Fractional and Volterra processes in Finance

Chapter 3 - Volatility modeling: the Volterra way

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Stylized facts

What is a stylized fact?

From a statistical point of view: the seemingly random variations of asset prices do share some quite nontrivial statistical properties. Such properties, common across a wide range of instruments, markets and time periods are called stylized empirical facts.

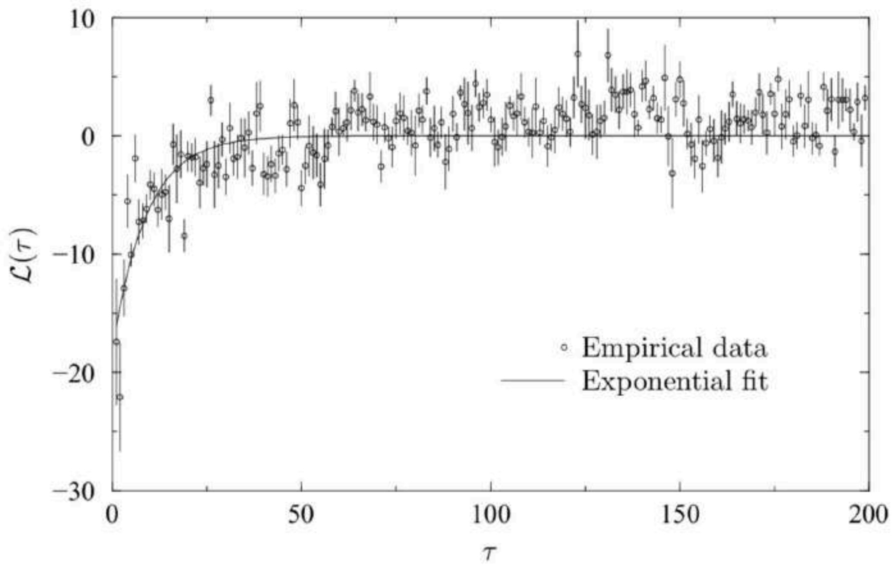
Indeed, stylized facts are usually formulated in terms of qualitative properties of asset returns and may not be precise enough to distinguish among different parametric models.

See Cont (2021) paper.

"Path-dependent" stylized facts

Levarage effect

$$\tau \mapsto \mathrm{Corr}(r_{t+\tau}^2, r_t)$$



decays rather quickly as exponential.

Volatility clustering?

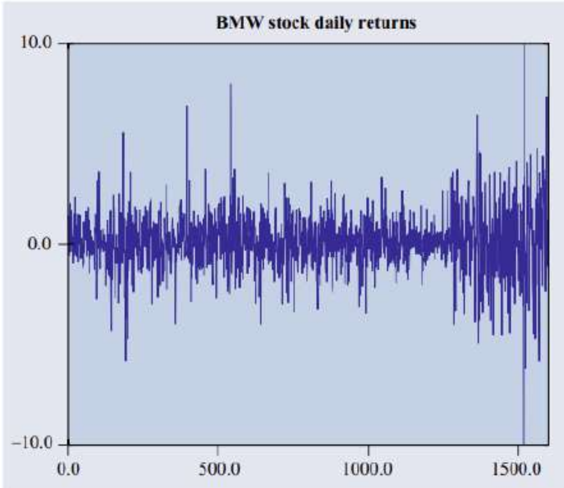


Figure 1. Daily returns of BMW shares on the Frankfurt Stock Exchange, 1992–1998.

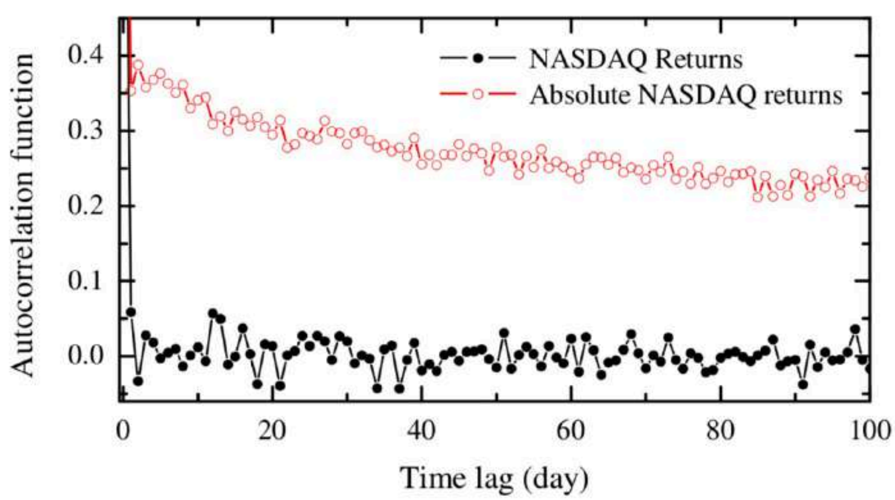
This is a quantitative signature of the well-known phenomenon of volatility clustering: large price variations are more likely to be followed by large price variations.

Squared returns, exhibit significant positive autocorrelation or persistence.

A quantity commonly used to measure volatility clustering is the autocorrelation function of the squared returns:

$$\tau \mapsto \mathrm{Corr}(r_{t+\tau}^2, r_t^2)$$

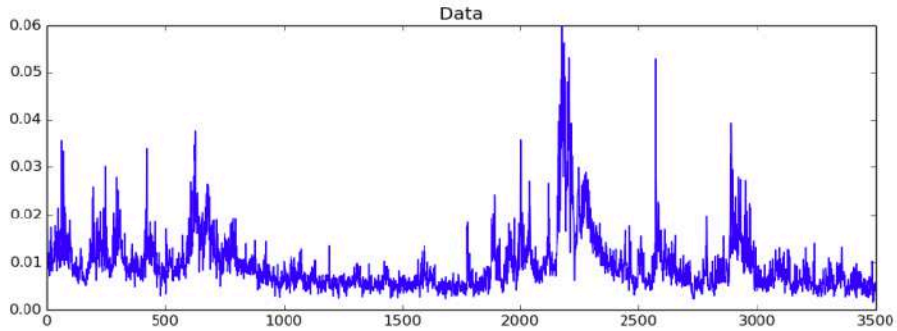
decays rather slowly:



Slowly, but at what rate/speed?

Monofractal scaling of volatility, rough volatility?

First: volatility process is not directly observable => proxy to estimate spot volatility values from intraday stock price:



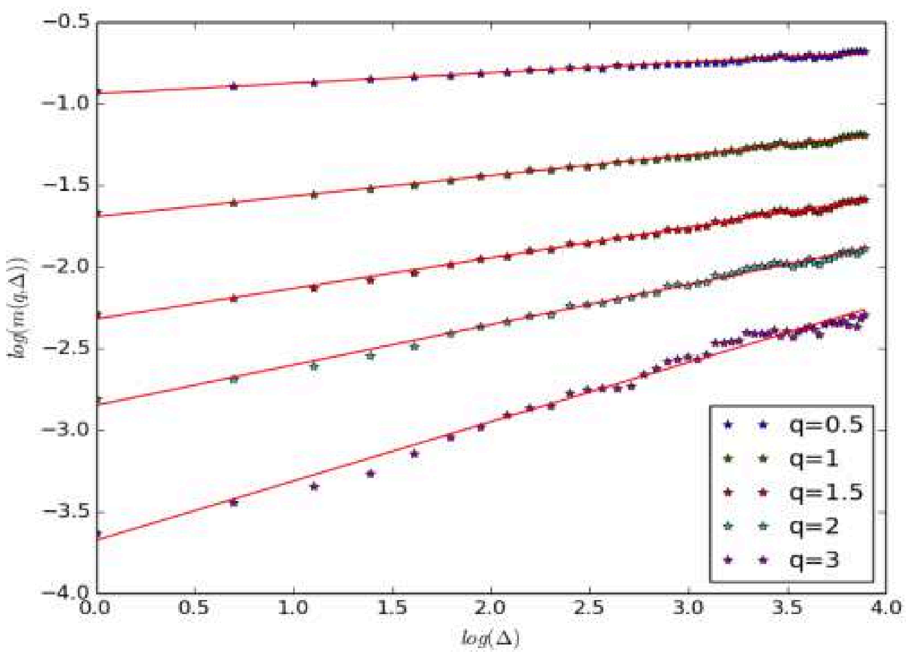
See Jaisson, Gatheral & Rosenbaum (2018).

> Erratic and spiky behavior.

Empirical estimate of

$$\mathbb{E}[|\log\sigma_{t+\tau}-\log\sigma_t|^q],\quad \tau\geq 0$$

$$m(q,\Delta)=\frac{1}{N}\sum_{k=1}^{N}|\log(\sigma_{k\Delta})-\log(\sigma_{(k-1)\Delta})|^q.$$



$\log m(q,\Delta)$ as a function of $\log\Delta$, DAX.

increments enjoy the following scaling property in expectation:

$$\mathbb{E}[|\log(\sigma_\Delta) - \log(\sigma_0)|^q] = K_q \Delta^{\zeta_q},$$

where $\zeta_q > 0$ is the slope of the line associated to q.

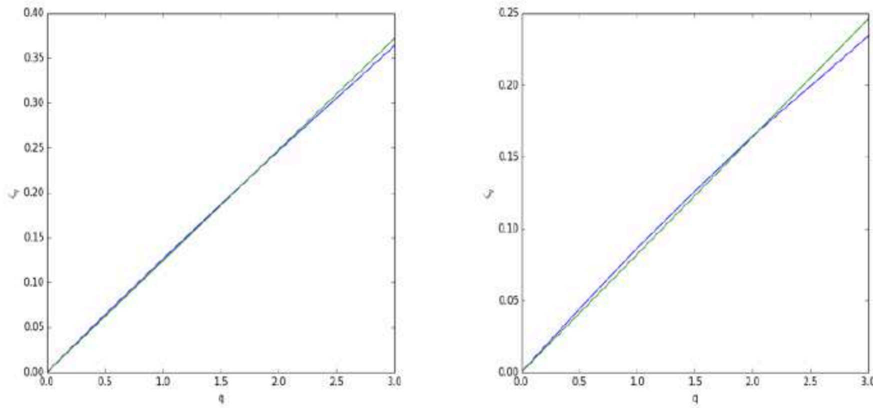


Figure 2.3: ζ_q (blue) and $0.125 \times q$ (green), DAX (left); ζ_q (blue) and $0.082 \times q$

This leads to a **monofractal type of scaling** across several timescales of the form

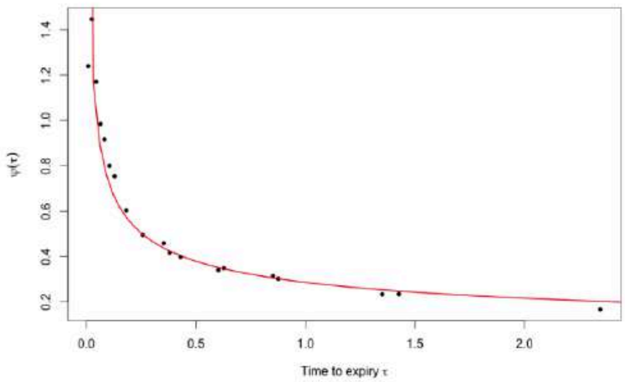
$$\mathbb{E}[|\log \sigma_{t+\tau} - \log \sigma_t|^q] \approx c_q \tau^{qH} \quad \text{for } \tau = 1 \text{ day}, 2 \text{ days}, \ldots$$

with $H \in (0, 0.15)$.

> Recall that a fractional Brownian motion W^H with Hurst parameter $H \in (0,1)$ satifies:
>
> $$\mathbb{E}[|W^H_{t+\tau} - W^H_t|^q] = K_q \tau^{qH}, \quad \tau \geq 0.$$

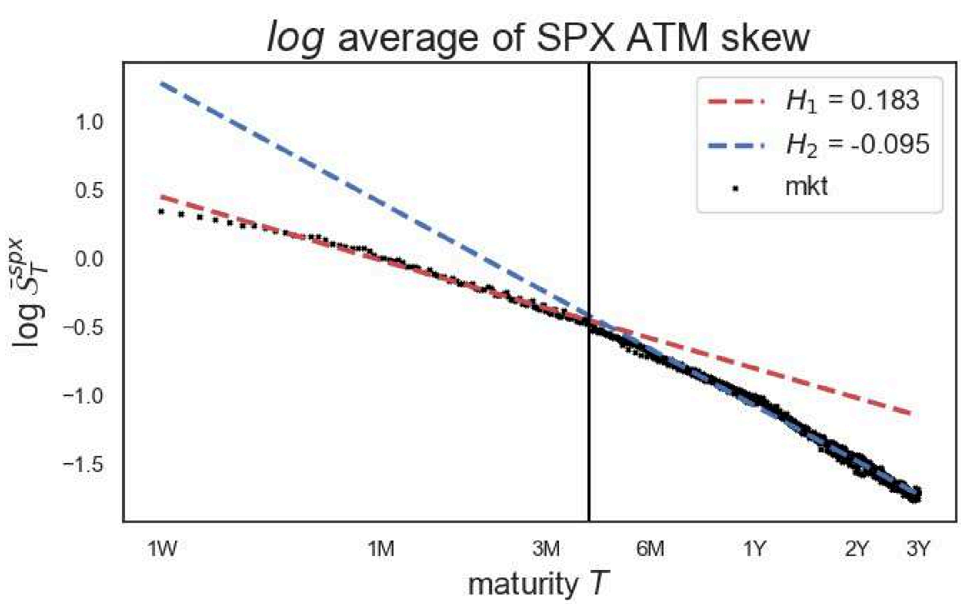
Can we conclude that volatility is rough?

Decay of at-the-money skew, explosive on short end?



The black dots are non-parametric estimates of the S&P at-the-money (ATM) volatility skews as of August 14, 2013; the red curve is the power-law fit $\psi(\tau) = A\,\tau^{-0.407}$, τ measured in years.

log-log plot:



Bergomi, L. (2015). Stochastic volatility modeling. CRC press.

Guyon, J., & El Amrani, M. (2022). Does the Term-Structure of Equity At-the-Money Skew Really Follow a Power Law?. Available at SSRN 4174538.

Delemotte, J., Marco, S. D., & Segonne, F. (2023). Yet Another Analysis of the SP500 At-The-Money Skew: Crossover of Different Power-Law Behaviours. Available at SSRN 4428407.

> Abi Jaber, E. & Li, S. (2024). Volatility models in practice: Rough, Path-dependent or Markovian?. arXiv preprint arXiv:2401.03345.

In summary

In practice, decrease of certain term structures, autocorrelations turns out to be much slower than it goes according to exponential decays, characetritic of standard 1 factor SV conventional stochastic volatility.

This evidence is clearly related to some type of path-dependency and mutiscale behaviour in the volatility, which can be better captured by introducing different kernels than the exponential kernel, hence Volterra models.

Memory matters!

Stochastic Volterra processes for modeling volatility

Consider stochastic Volterra equations:

$$X_t = X_0 + \int_0^t K(t,s)b(X_s)ds + \int_0^t K(t,s)\sigma(X_s)dW_s,$$

for some coefficients b, σ.

In a **Volterra volatility model** the dynamics of the asset are given by

$$dS_t = S_t f(X_t) dB_t,$$

for some specific functions f and

$$B = \rho W + \sqrt{1-\rho^2} W^{\perp}.$$

> Already discussed simulation techniques.
>
> **Advantages.** Very flexible class of models to encode path-dependencies of empirical realized variance and implied volatility.
>
> **Difficulties.** non-markovianity, non-semimartingality

Next sections Aim:

Introduce two tractable classes of Volterra volatility models in which fast pricing and calibration is possible via Fourier inversion techniques, i.e. where the characteristic function is known in analytic/semi-closed form:

- The Volterra Heston model
- The Volterra Stein--Stein model

=> Fast pricing and calibration in two tractable classes of Volterra volatility models.

Volterra Heston model

Heston model

The $\mathbf{Heston}$ model (1993):

$$
\begin{aligned}
dS_t &= S_t\sqrt{V_t}dB_t, \qquad d\langle B, W\rangle_t = \rho dt, \\
V_t &= V_0 + \int_0^t \lambda(\theta - V_s)ds + \int_0^t \eta\sqrt{V_s}dW_s.
\end{aligned}
$$

$\mathbf{Most\ popular\ model}$ closed form solutions for the characteristic function of the log-price $\Rightarrow$ Fast pricing and calibration by Fourier inversion techniques.

$$
\mathbb{E}\left[e^{u\log S_T}|\mathcal{F}_t\right] = e^{u\log S_t + \phi(T-t) + \psi(T-t)V_t},
$$

for $u \in i\mathbb{R}$ where (ϕ, ψ) are given in closed form.

> $\mathbf{(!)}$ (ϕ, ψ) solve the following system of Riccati ODEs
>
> $$
> \begin{aligned}
> \dot{\phi} &= \lambda\theta\psi, & \phi(0) = 0, \\
> \dot{\psi} &= \frac{1}{2}(u^2 - u) + (\rho\eta u - \lambda)\psi + \frac{\eta^2}{2}\psi^2, & \psi(0) = 0.
> \end{aligned}
> $$
>
> Recall Heston model

The Volterra Heston model

El Euch & Rosenbaum (2019) introduce the **rough Heston model** by replacing the standard square-root process in Heston by a fractional one:

$$
\begin{aligned}
dS_t &= S_t\sqrt{V_t}dB_t, \quad S_0 = 1, \\
V_t &= V_0 + \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}\left(\lambda(\theta - V_s)ds + \eta\sqrt{V_s}dW_s\right).
\end{aligned}
$$

> **(!)** V is not markovian, not a semimartingale (No Itô formula for $f(V_t)$)...

Remarkably, using scaling limits of Hawkes processes they obtain weak existence together with an exponential affine transform analogous to the classical case.

Definition More generally, fix a kernel $K \in L^2([0,T],\mathbb{R})$ and consider the **Volterra Heston model**:

$$dS_t = S_t\sqrt{V_t}dB_t, \quad S_0 = 1,$$
$$V_t = g_0(t) + \int_0^t K(t-s)\left(-\lambda V_s ds + \eta\sqrt{V_s}dW_s\right).$$

with $g_0(t)$ an input curve, e.g.

$$g_0(t) = V_0 + \int_0^t K(t-s)\lambda\theta ds.$$

For existence and uniqueness see Abi Jaber, Larsson & Pulido (2019).

1. Rough Heston:

$$K(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}$$

2. Standard Heston:

$$K(t) \equiv 1, \quad \text{or} \quad K(t) = e^{-\lambda t}$$

Back to standard Heston Standard Heston:

$$d\tilde{V}_t = \lambda(\theta - \tilde{V}_t)dt + \eta\sqrt{\tilde{V}_t}dW_t$$

By the variation of constant formula (Itô on $e^{\lambda t}\tilde{V}_t$)

$$\tilde{V}_t = \underbrace{V_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\lambda\theta ds}_{g_0(t)} + \int_0^t \underbrace{e^{-\lambda(t-s)}}_{K(t-s)}\eta\sqrt{\tilde{V}_s}dW_s$$

Characteristic function Back to the general Volterra Heston model. Let

$$g_t(u) = \mathbb{E}\left[V_u + \lambda\int_t^u V_s ds \mid \mathcal{F}_t\right] = g_0(u) + \int_0^t K(u-s)\left(-\lambda V_s ds + \eta\sqrt{V_s}dW_s\right)$$

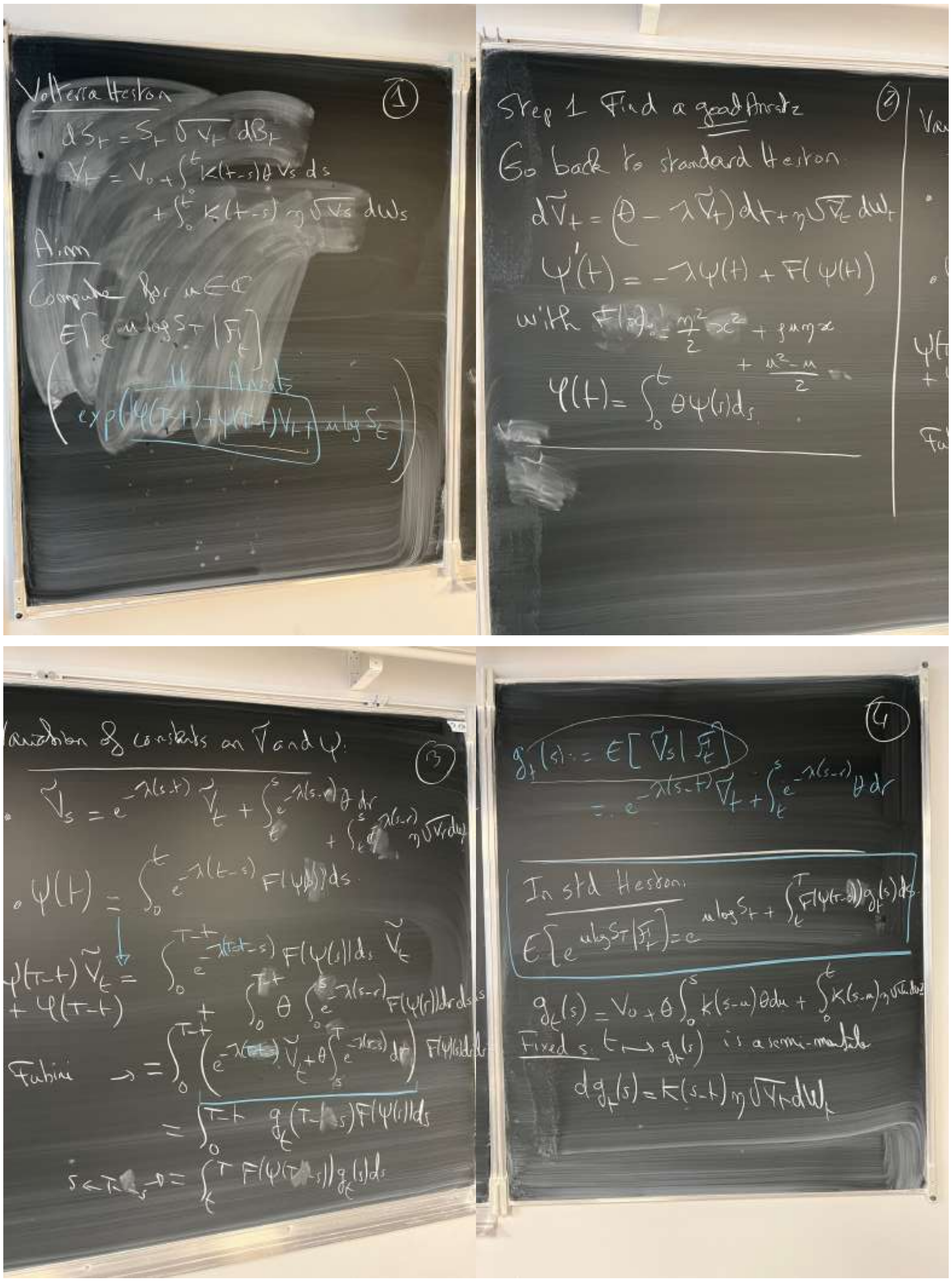
Theorem. (Characteristic function in the Volterra Heston model)

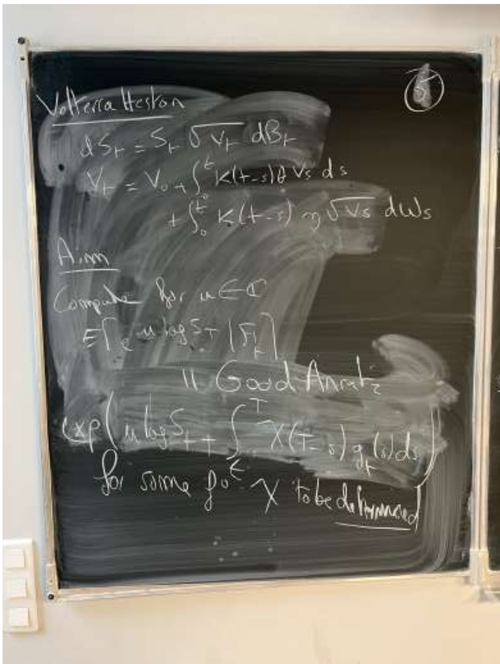
$$\mathbb{E}\left[e^{u\log S_T} \mid \mathcal{F}_t\right] = e^{u\log S_t + \int_t^T F(\psi)(T-s)g_t(s)ds},$$

where ψ solve the following system of Riccati Volterra equations

$$\psi(t) = \int_0^t K(t-s)F(\psi(s))ds$$

$$F(\psi) = \frac{1}{2}(u^2 - u) + (u\rho\eta - \lambda)\psi + \frac{\eta^2}{2}\psi^2.$$

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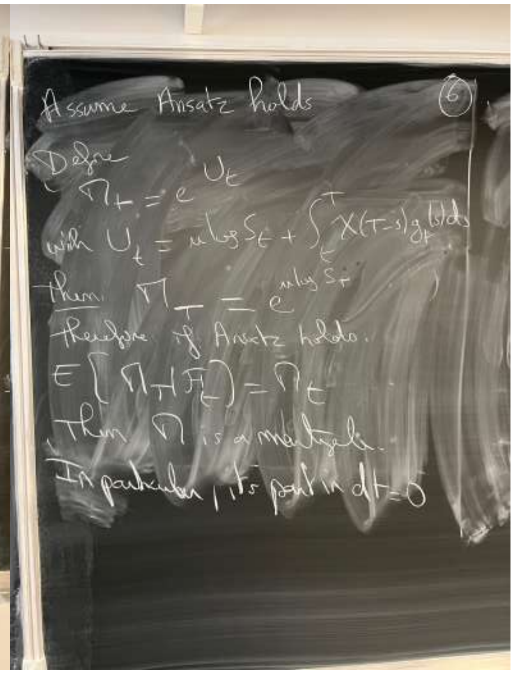
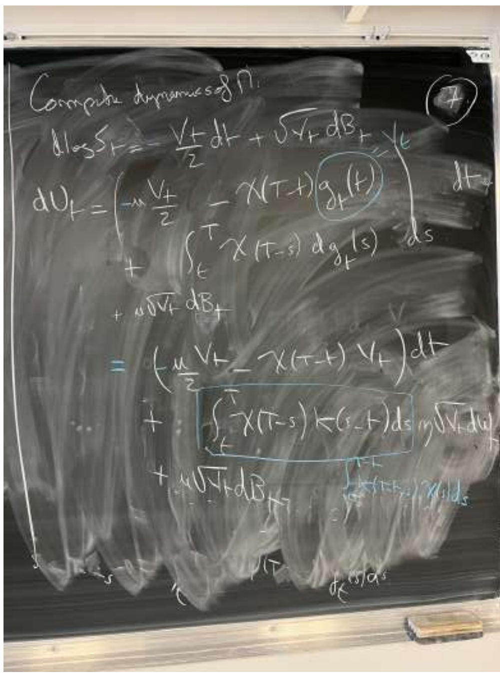
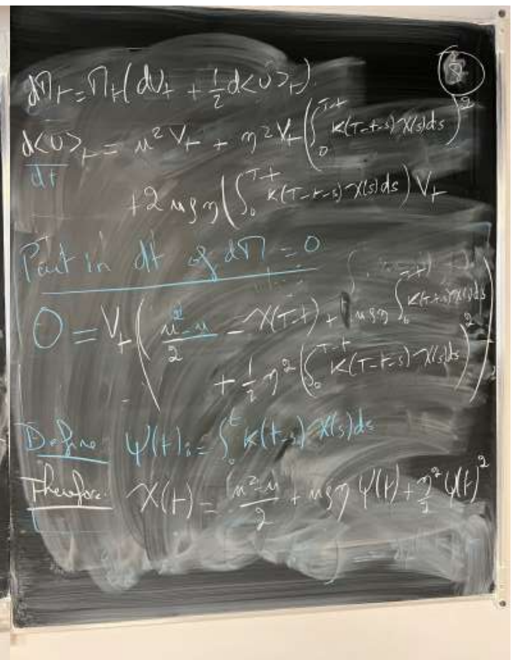
Volterra Heston
dS_t = S_t √V_t dB_t
V_t = V_0 + ∫_0^t K(t−s) θ V_s ds
+ ∫_0^t K(t−s) η √V_s dW_s
Aim
Compute for u ∈ ℂ
E[e^{u log S_T} | F_t]
Good Ansatz
exp(u log S_t + ∫_t^T χ(T−s) g_t(s) ds)
for some g_0^t, χ to be determined


Assume Ansatz holds
6
Define
Π_t = e^{U_t}
with U_t = u log S_t + ∫_t^T χ(T−s) g_t(s) ds
Then Π_T = e^{u log S_T}
Therefore if Ansatz holds
E[Π_T | F_t] = Π_t
Then Π is a martingale.
In particular, its part in dt = 0


Compute dynamics of Π
7
d log S_t = − V_t/2 dt + √V_t dB_t
dU_t = (−u V_t/2 − χ(T−t) g_t(t)
+ ∫_t^T χ(T−s) dg_t(s) ds) dt
+ u √V_t dB_t
= (−u V_t/2 − χ(T−t) V_t) dt
+ (∫_t^T χ(T−s) K(s−t) ds) η √V_t dW_t
+ u √V_t dB_t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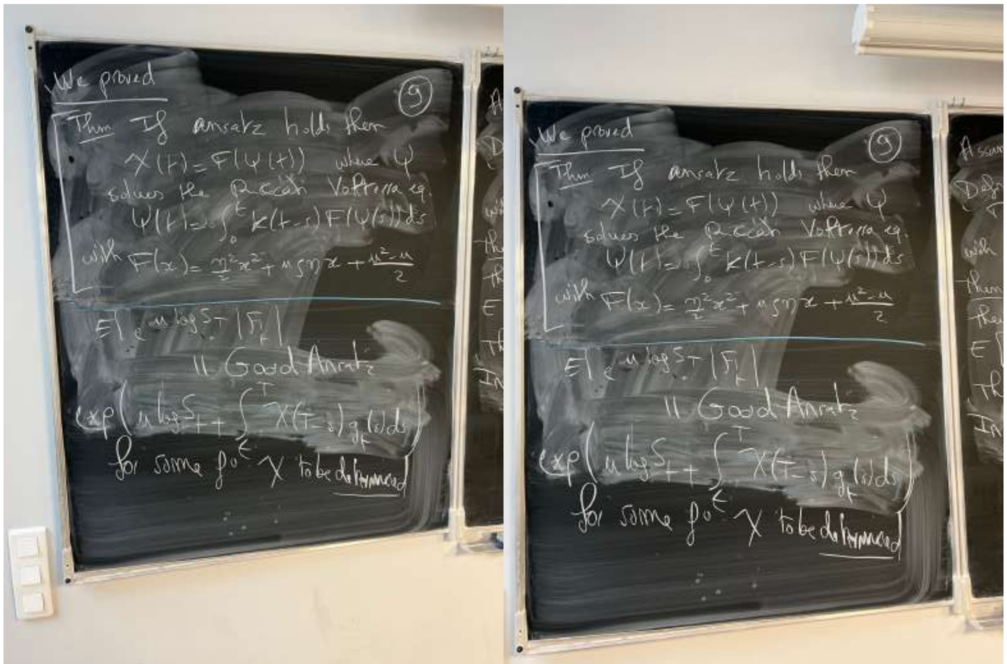
dΠ_t = Π_t (dU_t + 1/2 d⟨U⟩_t)
d⟨U⟩_t / dt = u² V_t + η² V_t (∫_0^{T−t} K(T−t−s) χ(s) ds)²
+ 2 u ρ η (∫_0^{T−t} K(T−t−s) χ(s) ds) V_t
Part in dt of dΠ = 0
0 = V_t ((u²−u)/2 − χ(T−t) + u ρ η ∫_0^{T−t} K(T−t−s) χ(s) ds
+ 1/2 η² (∫_0^{T−t} K(T−t−s) χ(s) ds)²)
Define ψ(t) := ∫_0^t K(t−s) χ(s) ds
Therefore χ(t) = (u²−u)/2 + u ρ η ψ(t) + η²/2 ψ(t)²



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> **Discuss** Adams Scheme + Multifactor for Volterra Riccati equation

In [ ]:

Volterra Stein-Stein model

Motivation

Back to standard Heston

Q Why do we have a closed form solution for the char. fun in standard Heston?

$\mathbf{Analytic}$

$$
\begin{aligned}
\dot{\phi} &= \lambda\theta\psi, & \phi(0) &= 0,\\
\dot{\psi} &= \frac{1}{2}(u^2-u) + (\rho\eta u - \lambda)\psi + \frac{\eta^2}{2}\psi^2, & \psi(0) &= 0.
\end{aligned}
$$

because the Riccati equation for ψ can be linearized by doubling the dimension $\psi = f^{-1}g$.

What about the fractional Riccati equation? Seems not possible:

- Can be seen as a correlated Riccati ODE in $\mathbb{R}^n$ as $n \to \infty$, and such Riccati ODEs cannot be linearized...
- No explicit solutions known to date.

$\textbf{(!)}$ Other Riccati equation that can be linearized? matrix Riccati equations in $\mathbb{R}^{n\times n}$ ({$n \to \infty$?}).

Why do we have a closed form solution for the char. fun in standard Heston?

$\textbf{Probabilistic}$ because the variance process has a generalized chi-squared distribution: Let $X = W$, by Itô $V = X^2$ has the square--root dynamics (squared Bessel processes)

$$dV_t = dt + 2\sqrt{V_t}dW_t.$$

$\textbf{Historically}$ to obtain his model, Heston took as starting point the Stein--Stein model (1991)

$$\begin{aligned} dS_t &= S_t X_t dB_t \\ X_t &= X_0 + \int_0^t \lambda(\theta - X_s)d_s + \eta W_t \end{aligned}$$

and observes that for $\theta = 0$, $V = X^2$ is a square-root process (main motivation introduce the leverage effect which was not present in Stein--Stein, see also Schobel-Zhu(1999)).

What about the fractional square-root process? $\textbf{Idea:}$ replace W with a fractional Brownian motion. Lack of It\^o $\Rightarrow V = (W^H)^2$ **does not** solve the frac. square-root equation.

Example:

$$X_t = \int_0^t K(t-s)dW_s, \quad g_t(s) := \mathbb{E}[X_s|\mathcal{F}_t] = \int_0^t K(s-u)dW_u$$

For fixed s we have on $[0, s)$:

$$dg_t(s) = K(s-t)dW_t$$

By Ito

$$g_t(s)^2 = \int_0^t K(s-u)^2 du + 2\int_0^t K(s-u)g_u(s)dW_u$$

Sending $t \to s$ and observing that $g_t(t) = X_t$, we get

$$X_t^2 = \int_0^t K(t-u)^2 du + 2\int_0^t K(t-u)g_u(t)dW_u$$

$$X_t^2 = \int_0^t K(t-u)^2 du + 2\int_0^t K(t-u)g_u(t)dW_u$$

In the standard case: ie $K \equiv 1$, $\mathbb{E}[X_s|\mathcal{F}_t] = X_t$ so that we recover for $V = X^2$

$$V_t = \int_0^t du + 2\int_0^t \sqrt{V_u} dW_u,$$

but in the general case, X^2 does not solve the Volterra square--root convolution equation:

$$V_t = V_0 + \int_0^t K(t-u)\sqrt{V_s} dW_s.$$

Towards another class of tractable models

$\textbf{Main idea:}$ take squares of Gaussian processes as building blocks to construct tractable (multi-dimensional) models for non-Markovian stochastic (rough) volatility models, short rate models with long memory, etc...

Volterra Stein-Stein

The (rough) Volterra Stein--Stein model Fix $K:[0,T]\to\mathbb{R}$, W Brownian motion and

$$X_t = X_0 + \int_0^t K(t,s)dW_s.$$

where $\int_0^T\int_0^T |K(t,s)|^2 dtds < \infty$. Define

$$g_t(s) := \mathbb{E}_t[X_s]1_{s\geq t} = \left(X_0 + \int_0^t K(s,u)dW_u\right)1_{s\geq t},$$
$$\Sigma_t(s,u) := \int_t^{s\wedge u} K(s,r)K(u,r)dr, \quad t \leq s,u \leq T.$$

Fix $t \leq T$. Conditional on $\mathcal{F}_t$, X_T follows a normal distribution

$$X_T \sim_{|\mathcal{F}_t} \mathcal{N}\left(g_t(T), \Sigma_t(T,T)\right)$$

and X_T^2 follows a χ^2-distribution.

the conditional Laplace transform reads

$$\mathbb{E}_t\left[\exp\left(uX_T^2\right)\right] = \frac{\exp\left(\frac{ug_t(T)^2}{1+2\Sigma_t(T,T)u}\right)}{\sqrt{1-2\Sigma_t(T,T)u}}$$

for all $u \leq 0$.

Aim: compute Laplace transform of integrated squared process

$$L_{t,T} = \mathbb{E}_t\left[\exp\left(w\int_t^T X_s^2 ds\right)\right], \quad w \in \mathbb{S}_-^d.$$

Idea: Exploit Gaussianity.

Laplace transform: main result

$$X_t = X_0 + \int_0^t K(t,s)dW_s \in \mathbb{R}^{d\times m}$$

$$\text{Assumption on } K\colon \sup_{t\le T}\int_0^T |K(t,s)|^2 ds < \infty, \quad \lim_{h\to 0}\int_0^T |K(t+h,s)-K(t,s)|^2 ds = 0$$

Theorem. (Laplace transform (Abi Jaber (2021))) Fix $w \in \mathbb{C}^{d\times d}$ such that $w \in \mathbb{S}^d_-$. Then,

$$\mathbb{E}_t\left[\exp\left(\int_t^T \mathrm{Tr}\left(X_s^\top w X_s\right) ds\right)\right] = \frac{\exp(\langle g_t, \mathbf{\Psi}_{t,T} g_t\rangle_{L^2})}{\det(\mathrm{id} - 2\sqrt{w}\mathbf{\Sigma}_t\sqrt{w})^{m/2}},$$

$$\text{where} \qquad \mathbf{\Psi}_{t,T} = \sqrt{w}\left(\mathrm{id} - 2\sqrt{w}\mathbf{\Sigma}_t\sqrt{w}\right)^{-1}\sqrt{w}.$$

Dynamic viewpoint $\dot{\mathbf{\Psi}}_t = 2\mathbf{\Psi}_t\dot{\mathbf{\Sigma}}_t\mathbf{\Psi}_t \quad \text{(Operator Riccati 'linearizable')}$

$\textbf{Quadratic ansatz} \exp(\phi_t + \langle g_t, \mathbf{\Psi}_{t,T} g_t\rangle_{L^2})$

Definition Fix $T > 0$. Consider the generalized version of the Stein--Stein model:

$$\begin{aligned} dS_t &= S_t X_t dB_t, \quad S_0 > 0,\\ X_t &= g_0(t) + \int_0^t K(t,s)\kappa X_s ds + \int_0^t K(t,s)\nu dW_s, \end{aligned}$$

with $B = \rho W + \sqrt{1-\rho^2}W^\perp$.

- Conventional mean reverting Stein-Stein model: $g_0(t) = X_0 - \kappa\theta t$ and $K(t,s) = 1$.
- The fractional Brownian motion with a Hurst index $H \in (0,1)$:

$$K(t,s) = \frac{(t-s)^{H-1/2}}{\Gamma(H+\frac{1}{2})}\,{}_2F_1\left(H-\frac{1}{2}; \frac{1}{2}-H; H+\frac{1}{2}; 1-\frac{t}{s}\right),$$

  where ${}_2F_1$ is the Gauss hypergeometric function
- Riemman-Liouville fBm:

$$K(t,s) = 1_{s<t}(t-s)^{H-1/2}/\Gamma(H+1/2).$$

The characteristic function For $u, w \in \mathbb{C}$ with $\Re(u) \in [0,1]$ and $\Re(w) \le 0$, because of the quadratic structure, **Ansatz**:

$$\mathbb{E}_t\left[\exp\left(u\log\frac{S_T}{S_t} + w\int_t^T X_s^2 ds\right)\right] = \exp(\phi_t + \langle g_t, \mathbf{\Psi}_t g_t\rangle_{L^2}),$$

with g_t the adjusted conditional mean given by

$$g_t(s) = 1_{t\le s}\mathbb{E}_t\left[X_s - \int_t^s K(s,r)\kappa X_r dr\right] = 1_{t\le s}\left(g_0(s) + \int_0^t K(s,u)\left(\kappa X_u du + \nu dW_u\right)\right).$$

$\Rightarrow M_t = \exp(u \log S_t + w \int_0^t X_s^2 ds + \phi_t + \langle g_t, \mathbf{\Psi}_t g_t \rangle_{L^2})$ is a martingale.By Itô $(\phi, \mathbf{\Psi})$:

$$\dot{\mathbf{\Psi}}_t = 2\mathbf{\Psi}_t \dot{\mathbf{\Sigma}}_t \mathbf{\Psi}_t,$$
$$(\mathbf{\Psi}_t f 1_t)(t) = (a\mathrm{id} + b\boldsymbol{K}^* \mathbf{\Psi}_t)(t)$$
$$\dot{\phi}_t = \mathrm{Tr}\left(\mathbf{\Psi}_t \dot{\mathbf{\Sigma}}_t\right)$$

$$a = w + \frac{1}{2}(u^2 - u), \quad b = \kappa + \rho\nu u$$

The characteristic function

$$\mathbb{E}_t\left[\exp\left(u \log \frac{S_T}{S_t} + w\int_t^T X_s^2 ds\right)\right] = \frac{\exp(\langle g_t, \mathbf{\Psi}_t g_t \rangle_{L^2})}{\det(\mathbf{\Phi}_t)^{1/2}},$$

and $\mathbf{\Psi}_t$ a linear operator

$$\mathbf{\Psi}_t = (\mathrm{id} - b\boldsymbol{K})^{-*} a \left(\mathrm{id} - 2a\tilde{\mathbf{\Sigma}}_t\right)^{-1} (\mathrm{id} - b\boldsymbol{K})^{-1}, \quad t \le T,$$

where $\boldsymbol{F}^{-*} := (\boldsymbol{F}^{-1})^*$ and $\tilde{\mathbf{\Sigma}}_t$ the adjusted covariance integral operator defined by

$$\tilde{\mathbf{\Sigma}}_t = (\mathrm{id} - b\boldsymbol{K})^{-1} \mathbf{\Sigma}_t (\mathrm{id} - b\boldsymbol{K})^{-*}, \quad \mathbf{\Phi}_t = \mathrm{id} - 2a\tilde{\mathbf{\Sigma}}_t.$$

Theorem. (Standard case) For $K(t,s) = 1_{s \le t}$ and an input curve of the form $g_0(t) = X_0 + \theta t$ one recovers the closed form exp. Stein-Stein, Schobel-Zhu and Heston (for $\theta = 0$):

$$\mathbb{E}_t\left[\exp\left(u \log \frac{S_T}{S_t} + w\int_t^T X_s^2 ds\right)\right] = \exp\big(A(t) + B(t)X_t + C(t)X_t^2\big)$$

where A, B, C solve the following system of Riccati equations

$$\dot{A} = -\theta B - \frac{1}{2}\nu^2 B^2 - \nu^2 C, \quad A(T) = 0,$$
$$\dot{B} = -2\theta C - (\kappa + \rho\nu u + 2\nu C)B, \quad B(T) = 0,$$
$$\dot{C} = -2\nu^2 C^2 - 2(\kappa + \rho\nu u)C - w - \frac{1}{2}(u^2 - u), \quad C(T) = 0.$$

In particular, (A, B, C) can be computed in closed form.

Numerical implementation

$$\mathbb{E}_t\left[\exp\left(u \log \frac{S_T}{S_t} + w\int_t^T X_s^2 ds\right)\right] = \frac{\exp(\langle g_t, \mathbf{\Psi}_t g_t \rangle_{L^2})}{\det(\mathbf{\Phi}_t)^{1/2}},$$

1. Discretize RHS: Fredholm, Nystrom method

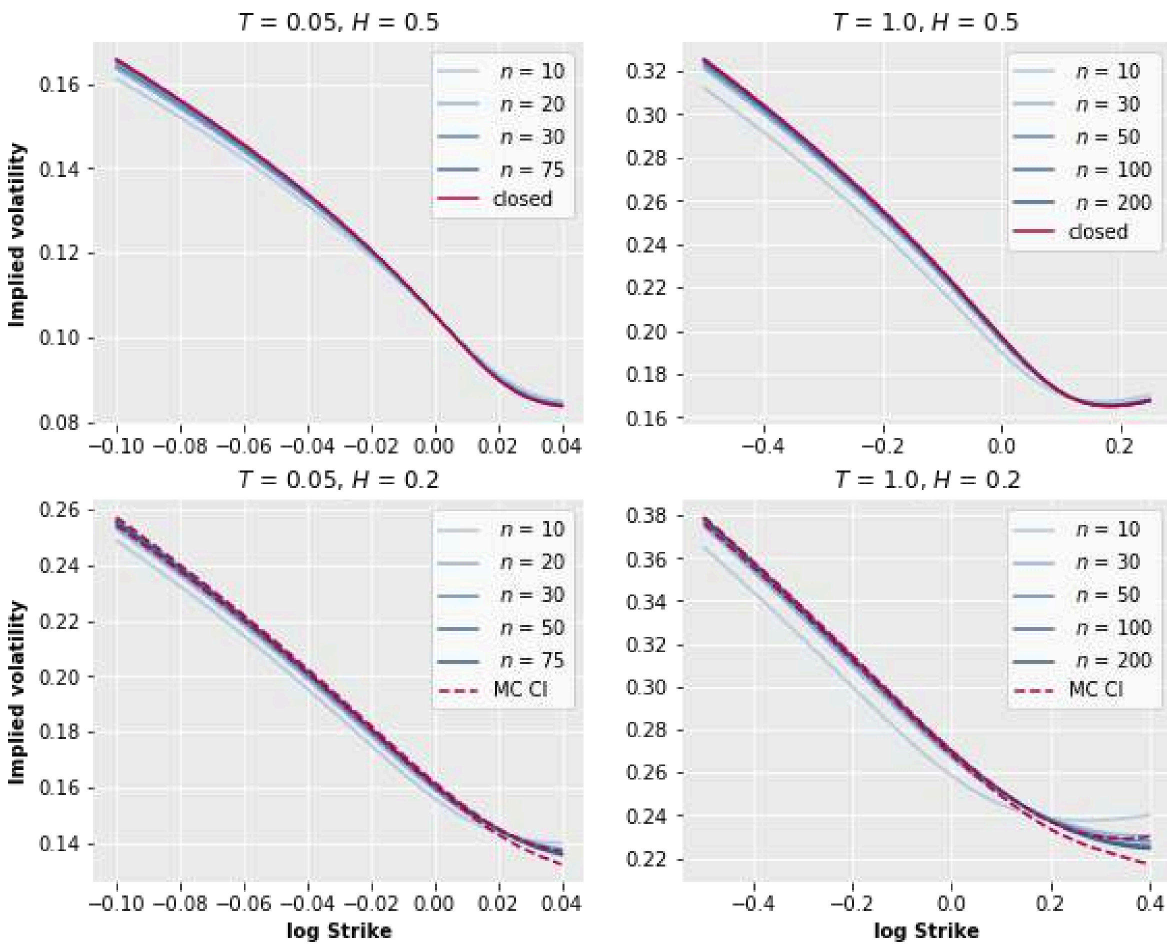
2. Discretize LHS: by projection argument
$\mathbb{E}\left[\exp\left(u\log\frac{S_T}{S_0}+w\int_0^T X_s^2 ds\right)\right]=\mathbb{E}\left[\exp\left(\alpha\int_0^T X_s^2 ds+\beta\int_0^T X_s dW_s\right)\right]$, with $\alpha = w+\frac{1}{2}(u^2-u)-\frac{\rho^2u^2}{2} \quad \text{and} \quad \beta=\rho u$. Euler discretization:

$$\alpha\int_0^T X_s^2 ds+\beta\int_0^T X_s dW_s \approx \sum_{i=1}^n \frac{\alpha T}{n}X_{t_i}^2+\beta X_{t_i}Y_i,$$

with

$$X_i = X_{t_{i-1}} \quad \text{and} \quad Y_i=\int_{t_{i-1}}^{t_i} dW_s, \quad i=1,\dots,n.$$

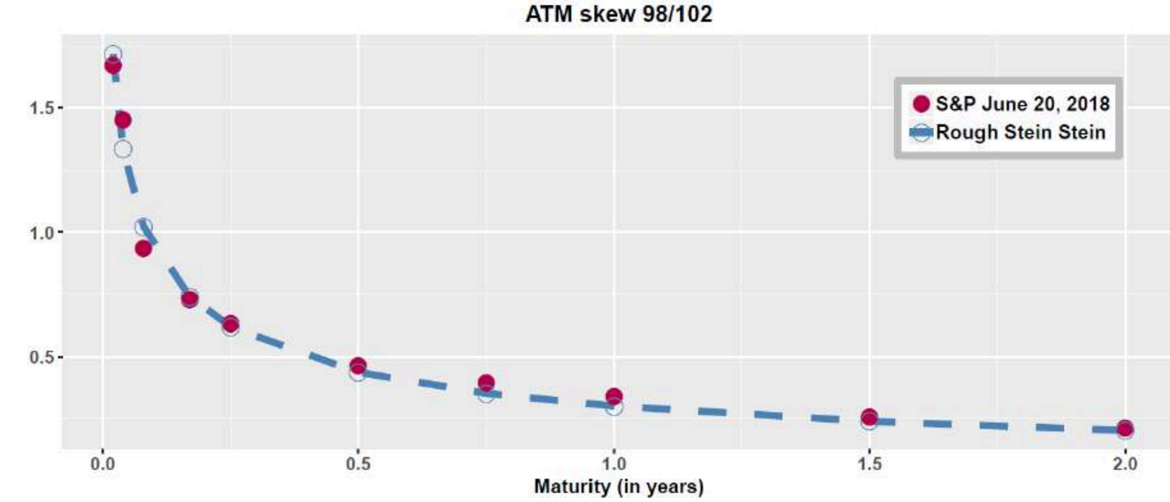
The $2n$-dimensional vector $Z_n=(X_1,\dots,X_n,Y_1,\dots Y_n)^\top$ is Gaussian with known convariance matrix and mean $\Rightarrow$ Wishart formulas are applicable.



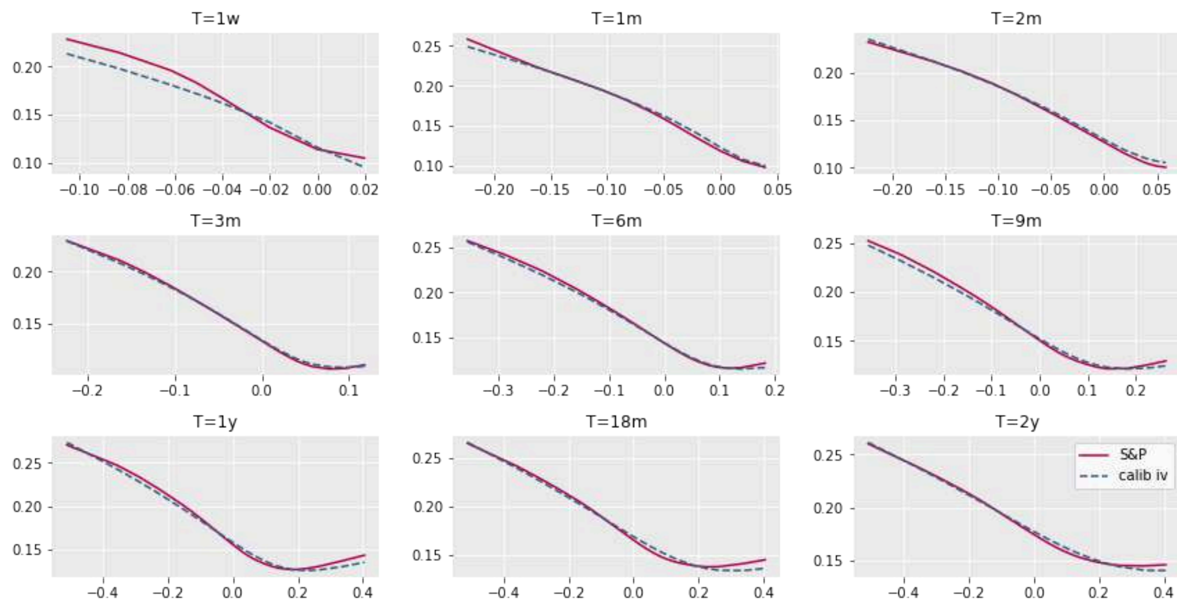
Calibration of ATM-Skew with fractional kernel

$$K(t,s)=1_{s<t}\frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}$$

$$\hat{\nu}=0.5231458, \quad \hat{\rho}=-0.9436174 \quad \text{and} \quad \hat{H}=0.2234273.$$



Calibration to full vol surface:

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