THE ELEMENTARY GEOMETRIC STRUCTURE OF COMPACT LIE GROUPS

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Abstract. We give geometric proofs of some of the basic structure theorems for compact Lie groups. The goal is to take a fresh look at these theorems, prove some that are difficult to find in the literature, and illustrate an approach to the theorems that can be imitated in the homotopy theoretic setting of p-compact groups.

1. Introduction

A *compact Lie group* G is a compact differentiable manifold together with a smooth multiplication map $G \times G \to G$ which gives G the structure of a group. (The inverse map is automatically smooth [8, p. 22].) For instance, G might be the multiplicative group of unit complex numbers, the multiplicative group of unit quaternions, or the group SO_n of rotations in Euclidean n-space. Compact Lie groups are ubiquitous in topology, algebra, and analysis. The aim of this paper is to study their basic structure from a geometric standpoint close to homotopy theory: homology groups, Euler characteristics, and Lefschetz numbers play a major role, but as much as possible we avoid infinitesimal constructions and the use of Lie algebras. There are a couple of reasons for this unusual approach. First of all, the arguments are relatively straightforward and might give some readers new insight into the structure theorems. Secondly, we are able to cover a few points which are not usually emphasized in the literature; for instance, we give an explicit calculation of the center of a connected group (8.2), a calculation of the group of components of the centralizer of a subgroup of the torus (8.4), and a lattice criterion for a connected group to split as a product (9.1). Finally, this treatment of compact Lie groups relies on the same ideas which, supported by additional machinery from homotopy theory, give structure theorems for *p-compact groups* (the homotopical analogues of compact Lie groups [10], [11], [12]). In dealing with p-compact groups, analytical objects like Lie algebras are not available.

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Organization of the paper. Section 2 introduces tori, which turn out to be exactly the connected abelian compact Lie groups. In §3 there is a discussion of Lefschetz numbers and Euler characteristics, and in §4 a proof of the existence and essential uniqueness of a maximal torus T for a compact Lie group G. Section 5 treats the Weyl group of T (or G) and shows that if G is connected the rational cohomology of the classifying space $\mathrm{B}G$ is isomorphic to the ring of invariants of the natural action of W on $\mathrm{H}^*(\mathrm{B}T; \mathbf{Q})$. It follows that W is a Coxeter group (5.16). Section 6 has a brief discussion of subgroups of maximal rank, while sections 7 and 8 collect together properties of the center of G, centralizers in G of subgroups of T, and connected components of such centralizers. Section 9 gives general conditions, involving T, under which a connected G decomposes as a product, and §10 exploits the results of §9 to show that any connected G in adjoint form can be written as a product of simple factors. The last section contains a short proof (due to J. Moore) of an algebraic theorem used in §5.

1.1. Notation and terminology. All of the manifolds we consider are smooth. A compact manifold is said to be *closed* if it does not have a boundary. A (smooth) action of a compact Lie group G on a manifold M is a group action in the ordinary sense such that the action map $G \times M \to M$ is differentiable. A homomorphism (resp. isomorphism) of compact Lie groups is a group homomorphism which is also a differentiable map (resp. a homomorphism which is also a diffeomorphism).

If G is a Lie group then T_eG denotes the tangent space to G at the identity element e. If $\phi : G \to H$ is homomorphism of Lie groups, the differential $d\phi$ gives a linear map $(d\phi)_e : T_eG \to T_eH$. Occasionally we treat a real vector space V as a (noncompact) Lie group [1, 1.4] with respect to vector addition; in this case the identity element e is the zero of V, and there is a natural identification $T_eV \cong V$.

If G is a group with subgroup K, then $\mathbf{C}_G(K)$ and $\mathbf{N}_G(K)$ denote respectively the centralizer and normalizer of K in G.

1.2. Geometric background. Of course, there are results from manifold theory and differential geometry that we have to take for granted. Most of these are mentioned in the text as they come up. Several times we refer to the fact that any closed subgroup of a compact Lie group is a compact Lie subgroup [1, 2.27], i.e., a subgroup of G which is also a compact smooth submanifold. We also use the fact that if G is a compact Lie group and H is a compact Lie subgroup, then the coset space G/H is a smooth manifold in such a way that the projection $G \to G/H$ is a smooth principal fibre bundle with structure group H

[8, p. 52]. If H is a normal subgroup of G, then G/H is a Lie group and $G \to G/H$ is a homomorphism.

Relationship to other work. All of the results in this paper are known, though some are less well-known than others. Up though §4 our arguments are roughly parallel to those of Adams [1, Ch. 4], which are themselves mostly classical [15]. At that point our treatment diverges from the customary ones. The scheme used in §5 to fashion a link between the Weyl group W and $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$ seems to be new. The analysis of centralizers in §8.4 is unusually detailed, and the geometric approach to product decompositions in §9 and §10 is one we have not seen before.

For extensive discussions of Lie groups and related topics see, for instance, [1], [20], [8], [14], [25], [17], [22], [13], or [7]. The authors would like to thank those who offered comments on a preliminary version of this paper, with special thanks to E. Dror-Farjoun.

2. Tori

2.1. Definition. A *torus* is a compact Lie group which is isomorphic to $\mathbf{R}^n/\mathbf{Z}^n \cong (\mathbf{R}/\mathbf{Z})^n$ for some $n \geq 0$. The number n is called the *rank* of the torus.

It is clear that the rank of a torus is equal to its dimension as a manifold. The following proposition explains our interest in tori.

Theorem 2.2. *Any connected compact abelian Lie group is isomorphic to a torus.*

To prove this we need the following theorem, which is a special case of Theorem 3 of [8, p. 33]; see also [1, Ch. 2].

Theorem 2.3. *Suppose that V is a real vector space of dimension n, that G is a Lie group, and that $\varphi : V \to T_eG$ is map of vector spaces (1.1). Then if $n = 1$ or G is abelian there is a unique homomorphism $\phi : V \to G$ such that $\varphi = (d\phi)_e$.*

Proof of 2.2. (See [8, p. 43, Exer. 18] or [1, 2.19].) Let G be a connected compact abelian Lie group, let $V = T_eG$, and let $\varphi : V \to V$ be the identity map. By 2.3 there is a homomorphism $\phi : V \to G$ such that $\varphi = (d\phi)_e$. The kernel $K = \phi^{-1}(e)$ is a closed subgroup of V. Since $(d\phi)_e$ is the identity map of V, the implicit function theorem implies that ϕ gives a diffeomorphism between some neighborhood U of the identity element in V and a neighborhood $\phi(U)$ of the identity element in G. In particular $K \cap U = \{e\}$, so that K is a discrete subgroup of V. Since G is connected the set $\phi(U)$ generates G [8, p. 22, Prop. 2], and so

$\phi : V \to G$ is surjective. It follows easily that ϕ is a covering map and that K is isomorphic to $\pi_1 G$. The group $K \subset V$ is finitely generated (because G is compact), abelian, and torsion free, and so $K \cong \mathbf{Z}^n$ for some n. A straightforward argument using the discreteness of K in V now shows that $n = \dim(V)$ and that a basis can be chosen for V such that K is the subgroup of V generated by the basis elements. $\square$

Proposition 2.4. *Suppose that T and T' are tori. Then the natural map*

$$\begin{aligned} \mathrm{Hom}(T, T') \to & \mathrm{Hom}(\pi_1 T, \pi_1 T') \\ & \cong \mathrm{Hom}(\mathrm{H}_1(T; \mathbf{Z}), \mathrm{H}_1(T'; \mathbf{Z})) \\ & \cong \mathrm{Hom}(\mathrm{H}^1(T'; \mathbf{Z}), \mathrm{H}^1(T; \mathbf{Z})) \end{aligned}$$

is an isomorphism. In particular, the space of homomorphisms $T \to T'$ is discrete.

2.5. Proof. This follows from the proof of 2.2, which in effect shows that if T is a torus with fundamental group π, then there is a natural isomorphism between T and the quotient $(\mathbf{R} \otimes \pi)/\pi$. $\square$

Proposition 2.6. *Suppose that T is a torus and that $f : T \to T$ is an injective homomorphism. Then f is an automorphism of T.*

Proof. Take $T = \mathbf{R}^n/\mathbf{Z}^n$. It follows from 2.5 that f lifts to a vector space homomorphism $\tilde{f} : \mathbf{R}^n \to \mathbf{R}^n$. It is easy to see that $\tilde{f}$ is injective and so, by linear algebra, surjective. This implies that f is surjective. $\square$

Proposition 2.7. *Suppose that*

$$1 \to T_1 \to G \to T_2 \to 1$$

is a short exact sequence of compact Lie groups in which both T_1 and T_2 are tori. Then G is also isomorphic to a torus.

Proof. The sequence $T_1 \to G \to T_2$ is a fibration sequence of spaces, and it is clear from the associated long exact homotopy sequence that G is connected. In order to show that G is a torus, it is enough to show that G is abelian (2.2). Since T_1 is a normal subgroup of G, the action of G on itself by conjugation gives a (continuous) homomorphism $G \to \mathrm{Aut}(T_1)$. But $\mathrm{Aut}(T_1)$ is discrete (2.4) and G is connected, so the homomorphism is trivial and T_1 is in the center of G. Now for each $g \in G$ construct a map $c_g : T_2 \to T_1$ by setting $c_g(x) = g\tilde{x}g^{-1}\tilde{x}^{-1}$, where $\tilde{x}$ is an element of G which projects to $x \in T_2$. It is easy to see that $c_g(x)$ does not depend on the choice of $\tilde{x}$, and that the correspondence $g \mapsto c_g$ gives a continuous map from G into the space of homomorphisms from

T_2 to T_1. But as before G is connected and the space of homomorphisms $T_2 \to T_1$ is discrete, so the homomorphism c_g does not depend on g. Checking what happens when $g \in G$ is the identity element shows that c_g is the trivial homomorphism for each $g \in G$. This immediately implies that G is abelian. □

Proposition 2.8. *Suppose that G is a compact Lie group which is not zero dimensional. Then G contains a nontrivial Lie subgroup isomorphic to a torus.*

Proof. Since G is not zero dimensional, T_eG is not trivial. Let $\varphi : \mathbf{R} \to T_eG$ be a nontrivial homomorphism. By 2.3 there is a homomorphism $\phi : \mathbf{R} \to G$ with $\varphi = (d\phi)_e$. Let K be the closure in G of $\phi(\mathbf{R})$; it is easy to see that K is a nontrivial subgroup of G, and that K is both abelian and connected (because $\mathbf{R}$ is). By 1.2 the group K is a Lie subgroup of G, and by 2.2, K is a torus. □

For a nice proof of the following proposition see [1, 4.3].

Lemma 2.9. *If T is a torus, then there is an element $x \in T$ such that the cyclic subgroup $\langle x \rangle$ is dense in T.*

3. Lefschetz numbers and Euler characteristics

Many of the arguments in this paper are based on elementary properties of Lefschetz numbers and Euler characteristics. In this section we describe these properties. Recall that a compact smooth manifold (with or without boundary) can be triangulated as a finite simplicial complex.

3.1. Definition. Suppose that X is a finite CW-complex. The *Euler characteristic of* X, denoted $\chi(X)$, is the sum $\sum_i (-1)^i \operatorname{rk}_{\mathbf{Q}} \mathrm{H}^i(X;\mathbf{Q})$. If $f : X \to X$ is a map, the *Lefschetz number* of f, denoted $\Lambda(f)$, is the sum $\sum_i (-1)^i \operatorname{tr}_{\mathbf{Q}} \mathrm{H}^i(f;\mathbf{Q})$.

Here $\operatorname{rk}_{\mathbf{Q}} \mathrm{H}^i(X;\mathbf{Q})$ denotes the dimension of the rational vector space $\mathrm{H}^i(X;\mathbf{Q})$, and $\operatorname{tr}_{\mathbf{Q}} \mathrm{H}^i(f;\mathbf{Q})$ the trace of the endomorphism of this vector space induced by f. It is clear that the Euler characteristic of X is equal to the Lefschetz number of the identity map of X, and that the Lefschetz number of f depends only on the homotopy class of f. The following is the classical Lefschetz fixed point theorem [23, 4.7.7].

Proposition 3.2. *Suppose that X is a finite CW-complex and that $f : X \to X$ is a map with no fixed points. Then $\Lambda(f) = 0$.*

This has an easy corollary.

Corollary 3.3. *If G is a compact Lie group, then $\chi(G) \neq 0$ if and only if G is discrete, that is, if and only if G is a finite group.*

Proof. If G is finite, then $\chi(G) = \#(G) \neq 0$. If G is not finite, then the identity component of G is nontrivial, and left multiplication by any nonidentity element in the identity component gives a self-map of G which is homotopic to the identity but has no fixed points. By 3.2, $\chi(G) = 0$. □

The Lefschetz number has the following patching property.

Proposition 3.4. *Suppose that X is a finite CW-complex which can be written as a union $A \cup_B C$, where A and C are subcomplexes of X which intersect in the subcomplex B. Let $f : X \to X$ be a map with $f(A) \subset A$ and $f(C) \subset C$ (so that $f(B) \subset B$). Then*

$$\Lambda(f) = \Lambda(f|_A) + \Lambda(f|_C) - \Lambda(f|_B)\ .$$

Proof. If $g = (g_U, g_V, g_W)$ is an endomorphism of an exact sequence $0 \to U \to V \to W \to 0$ of finite dimensional vector spaces over $\mathbf{Q}$, then $\operatorname{tr}(g_V) = \operatorname{tr}(g_U) + \operatorname{tr}(g_W)$; this is the additivity of the trace for short exact sequences, and can be proved by extending a basis for U to a basis for V and expressing g as a matrix with respect to this basis. Now f induces an endomorphism of the Mayer-Vietoris sequence

$$\to \mathrm{H}^i(X;\mathbf{Q}) \xrightarrow{j^*} \mathrm{H}^i(A;\mathbf{Q}) \oplus \mathrm{H}^i(C;\mathbf{Q}) \xrightarrow{k^*} \mathrm{H}^i(B;\mathbf{Q}) \to \mathrm{H}^{i+1}(X;\mathbf{Q}) \to$$

Let U^* be the image of j^*, V^* the image of k^* and W^* the kernel of j^*. The additivity of the trace and the definition of Lefschetz number lead directly to the following formulas. We write $\Lambda(f_{U^*})$, for instance, for $\sum(-1)^i \operatorname{tr}(f_{U^i})$, where f_{U^i} is the endomorphism of U^i induced by f.

$$\begin{aligned}
\Lambda(f|_A) + \Lambda(f|_C) &= \Lambda(f_{U^*}) + \Lambda(f_{V^*}) \\
\Lambda(f|_B) &= \Lambda(f_{V^*}) - \Lambda(f_{W^*}) \\
\Lambda(f) &= \Lambda(f_{W^*}) + \Lambda(f_{U^*})
\end{aligned}$$

The proposition is proved by subtracting the second equation from the first, and comparing the result with the third. □

3.5. Remark. Applying 3.4 to the identity map f gives the familiar additivity property of the Euler characteristic. This in turn leads to the statement that if $F \to E \to B$ is a suitable fibre bundle, for instance a smooth bundle over a compact manifold with compact fibres, then $\chi(E) = \chi(B)\chi(F)$. The argument for this involves filtering E by the inverse images E_i of the skeleta B_i of B, and proving by induction and additivity that $\chi(E_i) = \chi(B_i)\chi(F)$.

Proposition 3.6. *Suppose that M is a compact manifold, that G is a compact Lie group acting smoothly on M, and that $f : M \to M$ is a diffeomorphism provided by the action of some $g \in G$. Let N be the fixed point set of f. Then $\Lambda(f) = \chi(N)$.*

Proof. Let $H \subset G$ be the closure of the cyclic subgroup generated by g. Then H is a Lie subgroup of G (1.2), and so $N = M^H$ is a submanifold of M [24, 5.13]. Let A be a closed tubular neighborhood of N such that $f(A) = A$ (cf. [24, 5.6]), let C be the complement in M of the interior of A and let $B = A \cap C$. The space B is diffeomorphic to the total space of the normal sphere bundle of N in M. By 3.4,

$$\Lambda(f) = \Lambda(f|_A) + \Lambda(f|_C) - \Lambda(f|_B) \ .$$

By 3.2, though, $\Lambda(f|_C)$ and $\Lambda(f|_B)$ are both zero. Since the inclusion $N \to A$ is a homotopy equivalence which commutes with the action of f, and f acts as the identity on N, it is clear that $\Lambda(f|_A) = \Lambda(f|_N) = \chi(N)$. □

Remark. It is possible to show that a diffeomorphism f of a compact manifold M belongs to the action of a compact Lie group on M if and only if f is an isometry with respect to some Riemannian metric on M. The conclusion of Proposition 3.6 does not hold without some restriction on the diffeomorphism f; for instance, it is not hard to construct a diffeomorphism of the 2-sphere which is homotopic to the identity (in fact, belongs to a flow) but has only one fixed point. See [9] for a very general fixed point theorem that would cover the case of such a diffeomorphism.

Proposition 3.7. *Suppose that X is a finite CW-complex. If a torus T acts continuously on X without fixed points, then $\chi(X) = 0$.*

Proof. Let α be an element of T such that the subgroup $\langle\alpha\rangle$ generated by α is dense in T (2.9). Any point in X which is fixed by α is also fixed by the closure of $\langle\alpha\rangle$; it follows that α acts on X without fixed points. Since T is connected, the self-map of X given by the action of α is homotopic to the identity, and so the Lefschetz number of this self-map is $\chi(X)$. By 3.2, $\chi(X) = 0$. □

Proposition 3.8. *Suppose that M is a compact smooth manifold and that T is a torus which acts smoothly on M with fixed point set M^T. Then $\chi(M^T) = \chi(M)$.*

Proof. This is essentially the same as the proof of 3.6. Again M^T is a submanifold of M, and it is possible to find a closed tubular neighborhood A of M^T which is invariant under T. By 3.7 the boundary of A and the complement of the interior of A both have zero Euler characteristic. The proposition follows from the additivity of the Euler characteristic (3.5) and the fact that $\chi(A) = \chi(M^T)$. $\square$

4. The maximal torus

A *maximal torus* for a compact Lie group G is a closed subgroup T of G, isomorphic to a torus, such that $\chi(G/T) \neq 0$. The primary aim of this section is to prove the following theorem.

Theorem 4.1. *Any compact Lie group G contains a maximal torus.*

The arguments below are classical, but the particular line of reasoning we have chosen is designed to generalize to p-compact groups [10, §8]. Before attacking 4.1, we point out that maximal tori actually are maximal in an appropriate sense, and that they are unique up to conjugacy.

Lemma 4.2. *Suppose that T is a maximal torus for the compact Lie group G, and that T' is any other toral subgroup of G. Then there is an element $g \in G$ such that $T' \subset gTg^{-1}$.*

Proof. Let T' act on G/T by left translation. By 3.7, the fixed point set $(G/T)^{T'}$ is nonempty. If $g \in G$ projects to a point of $(G/T)^{T'}$, then $T' \subset gTg^{-1}$. $\square$

Theorem 4.3. *If T and T' are maximal tori in the compact Lie group G, then T and T' are conjugate; in other words, there exists an element $g \in G$ such that $gTg^{-1} = T'$.*

Proof. By 4.2 there are elements g, h in G such that $T' \subset gTg^{-1}$ and $T \subset hT'h^{-1}$. It follows from 2.6 that these two inclusions are equalities. $\square$

4.4. Remark. The *rank* of a compact Lie group G is defined to be the rank of T (2.1), where T is any maximal torus for G.

The proof of 4.1 depends on some lemmas.

4.5. Definition. An abelian subgroup A of a compact Lie group G is said to be *self-centralizing* (in G) if A equals its centralizer $\mathbf{C}_G(A)$. The subgroup A is said to be *almost self-centralizing* if A has finite index in $\mathbf{C}_G(A)$.

Proposition 4.6. *Suppose that G is a compact Lie group and that T is a toral subgroup of G. Then T is a maximal torus for G if and only if T is almost self-centralizing.*

Remark. We will show later on (5.5) that if G is connected the maximal torus is in fact self-centralizing.

Proof of 4.6. Let C denote the centralizer $\mathbf{C}_G(T)$ and N the normalizer $\mathbf{N}_G(T)$. Both C and N are closed subgroups of G and hence Lie subgroups of G (1.2). Consider the fixed point set $(G/T)^T$, where T acts on G/T by left translation. This is equal to S/T, where S is the set of all elements $g \in G$ such that $gTg^{-1} \subset T$. By 2.6, $S = N$ and $(G/T)^T$ is N/T. By 3.8, $\chi(N/T) = \chi(G/T)$. There is a principal fibration

$$C/T \to N/T \to N/C\,.$$

In this fibration the base space N/C is homeomorphic to a compact subspace of the discrete space $\mathrm{Aut}(T)$ (2.4) and is therefore a finite set, from which it is clear that (3.5)

$$\chi(G/T) = \chi(N/T) = \chi(C/T) \cdot \#(N/C)\ . \tag{4.7}$$

If T is almost self-centralizing, then C/T is a finite group and formula 4.7 shows that $\chi(G/T) \neq 0$. If T is not almost self-centralizing, then C/T is a compact Lie group with a nontrivial identity component, $\chi(C/T) = 0$ (3.3), and formula 4.7 gives that $\chi(G/T) = 0$. □

Proof of 4.1. The argument is by induction on the dimension of G. If G has dimension 0, in other words if G is finite, then the trivial subgroup is a maximal torus for G.

Suppose then that maximal tori exist for any compact Lie group of dimension less than the dimension of G. Let A be a nontrivial closed subgroup of G isomorphic to a torus (2.8), and C the centralizer $\mathbf{C}_G(A)$. Since C is a closed subgroup of G, it is a Lie subgroup of G (1.2). It is clear that A is a closed normal subgroup of C. Then

$$\dim(C/A) = \dim(C) - \dim(A) < \dim(C) \leq \dim(G),$$

so by induction we can find a maximal torus $\bar{T}$ for C/A. Let T be the inverse image of $\bar{T}$ in C. There is a short exact sequence

$$1 \to A \to T \to \bar{T} \to 1$$

and it follows from 2.7 that T is a toral subgroup of G. Since $\chi(C/T) = \chi((C/A)/\bar{T})$ is not zero, T is a maximal torus for C and hence almost self-centralizing in C (4.6). Inasmuch as T contains A, the centralizer of T in G is the same as the centralizer of T in $C = \mathbf{C}_G(A)$, so T is almost self-centralizing in G and is (4.6) a maximal torus for G. □

5. The Weyl group and the cohomology of BG

In this section we define a particular finite group associated to a compact Lie group G, called the *Weyl group*. If G is connected we compute the cohomology of BG in terms of an action of the Weyl group on a maximal torus T, and deduce from this calculation that the Weyl group has a remarkable algebraic property: its image in $\mathrm{Aut}(\mathbf{Q}\otimes\pi_1 T)$ is generated by reflections.

5.1. Definition. Let G be a compact Lie group with maximal torus T. The *Weyl group* of T in G is the quotient $\mathbf{N}_G(T)/T$.

5.2. Remark. The Weyl group W is obviously a compact Lie group (1.2); in fact, it is a finite group, because T is almost self-centralizing (4.6) and $\mathrm{Aut}(T)$ is discrete (2.4). The uniqueness property of T (4.3) implies that up to isomorphism W depends only on G, and it is sometimes convenient to take T for granted and refer to W as the Weyl group of G. Since T is abelian, the conjugation action of $\mathbf{N}_G(T)$ on T induces an action of W on T.

We begin by computing the order of the Weyl group, and showing that it is large enough to account for all of the components of G.

Proposition 5.3. *Suppose that G is a compact Lie group with maximal torus T and Weyl group W. Then $\#(W)=\chi(G/T)$.*

Proof. As in the proof of 4.6, W can be identified with the fixed point set $(G/T)^T$. The Euler characteristic calculation then comes from 3.8. □

If G is a compact Lie group, let G_e denote the identity component of G.

Proposition 5.4. *Suppose that G is a compact Lie group with maximal torus T and Weyl group W. Let N denote $\mathbf{N}_G(T)$, N' the intersection $N\cap G_e$, and $W'=N'/T$ the image of N' in W. Then the natural map $N/N'=W/W'\to G/G_e=\pi_0 G$ is an isomorphism.*

Remark. Note that the group N' above is the normalizer of T in G_e.

Proof of 5.4. By construction the map is an injection, so it is only necessary to prove that it is onto. The space G/T is diffeomorphic to a disjoint union of copies of G_e/T, one for each element of $\pi_0 G$. Since $0\neq\chi(G/T)=\#(\pi_0 G)\chi(G_e/T)$, each component X of G/T has a nonvanishing Euler characteristic. The left translation action of T on G/T restricts to an action of T on X, and by 3.7 this action has at least one fixed point. Any such fixed point gives an element of N/T which projects in $\pi_0 G$ to the component represented by X. □

To get much further we have to assume that G is connected.

Proposition 5.5. *Let G be a connected compact Lie group with maximal torus T and Weyl group W. Then T is self-centralizing in G; equivalently, the conjugation action of W on T is faithful.*

Proof. We must show that if $a \in G$ commutes with T then $a \in T$. Let $A = \langle a, T\rangle$ and consider the fixed point set

$$(G/T)^A = ((G/T)^{\langle a\rangle})^T \ .$$

Here as usual A acts on G/T by left translation. Since G is connected, the action of a on G/T is homotopic to the identity; by 3.6, $\chi((G/T)^{\langle a\rangle}) = \chi(G/T) \neq 0$. It then follows from 3.7 (applied with $M = (G/T)^{\langle a\rangle}$) that $(G/T)^A$ is not empty. If $g \in G$ projects to a point of $(G/T)^A$, then $gAg^{-1} \subset T$. By 2.6 it must be the case that $gTg^{-1} = T$, so the fact that $gAg^{-1} \subset T$ implies that $A = T$ and that $a \in T$. □

5.6. Remark. The proof of 5.5 shows that if G is connected, a is an element of G, and S is a torus in G which commutes with a, then the subgroup $\langle S, a\rangle$ of G is conjugate to a subgroup of the maximal torus T. Taking $S = \{e\}$ gives the statement that every element of G is conjugate to an element of T.

We will now look at the question of computing $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$.

Proposition 5.7. *If G is a connected compact Lie group, the rational cohomology ring $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$ is isomorphic to a polynomial algebra on a finite set of even-dimensional generators.*

5.8. Remark. The number of polynomial generators in $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$ is called the *rational rank* of G; we will show below (5.14) that this is the same as the rank (4.4) of G.

The proof of 5.7 depends upon a standard calculation.

Proposition 5.9. *Suppose that G is a connected topological group. Then the following are equivalent.*

1. $\mathrm{H}^*(G; \mathbf{Q})$ *is a finitely generated exterior algebra over* $\mathbf{Q}$.
2. $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$ *is a finitely generated polynomial algebra over* $\mathbf{Q}$.

If (1) and (2) hold and $\{n_i\}$ is the collection of degrees for any set of polynomial generators of $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$, then $\{n_i - 1\}$ is the collection of degrees for any set of exterior generators of $\mathrm{H}^*(G; \mathbf{Q})$.

Proof. The proposition can be derived from parallel applications of the Eilenberg-Moore spectral sequence [18] and the Rothenberg-Steenrod spectral sequence [21], or from a careful argument with the Serre spectral sequence (cf. [5, 13.1]). Another approach is to notice that under assumption (1) the space G has the rational homotopy type of a finite product of Eilenberg-MacLane spaces $\prod_i K(\mathbf{Q}, 2n_i - 1)$ (cf. [6, §19]). In particular $\mathbf{Q}\otimes\pi_i G$ is nonzero only for i odd, so $\mathbf{Q}\otimes\pi_i(\mathrm{B}G)$ is nonzero only for i even. It is now easy to prove by a Postnikov stage induction that any 1-connected space with rational homotopy concentrated in even dimensions is rationally equivalent to a product $\prod_i K(\mathbf{Q}, 2m_i)$ [6, 18.9, 18.12], and statement (2) follows. Conversely, given (2) it is clear that $\mathrm{B}G$ is rationally equivalent to a product $\prod K(Q, 2n_i)$, and hence that $G \sim \Omega\,\mathrm{B}G$ is rationally equivalent to $\prod K(Q, 2n_i - 1)$. □

Proof of 5.7. Given 5.9, it is only necessary to observe, as Hopf did, that the connected Hopf algebra $\mathrm{H}^*(G; \mathbf{Q})$ must for direct algebraic reasons be an exterior algebra on a finite number of odd dimensional generators [19, 7.20] [23, 5.8.12]. □

In the next two lemmas, G is a connected compact Lie group with maximal torus T. There is an inclusion map $\iota : T \to G$ and an induced map $\mathrm{B}\iota : \mathrm{B}T \to \mathrm{B}G$ of classifying spaces.

Lemma 5.10. *The rational cohomology map* $(\mathrm{B}\iota)^* : \mathrm{H}^*(\mathrm{B}G; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{B}T; \mathbf{Q})$ *is an injection.*

Proof. There is a fibration sequence

$$G/T \to \mathrm{B}T \xrightarrow{\mathrm{B}\iota} \mathrm{B}G$$

in which the fibre has the homotopy type of a finite complex. In this situation the transfer construction of Becker and Gottlieb [4] induces a map $\tau_* : \mathrm{H}^*(\mathrm{B}T; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$. The composite $\tau_* \cdot (\mathrm{B}\iota)^*$ is the self map of $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$ given by multiplication by $\chi(G/T)$. Since $\chi(G/T) \neq 0$, the map $(\mathrm{B}\iota)^*$ is injective. □

Lemma 5.11. $\mathrm{H}^*(\mathrm{B}T; \mathbf{Q})$ *is finitely generated as a module over the ring* $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$.

Remark. The proof works with T replaced by any other closed subgroup of G.

Proof of 5.11. Let R denote the ring $\mathrm{H}^*(\mathrm{B}G; \mathbf{Q})$. The rational cohomology Serre spectral sequence of the fibration $G/T \to \mathrm{B}T \to \mathrm{B}G$ is a spectral sequence of modules over R. Since $H^*(G/T; \mathbf{Q})$ is finite dimensional, the E_2-page of this spectral sequence is finitely generated over R. It follows by induction and the fact that R is noetherian that each page E_r, $r \geq 2$, is also finitely generated over R. Since the spectral sequence is concentrated in a horizontal band, $E_r = E_\infty$ for $r \gg 0$, and so E_∞ is finitely generated over R. However, E_∞ is the graded object associated to a finite filtration of $\mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$ by R-modules (the k'th filtration quotient is $E_\infty^{*,k}$). It follows that $\mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$ is finitely generated over R. □

The following algebraic statement is proved in §11.

Proposition 5.12. *Suppose that S is a connected nonnegatively graded polynomial algebra over $\mathbf{Q}$ on s indeterminates, and that R is a graded subalgebra of S which is isomorphic to a polynomial algebra on r indeterminates. Assume that S is finitely generated as a module over R. Then $r = s$, and S is free as a module over R.*

We need one general lemma about fibrations.

Lemma 5.13. *Suppose that $F \to E \to B$ is a fibration sequence such that B is 1-connected and the rational cohomology groups of all three spaces are finite dimensional in each degree. Suppose that $\mathrm{H}^*(E;\mathbf{Q})$ is free as a module over $\mathrm{H}^*(B;\mathbf{Q})$. Then the natural map*

$$\mathbf{Q} \otimes_{\mathrm{H}^*(B;\mathbf{Q})} \mathrm{H}^*(E;\mathbf{Q}) \to \mathrm{H}^*(F;\mathbf{Q})$$

is an isomorphism.

Proof. The quickest way to prove this is with the Eilenberg-Moore spectral sequence [18]. It is also possible to use multiplicative structures and a careful filtration argument to show that the Serre spectral sequence of the fibration collapses (otherwise $\mathrm{H}^*(E;\mathbf{Q})$ could not be free as a module over $\mathrm{H}^*(B;\mathbf{Q})$). The desired isomorphism follows by inspection. □

If G is a compact Lie group with maximal torus T and Weyl group W, then by construction the automorphisms of T given by the action of W on T (5.2) extend to inner automorphisms of G. Since inner automorphisms of a group act trivially on the cohomology of the classifying space, the restriction map $\mathrm{H}^*(\mathrm{B}G;\mathbf{Q}) \to \mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$ lifts to a map $\mathrm{H}^*(\mathrm{B}G;\mathbf{Q}) \to \mathrm{H}^*(\mathrm{B}T;\mathbf{Q})^W$.

Theorem 5.14. *Let G be a connected compact Lie group with maximal torus T and Weyl group W. Then*

1. *the conjugation action of W on $\mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$ is faithful,*
2. *the rank of T (2.1) is equal to the rational rank of G (5.8), and*
3. *the natural map $\mathrm{H}^*(\mathrm{B}G;\mathbf{Q}) \to \mathrm{H}^*(\mathrm{B}T;\mathbf{Q})^W$ is an isomorphism.*

Proof. By 5.5 the conjugation map $W \to \mathrm{Aut}(T)$ is a monomorphism, and so (by 2.4) W acts faithfully on $\mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$.

Let $R = \mathrm{H}^*(\mathrm{B}G;\mathbf{Q})$ and $S = \mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$. Then R is a polynomial ring on r generators, where r is the rational rank of G, and S is a polynomial ring on s generators, where s is the rank of T. The map $R \to S^W \to S$ is injective (5.10) and makes S into a finitely generated module over R (5.11). By 5.12, S is actually a *free* module over R, on, say, t generators. By 5.13 there is an isomorphism $\mathrm{H}^*(G/T; \mathbf{Q}) \cong \mathbf{Q} \otimes_R S$. It is clear from this that $\mathrm{H}^*(G/T, \mathbf{Q})$ is concentrated in even degrees, that $t = \chi(G/T)$, and, by 5.3, that $t = \#(W)$. By construction it is clear that t is the extension degree of the graded fraction field $F(S)$ of S over the graded fraction field $F(R)$ of R. Since $F(R) \subset F(S)^W$, the degree of $F(S)$ over $F(R)$ equals $\#(W)$, and W acts faithfully on $F(S)$, Galois theory shows that $F(R) = F(S)^W$ and hence that $S^W \subset F(R)$. The extension $R \subset S^W$ is integral in the sense of [2, 5.1]; in other words, for each $x \in S^W$ the ring $R[x]$ is finitely generated as an R-module. (One way to see this is to observe that R is noetherian and that $R[x]$ is contained in the finitely generated R-module S.) The isomorphism $R \cong S^W$ now follows from the fact that a polynomial algebra such as R is integrally closed in its field of fractions $F(R)$ [2, p. 63]. □

We are now in a position to the study the fundamental algebraic property of a Weyl group.

5.15. Definition. Suppose that V is a finite dimensional vector space over $\mathbf{Q}$. An element α of $\mathrm{Aut}(V)$ is said to be a *reflection* if it pointwise fixes a codimension one subspace of V. A subgroup W of $\mathrm{Aut}(V)$ is said to be a *reflection subgroup* if W is finite and W is generated by the reflections it contains.

In the situation of 5.15, an element $\alpha \in \mathrm{Aut}(V)$ of finite order is a reflection if and only if there is a basis for V with respect to which α has the form $\mathrm{diag}(-1, 1, \ldots, 1)$. In particular, α must have order 2.

Theorem 5.16. *Suppose that G is a connected compact Lie group with maximal torus T and Weyl group W, and let $V = \mathbf{Q} \otimes \pi_1 T$. Then the conjugation action of W on T presents W as a reflection subgroup of* $\mathrm{Aut}(V)$.

This is a consequence of a classical algebraic theorem; the part we use is due to Coxeter, the converse to Chevalley. See [16, 3.11] for a proof; note that the argument given there works over $\mathbf{Q}$ as well as over $\mathbf{R}$.

Theorem 5.17. *Suppose that V is a finite dimensional vector space over $\mathbf{Q}$ and W is a finite group of automorphisms of V. Let $V^\#$ be the dual space of V,* $\mathrm{Sym}(V^\#)$ *the symmetric algebra on $V^\#$, and* $\mathrm{Sym}(V^\#)^W$ *the fixed point set of the natural action of W on* $\mathrm{Sym}(V^\#)$.

Then $\mathrm{Sym}(V^{\#})^W$ *is isomorphic to a polynomial algebra over* $\mathbf{Q}$ *if and only if* W *is generated by reflections.*

Proof of 5.16. See 5.5 and 2.4 for the fact that map $W \to \mathrm{Aut}(V)$ is a monomorphism. The ring $\mathrm{Sym}(V^{\#})$ is naturally isomorphic to $\mathrm{H}^*(\mathrm{B}T;\mathbf{Q})$. By 5.14 and 5.7, $\mathrm{Sym}(V^{\#})^W \cong \mathrm{H}^*(\mathrm{B}G;\mathbf{Q})$ is a polynomial algebra. By 5.17, the group W (considered as a subgroup of $\mathrm{Aut}(V)$) is generated by reflections. □

6. Maximal rank subgroups

In this section we study a particular kind of subgroup of a compact Lie group G; these arise later on as centralizers of subgroups of G which are contained in a maximal torus.

6.1. Definition. A closed subgroup H of a compact Lie group G is said to have *maximal rank* if H contains a maximal torus for G.

It is not hard to see from 2.6 that a closed subgroup H of G is of maximal rank if and only if the rank of H (4.4) is the same as the rank of G; in this case any maximal torus for H is also a maximal torus for G. If H has maximal rank in G, then the Weyl group of H can be identified with a subgroup of the Weyl group of G.

Theorem 6.2. *Suppose that* G *is a connected compact Lie group and that* H *is a closed connected subgroup of maximal rank. Then* $H = G$ *if and only if the Weyl group of* H *is equal to the Weyl group of* G.

The proof we give relies on a simple dimension calculation. Suppose that V is a vector space over $\mathbf{Q}$ of rank r and that W is a reflection subgroup (5.15) of $\mathrm{Aut}(V)$. Let $V^{\#}$ denote the dual of V, and regard the symmetric algebra $\mathrm{Sym}(V^{\#})$ as a graded algebra in which $V^{\#}$ has degree 2. The fixed point subalgebra $\mathrm{Sym}(V^{\#})^W$ is isomorphic to a graded polynomial algebra on r generators of degrees, say, $d_1, \ldots, d_r$ (5.17). (Up to permutation the sequence $d_1, \ldots, d_r$ is independent of the choice of polynomial generators for $\mathrm{Sym}(V^{\#})^W$.) Define the integer $\delta(W)$, called the *dimension* associated to W, by

$$\delta(W) = \sum_{i=1}^{r}(d_i - 1) = \left(\sum_{i=1}^{r} d_i\right) - r\ .$$

Lemma 6.3. *Let* G *be a connected compact Lie group with maximal torus* T *and Weyl group* W*, where* W *is considered as a reflection subgroup of* $\mathrm{Aut}(\mathbf{Q}\otimes\pi_1 T)$ *(5.16). Then* $\delta(W) = \dim(G)$.

Proof. This is a combination of 5.14(3) and 5.9. Note that since G is an orientable (in fact parallelizable) compact manifold, $\dim(G)$ is equal to the largest integer i such that $\mathrm{H}^i(G;\mathbf{Q}) \neq 0$. $\square$

Proof of 6.2. By 6.3, $\dim(H) = \dim(G)$. Since G and H are closed manifolds and H is a submanifold of G, H must be equal to G. $\square$

6.4. Remark. Suppose that W_i is a reflection subgroup of $\mathrm{Aut}(V_i)$, $i =$ 1, 2. Then $W_1 \times W_2$ is a reflection subgroup of $\mathrm{Aut}(V_1 \times V_2)$, and there is an isomorphism

$$\mathrm{Sym}(V_1^\# \times V_2^\#)^{W_1\times W_2} \cong \mathrm{Sym}(V_1^\#)^{W_1} \otimes \mathrm{Sym}(V_2^\#)^{W_2} .$$

From this it follows that $\delta(W_1 \times W_2) = \delta(W_1) + \delta(W_2)$.

7. The center and the adjoint form

In this short section we prove the following two propositions.

Proposition 7.1. *If G is a connected compact Lie group, then the center of G is the intersection of all maximal tori in G.*

Proposition 7.2. *Suppose that G is a connected compact Lie group and that C is the center of G. Then the center of G/C is trivial.*

Remark. In the situation of 7.2, the quotient G/C is said to be the *adjoint form* of G. The name comes from the fact that G/C is the image of G in the *adjoint representation*, which is the action of G on T_eG induced by the conjugation action of G on itself. A connected compact Lie group is said to be *in adjoint form* if its center is trivial.

The above propositions depend on a result which is interesting in its own right.

Proposition 7.3. *Suppose that G is a connected compact Lie group and that S is a torus in G. Then the centralizer $\mathbf{C}_G(S)$ is connected.*

Proof. Let a be an element in $\mathbf{C}_G(S)$; we have to show that a is in the identity component of $\mathbf{C}_G(S)$. Let T be a maximal torus for G and let A be the subgroup $\langle a, S\rangle$ of G. The argument of 5.5 shows that A is conjugate to a subgroup of T, or, equivalently, that there is some conjugate T' of T which contains both S and a. Since $T' \subset \mathbf{C}_G(S)$ is connected, a must be in the identity component of $\mathbf{C}_G(S)$. $\square$

Proof of 7.1. Suppose that x is a central element of G and that T is a maximal torus for G. By 5.6, there exists some conjugate T' of T which contains x. Since x is central, T itself contains x, and so $x \in \cap_T T$.

Suppose on the other hand that x is not central in G. Then there exists an element $y \in G$ which does not commute with x, and (by

5.6) some maximal torus T which contains y. Clearly $x \notin T$, so that $x \notin \cap_T T$. □

Proof of 7.2. Let T be a maximal torus for G; by 7.1, T contains C. By 2.2, T/C is a torus, and the diffeomorphism

$$(G/C)/(T/C) \simeq G/T$$

shows that $\chi((G/C)/(T/C))$ is nonzero and thus that T/C is a maximal torus for G/C. Suppose that x is a nonidentity element of G/C and that $\tilde{x} \in G$ is an element which projects to x. Since $\tilde{x} \notin C$, there is a maximal torus T' for G which does not contain $\tilde{x}$ (7.1). Then T'/C is a maximal torus for G/C which does not contain x, and so x does not belong to the center of G/C. □

For future reference we record the following result.

Lemma 7.4. *If G is a connected compact Lie group, then any finite normal subgroup of G is central. In particular, if G is in adjoint form then G has no nontrivial finite normal subgroups.*

Proof. Let A be a finite normal subgroup of G. Since G is connected and A is discrete, the conjugation action of G on A must be trivial, so that A is in the center of G. □

8. Calculating the center

In a sense the previous section gives a formula for the center of a connected compact Lie group G: it is the intersection of all of the maximal tori in G. This is awkward to work with, and so in this section we calculate the center of G in terms of a *single* maximal torus and its normalizer. We do this by giving a general analysis of the centralizer $\mathbf{C}_G(A)$ of an arbitrary subgroup A of a maximal torus. In particular, we determine the group of connected components of $\mathbf{C}_G(A)$ (8.6).

It is necessary to set up some notation before stating the main theorems of the section.

8.1. Definition. Suppose that G is a connected compact Lie group with maximal torus T and Weyl group W. If $s \in W$ is a reflection (with respect to the conjugation action of W on $\mathbf{Q} \otimes \pi_1 T$) then

1. the *fixed point subgroup* $F(s)$ is the fixed set of the action of s on T,
2. the *singular torus* $H(s)$ is the identity component of $F(s)$,
3. the *singular coset* $K(s)$ is the subset of T given by elements of the form x^2, as x runs through the elements of $\mathbf{N}_G(T)$ which project to s in W, and

4. the *singular set* $\sigma(s)$ is the union $H(s) \cup K(s)$.

Theorem 8.2. *Suppose that G is a connected compact Lie group with maximal torus T and Weyl group W. Then the center of G is equal to the intersection $\cap_s \sigma(s)$, where s runs through the reflections in W.*

8.3. Remark. The subtlety in 8.2 has to do with determining the group of components of the center of G; it follows pretty directly from 8.2 that the identity component of the center is the same as the identity component of the fixed point set T^W, or equivalently, the same as the identity component of the center of $\mathbf{N}_G(T)$.

Theorem 8.2 follows from a more general calculation.

Theorem 8.4. *Suppose that G is a connected compact Lie group with maximal torus T and Weyl group W, and that A is a subgroup of T. Let G' be the identity component of $\mathbf{C}_G(A)$. Then G' is of maximal rank (6.1) in G, and the Weyl group W' of G' is the subgroup of W generated by the reflections $s \in W$ with $A \subseteq \sigma(s)$. The group W' contains no other reflections from W.*

In the situation of 8.4, describing the Weyl group of $\mathbf{C}_G(A)$ itself is quite a bit easier than describing the Weyl group of its identity component.

Proposition 8.5. *Suppose that G is a compact Lie group with maximal torus T and Weyl group W, and that A is a subgroup of T. Then $\mathbf{C}_G(A)$ is of maximal rank in G, and the Weyl group of $\mathbf{C}_G(A)$ is the subgroup of W consisting of those elements which under the conjugation action of W on T leave A pointwise fixed.*

Proof. Clearly $\mathbf{C}_G(A)$ contains a maximal torus for G, namely T itself. Calculating the Weyl group of $\mathbf{C}_G(A)$ amounts to noticing that the normalizer of T in $\mathbf{C}_G(A)$ is the centralizer of A in $\mathbf{N}_G(T)$. □

8.6. Remark. Let G, T, W, and A be as in 8.4. Combining 8.5 with 8.4 gives a formula for the finite group $\pi_0\mathbf{C}_G(A)$. Namely (see 5.4), $\pi_0\mathbf{C}_G(A)$ is isomorphic to the quotient $W(A)/W_e(A)$, where $W(A)$ is the group of elements in W which under conjugation leave A pointwise fixed, and $W_e(A)$ is the subgroup of $W(A)$ generated by the reflections $s \in W$ with $A \subseteq \sigma(s)$.

8.7. **Relatives of the singular set**. We now investigate the subsets of T defined in 8.1. First of all note, that $H(s)$ is a subgroup of $F(s)$ and that the index of $H(s)$ in $F(s)$ is either 1 or 2. This follows from the fact that (since s is a reflection) the quotient $T/H(s)$ is a circle

$C = \mathbf{R}/\mathbf{Z}$ on which s acts by inversion, so that there is an exact sequence

$$0 \to H(s) \to F(s) \to C^{\langle s\rangle} \to \cdots .$$

in which the group $C^{\langle s\rangle}$ is $Z/2$.

Next, since an element of $\mathbf{N}_G(T)$ representing s commutes with its own square, $K(s)$ is a subset of $F(s)$.

Finally, as the name suggests, the singular coset $K(s)$ is a coset of $H(s)$ in T. To see this, suppose that x and y are two elements of $\mathbf{N}_G(T)$ which project to $s \in W$. Then $x = ya$ for some $a \in T$, and so $x^2 = y^2\nu(a)$, where $\nu(a) = a\sigma(a)$. The image of $\nu : T \to T$ is a connected subgroup of T which is pointwise fixed by s and so is contained in $H(s)$. It follows that x^2 and y^2 are in the same coset of $H(s)$. Conversely, if x projects to s and $a \in H(s)$ then (because $H(s)$ is a divisible group) there exists $b \in H(s)$ such that $\nu(b) = b^2 = a$; then $y = xb$ also projects to s and $y^2 = (xb)^2 = x^2\nu(b) = x^2a$.

These remarks imply that the singular set $\sigma(s) = H(s) \cup K(s)$ equals either $H(s)$ or $F(s)$; in particular, $\sigma(s)$ is a subgroup of T. Here are the three possible alignments of these groups, together with examples of compact Lie groups in which the possibilities are realized. In each case the Weyl group is $\mathbf{Z}/2$, so there is only one choice for s.

1. $H(s) \subsetneq \sigma(s) = F(s)$: $SU(2)$.
2. $H(s) = \sigma(s) \subsetneq F(s)$: $SO(3)$.
3. $H(s) = \sigma(s) = F(s)$: $U(2)$

The following three lemmas provide the key information we need about the above subgroups of T. In all three lemmas G is a connected compact Lie group with maximal torus T and Weyl group W; $s \in W$ is a reflection.

Lemma 8.8. *Suppose that K is one of the groups $H(s)$, $\sigma(s)$, or $F(s)$. Then $\mathbf{C}_G(K)$ is of maximal rank in G, and the Weyl group of $\mathbf{C}_G(K)$ is the subgroup $\langle s\rangle$ of W.*

Proof. The Weyl group W' of $\mathbf{C}_G(K)$ is a subgroup of W which contains $\langle s\rangle$ (8.5), acts faithfully by conjugation on T (5.5), and fixes $H(s)$ pointwise. Let r denote the rank of T, and M the group of $r \times r$ invertible matrices over $\mathbf{Z}$ which agree with the identity matrix in the first $(r-1)$ columns. By 2.4, W' is isomorphic to a subgroup of M. It is an exercise to see that any finite subgroup of M is isomorphic via the determinant map to a subgroup of $\{\pm 1\}$, so W' cannot be larger than $\langle s\rangle$. □

Lemma 8.9. *The group $\mathbf{C}_G(\sigma(s))$ is connected.*

Proof. Let $G' = \mathbf{C}_G(H(s))$; this group is connected by 7.3 and has Weyl group $\langle s \rangle$ (8.8). If $\sigma(s) = H(s)$ then $\mathbf{C}_G(\sigma(s)) = G'$ is connected. Suppose that $\sigma(s) \neq H(s)$. Choose $\tilde{s} \in \mathbf{N}_{G'}(T)$ projecting to $s \in W$, and let $x = \tilde{s}^2 \in K(s) = \sigma(s) \setminus H(s)$ (see 8.7). Then $\mathbf{C}_{G'}(x) = \mathbf{C}_G(\sigma(s))$ and the Weyl group of $\mathbf{C}_{G'}(x)$ is $\langle s \rangle$ (8.5). In order to show that $\mathbf{C}_G(\sigma(s))$ is connected it is enough to show that $\tilde{s}$ belongs to the identity component of $\mathbf{C}_{G'}(x)$ (5.4). This follows from the fact that $\tilde{s}$ is contained inside some maximal torus T' of G' (5.6); T' is then a connected abelian subgroup of G' which contains both $\tilde{s}$ and $x = \tilde{s}^2$. □

Lemma 8.10. *If $F(s) \neq \sigma(s)$ then $\mathbf{C}_G(F(s))$ is not connected.*

Proof. If $F(s) \neq \sigma(s)$ then $K(s) = H(s)$, so it is possible to choose an element $\tilde{s}$ in $\mathbf{N}_G(T)$ which projects to $s \in W$ and has $\tilde{s}^2 = e$. Note that $H(s)$ is divisible and therefore injective as an abelian group, so there is an abelian group isomorphism

$$T \cong H(s) \times T/H(s) \cong H(s) \times \mathbf{R}/\mathbf{Z} \ .$$

Since $\#(F(s)/H(s)) = 2$, it follows immediately that $F(s)$ contains all of the elements of exponent 2 in T. In particular, all of the elements of exponent 2 in $\mathbf{C}_G(F(s)) \cap T$ are central. However $\tilde{s} \in \mathbf{C}_G(F(s))$, $\tilde{s} \notin T$, and $\tilde{s}$ has order 2. Clearly $\tilde{s}$ is not conjugate in $\mathbf{C}_G(F(s))$ to any element of T, which by 5.6 and 8.5 implies that $\tilde{s}$ lies outside the identity component of $\mathbf{C}_G(F(s))$. □

Proof of 8.4. By 5.16, W' is a subgroup of W which is generated by reflections; it remains to determine which reflections $s \in W$ actually belong to W'.

Suppose that $A \subseteq \sigma(s)$. By 8.9 there is a representative $\tilde{s} \in \mathbf{N}_G(T)$ for s which lies in the connected group $\mathbf{C}_G(\sigma(s))$ and therefore in the identity component of G'. Consequently, $s \in W'$.

Suppose that $A \not\subseteq \sigma(s)$. If $A \not\subseteq F(s)$ then s does not even belong to the Weyl group of $\mathbf{C}_G(A)$ (8.5), so that $s \notin W'$. If $A \subseteq F(s)$, the $H(s) = \sigma(s) \subsetneq F(s)$ and $\langle A, H(s) \rangle = F(s)$. Suppose that $s \in W'$; we will show that this leads to a contradiction. If $s \in W'$, then there exists an element $\tilde{s} \in \mathbf{N}_{G'}(T) \subseteq \mathbf{N}_G(T)$ which projects to $s \in W$. Since $\tilde{s}$ centralizes $H(s)$, $\tilde{s}$ belongs to the connected (7.3) subgroup $\mathbf{C}_{G'}(H(s))$ of $\mathbf{C}_G(F(s))$. By 8.10, this is impossible. □

Proof of 8.2. Let C denote $\cap_s \sigma(s)$ and Z the center of G; by 7.1, $Z \subset T$. Suppose that A is a subgroup of T. If $A \not\subseteq C$, then the Weyl group of the identity component of $\mathbf{C}_G(A)$ is strictly smaller than W (8.4), so that $A \not\subseteq Z$. On the other hand, by 8.4 the Weyl group of the identity component of $\mathbf{C}_G(C)$ contains all of the reflections in W, by 5.16 this

Weyl group is equal to W, and so, by 6.2, $\mathbf{C}_G(C) = G$ and $C \subset Z$. It follows that $C = Z$. □

Remark. Let $\Sigma = \cup_s \sigma(s)$, where s runs through the reflections in W; this is the *singular subset* of T. The reader may enjoy showing that Σ is the set of all elements in T which are contained in more than one maximal torus of G; equivalently, $T \setminus \Sigma$ is the set of all elements $x \in T$ with the property that the identity component of $\mathbf{C}_G(x)$ is equal to T.

9. Product Decompositions

Let G be a compact Lie group with maximal torus T and Weyl group W. The finitely generated free abelian group $L = \pi_1 T$ has a natural action of W; from now on we will call this W-module the *dual weight lattice* of G. (The name comes from the fact that in representation theory the module $\mathrm{Hom}(L, \mathbf{Z})$ is called the *weight lattice* of G.) If G can be written as a product $\prod_i G_i$, where each G_i is a connected compact Lie group with Weyl group W_i and dual weight lattice L_i, then W is isomorphic to the product group $\prod_i W_i$ and L is isomorphic to the product module $\prod_i L_i$. In other words, the product decomposition of the group G is reflected in a product decomposition of the W-module L. We will say that a product decomposition of L which arises in this way is *realized by a product decomposition of* G. The main theorem of this section is the following one.

Theorem 9.1. *Suppose that G is a connected compact Lie group with maximal torus T, Weyl group W, and dual weight lattice $L = \pi_1 T$. Then any product decomposition of L as a module over W can be realized by a product decomposition of G.*

The first step is to prove something like 9.1 in the special case in which one of the required factors of G is a torus.

Proposition 9.2. *Let G be a connected compact Lie group with Weyl group W and dual weight lattice L. Suppose that $L \cong L_1 \times L_2$ is a product decomposition of L as a module over W, and that the action of W on L_2 is trivial. Then this product decomposition of L is realized by a unique product decomposition $G \cong G_1 \times G_2$ of G. In this decomposition, the factor G_2 is a torus.*

Remark. "Uniqueness" in 9.2 means that G_1 and G_2 are uniquely determined as subgroups of G.

Lemma 9.3. *Suppose that G is a connected compact Lie group with maximal torus T, Weyl group W and dual weight lattice L. Then the* *inclusion $T \to G$ induces a map*

$$\mathrm{H}_0(W; L) = \mathrm{H}_0(W; \pi_1 T) \to \pi_1 G$$

which is surjective and has finite kernel.

Proof. Suppose that $p : \tilde{G} \to G$ is a connected finite covering space of G. Given a point $\tilde{e} \in \tilde{G}$ above the identity $e \in G$, there is a unique Lie group structure on $\tilde{G}$ with respect to which $\tilde{e}$ is the identity and p is a homomorphism. (This follows from elementary properties of covering spaces.) The kernel K of p is a finite normal subgroup of $\tilde{G}$, therefore central (7.4), and therefore contained in any maximal torus $\tilde{T}$ of $\tilde{G}$ (7.1). The quotient $\tilde{T}/K$ is a maximal torus for G (see the proof of 7.2), and so up to conjugation we can assume that $\tilde{T}/K = T$. Then $p^{-1}(T) = \tilde{T}$, so that in particular p restricts to a connected covering of T. However $\pi_1 G = \pi_2 \mathrm{B}G$ is a finitely generated abelian group, so if the quotient $\pi_1 G / \operatorname{im}(\pi_1 T)$ were nontrivial, there would be a nontrivial epimorphism from $\pi_1 G$ to a finite group which restricts to a trivial homomorphism on $\pi_1 T$. Such a homomorphism would classify a connected finite covering space of G which restricts to a trivial (product) covering of T. Since such covers of G do not exist, it must be the case that the natural map $u : L = \pi_1 T \to \pi_1 G$ is surjective.

By the Hurewicz theorem and the homology suspension isomorphism, u can be identified with the map $\mathrm{H}_2(\mathrm{B}T; \mathbf{Z}) \to \mathrm{H}_2(\mathrm{B}G; \mathbf{Z})$ induced by $\mathrm{B}T \to \mathrm{B}G$. Now the automorphisms of T provided by conjugation with elements of $\mathbf{N}_G(T)$ extend to inner automorphisms of G; since these inner automorphisms act trivially on $\mathrm{H}_*(\mathrm{B}G; \mathbf{Z})$, u extends to a surjection

$$\bar{u} : \mathrm{H}_0(W; L) \cong \mathrm{H}_0(W; \mathrm{H}_2(\mathrm{B}T; \mathbf{Z})) \to \mathrm{H}_2(\mathrm{B}G; \mathbf{Z}) \cong \pi_1 G\ .$$

The fact that $\bar{u}$ has finite kernel is a consequence of the fact that the dual map $\mathrm{H}^2(\mathrm{B}G; \mathbf{Q}) \to \mathrm{H}^2(\mathrm{B}T; \mathbf{Q})^W$ is surjective (5.14). □

We now need some way of understanding homomorphisms from a connected compact Lie group into $\mathbf{R}/\mathbf{Z}$.

Proposition 9.4. *Suppose that G is a connected compact Lie group, and that S^1 is the circle group $\mathbf{R}/\mathbf{Z}$. Then the natural map*

$$\alpha : \mathrm{Hom}(G, S^1) \to \mathrm{Hom}(\pi_1 G, \pi_1 S^1) = \mathrm{Hom}(\pi_1 G, \mathbf{Z}) = \mathrm{H}^1(G; \mathbf{Z})$$

is a bijection.

Remark. Proposition 9.4 is a partial generalization of 2.4. We have to work to prove this here, but the analogous statement for p-compact groups is very simple to show: a homomorphism between p-compact groups is a map between their classifying spaces, which, if the target is a (p-completed) circle, amounts to a cohomology class.

The proof of 9.4 depends on de Rham cohomology theory [6, §1]. Say that a closed 1-form ω on a smooth manifold M is *integral* if $\int_\gamma \omega \in \mathbf{Z}$ for any piecewise smooth loop γ in M.

Lemma 9.5. *Suppose that G is a connected compact Lie group. Let S^1 be the circle group (with unit element 1), and let $d\theta$ be the usual left invariant 1-form on S^1. Then the assignment $f \mapsto f^*(d\theta)$ gives a bijective correspondence between smooth functions $f : G \to S^1$ with $f(e) = 1$ and closed integral 1-forms on G. A smooth map $f : G \to S^1$ with $f(e) = 1$ is a homomorphism if and only if $f^*(d\theta)$ is invariant under left translation by elements of G.*

Proof. Given a closed integral 1-form ω on G, define $\alpha : G \to S^1 = \mathbf{R}/\mathbf{Z}$ by $\alpha(g) = \int_\gamma \omega$, where γ is any piecewise smooth path in G from the identity element to g. The map α is well defined because $\int_\gamma \omega \in \mathbf{Z}$ for any loop γ in G. Then $\alpha(e) = 1$, and $\alpha^*(d\theta) = \omega$. If $\beta : G \to S^1$ is any other smooth function with $\beta^*(d\theta) = \omega$, then $(\alpha/\beta)^*(d\theta) = 0$ and α/β is a constant function; if $\beta(e) = 1$, then $\beta = \alpha$.

If $f : G \to S^1$ is a homomorphism, $f^*(d\theta)$ is a left invariant closed integral 1-form on G because $d\theta$ is an left invariant closed integral 1-form on S^1. Conversely, suppose that ω is a left invariant closed integral 1-form on G. Let g and g' be two elements of G with corresponding paths γ and γ' connecting them to the identity element. Denote by $\gamma + g\gamma'$ the path obtained by splicing γ to $g \cdot \gamma'$; this is a path connecting gg' to the identity. Then (with α as above)

$$\alpha(gg') = \int_{\gamma + g\gamma'} \omega = \int_\gamma \omega + \int_{g\gamma'} \omega = \alpha(g) + \alpha(g')$$

where the last equality follows from the left invariance of ω. In particular, α is a homomorphism. □

Remark. We leave it as an exercise to check that two maps $f_1, f_2 : G \to S^1$ are homotopic if and only if the 1-forms $f_1^*(d\theta)$ and $f_2^*(d\theta)$ are cohomologous.

Lemma 9.6. *Suppose that G is a connected compact Lie group acting smoothly on a compact closed manifold M, and that ω is a closed form on M. Then there exists a closed form ω', cohomologous to ω, which is invariant under translation by elements of G.*

Sketch of proof. The manifold M can be provided with a G-invariant Riemannian metric. By Hodge theory, the cohomology class represented by ω has a unique representative ω' which is harmonic with respect to the Laplace-Beltrami operator derived from the metric [25, 6.11]. Since G is connected, translation L_g by any element $g \in G$ acts as the identity on the cohomology of M, so ω' is cohomologous to $L_g^*(\omega')$. Since L_g is an isometry, $L_g^*(\omega')$ is harmonic. By the uniqueness property of ω', $L_g^*(\omega') = \omega'$.

It is also possible to construct a suitable ω' directly by averaging (i.e. integrating) the forms $L_g^*(\omega)$, $g \in G$, with respect to Haar measure on G. □

Proof of 9.4. We first show that α is monic. Let f_1 and f_2 be two distinct homomorphisms $G \to S^1$. By 9.5, the 1-forms $\omega_i = f_i^*(d\theta)$, $i = 1\,,2$ are distinct closed left-invariant 1-forms on G. The difference $\omega = \omega_1 - \omega_2$ is then a nonzero closed left-invariant 1-form. The form ω is not cohomologous to zero, since any smooth function f on G takes on a maximum value at some point $x \in G$, and $0 = (df)_x \neq (\omega)_x$. Therefore ω_1 is not cohomologous to ω_2, and so f_1 is not homotopic to f_2.

Suppose now that $f : G \to S^1$ is a map; we have to show that f is homotopic to a homomorphism. We can assume that f is smooth [6, 17.8]. Let $\omega = f^*(d\theta)$; by 9.5, it is enough to show that ω is cohomologous to a left invariant form. This is a special case of 9.6. □

Proof of 9.2. Let T be the maximal torus of G and let $T_i = (\mathbf{R} \otimes L_i)/L_i$ $(i = 1, 2)$ so that $T_G \cong T_1 \times T_2$ (see 2.5). By assumption, the conjugation action of W of T_2 is trivial (2.4). According to 9.3 the inclusion $T \to G$ induces an epimorphism

$$\mathrm{H}_0(W; L) \cong \mathrm{H}_0(W; L_1 \times L_2) \cong \mathrm{H}_0(W; L_1) \times L_2 \to \pi_1 G$$

with finite kernel. In particular, the kernel is contained inside of $\mathrm{H}_0(W; L_1)$. Let $c \in \mathrm{H}^1(G; L_2) = \mathrm{Hom}(\pi_1 G, L_2)$ be the unique cohomology class which on $\mathrm{H}_0(W; L_1)$ restricts to zero and on L_2 to the identity map. Choosing a basis for L_2 over $\mathbf{Z}$ allows c to be represented by a collection $c_1, \ldots, c_{r_2}$ of elements of $\mathrm{H}^1(G; \mathbf{Z})$ or even by a collection $\omega_1, \ldots, \omega_{r_2}$ of integral closed 1-forms on G. By 9.4 we can assume that each form ω_i is left invariant. Integrating these forms (9.5) gives a homomorphism $G \to T_2$ which restricts to the identity map $T_2 \to T_2$ (see 2.4). On the other hand, by 8.3 the group T_2 lies in the center of G, so there is a quotient homomorphism $G \to G/T_2$. The induced map $T/T_2 \to G/T_2$ is a maximal torus for G/T_2 (see the proof of 7.2). The product map $f : G \to T_2 \times G/T_2$ is a monomorphism, because $\ker(f)$ contains no elements of T (5.6).

Since the domain and range of f have the same dimension, f is an isomorphism. (One can see explicitly that the differential $(df)_e$ is invertible, deduce that f is topologically a covering map, and conclude that f is a diffeomorphism because it is monic.) This gives the necessary splitting of G.

To check the uniqueness condition, suppose that $G = G/T_2 \times T_2 \cong G_1 \times G_2$ is a product decomposition which induces the given decomposition of L. Note first that $G_2 = T_2$. (Any appropriate factor G_2 must contain T_2, and therefore equals T_2 (6.2) because, by the assumption that W acts trivially on L_2, the Weyl group of G_2 is trivial.) Specifying the first factor G_1 then amounts to choosing a section s of the projection

$$G \cong (G/T_2) \times T_2 \to G/T_2$$

and setting $G_1 = s(G/T_2)$. The requirement that the product decomposition $G \cong G_1 \times T_2$ be compatible with the product decomposition of L entails that s restrict on $T_1 \subset G/T_2$ to the given inclusion $T_1 \subset G$. However, giving such a section s amounts to giving a homomorphism $\pi_1(G/T_2) \to \pi_1 T_2$ (cf. 9.4), and uniqueness is thus a consequence of the fact that any such homomorphism is determined by its restriction to $\pi_1 T_1$ (9.3). The only one which fits the criteria in the present situation is the trivial homomorphism. □

Going from 9.2 to 9.1 requires some more information about the relationship between conjugacy in the normalizer of the torus and conjugacy in the group itself.

Lemma 9.7. *Suppose that G is a connected compact Lie group with maximal torus T. Let N denote the normalizer $\mathbf{N}_G(T)$, and let A and B be subgroups of T. Then any isomorphism $A \to B$ realized by conjugation with an element of G is also realized by conjugation with an element of N. In particular, if two elements of T are conjugate in G, then they are conjugate in N.*

Proof. Suppose that g is an element of G with $gAg^{-1} = B$, so that g realizes the isomorphism $a \mapsto gag^{-1}$ from A to B. The tori T and gTg^{-1} both contain B and so both lie in the identity component of $\mathbf{C}_G(B)$. Since T and gTg^{-1} are maximal tori in G, they are also maximal tori in $\mathbf{C}_G(B)$. By 4.3, there is an element $h \in \mathbf{C}_G(B)$ such that $h(gTg^{-1})h^{-1} = T$. Then the product hg lies in N, and realizes the same isomorphism $A \to B$ as g does. □

Lemma 9.8. *Let G be a connected compact Lie group with maximal torus T, Weyl group W, and dual weight lattice L. Suppose that*

$$\mathbf{Q} \otimes L \cong M_1 \oplus \cdots \oplus M_n$$

is a direct sum decomposition of $\mathbf{Q} \otimes L$ *as a module over* W. *Let* W_i, *be the subgroup of* W *given by the elements which act trivially on* $M_1 \oplus \cdots \oplus \hat{M}_i \oplus \cdots \oplus M_n$. *Then*

1. W_i *commutes with* W_j *if* $i \neq j$, *and*
2. *the product map* $W_1 \times \cdots \times W_n \to W$ *is an isomorphism of groups.*

Proof. It is clear from the definitions that if $x \in W_i$ and $y \in W_j$, $i \neq j$, then the commutator $[x, y]$ belongs to $W_i \cap W_j$ and thus that $[x, y]$ acts trivially on L. Since the action of W on L is faithful (5.14), the element $[x, y]$ is trivial. By elementary rank considerations, if $s \in W$ is a reflection (5.15), then $s \in W_i$ for some i. (Let s_i be the restriction of s to M_i. The rank of $(s-1)$ is one, because s is a reflection, but on the other hand it is equal to $\sum_i \mathrm{rk}(s_i - 1)$. Clearly then, s_i is the identity map for all except one value of i). Since W is generated by reflections (5.16), the indicated product map is surjective. The map is trivially a monomorphism. □

Proof of 9.1. By induction, it is enough to deal with the case in which the decomposition of L has two factors, $L \cong L_1 \times L_2$. The problem is to produce a group splitting $G \cong G_1 \times G_2$. The natural isomorphism $T \cong \mathbf{R}/\mathbf{Z} \otimes L$ (see 2.5) gives a splitting $T \cong T_1 \times T_2$, where $T_i = \mathbf{R}/\mathbf{Z} \otimes L_i$. Let H' be the centralizer of T_2 in G and K' the centralizer of T_1. Denote the quotients H'/T_2 and K'/T_1 by H and K respectively. These groups are connected (7.3), and they can be arranged in the following commutative diagram:

$$\begin{tikzcd}
T_2 \arrow[r, "k_2"] \arrow[d, "="] & T \arrow[r, "="] \arrow[d, "i_{H'}"'] & T \arrow[d, "i"'] & T \arrow[l, "="'] \arrow[d, "i_{K'}"'] & T_1 \arrow[l, "k_1"'] \arrow[d, "="'] \\
T_2 \arrow[r, "k_2'"] & H' \arrow[r] \arrow[d, "q_{H'}"'] & G & K' \arrow[l] \arrow[d, "q_{K'}"'] & T_1 \arrow[l, "k_1'"'] \\
 & H = H'/T_2 & & K = K'/T_1 &
\end{tikzcd}$$

The group T is a maximal torus for H' and for K'. The composite $T_1 \to T \to H' \to H$ is a maximal torus inclusion, as is the corresponding composite $T_2 \to K$ (see the proof of 7.2). For simplicity, then, denote T_1 by T_H and T_2 by T_K.

The lattice $L = \pi_1 T$ splits as a product $L_1 \times L_2$ as a module over the Weyl group$W_{H'}$ of H' (§6), and by the definition of H' as a centralizer, $W_{H'}$ acts trivially on the factor $L_2 = \pi_1 T_2$. By 9.2 there is a unique homomorphism $H \to H'$ which is a right inverse to $q_{H'}$ and agrees on T_H with the existing homomorphism $T_H \to H'$; there is a similar section for $q_{K'}$. These sections give direct product decompositions $H' \cong H \times T_K$ and $K' \cong K \times T_H$. Composing the sections with the inclusions $H' \to G$ and $K' \to G$ give homomorphisms $H \to G$ and $K \to G$, which allow H and K to be treated as subgroups of G. The Weyl group W_H of H is isomorphic to the subgroup of W_G which pointwise fixes T_K; similarly, W_K is isomorphic to the subgroup of W_G which pointwise fixes T_H. These two subgroups of W_G commute, and the product map $W_H \times W_K \to W_G$ is an isomorphism (9.8).

Say that a group element is *proper* if it is not the identity element. By 9.7, no proper element of T_H is conjugate in G to an element of T_K, and so by 5.6 no proper element of H is conjugate in G to an element of K. Another way of expressing this is to say that the left action of H on G/K is free, which implies the double coset space $H\backslash G/K$ is a manifold and that there is a principal fibre bundle

$$H \times K \to G \to H\backslash G/K \ .$$

It follows from dimension counting (6.4, 6.3) that $\dim(H \times K) = \dim(G)$, so that $\dim(H\backslash G/K) = 0$, $H\backslash G/K$ is a point, and the multiplication map $H \times K \to G$ is a diffeomorphism. From this it is easy to see that the natural map

$$H \cong (H \times T_K)/T_K = H'/T_K \to G/K \cong H$$

is also a diffeomorphism. It is clear that this diffeomorphism respects the left actions of T_K on H'/T_K and on G/K. Since the left action of T_K on H'/T_K is trivial, so is the left action of T_K on G/K. The kernel of the left action of K on G/K is then a normal subgroup of K which contains T_K, and so must be equal to K itself. In other words, K is a normal subgroup of G; it is immediately clear that the composite homomorphism $H \to G \to G/K$ is an isomorphism. By symmetry, H is normal in G, and it follows easily that the product map $G \to G/H \times G/K \cong H \times K$ is an isomorphism. It is routine to check that this product decomposition of G induces the original splitting of L. □

10. Splitting into simple factors

A compact Lie group is said to be *simple* if it has no nontrivial closed normal subgroups. In this section we will prove the following theorem.

Theorem 10.1. *Any connected compact Lie group with a trivial center is isomorphic to a product of simple compact Lie groups.*

The reader may enjoy using arguments like the one below to prove various related results. For instance, call a connected compact Lie group *almost simple* if it has a finite center. Then any connected compact Lie group G has a finite cover which is isomorphic to a product $T \times G'$, where T is a torus and G' is a product of almost simple groups. In particular, any 1-connected compact Lie group is isomorphic to a product of almost simple groups.

In order to derive 10.1 from 9.1 we need two auxiliary results.

Proposition 10.2. *Let G be a connected compact Lie group with Weyl group W and dual weight lattice L (§9). Then G is simple if and only if the center of G is trivial and $\mathbf{Q} \otimes L$ is irreducible as a $\mathbf{Q}[W]$-module.*

Proposition 10.3. *Let G a connected compact Lie group with Weyl group W and dual weight lattice L. Suppose that G has trivial center. Then there is a unique splitting of $\mathbf{Q} \otimes L$*

$$\mathbf{Q} \otimes L \cong M_1 + \cdots + M_n$$

as an (internal) direct sum of irreducible $\mathbf{Q}[W]$-submodules. Let $L_i = L \cap M_i$. Then $\mathbf{Q} \otimes L_i \cong M_i$, and the addition map

$$\alpha : L_1 \times \cdots \times L_n \to L$$

is an isomorphism of W-modules.

Proof of 10.1 (given 10.2 and 10.3). Let L be the dual weight lattice of G and W the Weyl group. By 10.3, L splits as an internal direct sum $L_1 + \cdots + L_n$ of W-modules, where the action of W on $\mathbf{Q} \otimes L_i$ is irreducible. By 9.1, this splitting is realized by a product decomposition $\prod_i G_i$ of G. In particular, W splits as a product $\prod_i W_i$, where W_i is the Weyl group of G_i and L_i is the dual weight lattice of G_i. The action of W on L_i factors through the projection $W \to W_i$; this implies that the action of W_i on $\mathbf{Q} \otimes L_i$ is irreducible, and so, by 10.2, that G_i is simple. □

Proof of 10.2. The fact that any simple G has the stated properties is clear from 9.1 and 10.3. Conversely, suppose that G has the stated properties, and that K is a closed normal subgroup of G. If K is discrete, then K is finite by compactness and K is in the center of G by 7.4, so by assumption K is trivial. Suppose then that K has a nontrivial identity component, and therefore a nontrivial maximal torus T_K (6.2). We can assume $T_K \subset T$, where T is a maximal torus for G (4.2). It is clear that T_K is the identity component of $T \cap K$; otherwise this identity component would be a connected abelian subgroup of K strictly larger than T_K, which is impossible. Let $L_K = \pi_1 T_K$. Since K is normal in G, the conjugation action of W on T carries T_K to itself, and so $\mathbf{Q} \otimes L_K$ is a (nontrivial) W-invariant subspace of $\mathbf{Q} \otimes L$. It follows from the irreducibility of $\mathbf{Q} \otimes L$ that $\mathbf{Q} \otimes L_K = \mathbf{Q} \otimes L$. Let

T' be the quotient T/T_K, so that there is a short exact sequence $0 \to L_K \to L \to \pi_1 T' \to 0$. Tensoring this with $\mathbf{Q}$ shows that $\mathbf{Q} \otimes \pi_1 T' = 0$ and hence, since T' is a torus, that T' is trivial. In other words, $T_K = T$, $T \subset G$, and, since K is normal in G and every element in G is conjugate to an element of T (5.6), $K = G$. □

Proof of 10.3. By elementary representation theory, $\mathbf{Q} \otimes L$ can be written as a direct sum of irreducible $\mathbf{Q}[W]$-submodules. Now the short exact sequence

$$0 \to L \to \mathbf{R} \otimes L \to T \to 0 \tag{10.4}$$

gives an exact sequence

$$0 \to L^W \to (\mathbf{R} \otimes L)^W \to T^W \to \cdots .$$

Since the center of G is trivial, the identity component of T^W is trivial (8.3). This implies that $(\mathbf{R} \otimes L)^W = 0$, hence that $(\mathbf{Q} \otimes L)^W$ is trivial, and hence that the trivial representation of W does not appear among the list $\{M_i\}$ of summands of $\mathbf{Q} \otimes L$. In particular, for each k between 1 and n there exists a reflection s_k in W which operates nontrivially on M_k (5.16). By the definition of reflection (5.15), then, M_k can up to isomorphism appear once and only once in the list $\{M_i\}_{i=1}^n$. Again by elementary representation theory, since the irreducible $\mathbf{Q}[W]$-constituents of $\mathbf{Q} \otimes L$ appear in $\mathbf{Q} \otimes L$ with multiplicity one, the expression of $\mathbf{Q} \otimes L$ as a direct sum of irreducible submodules is unique.

Now consider the map α constructed in the statement of the proposition. By definition, α is a map of modules over W. Since α fits into a commutative diagram

$$\begin{tikzcd}
L_1 \times \cdots \times L_n \arrow[r] \arrow[d, "\alpha"'] & M_1 \times \cdots \times M_n \arrow[d, "\cong"'] \\
L \arrow[r] & \mathbf{Q} \otimes L
\end{tikzcd}$$

in which the other maps are injective, it is clear that α is injective. Let D denote $\operatorname{coker}(\alpha)$. If x is any element of $\mathbf{Q} \otimes L$ then some multiple of x lies in L; this implies that each quotient group M_i/L_i is torsion and hence that D is torsion. Homological algebra gives a short exact sequence

$$0 \to \operatorname{Tor}(\mathbf{Q}/\mathbf{Z}, D) \to \oplus_i(\mathbf{Q}/\mathbf{Z} \otimes L_i) \xrightarrow{\mathbf{Q}/\mathbf{Z} \otimes \alpha} \mathbf{Q}/\mathbf{Z} \otimes L \to 0$$

in which the kernel $\operatorname{Tor}(\mathbf{Q}/\mathbf{Z}, D)$ is isomorphic to D, so in order to show that D is zero it is enough to show that $\mathbf{Q}/\mathbf{Z} \otimes \alpha$ is injective. Note that, by the definition of L_i, the cokernel of the inclusion $L_i \to L$

is torsion free; this implies that the map $\mathbf{Q}/\mathbf{Z}\otimes L_i \to \mathbf{Q}/\mathbf{Z}\otimes L$ is a monomorphism and allows $\mathbf{Q}/\mathbf{Z}\otimes L_i$ to be treated as a subgroup of $\mathbf{Q}/\mathbf{Z}\otimes L$.

Let W_i, $i = 1, \ldots, n$ be as in 9.8, so that each reflection of W lies in W_i for a unique i. Choose $x \in \ker(\mathbf{Q}/\mathbf{Z}\otimes\alpha)$, so that $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbf{Q}/\mathbf{Z}\otimes L$ and $\sum x_i = 0$. We will be done if we can prove that each component x_k of x is trivial. We identify $\mathbf{Q}/\mathbf{Z}\otimes L$ with the subgroup of torsion elements in $\mathbf{R}/\mathbf{Z}\otimes L \cong T$ (see 10.4). Consider some component x_k of x, and pick a reflection $s \in W$ with $s \in W_i$. If $i \neq k$, then x_k belongs to the divisible subgroup $\mathbf{Q}/\mathbf{Z}\otimes L_k$ of $F(s)$, so that $x_k \in H(s)$ (8.1, 8.7). If $i = k$, then $x_k = -\sum_{j\neq k} x_j$ belongs to $\sum_{j\neq k} \mathbf{Q}/\mathbf{Z}\otimes L_j$, which is again a divisible subgroup of $F(s)$, so that $x_k \in H(s)$ in this case also. The conclusion is that $x_k \in \cap_s H(s)$, where the intersection is taken over all reflections in W. This intersection is trivial because it is contained in the center of G (8.2). $\square$

11. A theorem in commutative algebra

Suppose that k is a field, that S is a connected nonnegatively graded polynomial algebra over k on s indeterminates, and that R is a graded k-subalgebra of S which is isomorphic to a polynomial algebra over k on r indeterminates. Assume that S is finitely generated as a module over R. In this section we show that $r = s$ and that S is actually free as a module over R; this is Proposition 5.12. The proof is due to J. Moore and was explained to us by S. Halperin. The same result is true in the ungraded case, but the proof is more complicated and relies on theorems of Auslander and Buchsbaum [3] about the relationship between depth and projective dimension.

Let $T = k[x_1, \ldots, x_t]$ be a connected nonnegatively graded polynomial algebra over k (e.g. $T = S$, $T = R$ or $T = R\otimes_k S$). We make the convention that k is treated as a module over T by the augmentation $T \to T_0 = k$. Let $K(i)$ be the chain complex over $k[x_i]$ with a copy of $k[x_i]$ in dimension 0 and another copy of $k[x_i]$ (shifted up in internal grading by $|x_i|$) in dimension 1; the differential $K(i)_1 \to K(i)_0$ is multiplication by x_i. It is immediate that K_i is a free resolution of k over $k[x_i]$, and so $K = K(1)\otimes_k \cdots \otimes_k K(t)$ is a free resolution of k over T (see [26, 4.5.5]). There are a few standard lemmas.

Lemma 11.1. *Suppose that M is a module over T. Then*

1. $\mathrm{Tor}_0^T(k, M) = M/(x_1 M + \cdots + x_t M)$,
2. $\mathrm{Tor}_t^T(k, M) = \{y \in M \mid x_1 y = \cdots = x_t y = 0\}$, *and*
3. $\mathrm{Tor}_i^T(k, M) = 0$ *for* $i > t$.

Proof. These statements follow by inspection from the explicit resolution K. □

Lemma 11.2. *Let M be a graded module over T, and assume that M is zero in negative degrees. Then*

1. *M is zero if and only if $\mathrm{Tor}_0^T(k, M) = 0$, and*
2. *M is free if and only if $\mathrm{Tor}_1^T(k, M) = 0$.*

Proof. If M is not the zero module, let $m \in M$ be a nonzero element of minimal degree. It is immediate from 11.1(1) that m represents a nonzero element in $\mathrm{Tor}_0^T(k, M)$.

Suppose that $\mathrm{Tor}_1^T(k, M) = 0$. Choose a graded basis $\{\bar{y}_\alpha\}$ for $\mathrm{Tor}_0^T(k, M) = M/(x_1 M + \cdots + x_t M)$ and lift the elements $\{\bar{y}_\alpha\}$ to elements $\{y_\alpha\}$ of M. Let F be the free T-module generated by the elements $\{y_\alpha\}$, and consider the obvious map $f : F \to M$. Left exactness of Tor gives an exact sequence

$$\mathrm{Tor}_0^T(k, T) \xrightarrow{\mathrm{Tor}_0^T(k,f)} \mathrm{Tor}_0^T(k, M) \to \mathrm{Tor}_0^T(k, \mathrm{coker}(f)) \to 0 \ .$$

Since $\mathrm{Tor}_0^T(k, f)$ is by construction an isomorphism, $\mathrm{Tor}_0^T(k, \mathrm{coker}(f))$ vanishes and hence as above $\mathrm{coker}(f)$ vanishes and f is surjective. The exact fragment

$$\mathrm{Tor}_1^T(k, M) \to \mathrm{Tor}_0^T(k, \ker(f)) \to \mathrm{Tor}_0^T(k, F) \xrightarrow{\cong} \mathrm{Tor}_0^T(k, M)$$

now shows that $\mathrm{Tor}_0^T(k, \ker(f)) = 0$, so that $\ker(f) = 0$ and f is an isomorphism. □

Lemma 11.3. *Let M be a module over $R \otimes_k S = R \otimes S$. Then there are two first-quadrant spectral sequences of homological type*

$$\begin{aligned} E^2_{i,j} &= \mathrm{Tor}_i^S(k, \mathrm{Tor}_j^R(k, M)) \Rightarrow \mathrm{Tor}_{i+j}^{R\otimes S}(k, M) \\ E^2_{i,j} &= \mathrm{Tor}_i^R(k, \mathrm{Tor}_j^S(k, M)) \Rightarrow \mathrm{Tor}_{i+j}^{R\otimes S}(k, M) \end{aligned} \ .$$

Proof. These are composition-of-functors spectral sequences associated to the isomorphisms

$$k \otimes_S (k \otimes_R M) \cong k \otimes_{R\otimes S} M \cong k \otimes_R (k \otimes_S M) \ .$$

See [26, 5.6.2] and keep in mind that in our situation all rings are commutative, so that the distinction between left modules and right modules is not important. □

Proof of 5.12. Let $\{n_k\}$ be the degrees of the polynomial generators for S and $\{m_k\}$ the degrees of the generators for R. The Poincaré series $P_S(t) = \sum_i \dim_k(S_i) t^i$ is then $\prod_{k \le s} (1 - t^{m_k})^{-1}$, while $P_R(t) =$

$\prod_{k\le r}(1-t^{n_k})^{-1}$. Let N be the number of generators of S as an R-module. It is clear that for $0<t<1$ there is an inequality

$$P_R(t) \le P_S(t) \le NP_R(t)\ .$$

Since at $t=1$ the function $P_S(t)$ has a pole of order s and the function $P_R(t)$ has a pole of order r, an elementary estimate with the above inequality gives $r=s$.

Consider the two spectral sequences of 11.3 in the special case in which the module M is S itself, with both R and S acting on M by multiplication. The second spectral sequence collapses (because $M=S$ is free as a module over S) and gives (see 11.1(3))

$$\mathrm{Tor}_i^{R\otimes S}(k,S)=0,\quad i>r\ . \tag{11.4}$$

Let q be the largest integer such $\mathrm{Tor}_q^R(k,S)\neq 0$. By 11.2 we will be done if we can show that $q=0$. The second spectral sequence of 11.3 is concentrated in the rectangle $(0\le i\le s,\ 0\le j\le q)$, and the group which occupies the upper right hand corner of this rectangle on the E^2-page is $E^2_{s,q}=\mathrm{Tor}_s^S(k,\mathrm{Tor}_q^R(k,S))$. For positional reasons $E^2_{s,q}$ persists unchanged to $E^\infty_{s,q}$. It is enough for our purposes to show that $E^2_{s,q}$ is nonzero; it will follow that $\mathrm{Tor}^{R\otimes S}_{s+q}(k,S)\neq 0$ and hence, given 11.4 and the equality $r=s$ from above, that $q=0$.

Since R is noetherian and S is finitely generated over R, $\mathrm{Tor}_q^R(k,S)$ is finitely generated as a module over k. In particular, $\mathrm{Tor}_q^R(k,S)$ is nonzero in only a finite number of internal degrees. Let q' be the largest internal degree such that $\mathrm{Tor}_q^R(k,S)_{q'}\neq 0$, and let y be a nonzero element of $\mathrm{Tor}_q^R(k,S)_{q'}$. For degree reasons, $xy=0$ for each polynomial generator x of S. By 11.1(2), y represents a nonzero element of $E^2_{s,q}$. $\square$

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