Networks & Hierarchies

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Rankings and linear hierarchies

Many uses for models of large-scale structure

Treat the network like a system:

Extrapolation. Make predictions for as-yet unseen nodes (in "space" or time).
Interpolation. Identify missing links.
Generalization. Nodes of this type are like others of the same type.

Treat the network like an artifact:

Mechanisms. How did this network arise? What rules governed its assembly?
Explanations. Coarse-graining or compression.

Treat the network like a means to an end; an intermediate data structure:

Useful division. Need groups so that we can assign treatments in an A/B test.
Simplification. Downstream regression model needs ranks or groups.

intuition: compare this list with the list you would write for regression

The idea of rankings—pervasive!

Assumptions:

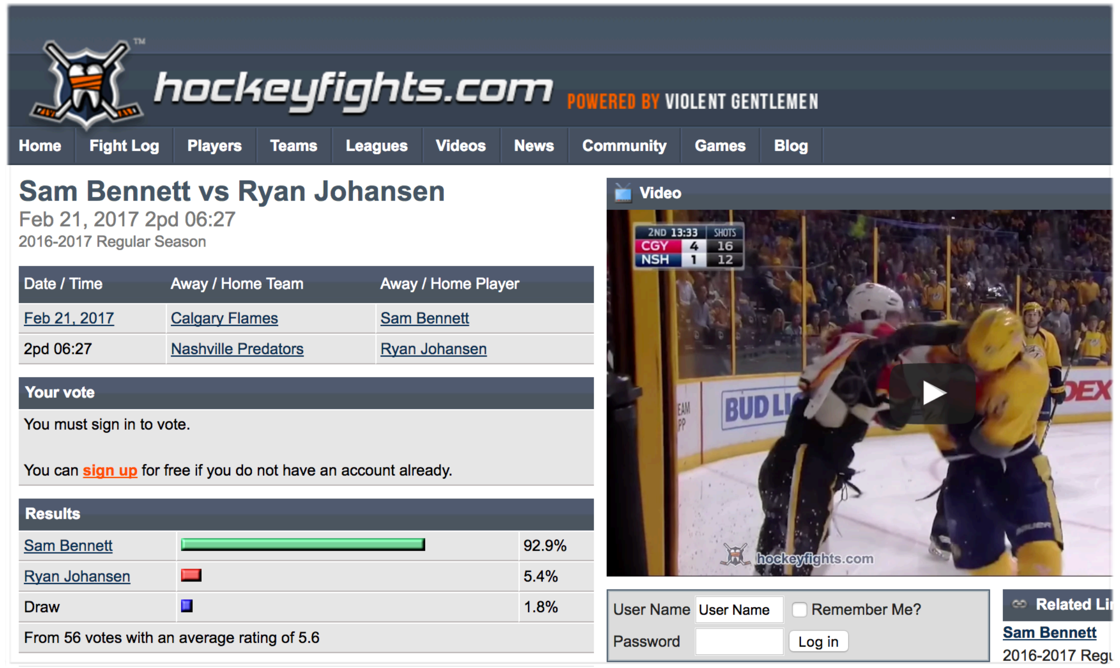
1. Competitors have some intrinsic quality (or vector of qualities).
2. Interactions can (stochastically) reveal differences in qualities.
3. Competitions are pair-wise. (Lee Sedol vs. AlphaGo; Astros vs. Dodgers)

In other words: outcomes are generated by a stochastic process, which is some function of the positions of the competitors.

Systems of dominance

social

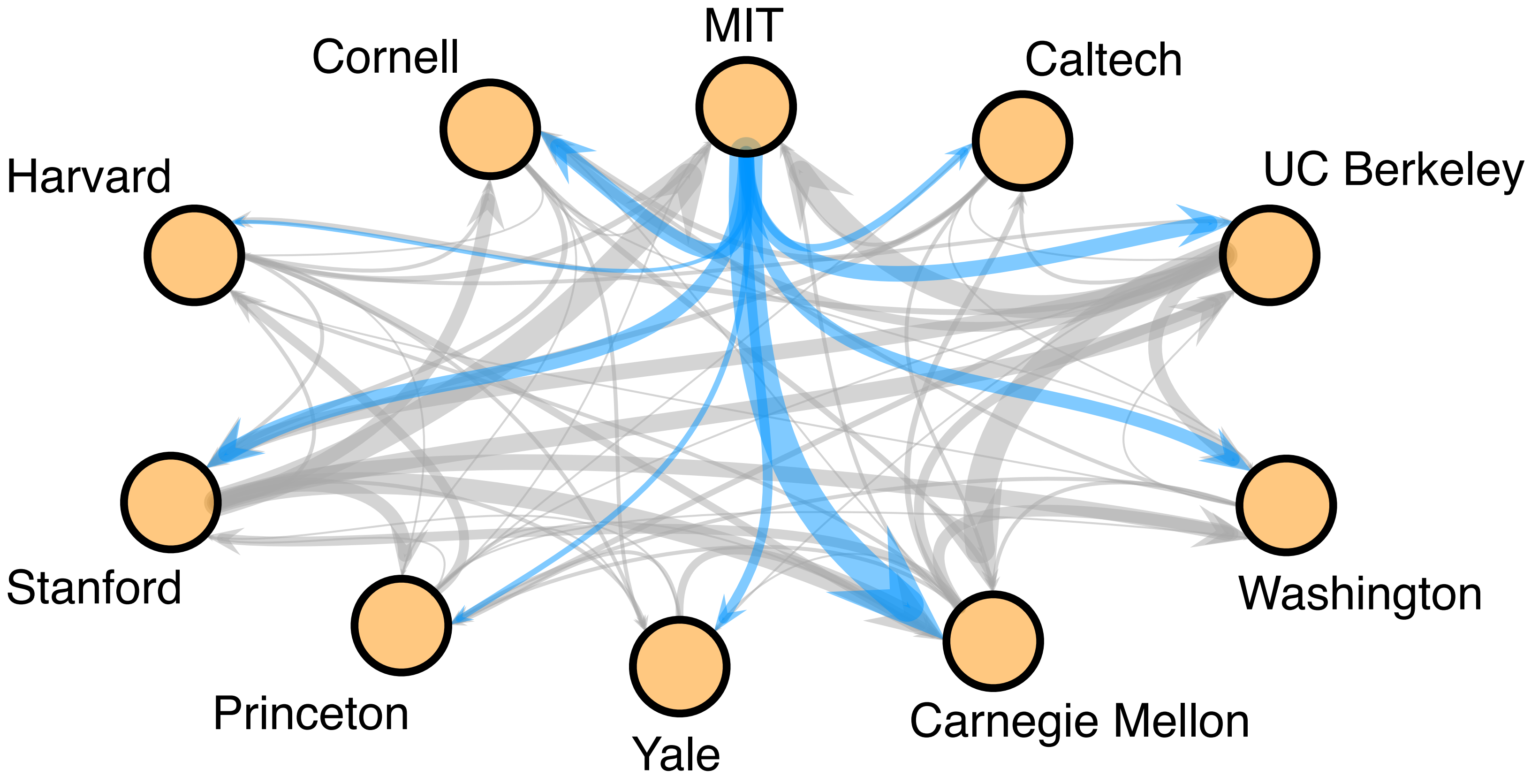
physical

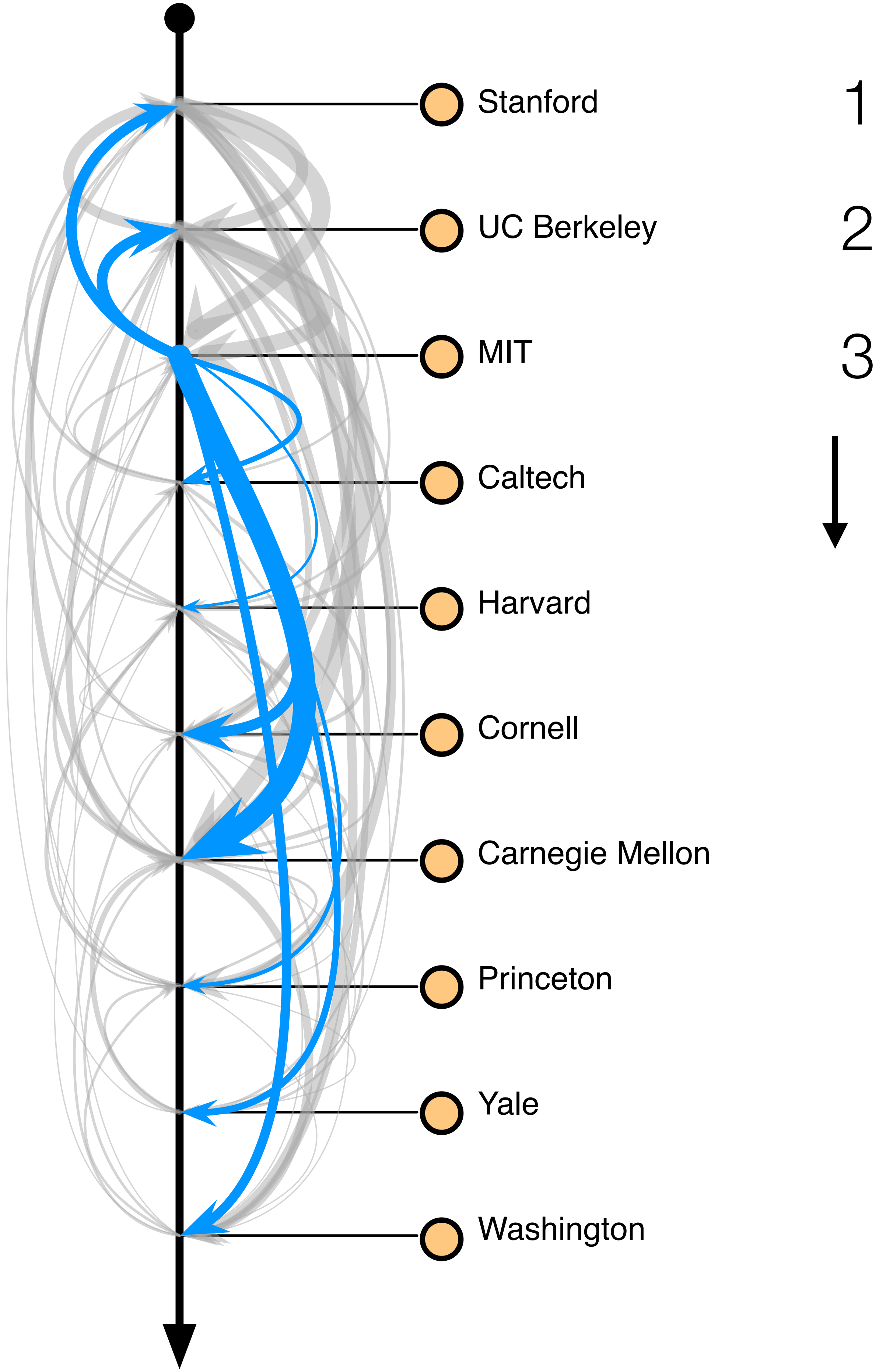


mental

financial

Systems of endorsement



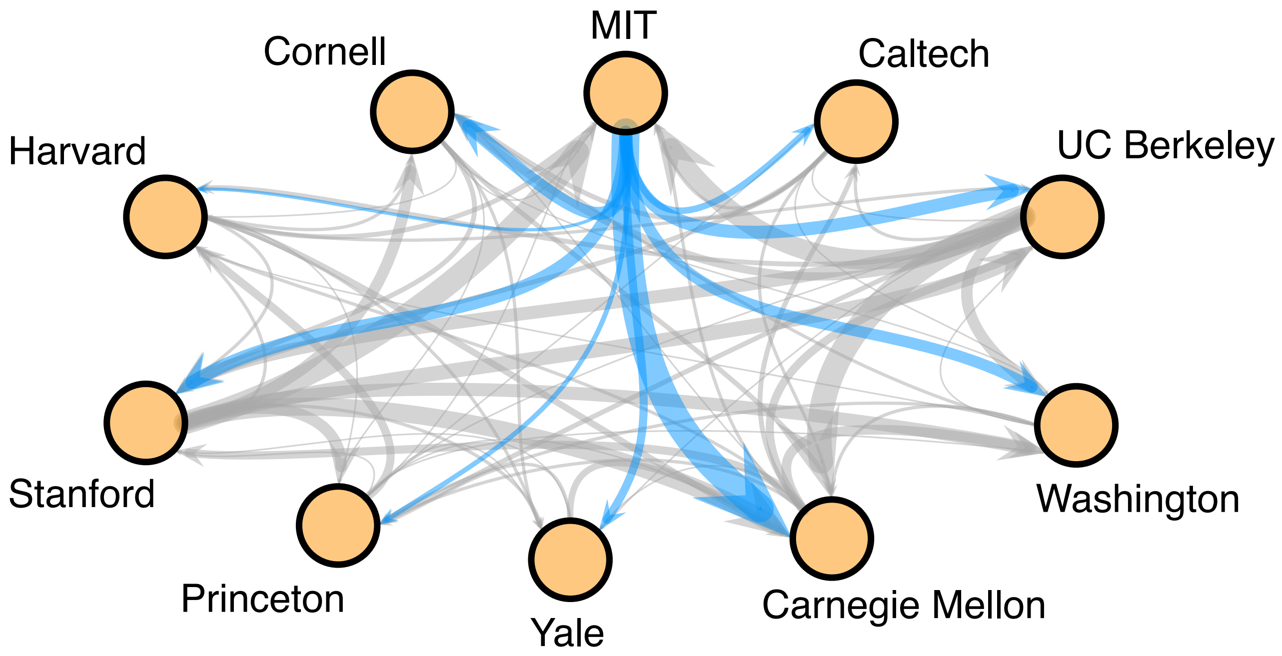


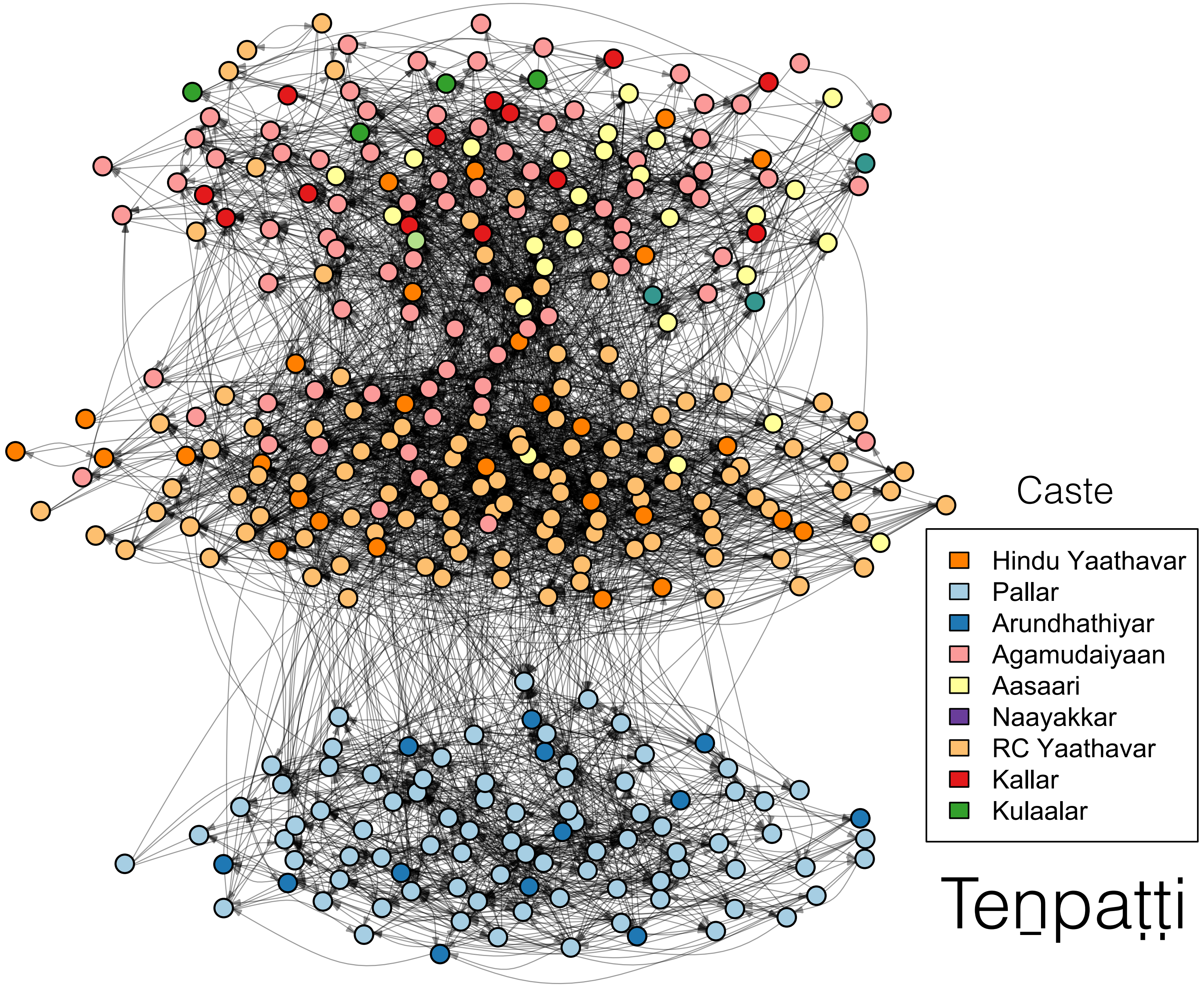
Assumptions:

1. Endorsers have some intrinsic quality.
2. Interactions can reveal differences in qualities.
3. Endorsements are pair-wise.

Clauset, Arbesman, Larremore. *Science Advances* 1, e1400005 (2015).

Systems of endorsement





Latent position can be revealed by **dominance** or **endorsement** interactions.

Figure: Larremore, Hébert-Dufresne, Power. *Draft.* Data: Power. *Nature Human Behaviour* 1, 0057 (2017)

The setup: suppose we have a *directed* network.

Its adjacency matrix is A .

$A_{ij} = A_{i \to j}$ means i beat j or i was endorsed by j

The problem: Rank the nodes.

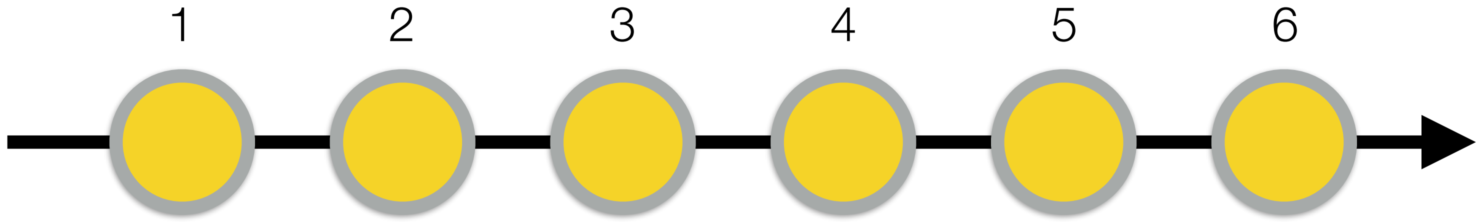
Alternative view: there might be **no network here.** In some cases we're just seeing a network in pairwise comparison data because **networks are a convenient data structure.**

Alternative problem: Which items should be compared next in order to most/best resolve our estimate of the ranks? (sequential tournament design)

Embeddings vs Orderings

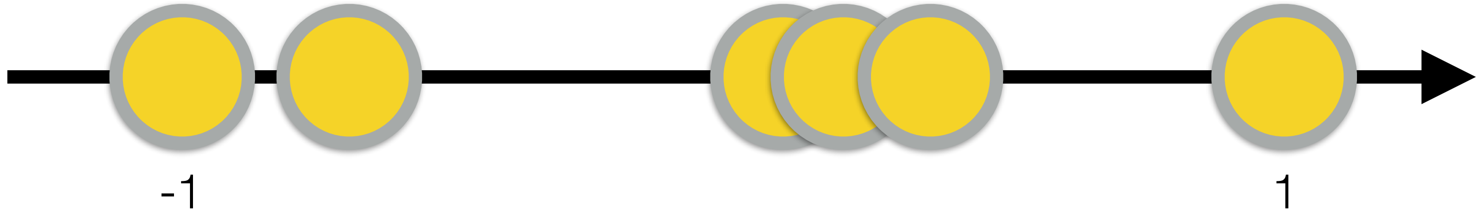
Ordering place the nodes in order:

1, 2, 3, …



Embedding assigns a position to each node:

1, 1.2, 7, 20, 21, 21.2, …



Which one should I use?

> Depends on the use case.

> Is it possible for two nodes to occupy the same rank or position? If so, an embedding is more appropriate. Also better when meaning of 1-rank Δ varies.

> Consider that you can always go from an embedding to an ordering, if you have a rule for breaking ties.

Win-Loss is not satisfactory: schedule matters

Beating the grandmaster counts for more than beating a novice.

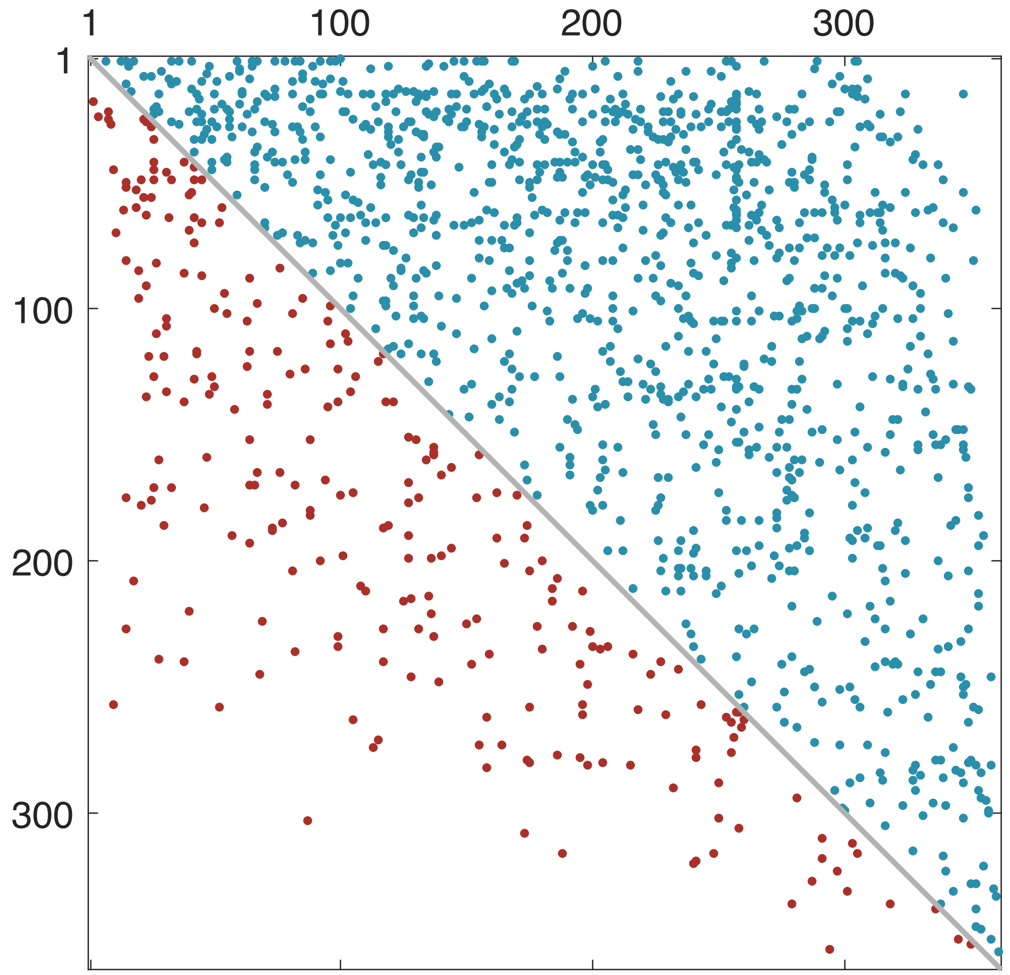
Win and loss tallies don't take this "schedule difficulty" into account. Put differently, win-loss records **leave information on the table.**

One way to make use of this information:

i **beats** j implies $s_i > s_j$

Therefore if we have a whole list of outcomes, we can try to find a total ordering that breaks as few of these implications as possible.

$A_{ij} =$ number of times that i beat j.



minimum violation ranking: sort A.

Win-Loss is not satisfactory: schedule matters

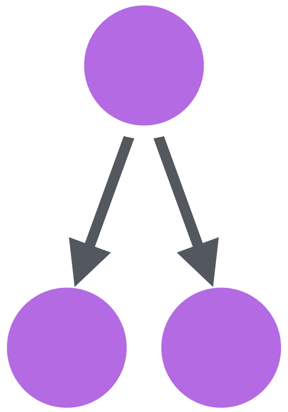
How do we find an ordering that minimizes the number of violations (or upsets) ?

Recipe (MCMC):

1. Order the nodes randomly.
2. Compute the number of violations. In expectation, this should be 50% of edges.
3. Pick two nodes at random and propose to swap their positions.
4. Compute the number of violations in this scenario.
5. If #violations decreases or stays the same, keep the swap. Otherwise, reject.
6. Repeat until….?

Notes:

* The number of violations is non-increasing over time.
* There may be no unique minimum. Consider this scenario:

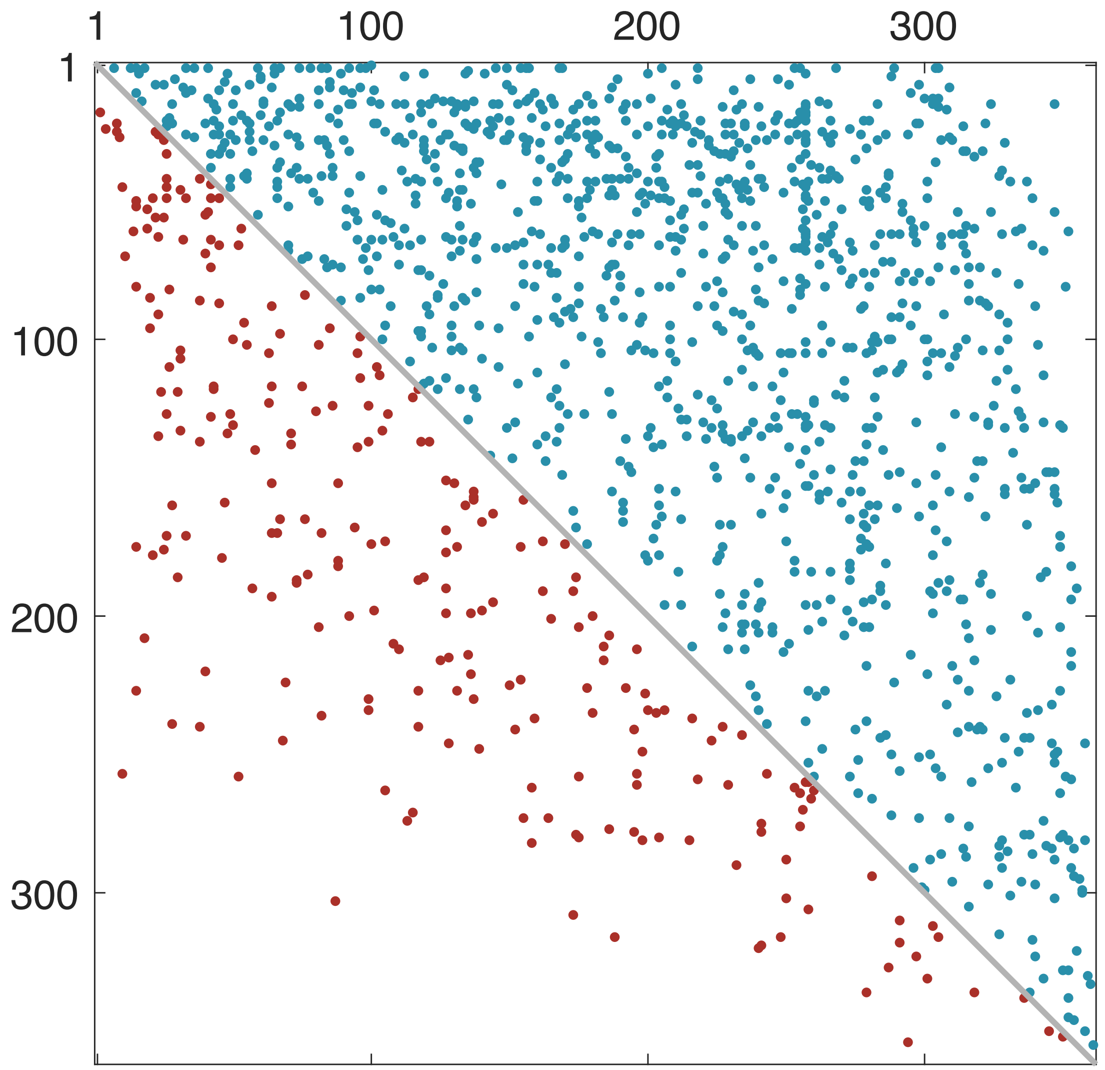


Embeddings & Orderings 0: MVR & Agony

- There is no guarantee of a unique minimizing ranking s.
- Space of ordinal rankings has n! elements, requiring slow search algorithms (e.g. MCMC).
- Ordinal. No ties. No interpretability of rank differences.

What if you allowed for **ties** and then ran Minimum Violation Ranking (MVR)? What would happen?

MVR: uniform cost (1 per edge).
Agony: generic cost function.
for example, difference in ranks.



minimum violation ranking: sort *A*.

What are other premises on which we can base a ranking model?

Gupte et al. Finding hierarchy in directed online social networks WWW (2011)

Embeddings and Orderings 1: Discrete choice models

Louis Leon Thurstone and Thelma Thurstone

Thelma: Prof. of Education & Psych UNC Chapel Hill. Louis: Worked with Edison.

Embeddings and Orderings 1: Discrete choice models

Instead of rating everything from 1 to 10, try *paired comparisons*.

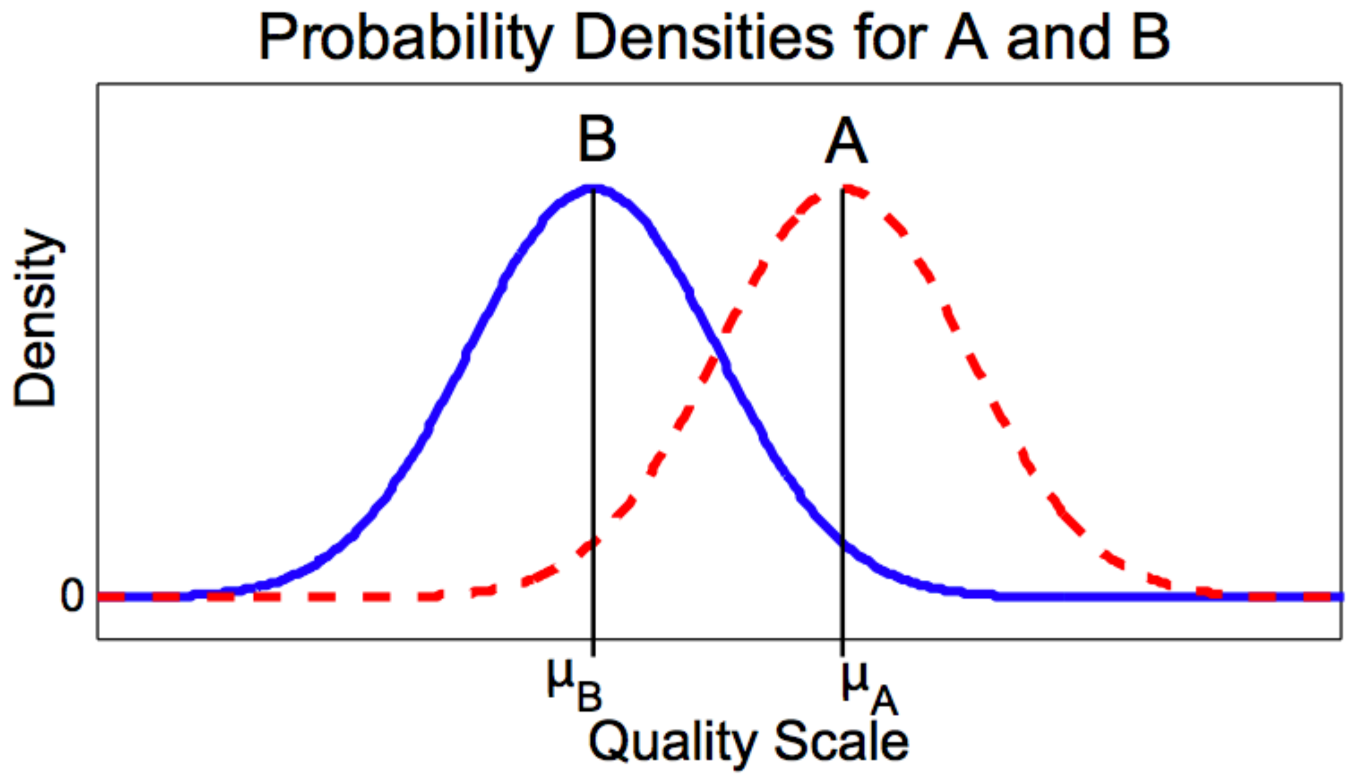
Do you prefer *i* or *j* ?

Why? Consider: My 3 is not your 3. What is 1 and what is 10?

https://xkcd.com/883/

Embeddings and Orderings 1: Discrete choice models

Thurstone: items have quality distributions. When a person judges whether A is better than B they draw from A's distribution and from B's distribution and see which is higher.



Thurstone modeled these as Gaussians.

$$P(A > B) = P(A - B > 0)$$

Difference of Gaussians is Gaussian.

$$\hat{\mu}_{AB} = \Phi^{-1}\left(\frac{C_{A\to B}}{C_{A\to B} + C_{B\to A}}\right)$$

Where $\Phi^{-1}(x)$ is the inverse CDF of standard normal, a.k.a. the *probit*.

Powerful idea: lots of pairwise comparisons = estimates of all the qualities! An embedding!

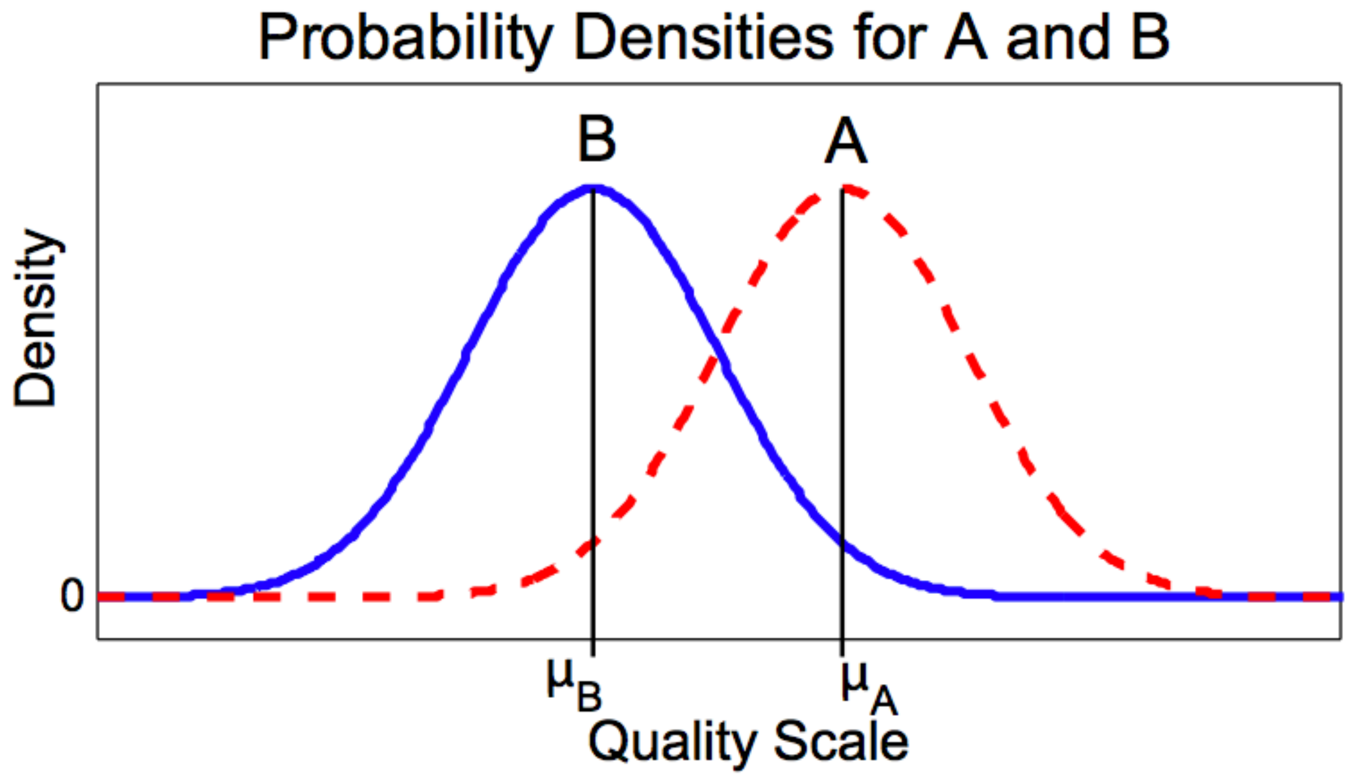
Choice modelers would instead say that we model utility, which has an observable component and a random component. They call these Random Utility Models.

Key: pairwise comparisons = directed network. i preferred to $j \;=\; i \to j$

Finding the qualities of items from pairwise comparisons = Finding embedding of nodes.

Embeddings and Orderings 1: Discrete choice models

Bradley-Terry & Luce: items have quality distributions. When a person judges whether A is better than B they draw from A's and from B's distribution and see which is higher.



BTL

$$P(A > B) = \frac{\pi_A}{\pi_A + \pi_B}$$

Or usually:

$$P(A > B) = \frac{e^{\mu_A/s}}{e^{\mu_A/s} + e^{\mu_B/s}} = \frac{1}{1 + e^{-(\mu_A - \mu_B)/s}}$$

Same idea; different distribution. (*logit* instead of *probit*; *Gumbel* instead of *Gaussian*)

Powerful idea: lots of pairwise comparisons = estimates of all the qualities! An embedding!

Choice modelers would instead say that we model utility, which has an observable component and a random component. They call these Random Utility Models.

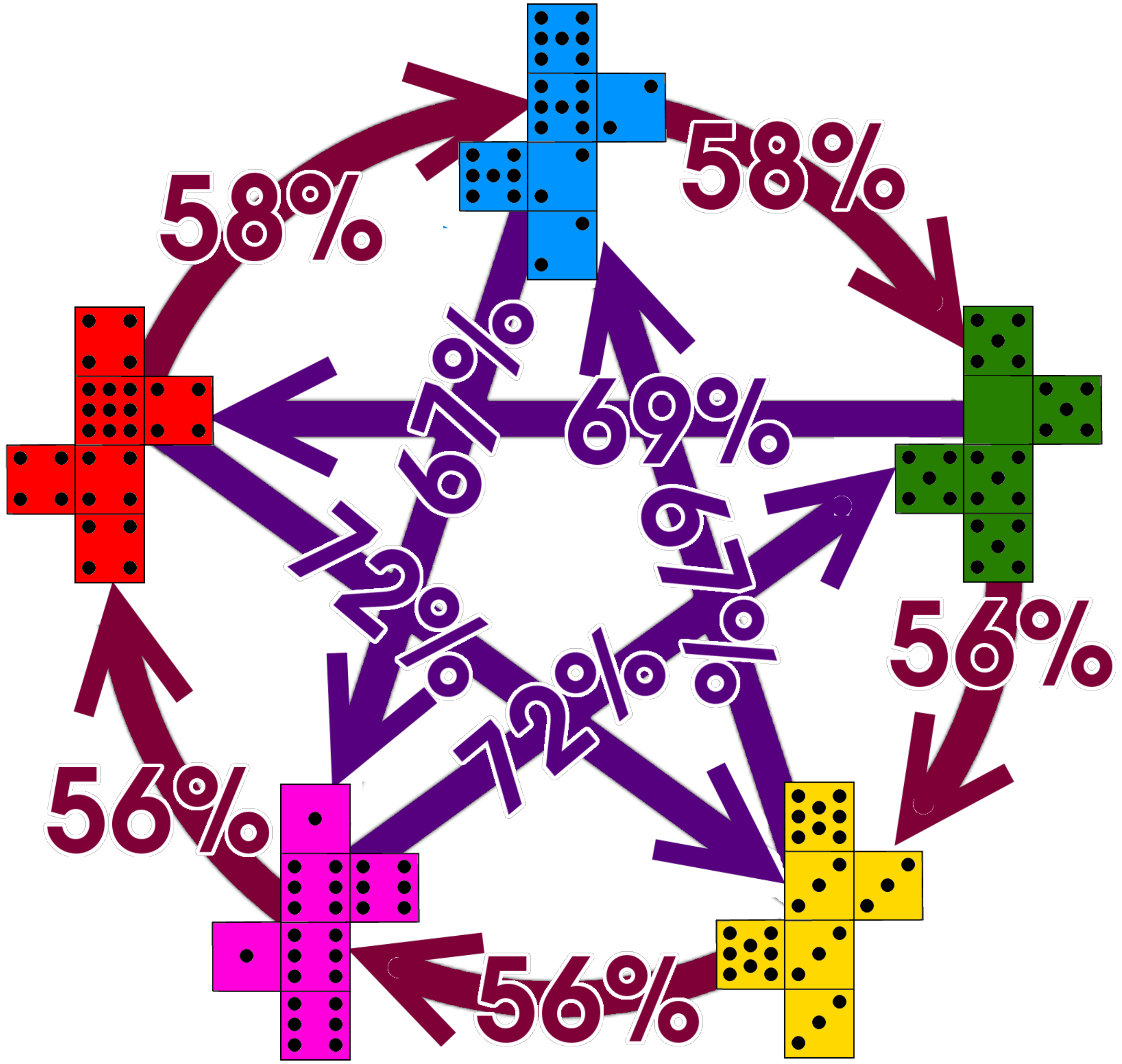
BTL avoids non-transitivities (aka rock-paper-scissors)

Introducing: **non-transitive dice**!

- 3 (or more) dice {A,B,C}
- faces chosen so that they have the property:
    - A>B more than half the time.
    - B>C more than half the time.
    - C>A more than half the time (?!)

https://en.wikipedia.org/wiki/Nontransitive_dice

A great gift for your favorite nerd's desk!
Go to the makerspace and laserbeam your own!



Bradley-Terry-Luce

These methods embed items or players in a 1D space.

- Provably avoids non-transitive properties
- Great when lots of data per interaction.

Pairwise ranking is really nice for ordering large sets of preferences too, and this model specifically models the probability that the preference will be for *i* over *j*.

Iterative algorithms exist. Needs a little regularization so the winningest winners don't fly off to infinity. [why?]

$$P(A \to B) = \frac{\pi_A}{\pi_A + \pi_B} = \frac{e^{\gamma_A}}{e^{\gamma_A} + e^{\gamma_B}}$$

Embeddings and Orderings 1: Discrete choice models

Introductory tutorial (Gupta):

http://mayagupta.org/publications/PairedComparisonTutorialTsukidaGupta.pdf

Discrete choice today (Ugander):

https://web.stanford.edu/~jugander/papers/nips16-pcmc-slides.pdf

The textbook (Train):

https://eml.berkeley.edu/books/train1201.pdf

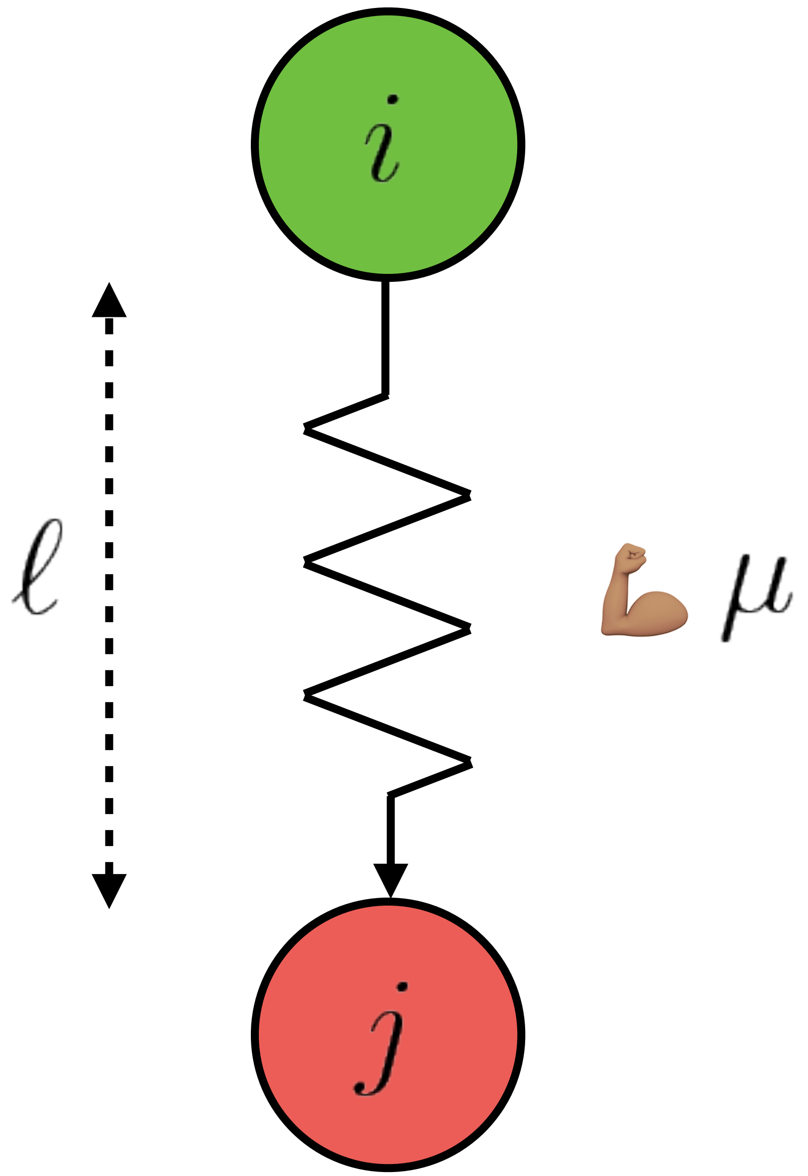
Nobel Lecture for roadtrip when you're out of podcasts (McFadden)

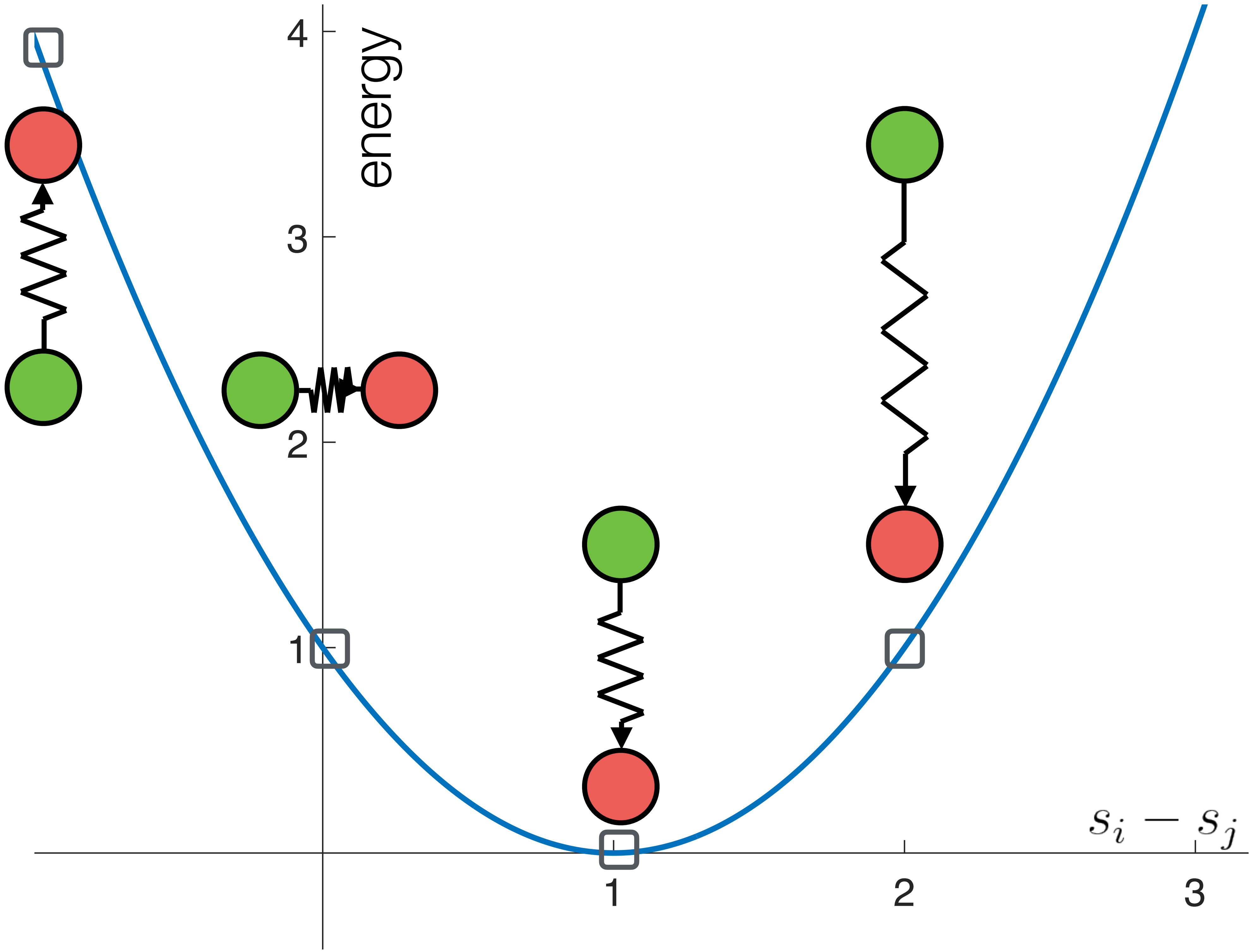
https://www.nobelprize.org/prizes/economic-sciences/2000/mcfadden/facts/

This is such good stuff, people! Lots of problems can be viewed through the lens of Choice Modeling, and powerful Random Utility Models. Dig in!

Embeddings & Orderings 2: SpringRank

Each directed edge = directed spring





How much energy is this system of springs?

SpringRank Hamiltonian = energy of the system, given the node positions *s.*

Relax and let the springs decide the ranks

$$H(s) = \frac{1}{2} \sum_{i,j=1}^{N} A_{ij}(s_i - s_j - 1)^2$$

SpringRank Hamiltonian = energy of the system, given the node positions *s.*

Because the springs are linear, the potential is quadratic.

The SR Hamiltonian is *convex* in s.

$$\nabla H(s) = 0$$

The solution is unique…up to an additive constant. (Why?)

Derivatives work out nicely

$$0 = \frac{\partial H}{\partial s_i} = \sum_j A_{ij}(s_i - s_j - 1) - A_{ji}(s_j - s_i - 1)$$

Rewrite as a linear algebra problem.

$$\left[D^{\text{out}} + D^{\text{in}} - \left(A + A^T\right)\right] s^* = \left[D^{\text{out}} - D^{\text{in}}\right] \mathbf{1}$$

We know *a priori* that the matrix on the left is singular: translational invariance of H(s).
[if *s* is a solution, then *s* + *k* is a solution for any constant *k;* eigenvalue 0, eigenvector **1**]

Notice: the matrix on the left is the *graph Laplacian* of the *undirected* network.

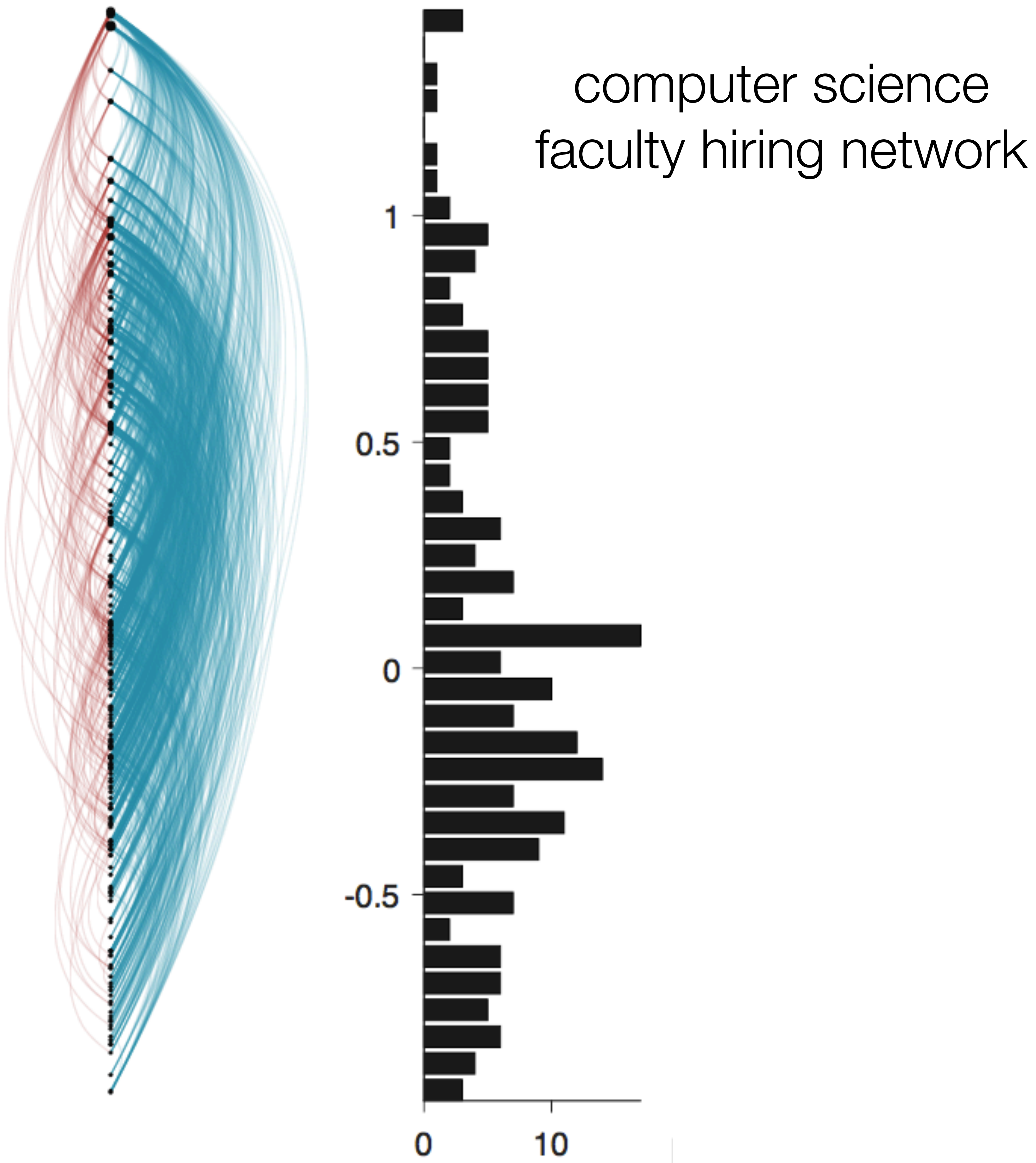
Uniqueness: Set s_1=0, min(*s*)=0, or mean(*s*)=0. Or use a pseudoinverse. Or regularize.

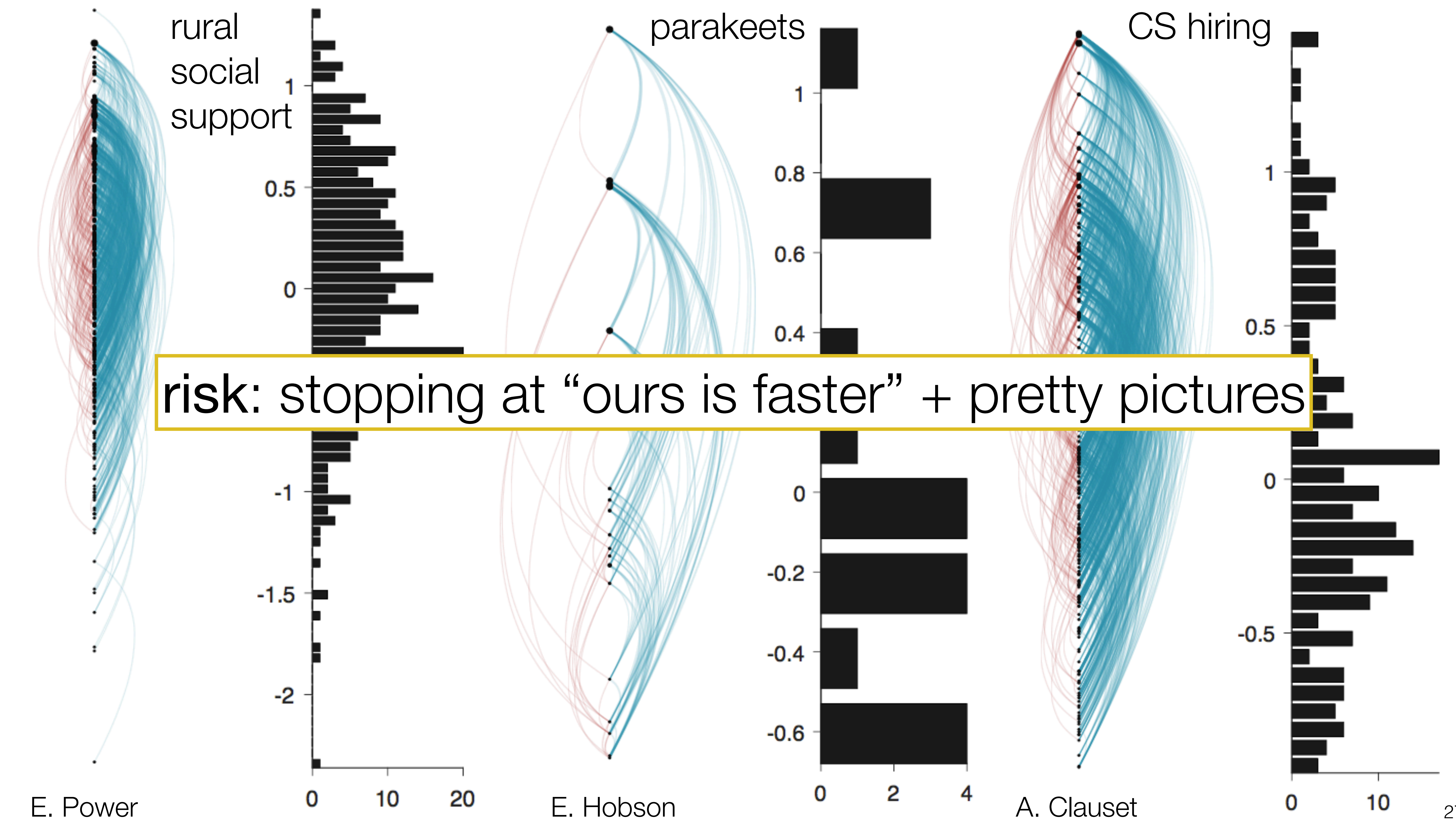
It works!

Real networks tend to be sparse…
  our linear algebra problem is sparse…
    we can use sparse iterative solvers…
      millions of edges in **seconds**.

Even better: it's a linear-Laplacian system.
🚀 Near-linear-time (in |edges|) solutions.

Note that node positions can be clumpy, since this is an *embedding*.





risk: stopping at “ours is faster” + pretty pictures

Cross validation: train on 80%, predict 20%

In a linear hierarchy the key quantity to predict is *edge direction,* given *edge existence*.

If i and j were to face off, who would win?

I'll give you *undirected(A)*, and you predict *directed(A).*

Setup: learn s from 80% of A. Then predict edge directions for remaining 20% of A.

SpringRank predicts edge direction based on the relative direction probabilities:

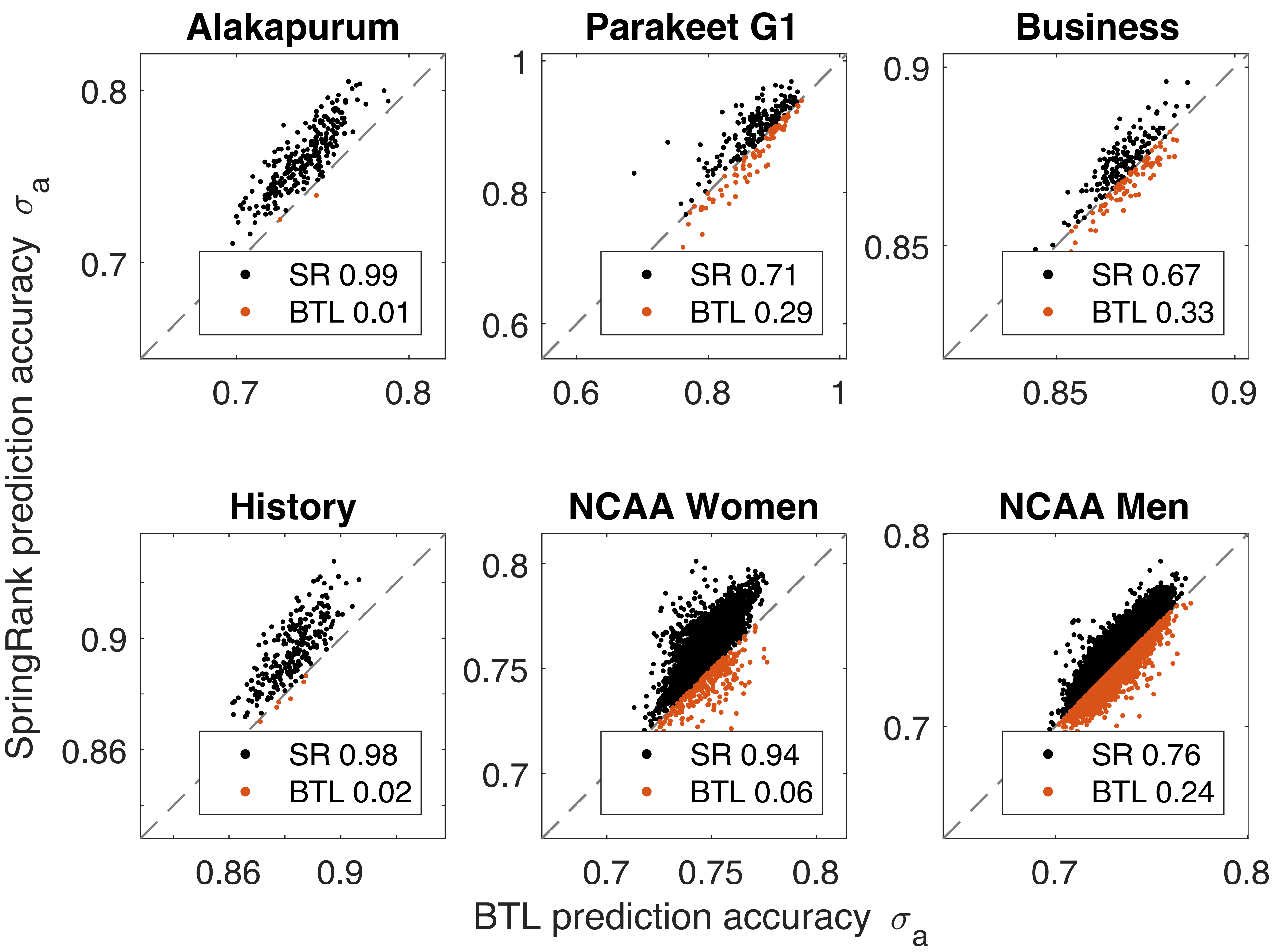
$$P_{ij}(\beta) = \frac{e^{-\beta H_{ij}}}{e^{-\beta H_{ij}} + e^{-\beta H_{ji}}} = \frac{1}{1 + e^{-2\beta(s_i - s_j)}}$$

Cross validation vs BTL: SR makes better predictions

Accuracy:

$$\sigma_a = 1 - \frac{1}{2M} \sum_{i,j} |A_{ij} - (A_{ij} + A_{ji})\, P_{ij}|$$

Goal: maximize the number of correctly predicted edge directions.



Sync Rank, Cucuringu et al. https://ieeexplore.ieee.org/document/7395367

50 independent trials of 5-fold cross validation (250 folds)

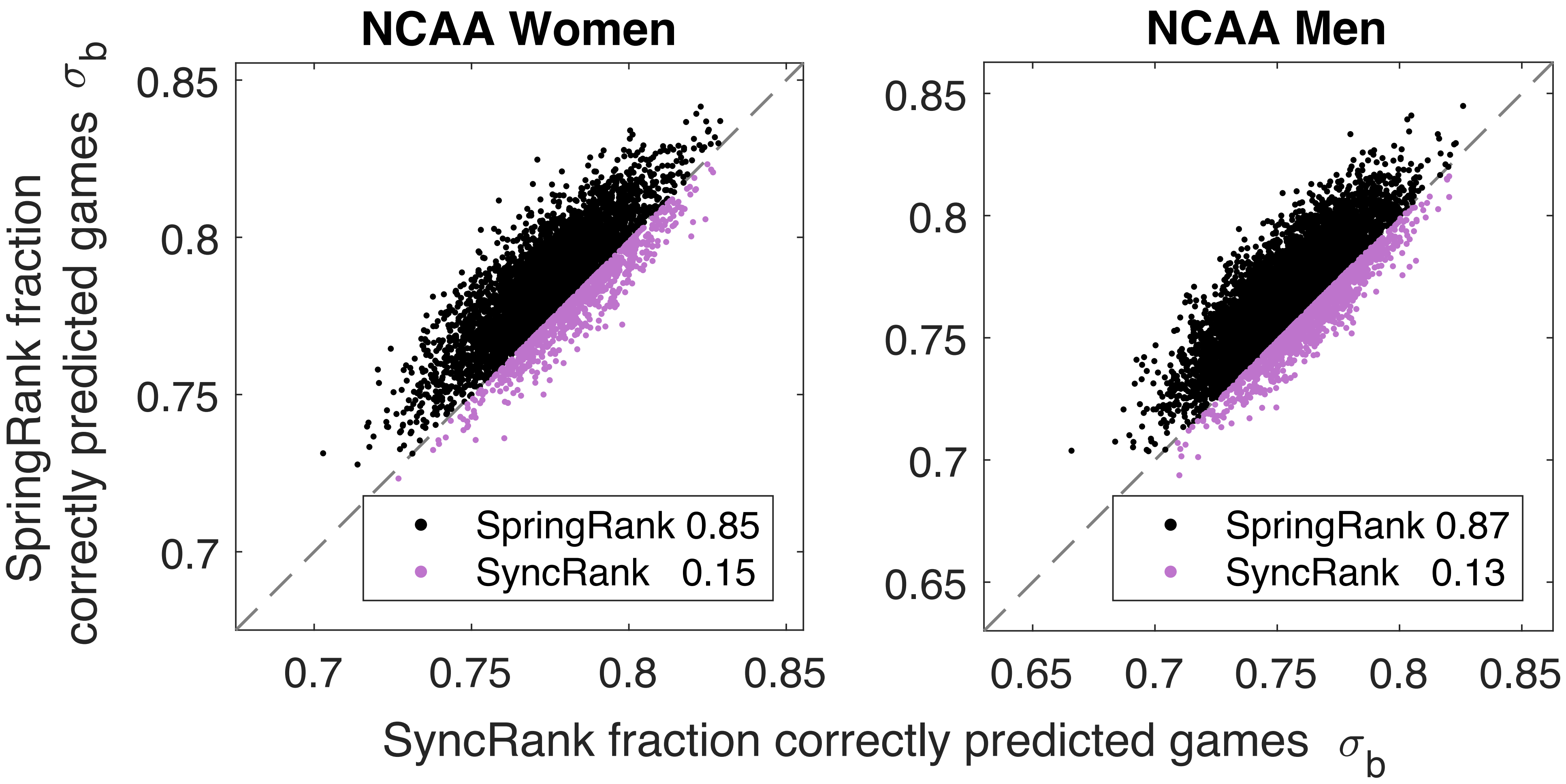
Cross validation vs SyncRank: SR makes better predictions

"One-bit" Accuracy:

Higher ranked player always wins.

- No probabilistic prediction.
- Bad for gambling.

Goal: maximize the number of correctly predicted edge directions.



50 independent trials of 5-fold cross validation (250 folds)

Why/when would a model of springs make better predictions
than a model of the choices themselves?🧐

It’s unclear *why* we get this result!
Both BTL and SpringRank make logistic predictions about preference.
Key Idea: SpringRank makes different regularization assumptions.

Embeddings and Orderings 3: PageRank

PageRank defines scalar rank recursively:

important pages are those that are linked to by important pages.

- Great at finding the top 3 but limited predictions available using the PageRank scores.

The PageRank Citation Ranking: Bringing Order to the Web

January 29, 1998

Abstract

The importance of a Web page is an inherently subjective matter, which depends on the readers interests, knowledge and attitudes. But there is still much that can be said objectively about the relative importance of Web pages. This paper describes PageRank, a method for rating Web pages objectively and mechanically, effectively measuring the human interest and attention devoted to them.

We compare PageRank to an idealized random Web surfer. We show how to efficiently compute PageRank for large numbers of pages. And, we show how to apply PageRank to search and to user navigation.

The Anatomy of a Large-Scale Hypertextual Web Search Engine

Sergey Brin and Lawrence Page

Computer Science Department,
Stanford University, Stanford, CA 94305, USA
sergey@cs.stanford.edu and page@cs.stanford.edu

Abstract

In this paper, we present Google, a prototype of a large-scale search engine which makes heavy use of the structure present in hypertext. Google is designed to crawl and index the Web efficiently and produce much more satisfying search results than existing systems. The prototype with a full

Embeddings and Orderings 3: PageRank

We imagine a web surfer who choose a starting webpage at random.

From that webpage, she looks at the links on the page, and either
(a) clicks on a random link or
(b) stops surfing; when she returns, she starts at a new random page.

What's the probability that she's at a particular page? *That's PageRank*.

$$\pi_{ji} = \frac{A_{ji}}{k_j}$$

define a transition matrix

$$p_i = \frac{1-d}{N} + d \sum_j p_j \pi_{ji}$$

write the equation

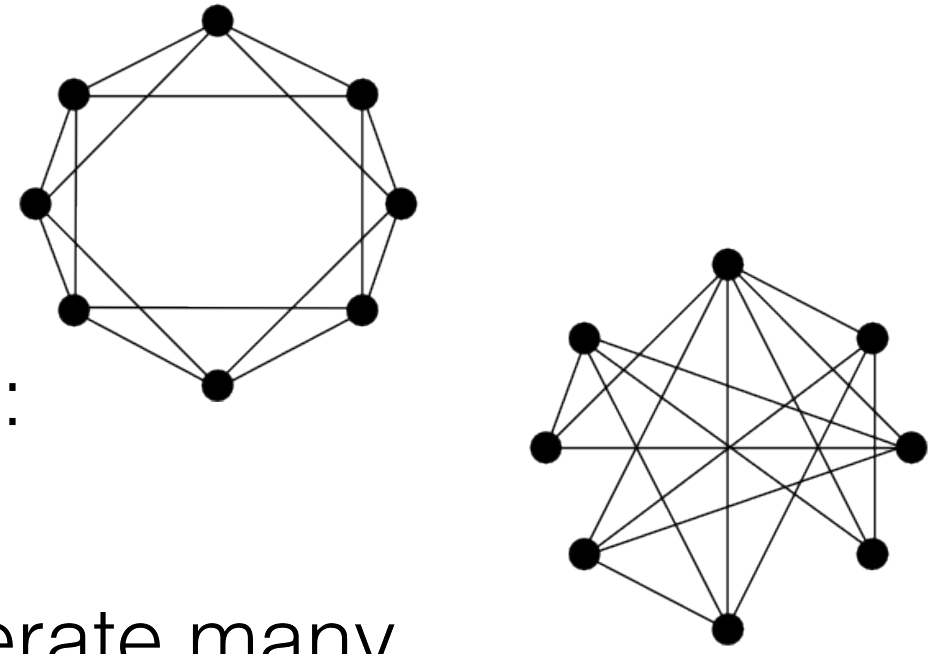
$$\mathbf{p} = \left(\frac{1-d}{N}\right)\mathbf{1} + d\pi^T\mathbf{p}$$

matrix-vector form

Alternative: stationary distribution of random walk on the network + weak all-to-all links

Jeremy Kun: http://www.infinitelooper.com/?v=K3pT0gTaDec&p=n

Stochastic models, sets, and distributions



- a model is just a recipe:
  choose parameters→make the network
- a **stochastic** (generative) model is also just a recipe:
  choose parameters→draw **a** network
- since a single stochastic generative model can generate many networks, the model itself corresponds to a **set of networks**.
- and since the generative model itself is some combination or composition of random variables, a **random graph model** is a set of possible networks, each with an associated probability, i.e., a distribution.

> By changing the recipe, we can change the *support* of the distribution and the *probability masses* themselves.

Consider the configuration model.

Generative models for network structure

Generate the structure you wish to infer.

We like generative models because they open the door to inference:

$$\text{GM + parameters} \xrightarrow{\text{stochastic draw}} \text{Data}$$

$$\text{GM + parameters} \xleftarrow{\text{inference}} \text{Data}$$

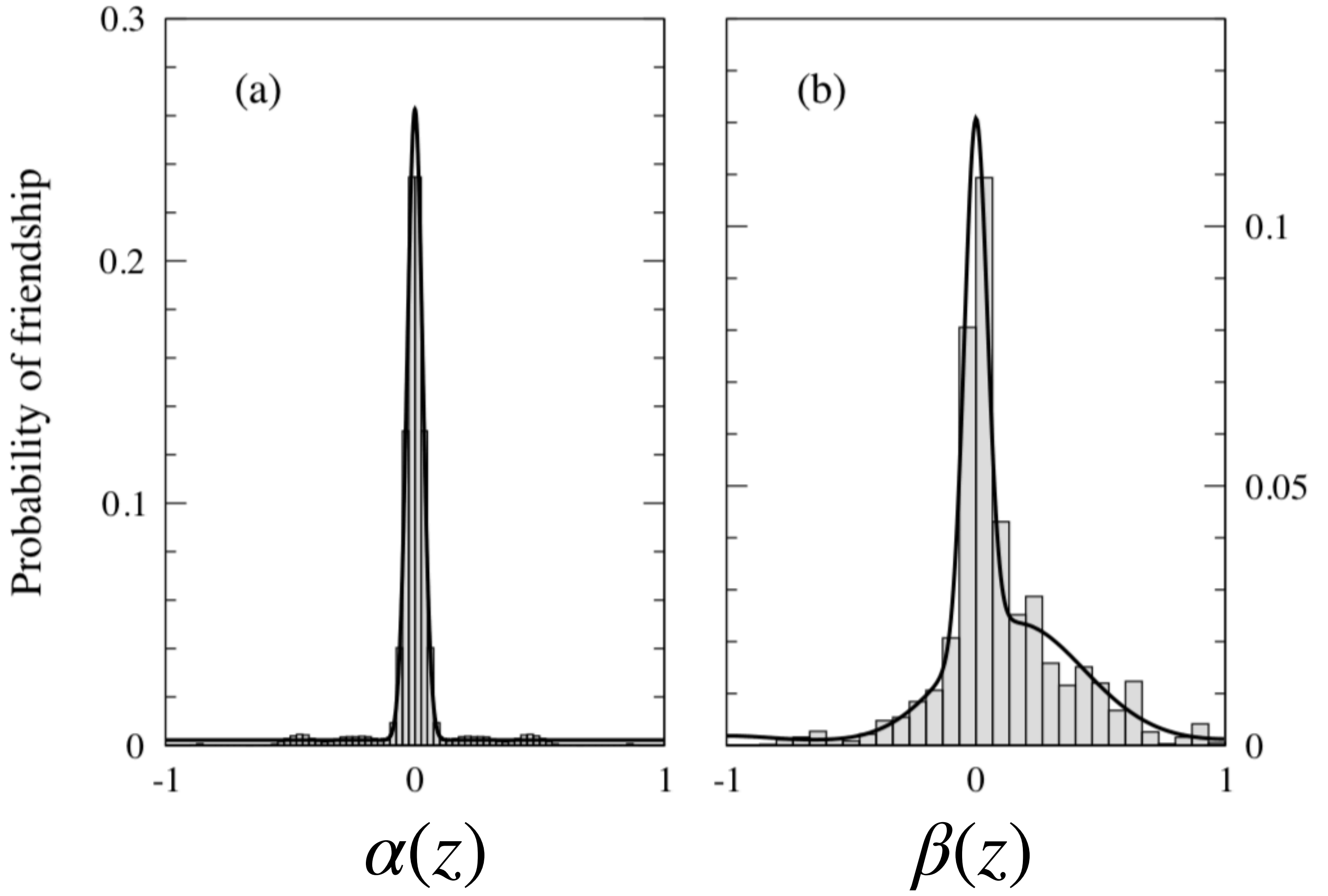
In other words: let's write down a model whose ensemble's distribution is not uniform but **highly peaked** around networks with structures that we want to see.

Embeddings and Orderings 4: Ball & Newman

Generative model:
Generate the patterns that you want to identify.

Create N nodes.
Assign each node an integer rank *r*, from 1 to N.

IRL, not all friendships are reciprocated 😭
So let's generate undirected AND directed edges:



$$P(i \leftrightarrow j) \; = \; \alpha(r_i - r_j)$$

A gaussian centered at 0

$$P(i \rightarrow j) \; = \; \beta(r_i - r_j)$$

Fourier cosine series, keeping five terms & squaring to enforce nonnegativity, plus an additional Gaussian peak at the origin.

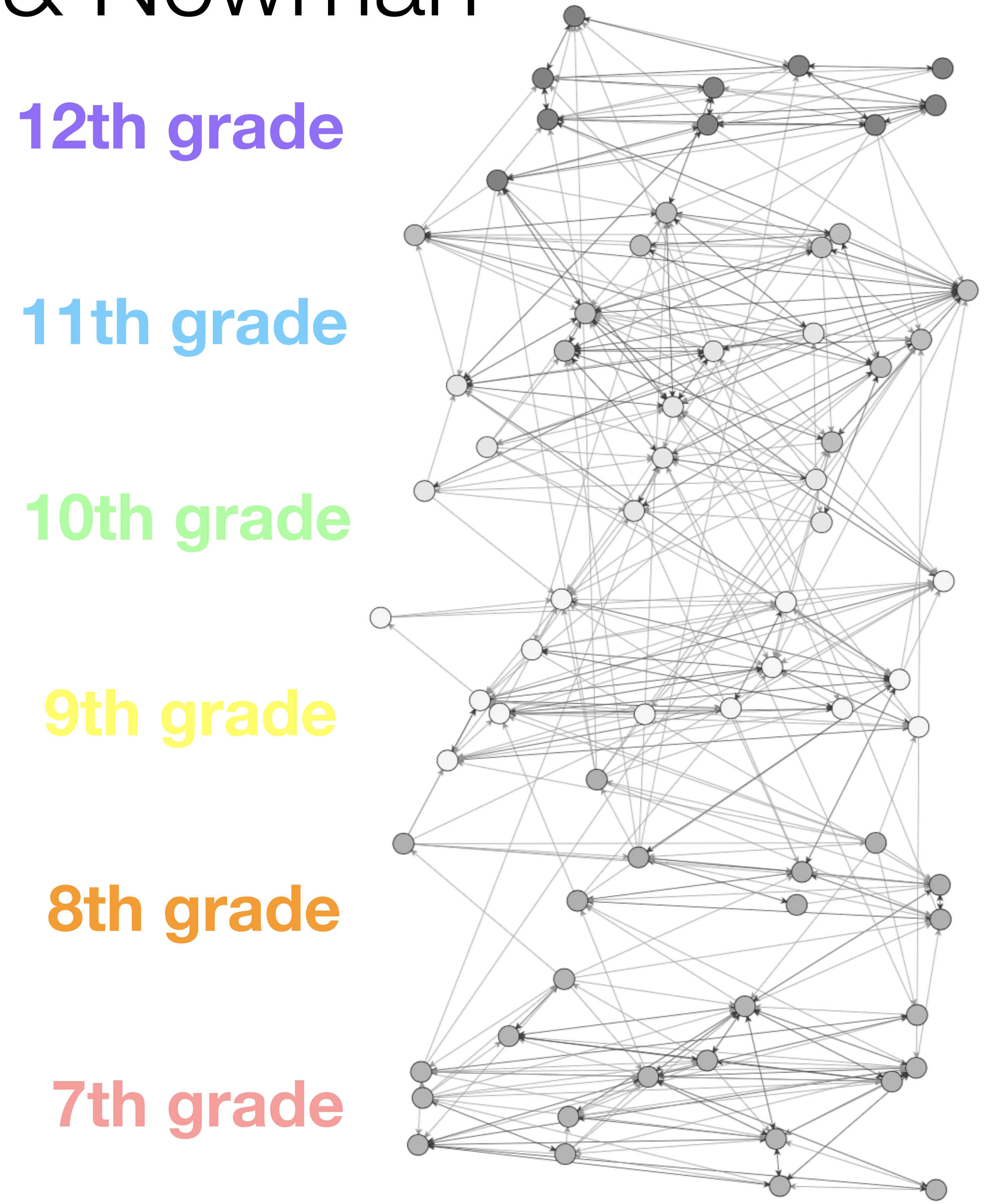
Ball, Newman. Network Science 1, 16-30 (2013)

Embeddings and Orderings 4: Ball & Newman

GM + parameters α β ranks ← inference — Data

Inferred parameters of people's attachment preferences & ranks.

- Identified the need to learn from reciprocated friendships.
- Found that in AddHealth data, teens link to others of *nearby* social status.



Ball, Newman. Network Science 1, 16-30 (2013)

Embeddings and Orderings 5: Niche Models

Niche Models embed species in a latent space based on feeding preferences:

most species feed from narrow range in a 1-dim. space (~body size).

- Great for food webs. Inference models v slow for all but small networks.

Want more? Jen Dunne, Cris Moore

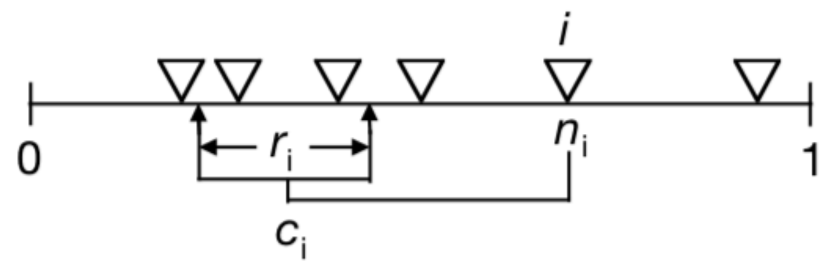


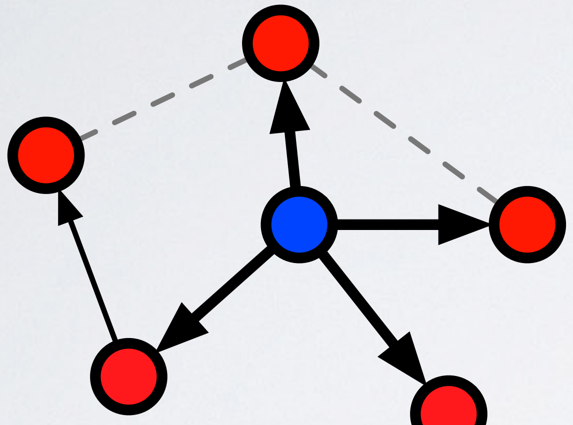
Figure 1 Diagram of the niche model. Each of ***S*** species (for example, ***S*** = 6, each shown as an inverted triangle) is assigned a 'niche value' parameter (n_i) drawn uniformly from the interval [0,1]. Species ***i*** consumes all species falling in a range (r_i) that is placed by uniformly drawing the centre of the range (c_i) from [r_i/2, n_i]. This permits looping and cannibalism by allowing up to half of r_i to include values $\geqslant n_i$. The size of r_i is assigned by using a beta function to randomly draw values from [0,1] whose expected value is 2***C*** and then multiplying that value by n_i [expected $E(n_i) = 0.5$] to obtain the desired ***C***. A beta distribution with $\alpha = 1$ has the form $f(x|1, \beta) = \beta(1-x)^{\beta-1}$, $0 < x < 1$, 0 otherwise, and $E(X) = 1/(1+\beta)$. In this case, $x = 1-(1-y)^{1/\beta}$ is a random variable from the beta distribution if y is a uniform random variable and β is chosen to obtain the desired expected value. We chose this form because of its simplicity and ease of calculation. The fundamental generality of species i is measured by r_i. The number of species falling within r_i measures realized generality. Occasionally, model-generated webs contain completely disconnected species or trophically identical species. Such species are eliminated and replaced until the web is free of such species. The species with the smallest n_i has $r_i = 0$ so that every web has at least one basal species.

Williams & Martinez. Nature 404.6774 (2000).

Embeddings and Orderings 6: Centrality Redux!

[Centrality Week]

describing networks



position = centrality:
structural vs. dynamical importance

geometric	harmonic centrality
	closeness centrality
	betweenness centrality
connectivity	degree centrality
	eigenvector centrality
	PageRank
	Katz centrality
	many many more…

structural importance = cheap estimate of dynamical importance (aka "influence")

Boldi & Vigna, arxiv:1308.2140 (2013)
Borgatti, *Social Networks* **27**, 55–71 (2005)

</methods>

<applications>

Many uses for the same techniques. cf regression

Treat the network like a system:

Extrapolation. Make predictions for as-yet unseen nodes (in "space" or time).

Interpolation. Identify missing links.

Generalization. Nodes of this type are like others of the same type.

Treat the network like an artifact:

Mechanisms. How did this network arise? What rules governed its assembly?

Explanations. Coarse-graining or compression.

Treat the network like a means to an end; an intermediate data structure:

Useful division. Need groups so that we can assign treatments in an A/B test.

Simplification. Downstream regression model needs ranks or groups.

Structure and inequality in academic hiring

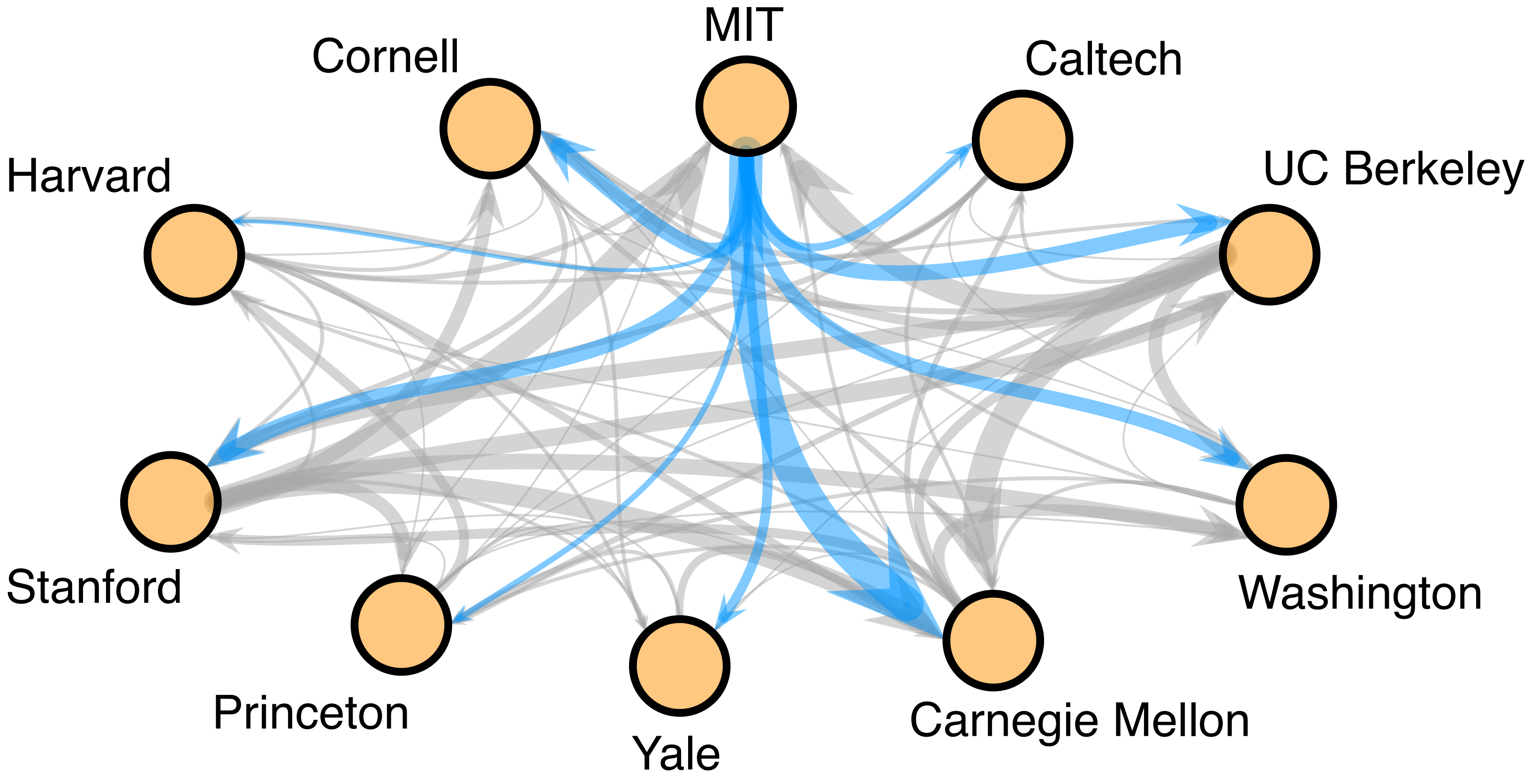
Collect the data (by hand 😭)

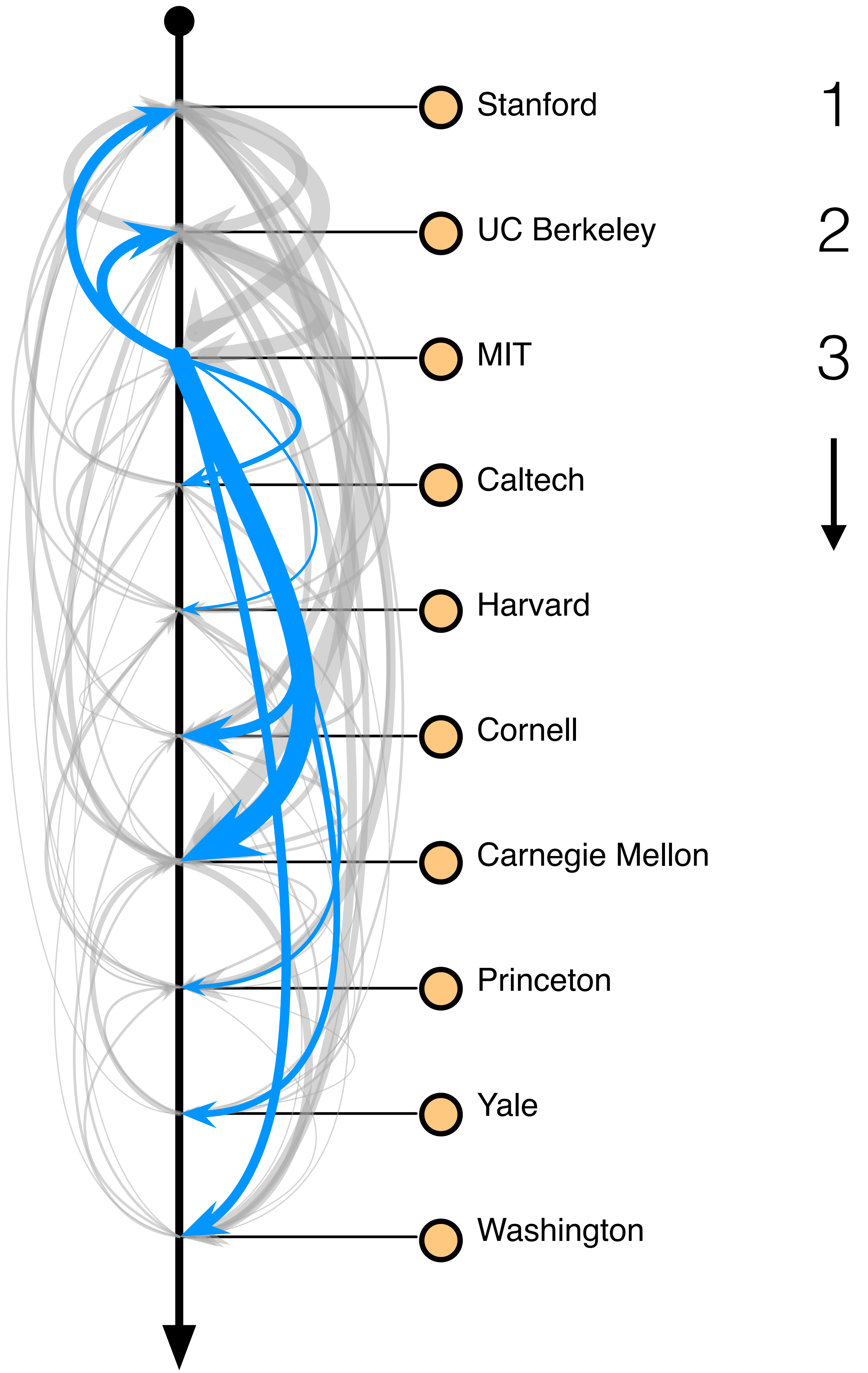
CVs of all US & Canadian tenure-track faculty in CS, Business, History: 2011-2013.

	Computer Science	Business	History
institutions	205	112	144
tenure-track faculty	5032	9336	4556
mean size	25	83	32
female	15%	22%	36%

total: **18,924** CVs

Faculty hiring networks





Premises:

1. Each hiring committee wants to hire **the best**.
2. Entire network reveals **collective preferences**.

Faculty hiring networks

systematic

90% of hiring movement
is “down” the hierarchy

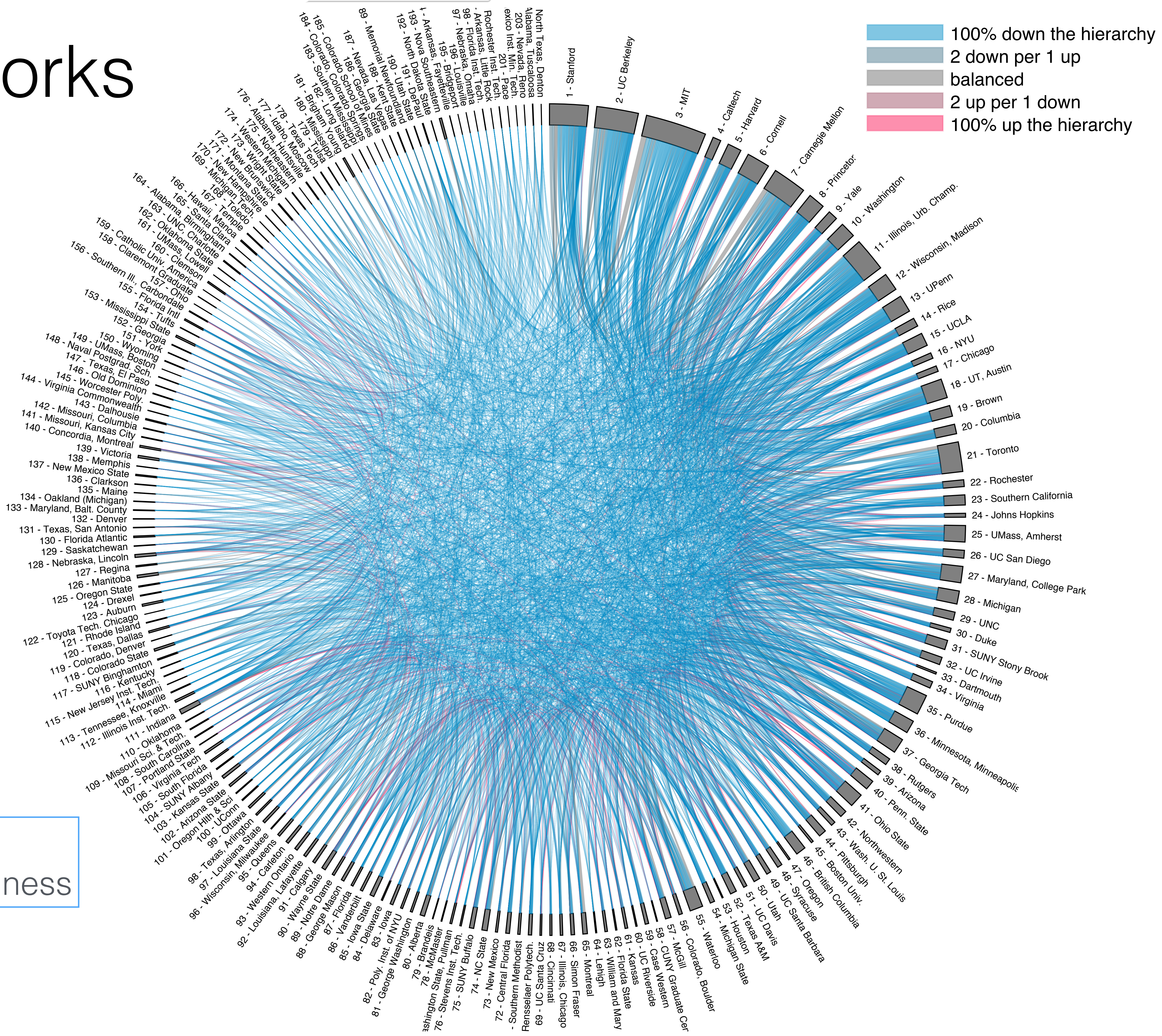
steep

< 7% of faculty have PhD
from lower 75% of universities

biased

median change for women
~3 ranks worse than men

danlarremore.com/**faculty**/
explore 19,000 hires for History, CS, Business



University of Colorado **Boulder**

Colorado
Sam Way
Aaron Clauset
Allison Morgan
Dimitrios Economou

Kauffman / Lux
Sam Arbesman

luxcapital

Ewing Marion KAUFFMAN Foundation

SANTA FE INSTITUTE

NSF

2018

Allie Morgan

2015

Sam Way

2003

Aaron Clauset

Productivity, prominence, and the effects of academic environment
Way, Morgan, Larremore, Clauset. *PNAS* (2019).

Prestige drives epistemic inequality in the diffusion of scientific ideas
Morgan, Economou, Way, Clauset. *EPJ Data Science* (2018).

The misleading narrative of the canonical faculty productivity trajectory
Way, Morgan, Clauset, Larremore. *PNAS* (2017).

Data-driven predictions in the science of science
Clauset, Larremore, Sinatra. *Science* (2017).

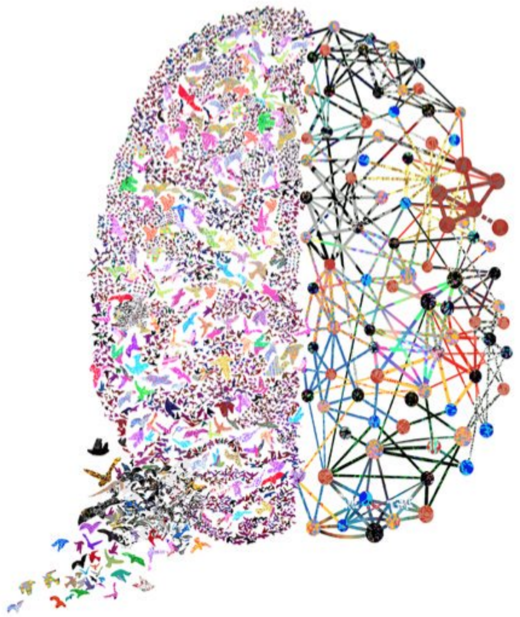
Gender, productivity, and prestige in computer science faculty hiring networks
Way, Larremore, Clauset. *Proc. WWW* (2016).

Systematic inequality and hierarchy in faculty hiring networks
Clauset, Arbesman, Larremore. *Science Advances*. (2015).

Conference on Complex Networks

COMPLENET '18

Hosted by Northeastern University Network Science Institute



BOSTON, MA | APRIL 2018

Xindi Wang

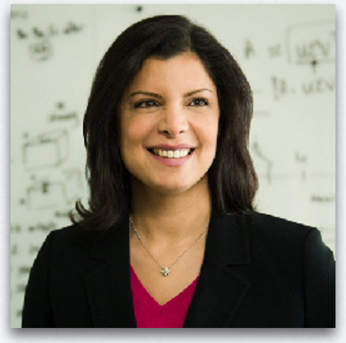
Northeastern University
Network Science Institute

LEARNING TO PLACE OBJECTS: A NETWORK-BASED APPROACH

Xindi Wang

Onur Varol

Tina Eliassi-Rad

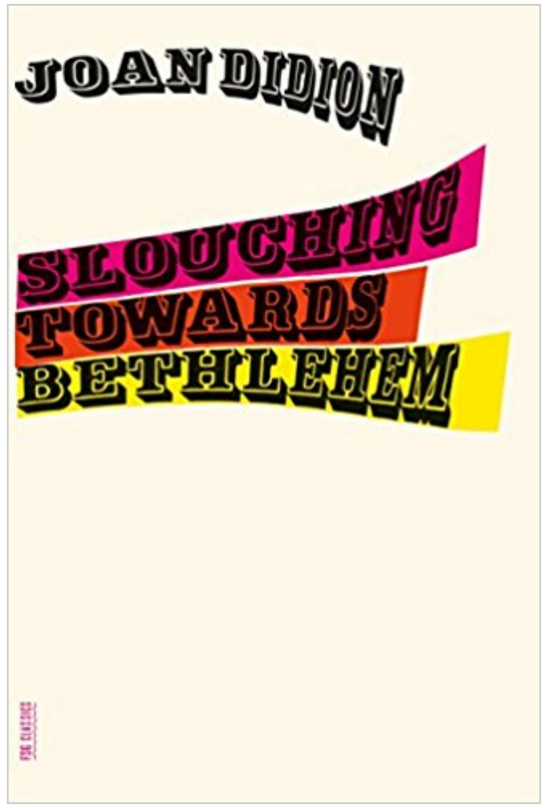


Albert-László Barabási

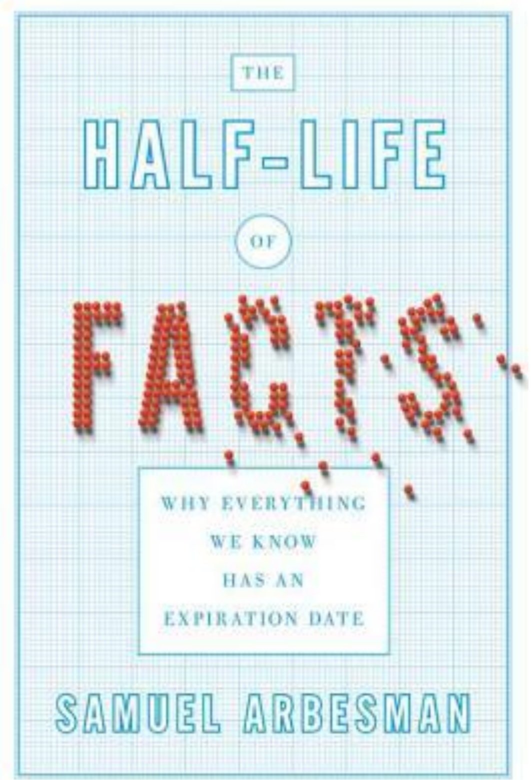
Suppose I give you a book. Predict its sales.

Existing data: books and their sales.

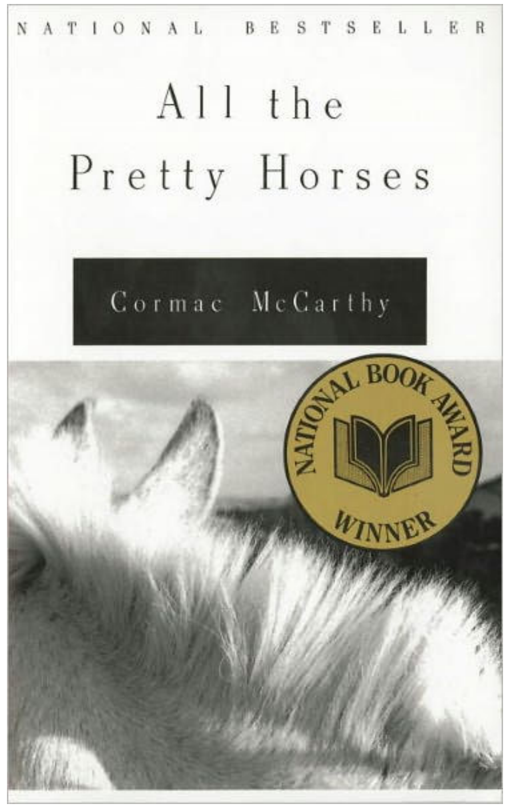
1. turn books into feature vectors.



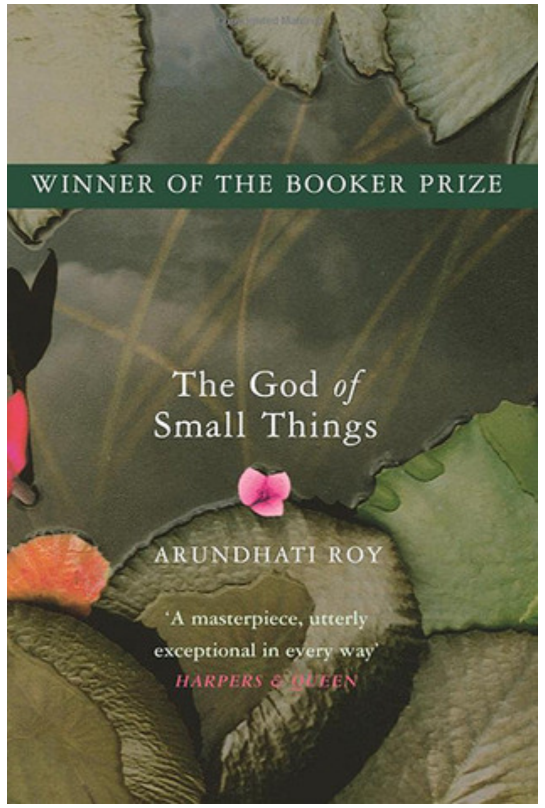
$\vec{x}_1$



$\vec{x}_2$



$\vec{x}_3$



$\vec{x}_4$

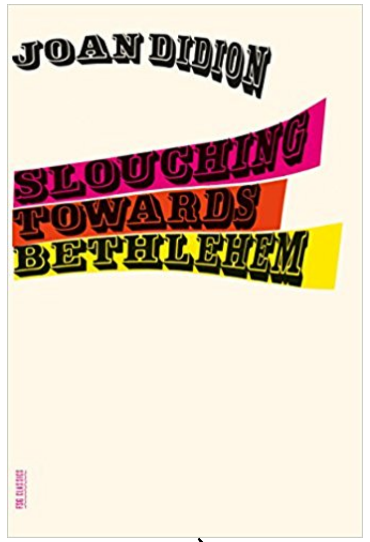
2. Train a model:

$$P(\text{book } i > \text{book } j \mid \vec{x}_i, \vec{x}_j, \theta)$$

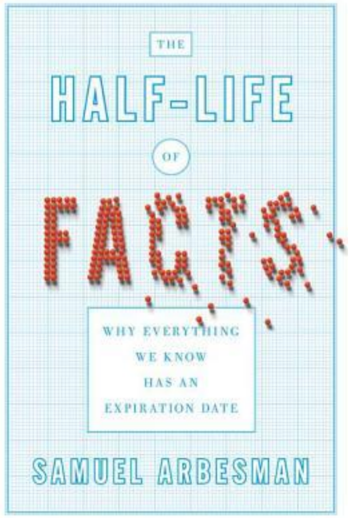
Suppose I give you a book. Predict its sales.

Existing data: books and their sales.

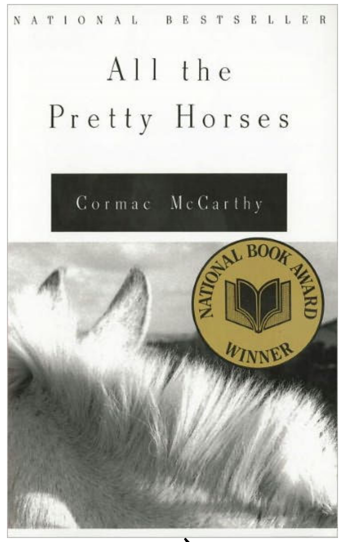
1. turn books into feature vectors.



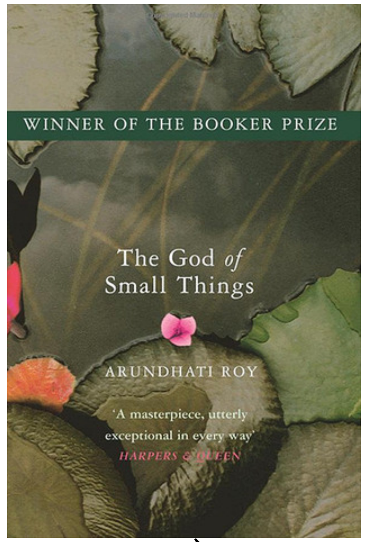
$\vec{x}_1$



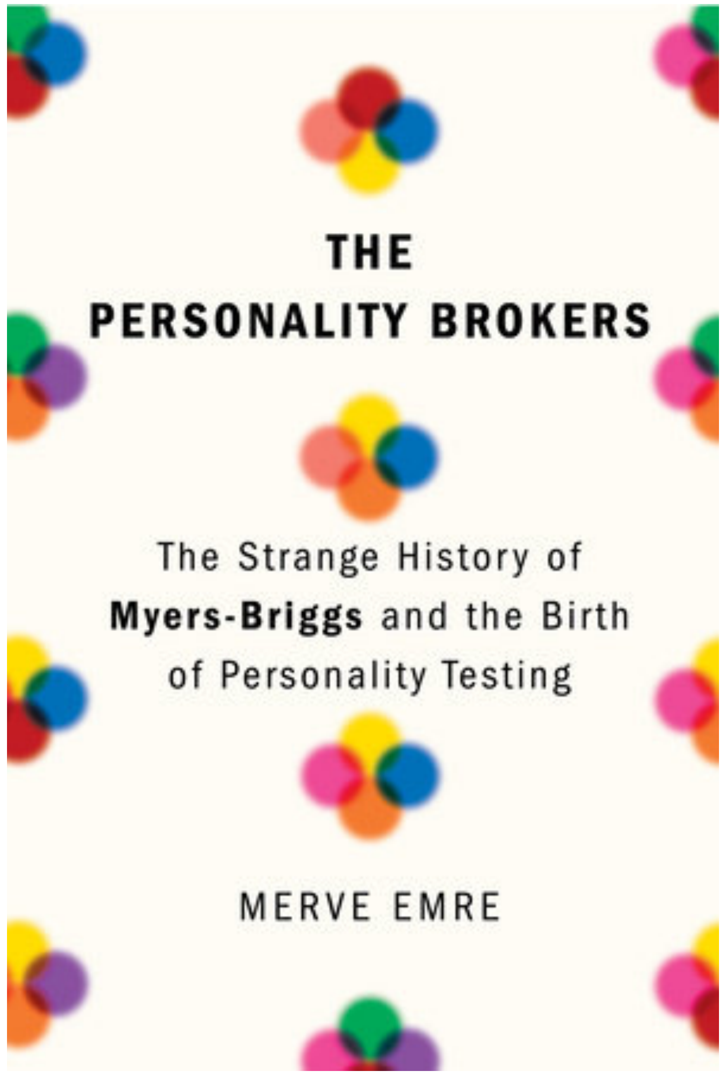
$\vec{x}_2$



$\vec{x}_3$



$\vec{x}_4$



$\vec{x}_5$

2. Train a model.

3. Use the model to simulate pairwise competitions.

   $P(\text{book } i > \text{book } 5 \mid \vec{x}_i, \vec{x}_5, \theta)$

4. Use [**your favo(u)rite algorithm**] to infer rank_5 from pairwise comparisons.

Rankings rankings

Area under the receiver-operator curve (AUC)

Method		AUC on Fiction	AUC on Biography
	KNN	0.759	0.815
Pairwise +	Cohen et al.	0.892	0.871
	WTG wave	**0.910**	**0.892**
	Voting	**0.915**	**0.891**
	FAS-PIVOT	**0.907**	**0.892**
	SpringRank	**0.908**	**0.893**

But actually… many methods performed well!

Now the question is: why do the top four algorithms perform similarly?

What does that tell us about the **structure of the problem** and the **structure of the space** over which we are ranking?

Use the consistency of the ranking results across algorithms to learn about the system itself.

What does it mean for a space or problem or set to be easily ordered or rankable?

Hierarchies of Desirability in Online Dating

Hierarchies are encoded in language about courtship

“She’s out of your league.”

asserts that:

1. Leagues or hierarchies of desirability exist.

   [How can we find them in data?]

2. The relative positions of individuals can be estimated.

3. Positions are predictive of something.

   [What behavior? And how noisy is the prediction?]

Data: a popular online dating service

Characteristics of the online dating service:

- Free.
- Around 4 million active self-identified heterosexual users.
- Ethnically diverse, urban, youngish.
- Approved and highly restricted data access through collaboration.

Characteristics of the data I work with:

- One month of anonymized and timestamped messaging.
- Self-identified heterosexual users with genders declared as M or F.

How can we use messaging data to answer these questions?

1. Do desirability hierarchies exist?
2. Are they predictive of behavior?

Intuition: Something like PageRank?

SCIENCE ADVANCES | RESEARCH ARTICLE

SOCIAL SCIENCES

Aspirational pursuit of mates in online dating markets

Elizabeth E. Bruch[1,2]* and M. E. J. Newman[2,3]

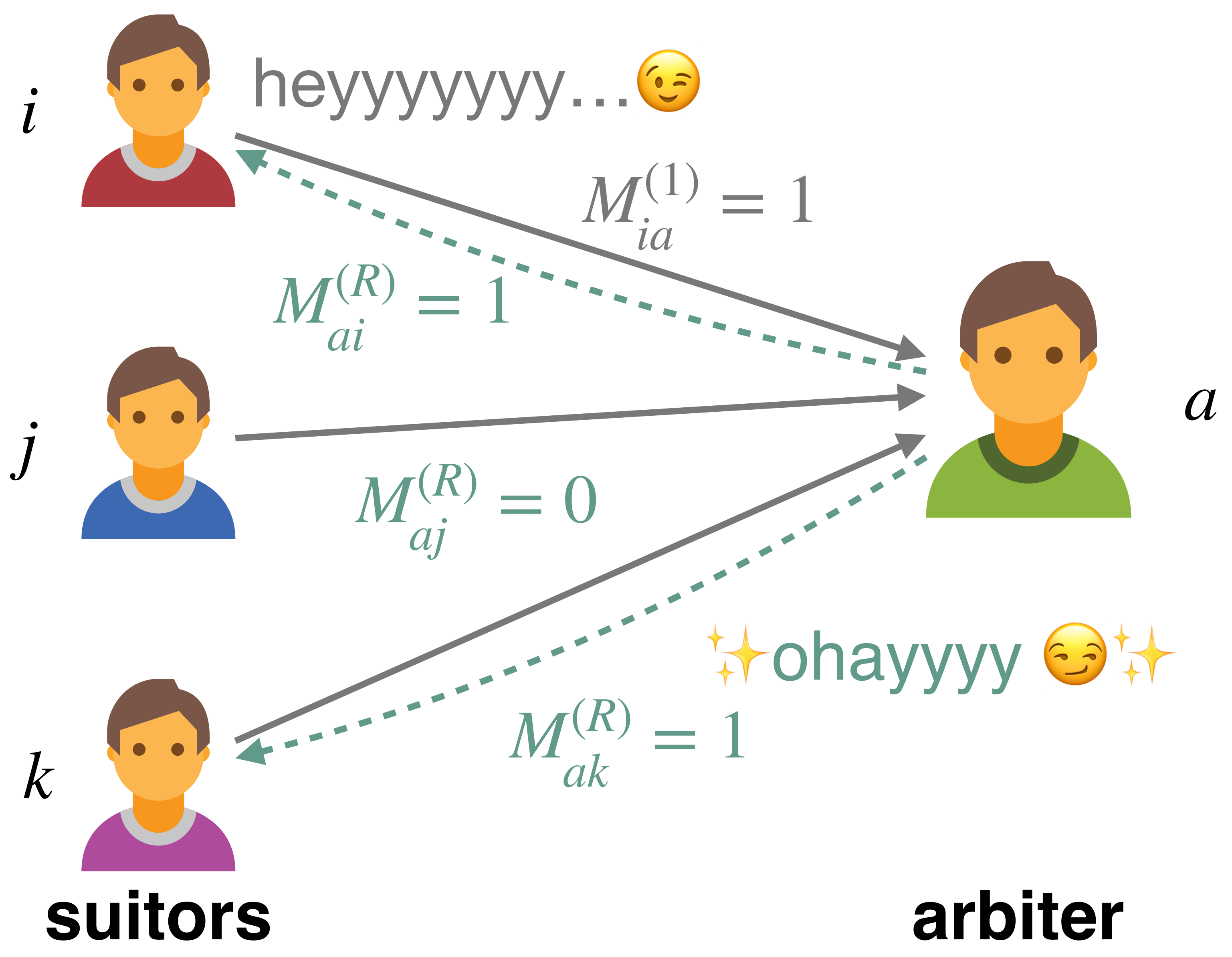
- Bruch & Newman 2018 used PageRank to analyze messaging network.
- Sorted individuals by **percentile PageRank scores**.
- **Aspirational pursuit** patterns: people message “up” the sorted network.

Recommended reading if you are interested in this subject!

Downside: PageRank scores aren’t *usefully* interpretable.
[Stationary distrib. of random walk + teleportation, on a network of message passing?]

Can we use the messaging data to find more **meaningfully interpretable ranks**?

Insight: messaging is a competition for attention

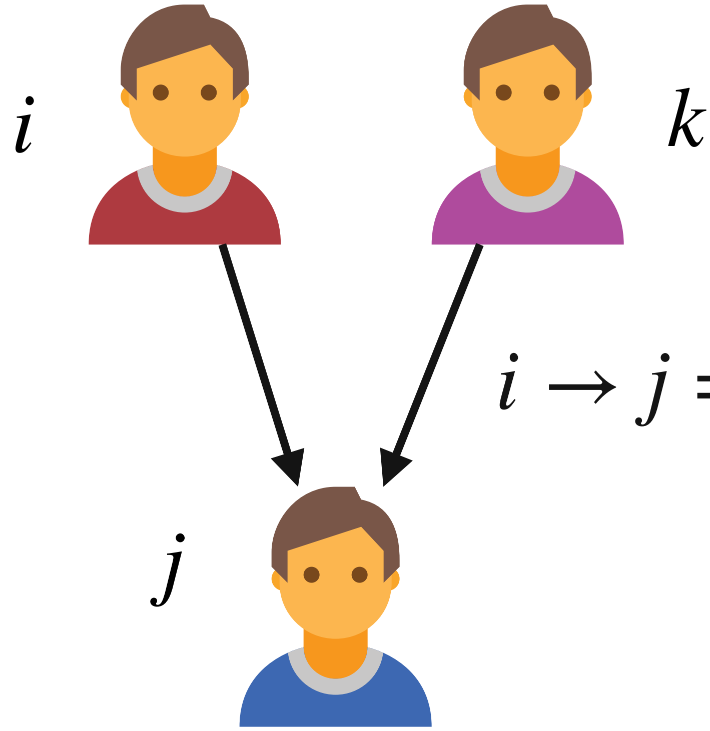


$M^{(1)}$: network layer of first messages

$M^{(R)}$: network layer of replies

Who won/lost this competition for attention?

Hierarchies in the competition for attention



$i \to j$ = i beats j

$C_{ij} = C_{kj} = 1$

Let C_{ij} be the number of times that i outcompetes j.

$$C_{ij} = \sum_a M_{ia}^{(1)} M_{ja}^{(1)} M_{ai}^{(R)} \left(1 - M_{aj}^{(R)}\right)$$

sum over all arbiters
(i.e. first-message receivers)

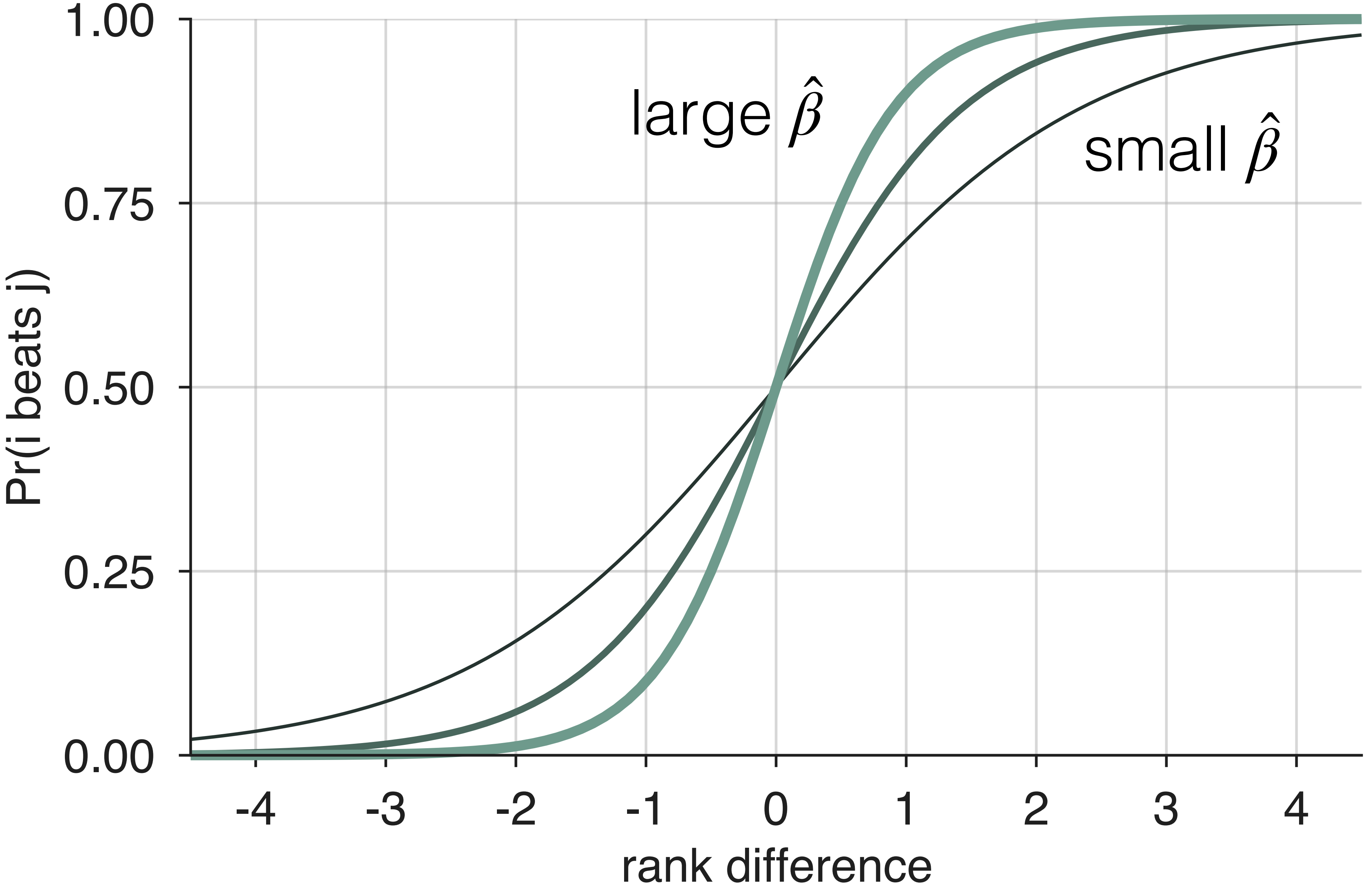
$i \mapsto a$
$j \mapsto a$

$i \hookleftarrow a$ $j \not\hookleftarrow a$

Each arbiter is presented with a choice set of suitors.
Collectively, arbiters' choices provide many *partial orderings* of suitors.

C is a one-component, directed network representing pairwise comparisons.
We will use **SpringRank** to find people's latent positions to *predict future behavior*.

Beta tells us how to interpret rank differences



$$P(i \to j \mid i \leftrightarrow j) = \frac{1}{1 + e^{-2\hat{\beta}(c_i - c_j)}}$$

$\hat{\beta}$ is the MLE inverse temperature of the SpringRank Boltzmann

It tells us the sensitivity/scale for predictions in the ranking space.

The Depths of Leagues

Dear Editors,

There are some questions I would like to ask. Firstly, how complex is backgammon compared to other games of skill such as chess or bridge?

Let's start with chess, which has evolved a well-developed rating system over the past 40 years. Chess ratings range from a high of about 2800 to theoretical lows of about 0 (a complete beginner who has just learned the moves). Chess ratings are also designed so that a 200-point rating difference between two players anywhere on the scale means that the higher-rated player has a 70-75% chance of defeating a lower-rated player (discounting draws, which are possible in chess but not in most of the other games we'll consider).

Now consider the following experiment:

(1) Take the best player in the world (in the case of chess, it's Gary Kasparov). Call him player 1.

(2) Find someone that the best player beats 70-75% of the time. Call him player 2.

(3) Call the difference between players 1 and 2 ***one skill differential.***

(4) Find someone that player 2 can beat 70-75% of the time. Call him player 3. The difference between players 2 and 3 is another skill differential.

(5) Continue this process until you have taken the chain down to an absolute beginner.

(6) Count the number of skill differentials involved. This is the ***complexity number*** *of the game.*

In the case of chess, this number is about 14.

Robertie, 1980 *Inside Backgammon.* h/t Johan Ugander, Brendan Tracey

The Depths of Leagues

COMPLEXITY NUMBERS

Go	*40*
Chess	*14*
Scrabble	*10*
Poker	*10*
Backgammon	*8*
Checkers	*8*
Hearts	*5*
Blackjack	*2*
Craps	*0.001*
Lotteries	*0.0000001*
Roulette	*0*
Online Dating	**???**

Now consider the following experiment:

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(2) Find someone that the best player beats 70-75% of the time. Call him player 2.

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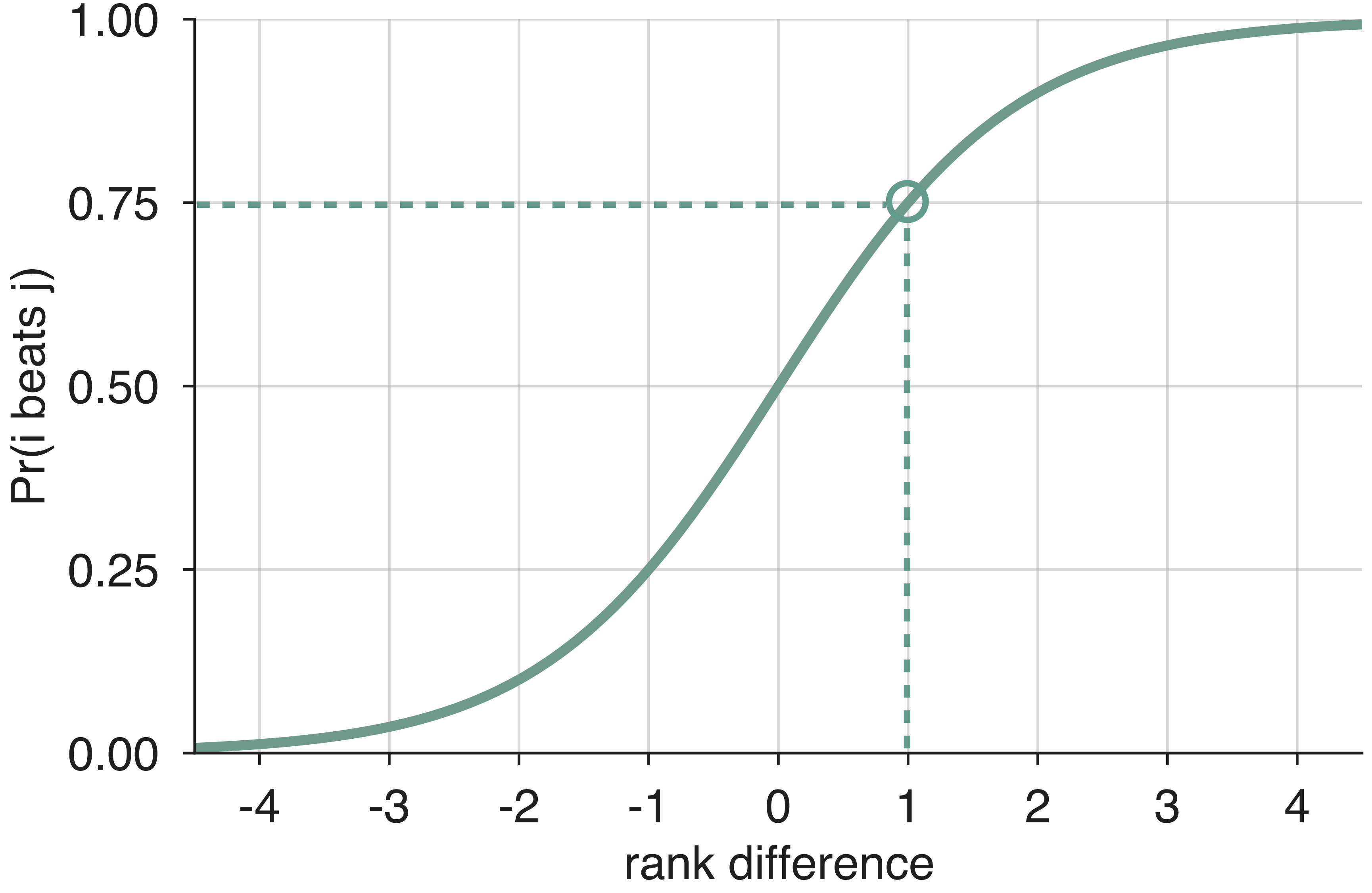
Robertie, 1980 *Inside Backgammon.* h/t Johan Ugander, Brendan Tracey

Choosing a scale for interpretability

scaling that we want



scaling that we have

$$P(i \to j \mid i \leftrightarrow j) = \frac{1}{1 + e^{-2\hat{\beta}(c_i - c_j)}}$$

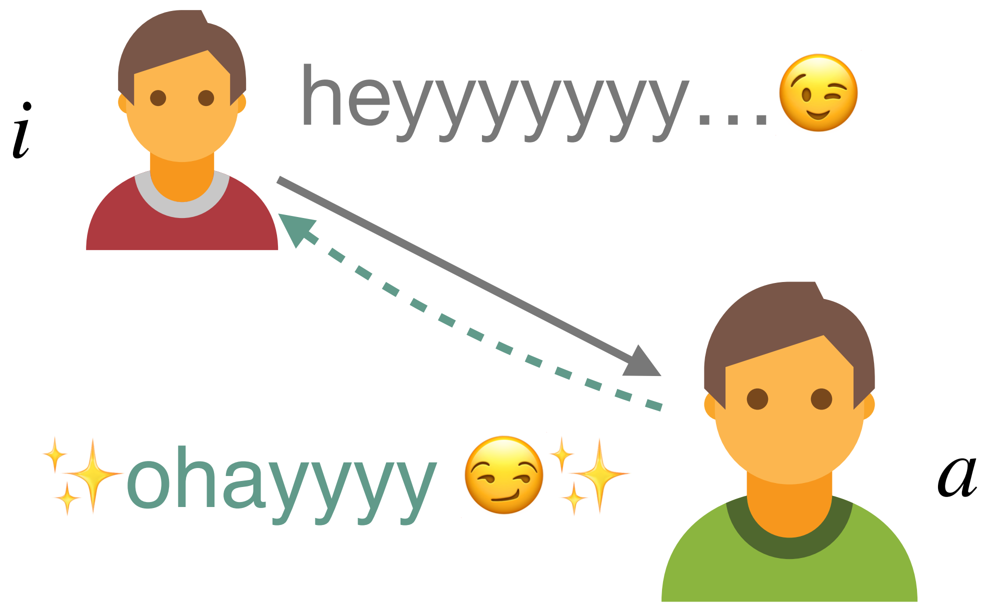
enforce the desired scale,
then solve for a rescaling constant *k*

let $c = k\bar{c}$

$$\bar{c}_i - \bar{c}_j = 1, \quad 0.75 = \frac{1}{1 + e^{-2\hat{\beta}(k\bar{c}_i - k\bar{c}_j)}}$$

$$k = \frac{\text{log odds } 0.75}{2\hat{\beta}}$$

Reviewing the approach.



1 $M^{(1)}$: network layer of first messages
$M^{(R)}$: network layer of replies

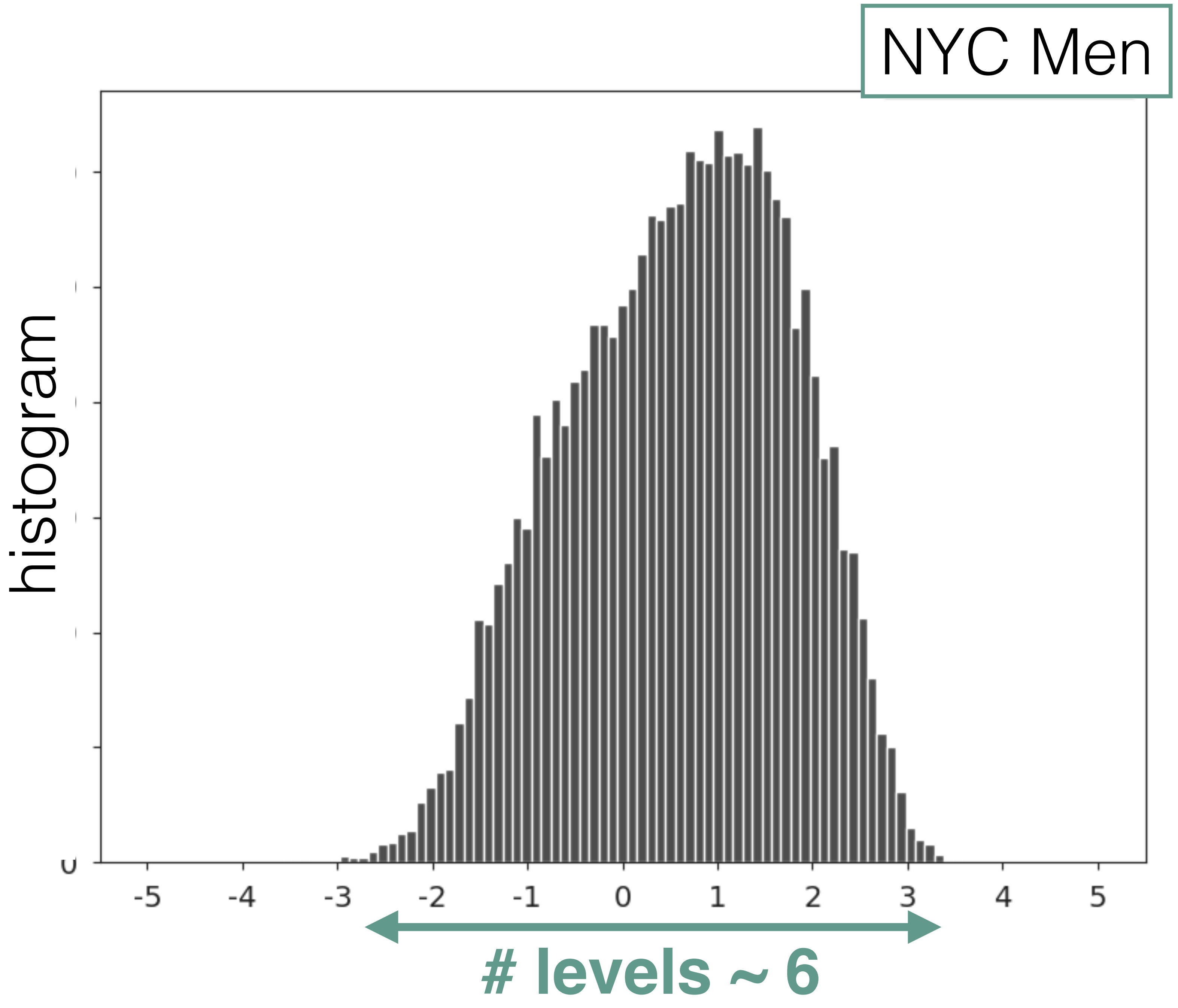
2 $$C_{ij} = \sum_a M_{ia}^{(1)} M_{ja}^{(1)} M_{ai}^{(R)} \left(1 - M_{aj}^{(R)}\right)$$

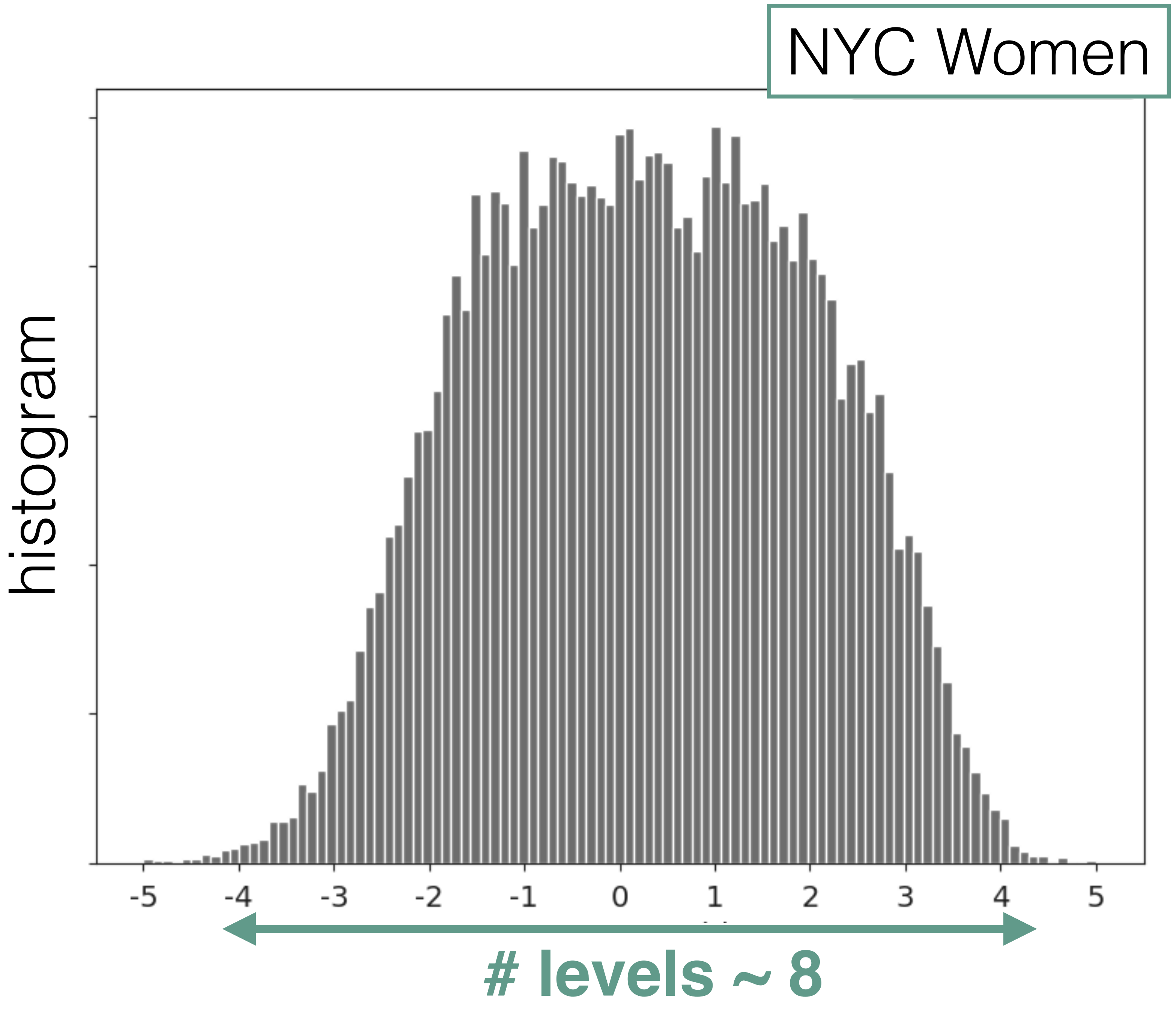
3 $$H(c) = \frac{1}{2} \sum_{ij} C_{ij} \left(c_i - c_j - 1\right)^2$$

4 $$\bar{c} = c \frac{\text{log odds } 0.75}{2\hat{\beta}}$$

Result:
An embedding of individuals in a linear hierarchy, such that a one-unit difference predicts a 75% “win rate” in the competition for attention.

Real data: New York City rank distributions





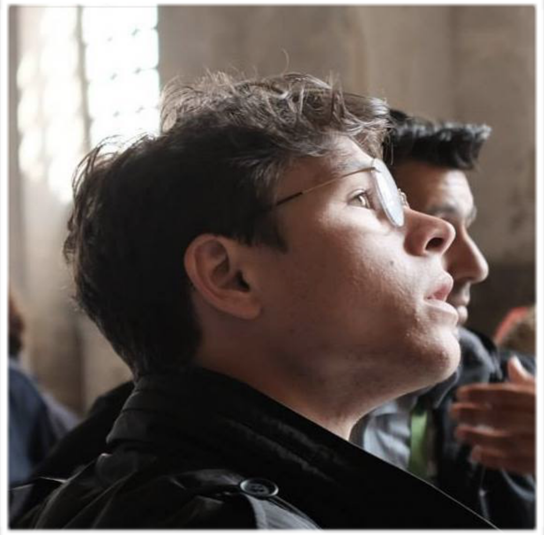
- Those who had 100% or 0% reply rates not shown.
- NYC Women's competition space has a deeper "strategic complexity" than NYC Men's.

Elizabeth Bruch

Swapnil Gavade

[empirical analyses]

UNIVERSITY OF MICHIGAN



K. Hunter Wapman

[simulations]

University of Colorado **Boulder**

SCIENCE ADVANCES | RESEARCH ARTICLE

SOCIAL SCIENCES

Aspirational pursuit of mates in online dating markets

Elizabeth E. Bruch[1,2]* and M. E. J. Newman[2,3]

Recommended reading if you are interested in this subject!

We talked about 6 methods:

1. Minimum Violations Rankings & Agony
2. Random Utility Models (economics & marketing)
3. SpringRank (physics)
4. PageRank (random walks & the www)
5. Generative models (social science)
6. Niche models (ecology)

Beyond pictures: these things matter.
Inequalities, forecasting, cognition, courtship, & social organization.

And 5 applications:

1. Faculty hiring networks and prestige (computational social science)
2. Sales predictions (e-commerce)
3. Bird hierarchies and cognition (animal behavior)
4. Online dating & desirability (sociology)
5. Group-level social hierarchies (anthropology)

Thank you

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daniel.larremore@colorado.edu