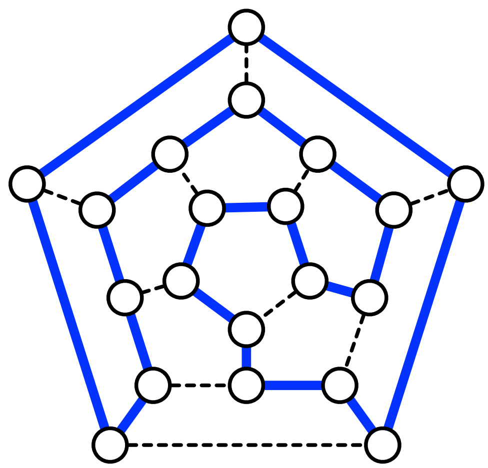
1. (35 pts total) A classic NP-complete problem is that of determining whether some unweighted, undirected graph $G = (V, E)$ contains a Hamiltonian cycle, i.e., a cycle that visits each vertex exactly once (except for the starting/ending vertex, which is visited exactly twice). The following figure gives a positive example; see Chapter 35.5.3 in CLRS for a proof of NP-completeness.



   (a) (5 pts) If we are given a "witness" for G in the form of a path P on its vertices, explain how we can verify in polynomial time whether P is or is not a Hamiltonian cycle. That is, demonstrate that Hamiltonian Cycle is in NP.

   (b) (30 pts) Suppose there is an oracle in a nearby cave. She will tell us, for the price of one tuppenny per question, whether a graph has a Hamiltonian cycle. If it does, show that by asking her a series of questions, perhaps involving modified versions of the original graph, we can find the Hamiltonian cycle after spending a number of tuppennies that grows polynomially as a function of the number of vertices. (Hence prove that if we can solve the Hamiltonian cycle decision problem in polynomial time, we can solve its search problem as well.)

2. (10 pts) The logical definition of NP requires that the number of bits of the witness is polynomial in the number of bits of the input, i.e., $|w| = poly(n)$. Suppose we have a property for which witnesses of logarithmic size exist, i.e., $|w| = O(\log n)$. (i) Show that any such property is in P, and (ii) explain why the logical definition of NP implies exponential time for brute force.

3. (15 pts) Two graphs G and H are isomorphic if their structure is the same, except that their vertices are permuted: formally, if there is a 1-to-1 and onto mapping ϕ from the vertices of G to the vertices of H such that u and v are connected in G if and only if $\phi(u)$ and $\phi(v)$ are connected in H.

   - Prove that the property Graph Isomorphism (that is, the language $\{\langle G, H\rangle : G$ and H are isomorphic graphs$\}$) is in NP.
   - What about Graph Non-Isomorphism?

4. (40 pts total) Download the PS10 data file on the class website, which is a plaintext ASCII file containing an edge list for the Facebook social network of students and faculty within a top-10 public university in 2005.[1]

   (a) (25 pts) In many social networks, we observe a surprising phenomenon called the *friendship paradox*. Let k_u denote the degree of some individual u, and let some edge $(u, v) \in E$. The paradox is that the average degree of the neighbor $\langle k_v\rangle$ is *greater* than the average degree $\langle k_u\rangle$ of the vertex. That is, on average, each friend of yours has more friends than you.

   - Using the graph provided, develop and implement an $O(V + E)$-time algorithm that computes and compares (i) the mean degree $\langle k_u\rangle$ and (ii) the mean neighbor degree $\langle k_v\rangle$. Describe the algorithm and justify how it achieves the given bound. Comment on the degree to which we observe a friendship paradox.
   - If we assume that edges are distributed randomly, conditioned on each vertex having its observed degree,[2] it can be shown that the expected mean neighbor degree is $\langle k^2\rangle/\langle k\rangle$, where k is the degree of a vertex. Compute this quantity using the graph provided, and compare it to the empirical value calculated above. Comment on the degree to which these two calculations agree or disagree.

   (b) (15 pts) Another common property of social networks is that they have very small diameters relative to their total size. This property is sometimes called the

[1] The data were kindly provided by A.L. Traud, P.J. Mucha and M.A. Porter, as part of their paper "Social Structure of Facebook Networks," available here `http://arxiv.org/abs/1102.2166` .

[2] This model is called the *configuration model*, and is a useful family of random graphs, often used as a reference point in network analysis.

"small-world phenomenon," or called having "six degrees of separation".[3]

- Compute and report the diameter of the largest component in the given graph (which contains 99.84% of V) and comment on the degree to which it agrees with the six-degrees of separation idea.
- Briefly discuss whether and why you think the diameter of Facebook has increased, stayed the same, or decreased relative to this value, since 2005. (Recall that Facebook now claims to have nearly 10^9 accounts.)

5. (10 pts extra credit) In Subgraph Isomorphism, we are given two undirected graphs G_1 and G_2, and we must determine whether G_1 is isomorphic to some subgraph of G_2. Show that the subgraph-isomorphism problem is NP-complete.

   Hint: Choose G_1 to be a very specific kind of graph.

[3]This term originated in a play written by John Guare in 1990, which was turned into a 1993 movie starring Will Smith. The concept, however, was originated by the sociologists Stanley Milgram, working in 1967, who was the first to measure the lengths of paths in large social networks.