MAT033 Pre-Algebra Lecture 1, part a: Integers

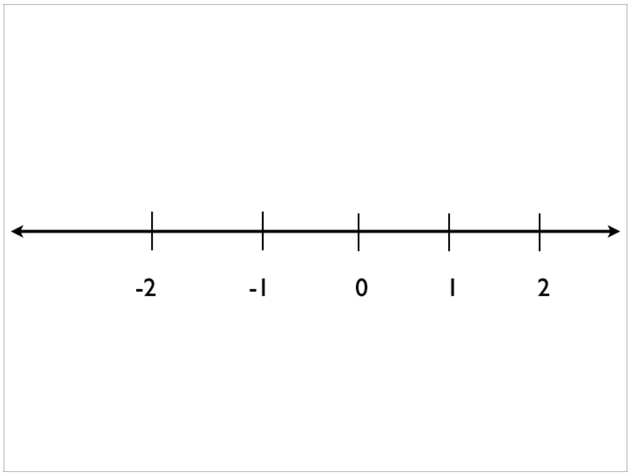
Today we will discuss **integers on the number line, i.e. negative and positive integers**.
Let's start with the example of a thermometer (*draw on board, somewhere where you do not have to erase*). Imagine it is 60 degrees F outside, but then the sun comes out and the temperature rises by 10 degrees. What is the new temperature? Now imagine a snow storm comes and the temperature drops by 40 degrees - what happens now? Finally, what happens when the sun goes down, and the temperature drops by 10, 20, 30 degrees?

Outline:

1. Number Line
2. Integers
3. Adding Negative Integers
4. Adding Integers of Opposite Sign
5. Subtracting Integers
6. Multiplying and Dividing Integers
7. Identity Properties
8. Order of Operations
9. Introducing Equations

1 The Number Line

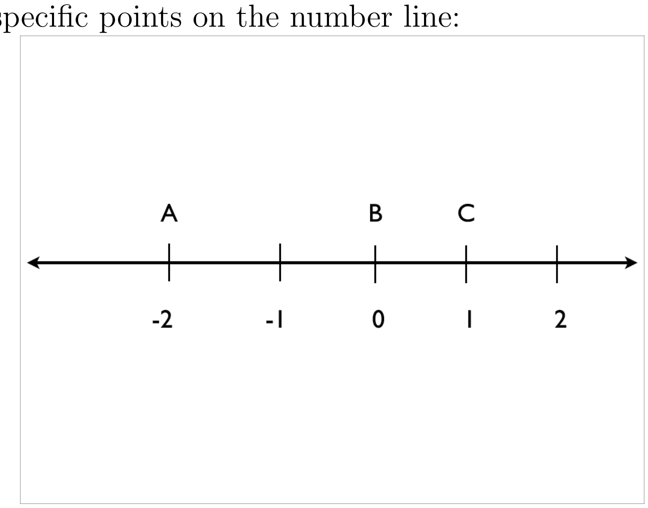
A thermometer is just one example of a number line. When we make horizontal number lines, we put negative numbers to the left of zero and positive numbers to the right.



Pairs of numbers that are the same distance from zero are called "opposites" or "additive inverses" of each other. For example, -4 and +4 are opposites.

2 Integers

Integers
The numbers "$1, 2, 3, 4...$" and their opposites "$-1, -2, -3, -4...$" and ZERO make up the integers.
Note that, strictly speaking, you should put a '+' sign in front of a positive number - for example, $+4 =$ positive 4. The $+$ sign is usually left out to avoid clutter, but its presence is always implicit.
Now, we can label specific points on the number line:



In this example $-2, 0, 1$ are the "coordinates" of A, B, C. On a number line the number to the right is always greater and numbers are always increasing to the right. So:

$$-5 < -2 \qquad -2 < 3 \qquad -5 < 3$$

or

$$-2 > -5 \qquad -1 > -3 \qquad 3 > -1$$

where $<$ means "is less than" and $>$ means "is greater than".
Here are some examples for you to try:

$$-3 \_\_\_\_\_\_ 3$$

$$-20 \_\_\_\_\_\_ -21$$

$$5 \_\_\_\_\_\_ 0$$

And now, name the opposite of each number:

$$-5 \qquad 10 \qquad -50$$

3 Adding Negative Integers

Now we will return to our example of the thermometer. Imagine it is -5 degrees Fahrenheit. Then the temperature drops by 10 degrees. What is the new temperature?

How can we use the number line to add -5 and -10? If we start at 0, which direction represents -5? What direction do we move to add -10?

Now we will practice using the number line to calculate sums:

$$-2 + -7$$

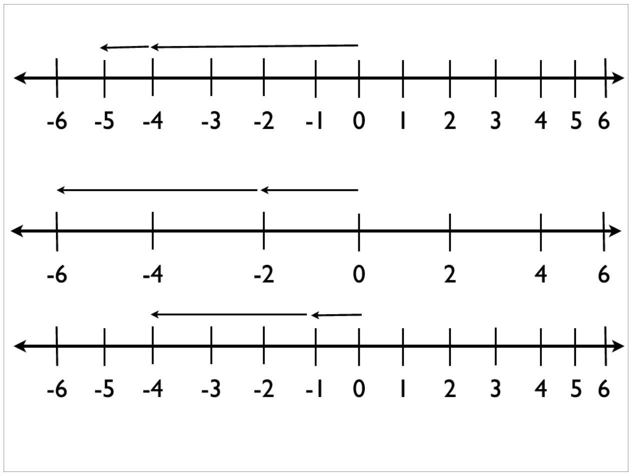
$$-1 + -10$$

$$-5 + -4$$

If we add two negative numbers, what will be the sign of the resulting number? How about if we add two positive numbers?

The sum of two positive numbers is a positive number.
The sum of two negative numbers is a negative number.

OK, for practice now I will draw some number lines, and we will decide what sum is represented.



4 Adding Integers of Opposite Sign

Going back to the thermometer, imagine that the winter is now ending. The temperature rises by 40 degrees from -10 degrees F. What is the final temperature?

We can use the number line to help us add integers of opposite sign.

Here are some examples:

$-3+2$ $\quad$ $-2+5$ $\quad$ $5+-7$ $\quad$ $6+-6$

This last example illustrates an important property:

The sum of two opposite numbers is 0.
$\mathrm{a}+-\mathrm{a}=0$ **and** $-\mathrm{a}+\mathrm{a}=0$

where "a" is shorthand for "any number". So for example

$$6 + \text{-}6 = 0$$
$$722 + \text{-}722 = 0$$

Note that here we have introduced the first algebraic concept. We're using "a" to stand for "any number". It is certainly easier to do this than to write out every number there is !

Here is a set of rules to help you add positive and negative integers without drawing a number line. First, we introduce the concept of *absolute value*. The *absolute value* of a number is its size without the sign. For example, the absolute value of 5 is 5. The absolute value of -5 is 5. The absolute value of a number is denoted by two vertical lines, like this: $|-5|=5$. Here are the rules:

1. First concentrate on the absolute value of each number, and subtract the larger from the smaller.

2. Notice the sign of the number that is farther from 0. This will be the *sign* of the sum.
3. Write the number found in (1) and the sign found in (2). That is the final answer.

Here is an example:

$$5 + -8$$

1. The difference between the absolute values of the numbers, 5 and 8, is $8 - 5 = 3$.
2. -8 is farther from 0 than 5, so the final answer will be negative.
3. $\boxed{5 + -8 = -3}$

Here is some practice. Draw a number line to represent each sum, then solve:

$$-4 + 2 \qquad\qquad -7 + 8 \qquad\qquad -5 + 5$$

Solve these without a number line:

$$-10 + -15 \qquad\qquad 0 + -3 \qquad\qquad 6 + -12$$

5 Subtracting Integers

Imagine the temperature in our thermometer starts at 20 degrees and then drops by 30 degrees. We can think of that temperature change *either* as adding 20 and -30 *or* as subtracting 30 degrees from 20 degrees.

Try solving each of these equations using the number line:

$$n = 3 - 5 \qquad\qquad n = 3 + -5$$

The two are equivalent! Here are a few more examples:

$$\begin{aligned} x &= 12 - 9 \qquad\qquad & x &= 12 + -9 \\ x &= 5 - 4 \qquad\qquad & x &= 5 + -4 \end{aligned}$$

Subtracting a number is equivalent to adding its opposite.

$$\mathrm{a} - \mathrm{b} = \mathrm{a} + -\mathrm{b}$$

Here are three examples:

$$8 - 3 \qquad\qquad -8 - 3 \qquad\qquad -8 - (-3)$$

The last example needs some more attention. What does it mean to “subtract a negative number”? Think about the number line: “+” means go to the right, “-” means go to the left. So “-” means “change direction”. If you have two negatives, as in

8 - (-3)

you change direction twice, so that

8 - (-3) = 8 + 3 = 11

In other words: two negative signs in front of the same number = positive.

8 + 3 = 11

8 - 3 = 5

8 + (-3) = 8 - 3 = 5

8 - (-3) = 8 + 3 = 11

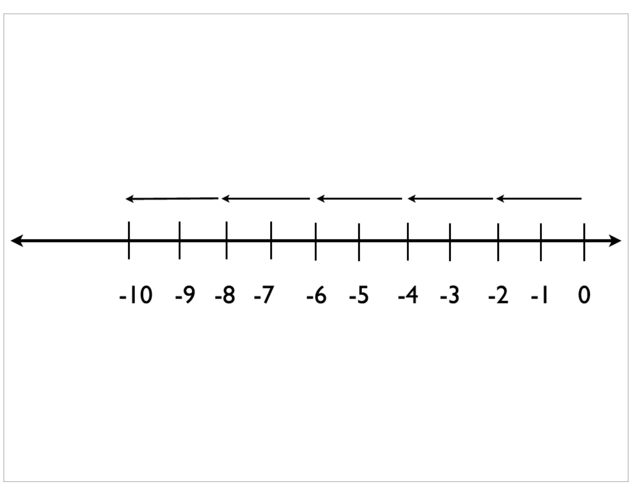
6 Multiplying and Dividing Integers

Returning to our thermometer example, imagine that the temperature starts at 0 and drops by two degrees every day for five consecutive days. Using addition, we have:

$$-2 + -2 + -2 + -2 + -2$$

This problem is easier to solve using multiplication:

$$5 \times (-2) = -10$$



We will consider now a sequence of multiplications:

$$
\begin{gathered}
5 \times -2 = -10 \\
4 \times -2 = -8 \\
3 \times -2 = -6 \\
2 \times -2 = -4 \\
1 \times -2 = -2 \\
0 \times -2 = 0 \\
-1 \times -2 = 2 \\
-2 \times -2 = 4
\end{gathered}
$$

What do you notice? How much does the answer increase with each step in the sequence? When we multiply a negative number and a positive number, what sign is the result? How about when we multiply two negative numbers together?
When we multiply two positive numbers together, the result is positive.

When we multiply two negative numbers, the result is positive.

When we multiply a positive number by a negative number, the product is negative.

Here is some practice:

$(-5)(4) =$ $\qquad$ $(-2)(0) =$ $\qquad$ $(-5)(-4) =$

$(4)(4)(-2) =$ $\qquad$ $(-6)(2)(-1) =$ $\qquad$ $(-2)(-3)(-1) =$

(note that writing the numbers in parentheses like this means: multiply the numbers. $(-2)(2)$ is the same as -2×2.

7 Dividing Integers

Every division is associated with a multiplication:

$$
\begin{gathered}
4 \times 3 = 12 - -- > 12 \div 3 = 4 \\
5 \times 2 = 10 - -- > 10 \div 2 = 5 \\
\mathrm{a} \times \mathrm{b} = \mathrm{c} - -- > \mathrm{c} \div \mathrm{b} = \mathrm{a}
\end{gathered}
$$

The same is true if we work with negative as well as positive integers:
$-5 \times 6 = -30 - -- > -30 \div 6 = -5$

A negative number divided by a positive number is negative.

$-5 \times -6 = 30 - -- > 30 \div -6 = -5$

A positive number divided by a negative number is negative.

$5 \times 6 = 30 - -- > 30 \div 6 = 5$

A positive number divided by a positive number is positive.

A negative number divided by a negative number is positive.

Practice:

$$x = -15 \div 5$$
$$x = 15 \div 5$$
$$x = -15 \div -5$$

8 Identity Properties

Zero and one are special numbers:

8.1 Additive Identity

Any number added to zero gives the number:

$$0 + 5 = 5$$
$$0 + 10 = 10$$
$$0 + -5 = -5$$
$$0 + -10 = -10$$

8.2 Multiplicative Identity

Any number multiplied by one gives the number:

$$1 \times 5 = 5$$
$$1 \times -5 = -5$$

Finally, multiplying by zero ALWAYS gives zero:

$$5 \times 0 = 0$$
$$500 \times 0 = 0$$

On the other hand, any number divided by zero is UNDEFINED.

9 Order of Operations

Sometimes it is important that we have specific rules to determine the order in which we perform certain operations. For instance:

$$3 + 4 \times 5$$

If we took (3+4) and then multiplied by 5, the answer would be 35. But if, instead, we take 4×5 and then add 3, the answer is 23. It is important to remember the rules for the order in which you perform different operations:

Parenthesis
Exponents
Multiplication
Division
Addition
Subtraction

One common way to remember is *Please excuse my dear aunt sally.* We'll be discussing exponents later in the class.
Practice:

$$(3+4)\times 5$$
$$3+3-5\times 2$$
$$(3\times 4)\times -2$$
$$6\times 5+2$$
$$6\times (5+2)$$

10 Equations

Since this is a pre-algebra class, we will practice working with equations as the class proceeds.

Here is an example of an equation:
$5 \quad = \quad 5$

The equation has three parts: the *left-hand side* (LHS), the equals sign, =, and the *right-hand side* (RHS).
Here is a slightly less trivial example:

$3 \quad + \quad 2 \quad = 5$

Now, if I add a number, let's say 7, to the RHS, the equation is no longer correct: 3+2 does not equal 5+7. To keep the equation correct, I have to add the 7 to *both* sides of the equation. i.e.

$3 \quad + \quad 2 \quad + 7 \quad = \quad 5 \quad + \quad 7$

So to keep the relationship between both sides, then ANYTHING YOU DO TO ONE SIDE OF THE EQUATION SHOULD BE DONE TO THE OTHER SIDE. So if for example you add 5 to the left hand side (LHS) of an equation you also have to add 5 to the right hand side (RHS). If you multiply the RHS by 2, you also have to multiply the LHS by 2. Doing the same operation to both sides of the equation keeps both sides equal.
Here are two examples:

$x - 3 = 4$

$x + 2 = 7$

In these equations, "x" (something denoted by a letter or a symbol rather than by a number) is the **unknown**, i.e. at the beginning, you do not know the numerical value of x. The equation tells you how to figure that out. **Solving the equation** means working out the numerical value of x which makes both sides of the equation equal. You can try guessing, of course, but that's not very efficient. To **solve** an equation means to manipulate the equation by doing the same operations to both sides of the equation until you have isolated the unknown on the LHS of the equation and a number on the RHS. This number is the value of x which makes both sides of the equation equal, i.e. is the **solution** to the equation. For example, consider the following equation:
$x - 3 = 4$
add 3 to both sides:
$x - 3 + 3 = 4 + 3$
$-3 + 3 = 0$, and $4 + 3 = 7$
so
$x = 7$
is the **solution**. But you're not quite done - you need to check your answer. Do this by going back to the original equation and replacing x by the value you have found, $x = 7$. The LHS is then 7 - 3 = 4. Look at the RHS. The RHS is indeed 4, so your answer is correct.

11 Solving Homework Problems

When you turn in your homework, please show the work you did to get to the answer. This is so that: (a) we can see that you know how to solve the problem; (b) if you made a mistake somewhere the mistake can be identified and corrected, and you are likely to get most of the credit for mostly-correct work; and (c) when you use your homeworks to study, careful and clear work is a lot easier to follow. So let's look at a couple of example problems and see what the answer should look like.

Example Problem 1: Solve the following equation:

$x - 4 = 7$

Answer; Solving this equation means finding the value of x which makes the left and right hand sides of the equation equal.

$x - 4 = 7$

$\quad +4 \quad +4$

$x = 7 + 4 = 11$

Check the answer: plug $x = 11$ into the LHS of the equation:

LHS= $x - 4 = 11 - 4 = 7$. Look at the RHS: RHS = 7.
So the answer is correct. Now write the answer:
$\boxed{\text{x = 11}}$

Example Problem 2:

I need to pay my rent (\$500) and car payment (\$200) this week, as well as buy groceries and gas (\$150). I have \$900 in my bank account. Can I afford to take my friend to the movies tonight? (the tickets are \$15 each).

Answer: How much am I going to spend this week?

Rent \$500
Car \$200
Groceries and gas \$150

Total: \$(500 + 200 + 150) = \$850. I therefore have

\$900 - \$850 = \$50

left over. The movie tickets cost me 2×\$15 = \$30. Since $50 > 30$,
$\boxed{\text{yes, I can afford to go to the movies tonight}}$.

Now this may all seem very tiresome, and it is, because you can do these simple problems in your head. But in the world we very quickly get into problems where the answers can't be figured out just by looking. However, the answers to those problems can be figured out using exactly the same methods as we used above, so let's get into the habit of writing out the work.