1. (50 pts) An induced subgraph is a graph G' that remains after deleting some vertices and all of the edges that are incident to the deleted vertices.

   Describe and analyze an efficient algorithm that, given an undirected, unweighted graph G, represented as an adjacency list, and an integer k, returns the largest induced subgraph G' such that the degree of every vertex in G' is greater than or equal to k.

   Hint: Determine which vertices to delete so that the remaining graph has the desired property, and think about how to keep track of how deleting a vertex may change the status of other vertices.

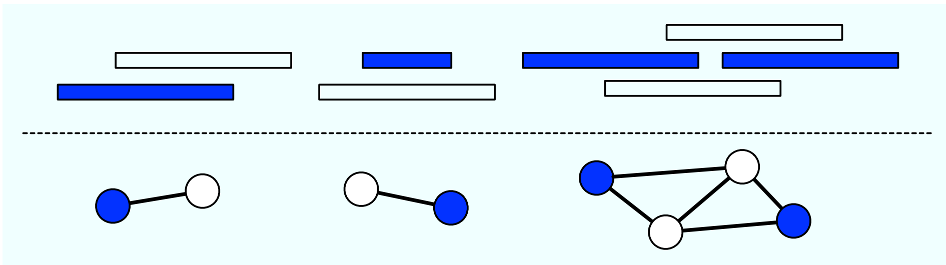
2. (50 pts total) Imagine that you own a long and contiguous stretch of beach-front property in a desirable area. Although you love the view, you decide to sell it. From various interested parties, you receive a set of bids for various intervals of the beach and now need to decide which subset to accept so as to maximize your profit from the set of sales.

   Let $[A, B]$ denote the left- and right-endpoints of the beach on some real number line. Let the n bids you receive be denoted by the set X. Each bid is composed of an interval of the beach $x_i = [\,L_i, R_i\,]$, where $A \leq L_i < R_i \leq B$, and a value $w(x_i) > 0$. The task is to find the largest subset of bids $Y \subseteq X$ such that its value $w(Y) = \sum_{x_i \in Y} w(x_i)$ is maximized. Note that if two intervals overlap, then they both cannot be in Y, i.e., you cannot sell the same piece of beach to two different bidders. See the upper half of the figure below.

   (a) (20 pts) Describe a non-trivial greedy approach to this problem and explain why it is not correct (does not maximize $w(Y)$).

   (b) (30 pts) Describe and analyze (describe in detail, derive its running time, and argue for its correctness) a dynamic programming algorithm that solves this problem correctly.

   Hint: We can transform the input intervals X into an "interval graph" $G = (V, E)$ by letting each node in V correspond to an interval $x_i \in X$ and let two nodes be connected $(x_i, x_j) \in E$ if and only if the corresponding intervals overlap. If we define a *valid* solution Y as a subset of interval such that no piece of land is sold twice, all valid solutions on the interval graph are then *independent sets* (do you see why?); see the lower half of the figure above. The goal, then, is reduced to finding the maximum-weight independent set on G.



3. (*25 pts extra credit*) Prove that finding the second largest element in an n-element array requires exactly $n - 2 + \lceil \lg n \rceil$ comparisons in the worst case. Prove the upper bound by describing and analyzing an algorithm; prove the lower bound using an adversary argument.

   Hint: An adversary argument is one that articulates exactly what the worst possible input could be, chosen by the adversary, regardless of the algorithm used. The adversary's strategy cannot depend on any predetermined order of operations or what the is done when bits are examined.

4. (*30 pts extra credit total*) Let S be a set of n points in the plane. A point p in S is called Pareto-optimal if no other point in S is both above and to the right of p.

   (a) (*15 pts*) Describe and analyze a deterministic algorithm that computes the Pareto-optimal points in S in $O(n \log n)$ time.

   (b) (*15 pts*) Suppose each point in S is chosen independently and uniformly at random from the unit square $[0, 1] \times [0, 1]$. What is the exact expected number of Pareto-optimal points in S?