QuickSort Analysis

Wait a min, we need a whole lecture for this?

QuickSort

- Easy to understand! (divide-and-conquer…)
- Moderately hard to implement correctly.
- Harder to analyze. (Randomization…)
- Challenging to optimize.

Let’s Revise Some Probability

Probability

Real Story

- Blackjack can be legally beaten by a skilled player.
- Since the early 1960s a large number of card counting schemes have been published, and casinos have adjusted the rules of play in an attempt to counter the most popular methods.
- The idea behind all card counting is that, because a low card is usually bad and a high card usually good, and as cards already seen since the last shuffle cannot be at the top of the deck and thus drawn, the counter can determine the high and low cards that have already been played. He or she thus knows the probability of getting a high card (10,J,Q,K,A) as compared to a low card (2,3,4,5,6).

Real History

- In 1980, six MIT students and residents of the Burton-Conner House at MIT taught themselves card-counting.
- They traveled to Atlantic City during the spring break to win their fortune.
- They offered a course on blackjack for MIT's January, 1980 Independent Activities Period (IAP), during which classes may be offered on almost any subject.

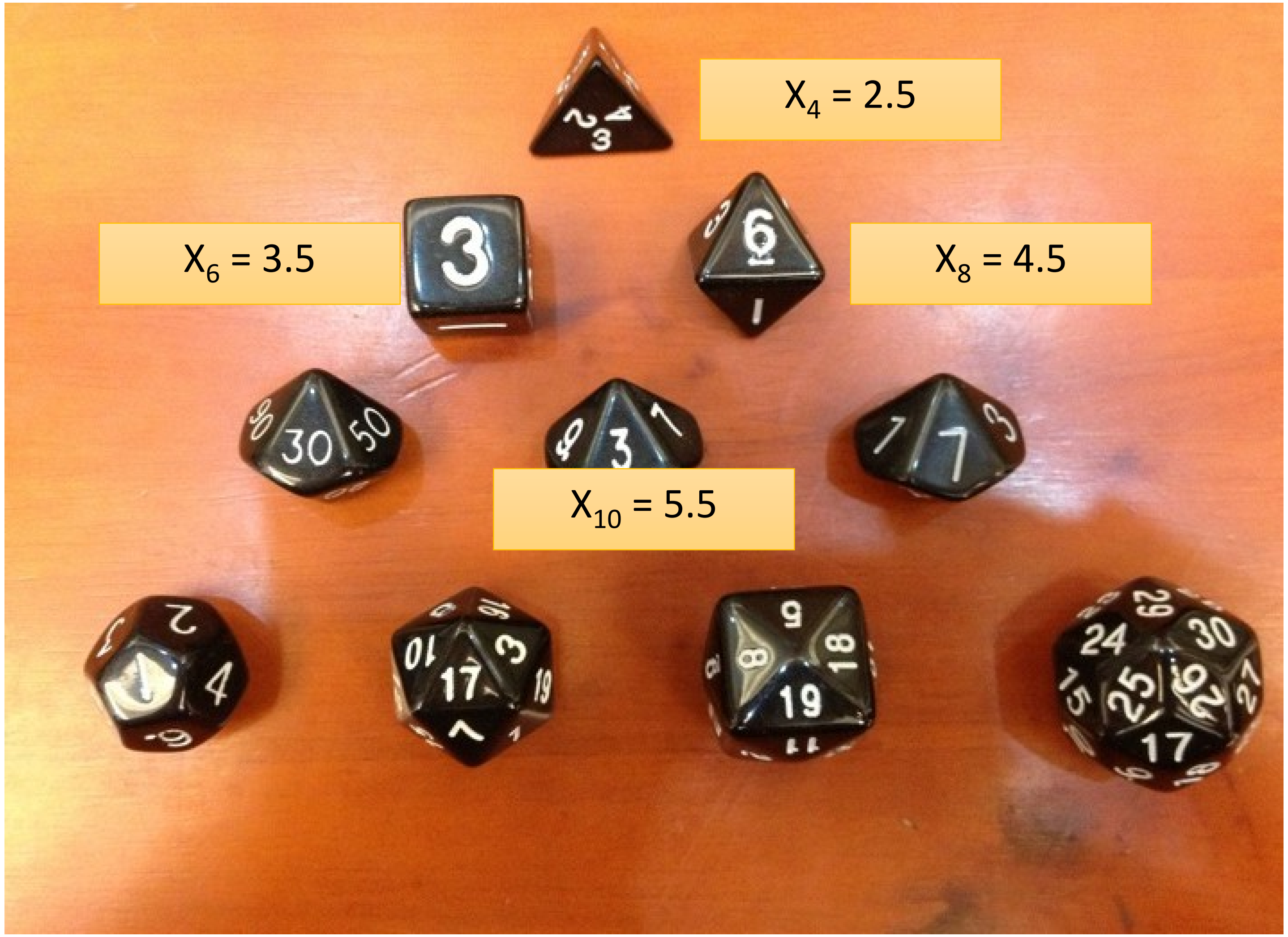
Real History

- Profits per hour played at the tables were $162.50, statistically equivalent to the projected rate of $170/hour detailed in the investor offering prospectus. (1980, ~x 5 for 2020 with inflation rate = 4%)
- Over the ten-week period of this first bank, players, mostly undergraduates, earned an average of over $80/hour while investors achieved an annualized return in excess of 250%.
- The MIT Blackjack Team ran at least 22 partnerships in the time period from late 1979 through 1989. At least 70 people played on the team in some capacity (either as counters, Big Players, or in various supporting roles) over that time span. Every partnership was profitable during this time period, after paying all expenses as well as the players' and managers' share of the winnings, with returns to investors ranging from 4%/year to over 300%/year.

Which Weapon will you pick?

1. Halberd; 2. Longbow; 3. Handaxe; 4. Short sword; 5. Shortbow; 6. Longsword; 7. Maul; 8. Greataxe; 9. War pick; 10. Bastard sword; 11. Warhammer; 12. Flail; 13. Battleaxe; 14. Throwing hammer; 15. Scimitar; 16. Glaive

Name	Source	Prof	Damage	
Falchion	PHB	+3	2d4	-
Glaive	PHB	+2	2d4	-
Greataxe	PHB	+2	1d12	-
Greatsword	PHB	+3	1d10	-
Halberd	PHB	+2	1d10	-
Heavy flail	PHB	+2	2d6	-
Heavy war pick	AV	+2	1d12	-


X_4 = 2.5
X_6 = 3.5
X_8 = 4.5
X_{10} = 5.5

Which Weapon will you pick?

bow; 6. Longsword; 7. Maul; 8. Greataxe; 9. War pick; 10. Bastard sword;
nmer; 15. Scimitar; 16. Glaive

Name	Source	Prof	Damage	Expected Damage
Falchion	PHB	+3	2d4	5
Glaive	PHB	+2	2d4	5
Greataxe	PHB	+2	1d12	6.5
Greatsword	PHB	+3	1d10	5.5
Halberd	PHB	+2	1d10	5.5
Heavy flail	PHB	+2	2d6	7
Heavy war pick	AV	+2	1d12	6.5

Expected #Times to Happen

How many times you need to bet until you win a lucky draw/Toto?

If you have a fair coin

- The Probability of getting a head = ½
- The Probability of getting a tail = ½

- If you flip the coin 10^{10} times, about how many heads do you expect?
  - $\frac{1}{2} \times 10^{10}$ heads
- If you flip the coin n times, about how many heads do you expect?
  - $\frac{1}{2} \times n$ heads

If you have a fair coin

- The Probability of getting a head = ½
- The Probability of getting a tail = ½

- If you flip the coin n times, you expect ½ × n heads
- How many times (n) do you need to flip so that you expect to have one head?
  - $\frac{1}{2} \times n = 1$
  - $n = (\frac{1}{2})^{-1} = 2$
- It means that if I flip the coins two times, I expect there will be one head

If an event has a probability of p

- The Probability of an event is p

- If you repeat n times, you the event appears $p \times n$ times
- How many times (n) do you need to repeat so that you the event to happen once?
  - $p \times n = 1$
  - $n = (p)^{-1} = 1/p$
- It means that if repeat $1/p$ times, I expect the event will happen once.
- If I want the event to happen once, the expected number of repetition is $1/p$.

If an event has a probability of p

- The Probability of an event is p
- If I want the event to happen once, the expected number of repetition is $1/p$.

- Example:
  - How many time do I need to draw a card randomly from a pile of 52 cards in order to get a King?
  - Probably of drawing a King is $p = 1/13$
  - If I draw $1/p = 13$ cards, I expect to get one King

QuickSort Time Complexity

Recap: Time Complexity?

```
QuickSort(A[1..n], n)
    if (n==1) then return;
    else
        p = ThreeWayPartition(A[1..n], n)       O(n)
        x = QuickSort(A[1..p-1], p-1)           T(p)
        y = QuickSort(A[p+1..n], n-p)           T(n-p)
```

- Lucky case
  - If $p = n/2$ all the time
- $T(n) = cn + 2\ T(n/2)$
- Same as MergeSort!

The pivot we picked is always the median of the array

Recap: Time Complexity?

```
QuickSort(A[1..n], n)
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        y = QuickSort(A[p+1..n], n-p)
```

$O(n)$

$T(p)$

$T(n\text{-p})$

- But what if $p = 1$ all the time?
- $T(n) = cn + T(n - 1) + T(1)$

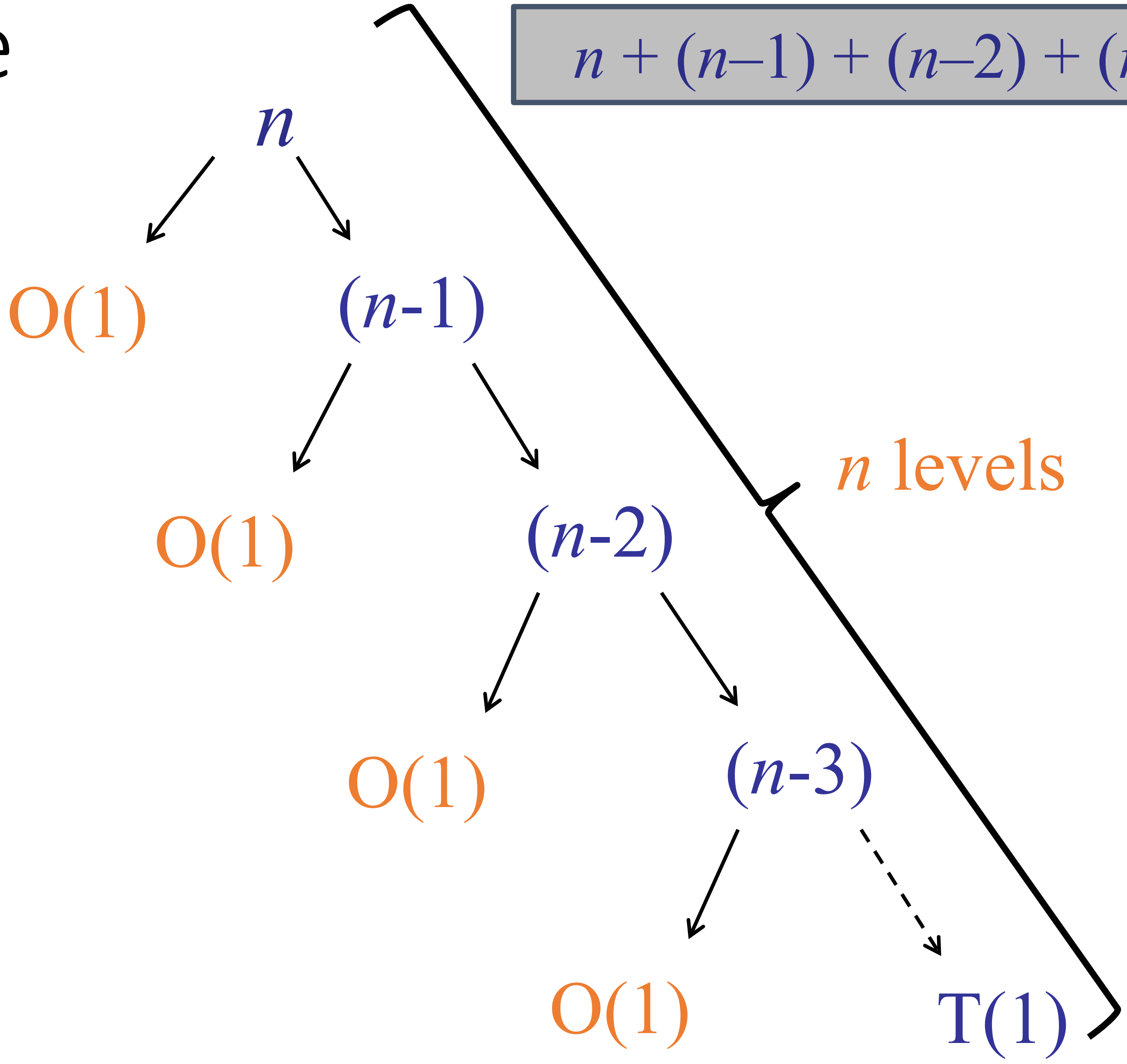
$= cn + c(n - 1) + T(n - 2) + T(1) + T(1)$

$= cn + c(n - 1) + c(n - 2) + T(n - 2) + T(1) + T(1) + T(1)$

$= c(n + (n - 1) + (n - 2) + (n - 3) + \ldots + 1) + T(n) = O(n^2)$

Worst-case

$$n + (n-1) + (n-2) + (n-3) + \ldots = \mathbf{O(n^2)}$$



Time Complexity

- Lucky case
  - If $p = n/2$ all the time
  - $T(n) = cn + 2\ T(n/2) = \mathrm{O}(n \log n)$
- Worst case
  - if $p = 1$ all the time
  - $T(n) = \ \mathrm{O}(n^2)$

Today

- How about choose something in the middle?
  - E.g. $n/10 > p > 9n/10$?
  - That will give $T(n) = \mathrm{O}(n \log n)$!!!

Ways to Choose a Pivot

- Choose the First

- Choose the Last

- Choose the Middle

- Choose the Median

- Choose randomly

Ways to Choose a Pivot

- Choose the First

- Choose the Last

- Choose the Middle

- All may results in $O(n^2)$
- Challenge: Try to reverse engineer such a input list?

Ways to Choose a Pivot

- Choosing the Median

  - $O(n \log n)$ Great!
  - But how to choose the Median?
- Choose randomly

  - Expected time $O(n \log n)$
  - Huh?
  - Isn’t it the best?
    - (Why do we still bother to find the median!?!)

Idea

```
Repeat
   Partition with a random pivot
Until the pivot is good.
```

- We said that a pivot is good if it divides the array into two pieces, each of which is size at least n/10.
- Or technically:
- After partitioning, let
    - $L =$ no. of elements that are smaller than the pivot
    - $H =$ no. of elements that are larger than the pivot
  - $L > n / 10$ and $H > n / 10$

Question 1
How many times we need to repeat?

Question 2
How does this lead to an $O(n \log n)$ algorithm

Paranoid QuickSort

```
ParanoidQuickSort(A[1..n], n)
     if (n == 1) then return;
     else
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               p = partition(A[1..n], n, pIndex)
          until p > (1/10)*n and p < (9/10)*n

          x = QuickSort(A[1..p-1], p-1)
          y = QuickSort(A[p+1..n], n-p)
```

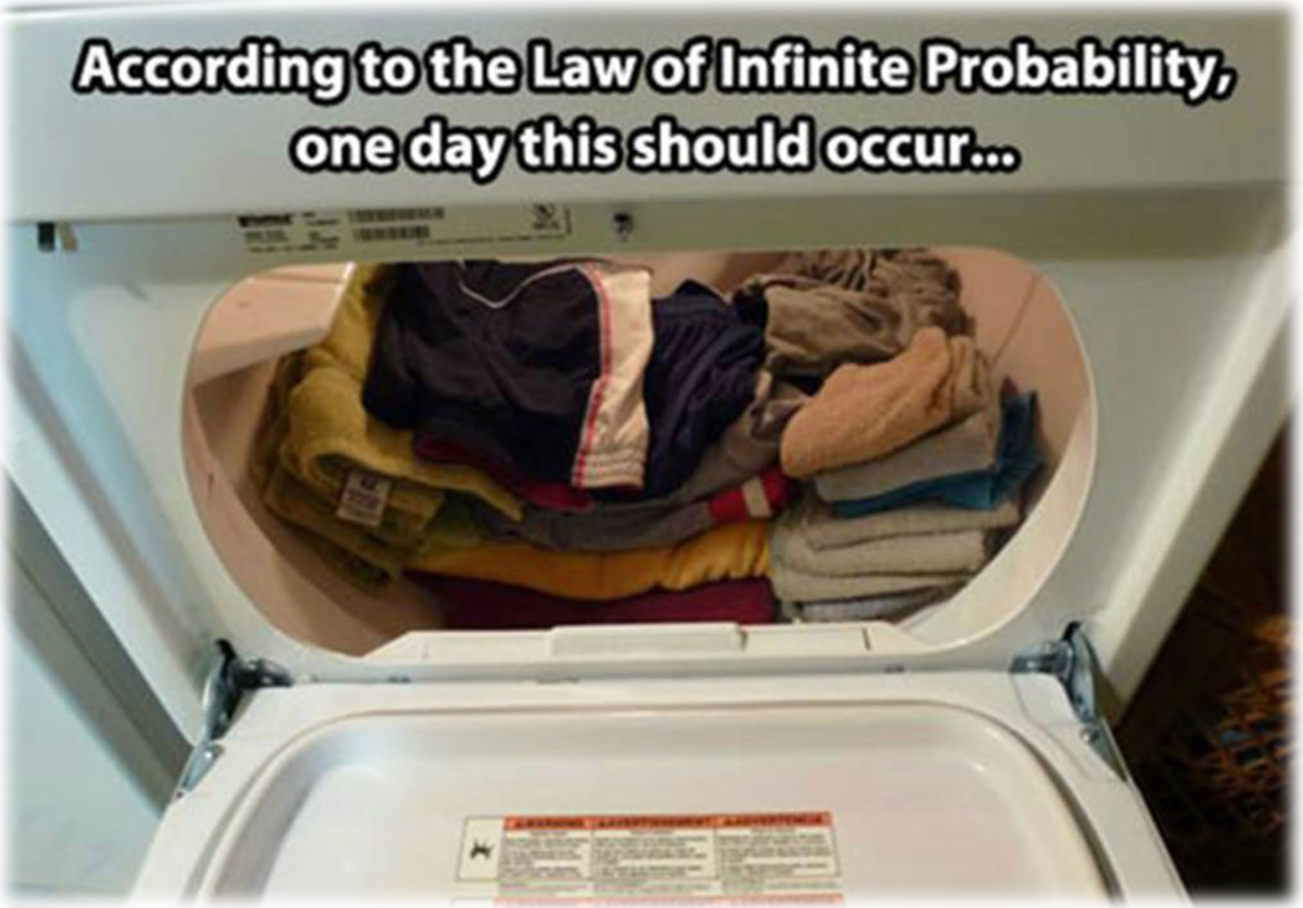
Question 1

How many times we need to repeat?

If we need to repeat $O(n)$ time, then it is $O(n^2)$ for ONE partitioning

Question 1

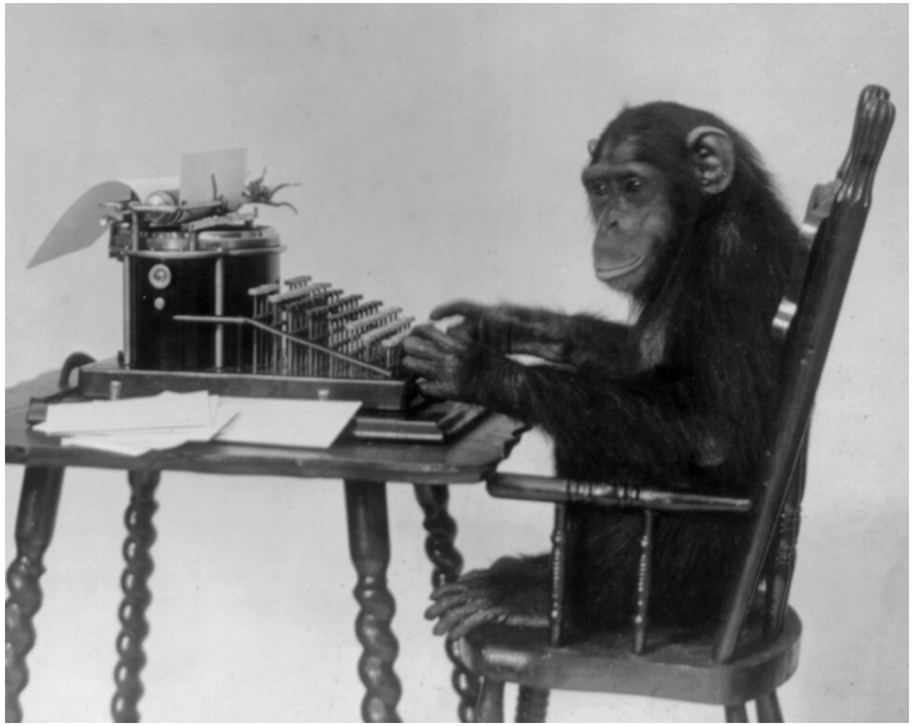
- How many repetitions until the partitioning is good?



Infinite Monkey Theorem

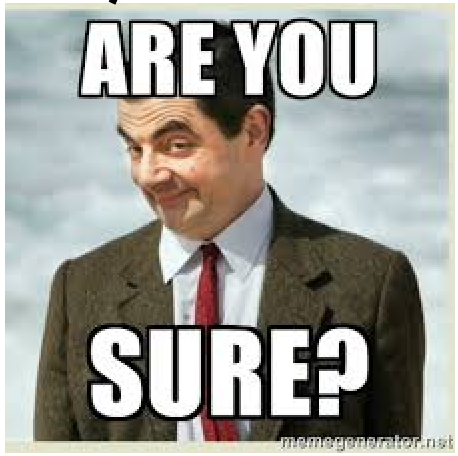
- The infinite monkey theorem states that a monkey hitting keys at random on a typewriter keyboard for an infinite amount of time will almost surely type any given text, such as the complete works of William Shakespeare.

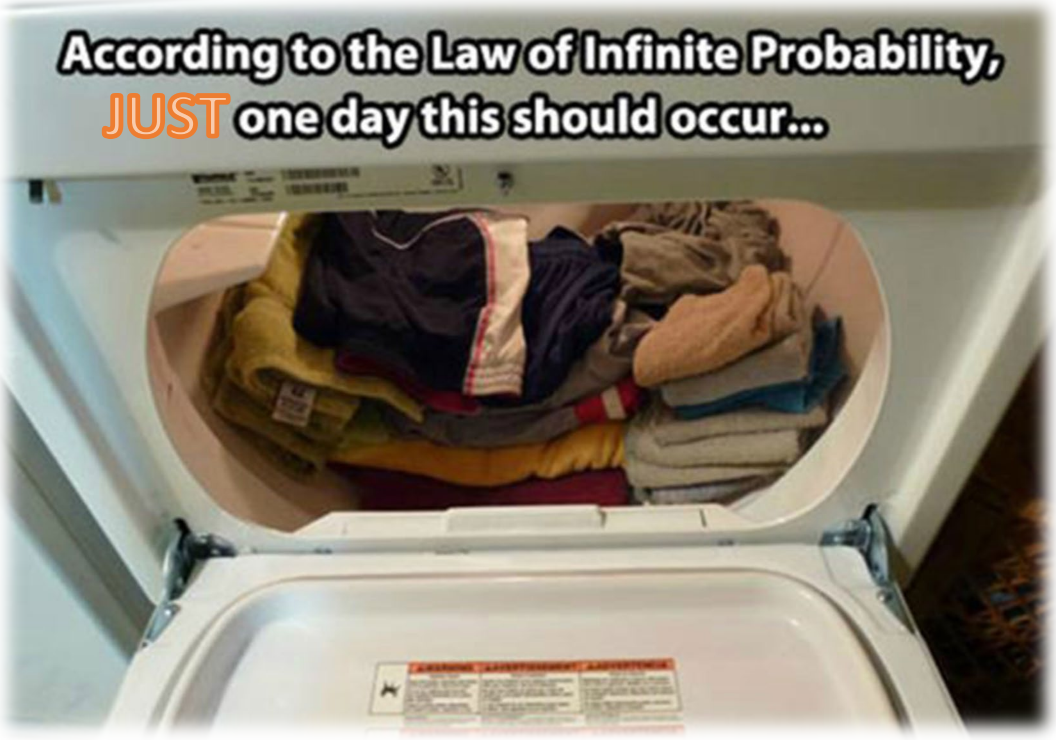
BUT THE WAITING TIME IS INFINITY!



Question 1

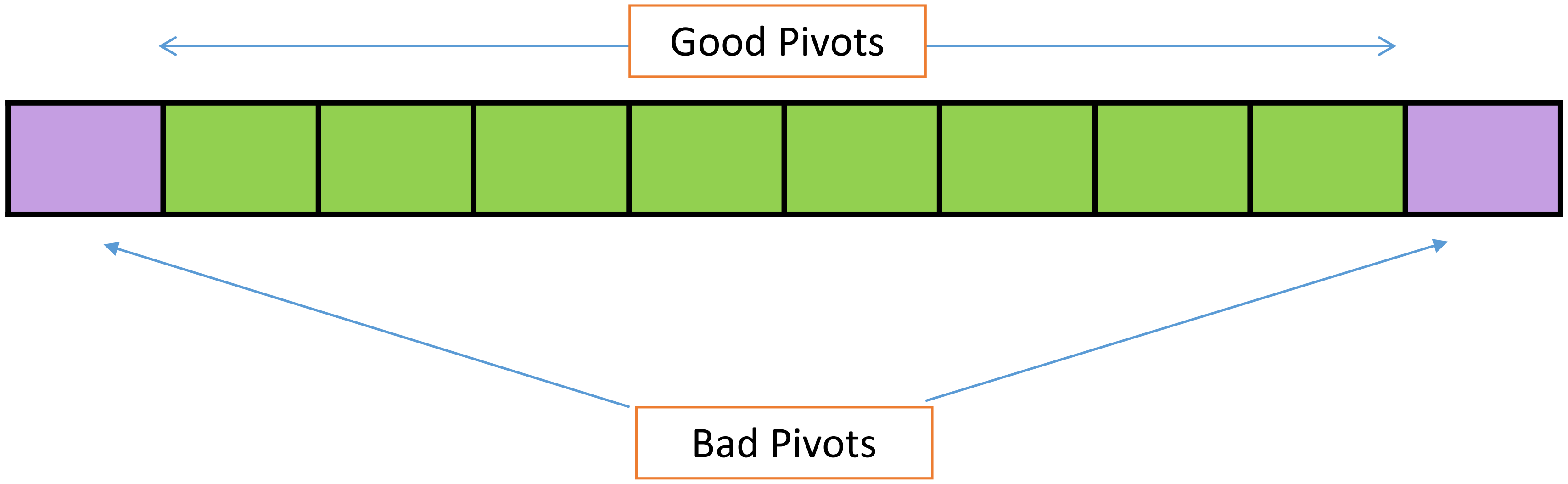
- How many repetitions until the partitioning is good?
- And we claim that
  - Only need to repeat $O(1)$ Time!
  - Yes, for n equals to any integer!





Question 1

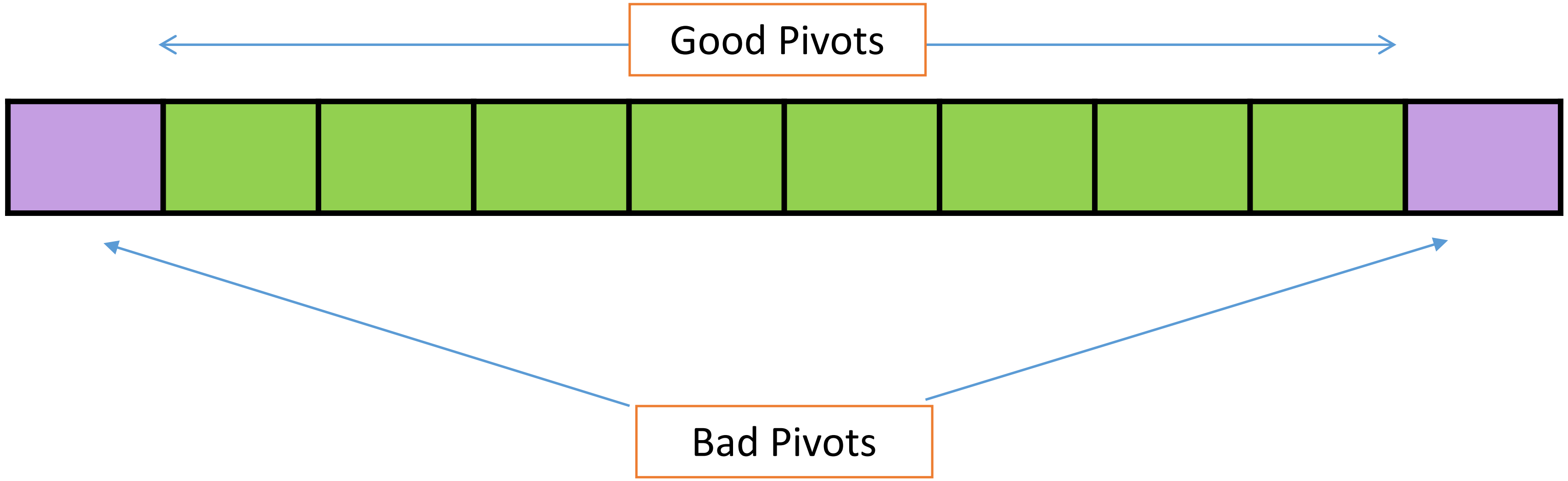
- How many repetitions until the partitioning is good?
- A pivot will make the partitioning good if
  - The pivot is larger than 10% of the numbers in the array, and
  - The pivot is smaller than 10% of the numbers in the array,
- If the array is sorted, and each of box below is 10% of the array



Question 1

- What is the probability of picking a good pivot randomly?
  - $p = 8/10$
- Will the probability be different when the list is not sorted?

- If the array is sorted, and each of box below is 10% of the array

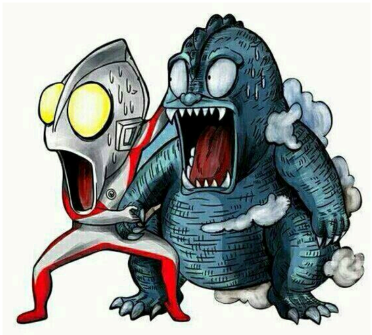


Question 1

- How many repetitions until the partitioning is good?
- The probability of picking a good pivot is $p = 8/10$
- If we pick a pivot randomly, how many times we need to pick such that we finally end up with a good pivot?

$$\#\text{time} = 1/p = 10/8 = 1.25$$

- It means, if pick a pivot randomly for two times (> 1.25), we will expect one of them is a good pivot!



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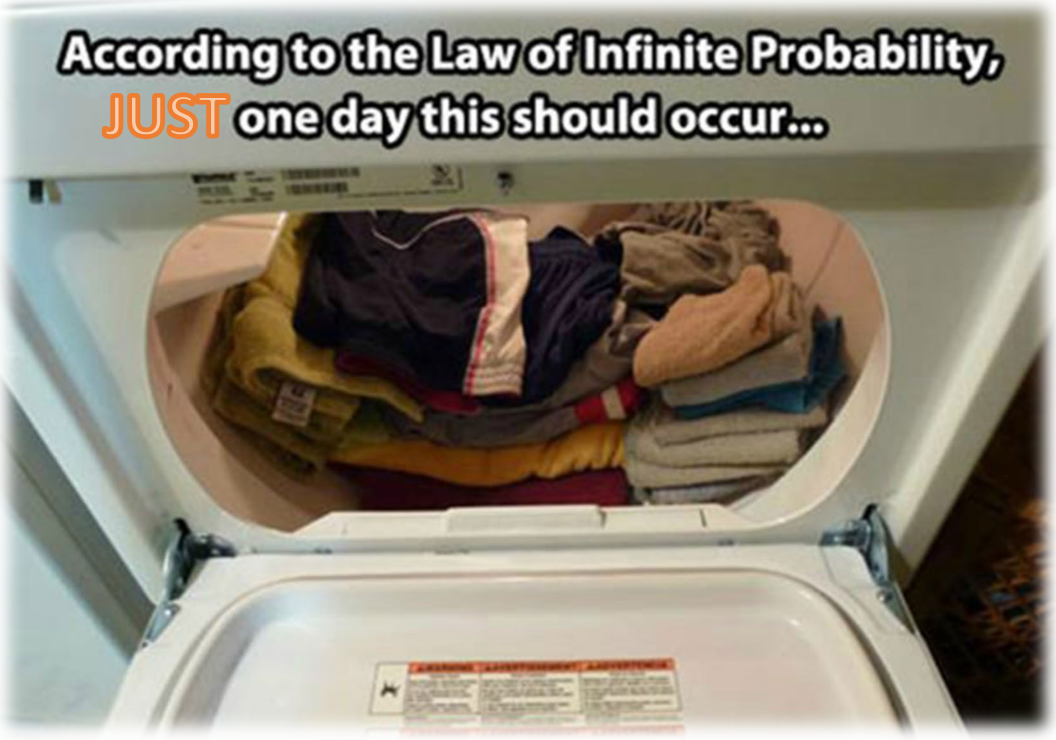

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        until p > (1/10)*n and p < (9/10)*n

        x = QuickSort(A[1..p-1], p-1)
        y = QuickSort(A[p+1..n], n-p)
```

Expected time = $O(n)$

Question 1

How many times we need to repeat?

We only need 2 = $O(1)$ times to expect a good pivot

Paranoid QuickSort

```
ParanoidQuickSort(A[1..n], n)
     if (n == 1) then return;
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          until p > (1/10)*n and p < (9/10)*n

          x = QuickSort(A[1..p-1], p-1)
          y = QuickSort(A[p+1..n], n-p)
```

Expected time = O(n)

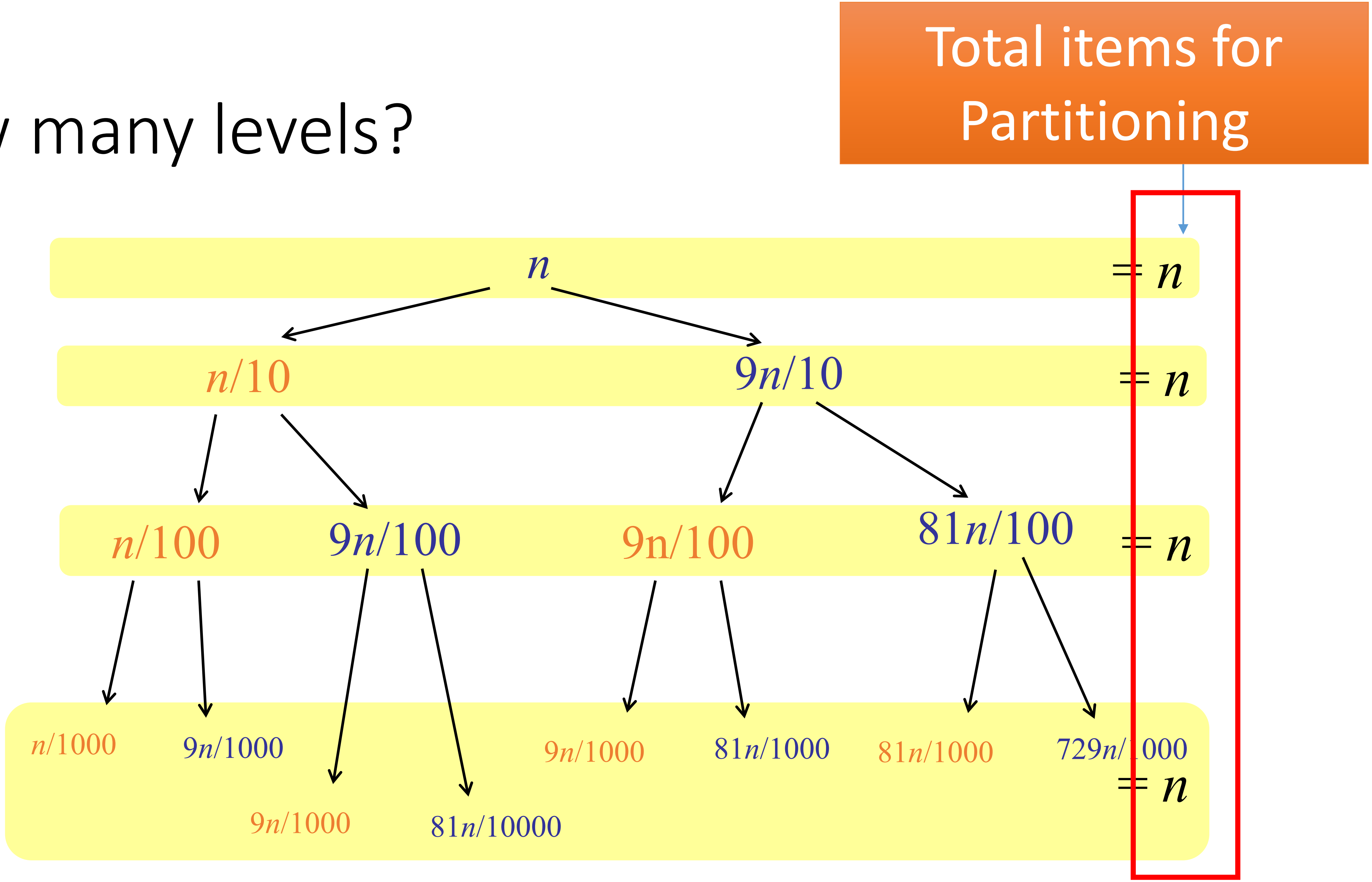
$$T(n) = cn + T(p) + T(n\text{-}p)$$

Paranoid QuickSort Time Complexity

- $T(n) = cn + T(p) + T(n\text{-}p)$
- And $p = n \times 1/10$ (or $p = n \times 9/10$)
- $T(n) = cn + T(n/10) + T(9n/10)$
- And we claim that

$$T(n) = \mathrm{O}(n \log n)$$

How many levels?



How many levels?

$$1 = n(9/10)^h$$

$$(10/9)^h = n$$

$$h = \log_{10/9}(n) = O(\log n)$$

n

$n/10$ $\quad$ $n(9/10)$

$n(9/10)^2$

$n(9/10)^3$

When the lowest level has only one element

$n(9/10)^h$

QuickSort Summary

- If we could split the array $(1/10) : (9/10)$
- Good performance: $O(n \log n)$

Paranoid QuickSort

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```

Expected time = $O(n)$

$$T(n) = O(n \log n)$$

QuickSort

- Key Idea:
    - Choose the pivot at random.
- Randomized Algorithms:
    - Algorithm makes decision based on randomness (coin flips)
    - Can “fool” the adversary (who provides bad input)
    - Running time is a random variable.

Randomization

Randomized algorithm:

- Algorithm makes random choices
- For every input, there is a good probability of success.

Average-case analysis:

- Algorithm (may be) deterministic
- “Environment” chooses random input
- Some inputs are good, some inputs are bad
- For most inputs, the algorithm succeeds

QuickSort Tips

- Optimize the partition routine
  - Most important aspect of a good QuickSort is partitioning.

- Choose a pivot carefully (e.g., at random)
  - Bad pivots lead to bad performance.

- Plan for arrays with duplicate values.
  - Equal elements can cause bad performance.

QuickSort Optimizations

- For small arrays, use InsertionSort.
  - Recursion has overhead.
  - QuickSort is slow on small arrays.
  - Idea: if the array is small, switch to InsertionSort

- Details:
  - Once recursion reaches a small array, use InsertionSort (instead of partition/recurse).
  - Once recursion reaches 8 elements, hand-code?

QuickSort Optimizations

- Two-pivot Quicksort
  - Recently shown that two pivots is faster than one!
  - Choose two pivots, partition around both.
  - What about three pivots? Four?
  - Experiment!

QuickSort Summary

- Algorithm basics: divide-and-conquer
- How to partition an array in O(n) time.
- How to choose a good pivot.
- Paranoid QuickSort.
- Randomized analysis.