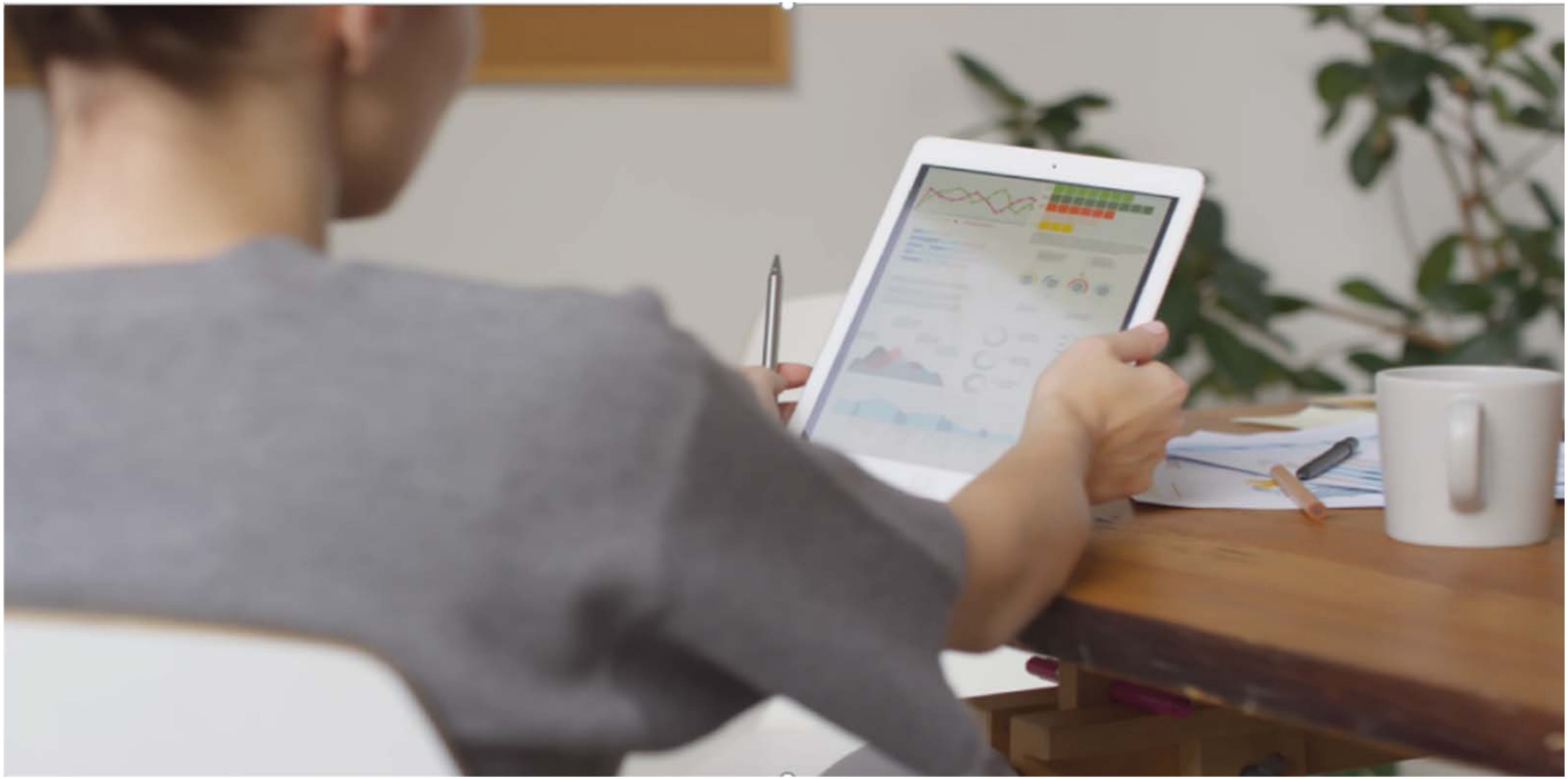

TBA2102 2020/2021 Semester 2
Tutorial 10: Linear Optimization

STRUCTURE OF TUTORIALS

Duration:

45 mins

Content:

- Linear optimization concepts
- Tutorial 10 (Questions 1& 2)

Linear Optimization Concepts

KEY CONCEPTS

01

Optimization

Fundamental tool in Prescriptive Analytics to identify best alternatives to minimize or maximize some objective.

- Linear optimization: linear program relaxation
- Integer optimization
- Non linear optimization

02

Decision variables

Unknown values that the model seeks to determine.

- Usually within the control of the decision maker.
- E.g., how many units of A & B to produce?

03

Objective function

The quantity we seek to minimize or maximize

- E.g., Maximize profit or revenue, or minimize cost or some measure of risk.
- Formulate the objective function in terms of a real world quantity (i.e., relevant to a business context) that we want to maximize or minimize, by choosing our X's.

04

Constraints

Limitations, requirements, or other restrictions imposed on any solution.

- Binding: value is equal to the right-hand side of the value of the constraint; shadow price is non-zero.
- Non-binding: shadow price is zero

05

Sensitivity analyses

How sensitive is the solution to changes in the constraints or optimization function.

- What will happen to the optimal solution if you change the constraints or optimization function by a little?
  - vary objective function coefficients
  - vary constraint values (shadow prices)

STEPS IN LINEAR OPTIMIZATION

1. Identify the decision variables and objective function

2. Identify all appropriate constraints

3. Write objective function & constraints as mathematical functions

4. Solve the linear program (manually, graphically or using R)

5. Conduct sensitivity analyses

6. Interpret results and write recommendation

Tutorial 10 Question 1

QUESTION 1A

So far, the examples we have discussed in lecture were all maximization problems. In this question we shall explore minimization.

FunToys is famous for 3 types of toys: Cars, Animals, and Robots. Each year, near the holiday season, it receives large bulk orders for these items. To meet these orders, FunToys operates three small toy-making factories, A, B and C.

Factory A
- Costs $1000 per day to operate, &
- Can produce 30 cars, 20 animals & 30 robots per day.

Factory B
- Costs $1200 per day to operate, &
- Can produce 40 cars, 50 animals & 10 robots per day.

Factory C
- Costs $1500 per day to operate, &
- Can produce 50 cars, 40 animals & 15 robots per day.

This Christmas, FunToys is required to deliver 5000 cars, 3000 animals & 2500 robots. Find out what is the most cost-efficient way to meet the order.

QUESTION 1A

In this question, IGNORE integer requirements, i.e., just use fractional answers if/when they come up.

Start by formulating the problem statement as an optimization problem.

- Write down your decision variables.
- Write your objective function in terms of your decision variables.
- Write down the constraints
  - What are the contractual requirements you need to fulfill. What other constraints are there? Write them down in terms of your decision variables.
- Summarize them nicely in a table.
- Include all non negativity-constraints at the bottom of the table.

QUESTION 1A

Let decision variables X_1, X_2, X_3 be the number of days to run Factories A, B, and C respectively.

Minimize total cost using decision variables X_1, X_2, X_3	Cost = 1000 X_1 + 1200 X_2 + 1500 X_3
Subject to	
Contract for cars	$30X_1 + 40X_2 + 50X_3 \geq 5000$
Contract for animals	$20X_1 + 50X_2 + 40X_3 \geq 3000$
Contract for robots	$30X_1 + 10X_2 + 15X_3 \geq 2500$
[redacted]	$X_1 + \quad + \quad \geq 0$
	$+ X_2 + \quad \geq 0$
	$+ \quad + X_3 \geq 0$

Specify non-negativity constraints at the end of the table.

QUESTION 1B

- Write the R code to solve this optimization problem.
- Report the optimal solution, and the value of the objective function at that solution.
- Interpret the solution: what do these numbers mean?

QUESTION 1B

Minimize total cost using decision variables X_1, X_2, X_3	Cost = 1000 X_1 + 1200 X_2 + 1500 X_3
Subject to	
Contract for cars	$30X_1 + 40X_2 + 50X_3 \geq 5000$
Contract for animals	$20X_1 + 50X_2 + 40X_3 \geq 3000$
Contract for robots	$30X_1 + 10X_2 + 15X_3 \geq 2500$
Non-Negativity Constraint 1	$X_1 + \quad + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 + \quad \geq 0$
Non-Negativity Constraint 3	$+ \quad + X_3 \geq 0$

```
#defining parameters
objective.fn <- c(1000, 1200, 1500)

const.mat <- matrix(c(30, 40, 50,
                      20, 50, 40,
                      30, 10, 15) ,
                      ncol=3 ,
                      byrow=TRUE)

const.dir <- c(">=", ">=", ">=")

const.rhs <- c(5000, 3000, 2500)

#solving model
lp.solution <- lp("min", objective.fn, const.mat, const.dir, const.rhs, compute.sens=TRUE)

lp.solution$solution #decision variables values
```

```
## [1] 47.61905  0.00000 71.42857
```

What do these mean?

```
lp.solution
```

```
## Success: the objective function is 154761.9
```

What does this mean?

QUESTION 1B

```
## [1] 47.61905  0.00000 71.42857
```

```
lp.solution
```

```
## Success: the objective function is 154761.9
```

Optimal solution is: X_1=47.6, X_2=0.00 and X_3=71.43.

That is, run Factory A for 47.6 days, Factory B for 0 days and Factory C for 71.4 days.

The Minimum cost is $154761.90.

```
<p style="color:blue">
Optimal solution is: $X_1 = 47.6$, $X_2 = 0.00$, $X_3 = 71.43$. That is, run Factory A for
47.6 days, Factory B for 0 days and Factory C for 71.4 days. The Minimum cost is \$154761.90.
</p>
```

QUESTION 1C

What if there is now an additional constraint that FunToys only has 60 days to complete the order? (Note that we can run all three factories simultaneously). What happens now?

Re-produce a new table summarizing the optimization problem (including the existing and new constraints).

Write the R code to solve it. What is the new solution, and what is the objective function value?

QUESTION 1C

Let decision variables X_1, X_2, X_3 be the number of days to run Factories A, B, and C respectively.

Minimize total cost using decision variables X_1, X_2, X_3	Cost = 1000 X_1 + 1200 X_2 + 1500 X_3
Subject to	
Contract for cars	$30X_1 + 40X_2 + 50X_3 \geq 5000$
Contract for animals	$20X_1 + 50X_2 + 40X_3 \geq 3000$
Contract for robots	$30X_1 + 10X_2 + 15X_3 \geq 2500$
Time Constraint 1	$X_1 + \quad + \quad \leq 60$
Time Constraint 2	$+ X_2 + \quad \leq 60$
Time Constraint 3	$+ \quad + X_3 \leq 60$
Non-Negativity Constraint 1	$X_1 + \quad + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 + \quad \geq 0$
Non-Negativity Constraint 3	$+ \quad + X_3 \geq 0$

Why do we add 3 time constraints?

QUESTION 1C

Minimize total cost using decision variables X_1, X_2, X_3	Cost = 1000 X_1 + 1200 X_2 + 1500 X_3
Subject to	
Contract for cars	$30X_1 + 40X_2 + 50X_3 \geq 5000$
Contract for animals	$20X_1 + 50X_2 + 40X_3 \geq 3000$
Contract for robots	$30X_1 + 10X_2 + 15X_3 \geq 2500$
Time Constraint 1	$X_1 + \quad + \quad \leq 60$
Time Constraint 2	$+ X_2 + \quad \leq 60$
Time Constraint 3	$+ \quad + X_3 \leq 60$
Non-Negativity Constraint 1	$X_1 + \quad + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 + \quad \geq 0$
Non-Negativity Constraint 3	$+ \quad + X_3 \geq 0$

```
#defining parameters
objective.fn <- c(1000, 1200, 1500)
const.mat <- matrix(c(30, 40, 50,
                      20, 50, 40,
                      30, 10, 15,
                      1, 0, 0,
                      0, 1, 0,
                      0, 0, 1) ,
                      ncol=3 ,
                      byrow=TRUE)
const.dir <- c(">=", ">=", ">=", "<=", "<=", "<=")
const.rhs <- c(5000, 3000, 2500, 60, 60, 60)

#solving model
lp.solution <- lp("min", objective.fn, const.mat, const.dir, const.rhs, compute.sens=TRUE)
```

QUESTION 1C

```
lp.solution$solution #decision variables values
```

```
## [1] 48.88889 13.33333 60.00000
```

```
lp.solution
```

```
## Success: the objective function is 154888.9
```

```
lp.solution$sens.coef.from
```

```
## [1]  9.000000e+02  1.190476e+03 -1.000000e+30
```

```
lp.solution$sens.coef.to
```

```
## [1] 3600.000 1333.333 1511.111
```

- The solution is now: X_1=48.89, X_2=13.33, X_3=60.
- That is, run Factory A for 48.89 days, Factory B for 13.33 days and Factory C for 60 days.
- The Minimum cost is $154888.90.

QUESTION 1D

- For the solution in 1c, which of the constraints are binding, and which are non-binding?

- During lecture, we identified this visually and through shadow prices.

- In this question, let's look for the binding constraints by calculating how many Cars, Animals, and Robots are produced and check if they are bound by their respective constraints.

- We can do the same for the factory usage Time constraints.

QUESTION 1D

```
num_cars <- sum(lp.solution$solution*c(30, 40, 50)) # 5000
num_animals <- sum(lp.solution$solution*c(20, 50, 40)) # 4044
num_robots <- sum(lp.solution$solution*c(30, 10, 15)) # 2500
```

- Binding: value is equal to the right-hand side of the value of the constraint; shadow price is non-zero.

```
#defining parameters
objective.fn <- c(1000, 1200, 1500)
const.mat <- matrix(c(30, 40, 50,
                      20, 50, 40,
                      30, 10, 15,
                      1, 0, 0,
                      0, 1, 0,
                      0, 0, 1) ,
                      ncol=3 ,
                      byrow=TRUE)
const.dir <- c(">=", ">=", ">=", "<=", "<=", "<=")
const.rhs <- c(5000, 3000, 2500, 60, 60, 60)

#solving model
lp.solution <- lp("min", objective.fn, const.mat, const.dir, const.rhs, compute.sens=TRUE)
lp.solution$solution #decision variables values
```

Minimize total cost using decision variables X_1, X_2, X_3	Cost = 1000 X_1 + 1200 X_2 + 1500 X_3
Subject to	
Contract for cars	$30X_1 + 40X_2 + 50X_3 \geq 5000$
Contract for animals	$20X_1 + 50X_2 + 40X_3 \geq 3000$
Contract for robots	$30X_1 + 10X_2 + 15X_3 \geq 2500$
Time Constraint 1	$X_1 + \quad + \quad \leq 60$
Time Constraint 2	$+ X_2 + \quad \leq 60$
Time Constraint 3	$+ \quad + X_3 \leq 60$
Non-Negativity Constraint 1	$X_1 + \quad + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 + \quad \geq 0$
Non-Negativity Constraint 3	$+ \quad + X_3 \geq 0$

```
## [1] 48.88889 13.33333 60.00000
```

Which constraint is binding here?

QUESTION 1E

- Using your solution in 1c, print out the Shadow Prices.
- Interpret these values –explain why each shadow price is zero or why it is positive/negative!

Minimize total cost using decision variables X_1, X_2, X_3	Cost = 1000 X_1 + 1200 X_2 + 1500 X_3
Subject to	
Contract for cars	$30X_1 + 40X_2 + 50X_3 \geq 5000$
Contract for animals	$20X_1 + 50X_2 + 40X_3 \geq 3000$
Contract for robots	$30X_1 + 10X_2 + 15X_3 \geq 2500$
Time Constraint 1	$X_1 + \quad + \quad \leq 60$
Time Constraint 2	$+ X_2 + \quad \leq 60$
Time Constraint 3	$+ \quad + X_3 \leq 60$
Non-Negativity Constraint 1	$X_1 + \quad + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 + \quad \geq 0$
Non-Negativity Constraint 3	$+ \quad + X_3 \geq 0$

```
lp.solution$duals
```

```
## [1]  28.888889   0.000000   4.444444   0.000000   0.000000 -11.111111   0.000000
## [8]   0.000000   0.000000
```

Tutorial 10 Question 2

QUESTION 2

- You work for a furniture company.

- Client A has a contract with your company, whereby Client A will purchase all the furniture that you can send them, but you have to deliver at least 10 chairs and 5 tables or risk violating the contract.

- They will pay $50 for each chair, and $120 for each table.

- A chair requires 3 production hours to build, while a table requires 7 production hours.

- You have a budget of 60 production hours.

- Your boss asks you to optimize the furniture production to maximize profit while keeping within budget.

QUESTION 2A

- Write out the decision variables, objective function and constraints.
- Please include the non-negativity constraints.

Let decision variables X_1 be number of chairs and X_2 be number of tables

Maximize total profit using decision variables X_1, X_2	Profit = $50\ X_1 + 120\ X_2$
Subject to	
Budget Constraint	$3X_1 + 7X_2 \leq 60$
Contract Constraint 1	$X_1 + \quad \geq 10$
Contract Constraint 2	$+ X_2 \geq 5$
Non-Negativity Constraint 1	$X_1 + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 \geq 0$

- What's missing?
- Specify non-negativity constraints.

QUESTION 2B

Are there any feasible solutions, and if so, how many chairs and tables should you build to maximize profit? What is the profit associated with this solution?

```
#defining parameters
objective.fn <- c(50, 120)
const.mat <- matrix(c(3, 7,
                      1, 0,
                      0, 1) ,
                      ncol=2 ,
                      byrow=TRUE)
const.dir <- c("<=", ">=", ">=")
const.rhs <- c(60, 10, 5)

#solving model
lp.solution <- lp("max", objective.fn, const.mat, const.dir, const.rhs, compute.sens=TRUE)
lp.solution$solution #decision variables values
```

Maximize total profit using decision variables X_1, X_2	Profit = $50\ X_1 + 120\ X_2$
Subject to	
Budget Constraint	$3X_1 + 7X_2 \leq 60$
Contract Constraint 1	$X_1 + \quad \geq 10$
Contract Constraint 2	$+ X_2 \geq 5$
Non-Negativity Constraint 1	$X_1 + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 \geq 0$

```
## [1] 0 0
```

```
lp.solution
```

```
## Error: no feasible solution found
```

```
# no solution
```

QUESTION 2C

- Now you decide to rent a neighboring factory, which adds another 60 production hours to your budget (for a total of 120).
- How many chairs and tables should you build with this new production hours budget? What is the profit associated with this solution?

```
#defining parameters
objective.fn <- c(50, 120)
const.mat <- matrix(c(3, 7,
                      1, 0,
                      0, 1) ,
                      ncol=2 ,
                      byrow=TRUE)
const.dir <- c("<=", ">=", ">=")
const.rhs <- c(120, 10, 5)

#solving model
lp.solution <- lp("max", objective.fn, const.mat, const.dir, const.rhs, compute.sens=TRUE)
lp.solution$solution #decision variables values
```

- **Optimal solution:** 10 chairs and 12.86 tables
- This will give a profit of $2042.86

```
## [1] 10.00000 12.85714
```

```
lp.solution
```

```
## Success: the objective function is 2042.857
```

QUESTION 2D

- Use sensitivity analysis to answer the following questions based on the answer in 2c):
- How much would you have to increase the selling price of chairs (to Client A) in order to make it more profitable to produce more chairs?
- How much more profit would you make if you are able to secure 1 more production hour?

```
lp.solution$sens.coef.from
```

```
## [1] -1.000000e+30  1.166667e+02
```

```
lp.solution$sens.coef.to
```

```
## [1] 5.142857e+01 1.000000e+30
```

```
lp.solution$duals
```

```
## [1] 17.142857 -1.428571  0.000000  0.000000  0.000000
```

Maximize total profit using decision variables X_1, X_2	Profit = 50 X_1 + 120 X_2
Subject to	
Budget Constraint	$3X_1 + 7X_2 \leq 60$
Contract Constraint 1	$X_1 + \quad \geq 10$
Contract Constraint 2	$+ X_2 \geq 5$
Non-Negativity Constraint 1	$X_1 + \quad \geq 0$
Non-Negativity Constraint 2	$+ X_2 \geq 0$

- Increase the selling price to above 51.43 for chair production to be profitable.
- Adding 1 more production hour will increase profit by $17.14.

THANK YOU.
BEST WISHES FOR YOUR FINALS.