Mastering Algebra

A Building a Strong Foundation for Future Math

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Chapter 1

Introduction to Algebra

1.1 What is a Variable?

In algebra, a **variable** is a symbol, usually a letter, that represents an unknown or changing value. Variables allow us to write general rules and solve problems systematically. Instead of working with specific numbers, we use variables to describe relationships and patterns.

For example, consider the equation:

$$x + 5 = 10 \tag{1.1}$$

Here, x is a variable because we do not yet know its value. Our goal is to find x by making the equation true. To do this, we need to isolate x on one side of the equation.

$$x + 5 = 10 \tag{1.2}$$

Since x is being added to 5, we need to "undo" this addition. In mathematics, we use **inverse operations** to cancel out terms. The inverse operation of addition is **subtraction**, so we subtract 5 from both sides:

$$x + 5 - 5 = 10 - 5$$
$$x = 5$$

By subtracting 5 from both sides, we eliminate the +5 on the left, leaving us with $x = 5$. This is the solution to the equation.

In this case, $x = 5$, but in other equations, x could represent a different number. Variables help us generalize mathematical concepts and apply them to various situations. Now that we know $x = 5$ we can plug 5 in for x in the original equation and confirm that x = 5 makes sense in the equation.

$$5 + 5 = 10.\checkmark$$

One of the fundamental concepts in algebra is solving for x in an equation. This means finding the value of x that makes the equation true.

1.2 Solving for x

One of the fundamental concepts in algebra is solving for x in an equation. This involves using **inverse operations** to isolate x. Inverse operations are mathematical operations that undo each other, such as addition and subtraction or multiplication and division.

Consider the equation:

$$2x + 5 = 11 \tag{1.3}$$

To solve for x, we need to isolate it by performing inverse operations in the correct order.

- The term $+5$ is added to $2x$. Since addition and subtraction are inverse operations, we subtract 5 from both sides to cancel out the $+5$:

$$2x + 5 - 5 = 11 - 5$$
$$2x = 6$$

- Now, x is being multiplied by 2. The inverse operation of multiplication is **division**, so we divide both sides by 2:

$$\frac{2x}{2} = \frac{6}{2}$$

Since $\frac{2}{2} = 1$, we simplify:

$$1x = 3$$

- In algebra, $1x$ is simply written as x since multiplying by 1 does not change the value. Therefore:

$$x = 3$$

Thus, the solution is $x = 3$. By applying inverse operations step by step, we successfully isolated x.

$$2(3) + 5 = 11 \tag{1.4}$$

$$6 + 5 = 11\checkmark$$

1.3 Introduction to Algebraic Terms and Operations

In algebra, when you multiply a number by a variable, the result is called a **term**. For example:

$$2 \cdot x = 2x \tag{1.5}$$

Similarly, when you divide a number by a variable, the result is also a term in the form of a fraction:

$$3 \div w = \frac{3}{w} \quad \text{or} \quad x \div 7 = \frac{x}{7} \tag{1.6}$$

It is important to note that we prefer to write terms like $\frac{x}{7}$ rather than $\frac{1}{7}x$, as the former is more commonly used in algebraic expressions.

Additionally, **numbers and variables cannot be added or subtracted directly unless they share the same base and exponent**. In other words, you cannot add or subtract different powers of variables. Here are some key rules for combining like terms:

- $2 + x = x + 2$: Addition is commutative, so the order does not matter. But we prefer to have the variable before the number.
- $x - 2 - y = x - y - 2$: Subtraction is also commutative within the terms.

When we add or subtract variables that have the same base and exponent, we follow the distributive property. For example:

- $w + w = 2w$: This is similar to saying you have one apple, and if you add one more, you get 2 apples.
- $w - w = 0w = 0$, If you subtract the same quantity, the result is zero.
- $5p + 20p = 25p$: Like adding 5 apples and 20 apples to get 25 apples.
- $4o - 2o = 2o$: If you have 4 apples and take away 2, you're left with 2 apples.

When dealing with fractions, if the denominators are the same, you can add the numerators. For example:

- $\frac{x}{2} + \frac{x}{2} = \frac{2x}{2} = x$
- $\frac{4}{x} + \frac{4}{x} = \frac{8}{x}$

However, it is crucial to remember that **different variables cannot be added together** unless they are part of the same term. For example:

$$x + w + p + a = a + p + x + w \tag{1.7}$$

Multiplication of variables follows specific rules as well. For example:

- $x \cdot x = x^2$
- $x \cdot x \cdot x \cdot x \cdot x = x^5$
- $x \cdot w \cdot y = xyw$
- $2 \cdot x \cdot y = 2xy$
- $x \div z = \frac{x}{z}$

When dividing variables, they "cancel out" because division is essentially a form of multiplication. For example:

$$\frac{x}{xy} = \frac{1}{y} \tag{1.8}$$

$$\frac{-h}{(x+h)} \cdot \frac{1}{h} = \frac{-1}{(x+h)} \tag{1.9}$$

$$\frac{2xwy}{xzy} = \frac{2w}{z} \tag{1.10}$$

In this case, the x terms cancel each other out because they are just multiplying, and there's no addition or subtraction involved in the fraction. This cancellation only works because multiplication is the operation at play, so it's simply reducing the expression to what remains after the cancellation.

However, **you cannot cancel variables when there is addition or subtraction involved** in the terms. For example:

$$\frac{x+y}{xy} \neq 1 \tag{1.11}$$

Similarly:

$$\frac{2+xy}{xy} \neq 2 \tag{1.12}$$

The rules for combining terms, handling exponents, and simplifying fractions are essential for building a strong algebraic foundation. These rules will become crucial in solving more complex algebraic expressions, which will be applied in later chapters on polynomials, functions, and calculus.

1.4 Solving for Any Variable

If an equation contains multiple variables, solving for a specific variable follows the same principles we have already learned. Just because an equation does not contain numbers does not mean we cannot apply inverse operations to isolate a variable. The same rules of addition, subtraction, multiplication, and division still apply, along with inverse operations for exponents and square roots.

1.4.1 Basic Example

Consider the equation:

$$ap + w = b \tag{1.13}$$

To isolate p, we follow these steps:

- w is added to ap, so we subtract w from both sides (inverse of addition):

$$\begin{aligned} ap + w - w &= b - w \\ ap &= b - w \end{aligned}$$

- p is multiplied by a, so we divide both sides by a (inverse of multiplication):

$$\begin{aligned} \frac{ap}{a} &= \frac{b-w}{a} \\ p &= \frac{b-w}{a} \end{aligned}$$

1.4.2 Undoing Division with Multiplication

Now, let's consider a case where we are solving for a variable that is divided by another value. For example, suppose we have the equation:

$$\frac{k}{m} = w \tag{1.14}$$

Here, k is divided by m. To solve for k, we undo the division by multiplying both sides by m, which is the inverse of division:

$$m \cdot \frac{k}{m} = m \cdot w$$

On the left side, m cancels out with the denominator, leaving:

$$k = m \cdot w$$

Note that multiplication between m and w can also be written as mw. This is simply a more compact form of notation, and no operation is lost. Therefore, we can rewrite the equation as:

$$k = mw$$

Thus, the solution is $k = mw$, where we successfully used multiplication to undo the division and isolate k.

1.4.3 Using Exponents and Square Roots

We use the square root as its inverse operation if a variable is squared. Consider the equation:

$$k^2 = a \tag{1.15}$$

To solve for k, we take the square root of both sides:

$$\begin{aligned} \sqrt{k^2} &= \sqrt{a} \\ k &= \pm\sqrt{a} \end{aligned}$$

The $\pm$ symbol indicates that there are two possible solutions: one positive and one negative.

Similarly, if a variable is under a square root, we use exponentiation to undo it. For example:

$$\sqrt{w} = b \tag{1.16}$$

To solve for w, we square both sides (inverse of square root):

$$\begin{aligned} (\sqrt{w})^2 &= b^2 \\ w &= b^2 \end{aligned}$$

1.4.4 General Case with Exponents

If a variable is raised to a power, we use roots to isolate it. Consider:

$$a^3 = k \tag{1.17}$$

To solve for a, we take the cube root (inverse of cubing):

$$\begin{aligned} \sqrt[3]{a^3} &= \sqrt[3]{k} \\ a &= \sqrt[3]{k} \end{aligned}$$

In summary, no matter which variable we are solving for, we apply inverse operations step by step. Whether dealing with basic arithmetic, exponents, or square roots, the key is always to "undo" operations in the correct order.

1.5 Power Rules

Exponents are a powerful tool in algebra, and understanding the rules for manipulating them will serve as the foundation for more advanced mathematics. In this section, we will explore the key power rules that are essential when working with exponents.

1.5.1 Multiplying Powers with the Same Base

When multiplying powers that have the same base, you add the exponents. This rule is based on the principle that when multiplying like terms, you are effectively adding their counts. Mathematically:

$$x^a \cdot x^b = x^{a+b} \tag{1.18}$$

For example:

$$x^3 \cdot x^4 = x^{3+4} = x^7 \tag{1.19}$$

1.5.2 Dividing Powers with the Same Base

When dividing powers with the same base, you subtract the exponents. This is the inverse of multiplication, as you are essentially canceling out parts of the base. The rule is:

$$\frac{x^a}{x^b} = x^{a-b} \tag{1.20}$$

For example:

$$\frac{x^5}{x^2} = x^{5-2} = x^3 \tag{1.21}$$

1.5.3 Adding and Subtracting Powers with Different Bases

It is important to note that you cannot add or subtract powers with different exponents unless they have the same base. The exponents need to be the same to combine the terms in addition or subtraction. For instance, the following is not correct:

$$x^3 + x^4 \neq x^7 \tag{1.22}$$

In this case, x^3 and x^4 are two different terms with different exponents, so they cannot be combined into a single term by simple addition. You can only combine like terms, which means terms with the same base and the same exponent. This is just like having 1 apple and being given another apple, how many apples do you have? 2 apples, hence $2x^3$

For example, the following is correct:

$$x^3 + x^3 = 2x^3 \tag{1.23}$$

This is because both terms have the same base and exponent, so they can be added together, with the coefficient (the number in front) added.

1.5.4 Square Root Properties

When using multiplication with square roots, you can combine the inside terms and write them under one square root.

For example, the following is an example of how to multiply square roots:

$$\sqrt{x} \cdot \sqrt{y} = \sqrt{xy} \tag{1.24}$$

When adding and subtracting square roots you cannot combine the inside terms, you have to treat the square roots like they are their own variables "apples" or "bananas"" etc..

You have 1 apple and 1 banana what do you have in total? 1 apple and 1 banana.

$$\sqrt{x} + \sqrt{y} = \sqrt{x} + \sqrt{y} \tag{1.25}$$

$$\sqrt{x} + \sqrt{y} \neq \sqrt{xy} \tag{1.26}$$
$$\sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$$

You have 5 apples and 2 more are given. You have 7 apples.

$$5\sqrt{x} + 2\sqrt{x} = 7\sqrt{x} \tag{1.27}$$

You have 5 apples and you are given 2 bananas. You have 5 apples and 2 bananas

$$5\sqrt{x} + 2\sqrt{y} = 5\sqrt{x} + 2\sqrt{y} \tag{1.28}$$

1.5.5 Rationalizing Roots

Rationalizing a root means rewriting an expression so that no radicals remain in the denominator. This is often done by multiplying both the numerator and denominator by a suitable expression that eliminates the radical.

Given the expression:

$$\sqrt{\frac{24x^3}{8y}}$$

We begin by rewriting the radical as:

$$\frac{\sqrt{24x^3}}{\sqrt{8y}}$$

Step 1: Multiply by the Denominator's Square Root To eliminate the square root in the denominator, we multiply both the numerator and denominator by $\sqrt{8y}$:

$$\frac{\sqrt{24x^3} \cdot \sqrt{8y}}{\sqrt{8y} \cdot \sqrt{8y}}$$

Since $\sqrt{8y} \cdot \sqrt{8y} = 8y$, the denominator simplifies to:

$$\frac{\sqrt{24x^3} \cdot \sqrt{8y}}{8y}$$

Step 2: Distribute the Square Root in the Numerator
Using the property $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$:

$$\frac{\sqrt{(24x^3)(8y)}}{8y}$$

$$\frac{\sqrt{(192x^3y)}}{8y}$$

Now, we simplify inside the square root.
Step 3: Factorizing 192 and x^3
We break down $192 = 64 \cdot 3$ using a factor tree:

$$192 = 64 \times 3$$

Similarly, we break down x^3 into:

$$x^3 = x^2 \cdot x$$

Thus, we rewrite:

$$\sqrt{192x^3y} = \sqrt{(64x^2)} \cdot \sqrt{(3xy)}$$

Rewriting into:

$$\sqrt{(64x^2)} = 8x$$

So the expression simplifies to:

$$\frac{8x\sqrt{3xy}}{8y}$$

Step 4: Final Simplification
Canceling the 8: $\boxed{\frac{x\sqrt{3xy}}{y}}$

1.5.6 Using Conjugates to Rationalize

When the denominator contains a binomial with a square root, we multiply by its conjugate to eliminate the radical.
For example, given:

$$\frac{1}{\sqrt{3}+2}$$

Multiply by the conjugate $\sqrt{3} - 2$:

$$\frac{1}{\sqrt{3}+2} \cdot \frac{\sqrt{3}-2}{\sqrt{3}-2}$$

In the denominator:

$$(\sqrt{3}+2)(\sqrt{3}-2) = 3 + 2\sqrt{3} - 2\sqrt{3} - 4 = 3 - 4 = -1$$

Thus, we obtain the following.

$$\frac{\sqrt{3}-2}{-1} = -\sqrt{3}+2$$

This process ensures a rational denominator.

Polynomials: An Introduction

When you have a sum or difference of terms like these, where each term is a power of the same or different base, this expression is known as a **polynomial**. A polynomial is an algebraic expression consisting of multiple terms, which may include constants, variables, and exponents.

For example, an expression like:

$$5x^3 + 3x^2 + 4x + 5 \tag{1.29}$$

is a polynomial, where $5x^3$, $3x^2$, $4x$, and 5 are the terms of the polynomial. In this expression, the terms have different exponents, but each term is a power of x. We will explore polynomials in more detail in a future chapter, where we will learn how to factor them, solve polynomial equations, and understand their graphing properties.

Polynomials are a fundamental part of algebra, and mastering how to work with them is essential for progressing to more advanced topics in mathematics, including calculus and beyond.

Simplify the polynomial:

$$\left(x^2+2\right) + (x+5) - \left(3x^3 - 4x\right)$$

Step 1: Distribute the negative sign:

$$x^2 + 2 + x + 5 - 3x^3 + 4x$$

Step 2: Rearrange the terms:

$$-3x^3 + x^2 + x + 4x + 2 + 5$$

Step 3: Combine like terms:

$$-3x^3 + x^2 + 5x + 7$$

Thus, the simplified polynomial is:

$$\boxed{-3x^3 + x^2 + 5x + 7}$$

1.5.7 Rewriting Roots as Fractional Exponents

Roots can be rewritten as fractional exponents. In general, the n-th root of a number is written as:

$$\sqrt[n]{x} = x^{\frac{1}{n}} \tag{1.30}$$

For example, the square root of x is written as:

$$\sqrt{x} = x^{\frac{1}{2}} \tag{1.31}$$

The cube root of x is written as:

$$\sqrt[3]{x} = x^{\frac{1}{3}} \tag{1.32}$$

More generally, the n-th root of x^a can be written as:

$$\sqrt[n]{x^a} = (x^a)^{\frac{1}{n}} \tag{1.33}$$

This can be understood as a power raised to another power. According to the power of a power rule:

$$(x^a)^{\frac{1}{n}} = x^{a \cdot \frac{1}{n}} \tag{1.34}$$

So, the n-th root of x^a simplifies to:

$$\sqrt[n]{x^a} = x^{\frac{a}{n}} \tag{1.35}$$

For instance, if you have the cube root of x^5, it is written as:

$$\sqrt[3]{x^5} = \left(x^5\right)^{\frac{1}{3}} = x^{\frac{5}{3}} \tag{1.36}$$

This shows that taking the root of an exponent is equivalent to applying the power-to-a-power rule, where the exponent is multiplied by the fractional exponent.

In summary, taking the root of a number can be viewed as raising that number to a fractional exponent, where the numerator of the fraction is the exponent of the base and the denominator is the root.

1.5.8 Rewriting Fractions as Negative Exponents

When a base is in the denominator of a fraction, we can rewrite it as a negative exponent. This is based on the rule that moving terms between the numerator and denominator changes the sign of the exponent. For example:

$$\frac{1}{x^a} = x^{-a} \tag{1.37}$$

helps simplify simplifying complex expressions with fractions in the denominator.

Example with a Fraction

Consider the expression:

$$\frac{1}{3x^5} \tag{1.38}$$

We can apply the rule of negative exponents. First, we note that x^5 is in the denominator, so we move it to the numerator and make the exponent negative. This gives us:

$$\frac{1}{3x^5} = \frac{1}{3} \cdot x^{-5} \tag{1.39}$$

In this form, x^{-5} is now in the numerator, and the fraction is simplified. This rule works for any base and exponent and is particularly useful when dealing with complex expressions in algebra. It helps avoid cumbersome fractions and makes algebraic manipulation much easier.

Another example is:

$$\frac{1}{3x^5} = \frac{x^{-5}}{3} \tag{1.40}$$

This shows that moving the x^5 term from the denominator to the numerator results in the exponent becoming negative, making the expression easier to work with in subsequent steps.

1.5.9 Combining fractions with variables

Common Denominator

One of the fundamental skills in algebra is finding a common denominator when adding or subtracting fractions with variables. This skill is essential not only in basic algebra but also in more advanced mathematics, including calculus. article amsmath

1.6 Combining Fractions with Variables

1.6.1 Example 1:

Consider the following fraction: $\frac{1}{x} - \frac{1}{x+y}$

To subtract these fractions, we first find a common denominator.
This will almost ALWAYS result in **multiplying the two denominators together** to make a common denominator.

Step 1: Determine the Least Common Denominator (LCD)

The denominators are x and $x + y$. The least common denominator (LCD) is:

$$\text{LCD} = x(x + y)$$

Step 2: Rewrite Each Fraction with the LCD

Remember we are multiplying the denominator by the LCD. The rule of common denominators states: What you do on the **bottom** you also have to do to the **top**.

"In the end, you are essentially multiplying your fraction by 1, represented as a form of variables.
Therefore, multiplying by $\frac{x+y}{x+y}$

$$\frac{1}{x} = \frac{x+y}{x(x+y)}$$

$$\frac{1}{x+y} = \frac{x}{x(x+y)}$$

Step 3: Perform the Subtraction

$$\frac{x+y}{x(x+y)} - \frac{x}{x(x+y)}$$

Since the denominators are the same, we subtract the numerators:

$$\frac{x+y-x}{x(x+y)}$$

$$\frac{y}{x(x+y)}$$

Thus, the simplified expression is:

$$\frac{y}{x(x+y)}$$

—

Summary

To recap, we have covered the following power rules:

- When multiplying powers with the same base, add the exponents: $x^a \cdot x^b = x^{a+b}$.
- When dividing powers with the same base, subtract the exponents: $\frac{x^a}{x^b} = x^{a-b}$.
- You cannot combine powers with different bases unless they have the same exponent. For example, $x^3 + x^4 \neq x^7$, but $x^3 + x^3 = 2x^3$.
- When raising a power to another power, multiply the exponents: $(x^a)^b = x^{a \cdot b}$.
- Roots can be written as fractional exponents: $\sqrt[n]{x} = x^{\frac{1}{n}}$, and the n-th root of x^a can be written as $\sqrt[n]{x^a} = x^{\frac{a}{n}}$.
- Fractions in the denominator can be rewritten as negative exponents: $\frac{1}{x^a} = x^{-a}$. For example, $\frac{1}{3x^5} = \frac{x^{-5}}{3}$.

Mastering these power rules is essential for progressing to more advanced topics in mathematics. Later in calculus, you will encounter power rules again when working with derivatives and integrals. Understanding these rules now will make it much easier to learn more complex concepts and avoid difficulties when dealing with them later.

Conclusion

In this chapter, we covered fundamental concepts in algebra that will form the foundation for all the mathematics you'll encounter in this book and beyond. We started with understanding variables and solving for x, the cornerstone of algebraic problem-solving. From there, we introduced inverse operations, which are essential tools for isolating variables and systematically manipulating equations.

The principles we have covered in solving for variables and using inverse operations are not just limited to basic algebraic equations. In fact, these concepts are foundational to all areas of mathematics and science. In physics, for example, we frequently encounter equations where we need to isolate a variable, much like the equations we solved in this section. Consider the following physics problem:

$$F = ma \tag{1.41}$$

Where:

- F is the force,
- m is the mass, and
- a is the acceleration.

If we know the force F and the mass m, and we want to find the acceleration a, we can solve for a using inverse operations. Dividing both sides of the equation by m, we get:

$$a = \frac{F}{m}$$

This is an example of using the same inverse operation—division—that we've used throughout our algebraic work. Understanding how to manipulate equations in this way allows us to solve real-world problems efficiently, whether we're working in physics, engineering, or economics.

It is also important to recognize that the concepts learned in algebra are used throughout more advanced mathematics, such as calculus, differential equations, and complex analysis. For example, in calculus, you'll encounter derivatives and integrals where you'll need to apply the inverse operations of differentiation and integration. In differential equations, you will often need to isolate terms and apply inverse operations, much like solving for x in a simple equation. Similarly, in complex analysis, working with complex numbers and functions involves manipulating equations in ways that build upon the basics of algebra.

The core ideas of algebra never go away. As you progress through higher-level mathematics, these fundamental concepts form the building blocks of more complex topics. Therefore, it is crucial to master them now so that you will not struggle later on. Without a solid understanding of these basic concepts, you will find it much more difficult to tackle the more advanced topics, which require a strong foundation in algebra.

In conclusion, mastering algebra not only prepares you for future mathematical studies but also equips you with the problem-solving skills needed to approach challenges in various scientific and engineering fields. So, take the time to fully understand these operations—whether it's solving for variables, using exponents, or manipulating equations—because they will continue to be useful throughout your academic and professional life.

We then explored key power rules, including how to handle multiplication and division with exponents, rewrite roots as fractional exponents, and handle fractions in the denominator by converting them to negative exponents. These principles are critical in solving algebraic equations, simplifying expressions, and working with more complex mathematical models.

The ability to manipulate algebraic expressions using these rules is central to your success in higher-level mathematics. These power rules, for example, will be revisited repeatedly in calculus, where they are essential for taking derivatives and integrals. Understanding how to rewrite roots and work with exponents will also be crucial when you study topics like differential equations and complex analysis.

By mastering the algebraic concepts covered in this chapter, you are building a strong foundation that will support your future studies. As we progress in this book, you'll see how these basic principles are applied in more advanced contexts, such as polynomials, rational expressions, and functions. The skills you develop here will also be invaluable for real-world applications in physics, engineering, economics, and beyond.

In conclusion, the ideas you've learned in Chapter 1 are not just isolated to the world of algebra. They are the building blocks for the more advanced topics you'll encounter throughout your academic career. Whether you're tackling derivatives in calculus or solving real-world problems, understanding these core concepts will make it easier for you to handle more complex subjects later on. Keep practicing these fundamental techniques, as they will continue to be useful and essential in your mathematical journey.

Inquisitive Activity: Chapter 1 Introduction to Algebra

Multiple Choice Questions

1. What is a variable in algebra?

   a) A number that changes its value.
   b) A symbol that represents a changing or unknown value.
   c) A mathematical operation.
   d) A constant value that cannot change.

2. What is the inverse operation of multiplication?

   a) Division
   b) Addition
   c) Subtraction
   d) Exponentiation

3. Which of the following expressions represents the correct way to isolate x in the equation $2x + 5 = 11$?

   a) Subtract 5 from both sides and divide by 2.
   b) Subtract 2 from both sides and divide by 5.
   c) Multiply both sides by 5 and subtract 2.
   d) Add 5 to both sides and multiply by 2.

4. What happens when you add $w + w$?

   a) $2w$
   b) w^2
   c) w
   d) It cannot be simplified.

5. Which of the following is the correct result when simplifying $\frac{x}{x}$?
   a) 1
   b) x
   c) 0
   d) Undefined

6. Which of the following is true about terms with different exponents?
   a) They can be combined in addition and subtraction.
   b) They cannot be combined unless they have the same base and exponent.
   c) They can always be combined by multiplying them.
   d) They cancel out.

7. What is the result of multiplying $x \cdot x \cdot x$?
   a) x^2
   b) x^3
   c) x^6
   d) $2x^2$

8. Which of the following expressions is equivalent to $\frac{2xwy}{xzy}$?
   a) $\frac{2w}{z}$
   b) $\frac{2w}{xz}$
   c) $\frac{2x}{zy}$
   d) $2xwy$

9. When dividing two variables with the same base and exponent, which rule applies?
   a) You subtract the exponents.
   b) You add the exponents.
   c) You cancel out the variables.
   d) You cannot divide variables with the same base and exponent.

10. What happens when you try to cancel variables that are added in a fraction, for example, $\frac{x+y}{xy}$?
    a) It simplifies to 1.
    b) It remains the same.
    c) It cannot be simplified.
    d) It simplifies to $\frac{x}{y}$.

True or False

1. $x \cdot x \cdot x \cdot x = x^4$. **(True/False)**
2. You can add or subtract terms with different exponents if they have the same base. **(True/False)**
3. The square root of x^2 is simply x. **(True/False)**
4. The expression $\frac{x+2}{x}$ simplifies to 2). **(True/False)**
5. When dividing powers with the same base, you subtract the exponents. **(True/False)**
6. $\frac{1}{x^a} = x^{-a}$. **(True/False)**

Short Answer

1. What is the inverse operation of addition, and how is it used to solve the equation $x + 3 = 8$?
2. Simplify the following expression: $3x + 5x$.
3. What is the result when x is divided by x^2?
4. Write $\sqrt{x^4}$ using exponents.
5. Explain why you cannot cancel out terms in an expression with addition or subtraction.

Problem Solving

1. Solve for x in the equation $4x - 7 = 9$.
2. Solve for y in the equation $3y + 5 = 20$.
3. If $\frac{m}{n} = p$, solve for m in terms of n and p.
4. Solve the equation $x^3 = 27$.

Reflection

- Reflect on the importance of understanding inverse operations in solving algebraic equations. Why are they necessary?
- Describe how the rules for combining terms, multiplying, and dividing exponents will be helpful in future math courses.

Answer Key

Multiple Choice Questions

1. b) A symbol that represents a changing or unknown value.
2. a) Division
3. a) Subtract 5 from both sides and divide by 2.
4. a) $2w$
5. a) 1 (assuming $x \neq 0$)
6. b) They cannot be combined unless they have the same base and exponent.
7. b) x^3
8. a) $\frac{2w}{z}$
9. a) You subtract the exponents.
10. c) It cannot be simplified.

True or False

1. **True**
2. **False**
3. **False** (The square root of x^2 is $|x|$ in general.)
4. **False**
5. **True**
6. **True**

Short Answer

1. The inverse operation of addition is subtraction. To solve $x + 3 = 8$, subtract 3 from both sides: $x = 5$.
2. $3x + 5x = 8x$.
3. $\frac{x}{x^2} = x^{1-2} = x^{-1} = \frac{1}{x}$.
4. $\sqrt{x^4} = x^{4/2} = x^2$.
5. Terms in an expression with addition or subtraction cannot be canceled unless they are factored properly because division only applies to factors, not separate terms.

Problem Solving

1. $4x - 7 = 9$
   Add 7 to both sides: $4x = 16$
   Divide by 4: $x = 4$.

2. $3y + 5 = 20$
   Subtract 5 from both sides: $3y = 15$
   Divide by 3: $y = 5$.

3. $\frac{m}{n} = p$
   Multiply both sides by n: $m = pn$.

4. $x^3 = 27$
   Take the cube root: $x = \sqrt[3]{27} = 3$.

Reflection

- Understanding inverse operations is crucial for solving equations because it allows us to efficiently isolate variables and determine their values. .

- The rules for combining terms, multiplying, and dividing exponents are essential for simplifying algebraic expressions, solving equations, and working with advanced mathematical concepts in calculus and beyond.

Chapter 2

Linear Equations

2.1 What is a Function?

In mathematics, a **function** is a relationship between a set of **inputs** and a set of possible **outputs**. Each input is related to exactly one output. The notation $f(x)$ is used to represent a function of x, where x is the independent variable, and $f(x)$ is the dependent variable or output. The expression $f(x)$ can be read as "the function of x" or "the value of the function at x."

For example, if $f(x) = 2x + 3$, the output of the function depends on the value of x. When $x = 1$, we can substitute it into the equation to find the output:

$$f(1) = 2(1) + 3 = 5$$

Thus, the function at $x = 1$ is 5.

2.2 The Equation of a Line: $y = mx + b$

A linear equation expresses a straight-line relationship between two variables, typically denoted x and y. The equation $y = mx + b$ is called the slope-intercept form of a linear equation, where:

- m represents the slope of the line, which indicates how steep the line is.
- b represents the y-intercept, which is the point where the line crosses the y-axis.

The slope m is calculated as the ratio of the change in y (vertical change) to the change in x (horizontal change). The y-intercept b is the value of y when $x = 0$.

Let's professionalize this with a specific example:

$$y = 2x + 1$$

In this equation:

- x and y are variables.
- x is the input variable, and y is the output variable.
- The slope $m = 2$, and the y-intercept $b = 1$.

2.2.1 Example: Solving for y

Question 1: If we are asked to replace x with 2, i.e., $x = 2$, what would the output for y be?

Substitute $x = 2$ into the equation:

$$y = 2(2) + 1 = 4 + 1 = 5$$

So, the output for y when $x = 2$ is 5.

2.3 Graphing Linear Equations Using an x-y Table

To graph a linear equation, we can create a table of values for x and solve for y using the equation. Here is an example with the equation $y = 2x + 1$:

x	$y = 2x + 1$
-2	-3
-1	-1
0	1
1	3
2	5

Now, plot these points on the coordinate plane and draw a straight line through them.

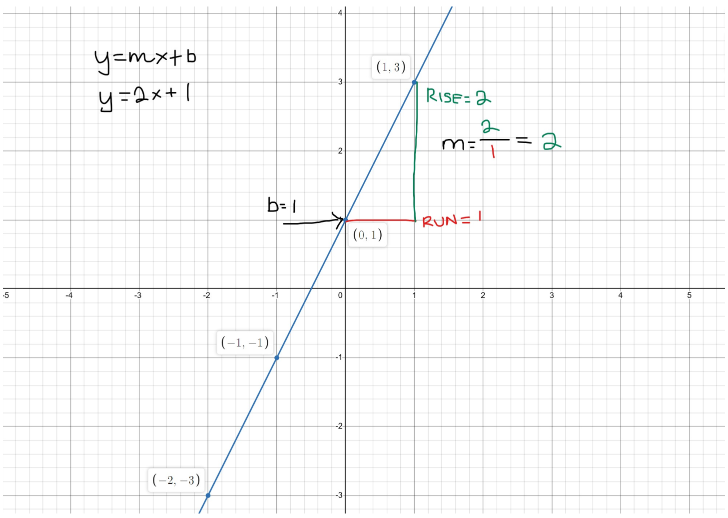


Figure 2.1: Graph of the linear equation $y = 2x + 1$

We can easily spot the slope m and the y-intercept b by looking atTots graph. To find the slope we can take the rise of the graph which is 2 units and divide it by the run which is 1 unit $2 \div 1 = 2$ so $m = 2$.

To find the y-intercept we can simply look at where the line intercepts the y-axis. We see that the line crosses the y-axis at $y = 1$ therefore our $b = 1$, completing $y = mx + b = 2x + 1$.

2.4 Standard Form of a Linear Equation

A linear equation can also be written in **standard form**, which is:

$$Ax + By = C$$

where A, B, and C are constants, and A and B are not both zero. In this form, the equation represents a straight line, and we can solve for y to rewrite it in slope-intercept form if necessary.

2.5 Solving for the equation of a line

As discussed earlier, the slope-intercept form of a linear equation is:

$$y = mx + b$$

where m is the slope, and b is the y-intercept.

2.5.1 Finding the Slope Between Two Points

When we are given two points, we can calculate the slope using the following formula:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

where (x_0, y_0) and (x_1, y_1) are the coordinates of the two points.

Example: Finding the Equation of a Line

Find the equation of the line in slope-intercept form passing through the points $(4, 5)$ and $(2, -1)$.

Step 1: Find the slope

Using the slope formula:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Substitute the values $x_0 = 4$, $y_0 = 5$, $x_1 = 2$, and $y_1 = -1$:

$$m = \frac{-1 - 5}{2 - 4} = \frac{-6}{-2} = 3$$

Step 2: Write the equation in slope-intercept form

We now have the slope $m = 3$. Use the slope-intercept form $y = mx + b$. Substitute $m = 3$:

$$y = 3x + b$$

Now, we will solve for b. Use the point $(4, 5)$, where $x = 4$ and $y = 5$, and substitute into the equation:

$$5 = 3(4) + b$$

$$5 = 12 + b$$

Subtract 12 from both sides:

$$-7 = b$$

Final Equation:

$$y = 3x - 7$$

We can now verify this equation by plotting the points $(4, 5)$ and $(2, -1)$ on a graph. Using a graphing calculator, such as Desmos, we can confirm that the line passes through both points.

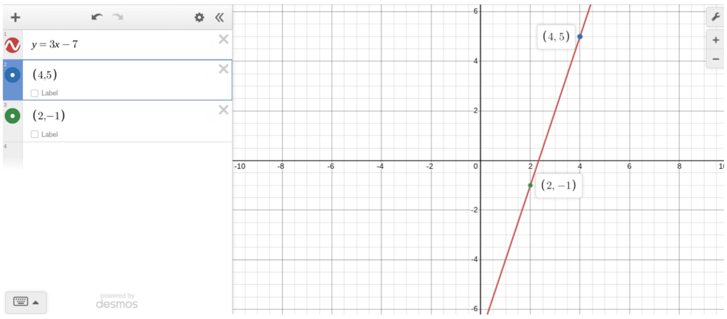


Figure 2.2: Graph of linear equation passing through $(4, 5)$ and $(2, -1)$

2.5.2 Finding the Equation of Parallel and Perpendicular Lines

Given a line with the equation:

$$y = mx + b$$

Where m is the slope, we want to find:

1. A line parallel to this line passing through a given point (x_1, y_1). 2. A line perpendicular to this line passing through the same point.

Finding the Parallel Line

A parallel line has the same slope m, so its equation is:

$$y - y_1 = m(x - x_1)$$

Which simplifies to:

$$\boxed{y = m(x - x_1) + y_1}$$

Finding the Perpendicular Line

A perpendicular line has a slope that is the negative reciprocal of m, which is:

$$m_\perp = -\frac{1}{m}$$

Thus, the equation of the perpendicular line passing through (x_1, y_1) is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Which simplifies to:

$$\boxed{y = -\frac{1}{m}(x - x_1) + y_1}$$

These equations allow us to determine parallel and perpendicular lines easily.

Given the line:

$$y = 2x + 3$$

We will find:
1. A line parallel to this line that passes through the point $(4, 5)$.

2. A line perpendicular to this line that passes through the same point.

Example 1: Parallel Line
A parallel line has the same slope $m = 2$. Using the point-slope formula:

$$y - y_1 = m(x - x_1)$$

Substituting $(x_1, y_1) = (4, 5)$ and $m = 2$:

$$y - 5 = 2(x - 4)$$

Expanding:

$$y - 5 = 2x - 8$$

Solving for y:

$$y = 2x - 3$$

Thus, the equation of the parallel line is:

$$\boxed{y = 2x - 3}$$

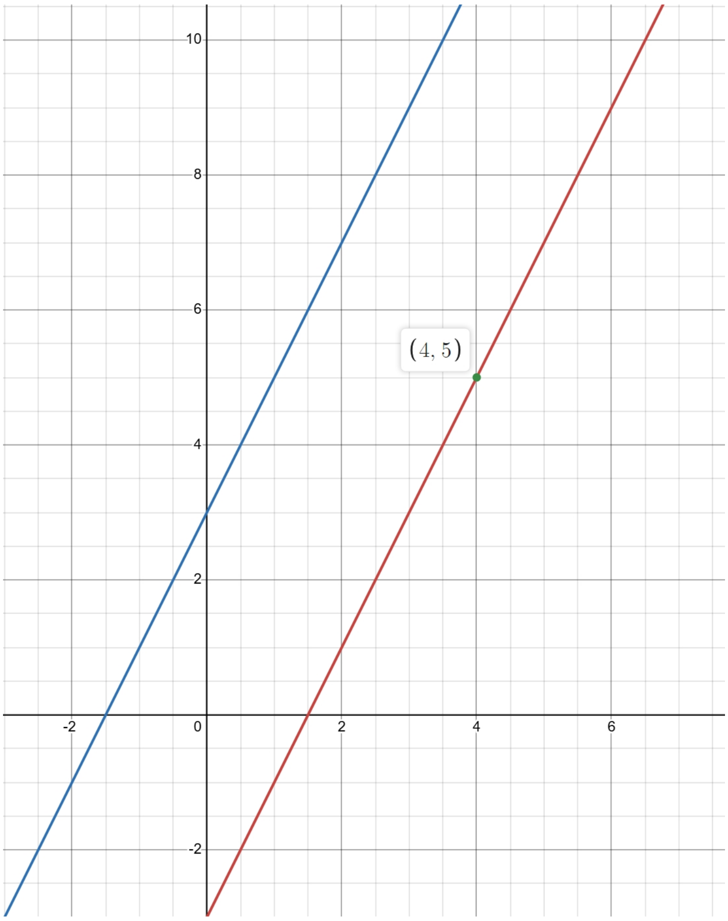


Figure 2.3: Visual proof that the linear equation meets the requirements

Example 2: Perpendicular Line A perpendicular line has a slope that is the negative reciprocal of $m = 2$:

$$m_\perp = -\frac{1}{2}$$

Using the point-slope formula:

$$y - y_1 = m_\perp(x - x_1)$$

Substituting $(4, 5)$ and $m_\perp = -\frac{1}{2}$:

$$y - 5 = -\frac{1}{2}(x - 4)$$

Expanding:

$$y - 5 = -\frac{1}{2}x + 2$$

Solving for y: $y = -\frac{1}{2}x + 7$

Thus, the equation of the perpendicular line is:

$$\boxed{y = -\frac{1}{2}x + 7}$$

These examples demonstrate how to find parallel and perpendicular lines given a point.

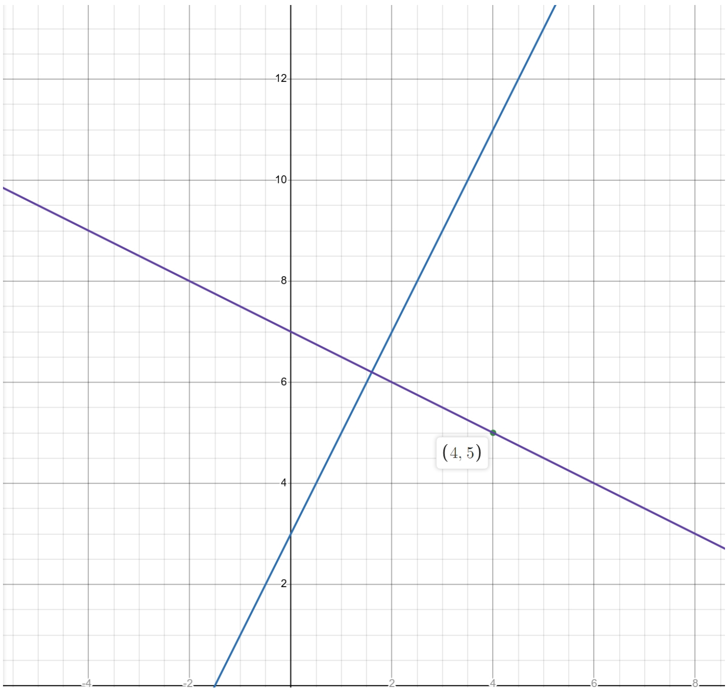


Figure 2.4: Visual proof that the linear equation meets the requirements

Recap

In this section, we explored the fundamental concepts of linear equations, including finding the slope between two points, determining the equation of a line, and identifying parallel and perpendicular lines. Using the slope-intercept form, we derived equations for given conditions and verified them graphically.

By applying these methods, we can analyze relationships between points and lines efficiently, providing a foundation for more advanced algebraic and geometric applications. Understanding these principles is essential for solving real-world problems involving rates of change, optimization, and spatial relationships.

2.6 System of Linear Equations

A system of linear equations consists of two or more equations involving the same set of variables. One of the most common methods for solving such systems is the **elimination method**, which involves eliminating one of the variables by adding or subtracting the equations.

2.6.1 Solving by Elimination

Consider the following system of equations:

$$2x + 3y = 8 \tag{2.1}$$

$$4x - y = 10 \tag{2.2}$$

To eliminate y, we multiply the second equation by 3 to make the coefficients of y opposites:
Remember! What we do to one side of the equation we always have to do to the other side! Learn how to distribute in section 3.2.4.

$$\begin{aligned} 2x + 3y &= 8 \\ 3(4x - y) &= 3(10) \end{aligned}$$

It is important to note that you can choose which variable to eliminate first. In this case, we chose to eliminate y, but we could have just as easily eliminated x instead. There is no wrong choice—either approach will lead to the correct solution as long as the algebra is performed correctly.

$$\begin{aligned} 2x + 3y &= 8 \\ 12x - 3y &= 30 \end{aligned}$$

Now, adding both equations and combining like results in:

$$\begin{aligned} (2x + 3y) + (12x - 3y) &= 8 + 30 \\ 14x &= 38 \\ x &= \frac{38}{14} = \frac{19}{7} \end{aligned}$$

Substituting $x = \frac{19}{7}$ into the first equation:

$$\begin{aligned} 2\left(\frac{19}{7}\right) + 3y &= 8 \\ \frac{38}{7} + 3y &= 8 \\ 3y &= 8 - \frac{38}{7} \\ 3y &= \frac{56}{7} - \frac{38}{7} \\ 3y &= \frac{18}{7} \\ y &= \frac{6}{7} \end{aligned}$$

Thus, the solution is $\left(\frac{19}{7}, \frac{6}{7}\right)$.

2.6.2 Special Cases

Sometimes a system of equations has no solution or infinitely many solutions. Consider the following cases:

No Solution

$$2x + 3y = 5 \tag{2.3}$$
$$4x + 6y = 12 \tag{2.4}$$

Multiplying the first equation by 2:

$$2(2x + 3y) = 2(5)$$

$$4x + 6y = 10$$
$$4x + 6y = 12$$

This contradicts the second equation $4x + 6y = 12$, leading to:

$$10 = 12,$$

which is a false statement. Hence, the system has **no solution** and is inconsistent.

Infinite Solutions

$$x - 2y = 4 \tag{2.5}$$
$$2x - 4y = 8 \tag{2.6}$$

Multiplying the first equation by 2:

$$\begin{aligned} 2(x - 2y) &= 2(4) \\ 2x - 4y &= 8 \\ 2x - 4y &= 8 \\ 8 &= 8 \end{aligned}$$

Since this is identical to the second equation, it means that both equations represent the same line. Thus, there are **infinitely many solutions**.

Conclusion

Although we may not use systems of equations frequently in upcoming topics, they will return in **Linear Algebra**. There, we will explore different techniques such as matrix row reduction, determinants, and Cramer's Rule to solve systems more efficiently.

2.7 Real World Systems Problem: Hot Dogs and Popcorn

Suppose a concession stand sells hot dogs for \$6 each and popcorn for \$4 each. A customer buys a total of 10 items, spending \$48 in total. We want to determine how many hot dogs and how much popcorn was purchased.

2.7.1 Defining the Variables

Let:

- x be the number of hot dogs purchased.
- y be the number of popcorns purchased.

2.7.2 Setting Up the System of Equations

We are given the following information:

1. The total number of items purchased is 10:

$$x + y = 10$$

2. The total cost of the items is \$48:

$$6x + 4y = 48$$

2.7.3 Subsitution Method

We solve for x and y using the substitution or elimination method.
In this problem we will teach you the **substitution method**.

Step 1: Solve for y in Terms of x
From the first equation:

$$y = 10 - x$$

Step 2: Substitute $10 - x$ into the Second Equation for y

$$6x + 4y = 48$$

$$6x + 4(10 - x) = 48$$

Distribute Learn in section 3.2.4:

$$6x + 40 - 4x = 48$$

$$2x + 40 = 48$$

Step 3: Solve for x

$$2x = 8$$

$$x = 4$$

Step 4: Solve for y

$$y = 10 - 4 = 6$$

Final Answer

The customer purchased:

4 hot dogs and **6** popcorns.

Item	Quantity Purchased
Hot Dogs	4
Popcorn	6

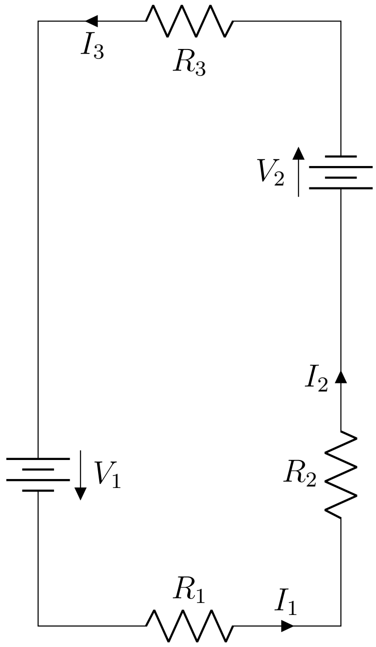
Thus, we have successfully solved the system of linear equations using substitution.

2.8 Application in Engineering: Kirchhoff's Rules in Electrical Circuits

In electrical circuit analysis, Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL) help us analyze complex circuits by forming a system of linear equations.

2.8.1 Example: Solving for Currents in a Circuit

Consider the following circuit, where we have two voltage sources and three resistors forming two loops. Our goal is to determine the unknown currents flowing through each resistor.



Defining the Variables

Let:

- I_1 be the current flowing through R_1.
- I_2 be the current flowing through R_2.
- I_3 be the current flowing through R_3.

Applying Kirchhoff's Current Law (KCL) at a junction, we know that:

$$I_1 = I_2 + I_3$$

Applying Kirchhoff's Voltage Law (KVL)

For the left loop containing V_1, R_1, and R_2:

$$V_1 - I_1 R_1 - I_2 R_2 = 0$$

For the right loop containing V_2, R_2, and R_3:

$$V_2 - I_2 R_2 - I_3 R_3 = 0$$

Solving the System

We now have three equations:

$$\begin{aligned} I_1 &= I_2 + I_3 \\ V_1 - I_1 R_1 - I_2 R_2 &= 0 \\ V_2 - I_2 R_2 - I_3 R_3 &= 0 \end{aligned}$$

Substituting $I_1 = I_2 + I_3$ into the first KVL equation:

$$V_1 - (I_2 + I_3) R_1 - I_2 R_2 = 0$$

Rearrange:

$$V_1 = I_2 (R_1 + R_2) + I_3 R_1$$

From the second KVL equation:

$$V_2 = I_2 R_2 + I_3 R_3$$

Thus, our system of equations is:

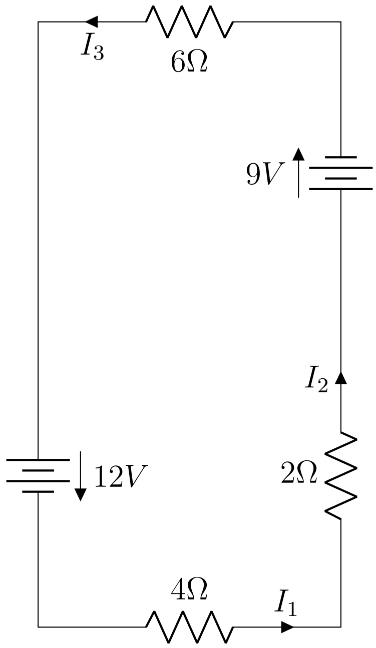
$$\begin{aligned} V_1 &= I_2 R_1 + I_2 R_2 + I_3 R_1 \\ V_2 &= I_2 R_2 + I_3 R_3 \end{aligned}$$

2.8.2 Example: Solving for Currents in a Circuit with Given Values

Consider the following values for the voltage sources and resistances:

$$V_1 = 12\,\text{V}, \quad V_2 = 9\,\text{V}$$

$$R_1 = 4\,\Omega, \quad R_2 = 2\,\Omega, \quad R_3 = 6\,\Omega$$



Substituting these values into the system of equations:

$$12 = I_2(4+2) + I_3(4)$$
$$9 = I_2(2) + I_3(6)$$

This simplifies to:

$$12 = 6I_2 + 4I_3 \quad \text{(Equation 1)}$$
$$9 = 2I_2 + 6I_3 \quad \text{(Equation 2)}$$

Now, we solve this system of equations.
First, solve Equation 2 for I_2:

$$2I_2 = 9 - 6I_3$$
$$I_2 = \frac{9 - 6I_3}{2}$$

Substitute this expression for I_2 into Equation 1:

$$12 = 6\left(\frac{9 - 6I_3}{2}\right) + 4I_3$$

$$12 = 3(9 - 6I_3) + 4I_3$$
$$12 = 27 - 18I_3 + 4I_3$$

$$12 = 27 - 14I_3$$

$$14I_3 = 15$$

$$I_3 = \frac{15}{14} \approx 1.07\,\text{A}$$

Now, substitute $I_3 = 1.07$ into the equation for I_2:

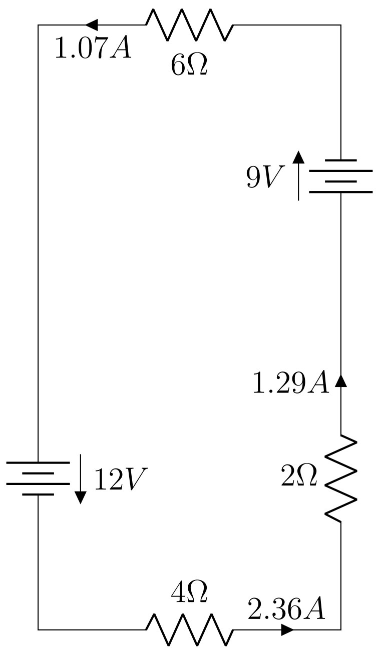
$$I_2 = \frac{9 - 6(1.07)}{2} = \frac{9 - 6.42}{2} = \frac{2.58}{2} = 1.29\,\text{A}$$

Finally, since $I_1 = I_2 + I_3$:

$$I_1 = 1.29 + 1.07 = 2.36\,\text{A}$$

Thus, the currents through the resistors are:

$$\boxed{I_1 = 2.36\,\text{A},} \boxed{I_2 = 1.29\,\text{A},} \boxed{I_3 = 1.07\,\text{A}}$$



This concludes the example of solving for currents in the given circuit.

2.9 Population Growth in Linear Equations

2.9.1 Example: Rabbit Population Growth

In some real-world applications, linear equations can be used to model population growth. Consider the population of rabbits in a particular region. We can represent the population at time t using a linear equation of the form:

$$P(t) = P_0 + kt$$

where:

$P(t)$ is the population at time t,
P_0 is the initial population at time $t = 0$,
k is the rate of change (slope) of the population per year, and
t is the number of years since the starting point (e.g., $t = 0$ could correspond to the year 2020).

Given Data

Let's say the population of rabbits in the year 2020 (when $t = 0$) was 100 rabbits. By 2023, the population increased to 130 rabbits. We can use this information to find the equation of the line and predict the population in future years.

Step 1: Find the Rate of Change (Slope)

The slope k represents the rate of change in the rabbit population per year. We can calculate the slope using the formula:

$$k = \frac{P_1 - P_0}{t_1 - t_0}$$

where:

$P_1 = 130$ (the population in 2023),
$P_0 = 100$ (the population in 2020),
$t_1 = 2023 - 2020 = 3$ years,
$t_0 = 0$ (since $t = 0$ corresponds to 2020).

Substituting the values into the formula:

$$k = \frac{130 - 100}{2023 - 2020} = \frac{30}{3} = 10$$

Thus, the rate of change $k = 10$, which means the rabbit population increases by 10 rabbits per year.

Step 2: Write the Population Equation

Now that we know the slope $k = 10$, we can write the equation of the population as a linear function:

$$P(t) = P_0 + kt$$

Substituting $P_0 = 100$ and $k = 10$:

$$P(t) = 100 + 10t$$

This equation represents the population of rabbits in terms of time t, where t is the number of years since 2020.

Step 3: Predict the Population for 2032
To predict the rabbit population in 2032, we need to find t for that year. Since $t = 0$ corresponds to 2020, the year 2032 is 12 years after 2020. Thus, $t = 2032 - 2020 = 12$.

Now, substitute $t = 12$ into the equation:

$$P(12) = 100 + 10(12) = 100 + 120 = 220$$

So, the predicted rabbit population in 2032 is 220 rabbits.

Conclusion

In summary, we have used the linear equation $P(t) = 100 + 10t$ to model the growth of the rabbit population. By calculating the slope based on the change in population over a 3-year period, we found that the population increases by 10 rabbits per year. Using this equation, we were able to predict that the population in 2032 will be 220 rabbits.

This is an example of how linear equations can be used to model real-world situations like population growth. Such equations are useful tools in fields like biology, economics, and environmental science, where we often need to predict changes over time.

Conclusion

In this chapter, we laid out the foundational concepts for understanding linear equations. We began by exploring the slope-intercept form, $y = mx + b$, where we identified key components such as the slope m and the y-intercept b. These components are essential for graphing linear equations and understanding the rate of change between two variables. We learned that the slope represents the rate at which one variable changes concerning another, and the y-intercept represents the value of y when $x = 0$.

We then moved on to how to find the slope between two points, using the formula $m = \frac{y_1 - y_0}{x_1 - x_0}$, which allowed us to calculate the slope when given two distinct points on a line. This is particularly useful when the equation of the line is not immediately available.

Graphing linear equations was also a key topic in this chapter. We used tables of values to plot points and draw the graph of a line, which helped us visually understand the relationship between x and y for linear equations. We also demonstrated how to use a graph to visually identify the slope and y-intercept, giving us another tool for analyzing linear relationships.

Additionally, we discussed the standard form of a linear equation, $Ax + By = C$, and how it differs from the slope-intercept form. We showed how to convert between these forms and the importance of recognizing both when solving real-world problems.

Finally, we applied linear equations to model real-world scenarios, such as predicting population growth. By using the example of a rabbit population, we demonstrated how to use the slope-intercept form to calculate the rate of change and predict future values based on known data.

In conclusion, understanding the components of linear equations and how to manipulate them is crucial for solving algebraic problems. These concepts form the building blocks for more advanced topics, such as systems of equations, functions, and calculus. Mastering linear equations will serve as a strong foundation for future mathematical studies and real-world problem-solving.

Inquisitive Activity: Chapter 2 Linear Equations

Multiple Choice Questions

1. Which of the following equations represents a linear function?

   a) $y = 5x - 3$

   b) $y = x^2 + 4$

   c) $2x + 3y = 6$

   d) $y = \frac{1}{x} + 2$

2. What is the slope of a horizontal line?

   a) 0

   b) Undefined

   c) 1

   d) -1

True or False

3. The equation $y = 3x - 4$ has a slope of 3 and a y-intercept of -4. (**True** / **False**)

4. The equation $4x - 3y = 12$ is already in slope-intercept form. (**True** / **False**)

5. The equation of a vertical line has an undefined slope. (**True** / **False**)

Short Answer Questions

6. Given the equation $y = 3x - 4$:

   - What is the slope m?
   - What is the y-intercept b?
   - At what point does the line cross the y-axis?

7. Convert the equation $4x - 3y = 12$ into slope-intercept form.

8. The total expenses for the trip were \$600. The three expenses—gas, food, and lodging—were split among them as follows:

   - Alice paid twice as much as Bob.
   - Charlie paid \$50 more than Bob.
   - Together, they covered the total cost of \$600.

   Let:

   - x be the amount Bob paid.
   - y be the amount Alice paid.
   - z be the amount Charlie paid.

   Construct a system of linear equations and solve for each variable.

9. A car rental service charges a flat fee of \$30 plus \$5 per day for the rental.

   - Write the equation that represents the total cost C for renting a car for d days.
   - What is the slope, and what does it represent in this context?
   - What is the y-intercept, and what does it represent in this context?

Problem-Solving Questions

10. Find the equation of the line passing through the points $(2, 5)$ and $(6, 13)$. Write the equation in slope-intercept form.

- Step 1: Find the slope using the formula $m = \frac{y_1 - y_0}{x_1 - x_0}$.
- Step 2: Use one of the points to substitute into the equation $y = mx + b$ and solve for b.
- Step 3: Write the final equation.

11. Consider the population model:

$$P(t) = 50 + 10t$$

where $P(t)$ represents the population at time t, and $t = 0$ corresponds to the year 2020.

- What is the population in 2025? (Hint: $t = 2025 - 2020$)
- How many rabbits are there in 2030?
- What is the rate of change (slope) in the population per year?

12. In 2020, the population of a particular species of rabbits was 150. By 2023, the population grew to 180. Assuming the population growth is linear:

- Calculate the rate of change (slope).
- Use one of the points to find the y-intercept.
- Write the equation of the population model.
- Predict the population in 2028.

13. In a particular ecosystem, the population of a species of frogs is modeled by the equation:

$$P(t) = 200 + 15t$$

where $P(t)$ represents the population at time t, and $t = 0$ corresponds to the year 2010.

- What is the population of frogs in 2015?
- How many frogs will there be in 2020?
- What is the rate of change in the frog population per year?

Answer Key

Multiple Choice Questions

Which of the following equations represents a linear function?
Answer: (a) $y = 5x - 3$ and (c) $2x + 3y = 6$

What is the slope of a horizontal line?
Answer: (a) 0

True or False

The equation $y = 3x - 4$ has a slope of 3 and a y-intercept of -4.
Answer: True

The equation $4x - 3y = 12$ is already in slope-intercept form.
Answer: False

The equation of a vertical line has an undefined slope.
Answer: True

Short Answer Questions

Given the equation

$y = 3x - 4$:

- Slope $m = 3$
- Y-intercept $b = -4$
- Point where the line crosses the y-axis $(0, -4)$

Convert the equation

$4x - 3y = 12$ into slope-intercept form. **Answer:** $y = \frac{4}{3}x - 4$

Solving Systems of Equations in a real-life example

$$y = 2x \quad \text{(Alice paid twice as much as Bob)} \tag{2.7}$$
$$z = x + 50 \quad \text{(Charlie paid \$50 more than Bob)} \tag{2.8}$$
$$x + y + z = 600 \quad \text{(Total cost)} \tag{2.9}$$

Substituting equation (2.7) and equation (2.8) into equation (2.9):

$$x + 2x + (x + 50) = 600$$

$$4x + 50 = 600$$

$$4x = 550$$

$$x = 137.5$$

Using $x = 137.5$ in equation (2.7):

$$y = 2(137.5) = 275$$

Using $x = 137.5$ in equation (2.8):

$$z = 137.5 + 50 = 187.5$$

So the final contributions of each person are:

- Bob paid \$137.50.
- Alice paid \$275.00.
- Charlie paid \$187.50.

This solution ensures that the total expense is evenly distributed according to their contributions.

Car Rental Cost Analysis

A car rental service charges a flat fee of \$30 plus \$5 per day for the rental.

Equation for Total Cost

Let C represent the total cost of renting a car for d days. The total cost consists of a fixed fee of \$30 and an additional charge of \$5 per day. The equation is:

$$C = 5d + 30$$

Slope and Its Meaning

The equation $C = 5d + 30$ is in slope-intercept form $C = md + b$, where the slope m represents the rate of change.

$$\text{Slope} = 5$$

Interpretation: The slope of 5 means that for each additional day the car is rented, the total cost increases by \$5.

Y-Intercept and Its Meaning

The y-intercept is the value of C when $d = 0$:

$$\text{Y-Intercept} = 30$$

Interpretation: The y-intercept of 30 represents the initial flat fee that must be paid even if the car is rented for 0 days.

Problem solving questions

Finding the Equation of a Line

We need to find the equation of the line passing through the points $(2, 5)$ and $(6, 13)$ in slope-intercept form $y = mx + b$.

Step 1: Find the Slope

The formula for the slope m of a line passing through two points (x_0, y_0) and (x_1, y_1) is:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Substituting $(x_0, y_0) = (2, 5)$ and $(x_1, y_1) = (6, 13)$:

$$m = \frac{13 - 5}{6 - 2} = \frac{8}{4} = 2$$

Step 2: Solve for b

Using the slope-intercept form of a line:

$$y = mx + b$$

Substituting $m = 2$ and using point $(2, 5)$:

$$5 = 2(2) + b$$

$$5 = 4 + b$$

$$b = 1$$

Step 3: Write the Final Equation

Thus, the equation of the line is:

$$y = 2x + 1$$

Consider the population model

$P(t) = 50 + 10t$:

- Population in 2025: $P(5) = 50 + 10(5) = 100$
- Population in 2030: $P(10) = 50 + 10(10) = 150$
- Rate of change (slope): 10 per year

In 2020, the population of rabbits was 150; in 2023, it was 180.

- Slope: $m = \frac{180-150}{2023-2020} = 10$
- Using point (2020,150): $150 = 10(2020) + b \Rightarrow b = -19850$
- Equation: $P(t) = 10t - 19850$
- Population in 2028: $P(2028) = 10(2028) - 19850 = 230$

Given the equation $P(t) = 200 + 15t$:

- Population in 2015: $P(5) = 200 + 15(5) = 275$
- Population in 2020: $P(10) = 200 + 15(10) = 350$
- Rate of change: 15 per year

Reflection

Take a moment to reflect on the material covered in this chapter. Think about how the concepts of slope and y-intercept apply to both mathematical problems and real-world scenarios, such as population modeling or cost analysis. Understanding these concepts will be essential for solving more complex equations and problems in future chapters.

Chapter 3

Quadratic Equations

3.1 General Form

A quadratic equation is expressed as:

$$ax^2 + bx + c = 0 \tag{3.1}$$

where a, b, c are constants and $a \neq 0$.

Here is the graph of quadratic equations illustrating the effect of varying a in $ax^2 + bx + c$.

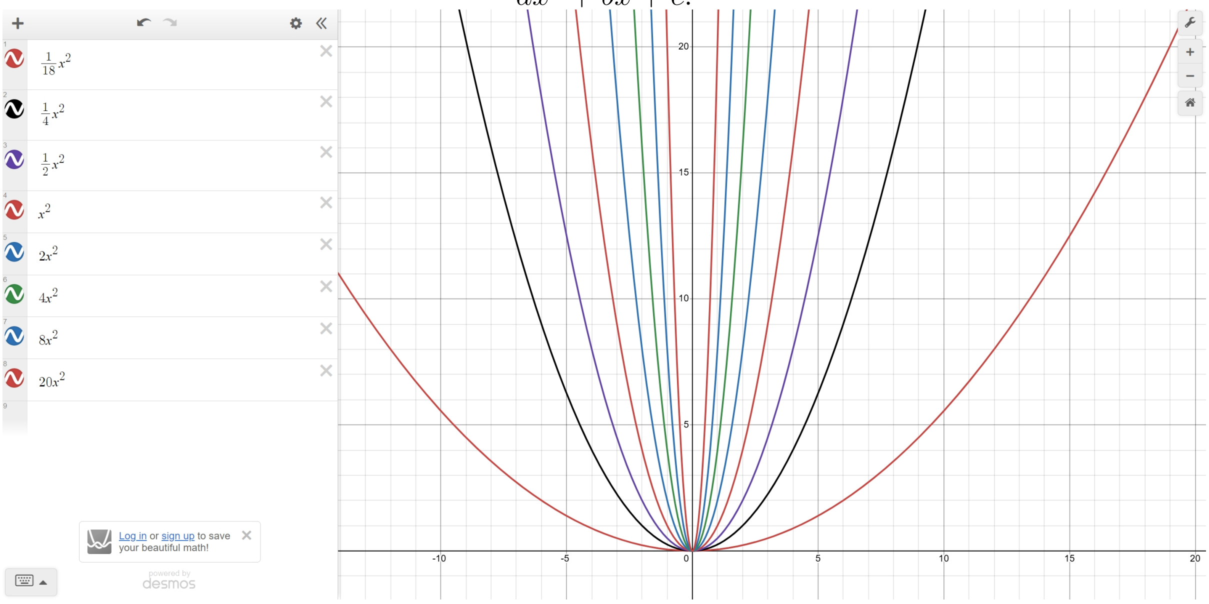


Figure 3.1: *

As shown in the figure, decreasing the value of a results in a wider parabola, while increasing a makes the parabola narrower.

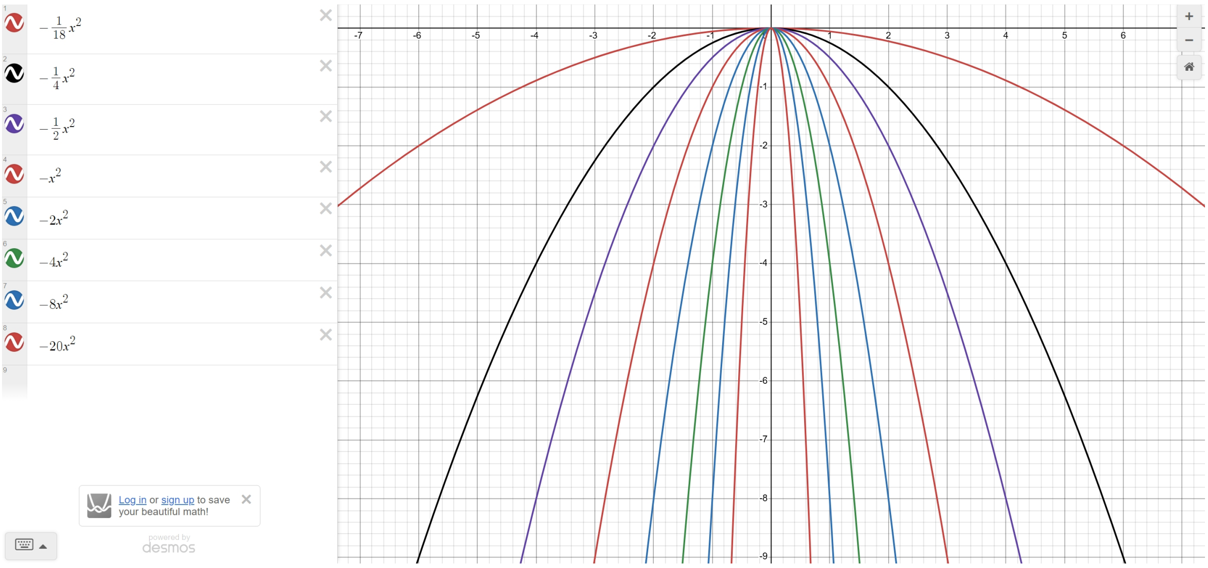


Figure 3.2: *

As shown in the figure, when a is negative, the parabola opens downward instead of upward.

Here is the graph of quadratic equations illustrating the effect of varying b in $ax^2 + bx + c$.

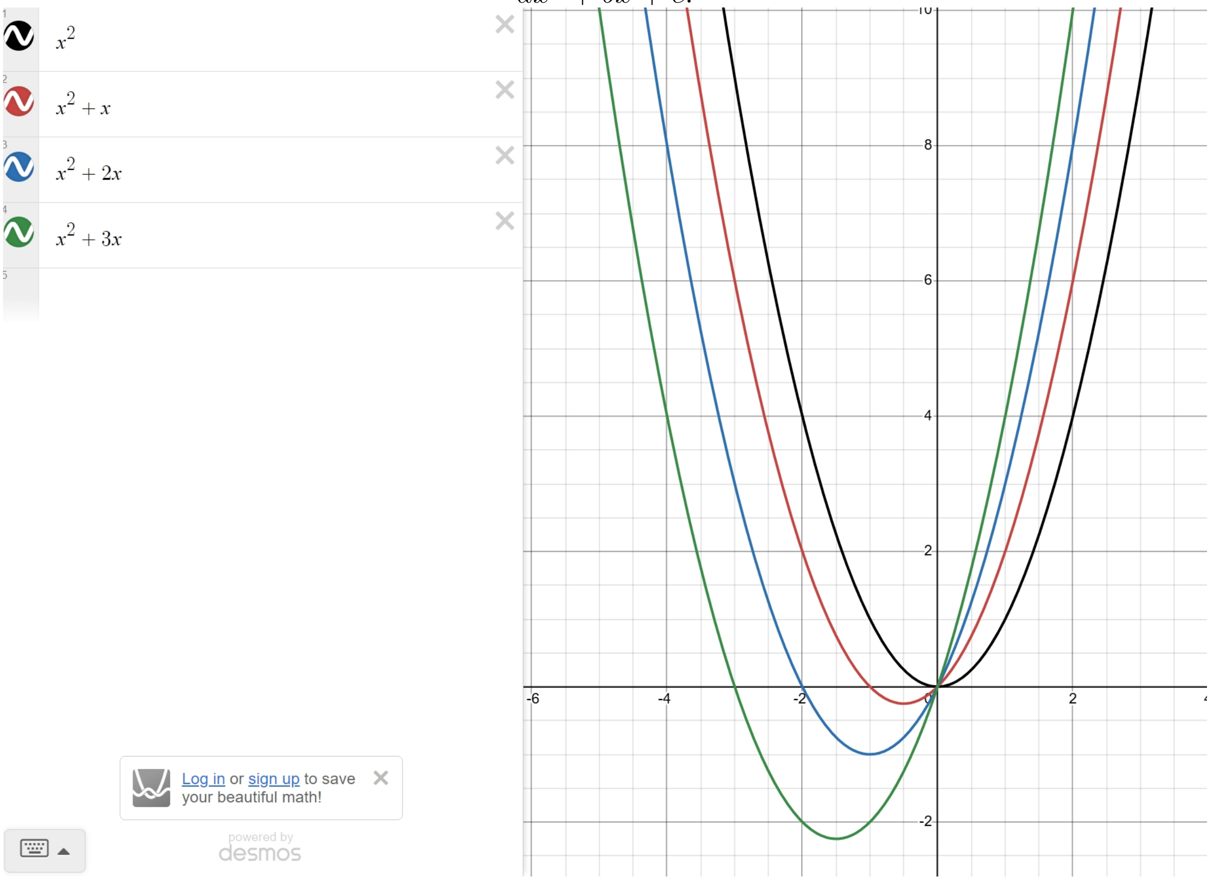


Figure 3.3: *

As shown in the figure, increasing b shifts the vertex of the parabola diagonally to the left.

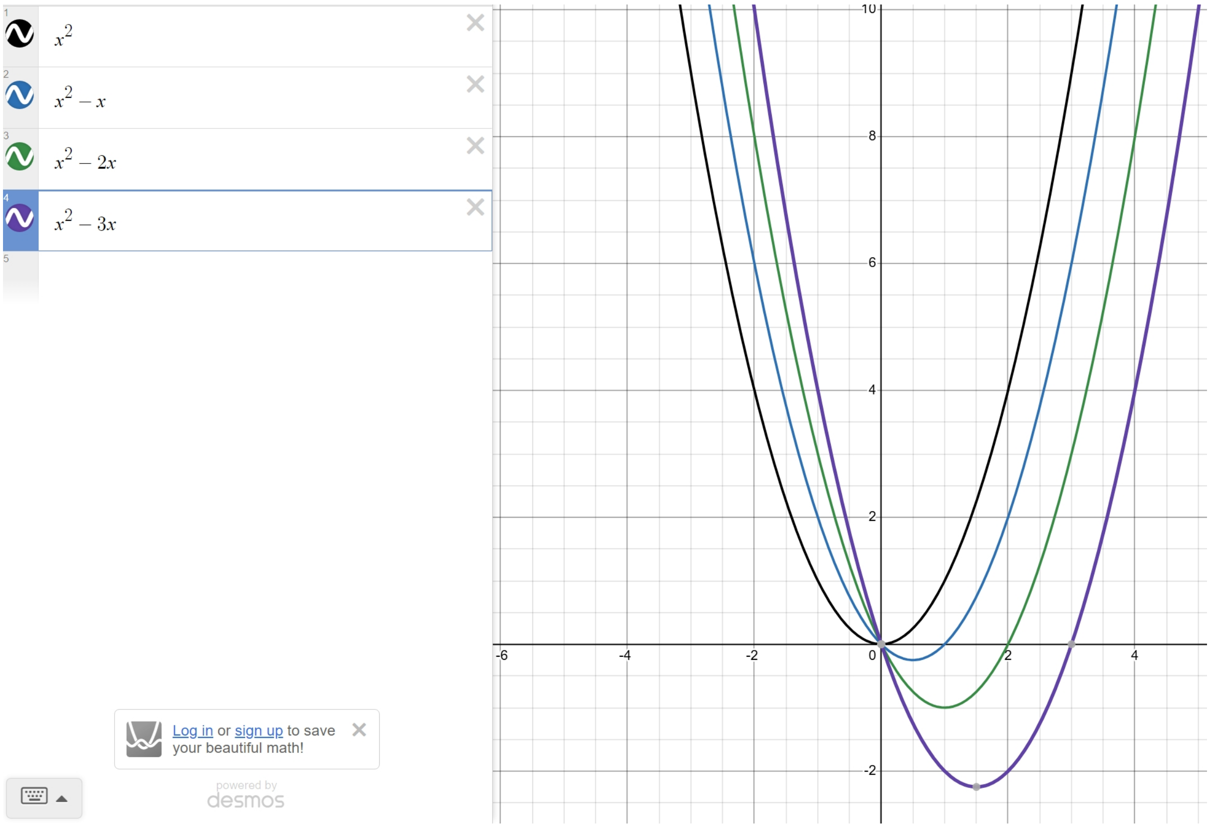


Figure 3.4: *

As shown in the figure, making b negative shifts the vertex of the parabola diagonally to the right.

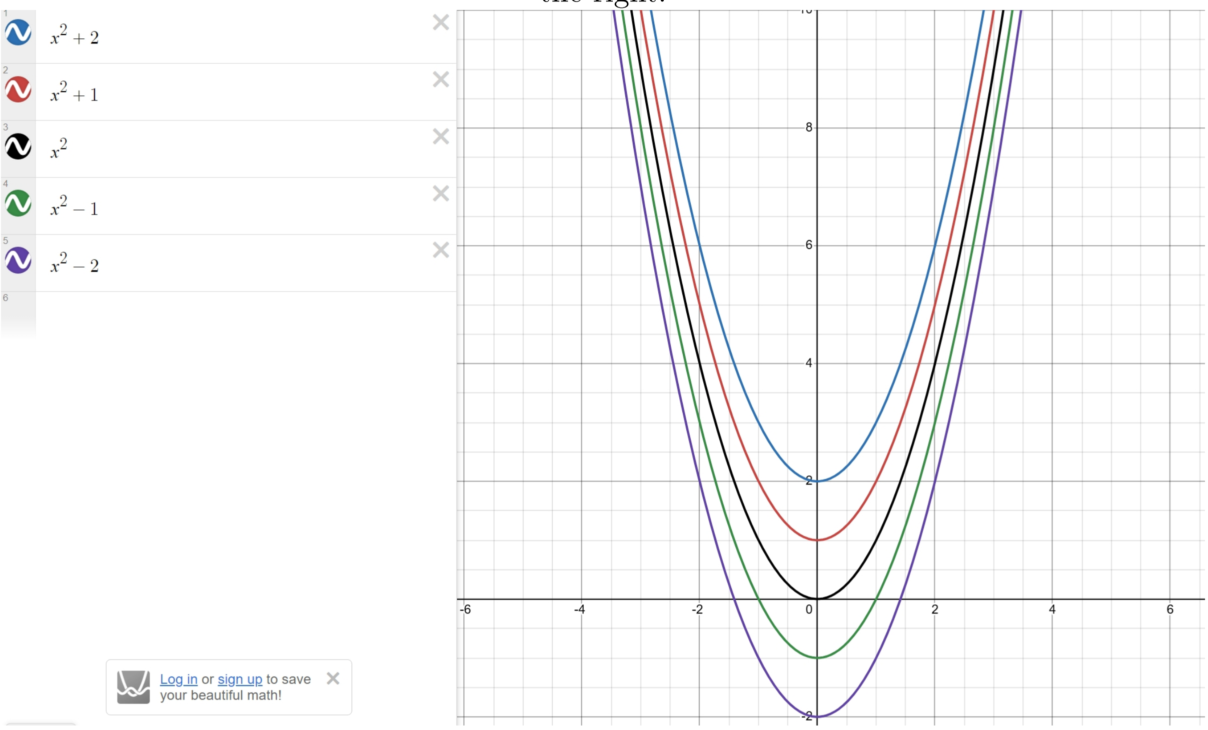


Figure 3.5: *

As shown in the figure, increasing c shifts the graph upward, while decreasing c shifts it downward.

3.2 Solving for x by finding x-intercepts

3.2.1 Factoring

Factoring is a method used to express a quadratic equation as a product of simpler binomial expressions. It is particularly useful for finding the x-intercepts (or roots) of a quadratic function, where the equation is set to zero.

Factoring a Quadratic Expression

A quadratic expression in standard form is given as:

$$ax^2 + bx + c. \tag{3.2}$$

Factoring rewrites this expression as the product of two binomials:

$$(a_1x + b_1)(a_2x + b_2). \tag{3.3}$$

Example 1: Factoring Out the Greatest Common Factor (GCF)
Consider the quadratic expression:

$$4x^2 + 2x. \tag{3.4}$$

First, factor out the greatest common factor (GCF), which is $2x$:

$$4x^2 + 2x = 2x(2x + 1). \tag{3.5}$$

Factoring Using the X-Factor Method

The X-Factor method is useful when factoring quadratics of the form $x^2 + bx + c$.
Example 2: Factoring $x^2 + x - 6$ Using the X-Factor Method
Consider the quadratic equation:

$$x^2 + x - 6 = 0. \tag{3.6}$$

To factor this, we find two numbers that multiply to -6 (the constant term) and add to 1 (the coefficient of x):

$$\text{Numbers: } (-2, 3) \quad \text{since } (-2) \cdot (3) = -6 \text{ and } (-2) + 3 = 1.$$

Thus, we can factor the expression as:

$$(x - 2)(x + 3) = 0. \tag{3.7}$$

Finding the X-Intercepts

To find the x-intercepts, we set each factor equal to zero:

$$\begin{aligned} x-2=0 \quad &\Rightarrow \quad x=2, \\ x+3=0 \quad &\Rightarrow \quad x=-3. \end{aligned}$$

3.2.2 Common Denominators in the form of Quadratic Equations:

To revisit common denominators refer back to section 1.6.1
Consider the following fraction: $\frac{x}{x^2+2x+1} + \frac{2}{x^2-x-2}$

Step 1: Factor the Denominators

We factor both denominators:

$$x^2 + 2x + 1 = (x+1)(x+1) = (x+1)^2$$

$$x^2 - x - 2 = (x-2)(x+1)$$

Step 2: Determine the LCD

The least common denominator (LCD) must include both factored forms:
Since $x+1$ is already a factor of x^2-x-2 we only need an additional $x+1$ to complete the LCD

$$\text{LCD} = (x+1)^2(x-2)$$

Step 3: Rewrite Each Fraction with the LCD

Rewriting the first fraction by multiplying by $(x-2)$ on the top and bottom:

$$\frac{x}{x^2+2x+1} = \frac{x}{(x+1)^2} = \frac{x(x-2)}{(x+1)^2(x-2)}$$

Rewriting the second fraction by multiplying by $(x+1)$: on the top and bottom

$$\frac{2}{x^2-x-2} = \frac{2}{(x-2)(x+1)} = \frac{2(x+1)}{(x+1)^2(x-2)}$$

Step 4: Perform the Addition

$$\frac{x(x-2)}{(x+1)^2(x-2)} + \frac{2(x+1)}{(x+1)^2(x-2)} = \frac{x(x-2)+2(x+1)}{(x+1)^2(x-2)}$$

Expanding the numerator:

$$x^2 - 2x + 2x + 2$$

$$x^2 + 2$$

Thus, the final simplified expression is: $\boxed{\dfrac{x^2+2}{(x+1)^2(x-2)}}$

3.2.3 Imaginary Solutions and Intro to Complex Numbers

Consider the equation:

$$x^2 + 1 = 0. \tag{3.8}$$

To solve for x, first move the constant term to the other side of the equation:

$$x^2 = -1. \tag{3.9}$$

Next, take the square root of both sides:

$$x = \pm\sqrt{-1}. \tag{3.10}$$

This is a problem since taking the square root of a negative number is undefined.

To address this mathematical challenge, we define a special quantity called the imaginary unit, denoted by i. So now $\sqrt{-1} = i$.

Since $\sqrt{-1} = i$, we can rewrite the solution as:

$$x = \pm i. \tag{3.11}$$

Thus, the "complex" solutions to the equation $x^2 + 1 = 0$ are:

$$x = i \quad \text{or} \quad x = -i. \tag{3.12}$$

What is a complex number?

An imaginary number is a number that, when squared, results in a negative value. The fundamental imaginary unit, denoted by i, is defined as:

$$i = \sqrt{-1}. \tag{3.13}$$

Imaginary numbers arise when we attempt to take the square root of a negative number, which is not possible within the set of real numbers. However, by introducing the imaginary unit i, we extend the number system to include solutions to such equations.

Complex Numbers

A complex number is any number that can be written in the form:

$$z = a + bi, \tag{3.14}$$

where a and b are real numbers, and i is the imaginary unit. The number a is called the real part, and bi is the imaginary part. For example, $3 + 4i$ is a complex number, where 3 is the real part and $4i$ is the imaginary part.

Conclusion

Imaginary numbers allow us to extend the real number system to include solutions for equations involving the square roots of negative numbers. In the case of the equation $x^2 + 1 = 0$, the solutions are purely imaginary, $x = i$ and $x = -i$, demonstrating how imaginary numbers help solve equations that would otherwise have no real solutions.

3.2.4 Distributing

The distributive property is a fundamental algebraic principle used to simplify expressions by distributing multiplication over addition or subtraction. It states that:

$$a(b + c) = ab + ac. \tag{3.15}$$

This property is essential when expanding expressions and working with polynomials.

Example 1: Distributing a Single Term

Consider the expression:

$$x(x + 5). \tag{3.16}$$

Using the distributive property, multiply x by each term inside the parentheses:

$$\begin{aligned} x(x + 5) &= x \cdot x + x \cdot 5 \\ &= x^2 + 5x. \end{aligned}$$

Thus, the expanded form of $x(x + 5)$ is:

$$x^2 + 5x. \tag{3.17}$$

Example 2: Distributing Two Binomials

Now, consider the multiplication of two binomials:

$$(x + 5)(2x + 7). \tag{3.18}$$

To expand this expression, apply the distributive property twice:

$$\begin{aligned} (x + 5)(2x + 7) &= x(2x + 7) + 5(2x + 7) \\ &= (x \cdot 2x + x \cdot 7) + (5 \cdot 2x + 5 \cdot 7) \\ &= 2x^2 + 7x + 10x + 35. \end{aligned}$$

Now, combine like terms:

$$2x^2 + 17x + 35. \tag{3.19}$$

Thus, the expanded form of $(x + 5)(2x + 7)$ is:

$$2x^2 + 17x + 35. \tag{3.20}$$

Conclusion

The distributive property is a fundamental tool in algebra that simplifies expressions and allows for the expansion of polynomial products. Mastering this technique is crucial for solving equations, factoring, and simplifying expressions efficiently. As previously mentioned, all the skills you learn, especially factoring and distributing, never go away. These two topics are used constantly very often in calculus, such as finding the difference quotient.

3.3 Vertex Form

3.3.1 What is a Vertex?

The vertex of a quadratic graph is the point representing its maximum or minimum value. It is the turning point of the parabola and provides key information on its position and orientation.

A quadratic function can be written in vertex form as:

$$y = a(x - h)^2 + k, \tag{3.21}$$

Where:

- (h, k) is the vertex of the parabola.
- a determines the direction and width of the parabola. If $a > 0$, the parabola opens upward; if $a < 0$, it opens downward.

3.3.2 Converting a Quadratic Equation to Vertex Form

To convert a quadratic function from standard form, $y = ax^2 + bx + c$, to vertex form, we complete the square.

Example: Convert $y = 2x^2 + 8x + 3$ to vertex form.

1. Factor out the coefficient of 2 from the first two terms:
$$y = 2(x^2 + 4x) + 3. \tag{3.22}$$
2. Complete the square inside the parentheses. You can find the term by performing $(\frac{b}{2})^2 = (\frac{4}{2})^2 = 4$
$$y = 2(x^2 + 4x + 4 - 4) + 3. \tag{3.23}$$
3. Rewrite the expression as a perfect square: $x^2 + 4x + 4 = (x + 2)^2$
$$y = 2((x + 2)^2 - 4) + 3. \tag{3.24}$$
4. Distribute the 2 and simplify:
$$y = 2(x + 2)^2 - 8 + 3, \tag{3.25}$$
$$y = 2(x + 2)^2 - 5. \tag{3.26}$$
Thus, the vertex form is:
$$y = 2(x + 2)^2 - 5. \tag{3.27}$$
The vertex of the parabola is $(-2, -5)$.

3.3.3 Finding the Vertex from Standard Form

If a quadratic function is given in standard form $y = ax^2 + bx + c$, the vertex can be found using the formula:

$$h = -\frac{b}{2a}, \quad k = f(h). \tag{3.28}$$

Example: Find the vertex of $y = 3x^2 - 6x + 1$.

1. Compute h:

$$h = -\frac{-6}{2(3)} = \frac{6}{6} = 1. \tag{3.29}$$

2. Compute k by substituting $h = 1$ into the equation:

$$k = 3(1)^2 - 6(1) + 1 = 3 - 6 + 1 = -2. \tag{3.30}$$

3. The vertex is $(1, -2)$.

By using these methods, we can efficiently determine the vertex of any quadratic function, providing insight into the function's graph and behavior.

3.4 Quadratic Formula

The Quadratic Formula is used to find the roots of quadratic equations when factoring is not possible:

3.4.1 Derivation of the Quadratic Formula

The quadratic formula is a fundamental tool for solving quadratic equations of the form:

$$ax^2 + bx + c = 0. \tag{3.31}$$

It provides a direct way to find the values of x without factoring or completing the square. The quadratic formula is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.32}$$

To derive the quadratic formula, we solve the general quadratic equation $ax^2 + bx + c = 0$ using the method of completing the square.

1. First, divide the entire equation by a to make the coefficient of x^2 equal to 1:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{3.33}$$

2. Move the constant term to the other side:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{3.34}$$

3. Complete the square by adding $\left(\frac{b}{2a}\right)^2$ to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2. \tag{3.35}$$

4. Rewrite the left-hand side as a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}. \tag{3.36}$$

5. Simplify the right-hand side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{3.37}$$

6. Take the square root on both sides:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}. \tag{3.38}$$

7. Finally, isolate x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.39}$$

This formula allows us to solve any quadratic equation, even when factoring is difficult or impossible.

3.4.2 Example: Solving a Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation:

$$2x^2 - 3x - 5 = 0. \tag{3.40}$$

Identify the coefficients: $a = 2$, $b = -3$, and $c = -5$. Substitute them into the quadratic formula:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}. \tag{3.41}$$

Simplify inside the square root:

$$x = \frac{3 \pm \sqrt{9 + 40}}{4}. \tag{3.42}$$

$$x = \frac{3 \pm \sqrt{49}}{4}. \tag{3.43}$$

Since $\sqrt{49} = 7$, we have:

$$x = \frac{3 \pm 7}{4}. \tag{3.44}$$

Now, solve for both possible values of x:

$$x_1 = \frac{3+7}{4} = \frac{10}{4} = \frac{5}{2},$$
$$x_2 = \frac{3-7}{4} = \frac{-4}{4} = -1.$$

Thus, the solutions to the equation $2x^2 - 3x - 5 = 0$ are:

$$x = \frac{5}{2} \quad \text{or} \quad x = -1. \tag{3.45}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.4.3 What is the Discriminant?

The discriminant is a key component of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.47}$$

The expression inside the square root,

$$D = b^2 - 4ac, \tag{3.48}$$

is called the **discriminant**. It determines the nature of the solutions (or roots) of the quadratic equation $ax^2+bx+c = 0$, which correspond to the x-intercepts of the parabola.

3.4.4 Interpreting the Discriminant

The value of the discriminant affects the number and type of solutions:

- If $D > 0$, the quadratic equation has **two distinct real solutions**. This means the parabola intersects the x-axis at two different points.
- If $D = 0$, the quadratic equation has **one real solution** (a repeated root). This means the parabola touches the x-axis at exactly one point (vertex).
- If $D < 0$, the quadratic equation has **no real solutions**. This means the parabola does not intersect the x-axis, and the roots are complex (imaginary).

3.4.5 Example: Finding the Nature of the Roots

Consider the quadratic equation:

$$x^2 - 6x + 9 = 0. \tag{3.49}$$

Identify the coefficients: $a = 1$, $b = -6$, and $c = 9$. Compute the discriminant:

$$\begin{aligned} D &= (-6)^2 - 4(1)(9) \\ &= 36 - 36 \\ &= 0. \end{aligned}$$

Since $D = 0$, the equation has exactly **one real solution**, meaning the parabola touches the x-axis at a single point.

3.4.6 Verifying the Solution

Using the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{0}}{2(1)}. \tag{3.50}$$

$$x = \frac{6 \pm 0}{2}. \tag{3.51}$$

$$x = \frac{6}{2} = 3. \tag{3.52}$$

Thus, the only x-intercept is $x = 3$, confirming that the parabola touches the x-axis at a single point.

3.5 Real World Applications

3.5.1 Projectile Motion: Finding the Maximum Height and Time

Quadratic equations frequently appear in physics, especially in projectile motion. Suppose a ball is thrown vertically upward from a height of 2 meters with an initial velocity of 15 m/s. We want to determine:

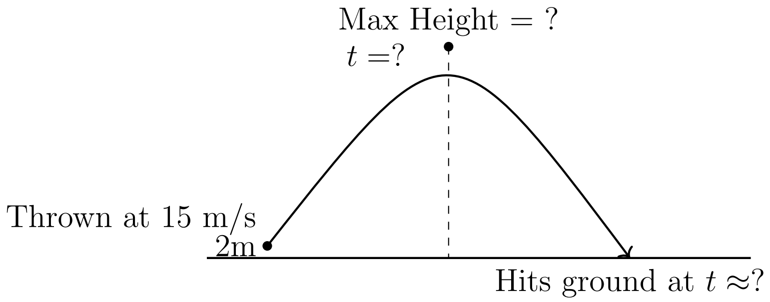
1. The time it takes to reach the highest point.
2. The maximum height.
3. The time when the ball hits the ground.

The equation for the height of the ball as a function of time is:

$$h(t) = -4.9t^2 + 15t + 2$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity.

Projectile Motion Diagram



Time to Reach Maximum Height

Since this is an algebra course, we use the fact that the maximum of a quadratic function $ax^2 + bx + c$ occurs at:

$$t = \frac{-b}{2a}$$

For $h(t) = -4.9t^2 + 15t + 2$, we identify $a = -4.9$ and $b = 15$:

$$t = \frac{-15}{2(-4.9)} = \frac{15}{9.8} \approx 1.53 \text{ seconds}$$

Finding Maximum Height

Substituting $t = 1.53$ into $h(t)$:

$$h_{\text{max}} = -4.9(1.53)^2 + 15(1.53) + 2$$

$$h_{\text{max}} \approx -4.9(2.34) + 22.95 + 2$$

$$h_{\text{max}} \approx -11.47 + 22.95 + 2 = 13.48 \text{ meters}$$

Time When the Ball Hits the Ground

Setting $h(t) = 0$:

$$-4.9t^2 + 15t + 2 = 0$$

Using the quadratic formula:

$$t = \frac{-15 \pm \sqrt{(15)^2 - 4(-4.9)(2)}}{2(-4.9)}$$

$$t = \frac{-15 \pm \sqrt{225 + 39.2}}{-9.8}$$

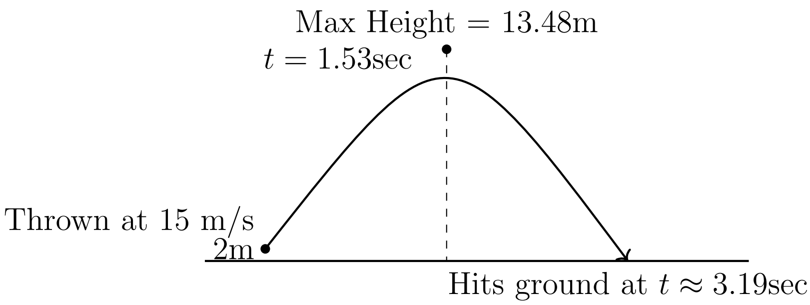
$$t = \frac{-15 \pm 16.25}{-9.8}$$

$$t_1 = \frac{-15 + 16.25}{-9.8} = \frac{1.25}{-9.8} \approx -0.13 \quad \text{(discarded, as time cannot be negative)}$$

$$t_2 = \frac{-15 - 16.25}{-9.8} = \frac{-31.25}{-9.8} \approx 3.19 \text{ seconds}$$

Thus, the ball hits the ground at $t \approx 3.19$ seconds.

Projectile Motion Diagram With Solutions



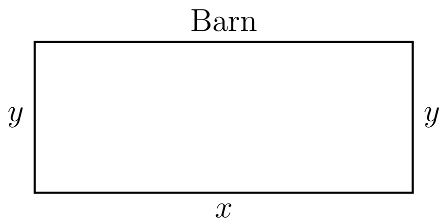
3.6 Optimization Problem: Maximizing the Area of a Pen

A farmer wants to build a rectangular pen using 186 feet of fencing. One side of the pen is against a barn, so fencing is only needed for the other three sides. We need to determine the dimensions that maximize the enclosed area.

3.6.1 Forming the Quadratic Equation

Let:

- x be the length of the pen (parallel to the barn).
- y be the width of the pen (perpendicular to the barn).
- The total fencing available is 186 feet.



Since fencing is used for one length and two widths, we set up the equation:

$$x + 2y = 186$$

Solving for x:

$$x = 186 - 2y$$

The area of the pen is given by:

$$A = x \cdot y$$

Substituting $x = 186 - 2y$:

$$A(y) = (186 - 2y) \cdot y$$

$$A(y) = 186y - 2y^2$$

3.6.2 Finding the Maximum Area

Since $A(y)$ is a quadratic function of the form:

$$A(y) = -2y^2 + 186y$$

The maximum area occurs at the vertex, given by Also found in section:3.53

$$y = \frac{-b}{2a} = \frac{-186}{2(-2)} \tag{3.53}$$

$$y = \frac{186}{4} = 46.5 \text{ feet}$$

Substituting $y = 46.5$ into $x = 186 - 2y$:

$$x = 186 - 2(46.5) = 93 \text{ feet}$$

Thus, the dimensions that maximize the area are:

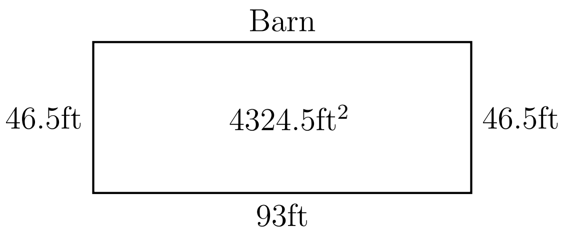
$$\mathbf{93 \times 46.5}$$

3.6.3 Calculating the Maximum Area

$$A(46.5) = -2(46.5)^2 + 186(46.5)$$

$$A(46.5) = -2(2162.25) + 8649$$

$$A(46.5) = -4324.5 + 8649 = 4324.5 \text{ ft}^2$$



Inquisitive Activity: Chapter 3 Quadratic Equations

1. Consider the quadratic equation $x^2 - 4x + 3 = 0$. Without solving explicitly, what do the signs of the coefficients tell you about the nature of its roots?

2. The graph of the quadratic function $y = ax^2 + bx + c$ is a parabola. How does the sign of a determine the orientation of the parabola?

3. A quadratic equation has exactly one real solution. What must be true about its discriminant, $b^2 - 4ac$?

4. The quadratic equation $x^2 + 6x + 9 = 0$ is already in factored form. What is the solution to this equation, and how does it relate to the concept of a repeated root?

5. Given the quadratic function $y = (x - 3)(x + 5)$, what are its x-intercepts? How does this relate to the Zero Product Property?

6. Consider the function $y = 2(x + 1)^2 - 7$. Identify its vertex and explain how this form of the equation makes it easy to determine.

7. If the vertex of a parabola is at (h, k) and it passes through the point $(2, 5)$, write an equation for the quadratic function in vertex form, assuming the leading coefficient a is 1.

8. Why does completing the square help in converting a quadratic equation from standard form to vertex form? Describe the key steps in this process.

9. How can you determine the axis of symmetry for a quadratic equation given in standard form $y = ax^2 + bx + c$?

10. If a projectile follows the path given by $h(t) = -16t^2 + 32t + 48$, what is the maximum height reached by the projectile, and at what time does it occur?

Answer Key

1. The signs of the coefficients suggest that the quadratic can be factored into two binomials with real roots, as $b^2 - 4ac > 0$.

2. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward.

3. The discriminant must be zero, i.e., $b^2 - 4ac = 0$.

4. The equation can be rewritten as $(x + 3)(x + 3) = 0$, meaning the only solution is $x = -3$, which is a repeated root.

5. The x-intercepts are $x = 3$ and $x = -5$, found by setting each factor to zero.

6. The vertex is $(-1, -7)$, easily identified from vertex form $y = a(x - h)^2 + k$.

7. Using the vertex form $y = (x - h)^2 + k$, substituting (h, k) and the given point yields $y = (x - h)^2 + k$ with the known values.

8. Completing the square isolates the squared term, allowing easy identification of the vertex (h, k).

9. The axis of symmetry is given by $x = -\frac{b}{2a}$.

10. The maximum height occurs at the vertex. Using $t = -\frac{b}{2a}$, we find $t = 1$ sec. Substituting $t = 1$ into $h(t)$ gives $h(1) = 64$ feet.

Chapter 4

Polynomial Equations

Chapter 5

Logarithms and Exponential Algebra

Chapter 6

Exponential Equations

Chapter 7

Trigonometry

Chapter 8

Complex Numbers