Lie groups and Lie algebras

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1. Terminology and notation

1.1. Lie groups.

Definition 1.1. A Lie group is a group G, equipped with a manifold structure such that the group operations

$$\mathrm{Mult}\colon G\times G\to G,\quad (g_1,g_2)\mapsto g_1g_2$$
$$\mathrm{Inv}\colon G\to G,\quad g\mapsto g^{-1}$$

are smooth. A morphism of Lie groups G,G' is a morphism of groups $\phi\colon G\to G'$ that is smooth.

Remark 1.2. Using the implicit function theorem, one can show that smoothness of Inv is in fact automatic. (Exercise)

The first example of a Lie group is the *general linear group*

$$\mathrm{GL}(n,\mathbb{R})=\{A\in\mathrm{Mat}_n(\mathbb{R})|\ \det(A)\neq 0\}$$

of invertible $n\times n$ matrices. It is an open subset of $\mathrm{Mat}_n(\mathbb{R})$, hence a submanifold, and the smoothness of group multiplication follows since the product map for $\mathrm{Mat}_n(\mathbb{R})$ is obviously smooth.

Our next example is the orthogonal group

$$\mathrm{O}(n)=\{A\in\mathrm{Mat}_n(\mathbb{R})|\ A^TA=I\}.$$

To see that it is a Lie group, it suffices to show that $\mathrm{O}(n)$ is an embedded submanifold of $\mathrm{Mat}_n(\mathbb{R})$. In order to construct submanifold charts, we use the exponential map of matrices

$$\exp\colon\ \mathrm{Mat}_n(\mathbb{R})\to\mathrm{Mat}_n(\mathbb{R}),\quad B\mapsto\exp(B)=\sum_{n=0}^{\infty}\frac{1}{n!}B^n$$

(an absolutely convergent series). One has $\frac{d}{dt}|_{t=0}\exp(tB)=B$, hence the differential of exp at 0 is the identity $\mathrm{id}_{\mathrm{Mat}_n(\mathbb{R})}$. By the inverse function theorem, this means that there is $\epsilon>0$ such that exp restricts to a diffeomorphism from the open neighborhood $U=\{B:\ ||B||<\epsilon\}$ of 0 onto an open neighborhood $\exp(U)$ of I. Let

$$\mathfrak{o}(n)=\{B\in\mathrm{Mat}_n(\mathbb{R})|\ B+B^T=0\}.$$

We claim that
$$\exp(\mathfrak{o}(n) \cap U) = \mathrm{O}(n) \cap \exp(U),$$
so that exp gives a submanifold chart for $\mathrm{O}(n)$ over $\exp(U)$. To prove the claim, let $B \in U$. Then
$$\begin{aligned} \exp(B) \in \mathrm{O}(n) &\Leftrightarrow \exp(B)^T = \exp(B)^{-1} \\ &\Leftrightarrow \exp(B^T) = \exp(-B) \\ &\Leftrightarrow B^T = -B \\ &\Leftrightarrow B \in \mathfrak{o}(n). \end{aligned}$$
For a more general $A \in \mathrm{O}(n)$, we use that the map $\mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ given by left multiplication is a diffeomorphism. Hence, $A\exp(U)$ is an open neighborhood of A, and we have
$$A\exp(U) \cap \mathrm{O}(n) = A(\exp(U) \cap \mathrm{O}(n)) = A\exp(U \cap \mathfrak{o}(n)).$$
Thus, we also get a submanifold chart near A. This proves that $\mathrm{O}(n)$ is a submanifold. Hence its group operations are induced from those of $\mathrm{GL}(n, \mathbb{R})$, they are smooth. Hence $\mathrm{O}(n)$ is a Lie group. Notice that $\mathrm{O}(n)$ is compact (the column vectors of an orthogonal matrix are an orthonormal basis of $\mathbb{R}^n$; hence $\mathrm{O}(n)$ is a subset of $S^{n-1} \times \cdots S^{n-1} \subset \mathbb{R}^n \times \cdots \mathbb{R}^n$).

A similar argument shows that the *special linear group*
$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathbb{R}) | \ \det(A) = 1\}$$
is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$, and hence is a Lie group. The submanifold charts are obtained by exponentiating the subspace
$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in \mathrm{Mat}_n(\mathbb{R}) | \ \mathrm{tr}(B) = 0\},$$
using the identity $\det(\exp(B)) = \exp(\mathrm{tr}(B))$.

Actually, we could have saved most of this work with $\mathrm{O}(n)$, $\mathrm{SL}(n, \mathbb{R})$ once we have the following beautiful result of E. Cartan:

> **Fact:** *Every closed subgroup of a Lie group is an embedded submanifold, hence is again a Lie group.*

We will prove this very soon, once we have developed some more basics of Lie group theory. A closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ (for suitable n) is called a *matrix Lie group*. Let us now give a few more examples of Lie groups, without detailed justifications.

Examples 1.3. (a) Any finite-dimensional vector space V over $\mathbb{R}$ is a Lie group, with product Mult given by addition.

(b) Let $\mathcal{A}$ be a finite-dimensional associative algebra over $\mathbb{R}$, with unit $1_{\mathcal{A}}$. Then the group $\mathcal{A}^\times$ of invertible elements is a Lie group. More generally, if $n \in \mathbb{N}$ we can create the algebra $\mathrm{Mat}_n(\mathcal{A})$ of matrices with entries in $\mathcal{A}$, and the *general linear group*
$$\mathrm{GL}(n, \mathcal{A}) := \mathrm{Mat}_n(\mathcal{A})^\times$$
is a Lie group. If $\mathcal{A}$ is *commutative*, one has a determinant map $\det\colon \mathrm{Mat}_n(\mathcal{A}) \to \mathcal{A}$, and $\mathrm{GL}(n, \mathcal{A})$ is the pre-image of $\mathcal{A}^\times$. One may then define a *special linear group*
$$\mathrm{SL}(n, \mathcal{A}) = \{g \in \mathrm{GL}(n, \mathcal{A}) | \ \det(g) = 1\}.$$

(c) We mostly have in mind the cases $\mathcal{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Here $\mathbb{H}$ is the algebra of *quaternions* (due to Hamilton). Recall that $\mathbb{H} = \mathbb{R}^4$ as a vector space, with elements $(a, b, c, d) \in \mathbb{R}^4$ written as

$$x = a + ib + jc + kd$$

with imaginary units i, j, k. The algebra structure is determined by

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j.$$

Note that $\mathbb{H}$ is non-commutative (e.g. $ji = -ij$), hence $\mathrm{SL}(n, \mathbb{H})$ is *not* defined. On the other hand, one can define complex conjugates

$$\overline{x} = a - ib - jc - kd$$

and

$$|x|^2 := x\overline{x} = a^2 + b^2 + c^2 + d^2.$$

defines a norm $x \mapsto |x|$, with $|x_1 x_2| = |x_1||x_2|$ just as for complex or real numbers. The spaces $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ inherit norms, by putting

$$||x||^2 = \sum_{i=1}^{n} |x_i|^2, \quad x = (x_1, \ldots, x_n).$$

The subgroups of $\mathrm{GL}(n, \mathbb{R}),\ \mathrm{GL}(n, \mathbb{C}),\ \mathrm{GL}(n, \mathbb{H})$ preserving this norm (in the sense that $||Ax|| = ||x||$ for all x) are denoted

$$\mathrm{O}(n),\ \mathrm{U}(n),\ \mathrm{Sp}(n)$$

and are called the *orthogonal, unitary, and symplectic group*, respectively. Since the norms of $\mathbb{C}, \mathbb{H}$ coincide with those of $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} = \mathbb{C}^2 \cong \mathbb{R}^4$, we have

$$\mathrm{U}(n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n), \quad \mathrm{Sp}(n) = \mathrm{GL}(n, \mathbb{H}) \cap \mathrm{O}(4n).$$

In particular, all of these groups are compact. One can also define

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}), \quad \mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}),$$

these are called the *special orthogonal* and *special unitary* groups. The groups $\mathrm{SO}(n),\ \mathrm{SU}(n), \mathrm{Sp}(n)$ are often called the *classical groups* (but this term is used a bit loosely).

(d) For any Lie group G, its universal cover $\widetilde{G}$ is again a Lie group. The universal cover $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is an example of a Lie group that is not isomorphic to a matrix Lie group.

1.2. **Lie algebras.**

Definition 1.4. A Lie algebra is a vector space $\mathfrak{g}$, together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying *anti-symmetry*

$$[\xi, \eta] = -[\eta, \xi] \text{ for all } \xi, \eta \in \mathfrak{g},$$

and the *Jacobi identity*,

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0 \text{ for all } \xi, \eta, \zeta \in \mathfrak{g}.$$

The map $[\cdot, \cdot]$ is called the Lie bracket. A morphism of Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is a linear map $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ preserving brackets.

The space

$$\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}_n(\mathbb{R})$$

is a Lie algebra, with bracket the commutator of matrices. (The notation indicates that we think of $\mathrm{Mat}_n(\mathbb{R})$ as a Lie algebra, not as an algebra.)

A Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, i.e. a subspace preserved under commutators, is called a *matrix Lie algebra*. For instance,

$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in \mathrm{Mat}_n(\mathbb{R})\colon \ \mathrm{tr}(B) = 0\}$$

and

$$\mathfrak{o}(n) = \{B \in \mathrm{Mat}_n(\mathbb{R})\colon \ B^T = -B\}$$

are matrix Lie algebras (as one easily verifies). It turns out that every finite-dimensional real Lie algebra is isomorphic to a matrix Lie algebra (*Ado's theorem*), but the proof is not easy.

The following examples of finite-dimensional Lie algebras correspond to our examples for Lie groups. The origin of this correspondence will soon become clear.

Examples 1.5. (a) Any vector space V is a Lie algebra for the zero bracket.

(b) Any associative algebra $\mathcal{A}$ can be viewed as a Lie algebra under commutator. Replacing $\mathcal{A}$ with matrix algebras over $\mathcal{A}$, it follows that $\mathfrak{gl}(n, \mathcal{A}) = \mathrm{Mat}_n(\mathcal{A})$, is a Lie algebra, with bracket the commutator. If $\mathcal{A}$ is commutative, then the subspace $\mathfrak{sl}(n, \mathcal{A}) \subset \mathfrak{gl}(n, \mathcal{A})$ of matrices of trace 0 is a Lie subalgebra.

(c) We are mainly interested in the cases $\mathcal{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Define an inner product on $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ by putting

$$\langle x, y \rangle = \sum_{i=1}^{n} \overline{x}_i y_i,$$

and define $\mathfrak{o}(n)$, $\mathfrak{u}(n)$, $\mathfrak{sp}(n)$ as the matrices in $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{H})$ satisfying

$$\langle Bx, y \rangle = -\langle x, By \rangle$$

for all x, y. These are all Lie algebras called the (infinitesimal) orthogonal, unitary, and symplectic Lie algebras. For $\mathbb{R}, \mathbb{C}$ one can impose the additional condition $\mathrm{tr}(B) = 0$, thus defining the special orthogonal and special unitary Lie algebras $\mathfrak{so}(n)$, $\mathfrak{su}(n)$. Actually,

$$\mathfrak{so}(n) = \mathfrak{o}(n)$$

since $B^T = -B$ already implies $\mathrm{tr}(B) = 0$.

Exercise 1.6. Show that $\mathrm{Sp}(n)$ can be characterized as follows. Let $J \in U(2n)$ be the unitary matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Then

$$\mathrm{Sp}(n) = \{A \in \mathrm{U}(2n) | \ \overline{A} = JAJ^{-1}\}.$$

Here $\overline{A}$ is the componentwise complex conjugate of A.

Exercise 1.7. Let $R(\theta)$ denote the 2×2 rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that for all $A \in \mathrm{SO}(2m)$ there exists $O \in \mathrm{SO}(2m)$ such that OAO^{-1} is of the block diagonal form

$$\begin{pmatrix} R(\theta_1) & 0 & 0 & \cdots & 0 \\ 0 & R(\theta_2) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & R(\theta_m) \end{pmatrix}.$$

For $A \in \mathrm{SO}(2m+1)$ one has a similar block diagonal presentation, with m 2×2 blocks $R(\theta_i)$ and an extra 1 in the lower right corner. Conclude that $\mathrm{SO}(n)$ is connected.

Exercise 1.8. Let G be a connected Lie group, and U an open neighborhood of the group unit e. Show that any $g \in G$ can be written as a product $g = g_1 \cdots g_N$ of elements $g_i \in U$.

Exercise 1.9. Let $\phi\colon G \to H$ be a morphism of connected Lie groups, and assume that the differential $\mathrm{d}_e\phi\colon T_eG \to T_eH$ is bijective (resp. surjective). Show that ϕ is a covering (resp. surjective). Hint: Use Exercise 1.8.

2. The covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$

The Lie group $\mathrm{SO}(3)$ consists of rotations in 3-dimensional space. Let $D \subset \mathbb{R}^3$ be the closed ball of radius π. Any element $x \in D$ represents a rotation by an angle $||x||$ in the direction of x. This is a 1-1 correspondence for points in the interior of D, but if $x \in \partial D$ is a boundary point then $x, -x$ represent the same rotation. Letting $\sim$ be the equivalence relation on D, given by antipodal identification on the boundary, we have $D^3/\sim = \mathbb{R}P(3)$. Thus

$$\mathrm{SO}(3) = \mathbb{R}P(3)$$

(at least, topologically). With a little extra effort (which we'll make below) one can make this into a diffeomorphism of manifolds.

By definition

$$\mathrm{SU}(2) = \{A \in \mathrm{Mat}_2(\mathbb{C}) | \ A^\dagger = A^{-1}, \ \det(A) = 1\}.$$

Using the formula for the inverse matrix, we see that $\mathrm{SU}(2)$ consists of matrices of the form

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} | \ |w|^2 + |z|^2 = 1 \right\}.$$

That is, $\mathrm{SU}(2) = S^3$ as a manifold. Similarly,

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} it & -\overline{u} \\ u & -it \end{pmatrix} | \ t \in \mathbb{R}, \ u \in \mathbb{C} \right\}$$

gives an identification $\mathfrak{su}(2) = \mathbb{R} \oplus \mathbb{C} = \mathbb{R}^3$. Note that for a matrix B of this form, $\det(B) = t^2 + |u|^2$, so that det corresponds to $|| \cdot ||^2$ under this identification.

The group SU(2) acts linearly on the vector space $\mathfrak{su}(2)$, by matrix conjugation: $B \mapsto ABA^{-1}$. Since the conjugation action preserves det, we obtain a linear action on $\mathbb{R}^3$, preserving the norm. This defines a Lie group morphism from SU(2) into O(3). Since SU(2) is connected, this must take values in the identity component:

$$\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3).$$

The kernel of this map consists of matrices $A \in \mathrm{SU}(2)$ such that $ABA^{-1} = B$ for all $B \in \mathfrak{su}(2)$. Thus, A commutes with all skew-adjoint matrices of trace 0. Since A commutes with multiples of the identity, it then commutes with all skew-adjoint matrices. But since $\mathrm{Mat}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ (the decomposition into skew-adjoint and self-adjoint parts), it then follows that A is a multiple of the identity matrix. Thus $\ker(\phi) = \{I, -I\}$ is discrete. Since $\mathrm{d}_e\phi$ is an isomorphism, it follows that the map ϕ is a double covering. This exhibits $\mathrm{SU}(2) = S^3$ as the double cover of SO(3).

Exercise 2.1. Give an explicit construction of a double covering of SO(4) by SU(2) × SU(2). Hint: Represent the quaternion algebra $\mathbb{H}$ as an algebra of matrices $\mathbb{H} \subset \mathrm{Mat}_2(\mathbb{C})$, by

$$x = a + ib + jc + kd \mapsto x = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}.$$

Note that $|x|^2 = \det(x)$, and that $\mathrm{SU}(2) = \{x \in \mathbb{H}|\ \det(x) = 1\}$. Use this to define an action of SU(2) × SU(2) on $\mathbb{H}$ preserving the norm.

3. The Lie algebra of a Lie group

3.1. **Review: Tangent vectors and vector fields.** We begin with a quick reminder of some manifold theory, partly just to set up our notational conventions.

Let M be a manifold, and $C^\infty(M)$ its algebra of smooth real-valued functions. For $m \in M$, we define the tangent space T_mM to be the space of directional derivatives:

$$T_mM = \{v \in \mathrm{Hom}(C^\infty(M), \mathbb{R})|\ v(fg) = v(f)g + v(g)f\}.$$

Here $v(f)$ is local, in the sense that $v(f) = v(f')$ if $f' - f$ vanishes on a neighborhood of m.

Example 3.1. If $\gamma\colon J \to M$, $J \subset \mathbb{R}$ is a smooth curve we obtain tangent vectors to the curve,

$$\dot\gamma(t) \in T_{\gamma(t)}M, \quad \dot\gamma(t)(f) = \frac{\partial}{\partial t}|_{t=0} f(\gamma(t)).$$

Example 3.2. We have $T_x\mathbb{R}^n = \mathbb{R}^n$, where the isomorphism takes $a \in \mathbb{R}^n$ to the corresponding velocity vector of the curve $x + ta$. That is,

$$v(f) = \frac{\partial}{\partial t}|_{t=0} f(x+ta) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

A smooth map of manifolds $\phi\colon M \to M'$ defines a *tangent map*:

$$\mathrm{d}_m\phi\colon T_mM \to T_{\phi(m)}M', \quad (\mathrm{d}_m\phi(v))(f) = v(f \circ \phi).$$

The locality property ensures that for an open neighborhood $U \subset M$, the inclusion identifies $T_mU = T_mM$. In particular, a coordinate chart $\phi\colon U \to \phi(U) \subset \mathbb{R}^n$ gives an isomorphism

$$\mathrm{d}_m\phi\colon T_mM = T_mU \to T_{\phi(m)}\phi(U) = T_{\phi(m)}\mathbb{R}^n = \mathbb{R}^n.$$

Hence T_mM is a vector space of dimension $n = \dim M$. The union $TM = \bigcup_{m\in M} T_mM$ is a vector bundle over M, called the tangent bundle. Coordinate charts for M give vector bundle charts for TM. For a smooth map of manifolds $\phi\colon M \to M'$, the entirety of all maps $\mathrm{d}_m\phi$ defines a smooth vector bundle map

$$\mathrm{d}\phi\colon TM \to TM'.$$

A *vector field* on M is a derivation $X\colon C^\infty(M) \to C^\infty(M)$. That is, it is a linear map satisfying

$$X(fg) = X(f)g + fX(g).$$

The space of vector fields is denoted $\mathfrak{X}(M) = \mathrm{Der}(C^\infty(M))$. Vector fields are local, in the sense that for any open subset U there is a well-defined restriction $X|_U \in \mathfrak{X}(U)$ such that $X|_U(f|_U) = (X(f))|_U$. For any vector field, one obtains tangent vectors $X_m \in T_mM$ by $X_m(f) = X(f)|_m$. One can think of a vector field as an assignment of tangent vectors, depending smoothly on m. More precisely, a vector field is a smooth section of the tangent bundle TM. In local coordinates, vector fields are of the form $\sum_i a_i \frac{\partial}{\partial x_i}$ where the a_i are smooth functions.

It is a general fact that the commutator of derivations of an algebra is again a derivation. Thus, $\mathfrak{X}(M)$ is a Lie algebra for the bracket

$$[X, Y] = X \circ Y - Y \circ X.$$

In general, smooth maps $\phi\colon M \to M'$ of manifolds do not induce maps of the Lie algebras of vector fields (unless ϕ is a diffeomorphism). One makes the following definition.

Definition 3.3. Let $\phi\colon M \to N$ be a smooth map. Vector fields X, Y on M, N are called ϕ-related, written $X \sim_\phi Y$, if

$$X(f \circ \phi) = Y(f) \circ \phi$$

for all $f \in C^\infty(M')$.

In short, $X \circ \phi^* = \phi^* \circ Y$ where $\phi^*\colon C^\infty(N) \to C^\infty(M),\ f \mapsto f \circ \phi$.

One has $X \sim_\phi Y$ if and only if $Y_{\phi(m)} = \mathrm{d}_m\phi(X_m)$. From the definitions, one checks

$$X_1 \sim_\phi Y_1,\ X_2 \sim_\phi Y_2 \ \Rightarrow\ [X_1, X_2] \sim_\phi [Y_1, Y_2].$$

Example 3.4. Let $j\colon S \hookrightarrow M$ be an embedded submanifold. We say that a vector field X is *tangent to* S if $X_m \in T_mS \subset T_mM$ for all $m \in S$. We claim that if two vector fields are tangent to S then so is their Lie bracket. That is, the vector fields on M that are tangent to S form a Lie subalgebra.

Indeed, the definition means that there exists a vector field $X_S \in \mathfrak{X}(S)$ such that $X_S \sim_j X$. Hence, if X, Y are tangent to S, then $[X_S, Y_S] \sim_j [X, Y]$, so $[X_S, Y_S]$ is tangent.

Similarly, the vector fields vanishing on S are a Lie subalgebra.

Let $X \in \mathfrak{X}(M)$. A curve $\gamma(t),\ t \in J \subset \mathbb{R}$ is called an *integral curve* of X if for all $t \in J$,

$$\dot\gamma(t) = X_{\gamma(t)}.$$

In local coordinates, this is an ODE $\frac{\mathrm{d}x_i}{\mathrm{d}t} = a_i(x(t))$. The existence and uniqueness theorem for ODE's (applied in coordinate charts, and then patching the local solutions) shows that for any $m \in M$, there is a unique maximal integral curve $\gamma(t),\ t \in J_m$ with $\gamma(0) = m$.

Definition 3.5. A vector field X is *complete* if for all $m \in M$, the maximal integral curve with $\gamma(0) = m$ is defined for all $t \in \mathbb{R}$.

In this case, one obtains a *smooth* map

$$\Phi \colon \mathbb{R} \times M \to M,\ (t, m) \mapsto \Phi_t(m)$$

such that $\gamma(t) = \Phi_{-t}(m)$ is the integral curve through m. The uniqueness property gives

$$\Phi_0 = \mathrm{Id},\quad \Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$$

i.e. $t \mapsto \Phi_t$ is a group homomorphism. Conversely, given such a group homomorphism such that the map Φ is smooth, one obtains a vector field X by setting

$$X = \frac{\partial}{\partial t}|_{t=0} \Phi^*_{-t},$$

as operators on functions. That is, $X(f)(m) = \frac{\partial}{\partial t}|_{t=0} f(\Phi_{-t}(m))$. [1]

The Lie bracket of vector fields measure the non-commutativity of their flows. In particular, if X, Y are complete vector fields, with flows $\Phi^X_t,\ \Phi^Y_s$, then $[X, Y] = 0$ if and only if

$$\Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t.$$

In this case, $X + Y$ is again a complete vector field with flow $\Phi^{X+Y}_t = \Phi^X_t \circ \Phi^Y_t$. (The right hand side defines a flow since the flows of X, Y commute, and the corresponding vector field is identified by taking a derivative at $t = 0$.)

3.2. **The Lie algebra of a Lie group.** Let G be a Lie group, and TG its tangent bundle. For all $a \in G$, the left,right translations

$$L_a \colon G \to G,\ g \mapsto ag$$

$$R_a \colon G \to G,\ g \mapsto ga$$

are smooth maps. Their differentials at e define isomorphisms $\mathrm{d}_g L_a \colon T_g G \to T_{ag} G$, and similarly for R_a. Let

$$\mathfrak{g} = T_e G$$

be the tangent space to the group unit.

A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if

$$X \sim_{L_a} X$$

for all $a \in G$, i.e. if it commutes with L^*_a. The space $\mathfrak{X}^L(G)$ of left-invariant vector fields is thus a Lie subalgebra of $\mathfrak{X}(G)$. Similarly the space of right-invariant vector fields $\mathfrak{X}^R(G)$ is a Lie subalgebra.

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[1] The minus sign is convention, but it is motivated as follows. Let $\mathrm{Diff}(M)$ be the infinite-dimensional group of diffeomorphisms of M. It acts on $C^\infty(M)$ by $\Phi.f = f \circ \Phi^{-1} = (\Phi^{-1})^* f$. Here, the inverse is needed so that $\Phi_1.\Phi_2.f = (\Phi_1\Phi_2).f$. We think of vector fields as 'infinitesimal flows', i.e. informally as the tangent space at id to $\mathrm{Diff}(M)$. Hence, given a curve $t \mapsto \Phi_t$ through $\Phi_0 = \mathrm{id}$, smooth in the sense that the map $\mathbb{R} \times M \to M,\ (t, m) \mapsto \Phi_t(m)$ is smooth, we define the corresponding vector field $X = \frac{\partial}{\partial t}|_{t=0} \Phi_t$ in terms of the action on functions: as

$$X.f = \frac{\partial}{\partial t}|_{t=0} \Phi_t.f = \frac{\partial}{\partial t}|_{t=0} (\Phi_t^{-1})^* f.$$

If Φ_t is a flow, we have $\Phi_t^{-1} = \Phi_{-t}$.

Lemma 3.6. *The map*

$$\mathfrak{X}^L(G) \to \mathfrak{g},\ X \mapsto X_e$$

is an isomorphism of vector spaces. (Similarly for $\mathfrak{X}^R(G)$.)

Proof. For a left-invariant vector field, $X_a = (\mathrm{d}_e L_a)X_e$, hence the map is injective. To show that it is surjective, let $\xi \in \mathfrak{g}$, and put $X_a = (\mathrm{d}_e L_a)\xi \in T_aG$. We have to show that the map $G \to TG,\ a \mapsto X_a$ is smooth. It is the composition of the map $G \to G \times \mathfrak{g},\ g \mapsto (g, \xi)$ (which is obviously smooth) with the map $G \times \mathfrak{g} \to TG,\ (g, \xi) \mapsto \mathrm{d}_e L_g(\xi)$. The latter map is the restriction of $\mathrm{d}\,\mathrm{Mult}\colon TG \times TG \to TG$ to $G \times \mathfrak{g} \subset TG \times TG$, and hence is smooth. $\square$

We denote by $\xi^L \in \mathfrak{X}^L(G),\ \xi^R \in \mathfrak{X}^R(G)$ the left,right invariant vector fields defined by $\xi \in \mathfrak{g}$. Thus

$$\xi^L|_e = \xi^R|_e = \xi$$

Definition 3.7. The Lie algebra of a Lie group G is the vector space $\mathfrak{g} = T_eG$, equipped with the unique bracket such that

$$[\xi, \eta]^L = [\xi^L, \eta^L], \quad \xi \in \mathfrak{g}.$$

Remark 3.8. If you use the right-invariant vector fields to define the bracket on $\mathfrak{g}$, we get a minus sign. Indeed, note that $\mathrm{Inv}\colon G \to G$ takes left translations to right translations. Thus, ξ^R is Inv-related to some left invariant vector field. Since $\mathrm{d}_e \mathrm{Inv} = -\mathrm{Id}$, we see $\xi^R \sim_{\mathrm{Inv}} -\xi^L$. Consequently,

$$[\xi^R, \eta^R] \sim_{\mathrm{Inv}} [-\xi^L, -\eta^L] = [\xi, \eta]^L.$$

But also $-[\xi, \eta]^R \sim_{\mathrm{Inv}} [\xi, \eta]^L$, hence we get

$$[\xi^R, \zeta^R] = -[\xi, \zeta]^R.$$

The construction of a Lie algebra is compatible with morphisms. That is, we have a *functor* from Lie groups to finite-dimensional Lie algebras.

Theorem 3.9. *For any morphism of Lie groups $\phi\colon G \to G'$, the tangent map $d_e\phi\colon \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras. For all $\xi \in \mathfrak{g},\ \xi' = d_e\phi(\xi)$ one has*

$$\xi^L \sim_\phi (\xi')^L,\ \xi^R \sim_\phi (\xi')^R.$$

Proof. Suppose $\xi \in \mathfrak{g}$, and let $\xi' = \mathrm{d}_e\phi(\xi) \in \mathfrak{g}'$. The property $\phi(ab) = \phi(a)\phi(b)$ shows that $L_{\phi(a)} \circ \phi = \phi \circ L_a$. Taking the differential at e, and applying to ξ we find $(\mathrm{d}_e L_{\phi(a)})\xi' = (\mathrm{d}_a\phi)(\mathrm{d}_e L_a(\xi))$ hence $(\xi')^L_{\phi(a)} = (\mathrm{d}_a\phi)(\xi^L_a)$. That is $\xi^L \sim_\phi (\xi')^L$. The proof for right-invariant vector fields is similar. Since the Lie brackets of two pairs of ϕ-related vector fields are again ϕ-related, it follows that $\mathrm{d}_e\phi$ is a Lie algebra morphism. $\square$

Remark 3.10. Two special cases are worth pointing out.

(a) Let V be a finite-dimensional (real) vector space. A representation of a Lie group G on V is a Lie group morphism $G \to \mathrm{GL}(V)$. A representation of a Lie algebra $\mathfrak{g}$ on V is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$. The Theorem shows that the differential of any Lie group representation is a representation of its a Lie algebra.

(b) An *automorphism of a Lie group* G is a Lie group morphism $\phi\colon G \to G$ from G to itself, with ϕ a diffeomorphism. An *automorphism of a Lie algebra* is an invertible morphism from $\mathfrak{g}$ to itself. By the Theorem, the differential of any Lie group automorphism is an automorphism of its Lie algebra. As an example, $\mathrm{SU}(n)$ has a Lie group automorphism given by complex conjugation of matrices; its differential is a Lie algebra automorphism of $\mathfrak{su}(n)$ given again by complex conjugation.

Exercise 3.11. Let $\phi\colon G \to G$ be a Lie group automorphism. Show that its fixed point set is a closed subgroup of G, hence a Lie subgroup. Similarly for Lie algebra automorphisms. What is the fixed point set for the complex conjugation automorphism of $\mathrm{SU}(n)$?

4. The exponential map

Theorem 4.1. *The left-invariant vector fields ξ^L are complete, i.e. they define a flow Φ_t^ξ such that*

$$\xi^L = \frac{\partial}{\partial t}|_{t=0}(\Phi^\xi_{-t})^*.$$

Letting $\phi^\xi(t)$ denote the unique integral curve with $\phi^\xi(0) = e$. It has the property

$$\phi^\xi(t_1 + t_2) = \phi^\xi(t_1)\phi^\xi(t_2),$$

and the flow of ξ^L is given by right translations:

$$\Phi_t^\xi(g) = g\phi^\xi(-t).$$

Similarly, the right-invariant vector fields ξ^R are complete. $\phi^\xi(t)$ is an integral curve for ξ^R as well, and the flow of ξ^R is given by left translations, $g \mapsto \phi^\xi(-t)g$.

Proof. If $\gamma(t),\ t \in J \subset \mathbb{R}$ is an integral curve of a left-invariant vector field ξ^L, then its left translates $a\gamma(t)$ are again integral curves. In particular, for $t_0 \in J$ the curve $t \mapsto \gamma(t_0)\gamma(t)$ is again an integral curve. Hence it coincides with $\gamma(t_0 + t)$ for all $t \in J \cap (J - t_0)$. In this way, an integral curve defined for small $|t|$ can be extended to an integral curve for all t, i.e. ξ^L is complete.

Since ξ^L is left-invariant, so is its flow Φ_t^ξ. Hence

$$\Phi_t^\xi(g) = \Phi_t^\xi \circ L_g(e) = L_g \circ \Phi_t^\xi(e) = g\Phi_t^\xi(e) = g\phi^\xi(-t).$$

The property $\Phi^\xi_{t_1+t_2} = \Phi^\xi_{t_1}\Phi^\xi_{t_2}$ shows that $\phi^\xi(t_1+t_2) = \phi^\xi(t_1)\phi^\xi(t_2)$. Finally, since $\xi^L \sim_{\mathrm{Inv}} -\xi^R$, the image

$$\mathrm{Inv}(\phi^\xi(t)) = \phi^\xi(t)^{-1} = \phi^\xi(-t)$$

is an integral curve of $-\xi^R$. Equivalently, $\phi^\xi(t)$ is an integral curve of ξ^R. □

Since left and right translations commute, it follows in particular that

$$[\xi^L, \eta^R] = 0.$$

Definition 4.2. A 1-parameter subgroup of G is a group homomorphism $\phi\colon \mathbb{R} \to G$.

We have seen that every $\xi \in \mathfrak{g}$ defines a 1-parameter group, by taking the integral curve through e of the left-invariant vector field ξ^L. Every 1-parameter group arises in this way:

Proposition 4.3. *If ϕ is a 1-parameter subgroup of G, then $\phi = \phi^\xi$ where $\xi = \dot{\phi}(0)$. One has*
$$\phi^{s\xi}(t) = \phi^\xi(st).$$
The map
$$\mathbb{R} \times \mathfrak{g} \to G,\ (t, \xi) \mapsto \phi^\xi(t)$$
is smooth.

Proof. Let $\phi(t)$ be a 1-parameter group. Then $\Phi_t(g) := g\phi(-t)$ defines a flow. Since this flow commutes with left translations, it is the flow of a left-invariant vector field, ξ^L. Here ξ is determined by taking the derivative of $\Phi_{-t}(e) = \phi(t)$ at $t = 0$: $\xi = \dot{\phi}(0)$. This shows $\phi = \phi^\xi$. As an application, since $\psi(t) = \phi^\xi(st)$ is a 1-parameter group with $\dot{\psi}^\xi(0) = s\dot{\phi}^\xi(0) = s\xi$, we have $\phi^\xi(st) = \phi^{s\xi}(t)$. Smoothness of the map $(t, \xi) \mapsto \phi^\xi(t)$ follows from the smooth dependence of solutions of ODE's on parameters. $\square$

Definition 4.4. The *exponential map* for the Lie group G is the smooth map defined by
$$\exp\colon \mathfrak{g} \to G,\ \xi \mapsto \phi^\xi(1),$$
where $\phi^\xi(t)$ is the 1-parameter subgroup with $\dot{\phi}^\xi(0) = \xi$.

Proposition 4.5. *We have*
$$\phi^\xi(t) = \exp(t\xi).$$
If $[\xi, \eta] = 0$ then
$$\exp(\xi + \eta) = \exp(\xi)\exp(\eta).$$

Proof. By the previous Proposition, $\phi^\xi(t) = \phi^{t\xi}(1) = \exp(t\xi)$. For the second claim, note that $[\xi, \eta] = 0$ implies that ξ^L, η^L commute. Hence their flows $\Phi_t^\xi,\ \Phi_t^\eta$, and $\Phi_t^\xi \circ \Phi_t^\eta$ is the flow of $\xi^L + \eta^L$. Hence it coincides with $\Phi_t^{\xi+\eta}$. Applying to e, we get $\phi^\xi(t)\phi^\eta(t) = \phi^{\xi+\eta}(t)$. Now put $t = 1$. $\square$

In terms of the exponential map, we may now write the flow of ξ^L as $\Phi_t^\xi(g) = g\exp(-t\xi)$, and similarly for the flow of ξ^R. That is,
$$\xi^L = \frac{\partial}{\partial t}|_{t=0} R^*_{\exp(t\xi)},\quad \xi^R = \frac{\partial}{\partial t}|_{t=0} L^*_{\exp(t\xi)}.$$

Proposition 4.6. *The exponential map is functorial with respect to Lie group homomorphisms $\phi\colon G \to H$. That is, we have a commutative diagram*
$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d_e\phi"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Proof. $t \mapsto \phi(\exp(t\xi))$ is a 1-parameter subgroup of H, with differential at e given by
$$\frac{d}{dt}\Big|_{t=0} \phi(\exp(t\xi)) = \mathrm{d}_e\phi(\xi).$$
Hence $\phi(\exp(t\xi)) = \exp(t\mathrm{d}_e\phi(\xi))$. Now put $t = 1$. $\square$

Proposition 4.7. *Let $G \subset \mathrm{GL}(n,\mathbb{R})$ be a matrix Lie group, and $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ its Lie algebra. Then $\exp\colon \mathfrak{g} \to G$ is just the exponential map for matrices,*

$$\exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n.$$

Furthermore, the Lie bracket on $\mathfrak{g}$ is just the commutator of matrices.

Proof. By the previous Proposition, applied to the inclusion of G in $\mathrm{GL}(n,\mathbb{R})$, the exponential map for G is just the restriction of that for $\mathrm{GL}(n,\mathbb{R})$. Hence it suffices to prove the claim for $G = \mathrm{GL}(n,\mathbb{R})$. The function $\sum_{n=0}^{\infty} \frac{t^n}{n!}\xi^n$ is a 1-parameter group in $\mathrm{GL}(n,\mathbb{R})$, with derivative at 0 equal to $\xi \in \mathfrak{gl}(n,\mathbb{R})$. Hence it coincides with $\exp(t\xi)$. Now put $t = 1$. □

Proposition 4.8. *For a matrix Lie group $G \subset \mathrm{GL}(n,\mathbb{R})$, the Lie bracket on $\mathfrak{g} = T_I G$ is just the commutator of matrices.*

Proof. It suffices to prove for $G = \mathrm{GL}(n,\mathbb{R})$. Using $\xi^L = \frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(t\xi)}$ we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} (R^*_{\exp(-t\xi)} R^*_{\exp(-s\eta)} R^*_{\exp(t\xi)} R^*_{\exp(s\eta)}) \\
&= \frac{\partial}{\partial s}\Big|_{s=0} (R^*_{\exp(-s\eta)} \xi^L R^*_{\exp(s\eta)} - \xi^L) \\
&= \xi^L \eta^L - \eta^L \xi^L \\
&= [\xi, \eta]^L.
\end{aligned}$$

On the other hand, write

$$R^*_{\exp(-t\xi)} R^*_{\exp(-s\eta)} R^*_{\exp(t\xi)} R^*_{\exp(s\eta)} = R^*_{\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta)}.$$

Since the Lie group exponential map for $\mathrm{GL}(n,\mathbb{R})$ coincides with the exponential map for matrices, we may use Taylor's expansion,

$$\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta) = I + st(\xi\eta - \eta\xi) + \ldots = \exp(st(\xi\eta - \eta\xi)) + \ldots$$

where $\ldots$ denotes terms that are cubic or higher in s, t. Hence

$$R^*_{\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta)} = R^*_{\exp(st(\xi\eta - \eta\xi)} + \ldots$$

and consequently

$$\frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(-t\xi)\exp(-s\eta)\exp(t\xi)\exp(s\eta)} = \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(st(\xi\eta - \eta\xi))} = (\xi\eta - \eta\xi)^L.$$

We conclude that $[\xi, \eta] = \xi\eta - \eta\xi$. □

Remark 4.9. Had we defined the Lie algebra using right-invariant vector fields, we would have obtained *minus* the commutator of matrices. Nonetheless, some authors use that convention.

The exponential map gives local coordinates for the group G on a neighborhood of e:

Proposition 4.10. *The differential of the exponential map at the origin is $d_0 \exp = \mathrm{id}$. As a consequence, there is an open neighborhood U of $0 \in \mathfrak{g}$ such that the exponential map restricts to a diffeomorphism $U \to \exp(U)$.*

Proof. Let $\gamma(t) = t\xi$. Then $\dot\gamma(0) = \xi$ since $\exp(\gamma(t)) = \exp(t\xi)$ is the 1-parameter group, we have

$$(\mathrm{d}_0 \exp)(\xi) = \frac{\partial}{\partial t}|_{t=0} \exp(t\xi) = \xi.$$

∎

Exercise 4.11. Show hat the exponential map for $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ $\mathrm{U}(n)$ are surjective. (We will soon see that the exponential map for any compact, connected Lie group is surjective.)

Exercise 4.12. A matrix Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$ is called *unipotent* if for all $A \in G$, the matrix $A - I$ is nilpotent (i.e. $(A - I)^r = 0$ for some r). The prototype of such a group are the upper triangular matrices with 1's down the diagonal. Show that for a connected unipotent matrix Lie group, the exponential map is a diffeomorphism.

Exercise 4.13. Show that $\exp\colon \mathfrak{gl}(2, \mathbb{C}) \to \mathrm{GL}(2, \mathbb{C})$ is surjective. More generally, show that the exponential map for $\mathrm{GL}(n, \mathbb{C})$ is surjective. (Hint: First conjugate the given matrix into Jordan normal form).

Exercise 4.14. Show that $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective, by proving that the matrices $\begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ are not in the image. (Hint: Assuming these matrices are of the form $\exp(B)$, what would the eigenvalues of B have to be?) Show that these two matrices represent *all* conjugacy classes of elements that are not in the image of exp. (Hint: Find a classification of the conjugacy classes of $\mathrm{SL}(2, \mathbb{R})$, e.g. in terms of eigenvalues.)

5. Cartan's theorem on closed subgroups

Using the exponential map, we are now in position to prove Cartan's theorem on closed subgroups.

Theorem 5.1. *Let H be a closed subgroup of a Lie group G. Then H is an embedded submanifold, and hence is a Lie subgroup.*

We first need a Lemma. Let V be a Euclidean vector space, and $S(V)$ its unit sphere. For $v \in V \backslash \{0\}$, let $[v] = \frac{v}{||v||} \in S(V)$.

Lemma 5.2. *Let $v_n, v \in V \backslash \{0\}$ with $\lim_{n\to\infty} v_n = 0$. Then*

$$\lim_{n\to\infty} [v_n] = [v] \Leftrightarrow \exists a_n \in \mathbb{N}\colon \lim_{n\to\infty} a_n v_n = v.$$

Proof. The implication $\Leftarrow$ is obvious. For the opposite direction, suppose $\lim_{n\to\infty} [v_n] = [v]$. Let $a_n \in \mathbb{N}$ be defined by $a_n - 1 < \frac{||v||}{||v_n||} \leq a_n$. Since $v_n \to 0$, we have $\lim_{n\to\infty} a_n \frac{||v_n||}{||v||} = 1$, and

$$a_n v_n = \left(a_n \frac{||v_n||}{||v||} \right) [v_n] \, ||v|| \to [v] \, ||v|| = v.$$

∎

Proof of E. Cartan's theorem. It suffices to construct a submanifold chart near $e \in H$. (By left translation, one then obtains submanifold charts near arbitrary $a \in H$.) Choose an inner product on $\mathfrak{g}$.

We begin with a candidate for the Lie algebra of H. Let $W \subset \mathfrak{g}$ be the subset such that $\xi \in W$ if and only if either $\xi = 0$, or $\xi \neq 0$ and there exists $\xi_n \neq 0$ with

$$\exp(\xi_n) \in H, \quad \xi_n \to 0, \quad [\xi_n] \to [\xi].$$

We will now show the following:

(i) $\exp(W) \subset H$,

(ii) W is a subspace of $\mathfrak{g}$,

(iii) There is an open neighborhood U of 0 and a diffeomorphism $\phi\colon U \to \phi(U) \subset G$ with $\phi(0) = e$ such that

$$\phi(U \cap W) = \phi(U) \cap H.$$

(Thus ϕ defines a submanifold chart near e.)

Step (i). Let $\xi \in W\backslash\{0\}$, with sequence ξ_n as in the definition of W. By the Lemma, there are $a_n \in \mathbb{N}$ with $a_n\xi_n \to \xi$. Since $\exp(a_n\xi_n) = \exp(\xi_n)^{a_n} \in H$, and H is closed, it follows that

$$\exp(\xi) = \lim_{n\to\infty} \exp(a_n\xi_n) \in H.$$

Step (ii). Since the subset W is invariant under scalar multiplication, we just have to show that it is closed under addition. Suppose $\xi, \eta \in W$. To show that $\xi + \eta \in W$, we may assume that $\xi, \eta, \xi + \eta$ are all non-zero. For t sufficiently small, we have

$$\exp(t\xi)\exp(t\eta) = \exp(u(t))$$

for some smooth curve $t \mapsto u(t) \in \mathfrak{g}$ with $u(0) = 0$. Then $\exp(u(t)) \in H$ and

$$\lim_{n\to\infty} n\, u(\frac{1}{n}) = \lim_{h\to 0} \frac{u(h)}{h} = \dot{u}(0) = \xi + \eta.$$

hence $u(\frac{1}{n}) \to 0$, $\exp(u(\frac{1}{n}) \in H$, $[u(\frac{1}{n})] \to [\xi + \eta]$. This shows $[\xi + \eta] \in W$, proving (ii).

Step (iii). Let W' be a complement to W in $\mathfrak{g}$, and define

$$\phi\colon \mathfrak{g} \cong W \oplus W' \to G, \quad \phi(\xi + \xi') = \exp(\xi)\exp(\xi').$$

Since $\mathrm{d}_0\phi$ is the identity, there is an open neighborhood $U \subset \mathfrak{g}$ of 0 such that $\phi\colon U \to \phi(U)$ is a diffeomorphism. It is automatic that $\phi(W \cap U) \subset \phi(W) \cap \phi(U) \subset H \cap \phi(U)$. We want to show that we can take U sufficiently small so that we also have the opposite inclusion

$$H \cap \phi(U) \subset \phi(W \cap U).$$

Suppose not. Then, any neighborhood $U_n \subset \mathfrak{g} = W \oplus W'$ of 0 contains an element (η_n, η_n') such that

$$\phi(\eta_n, \eta_n') = \exp(\eta_n)\exp(\eta_n') \in H$$

(i.e. $\exp(\eta_n') \in H$) but $(\eta_n, \eta_n') \notin W$ (i.e. $\eta_n' \neq 0$). Thus, taking U_n to be a nested sequence of neighborhoods with intersection $\{0\}$, we could construct a sequence $\eta_n' \in W' - \{0\}$ with $\eta_n' \to 0$ and $\exp(\eta_n') \in H$. Passing to a subsequence we may assume that $[\eta_n'] \to [\eta]$ for some $\eta \in W'\backslash\{0\}$. On the other hand, such a convergence would mean $\eta \in W$, by definition of W. Contradiction. □

As remarked earlier, Cartan's theorem is very useful in practice. For a given Lie group G, the term 'closed subgroup' is often used as synonymous to 'embedded Lie subgroup'.

Examples 5.3. (a) The matrix groups $G = \mathrm{O}(n), \mathrm{Sp}(n), \mathrm{SL}(n,\mathbb{R}), \ldots$ are all closed subgroups of some $\mathrm{GL}(N,\mathbb{R})$, and hence are Lie groups.

(b) Suppose that $\phi\colon G \to H$ is a morphism of Lie groups. Then $\ker(\phi) = \phi^{-1}(e) \subset G$ is a closed subgroup. Hence it is an embedded Lie subgroup of G.
(c) The center $Z(G)$ of a Lie group G is the set of all $a \in G$ such that $ag = ga$ for all $a \in G$. It is a closed subgroup, and hence an embedded Lie subgroup.
(d) Suppose $H \subset G$ is a closed subgroup. Its *normalizer* $N_G(H) \subset G$ is the set of all $a \in G$ such that $aH = Ha$. (I.e. $h \in H$ implies $aha^{-1} \in H$.) This is a closed subgroup, hence a Lie subgroup. The *centralizer* $Z_G(H)$ is the set of all $a \in G$ such that $ah = ha$ for all $h \in H$, it too is a closed subgroup, hence a Lie subgroup.

The E. Cartan theorem is just one of many 'automatic smoothness' results in Lie theory. Here is another.

Theorem 5.4. *Let G, H be Lie groups, and $\phi\colon G \to H$ be a continuous group morphism. Then ϕ is smooth.*

As a corollary, a given topological group carries at most one smooth structure for which it is a Lie group. For profs of these (and stronger) statements, see the book by Duistermaat-Kolk.

6. The adjoint representation

6.1. **Automorphisms.** The group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of a Lie algebra $\mathfrak{g}$ is closed in the group $\mathrm{End}(\mathfrak{g})^\times$ of vector space automorphisms, hence it is a Lie group. To identify its Lie algebra, let $D \in \mathrm{End}(\mathfrak{g})$ be such that $\exp(tD) \in \mathrm{Aut}(\mathfrak{g})$ for $t \in \mathbb{R}$. Taking the derivative of the defining condition $\exp(tD)[\xi, \eta] = [\exp(tD)\xi, \exp(tD)\eta]$, we obtain the property

$$D[\xi, \eta] = [D\xi, \eta] + [\xi, D\eta]$$

saying that D is a *derivation* of the Lie algebra. Conversely, if D is a derivation then

$$D^n[\xi, \eta] = \sum_{k=0}^{n} \binom{n}{k} [D^k\xi,\ D^{n-k}\eta]$$

by induction, which then shows that $\exp(D) = \sum_n \frac{D^n}{n!}$ is an automorphism. Hence the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$.

Exercise 6.1. Using similar arguments, verify that the Lie algebra of $\mathrm{SO}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \ldots$ are $\mathfrak{so}(n),\ \mathfrak{su}(n),\ \mathfrak{sp}(n), \ldots$.

6.2. **The adjoint representation of G.** Recall that an automorphism of a Lie group G is an invertible morphism from G to itself. The automorphisms form a group $\mathrm{Aut}(G)$. Any $a \in G$ defines an 'inner' automorphism $\mathrm{Ad}_a \in \mathrm{Aut}(G)$ by conjugation:

$$\mathrm{Ad}_a(g) = aga^{-1}$$

Indeed, Ad_a is an automorphism since $\mathrm{Ad}_a^{-1} = \mathrm{Ad}_{a^{-1}}$ and

$$\mathrm{Ad}_a(g_1g_2) = ag_1g_2a^{-1} = ag_1a^{-1}ag_2a^{-1} = \mathrm{Ad}_a(g_1)\,\mathrm{Ad}_a(g_2).$$

Note also that $\mathrm{Ad}_{a_1a_2} = \mathrm{Ad}_{a_1}\mathrm{Ad}_{a_2}$, thus we have a group morphism

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$$

into the group of automorphisms. The kernel of this morphism is the center $Z(G)$, the image is (by definition) the subgroup $\mathrm{Int}(G)$ of inner automorphisms. Note that for any $\phi \in \mathrm{Aut}(G)$, and any $a \in G$,

$$\phi \circ \mathrm{Ad}_a \circ \phi^{-1} = \mathrm{Ad}_{\phi(a)} \, .$$

That is, $\mathrm{Int}(G)$ is a *normal* subgroup of $\mathrm{Aut}(G)$. (I.e. the conjugate of an inner automorphism by any automorphism is inner.) It follows that $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ inherits a group structure; it is called the *outer automorphism group.*

Example 6.2. If $G = \mathrm{SU}(2)$ the complex conjugation of matrices is an inner automorphism, but for $G = \mathrm{SU}(n)$ with $n \geq 3$ it cannot be inner (since an inner automorphism has to preserve the spectrum of a matrix). Indeed, one know that $\mathrm{Out}(\mathrm{SU}(n)) = \mathbb{Z}_2$ for $n \geq 3$.

The differential of the automorphism $\mathrm{Ad}_a \colon G \to G$ is a Lie algebra automorphism, denoted by the same letter: $\mathrm{Ad}_a = \mathrm{d}_e \mathrm{Ad}_a \colon \mathfrak{g} \to \mathfrak{g}$. The resulting map

$$\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$$

is called the *adjoint representation of* G. Since the Ad_a are Lie algebra/group morphisms, they are compatible with the exponential map,

$$\exp(\mathrm{Ad}_a \xi) = \mathrm{Ad}_a \exp(\xi).$$

Remark 6.3. If $G \subset \mathrm{GL}(n, \mathbb{R})$ is a matrix Lie group, then $\mathrm{Ad}_a \in \mathrm{Aut}(\mathfrak{g})$ is the conjugation of matrices

$$\mathrm{Ad}_a(\xi) = a \xi a^{-1}.$$

This follows by taking the derivative of $\mathrm{Ad}_a(\exp(t\xi)) = a \exp(t\xi) a^{-1}$, using that exp is just the exponential series for matrices.

6.3. **The adjoint representation of $\mathfrak{g}$.** Let $\mathrm{Der}(\mathfrak{g})$ be the Lie algebra of derivations of the Lie algebra $\mathfrak{g}$. There is a Lie algebra morphism,

$$\mathrm{ad} \colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), \quad \xi \mapsto [\xi, \cdot].$$

The fact that ad_ξ is a derivation follows from the Jacobi identity; the fact that $\xi \mapsto \mathrm{ad}_\xi$ it is a Lie algebra morphism is again the Jacobi identity. The kernel of ad is the center of the Lie algebra $\mathfrak{g}$, i.e. elements having zero bracket with all elements of $\mathfrak{g}$, while the image is the Lie subalgebra $\mathrm{Int}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$ of *inner* derivations. It is a normal Lie subalgebra, i.e $[\mathrm{Der}(\mathfrak{g}), \mathrm{Int}(\mathfrak{g})] \subset \mathrm{Int}(\mathfrak{g})$, and the quotient Lie algebra $\mathrm{Out}(\mathfrak{g})$ are the *outer automorphims.*

Suppose now that G is a Lie group, with Lie algebra $\mathfrak{g}$. We have remarked above that the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$. Recall that the differential of any G-representation is a $\mathfrak{g}$-representation. In particular, we can consider the differential of $G \to \mathrm{Aut}(\mathfrak{g})$.

Theorem 6.4. *If $\mathfrak{g}$ is the Lie algebra of G, then the adjoint representation* $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ *is the differential of the adjoint representation* $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$*. One has the equality of operators*

$$\exp(\mathrm{ad}_\xi) = \mathrm{Ad}(\exp \xi)$$

for all $\xi \in \mathfrak{g}$.

Proof. For the first part we have to show $\frac{\partial}{\partial t}\big|_{t=0} \mathrm{Ad}_{\exp(t\xi)}\,\eta = \mathrm{ad}_\xi\,\eta$. This is easy if G is a matrix Lie group:

$$\frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Ad}_{\exp(t\xi)}\,\eta = \frac{\partial}{\partial t}\Big|_{t=0} \exp(t\xi)\eta\exp(-t\xi) = \xi\eta - \eta\xi = [\xi,\eta].$$

For general Lie groups we compute, using

$$\exp(s\,\mathrm{Ad}_{\exp(t\xi)}\,\eta) = \mathrm{Ad}_{\exp(t\xi)}\exp(s\eta) = \exp(t\xi)\exp(s\eta)\exp(-t\xi),$$

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0}(\mathrm{Ad}_{\exp(t\xi)}\,\eta)^L &= \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0} R^*_{\exp(s\,\mathrm{Ad}_{\exp(t\xi)}\,\eta)} \\
&= \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0} R^*_{\exp(t\xi)\exp(s\eta)\exp(-t\xi)} \\
&= \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0} R^*_{\exp(t\xi)} R^*_{\exp(s\eta)} R^*_{\exp(-t\xi)} \\
&= \frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(t\xi)}\ \eta^L\ R^*_{\exp(-t\xi)} \\
&= [\xi^L, \eta^L] \\
&= [\xi,\eta]^L = (\mathrm{ad}_\xi\,\eta)^L.
\end{aligned}$$

This proves the first part. The second part is the commutativity of the diagram

$$\begin{array}{ccc}
G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}) \\
{\scriptstyle\exp}\big\uparrow & & \big\uparrow{\scriptstyle\exp} \\
\mathfrak{g} & \xrightarrow[\mathrm{ad}]{} & \mathrm{Der}(\mathfrak{g})
\end{array}$$

which is just a special case of the functoriality property of exp with respect to Lie group morphisms. $\square$

Remark 6.5. As a special case, this formula holds for matrices. That is, for $B, C \in \mathrm{Mat}_n(\mathbb{R})$,

$$e^B\,C\,e^{-B} = \sum_{n=0}^\infty \frac{1}{n!}[B,[B,\cdots[B,C]\cdots]].$$

The formula also holds in some other contexts, e.g. if B, C are elements of an algebra with B nilpotent (i.e. $B^N = 0$ for some N). In this case, both the exponential series for e^B and the series on the right hand side are finite. (Indeed, $[B,[B,\cdots[B,C]\cdots]]$ with n B's is a sum of terms B^jCB^{n-j}, and hence must vanish if $n \geq 2N$.)

7. The differential of the exponential map

We had seen that $\mathrm{d}_0\exp = \mathrm{id}$. More generally, one can derive a formula for the differential of the exponential map at arbitrary points $\xi \in \mathfrak{g}$,

$$\mathrm{d}_\xi\exp\colon \mathfrak{g} = T_\xi\mathfrak{g} \to T_{\exp\xi}G.$$

Using left translation, we can move $T_{\exp\xi}G$ back to $\mathfrak{g}$, and obtain an endomorphism of $\mathfrak{g}$.

Theorem 7.1. *The differential of the exponential map* $\exp\colon \mathfrak{g} \to G$ *at* $\xi \in \mathfrak{g}$ *is the linear operator* $d_\xi \exp\colon \mathfrak{g} \to T_{\exp(\xi)}\mathfrak{g}$ *given by the formula,*

$$d_\xi \exp = (d_e L_{\exp \xi}) \circ \frac{1 - \exp(-\operatorname{ad}_\xi)}{\operatorname{ad}_\xi}.$$

Here the operator on the right hand side is defined to be the result of substituting ad_ξ in the entire holomorphic function $\frac{1-e^{-z}}{z}$. Equivalently, it may be written as an integral

$$\frac{1 - \exp(-\operatorname{ad}_\xi)}{\operatorname{ad}_\xi} = \int_0^1 \mathrm{d}s \;\exp(-s \operatorname{ad}_\xi).$$

Proof. We have to show that for all $\xi, \eta \in \mathfrak{g}$,

$$(\mathrm{d}_\xi \exp)(\eta) \circ L^*_{\exp(-\xi)} = \int_0^1 \mathrm{d}s \;(\exp(-s \operatorname{ad}_\xi)\eta)$$

as operators on functions $f \in C^\infty(G)$. To compute the left had side, write

$$(\mathrm{d}_\xi \exp)(\eta) \circ L^*_{\exp(-\xi)}(f) = \frac{\partial}{\partial t}\Big|_{t=0} (L^*_{\exp(-\xi)}(f))(\exp(\xi + t\eta)) = \frac{\partial}{\partial t}\Big|_{t=0} f(\exp(-\xi)\exp(\xi + t\eta)).$$

We think of this as the value of $\frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(-\xi)} R^*_{\exp(\xi+t\eta)} f$ at e, and compute as follows: [2]

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(-\xi)} R^*_{\exp(\xi+t\eta)} &= \int_0^1 \mathrm{d}s \;\frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s} R^*_{\exp(-s\xi)} R^*_{\exp(s(\xi+t\eta))} \\
&= \int_0^1 \mathrm{d}s \;\frac{\partial}{\partial t}\Big|_{t=0} R^*_{\exp(-s\xi)} (t\eta)^L R^*_{\exp(s(\xi+t\eta))} \\
&= \int_0^1 \mathrm{d}s \; R^*_{\exp(-s\xi)} \;\eta^L\; R^*_{\exp(s(\xi))} \\
&= \int_0^1 \mathrm{d}s \;(\operatorname{Ad}_{\exp(-s\xi)} \eta)^L \\
&= \int_0^1 \mathrm{d}s \;(\exp(-s \operatorname{ad}_\xi)\eta)^L.
\end{aligned}$$

Applying this result to f at e, we obtain $\int_0^1 \mathrm{d}s \;(\exp(-s \operatorname{ad}_\xi)\eta)(f)$ as desired. ☐

Corollary 7.2. *The exponential map is a local diffeomorphism near* $\xi \in \mathfrak{g}$ *if and only if* ad_ξ *has no eigenvalue in the set* $2\pi i\mathbb{Z}\backslash\{0\}$.

Proof. $\mathrm{d}_\xi \exp$ is an isomorphism if and only if $\frac{1-\exp(-\operatorname{ad}_\xi)}{\operatorname{ad}_\xi}$ is invertible, i.e. has non-zero determinant. The determinant is given in terms of the eigenvalues of ad_ξ as a product, $\prod_\lambda \frac{1-e^{-\lambda}}{\lambda}$. This vanishes if and only if there is a non-zero eigenvalue λ with $e^\lambda = 1$. ☐

[2] We will use the identities $\frac{\partial}{\partial s} R^*_{\exp(s\zeta)} = R^*_{\exp(s\zeta)}\, \zeta^L = \zeta^L\, R^*_{\exp(s\zeta)}$ for all $\zeta \in \mathfrak{g}$. Proof: $\frac{\partial}{\partial s} R^*_{\exp(s\zeta)} = \frac{\partial}{\partial u}|_{u=0} R^*_{\exp((s+u)\zeta)} = \frac{\partial}{\partial u}|_{u=0} R^*_{\exp(u\zeta)} R^*_{\exp(s\zeta)} = \zeta^L R^*_{\exp(s\zeta)}$.

As an application, one obtains a version of the *Baker-Campbell-Hausdorff formula.* Let $g \mapsto \log(g)$ be the inverse function to exp, defined for g close to e. For $\xi, \eta \in \mathfrak{g}$ close to 0, the function

$$\log(\exp(\xi)\exp(\eta))$$

The BCH formula gives the Taylor series expansion of this function. The series starts out with

$$\log(\exp(\xi)\exp(\eta)) = \xi + \eta + \tfrac{1}{2}[\xi,\eta] + \cdots$$

but gets rather complicated. To derive the formula, introduce a t-dependence, and let $f(t,\xi,\eta)$ be defined by $\exp(\xi)\exp(t\eta) = \exp(f(t,\xi,\eta))$ (for ξ, η sufficiently small). Thus

$$\exp(f) = \exp(\xi)\exp(t\eta)$$

We have, on the one hand,

$$(\mathrm{d}_e L_{\exp(f)})^{-1}\frac{\partial}{\partial t}\exp(f) = \mathrm{d}_e L^{-1}_{\exp(t\eta)}\frac{\partial}{\partial t}\exp(t\eta) = \eta.$$

On the other hand, by the formula for the differential of exp,

$$(\mathrm{d}_e L_{\exp(f)})^{-1}\frac{\partial}{\partial t}\exp(f) = (\mathrm{d}_e L_{\exp(f)})^{-1}(\mathrm{d}_f \exp)(\frac{\partial f}{\partial t}) = \frac{1 - e^{-\operatorname{ad}_f}}{\operatorname{ad}_f}(\frac{\partial f}{\partial t}).$$

Hence

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\operatorname{ad}_f}{1 - e^{-\operatorname{ad}_f}}\eta.$$

Letting χ be the function, holomorphic near $w = 1$,

$$\chi(w) = \frac{\log(w)}{1 - w^{-1}} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}(w-1)^n,$$

we may write the right hand side as $\chi(e^{\operatorname{ad}_f})\eta$. By Applying Ad to the defining equation for f we obtain $e^{\operatorname{ad}_f} = e^{\operatorname{ad}_\xi}e^{t\operatorname{ad}_\eta}$. Hence

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \chi(e^{\operatorname{ad}_\xi}e^{t\operatorname{ad}_\eta})\eta.$$

Finally, integrating from 0 to 1 and using $f(0) = \xi,\ f(1) = \log(\exp(\xi)\exp(\eta))$, we find:

$$\log(\exp(\xi)\exp(\eta)) = \xi + \Big(\int_0^1 \chi(e^{\operatorname{ad}_\xi}e^{t\operatorname{ad}_\eta})\mathrm{d}t\Big)\eta.$$

To work out the terms of the series, one puts

$$w - 1 = e^{\operatorname{ad}_\xi}e^{t\operatorname{ad}_\eta} - 1 = \sum_{i+j\geq 1}\frac{t^j}{i!j!}\operatorname{ad}_\xi^i\operatorname{ad}_\eta^j$$

in the power series expansion of χ, and integrates the resulting series in t. We arrive at:

Theorem 7.3 (Baker-Campbell-Hausdorff series)**.** *Let G be a Lie group, with exponential map $\exp\colon \mathfrak{g} \to G$. For $\xi, \eta \in \mathfrak{g}$ sufficiently small we have the following formula*

$$\log(\exp(\xi)\exp(\eta)) = \xi + \eta + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\Big(\int_0^1 dt\ \Big(\sum_{i+j\geq 1}\frac{t^j}{i!j!}\operatorname{ad}_\xi^i\operatorname{ad}_\eta^j\Big)^n\Big)\eta.$$

An important point is that the resulting Taylor series in ξ, η is a *Lie series*: all terms of the series are of the form of a constant times $\mathrm{ad}_\xi^{n_1}\mathrm{ad}_\eta^{m_2}\cdots\mathrm{ad}_\xi^{n_r}\eta$. The first few terms read,

$$\log(\exp(\xi)\exp(\eta)) = \xi+\eta+\tfrac{1}{2}[\xi,\eta]+\frac{1}{12}[\xi,[\xi,\eta]]-\frac{1}{12}[\eta,[\xi,\eta]]+\frac{1}{24}[\eta,[\xi,[\eta,\xi]]]+\dots.$$

Exercise 7.4. Work out these terms from the formula.

There is a somewhat better version of the BCH formula, due to Dynkin. A good discussion can be found in the book by Onishchik-Vinberg, Chapter I.3.2.

8. Actions of Lie groups and Lie algebras

8.1. **Lie group actions.**

Definition 8.1. An *action of a Lie group G on a manifold M* is a group homomorphism

$$\mathcal{A}\colon G\to \mathrm{Diff}(M),\ g\mapsto \mathcal{A}_g$$

into the group of diffeomorphisms on M, such that the *action map*

$$G\times M\to M,\quad (g,m)\mapsto \mathcal{A}_g(m)$$

is smooth.

We will often write $g.m$ rather than $\mathcal{A}_g(m)$. With this notation, $g_1.(g_2.m) = (g_1g_2).m$ and $e.m = m$. A map $\Phi\colon M_1\to M_2$ between G-manifolds is called G-equivariant if $g.\Phi(m) = \Phi(g.m)$ for all $m\in M$, i.e. the following diagram commutes:

$$\begin{array}{ccc} G\times M_1 & \longrightarrow & M_1\\ \Big\downarrow{\scriptstyle \mathrm{id}\times\Phi} & & \Big\downarrow{\scriptstyle \Phi}\\ G\times M_2 & \longrightarrow & M_2\end{array}$$

where the horizontal maps are the action maps.

Examples 8.2. (a) An $\mathbb{R}$-action on M is the same thing as a global flow.

(b) The group G acts $M = G$ by right multiplication, $\mathcal{A}_g = R_{g^{-1}}$, left multiplication, $\mathcal{A}_g = L_g$, and by conjugation, $\mathcal{A}_g = \mathrm{Ad}_g = L_g\circ R_{g^{-1}}$. The left and right action commute, hence they define an action of $G\times G$. The conjugation action can be regarded as the action of the diagonal subgroup $G\subset G\times G$.

(c) Any G-representation $G\to \mathrm{End}(V)$ can be regarded as a G-action, by viewing V as a manifold.

(d) For any closed subgroup $H\subset G$, the space of right cosets $G/H = \{gH|\ g\in G\}$ has a unique manifold structure such that the quotient map $G\to G/H$ is a smooth submersion, and the action of G by left multiplication on G descends to a smooth G-action on G/H. (Some ideas of the proof will be explained below.)

(e) The defining representation of the orthogonal group $\mathrm{O}(n)$ on $\mathbb{R}^n$ restricts to an action on the unit sphere S^{n-1}, which in turn descends to an action on the projective space $\mathbb{R}P(n-1)$. One also has actions on the Grassmann manifold $\mathrm{Gr}_\mathbb{R}(k,n)$ of k-planes in $\mathbb{R}^n$, on the flag manifold $\mathrm{Fl}(n)\subset \mathrm{Gr}_\mathbb{R}(1,n)\times\cdots\mathrm{Gr}_\mathbb{R}(n-1,n)$ (consisting of sequences of subspaces $V_1\subset\cdots V_{n-1}\subset\mathbb{R}^n$ with $\dim V_i = i$), and various types of 'partisl' flag manifolds. These examples are all of the form $\mathrm{O}(n)/H$ for various choices of H. (E.g, for $\mathrm{Gr}(k,n)$ one takes H to be the subgroup preserving $\mathbb{R}^k\subset\mathbb{R}^n$.)

8.2. **Lie algebra actions.**

Definition 8.3. An *action of a finite-dimensional Lie algebra* $\mathfrak{g}$ on M is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto \mathcal{A}_\xi$ such that the action map

$$\mathfrak{g} \times M \to TM, \quad (\xi, m) \mapsto \mathcal{A}_\xi|_m$$

is smooth.

We will often write $\xi_M =: \mathcal{A}_\xi$ for the vector field corresponding to ξ. Thus, $[\xi_M, \eta_M] = [\xi, \eta]_M$ for all $\xi, \eta \in \mathfrak{g}$. A smooth map $\Phi\colon M_1 \to M_2$ between $\mathfrak{g}$-manifolds is called equivariant if $\xi_{M_1} \sim_\Phi \xi_{M_2}$ for all $\xi \in \mathfrak{g}$, i.e. if the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} \times M_1 & \longrightarrow & TM_1 \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \Phi} & & \Big\downarrow{\scriptstyle \mathrm{d}\Phi} \\ \mathfrak{g} \times M_2 & \longrightarrow & TM_2 \end{array}$$

where the horizontal maps are the action maps.

Examples 8.4.
(a) Any vector field X defines an action of the Abelian Lie algebra $\mathbb{R}$, by $\lambda \mapsto \lambda X$.
(b) Any Lie algebra representation $\phi\colon \mathfrak{g} \to \mathrm{gl}(V)$ may be viewed as a Lie algebra action

$$(\mathcal{A}_\xi f)(v) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(v - t\phi(\xi)v) = -\langle \mathrm{d}_v f,\ \phi(\xi)v\rangle, \quad f \in C^\infty(V)$$

defines a $\mathfrak{g}$-action. Here $\mathrm{d}_v f\colon T_vV \to \mathbb{R}$ is viewed as an element of V^*. Using a basis e_a of V to identify $V = \mathbb{R}^n$, and introducing the components of $\xi \in \mathfrak{g}$ in the representation as $\phi(\xi).e_a = \sum_b \phi(\xi)^b_a e_b$ the generating vector fields are

$$\xi_V = -\sum_{ab} \phi(\xi)^b_a x^a \frac{\partial}{\partial x^b}.$$

Note that the components of the generating vector fields are homogeneous linear functions in x. Any $\mathfrak{g}$-action on V with this property comes from a linear $\mathfrak{g}$-representation.
(c) For any Lie group G, we have actions of its Lie algebra $\mathfrak{g}$ by $\mathcal{A}_\xi = \xi^L$, $\mathcal{A}_\xi = -\xi^R$ and $\mathcal{A}_\xi = \xi^L - \xi^R$.
(d) Given a closed subgroup $H \subset G$, the vector fields $-\xi^R \in \mathfrak{X}(G)$, $\xi \in \mathfrak{g}$ are invariant under the right multiplication, hence they are related under the quotient map to vector fields on G/H. That is, there is a unique $\mathfrak{g}$-action on G/H such that the quotient map $G \to G/H$ is equivariant.

Definition 8.5. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given a G-action $g \mapsto \mathcal{A}_g$ on M, one defines its *generating vector fields* by

$$\mathcal{A}_\xi = \frac{d}{dt}\Big|_{t=0} \mathcal{A}^*_{\exp(-t\xi)}.$$

Example 8.6. The generating vector field for the action by right multiplication $\mathcal{A}_a = R_{a^{-1}}$ are the left-invariant vector fields,

$$\mathcal{A}_\xi = \frac{\partial}{\partial t}|_{t=0} R^*_{\exp(t\xi)} = \xi^L.$$

Similarly, the generating vector fields for the action by left multiplication $\mathcal{A}_a = L_a$ are $-\xi^R$, and those for the conjugation action $\mathrm{Ad}_a = L_a \circ R_{a^{-1}}$ are $\xi^L - \xi^R$.

Observe that if $\Phi\colon M_1 \to M_2$ is an equivariant map of G-manifolds, then the generating vector fields for the action are Φ-related.

Theorem 8.7. *The generating vector fields of any G-action $g \to \mathcal{A}_g$ define a $\mathfrak{g}$-action $\xi \to \mathcal{A}_\xi$.*

Proof. Write $\xi_M := \mathcal{A}_\xi$ for the generating vector fields of a G-action on M. We have to show that $\xi \mapsto \xi_M$ is a Lie algebra morphism. Note that the action map

$$\Phi\colon G \times M \to M,\ (a, m) \mapsto a.m$$

is G-equivariant, relative to the given G-action on M and the action $g.(a, m) = (ga, m)$ on $G \times M$. Hence $\xi_{G\times M} \sim_\Phi \xi_M$. But $\xi_{G\times M} = -\xi^R$ (viewed as vector fields on the product $G \times M$), hence $\xi \mapsto \xi_{G\times M}$ is a Lie algebra morphism. It follows that

$$0 = [(\xi_1)_{G\times M}, (\xi_1)_{G\times M}] - [\xi_1, \xi_2]_{G\times M} \sim_\Phi [(\xi_1)_M, (\xi_2)_M] - [\xi_1, \xi_2]_M.$$

Since Φ is a surjective submersion (i.e. the differential $\mathrm{d}\Phi\colon T(G \times M) \to TM$ is surjective), this shows that $[(\xi_1)_M, (\xi_2)_M] - [\xi_1, \xi_2]_M = 0$. $\square$

8.3. **Integrating Lie algebra actions.** Let us now consider the inverse problem: For a Lie group G with Lie algebra $\mathfrak{g}$, integrating a given $\mathfrak{g}$-action to a G-action. The construction will use some facts about *foliations.*

Let M be a manifold. A *rank k distribution* on M is a $C^\infty(M)$-linear subspace $\mathfrak{R} \subset \mathfrak{X}(M)$ of the space of vector fields, such that at any point $m \in M$, the subspace

$$E_m = \{X_m |\ X \in \mathfrak{R}\}$$

is of dimension k. The subspaces E_m define a rank k vector bundle $E \subset TM$ with $\mathfrak{R} = \Gamma(E)$, hence a distribution is equivalently given by this subbundle E. An *integral submanifold* of the distribution $\mathfrak{R}$ is a k-dimensional submanifold S such that all $X \in \mathfrak{R}$ are tangent to S. In terms of E, this means that $T_mS = E_m$ for all $m \in S$. The distribution is called *integrable* if for all $m \in M$ there exists an integral submanifold containing m. In this case, there exists a maximal such submanifold, $\mathcal{L}_m$. The decomposition of M into maximal integral submanifolds is called a k-dimensional foliation of M, the maximal integral submanifolds themselves are called the *leaves* of the foliation.

Not every distribution is integrable. Recall that if two vector fields are tangent to a submanifold, then so is their Lie bracket. Hence, a *necessary* condition for integrability of a distribution is that $\mathfrak{R}$ is a Lie subalgebra. Frobenius' theorem gives the converse:

Theorem 8.8 (Frobenius theorem)**.** *A rank k distribution $\mathfrak{R} \subset \mathfrak{X}(M)$ is integrable if and only if $\mathfrak{R}$ is a Lie subalgebra.*

The idea of proof is to show that if $\mathfrak{R}$ is a Lie subalgebra, then the $C^\infty(M)$-module $\mathfrak{R}$ is spanned, near any $m \in M$, by k *commuting* vector fields. One then uses the flow of these vector fields to construct integral submanifold.

Exercise 8.9. Prove Frobenius' theorem for distributions $\mathfrak{R}$ of rank $k = 2$. (Hint: If $X \in \mathfrak{R}$ with $X_m \neq 0$, one can choose local coordinates such that $X = \frac{\partial}{\partial x_1}$. Given a second vector field $Y \in \mathfrak{R}$, such that $[X, Y] \in \mathfrak{R}$ and X_m, Y_m are linearly independent, show that one can replace Y by some $Z = aX + bY \in \mathfrak{R}$ such that $b_m \neq 0$ and $[X, Z] = 0$ on a neighborhood of m.)

Exercise 8.10. Give an example of a non-integrable rank 2 distribution on $\mathbb{R}^3$.

Given a Lie algebra of dimension k and a *free* $\mathfrak{g}$-action on M (i.e. $\xi_M|_m = 0$ implies $\xi = 0$), one obtains an integrable rank k distribution $\mathfrak{R}$ as the span (over $C^\infty(M)$) of the ξ_M's. We use this to prove:

Theorem 8.11. *Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. A Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto \xi_M$ integrates to an action of G if and only if the vector fields ξ_M are all complete.*

Proof of the theorem. The idea of proof is to express the G-action in terms of a foliation. Given a G-action on M, consider the diagonal G-action on $G \times M$, where G acts on itself by left multiplication. The orbits of this action define a foliation of $G \times M$, with leaves indexed by the elements of m:

$$\mathcal{L}_m = \{(g, g.m) |\, g \in G\}.$$

Let $\mathrm{pr}_1, \mathrm{pr}_2$ the projections from $G \times M$ to the two factors. Then pr_1 restricts to diffeomorphisms $\pi_m \colon \mathcal{L}_m \to G$, and we recover the action as

$$g.m = \mathrm{pr}_2(\pi_m^{-1}(g)).$$

Given a $\mathfrak{g}$-action, our plan is to construct the foliation from an integrable distribution.

Let $\xi \mapsto \xi_M$ be a given $\mathfrak{g}$-action. Consider the diagonal $\mathfrak{g}$ action on $G \times M$,

$$\xi_{G\times M} = (-\xi^R, \xi_M) \in \mathfrak{X}(G \times M).$$

Note that the vector fields $\xi_{\widehat{M}}$ are complete, since it is the sum of commuting vector fields, both of which are complete. If Φ_t^ξ is the flow of ξ_M, the flow of $\xi_{\widehat{M}} = (-\xi^R, \xi_M)$ is given by

$$\widehat{\Phi}_t^\xi = (L_{\exp(t\xi)}, \Phi_t^\xi) \in \mathrm{Diff}(G \times M).$$

The action $\xi \mapsto \xi_{G\times M}$ is free, hence it defines an integrable $\dim G$-dimensional distribution $\mathfrak{R} \subset \mathfrak{X}(G \times M)$. Let $\mathcal{L}_m \hookrightarrow G \times M$ be the unique leaf containing the point (e, m). Projection to the first factor induces a smooth map $\pi_m :\ \mathcal{L}_m \to G$.

We claim that π_m is *surjective.* To see this, recall that any $g \in G$ can be written in the form $g = \exp(\xi_r) \cdots \exp(\xi_1)$ with $\xi_i \in \mathfrak{g}$. Define $g_0 = e,\ m_0 = m$, and

$$g_i = \exp(\xi_i) \cdots \exp(\xi_1), \quad m_i = (\Phi_1^{\xi_i} \circ \cdots \circ \Phi_1^{\xi_1})(m)$$

for $i = 1, \ldots, r$. Each path

$$\widehat{\Phi}_t^{\xi_i}\big(g_{i-1},\, m_{i-1}\big) = (\exp(t\xi_i) g_{i-1},\ \Phi_t^{\xi_i}(m_{i-1})), \quad t \in [0, 1]$$

connects (g_{i-1}, m_{i-1}) to (g_i, m_i), and stays within a leaf of the foliation (since it is given by the flow). Hence, by concatenation we obtain a (piecewise smooth) path in $\mathcal{L}_m$ connecting (e, m) to $(g_r, m_r) = (g, m_r)$. In particular, $\pi_m^{-1}(g) \neq \emptyset$.

For any $(g, x) \in \mathcal{L}_m$ the tangent map $\mathrm{d}_{(g,x)}\pi_m$ is an isomorphism. Hence $\pi_m \colon \mathcal{L}_m \to G$ is a (surjective) covering map. Since G is simply connected by assumption, we conclude that $\pi_m \colon \mathcal{L}_m \to G$ is a diffeomorphism. We now define $\mathcal{A}_g(m) = \mathrm{pr}_2(\pi_m^{-1}(g))$. Concretely, the construction above shows that if $g = \exp(\xi_r) \cdots \exp(\xi_1)$ then

$$\mathcal{A}_g(m) = (\Phi_1^{\xi_r} \circ \cdots \circ \Phi_1^{\xi_1})(m).$$

From this description it is clear that $\mathcal{A}_{gh} = \mathcal{A}_g \circ \mathcal{A}_h$. $\square$

Let us remark that, in general, one cannot drop the assumption that G is simply connected. Consider for example $G = \mathrm{SU}(2)$, with $\mathfrak{su}(2)$-action $\xi \mapsto -\xi^R$. This exponentiates to an action of SU(2) by left multiplication. But $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ as Lie algebras, and the $\mathfrak{so}(3)$-action does not exponentiate to an action of the group SO(3).

As an important special case, we obtain:

Theorem 8.12. *Let H, G be Lie groups, with Lie algebras $\mathfrak{h}, \mathfrak{g}$. If H is connected and simply connected, then any Lie algebra morphism $\phi\colon \mathfrak{h} \to \mathfrak{g}$ integrates uniquely to a Lie group morphism $\psi\colon H \to G$.*

Proof. Define an $\mathfrak{h}$-action on G by $\xi \mapsto -\phi(\xi)^R$. Since the right-invariant vector fields are complete, this action integrates to a Lie group action $\mathcal{A}\colon H \to \mathrm{Diff}(G)$. This action commutes with the action of G by right multiplication. Hence, $\mathcal{A}_h(g) = \psi(h)g$ where $\psi(h) = \mathcal{A}_h(e)$. The action property now shows $\psi(h_1)\psi(h_2) = \psi(h_1h_2)$, so that $\psi\colon H \to G$ is a Lie group morphism integrating ϕ. $\square$

Corollary 8.13. *Let G be a connected, simply connected Lie group, with Lie algebra $\mathfrak{g}$. Then any $\mathfrak{g}$-representation on a finite-dimensional vector space V integrates to a G-representation on V.*

Proof. A $\mathfrak{g}$-representation on V is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$, hence it integrates to a Lie group morphism $G \to \mathrm{GL}(V)$. $\square$

Definition 8.14. A *Lie subgroup* of a Lie group G is a subgroup $H \subset G$, equipped with a Lie group structure such that the inclusion is a morphism of Lie groups (i.e. , is smooth).

Note that a Lie subgroup need not be closed in G, since the inclusion map need not be an embedding. Also, the one-parameter subgroups $\phi\colon \mathbb{R} \to G$ need not be subgroups (strictly speaking) since ϕ need not be injective.

Proposition 8.15. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. For any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there is a unique connected Lie subgroup H of G such that the differential of the inclusion $H \hookrightarrow G$ is the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$.*

Proof. Consider the distribution on G spanned by the vector fields $-\xi^R$, $\xi \in \mathfrak{g}$. It is integrable, hence it defines a foliation of G. The leaves of any foliation carry a unique manifold structure such that the inclusion map is smooth. Take $H \subset G$ to be the leaf through $e \in H$, with this manifold structure. Explicitly,

$$H = \{g \in G|\ g = \exp(\xi_r)\cdots\exp(\xi_1), \xi_i \in \mathfrak{h}\}.$$

From this description it follows that H is a Lie group. $\square$

By Ado's theorem, any finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a matrix Lie algebra. We will skip the proof of this important (but relatively deep) result, since it involves a considerable amount of structure theory of Lie algebras. Given such a presentation $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, the Lemma gives a Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$ integrating $\mathfrak{g}$. Replacing G with its universal covering, this proves:

Theorem 8.16 (Lie's third theorem)**.** *For any finite-dimensional real Lie algebra $\mathfrak{g}$, there exists a connected, simply connected Lie group G, unique up to isomorphism, having $\mathfrak{g}$ as its Lie algebra.*

The book by Duistermaat-Kolk contains a different, more conceptual proof of Cartan's theorem. This new proof has found important generalizations to the integration of *Lie algebroids*. In conjunction with the previous Theorem, Lie's third theorem gives an equivalence between the categories of finite-dimensional Lie algebras $\mathfrak{g}$ and connected, simply-connected Lie groups G.

9. Universal covering groups

Given a connected topological space X with base point x_0, one defines the covering space $\widetilde{X}$ as equivalence classes of paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$. Here the equivalence is that of homotopy relative to fixed endpoints. The map taking $[\gamma]$ to $\gamma(1)$ is a covering $p\colon \widetilde{X} \to X$. The covering space carries an action of the fundamental group $\pi_1(X)$, given as equivalence classes of paths with $\gamma(1) = x_0$, i.e. $\pi_1(X) = p^{-1}(x_0)$. The group structure is given by concatenation of paths

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases},$$

i.e. $[\gamma_1][\gamma_2] = [\gamma_1 * \gamma_2]$ (one shows that this is well-defined). If $X = M$ is a manifold, then $\pi_1(X)$ acts on $\widetilde{X}$ by *deck transformations*, this action is again induced by concatenation of paths:

$$\mathcal{A}_{[\lambda]}([\gamma]) = [\lambda * \gamma].$$

A continuous map of connected topological spaces $\Phi\colon X \to Y$ taking x_0 to the base point y_0 lifts to a continuous map $\widetilde{\Phi}\colon \widetilde{X} \to \widetilde{Y}$ of the covering spaces, by $\widetilde{\Phi}[\gamma] = [\Phi \circ \gamma]$, and it induces a group morphism $\pi_1(X) \to \pi_1(Y)$.

If $X = M$ is a manifold, then $\widetilde{M}$ is again a manifold, and the covering map is a local diffeomorphism. For a smooth map $\Phi\colon M \to N$ of manifolds, the induced map $\widetilde{\Phi}\colon \widetilde{M} \to \widetilde{N}$ of coverings is again smooth. This construction is functorial, i.e. $\widetilde{\Psi \circ \Phi} = \widetilde{\Psi} \circ \widetilde{\Phi}$. We are interested in the case of connected Lie groups G. In this case, the natural choice of base point is the group unit $x_0 = e$. We have:

Theorem 9.1. *The universal covering $\widetilde{G}$ of a connected Lie group G is again a Lie group, and the covering map $p\colon \widetilde{G} \to G$ is a Lie group morphism. The group $\pi_1(G) = p^{-1}(\{e\})$ is a subgroup of the center of $\widetilde{G}$.*

Proof. The group multiplication and inversion lifts to smooth maps $\widetilde{Mult}\colon \widetilde{G \times G} = \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ and $\widetilde{\mathrm{Inv}}\colon \widetilde{G} \to \widetilde{G}$. Using the functoriality properties of the universal covering construction, it is clear that these define a group structure on $\widetilde{G}$. A proof that $\pi_1(G)$ is central is outlined in the following exercise. $\square$

Exercise 9.2. Recall that a subgroup $H \subset G$ is normal in G if $\mathrm{Ad}_g(H) \subset H$ for all $g \in G$.

a) Let G be a connected Lie group, and $H \subset G$ a normal subgroup that is discrete (i.e. 0-dimensional). Show that H is a subgroup of the center of G.

b) Prove that the kernel of a Lie group morphism $\phi\colon G \to G'$ is a closed normal subgroup.

The combination of these two facts shows that if a Lie group morphism is a covering, then its kernel is a central subgroup.

Example 9.3. The universal covering group of the circle group $G = \mathrm{U}(1)$ is the additive group $\mathbb{R}$.

Example 9.4. SU(2) is the universal covering group of SO(3), and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is the universal covering group of SO(4). In both cases, the group of deck transformations is $\mathbb{Z}_2$.

For all $n \geq 3$, the fundamental group of $\mathrm{SO}(n)$ is $\mathbb{Z}_2$. The universal cover is called the *Spin group* and is denoted $\mathrm{Spin}(n)$. We have seen that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. One can also show that $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ and $\mathrm{Spin}(6) = \mathrm{SU}(4)$. (See e.g. by lecture notes on 'Lie groups and Clifford algebras', Section III.7.6.) Starting with $n = 7$, the spin groups are 'new'.

We will soon prove that the universal covering group $\widetilde{G}$ of a Lie group G is compact if and only if G is compact with finite center.

If $\Gamma \subset \pi_1(G)$ is any subgroup, then Γ (viewed as a subgroup of $\widetilde{G}$) is central, and so $\widetilde{G}/\Gamma$ is a Lie group covering G, with $\pi_1(G)/\Gamma$ as its group of deck transformations.

10. The universal enveloping algebra

As we had seen any algebra[3] $\mathcal{A}$ can be viewed as a Lie algebra, with Lie bracket the commutator. This correspondence defines a functor from the category of algebras to the category of Lie algebras. There is also a functor in the opposite direction, associating to any Lie algebra an algebra.

Definition 10.1. The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is the algebra $U(\mathfrak{g})$, with generators $\xi \in \mathfrak{g}$ and relations, $\xi_1\xi_2 - \xi_2\xi_1 = [\xi_1, \xi_2]$.

Elements of the enveloping algebra are linear combinations words $\xi_1 \cdots \xi_r$ in the Lie algebra elements, using the relations to manipulate the words. Here we are implicitly using that the relations don't annihilate any Lie algebra elements, i.e. that the map $\mathfrak{g} \to U(\mathfrak{g}),\ \xi \mapsto \xi$ is injective. This will be justified by the Poincaré-Birkhoff-Witt theorem to be discussed below.

Example 10.2. Let $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{R})$ be the Lie algebra with basis e, f, h and brackets

$$[e, f] = h,\ \ [h, e] = 2e,\ \ [h, f] = -2f.$$

It turns out that any element of $U(\mathfrak{sl}(2, \mathbb{R}))$ can be written as a sum of products of the form $f^k h^l e^m$ for some $k, l, m \geq 0$. Let us illustrate this for the element ef^2 (just to get used to some calculations in the enveloping algebra). We have

$$\begin{aligned} ef^2 &= [e, f^2] + f^2 e \\ &= [e, f]f + f[e, f] + f^2 e \\ &= hf + fh + f^2 e \\ &= [h, f] + 2fh + f^2 e \\ &= -2f + 2fh + f^2 e. \end{aligned}$$

[3]Unless specified differently, we take algebra to mean associative algebra with unit.

More formally, the universal enveloping algebra is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal $\mathcal{I}$ generated by all $\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 - [\xi_1, \xi_2]$. The inclusion map $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$ descends to a map $j \colon \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$. By construction, this map j is a Lie algebra morphism.

The construction of the enveloping algebra $U(\mathfrak{g})$ from a Lie algebra $\mathfrak{g}$ is functorial: Any Lie algebra morphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ induces a morphism of algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, in a way compatible with the composition of morphisms. As a special case, the zero map $\mathfrak{g} \to 0$ induces an algebra morphism $U(\mathfrak{g}) \to \mathbb{R}$, called the *augmentation map.*

Theorem 10.3 (Universal property)**.** *If $\mathcal{A}$ is an associative algebra, and $\kappa \colon \mathfrak{g} \to \mathcal{A}$ is a homomorphism of Lie algebras, then there is a unique morphism of algebras $\kappa_U \colon U(\mathfrak{g}) \to \mathcal{A}$ such that $\kappa = \kappa_U \circ j$.*

Proof. The map κ extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathcal{A}$. This algebra homomorphism vanishes on the ideal $\mathcal{I}$, and hence descends to an algebra homomorphism $\kappa_U \colon U(\mathfrak{g}) \to \mathcal{A}$ with the desired property. This extension is unique, since $j(\mathfrak{g})$ generates $U(\mathfrak{g})$ as an algebra. □

By the universal property, any Lie algebra representation $\mathfrak{g} \to \mathrm{End}(V)$ extends to a representation of the algebra $U(\mathfrak{g})$. Conversely, given an algebra representation $U(\mathfrak{g}) \to \mathrm{End}(V)$ one obtains a $\mathfrak{g}$-representation by restriction. That is, there is a 1-1 correspondence between Lie algebra representations of $\mathfrak{g}$ and algebra representations of $U(\mathfrak{g})$.

Let $\mathrm{Cent}(U(\mathfrak{g}))$ be the center of the enveloping algebra. Given a $\mathfrak{g}$-representation $\pi \colon \mathfrak{g} \to \mathrm{End}(V)$, the operators $\pi(x),\ x \in \mathrm{Cent}(U(\mathfrak{g}))$ commute with all $\pi(\xi), \xi \in \mathfrak{g}$:

$$[\pi(x), \pi(\xi)] = \pi([x, \xi]) = 0.$$

It follows that the eigenspaces of $\pi(x)$ for $x \in \mathrm{Cent}(U(\mathfrak{g}))$ are $\mathfrak{g}$-invariant.

Exercise 10.4. Let $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{R})$ be the Lie algebra with basis e, f, h and brackets $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$. Show that

$$x = 2fe + \frac{1}{2}h^2 + h \in U(\mathfrak{sl}(2, \mathbb{R}))$$

lies in the center of the enveloping algebra.

The construction of the enveloping algebra works for any Lie algebra, possibly of infinite dimension. It is a non-trivial fact that the map j is always an inclusion. This is usually obtained as a corollary to the Poincaré-Birkhoff-Witt theorem. The statement of this Theorem is as follows. Note that $U(\mathfrak{g})$ has a filtration

$$\mathbb{R} = U^{(0)}(\mathfrak{g}) \subset U^{(1)}(\mathfrak{g}) \subset U^{(2)}(\mathfrak{g}) \subset \cdots,$$

where $U^{(k)}(\mathfrak{g})$ consists of linear combinations of products of at most k elements in $\mathfrak{g}$. That is, $U^{(k)}(\mathfrak{g})$ is the image of $T^{(k)}(\mathfrak{g}) = \bigoplus_{i \le k} T^i(\mathfrak{g})$.

The filtration is compatible with the product, i.e. the product of an element of filtration degree k with an element of filtration degree l has filtration degree $k + l$. Let

$$\mathrm{gr}(U(\mathfrak{g})) = \bigoplus_{k=0}^{\infty} \mathrm{gr}^k(U(\mathfrak{g}))$$

be the associated graded algebra, where $\mathrm{gr}^k(U(\mathfrak{g})) = U^{(k)}(\mathfrak{g})/U^{(k-1)}(\mathfrak{g})$.

Lemma 10.5. *The associated graded algebra* $\mathrm{gr}(U(\mathfrak{g}))$ *is commutative. Hence, the map* $j\colon \mathfrak{g} \to U(\mathfrak{g})$ *defines an algebra morphism*

$$j_S\colon S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$$

Proof. If $x = \xi_1 \cdots \xi_k \in U^{(k)}(\mathfrak{g})$, and $x' = \xi_{s(1)} \cdots \xi_{s(k)}$ for some permutation s, then $x' - x \in U^{(k-1)}(\mathfrak{g})$. (If s is a transposition of adjacent elements this is immediate from the definition; but general permutations are products of such transpositions.) As a consequence, the products of two elements of filtration degrees k, l is independent of their order modulo terms of filtration degree $k + l - 1$. Equivalently, the associated graded algebra is commutative. □

Explicitly, the map is the direct sum over all

$$j_S\colon S^k(\mathfrak{g}) \to U^{(k)}(\mathfrak{g})/U^{(k-1)}(\mathfrak{g}),\ \xi_1 \cdots \xi_k \mapsto \xi_1 \cdots \xi_k \mod U^{(k-1)}(\mathfrak{g}).$$

Note that the map j_S is surjective: Given $y \in U^{(k)}(\mathfrak{g})/U^{(k-1)}(\mathfrak{g})$, choose a lift $\tilde{y} \in U^{(k)}(\mathfrak{g})$ given as a linear combination of k-fold products of elements in $\mathfrak{g}$. The same linear combination, with the product now interpreted in the symmetric algebra, defines an element $x \in S^k(\mathfrak{g})$ with $j_S(x) = y$. The following important result states that j_S is also injective.

Theorem 10.6 (Poincaré-Birkhoff-Witt theorem)**.** *The map*

$$j_S\colon S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g})$$

is an isomorphism of algebras.

Corollary 10.7. *The map* $j\colon \mathfrak{g} \to U(\mathfrak{g})$ *is injective.*

Corollary 10.8. *Suppose* $f\colon S\mathfrak{g} \to U(\mathfrak{g})$ *is a filtration preserving linear map whose associated graded map* $\mathrm{gr}(f)\colon S\mathfrak{g} \to \mathrm{gr}(U(\mathfrak{g}))$ *coincides with* j_S*. Then* f *is an isomorphism.*

Indeed, a map of filtered vector spaces is an isomorphism if and only if the associated graded map is an isomorphism. One typical choice of f is *symmetrization*, characterized as the unique linear map $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ such that $\mathrm{sym}(\xi^k) = \xi^k$ for all k. That is,

$$\mathrm{sym}(\xi_1, \ldots, \xi_k) = \frac{1}{k!} \sum_{s \in S_k} \xi_{s(1)} \cdots \xi_{s(k)};$$

for example,

$$\mathrm{sym}(\xi_1 \xi_2) = \tfrac{1}{2}(\xi_1 \xi_2 + \xi_2 \xi_1) = \xi_1 \xi_2 - \tfrac{1}{2}[\xi_1, \xi_2].$$

Corollary 10.9. *The symmetrization map* $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ *is an isomorphism of vector spaces.*

Another choice for f is to pick a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, and define f by

$$f(e_1^{i_1} \cdots e_n^{i_n}) = e_1^{i_1} \cdots e_n^{i_n}.$$

Hence we obtain,

Corollary 10.10. *If* $e_1, \ldots, e_n$ *is a basis of* $\mathfrak{g}$*, the products* $e_1^{i_1} \cdots e_n^{i_n} \in U(\mathfrak{g})$ *with* $i_j \geq 0$ *form a basis of* $U(\mathfrak{g})$*.*

Corollary 10.11. *Suppose $\mathfrak{g}_1, \mathfrak{g}_2$ are two Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces. Then the multiplication map*

$$U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \to U(\mathfrak{g})$$

is an isomorphism of vector spaces.

Indeed, the associated graded map is the multiplication $S(\mathfrak{g}_1) \otimes S(\mathfrak{g}_2) \to S(\mathfrak{g})$, which is well-known to be an isomorphism. The following is left as an exercise:

Corollary 10.12. *The algebra $U(\mathfrak{g})$ has no (left or right) zero divisors.*

We will give a proof of the PBW theorem for the special case that $\mathfrak{g}$ is the Lie algebra of a Lie group G. (In particular, $\mathfrak{g}$ is finite-dimensional.) The idea is to relate the enveloping algebra to differential operators on G. For any manifold M, let

$$\mathrm{DO}^{(k)}(M) = \{D \in \mathrm{End}(C^\infty(M)) |\ \forall f_0, \ldots, f_k \in C^\infty(M),\ \mathrm{ad}_{f_0} \cdots \mathrm{ad}_{f_k} D = 0\}$$

be the differential operators of degree k on M. Here $\mathrm{ad}_f = [f, \cdot]$ is commutator with the operator of multiplication by f. [4] By polarization, $D \in \mathrm{DO}^{(k)}(M)$ if and only if $\mathrm{ad}_f^{k+1} D = 0$ for all f.

Remark 10.13. We have $\mathrm{DO}^{(0)}(M) \cong C^\infty(M)$ by the map $D \mapsto D(1)$. Indeed, for $D \in \mathrm{DO}^{(0)}(M)$ we have $D(f) = D(f \cdot 1) = [D, f]1 + fD(1) = fD(1)$. Similarly

$$\mathrm{DO}^{(1)}(M) \cong C^\infty(M) \oplus \mathfrak{X}(M),$$

where function component of D is $D(1)$ and the vector field component is $[D, \cdot]$. Note that $[D, \cdot]$ is a vector field since $[D, f_1 f_2] = [D, f_1]f_2 + f_1[D, f_2]$ with $[D, f_i] \in C^\infty(M)$. The isomorphism follows from $D(f) = D(f \cdot 1) = [D, f] \cdot 1 + f\, D(1)$.

The algebra $\mathrm{DO}(M)$ given as the union over all $\mathrm{DO}^{(k)}(M)$ is a filtered algebra: the product of operators of degree k, l has degree $k + l$. Let $\mathrm{gr}(\mathrm{DO}(M))$ be the associated graded algebra.

Proof of the PBW theorem. (for the special case that $\mathfrak{g}$ is the Lie algebra of a Lie group G). The map $\kappa \colon \mathfrak{g} \to \mathrm{DO}(G), \xi \mapsto \xi^L$ is a Lie algebra morphism, hence by the universal property it extends to an algebra morphism

$$\kappa_U \colon U(\mathfrak{g}) \to \mathrm{DO}(G).$$

The map κ_U preserves filtrations Let $S\mathfrak{g} \cong \mathrm{Pol}(\mathfrak{g}^*),\ x \mapsto p_x$ be the identification with the algebra of polynomials on $\mathfrak{g}^*$, in such a way that $x = \xi_1 \cdots \xi_k \in S^k(\mathfrak{g})$ corresponds to the polynomial $p_x(\mu) = k! \langle \mu, \xi_1 \rangle \cdots \langle \mu, \xi_k \rangle$.

Given $x \in S^k(\mathfrak{g})$, $\mu \in \mathfrak{g}^*$, choose $f \in C^\infty(G)$ and $y \in U^{(k)}(\mathfrak{g})$ such that

$$\mu = \mathrm{d}_e f \colon \mathfrak{g} \to \mathbb{R},\ \ j_S(x) = y \mod U^{(k)}(\mathfrak{g}).$$

[4]In local coordinates, such operators are of the form

$$D = \sum_{i_1 + \ldots + i_n \le k} a_{i_1 \cdots i_n} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}$$

with smooth functions $a_{i_1 \cdots i_n}$, but for our purposes the abstract definition will be more convenient.

The differential operator $D = \kappa_U(y) \in \mathrm{DO}^{(k)}(G)$ satisfies

$$\mathrm{ad}_f^k(D)|_e = (-1)^k p_x(\mu),$$

by the calculation

$$\mathrm{ad}_f^k(\xi_1^L \cdots \xi_k^L)|_e = (-1)^k k! \xi_1^L(f) \cdots \xi_k^L(f)|_e = (-1)^k k!\ \langle \xi_1, \mu \rangle \cdots \langle \xi_k, \mu \rangle.$$

Hence, if $j_S(x) = 0$ so that $y \in U^{(k-1)}(\mathfrak{g})$ and hence $D \in \mathrm{DO}^{(k-1)}(G)$, i.e. $\mathrm{ad}_f^k(D) = 0$, we find $p_x = 0$ and therefore $x = 0$. □

Exercise 10.14. For any manifold M, the inclusion of vector fields is a Lie algebra morphism $\mathfrak{X}(M) = \Gamma(TM) \to \mathrm{DO}(M)$. Hence it extends to an algebra morphism $U(\mathfrak{X}(M)) \to \mathrm{DO}(M)$, which in turn gives maps

$$S^k(\mathfrak{X}(M)) \to \mathrm{gr}^k(U(\mathfrak{X}(M))) \to \mathrm{gr}^k(\mathrm{DO}(M)).$$

Show that this map descends to a map $\Gamma(S^k(TM)) \to \mathrm{gr}^k(\mathrm{DO}(M))$. Consider $\mathrm{ad}_f^k(D)$ to construct an inverse map, thus proving

$$\Gamma(S(TM)) \cong \mathrm{gr}(\mathrm{DO}(M)).$$

(This is the *principal symbol* isomorphism.)

The following is a consequence of the proof, combined with the exercise.

Theorem 10.15. *For any Lie group G, with Lie algebra $\mathfrak{g}$, the map $\xi \mapsto \xi^L$ extends to an isomorphism*

$$U(\mathfrak{g}) \to \mathrm{DO}^L(G)$$

where $\mathrm{DO}^L(G)$ is the algebra of left-invariant differential operators on G.