MAT033 Pre-Algebra Lecture 2, part a: Fractions I

Contents

In this lecture we are going to go over what fractions and rational numbers are, how we represent them in writing. A few terms we'll learn along the way are:

fraction	*numerator*	*denominator*	*improper fraction*
fraction in lowest terms	*mixed number*	*rational number*	*lowest common denominator*

1 What is a fraction?

Generally speaking, a fraction represents a portion of a whole. We will come back to this description in a little while to make it more precise, but this is the basic idea: if a whole can be divided up into an integer number of parts, then a group of those parts can be described as a fraction of the whole.

A fraction is typically written as a/b or

$$\frac{a}{b},$$

where a and b are integers (positive or negative whole numbers) and $b \neq 0$.

Example: For instance, $1/3$, $3/5$, $2/4$, and $0/6$ are all perfectly fine fractions. $3/0$, however, is not.

The upper number in a fraction is called the *numerator*, and the lower number is called the *denominator*. Another way to phrase the rule that $b \neq 0$ is to say **the denominator of a fraction cannot be 0**.

Example: The following line is divided into 6 segments:

$$- - - - - -$$

If I now divide the line into 2 pieces as follows:

$$- - - - \qquad - -$$

then what fraction of the whole line is represented by each piece?
Answer: The left part has 4 of the 6 segments, so it represents $4/6$ or $\frac{4}{6}$ of the whole, while the right part has $2/6$ of the whole.
(Note that $4/6$ can also be written as $2/3$ and $2/6$ can also be written as $1/3$. In just a little while, we'll return to this idea of how to rewrite the same fraction in different ways.)

Example: What fraction of an hour is 36 minutes?
Answer: An hour is divided into 60 minutes, so 36 minutes represents $36/60$ of an hour.

Example: First John eats his lunch, and then Susan eats hers immediately afterward. It takes the two of them an hour to finish lunch, and it took John 3 times as long to finish as it took Susan. How long did John take, and how long did Susan take?

Answer: Clearly the hour is divided into 4 pieces among the two of their lunches – 3 parts of the hour are devoted to John's lunch, and 1 part for Susan's. So, John's lunch takes $\frac{3}{4}$ of an hour, and Susan's takes $\frac{1}{4}$. $\frac{3}{4}$ of an hour is 45 minutes, and $\frac{1}{4}$ of an hour is 15 minutes.

1.1 Fractions as division

Notice that in each of the descriptions of fractions that we have had so far, the word "divide" came up? The definition of a fraction that we used was, *If a whole can be* **divided** *up into an integer number of parts, then a group of those parts can be described as a fraction of the whole.* Then we talked about a line being divided into segments, and an hour being divided into minutes. The reason this word keeps coming up is that a fraction is another way to represent a division problem.

$1 \div 2$ may be expressed as the fraction $1/2$. $2 \div 3$ may be expressed as the fraction $2/3$.

Can anyone now explain why the denominator (the "downstairs number") of a fraction cannot be 0?
Answer: The fact that fractions are representations of division problems should now make it clear why the denominator of a fraction cannot be 0 – you cannot divide by zero!

2 Representations of fractions

2.1 Improper fractions

Also, $2 \div 1$ may be expressed as the fraction $2/1$. Of course, $2 \div 1$ is just 2, so this shows that integers can be represented as fractions.

Example: $3/1 = 3$, because $3 \div 1 = 3$.

Example: $6/2 = 3$, because $6 \div 2 = 3$.

Example: $4 = 24/6$, because $24 \div 6 = 4$.

Example: $-5 = (-5)/1 = 5/(-1)$, because $-5 \div 1 = 5 \div (-1) = -5$.

These fractions might look a little funny, because the numerator (the "upstairs" number) is bigger than the denominator (the downstairs number). Such fractions are called *improper fractions*, but there's really nothing at all wrong with them.

These examples show that there are multiple ways to write a fraction. The simplest way to write $24/6$ as a fraction is $4/1$, because $24/6 = (6 \cdot 4)/(6 \cdot 1)$, so 6 goes into both the numerator and the denominator and may therefore be divided out of both, but 4 and 1 don't share any factors in common (besides 1).

There are other improper fractions, too, besides just integers written as fractions. For instance, $3 \div 2 = 3/2$ is an improper fraction, and $10 \div 8 = 10/8$ is an improper fraction. Again, despite the name, there's nothing at all wrong with improper fractions.

2.2 Fractions in lowest terms

This leads to the idea of representing fractions in lowest terms. The division problem $4 \div 8$ may be written as

$$\begin{aligned} 4 \div 8 = \frac{4}{8} & \\ &= \frac{4}{4 \cdot 2} \\ &= (4) \div (4 \cdot 2) \\ &= 1 \div 2 \\ &= \frac{1}{2}, \end{aligned}$$

because 4 goes into both the numerator and the denominator. When represented as $1/2$, the fraction is said to be in *lowest terms*, because there is no simpler way to write it. When represented as $4/8$, the fraction is not in lowest terms. But how can we find lowest terms for a fraction?

2.2.1 Multiplying and dividing fractions

Here's a preview of something you'll learn in future classes about doing arithmetic with fractions: When you multiply two fractions together, the solution is the product of the numerators divided by the product of the denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot b}{c \cdot d}.$$

Example:

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1 \cdot 1}{2 \cdot 2} = \frac{1}{4},$$

which should make sense, because half of a half is a quarter. (Half of half-an-hour is 15 minutes, or a quarter of an hour.)

And when you divide one fraction by another, you can take the quotient of the numerators and divide by the quotient of the denominators:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}.$$

Example:

$$\frac{4}{8} \div \frac{4}{4} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2},$$

which should make sense since we already saw above that $4/8 = 1/2$.

Again, you'll go over arithmetic with fractions in much more detail in the next couple classes, and these rules should make more sense in the future. For now, the important thing to remember is since dividing a number by 1 does not change its value, you can always divide a fraction by another fraction that is equal to 1 without changing the result.

2.2.2 Calculating the lowest terms versions of fractions

This can be very useful when trying to find the lowest terms representation of a fraction.

Any time the numerator and denominator share a common factor (besides 1), the fraction is not in lowest terms. You can always convert a fraction to lowest terms by dividing it by the number 1, represented as the greatest common factor of the numerator and denominator (the *gcf*) divided by itself (gcf/gcf).

Example: Consider the fraction 14/35. The factors of 14 are 2 and 7; the factors of 35 are 5 and 7; so the greatest common factor of 14 and 35 is 7. $14/35 \div 1 = 14/35$, so we will not change the value if we divide by 1, but we can change how we write the fraction.

$$\frac{14}{35} \div \frac{7}{7} = \frac{14 \div 7}{35 \div 7} = \frac{2}{5}.$$

So, 14/35 written in lowest terms is 2/5.

2.3 Mixed numbers

Sometimes, instead of writing improper fractions such as 3/2, people prefer to write another version of remainder form, called *mixed numbers*.

In the case of $3/2$, it may also be written as $1\frac{1}{2}$, read "one and one half", because 2 goes into 3 1 time, and the remainder is 1. The expression $1\frac{1}{2}$ really represents $1+\frac{1}{2}$. Another way to understand that

$$\frac{3}{2} = 1 + \frac{1}{2}$$

is to notice that 1 may be rewritten as $\frac{2}{2}$, and of course $\frac{2}{2}+\frac{1}{2}=\frac{3}{2}$.
(Note that when you add fractions that have like denominators, you add only the numerators, not the denominators. Otherwise, you'd get that $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$, which is the same as $\frac{1}{2}$, instead of getting $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}=1$, as you should.)

In general, the mixed number

$$N\frac{a}{b} = N + \frac{a}{b}$$

is equivalent to the fraction

$$\frac{(N \cdot b) + a}{b},$$

because

$$N + \frac{a}{b} = \frac{N \cdot b}{b} + \frac{a}{b} = \frac{(N \cdot b) + a}{b};$$

and the fraction

$$\frac{c}{d}, \qquad \text{where } c > d$$

may be represented as the mixed number

$$M\frac{e}{d},$$

where d goes into c M times with a remainder of e.

> **Example:** The improper fraction $36/16$ may be written as a mixed number. In order to do so, we first write it in lowest terms (this isn't necessary, but it's a good idea). The *gcf* of 36 and 16 is 4, and after dividing numerator and denominator by 4 we get $9/4$. To write this in lowest terms, we see that 4 goes into 9 2 times, with a remainder of 1, so we write
>
> $$\frac{36}{16} = \frac{9}{4} = 2\frac{1}{4}$$
>
> **Example:** The mixed number $-3\frac{5}{8}$ may be converted to an improper fraction as follows:
>
> $$-3\frac{5}{8} = -\frac{(3 \cdot 8) + 5}{8} = -\frac{24+5}{8} = -\frac{29}{8}.$$

Warning: People are sometimes sloppy and write multiplication of fractions without including the multiplication sign ($\times$ or $\cdot$). This isn't a good idea, and you shouldn't do it, but it happens sometimes. As a result, sometimes expressions like $2\frac{1}{4}$ might actually mean $2 \times \frac{1}{4}$ (which equals $1/2$, not $9/4$). In this class, we won't write expressions like that (at least not on purpose!), but you should be aware that you need to decide from context if such an expression represents a mixed number or a multiplication expression.

3 Rational numbers

3.1 Fractions as a class of numbers

We have seen that there are various classes of numbers. The natural numbers are positive integers/whole numbers, such as 1, 2, 3,.... The integers, which are positive or negative integers, and 0 (examples: -37, -20, -5, 0, 6, 18, 343, ...). Another class of numbers is the set of all numbers that can be written as fractions. These are called *rational numbers*. The term *rational* comes from the notion that fractions are *ratios* of integers, so all rational numbers are ratios of integers.

3.1.1 Densely packed

The rational numbers are said to be *densely* packed on the number line. This means that between any two rational numbers there is another rational number.

Can anyone tell me if this is the case with integers? Are the integers densely packed?
Answer: Let's sketch a number line. Between 3 and 5 we can find another integer – 4. But between 3 and 4, there are no other integers. So the integers are not densely packed.

Between 1/2 and 3/4 there are infinitely many other rational numbers. Let's sketch on a number line what this means. In a moment, we'll see how to find one.

We will want a fraction that is bigger than 1/2 but smaller than 3/4. To find such a number, we will first want to find the *lowest common denominator* (*lcd*) of 1/2 and 3/4, and represent both as fractions over double that denominator ($2\times lcd$). We'll return to this problem in a moment.

3.1.2 Lowest common denominators

A *common denominator* (or a "like denominator") of two fractions is a number that could be the denominator of both. In order for this to be the case, both denominators have to be factors of the common denominator. You can always find a common denominator between two fractions by multiplying the two denominators together.

Finding a common denominator, or a like denominator, of two fractions is a way to determine which fraction is larger. If both fractions are positive, then if you put them over a common denominator, the one with the larger numerator is the larger one. If both are negative, the opposite is true (the one with the larger numerator is the *smaller* one).

What if one fraction is positive and the other is negative? Can anyone tell me how we can figure out which is the larger one in that case? For instance, $-\frac{2}{3}$ compared with $\frac{5}{8}$, which is bigger?
Answer: The positive one is greater than the negative one!

Example: Find a common denominator between of $\frac{3}{10}$ and $\frac{4}{15}$, and determine which one is greater.
Answer: One way to find a common denominator is to multiply the denominators together. $10 \times 15 = 150$, so 150 will do as a common denominator. If we represent these two fractions with the lowest common denominator, we get

$$\frac{3}{10} = \frac{3}{10} \cdot \frac{15}{15} = \frac{3 \cdot 15}{10 \cdot 15} = \frac{45}{150}$$

and

$$\frac{4}{15} = \frac{4}{15} \cdot \frac{10}{10} = \frac{4 \cdot 10}{15 \cdot 10} = \frac{40}{150}.$$

It is now clear that 3/10 is larger than 4/15, because 45/150 is larger than 40/150. In both of these instances, we transformed to a common denominator form of the fractions by multiplying by 1 (written as 15/15 or 10/10), so as not to change the value of the fraction (multiplying by anything except 1 would have changed the value).

The *lowest common denominator* of two fractions is the smallest number that has the denominators of both fractions as factors. In other words, it's the *least common multiple* of the two denominators. In the case of 1/2 and 3/4, the *lcd* is 4, because 4 is the smallest number that has both 2 and 4 as factors.

Example: Find the *lcd* of $\frac{3}{10}$ and $\frac{4}{15}$, and express both as fractions over the *lcd*.
Answer: The least common multiple of 10 and 15 is 30, so 30 is the lowest common denominator of these two fractions. If we represent these two fractions with the lowest common denominator, we get

$$\frac{3}{10} = \frac{3}{10} \cdot \frac{3}{3} = \frac{3 \cdot 3}{10 \cdot 3} = \frac{9}{30}$$

and

$$\frac{4}{15} = \frac{4}{15} \cdot \frac{2}{2} = \frac{4 \cdot 2}{15 \cdot 2} = \frac{8}{30}.$$

Again, in both of these instances, we transformed to the lowest common denominator form of the fractions by multiplying by 1 (written as 3/3 or 2/2), so as not to change the value of the fraction.

Returning to the problem of finding a rational number between 1/2 and 3/4:
We first write both as fractions with a denominator that is double the lowest common denominator 4 (i.e., with a denominator of $2 \cdot 4 = 8$): $1/2 \mapsto 4/8$ and $3/4 \mapsto 6/8$. So, we want a number between 4/8 and 6/8. Clearly, 5/8 is such a number.

> **Example:** Find a rational number between 4/15 and 3/10.
> **Answer:** First, we write both as fractions whose denominators are twice the *lcd*, or twice 30, which is 60: $4/15 \mapsto 16/60$ and $3/10 \mapsto 18/60$. So, a rational number between these two rational numbers is clearly 17/60.

3.2 Rationals as decimals

It turns out that if you express a rational number as a decimal number, say by doing a division calculation on a calculator, you'll get either a terminating decimal or a repeating decimal.

For instance, $1/4 = 0.25$, which is a terminating decimal. $1/9 = 0.1111\ldots$, which is a repeating decimal. $3/11 = 0.27272727\ldots$, which is a repeating decimal.

A repeating decimal is often written with a bar over the repeating set of digits. For instance, $1/9 = 0.11\bar{1} = 0.\bar{1}$, and $3/11 = 0.\overline{27}$.

Conversely, any decimal that repeats forever is a rational number, and may be represented as a fraction, or a ratio of integers.

Decimals that do not repeat, however, are not rational numbers, and are called "irrational numbers." $\sqrt{2} \approx 1.41421356\ldots$ and $\pi \approx 3.1415926535\ldots$ are examples of irrational numbers. No matter how far out you follow these decimal representations, they will never settle into a repeating pattern.

3.3 Properties of rationals

Rational numbers have some of the same arithmetic properties as integers:

- The commutative property of addition and multiplication: $a/b + c/d = c/d + a/b$; $a/b \cdot c/d = c/d \cdot a/b$.

- The associative properties of each addition and multiplication: a/b + (c/d + e/f) = (a/b + c/d) + e/f; and likewise for multiplication.
- Additive identity: a/b + 0 = a/b.
- Additive inverse: a/b + (-a/b) = 0.

There are some other arithmetical properties of rational numbers, too, which we'll learn in coming lessons.