NUS National University of Singapore | School of Computing

Leading The World With Asia's Best

Introduction to Business Analytics

Linear Optimization

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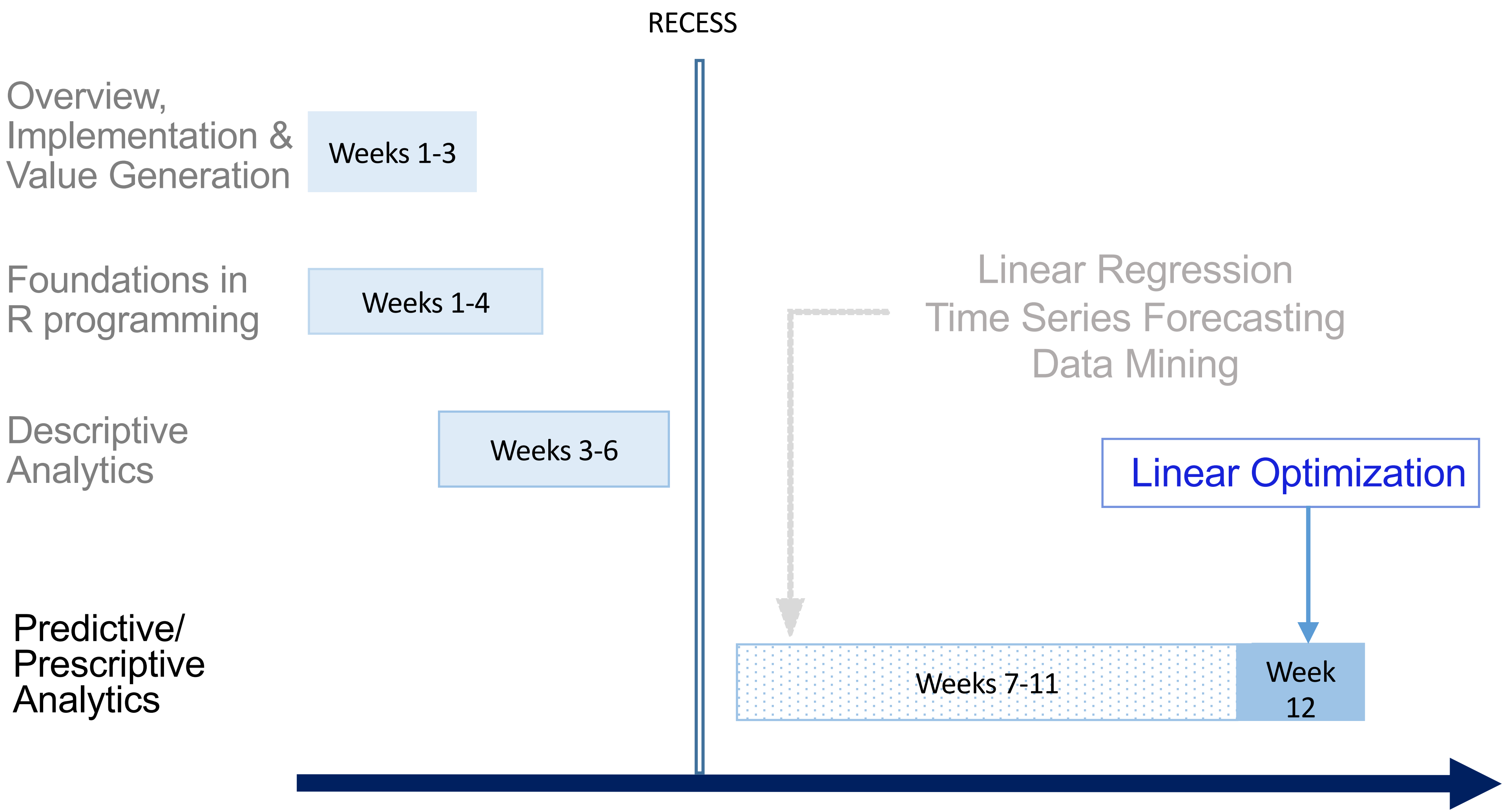
Course Map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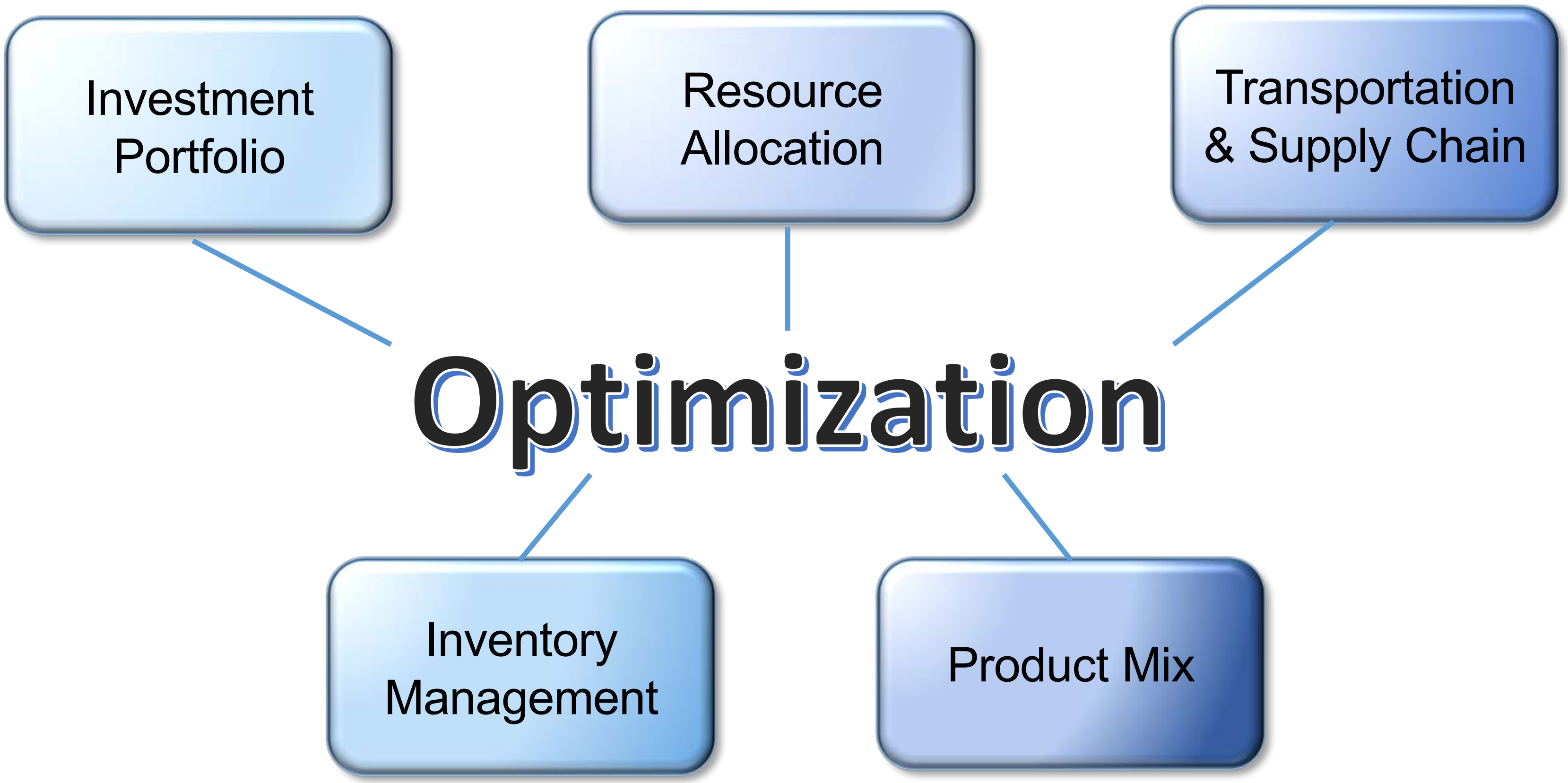
Learning outcomes

- Able to identify an objective function and relevant constraints from a given optimization problem statement
- Understand linear optimization concepts like the feasible region and finding optimal solutions as corner points of the feasible region
- Able to solve simple linear optimization problems
- Able to interpret the results of the optimization and of sensitivity analyses, and make a recommendation with justifications

Optimization

- Fundamental tool in Prescriptive Analytics to identify best alternatives to minimize or maximize some objective.



Linear Optimization

Linear optimization is one of the most common class of optimization models.

<u>Steps in Linear Optimization</u>:

1. Identify the decision variables and objective function.
2. Identify all appropriate constraints
3. Write objective function and constraints as mathematical functions
4. Solve the linear program (manually, graphically or using R)
5. Conduct sensitivity analyses
6. Interpret results and write recommendation

Decision Variables & Objective Functions

Decision variables (Xs) are unknown values that the model seeks to determine.

- Example:
  - How many units of A & B to produce?
  - How many delivery trucks and drivers do we need?

Objective function is the quantity we seek to minimize or maximize.

- Example:
  - Minimize cost of production; Minimize customer waiting time
  - Maximize profits; Maximize number of trips made per day

We formulate the objective function in terms of a real world quantity (i.e., relevant to a business context) that we want to maximise or minimise, by choosing our X's.

Advertising Example: Obj fn & Decision vars

A marketing manager must decide how much time to allocate between prime time and nonprime time TV advertisement for the next quarter. Her goal is to maximize audience exposure. She has a budget of $25,000. Research tells her that TV ads during prime time are more effective than those during nonprime time. Hence she would like to have at least 70% of time allocated to prime-time.

Below is the data she has:

Timing of TV Ad	Audience Exposure per min	Cost per min
Nonprime	350	400
Prime	800	2000

Decision variables:

X_1: time allocated to nonprime tv ads

X_2: time allocated to prime tv ads

Objective Function:

Maximize audience exposure = $350*X_1 + 800*X_2$

Constraints

Constraints are limitations, requirements. or other restrictions imposed on any solution (could be due to practical/technological considerations or management policy)

Formally, constraints are represented as mathematical inequalities or equations. They can be less than (<, ≤), greater than (>, ≥), or strictly equal (=).

We formulate our decision variables on the left-hand side, and the constants on the right-hand side of the constraints.

<u>Examples</u>	<u>Translate to:</u>
"Deliver within budget"	Total cost ≤ Budget
"Deliver at least 50 units of product"	Product ≥ 50
"Allocation must contain exactly 40 hours of work"	Working hours = 40
Implicit assumption: Product and hours of work must be non-negative (cannot be negative)	Product ≥ 0 Working hours ≥ 0

Advertising Example: Constraints

A marketing manager must decide how much time to allocate between prime time and nonprime time TV advertisement for the next quarter. Her goal is to maximize audience exposure. She has a budget of $25,000. Research tells her that TV ads during prime time are more effective than those during nonprime time. Hence she would like to have at least 70% of time allocated to prime-time.

Below is the data she has:

Timing of TV Ad	Audience Exposure per min	Cost per min
Nonprime	350	400
Prime	800	2000

Decision variables:

X_1: time allocated to nonprime tv ads;

X_2: time allocated to prime tv ads

What are the constraints?

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $X_2>=0.7*(X_2+X_1)$; translates to $0.7*X_1 - 0.3*X_2 \leq 0$

Non-negativity constraints: $X_1 \geq 0$; $X_2 \geq 0$

Translate model information to mathematical expressions

Let's summarise all the information in the problem into one system of equations to solve.

Note that a linear optimization model is often referred to as a linear program (LP).

Two **properties** of LP:
1) the objective function and all constraints are linear functions of the decision variables
2) all variables are continuous and may assume any real value (typically non-negative)

Continuing our advertising example:

Maximize audience exposure using decision variables X_1 & X_2 $= 350*X_1 + 800*X_2$

Subject to:

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$

Non-negativity constraints: $X_1 \geq 0$; $X_2 \geq 0$

Any solution that satisfies all the constraints is a **feasible** solution.
We want to find the best / optimal feasible solution.

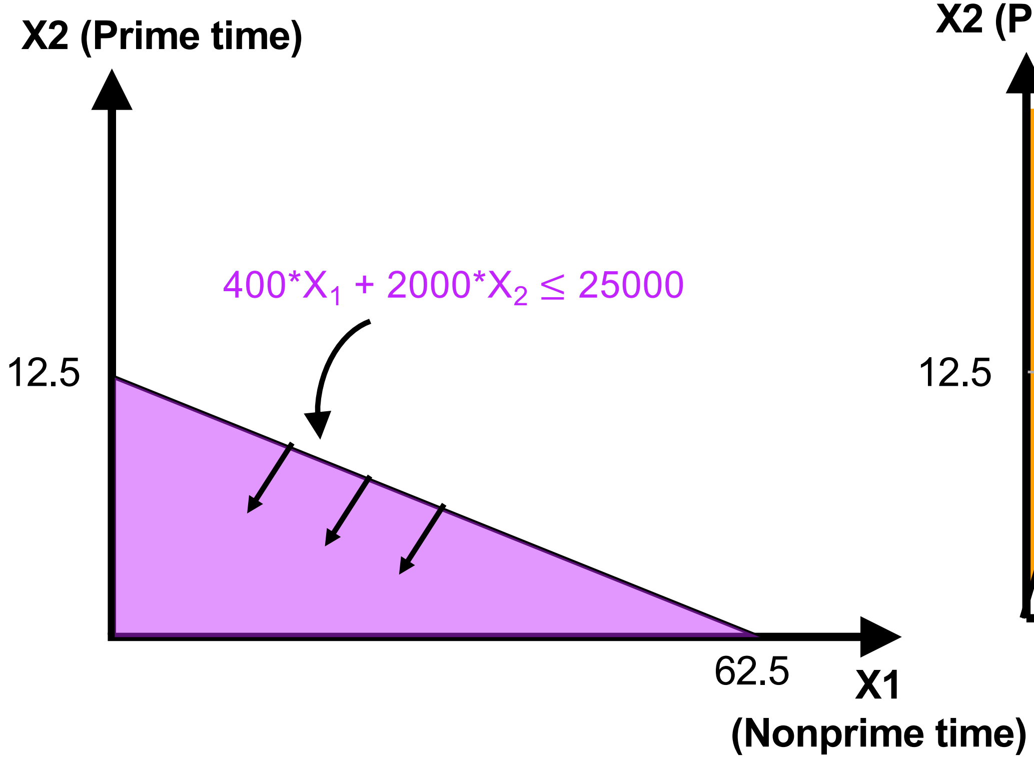
Solving the linear program – Graphically

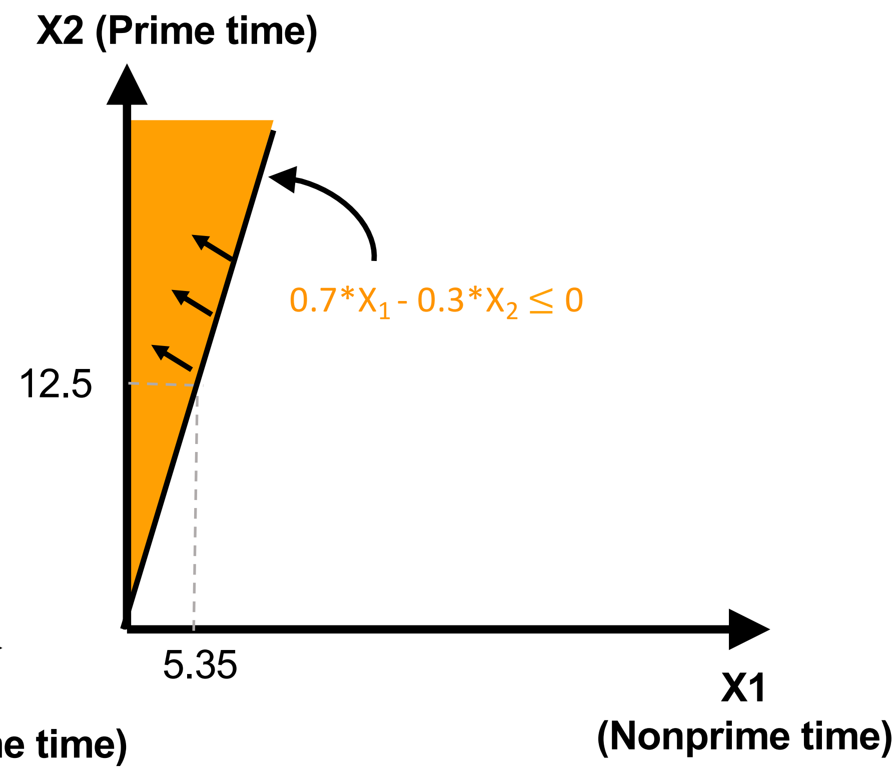
Let's visualize this problem better by graphing the constraints as regions on an X1-X2 graph.

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$

Non-negativity constraints: $X_1 \geq 0;$
$X_2 \geq 0$





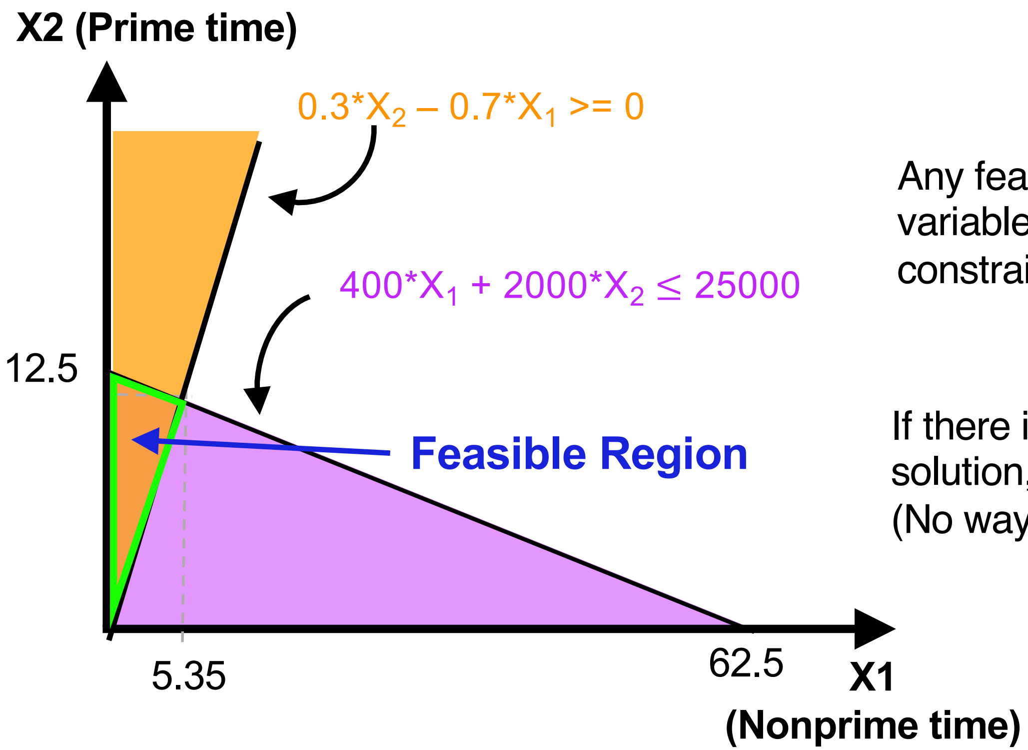
Solving the linear program – Graphically

Feasible Region: Intersection of all constraint regions

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$

Non-negativity constraints: $X_1 \geq 0;$ $X_2 \geq 0$

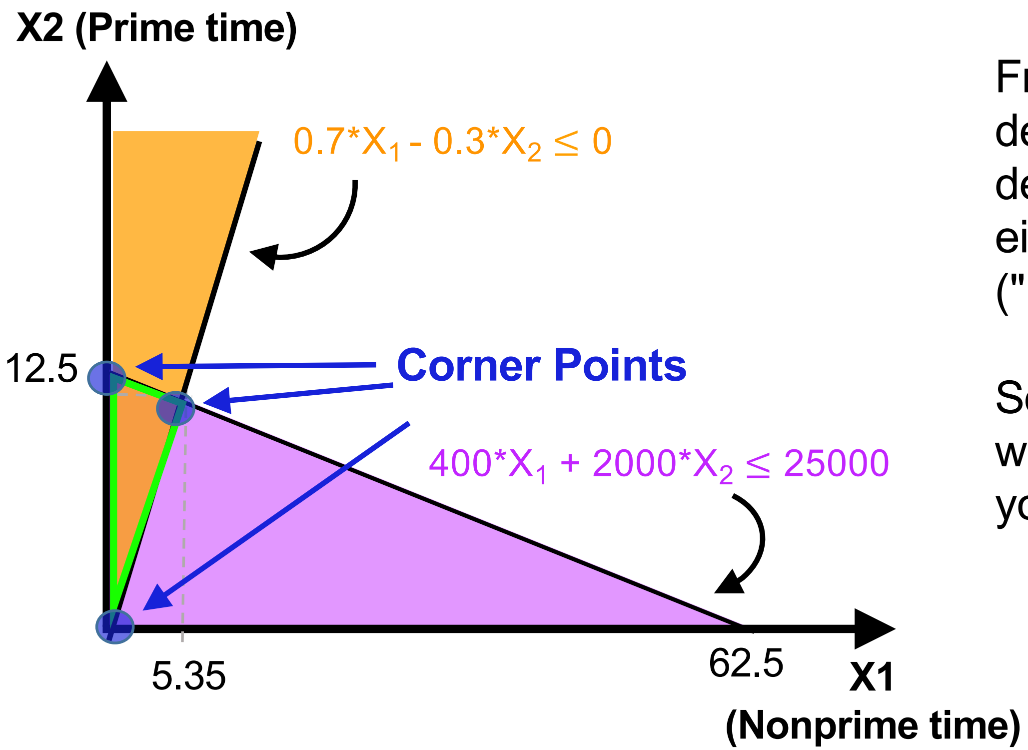


Any feasible solution (i.e., an allocation of decision variables like $X_1 = 3$, $X_2 = 10$, that satisfies all the constraints) will lie in the feasible region.

If there is no feasible region, then there is no feasible solution, and we say that the problem is infeasible. (No way to solve while satisfying all the constraints).

Solving the linear program – Graphically

Corner Points: While any point in the feasible region is a possible solution, it turns out that the optimal solution(s), if it exists, lies at a "corner" of the feasible region.



Why? Imagine that you have a solution in the center of the feasible region, not near a boundary.

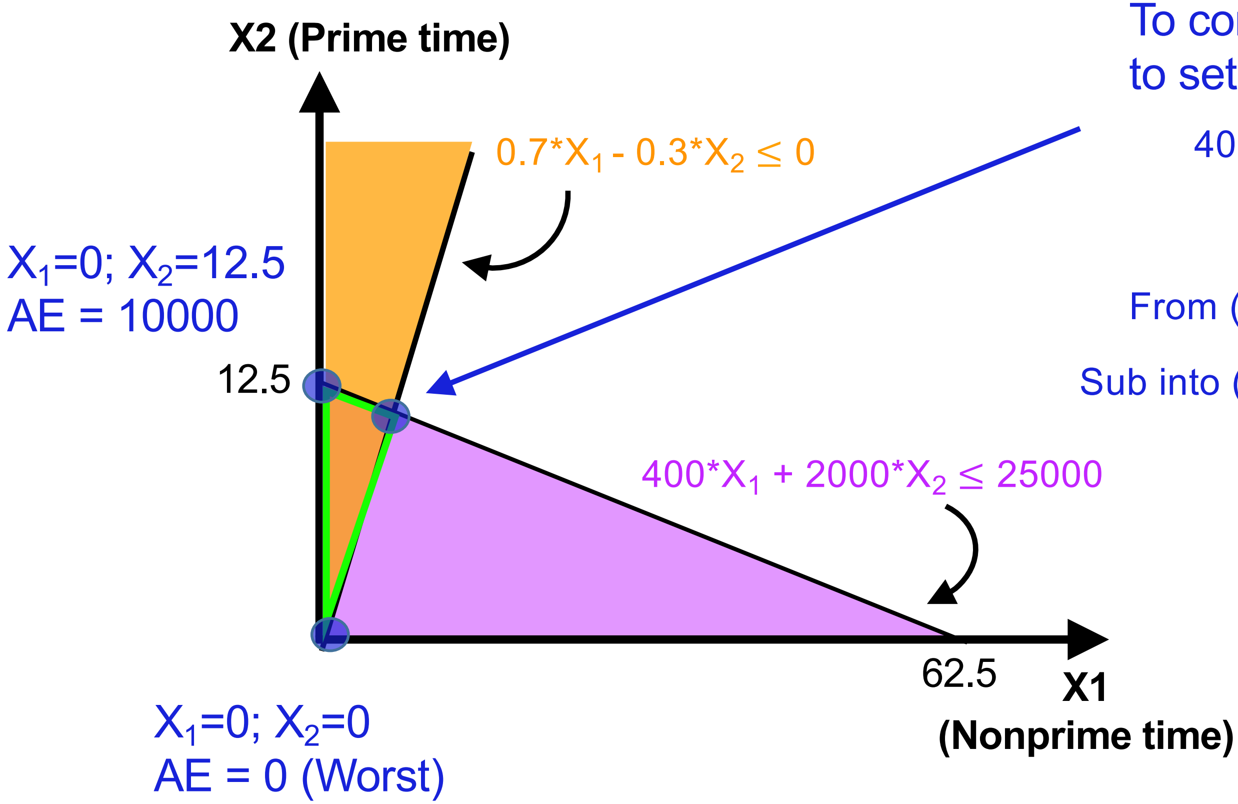
From here, you can still increase or decrease X1; or you can increase or decrease X2. These changes would either increase or decrease your profit ("better" or "worse").

So the best (and worst) solutions are when you cannot increase or decrease your decision variables any more!

Solving the linear program – Graphically

Let's compute the audience exposure for each corner point:

Audience Exposure (AE) = $350*X_1 + 800*X_2$



To compute this corner point, we need to set the inequalities to equalities.

$$400*X_1 + 2000*X_2 = 25000 \quad (1)$$

$$0.7*X_{!} - 0.3*X_2 = 0 \quad (2)$$

From (2), $0.3*X_2 = 0.7*X_1$; $X_1 = (0.3/0.7)*X_2$

Sub into (1), $400* (0.3/0.7)*X_2 + 2000*X_2 = 25000$

$$400* (0.3/0.7)*X_2 + 2000*X_2 = 25000$$

$$X_2 = 11.5132$$

$$X_1 = (0.3/0.7)*X_2$$

$$= (0.3/0.7)*11.51$$

$$= 4.9342$$

$$AE = 350(4.9342) + 800(11.5132)$$

$$= 10937.5$$

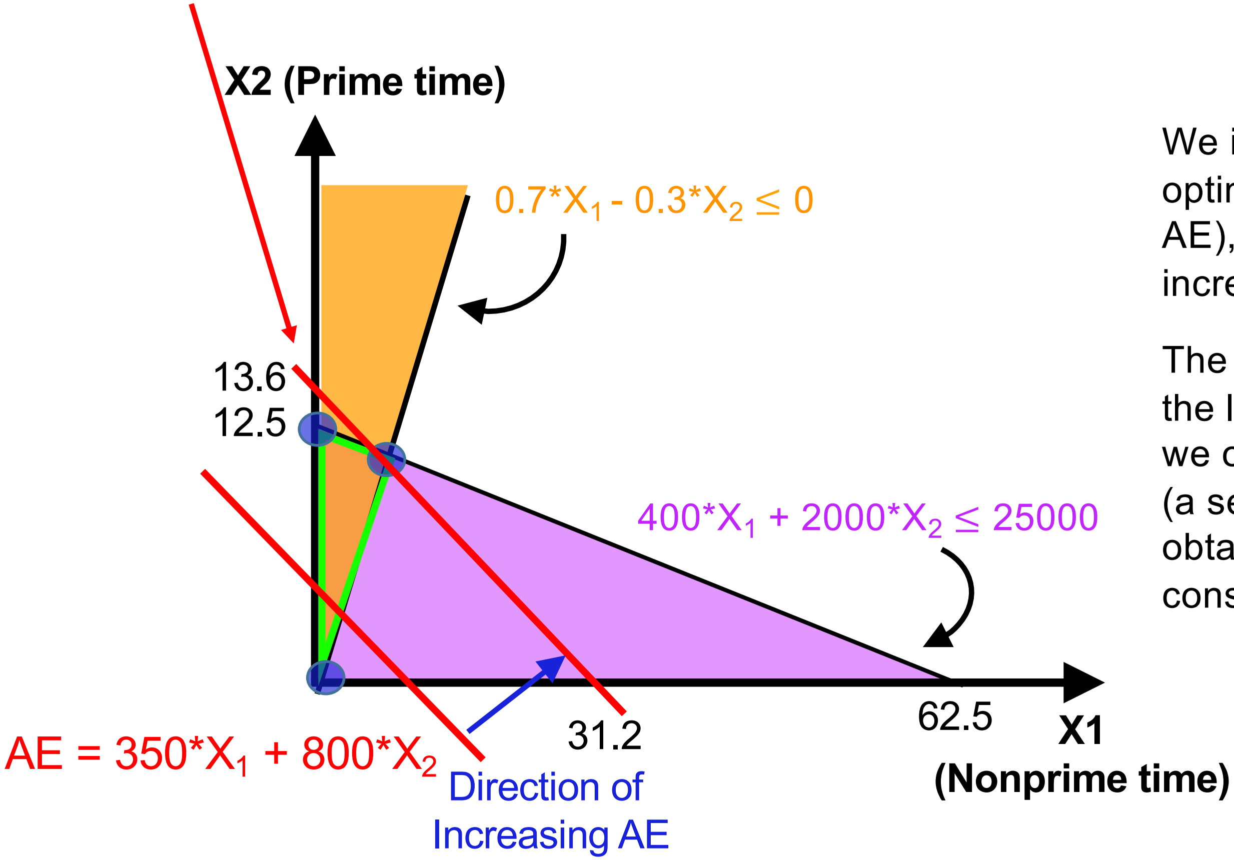
Hence, the optimal solution is $X_1 = 4.93$ & $X_2 = 11.51$

(Time for nonprime tv ads) (Time for prime tv ads)

Solving the linear program – Graphically

Alternatively, we can draw the level set of the objective function.

The level set is the set of all the points that give the same audience exposure.
(To draw this line, use AE=10915, set X_1=0; X_2=10915/800=13.6; set X_2=0, X_1=10915/350=31.2)



We indicate the direction in which we are optimising (i.e., direction of increasing AE), and we "shift" the level set as we increase AE.

The optimal solution is the last point before the level set leaves the feasible region as we optimise our objective function.
(a set of optimal solutions could also be obtained if the level set is parallel to a constraint line)

Types of Solutions

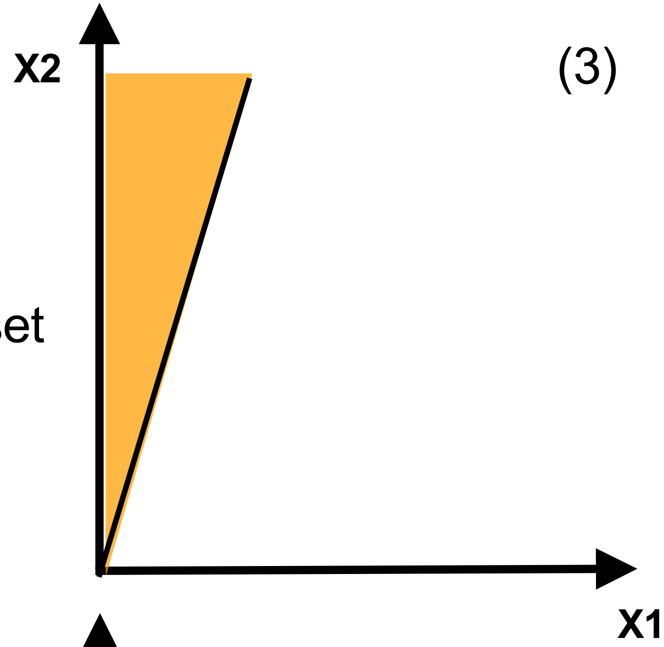
There are 4 possible types of solutions:

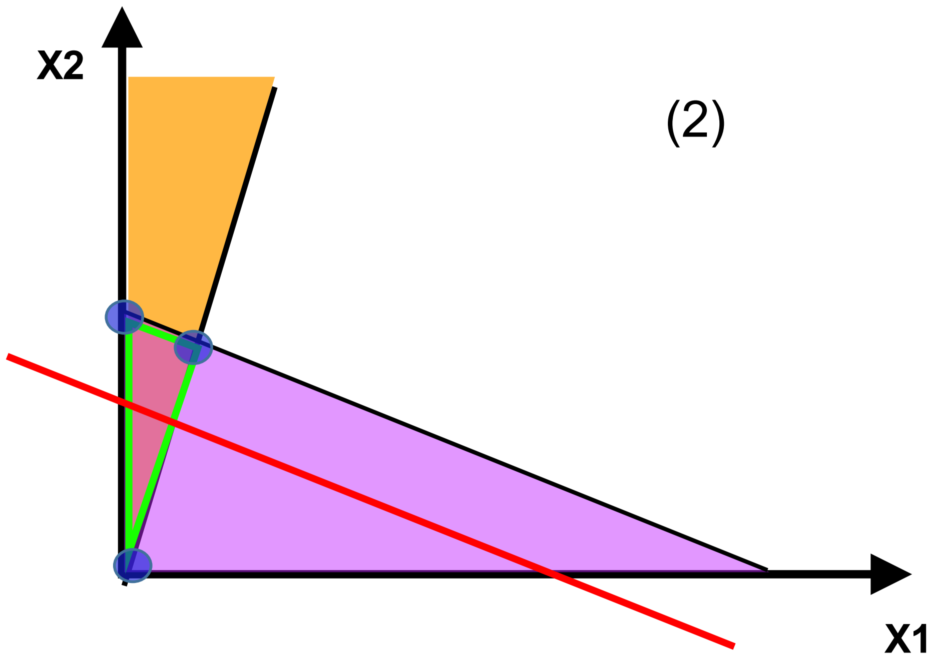
1) There exists on unique optimal solution

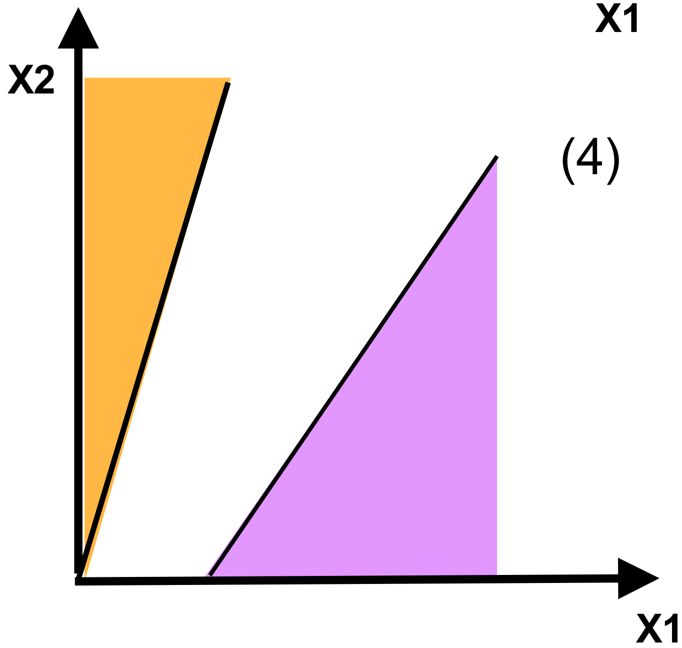
2) There exists multiple optimal solutions (level set is parallel to or lie along a constraint line)

3) The solution is unbounded

4) There exists no feasible solution





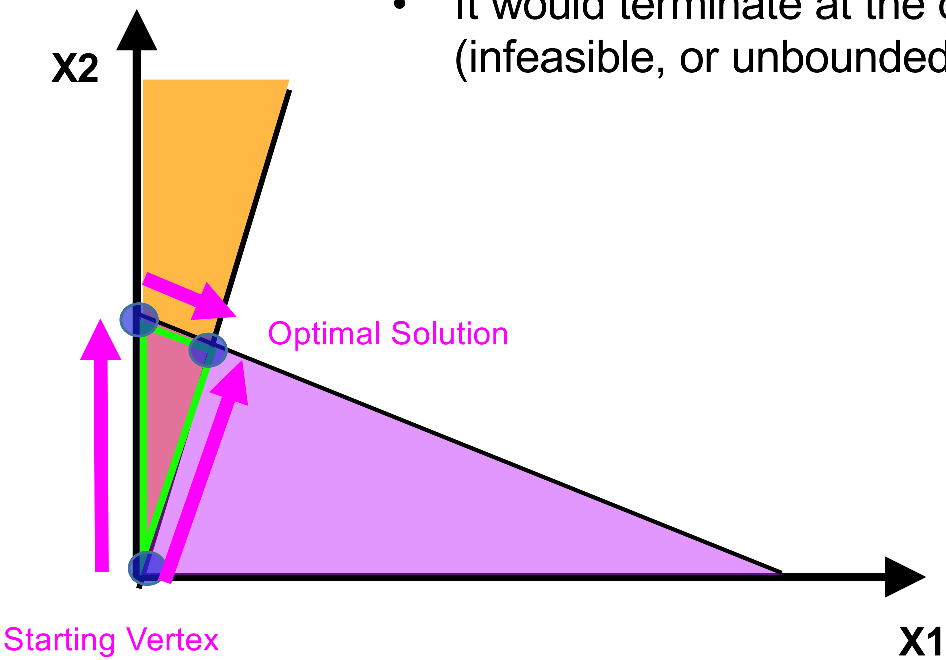


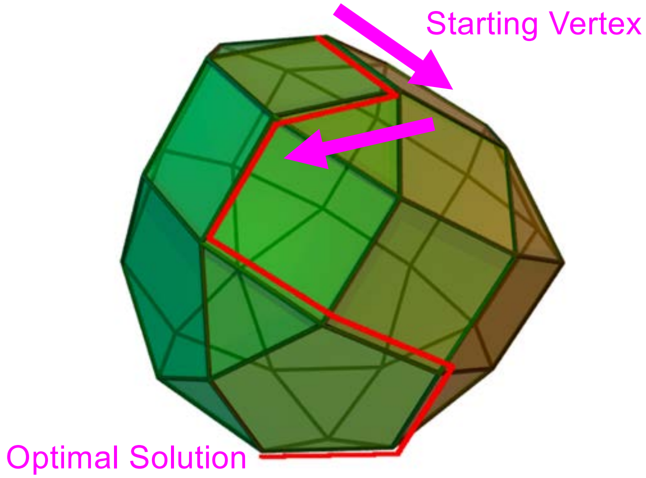
Simplex Algorithm

Given that the optimal point must lie on a corner point, how can we systematically search through all corner points to find the optimal point? (esp when there are > 2 decision variables and dimensions)

The Simplex Algorithm moves systematically from one corner point to another to improve the objective function.

- Start at a vertex (corner):
- Continue along an edge ("side of polyhedron") to another vertex (only if the objective function is strictly increasing along that edge).
- Repeat until no such edges are found.
- It would terminate at the optimal solution, or report that none are found (infeasible, or unbounded)





Simplex algorithm on a 3D Polyhedron
Source: Wikipedia

Using R to solve the linear program

We can install and use the “lpSolve” package to solve the linear program in R.

Maximize audience exposure using decision variables X_1 & X_2 = $350*X_1 + 800*X_2$

Subject to:

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$

Non-negativity constraints: $X_1 \geq 0; X_2 \geq 0$

lpSolve assumes all decision vars are non-negative so non-negativity constraints do not have to be specified in R codes.

number of decision variables

```
#defining parameters
objective.fn <- c(350, 800)
const.mat <- matrix(c(400, 2000, 0.7, -0.3) , ncol=2 , byrow=TRUE)
const.dir <- c("<=", "<=")
const.rhs <- c(25000, 0)
```

specify objective function as min or max

```
#solving model
lp.solution <- lp("max", objective.fn, const.mat, const.dir, const.rhs,
compute.sens=TRUE)
```

Compute sensitivity

```
#decision variables values
lp.solution$solution
[1] 4.934211 11.513158
```

```
# objective function value
lp.solution
Success: the objective function is 10937.5
```

Sensitivity Analyses

After you have found the optimal solution, you may want to know how sensitive is your solution to changes in the constraints or optimization function. That is, you want to ask, what will happen to the optimal solution if you change the constraints or optimization function by a little.

We can ask this question systematically by performing sensitivity analyses.

There are two types of sensitivity analyses:

1) vary objective function coefficients

2) vary constraint values (shadow prices)

Sensitivity Analyses

Varying objective function coefficients

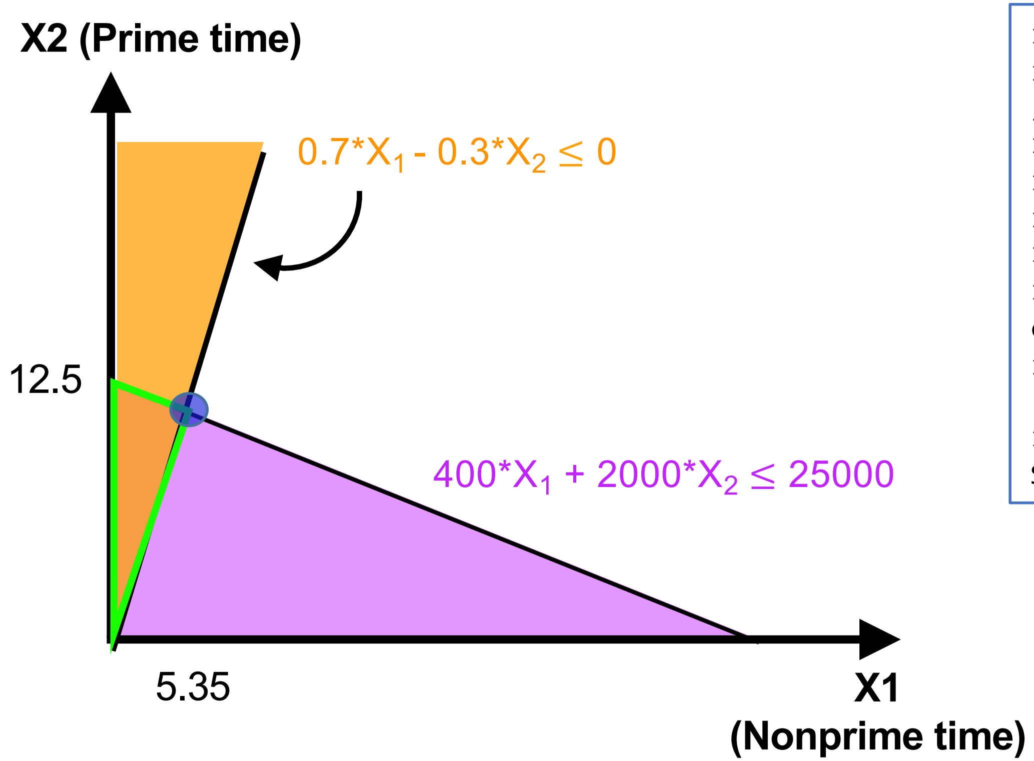
In our advertising example, the optimal solution is $X_1 = 4.93$ & $X_2 = 11.51$

and the objective function is: Audience Exposure (AE) = $350*X_1 + 800*X_2$

What if a new research shows that the exposure for prime tv ad is 1000 audience per min? (changing the objective function to:

Audience Exposure (AE) = $350*X_1 + 1000*X_2$

What is the impact on the optimal solution? Run lpSolve and check.




```
> objective.fn <- c(350, 1000)
> const.mat <- matrix(c(400, 2000, 0.7, -0.3), ncol=2
, byrow=TRUE)
> const.dir <- c("<=", "<=")
> const.rhs <- c(25000, 0)
>
> #solving model
> lp.solution <- lp("max", objective.fn, const.mat,
const.dir, const.rhs, compute.sens=TRUE)
> lp.solution$solution #decision variables values
[1]  4.934211 11.513158
> lp.solution
Success: the objective function is 13240.13
```

Turns out that the optimal solution remains the same at $X_1 = 4.93$ & $X_2 = 11.51$ with total audience exposure = 13240

Sensitivity Analyses

Varying objective function coefficients

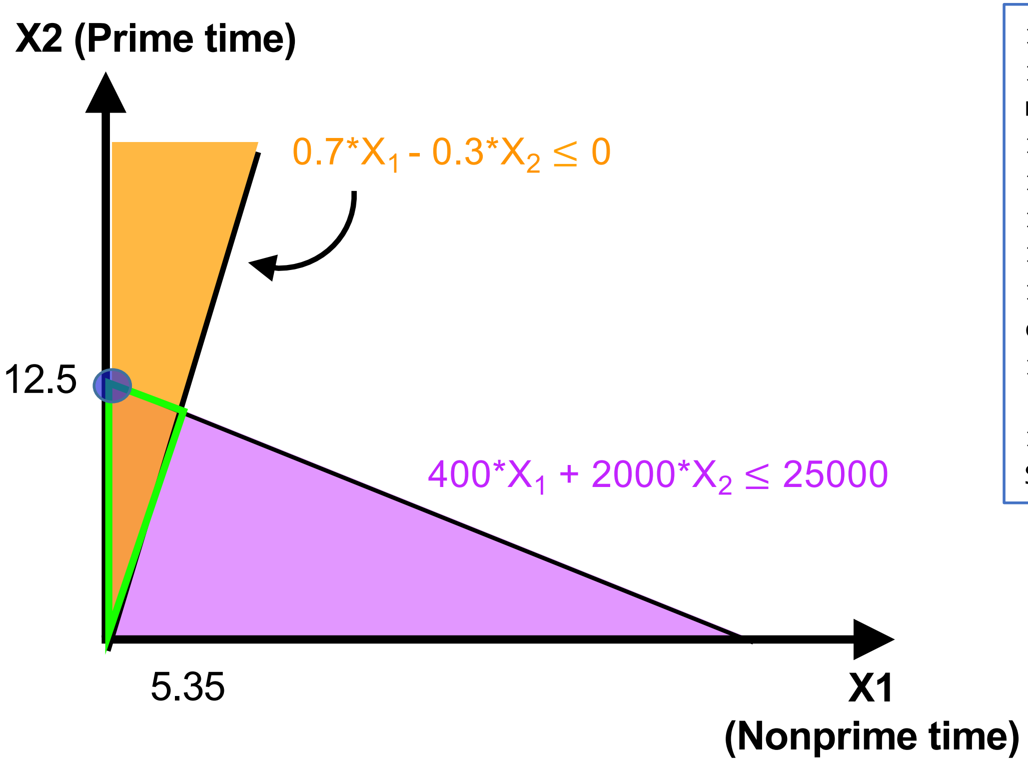
In our advertising example, the optimal solution is $X_1 = 4.93$ & $X_2 = 11.51$

and the objective function is: Audience Exposure (AE) = $350*X_1 + 800*X_2$

What if a new research shows that the exposure for nonprime tv ad is 150 audience per min? (changing the objective function to:

Audience Exposure (AE) = $150*X_1 + 800*X_2$

What is the impact on the optimal solution? Run lpSolve and check.




```
> objective.fn <- c(150, 800)
> const.mat <- matrix(c(400, 2000, 0.7, -0.3) ,
ncol=2 , byrow=TRUE)
> const.dir <- c("<=", "<=")
> const.rhs <- c(25000, 0)
>
> #solving model
> lp.solution <- lp("max", objective.fn, const.mat,
const.dir, const.rhs, compute.sens=TRUE)
> lp.solution$solution #decision variables values
[1]  0.0 12.5
> lp.solution
Success: the objective function is 10000
```

Now optimal solution has changed to $X_1 = 0$ & $X_2 = 12.5$ with total audience exposure = 10000

It is no longer optimal to have nonprime TV ad.

Sensitivity Analyses

Varying objective function coefficients

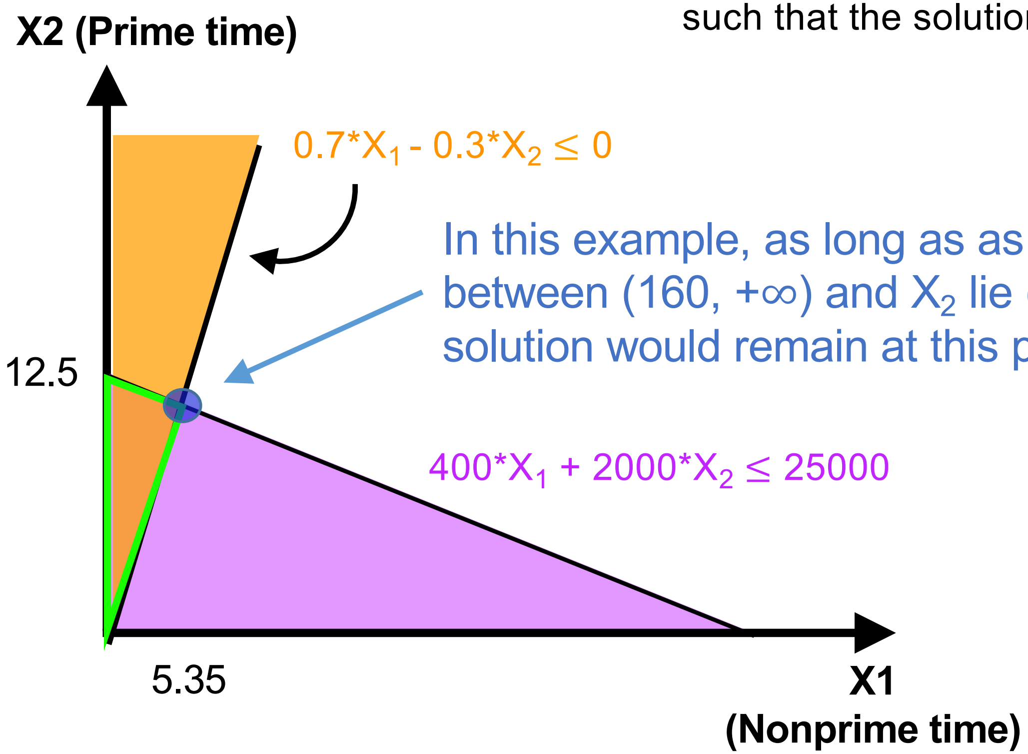
R's lpSolve::ip() function calculates the range of coefficient values for which the given solution is optimal.

Audience Exposure (AE) = $350*X_1 + 800*X_2$

```
> lp.solution$sens.coef.from
[1]  160 -150
```

```
> lp.solution$sens.coef.to
[1] 1.00e+30 1.75e+03
```

The `sens.coef.from` and `sens.coef.to` variables from the lpSolve solution tells us the range in which the coeffcients of X_1 and X_2 lie such that the solution is still optimal.



Once the coefficients shift out of these ranges, the optimal solution will change to a different vertex (like in the previous slide).

Note: These ranges assume coefficients are shifted for one variable at a time.

Sensitivity Analyses

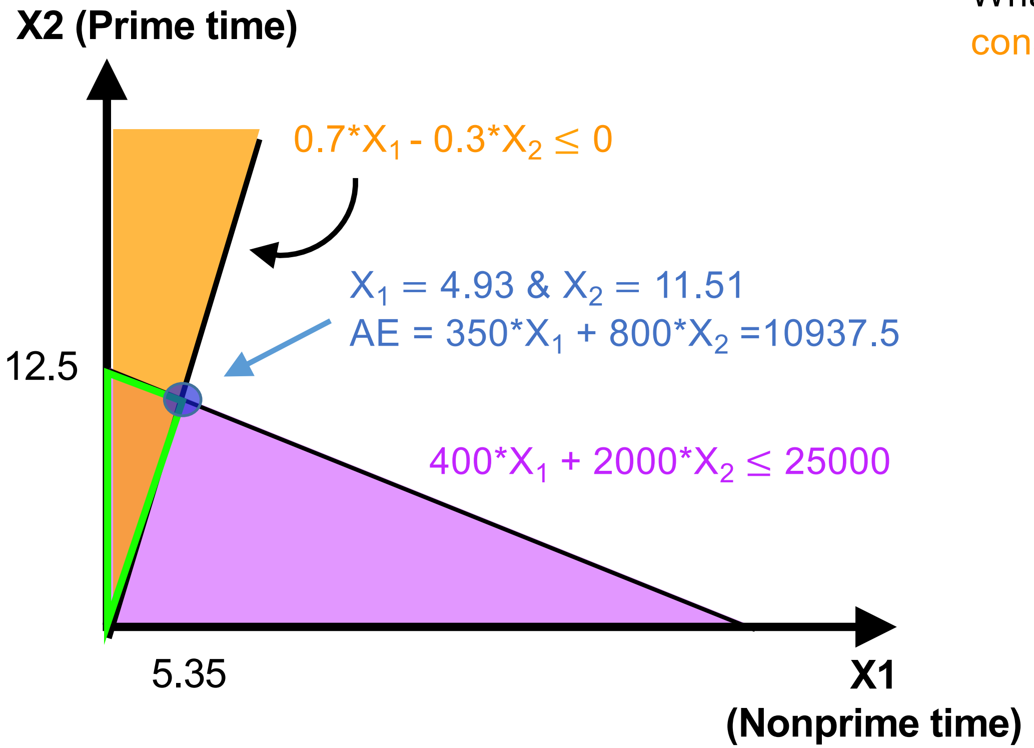
Varying constraint values (Shadow Prices)

Question: What if we change the values of the constraints, how will it affect the optimal solution?

The Shadow Price of a constraint is the change in the objective function value per unit-increase in the right-hand-side value of that constraint (holding all else equal).

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$



What if we increase the RHS of budgetary constraint from 25000 to 25001?

Sensitivity Analyses

Varying constraint values (Shadow Prices)

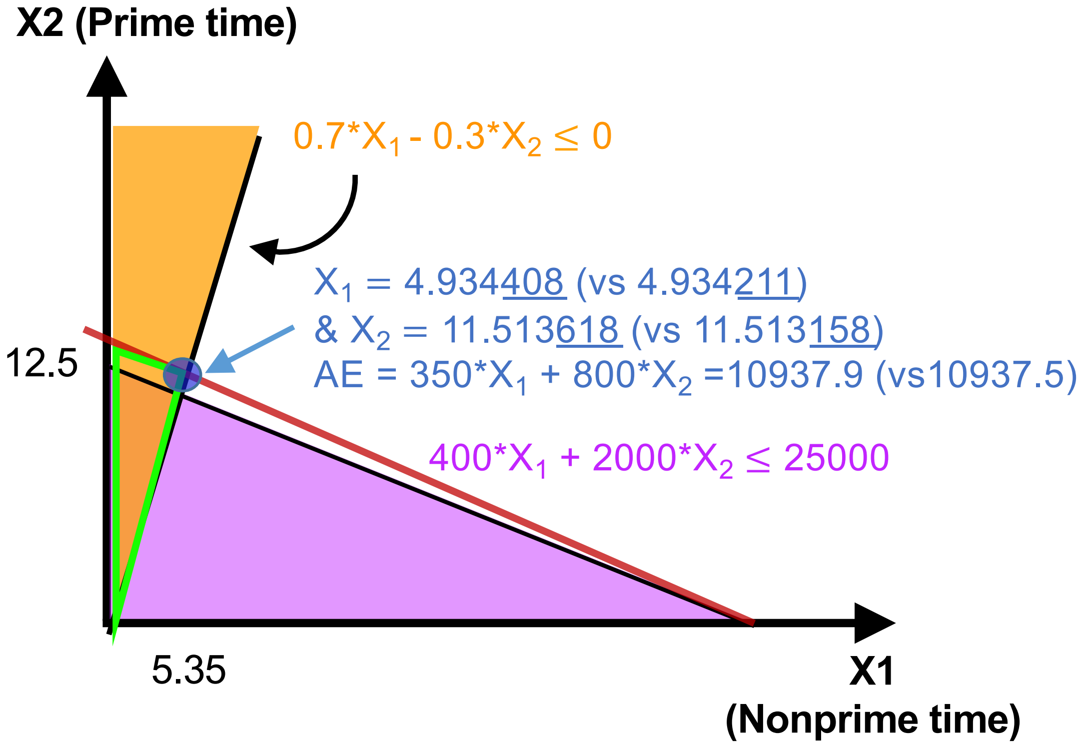
Question: What if we change the values of the constraints, how will it affect the optimal solution?

The Shadow Price of a constraint is the change in the objective function value per unit-increase in the right-hand-side value of that constraint (holding all else equal).

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$

What if we increase the RHS of budgetary constraint from 25000 to 25001? (try use lpSolve)



The feasible region was pushed out a little (up), which moved the optimal vertex up and to the right slightly.

The new solution has only an increase in 0.4. (note that the change of 1 is very small relative to 25000, hence the changes in values are all in small decimal places. can compare better if you run lpSolve)

Thus, increasing the budgetary constraint by 1 unit (25000 to 25001) increases the AE by 0.4.

Thus, the Shadow Price of the budgetary constraint is 0.4.

Sensitivity Analyses

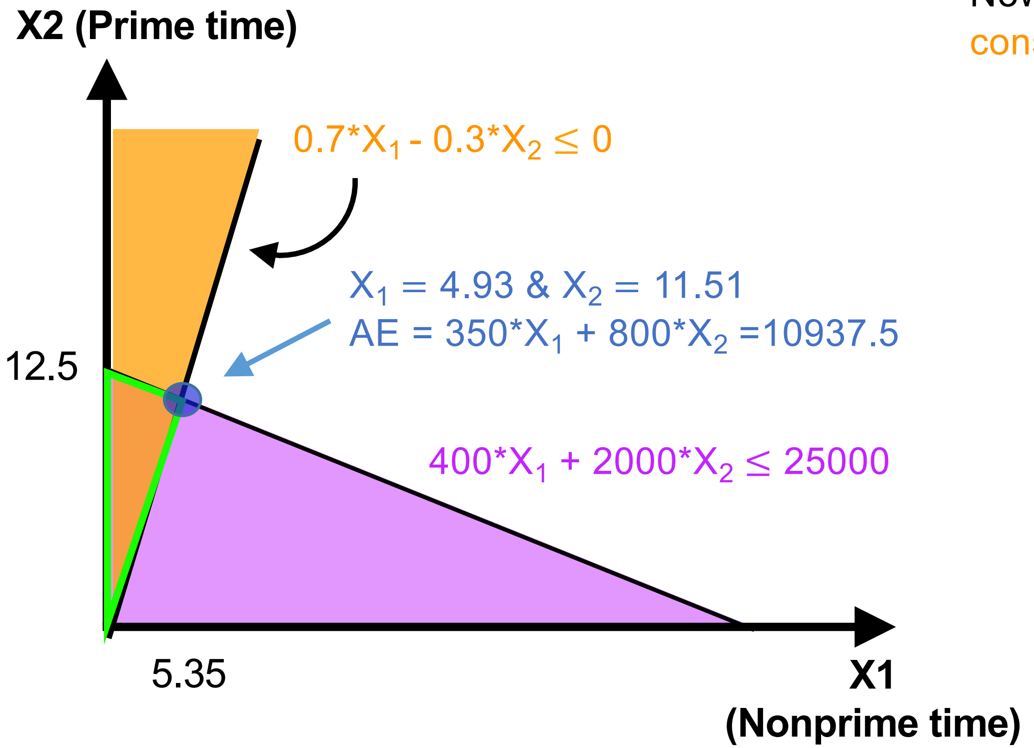
Varying constraint values (Shadow Prices)

Question: What if we change the values of the constraints, how will it affect the optimal solution?

The Shadow Price of a constraint is the change in the objective function value per unit-increase in the right-hand-side value of that constraint (holding all else equal).

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$



Now, what if we increase the RHS of allocation constraint from 0 to 1?

Sensitivity Analyses

Varying constraint values (Shadow Prices)

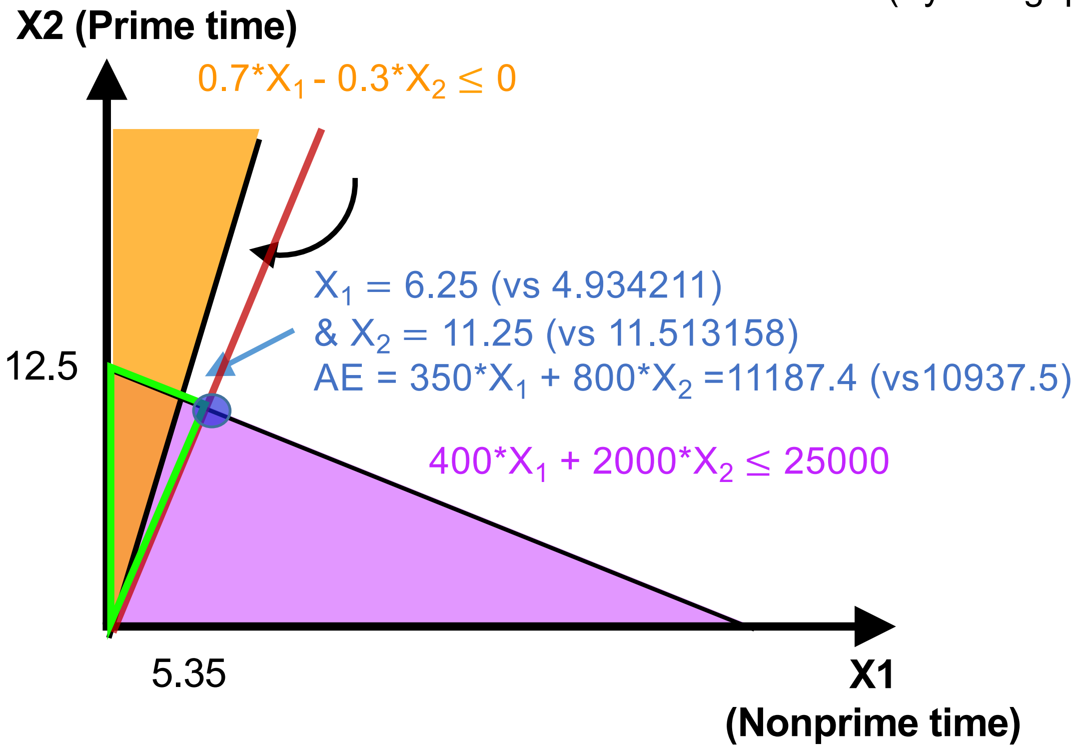
Question: What if we change the values of the constraints, how will it affect the optimal solution?

The Shadow Price of a constraint is the change in the objective function value per unit-increase in the right-hand-side value of that constraint (holding all else equal).

Budgetary constraint: $400*X_1 + 2000*X_2 \leq 25000$

Allocation constraint: $0.7*X_1 - 0.3*X_2 \leq 0$

What if we increase the RHS of allocation constraint from 0 to 1? (try using lpSolve)



The feasible region is now pushed out quite a bit, moving the optimal vertex down and to the right.

The new solution has an increase of 249.9 (~250) AE.

Thus, increasing the allocation constraint by 1 unit (0 to 1) increases the AE by 249.9.

Thus, the Shadow Price of the allocation constraint is 249.9.

Sensitivity Analyses

Varying constraint values (Shadow Prices)

In R, we can use the `duals` variable of the lpSolve solution to obtain the shadow prices.
Note: in other fields like computer scence, shadow prices are also called duals.

```
> objective.fn <- c(350, 800)
> const.mat <- matrix(c(400, 2000, 0.7, -0.3) , ncol=2 , byrow=TRUE)
> const.dir <- c("<=", "<=")
> const.rhs <- c(25000, 0)
>
> #solving model
> lp.solution <- lp("max", objective.fn, const.mat, const.dir, const.rhs,
compute.sens=TRUE)
> lp.solution$solution #decision variables values
[1]  4.934211 11.513158
> lp.solution
Success: the objective function is 10937.5
> lp.solution$duals
[1]   0.4375 250.0000   0.0000   0.0000
```

Shadow price of budgetary constraint
($400*X_1 + 2000*X_2 \leq 25000$)

Shadow price of allocation constraint
($0.7*X_1 - 0.3*X_2 \leq 0$)

Shadow prices of non-negativity constraints ($X_1 \geq 0$; $X_2 \geq 0$); sometimes referred to as *reduced costs*

Reduced costs is the amount of penalty you pay for adding one unit of the variable.

Another Example: Manufacturing Chairs

Dining

Selling Price: $225

Cost(hrs): 5

Storage space(ft^3): 30

Sofa

Selling Price: $300

Cost(hrs): 8

Storage space (m^3): 40

Bar

Selling Price:$250

Cost(hrs): 7

Storage space (m^3): 15

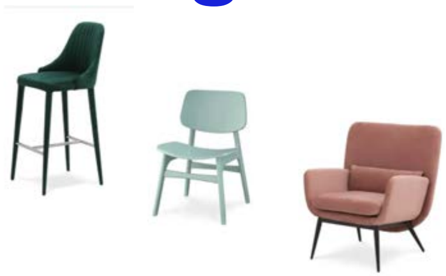
The selling price of each type of chair, as well as the production-hour cost of producing each chair, are listed above.

Each week, the manufacturer has a budget of 60 production hours. Additionally, there is warehouse storage of only 200 ft^3 (cubic-feet). Furthermore, from past trends, you can only sell a maximum of 7 bar chairs.

How many of each type of chairs should you produce to maximize revenue from the sales of chairs?

Another Example: Manufacturing Chairs

Decision Variables:
X_1 = number of bar chairs made
X_2 = number of dining chairs made
X_3 = number of sofa chairs made



Maximize total revenue using decision variables X_1, X_2, X_3

Revenue = $250\ X_1 + 225\ X_2 + 300\ X_3$

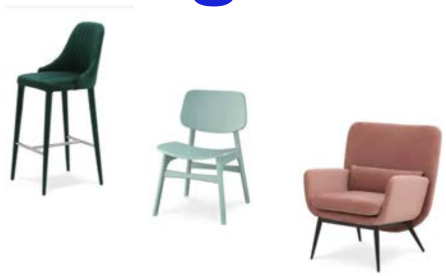
Subject to:

Production Hour Budget Constraints	$7X_1 + 5X_2 + 8X_3 \leq 60$
Storage Constraints	$15X_1 + 30X_2 + 40X_3 \leq 200$
Demand Constraints	$X_1 + \quad + \quad \leq 7$
Non-negativity Constraints	$X_1 \geq 0;\ X_2 \geq 0;\ X_3 \geq 0$

```
> objective.fn <- c(250, 225, 300)
> const.mat <- matrix(c(7, 5, 8, 15, 30, 40, 1, 0, 0) , ncol=3 , byrow=TRUE)
> const.dir <- c("<=", "<=","<=")
> const.rhs <- c(60, 200, 7)
>
> #solving model
> lp.solution <- lp("max", objective.fn, const.mat, const.dir, const.rhs,
compute.sens=TRUE)
```

Another Example: Manufacturing Chairs

Decision Variables:
X_1 = number of bar chairs made
X_2 = number of dining chairs made
X_3 = number of sofa chairs made



```
> lp.solution$solution
[1] 5.925926 3.703704 0.000000
```

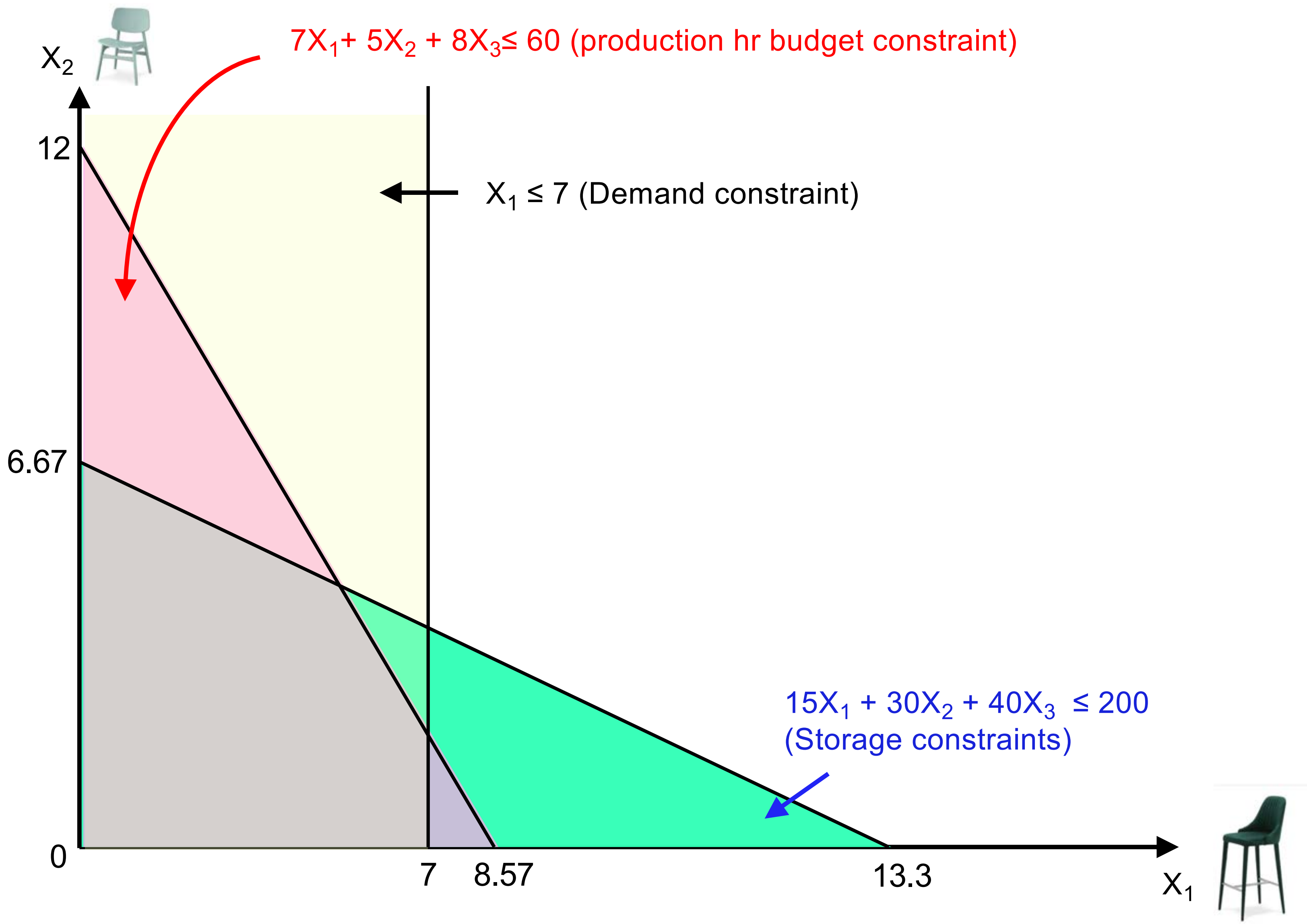
```
> lp.solution
Success: the objective function is
2314.815
```

Thus, the optimal solution is:
X_1 = 5.93, X_2 = 3.70, X_3 = 0; Maximized revenue = $2314.82

The optimal solution is to not produce any sofa chairs at all...

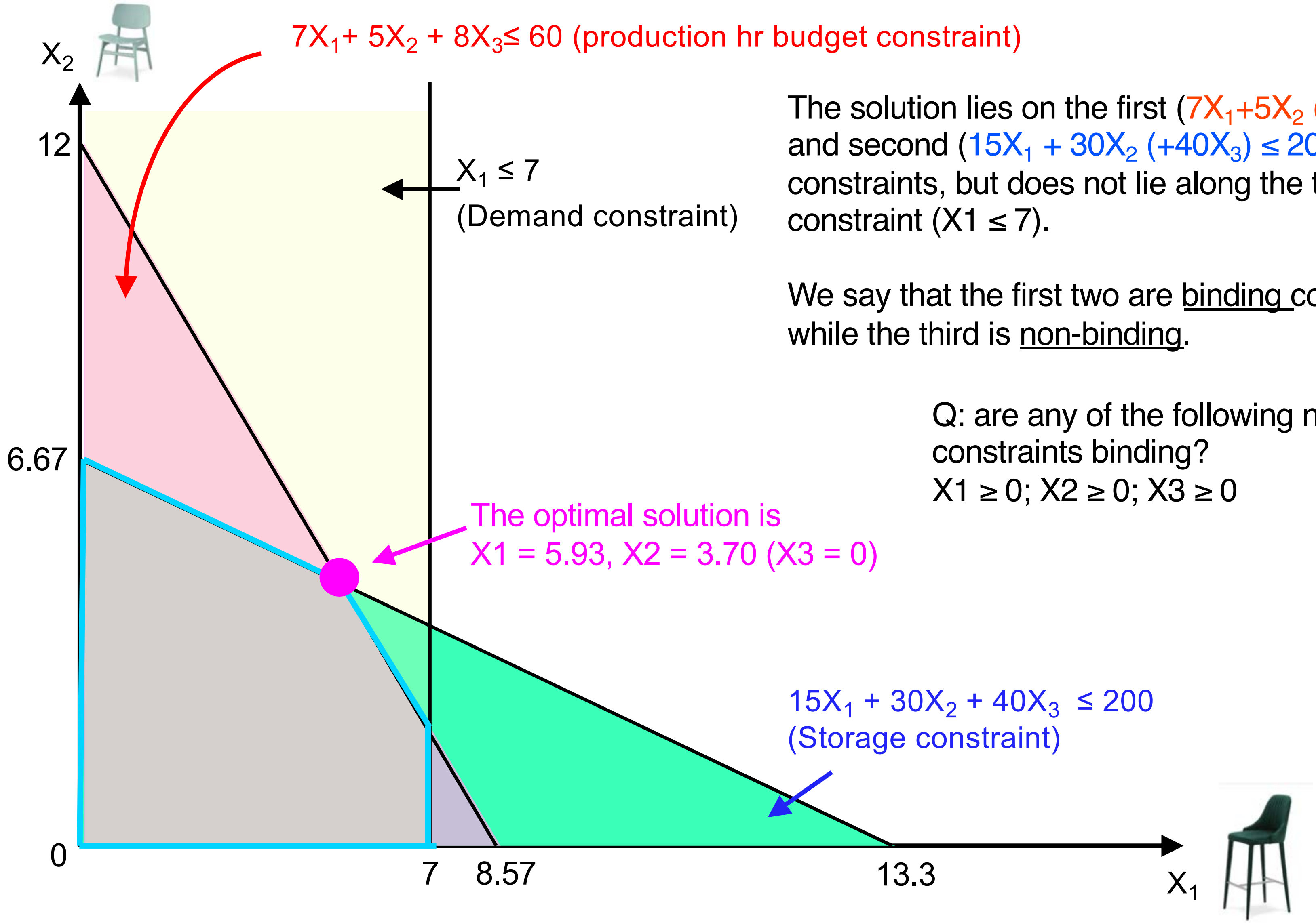
Another Example: Manufacturing Chairs

Ignoring X_3 for now, plot the 3 constraints onto X_1 and X_2



Another Example: Manufacturing Chairs

The feasible region is indicated in blue.



The solution lies on the first ($7X_1+5X_2$ $(+8X_3) \leq 60$) and second ($15X_1 + 30X_2$ $(+40X_3) \leq 200$) constraints, but does not lie along the third constraint (X1 ≤ 7).

We say that the first two are <u>binding</u> constraints, while the third is <u>non-binding</u>.

Q: are any of the following non-negativity constraints binding?
X1 ≥ 0; X2 ≥ 0; X3 ≥ 0

Another Example: Manufacturing Chairs

Sensitivity Analyses: Varying the objective function coefficients

$$\text{Revenue} = 250\ X_1 + 225\ X_2 + 300\ X_3$$

The optimal solution is $X_1 = 5.93$, $X_2 = 3.70$, $X_3 = 0$

```
> lp.solution$solution
[1] 5.925926 3.703704 0.000000
lp.solution$sens.coef.from
[1]  1.12500e+02  1.90625e+02 -1.00000e+30
> lp.solution$sens.coef.to
[1] 315.0000 500.0000 340.7407
```

This means that the optimal solution remains the same if:
- The price we can sell bar chairs (coefficient on X_1) lies between 112 and 315
- The price we can sell dining chairs (coefficient on X_2) lies between 191 and 500
- The price we can sell sofa (coefficient on X_3) lies between -Infinity and 341

Our optimal solution is $X_1 = 5.93$, $X_2 = 3.70$, $X_3 = 0$

That is, in order for the sofa to be profitable, we would need to increase the selling price to at least $341 per sofa!

Another Example: Manufacturing Chairs

Sensitivity Analyses: Shadow Prices

Revenue = $250\ X_1 + 225\ X_2 + 300\ X_3$

The optimal solution is $X_1 = 5.93, X_2 = 3.70, X_3 = 0$



```
> lp.solution$duals
[1]  30.555556   2.407407   0.000000   0.000000   0.000000 -40.740741
```

shadow prices of constraints

shadow prices of non-negativity constraints

Increasing RHS of production budget hr constraint by 1 unit, increases revenue by $30.56.

Increasing RHS of storage constraint by 1 unit, increases revenue by $2.41.

Q: Why is the shadow price of non-binding constraint zero?

Q: Why is the last shadow price non-zero? And why is it negative?

Summary

- In this lecture you have learnt the basics of formulating and solving simple linear optimisation problems, which help us "prescribe" what business choices to make.
- If you can formulate your real-world business problem as a linear optimisation problem, then there are very efficient solvers and algorithms that can solve the problem, and to help you gain further insight into your problem (e.g., via sensitivity analyses).
- In the models we've covered this week, we've assumed that all the decision variables are continuous, real-valued numbers. In future, you may learn about integer optimization where decision variables have to be integers or binary yes/no decision.