Connection between allocation scoring rules an CRPS: example with exponential forecasts

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```
library(tidyr)
library(dplyr)
library(ggplot2)
```

Consider a simplified setting where $O_i = 0$ and $U_i = U$ are shared for all i, and $g_i(x_i) = x_i$. This is the setting we have laid out in the initial alloscore manuscript. We consider three settings:

1. The decision maker is fixed with a fixed, known, constraint on the total allocation, K.
2. The decision maker has some uncertainty about the total constraint K.
3. We would like to understand the relationship between the allocation score and CRPS by exhibiting the distribution on K that would lead to an "equally-weighted" CRPS.

Throughout, we will suppose there are $n = 2$ locations, and a forecaster produces the forecasts $Y_1 \sim Exp(1/\sigma_1)$ with $\sigma_1 = 1$ and $Y_2 \sim Exp(1/\sigma_2)$ with $\sigma_2 = 5$. The quantile functions corresponding to these forecasts are given by $F_i^{-1}(\tau) = -\sigma_i \log(1-\tau)$, where τ is a probability level which we take to be in $(0, 1)$.

The expected loss function is

$$\bar{s}_F(\mathbf{x}) = \sum_{i=1}^{2} \mathbb{1}\{x_i < 0\}(\sigma - x_i) + \mathbb{1}\{0 \le x_i\}\sigma e^{-x_i/\sigma}.$$

Fixed K

At a fixed constraint K, in our simplified setting the solution of the allocation problem is given by the quantiles $(F_1^{-1}(\tau), F_2^{-1}(\tau))$ at a probability level τ (connecting to notation elsewhere, $\tau = 1 - \lambda^\star$) such that

$$\begin{aligned} K &= F_1^{-1}(\tau) + F_2^{-1}(\tau) \\ &= -\log(1-\tau)(\sigma_1 + \sigma_2) \\ &= h(\tau) \end{aligned}$$

We can rearrange to obtain

$$\tau = h^{-1}(K) = 1 - \exp[-K/(\sigma_1 + \sigma_2)]. \tag{1}$$

We can visualize this in terms of the expected loss function (shown in shades of blue and yellow) as follows:

```
sigma_1 <- 1
sigma_2 <- 5

K <- 5

tau <- 1 - exp(-K/(sigma_1 + sigma_2))

x_1star <- qexp(tau, rate = 1 / sigma_1)
x_2star <- qexp(tau, rate = 1 / sigma_2)
```

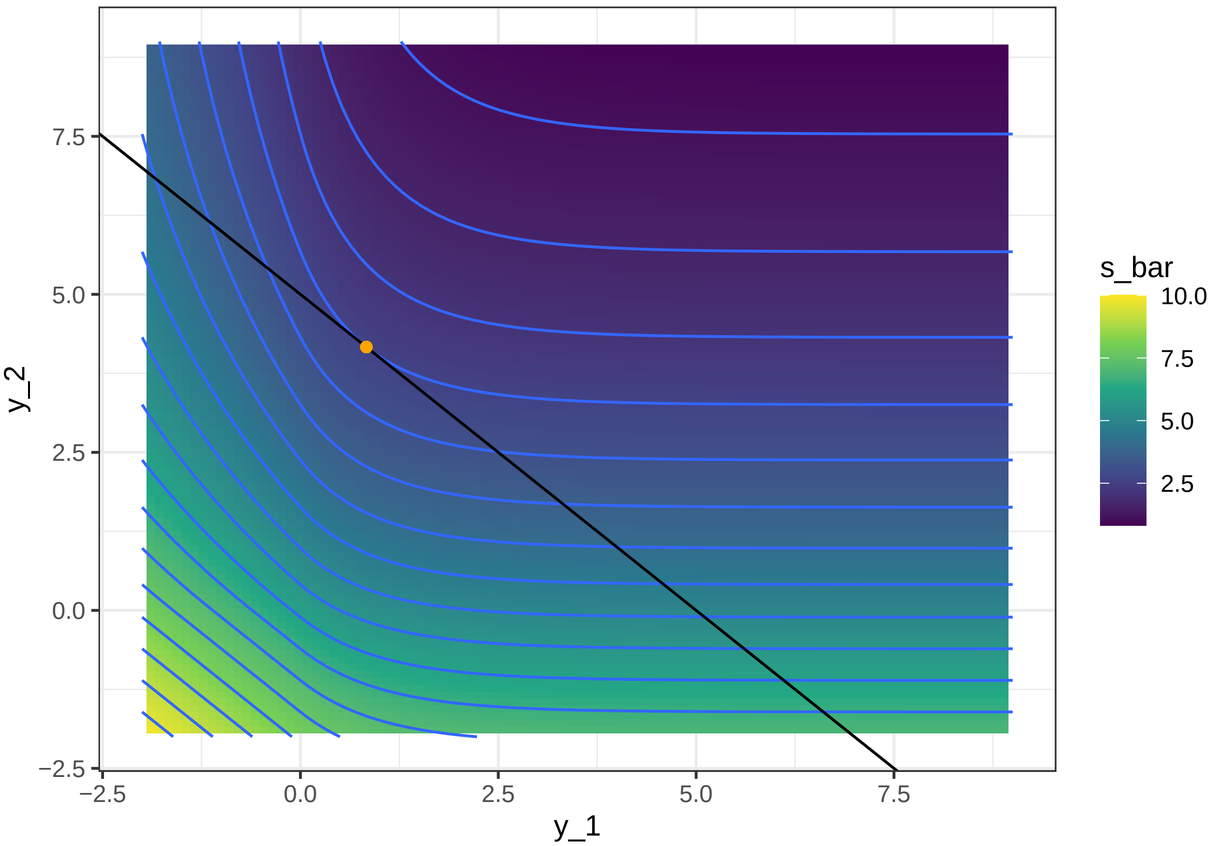
```r
n_grid <- 101
grid_l <- -2.0
grid_u <- 9.0
y_grid <- tidyr::expand_grid(
    y_1 = seq(from = grid_l, to = grid_u, length.out = n_grid),
    y_2 = seq(from = grid_l, to = grid_u, length.out = n_grid)
)

s_bar <- function(x_1, x_2) {
    sigma_1 * exp(-x_1 / sigma_1)*(x_1>=0) + sigma_2 * exp(-x_2 / sigma_2)*(x_2>=0) +
    (sigma_1 - x_1)*(x_1<0) + (sigma_2 - x_2)*(x_2<0)
}

joint_dist <- y_grid %>%
    dplyr::mutate(
        s_bar = s_bar(y_1, y_2)
    )

ggplot(data = joint_dist) +
    geom_raster(aes(x = y_1, y = y_2, fill = s_bar)) +
    geom_contour(mapping = aes(x = y_1, y = y_2, z = s_bar), breaks = s_bar(x_1star, x_2star)%%1 + seq(
    geom_abline(intercept = K, slope = -1) +
    geom_point(x = x_1star, y = x_2star, color = "orange") +
    scale_fill_viridis_c() +
    xlim(grid_l, grid_u) +
    ylim(grid_l, grid_u) +
    theme_bw()
```

```
## Warning: Removed 400 rows containing missing values (`geom_raster()`).
```



Suppose we adjust the scale parameters in both locations by the multiplicative factor c, so that our updated forecast distributions are $Y_1 \sim Exp(1/(c\sigma_1))$ and $Y_2 \sim Exp(1/(c\sigma_2))$. As a function of c, the optimal allocations are at the quantile level

$$\tau(c) = h^{-1}(K) = 1 - \exp[-K/(c\sigma_1 + c\sigma_2)]$$

The optimal allocations are then

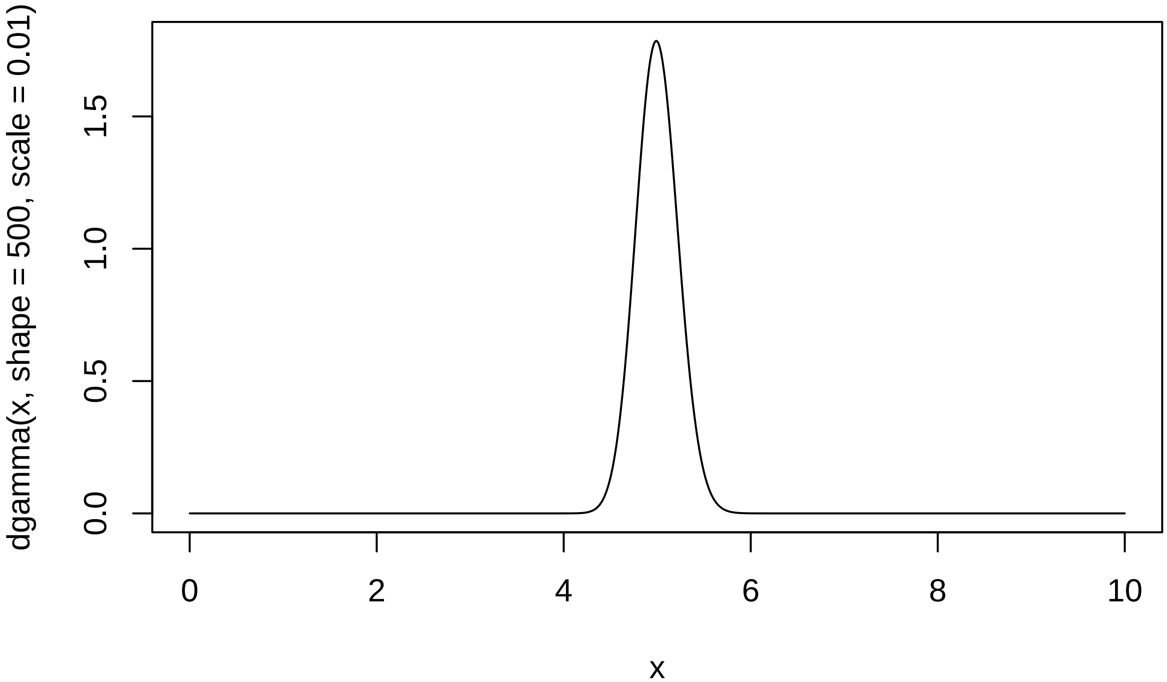
$$\begin{aligned} F_i^{-1}(\tau(c)) &= -c\sigma_i \log(1 - \tau(c)) \\ &= -c\sigma_i - K/(c\sigma_1 + c\sigma_2) \\ &= \frac{\sigma_i}{\sigma_1 + \sigma_2} K \end{aligned}$$

Note that these allocations do not depend on the parameter c, only the relative scales in the two locations. Since the allocations are the same for any pairs of forecast distributions with the same relative scales, the allocation score is also the same. While this result is specific to exponentially-distributed forecasts, it illustrates an important feature of the allocation score: in general, the allocation score does not measure whether forecast distributions in individual locations are correct in an absolute sense; instead, it measures whether the relative magnitudes in different locations are forecasted correctly. It is these relative magnitudes that determine whether or not forecast-informed allocations send resources to the correct locations.

Obtaining the CRPS weighting associated with a distribution on K

Suppose that the decision maker has some uncertainty about the level of the constraint K, expressed by the distribution F_K. For concreteness, here we take this distribution to be $Gamma(500, 0.01)$ using a shape and scale parameterization. Here's a picture of this distribution, which is concentrated near $K = 5$:

```r
curve(dgamma(x, shape = 500, scale = 0.01), from = 0, to = 10, n = 1001)
```



As discussed above, given fixed forecasts F_1 and F_2, for each value of the constraint K, a quantile probability level $\tau = h^{-1}(K) = 1 - \exp[-K/(\sigma_1 + \sigma_2)]$ is determined. We can therefore use a change of variables to obtain a density for τ from the density for K as follows:

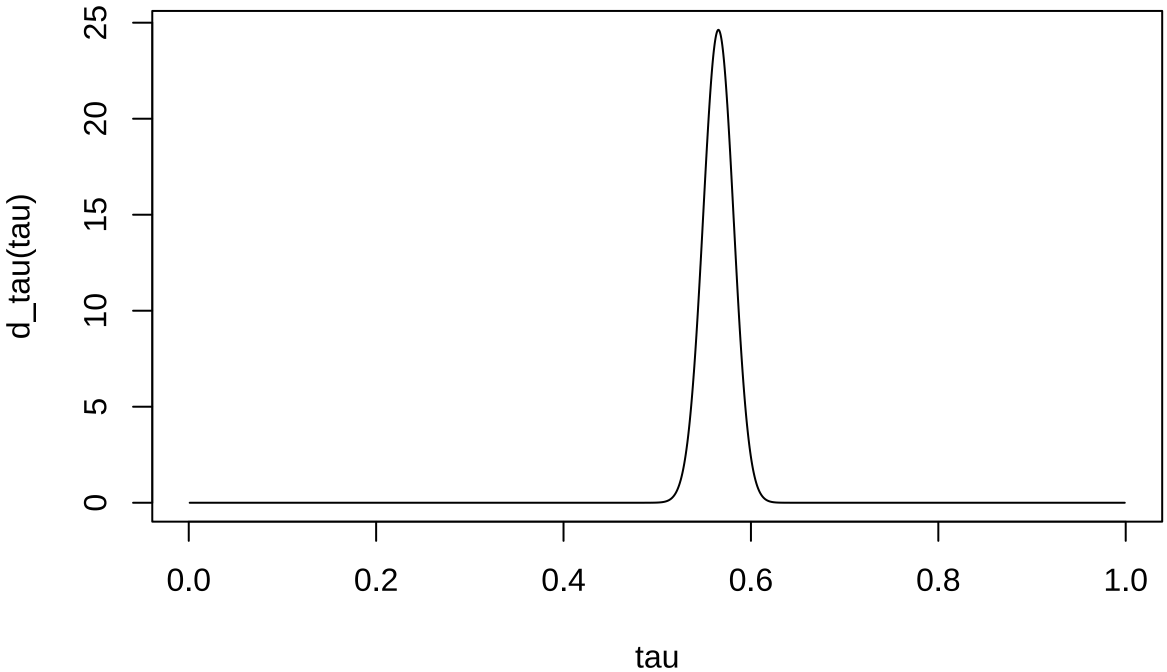
$$\begin{aligned} f_T(\tau) &= f_K(h(\tau)) \left| \frac{d}{d\tau} h(\tau) \right| \\ &= f_K(-\log(1-\tau)(\sigma_1 + \sigma_2)) \frac{(\sigma_1 + \sigma_2)}{1 - \tau} \end{aligned}$$

Here's a plot of this induced density on τ:

```r
tau <- seq(from = 0.001, to = 0.999, length.out = 10000)

d_tau <- function(tau) {
    dgamma(-log(1 - tau) * (sigma_1 + sigma_2), shape = 500, scale = 0.01) *
                ((sigma_1 + sigma_2) / (1 - tau))
}

plot(tau, d_tau(tau), "l")
```




```
# note that this is a density
sum(d_tau(tau) * diff(tau)[1])
```

```
## [1] 1
```

We can think of the allocation score determined by F_K as corresponding to a weighted CRPS with the weighting expressed by the above density on quantile levels. However, note that this interpretation is specific to this forecast. A different forecast would translate to a different weighting on quantile levels.

Reproducing equally weighted CRPS

Going in the other direction, given forecasts F_1 and F_2, we can determine the distribution on values of the constraint K that corresponds to an equal weighting of all quantile levels, as is done in CRPS. Again, this implied distribution on K depends on the forecasts and will be different for different forecasters.

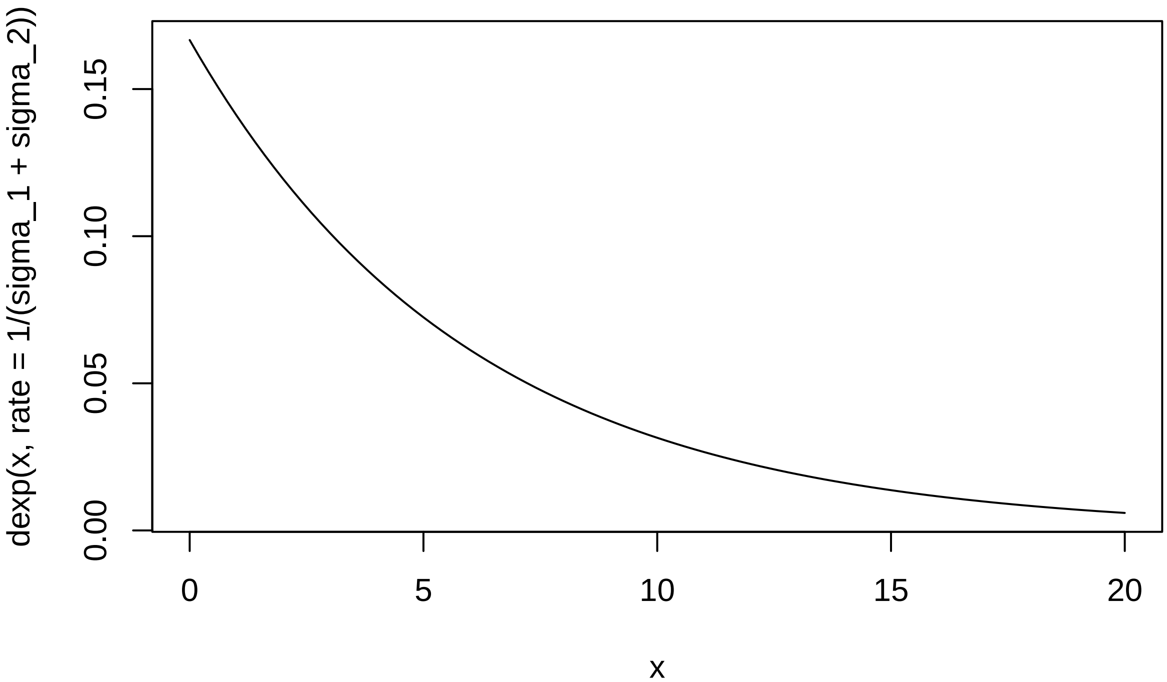
In this setting, we start with $\tau \sim Unif(0,1)$, and note that given a value of τ we can calculate $K = h(\tau) = -\log(1-\tau)(\sigma_1 + \sigma_2)$. This induces a distribution on K with density

$$\begin{aligned} f_K(K) &= f_T(h^{-1}(K)) \left| \frac{d}{dK} h^{-1}(K) \right| \\ &= 1 \cdot (\sigma_1 + \sigma_2)^{-1} \exp[-K/(\sigma_1 + \sigma_2)] \end{aligned}$$

This is the density of an $Exp(\sigma_1 + \sigma_2)$ random variable.

Thus, with our forecasts, CRPS corresponds to the weighting on resource constraints illustrated in the following figure:

```r
curve(dexp(x, rate = 1 / (sigma_1 + sigma_2)), 0, 20)
```



This corresponds to placing large mass on small values of the constraint K, which may not correspond well to actual knowledge about the resource constraints.