Introduction

Diff Eq:

Let $x(t) = f(t)$

$$x(t) = f\left(x, \alpha, \underbrace{\frac{d}{dt}x, \frac{d^2}{d^2t}x^2}_{\text{Diff.}} \ldots\right)$$

"Solve" a DE?

↳ Take DE ⇒ find original definition of dep. var. in terms of indep. variables.

Signals:

Carries info

System:

Signal $\longrightarrow \boxed{f(..)} \longrightarrow$ Signal..

Algebraic:

Ex://
$$\frac{dx}{dt} = \alpha x(t)$$

① Separate:
$$\frac{dx}{x} = \alpha\, dt$$

② Integrate:
$$\int \frac{1}{x}\, dx = \int \alpha\, dt$$

$$\ln x + c_1 = \alpha t + \boxed{c_2}$$

Constants matter!

$$\ln x = \alpha t + c_3 \qquad c_3 = c_2 - c_1$$

③ Solve for x:
$$x = e^{\alpha t + c_3}$$
$$x = c e^{\alpha t} \Rightarrow c = e^{c_3}$$

Numerically: out-of-scope.

Important: constants b/c they encaps. initial cond.

↳ Initial cond: $x(t_0) = x_0$ (t_0 doesn't have to be 0)

or $x'(t_0)$, or $x''(t_0)$

↳ Boundary cond: $y(x = x_1)$, $y''(x = x_2)$, ...

Ex://
$$\frac{dx}{dt} = ax - bx^2 \qquad (a, b > 0)$$

↳ $a < 0$ ⇒ exp. decay

① Separation:
$$\int \frac{dx}{ax - bx^2} = \int dt$$

② PFD:

$$\int \frac{1/a}{x} + \int \frac{b/a}{a-bx} = \int dt$$

③ Integral:

$$\tfrac{1}{a} \ln x + \tfrac{b}{a}\left(-\tfrac{1}{b}\right) \ln(a-bx) = t + c$$

$$\tfrac{1}{a}\left(\ln x - \ln(a-bx)\right) = t+c$$

$$\ln\left(\frac{x}{a-bx}\right) = a(t+c)$$

$$\frac{x}{a-bx} = e^{a(t+c)}$$

④ Solve for x:

$$x = e^{a(t+c)}\,(a-bx)$$

$$= ae^{a(t+c)} - bx\,e^{a(t+c)}$$

$$x + bx\,e^{a(t+c)} = ae^{a(t+c)}$$

$$x = \frac{ae^{a(t+c)}}{1+be^{a(t+c)}}$$

$$= \vdots$$

$$= \frac{a}{b} \cdot \frac{1}{1+e^{-a(t-t_0)}}$$

You can do simple analysis: $\boxed{\lim_{t\to\infty} x = \frac{a}{b}}$ → Steady state.

Easier way to get S.S.:

$$\frac{df}{dx} = 0$$

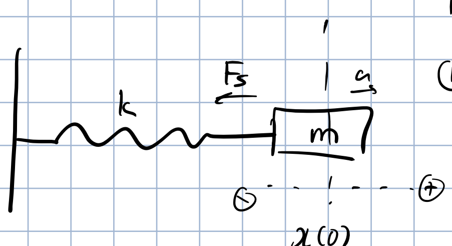
$$\frac{dx}{dt} = ax - bx^2 = 0$$

$$x(a-bx) = 0$$

$$\therefore x = 0,\ a/b \implies 2 \text{ s.s. soln.}$$

Ex:// Spring

Model via D.E.



① Equations using defn.

$$F_s = ma$$

$$-kx = m \cdot \frac{d^2x}{dt^2}$$

Taylor Expansion

Problem: function are hard to integrate

Soln: approx. func via polynomials (via Taylor method)

Taylor Theorem

Say that approx. func $f(x)$ at $x = x_0$.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

i^{th} order: $\frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$

Ex:// Prove $\sin\theta = \theta$ for small θ.

① Function + center

$f(\theta) = \sin\theta$

Center: small $\theta \sim 0$

② Taylor exp.

$$f(\theta) = \sin(0) + \frac{f'(0)}{1!}\theta + \frac{f''(0)}{2!}\theta^2 + \dots$$

$$= 0 + \theta + 0 + \dots$$

$$\sim \theta$$

Q: n^{th} order Taylor polynomial

↳ $f(x) \rightarrow$ T.E $\rightarrow$ only go up to n^{th} derivative $(x - x_0)^n$

Useful expansions:

1. $\sin\theta \sim \theta$
2. $\cos\theta \sim 1 - \theta^2/2$
3. $e^x \sim 1 + x$
4. $\sqrt{1+x} \sim 1 + x/2$
5. $\frac{1}{1 \pm x} \sim 1 \mp x$
6. $\frac{1}{\sqrt{1+x}} \sim 1 - x/2$

Use in compound T. E.

Ex:// T. E. of $\frac{1}{\sqrt{R^2 + D^2}}$

① Component func:

Looks like $\frac{1}{\sqrt{1+x}} \Rightarrow$ variable sub

$$\frac{1}{\sqrt{R^2 + D^2}} = \frac{1}{R\sqrt{1 + D^2/R^2}} = \frac{1}{R} \cdot \frac{1}{\sqrt{1 + D^2/R^2}}$$

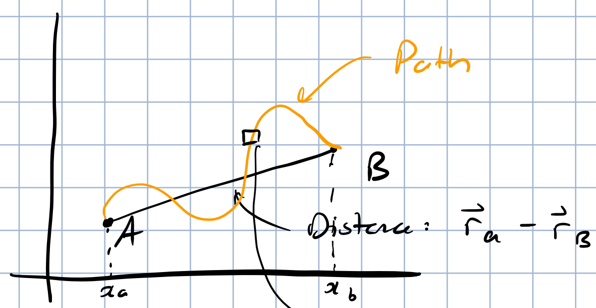
Let $x = D^2/R^2$

② T.F. of comp. of fn.:

$$\frac{1}{R}\cdot\frac{1}{\sqrt{1+D^2/R^2}} = \frac{1}{R}\left(1-\frac{D^2}{2R^2}\right)$$

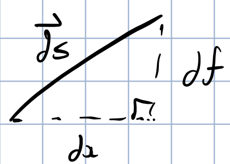
Important note: make sure dep. variable in expansion is <u>unitless</u>

Distance vs Path Length



Q: Path length?

If s is a function of x ($f(x)$)



$\therefore\ ds^2 = dx^2 + df^2$

$ds = \sqrt{dx^2 + df^2}$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{df}{dx}\right)^2}$

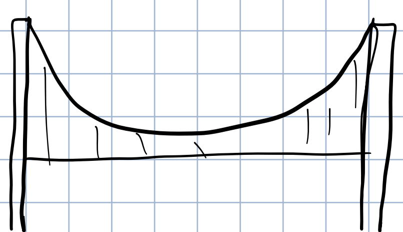
$$S = \int ds$$

$$= \int_{x_a}^{x_b} \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx$$

$$= \boxed{\int_{x_a}^{x_b} \sqrt{1+(f'(x))^2}\, dx}$$

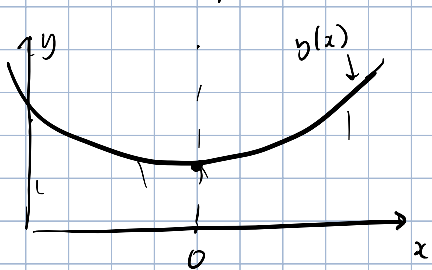
Hanging cable problem

Problem:



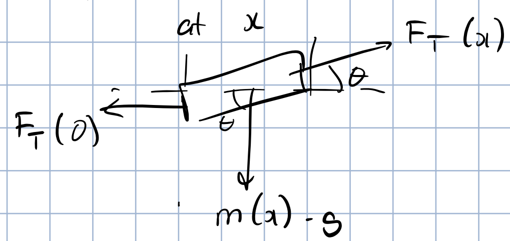
What is shape of cable?

① Model problem via physics:



Ls

a. Consider differential:



b. Use physics

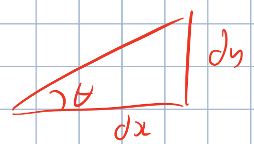
$\hat{x}$: $F_T(x)\cos\theta = F_T(0)$ — ①

$\hat{y}$: $F_T(x)\sin\theta = m(x)\cdot g$ — ②

② ÷ ①: $\tan\theta = \dfrac{m(x)\cdot g}{F_T(0)}$

② Introduce derivative:

Trick:

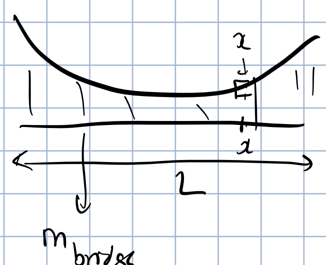


$\Rightarrow \tan\theta = \dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{m(x)\cdot g}{F_T(0)}$$

③ Classify $m(x)$

1) Case #1: Cable weight is negligible



Assuming linear density

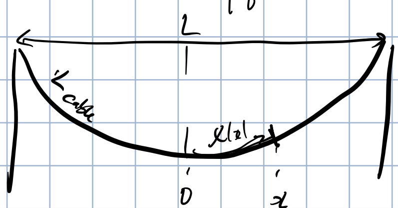
$$m(x) = \frac{m_{bridge}}{L}\cdot x$$

$$\therefore \frac{dy}{dx} = \underbrace{\left[\frac{m_{bridge}\cdot g}{F_T(0)\cdot L}\right]}_{c_1}\cdot x$$

$$\frac{dy}{dx} = c_1 x$$

$$y = \frac{c_1}{2}\cdot x^2 \sim \text{Parabola}$$

2) Case #2: Cable supports own weight



$\Rightarrow$ Weight at pt. x?

Assume linear density:

$$m(x) = \frac{m_{cable}}{L_{cable}}\cdot \ell(x)$$

$$\therefore \frac{dy}{dx} = \underbrace{\left[\frac{m_{cable}\cdot g}{L_{cable}\cdot F_T(0)}\right]}_{c_1}\cdot \ell(x)$$

$$dy = c_1 \ell(x) \cdot dx$$

① Take 2nd derivative

2nd order diff eqn. $\Longrightarrow$

$$\frac{d^2y}{dx^2} = c_1 \cdot \frac{d\ell(x)}{dx} \longrightarrow$$ Path differential!

$$= c_1 \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

② Variable sub:

$$s = \frac{dy}{dx}$$

$$\therefore \frac{ds}{dx} = c_1 \cdot \sqrt{1 + s^2}$$

③ Integrate:

$$\frac{dy}{dx} = s = \frac{e^{c_1 x} - e^{-c_1 x}}{2}$$

$$y = c_1 \cdot \frac{e^{c_1 x} + e^{-c_1 x}}{2} + c_3$$

$$\boxed{y = c_1 \cdot \cosh(c_1 x) + c_3}$$

- Partial diff eqns:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \cdot \frac{\partial^2 y}{\partial^2 x} \Rightarrow y = f(t, x)$$

$$\vdots$$

$$y = A \sin k\left(x - \sqrt{\frac{T}{\rho}}\, t\right)$$

LINEAR ODE W/ CONSTANT COEFFICIENTS

- Linear:

$$L y = f(x)$$

Linear operator

$$\frac{d^2y}{dt^2} + ay = f(x) \Rightarrow \overbrace{\left(\frac{d^2}{dt^2} + a\right)}^{\text{Linear}} y = f(x) \quad ✓$$

$$y\frac{d^2y}{dt^2} + ay^2 = f(x) \Rightarrow \overbrace{\left(\frac{d^2y}{dt^2} + ay\right)}^{\text{nonlinear}} y = f(x) \quad ✗$$

- Ordinary $\Rightarrow$ no partial derivatives
- Homogeneous vs inhomog:

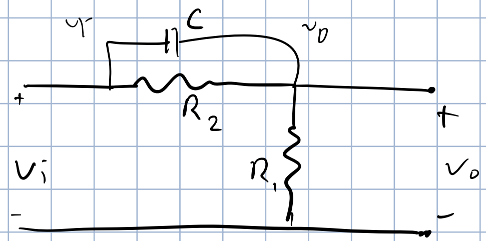
$$\underbrace{Ly = f(x)}_{\text{Inhomo..}} \qquad \underbrace{Ly = 0}_{\text{homogeneous}}$$

- Principle of superposition:
  - How do we take multiple solns. into account?

    If $f_1(t)$ is 1 soln. of y, $f_2(t)$ is another soln $\Rightarrow$ for any constants,

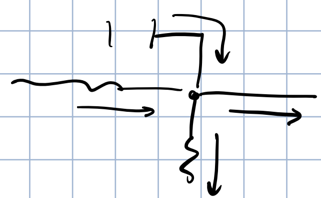
    c_1, c_2, $c_1 f_1(t) + c_2 f_2(t)$ is also a soln for y

- Setting up D.E.:
  - Physics (Newton's law / electric laws)
  - Ex://



Set up D. E.

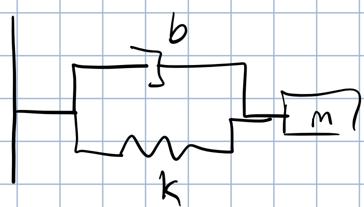
Kirchhoff's Law:



$$\Rightarrow I_{R_1} + I_{V_0} = I_C + I_{R_2}$$

$$\frac{V_0(t)}{R_1} = C \cdot \frac{d}{dt}(V_0 - V_i) + \frac{1}{R_2}(V_0 - V_i)$$

  - Ex://



Set up D. E.

Newton's Law $\Rightarrow$ Hooke's law.

$$F = ma$$

$$-kx - bv = ma$$

$$ma + kx + bv = 0$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

- If $Ly = f(t)$; $y_1(t)$ is soln of $Ly = 0$, y_2 is soln. of $Ly = f(t)$, then we might consider soln of $c_1 y_1 + c_2 y_2$
  - y_1 : transient state
  - y_2 : steady / driving state

- How to solve a D.E. in "guess" method
  1. Guess: $y = e^{\lambda t}$
  2. Substitute into DE $\Rightarrow$ characteristic equation
  3. Find values of λ
  4. Combine via superposition

- Ex://

$$\frac{d^2y}{dt^2} + \frac{b}{m}\left(\frac{dy}{dt}\right) + \frac{k}{m}y = 0$$

1. Guess: $y = e^{\lambda t}$

2. Sub + char. eqn:

$$\lambda^2 e^{\lambda t} + \frac{b}{m}\cdot\lambda e^{\lambda t} + \frac{k}{m}e^{\lambda t} = 0$$

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

3. Find λ values:

Quadratic formula:

$$\lambda = \frac{-b/m \pm \sqrt{b^2/m^2 - \frac{4k}{m}}}{2}$$

Cases:

a) $b^2/m^2 - 4k/m = 0 \Rightarrow$ Critically damped, 1 soln.

b) $\quad$ " $\quad > 0 \Rightarrow$ 2 real solns. (overdamped)

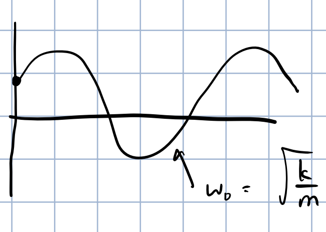
c) $\quad$ " $\quad < 0 \Rightarrow$ 2 imag. solns. (underdamped)

Solns: s_1, s_2 for λ

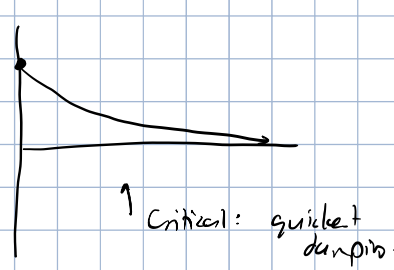
4. Superposition:

$$y = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

No damping



Critical/overdamped



Underdamped

Laplace Transform

- Laplace transform:

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$F(s) := \int_{-\infty}^{\infty} f(t) e^{-st} \, dt$$

$t \to$ domain s, diff $\Rightarrow$ analysis much easier

- Inv. laplace:

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t)$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$$

$\to$ $f(t)$ is a combination of the exp. function!

- Useful: $f(t) \to$ exponentials, don't mangle from derivative $\left(\frac{d}{dt} e^{at} = a \cdot \underline{e^{at}} \right)$

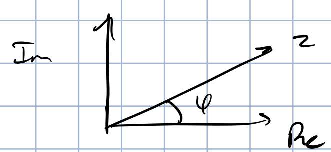
- Ex.//

$$\mathcal{L}\{u(t)\} = \mathcal{L}\left\{ u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \right\}$$

$$\mathcal{L} = \int_0^{\infty} e^{-st} = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \begin{cases} 1/s & \text{if } Re(s) > 0 \\ \infty & \text{o.w.} \end{cases}$$

- Use table lookup to find results

- Know complex #
  - Euler's formula: $z = |z| e^{i\varphi} = |z| \cdot (\cos\varphi + i \sin\varphi)$



- Unit step function: $u_{-1}(t)$
  - Transform: $F(s) = s^{-1}$
- Time shift property:

$$u_{-1}(t - T_0) = \underbrace{e^{-sT_0}}_{\text{Exp. shift}} \cdot \frac{1}{s}$$

- Sine property:

$$\mathcal{L}\{\sin(\omega t) \cdot u_{-1}(t)\} = \frac{\omega}{s^2 + \omega^2}$$

- Proof:

$$\sin(\omega t) \cdot u_{-1}(t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \cdot u_{-1}(t)$$

identities! Euler's

$$= \left(\frac{1}{2j} e^{j\omega t} - \frac{1}{2j} e^{-j\omega t} \right) u_{-1}(t)$$

$$\mathcal{L}\{ \quad '' \quad \} = \frac{1}{2j} \mathcal{L}\{e^{j\omega t}\} - \frac{1}{2j} \mathcal{L}\{e^{-j\omega t}\}$$

$$= \frac{1}{2j} \cdot \frac{1}{s-j\omega} - \frac{1}{2j} \frac{1}{s+j\omega}$$

$$= \frac{\omega}{s^2+\omega^2}$$

- Exponential property:

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$$

- Exponential modulation:

$$\mathcal{L}\{e^{\alpha t} \cdot f(t)\} = F(s-\alpha) \longrightarrow \quad F(s) = \mathcal{L}\{f(t)\}$$

- Time shifting of any function

$$\mathcal{L}\{f(t-T_0)\} = e^{-sT_0} \cdot F(t-T_0)$$

- Proof:

$$\int_0^\infty f(t-T_0)e^{-st}\,dt = \int_0^\infty f(u)e^{-s(u+T_0)}\,du = e^{-sT_0}\int_0^\infty f(u)e^{-su}\,du$$

$$\boxed{u = t - t_0}$$

$$= e^{-sT_0} \cdot F(u)$$

$$= e^{-sT_0} \cdot F(t-t_0)$$

- Multip. by t:

$$\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds} F(s) \quad \hookrightarrow \quad F(s) = \mathcal{L}\{f(t)\}$$

- Proof:

$$\int_{-\infty}^{\infty} t\, f(t)\, e^{-st}\,dt = \int_{-\infty}^{\infty} -\frac{\partial}{\partial s} e^{-st} \cdot f(t)\,dt \qquad -\frac{\partial}{\partial s}\left(e^{-st} f(t)\right) = t \cdot e^{-st} f(t)$$

$$= -\frac{\partial}{\partial s} \underbrace{\int_{-\infty}^{\infty} e^{-st} f(t)\,dt}$$

$$= -\frac{d}{ds} F(s)$$

- Extension: $\mathcal{L}\{t^{n-1} \cdot u_{-1}(t)\} = \frac{(n-1)!}{s}$

- Differentiation:

a) $F(s) = \frac{1}{s} f(0^-) + \frac{1}{s} \mathcal{L}\{f'(t)\}$

b) $\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)$

c) $\mathcal{L}\{f''(t)\} = s^2 F(s) - f(0^-) - s(f''(0^-))$

- Ex://

$$\mathcal{L}\{t \cdot u_{-1}(t)\} = -\frac{d}{ds}\left(1/s\right)$$

$$= -\left(-\frac{1}{s^2}\right)$$

$$= 1/s^2$$

- Using Laplace to solve D.E.
  - Ex://

Solve $\dot{y} + y = t + e^{t}$, $y(0^-) = 1$

① Apply $\mathcal{L}$ to both sides

$$\mathcal{L}\{\dot{y} + y\} = \mathcal{L}\{t + e^{t}\}$$

$$sF(s) - y(0^-) + F(s) = \frac{1}{s^2} + \frac{1}{s-1}$$

② Solve for $F(s)$ + simple

$$F(s)\,(s+1) = \frac{1}{s^2} + \frac{1}{s-1} + 1$$

$$F(s) = \frac{1}{s+1} + \frac{1}{s^2(s+1)} + \frac{1}{(s+1)(s-1)}$$

PFD

$$F(s) = \frac{3/2}{s+1} + \frac{1}{s^2} + \frac{1/2}{s-1} - 1/s$$

③ Inv. Laplace:

$$y(t) = 3/2\, \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 1/2\, \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= 3/2 \cdot e^{-t} + t + 1/2\, e^{t} - 1$$

- Convolution:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau)\, d\tau$$

  - Commutative:

$$(f * g)(t) = (g * f)(t)$$

  - Laplace:

$$\mathcal{L}\{(f * g)(t)\} = F(s)\, G(s)$$

- Initial value theorem: if $f(t) = 0$ for $t < 0$ & no impulses at 0

$$f(0^+) = \lim_{s \to \infty} s \cdot F(s)$$

  - Ex:// Find initial value of $F(s) = \frac{s}{s^2 + \omega^2}$.

Method #1: $F(s) \to \mathcal{L}^{-1} \to f(t)$, $t \to 0$

Method #2: IVT

$$\lim_{s \to \infty} \frac{s^2}{s^2 + \omega^2} = \lim_{s \to \infty} \frac{1}{1 + \cancel{\frac{\omega^2}{s^2}}^{\,0}}$$

$$= \underline{1}$$

  - Pro tip: wb $f(1)$, $f(3)$ ... (not init val.)

Time shifting $\Rightarrow$ $e^{-st_0} \cdot F(s)$ $\Rightarrow$ IVT $\Rightarrow$ value at $f(t_0)$

$\uparrow$ shifted by t_0

Fractions in Freq. Domain:

$$\frac{(x-a)(x-b)}{(x-c)(x-d)} \quad \begin{array}{l} \Rightarrow x = a, b, \text{ zeros of num.} \\ \Rightarrow x = c, d, \text{ poles of denom.} \end{array}$$

Cancel out common factors.

Proper rational fnc:
deg (num.) $\leq$ deg (denom.)
Strictly prop.: " $<$ "

Final value theorem:

① $F(s)$ must be a proper rational function.

② $F(s)$ poles must have real negative parts

If ① & ② satisfied:

$$\underbrace{\lim_{t \to \infty} f(t)}_{\text{steady state.}} = \lim_{s \to 0} s F(s)$$

If ① || ② not satisfied $\Rightarrow$ no final value.

Ex.: // Find final value of $F(s) = \frac{s}{s^2 + \omega^2}$

① Is this a proper rational fnc? Yup!

② Negative real parts of poles?

Poles $\Rightarrow$ $s^2 = -\omega^2$

$s = i\omega$

Negative real part $\Rightarrow$ N

$\therefore \lim_{t \to \infty} f(t)$ DNE.

Integration in time domain:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau, \quad x(t) \xrightarrow{\mathcal{L}} X(s) \Longrightarrow Y(s) = \frac{1}{s} X(s)$$

Time scaling:

$$x(at) \xrightarrow{\mathcal{L}} X(s) = \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{if } a < 0$$

Complex Conjugation:

$$x^*(t) \xrightarrow{\mathcal{L}} X^*(s^*)$$

- Ex://

$$e^{j\omega t} \xrightarrow{\mathcal{L}} \frac{1}{s - j\omega}$$

$$e^{-j\omega t} \xrightarrow{\mathcal{L}_s} \frac{1}{s + j\omega}$$

Zero input & zero state

Assume:

$$\ddot{y} + \alpha \dot{y} + \beta = s(t)$$

Solving:

all initial condition

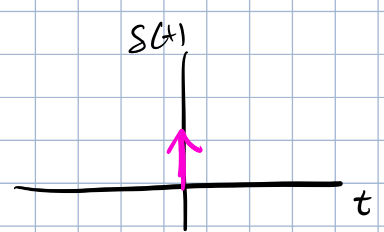
$$Y(s) = \underbrace{\gamma\, G(s)}_{\text{Zero-state}} + \underbrace{\Omega}_{\text{Zero-input (no driving term)}} \Rightarrow \text{General form.}$$

↳ Assumes all init. conditions / dep. vars @ 0

Dirac Delta Function

Defn:

$$a < 0,\ b > 0 \implies \int_a^b \delta(t)\, dt = 1$$



Properties:

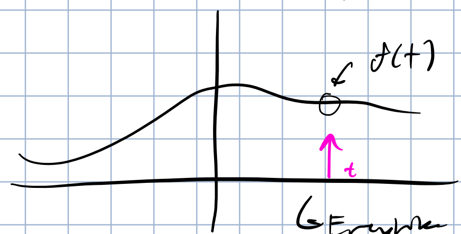
① Sifting

$$\underbrace{\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau}_{\text{Convolution}} = f(t)$$

$$\boxed{(f * \delta)(t) = f(t)}$$

* Proof:

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} f(t)\,d\tau = f(t)$$



↳ Everywhere bounds $t=\tau \Rightarrow \delta(t) = 0$

① Laplace:

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= e^{-s0}$$

$$= \underline{\underline{1}}$$

Impulse Response

$$Y(s) = \gamma\, G(s) + \Omega$$

At $f(t) = \delta(t) \rightarrow$ zero-state $(\dot{y} = \ddot{y} = \dddot{y} \dots = y = 0)$

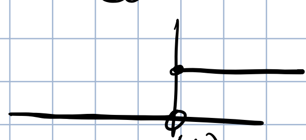
$$Y(s) = \gamma\, G(s)$$

↳ $G(s) = \mathcal{L}\{\delta(t)\} = 1$

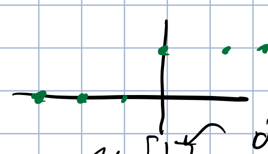
Systems & Signal: Intro & Properties

Signal

Continuous — Discrete



$u(t)$



$u[k]$ ← discrete variable

Derivative? Difference eq.

$\nabla y[k] = y[k] - y[k-1]$

$\nabla^2 y[k] = \nabla y[k] - \nabla y[k-1]$

⋮

System

Continuous ⇓ ct. sgn

Discrete ⇓ dis. sgn

Hybrid ⇓ both disc & ct. signals

Properties

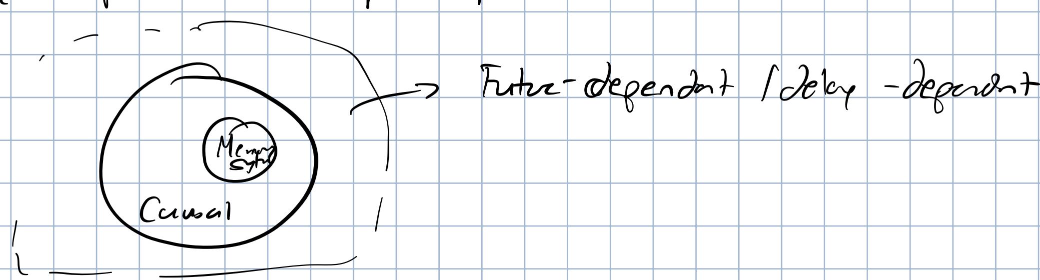
① Memoryless vs. dynamic

Memoryless: output of system $\propto$ current input

Dynamic: " $\propto$ past / current / future input

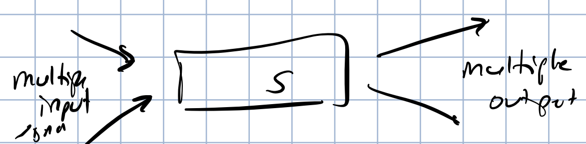
② Causality:

System only depends on past / present input

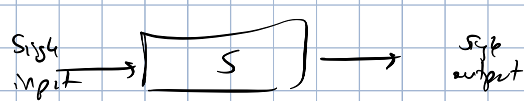


③ Multivariable vs. scalar:

Multivariable



Scalar



④ Linearity:

L.C. of input → system → linear combo of output.

Ex.// System: $y(t) = f(t) + 1$.

Linear?

① Find output of $c_1 f_1(t)$, $c_2 f_2(t)$

$y_1(t) = c_1 f_1(t) + 1$

$y_2(t) = c_2 f_2(t) + 1$

② For output of $c_1 f_1(t) + c_2 f_2(t)$

$$y_3(t) = c_1 f_1(t) + c_2 f_2(t) + 1$$

③ Check if $y_3(t) = c_1 y_1(t) + c_2 y_2(t)$

Nope $\Rightarrow$ not linear.

⑤ Time invariance

Defn: $f(t) \xrightarrow{S} y(t) \xRightarrow{\text{time invariant}} f(t-T) \xrightarrow{S} y(t-T)$

⑥ Stability:

System is stable if bounded input $\Rightarrow$ bounded output for all time.

Ex.// $y(t) = t \cdot x(t)$

Not stable: $x(t) = u_{-1}(t)$. At $t \to \infty$, $y(t) \longrightarrow \infty$ (not bounded even though $0 \le u_{-1}(t) < 1$)

⑦ Invertibility:

Distinct inputs $\xrightarrow{\text{System}}$ distinct outputs.

"Injective"

Ex.// $y(t) = 2 \cdot x(t)$. Invertible?

① Find inverse:

From output, can we define input?

$x(t) = \frac{1}{2} y(t)$ $\Rightarrow$ injective input

Linear Time Invariant Systems

2 interesting properties:

① Impulse response & convolution

② Response to exponential inputs

Impulse response:

a) Discrete time:

Every input $f[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\,\delta[t-\tau]$ (sifting property)

Now:

$$\delta[t] \longrightarrow \boxed{S} \longrightarrow h[t]$$

By linearity & time invariance:

Output $y[t] = \sum_{\tau=0}^{\infty} f[\tau] \underbrace{h[t-\tau]}_{\text{impulse respon.}} = f[t] * h[t]$

All you need is impulse response

b) Continous time:

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$

$$\downarrow S$$

$$y(t) = \int_{0}^{\infty} f(\tau)\,h(t-\tau)\,d\tau = f(t) * h(t)$$

Exponential inputs

Suppose S is an LTI system

$$e^{st} \xrightarrow{S} y(t)$$

By T.I.:

$$e^{s(t-T)} \xrightarrow{S} y(t-T)$$

$$e^{st} e^{-sT} \xrightarrow{S} y(t-T)$$

By linearity:

$$e^{st} e^{-sT} \longrightarrow e^{-sT} y(t) \qquad \text{b/c } e^{-sT} \text{ is a constant}$$

$\therefore$ Equality:

$$y(t-T) = e^{-sT} y(t)$$

If $T = t$

$$\boxed{y(t) = y(0)\,e^{st}}$$

We also know:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau \qquad \text{from above}$$

$$= \int_{-\infty}^{\infty} h(\tau)\,f(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau$$

Transfer functn. $(\mathcal{L}\{h(t)\})$

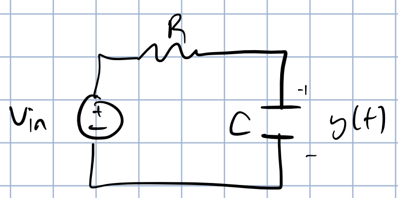
$$\therefore \quad e^{st} \xrightarrow{S} y(t) = e^{st} \cdot H(s)$$
$$= e^{st} \cdot \underline{\underline{y(0)}}$$

Transfer Function

$$H(s) = \mathcal{L}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

You don't need $h(t)$ to find $H(s)$ $\Rightarrow$ $H(s)$ is zero-state response.

Ex: 1) Find transfer func. of RC circuit



① Differential equation:

$$x(t) = R \cdot i + y(t)$$

Current:
$$i = C \cdot \frac{dy}{dt}$$

$$\therefore \; RC \cdot \frac{dy}{dt} + y = x$$

② Laplace:

$$RC\,(sY(s) - y(0^-)) + Y(s) = X(s)$$

$$Y(s) = \frac{1}{sRC+1} X(s) + RC\,y(0^-)$$

③ Zero state: transfer func.

$$Y(s) = \boxed{\frac{1}{sRC+1}} X(s)$$

$$H(s) = \frac{1}{sRC+1} = \frac{1}{RC} \cdot \frac{1}{s + 1/RC}$$

④ Find $h(t)$:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{RC} \cdot e^{-\frac{1}{RC}t}$$

Poles of transfer func $\longrightarrow$ natural response of system.

Standard First-Order System

Form:
$$\dot{y} + \frac{1}{\tau} y = f(t) \cdot \frac{k}{\tau} \quad \Longrightarrow \quad H(s) = \frac{k}{s\tau + 1}$$

∴ Impulse response:

$$\mathcal{L}^{-1}\{H(s)\} = \frac{k}{\tau} e^{-t/\tau}$$

∴ Step response:

$$Y(s) = H(s) \times \overset{\text{step}}{X(s)}$$

$$= H(s) \times 1/s$$

$$\downarrow \mathcal{L}^{-1}$$

$$y(t) = k\left(1 - e^{-t/\tau}\right)$$

∴ $\lim_{t \to \infty} y(t) = k$. We call k our "$\underbrace{\text{D.C.}}_{\approx u(t)}$" gain

Standard 2nd Order System

Form:

$$\ddot{y} + 2\omega_n \zeta \dot{y} + y\,\omega_n^2 = f(t) \Longrightarrow H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n \geq 0$$

Poles of $H(s)$:

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Cases:

① $\zeta > 1 \Rightarrow$ 2 distinct, real poles.

② $\zeta = 1 \Rightarrow$ repeated pole at $\omega_n \Rightarrow$ decaying exponential $\times t$

③ $\zeta < 1 \Rightarrow$ underdamped

↳ $y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\right) u(t)$

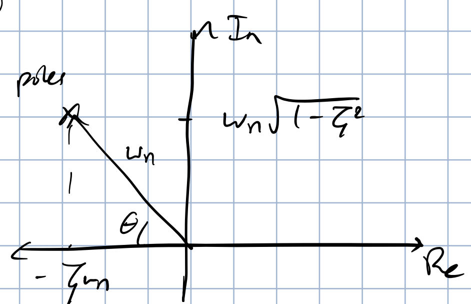
↳ Unit step response: $\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} + \overset{\cos^{-1}(\zeta)}{\theta}\right) u(t)$

Note: $\zeta \to 0 \Rightarrow \omega_n e^{-\omega_n t} \sin(\underbrace{\omega_n}_{\text{Natural freq.}})$

$\downarrow$

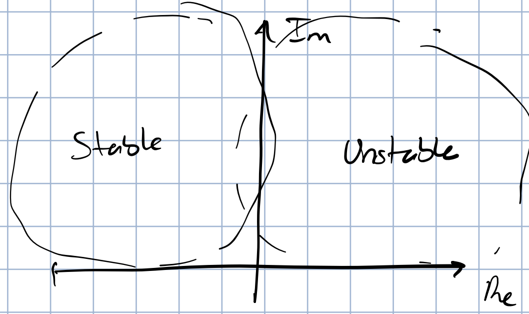
ζ is damping ratio

On imaginary plane:



Stable Systems

Function is stable ← location of poles of transfr fnc.



Theorem:

① If all poles of transfr fnc. is strictly left of Im ⇒ transfr fnc. is stable

↳ Transfr fnc. should be rational

② SISO LTI system w/ rational transfr is stable ⟺ transfr fnc. is stable & proper.

Frequency Response of LTI System

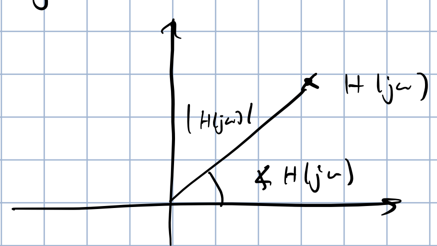
Recall

$$x(t) = e^{st} \longrightarrow \boxed{H(s)} \longrightarrow y(t) = H(s)\, e^{st}$$

For sinusoidal signals → exponential input

$$x(t) = e^{j\omega t} \longrightarrow \boxed{H(s)} \longrightarrow y(t) = H(j\omega)\, e^{j\omega t}$$

What exactly is $H(j\omega)$?



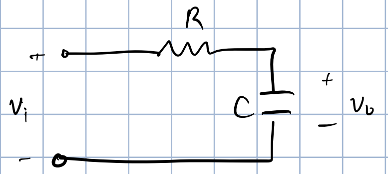
$$\therefore \quad y(t) = |H(j\omega)|\, e^{j \measuredangle H(j\omega)} \cdot e^{j\omega t}$$

$$= |H(j\omega)|\, e^{j(\omega t + \measuredangle H(j\omega))}$$

Rescaled & shifted input as output! All based off transfr fnc.

Note: if $e^{j\omega t} u_{-1}(t)$ → led to non-exponential transients.

Ex://



$H(s) = \frac{1}{RCs + 1}$, $RC = 0.01$

If sinusoidal signal $e^{j\omega}$ sent $\Rightarrow H(j\omega) = \frac{1}{RCj\omega + 1}$

Q: What is v_o if $v_i = 0.5 \sin 100t$?

① Rescale factor:

$\omega = 100$

$\therefore H(j\omega) = \frac{1}{0.01 \cdot j \cdot 100 + 1} = \frac{1}{j+1}$

Make it into exponential:

$\frac{1+j}{2} = \frac{1}{\sqrt{2}} e^{-j\,45°}$

$\therefore |H(j\omega)| = 1/\sqrt{2}$. $\measuredangle H(j\omega) = -45°$

② Output:

$y(t) = |H(j\omega)| \, e^{(\omega t + \measuredangle H(j\omega))}$

$= \frac{0.5}{\sqrt{2}} \sin(100t - 45)$

Bode Plots

2 plots:

① $|H(j\omega)|_{dB}$ vs. $\log_{10} \omega$ $\Rightarrow$ gain of output

② $\measuredangle H(j\omega)$ vs. $\log_{10} \omega$ $\Rightarrow$ phase slip

Decibels:

$$|H(j\omega)|_{dB} = 20 \log_{10} |H(j\omega)|$$ $\Rightarrow$ defn. of dB

"From Scratch" Method:

① Find $H(j\omega) \rightarrow |H(j\omega)|$, $\measuredangle H(j\omega)$

② Look at asymptotes: $\omega \rightarrow 0$, $\omega \rightarrow \infty$

Ex://

$H(s) = \frac{k}{1 + s\tau}$, $k, \tau > 0$. Draw Bode Plots for 1st order system

① Frequency response:

$$H(j\omega) = \frac{k}{1 + j\omega\tau}$$

Note: for easier graphing, make denominator in form of $j\omega - \ldots$

$$H(j\omega) = \frac{k/\tau}{j\omega - (-1/\tau)}$$

② Find magnitude & phase shift

Recall:
$$\frac{z_1}{z_2} = \frac{|z_1| e^{i\phi_1}}{|z_2| e^{i\phi_2}} = \left|\frac{z_1}{z_2}\right| e^{i(\phi_1 - \phi_2)}$$

A: Magnitude:

Numerator: k

Denominator: $|j\omega\tau + 1|^2 = (j\omega\tau + 1)(j\omega\tau - 1)$

$= \sqrt{1 + (\omega\tau)^2}$ → Simpler form of complex

B: Phase:

Phase of numerator: 0

Phase of denominator: $\angle(j\omega - (-1/\tau)) = \tan^{-1}\left(\frac{\omega}{1/\tau}\right)$

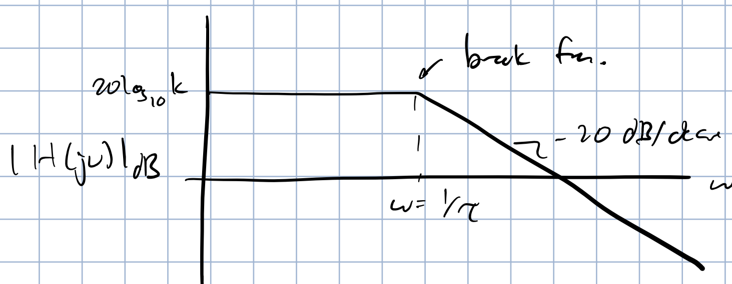
③ Plot:

Magnitude in dB: $20\log_{10} k - 10\log(1 + (\omega\tau)^2)$

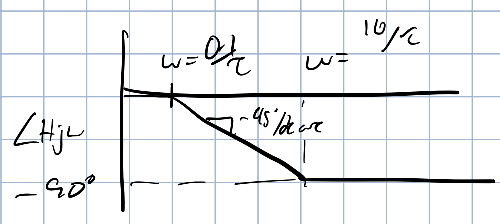
Asymptotes of magnitude:

$\omega \to 0$: $20\log_{10} k$

$\omega \to \infty$: roughly -20 dB / decade



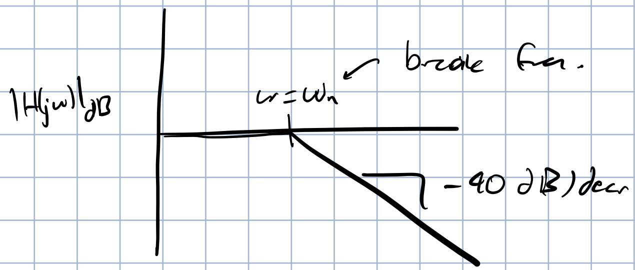
Phase:

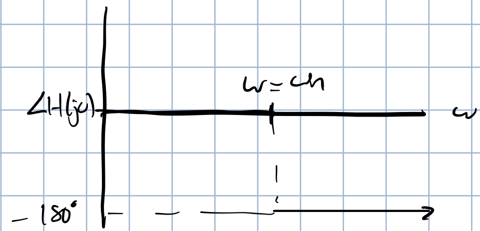


Second order system:

- Freq. respons:

$$H(j\omega) = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + (2j\zeta\omega_n\omega)}$$





Simpler method for Bode Plots:

① Know 1st order & 2nd order graphs:

1st order: $\frac{1}{j\omega + \omega_n}$ break freq.

2nd order: $\frac{1}{(j\omega + \omega_n)^2}$

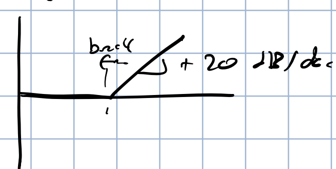
② Break up transfer func. in terms of 1st & 2nd order (normal inverse)

③ Superposition of plots.

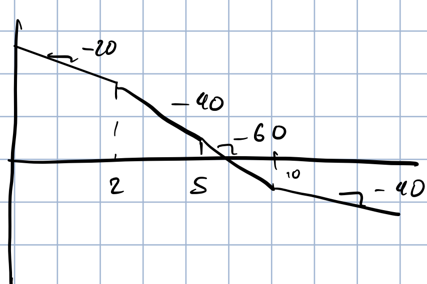
Ex:// $H(s) = \frac{10(s+10)}{s(s+2)(s+5)}$ Draw magnitude Bode plot.

① Break up into 1st & 2nd order:

$$H(j\omega) = 10 \cdot \underbrace{\frac{1}{j\omega} \cdot \frac{1}{j\omega + 2} \cdot \frac{1}{j\omega + 5}}_{\text{1st order}} \cdot \underbrace{(j\omega + 10)}_{\text{1st order but inverse}}$$



② Superposition:



Fourier Series

Intro

Periodic, piecewise cts. function → infinite sum of sinusoidals

$f(t+T) = f(t)$ — func. pieces are cts.

Any function can be expressed as a sum of complex exponentials ⇒ "basis" function.

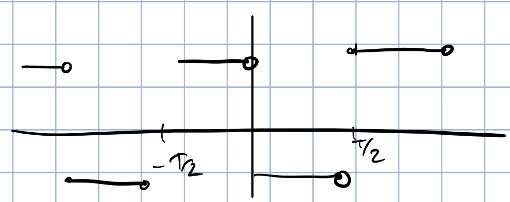
Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

To solve problem, solve general c_n integral & c_0 ⇒ Fourier series w/ results.

Ex://



① c_n:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

$$= \dots$$

$$= \frac{1}{j\pi n}\left(1-(-1)^n\right)$$

② c_0:

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt$$

$$= 0$$

③ Fourier:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}, \quad c_n = \begin{cases} \text{—} \quad \text{—}, & n \neq 0 \\ \text{\_\_\_\_\_\_}, & n = 0 \end{cases}$$