1 Greedy algorithms

In algorithm design, a *greedy algorithm* is one that always takes the most obvious step toward the goal. For instance, if we have some objective function that scores the intermediate state of our algorithm, a greedy approach is one that, for all possible incremental changes, makes the one that increases the objective function the most.

Greedy algorithms are very most common algorithms because they are so simple. In general, the greedy approach will work well whenever a problem has the following structure:

1. Every solution to the problem can be assigned or translated in a numerical value or score. Let x be a candidate solution and let score(x) be the value assigned to that solution.

2. The "best" (optimal) solution has the highest value among all solutions (in the case of maximization; lowest, in the case of minimization); i.e., it is the global optimum, and its contains optimal solutions to its subproblems ("optimal substructure" property).

3. A solution can be constructed incrementally (and a partial solution assigned a score) without reference to future decisions, past decisions or the set of possible solutions to the problem ("greedy choice" property).

4. At each intermediate step in constructing or finding a solution, there are a set of options for which piece to add next.

A greedy solution always chooses the option that yields the largest improvement in the intermediate solution's score.

Although greedy approaches are often very intuitive and easy to come up with, they are not necessarily guaranteed to return the optimum solution (the x corresponding to the global optimum) nor are they necessarily fast. As with other types of algorithms, it is necessary to show that the algorithm is *correct*, i.e., it is guaranteed to return the correct solution for all possible inputs, and to analyze its time and space usage.

1.1 Insertion sort is an optimal greedy algorithm

To give you a sense of how a greedy algorithm can be correct but not efficient, let's analyze the behavior of *insertion sort*, which is in fact a greedy algorithm, albeit not a very efficient one. (Recall that insertion sort takes $\Theta(n^2)$ time to sort n numbers.)

Insertion sort is a simple loop. It starts with the first element of the input array A. For each subsequent element j, it then inserts $A[j]$ into the sorted list $A[1..j-1]$.

```
INSERTION-SORT(A)
  for j=2 to n
     // assert: A[1..j-1] is sorted
     insert A[j] into the sorted sequence A[1..j-1]
     // assert: A[1..j] is sorted
}}
```

To see that insertion sort is correct, we observe that the following loop invariant, i.e., a property of the algorithm (either its behavior or its internal state) that is true each time we begin or end the loop.

Note that at the start of the `for` loop, the subarray $A[1..j-1]$ contains the original elements of $A[1..j-1]$ but now in sorted order. When $j = 2$ ("initialization") this fact is true because a list with one element is, by definition, sorted. When we insert the jth item into the subarray $A[1..j-1]$, we do so in a way that $A[1..j]$ is now sorted; thus, if the invariant is true at the beginning of the loop, it will also be true at the end of the loop ("maintenance").[1] Finally, when the loop terminates, $j = n$ and we have inserted the last element correctly into the sorted subarray ("termination"); thus, the entire array is sorted.

Now that we know insertion sort is correct, we can show that it is, in fact, an *optimal greedy algorithm*, meaning that (i) at any intermediate step, the algorithm always makes the choice that increases the quality of the intermediate state (greedy property) and (ii) it returns the correct answer upon termination (optimum behavior).

To begin, we first define a score function that allows us to build sorted sequences one element at a time. Clearly, if A is already sorted, score(A) must yield a maximal value, but it must also give partial credit if A is partially sorted and that credit should be larger the more sorted A is.

A sufficient score function is to count the number of sequential comparisons that violate the sorting requirement, i.e.,

$$\text{score}(A) = \sum_{i=1}^{n-1} (A[i] \leq A[i+1]) \quad ,$$

where we assume that the comparison operator is a binary function that returns 1 if the comparison yields true and 0 if it yields false. By definition, this property is true for all i in a fully sorted array, i.e., $A[1] \leq A[2] \leq \cdots \leq A[n]$, and so a fully sorted sequence will receive the maximal score of $n-1$. A sequence in reverse order will receive the lowest score of 0, and every other sequence

[1] For those of you unfamiliar with induction, this is a verbal "proof by induction" using the loop invariant. Loop invariants are nice precisely because they make proofs by induction easy.

can be assigned something between these two extremes.

With the score function selected, let's analyze the behavior of insertion sort under this function. An intermediate solution here is the partially sorted array. Given such an array, our "local" move is to take one element $A[j]$ and insert it into the subarray $A[1..j-1]$. Not all choices of where within $A[1..j-1]$ we put $A[j]$ will lead to a sorted list upon termination, and most of the possible choices of where to insert $A[j]$ are suboptimal.

Recall our loop invariant from above. For any input sequence A, when the loop initializes, it has $\text{score}(A) \geq 0$, where the lower bound is achieved by a strict reverse ordering. Because the sorted subarray's size grows by 1 each time we pass through the loop, so too does the number of correct sequential comparisons. That is, our loop invariant is equivalent to $\text{score}(A) \geq j-1$ where j is the loop index. And, when the loop completes, $j = n$ and $\text{score}(A) \geq n-1$. Thus, insertion sort is a kind of optimally greedy algorithm.

1.2 Linear storage media

Now let's consider a related problem. Suppose we have a set of n files and that we want to store these files on a tape, or some other kind of linear storage media.[2] Once we've written these files to the tape, the cost of accessing any particular file is proportional to the length of the files stored ahead of it on the tape. In this way, tape access is very slow and costly relative to either magnetic disks or RAM. Let $L[i]$ be the length of the ith file. If the files are stored on the tape in order of their indices, the cost of accessing the jth file is

$$\text{cost}(j) = \sum_{i=1}^{j} L[i] \ .$$

That is, we first have to scan past (which takes the same time as reading) the first $j-1$ files, and then we read the jth file.

If files are requested uniformly at random, then the expected cost for reading one is

$$E[\text{cost}] = \sum_{j=1}^{n} \Pr(j) \cdot \text{cost}(j) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{j} L[i] \ .$$

What if we change the ordering of the files on the tape? If not all files are the same size, this will change the cost of accessing some files versus others. For instance, if the first file is very large,

[2] Although tape memory is not often used by individuals, it remains one of the most efficient storage media for both very large files and for archival purposes. A linked list can be used to simulate a linear storage medium if the amount of data that can be stored in each node is limited; in fact, this kind of abstraction is precisely how files are stored on magnetic media.

then the cost of accessing every other file will be larger by an amount equal to its length. We can formalize and analyze the impact of a given ordering by letting $\pi(i)$ give the index of the file stored at location i on the tape. The expected cost of accessing a file is now simply

$$E[\text{cost}(\pi)] = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{j} L[\pi(i)] \ .$$

What ordering π should we choose to minimize the expected cost? Intuitively, we should order the files in order of their size, smallest to largest. Let's prove this.

Lemma 1: $E[cost(\pi)]$ is minimized when $L[\pi(i)] \leq L[\pi(i+1)]$ for all i.

Proof: Suppose $L[\pi(i)] > L[\pi(i+1)]$ for some i. If we swapped the files at these locations, then the cost of accessing the first file $\pi(i)$ increases by $L[\pi(i+1)]$ and the cost of accessing $\pi(i+1)$ decreases by $L[\pi(i)]$. Thus, the total change to the expected cost is $(L[\pi(i+1)] - L[\pi(i)])/n$, which is negative because, by assumption, $L[\pi(i)] \leq L[\pi(i+1)]$. Thus, we can always improve the expected cost by swapping some out-of-order pair, and the globally minimum cost is achieved when the files are sorted. □

Thus, any greedy algorithm that repeatedly swaps out-of-order pairs on the tape will lead us to the globally optimal ordering. (At home exercise: can you bound the expected cost in this case?)

Suppose now that files are not accessed with equal probability, but instead the ith file will be accessed $f(i)$ times over the lifetime of the tape. Now, the total cost of these accesses is

$$\text{total-cost}(\pi) = \sum_{j=1}^{n}\sum_{i=1}^{j} f(\pi(j)) \cdot L[\pi(i)] \ .$$

What ordering π should we choose now? Just as when the access frequencies were the same but the lengths were different we would sort the files by their lengths, if the lengths are all the same but the access frequencies different, we should sort the files in decreasing order of their access frequencies. (Can you prove this by modifying Lemma 1?) But, what if the sizes and frequencies are both non-uniform? The answer is to sort by the length-frequency ratio L/f.

Lemma 2: total-cost(π) is minimized when $\frac{L[\pi(i)]}{f(\pi(i))} \leq \frac{L[\pi(i+1)]}{f(\pi(i+1))}$ for all i.

Proof: Suppose $\frac{L[\pi(i)]}{f(\pi(i))} > \frac{L[\pi(i+1)]}{f(\pi(i+1))}$. The proof follows the same structure as Lemma 1, but where we observe that the proposed swap changes the total cost by $L[\pi(i+1)] \cdot f(\pi(i)) - L[\pi(i)] \cdot f(\pi(i+1))$, which is negative. □

Thus, the same class of greedy algorithms is optimal for non-uniform access frequencies.

1.3 Huffman codes

Continuing on the theme of storing and accessing information, consider the problem of *compression*. We are given a "message" or file x that is a sequence of characters (symbols) drawn from an n-character alphabet Σ. The task is to identify an encoding (mapping) of the original alphabet into a *prefix-free binary code*. A binary code is one built on the binary alphabet $\Sigma_{01} = \{0, 1\}$, and a prefix-free code is one in which no codeword is the prefix of any other. If a (binary) code is prefix-free, then it can be represented by a (binary) tree in which codewords are only located at the bottom of the tree, at the leaves.[3,4] The length of a codeword is given by the depth of its associated leaf. The goal of compression is to identify an encoding tree such that the encoded message is as small as possible. If f_i is the frequency of the ith character in the original message, we seek to minimize the function

$$\text{cost}(x) = \sum_{i=1}^{n} f_i \cdot \text{depth(i)} \ .$$

In 1952 David Huffman developed a greedy algorithm that produces an encoding that minimizes this function. In a compressed form, the idea is thus: merge the two least frequent characters and recurse.

Let's use a cute "self-descriptive" example from Lee Sallows[5]

> This sentence contains three a's, three c's, two d's, twenty-six e's, five f's, three g's, eight h's, thirteen i's, two l's, sixteen n's, nine o's, six r's, twenty-seven s's, twenty-two t's, two u's, five v's, eight w's, four x's, five y's, and only one z.

To keep things simple, let's ignore capitalization, the spaces (44), apostrophes (19), commas (19), hyphens (3) and the one period and focus on encoding the letters alone. The frequencies of the 26 letters are

A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1

Below is the encoding tree produced by applying Huffman's rule to this histogram: after 19 merges, all 20 characters have been merged and the history of merges gives the encoding tree. (Note: if

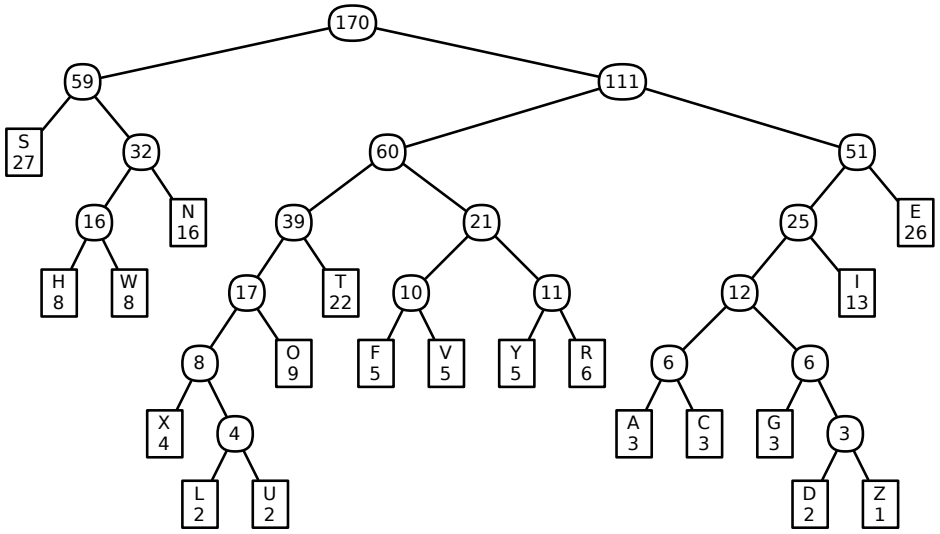
[3] Note that this is different from the *trie* data structure, which explicitly permits words to be prefixes of each other. Tries are very handy data structures for storing character strings in a space-efficient manner that allows fast lookups based on partial (prefix) matchings.

[4] Also note that this kind of tree is very different from a search tree. In fact, the ordering of the leaves in an encoding is very unlikely to be ordered in any way.

[5] A. K. Dewdney. Computer recreations. *Scientific American*, October 1984. Other examples appeared a few years earlier in some of Douglas Hofstadter's columns. My credit goes to Jeff Erickson, who also produced the figure below.

there is a tie for which pair of characters to merge, there will not be a unique Huffman code.)

To read the encoding of a character, place a 0 on each left-branch and a 1 on each right-branch, then write the sequence of 1s and 0s backwards as you read up the tree. To decode an encoded letter, do the reverse: start at the root, and take the left-right path given by the encoding down the tree to arrive at the decoded character. For example, the encoding of the letter "a" is 110000. In our example, the encoded message is 646 bits long.



Now let's prove that Huffman encoding is optimal.

Lemma 3: Let x and y be the two least frequency characters (with ties broken arbitrarily). There is an optimal code tree in which x and y are siblings and their parent is the largest depth of any leaf.

Proof: Let T be an optimal code tree with depth d. Because T is a "full" binary tree (every tree node has either 0 or 2 children), it must have at least two leaves at depth d that are siblings. Assume that these leaves are not x and y, but rather some other pair a and b.

Let T' by the code tree if we swapped x and a. The depth of x increases by some amount at most Δ, and the depth of a decreases by the same amount. Thus,

$$\text{cost}(T') = \text{cost}(T) - (f_a - f_x)\Delta \ .$$

But, by assumption, x is one of the two least frequency characters while a is not. This implies

$f_a \geq f_x$, and thus swapping x and a does not increase the total cost of the code. Since T was an optimal code tree, swapping x and a cannot decrease their cost. Thus, T' is also an optimal code tree (and $f_a = f_x$). Similarly for swapping y and b. Thus, doing both swaps yields an optimal tree with x and y as siblings and at maximum depth. □

We can now prove that the greedy encoding rule of Huffman is optimal.

Theorem 1: Huffman codes are optimal prefix-free binary codes.

Proof: If the message has only one or two different characters, the theorem is trivially true.

Otherwise, let $f_1, \ldots, f_n$ be the frequencies in the original message. Without loss of generality, let f_1 and f_2 be the smallest frequencies. When we recurse, we create a new frequency at the end of the list, e.g., $f_{n+1} = f_1 + f_2$. By Lemma 3, we know that the optimal tree T has characters 1 and 2 as siblings.

Let T' be the Huffman tree for $f_3, \ldots, f_{n+1}$. The inductive hypothesis implies that T' is an optimal code tree for this smaller set of frequencies. In the final optimal code tree T, we replace the frequency f_{n+1} with an internal node that has 1 and 2 as children.

Now, we can write the cost of T in terms of the cost of T'; let d_i be the depth of node i:

$$\begin{aligned}
\text{cost}(T) &= \sum_{i=1}^{n} f_i d_i \\
&= \left(\sum_{i=3}^{n+1} f_i d_i \right) + f_1 d_i + f_2 d_2 - f_{n+1} d_{n+1} \\
&= \text{cost}(T') + f_1 d_i + f_2 d_2 - f_{n+1} d_{n+1} \\
&= \text{cost}(T') + (f_1 + f_2) d_T - f_{n+1}(d_T - 1) \\
&= \text{cost}(T') + f_1 + f_2
\end{aligned}$$

Where we arrive at the last line by observing that the depth d_T relative to T' is 1. Thus, minimizing the cost of T is the same as minimizing the cost of T', and attaching the leaves for 1 and 2 to the leaf in T' labeled $n+1$ gives an optimal code tree for the original message. Thus, by induction, we have proved the theorem. □

(At home exercise: how long does it take to build a Huffman code on an n-character message? Hint: how can we efficiently find the minimum-frequency character at each step of the recursion?)

2 Next Time

1. Guest lecture by Prof. Sankaranarayanan on Wednesday, February 8
2. No office hours this week (out of town)
3. Reminder: Problem Set 1 due Tuesday, February 7 by 11:59pm via email