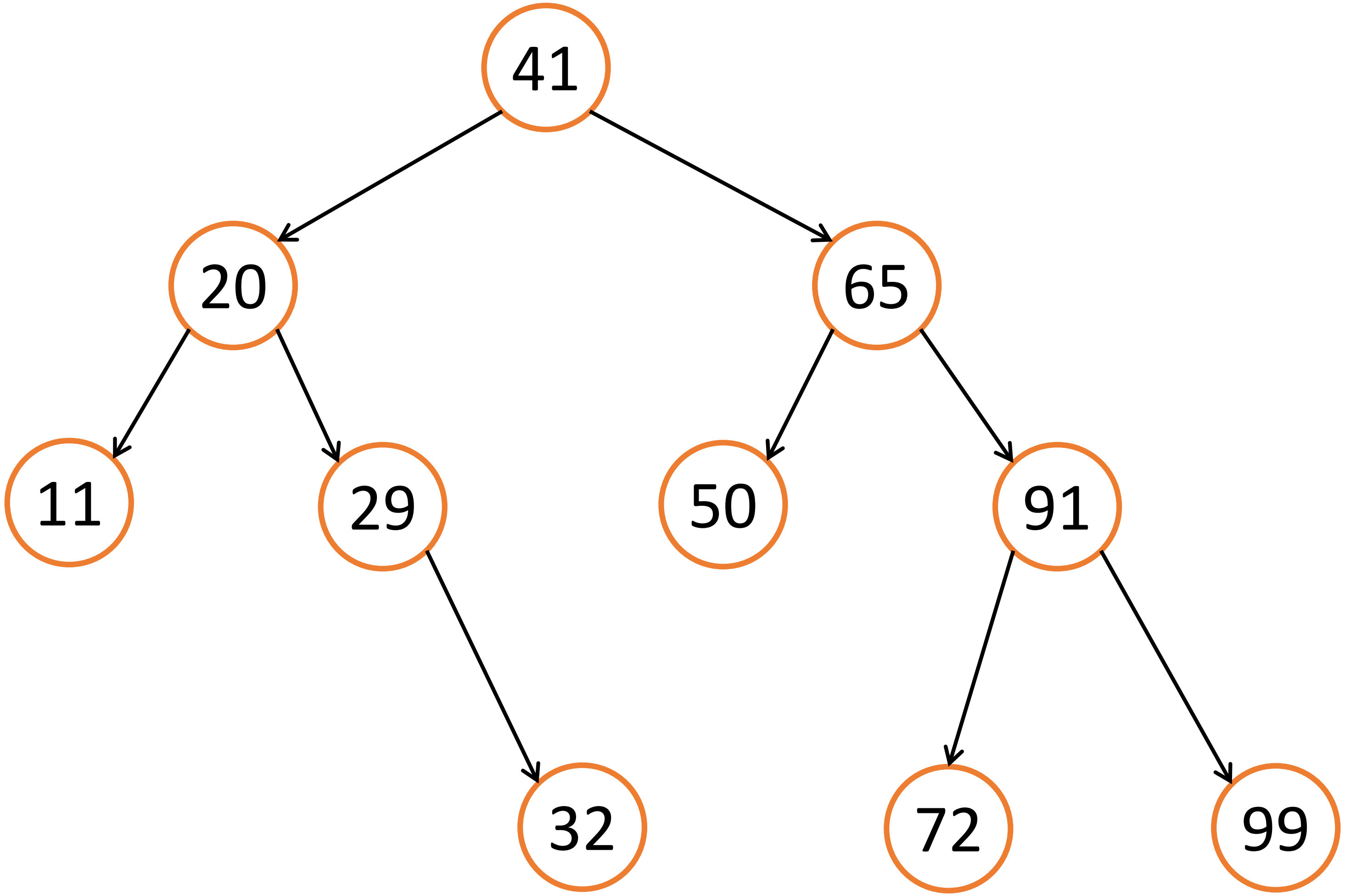
BST

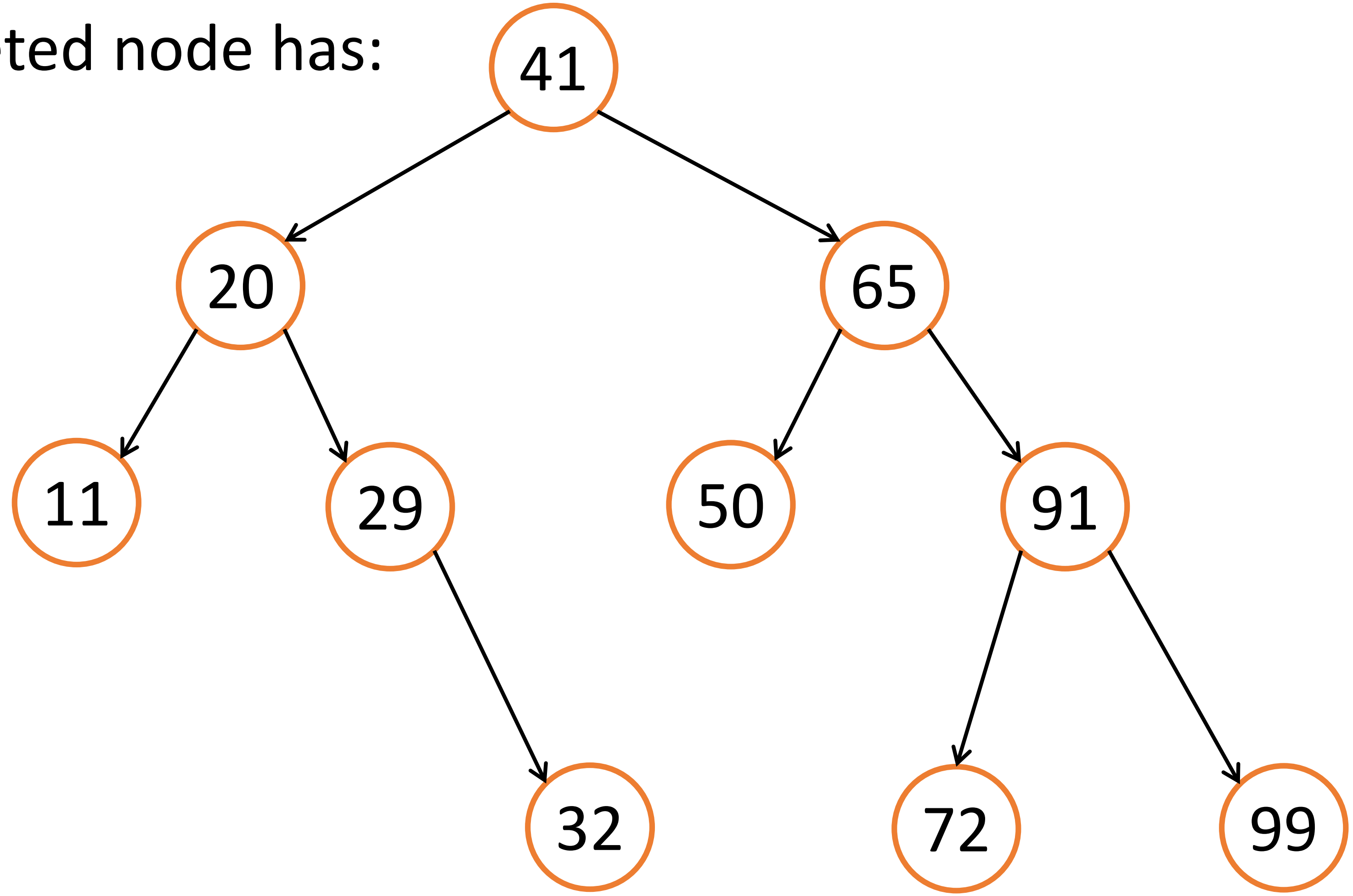
(Deletion of a node)

Deletion of a node



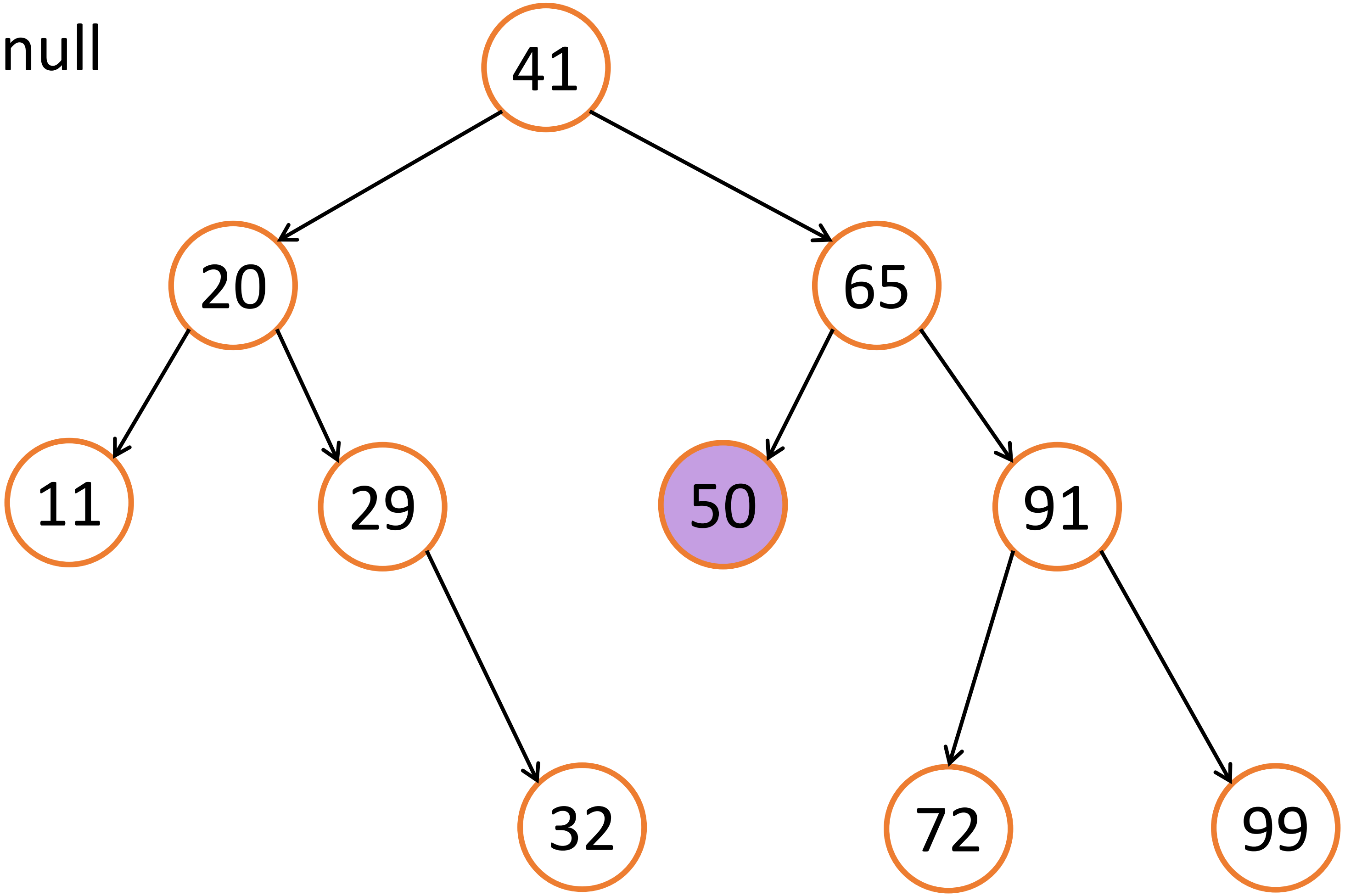
Three cases of Deleting a Node

- If the going-to-be-deleted node has:
  - No children
  - 1 child
  - 2 children



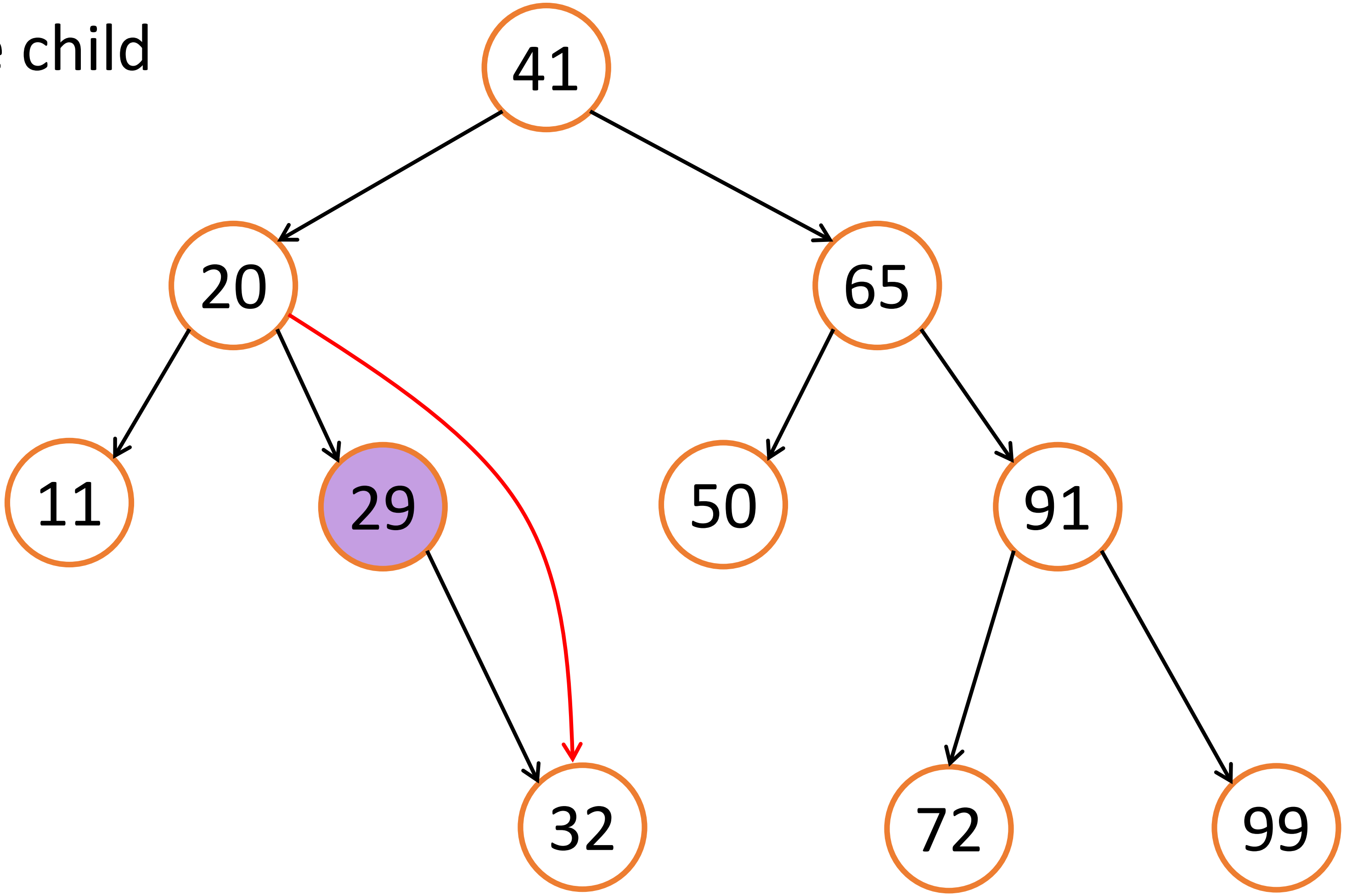
Case 1: No Children (delete(50))

- Set your parent's child to null
- Delete yourself



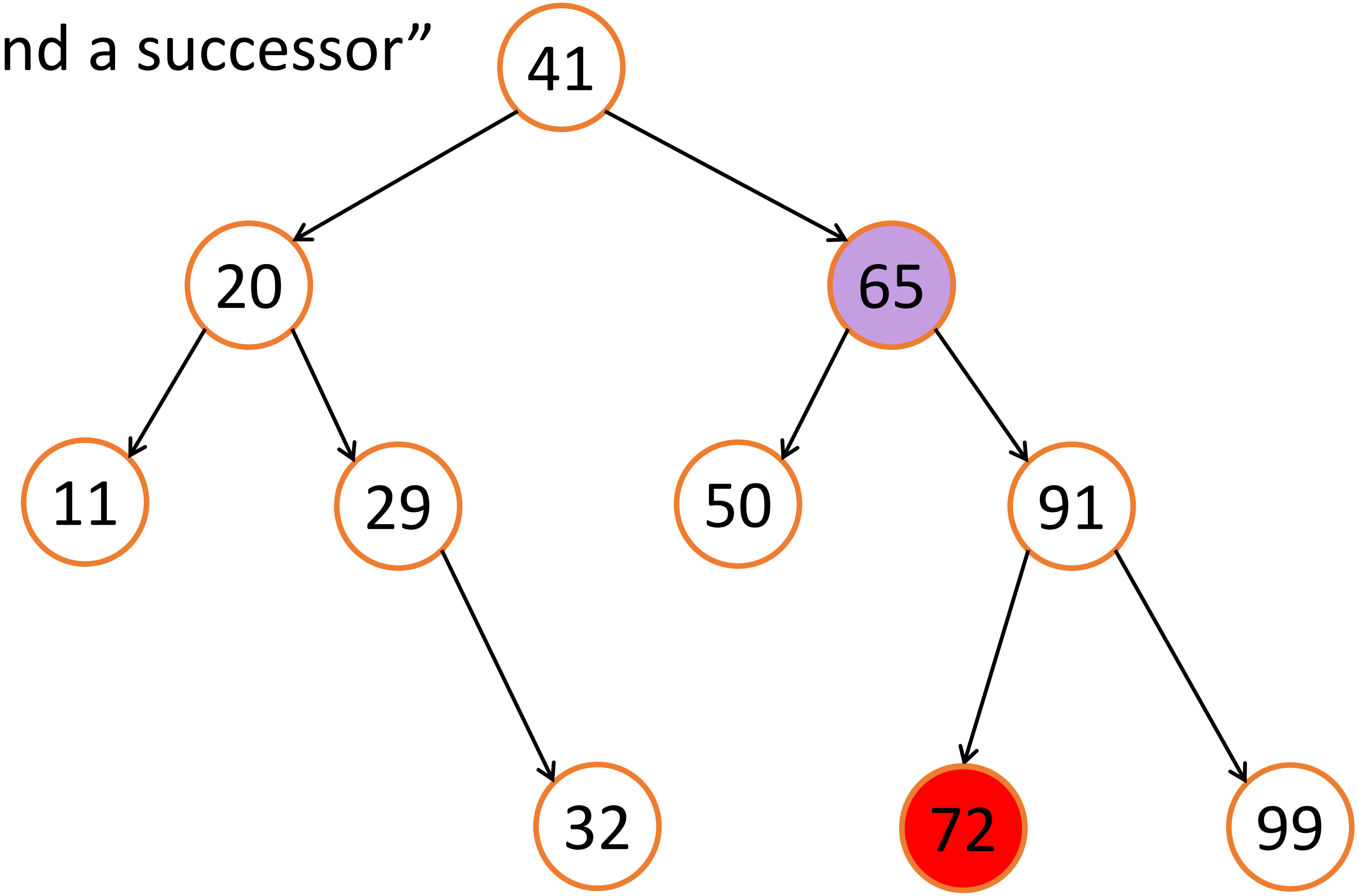
Case 2: One Child (delete(29))

- Link up the parent and the child
- Delete yourself



Case 3: 2 Children (delete(65))

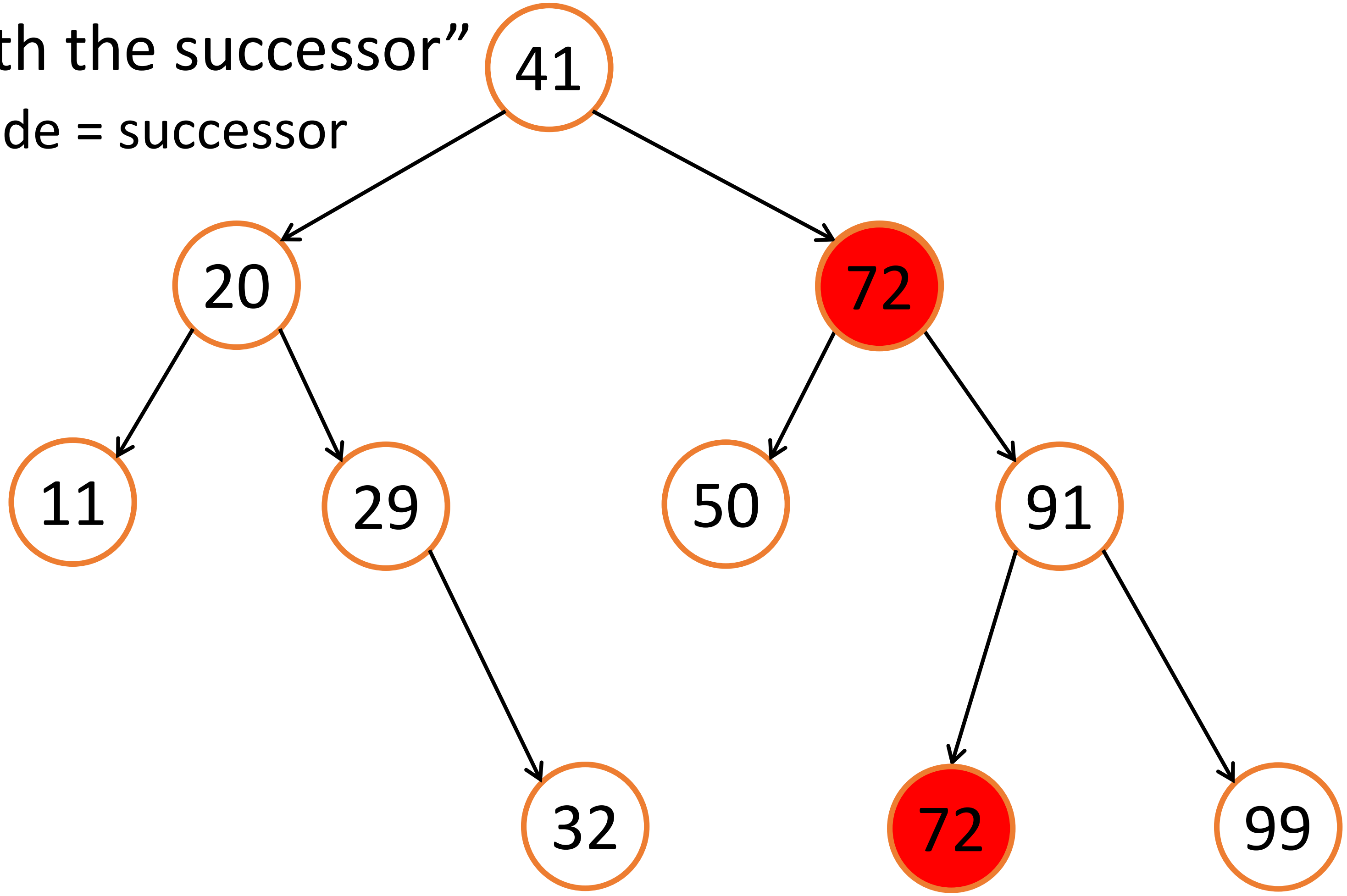
- "Before a king is gone, find a successor"



Case 3: 2 Children (delete(65))

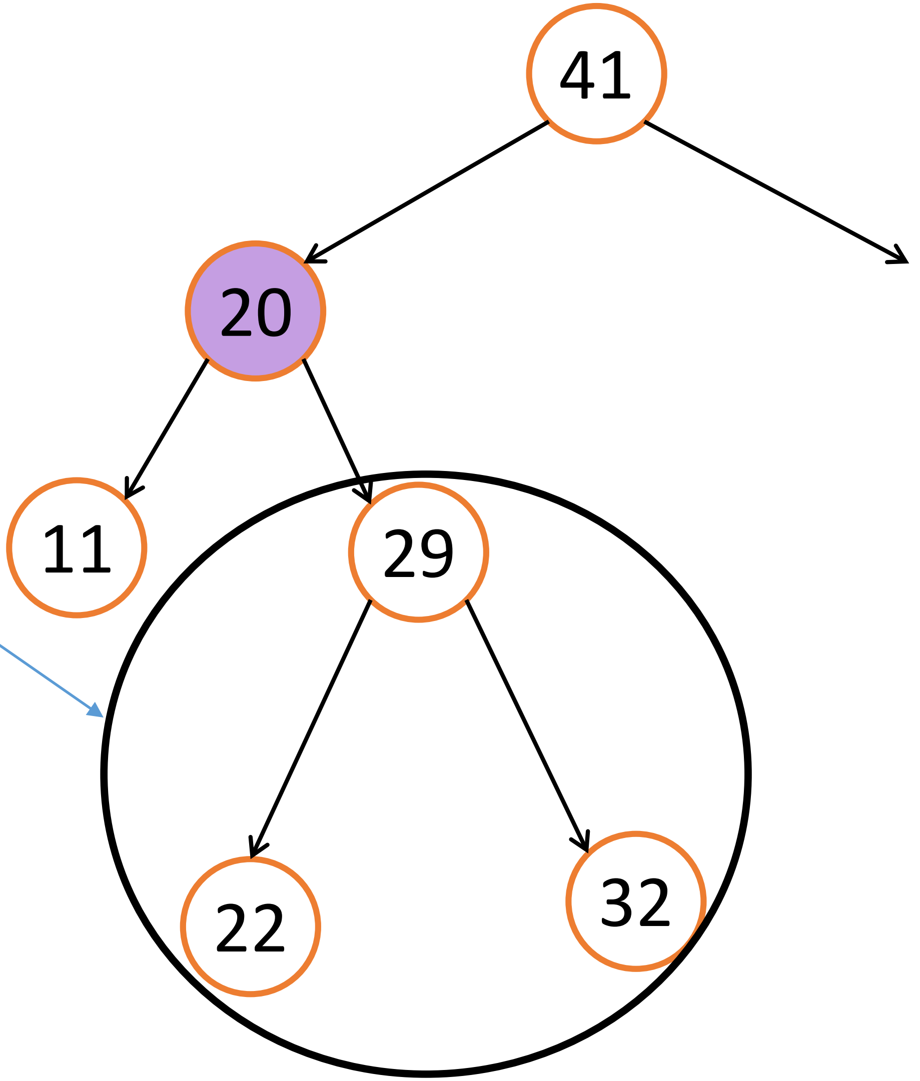
- “And replace the king with the successor”
  - “going-to-be-deleted” node = successor
  - delete successor

Wait! what if the successor has two children?



Recap: Successor Queries

- Under the assumption that
  - The node is in the tree
  - The node has two children
- Which one is the successor?
- The *minimum* of its right subtree!



Will this successor has two children?

Three Cases of Deleting a Node in BST

- No children:
  - remove v
- 1 child:
  - remove v
  - connect child(v) to parent(v)
- 2 children
  - x = successor(v)
  - replace v with x
  - remove x

AVL Tree Balancing

"The importance of being balanced"

Binary Search Tree (BST)

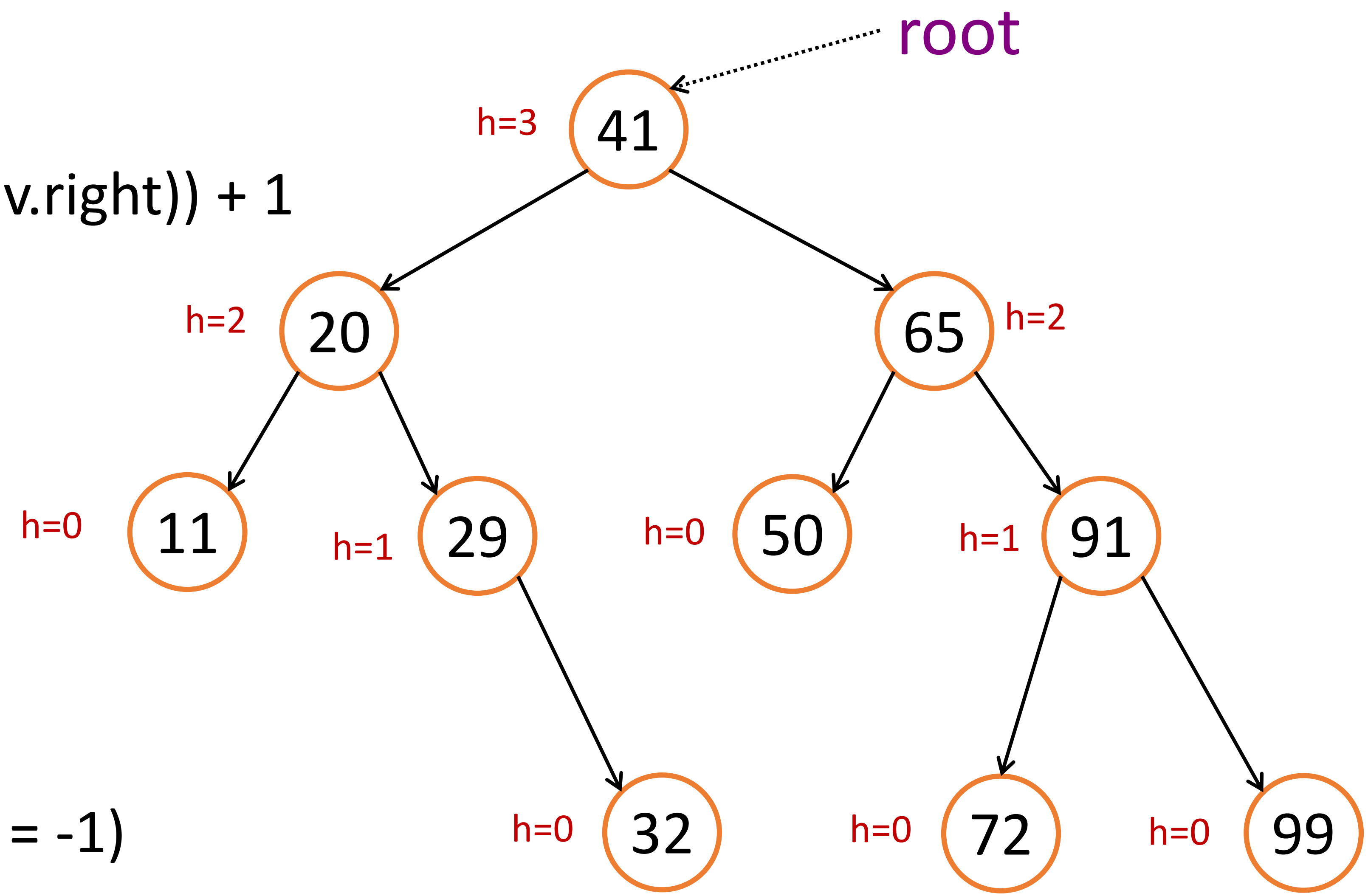
- Modifying Operations
  - insert: $O(h)$
  - delete: $O(h)$
- Query Operations:
  - search: $O(h)$
  - predecessor, successor: $O(h)$
  - findMax, findMin: $O(h)$
  - in-order-traversal: $O(n)$

Plan

- On the importance of being balanced
  - Height-balanced binary search trees
  - AVL trees

Recap: Tree Heights

- For a tree node v
- h(v) = 0 if v is a leaf
- h(v) = max(h(v.left), h(v.right)) + 1

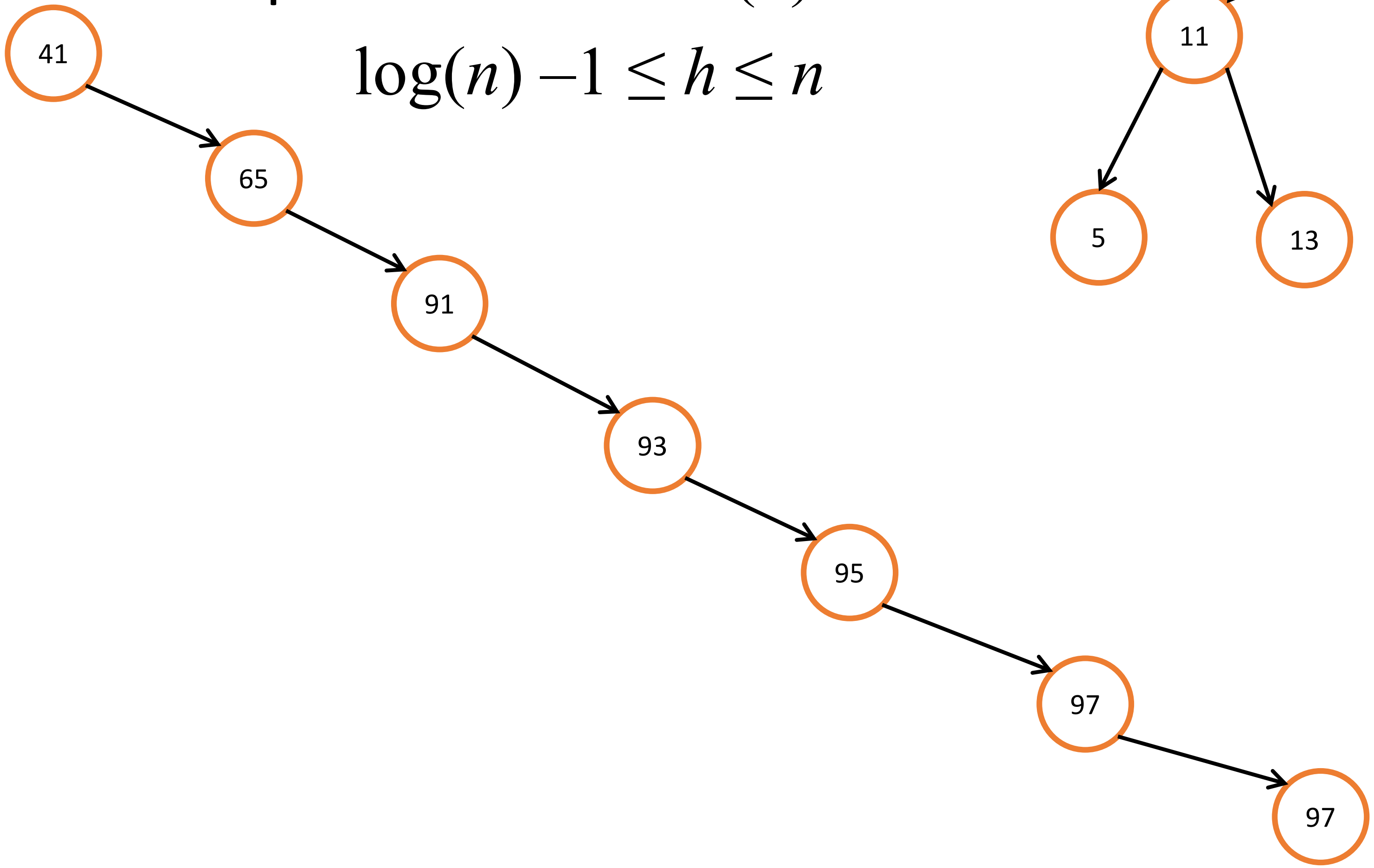


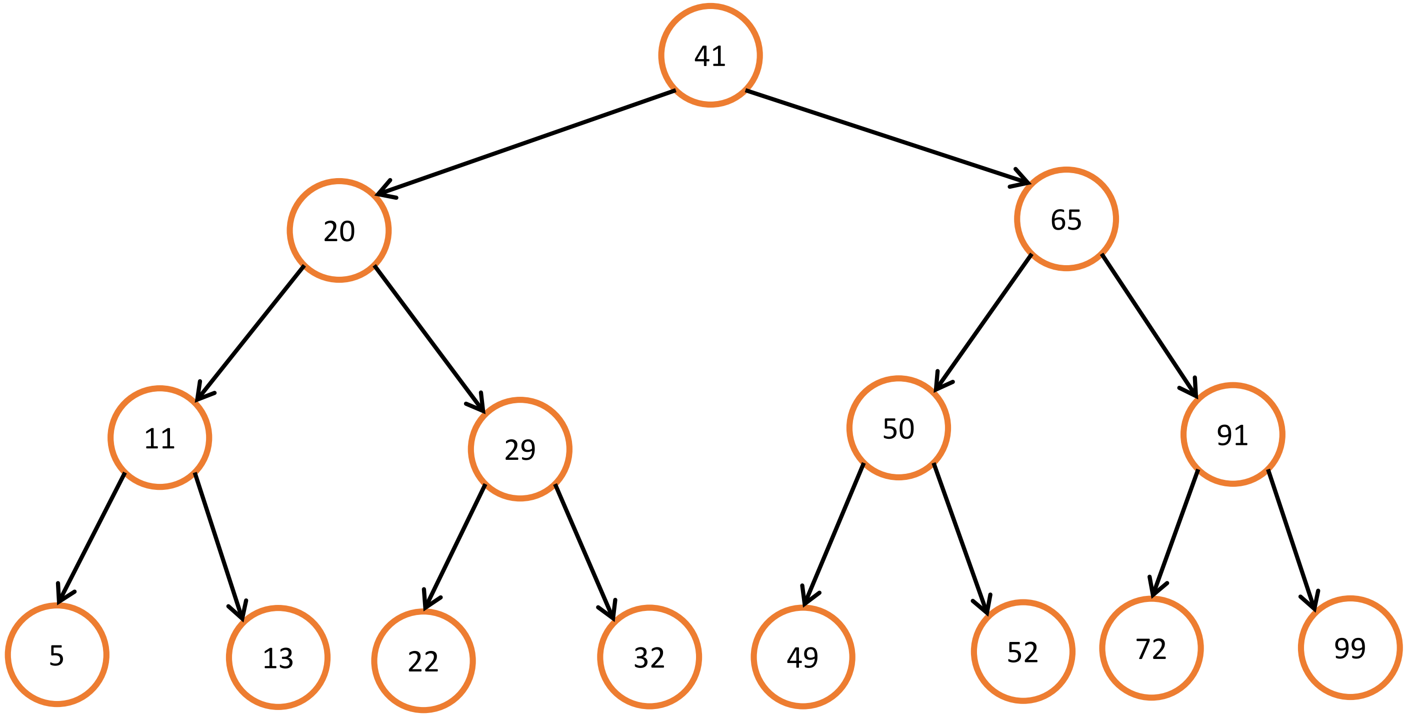
- (For simplicity: h(null) = -1)

Tree Height

- Operations take O(h) time:

$$\log(n) - 1 \le h \le n$$



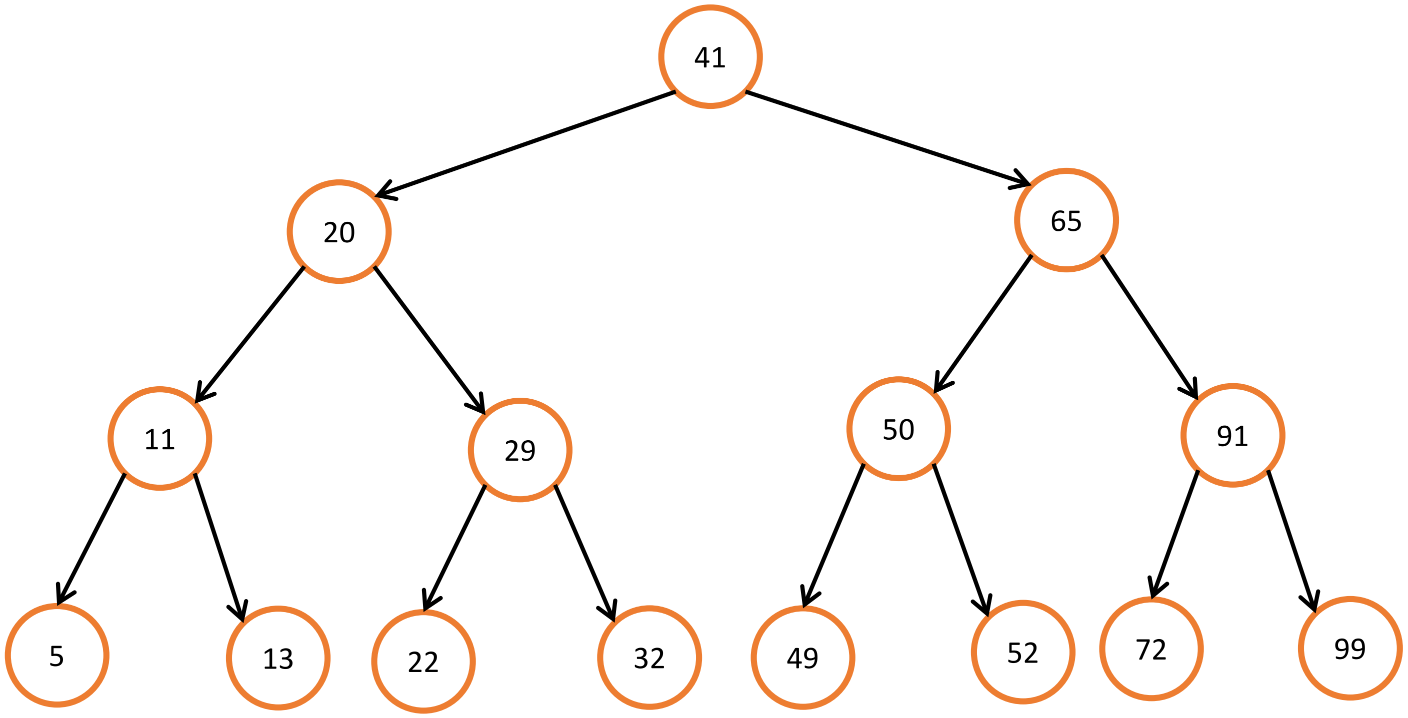


Tree Height

- Operations take $O(h)$ time:

$$\log(n) - 1 \le h \le n$$

- Operations take $O(\log n)$ time if the tree is balanced



Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)
- Scapegoat Trees (Anderson 1989)

Step 1: Augment the nodes

- In every node v, store height:

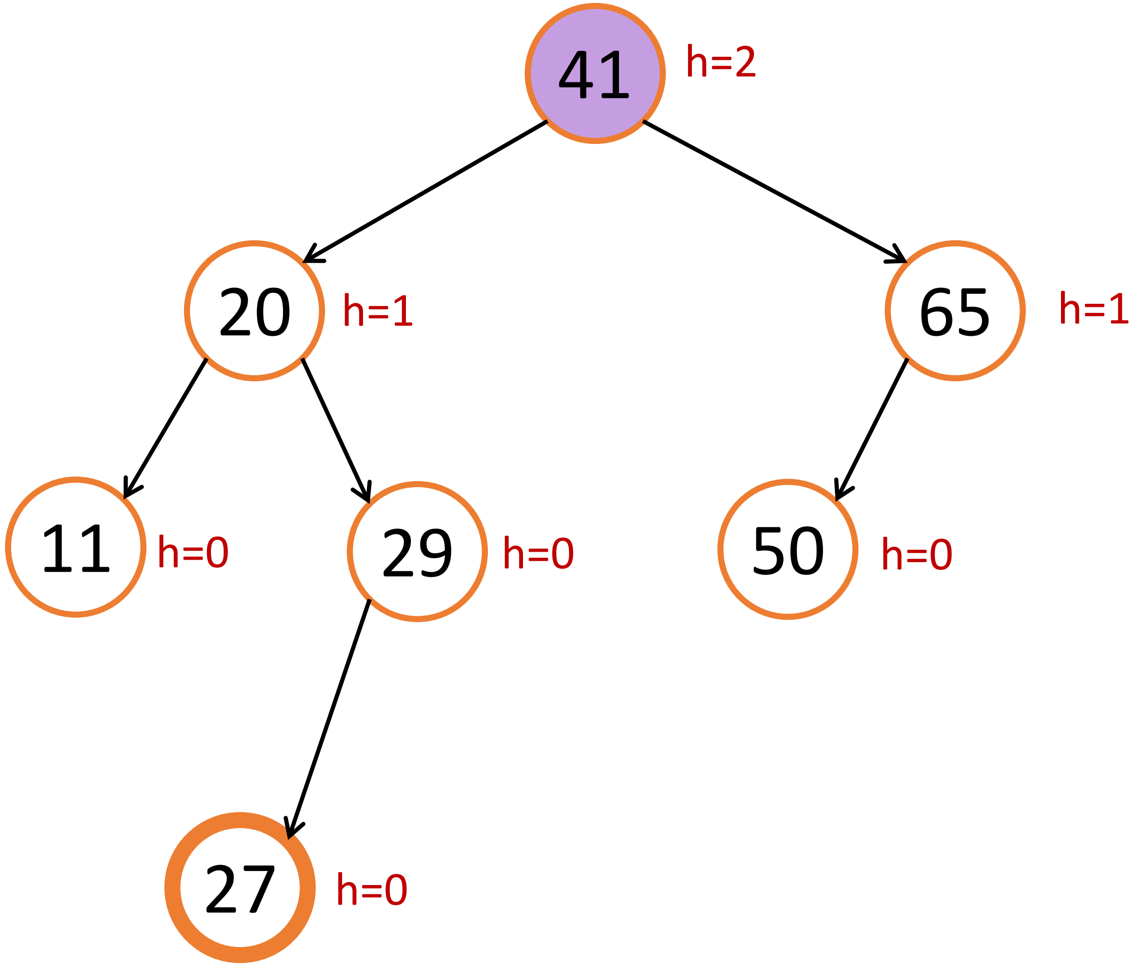
```
v.height = h(v)
```

- When we do insert/delete, we update the heights

```
insert(x)
    if (x < key)
        left.insert(x)
    else right.insert(x)
    height = max(left.height, right.height)+1
```

Example: Insert(27)

- Which node's height is not correct anymore after insertion?



Step 1: Augment the nodes

- In every node v, store height:

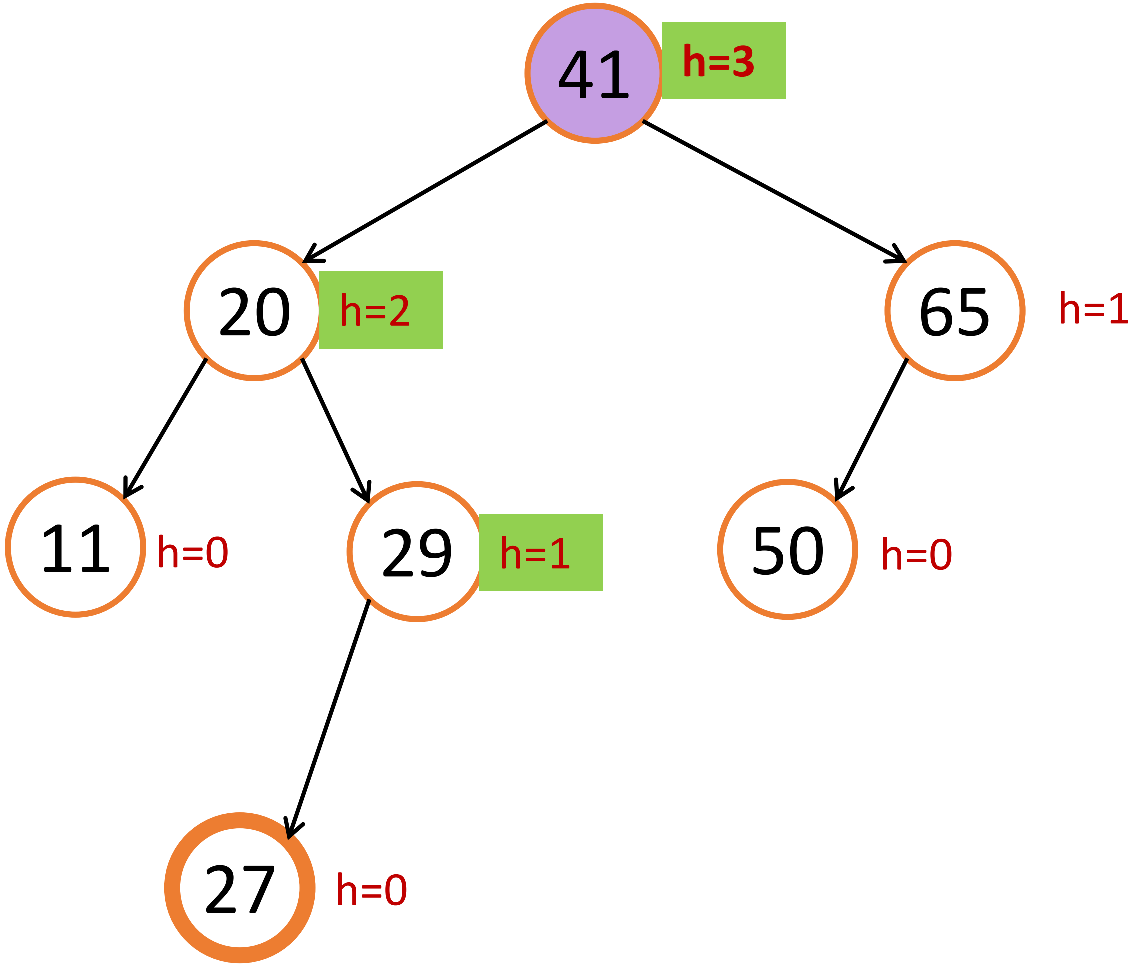
```
v.height = h(v)
```

- When we do insert/delete, we update the heights

```
insert(x)
    if (x < key)
        left.insert(x)
    else right.insert(x)
    height = max(left.height, right.height)+1
```

Example: Insert(27)

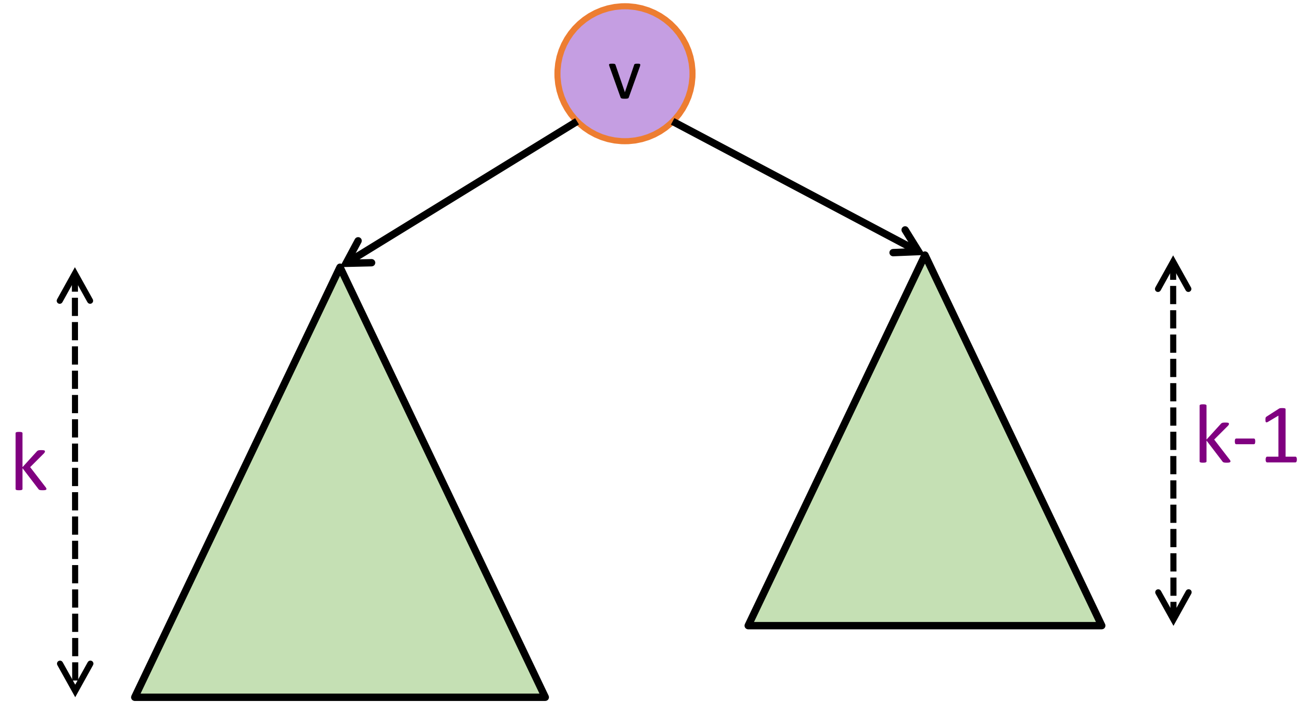
- Updating after each recursion



Step 2: Define Invariant

- AVL Trees [Adelson-Velskii & Landis 1962]
- A node v is ***height-balanced*** if:

```
abs(v.left.height - v.right.height) ≤ 1
```



Step 2: Define Invariant

- AVL Trees [Adelson-Velskii & Landis 1962]
- A node v is ***height-balanced*** if:

```
abs(v.left.height – v.right.height) ≤ 1
```

- A binary search tree is ***height-balanced*** if every node in the tree is height-balanced.

Height-Balanced Trees

- A height-balanced binary search tree with n nodes has at most height:

  $h < 2 \log n$

- But I don't know how to prove this
- Instead, I tried to prove…

That makes our tree operations $O(h) = O(\log n)$

Height-Balanced Trees

- A height balanced binary search tree with n nodes has at most height:

  $h < 2 \log n$

  $\Leftrightarrow h/2 < \log(n)$

  $\Leftrightarrow 2^{h/2} < 2^{\log(n)}$

  $\Leftrightarrow 2^{h/2} < n$

  $\Leftrightarrow$ For a tree with height h, a height-balanced tree contains at least $n > 2^{h/2}$ nodes

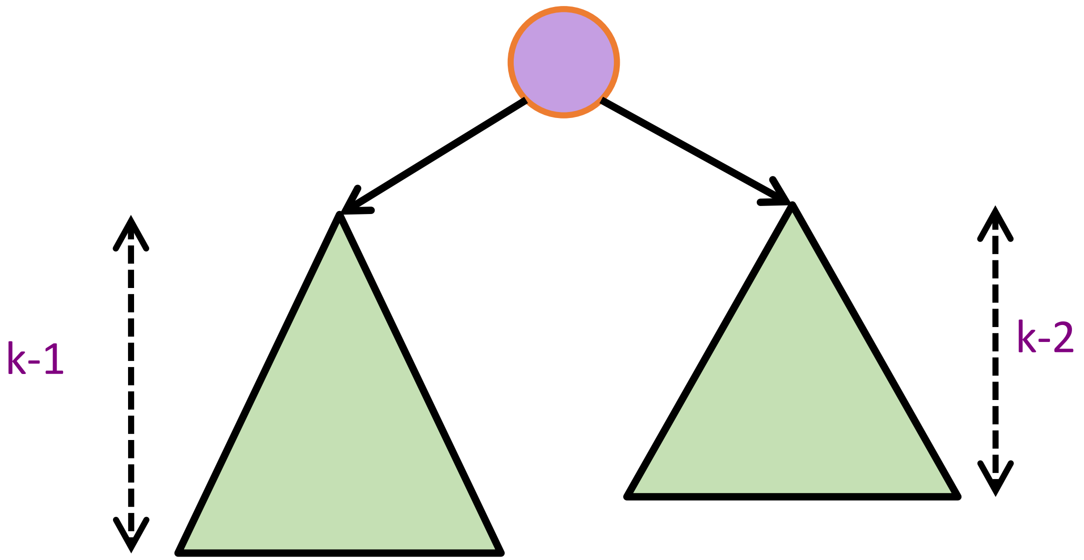
Proof of Height Balanced Tree h = O(log n)

- Claim:
  - ~~A height-balanced tree is balanced, i.e., has height $h = O(\log n)$.~~
  - For a tree with height h, a height-balanced tree contains at least $n > 2^{h/2}$ nodes
- Prove by Induction:
  - Denote as n_h the minimal no. of nodes of a height-balanced tree of height h
  - Assuming it is true for $h < k$.
    - Namely, $n_h > 2^{h/2}$ for $h < k$
  - Base case:
    - For $h = 0$, $n_h = 1$, it is a perfectly balanced tree with one node

Proof of Height Balanced Tree h = O(log n)

- Claim:
  - For a tree with height h, a height-balanced tree contains at least $n > 2^{h/2}$ nodes
- Prove by Induction:
  - Assuming it is true for $h < k$.
  - For $h = k$,

$$
\begin{aligned}
n_k &\geq 1 + n_{k\text{-}1} + n_{k\text{-}2} \\
&\geq 2\, n_{k\text{-}2} \\
&\geq 2 \times 2\, n_{k\text{-}4} \\
&\geq 2 \times 2 \times 2\, n_{k\text{-}6} \\
&\geq 2 \times 2 \times 2 \times 2\, n_{k\text{-}8}
\end{aligned}
$$



Proof of Height Balanced Tree h = O(log n)

- For $h = k$, the minimal no. of node will be at least

$$
\begin{aligned}
n_k &\geq 1 + n_{k\text{-}1} + n_{k\text{-}2} \geq \ 2\ n_{k\text{-}2} \\
&\geq \ 2 \times 2\ n_{k\text{-}4} \\
&\geq \ 2 \times 2 \times 2\ n_{k\text{-}6} \\
&\geq \ 2 \times 2 \times 2 \times 2\ n_{k\text{-}8} \\
&\geq \ 2^i\ n_{k\text{-}2i} \\
&\geq \ 2^{k/2}\ n_0 = 2^{k/2}
\end{aligned}
$$

- For $h = k$, $n_k \geq 2^{k/2}$

Height-Balanced Trees

- A height balanced binary search tree with n nodes has at most height:

$$h < 2\log n$$

$$\Leftrightarrow h/2 < \log(n)$$

$$\Leftrightarrow 2^{h/2} < 2^{\log(n)}$$

$$\Leftrightarrow 2^{h/2} < \mathrm{n}$$

$\Leftrightarrow$ For a tree with height h, a height-balanced tree contains at least $n > 2^{h/2}$ nodes

Proven

Height-Balanced Trees

- A height balanced binary search tree with n nodes has at most height $h < 2 \log n = \mathrm{O}(\log n)$

Q.E.D.

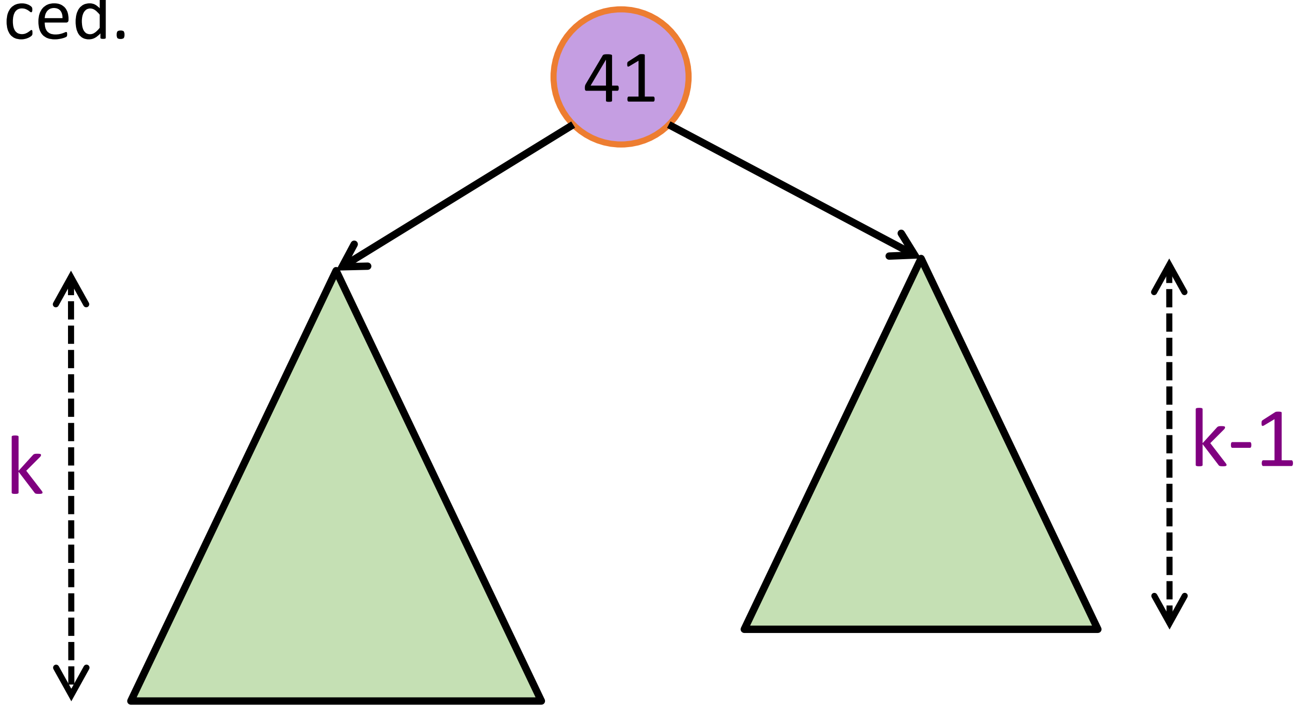
Step 3: How to Maintain Height-balance

Step 3: How to Maintain Height-balance

- A node v is ***height-balanced*** if:

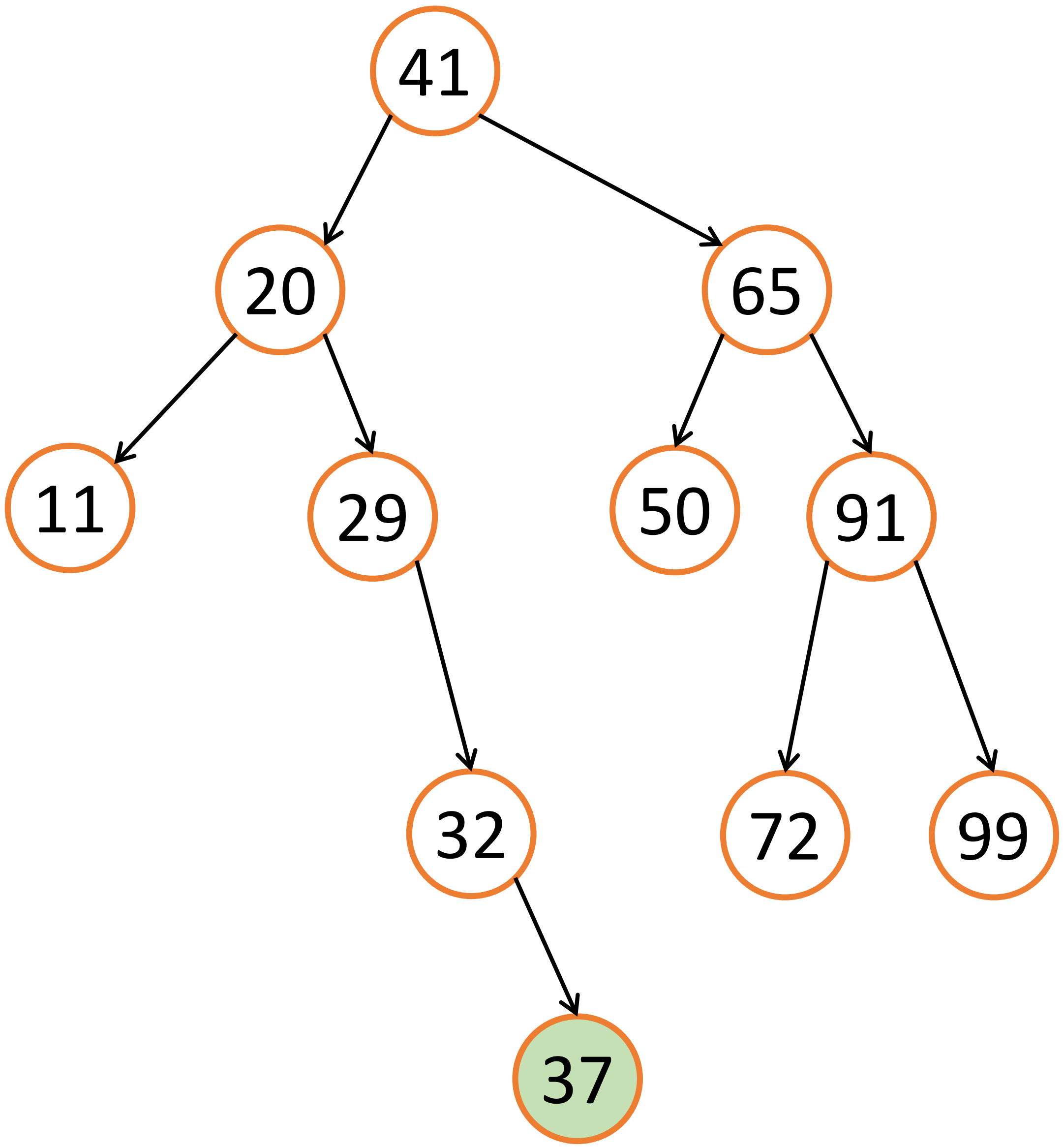
```
abs(v.left.height – v.right.height) ≤ 1
```

- A binary search tree is ***height balanced*** if every node in the tree is height-balanced.



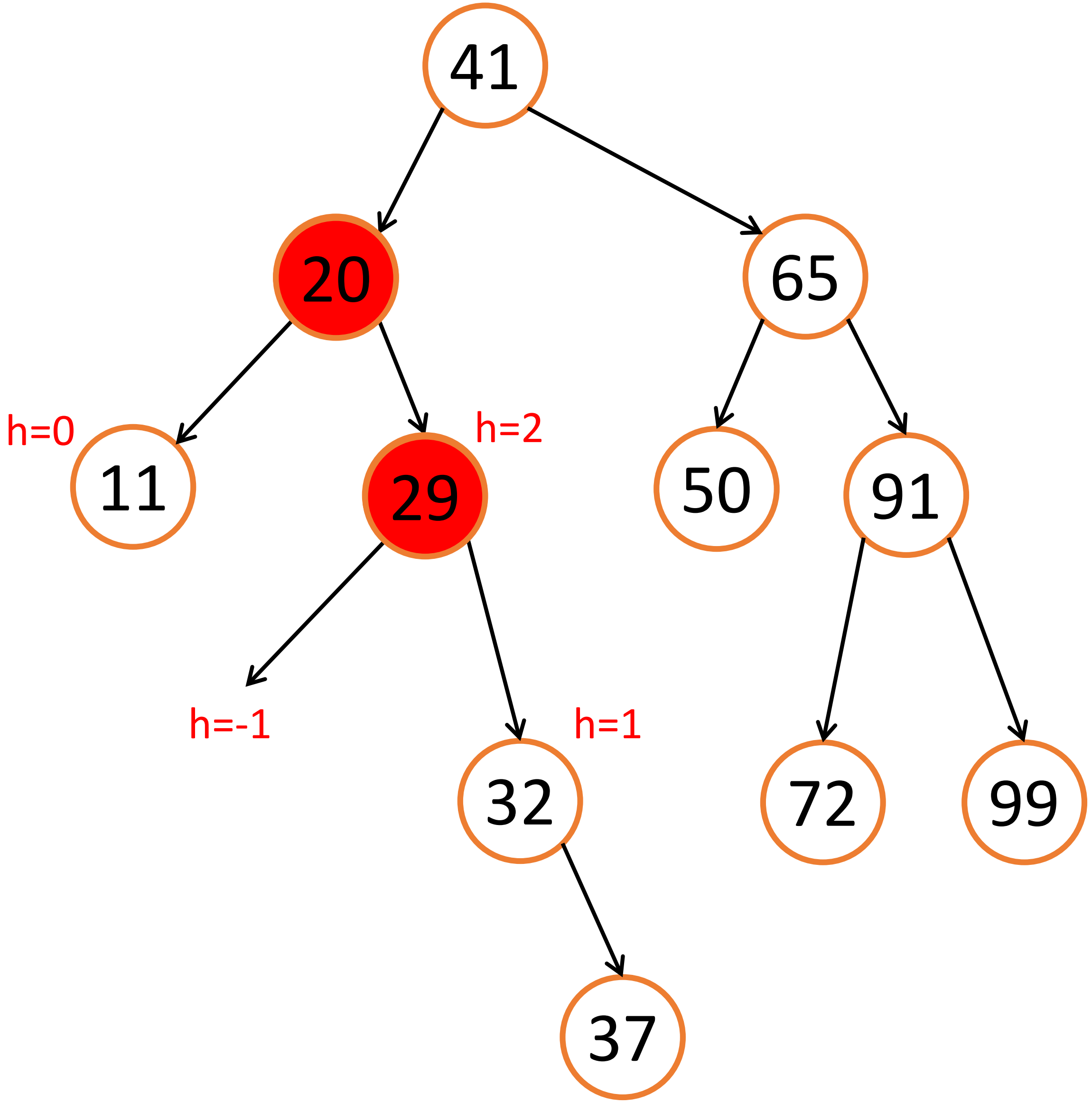
Example: Insert(37)

- Before insertion, the tree is balanced
- But not anymore after inserting 37
- Need to do something to make it balanced



Example: Insert(37)

- What are the nodes that are NOT balanced?

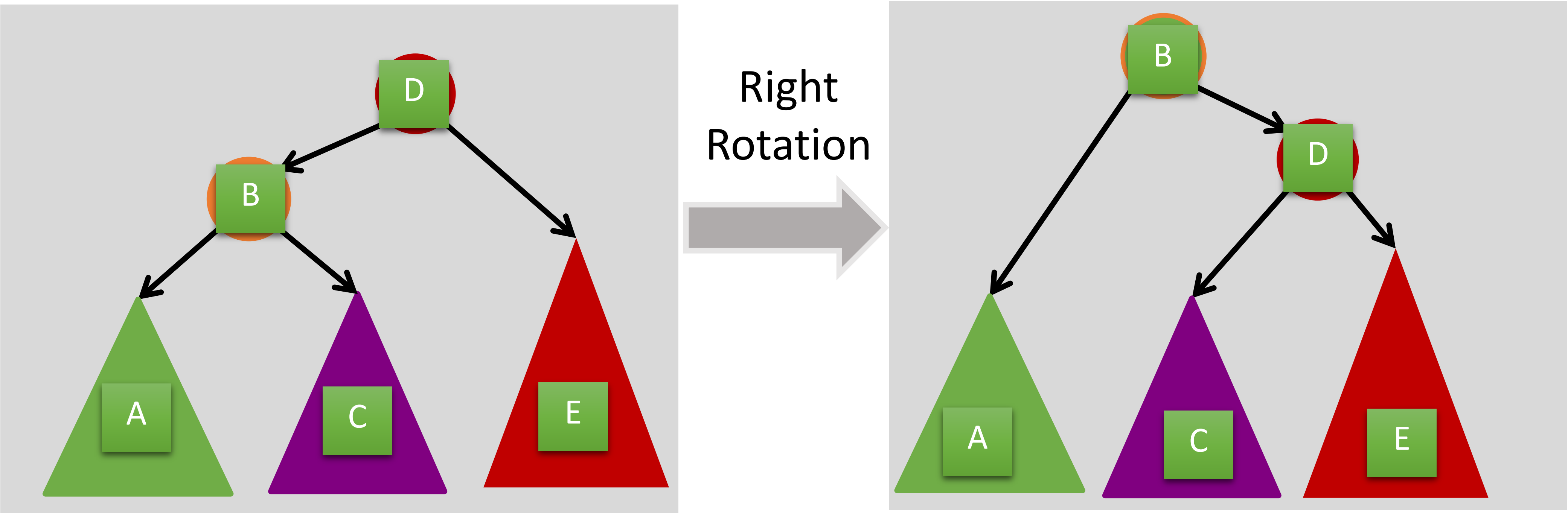


Tricks to Rebalance the Tree

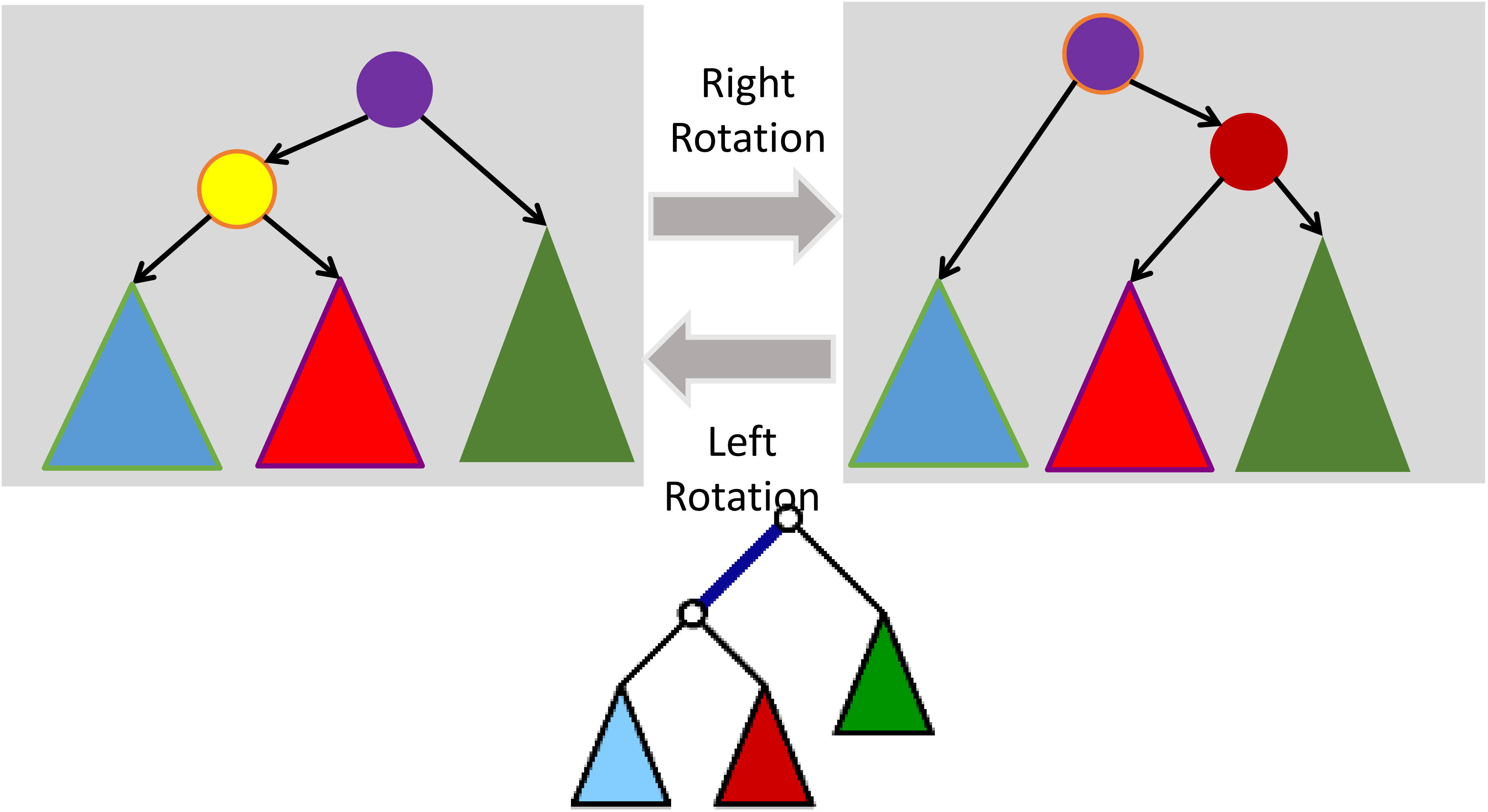
- Tree Rotation!

Tree Rotation

- A < B < C < D < E
- Rotations maintain ordering of keys. ⇒ Maintains BST property.

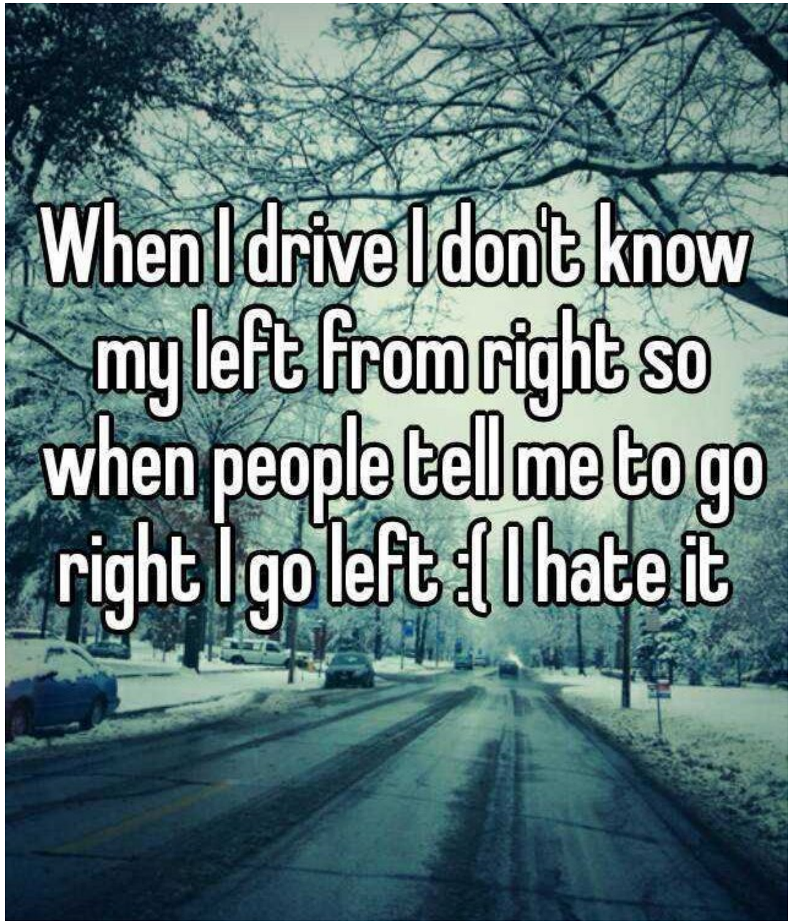


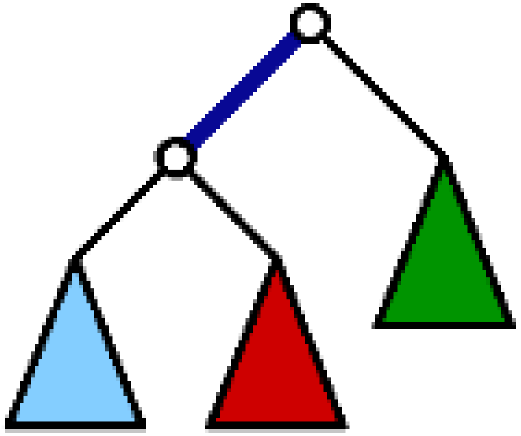
Left and Right Tree Rotations



I cannot tell the left from the right

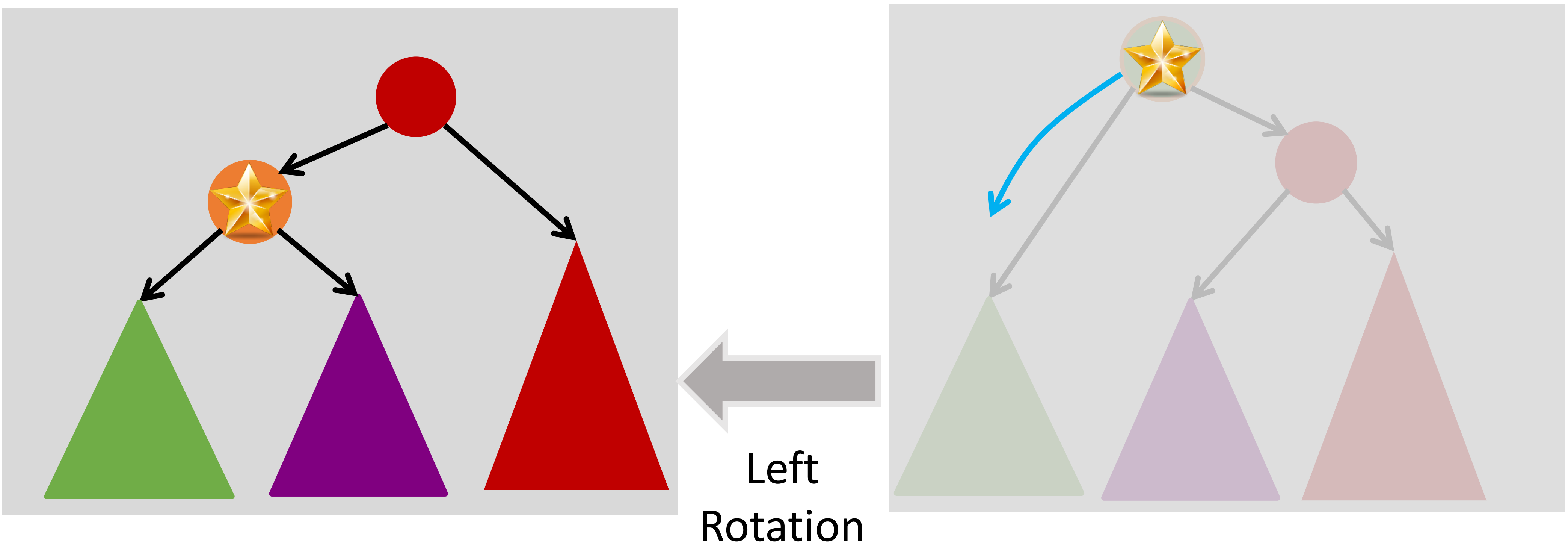
- Wait, what is a left rotation and what is a right rotation?





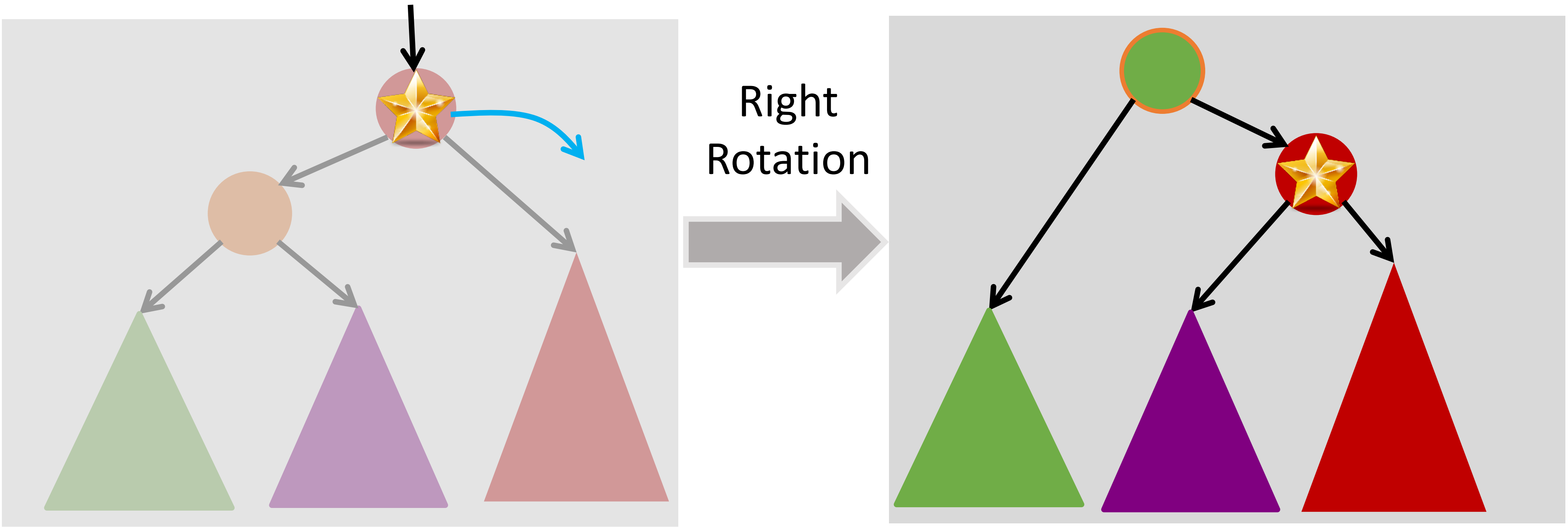
My way to remember it

- Left rotation = The original root of the subtree moves left



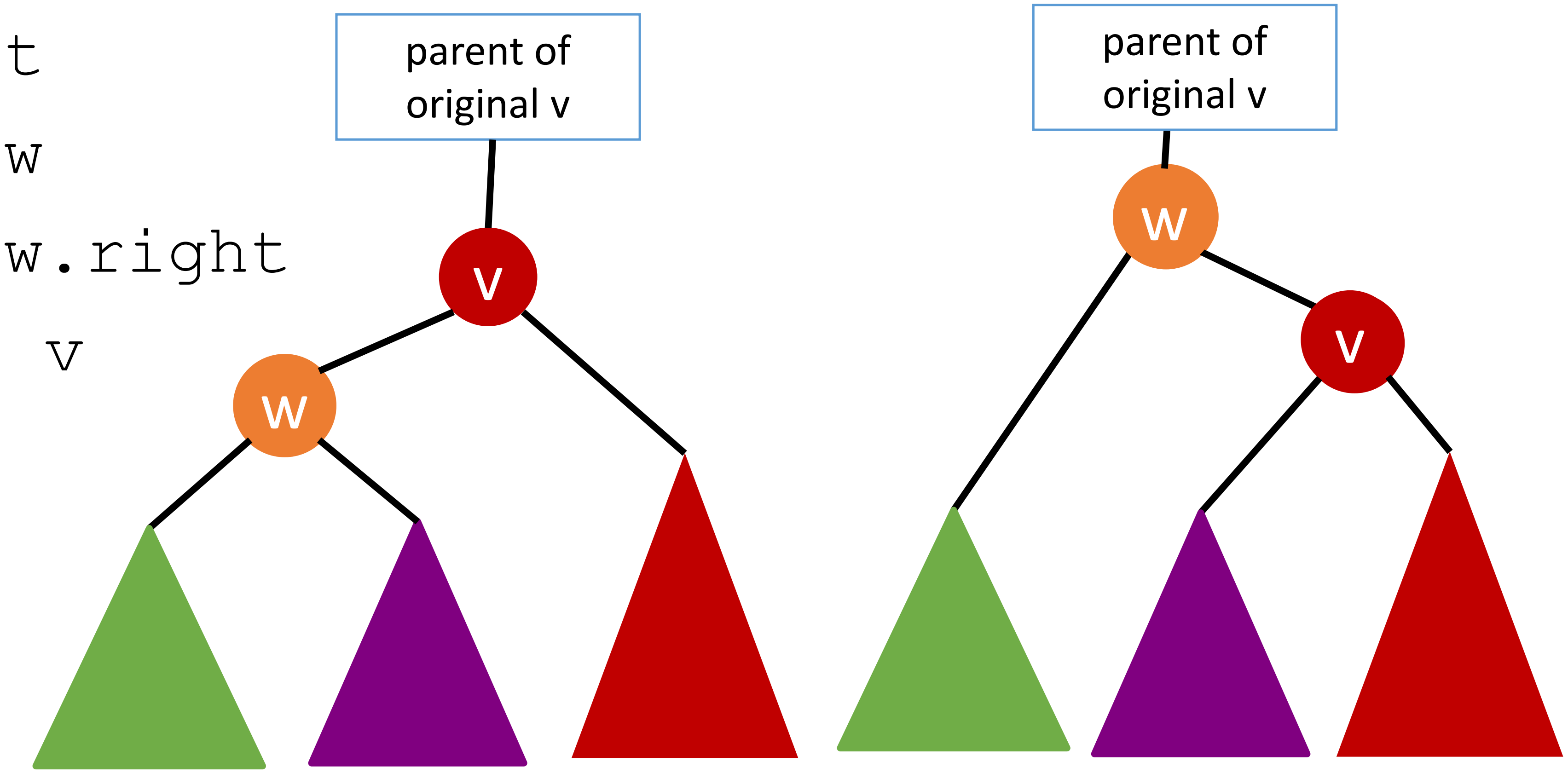
My way to remember it

- Right rotation = The original root of the subtree moves right



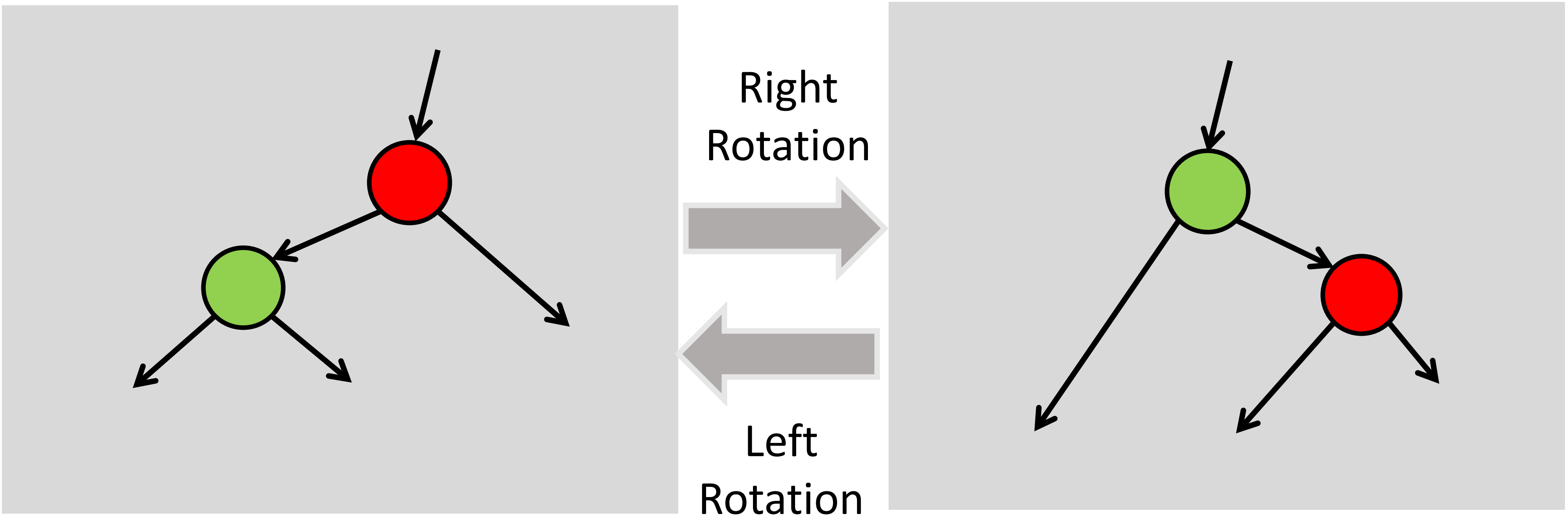
Right Rotation

```
right-rotate(v)  // assume v has left != null
    Let the parent of v is the pointer "parent"
    w = v.left
    parent = w
    v.left = w.right
    w.right = v
```



Children Requirements

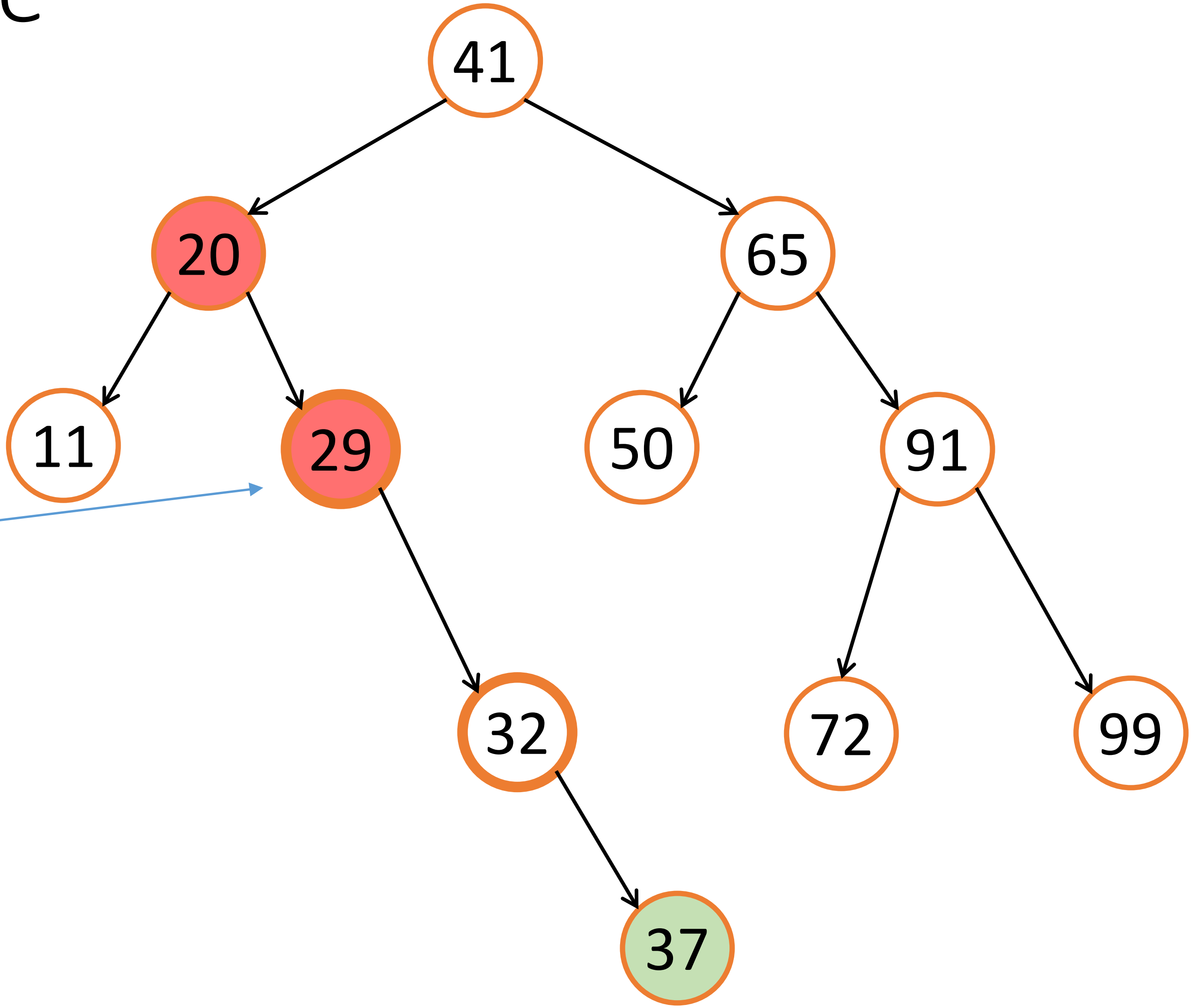
- rotate-right requires a left child
- rotate-left requires a right child



Balancing by Rotations

Insertion in AVL Tree

- After inserting 37, the tree is no longer balanced
- A node that is out of balance can be
  - left heavy or
  - right heavy



If a Node is Left Heavy and Needs Balancing

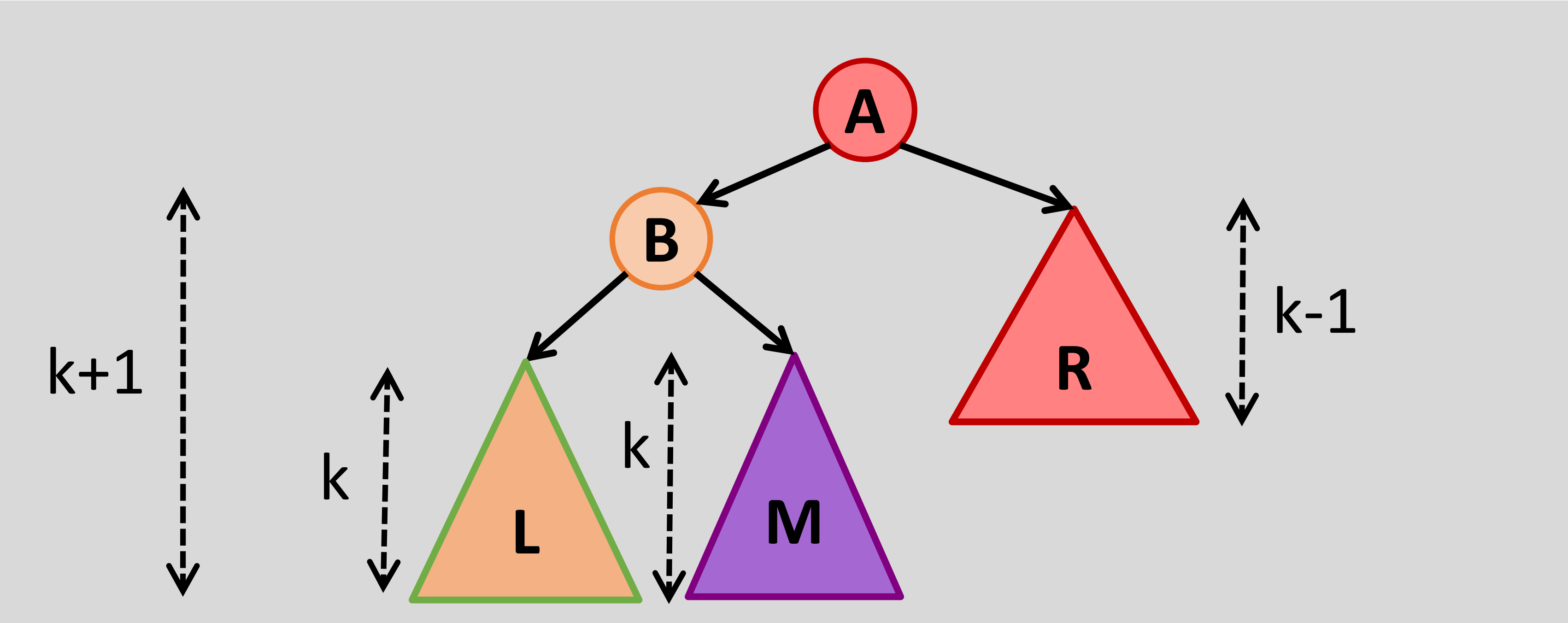
```
If v is out of balance and left heavy:
   v.left is balanced: right-rotate(v)
   v.left is left-heavy: right-rotate(v)
   v.left is right-heavy:
             left-rotate(v.left) then right-rotate(v)
```

- Or

```
If v is out of balance and left heavy:
   if v.left is right-heavy: left-rotate(v.left)
   right-rotate(v)
```

Proof of Correctness (A is left heavy)

- Assume A is the **lowest** node in the tree violating balance property

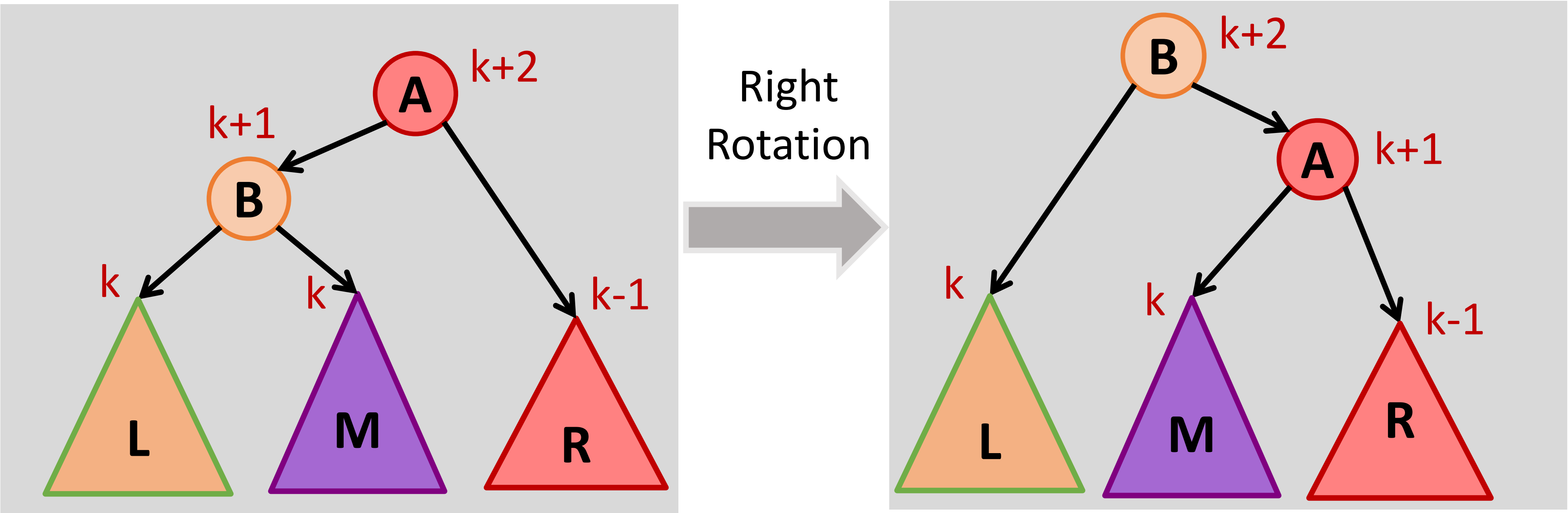


- Case 1: B is balanced

Note: The total height of the subtree doesn't change

Case 1: A is left-heavy, B is balanced

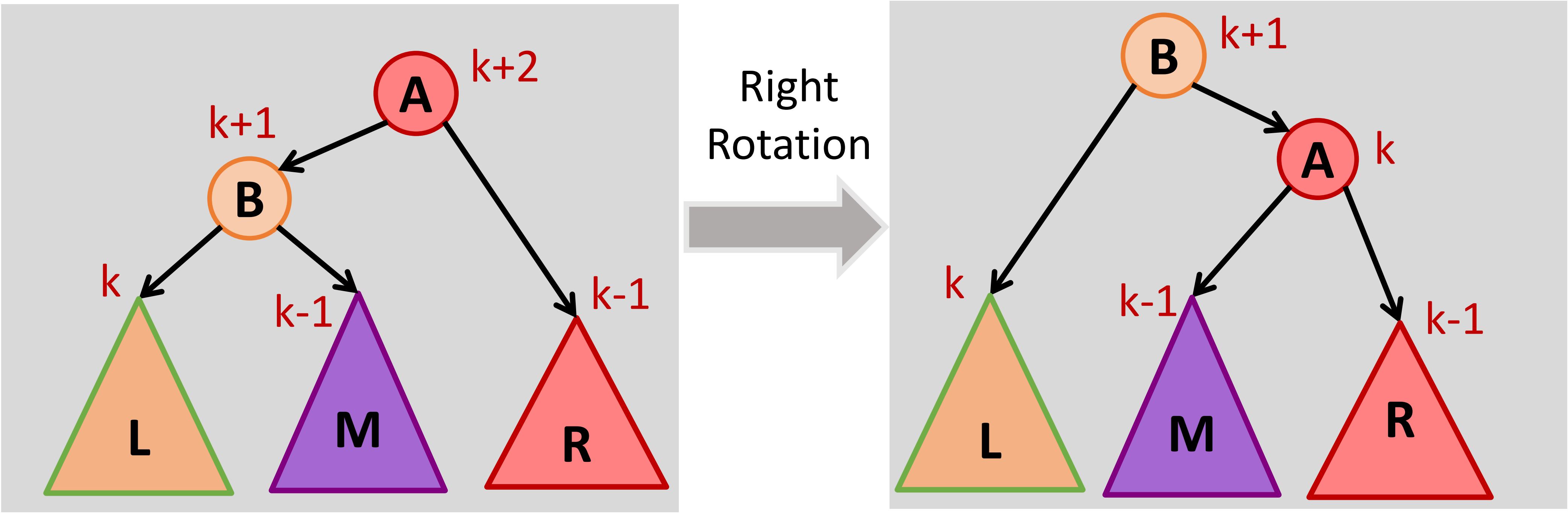
- After right-rotate(A)



Note: The total height of the subtree reduced by 1

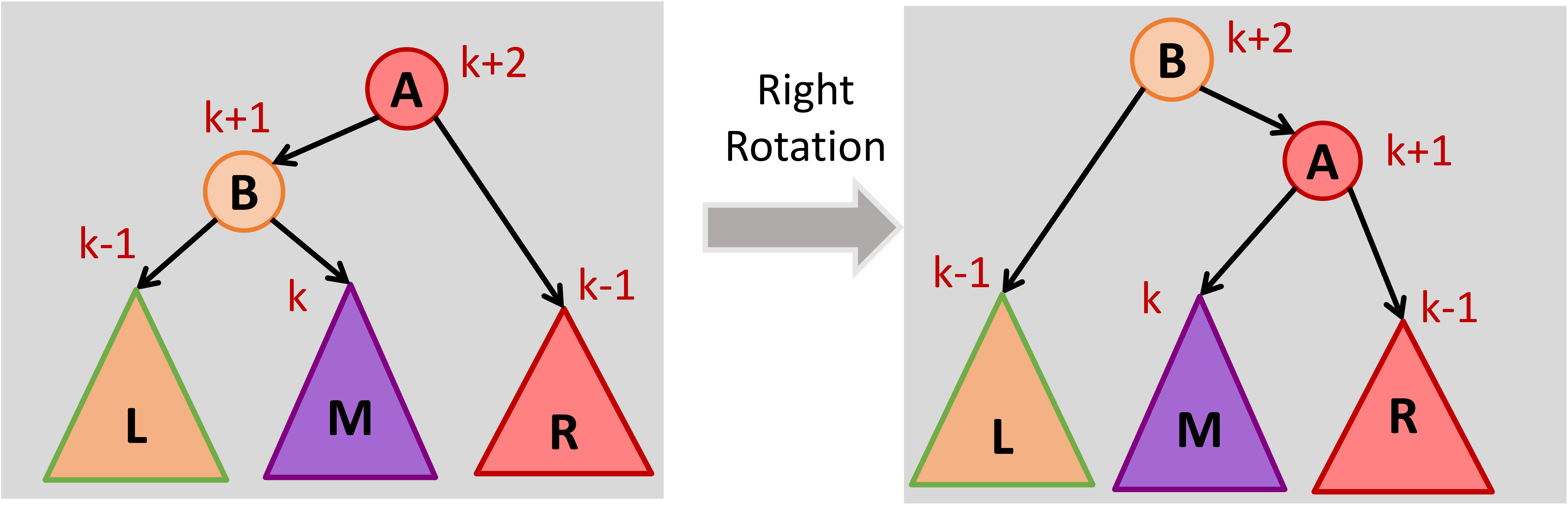
Case 2: A is left-heavy, B is left-heavy

- After right-rotate(A)



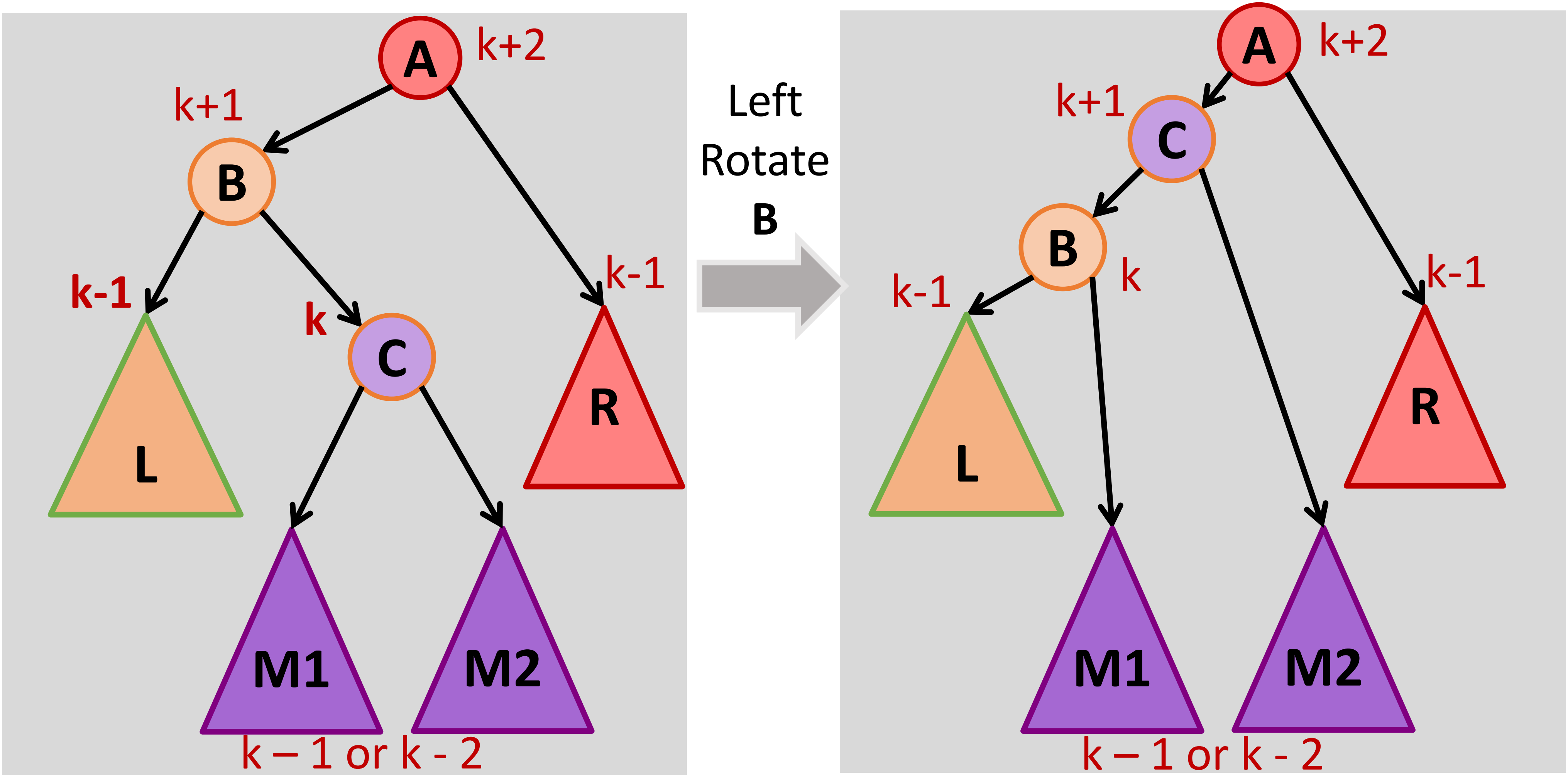
Case 3: A is left-heavy, B is right-heavy

- Just do a right-rotate(B)?



Case 3: A is left-heavy, B is right-heavy

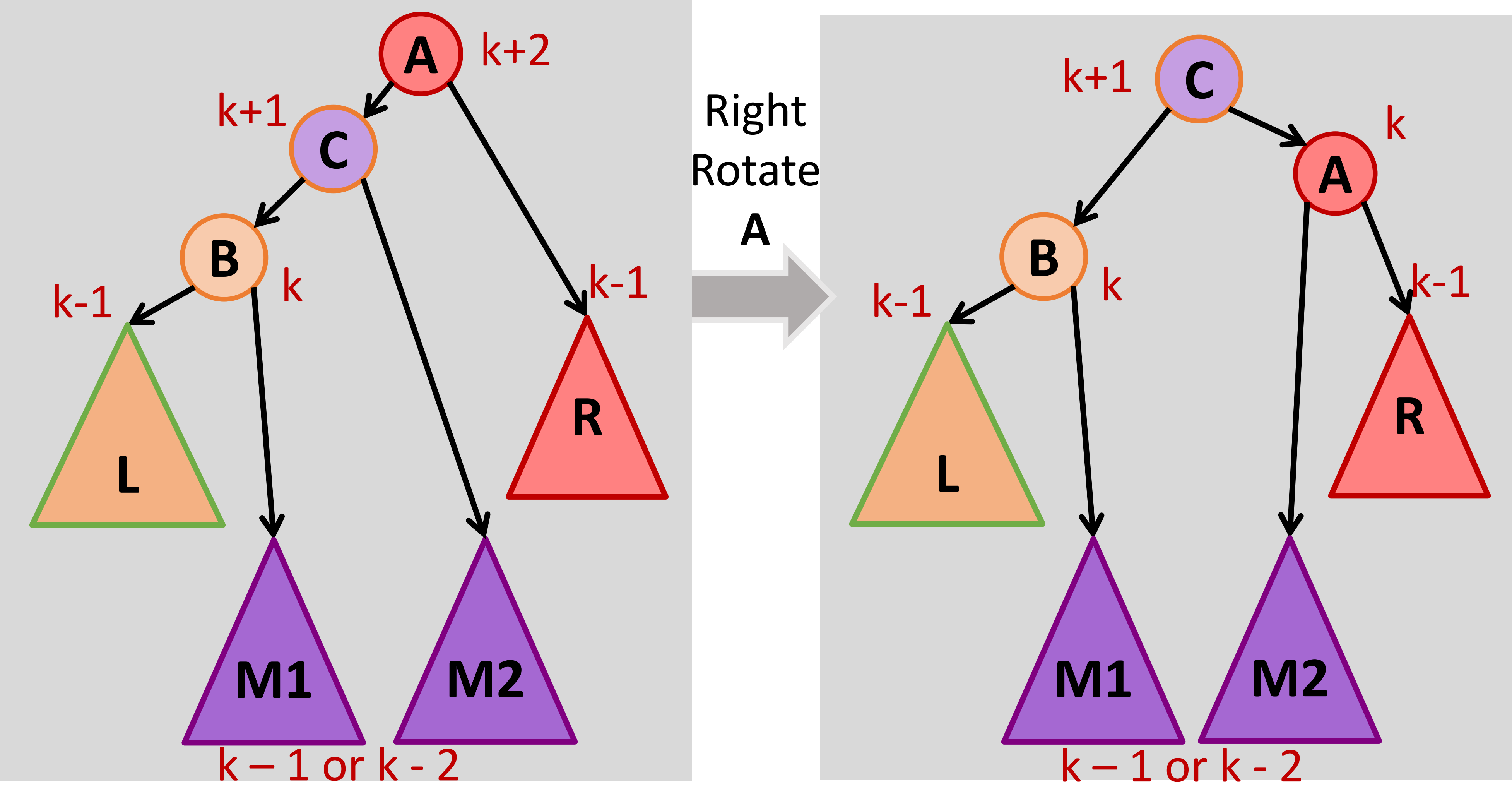
- Left-rotate(B) first



Note: The total height of the subtree decreased by 1

Case 3: A is left-heavy, B is right-heavy

- Then right-rotate(A)



If a Node is Left Heavy and Needs Balancing

```
If v is out of balance and left heavy:
   v.left is balanced: right-rotate(v)
   v.left is left-heavy: right-rotate(v)
   v.left is right-heavy:
            left-rotate(v.left) then right-rotate(v)
```

- Or

```
If v is out of balance and left heavy:
   if v.left is right-heavy: left-rotate(v.left)
   right-rotate(v)
```

If a Node is Right Heavy and Needs Balancing

```
If v is out of balance and right heavy:
   v.right is balanced: left-rotate(v)
   v.right is right-heavy: left-rotate(v)
   v.right is left-heavy:
            right-rotate(v.right) then left-rotate(v)
```

- Or

```
If v is out of balance and right heavy:
   if v.right is left-heavy: right-rotate(v.right)
   left-rotate(v)
```

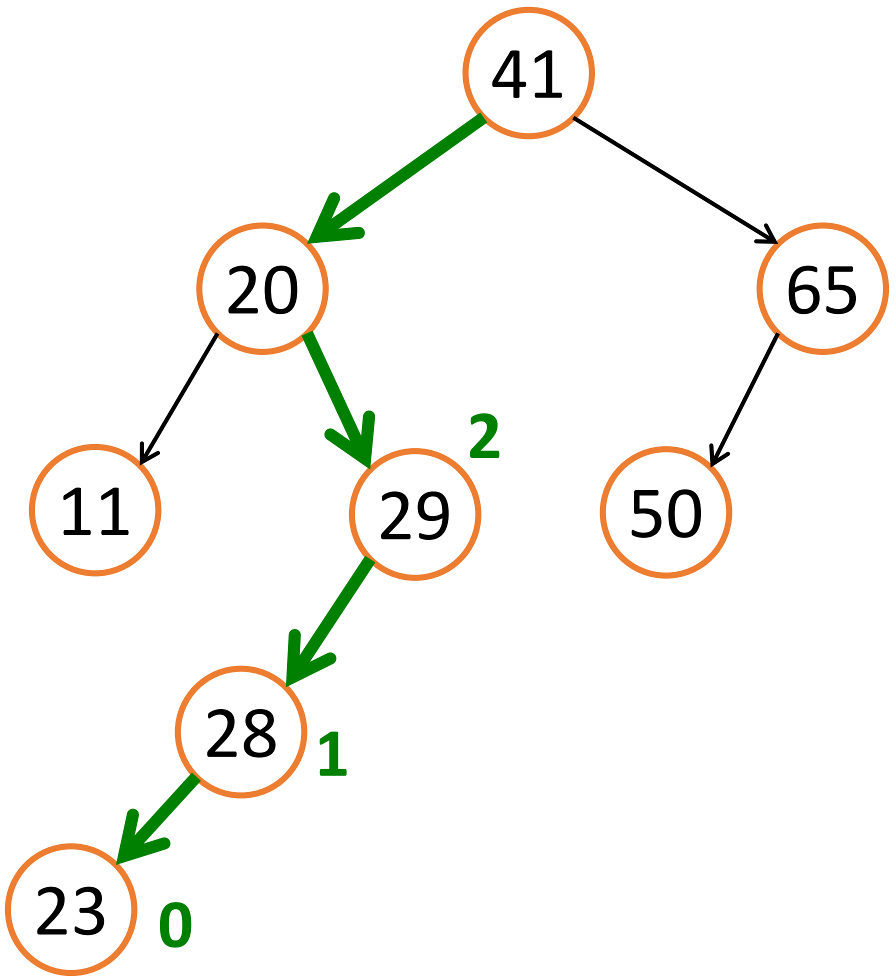
AVL Tree Insertion with Balancing

- Steps:
  - Insert key in BST.
  - Walk up tree:
    - At every step, check for balance.
    - If out-of-balance, use rotations to rebalance.

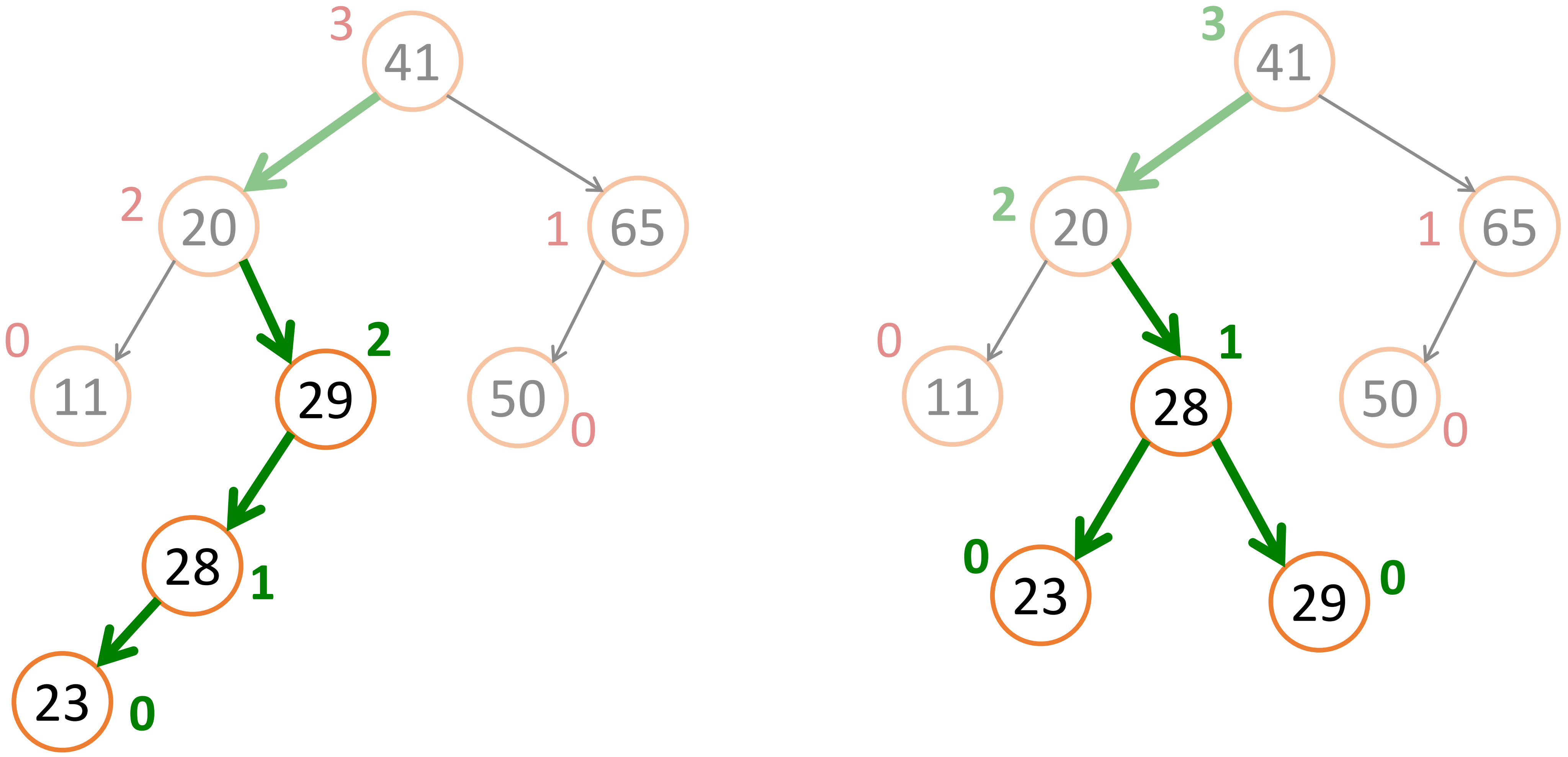
- Note: only need to perform at most two rotations
  - Why?
  - In each case, reduce height of sub-tree by 1
  - What about Case 1, above?

Example: Insert(23)

- Which node should we rotate?
- Out of the 3 cases, which case is it?
  - Case2: Left heavy and Left.left heavy
- Just do one right-rotate!

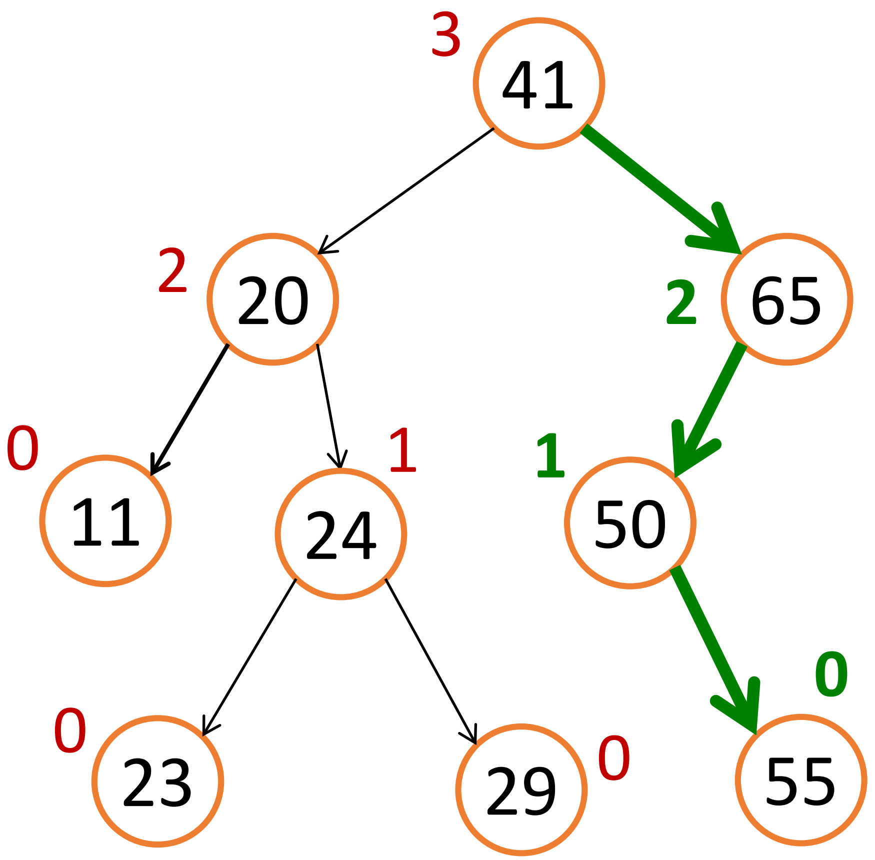


right-rotate(29)

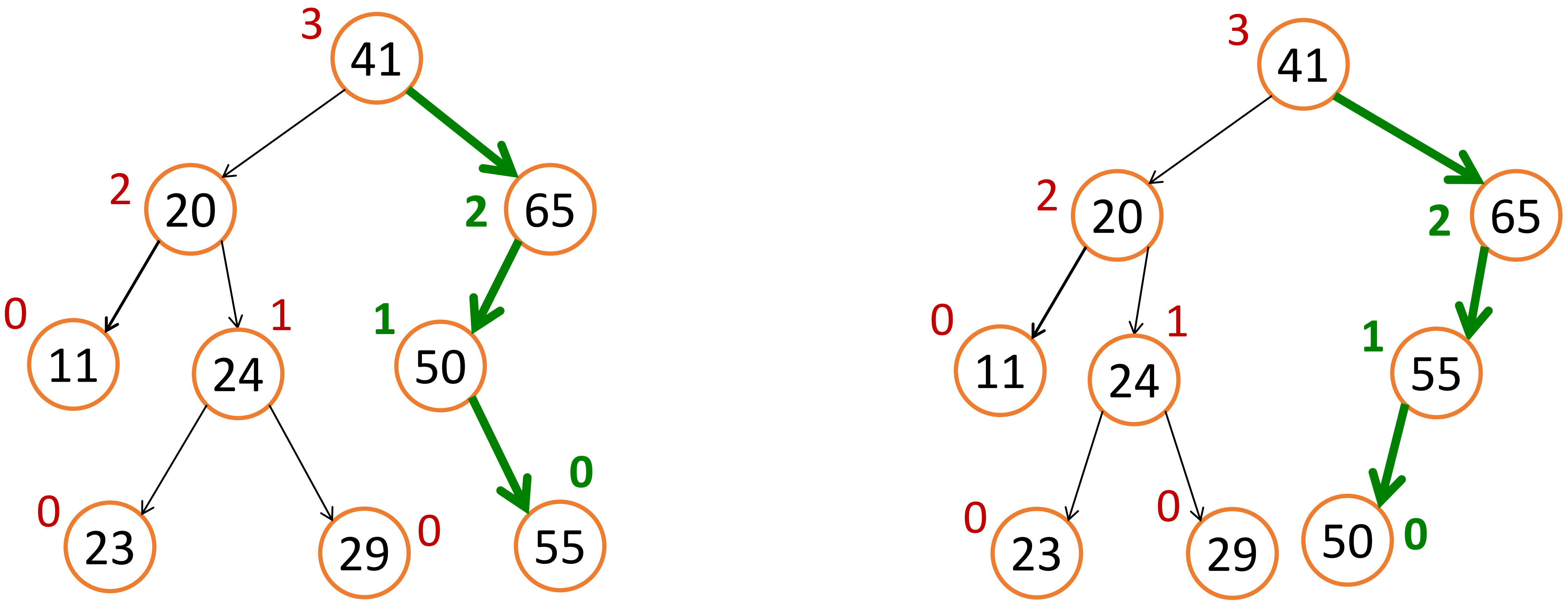


Another Example: Insert(55)

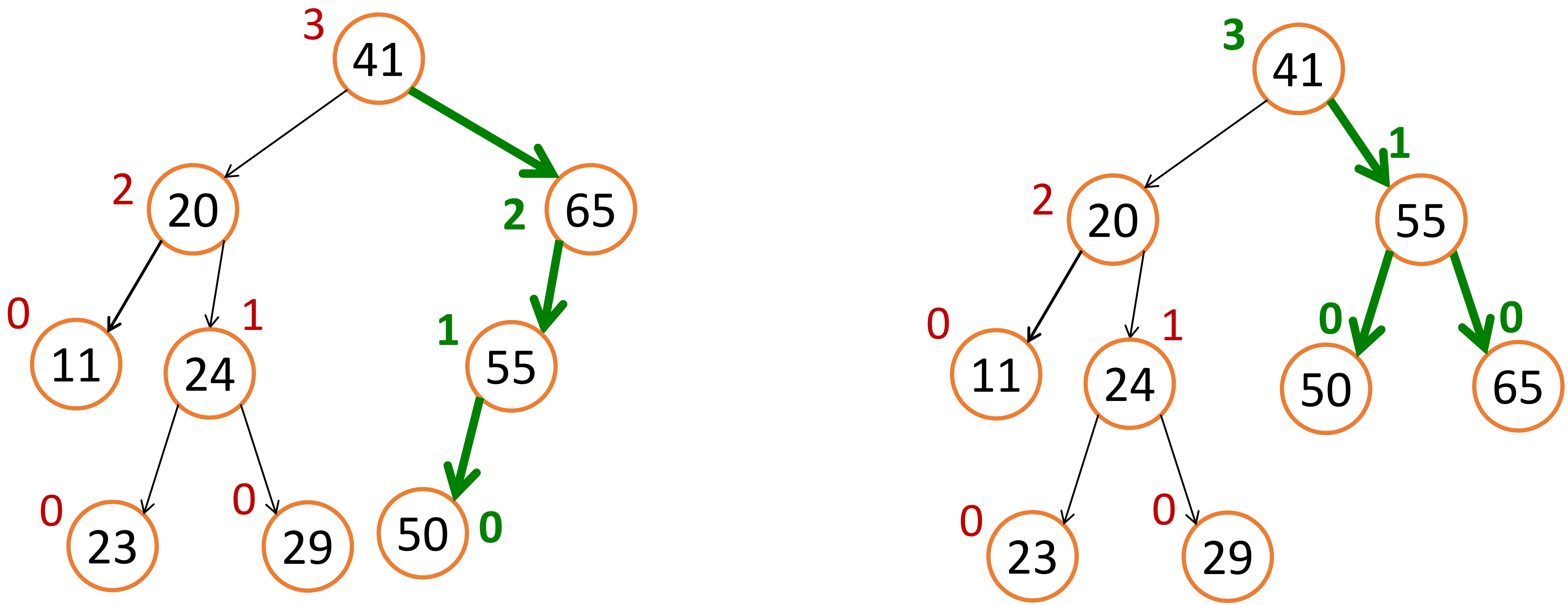
- Which node should we rotate?
- Out of the 3 cases, which case is it?
  - Case3: Left heavy and Left.right heavy
- Do two rotations!
  - left-rotate(50)
  - right-rotate(65)



left-rotate(50)



right-rotate(65)

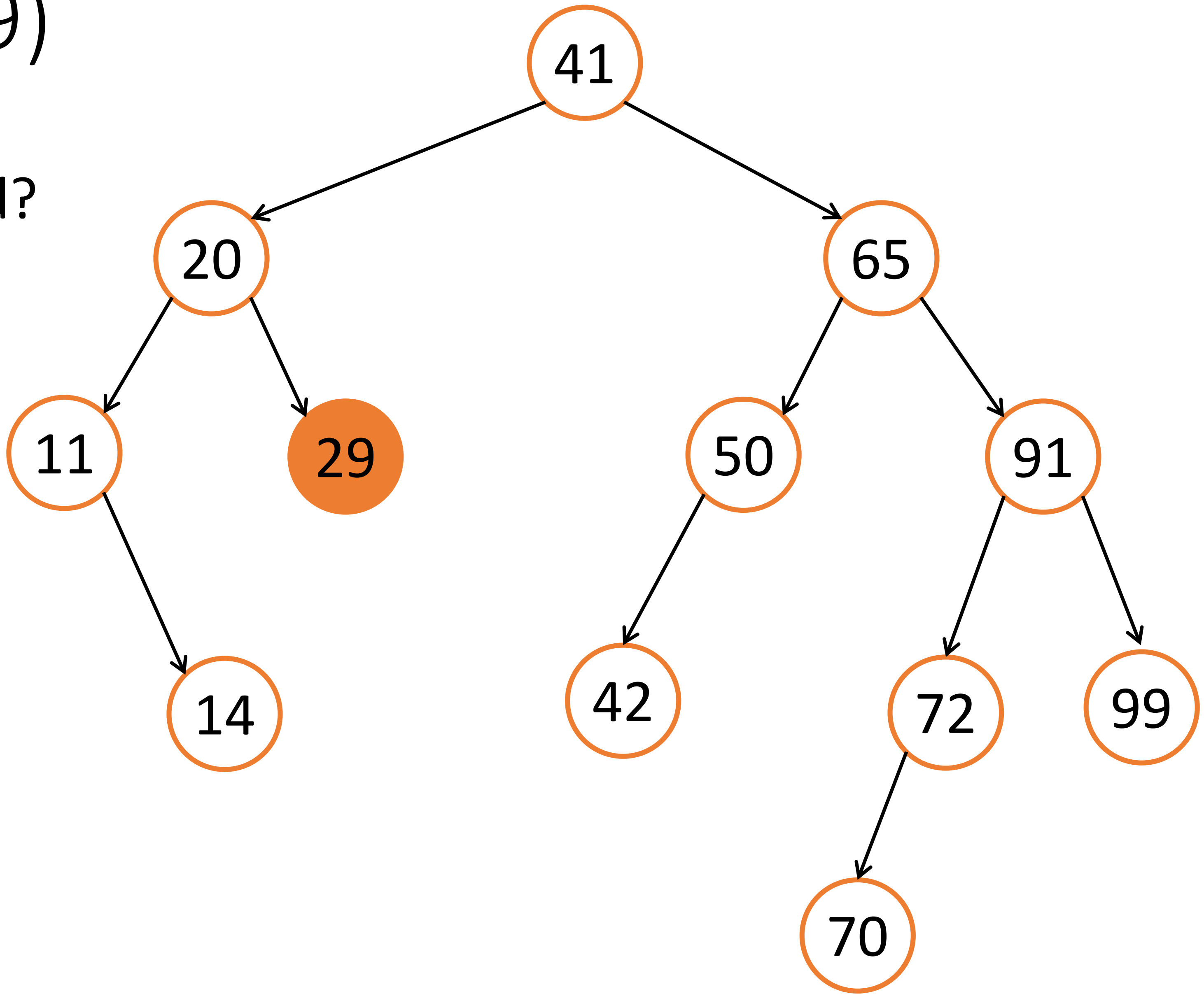


How about balancing after deletion?

- From the deleted note, walk up to the root to check every parent if they are balance
- If not balanced, perform the balancing like insertion

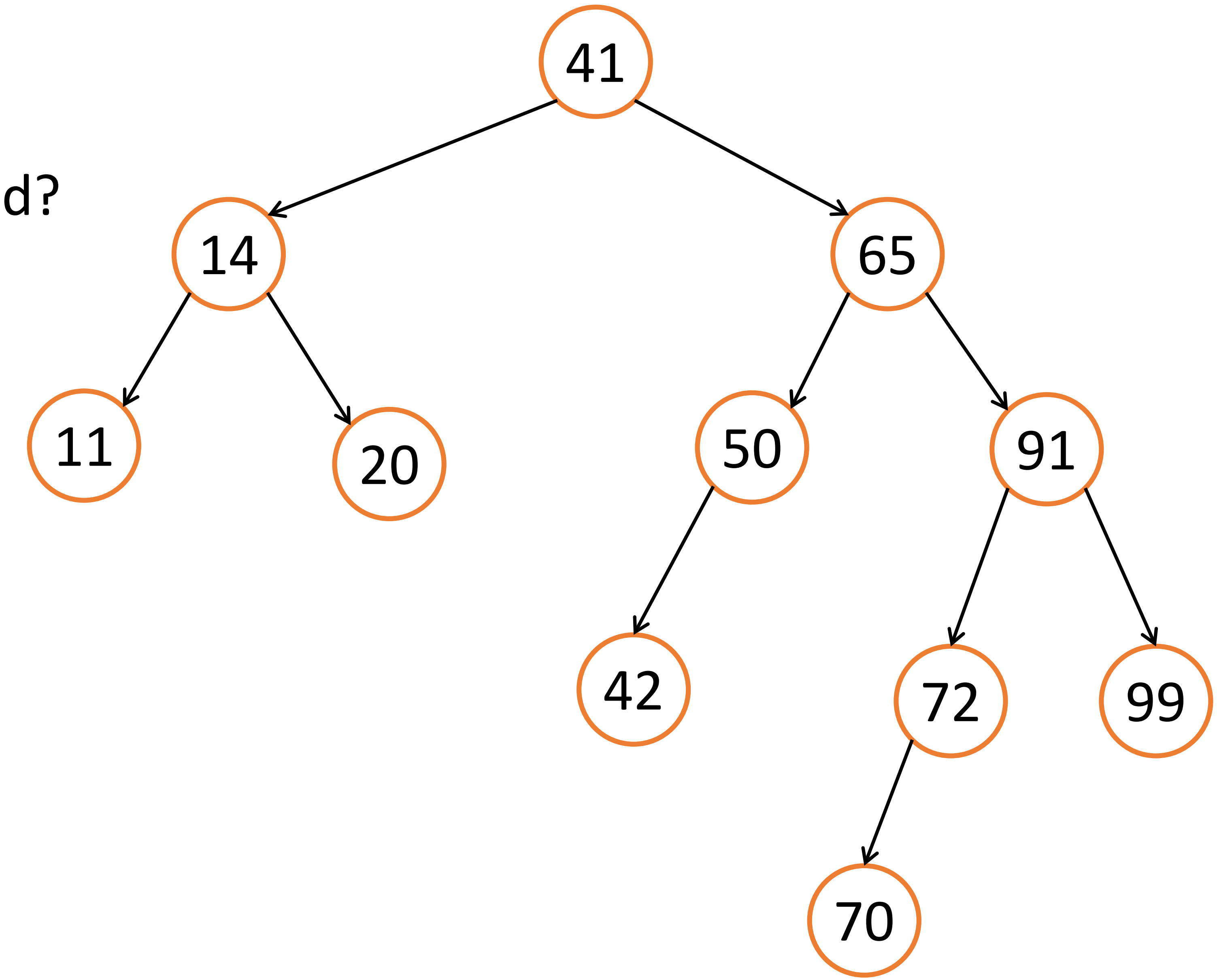
Example: delete(29)

- Which node is not balanced?
- Which case is that?
  - Case 3
- left-rotate(11)
- right-rotate(20)



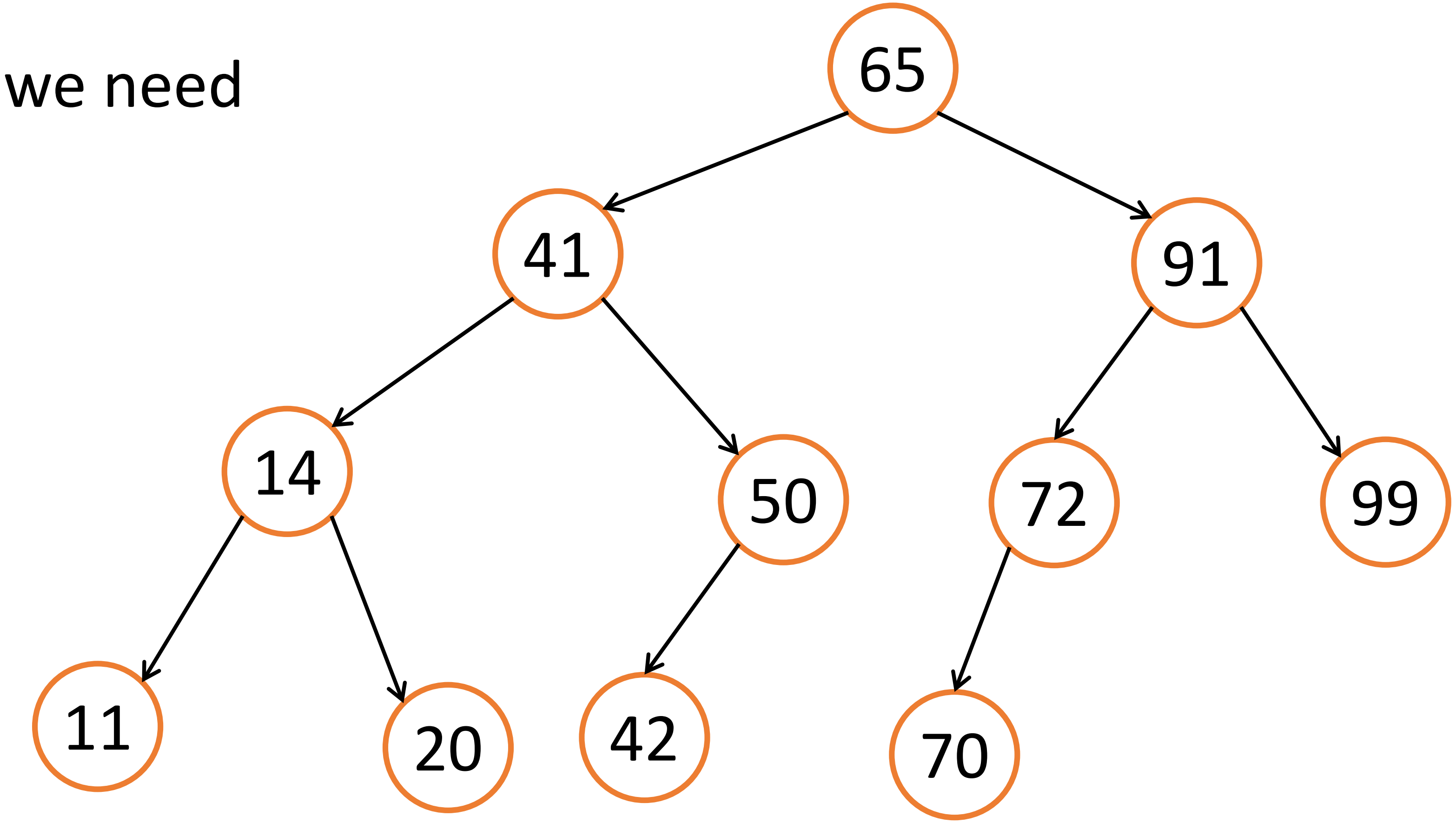
Are we done?

- Which node is not balanced?
- Which case is that?
  - Case 2: (RR case)
- left-rotate(41)



Finally

- Are we done?
- How many rotations do we need for *any* deletion?



AVL Tree Balancing

- Insertion:
  - Needs 0 to 2 rotations to balance
- Deletion:
  - Needs up to $O(\log n)$ rotations to balance
  - Tricks to deal with it:
    - Just mark the deleted note as “deleted” and leave it in the tree, instead of removing it from memory
    - Performance degrades over time
    - Clean up later? (Amortized performance…)

Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)
- Scapegoat Trees (Anderson 1989)

Red-Black trees

- More loosely balanced
- Rebalance using rotations on insert/delete
- O(1) rotations for all operations.
- Java TreeSet implementation
- Faster (than AVL) for insert/delete
- Slower (than AVL) for search