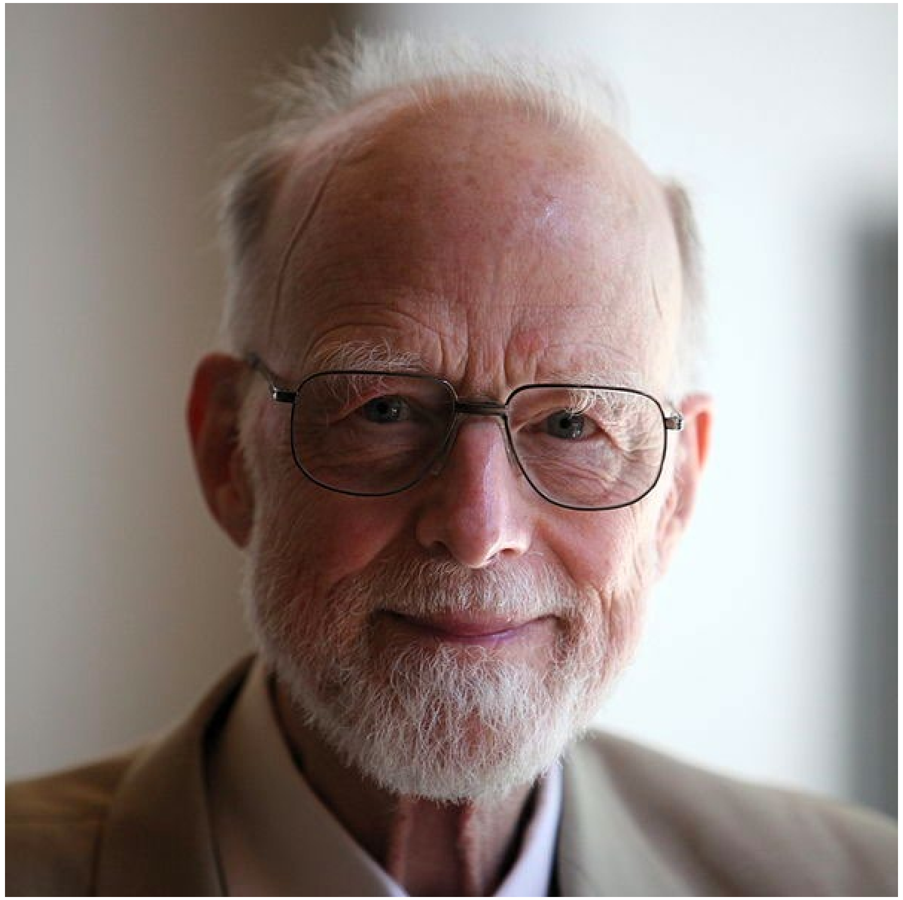
QuickSort

QuickSort History

- Invented by C.A.R. Hoare in 1960
  - Turing Award: 1980
- Visiting student at Moscow State University
  - Used for machine translation



Hoare Quote

- “There are two ways of constructing a software design:

  One way is to make it so simple that there are obviously no deficiencies,

  And the other way is to make it so complicated that there are no obvious deficiencies.
- The first method is far more difficult.”

My 40 Years of Experience

QuickSort

- History:
  - Invented by C.A.R. Hoare in 1960
  - Used for machine translation (English/Russian)
- In practice:
  - Very fast
  - Many optimizations
  - In-place (i.e., no extra space needed)
  - Good caching performance
  - Good parallelization

QuickSort Today

- 1960: Invented by Hoare
- 1979: Adopted everywhere (e.g., Unix qsort)
- 1993: Bentley & McIlroy improvements
- 2009: Vladimir Yaroslavskiy
  - Dual-pivot Quicksort !!!
  - Now standard in Java 7
  - 10% faster!
- 2012: Sebastian Wild and Markus E. Nebel
  - "Average Case Analysis of Java 7's Dual Pivot…"
  - Best paper award at ESA

QuickSort

- Easy to understand! (divide-and-conquer…)
- Moderately hard to implement correctly.
- Harder to analyze. (Randomization…)
- Challenging to optimize.

QuickSort First Assumption

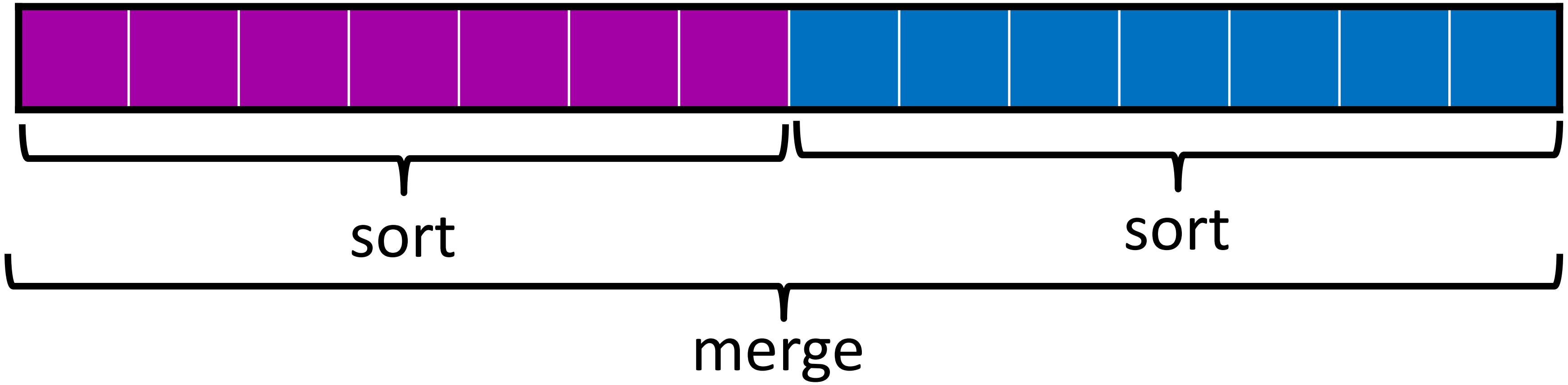
- For starter, let's assume the world is beautiful….
- For a lot of algorithms, it's better to be explained in a simplified problem first
- But it doesn't mean it cannot work on the "original" problem

Let’s Assume that

- Every element in the array is unique

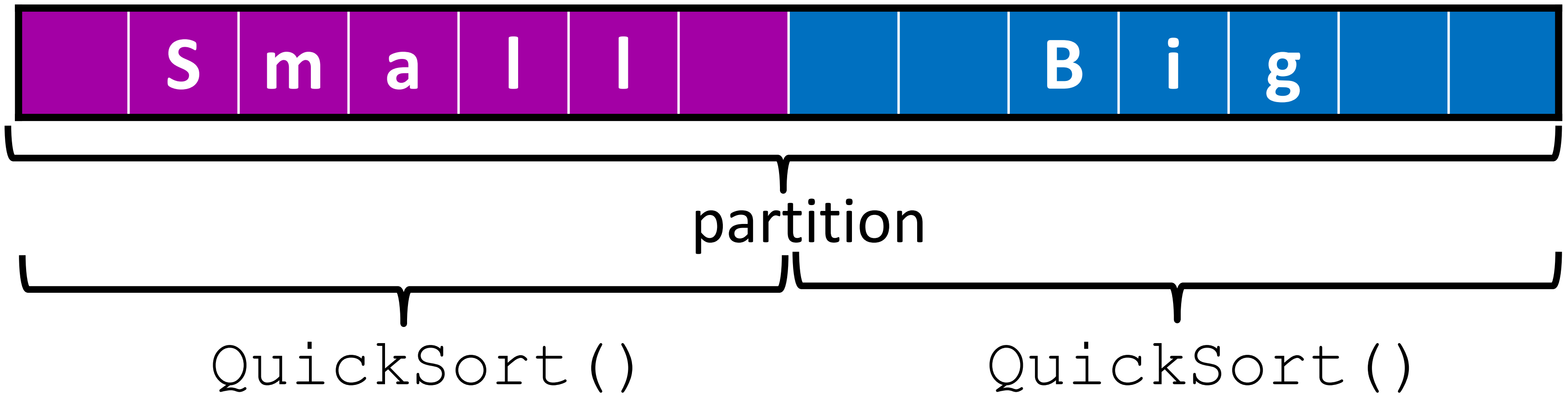
Recall: MergeSort

```
MergeSort(A, n)
    if (n=1) then return;
    else:
     X ←MergeSort(A[1..n/2],  n/2);
     Y ←MergeSort(A[n/2+1, n],  n/2);
    return Merge (X,Y, n/2);
```

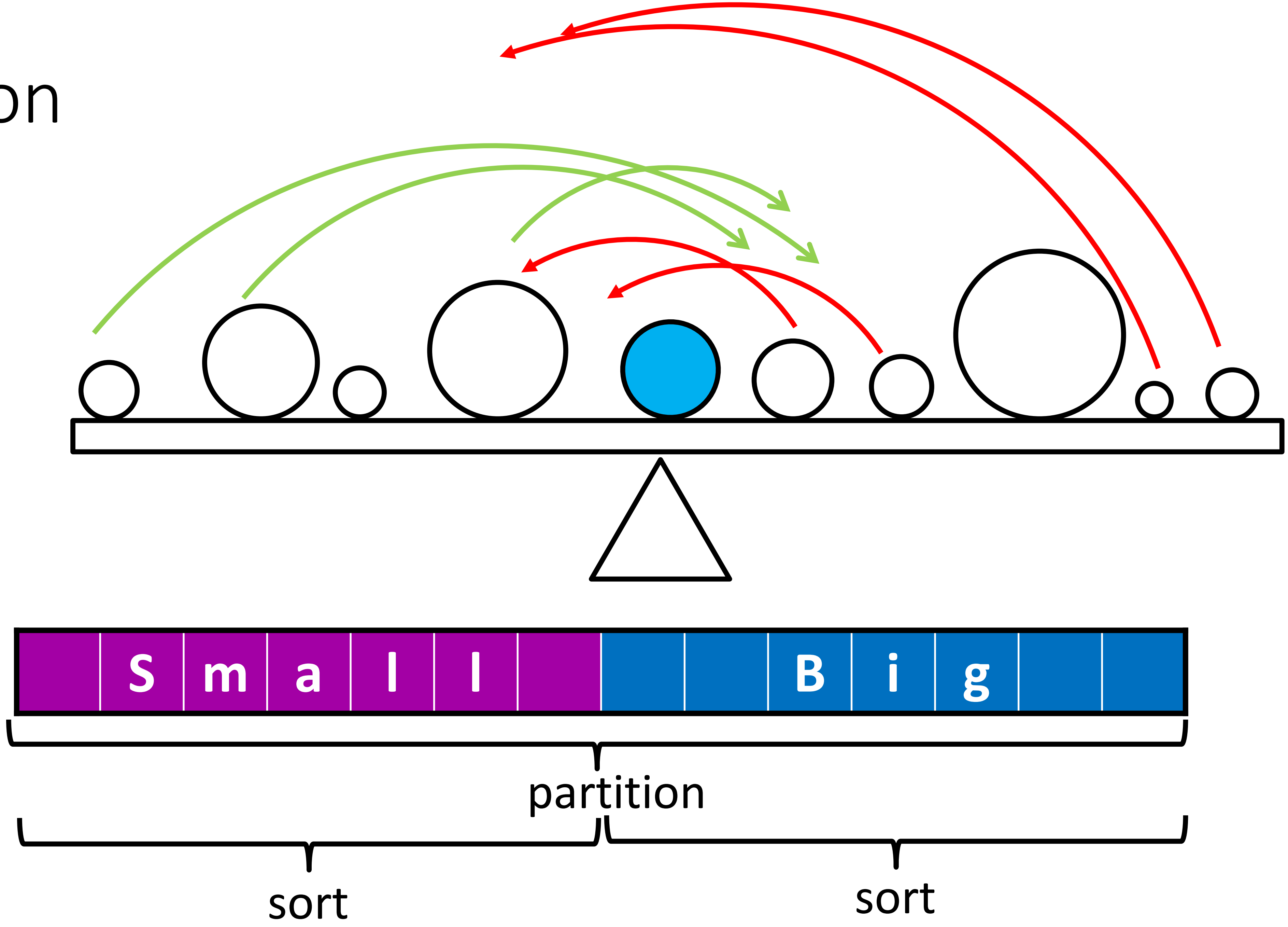


QuickSort

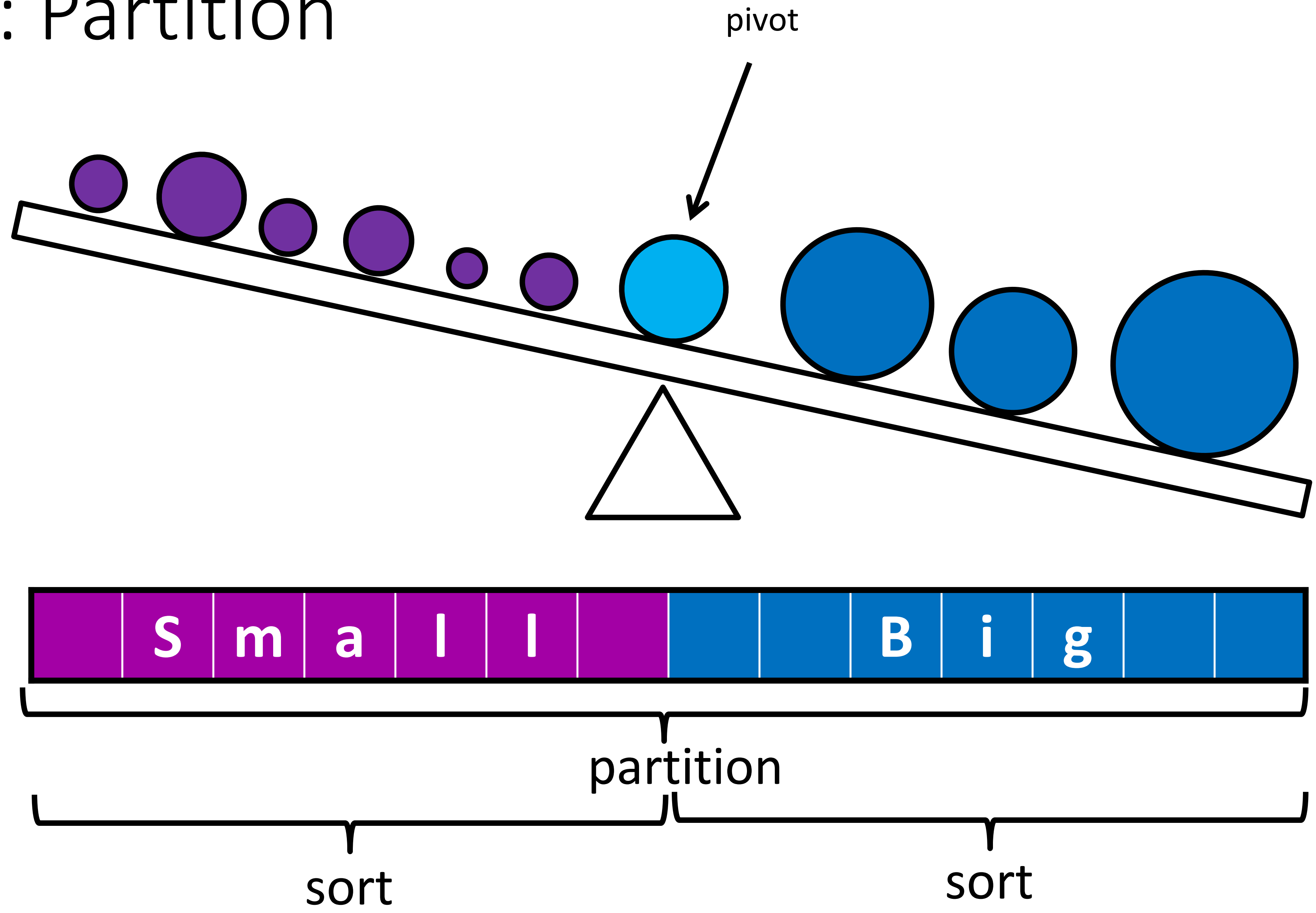
```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = partition(A[1..n], n)
        x = QuickSort(A[1..p-1], p-1)
        y = QuickSort(A[p+1..n], n-p)
```



Partition



QuickSort: Partition



QuickSort

Given: n element array $A[1..n]$

1. Divide: Partition the array into two sub-arrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper sub-array.

$< x$	x	$> x$

2. Conquer: Recursively sort the two sub-arrays.
3. Combine: Trivial, do nothing.

Key: efficient *partition* sub-routine

Partitioning

- Three steps:
  1. Choose a pivot, e.g. the first element.*
  2. Find all elements smaller than the pivot.
  3. Find all elements larger than the pivot.

* a lot of rooms to discuss

Let’s Assume We Got the Magic to Partition

- Given

6	3	9	8	4	1

- Pick a pivot, say the first item “6”

3	4	1	6	9	8

- “6” is “sorted”
- QuickSort the left and the right

Let’s Assume We Got the Magic to Partition

- “6” is sorted, QuickSort the left ~~and the right~~

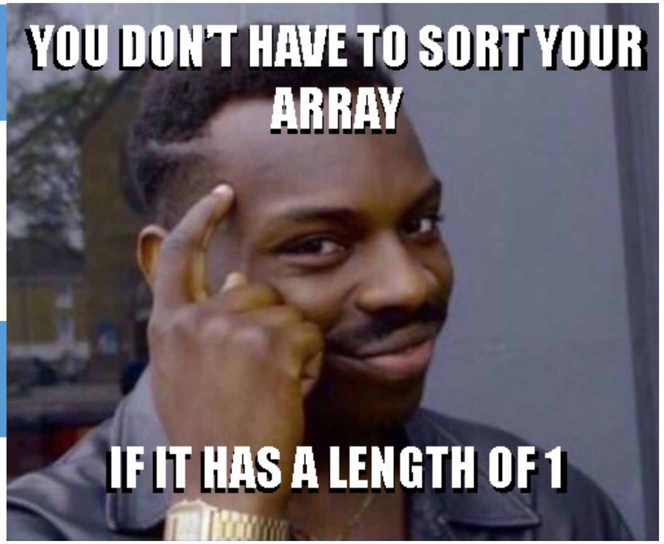
3	4	1	6	9	8

- Pick a pivot, say 3

3	4	1			

- After partition on the left array, “3” is sorted

1	3	4			



- And the two “arrays” with one element are sorted

Let's Assume We Got the Magic to Partition

- QuickSort the right

1	3	4	6	9	8

- Pick a pivot, say 9

1	3	4	6	8	9

- After partition on the left array, "9" is sorted
- And the "array" with "8" is also sorted
- DONE!

Which one is the pivot?

- If the following array is partitioned before further recursion, which one is the pivot?

18	5	6	1	10	22	40	32	50

Partitioning

- Three steps:
  1. Choose a pivot, e.g. the first element.*
  2. Find all elements smaller than the pivot.
  3. Find all elements larger than the pivot.

- What is the time complexity for partitioning once?

Partitioning

22	1	6	40	32	10	18	4	50

low
< 22

high
> 22

Move until it's bigger than the pivot

Move until it's less than the pivot

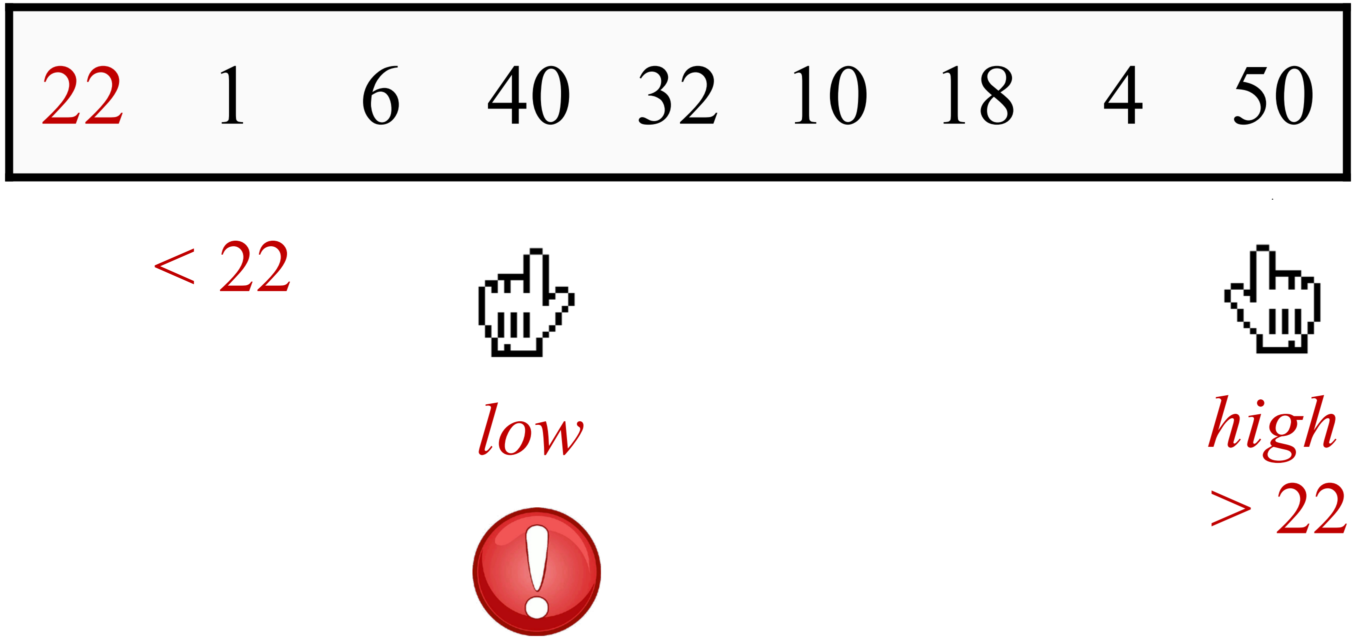
Partitioning

22	1	6	40	32	10	18	4	50

< 22 *low*

high > 22

Partitioning



Partitioning

22	1	6	40	32	10	18	4	50

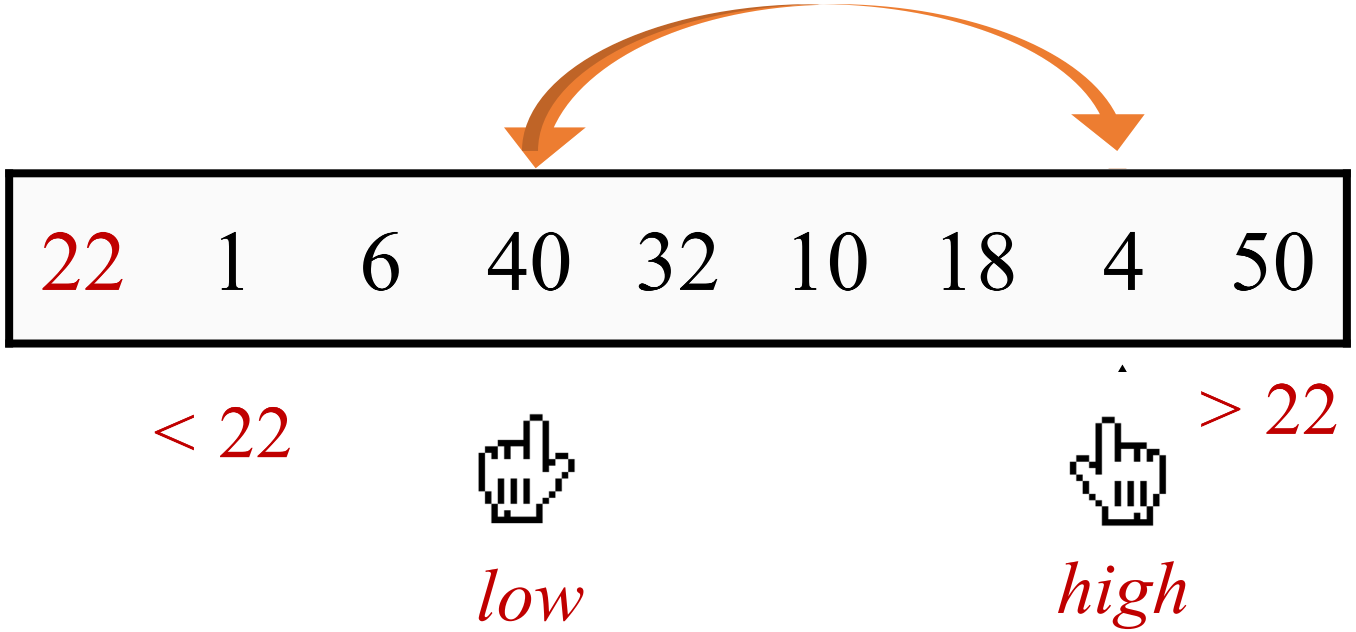
< 22

> 22

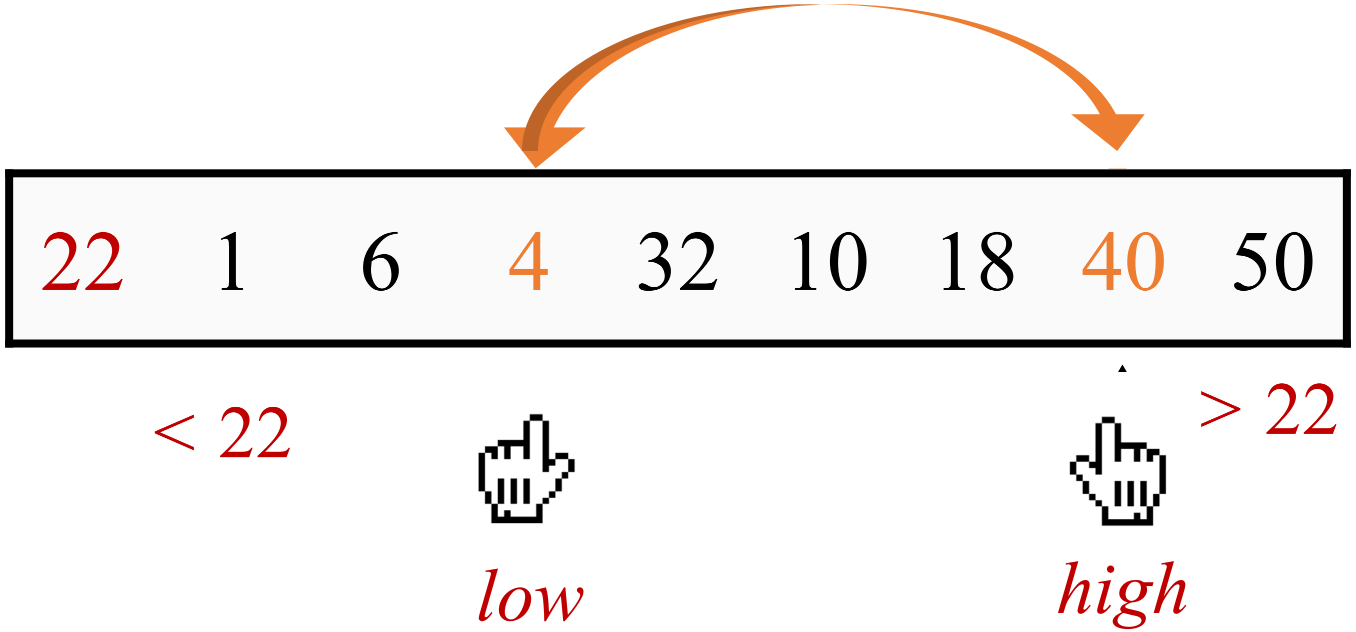
low

high

Partitioning



Partitioning



Partitioning

22	1	6	4	32	10	18	40	50

< 22

> 22

low

high

Partitioning

22	1	6	4	32	10	18	40	50

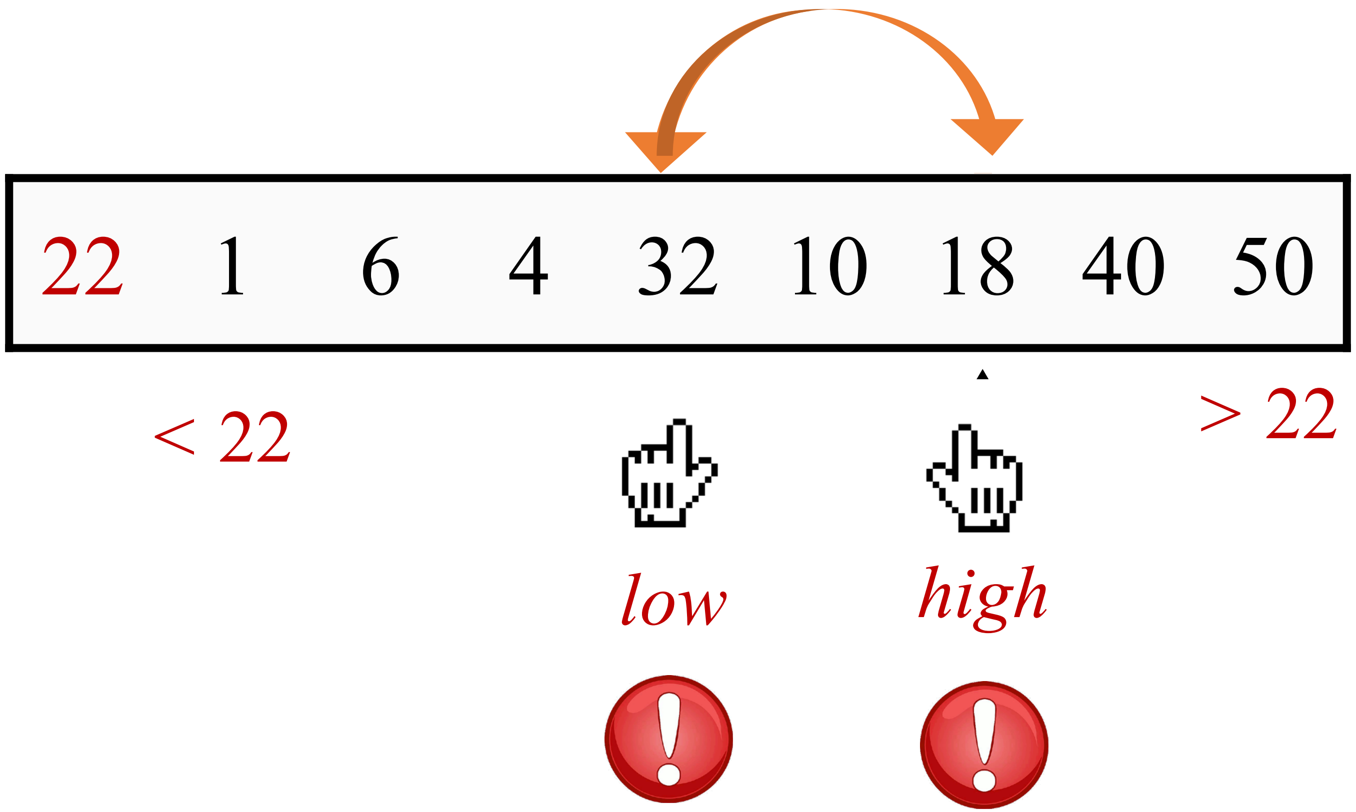
< 22

> 22

low

high

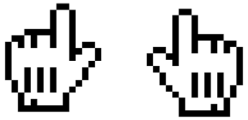
Partitioning



Partitioning

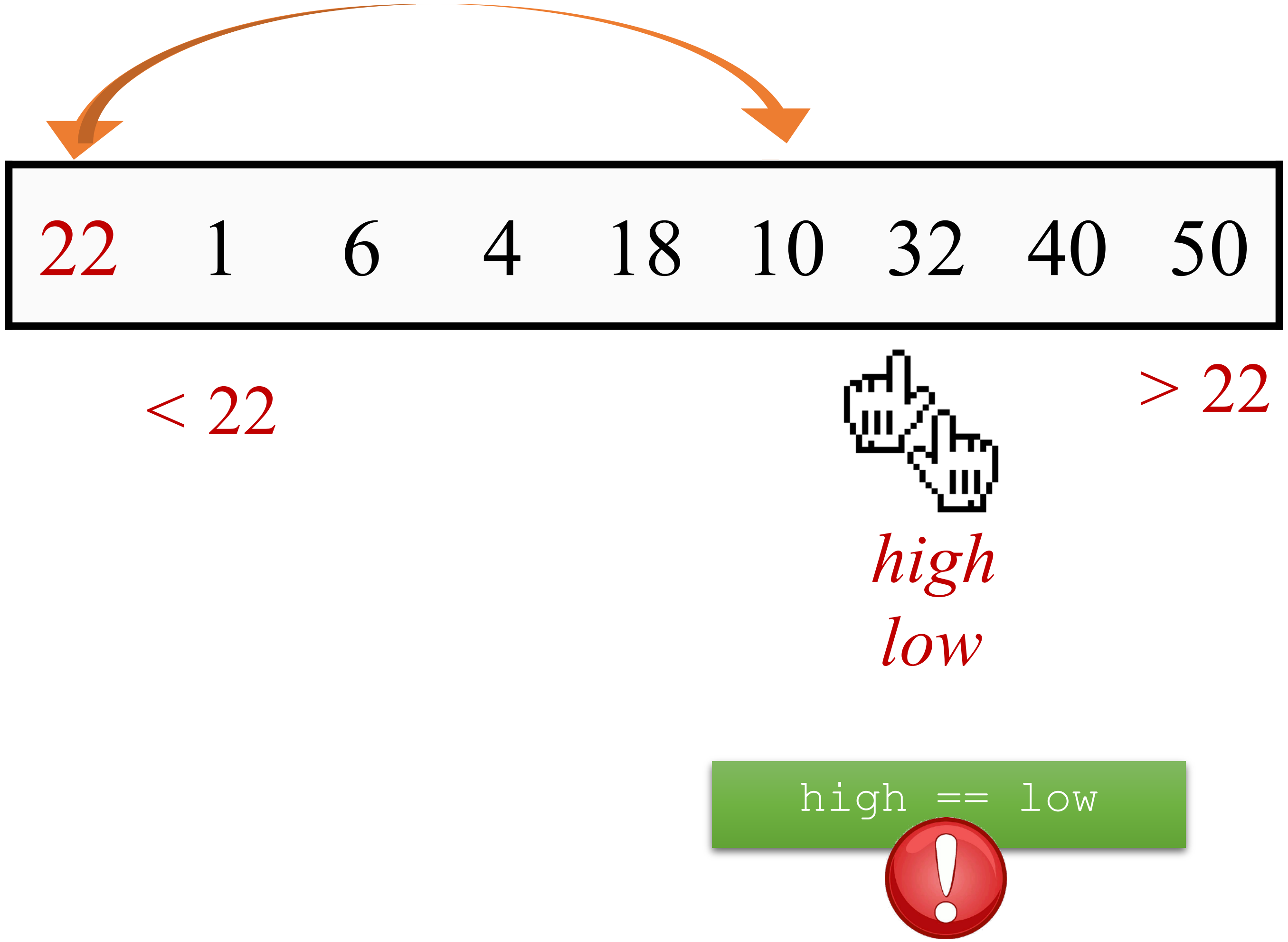
22	1	6	4	18	10	32	40	50

< 22 > 22

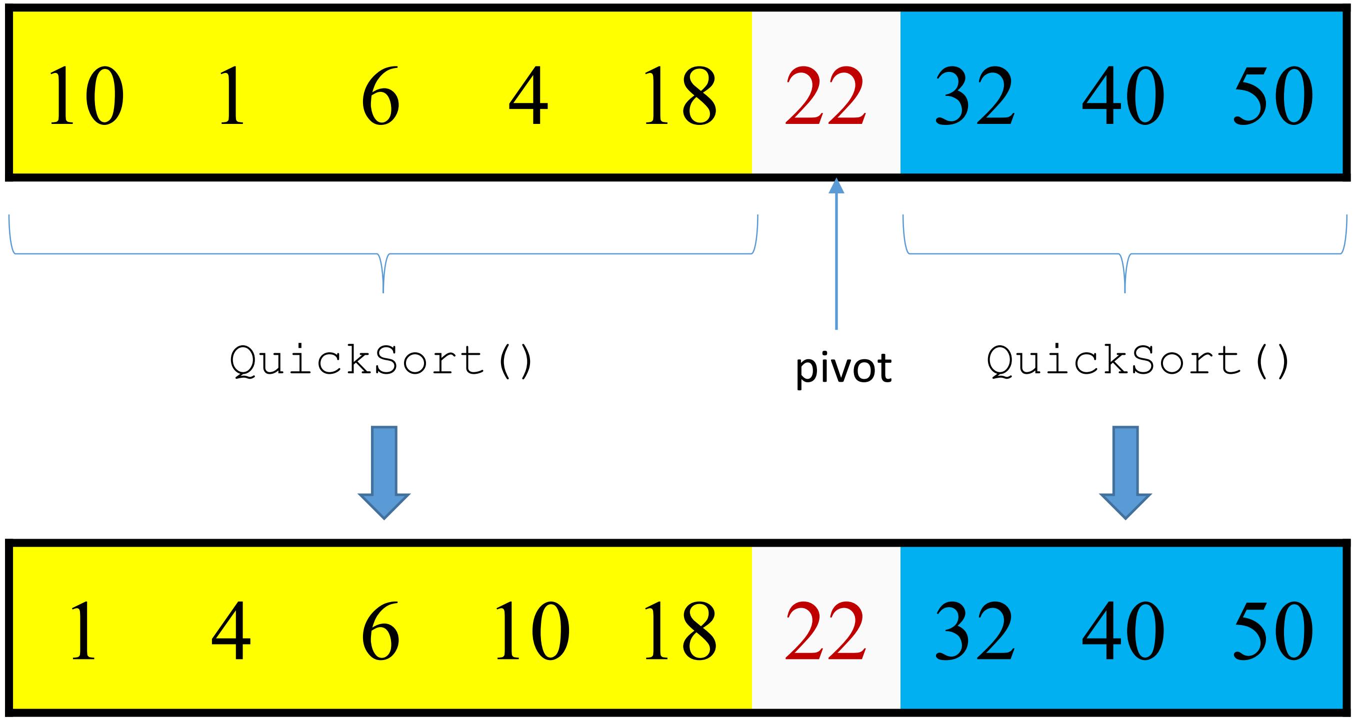


low *high*

Partitioning Final Step



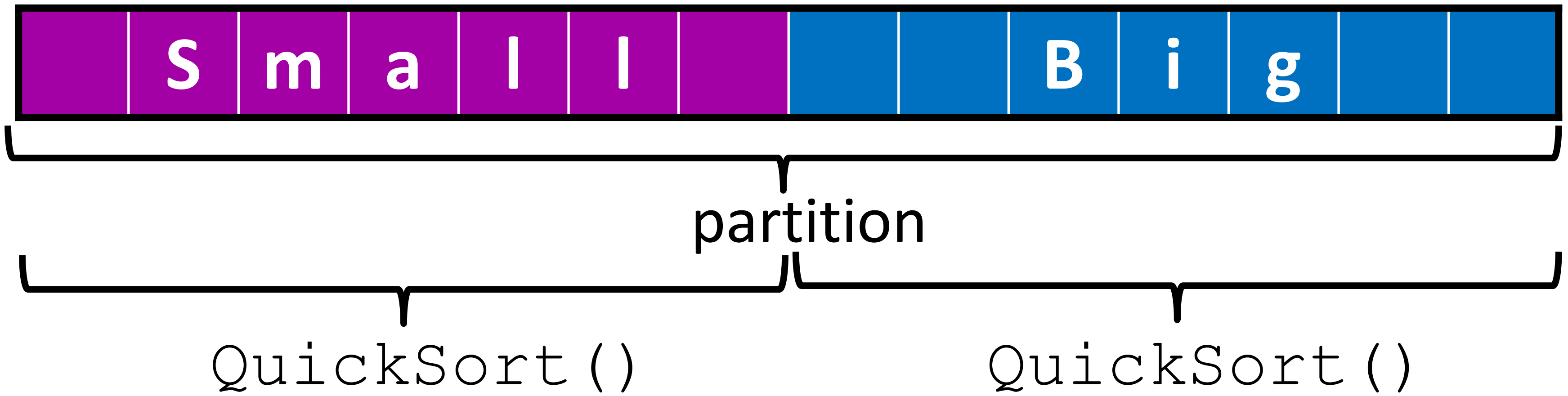
Partitioning Done



```
partition(A[1..n], n)
    pivot = 1
    low = 2;                         // start after pivot in A[1]
    high = n+1;                      // Define: A[n+1] = ∞
    while (low < high) {
      while (A[low]  < pivot) and (low < high) do low++;
      while (A[high] > pivot) and (low < high) do high--;
      if (low < high) then swap(A[low], A[high]);
    }
    swap(A[1], A[low-1]);
    return  low - 1;
```

QuickSort

```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = partition(A[1..n], n)
        x = QuickSort(A[1..p-1], p-1)
        y = QuickSort(A[p+1..n], n-p)
```



What if there are duplicates?

Where will it go wrong?

```
partition(A[1..n], n)
    pivot = 1
    low = 2;                        // start after pivot in A[1]
    high = n+1;                     // Define: A[n+1] = ∞
    while (low < high)
      while (A[low]  < pivot) and (low < high) do low++;
      while (A[high] > pivot) and (low < high) do high--;
      if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low-1]);
    return  low - 1;
```

Duplicates will get "stuck"

22	1	6	22	32	10	18	22	50

< 22 > 22

low *high*

Where will it go wrong?

```
partition(A[1..n], n)
    pivot = 1
    low = 2;                          // start after pivot in A[1]
    high = n+1;                       // Define: A[n+1] = ∞
    while (low < high)
      while (A[low]  < pivot) and (low < high) do low++;
      while (A[high] > pivot) and (low < high) do high--;
      if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low-1]);
    return  low - 1;
```

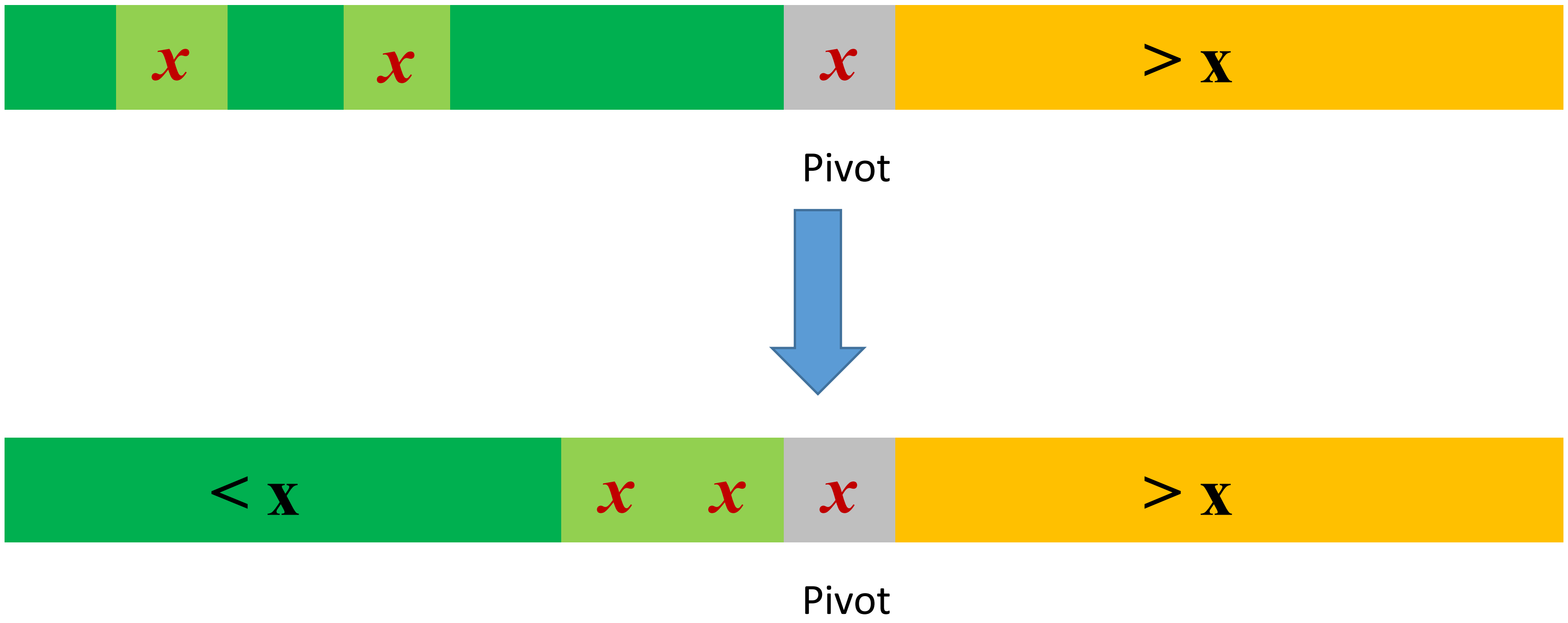
Nothing changed

Let it go, let it go…

```
partition(A[1..n], n)
    pivot = 1
    low = 2;                           // start after pivot in A[1]
    high = n+1;                        // Define: A[n+1] = ∞
    while (low < high)
      while (A[low]  ≤ pivot) and (low < high) do low++;
      while (A[high] > pivot) and (low < high) do high--;
      if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low-1]);
    return  low - 1;
```

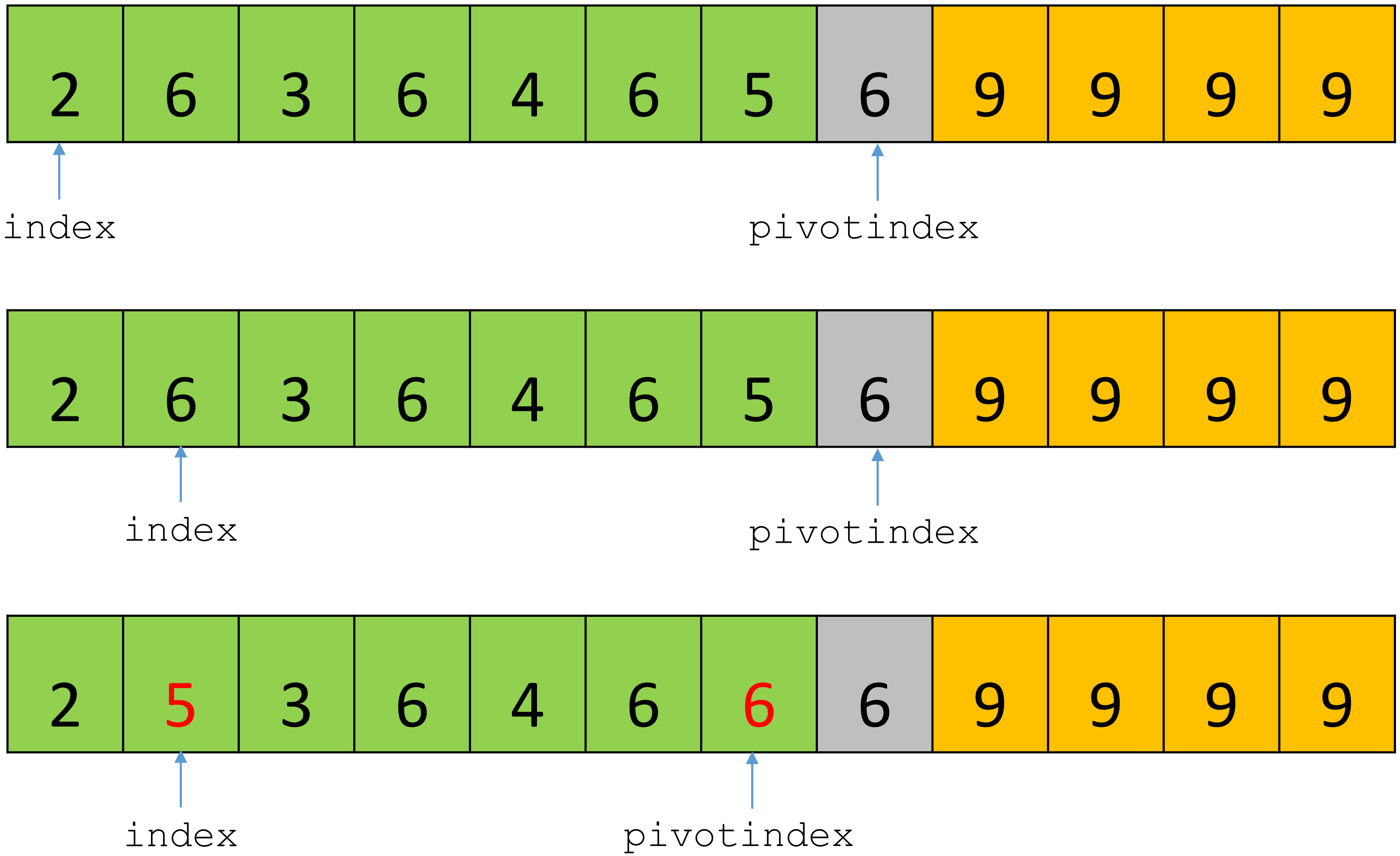
Pivot

Pack Duplicates

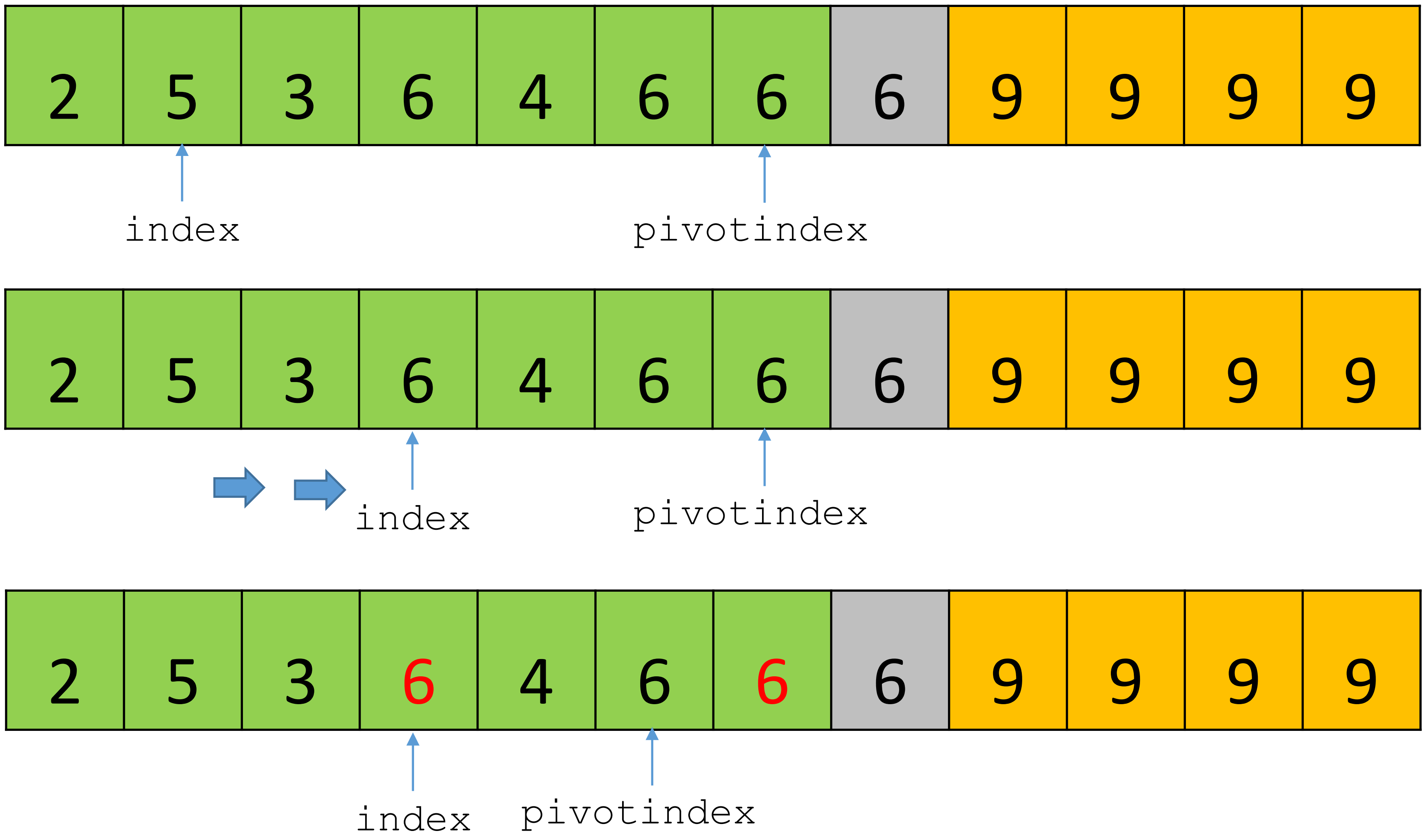


Pack Duplicates

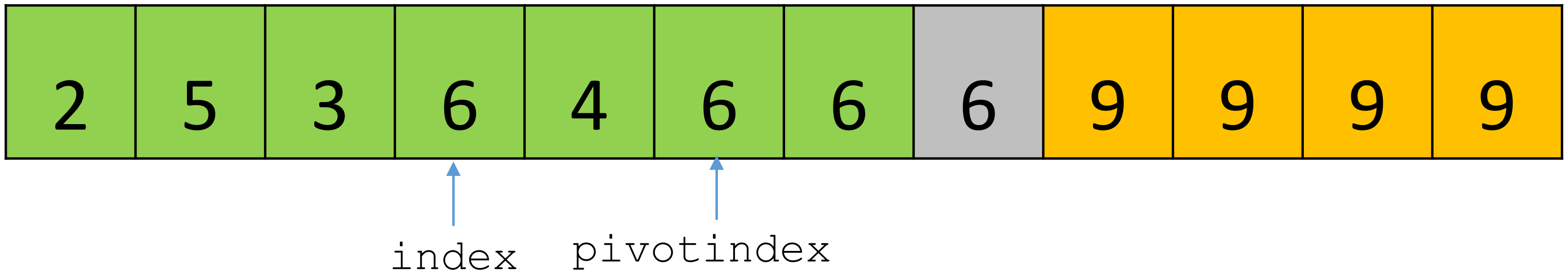
```
packDuplicates(A[1..n], n, pivotIndex)
      pivot = A[pivotIndex];
      index = 1;
      while (index  <  pivotIndex)
            if (A[index] == pivot) {
                  pivotIndex--;
                  swap(A[index], A[pivotIndex]);
            }
            else
                  index ++;
      }
```

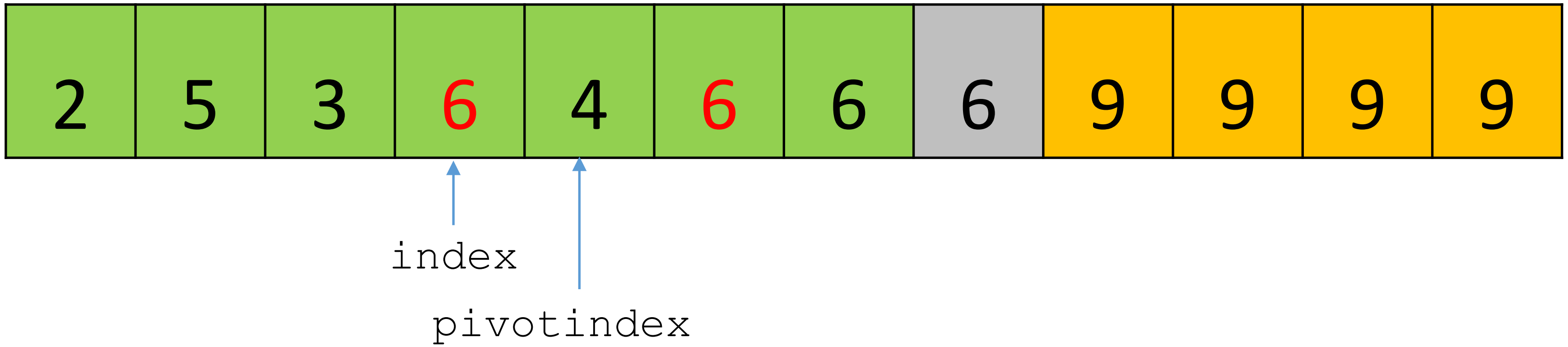


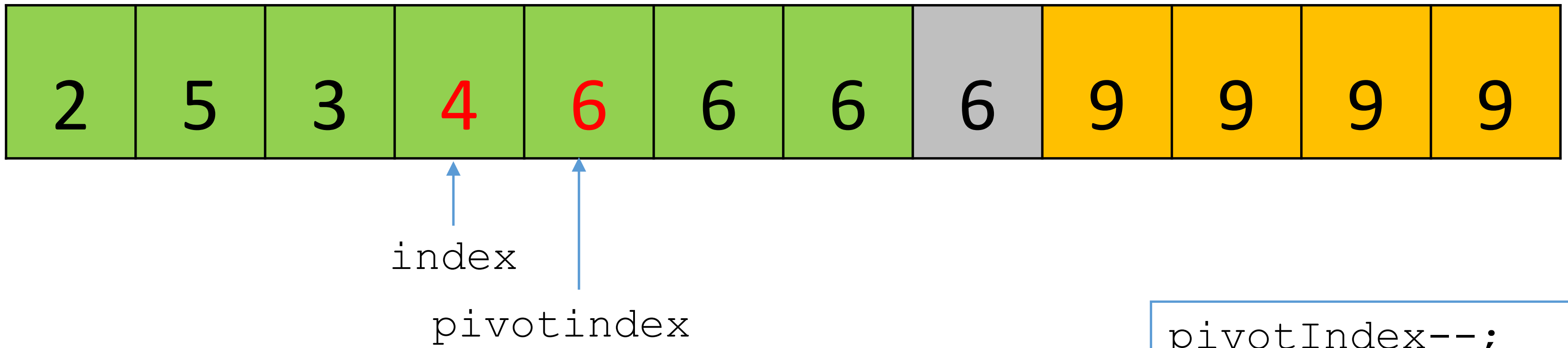

```
pivotIndex--;
swap(A[index], A[pivotIndex]);
```




```
pivotIndex--;
swap(A[index], A[pivotIndex]);
```








```
pivotIndex--;
swap(A[index], A[pivotIndex]);
```

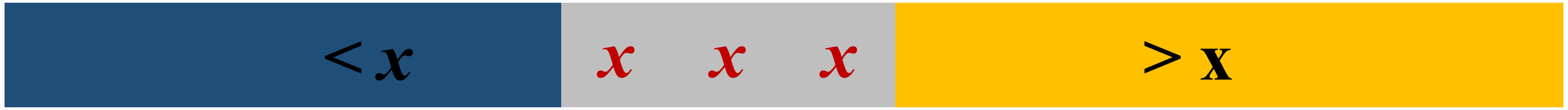
Pack Duplicates

```
packDuplicates(A[1..n], n, pivotIndex)
      pivot = A[pivotIndex];
      index = 1;
      while (index  <  pivotIndex)
            if (A[index] == pivot) {
                  pivotIndex--;
                  swap(A[index], A[pivotIndex]);
            }
            else
                  index ++;
      }
```

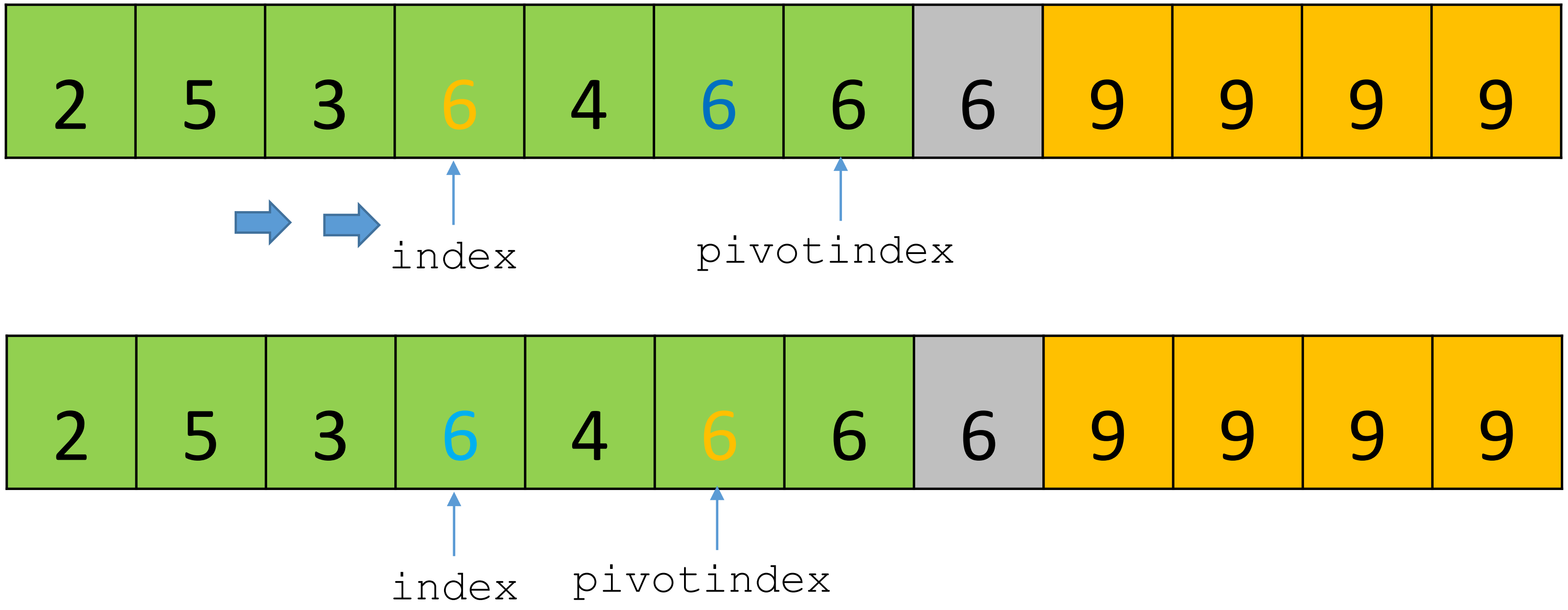
2	5	3	4	6	6	6	6	9	9	9	9

QuickSort

```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = ThreeWayPartition(A[1..n], n)
        x = QuickSort(A[1..p-1], p-1)
        y = QuickSort(A[p+1..n], n-p)
```



Is QuickSort Stable?




```
pivotIndex--;
swap(A[index], A[pivotIndex]);
```

Time Complexity?

```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = ThreeWayPartition(A[1..n], n)        O(n)
        x = QuickSort(A[1..p-1], p-1)            T(p)
        y = QuickSort(A[p+1..n], n-p)            T(n-p)
```

- Lucky case
  - If $p = n/2$ all the time
- $T(n) = cn + 2\ T(n/2)$
- Same as MergeSort!

The pivot we picked is always the median of the array

Time Complexity?

```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = ThreeWayPartition(A[1..n], n)
        x = QuickSort(A[1..p-1], p-1)
        y = QuickSort(A[p+1..n], n-p)
```

$O(n)$

$T(p)$

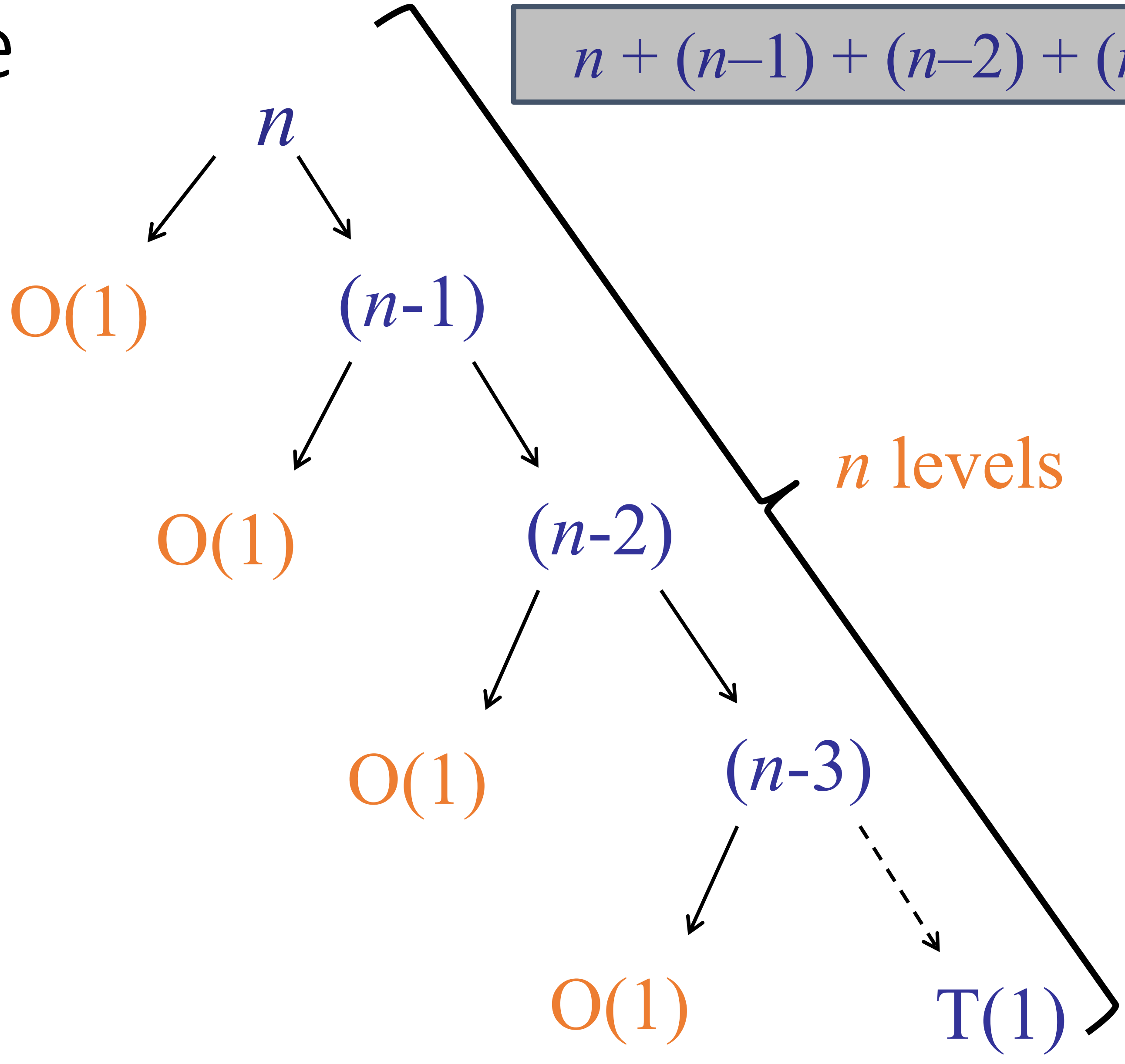
$T(n\text{-p})$

- But what if $p = 1$ all the time?
- $T(n) = cn + T(n - 1) + T(1)$

$$= cn + c(n - 1) + T(n - 2) + T(1) + T(1)$$
$$= cn + c(n - 1) + c(n - 2) + T(n - 2) + T(1) + T(1) + T(1)$$
$$= c(n + (n - 1) + (n - 2) + (n - 3) + \ldots + 1) + T(n) = O(n^2)$$

Worst-case

$$n + (n-1) + (n-2) + (n-3) + \ldots = \mathbf{O(n^2)}$$



Time Complexity

- Lucky case
  - If $p = n/2$ all the time
  - $T(n) = cn + 2\ T(n/2) = \mathrm{O}(n \log n)$
- Worst case
  - if $p = 1$ all the time
  - $T(n) = \ \mathrm{O}(n^2)$

Can we always choose the median as pivot?!

Time Complexity

- Lucky case
  - If $p = n/2$ all the time
  - $T(n) = cn + 2\ T(n/2) = \mathrm{O}(n \log n)$
- Worst case
  - if $p = 1$ all the time
  - $T(n) = \mathrm{O}(n^2)$
- Next: How about choose something in the middle?
  - E.g. $n/10 > p > 9n/10$?
  - That will give $T(n) = \mathrm{O}(n \log n)$!!!

See You Next Week!