Math 1540
Hastings section
Spring 2011
Notes 13
Differential 2-forms on R^2.

1 Wedge products

Above we defined the 2-form $dx \wedge dy$, etc. These are called the "wedge products" of the one-forms dx and dy. If we have general one-forms on R^2, say $p\ dx + q\ dy$ and $r\ dx + s\ dy$, then their wedge product is

$$(ps - qr)\, dx \wedge dy. \tag{1}$$

Once we realize that $ps - qr$ is a function from R^2 to R we see that this was defined earlier, as a 2-form , in equations (3) and (4) of notes 12. The formula (1) can be obtained from ordinary algebra:

$$(p\ dx + q\ dy) \wedge (r\ dx + s\ dy) = pr\ dx \wedge dx + ps\ dx \wedge dy + qr\ dy \wedge dx + qs\ dy \wedge dy,$$

where we used the previously established formulas $dx \wedge dx = dy \wedge dy = 0$ and $dy \wedge dx = -dx \wedge dy$.

2 The exterior derivative of differential forms

We first add to our list of differential forms:

Definition 1 *A* 0*-cell on* R^n *is a function* $\phi : \{0\} \to R^n$.

In practice, we will think of a 0-cell as a point in R^n.

Definition 2 *A* 0*-form on* R^n *is a function* $F : R^n \to R$. *(There is a reason I used* F *instead of* f*, as you will see.*

Thus, if ϕ is a 0-cell and F is a 0-form, then $F(\phi(0))$ is defined as a real number. Further on we will use our usual integral notation for the value of a form on a cell, giving

$$\int_{\phi(0)} F = F(\phi(0)).$$

We will probably not use ω to denote a 0-form, but instead the name of the function, such as F.

Definition 3 *The exterior derivative of a* 0*-form* F *is the functional* $dF : S_{1,n} \to R$ *given by*

$$dF(\phi) = F(\phi(1)) - F(\phi(0)). \tag{2}$$

We know that not all functionals on $S_{1,n}$ are one-forms. For example, the length of the curve ϕ is not a 1-form because it cannot be expressed as an integral in terms of $\phi' = (\phi_1', \phi_2')$. However, considering $n = 2$, if $F : R^2 \to R$ is smooth, then

$$\begin{aligned} &\int_0^1 \frac{\partial F}{\partial x}(\phi_1(t), \phi_2(t))\, \phi_1'(t) + \frac{\partial F}{\partial y}(\phi_1(t), \phi_2(t))\, \phi_2'(t)\, dt \\ &= \int_0^1 \frac{d}{dt} F(\phi_1(t), \phi_2(t))\ dt = F(\phi(1)) - F(\phi(0)) = dF(\phi) \end{aligned} \tag{3}$$

Hence, dF is a 1-form .

Continuing our use of integral notation for 1-forms, we now can write

$$dF(\phi) = \int_\phi dF = F(\phi(1)) - F(\phi(0)). \tag{4}$$

In fact, comparing the formula above with the formula for a general 1-form with name $\omega = pdx + qdy$, namely,

$$\omega(\phi) = \int_0^1 (p(x(t), y(t))\, x'(t) + q(x(t), y(t))\, y'(t))\, dt, \tag{5}$$

we see that

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy. \tag{6}$$

Definition 4 *Suppose that* $\omega = p\ dx + q\ dy$ *is a* 1*-form on* R^2*. Then the exterior derivative of* ω *is*

$$d\omega = dp \wedge dx + dq \wedge dy, \tag{7}$$

where dp *and* dq *are the exterior derivatives of the zero forms* p *and* q*. We have no difficulty knowing this is a form, since the wedge product was defined so as always to be a form; that is, an integral involving a determinant.*

We can easily get a more useful formula. Note that p and q are 0-forms on R^2. We can use equation (6) to give $dp = \frac{\partial p}{dx} dx + \frac{\partial p}{\partial y} dy$, and a similar expression for dq. The, substituting these in (7) and using the rules for wedge products, we get

$$d\omega = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx \wedge dy. \tag{8}$$

3 Chains and boundaries.

Recall that a k-cell in R^n is a smooth function $\phi : I^k \to R^n$.

Definition 5 *A k-chain (on R^n) is a set Φ of pairs (a_i, ϕ_i) of numbers a_i and k-cells ϕ_i (on R^n):*

$$\Phi = \{(a_1, \phi_1), ..., (a_N, \phi_N)\}.$$

Given a k-chain Φ, and a k-form ω, we define the integral of ω over Φ as follows:

$$\int_\Phi \omega = \sum_{i=1}^N a_i \int_{\phi_i} \omega. \tag{9}$$

This is a real number, and we can consider $\int_\Phi \omega$ as a functional on the set of all k-chains.

As an example, consider the following four 1-cells on R^2:

$$\phi_1(t) = (1, t), \phi_2(t) = (0, t), \phi_3(t) = (t, 1), \phi_4(t) = (t, 0), \tag{10}$$

for $0 \le t \le 1$. ($t \in I^1$.) Let

$$\Phi = \{(1, \phi_1), (-1, \phi_2), (-1, \phi_3), (1, \phi_4)\} \tag{11}$$

Notice that ϕ_1 maps $[0, 1]$ onto the right edge of the unit square, with the orientation given by the unit tangent vector $(0, 1)$. This points upward. Also, ϕ_2 maps $[0, 1]$ onto the left edge of the unit square, also with upward orientation. By pairing -1 with ϕ_2, when we compute $\int_\Phi \omega$ for some 1-form $\omega = pdx + qdy$, we get a term

$$-\int_{\phi_2} \omega.$$

This is the value of ω on the 1-form $(0, 1 - t)$, which goes along the left side of the unit square from top to bottom. We have not changed the image of ϕ_2, just its orientation.[1] Similar considerations apply to $(-1, \phi_3)$ and to $(1, \phi_4)$, and putting these together, we see that we have computed a line integral around the boundary of I^2 going in the counterclockwise direction.

Next we wish to define the "boundary" of a k-cell, for $k = 1$ and 2. We start with a 1-cell $\phi : I^1 \to R^n$.

Definition 6 *The boundary of a 1-cell ϕ is the 0-chain*

$$\partial\phi = \{(1, \phi(1)), (-1, \phi(0))\} \tag{12}$$

[1] Contrast this with the usual definition of $-\phi_2$, which would change its image.

3.1 The fundamental theorem of calculus.

Suppose that $F : R^1 \to R$ (which equals R^1). Then as we saw in Definition 2, F is a 0-form on R^1. And in Definition 3 we saw that the exterior derivative of F is

$$dF(\phi) = \int_\phi dF = F(\phi(1)) - F(\phi(0)). \tag{13}$$

This is a 1-form, as seen in equation (3), making allowances for the fact that there we were discussing a 0-form on R^2.

Also, if ϕ is a 0-cell, we defined the boundary of ϕ, $\partial\phi$, in equation (12). Using (9) we get

$$\int_{\partial\phi} F = \int_{\phi(1)} F - \int_{\phi(0)} F. \tag{14}$$

We have to interpret the right side correctly. Recall that a 0-cell is a point, and a 0-form is a functional on the set of zero cells. As we saw earlier, the usual integral notation for forms gives

$$\int_{\phi(1)} F = F(\phi(1)), \quad \int_{\phi(0)} F = F(\phi(0)).$$

Using (13) and (14) we get

$$\int_\phi dF = \int_{\partial\phi} F. \tag{15}$$

This is purely from the notation; no deep mathematics is involved.

However, if F is continuously differentiable, with derivative $F' = f$, then we can consider the 1-form $f\ dx$, which by definition is given by

$$\int_\phi f\ dx = \int_0^1 f(\phi(t))\,\phi'(t)\,dt.$$

(We are still in one dimension.) Using the change of variable theorem and the fundamental theorem of calculus, we get

$$\int_\phi f = \int_{\phi(0)}^{\phi(1)} f(x)\,dx = F(\phi(1)) - F(\phi(0)).$$

From this we see that the 1-form dF is the same as the 1-form $F'\ dx$. This result does require nontrivial mathematics, namely the fundamental theorem of calculus.

3.2 Boundary of a 2-form, Green's theorem for the unit square.

We can now define the "boundary" of a 2-cell.

Definition 7 *The boundary of a 2-cell ϕ is the 1-chain*

$$\partial\phi = \{(1,\phi(1,t)), (-1,\phi(0,t)), (1,\phi(t,0)), (-1,\phi(t,1))\}. \tag{16}$$

One example is the 1-chain I gave earlier (equations (10) and (11)), which is the boundary of the 2-cell $\phi(x,y) = (x,y)$.

Suppose that $n = 2$, and $pdx + qdy$ is a 1-form on R^2. We can ask: What is $\int_{\partial\phi} pdx + qdy$? This is the integral over a chain, so we have to use equation (9). In this section we will only consider the case where $\phi(x,y) = (x,y)$. Thus, the image of ϕ is the unit square I^2. Then

$$\partial\phi(t) = \{(1,(1,t)), (-1,(0,t)), (1,(t,0)), (-1,(t,1))\}.$$

In that case, (9) becomes

$$\begin{aligned}\int_{\partial\phi} pdx + qdy = &\int_{\phi_1} pdx + qdy - \int_{\phi_2} pdx + qdy \\ &- \int_{\phi_3} pdx + qdy + \int_{\phi_4} pdx + qdy,\end{aligned}$$

where ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are 1-forms in (10). I will evaluate one of these integrals and then give the final answer. Since $\phi_1(t) = (1,t)$, the standard line integral formula, as in (5) above, gives

$$\int_{\phi_1} pdx + qdy = \int_0^1 (0 + q(1,t)(1))\,dt = \int_0^1 q(1,t)\,dt.$$

Evaluating the other integrals similarly, we get

$$\int_{\partial\phi} pdx + qdy = \int_0^1 (q(1,t) - q(0,t) + p(t,0) - p(t,1))\,dt \tag{17}$$

We will now evaluate

$$\int_\phi d\omega,$$

where $\omega = pdx + qdy$ and where we again choose ϕ to be the 2-form $\phi(x, y) = (x, y)$. Recall from equation (8) that

$$d\omega = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy.$$

We then apply the definition of the 2-form in Definition 17 from Notes 12 (equation (3) of those notes). Recall that we are taking $\phi(x, y) = (x, y)$. Hence

$$\det \begin{pmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

and using Fubini's theorem,

$$\begin{aligned} \int_\phi \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy &= \int_0^1 \left(\int_0^1 \frac{\partial q}{\partial x} dx\right) dy - \int_0^1 \left(\int_0^1 \left(\frac{\partial p}{\partial y}\right) dy\right) dx. \\ &= \int_0^1 \left((q(1, s) - q(0, s)) - (p(s, 1) - p(s, 0))\right) ds \end{aligned}$$

which is the same as the right side of equation (17). This gives the formula

$$\int_\phi d\omega = \int_{\partial \phi} \omega. \tag{18}$$

This is Green's theorem for the unit square. Compare with equation (15).

4 Green's theorem for a general 2-cell on R^2.

4.1 Brute force method

It is often useful to see more than one proof of an important theorem. Green's theorem is simple enough, because it is set in R^2, to give a direct computational proof, in which both sides of equation (18), for a general 1-form ω and general 2-cell ϕ, are evaluated from their definitions and shown to be equal. Most mathematicians prefer a proof which is more easily generalized to higher k and n, but I will give this special case proof first.

Proof. Recall that a general 1-form on R^2 can be written as

$$\omega = p(x, y)\, dx + q(x, y)\, dy.$$

Suppose that ϕ is a 2-cell. Then the boundary $\partial\phi$ was given in equation (16):

$$\partial\phi = \{(1, \phi(1,t)), (-1, \phi(0,t)), (1, \phi(t,0)), (-1, \phi(t,1))\}.$$

Here $\phi(x,y) = (\phi_1(x,y), \phi_2(x,y))$. Then $\int_{\partial\phi}\omega$ is, by definition, the sum of four standard line integrals, the first two of which add up to

$$\begin{aligned}
&\int_{\phi(1,t)} pdx + qdy - \int_{\phi(0,t)} pdx + qdy = \\
&\int_0^1 \left\{ p(\phi_1(1,t), \phi_2(1,t)) \frac{\partial\phi_1}{\partial y}(1,t) - p(\phi_1(0,t), \phi_2(0,t)) \frac{\partial\phi_1}{\partial y}(0,t) \right\} dt \\
&+ \int_0^1 \left\{ q(\phi_1(1,t), \phi_2(1,t)) \frac{\partial\phi_2}{\partial y}(1,t) - q(\phi_1(0,t), \phi_2(0,t)) \frac{\partial\phi_2}{\partial y}(0,t) \right\} dt.
\end{aligned}$$

These terms involve integrals along the left and right sides of I^2. From now on we will just discuss the terms with p. The terms with q work out in the same way. Also, the terms in p and q along the top and bottom of I^2 work out similarly and will not be given here. So we consider

$$\int_0^1 \left\{ p(\phi_1(1,t), \phi_2(1,t)) \frac{\partial\phi_1}{\partial y}(1,t) - p(\phi_1(0,t), \phi_2(0,t)) \frac{\partial\phi_1}{\partial y}(0,t) \right\} dt. \tag{19}$$

We now consider the corresponding terms in $\int_\phi d\omega$. We gave $d\omega$ in equation (7):

$$d\omega = dp \wedge dx + dq \wedge dy.$$

Further, in (**??**) we saw that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy,$$

and so using the rules for wedge product, the term in $d\omega$ involving p is $-\frac{\partial p}{\partial y}(x,y)\, dx \wedge dy$. Hence, using the definition of a 2-form, the terms involving p in $\int_\phi d\omega$ are

$$\begin{aligned}
&\int_0^1 \int_0^1 \left(-\frac{\partial p}{\partial y}(\phi_1(x,y), \phi_2(x,y)) \right) \det \begin{pmatrix} \frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial y} \\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial y} \end{pmatrix} dxdy \\
&= \int_0^1 \int_0^1 \left(-\frac{\partial p}{\partial y}(\phi_1(x,y), \phi_2(x,y)) \right) \left(\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial y} - \frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial x} \right) dxdy.
\end{aligned} \tag{20}$$

We wish to evaluate this integral. We note that

$$\frac{\partial}{\partial x}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)=\frac{\partial p}{\partial x}\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\frac{\partial \phi_1}{\partial x}\left(x,y\right)+\frac{\partial p}{\partial y}\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\frac{\partial \phi_2}{\partial x}\left(x,y\right)$$
$$\frac{\partial}{\partial y}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)=\frac{\partial p}{\partial x}\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\frac{\partial \phi_1}{\partial y}\left(x,y\right)+\frac{\partial p}{\partial y}\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\frac{\partial \phi_2}{\partial y}\left(x,y\right).$$

(Be sure to notice the difference between $\frac{\partial}{\partial y}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)$ and $\frac{\partial p}{\partial y}\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)$.)

Multiply the first of these equations by $\frac{\partial \phi_1}{\partial y}$, the second by $\frac{\partial \phi_1}{\partial x}$ and subtract the second resulting equation from the first. The first terms on the right cancel, and we are left with

$$\begin{aligned}&\frac{\partial \phi_1}{\partial y}\frac{\partial}{\partial x}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)-\frac{\partial \phi_1}{\partial x}\frac{\partial}{\partial y}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\\&=\frac{\partial p}{\partial y}\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\left(\frac{\partial \phi_2}{\partial x}\frac{\partial \phi_1}{\partial y}-\frac{\partial \phi_2}{\partial y}\frac{\partial \phi_1}{\partial x}\right)\end{aligned}$$

The term on the right is the integrand in (20), and so we are left to evaluate

$$\int_0^1\int_0^1\left\{\frac{\partial \phi_1}{\partial y}\ \frac{\partial}{\partial x}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)-\frac{\partial \phi_1}{\partial x}\ \frac{\partial}{\partial y}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\right\}dxdy. \quad (21)$$

Considering the first term, we integrate the x integral by parts, getting

$$\begin{aligned}\int_0^1\frac{\partial \phi_1}{\partial y}\left(x,y\right)\frac{\partial}{\partial x}p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)dx&=\frac{\partial \phi_1}{\partial y}\left(x,y\right)p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)|_{x=0}^{x=1}\\&\quad-\int_0^1 p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\frac{\partial^2\phi_1}{\partial y\partial x}dx\\&=p\left(\phi_1\left(1,y\right),\phi_2\left(1,y\right)\right)\frac{\partial \phi_1}{\partial y}\left(1,y\right)-p\left(\phi_1\left(0,y\right),\phi_2\left(0,y\right)\right)\frac{\partial \phi_1}{\partial y}\left(0,y\right)\\&\qquad-\int_0^1 p\left(\phi_1\left(x,y\right),\phi_2\left(x,y\right)\right)\frac{\partial^2\phi_1}{\partial y\partial x}dx \quad (22)\end{aligned}$$

The first two terms are the integrand in (19), and so are the integrands in the line integrals involving p along the left and right sides of I^2. This means that the line integrals along the left and right sides of I^2 involving p are equal to two of the terms in $\int_\phi d\omega$. The last term, (22), will cancel a similar term obtained from the second integrand in (21), where we do the y integral first, again using integration by parts. The rest of the second integral in (21) will contribute the p terms for line integrals over the top and bottom of I^2. All q terms are similar, giving the result. ■

In the next set of notes we will give a less computational proof which can more easily be modified to apply to higher dimensions.

5 Homework due April 6

1. Suppose that $\phi(x,y) = \left(\left(\frac{3}{2}x-1\right)^2, y\right)$ for $(x,y) \in I^2$. Suppose that $\omega = dx \wedge dy$. Find $\int_\phi \omega$ from the definition of this integral, and explain your answer in terms of area.

2. Last week you were asked to find 2-cells on R^3 whose images were the unit sphere and the unit hemisphere. While there are many possible answers, here are two that work:

   Sphere: $\phi(x,y) = (\sin \pi x \cos 2\pi y, \sin \pi x \sin 2\pi y, \cos \pi x)$ for $(x,y) \in I^2$

   Hemisphere: $\phi(x,y) = \left(\sin \frac{\pi}{2} x \cos 2\pi y, \sin \frac{\pi}{2} x \sin 2\pi y, \cos \frac{\pi}{2} x\right)$ for $(x,y) \in I^2$.

   Find the boundaries of each of these 2-cells, describing the image and orientation of each section.

3. Find a 2-cell ϕ on R^2 which is $1:1$ and whose image is the region in the (u,v) plane between concentric circles with radii 1 and 2 and center the origin. (If you wish, the domain of ϕ can be some more convenient rectangle than the unit square.) Find the boundary $\partial\phi$.

4. (10 pts.) Continuing from (3), let $\omega = 3x^2 dx + xy^2 dy$, and verify Green's theorem in this case by explicitly calculating both sides of equation (18). You can use a computer (for example, with Maple or Mathematica) to evaluate the integrals. Or you can Google "evaluate integrals" and go the first link, Wolfram Mathematica online integrator.