Mathematics II

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Contents

1 Pre - Integration

1.1 Riemann Sums

The area under a curve can be split into multiple different rectangles. Adding the rectangles results in the total area.

$$y = f(x);\ a \leq x \leq b;\ \text{with } n \text{ sub intervals}$$

Using Left Endpoints

$$R_l = \sum_{i=1}^{n} f(x_i)\Delta x \qquad x_i = a + i\Delta x - \Delta x \qquad \Delta x = \frac{b-a}{n}$$

Using Right Endpoints

$$R_r = \sum_{i=1}^{n} f(x_i)\Delta x \qquad x_i = a + i\Delta x \qquad \Delta x = \frac{b-a}{n}$$

Using Midpoints (Average)

$$R_m = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x \qquad \bar{x}_i = a + i\Delta x - \frac{1}{2}\Delta x \qquad \Delta x = \frac{b-a}{n}$$

When n is small, this results in areas that are not taken into account resulting in an inaccurate area.

1.2 Definite Integral

When there are an infinitesimal amount of sub intervals, this results in the actual area under the curve.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) = \int_a^b f(x)dx$$

Where $\int_a^b f(x)dx$ is the definite integral of the functions running from a to b.

2 Fundamental Theorems of Calculus (FTC)

2.1 Part 1

If: $g(x) = \int_a^x f(t)dt$ Then: $g'(x) = f(x)$

Alternatively: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Take note that the upper limit is in the form of $\int_a^x f(t)dt$ and not any other form.

Example: Evaluate: $g'(x) = \int_a^{x^2} \sin(t)dt$

Due to the the integral not being the form of $\int_a^x f(t)dt$, we need to do a substitution.

$$
\begin{aligned}
g(x) &= \int_a^{x^2} \sin(t)dt & u = x^2 \\
g(x) &= \int_a^{u} \sin(t)dt \\
g'(u) &= \sin(u)
\end{aligned}
$$

We need to undo our substitution, however cannot directly undo it. Therefore, we will need to use the chain rule.

$$
\begin{aligned}
\frac{dg}{du} &= \sin(u) \\
\frac{dg}{dx}\frac{dx}{du} &= \sin(u) \\
\frac{dg}{dx} &= \sin(u)\ \frac{du}{dx} & du = 2x\ dx
\end{aligned}
$$

$$\boxed{\frac{dg}{dx} = \sin(x^2)\ 2x}$$

2.2 Part 2

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Where F is the anti derivative (integral) of f

Applications
If a particle is defined by a speed of $v(x)$, Then the displacement and speed through the time $a \leq t \leq b$ are:

$$\vec{d}\ \text{(Displacement)} = \int_a^b v(t)\ dt$$
$$d\ \text{(Distance)} = \int_a^b |v(t)|\ dt$$

3 Integration Techniques

3.1 Integration by Substitution

$$\int f(g(x)\ g'(x) = \int f(u)\ du \qquad\qquad \text{Where } u = g(x)$$

Indefinite Integrals need to have their substitution undone

Example: Evaluate: $\displaystyle\int \frac{x+3}{\sqrt{x^2+6x}}\ dx$
We'll have to do a substitution for $x^2 + 6x$ as we don't know the integral of a function in a radical

$$\begin{aligned} &\int \frac{x+3}{\sqrt{x^2+6x}}\ dx && u = x^2 + 6x \\ &= \int \frac{1}{2}\frac{du}{\sqrt{u}} && du = (2x+6)\ dx \\ &= \frac{1}{2}\int u^{-\frac{1}{2}}\ du && \frac{1}{2}\ du = (x+3)\ dx \\ &= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C \\ &= u^{\frac{1}{2}} + C \\ &= (x^2+6x)^{\frac{1}{2}} + C \end{aligned}$$

- Remember to substitute for dx as well
- You have to replace all occurrences of the original variables (No mixing of variables)

Definite Integrals need to have their bounds changed

Example: Evaluate: $\displaystyle\int_0^2 \frac{x+3}{\sqrt{x^2+6x}}\,dx$
We've already done this integral from the last example. However, this time, it is a definite integral.

$$\int_0^2 \frac{x+3}{\sqrt{x^2+6x}}\,dx \qquad u = x^2+6x \qquad \frac{1}{2}\,du = (x+3)\,dx$$

We must change the bounds of the integral using $u = x^2 + 6x$.

x	u
0	0
2	16

$$\begin{aligned} &\frac{1}{2}\int_0^{16} \frac{du}{\sqrt{u}}\,dx \\ &= u^{\frac{1}{2}}\Big|_0^{16} \\ &= 16^{\frac{1}{2}} - 0^{\frac{1}{2}} \\ &= 4 \end{aligned}$$

3.2 Integration of Symmetric Functions

If f is an even function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

If f is an odd function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

3.3 Integration by Parts

$$\int u\ dv = uv - \int v\ du \qquad\qquad \text{If } u = f(x) \text{ and } v = g(x)$$

$$\text{Alternatively: } \int uv = u \int \Big[v\Big] - \int \Big[du \int v\Big]$$

Example: Find the integral of: $\int e^x \cos 2x\ dx$

Determine what should be u and what should dv:

$$u = e^x \qquad\qquad dv = \cos 2x\ dx$$

$$du = e^x\ dx \qquad\qquad v = \frac{\sin 2x}{2}$$

$$\begin{aligned} &e^x\ \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\ e^x\ dx \\ &= \frac{1}{2}\ e^x \sin 2x - \frac{1}{2} \int \sin 2x\ e^x\ dx \end{aligned}$$

We must do integration by parts once again.

$$u = e^x \qquad\qquad dv = \sin 2x\ dx$$

$$du = e^x\ dx \qquad\qquad v = -\frac{\cos 2x}{2}$$

$$\begin{aligned} &\frac{1}{2}\ e^x \sin 2x - \frac{1}{2}\left(e^x \cdot -\frac{\cos 2x}{2} - \int -\frac{\cos 2x}{2}\ e^x\ dx\right) \\ &= \frac{1}{2}\ e^x \sin 2x + \frac{1}{4}\ e^x \cos 2x + \frac{1}{4} \int \cos 2x\ e^x\ dx \end{aligned}$$

We have the original integral so we are now in a position to subtract both side by it.

$$\begin{aligned} \frac{3}{4} \int e^x \cos 2x\ dx &= \frac{1}{2}\ e^x \sin 2x + \frac{1}{4}\ e^x \cos 2x \\ \int e^x \cos 2x\ dx &= \frac{4}{3}\left(\frac{1}{2}\ e^x \sin 2x + \frac{1}{4}\ e^x \cos 2x\right) \end{aligned}$$

$$\boxed{\frac{2}{3}\ e^x \sin 2x + \frac{1}{3}\ e^x \cos 2x}$$

3.4 Trigonometric Integrals

If in the form of $\int \sin^m x \cos^n x \; dx$

Odd Cosine Powers

Using the identity $\cos^2 x = 1 - \sin^2 x$

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x \; dx &= \int \sin^m x (\cos^2 x)^k \cos x \; dx \\ &= \int \sin^m x (1 - \sin^2 x)^k x \cos x \; dx\end{aligned}$$

Then substitute $u = \sin x$

Odd Sine Powers

Using the identity $\sin^2 x = 1 - \cos^2 x$

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \; dx &= \int (\sin^2 x)^k \cos^n x \sin x \; dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \; dx\end{aligned}$$

Then substitute $u = \cos x$

Both Even Sine & Cosine Powers

These can be reduced using the following identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

If in the form of (1) $\int \sin mx \; \cos nx \; dx$, (2) $\int \sin mx \; \sin nx \; dx$, or (3) $\int \cos mx \; \cos nx \; dx$

$$\sin A \; \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)] \tag{1}$$

$$\sin A \; \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)] \tag{2}$$

$$\cos A \; \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{3}$$

If in the form of $\int \tan^m x \sec^n x \; dx$

Even Secant Powers

Using the identity $\sec^2 x = 1 + \tan^2 x$

$$\begin{aligned}\int \tan^m x \sec^{2k} x \; dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \; dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec x \; dx\end{aligned}$$

Then substitute $u = \tan x$

Odd Tangent Powers

Using the identity $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \; dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \; dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \; dx\end{aligned}$$

Then substitute $u = \sec x$

Odd Secant & Even Tangent

The only way to solve these is to use integration by parts

Example: Evaluate: $\int \sin^5 x \cos^2 x \; dx$

Since the power of sin is odd we will have to substitute $u = \cos x$ after some manipulation to get it into the right form

$$\begin{aligned}&= \int \sin^4 x \cos^2 x \sin x \; dx && u = \cos x \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \; dx && du = -\sin x \; dx \\ &= -\int (1 - u^2)^2 \; u^2 \; du && -du = \sin x \; dx \\ &= -\int (1 - 2u^2 + u^4) \; u^2 \; du \\ &= -\int (u^2 - 2u^4 + u^6) \; du \\ &= -(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7) + C\end{aligned}$$

$$\boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}$$

3.5 Trigonometric Substitution

For when the integral contains $\sqrt{a^2+x^2}$ or similar comes up.

Table of Different Forms and Their Substitution

Form	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

Example: Evaluate: $\displaystyle\int \frac{\sqrt{16-x^2}}{x^2}\,dx$

We can immediate recognise that the radical is in the form of $\sqrt{a^2-x^2}$ so therefore, we will substitute $x=a\sin\theta$

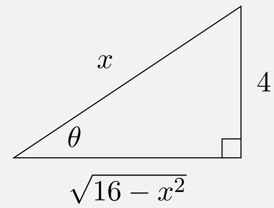
$$\begin{aligned}
&\sqrt{4^2-x^2} \\
&=\sqrt{4^2-(4\sin\theta)^2} \\
&=\sqrt{4^2(1-\sin^2\theta)} \\
&=4\sqrt{\cos^2\theta} \\
&=4\cos\theta
\end{aligned}
\qquad\qquad
\begin{aligned}
x&=4\sin\theta \\
dx&=4\cos\theta\ d\theta
\end{aligned}$$

Substitute back into the the original equation.

$$\begin{aligned}
&\int \frac{4\cos\theta}{(4\sin\theta)^2}\ 4\cos\theta\ d\theta \\
&=\int \frac{16\cos^2\theta}{16\sin^2\theta}\ d\theta \\
&=\int \cot^2\theta\ d\theta \\
&=\int (\csc^2\theta-1)\ d\theta \\
&=-\cot\theta-\theta+C
\end{aligned}$$

We have to undo our substitution turning θ back into x. We can use a right triangle to help.

$$x = 4\sin\theta$$
$$\frac{x}{4} = \sin\theta$$



$$\boxed{-\frac{\sqrt{16-x^2}}{4} - \arcsin\frac{x}{4} + C}$$

3.6 Integration by Partial Fraction Decomposition

These come in the form of $f(x) = \dfrac{P(x)}{Q(x)}$ where both $P(x)$ and $Q(x)$ are polynomials

Step 1

- If the degree of $P < Q$ then this is called **Proper** and we can move onto the next step
- If the degree of $P \geq Q$ then this is called **Improper** and we must reduce it to a proper fraction using either long or short division to obtain a function in the form of:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Step 2
Reduce the denominator $Q(x)$ into either linear or irreducible quadratic factors:

$$ax + b \quad \text{or} \quad ax^2 + bx + c \text{ where } b^2 - 4ac < 0$$

Step 3 (Denominator)
We can split the the whole function into forms of either:

$$\frac{A}{ax+b} \quad \text{or} \quad \frac{Ax+B}{ax^2+bx+c}$$

Case 1: We can express $Q(x)$ as a product of linear factors

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{a_i x + b_i}$$

Case 2: We can express $Q(x)$ as a product of linear factors where some a repeated (Where $a_i x + b_i$ is repeated r times)

$$\frac{A_1}{a_i x + b_i} + \frac{A_i}{(a_i x + b_i)^2} + \cdots + \frac{A_i}{(a_i x + b_i)^r}$$

Case 3: We can express $Q(x)$ as a product of linear factors and quadratic factors

$$\frac{A_i x + B}{a_i x^2 + b_i x + c_i}$$

Case 4: We can express $Q(x)$ as a product of linear factors and quadratic factors where some a repeated (Where $a_i x^2 + b_i x + c_i$ is repeated r times)

$$\frac{A_i x + B}{a_i x^2 + b_i x + c_i} + \frac{A_i x + B}{(a_i x^2 + b_i x + c_i)^2} + \cdots + \frac{A_i x + B}{(a_i x^2 + b_i x + c_i)^r}$$

These terms can be integrated by substitution and **completing the square.**

Step 3 (Numerator) To determine the coefficients given $\dfrac{R(x)}{Q(x)}$ follow this procedure.

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{a_i x + b_i}$$
$$R(x) = \sum_{i=1}^{n} A_i \, \frac{Q(x)}{a_i x + b_i}$$

After dividing, distribute A_i, and then collect like terms in a matrix from and solve for the coefficients. We can also set x in a way that makes the other factors 0.

Example: Evaluate: $\displaystyle\int \frac{7x^2 - 5x + 6}{x^3 - 10x^2 + 21x}\,dx$
The first step is to factor the denominator and then split into different fractions with numerators undetermined for now.

$$\frac{7x^2 - 5x + 6}{x(x-3)(x-7)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-7}$$

Multiply by the original (factored) denominator

$$\frac{A}{x}\,x(x-3)(x-7) \qquad \frac{B}{x-3}\,x(x-3)(x-7) \qquad \frac{C}{x-7}\,x(x-3)(x-7)$$
$$= A(x-3)(x-7) \qquad = Bx(x-7) \qquad = Cx(x-3)$$

Set them equal to the denominator

$$7x^2 - 5x + 6 = A(x-3)(x-7) + Bx(x-7) + Cx(x-3)$$

Solve for the coefficients by letting $x = 0, 3, 7$

$$\begin{array}{lll} x = 0 & 7(0)^2 - 5(0) + 6 = A[(0)-3][(0)-7] & A = \dfrac{2}{7} \\ x = 3 & 7(3)^2 - 5(3) + 6 = B(3)[(3)-7] & B = -\dfrac{9}{2} \\ x = 7 & 7(7)^2 - 5(7) + 6 = C(7)[(7)-3] & C = \dfrac{157}{14} \end{array}$$

We can now go back to the original integral

$$\int \frac{7x^2 - 5x + 6}{x^3 - 10x^2 + 21x}\ dx = \frac{2}{7}\int \frac{dx}{x} - \frac{9}{2}\int \frac{dx}{x-3} + \frac{157}{14}\int \frac{dx}{x-7}$$

$$\boxed{\frac{2}{7}\ln|x| - \frac{9}{2}\ln|x-3| + \frac{157}{14}\ln|x-7| + C}$$

Example: Evaluate: $\displaystyle\int \frac{4x^2+5x+2}{4x^2+4x+3}\ dx$
Since the degree of the numerator is greater or equal to the denominator, we must first use long division

$$\begin{array}{r} 1 \\ 4x^2+4x+3 \overline{\big)\ 4x^2+5x+2} \\ \underline{-\,4x^2-4x-3} \\ x-1 \end{array}$$

$$\frac{4x^2+5x+2}{4x^2+4x+3} = 1 + \frac{x-1}{4x^2+4x+3} \qquad \int 1dx = x$$

Complete the square of the denominator

$$\begin{aligned} 4x^2+4x+3 &= 4\left(x^2+x+\frac{3}{4}\right) \\ &= 4\left(x^2+x+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \frac{3}{4}\right) \\ &= 4\left(x+\frac{1}{2}\right)^2 + 2 \end{aligned}$$

Make the substitution of the square (ie $u = x + \dfrac{1}{2}$)

$$\begin{aligned} &\int \frac{x-1}{4x^2+4x+3}\ dx && u = x+\frac{1}{2} \\ =&\int \frac{u-\frac{1}{2}-1}{4u^2+2}\ du && du = dx \\ =&\int \frac{u-\frac{3}{2}}{4u^2+2}\ du && x = u-\frac{1}{2} \end{aligned}$$

Now it is in a form we can work with

$$\frac{u-\frac{3}{2}}{4u^2+2} = \frac{u}{4u^2+2} - \frac{\frac{3}{2}}{4u^2+2}$$

Take the integral for what we have now, using what techniques we know

$$\frac{1}{2}\int \frac{u}{2u^2+1}\ du - \frac{3}{4}\int \frac{1}{2u^2+1}\ du$$
$$= \frac{1}{2}\ \frac{1}{4}\ln|2u^2+1| - \frac{3}{8}\ \frac{1}{\sqrt{2}}\arctan(\sqrt{2}u) + C$$

Substitute back in what we have

$$\boxed{x + \frac{1}{8}\ln|2(x+\frac{1}{2})^2+1| - \frac{3}{8\sqrt{2}}\arctan(\sqrt{2}(x+\frac{1}{2})) + C}$$

4 Applications of Integration