alloscore talk

Motivation

Gov to allocate scarce hospital supplies across 50 states

- given many (probabilistic) forecasts of need, which should they use?

Raises question: how will chosen forecast be used?

One possibility: to minimize $E_F \sum_{l=1}^{50} \max(0, Y_i - x_i)$

- $Y_{1:50} \sim F$ are future needs (F unknown)
- $x_{1:50}$ are allocated quantities
- only K units of supplies are available

A constrained stochastic optimization problem

Let $Z_F(\mathbf{x}) = E_F \sum \max(0, Y_i - x_i)$

Lagrange multiplier equation: $\nabla Z(\mathbf{x}) + \lambda = 0$

$$\frac{d}{dx_i} E_F \max(0, Y_i - x_i) = \frac{d}{dx_i} \int_{x_i}^{\infty} (y - x_i) f_i(y) dy = F_i(x_i) - 1 = -\lambda$$

$$x_i = F_i^{-1}(1 - \lambda)$$

Solve $\sum_{l=1}^{50} F_i^{-1}(1 - \lambda) = K$ for $\lambda^\star \in (0, 1)$ to get best allocation $x_i^\star = F_i^{-1}(1 - \lambda^\star)$

Solution for location-scale family F_i's

$$\sum F_i^{-1}(1-\lambda) = \sum \mu_i + \sigma_i \Phi^{-1}(1-\lambda) = K$$

$$\implies \lambda^\star = 1 - \Phi\left(\frac{K - \sum \mu_j}{\sum \sigma_j}\right)$$

$$\implies x_i^\star = F_i^{-1} \circ \Phi\left(\frac{K - \sum \mu_j}{\sum \sigma_j}\right) = \mu_i + \sigma_i \left(\frac{K - \sum \mu_j}{\sum \sigma_j}\right)$$

Interpretation: underdispersion at loc i -> too little of “excess or shortage in mean” $K - \sum \mu_j$ given to or removed from i

Allocation as forecast perfomance

$\mathbf{x}_{\mathrm{F}}^{\star}$ is a **Bayes act** for F under loss

$$\mathrm{s}(\mathbf{x}, \mathbf{y}) = \sum \max(0, \mathrm{y_i} - \mathrm{x_i})$$

The quality of forecast F then can be estimated by how much loss using it actually incurs, i.e.,

$$\mathrm{S}(\mathbf{F}, \mathbf{y}) = \sum \max(0, \mathrm{y_i} - \mathrm{x}_{\mathrm{F,i}}^{\star})$$

- or perhaps $\mathrm{S}(\mathbf{F}, \mathbf{y}) - \max(0, \sum \mathrm{y_i} - \mathrm{K})$, perfomance against an oracle

Scoring function for LS case

$$s_K(\mathbf{Q}, \mathbf{y}) = \sum \mathbb{1}\left\{ \frac{K - \sum \mu_i}{\sum \sigma_i} \leq \frac{y_i - \mu_i}{\sigma_i} \right\} \left(y_i - x^{\star}_{F,i} \right)$$

Scoring Functions in general

Forecaster A asked to estimate a **functional** $T(F_Y)$ for a RV $Y \sim F_Y$.

- mean, median, variance, etc.
- A believes that $Y \sim F$

After A gives us x and nature realizes $Y = y$, we judge performance with a **scoring function** $s(x, y)$

s is **(strictly) consistent** for T if A cannot expect to do better than offer $x = T(F)$ as a forecast of $T(F_Y)$, i.e,

$$E_F[s(T(F)), Y](<) \leq E_F[s(x, Y)] \text{ for all } F, x$$

Elicitability

A functional is **elicitable** if it has a strictly consistent scoring function.

Basic example are α quantiles, with SF:

$$
\begin{aligned}
s(x,y) = s_{O,U}(x,y) &= O(g(x)-g(y))_+ + U(g(x)-g(y))_- . \\
&= \varkappa\left((1-\alpha)(g(x)-g(y))_+ + \alpha(g(x)-g(y))_-\right) \\
&= \varkappa(\mathbb{1}\{x>y\}-\alpha)(g(x)-g(y)) \\
&:= s_{\varkappa,\alpha}(x,y) := \varkappa s_\alpha(x,y)
\end{aligned}
$$

$\alpha = U/(U+O), \varkappa = O+U$

Reformulation of allocation problem

Hosp allocation problem of this form in each location, with $O_i = 0, U_i = 1$

With constraint $x \leq K$ Bayes act is just $Q = \min(K, F^{-1}(U/(U+O))) = K$

But taken jointly, have non-trivial BA $Q_i = F_i^{-1}(\alpha_i(1 - \lambda_K/U_i))$ or(!) 0

- $\lambda_K = 0$ corresponds to case when $\sum Q_i(\alpha_i) \leq K$

Domain versions

Newsvendor problem

Ordering decision faced by a retailer of a perishable good (such as newspapers) when customer demand is uncertain.

- Allocation problem appears as **multi-product NVP with stocking constraint**
- Well-studied in OR literature

Meteoroligist/Epidemiologist Cost/Loss Problem

- have not seen much in allocation form

No literature regarding allocation solution as scoring rule

Solving Allocation Problem

Straightforward in simple situations

- all F_i smooth with same support starting at 0
- K not too much smaller than $\mathbf{w}^T\mathbf{Q}(\alpha)$

$$\begin{aligned}
\nabla Z(\mathbf{x}) + \lambda\mathbf{w} &= 0 \\
\implies \varkappa_i(F_i(Q_i) - \alpha) + \lambda w_i &= 0 \\
\implies Q_i(\lambda) &= F_i^{-1}(\alpha_i(1 - \lambda w_i/U_i))
\end{aligned}$$

Now solve $\sum_{i=1}^{N} w_i Q_i(\lambda) = K$ for $\lambda^\star$ to get optimal allocation $\mathbf{Q}(\lambda^\star)$.

Complications

If some Q_i need to be 0, we don't know in general which set of 0's is best

- problem becomes intractable for large N
- need a non-linear programming algorithm… `alloscore` implements one

Scoring Rules

- Should be regret-based, i.e., how well you do against an oracle that knows y's and never wastes any resources

Next Steps

- Use Evan's package `distfromq` to interpolate quantile functions and distribtuions from hub quantile forecasts and score across K range
- extend this to no overprediction penalty i.e. $\alpha_i = 1$ scenario

Aspirations

- Allocation skill could be an indicator for peak/outbreak detection skill