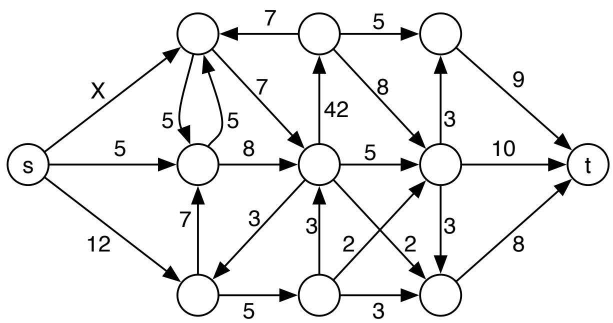
1. (25 pts total) The Engineers of Gondor have installed a set canals that convey water from the spring s to the town t. (They couldn't install just one big canal for technical reasons. Specifically: they're not dwarves.) Now, they are considering adding a new canal connecting the spring to their distribution network G. However, the Engineers are not sure how much additional water they will be able to push through G after adding the proposed canal; they need your help to figure it out. The diagram below shows G', the network G plus the proposed canal X; edge labels indicate edge capacities.



   (a) (10 pts) Make a diagram showing the minimum cut corresponding to the maximum flow for G (where $X = 0$). What is the weight of that cut?

   (b) (8 pts) If the Engineers add the canal X, what is the smallest capacity that would maximize the increase in the water flow across the network? Explain.

   (c) (7 pts) Describe how the Engineers could use a min-cut/max-flow algorithm to decide what capacity X should be used for an arbitrary graph $G = (V, E)$ and arbitrary proposed edge $(u, v) \notin E$ with capacity X.

2. (15 pts) Given a graph G and a minimum spanning tree T, suppose that we decrease the weight of one of the edges in T. Show that T is still a minimum spanning tree for G. More formally, let T be a minimum spanning tree for G with edge weights given by weight function w. Choose one edge $(x, y) \in T$ and a positive number k, and define the weight function w' by

$$w'(u,v) = \begin{cases} w(u,v) & \text{if } (u,v) \neq (x,y) \\ w(x,y) - k & \text{if } (u,v) = (x,y) \end{cases} .$$

Show that T is a minimum spanning tree for G with edge weights given by w'.

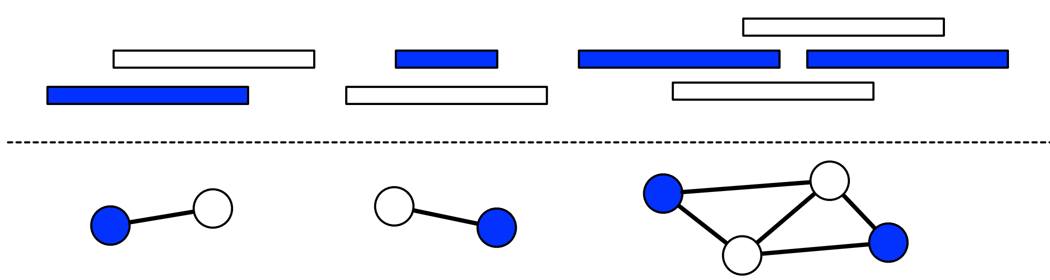
3. (40 pts total) Returning to the Shire after your long trek back from Mordor, you decide that you want to sell your stretch of river-front property (and move into a nice hobbit house). From various interested hobbits (and wizards?), you receive a set of bids for various intervals of the property. Wanting to maximize your profit across the set of sales, you must now decide which subset of bids to accept.

   Let $[A, B]$ denote the left- and right-endpoints of the river-front property on some real number line. Let the n bids you receive be denoted by the set X. Each bid is composed of (i) an interval $x_i = [\,L_i, R_i\,]$, where $A \leq L_i < R_i \leq B$, and (ii) a value $w(x_i) > 0$. Your task is to find the largest subset of bids $Y \subseteq X$ such that its value $w(Y) = \sum_{x_i \in Y} w(x_i)$ is maximized. Note that if two intervals overlap, then they both cannot be in Y, i.e., you cannot sell the same piece of land to two different bidders. See the upper half of the figure below.

   (a) (10 pts) Describe a naïve greedy approach to solving this problem. Explain what properties of X lead this approach to produce a suboptimal solution (a non-maximum $w(Y)$). Provide an example of X for which your algorithm returns a suboptimal solution, and identify the optimal solution Y.

   (b) (30 pts) Describe and analyze a dynamic programming algorithm that solves this problem correctly.

   Gandalf's hint: We can transform the input intervals X into an "interval graph" $G = (V, E)$ by letting each node in V correspond to an interval $x_i \in X$ and let two nodes be connected $(x_i, x_j) \in E$ if and only if the corresponding intervals overlap. If we define a *valid* solution Y as a subset of interval such that no piece of land is sold twice, all valid solutions on the interval graph are then *independent sets* (do you see why?); see the lower half of the figure above. The goal, then, is reduced to finding the maximum-weight independent set on G.



4. (20 pts total) Although your hobbit friends Meriadoc and Peregrin are staying with you, after a brilliant prank goes awry, they have bitter argument. You intervene to keep the peace and they agree to stay away from each other for the time being. In particular, they have agreed that when navigating the paths of the Shire, each will not walk on any section of dirt that the other hobbit has stepped on that day. The hobbits have no problem with their paths crossing at an intersection. The problem, however, is that they both still need to get to the market each day to buy supplies. Fortunately, both your house and the market are at intersections. You have a map of the Shire's paths. Show how to formulate the problem of determining whether both of your friends can go to the market as a max-flow problem.

5. (10 pts extra credit) Preparing for a big end-of-semester party in The Shire, you open your cellar and count n bottles of fine wine. Gandalf has previously warned you that exactly k of these bottles have been poisoned, and consuming poisoned wine will result in an unpleasant death. The party starts in one hour, and you do not want to poison any of your guests.

   Luckily, a family of ℓ docile rats occupies a corner of the cellar, and they have graciously volunteered to be test subjects for identifying the poisoned bottles. Let $\ell = o(n)$ and $k = 1$, and assume it takes just under one hour for poisoned wine to kill a rat.

   Describe a scheme by which you can feed wine to rats and identify with complete certainty the poisoned bottle, prove that the scheme is correct and give a tight bound on the number of rats ℓ necessary to solve the problem.

   Gandalf's hint: 1010101111