{w}\right]\right)$

POLYNOMIAL INTEROPOLATION



$\Rightarrow$ How do we find a polynomial s.t. it goes through all points?

Let there be n distinct observations. Let $p(x)$ be in:

$$p(x) = a_0 + a_1 x_1 + \dots + a_n x_1^{n-1} = y_1 \quad -①$$
$$a_0 + a_1 x_2 + \dots + a_n x_2^{n-1} = y_2 \quad -②$$
$$\vdots$$

$$\underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & \cdots & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & x_n^{n-1} \end{bmatrix}}_{\text{Vandermonde}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\det(\text{Vandermonde}) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Find all possible combos within n s.t. $j > i$.

Ex: If $n = 5$

$$\det(V) = (x_5 - x_4)(x_5 - x_3)(x_5 - x_2)(x_5 - x_1)(x_4 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$\therefore \det(V) \neq 0 \iff x_i$ are all distinct.

EIGENVALUES

$A\vec{x} = \lambda\vec{x} \Rightarrow$ Which pair of $(\lambda, \vec{x})$ will this work.

$(\lambda, \vec{x})$

eigenvalue ↗ ↖ eigenvector.

How to solve: theory:

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} - \lambda\vec{x} = 0$$
$$(A - I\lambda)\vec{x} = 0 \quad - ①$$

We also know $A - I\lambda$ is impossible to invert
↳ ($\lambda \to$ free variables!)

$$\therefore \det(A - I\lambda) = 0 \quad - ②$$

Actually, how do we solve?

1. Solve characteristic polynomial: $\det(A - I\lambda) = 0 \Rightarrow$ get λ values.
2. Substitute λ into ① and solve for $\vec{x}$ via augmented system.
   - Likely to get free variables $\to$ find bases.
3. Check: $A\vec{x} = \lambda\vec{x}$

Eigenspace: $\text{null}(A - I\lambda) \Rightarrow$ same as ①.
↳ Unique to each λ

Multiplicities:

- Algebraic: # of times λ is repeated in $C_A(\lambda)$
- Geometric: $\dim(\text{null}(A - I\lambda)) \Rightarrow$ # of bases.

$$1 \le g_\lambda \le a_\lambda$$

Complex system:

$(\lambda, \vec{x}) \Rightarrow (\bar{\lambda}, \bar{\vec{x}})$ is also valid.

- Ex:// Find eigenpairs for

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

① : Characteristic polynomial.

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$\therefore \det(A - \lambda I) = (1-\lambda)(1-\lambda) + 1 = 0$$

$$1 - 2\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

No way of factoring ⇒ quadratic formula.

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \cancel{1+2i}, \cancel{1-2i} \quad 1+i,\ 1-i$$ → Conjugate pairs

② : Solve original equation:

$$(A - \lambda I)\vec{x} = \vec{0}$$

Let $\lambda = 1 + i$.

$$\left[\begin{array}{cc|c} 1-1-i & -1 & 0 \\ 1 & 1-1-i & 0 \end{array}\right] = \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{ir_1 + r_2} \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$$\therefore -ix_1 = x_2$$

$$x_1 = -\frac{1}{i}x_2 = -\frac{i}{i^2} = ix_2$$

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\therefore \left(1+j, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$$

Other one: $\left(1-j, \begin{bmatrix} -j \\ 1 \end{bmatrix}\right)$

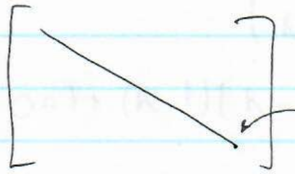
DIAGONALIZATION

$$[L] \longrightarrow [L']$$ (different matrix, same transformation)

Let $A + B$ be 2 different linear transformation.

$$\boxed{A = P^{-1} B P}$$ (P is some invertible

How to find P? $\Rightarrow$ Diagonal matrix



Non zero

$D = \text{diag}(a_1, \ldots, a_n)$ — element of diagonal.

Properties:

- $D_1 + D_2$: $\text{diag}(a_1 + b_1, \ldots, a_n + b_n)$
- $D_1 \cdot D_2$: $\text{diag}(a_1 b_1, \ldots, a_n b_n)$
- D^k : $\text{diag}(a_1^k, \ldots, a_n^k)$ if D is invertible

Can I change a matrix P (invertible) s.t. $A \rightarrow$ Diagonal?

$$D = P^{-1} A P$$

$$P = [\vec{v}_1 \ldots \vec{v}_k], \quad D = [\lambda_1 \vec{e}_1 \ldots \lambda_n \vec{e}_n]$$

$$PD = AP$$

$$P[\lambda_1 \vec{e}_1 \ldots \lambda_n \vec{e}_n] = A[\vec{v}_1 \ldots \vec{v}_k]$$

$$[\lambda_1 P\vec{e}_1 \ldots P\lambda_n \vec{e}_n] = [A\vec{v}_1 \ldots A\vec{v}_k]$$

$$[\lambda_1 \vec{v}_1 \ldots \lambda_n \vec{v}_n] = [A\vec{v}_1 \ldots A\vec{v}_k]$$

$\therefore A\vec{v}_i = \lambda_i \vec{v}_i \Leftarrow$ Each column! Eigenvalue problem.

Theorem:

If $a_\lambda = g_\lambda \Rightarrow P$ contains eigenvectors + D has eigenvalues across diagonal.

How to get a matrix into diagonal form:

1. Solve the characteristic polynomial
2. Get set of eigenvalues + eigenvectors.

$$P = [\vec{v}_1 \ldots \vec{v}_k], \quad D = [\lambda_1 \vec{e}_1 \ldots \lambda_n \vec{e}_n]$$

* Note: Can repeat eigenvalues if multiple eigenvectors.

- Application for diagonalization: powers.

$$A^k = P D^k P^{-1}$$

- Ex: Find A^k for

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}.$$

1. Find eigenvalues + eigenvectors.

   a) Eigenvalues.

$$\det(A - \lambda \mathbb{I}) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2, 3$$

   b) Eigenvectors:

$$(A - \lambda \mathbb{I})\vec{x} = 0$$

$\lambda = 2$:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ ← 1 basis found

$\lambda = 3$:

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\therefore D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. Raise to power:

Really easy w/ diagonals.

$$A^k = P^{-1} \cdot D^k \cdot P$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{k+1} - 3^k & 2 \cdot 3^k - 2^{k+1} \\ 2^k - 3^k & 2\, 3^k - 2^k \end{bmatrix}$$

- Application: finding determinant

$$\det A = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n$$

- Trace: sum of diagonal:

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{i} \lambda_i$$

① : $trace(AB) = tr(BA)$

② : $tr(D) = tr(A)$

ABSTRACT VECTOR SPACES

Set where vector addition + multiplication are defined.

Addition:

$\vec{u} + \vec{v} \in V$

$u + (\vec{v} + \vec{w}) = (\vec{u} + v) + \vec{w}$

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\vec{u} + \vec{0} = \vec{u}$ ⇒ Not always 0!

Multiplication

$c \cdot \vec{u} \in V$

$c \cdot \vec{u} = \vec{u} \cdot c$

$d(c\vec{u}) = c(d\vec{u})$

$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

$1\vec{u} = \vec{u}$ ⇒ Not always 1!

1. Operation changes
2. No longer vectors (matrices, polynomials)

$\Big\} \rightarrow$ Vector space.

1. ⇒ Stick to the definition! Esp. for additive inverse, identity, + multiplication inverse + identity.

Tip: Define what 0 and 1 are in vector space.

2. ⇒ Matrices + polynomials
   1. Subspaces
   2. Basis
   3. Linear independence.
   4. Span
   5. Subset of bases.
   6. Ker + Range.

Can be applied w/ defn you already know!

Consider entity exactly like a vector.

a) Subspace:

Use subspace test.

1. Show $\vec{O} \in V$
2. Scaler multiplication clousre.
3. Vector addition clousre.

$\Big]\rightarrow$ Satisfy condition of set!

Ex:// Let $V = M_{2\times2}(\mathbb{R})$ and $A \in M_{2\times2}(\mathbb{R})$ and $W = \{B \in M_{2\times2} \mid AB = [0]\}$. Show W is a subspace of V.

1. Nonempty:

$$\text{Consider } B = [0] \Rightarrow AB = A[0] = [0] \checkmark$$

$$\therefore \vec{O} \in W$$

2. Vector addition + scaler multiplication.

Let $X \in W \wedge Y \in W$, $\alpha \in \mathbb{R}$

$$\therefore AX = [0], \; AY = [0].$$

$$\therefore A(X+Y) = AX + AY = [0] \checkmark \rightarrow \text{Vector addition}$$

We know $AX = [0]$.

$$A(cX) = c(AX) = [0] \checkmark \rightarrow \text{Scaler multiplication}$$

b) Linear independence:

- Ex://

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Show that B is linearly independent.

Consider:

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = [0]$$

Show that $c_1 = c_2 = c_3 = c_4 = 0$!

c) Basis:

1. Show linear independence.
2. span (set) = subspace $\Leftrightarrow$ dim (span) = dim (subspace).

$$\text{Remember: } \dim(M_{m\times n}) = \underline{\underline{mn}}$$

OR

Show $x \in$ set can be repped as combo of bases.