SUFFICIENT STATISTICS FOR NONLINEAR TAX SYSTEMS WITH PREFERENCE HETEROGENEITY

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December 2021

Abstract

This paper provides general and empirically implementable sufficient statistics formulas for optimal nonlinear tax systems in the presence of preference heterogeneity. We study unrestricted tax systems on income and savings (or other commodities) that implement the optimal direct-revelation mechanism, as well as simpler tax systems that impose common restrictions like separability between earnings and savings taxes. We characterize the optimum using familiar elasticity concepts and a sufficient statistic for across-income preference heterogeneity: the difference between the cross-sectional variation of savings with income, and the causal effect of income on savings. The Atkinson-Stiglitz Theorem is a knife-edge case corresponding to zero difference, and a number of other key results in optimal tax theory are subsumed as special cases. Our formulas also apply to other sources of across-income heterogeneity, including heterogeneity in rates of return on savings, inheritances, and the ability to shift income between tax bases. We provide tractable extensions of these results that include multidimensional heterogeneity, additional efficiency rationales for taxing heterogeneous returns, and corrective motives to encourage more saving. Applying these formulas in a calibrated model of the U.S. economy, we find that the optimal savings tax is positive and progressive. *JEL Codes:* D61, H21, H24

*We are grateful to Afras Sial for excellent research assistance. This project was supported through a Quartet Pilot Research award and was funded by the Center for Health Incentives and Behavioral Economics at the University of Pennsylvania, by The Research Council of Norway 315765, and by the Alfred P. Sloan Foundation. Ferey gratefully acknowledges the financial support of Labex ECODEC at CREST, and of the Deutsche Forschungsgemeinschaft through CRC TRR 190 at LMU Munich. The content is solely the responsibility of the authors and does not necessarily represent the official views of the University of Pennsylvania. We thank Pierre Boyer, Philippe Choné, Ashley Craig, Nathan Hendren, Laurence Jacquet, Dirk Krueger, Etienne Lehmann, Jonas Loebbing, Jean-Baptiste Michau, Andreas Peichl, Dominik Sachs, Emmanuel Saez, Florian Scheuer, Stefanie Stantcheva, Joseph Stiglitz, Aleh Tsyvinski, Nicolas Werquin, Eric Zwick, participants at NBER Macro Public Finance, NBER Public Economics, IIPF, LAGV, NTA, and audiences at UC Berkeley, CREST - Ecole Polytechnique and LMU Munich for their valuable comments. Ferey: LMU Munich & CESIFO (antoine.ferey@econ.lmu.de). Lockwood: Wharton & NBER (ben.lockwood@wharton.upenn.edu). Taubinsky: UC Berkeley & NBER (dmitry.taubinsky@berkeley.edu).

1 Introduction

Taxes on capital income, estates, inheritances, and certain categories of consumption are a widespread feature of modern tax systems. Yet there is considerable debate, both among economists and in policy circles, about their optimal design. The celebrated theorem of Atkinson and Stiglitz (1976) is sometimes interpreted to suggest that such taxes should be eliminated. The theorem states that if preferences are homogeneous and weakly separable in consumption and labor, then differential taxes on commodities—including on future consumption in the form of savings—are suboptimal, and welfare is maximized when redistribution is carried out solely through an income tax. However, as was appreciated by contemporaneous work (Mirrlees, 1976) and emphasized by the authors themselves (Stiglitz, 2018), the assumptions underpinning the Atkinson-Stiglitz Theorem are strong, and the theorem does not apply in settings where earnings ability co-varies with commodity preferences, or with other attributes that affect saving levels, such as heterogeneous inheritances, rates of return, or income-shifting abilities.

As a result, an active literature has developed to demonstrate that non-zero commodity and savings taxes may be optimal when the Atkinson-Stiglitz assumptions are relaxed. Yet general, elasticity-based "sufficient statistics" formulas for optimal nonlinear commodity and savings taxes, of the kind common in the optimal income tax literature (e.g., Saez, 2001), have remained elusive. Existing results have instead studied settings with restrictions to a small number of discrete "types," restrictions on the form of utility functions or tax functions, or they have focused on qualitative insights.[1]

In this paper, we derive generally applicable sufficient statistics formulas for optimal linear and nonlinear commodity taxes in a setting where preferences or other consumer attributes, such as inheritances or rates of return, vary with income-earning ability. We study a general version of standard models where consumers with heterogeneous earning abilities and tastes choose labor supply and a consumption and savings bundle that exhausts their after-tax income.[2] Our formulas nest prior results in this setting, as well as the Atkinson-Stiglitz Theorem itself, as special cases. For concreteness in what follows, we describe results in terms of taxes on savings, although they also apply to other commodities.

We organize the paper around the following key contributions.

The first is a set of results about the optimal unrestricted, nonlinear tax system on earnings and savings. We begin with the question of implementation: Can the optimal allocation be implemented by a smooth (i.e., differentiable) tax on earnings and savings? A smooth tax system allows for *double deviations*, where individuals can jointly alter their earnings and savings to reach bundles not chosen

[1] Of particular note, Saez (2002) used a model like the one in this paper to answer the qualitative question of when a "small" *linear* commodity (savings) tax can increase welfare in the presence of preference heterogeneity, but left to future work the task of deriving an expression for the optimal tax, writing "It would of course be extremely useful to obtain optimal commodity tax formulas" in such a framework.

[2] See, e.g., Atkinson and Stiglitz (1976); Saez (2002); Farhi and Werning (2010); Diamond and Spinnewijn (2011); Golosov et al. (2013); Piketty and Saez (2013); Scheuer and Wolitzky (2016); Saez and Stantcheva (2018); Allcott et al. (2019); Gaubert et al. (2021)

by any other type; such deviations can simply be disallowed under the optimal mechanism. This introduces a complication not present in the standard income taxation model of Mirrlees (1971), nor in the Atkinson-Stiglitz setting: in the presence of preference heterogeneity, double deviations are generally the most attractive direction of adjustment. Nevertheless, we show that under modest regularity conditions, it is possible to construct a smooth tax system, dependent only on earnings and savings, that implements the optimal direct-revelation mechanism.

We then present new elasticity-based formulas for the optimal nonlinear tax on savings and earnings. We show that these formulas can be written entirely in terms of welfare weights and empirically measurable statistics, including a key sufficient statistic for preference heterogeneity: the difference between the cross-sectional variation of savings s with earnings z, denoted $s'(z)$, and the causal effect of income changes on savings, which we denote $s'_{inc}(z)$. The residual, $s'_{pref}(z) := s'(z) - s'_{inc}(z)$, is a sufficient statistic for (local) preference heterogeneity.[3] Intuitively, the total derivative of s with respect to z is the sum of two partial derivatives: (i) the causal income effect s'_{inc}, holding preferences constant and (ii) the degree to which higher-ability types prefer more s, holding earnings constant. The second component is captured by s'_{pref}, which we show can be estimated from existing data on the correlational and causal associations with earnings, avoiding the need to explicitly measure or model the relationship between unobserved preferences and ability.

The formula for optimal savings tax rates is a product of s'_{pref} and a term that resembles the optimal income tax formula in Saez (2001), with the elasticity of earnings replaced by the elasticity of savings with respect to the savings tax rate. This result provides an immediate generalization of the Atkinson-Stiglitz Theorem, as it implies that the optimal savings tax rate is everywhere zero when $s'_{pref}(z) = 0$ for all earnings levels z. We also present Pareto-efficiency conditions that use the same sufficient statistics and that can be used to test for (and address) inefficiencies in existing tax systems, without additional assumptions about social marginal welfare weights.

We show that these formulas apply in a variety of other settings that depart from the Atkinson-Stiglitz assumptions, including heterogeneous endowments or inheritances, differential rates of return on investments, human capital investments that enhance productivity, and the ability to engage in income shifting (Slemrod, 1995). In each case, the difference between the cross-sectional profile of savings and the causal income effect on savings, $s'(z) - s'_{inc}(z)$, is the key sufficient statistic for across-income heterogeneity. Consequently, these formulas can be viewed both as a synthesis of prior work that qualitatively studied these extensions in isolation, and as a method for quantifying optimal tax rates when several of these forces are at play simultaneously. For simplicity, we still refer to $s'_{pref}(z)$ as a measure of preference heterogeneity, but we emphasize that it is a sufficient statistic for many other forms of heterogeneity across incomes, or for deviations from weak separability as in, e.g., Corlett and Hague (1953).

Our second contribution is a characterization of what we call "simple tax systems." We document that across a large number of countries, the tax system consists of a nonlinear tax on income,

[3] To our knowledge, this statistic was first employed in Allcott et al. (2019), in a setting restricted to a separable linear commodity tax, which of course cannot implement the optimal mechanism.

accompanied by taxes on savings vehicles that can be classified as one of three types: (i) a separable linear (SL) savings tax; (ii) a separable nonlinear (SN) savings tax; or (iii) a system with a linear earnings-dependent (LED) savings tax, which allows, for example, lower-income people to have their savings taxed at a lower linear rate, as is the case for long-term capital gains in the U.S. We show that the optimal tax policy within each of these classes of simple systems can be expressed using the same sufficient statistics that appear in our formulas for the optimal smooth unrestricted tax system. Moreover, we provide sufficient conditions for the SN and LED systems to implement the optimal mechanism. Finally, an advantage of focusing on simple tax systems is that we can extend our results tractably to multidimensional heterogeneity and to a potentially suboptimal income tax. In this more general setting, the causal effect of income on savings, together with the joint distribution of savings and income, remain sufficient statistics for characterizing the optimal savings tax.

We provide further generality in three tractable extensions of our baseline results. First, we consider many dimensions of consumption and savings. For example, different categories of savings might be taxed differently. In this case, the additional necessary sufficient statistics are cross-price elasticities, which allow us to compute *tax diversion ratios*—the fiscal spillovers to taxes collected on goods $j \neq i$ relative to the reduction in taxes collected on good i, when the price of good i is increased. The optimal tax rate on good s_i is the sum of the formula in our baseline result and the tax diversion ratios.

In our second extension, we consider situations where the government wants to alter or correct individual behavior. Our model generalizes the setup of Farhi and Werning (2010), in which the government puts more weight on future generations than the parents, to allow for heterogeneous preferences. Our results also cover the case where individuals under-save due to behavioral biases such as myopia or lack of self control, as in Moser and Olea de Souza e Silva (2019).

In our third extension, we study settings in which there is an additional efficiency rationale for taxing savings, because the government can collect savings taxes either before or after returns are earned, and therefore can arbitrage heterogeneous private rates of return by shifting tax collections onto post-returns savings for high earners. This extension relates to independent work by Gerritsen et al. (2020), who study the special case where all across-income heterogeneity is from differences in rates of return, characterizing and quantifying the optimal separable nonlinear savings tax in terms of model primitives.

In the final part of the paper, we apply these sufficient statistics formulas to study the optimal tax treatment of savings in the U.S. We calibrate the distribution of savings across the income distribution using the Distributional National Accounts micro-files of Piketty et al. (2018). This evidence suggests that savings are approximately constant at low incomes but increase convexly at higher incomes, so that the cross-sectional slope $s'(z)$ is increasing with income. To calibrate the causal income effect on savings, we draw on two sources. The first is Fagereng et al. (2021), which estimates the medium-run marginal propensity to save out of windfall income using lottery prizes. The second is a new probability-based survey representing the U.S. adult population, conducted on the AmeriSpeak panel, which asked respondents about their savings behavior in response to a possible raise. The two sources are consistent in suggesting similar magnitudes for $s'_{inc}(z)$, with little variation across incomes. Together, these findings yield a positive and increasing value of the residual $s'(z) - s'_{inc}(z) = s'_{pref}(z)$, our sufficient statistic for heterogeneity, across most of the income distribution. Incorporated into our formulas, this implies a (mostly) positive and progressive optimal tax on savings. Our baseline estimates of optimal savings tax rates are somewhat higher than those currently in place in the U.S. across much of the income distribution, although as in other work, these results are sensitive to the elasticity of savings with respect to tax rates, about which there is still substantial uncertainty.

Our paper contributes to a number of literatures. The first is the literature studying optimal commodity and savings taxation in the presence of preference heterogeneity. Saez (2002) considers the special case of a separable linear commodity tax and derives conditions under which its optimal value is non-zero, but does not provide a formula for the magnitude. Golosov et al. (2013) derive conditions characterizing the optimal mechanism in a model like the one we study, but formulate their results in terms of first-order conditions on structural primitives rather than empirically estimable sufficient statistics. Their empirical estimates suggest substantially less across-income heterogeneity than ours do, resulting in much lower optimal savings tax rates. This difference could be because they study heterogeneity in time discounting only, rather than the broader set of forces that can contribute to $s'_{pref}(z)$ and that we allow in our general characterization.[4] Saez and Stantcheva (2018) study nonlinear capital taxation in a setting without income effects, which corresponds to the special case of our model where $s'_{inc}(z) = 0$ and $s'_{pref}(z) = s'(z)$. They consider multidimensional heterogeneity when tax systems are restricted to be either separable linear or separable nonlinear, so their results can be viewed as a special case of our extension characterizing optimal simple tax systems with multidimensional heterogeneity. Allcott et al. (2019) derive a sufficient statistics formula for the optimal separable linear commodity tax in the presence of preference heterogeneity across incomes.[5] Our results build on these insights by developing methods to characterize and implement the optimal mechanism using an unrestricted smooth nonlinear tax system, by studying other more restricted but still nonlinear tax systems that are commonly used in practice, and by incorporating forms of across-income heterogeneity that are not just preference-based.

Second, we contribute to the literature studying structural models with various departures from the Atkinson-Stiglitz assumptions. As noted above, our sufficient statistics strategy for quantifying

[4]The lower measured heterogeneity in Golosov et al. (2013) could also be driven by attenuation bias. They measure preference heterogeneity by regressing a structural estimate of time preferences on a plausibly noisy proxy of earnings ability (performance on the Armed Forces Qualication Test), which may be biased toward zero due to a noisy right-hand-side variable.

[5]The application of separable *linear* savings taxes in the presence of multidimensional heterogeneity is also considered in Piketty and Saez (2013), Diamond and Spinnewijn (2011), and Gauthier and Henriet (2018). Piketty and Saez (2013) derive sufficient statistics formulas but make the additional restriction of a linear income tax. Diamond and Spinnewijn (2011) and Gauthier and Henriet (2018) allow for a nonlinear income tax but assume a finite number of possible earnings levels, and derive results in terms of model primitives. Jacquet and Lehmann (2021a) provide a generalization to a separable sum of many one-dimensional nonlinear tax schedules.

preference heterogeneity spans several other sources of across-income heterogeneity, which have been studied separately from one another. Gahvari and Micheletto (2016) and Gerritsen et al. (2020) study heterogeneous rates of return, Boadway et al. (2000) and Cremer et al. (2003) study heterogeneous endowments, Christiansen and Tuomala (2008) study income shifting, and Bovenberg and Jacobs (2005) and Bovenberg and Jacobs (2011) study human capital investments.[6] Our methods can be viewed as providing a unified treatment of these different sources of across-income heterogeneity., but also a unified approach that can address—using the same set of sufficient statistics—both the growing literature on simpler tax systems with multidimensional heterogeneity and the smaller literature on optimal mechanisms with unidimensional heterogeneity.

Third, this paper complements the literature on dynamic taxation (see overviews by Golosov and Tsyvinski, 2006; Stantcheva, 2020), which typically assumes homogeneous preferences, but derives a theoretically robust role for capital taxation via the inverse Euler equation (e.g., Golosov et al., 2003; Farhi and Werning, 2013). Our work is complementary in relaxing the assumption of homogeneous and weakly separable preferences, but using a static (2-period) framework. Quantitatively, the dynamic taxation literature tends to find optimal savings "wedges" of only several percentage points (see, e.g., Golosov and Tsyvinski, 2015; Golosov et al., 2016; Farhi and Werning, 2013)—substantially lower than those suggested by our baseline calibrations at the same assumed values of elasticities. This suggests that across-income heterogeneity may play a quantitatively larger role in determining optimal savings tax policy than do the social insurance motives analyzed in the dynamic taxation literature, and it motivates future research incorporating our method of measuring and incorporating across-income heterogeneity into fully dynamic models.

The rest of this paper proceeds as follows. Section 2 presents our model and assumptions. Section 3 shows that smooth tax systems can implement the optimal mechanism, and provides sufficient statistics for optimal smooth tax systems. Section 4 studies simple tax systems. Section 5 presents extensions to our results. Section 6 applies our formulas to quantify optimal savings tax rates in the United States. Section 7 concludes. All proofs are gathered in the Appendix.

2 Model and Assumptions

Agents. There is a population of heterogeneous agents who differ in earnings ability and preferences for s, with their types denoted by θ. We begin with the common assumption that $\theta \in \Theta \subset \mathbb{R}$, where Θ is compact; Section 4.2 considers multidimensional heterogeneity. We assume that θ has a continuously differentiable cumulative distribution function $F(\theta)$.

Agents choose earnings z, and a consumption bundle (c, s), and derive utility $U(c, s, z; \theta)$. One application is where c is period-1 consumption and s is the realized savings in period 2, as in Saez (2002), Golosov et al. (2013), and many others. A second application is where c is period-1 consumption by the parents, while s is the wealth bequeathed to their children and consumed in period 2, as in Farhi and Werning (2010). A third application is where c is numeraire consumption

[6] See Stantcheva (2017) for an analysis of human capital policies in a dynamic setting. Our framework spans the static models in Bovenberg and Jacobs (2005) and Bovenberg and Jacobs (2011) but not more dynamic models.

and s is another dimension of commodity consumption that could be taxed nonlinearly, such as energy or housing.

We assume a linear production technology with marginal rate of transformation p between s and c. In the savings and inheritance interpretations of the model, $p = 1/R$, where R is the gross rate of return in a linear savings technology between the two periods.

Throughout the paper, we assume that:

Assumption 1. *$U(c, s, z; \theta)$ is twice continuously differentiable, increasing and weakly concave in c and s, and decreasing and strictly concave in z. The first derivatives U_c' and U_s' are bounded.*

For example, a frequently used functional form (e.g. Saez, 2002; Golosov et al., 2013) involves additively separable utility and heterogeneity in agents' productivity w and discount factor δ:

$$U(c, s, z; \theta) = u(c) + \delta(\theta)u(s) - k(z/w(\theta)), \tag{1}$$

with $u(.)$ the utility from consumption and $k(z/w)$ the disutility from work. There is preference heterogeneity across income-earning ability when the discount factor $\delta(\theta)$ covaries with productivity $w(\theta)$.

More generally, we say that there is across-ability preference heterogeneity when marginal rates of substitution between c and s vary with earnings ability. We denote the marginal rate of substitution by

$$\mathcal{S}(c, s, z; \theta) := \frac{U_s'(c, s, z; \theta)}{U_c'(c, s, z; \theta)}, \tag{2}$$

and we use the shorthand $S_\theta'(c, s, z; \theta_0) := \frac{\partial}{\partial \theta}\mathcal{S}_\theta(c, s, z; \theta)|_{\theta=\theta_0}$. We define across-ability preference heterogeneity as follows:

Definition 1. *There is* across-ability preference heterogeneity *for s if some agents prefer different (c, s) bundles conditional on having the same earnings level z; i.e.,*

$$\exists\theta_0, \forall(c, s, z), \ S_\theta'(c, s, z; \theta_0) \neq 0. \tag{3}$$

For instance, in the formulation in (1), $S_\theta'(c, s, z; \theta) > 0$ whenever $\delta'(\theta) > 0$. Such across-ability preference heterogeneity is the focus of our results to follow, and for the rest of the paper we will refer to it simply as "preference heterogeneity."

We similarly define $\mathcal{Z}$ as the marginal rate of substitution between consumption c and earnings z,

$$\mathcal{Z}(c, s, z; \theta) := \frac{U_z'(c, s, z; \theta)}{U_c'(c, s, z; \theta)}. \tag{4}$$

Government. An agent's type θ is private information and cannot be observed by the government; only the distribution of types, $F(\theta)$, is known. The government must design a tax and transfer system that depends only on the observable variables (c, s, z).

Without any restrictions on the form of the optimal tax system, the resulting optimal allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ must solve the following program:

$$\max \int_\theta \alpha(\theta) U(c(\theta), s(\theta), z(\theta); \theta) \, dF(\theta), \tag{5}$$

where $\alpha(\theta)$ represents some set of Pareto weights across types, subject to the resource constraint

$$\int_\theta [z(\theta) - ps(\theta) - c(\theta)] \, dF(\theta) \geq E, \tag{6}$$

where E is an exogenous revenue requirement, and incentive compatibility constraints

$$\forall \, (\theta, \theta') \in \Theta^2, \; U(c(\theta), s(\theta), z(\theta); \theta) \geq U(c(\theta'), s(\theta'), z(\theta'); \theta). \tag{7}$$

We refer to an allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ that maximizes (5) subject to (6) and (7) as the *optimal incentive-compatible allocation.*

3 Optimal Smooth Tax Systems

In this section, we provide two key results about *smooth tax systems*, by which we mean twice continuously differentiable tax functions $\mathcal{T}(s, z)$.[7] First, we show that the optimal incentive-compatible allocation is implementable by a smooth tax system, under intuitive regularity conditions. Second, we leverage our first result to derive a sufficient statistics characterization of optimal smooth tax systems.

We maintain the following assumptions throughout the rest of our analysis.

Assumption 2. *In the optimal incentive-compatible allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$, c, s, and z are smooth functions of θ. Any type θ strictly prefers its allocation $(c(\theta), s(\theta), z(\theta))$ to the allocation $(c(\theta'), s(\theta'), z(\theta'))$ of another type $\theta' \neq \theta$.*

Assumption 3. *Along the path of $\{c, s, z\}$ offered in the optimal incentive-compatible allocation $\mathcal{A}$, c and s are smooth functions of z, with c weakly increasing, and the following extended Spence-Mirrlees condition holds:*

$$\mathcal{S}'_\theta(c, s, z; \theta) \frac{ds}{dz} + \mathcal{Z}'_\theta(c, s, z; \theta) > 0. \tag{8}$$

Assumption 2 is a standard assumption required to apply optimal control methods to characterize the optimal allocation.

The main component of Assumption 3 is the extended Spence-Mirrlees condition, which generalizes the standard assumption, first stated in Mirrlees (1971), that $\mathcal{Z}'_\theta(c, s, z; \theta) > 0$. If $\mathcal{Z}'_\theta(c, s, z; \theta) > 0$ and s is *increasing* in z, this condition states that the relationship between earnings ability and

[7] Expressing the tax more generally as $\mathcal{T}(c, s, z)$ is redundant. Given such a tax function, any choice of s and z implies a consumption value given by $\mathcal{C}(s, z) := \max\{c | c = z - s - \mathcal{T}(c, s, z)\}$; thus, one can re-express the tax as a function of only savings and earnings: $\tilde{\mathcal{T}}(s, z) = \mathcal{T}(\mathcal{C}(s, z), s, z)$.

preferences for s isn't "too negative." If $\mathcal{Z}'_\theta(c,s,z;\theta) > 0$ and s is *decreasing* in z, this condition states that the relationship between earnings ability and preferences for s isn't "too positive." In the savings applications we consider, where evidence suggests that the preference for saving rises with income-earning ability ($\mathcal{S}'_\theta(c,s,z;\theta) > 0$) this assumption would be violated only if the optimal mechanism featured a savings allocation that is decreasing with earnings. We have not found examples of such counterintuitive mechanisms in our numerical applications.[8]

One consequence of Assumption 3 and the extended Spence-Mirrlees condition is that $z(\theta)$ is strictly increasing in θ (Appendix Lemma A.1). This allows us to define the function $\vartheta(z)$, which maps each earnings level z to the type to which it is assigned in the optimal incentive-compatible allocation.[9]

3.1 Implementability with Smooth Tax Systems

Definition 2. *We say that an allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ is implementable with a tax system $\mathcal{T}$ if*

1. *$\mathcal{T}$ satisfies type-specific feasibility: $c(\theta) + ps(\theta) + \mathcal{T}(s(\theta), z(\theta)) = z(\theta)$ for all $\theta \in \Theta$, and*
2. *$\mathcal{T}$ satisfies individual optimization: $(c(\theta), s(\theta), z(\theta))$ maximizes $U(c,s,z;\theta)$ for all $\theta \in \Theta$, subject to the constraint $c + ps + \mathcal{T}(s,z) \leq z$.*

Our first result shows that the optimal incentive-compatible allocation is implementable by some smooth tax system.

Theorem 1. *Under Assumption 1, 2, and 3, the optimal incentive-compatible allocation is implementable by a smooth tax system. In this smooth tax system, agents' choices are interior (first-order conditions hold), and their local optima are strict (strict second-order conditions).*

Although it is clear that the optimal incentive-compatible allocation $\{(c(\theta), s(\theta), z(\theta))\}_\theta$ can always be implemented by *some* two-dimensional tax system—for example, by defining $\mathcal{T}(s(\theta), z(\theta)) = z(\theta) - c(\theta) - s(\theta)$ for $\theta \in \Theta$ and letting $\mathcal{T}(s,z) \to \infty$ for $(c,s,z) \notin \{(c(\theta), s(\theta), z(\theta))\}_\theta$—such a tax system is not guaranteed to be smooth. A smooth tax system allows agents to independently adjust s and z locally to points not chosen by any other type in the optimal allocation, and thus the set of possible deviations is much larger than when the optimal mechanism can simply disallow certain allocations.

[8] Note that the assumption that c is increasing could alternatively be characterized not as an assumption about the nature of the optimal mechanism, but rather as a modest assumption on the space of allowable mechanisms. Since the space of weakly increasing functions is compact, an optimal mechanism within this space is guaranteed to exist.

[9] Incentive compatibility implies $\mathcal{S}'_\theta(c(\theta), s(\theta), z(\theta); \theta)s'(\theta) + \mathcal{Z}'_\theta\,(c(\theta), s(\theta), z(\theta); \theta)\,z'(\theta) \geq 0$ for any type θ. Absent preference heterogeneity, $\mathcal{S}'_\theta = 0$, and the standard Spence-Mirrlees condition $\mathcal{Z}'_\theta > 0$ implies that earnings z increase with type θ in any incentive compatible allocation. With preference heterogeneity, this no longer holds.

Starting from any given allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$, a smooth tax system can implement the allocation only by satisfying each type's θ first-order conditions:

$$\mathcal{T}_s'\left(s\left(\theta\right), z\left(\theta\right)\right) = \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta\right) - 1 \tag{9}$$

$$\mathcal{T}_z'\left(s\left(\theta\right), z\left(\theta\right)\right) = \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta\right) + 1. \tag{10}$$

In the presence of preference heterogeneity, individuals' incentives to deviate from their assigned allocation $(c(\theta), s(\theta), z(\theta))$ are higher under a smooth tax system than under an optimal incentive-compatible mechanism. For example, suppose that higher types θ have a stronger relative preference for s. If they deviate downward to some other earnings level $z(\theta') < z(\theta)$, then under the optimal mechanism they will be forced to choose $s(\theta')$. Under a smooth tax system, however, the deviating type θ will choose $s' > s(\theta')$ at earnings level $z(\theta')$, making this *double deviation* more appealing.

Tax implementation results that involve multidimensional consumption bundles and multidimensional tax systems typically avoid the difficulties associated with double deviations by ruling out the type of heterogeneity that we consider here. Thus, to our knowledge, our proof of Theorem 1 is different from typical implementation proofs in the optimal tax literature. The proof, contained in Appendix B.2, proceeds in three steps. The first step is to construct a sequence of tax systems $\mathcal{T}_k$ such that each element in the sequence satisfies type-specific feasibility and the first-order conditions above. The sequence is ordered such that successive elements are increasingly convex around the bundles $(s(\theta), z(\theta))$ offered in the optimal mechanism.

In the second step of the proof, we show that for each type θ there exists a k sufficiently large such that this type's second-order conditions hold at the point $(c(\theta), s(\theta), z(\theta))$. In other words, for each type there is a sufficiently large k such that $(c(\theta), s(\theta), z(\theta))$ is a *local* optimum under the tax system $\mathcal{T}_k$. This step requires auxiliary Lemmas B.1 and B.2, which characterize individuals' budget constraints and second derivatives of indirect utility functions for any tax system $\mathcal{T}$ that preserves only the first-order conditions of the optimal mechanism.

In the third step, we show that there exists a sufficiently large k such that $(c(\theta), s(\theta), z(\theta))$ is a *global* optimum for *all* types θ under $\mathcal{T}_k$. We complete this step via a proof by contradiction. Under the assumption that such a k does not exist, there exists an infinite sequence of values k and types θ_k such that type θ_k prefers to deviate from $(c(\theta_k), s(\theta_k), z(\theta_k))$ under $\mathcal{T}_k$. Because the type space is compact, the Bolzano-Weierstrass Theorem allows us to extract a convergent subsequence of types θ_j who all prefer to deviate from the allocation assigned to them under the optimal mechanism. We show that this implies a contradiction because, roughly speaking, the limit type of this sequence, $\hat{\theta}$, must then prefer to deviate from $(c(\hat{\theta}), s(\hat{\theta}), z(\hat{\theta}))$ to some other allocation $(c(\theta'), s(\theta'), z(\theta'))$ offered in the optimal mechanism.

Theorem 1 is an existence result, and our proof of the theorem does not offer insight into the structure of an optimal tax system. However, because agents' choices are shown to satisfy first-order and second-order conditions in a smooth tax system, we can use variational methods to characterize optimal tax systems. We now proceed by deriving optimal tax formulas expressed in terms of empirically estimable sufficient statistics that transparently highlight the key economic forces governing the optimal tax system.

3.2 Sufficient Statistics for Smooth Tax Systems

3.2.1 Definitions

To define the sufficient statistics we use to characterize the optimal tax system, it is helpful to write agents' optimization problem under a tax system $\mathcal{T}(s,z)$ as

$$\max_{z}\left\{\max_{c,s} U(c,s,z;\theta) \text{ s.t. } c\leq z-s-\mathcal{T}(s,z)\right\}, \tag{11}$$

where the inner problem represents the optimal choices of $c(z;\theta)$ and $s(z;\theta)$ for a given earnings level z, and the outer problem represents the optimal choice of earnings $z(\theta)$ taking into account endogenous choices of c and s.

Earnings responses to tax reforms are captured through ζ_z^c, the compensated elasticity of labor income with respect to the marginal labor income tax rate, and η_z, the income effect parameter. Formally, for each level of earnings $z(\theta)$ chosen by a type θ, we define

$$\zeta_z^c(z(\theta)) := -\frac{1-\mathcal{T}_z'(s(\theta),z(\theta))}{z(\theta)}\frac{\partial z(\theta)}{\partial \mathcal{T}_z'(s(\theta),z(\theta))} \tag{12}$$

$$\eta_z(z(\theta)) := -(1-\mathcal{T}_z'(s(\theta),z(\theta)))\frac{\partial z(\theta)}{\partial \mathcal{T}(s(\theta),z(\theta))} \tag{13}$$

where $\mathcal{T}(s(\theta),z(\theta))$ is the tax liability and $\mathcal{T}_z'(s(\theta),z(\theta))$ is the marginal labor income tax rate. Since the earnings choice takes into account endogenous choices of c and s, these elasticity concepts take into account the full sequence of adjustments due to changes in choices of c and s, as well as those due to any nonlinearities in the tax system.[10]

Changes in s in response to tax reforms are captured through $\zeta_{s|z}^c$, the compensated elasticity of s with respect to the marginal tax rate on s, $\eta_{s|z}$, the income effect parameter, and s_{inc}', the causal effect on consumption of s from a marginal change in gross pre-tax income z. These are formally defined as follows:

$$\zeta_{s|z}^c(z(\theta)) := -\frac{1+\mathcal{T}_s'(s(z;\theta),z)}{s(z;\theta)}\frac{\partial s(z;\theta)}{\partial \mathcal{T}_s'(s(z;\theta),z)}\Big|_{z=z(\theta)} \tag{14}$$

$$\eta_{s|z}(z(\theta)) := -\left(1+\mathcal{T}_s'(s(z;\theta),z)\right)\frac{\partial s(z;\theta)}{\partial \mathcal{T}(s(z;\theta),z)}\Big|_{z=z(\theta)} \tag{15}$$

$$s_{inc}'(z(\theta)) := \frac{\partial s(z;\theta)}{\partial z}\Big|_{z=z(\theta)} \tag{16}$$

where $\mathcal{T}_s'(s(z;\theta),z)$ is the marginal tax rate on s of of a type θ who earns labor income z. Elasticity concepts $\zeta_{s|z}^c$ and $\eta_{s|z}$ are conditional on z. They measure responses to tax reforms and nonlinearities

[10] This corresponds to the type of circular adjustment process described in e.g. Jacquet and Lehmann (2021b).

in the tax system, holding labor income z fixed fixed at $z(\theta)$. Note that we define the elasticity of s with respect to one plus the marginal tax rate, rather than with respect to $p + \mathcal{T}_s'$. This choice is natural in applications where s represents savings. However, defining the elasticity with respect to $p + \mathcal{T}_s'$ may be more natural in applications where s is a commodity sold at after-tax price of $q = p + \mathcal{T}_s'$. It is straightforward to convert our results between these elasticity definitions: the key results in Theorem 2 and Proposition 2 can be obtained by multiplying $\zeta_{s|z}^c$ by $(p + \mathcal{T}_s')/(1 + \mathcal{T}_s')$.[11] For all elasticity concepts, we use the "bar" notation, as in $\overline{\zeta_{s|z}^c}$, to denote a population elasticity.

To quantify preference heterogeneity, we decompose the cross-sectional profile of s. Intuitively, $s'(z)$, the cross-sectional change in s with respect to z comprises both the causal income effect and the degree to which preferences are changing with earnings z. We thus define our measure of local across-income preference heterogeneity, $s'_{pref}(z)$, as the difference between the cross-sectional variation of s along the earnings distribution and the causal income effect $s'_{inc}(z)$:

$$s'_{pref}(z(\theta)) := s'(z(\theta)) - s'_{inc}(z(\theta)) \tag{17}$$

Formally, if we denote by $\vartheta(z)$ the type θ that earns z, $s'(z)$ is a total derivative equal to the sum of two partial derivatives:

$$\frac{ds(z, \vartheta(z))}{dz} = \underbrace{\frac{\partial s(z'; \vartheta(z))}{\partial z'}\Big|_{z'=z}}_{s'_{inc}(z)} + \underbrace{\frac{\partial s(z; \vartheta(z'))}{\partial z'}\Big|_{z'=z}}_{s'_{pref}(z)} \tag{18}$$

The first term, which is equivalent to the definition of $s'_{inc}(z)$ in equation (16), measures how a change in z affects s consumption, holding the type θ constant. The second term, $s'_{pref}(z)$, measures how a change in type affects s consumption, holding earnings z constant.

To illustrate how $s'_{pref}(z)$ relates to model parameters, suppose, as in example (1) above, that $U = \ln c + \delta(\theta) \ln s - k(z/w(\theta))$, where s is period-2 consumption, δ is the discount factor, and that $\mathcal{T}(s, z)$ is a *simple* tax system (see Section 4) that is separable in s and z. Then a z-earner chooses

$$s(z) = \frac{1}{p} \frac{\delta(z)}{1 + \delta(z)} \Big(z - \mathcal{T}(s(z), z) \Big) \tag{19}$$

where $\delta(z)$ is used to denote the discount factor of the z-earner. Cross-sectional heterogeneity in s is then given by

$$s'(z) = \underbrace{\frac{1}{p} \frac{\delta(z)}{1 + \delta(z)} \Big(1 - \mathcal{T}_z'(s(z), z) \Big)}_{s'_{inc}(z)} + \underbrace{\frac{1}{p} \Big(z - \mathcal{T}(s(z), z) \Big) \frac{d}{dz} \left(\frac{\delta(z)}{1 + \delta(z)} \right)}_{s'_{pref}(z)} \tag{20}$$

The causal income effect s'_{inc} is obtained by differentiating (19) with respect to z while holding the discount factor $\delta(z)$ constant. The local preference heterogeneity term $s'_{pref}(z)$ is obtained by

[11]In this case, the only change in Theorem 2 is that the left-hand-side in equation (24) becomes $\frac{\mathcal{T}_s'(s(z),z)}{p + \mathcal{T}_s'(s(z),z)}$, and analogously for Proposition 2.

differentiating (19) with respect to z while holding after-tax income $z - \mathcal{T}$ constant.

The key insight that facilitates measurement is that $s'(z)$ and $s'_{inc}(z)$ are empirically measurable statistics that can be used to indirectly estimate $s'_{pref}(z)$, which is arguably more tractable than directly estimating how time preferences $\delta(z)$ vary with earnings.

Remarks. While we have referred to s'_{pref} as capturing across-income preference heterogeneity, we clarify that this term also captures failures of weak separability as in, e.g., Corlett and Hague (1953). For example, suppose that $z = w(\theta)l$, where l is labor and $w(\theta)$ is the wage, with $w'(\theta) > 0$. Suppose also that consumption of s and leisure are complements. Then, because higher types θ obtain more leisure for a fixed level of earnings z, higher types will have a stronger preference for s holding z constant. Thus, $s'_{pref} > 0$ in this example.

We also note that s'_{pref} does not capture the direct dependence of preferences for s on z. Because we leave the utility function $U(c, s, z; \theta)$ general, our framework accommodates preferences where, for example, the discount factor δ is directly a function of earnings z, but does not depend on earnings ability. In this case, preferences are heterogeneous and vary with earnings in the cross-section, but $s'_{pref} \equiv 0$. The direct dependence of δ on earnings z is fully captured by the s'_{inc} term.

3.2.2 Measurement

Because $s'(z)$ is the cross-sectional variation of s along the income distribution, it can be directly measured using standard data sources. The statistic $s'_{inc}(z)$ can be measured using a variety of strategies. Here we present three different methods of measuring $s'_{inc}(z)$, which rely on different sources of quasi-experimental variation.

Proposition 1. *Define $\xi^s_w(z)$ as the elasticity of s with respect to the wage rate w, $\xi^h_w(z)$ as the elasticity of hours with respect to the wage rate, and $\chi^c_s(z)$ as the elasticity of s with respect to the marginal net of tax rate on labor income. The sufficient statistic $s'_{inc}(z)$ can be measured as follows:*

- *If preferences are weakly separable and the tax system is separable in s and z, $s'_{inc}(z) = \frac{1-T'_z(z)}{1+T'_s(s(z))}\eta_{s|z}(z)$*
- *If wage rates w and hours h are observable, $s'_{inc}(z) = s(z)\frac{\xi^s_w(z)}{w(z)+h(z)\xi^h_w(z)}$*
- *If responses to tax reforms are measurable, $s'_{inc}(z) = \frac{s(z)}{z}\frac{\chi^c_s(z)}{\zeta^c_z(z)}$,*

If individuals' preferences are weakly separable as in example (1) above, and if the tax system is separable in s and z, then $s'_{inc}(z)$ is proportional to the income effect parameter for s. If individuals' preferences are not weakly separable but wage rates w and hours h are observable, $s'_{inc}(z)$ can be related to the elasticity of s with respect to the wage rate and to the elasticity of hours with respect to the wage rate. If the elasticities of both s and z with respect to the marginal tax rate on z are observable, $s'_{inc}(z)$ can be recovered from these elasticities.

A key question for empirical implementation is the time horizon over which the statistics must be measured. Interpreting our static model to represent a steady-state economy, $s'_{inc}(z)$ corresponds to the causal effect of a change in steady-state labor income on steady-state consumption of s.[12] Under the weak separability assumption, it is therefore necessary to measure the *long-run* marginal propensity to consume s. In the case of savings, this is the long-run marginal propensity to save, as estimated by Fagereng et al. (2021) for example, in response to a change in unearned income.[13]

3.2.3 Social marginal welfare weights

To encode the policymaker's redistributive objective, we follow the literature in defining social marginal welfare weights as the marginal social welfare derived from an increase in consumption for an individual at a given point in the savings and earnings distribution:

$$g(s(\theta), z(\theta)) := \frac{\alpha(\theta)}{\lambda} U'_c(c(\theta), s(\theta), z(\theta); \theta). \tag{21}$$

We define $\hat{g}(s, z)$ as the social marginal welfare weights augmented with the fiscal impact of income effects. This represents the social value of marginally increasing the disposable income of individuals with savings s and earnings z. Formally,

$$\hat{g}(s, z) := g(s, z) + \mathcal{T}'_z(s, z) \frac{\eta_z(z)}{1 - \mathcal{T}'_z(s, z)} + \mathcal{T}'_s(s, z) \left(\frac{\eta_{s|z}(z)}{1 + \mathcal{T}'_s(s, z)} + s'_{inc}(z) \frac{\eta_z(z)}{1 - \mathcal{T}'_z(s, z)} \right), \tag{22}$$

where the last term comes from the fact that income effects on earnings, proportional to $\eta_z(z)$, induce changes in savings proportional to $s'_{inc}(z)$ affecting savings tax revenues.

Social marginal welfare weights embed judgments about interpersonal utility comparisons. These are usually treated as normative assumptions, although some research has utilized survey data to estimate these weights (see Saez and Stantcheva, 2016, Appendix C) or estimated them from existing policies via an "inverse optimum" procedure (e.g., Bourguignon and Spadaro 2012; Lockwood and Weinzierl 2016). Such normative assumptions are particularly strong when there is preference heterogeneity, because individuals prefer different bundles—and face different tax burdens—even when they have identical budget sets. Lockwood and Weinzierl (2015) show that this

[12] A natural question is whether the effect of income received earlier in life—e.g., family income in childhood—should be used to measure the long-run income effect $s'_{inc}(z)$. It should not. As shown by Lemma 1 below, the role of $s'_{inc}(z)$ is to quantify the distortion in work-life income induced by a change in the steady-state tax on s, and this distortion depends on the causal effect of earnings *during work-life* on s. To the extent that income earlier in life affects s consumption differently from income during work-life, the former behaves like a component of preference heterogeneity.

[13] Fagereng et al. (2021) use lottery winnings as a source of exogenous variation in unearned income. However, if individuals respond differently to a one-time change in unearned income than to a persistent change of equal present value, then $s'_{inc}(z)$ should be measured based on the latter. We discuss this issue, and an alternative measure of $s'_{inc}(z)$ based on survey data, in Section 6.

There is some evidence that mental accounting and other behavioral frictions affect people's propensity to consume and save out of windfalls. For example, Thakral and To (2021) show that people save more out of long-anticipated windfalls. Since steady-state changes in earnings correspond to anticipated changes earnings, unanticipated windfalls could lead to an under-estimate of s'_{inc} when s corresponds to savings, and an over-estimate when s corresponds to immediate consumption.

difficulty arises even in the standard Mirrlees (1971) model, since it is observationally equivalent to a model with preference heterogeneity over leisure and consumption.

We write our theoretical results in terms of flexible welfare weights that span the degree of heterogeneity in individuals' types, so that optimal policy can be computed using whatever welfare weights the policymaker prefers.[14] For results in the case of unidimensional heterogeneity, welfare weights are written as a function only of income, $g(z(\theta))$, without loss of generality. For results involving multidimensional heterogeneity, in which savings are heterogeneous conditional on income, we write social marginal welfare weights as a function of both savings and income, $g(s(\theta), z(\theta))$.

3.3 Sufficient Statistics Characterization of Optimal Smooth Tax Systems

A key result result used to derive our sufficient statistics formula is an equivalence result for tax reforms affecting marginal tax rates on s versus z. The result is a generalization of Lemma 1 in Saez (2002) to arbitrarily nonlinear smooth tax systems.

Lemma 1. *A small increase $d\tau$ in the marginal tax rate on s faced by an individual earning z induces the same earnings change as a small increase $s'_{inc}(z)\, d\tau$ in the marginal tax rate on z.*

Lemma 1 relates the labor supply distortions induced by increasing taxes on s to the labor supply distortions induced by increasing taxes on earnings z. Intuitively, if the marginal tax rate on earnings z increases by $d\tau_z$, an individual realizes they must now pay an additional $d\tau_z$ on each marginal dollar of earnings, so they earn less in response. Alternatively, if the marginal tax rate on commodity s increases by $d\tau_s$, and the individual adjusts s by s'_{inc} for every dollar adjustment in earnings, then the individual realizes they must now effectively pay an additional $s'_{inc} d\tau_s$ more for each marginal dollar of earnings, accounting for the way in which they will also adjust s. If $d\tau_z = s'_{inc} d\tau_s$, then the induced earnings distortion will be the same from both reforms.

We are now in a position to write formulas characterizing necessary conditions for the optimal smooth tax system in terms of sufficient statistics. In the results that follow, we use $H(s, z)$ and $h(s, z)$ to denote the cumulative and density functions over (s, z), with h_s and h_z denoting the marginal density over s and z, respectively.

Theorem 2. *Under the assumptions of Theorem 1, at each bundle (c, s, z) chosen by a type θ, an optimal smooth tax system satisfies the following conditions on marginal tax rates on z and s, respectively:*

$$\frac{\mathcal{T}'_z(s,z)}{1-\mathcal{T}'_z(s,z)} = \frac{1}{z\,\zeta^c_z(z)} \frac{1}{h_z(z)} \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) - s'_{inc}(z) \frac{\mathcal{T}'_s(s,z)}{1-\mathcal{T}'_z(s,z)} \tag{23}$$

$$\frac{\mathcal{T}'_s(s,z)}{1+\mathcal{T}'_s(s,z)} = s'_{pref}(z) \frac{1}{s\,\zeta^c_{s|z}(z)} \frac{1}{h_z(z)} \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) \tag{24}$$

[14] Our empirical application in Section 6 employs a version of the inverse optimum approach, estimating optimal savings taxes consistent with the current U.S. taxes on labor income.

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}_s'(s,z)}{1+\mathcal{T}_s'(s,z)} = s_{pref}'(z)\frac{z\,\zeta_z^c(z)}{s\,\zeta_{s|z}^c(z)}\frac{\mathcal{T}_z'(s,z)+s_{inc}'(z)\mathcal{T}_s'(s,z)}{1-\mathcal{T}_z'(s,z)} \tag{25}$$

Formula (23) constitutes a familiar "ABC" condition analogous to Saez (2001), with one modification: when tax rates on s are non-zero, the formula also accounts for how changes in earnings affect consumption of s, and therefore the revenue from taxes on s.

Formula (24) is one of our key results about optimal marginal tax rates on s. Optimal tax rates on s satisfy a condition that is remarkably similar to the standard "ABC" formula for optimal income tax rates, as presented in equation (23). When $s_{pref}' > 0$, the magnitude of the optimal tax rate at point (s,z) is decreasing in the elasticity of s with respect to the tax rate, increasing in the strength of redistributive motives, and decreasing in the density of individuals at point (s,z).

Formula (24) also gives a transparent generalization of the Atkinson-Stiglitz Theorem. When the sufficient statistic for preference heterogeneity, s_{pref}', is equal to zero, the condition implies that the optimal tax on s must equal zero as well. When $s_{pref}' > 0$, implying that higher earners have a stronger preference for s, the condition implies that the optimal tax rate on s must be positive.

We can combine conditions (23) and(24) to derive the Pareto-efficiency condition in (25).[15] Because the condition in (25) does not feature social marginal welfare weights, it is an efficiency condition that must hold for any tax system that is not Pareto dominated. Intuitively, it quantifies the efficient balance between taxing s and taxing z, given the measure s_{pref}' of how tastes for s vary with earnings ability. The stronger the association between preferences for s and earnings ability, the more efficient it is to tax s instead of z.

An implication of the Pareto-efficiency condition in (25) is that in the absence of preference heterogeneity, positive tax rates on s are Pareto dominated, providing an extension of the Atkinson-Stiglitz Theorem to smooth tax systems $\mathcal{T}(s,z)$. On the other hand, any Pareto-efficient tax system must feature non-zero tax rates on s in the presence of preference heterogeneity.

3.4 Other Determinants of Taxes on s Captured by the Sufficient Statistics Formulas

In practice, the elasticities and the measure of across-income preference heterogeneity $s_{pref}'(z)$ in our formulas above may be affected by forces other than pure preferences for s. For example, across-income heterogeneity in prices of s (Gahvari and Micheletto, 2016; Gerritsen et al., 2020), income shifting (Slemrod, 1995; Christiansen and Tuomala, 2008), and heterogeneity in endowments (Boadway et al., 2000; Cremer et al., 2003) may all contribute to differences between the cross-sectional profile $s'(z)$ and the causal income effect $s_{inc}'(z)$. A key feature of our sufficient statistics approach is that the model can be reinterpreted so that $s_{pref}'(z)$ represents these alternative sources of heterogeneity—or a combination of them—and the characterization of optimal tax schedules and

[15] See Konishi (1995), Laroque (2005), Kaplow (2006) for Pareto-optimality conditions under the more restrictive assumptions of the Atkinson-Stiglitz theorem. These results are a special case of ours.

"simple" tax systems in Section 4 remain intact. The intuition for this generality stems from the logic of Feldstein (1999), which shows that the elasticity of taxable income is a sufficient statistic for efficiency losses irrespective of whether it is due to real labor supply responses or costly avoidance behavior.[16]

While these other sources of across-income differences have previously been studied in isolation to qualitatively assess the robustness of the Atkinson-Stiglitz Theorem, our sufficient statistics techniques can be applied to account for them in a quantitative and general manner. For each of these sources of heterogeneity, we show how it is possible to re-express a model with such heterogeneity in the form of our general model in Section 2, with an appropriately interpreted utility function $\tilde{U}$. Provided that $\tilde{U}$ satisfies the regularity assumptions in Section 2, our results will carry through.

Heterogeneous Prices. Suppose that individuals face prices $p(e, z, s, \theta)$ that might depend on effort e to seek out lower prices, types θ, earnings z, and the level of s. For example, higher ability types may be better at finding lower prices or higher returns on investments; higher income z might generate beneficial network effects that expose individuals to better opportunities; and higher levels of s might allow individuals to lock in better prices or interest rates (so-called "scale effects") . Modifying our model to allow for these channels, the individual utility function becomes $U(c, s, z, e; \theta)$ and the budget constraint becomes

$$c + p(e, z, s, \theta)s \leq z - \mathcal{T}(s, z). \tag{26}$$

This economy is then equivalent to an economy where $p \equiv 1$ and where individuals maximize the utility function

$$\tilde{U}(c, s, z; \theta) = \max_{e} \; U\left(c + (1 - p(e, z, s, \theta))s, s, z, e; \theta\right) \tag{27}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$. This is because with a price of $p = p'$ instead of $p = 1$, individuals receive $(1 - p')s$ more consumption c at a given choice s. The feasibility constraint $\int_{\theta} \mathcal{T}\left(s\left(\theta\right), z\left(\theta\right)\right) dF\left(\theta\right) \geq E$ is independent of the price p because the tax is deducted directly from an individual's earnings z.[17] This equivalence shows that it is without loss of generality to assume that $p \equiv 1$, which is a normalization we adopt in our proofs.

An insight from this reinterpretation is that some sources of across-income price heterogeneity justify taxing s, while others do not, and the decomposition of the cross-sectional profile $s'(z)$ into

[16] Chetty (2009) suggests limitations to Feldstein's (1999) results due to some avoidance behaviors generating new types of fiscal externalities, or due to behavioral biases. Variations of these considerations are relevant in our setting as well, as explored in Sections 5.2 and 5.3, respectively.

[17] For example, suppose that s represents liters of soda purchased. In our baseline setup, the tax is measured in dollars and is paid out of earnings. If instead the tax were to have a two-part structure where the individual pays some amount T_1 in units of c (dollars) and T_2 in units of s (liters of soda), then the equivalence above would not apply. Although such a system might seem unnatural for a commodity like soda, it is common in the setting of savings, where taxes are often paid in units of "period 2" dollars, after returns have been realized. This creates an additional arbitrage motive to tax some individuals more heavily in units of c and others in units of s. We explore such arbitrage motives in Section 5.3.

$s'_{inc}(z)$ and $s'_{pref}(z)$ correctly distinguishes between them. *Type*-dependent heterogeneity in prices will generally lead to $s'_{pref}(z) \neq 0$ and thus to deviations from the Atkinson-Stiglitz result. For example, consider again the illustrative case from Section 3.2 with $U = \ln c + \delta \ln s - k(z/w(\theta))$ and additively separable $\mathcal{T}$, and suppose that types θ face heterogeneous prices $p(\theta)$. Then heterogeneity in prices functions like heterogeneity in discount rates, with $s'_{pref}(z) = \frac{\delta}{1+\delta}(z - \mathcal{T}) \frac{d}{dz}\left(\frac{1}{p(z)}\right)$, where $p(z)$ is the price faced by z-earners. In contrast, heterogeneity in prices that is due to *scale* effects does not contribute to s'_{pref}. For example, when $U = \ln c + \delta \ln s - k(z/w(\theta))$, allowing scale effects where p is an increasing function of s or z (but not θ) would affect s'_{inc}, while leaving $s'_{pref} \equiv 0$ when δ is homogeneous.

Income Shifting. Slemrod (1995) argues that some tax systems may provide incentives for individuals to "convert ordinary income into preferentially taxed capital gains," "convert corporations into pass-through legal entities such as partnerships, or retain labor compensation within the corporation." Generalizing the two-type model in Christiansen and Tuomala (2008), suppose that individuals can shift some of their labor income to capital income or savings.

Formally, let $\tilde{z}$ and $\tilde{s}$ be the individuals' true labor income and savings, which are unobserved by the tax authority. Let χ denote the amount of labor income $\tilde{z}$ that individuals shift to be realized as s, and let $m(\chi;\theta)$ denote any financial costs involved in income shifting. Individuals' taxable labor income is thus $\tilde{z} - \chi$, their taxable realized savings are $\tilde{s} + \chi - m(\chi;\theta)$, and their utility function is $U(c, \tilde{s} + \chi - m(\chi;\theta), \tilde{z}, \chi; \theta)$, where χ might directly influence utility because of effort or psychic costs. This formulation allows for the possibility that individuals with higher earnings ability may be better able to engage in income shifting.

Individuals then choose c, $\tilde{s}$, $\tilde{z}$, and χ to maximize their utility subject to the budget constraint

$$c + \tilde{s} + \chi - m(\chi;\theta) \leq \tilde{z} - \mathcal{T}(\tilde{s} + \chi - m(\chi;\theta), \tilde{z} - \chi). \tag{28}$$

Setting $z = \tilde{z} - \chi$, $s = \tilde{s} + \chi - m(\chi;\theta)$, individuals equivalently choose c, s, z, χ to maximize $U(c, s, z + \chi, \chi; \theta)$ subject to the budget constraint $c + s \leq z + \chi - \mathcal{T}(s, z)$. Since individuals exhaust this budget constraint, we can alternatively rewrite this as an economy where individuals choose c, s, z to maximize

$$\tilde{U}(c, s, z; \theta) = \max_{\chi} \; U(c + \chi, s, z + \chi, \chi; \theta) \tag{29}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$, in which χ no longer appears. Again, the feasibility constraint $\int_\theta \mathcal{T}(s(\theta), z(\theta))\, dF(\theta) \geq E$ is unaffected, because taxes are always based on observed s and z. In this case, $s'_{pref}(z)$ corresponds to the heterogeneity in taxable savings across taxable labor income that arises from differences in types.

Heterogeneous Endowments. Suppose that individuals have endowments $y_0(\theta)$, from inheritance or other sources, such that their budget constraint is given by $c + s \leq y_0(\theta) + z - \mathcal{T}(s, z)$.

This economy is then equivalent to an economy where $y_0 \equiv 0$ and where individuals maximize the utility function

$$\tilde{U}(c, s, z; \theta) = U\left(c + y_0(\theta), s, z; \theta\right) \tag{30}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$. This is because at a given choice s, individuals receive y_0 more consumption of c. Again, the feasibility constraint $\int_\theta \mathcal{T}\left(s\left(\theta\right), z\left(\theta\right)\right) dF\left(\theta\right) \geq E$ is unaffected, because it is independent of the endowments.

Human Capital Investments. Our framework can also be interpreted to a class of models studying human capital investments. Suppose that s is human capital investments such as education, which affect productivity during working life (Bovenberg and Jacobs, 2005; Stantcheva, 2017). Following the static framework in Bovenberg and Jacobs (2005), let $\phi(s)$ denote the productivity-enhancing effect of human capital and assume it scales down the disutility to generating earnings z. Thus, individuals maximize utility $U(c, s, \frac{z}{\phi(s)}; \theta)$ subject to the budget constraint $c+s \leq z - \mathcal{T}(s, z)$. We can remap this to our model by simply rewriting the utility function:

$$\tilde{U}(c, s, z; \theta) = U\left(c, s, \frac{z}{\phi(s)}; \theta\right) \tag{31}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$. This reflects the fact that human capital investments can be viewed as a specific form of non-separability in preferences that our model accommodates. Again, the feasibility constraint $\int_\theta \mathcal{T}\left(s\left(\theta\right), z\left(\theta\right)\right) dF\left(\theta\right) \geq E$ is unaffected. The functional form assumptions in Bovenberg and Jacobs (2005) imply that $s'_{pref}(z) < 0$ and thus that human capital investments are subsidized at the optimum; Bovenberg and Jacobs (2011) consider more general settings in which these assumptions are relaxed.

4 Optimal "Simple" Tax Systems and Multidimensional Heterogeneity

In practice, tax systems must be defined by policymakers and implemented by institutions that may impose constraints on the degree of complexity in the tax function, such as separability or linearity. Although the details of these restrictions vary across settings, most taxes on savings and capital income can be classified into a few categories of functional restrictions that we call "simple" tax systems. In this section, we apply our sufficient statistics methods to characterize the optimal policy within each class of simple tax systems. These results can be used to provide policy guidance in settings with practical constraints on tax complexity.

By focusing on restricted tax systems, we can also characterize optimal policy in more general settings with multiple dimensions of heterogeneity. As is well-known, in settings with multidimensional heterogeneity, optimal fully flexible mechanisms tend to feature bunching, randomization, and other irregularities, which are sensitive to model assumptions. As a result, a common approach is to characterize the optimal policy within a restricted class of tax systems using conventional perturbation methods.[18] This section extends this literature by considering a more varied set of simple tax systems, and by expressing all results in terms of the sufficient statistics discussed in the previous section. In all cases, we show that $s'(z)$ and $s'_{inc}(z)$, together with standard elasticity concepts, are sufficient to characterize optimal taxes on s.

4.1 A Taxonomy of Common Simple Savings Tax Systems

Many governments tax both labor income and savings (or capital interest income). While these tax systems take a variety of forms, the details of which depend on specifics such as timing, many of these tax policies can be interpreted as a function of earnings and savings, analogous to our function $\mathcal{T}(s,z)$. Table I presents three classes of simple tax systems: separable linear (SL), separable nonlinear (SN), and linear earnings-dependent (LED). Table II categorizes the tax policies on each of five classes of savings-related tax bases—wealth, capital gains, property taxes, pensions, and inheritances— for 21 countries, showing that most of these taxes can be understood as fitting into one of the three simple tax system classes from Table I. In cases where there is ambiguity—such as the distinction between short-term and long-term capital gains in the U.S.—we provide supplementary information in Appendix D.[19]

We can see examples of each of the three types of simple tax systems within the United States. Most property taxes, levied at the state and local level, take the form of a separable linear tax, with a flat tax rate, independent of one's labor earnings, applied to the assessed value of the total property. The estate tax takes the form of a separable nonlinear tax: tax rates rise progressively with the value of the estate, but they do not vary with labor income of the donor or the recipient. And taxes on long-term capital gains and qualified dividends take the form of linear earnings-dependent taxes. In 2020, for example, an individual with \$50,000 in labor earnings faced a tax rate of 15% on long-term capital gains , whereas an individual earning \$500,000 faced a tax rate of 20%. Finally, although we focus on savings tax policies, these classes of simple tax systems are also relevant for other classes of commodities. Separable linear commodity taxes are ubiquitous (e.g., on lodging, airfare, and sales taxes that apply to specific classes of consumption); while separable nonlinear and linear income-dependent tax structures are often used for subsidies on goods like energy and education. [20]

[18] See Piketty and Saez (2013), Diamond and Spinnewijn (2011), and Gauthier and Henriet (2018) for examples restricting to a linear tax on s, and Saez and Stantcheva (2018) for a nonlinear separable tax on s and z.

[19] We impose several simplifications for our interpretation of the tax codes. First, we treat ordinary income as consisting primarily of labor income (earnings), written as z in our notation. Second, we separately consider taxes on five broad categories of savings vehicles: wealth, capital gains, real property, private pensions, and inheritances. These categories may overlap—real property is a component of wealth, for example—but we use these groups to reflect the tax instruments that many governments use in practice.

[20] One practical distinction between taxes on savings and on other commodities involves the measurement of the tax base. In our baseline model, the argument of the tax function s represents the amount of the commodity s consumed. This is natural for many commodities, but in the setting of savings, it is common for the tax system to be written as a function of gross savings before taxes, e.g., a tax $T_1(z)$ in period 1, and a tax $T_2(s_g, z)$ on gross pre-tax savings $s_g = (1+r)(z - T_1(z) - c)$ in period 2, so that period-2 consumption is given by $s = s_g - T_s(s_g, z)$, where r is the compounded rate of return. In this formulation, a SL structure is one where $T_2(s_g, z) = \tau_s s_g$, a SN structure is one where $T_2(s_g, z)$ is a function of s_g only, and a LED structure is one where $T_2(s_g, z) = \tau_s(z) s_g$. Fortunately,

4.2 Optimal Simple Tax Systems

We now present optimality conditions for SL, SN and LED tax systems. We focus on marginal tax rates on s in the body of the paper, and present conditions for marginal tax rates on z in Appendix A.4. The preference heterogeneity statistic s'_{pref} remains the key sufficient statistic for the marginal tax rate on s. For SL systems, it is convenient to introduce the term $s_{pref}(z) = \int_{x=z_{min}}^{z} s'_{pref}(x)dx$, which integrates local preference heterogeneity across incomes to obtain a measure of total preference heterogeneity up to earnings level z.

Proposition 2. *Suppose that the optimal SL, SN, and LED systems are smooth and that at the optimum: (i) agents' optima are unique and their first-order and second-order conditions strictly hold, (ii) there is no bunching, (iii) c and s are smooth functions of z, and (iv) in the SN system s is strictly monotonic (increasing or decreasing) in z. Then, at each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions:*

$$SL: \frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s}\overline{\zeta^c_{s|z}}} \int_z s'_{pref}(z) \left[\int_{x\geq z} (1-\hat{g}(x))dH_z(x)\right] dz \tag{32}$$

$$= -\frac{1}{\overline{s}\overline{\zeta^c_{s|z}}} Cov[s_{pref}(z), \hat{g}(z)] \tag{33}$$

$$SN: \frac{T'_s(s)}{1+T'_s(s)} = \frac{1}{s\,\zeta^c_{s|z}(z)} \frac{1}{h_z(z)} s'_{pref}(z) \int_{x\geq z} (1-\hat{g}(x))\, dH_z(x) \tag{34}$$

$$LED: \frac{\tau_s(z)}{1+\tau_s(z)} = \frac{1}{s\,\zeta^c_{s|z}(z)} \frac{1}{h_z(z)} s'_{pref}(z) \int_{x\geq z} (1-\hat{g}(x))\, dH_z(x) \tag{35}$$

Moreover, if a SL/SN/LED tax system is not Pareto dominated by another SL/SN/LED system, then it must satisfy the following conditions:

$$SL: \frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s}\overline{\zeta^c_{s|z}}} \int_z s'_{pref}(z)\; z\zeta^c_z(z)\; \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)}\, dH_z(z) \tag{36}$$

$$SN: \frac{T'_s(s)}{1+T'_s(s)} = s'_{pref}(z)\; \frac{z\zeta^c_z(z)}{s\zeta^c_{s|z}(z)}\; \frac{T'_z(z) + s'_{inc}(z)T'_s(s)}{1-T'_z(z)} \tag{37}$$

$$LED: \frac{\tau_s(z)}{1+\tau_s(z)} = s'_{pref}(z)\; \frac{z\zeta^c_z(z)}{s\zeta^c_{s|z}(z)}\; \frac{T'_z(z) + \tau'_s(z)\,s + s'_{inc}(z)\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\,s} \tag{38}$$

The optimal tax formulas and the Pareto-efficiency conditions for SN and LED systems are analogous to the conditions for $\mathcal{T}'_s$ in the fully flexible smooth tax systems derived in Theorem 2.

there is an equivalence between these formulations of two-period tax systems and the type of "static" tax function $\mathcal{T}$ considered in our baseline model, allowing us to translate results between them. Appendix A.6 shows the nature of this equivalence in two steps. First, the set of allocations implementable by these systems is identical, as there is a simple and natural translation between the "static" tax function $\mathcal{T}$ we consider and the two-period function. Second, if $T_1(z) + T_2(s_g, z)$ implements the same allocation as $\mathcal{T}(s, z)$, then $T_1(z) + T_2(s_g, z)$ will be SL/SN/LED if and only if $\mathcal{T}(s, z)$ is SL/SN/LED.

Appendix A.2 derives sufficiency conditions under which SN and LED systems can implement the optimal allocation. In the case of SN systems, these conditions are relatively weak, although they do require that s is weakly increasing with z in the optimal allocation. In contrast, the sufficiency conditions for LED systems are more restrictive, loosely requiring that the local preference for s must not increase "too quickly" across incomes, or else the the second-order condition for earnings may be violated. Yet LED systems do not require a non-decreasing schedule of $s(z)$ in the optimal allocation. Thus these results illustrate that SN and LED systems have distinct advantages in different settings.

The SL system is the most restrictive, and generically cannot implement the same allocation as the optimal smooth tax system. This is because the optimal smooth tax system does not generally feature constant marginal tax rates on s. Correspondingly, the optimal tax formula for the SL systems takes a different form. As shown in expression (32), the constant marginal tax rate τ_s for SL systems is in a certain sense an average of the z-earner specific marginal tax rates in expressions (34) and (35). Intuitively, the constant marginal tax rate is a population aggregate of the tax rates that would be optimal for individuals with different earnings levels. This is mirrored in the Pareto-efficiency condition (36). Expression (32) expresses this optimality condition in an alternative way, which was first derived by Allcott et al. (2019). This formulation has a familiar form resembling the Diamond (1975) "Many-person Ramsey tax rule." The expression is identical to the Diamond (1975) formula when $s_{pref}(z) \equiv s(z)$; i.e., when all consumption differences are driven by preference heterogeneity. This illustrates the relevance of Ramsey taxation principles even in the presence of nonlinear income taxation, as well as their limitations. The SL formula when $s_{pref}(z) \equiv 0$ reduces to the original statement of the Atkinson-Stiglitz Theorem. More generally, even for arbitrarily nonlinear taxes on s, the optimal tax rate is always inversely proportional to the elasticity $\zeta^c_{s|z}(z)$, consistent with standard Ramsey principles, as long as $s'_{pref}(z) \neq 0$.

4.3 Multidimensional Heterogeneity and Suboptimal Earnings Taxes

Our next result generalizes Proposition 2 in two key ways. First, it allows for multidimensional heterogeneity, where types θ belong to a subset of $\mathbb{R}^n$ for $n \geq 2$, so that the support of the distribution of (s, z) is two-dimensional. Second, it characterizes optimal taxes on s even when the earnings tax $T_z(z)$ is not necessarily optimal. Proposition A.5 in the Appendix characterizes optimal earnings tax schedules. The combination of Proposition A.5 and the result below provides a complete characterization of optimal simple tax schedules with multidimensional heterogeneity.

In settings with multidimensional heterogeneity, the relevant sufficient statistics may vary across the joint distribution of s and z. We use $\zeta^c_z(s, z)$ and $s'_{inc}(s, z)$ to denote the compensated elasticity of s and the causal income effect on s for an individual choosing the bundle (s, z). The formulas below also allow social marginal welfare weights g to be functions of both s and z.

Proposition 3. *Consider smooth simple tax systems with (potentially suboptimal) earnings tax schedules $T_z(z)$ and optimally set marginal tax rates on s. Assume that agents' first- and second-order conditions hold in these tax systems, and that there is no bunching. Then, at each bundle*

(c^0, s^0, z^0) chosen by some type θ, the marginal tax rates on s in SL/SN/LED systems must satisfy the following optimality conditions:

$$\begin{aligned}
SL: \frac{\tau_s}{1+\tau_s} \int_z \left\{ \mathbb{E}\left[s\zeta^c_{s|z}(s,z) \Big| z \right] \right\} dH_z(z) = \int_z \Bigg\{ \mathbb{E}\left[(1-\hat{g}(s,z))\, s \Big| z \right] \\
- \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z)\, s'_{inc}(s,z) \Big| z \right] \Bigg\} dH_z(z) \qquad (39) \\
SN: \frac{T'_s(s^0)}{1+T'_s(s^0)} \int_z \left\{ s^0\zeta^c_{s|z}(s^0,z) \right\} h\left(s^0,z\right) dz = \int_z \left\{ \mathbb{E}\Big[1-\hat{g}(s,z) \Big| z, s \geq s^0 \Big] \right\} dH_z(z) \\
- \int_z \left\{ \frac{T'_z(z) + s'_{inc}\left(s^0,z\right) T'_s\left(s^0\right)}{1-T'_z(z)} z\zeta^c_z(s^0,z) s'_{inc}\left(s^0,z\right) \right\} h\left(s^0,z\right) dz \qquad (40) \\
LED: \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)\, s + s'_{inc}(s,z)\,\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\, s} z\zeta^c_z(s,z) s \Big| z^0 \right] h_z\left(z^0\right) = \int_{z \geq z^0} \mathbb{E}\Big[(1-\hat{g}(s,z))\, s \Big| z \Big] dH_z(z) \\
- \int_{z\geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta^c_{s|z}(s,z) \Big| z \right] + \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)\, s + s'_{inc}(s,z)\,\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\, s} z\zeta^c_z(s,z) s'_{inc}(s,z) \Big| z \right] \right\} dH_z(z) \\
(41)
\end{aligned}$$

As in the unidimensional case, the s'_{inc} statistic, together with standard elasticity concepts, allow us to characterize optimal taxes on s in terms of observables. The main difference between multidimensional and unidimensional heterogeneity is that with unidimensional heterogeneity, the expectation operators $\mathbb{E}[\cdot|z]$ are not needed. In expression (39), $\hat{g}(s,z)$, $\zeta^c_z(s,z)$, and $s'_{inc}(s,z)$ are functions only of z in the unidimensional case. In expression (40), the term $\mathbb{E}\Big[1-\hat{g}(s,z)\Big|z, s \geq s^0\Big]$ reduces to $\mathbb{E}\Big[1-\hat{g}(z)\Big|z \geq z^0\Big]$ in the unidimensional case, and the functions s'_{inc} and h can be written as functions of z only. Analogous simplifications apply to expression (41).[21]

If all terms inside the expectation operators $\mathbb{E}[\cdot|z]$ in Proposition 3 are independent of each other, then the expectation can be applied to each statistic separately, and thus the unidimensional formulas are similar to the multidimensional formulas provided that all statistics are interpreted as averages conditional on z. For example, the first term in the integral in expression (39) can be written as $\left(1-\overline{\hat{g}(z)}\right)\overline{s(z)}$, where the "bar" notation denotes income-conditional averages. The second term in the integral can be written as

$$\frac{T'_z(z)\,\overline{s'_{inc}(z)} + \tau_s \overline{s'_{inc}(s,z)^2}}{1-T'_z(z)} z\overline{\zeta^c_z(z)}. \qquad (42)$$

The main new effect is the square of s'_{inc} inside the integral. Because $\int (s'_{inc})^2 dH > \left(\int s'_{inc} dH\right)^2$ and because the square enters into the optimal tax expression negatively, this implies that ignoring multidimensional heterogeneity can lead to over-estimates of optimal marginal tax rates on s. The formulas in (40) and (41) also involve squares of s'_{inc}, and thus also imply that multidimensional

[21] For reference, we provide a characterization of optimal taxes on s assuming unidimensional heterogeneity and a given (potentially suboptimal) earnings tax $T_z(z)$, in Appendix A.3.

heterogeneity can lower the optimal tax rate on s through $(s'_{inc})^2$. We quantify the importance of this insight in our empirical application in Section 6. More generally, positive covariances between pairs of statistics inside the expectation operator will tend to lower the optimal tax rate on s, while negative covariances will tend to increase it.

5 Extensions

In this section we provide three key extensions. First, we generalize our results to more than two dimensions of consumptions. This allows us to cover settings where, for example, individuals have access to multiple saving vehicles that are taxed differentially. Second, we allow for the possibility that the government wants people to save more than their perceived private optima, either because of a misalignment between private and social inter-generational preferences or because of individuals' behavioral biases. Third, we consider the case where taxes can be collected both in units of c and in units of s, as is often the case for savings taxes. These extensions highlight that s'_{pref} remains a key sufficient statistic for optimal taxes, and that our previous formulas readily extend to these settings.

5.1 Multiple Goods

We now extend our analysis to a setting where agents consume $n+1$ goods, c and $\boldsymbol{s} = (s_1, s_2, ..., s_n)$. For example, $\boldsymbol{s}$ might correspond to different categories of saving, which the government might choose to tax in different ways. We consider a tax system $\mathcal{T}(\boldsymbol{s}, z) = \mathcal{T}(s_1, s_2, ..., s_n, z)$, where we retain the normalization that the numeraire c is untaxed. We normalize $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ to measure consumption in units of the numeraire. An individual of type θ then maximizes $U(c, \boldsymbol{s}, z; \theta)$ subject to the budget constraint $c + \sum_{i=1}^{n} s_i \leq z - \mathcal{T}(\boldsymbol{s}, z)$.

We denote own-price elasticities of goods by $\zeta^c_{s_i|z}(z)$, and we define cross-substitution elasticities by $\xi^c_{s_{j,i}|z}(z) := -\frac{\mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta),z)}{s_j(z;\theta)} \frac{\partial s_j(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta),z)}\Big|_{\theta=\vartheta(z)}$, where $s_j(z;\theta)$ denotes type θ consumption of good j when earning labor income z. We denote causal income effects on good s_j by $s'_{j,inc}(z) := \frac{\partial s_j(z;\theta)}{\partial z}\Big|_{\theta=\vartheta(z)}$. We continue using $\hat{g}(z)$ to denote the social marginal welfare effect of increasing a z-earner's consumption of c by one unit.[22]

For the result below, as well as for Propositions 5, 6, and the supplementary results in Appendices A.3 and A.4, we assume that

Assumption 4. *The tax systems under consideration are such that at the optimum: (i) these tax systems are smooth, (ii) agents' optima are unique and their first-order and second-order conditions strictly hold, (iii) there is no bunching, (iv) c and s are smooth functions of z, and (v) when the SN system is studied, s is strictly monotonic (increasing or decreasing) in z.*

[22] The formula for $\hat{g}(z)$ in this more general setting is in Appendix B.9.

Proposition 4. *With n taxed goods $s_1, ..., s_n$, for each good i and at each bundle $(c, \boldsymbol{s}, z)$ chosen by a type θ, an optimal smooth tax system satisfies*

$$\frac{\mathcal{T}'_{s_i}(\boldsymbol{s},z)}{1+\mathcal{T}'_{s_i}(\boldsymbol{s},z)} = s'_{i,pref}(z)\frac{1}{s_i\zeta^c_{s_i|z}(z)}\frac{1}{h_z(z)}\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x) + \underbrace{\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}(\boldsymbol{s},z)}{\mathcal{T}'_{s_i}(\boldsymbol{s},z)}\frac{s_j\xi^c_{s_{j,i}|z}(z)}{s_i\zeta^c_{s_i|z}(z)}}_{\textit{Tax diversion ratio}}. \quad (43)$$

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}'_{s_i}(\boldsymbol{s},z)}{1+\mathcal{T}'_{s_i}(\boldsymbol{s},z)} = s'_{i,pref}(z)\frac{z\zeta^c_z(z)}{s_i\zeta^c_{s_i|z}(z)}\frac{\mathcal{T}'_z(\boldsymbol{s},z)+\sum_{j=1}^n s'_{j,inc}(z)\mathcal{T}'_{s_j}(\boldsymbol{s},z)}{1-\mathcal{T}'_z(\boldsymbol{s},z)} + \underbrace{\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}(\boldsymbol{s},z)}{\mathcal{T}'_{s_i}(\boldsymbol{s},z)}\frac{s_j\xi^c_{s_{j,i}|z}(z)}{s_i\zeta^c_{s_i|z}(z)}}_{\textit{Tax diversion ratio}}. \quad (44)$$

Proposition 4 features all of the familiar terms of Theorem 2, and includes a novel term that captures the tax implications of substitution effects between the different goods. Intuitively, substituting from s_i to higher-taxed goods generates positive fiscal externalities that motivate higher marginal tax rates on s_i, while substitution to lower-taxed goods generates negative fiscal externalities that motivate lower marginal tax rates on s_i. These effects are summarized by what we call the tax diversion ratio—the impact on taxes collected on goods $j \neq i$ relative to the impact on taxes collected on good i, when the price of good i is increased. The higher is the diversion ratio, the more favorable are the fiscal externalities associated with substitution away from good i, and thus the higher is the optimal tax rate on good i.

5.2 Optimal Taxation when the Government Wants Agents to Save More

Our framework can be interpreted as a bequest model in which parents work and consume in the first period, and leave a bequest to their heirs in the second period. Under this interpretation, our baseline model makes the implicit assumption that the government values bequests in the same way as parents. Farhi and Werning (2010) consider a model where the weight that parents attach to the wellbeing of future generations is too low from a normative perspective. This misalignment introduces a motive to encourage bequests, which we consider in this extension.

Following Farhi and Werning (2010), we assume additively separable preferences given by

$$U\big(c,s,z;\theta\big) = u\left(c;\theta\right) - k\left(z;\theta\right) + \beta v\left(s;\theta\right), \quad (45)$$

where $u\left(c;\theta\right)$ is the utility parents derive from consumption c, $k\left(z;\theta\right)$ is the disutility parents incur to obtain earnings z, $v\left(s;\theta\right)$ is the utility heirs derive from a bequest s, and β is the weight parents attach to the wellbeing of their heirs. As in Farhi and Werning (2010), the government maximizes

$$\int_\theta\Big[U\left(c\left(\theta\right),s\left(\theta\right),z\left(\theta\right);\theta\right)+\nu v\left(s\left(\theta\right);\theta\right)\Big]dF\left(\theta\right), \quad (46)$$

where ν parametrizes the degree of misalignment. Farhi and Werning (2010) microfound ν as the Lagrange multiplier associated with a constraint that the future generation attains a required level of well-being $\int_\theta v\left(s\left(\theta\right);\theta\right)dF\left(\theta\right) \geq \underline{V}$.

The formal model above can be interpreted more generally beyond the bequest application, and can be used to analyze behavioral biases as a motivation for encouraging savings. In particular, suppose that $v(s;\theta) = \delta(\theta)u(s;\theta)$, where δ is the "exponential discount factor" and β is "present focus," as in Laibson (1997). If the government utilizes the "long-run criterion" for welfare, then the degree of misalignment is given by $\nu = (1-\beta)$.[23] More generally, β may be heterogeneous, so that misalignment is type-dependent and given by $\nu(\theta) = (1-\beta(\theta))$. For example, Lockwood (2020) summarizes evidence suggesting that individuals with higher earnings ability have lower degrees of present focus.

Below, we characterize optimal taxation with heterogeneous misalignment, where $\beta(z)$ and $\nu(z)$ denote the parameters corresponding to a z-earner. This generalizes the result in Farhi and Werning (2010) by (i) allowing heterogeneity in preferences for s, and by (ii) allowing heterogeneity in the misalignment parameter ν.

Proposition 5. *At each bundle (c,s,z) chosen by a type θ, an optimal smooth tax system satisfies the following marginal tax rate condition*

$$\frac{\mathcal{T}_s'\left(s,z\right)}{1+\mathcal{T}_s'\left(s,z\right)} = s_{pref}'\left(z\right)\frac{1}{s\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left[1-\hat{g}(x)\right]dH_z(x) - \frac{\nu(z)}{\beta(z)}g(z). \tag{47}$$

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}_s'\left(s,z\right)}{1+\mathcal{T}_s'\left(s,z\right)} = s_{pref}'\left(z\right)\frac{z\zeta_z^c(z)}{s\zeta_{s|z}^c(z)}\left[\frac{\mathcal{T}_z'\left(s,z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s,z\right)}{1-\mathcal{T}_z'\left(s,z\right)} + s_{inc}'(z)\frac{\nu(z)}{\beta(z)}g(z)\right] - \frac{\nu\left(z\right)}{\beta\left(z\right)}g(z). \tag{48}$$

This is an intuitive generalization of Theorem 2, where the key new term is a form of Pigovian correction, given by $\frac{\nu(z)}{\beta(z)}g(z)$. As equation (47) shows, the presence of misalignment motivates the government to lower the tax rate on s. The degree by which the government lowers the tax rate depends on the degree of misalignment (relative to the discount factor β), and on the social marginal welfare weight. Because social marginal welfare weights decline with z, equation (47) gives the "progressive estate taxation" result of Farhi and Werning (2010)—i.e., savings subsidies that decline with income—under the special assumptions that (i) $s_{pref}' \equiv 0$ and (ii) $\beta(z) \equiv \beta \in \mathbb{R}$, $\nu(z) \equiv \nu \in \mathbb{R}$. This core result of Farhi and Werning (2010) extends the standard Pigovian taxation logic to optimal screening of distortions with a nonlinear tax.

More generally, Proposition 5 provides a simple formula for balancing the "corrective motives" studied by Farhi and Werning (2010) with the additional motives to tax s in the presence of preference heterogeneity studied in this paper. This extends the Allcott et al. (2019) results for linear commodity taxes with biased consumers to study optimal screening of biases with a nonlinear

[23]See Bernheim and Taubinsky (2018) for a detailed discussion of such a criterion, as well as alternative normative approaches to studying the implications of present focus.

tax. If $s'_{pref}(z) > 0$ and $\nu(z)/\beta(z)$ and $g(z)$ are decreasing with z, Proposition 5 suggests a progressive tax on s that can feature subsidies at low incomes and taxes at high incomes.

5.3 Tax Arbitrage with Heterogeneous Prices

Thus far we have considered tax functions where the tax is always paid in units of the numeraire commodity c. In some applications it is also natural to consider tax systems with *multidimensional range*, which include taxes collected in units of c and also in units of s. This is natural, for example, if c and s correspond to period 1 and period 2 consumption, respectively, and taxes must be paid in both periods. The additional richness in the range does not alter the optimal tax implications when the rates of transformation p are homogeneous; in equilibrium, the government's rate of transformation is the same as the homogenous rate for individuals, and it does not matter what portion of the total tax bill is collected in units of s—individuals will simply purchase sufficient s to cover that portion. However, we shall show that when prices are heterogeneous, there is an additional efficiency rationale for differentially taxing s. Intuitively, heterogeneity in prices motivates a form of "tax arbitrage," in which the government collects relatively more taxes in units of s from individuals who can obtain s at a low price—or in the setting of savings, it imposes relatively higher savings taxes (and lower earnings taxes) on individuals with high rates of returns. This extension provides a bridge between our baseline results and the independent work of Gerritsen et al. (2020), which also studies the role of such efficiency effects.

Formally, suppose that the government uses a two-part tax structure, where individuals pay a tax $T_1(z)$ in units of c and a tax $T_2(s, z)$ in units of s. For instance, in a two-period model where s is savings, T_1 represents the earnings tax levied in period 1 and paid in period-1 dollars, while T_2 represents the savings tax levied in period 2 and paid in period-2 dollars, and $p = 1/(1 + r)$ is a function of the rate of return r. For concreteness, we refer to T_1 as period-1 taxes and to T_2 as period-2 taxes, though we emphasize that the presence of efficiency effects is not about dynamics per se, but rather that T_2 is collected in units of s.

Following Gahvari and Micheletto (2016), we consider heterogeneous prices $p(z, \theta)$ which are a function of gross earnings and type. For example, wealthier individuals may have access to better rates of return on savings or prices of commodities. Alternatively, higher earnings ability may be associated with a better ability to obtain high rates of return or to find better prices.

Individuals maximize their utility $U(c, s, z; \theta)$ subject to the budget constraint $c + p(z, \theta)s \leq z - T_1(z) - p(z, \theta)T_2(s, z)$. Denoting by $\vartheta(z)$ the type θ of individuals who choose earnings z, we slightly abuse notation to define $p(z) := p(z, \vartheta(z))$. The government, as before, maximizes a weighted average of utilities,

$$\int_z \left\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z), z)\big), s(z), z; \vartheta(z)\Big) \right\} dH_z(z), \tag{49}$$

subject to the constraints

$$\int_z T_1(z) dH_z(z) \geq E_1 \tag{50}$$

$$\int_z T_2(s(z), z) dH_z(z) \geq E_2, \tag{51}$$

which generate marginal values of public funds λ_1 and λ_2 . We continue using $\hat{g}(z)$ to denote the social marginal welfare effect of increasing a z-earner's consumption of c by one unit, and it is here normalized by the marginal value of public funds λ_1.[24]

Heterogeneity in p generates efficiency effects through two channels. First, for individuals with relatively low $p(z)$, it is efficient for the government to decrease T_1 and increase T_2. This efficiency effect is present irrespective of the mechanism for the cross-sectional variation of p with z, and leads to a deviation from the Atkinson-Stiglitz Theorem.

Second, lump-sum changes in T_2 trigger novel substitution effects. This is because a lump-sum increase dT in T_2 has the same effect on an agent's utility as a $p(z)dT$ increase in T_1, and thus changes behavior as much as a marginal tax rate change of $\frac{\partial p}{\partial z} dT$ in T_1. We denote by $\varphi(z) := -\left(T_1'(z) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s'_{inc}(z)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{z\zeta_z^c(z)}{1-T_1'(z)}\frac{\partial p}{\partial z}$ the fiscal impacts of this substitution effect at earnings z. The impact of a uniform lump-sum change in T_2 is then $\overline{\hat{g}p - \varphi}$, where the "bar" notation is used to denote a population average across all earnings levels. Thus, $\lambda_2/\lambda_1 = \overline{\hat{g}p - \varphi}$, as we formally show in Appendix B.11.3.

Taking these new considerations into account, we characterize optimal taxes on s for SN tax systems where $T_2(s, z) = T_2(s)$, and for LED tax systems where $T_2(s, z) = \tau_s(z)\, s$. We supplement these results with a characterization of the optimal earnings tax schedule $T_1(z)$ in Appendix B.11.4.

Proposition 6. *With heterogeneous prices, at each bundle (c, s, z) chosen by a type θ, an optimal SN two-part tax system $\{T_1(z), T_2(s)\}$ satisfies*

$$\frac{\lambda_2/\lambda_1 T_2'(s)}{1 + p(z)T_2'(s)} = \frac{1}{s\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\left\{s'_{pref}(z)\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x) + \frac{s'(z)}{p(z)}\Big(\Psi(z) + \int_{x\geq z}\Big[\varphi(x) - \overline{\varphi}\Big]dH_z(x)\Big)\right\} \tag{52}$$

where

$$\Psi(z) := \Big(1 - H_z(z)\Big)\int_{x\leq z}\hat{g}(x)\Big(p(x) - p(z)\Big)dH_z(x) + H_z(z)\int_{x\geq z}\hat{g}(x)\Big(p(z) - p(x)\Big)dH_z(x), \tag{53}$$

An optimal LED two-part tax system $\{T_1(z), \tau_s(z)s\}$ satisfies

$$\frac{\lambda_2/\lambda_1 \tau_s(z)}{1 + p(z)\tau_s(z)} = \frac{1}{s(z)\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\left\{s'_{pref}(z)\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x) + \frac{p'(z)}{p(z)}s(z)\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x)\right\}$$
$$+ \frac{1}{\zeta_{s|z}^c(z)}\frac{1}{p(z)}\Big\{\overline{\hat{g}p} - \overline{\hat{g}}p(z) + \varphi(z) - \overline{\varphi}\Big\}. \tag{54}$$

[24]The formula for $\hat{g}(z)$ in this more general setting is in Appendix B.11.3.

Proposition 6 shows that the sufficient statistic $s'_{pref}(z)$ remains critical for optimal marginal tax rates on s. On the left-hand side of (52) and (54), the presence of $p(z)$ in the denominator is because an agent's marginal tax rate on s, translated to units of c, is $p(z)\frac{\partial T_2}{\partial s}$. The presence of λ_2/λ_1 in the numerator of the left-hand side is because fiscal externalities generated by substitution away from s must be weighted by the "period 2" marginal value of public funds.

Proposition 6 also introduces novel efficiency terms that to lead to taxes on s, even when $s'_{pref} \equiv 0$. In the SN formula, there are two additional efficiency effects. These terms are both positive and thus push toward taxing s when higher earners (i) face lower prices p (e.g., higher rates of returns on savings) and choose higher levels of s, and (ii) exhibit larger substitution effects φ. The first term, proportional to $\Psi(z)$, captures the efficiency effects of increasing period-2 taxes. This term is unambiguously positive when p decreases cross-sectionally with z, and captures the intuition that with a SN system, increasing marginal tax rates on s at any point $z > z_{\min}$ increases period-2 taxes on individuals with below-average p. The second term, proportional to $\int_{x \geq z} \left[\varphi(x) - \overline{\varphi}\right] dH_z(x)$, captures the fact that increasing marginal tax rates on s motivates individuals to increase labor supply in order to get lower prices p when $\frac{\partial p}{\partial z} < 0$. The SN formula generalizes the result in Gerritsen et al. (2020) to incorporate other forms of across-income heterogeneity and makes transparent the sign of these terms in a formula employing measurable sufficient statistics.

The implications for LED tax systems are somewaht different. Assume again that p declines cross-sectionally with z (i.e., $p'(z) < 0$). The first novel term in equation (54), proportional to $p'(z)/p(z)$, reflects the fact that higher earners are less responsive to marginal changes in T_2 when $p(z)$ declines with income, since period-2 consumption is "cheaper" for them than period-1 consumption. The second term, proportional to $\overline{\hat{g}p} - \overline{\hat{g}}p(z) + \varphi(z) - \overline{\varphi}$, is also negative for sufficiently low values of z, as in this case both $\overline{\hat{g}p} - \overline{\hat{g}}p(z)$ and $\varphi(z) - \overline{\varphi}$ are negative. However, this term is positive for sufficiently high values of z. Thus, when $s'_{pref}(z) \equiv 0$, the optimal LED system features subsidies on s for lower-income individuals and taxes on s for higher-income individuals.

The contrast in implications for SN versus LED tax systems—everywhere-positive tax rates in the former, subsidies followed by taxes in the latter—highlights that the new efficiency considerations from heterogeneous rates of return depend on the types of restrictions imposed on the tax system. The reason for this dependence is because positive tax rates on s are a consequence of a missing instrument problem. In a fully flexible tax system, the efficiency gains of taxing a person in period 2 instead of period 1 could be obtained by shifting each individual's total tax burden onto their lowest-cost tax base up to the point that heterogeneous prices are arbitraged away, without the distortion of increasing marginal tax rates on s. But less flexible tax systems can only generate this shifting of the tax burden by altering marginal tax rates on s, and the optimal means of doing this depend on the nature of the restricted tax system.

6 Empirical Application

We now apply our formulas to the question of savings taxes in the United States. We first calibrate the relevant sufficient statistics from micro data and empirical studies, devoting particular attention to the calibration of the sufficient statistic $s'_{pref}(z)$. We then use the Pareto-efficiency conditions derived in Proposition 2 to compute the SL, SN and LED savings tax schedules that would be consistent with the status quo income tax schedule. This allows us to study the welfare-improving reforms that could be made to the existing tax system, taking as given the distributional preferences already embedded in the existing income tax. As is typical for calculations based on sufficient statistics formulas, these results are approximations, as they do not account for changes in the underlying distributions and sufficient statistics that might arise if the savings tax were reformed. These results suggest that across-income heterogeneity leads to a (mostly) positive and progressive schedule of savings tax rates, which range from approximately 0% at the bottom of the income distribution up to between 15% and 20% at higher incomes in our baseline calibration.

6.1 Calibration

We calibrate a model of the U.S. economy that can be interpreted through the lens of our model with a joint savings and income tax function $\mathcal{T}(s, z)$, expressed in terms of the three simple tax systems described in Table I. Appendix C discusses details of this calibration; here, we summarize the key steps. We calibrate a two-period model economy with a fine grid of incomes, where the first period corresponds to work-life and the second to retirement. We assume that these periods are separated by 20 years, and we assume a constant (and, in our baseline, homogeneous) annual rate of return of 3.8% before taxes, drawing from Fagereng et al. (2020). We calibrate a version of the economy with unidimensional heterogeneity (i.e., a single level of savings at each income) and a version with multidimensional heterogeneity, reporting results for each below.

Note that because our model builds on standard models of commodity taxation, it implicitly assumes that z and $\mathcal{T}(s, z)$ are measured in the same units as consumption, which in a dynamic setting corresponds to "period-1" dollars. In practice, savings taxes are typically levied after returns, and they are thus measured in "period-2" dollars. We accordingly translate all tax rates into units of period-2 dollars when reporting results, so that a marginal savings tax rate of 10% indicates that if an individual's total wealth at retirement increases by \$1, then they must pay an additional \$0.10 in taxes when they retire. Appendix C.1 discusses details of our calibration and this conversion.

To calibrate the earnings and savings distributions—and thus the across-income savings profile $s(z)$—we use the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65). Discretizing the income distribution into percentiles by age group, our measure of annualized earnings during the working life z at the nth percentile is constructed by averaging earnings at the nth percentile across those aged 20 to 44 and those aged 45 to 64. Our measure of s is the average value of net personal wealth, *hweal*, projected forward to age 65 based on the value observed at each income percentile in the 45-64 age bucket. This measure of wealth includes housing assets, business assets, and financial assets, net of liabilities, as well as defined-contribution pension and life insurance assets.[25] We normalize both labor earnings and retirement savings by the number of years worked.

Figure I plots our estimate of gross (i.e., after-returns and before-tax) savings per year worked, across the income distribution. This does not include Social Security, which we model as lump-sum forced savings that are received during retirement. The figure shows that savings upon retirement are approximately zero at low incomes, but increase substantially with income. We convert this to net-of-tax savings using a calibration of savings tax rates across the earnings distribution in the U.S., derived by computing the weighted average of savings tax rates using the asset composition of savings portfolios reported in Bricker et al. (2019); see Appendix C.1.2 for details. The convex shape of the savings profile, which persists after accounting for taxes, indicates that the cross-sectional slope $s'(z)$ rises with income, as shown by the solid blue line in Figure II.

To calibrate the causal income effect on savings, $s'_{inc}(z)$, and thus our measure of local preference heterogeneity $s'_{pref}(z) = s'(z) - s'_{inc}(z)$, we draw from two sources. The first is Fagereng et al. (2021), who estimate the marginal propensity to consume (MPC) out of windfall income across the earnings distribution using information on lottery prizes linked with administrative data in Norway.[26] Lottery consumption is widespread in Norway—over 70% of adults from all income groups participated in 2012—and administrative records of asset and wealth holdings allow for direct measures of savings and consumption responses to lottery winnings. They find that individuals' consumption peaks during the winning year and gradually reverts to their previous value afterwards. Over a 5-year horizon, they estimate that winners consume close to 90% of the prize (see their Figure 2, "aggregate consumption response") which translates into a long-run MPC of 0.9, and a marginal propensity to save of 0.1. They do not find significant heterogeneity across incomes in this MPC. We convert this MPC into a response of net retirement savings to changes in pre-tax labor income using our calibrated schedules of income and savings tax rates.

Our second source of data on $s'_{inc}(z)$ is a new probability-based survey representing the U.S. adult population, conducted on the AmeriSpeak panel in the spring of 2021. In the survey we asked respondents to report how much more they would save each year if they received a hypothetical raise that increased their household's income by $1000 over the coming years. The relative advantages of this survey are that is based on the U.S. population and that it asks directly about a modest, persistent change in pre-tax income, rather than a large one-time windfall. The survey results

[25] The ongoing methodological discussion regarding the different ways to measure wealth (See e.g. Saez and Zucman, 2020; Smith et al., 2021) has important implications for estimates of wealth in the top 1% , but has little impact on the wealth distribution of the rest of the population that we are using here.

[26] Two other recent studies point to the promise of estimating such causal marginal propensities in a variety of settings. Golosov et al. (2021) study the response to lottery prize winnings in the U.S., although the absence of third-party administrative reporting of wealth in the U.S. complicates the measurement of marginal propensities to save. Straub (2018) estimates the propensity to save out of permanent income, although the absence of quasi-experimental variation in earnings makes it difficult to separate causal income effects from across-income heterogeneity.

suggest a short-run MPC close to that reported in Fagereng et al. (2021), with little variation across incomes. We translate this into a long-run MPC using the response profile of Fagereng et al. (2021). Figure II reports these two schedules of $s'_{inc}(z)$. There is a substantial difference between $s'(z)$ and $s'_{inc}(z)$, which is positive across most of the income distribution and rises with income.[27] This is the key force that pushes toward a progressive and mostly positive schedule of optimal savings taxes.

We assume a constant compensated earnings elasticity of $\bar{\zeta}_z^c = 0.33$, drawn from the meta-analysis of Chetty (2012). The value of the savings elasticity $\zeta_{s|z}^c$ is related to the elasticity of taxable wealth (e.g., Jakobsen et al. (2020)) and to the elasticity of capital gains realizations with respect to the capital gains tax (e.g., Agersnap and Zidar (2021)). However, studies that use tax reforms as quasi-experimental variation for identification estimate elasticities that are likely inflated by cross-base responses, as taxpayers re-optimize their savings portfolio towards savings vehicles that are relatively less taxed after the reform.[28] We report results for a broad range of values spanning $\zeta_{s|z}^c = 0.7$ to $\zeta_{s|z}^c = 3$, with a baseline of $\zeta_{s|z}^c = 1$, which approximately aligns with the baseline calibration considered in Golosov et al. (2013), in which the intertemporal elasticity of substitution is set to one. Appendix C.1.4 discusses this conversion.

By way of comparison, Golosov et al. (2013) estimate preference heterogeneity by estimating differences in discount factors across ability levels. They infer discount factors from a simple parametric model of savings choice applied to survey data on individuals' household income paths and net worth, and they use survey respondents' results to the Armed Forces Qualification Test (AFQT) as a proxy for ability. In contrast to our findings, their estimation strategy finds very little measured preference heterogeneity, amounting to less than 1% of the cross-sectional variation in savings (see Appendix C.1.3). This discrepancy could be driven by attenuation bias due to measurement error in their proxy for ability—an issue we avoid by computing preference heterogeneity directly as a difference of two statistics rather than from regression analysis. It could also be driven by their use of a narrower measure of across-income heterogeneity based only on time preferences, as opposed to all of the possible forms of across-income heterogeneity that our statistic comprises.

6.2 Results

Figure III reports the schedule of marginal tax rates for SL, SN and LED tax systems that satisfy the Pareto efficiency formulas in Proposition 2, taking the existing U.S. income tax schedule and income distribution as given. In each case, we translate the tax into a marginal tax rate on gross savings at retirement, measured in period-2 dollars. Each panel reports results for a different value of the savings elasticity. For SL tax systems, the linear savings tax rate τ_s is by definition

[27] Our measure of $s'_{pref}(z)$ appears to be slightly negative at low incomes, which in our simulations gives rise to slight savings subsidies at low incomes. However we note that this could be driven by limitations in our ability to measure $s'_{inc}(z)$ at low incomes. This emphasizes the value of additional empirical research on this statistic.

[28] Our extension to many goods (Section 5.1) shows how the inclusion of cross-base responses affect optimal savings tax formulas. It could be used to compute the optimal savings tax on different savings vehicles, if there was a larger body of empirical evidence on savings elasticities and cross-base responses.

constant across earnings levels. For LED tax systems, the linear savings tax rate $\tau_s(z)$ is earnings-dependent and we thus report the linear savings tax rate at each earnings level. For SN tax systems, the nonlinear savings tax schedule $T_s(s)$ depends on the value of savings s, and not on earnings z. But to make the SN system visually comparable to the other systems, we plot the marginal savings tax rate faced at the margin by each earner, given their level of saving (represented on Figure I).

In each panel, marginal savings tax rates are mostly positive, and the nonlinear tax schedules are progressive, with marginal rates increasing with income. The magnitudes depend on the value of the savings elasticity parameter. In the baseline case of $\zeta^c_{s|z} = 1$, savings tax rates in SN and LED tax systems average approximately 0% for annual incomes below \$50,000, then steadily increase up to nearly 20% for annual incomes around \$200,000, remaining stable thereafter. The savings tax rate in a SL tax system is approximately 6%. Changing the savings elasticity parameter scales the efficient savings tax rates without affecting the overall pattern: across-income heterogeneity calls for (mostly) positive and progressive savings tax rates. At the lower elasticity values, our estimates of optimal tax rates are substantially higher than the prevailing savings tax rates in the U.S., which are also shown in Figure III.

Figure IV considers two key extensions to these results: multidimensional heterogeneity, and heterogeneous rates of return with "tax arbitrage" efficiency effects, as discussed in Section 5.3. For comparability with our baseline results, all other parameters, including elasticity parameters and income-dependent welfare weights, are held fixed at the values from our baseline calibration. These results are computed using our baseline savings elasticity of $\zeta^c_{s|z} = 1$. We plot both types of nonlinear tax schedules, LED and SN, omitting the separable linear plots for legibility.

In the case of multidimensional heterogeneity, we use the same measure of gross savings, but rather than compute average savings at each income, we partition the population into four levels of savings at each level of income, representing quartiles of the income-conditional savings distribution.

In the case of heterogeneous rates of return, we follow Gerritsen et al. (2020) who, relying on empirical work by Fagereng et al. (2020), assume that rates of return rise by 1.4 percentage points from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return.

Consistent with the intuition described in Section 4.3, the top two panels of Figure IV show that incorporating multidimensional heterogeneity reduces the magnitude of optimal tax rates LED systems (top left panel) and in SN systems (top right panel). The effect is particularly pronounced for SN systems, where savings tax rates are plotted as a function of total savings at the time of retirement, since agents with the same income save different amounts and thus face different savings tax rates. In this extension, marginal savings tax rates are still progressive, and are above status quo savings tax rates across high incomes in our baseline specification.

The bottom two panels show that the presence of heterogeneous rates of returns tends to significantly raise optimal savings tax rates, reflecting the efficiency effects of tax arbitrage highlighted in Proposition 6.[29] The bottom right panel shows that tax rates in the SN system are higher at

[29] Consistent with the tax arbitrage interpretation, these efficiency effects are (almost) unaffected by whether return

all levels of income, consistent with our discussion of the formula for SN systems in Proposition 6. On the other hand, recall that the formula for LED systems implied lower savings tax rates at low incomes and higher tax rates at higher incomes. Consistent with this, the bottom left panel shows that relative to the baseline, the optimal savings tax rates with heterogeneous rates of return are even more progressive. For example, substantial savings subsidies are optimal for incomes below about $\$40,000$, whereas savings taxes are substantially higher at higher incomes.

Taken together, our empirical results show a robust role for progressive savings taxes, stemming from across-income heterogeneity captured in the s'_{pref} statistic. This highlights the importance of this sufficient statistic and motivates additional empirical work estimating the long-run marginal propensity to save out of earned income, as well as across-income consumption profiles and causal income effects in other applications. Moreover, our empirical results show that policy implications depend in important ways on the type of simple tax system in question, and they demonstrate the quantitative role of multidimensional heterogeneity and additional tax arbitrage efficiency effects.

7 Conclusion

This paper characterizes optimal smooth tax systems on earnings and savings (or other dimensions of consumption) in the presence of across-income heterogeneity. We first prove that with unidimensional heterogeneity, the optimal allocation can be implemented by a smooth tax on earnings and savings. We then derive formulas which characterize the optimal smooth tax system, expressed using familiar empirical statistics, as well as a key sufficient statistic for preference heterogeneity, $s'_{pref}(z)$. This statistic can be estimated from empirical data, and can also accommodate other dimensions of heterogeneity such as heterogeneous rates of return, endowments, or income-shifting abilities. We then consider a set of “simple” separable tax systems that are widely used in practice. We derive intuitive sufficient statistics formulas for these separable tax systems, under both unidimensional and multidimensional heterogeneity. We also provide tractable extensions to multiple goods, corrective motives, and heterogeneous prices with “tax arbitrage” efficiency effects. Finally, we apply our theoretical formulas to the setting of savings taxes in the U.S.. Results suggest that the savings tax rates that would be consistent with the existing income tax are progressive and (mostly) positive. Together, these results provide a practical and general method for quantifying optimal tax systems for savings, inheritances, and other commodities in the presence of across-income heterogeneity.

heterogeneity is driven by income (scale-dependence) or by ability (type-dependence).

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Tables & Figures

Table I: Types of simple tax systems

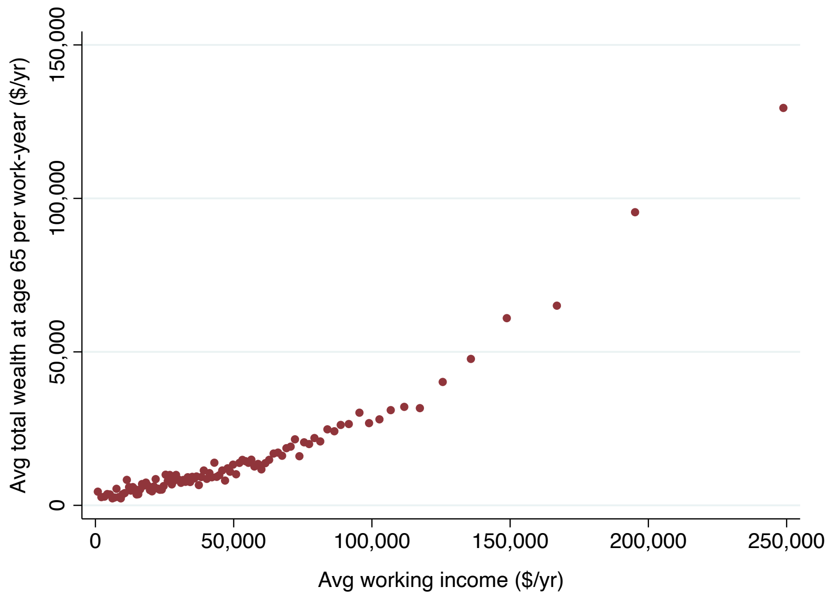
Type of simple tax system	$\mathcal{T}(s,z)$	$\mathcal{T}'_s(s,z)$	$\mathcal{T}'_z(s,z)$
SL: separable linear	$\tau_s\, s + T_z(z)$	τ_s	$T'_z(z)$
SN: separable nonlinear	$T_s(s) + T_z(z)$	$T'_s(s)$	$T'_z(z)$
LED: linear earnings-dependent	$\tau_s(z)\, s + T_z(z)$	$\tau_s(z)$	$T'_z(z) + \tau'_s(z)\, s$

Table II: Tax systems applied to different savings vehicles, by country.

Country	Wealth	Capital Gains	Property	Pensions	Inheritance
Australia	–	Other	SL, SN	SL	–
Austria	–	Other	SL, SN	SN	–
Canada	–	Other	SL	SN	–
Denmark	–	SN	SL, SN	SL, SN	SN
France	–	Other	Other	SL, SN	SN
Germany	–	Other	SL	SN	SN
Ireland	–	SN	SL, SN	SN	SN
Israel	–	Other	Other	SN	–
Italy	SL, SN	SL	SL	SL	SL, SN
Japan	–	SL, SN	SN	SN	SN
Netherlands	SN	SL	SL, SN	SN	SN
New Zealand	–	Other	SN	SL, LED	–
Norway	SN	SL	SL	SN	–
Portugal	–	SL	Other	SN	SL
Singapore	–	Other	SN	SN	–
South Korea	–	SN	SN	SN	SN
Spain	SN	SN	SL, SN	SN	SN
Switzerland	SN	SN	SL, SN	SN	SN
Taiwan	–	SL, SN	SL, SN	SN	SN
United Kingdom	–	Other	SN	SN	SN
United States	–	LED	SL	SN	SN

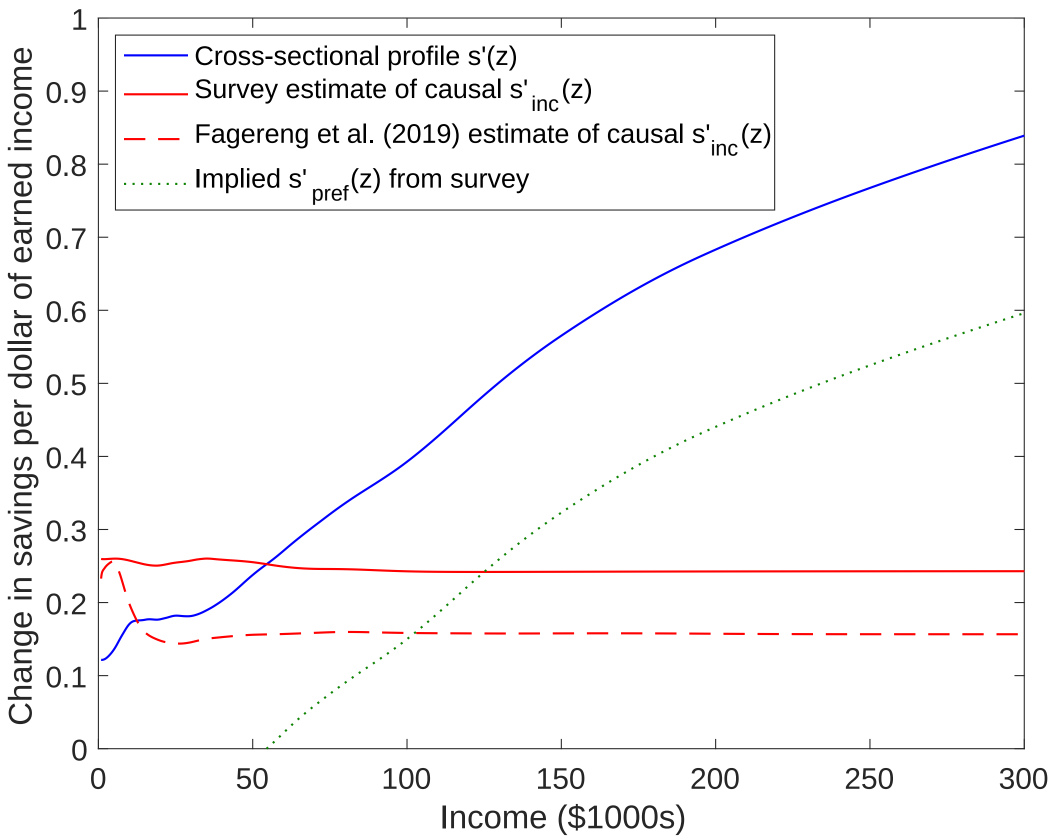
Notes: This table classifies tax systems applied to different savings vehicles across countries in 2020 according to the types in Table I. See Appendix D for further details.

Figure I: Savings Across Incomes in the United States



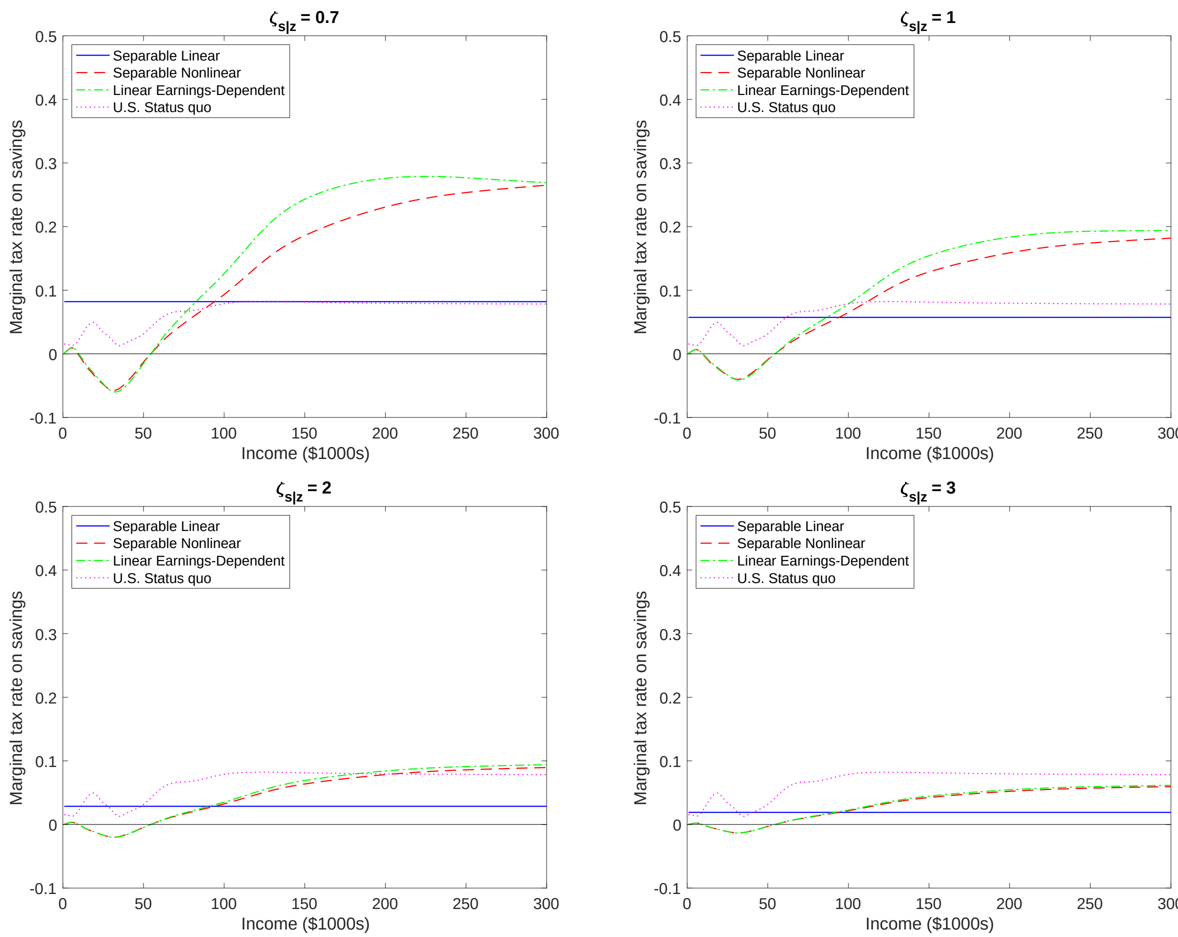
Notes: The earnings and savings distribution in the U.S. is calibrated based on the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65) to impute gross savings at the time of retirement, which we normalize by the number of work years. See Appendix C.1 for further details.

Figure II: Decomposition of Cross-Sectional Savings Profile into Income Effects and Preference Heterogeneity



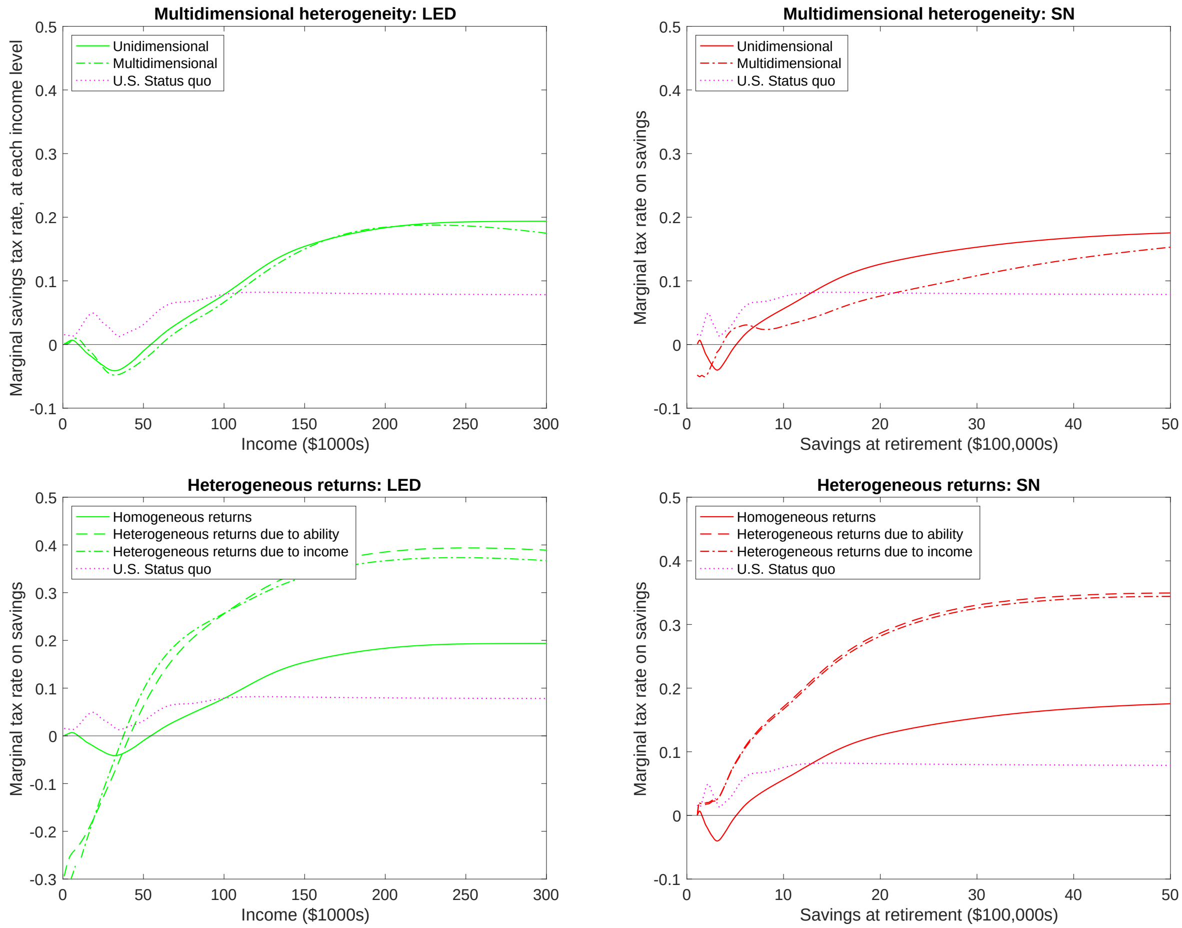
Notes: This figure reports the slope of the cross-sectional profile of savings $s'(z)$ (blue), as well as our calibrations of $s'_{inc}(z)$ based on causal income effects, derived from Fagereng et al. (2021) and from a new nationally representative survey. See Section 6 and Appendix C.1 for details.

Figure III: Savings Tax Rates Implied by Pareto-Efficiency Formulas



Notes: This figure presents the marginal savings tax rates values that satisfy the Pareto-efficiency formulas in Proposition 2, plotted against the earnings level to which they apply. We plot these schedules for four different values of the savings elasticity $\zeta_{s|z}$, with $\zeta_{s|z} = 1$ representing our baseline case.

Figure IV: Effects of Multidimensional Heterogeneity and Heterogeneous Returns



Notes: This figure plots the marginal savings tax rate schedules which are optimal, according to the first-order condition formulas presented in the text, for two extensions discussed in Section 5: multidimensional heterogeneity (top row), and heterogeneous returns (bottom row). All plots also reproduce the Pareto-efficient savings schedules from Figure III for comparison, as well as the status quo U.S. savings taxes. These plots use the same set of social welfare weights, calibrated to rationalize the status quo income tax in the unidimensional model. The Linear Earnings-Dependent (LED) schedules, in the left column, are plotted across earnings during work-life. The Separable Nonlinear (SN) schedules, in the right column, cannot be plotted this way, because individuals with a given income have different levels of savings and are thus subject to different savings taxes. We therefore plot them over total savings at the time of retirement. See Section 6 and Appendices C.2 and C.3 for details.

Online Appendix

Sufficient Statistics for Nonlinear Tax Systems With Preference Heterogeneity

Antoine Ferey, Benjamin B. Lockwood, and Dmitry Taubinsky

Table of Contents

A Supplementary Theoretical Results

A.1 Monotonicity with Preference Heterogeneity

Lemma A.1. *Under Assumption 1, 2, and 3, earnings z are strictly increasing with type θ in the optimal incentive-compatible allocation $\mathcal{A}$.*

A.2 Implementation Results for Simple Tax Systems

We proceed in three steps to provide sufficient conditions under which some SN and LED tax systems decentralize the optimal incentive-compatible allocation, here denoted $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$.

First, we define candidate SN and LED tax systems that satisfy type-specific feasibility and agents' first-order conditions at the optimal allocation. Second, in Proposition A.1, we present sufficient conditions under which these SN and LED tax systems also satisfy agents' second-order conditions at the optimal allocation, implying local optimality. Third, in Proposition A.2, we present sufficient conditions under which local optima are ensured to be global optima, implying that the candidate SN and LED systems are indeed implementing the optimal allocation.

There are interesting differences between SN and LED tax systems in their ability to implement the optimal allocation. Under our baseline assumptions, we have shown that $z^*(\theta)$ is strictly increasing with type (Lemma A.1). However, $s^*(\theta)$ may not be monotonic. When the optimal incentive-compatible allocation $\mathcal{A}$ features a monotonic $s^*(\theta)$, we show that implementation by a SN tax system requires relatively weaker conditions than implementation by a LED tax system. However, when the optimal incentive-compatible allocation $\mathcal{A}$ features non-monotonicity in $s^*(\theta)$, we show that a LED tax system may be able to implement the optimal allocation, whereas a SN tax system cannot – the candidate SN tax system is not even well defined. Hence, all implementation results for SN tax systems are made under the following assumption:

Assumption 5. *When the SN system is studied, $s^*(\theta)$ is assumed strictly monotonic (increasing or decreasing) in type θ.*

Step 1: Defining candidate tax systems. We first define a candidate SN tax system $\mathcal{T}(s, z) = T_s(s) + T_z(z)$, with the nonlinear functions T_s and T_z defined across all savings and earnings bundles of the optimal allocation $\mathcal{A} = (c^*(\theta), s^*(\theta), z^*(\theta))_\theta$ as follows:

$$T_s(s^*(\theta)) := \int_{\vartheta=\theta_{min}}^{\theta} \left(U_s'(\vartheta)/U_c'(\vartheta) - 1\right) s^{*\prime}(\vartheta) d\vartheta, \tag{55}$$

$$T_z(z^*(\theta)) := z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) + \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta \tag{56}$$

where θ_{min} denotes the lowest earning type of the compact type space Θ, and the derivatives are evaluated at the bundle assigned in the optimal allocation (e.g., $U_s'(\vartheta) = U_s'(c^*(\vartheta), s^*(\vartheta), z^*(\vartheta); \vartheta)$). Under this tax system, the optimal allocation satisfies by definition each type's first-order conditions for individual optimization given in Equations (9) and (10). By Lemma B.1, this tax system thus satisfies type-specific feasibility.

We similarly define a candidate LED tax system $\mathcal{T}(s,z) = \tau_s(z) \cdot s + T_z(z)$ as follows:

$$\tau_s(z^*(\theta)) := U_s'(\theta)/U_c'(\theta) - 1, \tag{57}$$

$$\begin{aligned} T_z(z^*(\theta)) :=& z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) \\ &+ \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta - s^*(z) \cdot \left(\tau_s(z) - \tau_s(z^*(\theta_{min}))\right). \end{aligned} \tag{58}$$

This tax system also satisfies local first-order conditions for individual optimization and type-specific feasibility.

Step 2: Local maxima. We can now derive sufficient conditions under which the above candidate SN and LED tax systems satisfy the second-order conditions for individual optimization, implying that under these conditions assigned bundles are local optima . These conditions can be simply stated in terms of the marginal rates of substitution between consumption and, respectively, savings $\mathcal{S}(c,s,z;\theta)$ and earnings $\mathcal{Z}(c,s,z;\theta)$. These marginal rates of substitutions are smooth functions of c, s, z, and θ by the smoothness of the allocation and the utility function, and sufficient conditions for local second-order conditions are given by the following proposition.

Proposition A.1. *Suppose that an allocation satisfies the conditions in Theorem 1. Under the SN tax system defined by Equations* (55) *and* (56)*, each agent's assigned choice of savings and earnings is a local optimum if the following conditions hold at each point in the allocation:*

$$\mathcal{S}_c' \geq 0,\ \mathcal{S}_z' \geq 0,\ \mathcal{S}_\theta' \geq 0 \tag{59}$$

and

$$\mathcal{Z}_c' \leq 0,\ \mathcal{Z}_s' \geq 0,\ \mathcal{Z}_\theta' \geq 0. \tag{60}$$

Under the LED tax system defined by Equations (57) *and* (58)*, each agent's assigned choice of savings and earnings is a local optimum if the utility function is additively separable in consumption, savings, and earnings ($U_{cs}'' = 0$, $U_{cz}'' = 0$, and $U_{sz}'' = 0$), and additionally the following conditions hold at each point in the allocation:*

$$\mathcal{S}_\theta' \geq 0,\ \mathcal{S}_\theta' \leq \frac{z^{*\prime}(\theta)}{s^{*\prime}(\theta)} \mathcal{Z}_\theta',\ \mathcal{S}_\theta' \leq s^{*\prime}(\theta) \left(\mathcal{S} \cdot \mathcal{S}_c' - \mathcal{S}_s'\right). \tag{61}$$

The sufficiency conditions (59) and (60) are quite weak; they are satisfied under many common utility functions used in calibrations of savings and income taxation models, including the simple example function in Equation (1). Conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{Z}_\theta' \geq 0$ are single crossing conditions for savings and earnings, while other conditions intuitively relate to the concavity of preferences.

The sufficiency conditions for LED systems are more restrictive. Beyond the single-crossing conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{Z}_\theta' \geq 0$, they place a constraint on the extent of local preference heterogeneity for savings, as compared with preference heterogeneity in earnings. In words, the preference for savings must not increase "too quickly" across types, or else the second-order condition for earnings may be violated. The intuition for this result can be seen from the definition of the potentially optimal LED system. If the marginal rate of substitution for saving, $\mathcal{S}$, increases very quickly with income at some point in the allocation, then the savings tax rate $\tau_s(z)$ must rise very quickly with z at that point, by Equation (57). Since the savings tax rate $\tau_s(z)$ applies to total savings (including inframarginal savings), this increase in $\tau_s(z)$ must be offset by a sharp decrease in $T_z(z)$ at the same point in the distribution, by Equation (58). Yet a sufficiently steep decrease in $T_z(z)$ will cause the second-order condition for earnings choice—holding fixed savings choice—to be violated.

Step 3: Global maxima. Having presented conditions under which the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ assigned to type θ is a local optimum under the candidate SN and LED tax systems, we now present a set of regularity conditions ensuring that these local optima are also global optima.

Proposition A.2. *Assume single-crossing conditions for earnings and savings ($\mathcal{Z}'_\theta \geq 0$ and $\mathcal{S}'_\theta \geq 0$), that preferences are weakly separable ($U''_{cz} = 0$ and $U''_{sz} = 0$), and that commodities c and s are weak complements ($U''_{cs} \geq 0$). If $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, local optima correspond to global optima when:*

1. *$\mathcal{T}$ is a SN system, and we have that for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{-T''_{ss}(s)}{1+T'_s(s)}$.*

2. *$\mathcal{T}$ is a LED system, and we have that*

   (a) for all $s < s^*(\theta)$ and θ, $\frac{-U''_{cc}(c(s,\theta),s,z^*(\theta);\theta)}{U'_c(c(s,\theta),s,z^*(\theta);\theta)} > \frac{1}{1+\tau_s(z^*(\theta))} \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s - T'_z(z^*(\theta))}$,

   (b) for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s - T'_z(z^*(\theta))}$.

where $c(s,\theta) := z^*(\theta) - s - \mathcal{T}(s, z^*(\theta))$

In essence, global optimality is ensured under the following assumptions. First, higher types θ derive higher gains from working and allocating those gains to consumption or savings — generalized single-crossing conditions. Second, additive separability of consumption and savings from labor. Third, the utility function U is sufficiently concave in consumption and savings.

For the case of SN tax systems, condition 1 imposes a particular concavity requirement with respect to savings. For the case of LED tax systems, condition 2 imposes particular concavity requirements with respect to both consumption and savings. Notably, these concavity conditions need only be checked when earnings are fixed at each type's assigned earnings level $z^*(\theta)$.

We can naturally apply this result to the candidate SN tax system defined in Equations (55) and (56), and to the candidate LED tax system defined in Equations (57) and (58). Because these candidate tax systems are defined in terms of agents' preferences and optimal allocations, we can then express conditions 1 and 2 fully in terms of agents' preferences and optimal allocations.

A.3 Optimal Taxes on s in Simple Tax Systems

We present optimal savings tax formulas for simple tax systems, which characterize the optimal savings tax schedule for *any* given earnings tax schedule—including a potentially suboptimal one. These formulas are derived assuming unidimensional heterogeneity and are similar to those presented in Proposition 3, where heterogeneity is allowed to be multidimensional.

Proposition A.3. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$, and assume that under each simple tax system (i) agents first-order and second-order conditions strictly hold, (ii) there is no bunching, (iii) c and s are smooth functions of z, and (iv) in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, SL, SN, and LED systems satisfy the following optimality conditions on the savings tax rates:*

$$SL: \frac{\tau_s}{1+\tau_s}\int_{x=z_{min}}^{z_{max}} s(x)\zeta_{s|z}^c(x)dH_z(x) = \int_{x=z_{min}}^{z_{max}}\left\{s\left(x\right)\left(1-\hat{g}(x)\right) - \frac{T_z'(x)+s_{inc}'(x)\,\tau_s}{1-T_z'\left(x\right)}x\zeta_z^c(x)\,s_{inc}'(x)\right\}dH_z(x) \tag{62}$$

$$SN: \frac{T_s'(s)}{1+T_s'(s)}s\zeta_{s|z}^c(z)h_z(z) = s'\left(z\right)\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - \frac{T_z'(z)+s_{inc}'(z)T_s'(s)}{1-T_z'(z)}z\zeta_z^c(z)s_{inc}'(z)h_z(z) \tag{63}$$

$$\begin{aligned} LED: &\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'(z)\tau_s(z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(z)\,s\,h_z(z)+\int_{x\geq z}\frac{\tau_s(x)}{1+\tau_s(x)}s(x)\zeta_{s|z}^c(x)\,dH_z(x) \\ &= \int_{x\geq z}\left\{\left(1-\hat{g}(x)\right)s\left(x\right)-\frac{T_z'\left(x\right)+\tau_s'\left(x\right)s\left(x\right)+s_{inc}'(x)\tau_s(x)}{1-T_z'\left(x\right)-\tau_s'\left(x\right)s\left(x\right)}x\zeta_z^c(x)s_{inc}'(x)\right\}dH_z(x)\end{aligned} \tag{64}$$

These optimal savings tax formulas are all different, reflecting differences between the savings tax instruments that we consider. Yet, they share common elements. First, the preference heterogeneity term $s_{pref}'(z)$ no longer appears in the formulas. The intuition is that outside of the full optimum, it may still be desirable to tax savings in the absence of preference heterogeneity, implying that optimality may clash with Pareto efficiency when the earnings tax is suboptimal. Second, $s_{inc}'(z)$ is a key sufficient statistic for optimal savings tax schedules. Indeed, by Lemma 1, a larger $s_{inc}'(z)$ means that savings tax reforms impose higher distortions on earnings and thus generally calls for lower savings tax rate.

A.4 Optimal Taxes on z in Simple Tax Systems

We now present optimal earnings tax formulas for simple tax systems.

Proposition A.4. *Consider given (and potentially suboptimal) SL, SN and LED savings tax schedules, and assume that under each simple tax system (i) agents first-order and second-order conditions strictly hold, (ii) there is no bunching, (iii) c and s are smooth functions of z, and (iv) in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, SL, SN, and LED systems satisfy the following optimality conditions on the earnings* *tax rates:*

$$SL: \frac{T_z'(z)}{1-T_z'(z)} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z)\frac{\tau_s}{1-T_z'(z)} \tag{65}$$

$$SN: \frac{T_z'(z)}{1-T_z'(z)} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z)\frac{T_s'(s)}{1-T_z'(z)} \tag{66}$$

$$LED: \frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z)\frac{\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s} \tag{67}$$

These conditions pinning down the optimal schedule of marginal earnings tax rates are a direct application of Equation (23) presented in Theorem 2 for smooth tax systems. While formulas for SL and SN tax systems look almost identical to the general condition, the formula for LED tax system looks a bit different. This difference only reflects the fact that for a LED tax system the marginal earnings tax rate is given by $\mathcal{T}_z'(s,z) = T_z'(z) + \tau_s'(z)\,s(z)$, accounting for the earnings-dependent nature of savings taxes.

A.5 Optimal Taxes on z in Simple Tax Systems with Multidimensional Heterogeneity

Proposition A.5. *Consider given (and potentially suboptimal) SL, SN, and LED savings tax schedule, and assume that under each simple tax system agents first-order and second-order conditions strictly hold. Then, at each bundle (c^0, s^0, z^0) chosen by a type θ^0, marginal tax rates on z in SL/SN/LED systems must satisfy the following optimality conditions:*

$$\begin{aligned} SL: \frac{T_z'(z^0)}{1-T_z'(z^0)}\mathbb{E}\left[\zeta_z^c(s,z)\middle|z^0\right] &= \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\}dH_z(z) \\ &\quad - \mathbb{E}\left[s_{inc}'(s,z)\frac{\tau_s}{1-T_z'(z)}\zeta_z^c(s,z)\middle|z^0\right] \end{aligned} \tag{68}$$

$$\begin{aligned} SN: \frac{T_z'(z^0)}{1-T_z'(z^0)}\mathbb{E}\left[\zeta_z^c(s,z)\middle|z^0\right] &= \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\}dH_z(z) \\ &\quad - \mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\zeta_z^c(s,z)\middle|z^0\right] \end{aligned} \tag{69}$$

$$\begin{aligned} LED: \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}\zeta_z^c(s,z)\middle|z^0\right] &= \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\}dH_z(z) \\ &\quad - \mathbb{E}\left[s_{inc}'(s,z)\frac{\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}\zeta_z^c(s,z)\middle|z^0\right] \end{aligned} \tag{70}$$

These conditions are similar to those presented above for optimal marginal earnings tax rates under unidimensional heterogeneity (Proposition A.4). Indeed, Lemma 1 still applies such that the previous derivations carry over when adding an expectation with respect to savings. Proofs are thus omitted.

A.6 Equivalences with Tax Systems Involving Gross Period-2 Savings

Suppose that there are two periods, and set $1 + r = 1/p$. In period 1 the individual earns z and consumes c, and pays income taxes $T_1(z)$. In period 2 the individual realizes *gross pre-tax savings* $s_g = (z - c - T_1(z))(1 + r)$, and pays income taxes $T_2(s_g, z)$. The realized savings s are given by $s_g - T_2(s_g, z)$. The total tax paid in "period-1 dollars" is given by $T_1(z) + T_2(s_g, z)/(1 + r)$. The individual maximizes $U(c, s, z)$ subject to the constraint

$$\begin{aligned} s &\leq (z - c - T_1(z))(1 + r) - T_2(s_g, z) \\ \Leftrightarrow c + \frac{s}{1 + r} &\leq z - T_1(z) - \frac{T_2((z - c - T_1(z))(1 + r), z)}{1 + r}. \end{aligned}$$

In our baseline formulation with period-1 tax function $\mathcal{T}(s, z)$, individuals choose s and z to maximize $U(z - s - \mathcal{T}(s, z), s, z; \theta)$. To convert from the formulation with period-2 taxes to our baseline formulation, define a function $\tilde{s}_g(s, z)$ implicitly to satisfy the equation

$$\tilde{s}_g - T_2(\tilde{s}_g, z) = s$$

Note that $\tilde{s}_g$ is generally uniquely defined if we have a system with monotonic realized savings s. Then the equivalence in tax schedules is given by

$$\mathcal{T}_s'(s, z) = \frac{1}{1 + r} \frac{\partial}{\partial s_g} T_2(s_g, z)|_{s_g = \tilde{s}_g} \frac{\partial}{\partial s} \tilde{s}_g \tag{71}$$

and $\mathcal{T}_z' = T_z'$. Equation (71) simply computes how a marginal change in s changes the tax burden in terms of period-1 units of money, and the division by $1 + r$ is to convert to period-1 units. Now differentiating the definition of $\tilde{s}_g$ gives

$$\frac{\partial}{\partial s} \tilde{s}_g - \frac{\partial}{\partial s_g} T_2(s_g, z) \frac{\partial}{\partial s} \tilde{s}_g = 1$$

and thus

$$\frac{\partial}{\partial s} \tilde{s}_g = \frac{1}{1 - \frac{\partial}{\partial s_g} T_2(s_g, z)}$$

from which it follows that

$$\mathcal{T}_s'(s, z) = \frac{1}{1 + r} \frac{\frac{\partial}{\partial s_g} T_2(s_g, z)|_{s_g = \tilde{s}_g}}{1 - \frac{\partial}{\partial s_g} T_2(s_g, z)|_{s_g = \tilde{s}_g}}. \tag{72}$$

We can also start with a schedule $\mathcal{T}$ and converts it to the two-period tax schedule. Now if s is the realized savings, we can define gross savings in period 2 as $s_g = s + \mathcal{T}(z, s)(1 + r) - \mathcal{T}(z, 0)$, and we define the function $\tilde{s}(s_g, z)$ to satisfy

$$s_g = \tilde{s} + (1 + r) \left(\mathcal{T}(\tilde{s}, z) - \mathcal{T}(0, z) \right).$$

Then

$$\frac{\partial}{\partial s_g} T_2(s_g, z) = (1 + r)\mathcal{T}_s'(\tilde{s}, z)\frac{\partial}{\partial s_g}\tilde{s}$$
$$= \frac{(1 + r)\mathcal{T}_s'(\tilde{s}, z)}{1 + (1 + r)\mathcal{T}_s'(\tilde{s}, z)} \tag{73}$$

A.6.1 Separable tax systems (SN).

Now if T_2 is a function of s_g only (a separable tax system), then s_g will be a function of s only, and thus $\mathcal{T}_s'$ will only depend on s. Conversely, note that if $\mathcal{T}$ is a separable system, so that $\mathcal{T}_s'$ does not depend on z, then (73) implies that $\frac{\partial}{\partial s_g} T_2(s_g, z)$ does not depend on z either. Thus, separability is a property preserved under these transformations.

Now if we start with a separable $\mathcal{T}$, then T_2 is given by

$$T_2'(s_g) = (1 + r)\frac{\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}{1 + \frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}$$

where $\tilde{s}$ is the value that solves $s_g = \tilde{s} + \mathcal{T}(\tilde{s})$.

A.6.2 Linear tax systems (LED and SL).

If $T_2 = s_g\,\tau(z)$, a linear earnings-dependent system, then $s = s_g(1 - \tau(z))$ and $s_g = \frac{s}{1-\tau(z)}$. Moreover, $\frac{\partial}{\partial s} s_g = \frac{1}{1-\tau(z)}$, and thus we have that

$$\mathcal{T}_s' = \frac{1}{1 + r}\frac{\tau(z)}{1 - \tau(z)}$$

which again implies that we have a linear earnings-dependent system with rate $\tilde{\tau}(z) = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$.

Conversely, if we start with a LED system $\mathcal{T}$ with $\mathcal{T}_s' = \tau(z)$, then

$$\frac{\partial}{\partial s_g} T_2(s_g, z) = (1 + r)\frac{\tau(z)}{1 + \tau(z)}.$$

When the tax rates τ are not functions of z, the calculations above show that the conversions preserve not just linearity, but also independence of z.

B Proofs

B.1 Proof of Lemma A.1 (Monotonicity with Preference Heterogeneity)

By Assumption 3, any (c, s) offered in the optimal incentive-compatible allocation can be written as functions of z: $c(z)$ and $s(z)$. By Assumption 2, c, s, z are differentiable in θ which implies that c and s are differentiable in z as well. The total derivative of agent θ utility $U(c(z), s(z), z; \theta)$ with respect to earnings z is

$$\begin{aligned}\frac{dU\left(c(z),s(z),z;\theta\right)}{dz} &= U_c'\left(c(z),s(z),z;\theta\right)c'(z)+U_s'\left(c(z),s(z),z;\theta\right)s'(z)+U_z'\left(c(z),s(z),z;\theta\right)\\ &= U_c'\left(c(z),s(z),z;\theta\right)\left(c'(z)+\mathcal{S}\left(c(z),s(z),z;\theta\right)s'(z)+\mathcal{Z}\left(c(z),s(z),z;\theta\right)\right)\end{aligned}$$

The first-order condition for an agent θ choosing earnings z therefore implies[30]

$$c'(z)+\mathcal{S}\left(c(z),s(z),z;\theta\right)s'(z)+\mathcal{Z}\left(c(z),s(z),z;\theta\right)=0.$$

As a result, the extended Spence-Mirrlees condition implies that each type θ chooses a different earnings level z since it implies

$$\mathcal{S}_\theta'\left(c(z),s(z),z;\theta\right)s'(z)+\mathcal{Z}_\theta'\left(c(z),s(z),z;\theta\right)>0,$$

meaning that the first-order condition cannot hold for two different θ.

Moreover, the extended Spence-Mirrlees condition also implies that type $\theta_2>\theta_1$ chooses earnings $z_2>z_1$. To prove this result, we proceed by contradiction. Assume (without loss of generality) that there is an open set $(\theta_1,\theta_2)\in\Theta$ where z is decreasing with θ, and denote $z_1:=z(\theta_1)$ and $z_2:=z(\theta_2)$ where $z_2<z_1$. Then,

$$\begin{aligned}&U\left(c(z_2),s(z_2),z_2;\theta_2\right)-U\left(c(z_1),s(z_1),z_1;\theta_2\right)\\ &=\int_{z=z_1}^{z_2}\frac{dU\left(c(z),s(z),z;\theta_2\right)}{dz}dz\\ &=-\int_{z=z_2}^{z_1}U_c'\left(c(z),s(z),z;\theta_2\right)\left(c'(z)+\mathcal{S}\left(c(z),s(z),z;\theta_2\right)s'(z)+\mathcal{Z}\left(c(z),s(z),z;\theta_2\right)\right)dz.\end{aligned}$$

But for each $\theta\in(\theta_1,\theta_2)$ choosing earnings $z\in[z_2,z_1]$ the first-order condition is

$$c'(z)+\mathcal{S}\left(c(z),s(z),z;\theta\right)s'(z)+\mathcal{Z}\left(c(z),s(z),z;\theta\right)=0$$

such that the extended Spence-Mirrlees condition implies that with $\theta_2>\theta$

$$c'(z)+\mathcal{S}\left(c(z),s(z),z;\theta_2\right)s'(z)+\mathcal{Z}\left(c(z),s(z),z;\theta_2\right)>0$$

Since $U_c'>0$, this means that the integral above is positive and thus that

$$U\left(c(z_2),s(z_2),z_2;\theta_2\right)<U\left(c(z_1),s(z_1),z_1;\theta_2\right).$$

This is a contradiction with the assumption that type $\theta_2>\theta_1$ chooses earnings $z_2<z_1$, which concludes the proof.

[30] The first-order condition has to be satisfied for any type θ in the interior of the compact space of types Θ since we assume smooth allocations (Assumption 2) and smooth preferences (Assumption 1).

B.2 Proof of Theorem 1 (Implementation with a Smooth Tax System)

In the appendix, we adopt the notation that agent's allocations in the optimal mechanism are labeled with a "star"; i.e., $(c^*(\theta), s^*(\theta), z^*(\theta))$. We construct a smooth tax system that implements the optimal incentive-compatible allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ by introducing penalties for deviations away from these allocations. This proof relies on Lemma B.1 and Lemma B.2, which we derive at the end of this subsection.

With unidimensional heterogeneity, type θ belongs to the compact space $\Theta = [\theta_{min}, \theta_{max}]$. Moreover, there is always a mapping $s^*(z)$ that denotes the savings level associated with earnings level $z = z^*(\theta)$ at the optimal incentive-compatible allocation.[31] Let $z_{max} := z^*(\theta_{max})$ and $z_{min} := z^*(\theta_{min})$ denote the maximal and minimal, respectively, earnings levels in the allocation. Let $s_{max} := \max_z s^*(z)$ and $s_{min} := \min_z s^*(z)$ denote the maximal and minimal savings levels.

Step 1: Defining the smooth tax system. We start from a separable and smooth tax system $T_s(s) + T_z(z)$ that satisfies type-specific feasibility and agents' first-order conditions at the optimal incentive-compatible allocation. We then add quadratic penalty terms parametrized by k for deviations from this allocation. For a given deviation, this allows to make the penalty arbitrarily large and enables us to make agents problem locally concave around the optimal incentive-compatible allocation. The other terms that we add are there to guarantee the smoothness of the penalized tax system $\mathcal{T}(s, z; k)$ at the boundaries of the set of optimal allocations.

Formally, $\mathcal{T}_k = \mathcal{T}(s, z; k)$ is defined by:

1. $T_s(s_{min}) = 0$ and $T_z(z_{min}) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\forall z \in [\, z_{min}\, ;\, z_{max}\,], T_z'(z) = \mathcal{Z}(c^*(\theta_z), s^*(\theta_z), z^*(\theta_z); \theta_z) + 1$ with θ_z such that $z = z^*(\theta_z)$
3. $\forall s \in [\, s_{min}\, ;\, s_{max}\,], T_s'(s) = \mathcal{S}(c^*(\theta_s), s^*(\theta_s), z^*(\theta_s); \theta_s) - 1$ with θ_s such that $s = s^*(\theta_s)$

[31] We consider (without loss of generality) the case in which s is strictly increasing, the proof is similar if s is strictly decreasing instead.

4. $$\mathcal{T}(s,z;k)=\begin{cases} T_s(s)+T_z(z)+k(s-s^*(z))^2 & \text{if } z_{min}\leq z\leq z_{max}, s_{min}\leq s\leq s_{max} \\ T_s(s_{min})+T_z(z)+k(s-s^*(z))^2+T_s'(s_{min})(s-s_{min}) & \text{if } z_{min}\leq z\leq z_{max}, s< s_{min} \\ T_s(s_{max})+T_z(z)+k(s-s^*(z))^2+T_s'(s_{max})(s-s_{max}) & \text{if } z_{min}\leq z\leq z_{max}, s> s_{max} \\ T_s(s)+T_z(z_{min})+k(s-s_{min})^2+k(z-z_{min})^2 & \text{if } z<z_{min}, s_{min}\leq s\leq s_{max} \\ \qquad +T_z'(z_{min})(z-z_{min}) & \\ T_s(s_{min})+T_z(z_{min})+k(s-s_{min})^2+k(z-z_{min})^2 & \text{if } z<z_{min}, s<s_{min} \\ \qquad +T_z'(z_{min})(z-z_{min})+T_s'(s_{min})(s-s_{min}) & \\ T_s(s_{max})+T_z(z_{min})+k(s-s_{min})^2+k(z-z_{min})^2 & \text{if } z<z_{min}, s>s_{max} \\ \qquad +T_z'(z_{min})(z-z_{min})+T_s'(s_{max})(s-s_{max}) & \\ T_s(s)+T_z(z_{max})+k(s-s_{max})^2+k(z-z_{max})^2 & \text{if } z>z_{max}, s_{min}\leq s\leq s_{max} \\ \qquad +T_z'(z_{max})(z-z_{max}) & \\ T_s(s_{max})+T_z(z_{max})+k(s-s_{max})^2+k(z-z_{max})^2 & \text{if } z>z_{max}, s>s_{max} \\ \qquad +T_z'(z_{max})(z-z_{max})+T_s'(s_{max})(s-s_{max}) & \\ T_s(s_{min})+T_z(z_{max})+k(s-s_{max})^2+k(z-z_{max})^2 & \text{if } z>z_{max}, s<s_{min} \\ \qquad +T_z'(z_{max})(z-z_{max})+T_s'(s_{min})(s-s_{min}) & \end{cases}$$

Assumptions 1 and 2 guarantee that the separable tax system $T_s(s)+T_z(z)$ is smooth, i.e., a twice continuously differentiable function. Our construction of the penalized tax system $\mathcal{T}_k=\mathcal{T}(s,z;k)$ guarantees that it inherits this smoothness property.

Step 2: Local maxima for sufficiently large k. For a given agent θ, we show that the bundle $(c^*(\theta),s^*(\theta),z^*(\theta))$ is a local optimum under the tax system $\mathcal{T}_k=\mathcal{T}(s,z;k)$ for sufficiently large k. To do so, we first establish that type-specific feasibility is satisfied together with the first-order conditions of agent θ maximization problem. We then show that for sufficiently large k, second-order conditions are also satisfied implying that the intended bundle is a local maximum.

The previous definition of the tax system implies

$$\mathcal{T}_z'(s^*(\theta),z^*(\theta);k)=T_z'(z^*(\theta))=\mathcal{Z}(c^*(\theta),s^*(\theta),z^*(\theta);\theta)+1$$
$$\mathcal{T}_s'(s^*(\theta),z^*(\theta);k)=T_s'(s^*(\theta))=\mathcal{S}(c^*(\theta),s^*(\theta),z^*(\theta);\theta)-1$$

meaning type-specific feasibility is satisfied by Lemma B.1 (see below).

Now, defining

$$V(s,z;\theta,k):=U(z-s-\mathcal{T}(s,z;k),s,z;\theta), \tag{74}$$

first-order conditions of agent θ choice of savings s and earnings z are

$$V_s'(s,z;\theta,k)=-(1+\mathcal{T}_s'(s,z;k))U_c'(z-s-\mathcal{T}(s,z;k),s,z;\theta)+U_s'(z-s-\mathcal{T}(s,z;k),s,z;\theta)=0$$
$$V_z'(s,z;\theta,k)=(1-\mathcal{T}_z'(s,z;k))U_c'(z-s-\mathcal{T}(s,z;k),s,z;\theta)+U_z'(z-s-\mathcal{T}(s,z;k),s,z;\theta)=0$$

and they are by construction satisfied at $(s^*(\theta), z^*(\theta))$ for each type θ.

Using Lemma B.2 (see below), second-order conditions at $(s^*(\theta), z^*(\theta))$ are

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \leq 0 \tag{75}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \leq 0 \tag{76}$$

$$\begin{aligned}(V''_{sz})^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg] \leq 0\end{aligned} \tag{77}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Here, U, $\mathcal{S}$, and $\mathcal{Z}$ are smooth functions implying that their derivatives are continuous functions over compact spaces, and are thus bounded. Now, by definition of $\mathcal{T}_k = \mathcal{T}(s, z; k)$, we have $\mathcal{T}''_{sz} = -2ks^{*\prime}(z)$ which is negative for any $k \geq 0$ and increasing in magnitude with k.

Noting $U'_c \geq 0$ and $s^{*\prime}(z) \geq 0$, it implies that V''_{ss} and V''_{zz} must be negative for sufficiently large k thanks to the last term since other terms are bounded and do not depend on k. By the same reasoning, under the extended Spence-Mirrlees single-crossing assumption that $\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta \geq 0$, we also have that $(V''_{sz})^2 - V''_{ss}V''_{zz}$ must be negative for sufficiently large k.

This shows that for a given agent θ, there exists a k_0 such that for all $k \geq k_0$ the allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum to agent θ maximization problem under the smooth penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$.

Step 3: Global maxima for sufficiently large k. Let $s_{\mathcal{T}_k}(\theta)$ and $z_{\mathcal{T}_k}(\theta)$ denote the level of savings and earnings, respectively, that a type θ chooses given a smooth penalized tax system $\mathcal{T}_k$. To prove implementability of optimal incentive-compatible allocations, we show that there exists a sufficiently large k such that for all θ, $s_{\mathcal{T}_k}(\theta) = s^*(\theta)$ and $z_{\mathcal{T}_k}(\theta) = z^*(\theta)$.

Let's proceed by contradiction, and suppose that it is not the case. Then, there exists an infinite sequence of types θ_k, choosing savings $s_{\mathcal{T}_k}(\theta_k) \neq s^*(\theta_k)$ and earnings $z_{\mathcal{T}_k}(\theta_k) \neq z^*(\theta_k)$ under tax system $\mathcal{T}_k$, and enjoying utility gains from this "deviation" to allocation $(s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$.

First, the fact that we impose quadratic penalties for earnings choices outside of $[\, z_{min}\,;\, z_{max}\,]$ guarantees that for any $\varepsilon > 0$, there exists k_1, such that for all $k \geq k_1$ and for all types θ, agents' earnings choices belong to the compact set $[\, z_{min} - \varepsilon\,;\, z_{max} + \varepsilon\,]$. Indeed, starting from a given earnings level $z > z_{max} + \varepsilon$, the utility change associated with an earnings change is $[(1 - \mathcal{T}'_z)\, U'_c + U'_z]\, dz$. By construction, the marginal earnings tax rate in this region is $\mathcal{T}'_z = 2k(z - z_{max}) + T'_z(z_{max})$. Noting that $U'_c > 0$, $U'_z < 0$, and that the type space is compact, we can make for all individuals the utility change from a decrease in earnings arbitrarily positive for sufficiently large k. This shows that all individuals choose earnings $z \leq z_{max} + \varepsilon$ for sufficiently large k. Symetrically, we can show that all individuals choose earnings $z \geq z_{min} - \varepsilon$ for sufficiently large k.

Second, since earnings shape agents' disposable income, the fact that earnings belong to a compact set for sufficiently large k implies that agents' allocation choices also belong to a compact set. Indeed, for sufficiently large k, agents' allocation choices must belong to the set of (c, s, z) such that $c \geq 0$, $s \geq 0$, $z \in [\, z_{min} - \varepsilon\, ;\, z_{max} + \varepsilon\,]$, and $c + s \leq z - \mathcal{T}(s, z; k)$ where the tax function is smooth and finite. These constraints make the space of allocations compact.

As a result, the infinite sequence $(\theta_k, s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$ belongs to a compact space of allocations and types, it is thus bounded. By the Bolzano–Weierstrass theorem, this means that there exists a convergent subsequence $\left(\theta_j, s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{\theta}, \hat{s}, \hat{z})$. Since all types θ_j belong to $[\,\theta_{min}\,;\,\theta_{max}\,]$, we know that the limit type $\hat{\theta}$ must belong to $[\,\theta_{min}\,;\,\theta_{max}\,]$. Now, from the previous paragraph, agents' earnings choices have to be arbitrarily close to $[\,z_{min}\,;\,z_{max}\,]$ as the penalty grows. This implies that the limit $\hat{z}$ must belong to $[\,z_{min}\,;\,z_{max}\,]$.

Next, we establish that the limit $\hat{s}$ must be such that $\hat{s} = s^*(\hat{z})$. Fix an earnings level $z \in [\,z_{min}\,;\,z_{max}\,]$, starting from a savings level $s \neq s^*(z)$, the utility change associated with a savings change is $[-(1 + \mathcal{T}'_s)\, U'_c + U'_s]\, ds$. Assuming without loss of generality that s belongs to $[\,s_{min}\,;\,s_{max}\,]$, the marginal savings tax rate in this region is $\mathcal{T}'_s = T'_s(s) + 2k(s - s^*(z))$. Noting that $U'_c > 0$, and that U'_s is bounded, we can make the utility gains from a savings change towards $s^*(z)$ arbitrarily large for sufficiently large k. As a result, for any $\varepsilon > 0$, there exists k_2 such that for all $k \geq k_2$, agent $\hat{\theta}$ chooses savings $s \in [\,s^*(z) - \varepsilon\,;\,s^*(z) + \varepsilon\,]$ for a fixed z.[32] Since agent $\hat{\theta}$ savings choice can be made arbitrarily close to $s^*(z)$ for sufficiently large k, we must have at the limit $s = s^*(z)$. Now, because earnings z converge to $\hat{z}$ and the function $s^*(z)$ is by assumption continuous, we must have at the limit $\hat{s} = s^*(\hat{z})$.

We have thus established that the limit $(\hat{\theta}, \hat{s}, \hat{z})$ is such that $\hat{\theta} \in [\,\theta_{min}\,;\,\theta_{max}\,]$, $\hat{z} \in [\,z_{min}\,;\,z_{max}\,]$, and $\hat{s} = s^*(\hat{z})$. This means that the limit allocation $(\hat{c}, \hat{s}, \hat{z})$ belongs to the set of optimal incentive-compatible allocations. Given our continuity and monotonicity assumptions, this implies that it is the optimal allocation of some type θ and it has to be by definition that of agent $\hat{\theta}$. Hence, $(\hat{c}, \hat{s}, \hat{z}) = (c^*(\hat{\theta}), s^*(\hat{\theta}), z^*(\hat{\theta}))$.

To complete the proof and find a contradiction, fix a value $k^\dagger$ that is large enough such that second-order conditions hold for type $\hat{\theta}$ at allocation $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ under tax system $\mathcal{T}_{k^\dagger}$ – this $k^\dagger$ exists by step 2. This implies that there exists an open set N containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ such that $V(s, z; \hat{\theta}, k^\dagger)$ is strictly concave over $(s, z) \in N$.

Now, consider types θ^j with $j \geq k^\dagger$. Since these agents belong to the previously defined subsequence, they prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_j$. Because penalties are increasingly large and $j \geq k^\dagger$, this implies that agents θ^j also prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_{k^\dagger}$.

Yet, by continuity, the function $V\left(s, z; \theta_j, k^\dagger\right)$ gets arbitrarily close to the function $V(s, z; \hat{\theta}, k^\dagger)$ for sufficiently large j since $\theta_j \to \hat{\theta}$. For the same reason, $(s^*(\theta_j), z^*(\theta_j)) \to (s^*(\hat{\theta}), z^*(\hat{\theta}))$. Moreover, by definition $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{s}, \hat{z})$. As a result, for any open set $N' \subsetneq N$ containing

[32] A way to see this is that the marginal rate of substitution between consumption and savings $\mathcal{S}$ is continuous on a compact space and thus bounded, whereas the marginal tax rate which is parametrized by k can be made arbitrarily large. As a result, agents' first-order condition can never hold for sufficiently large k, while we can similarly exclude corner solutions for sufficiently large k.

$(s^*(\hat{\theta}), z^*(\hat{\theta}))$, there exists a $j^\dagger \geq k^\dagger$ such that $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ and such that both $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ and $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ belong to the set N'.

Since $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ it has a unique optimum on N'. By definition of $\mathcal{T}_{k^\dagger}$, agent $\theta_{j^\dagger}$ first-order conditions are satisfied at $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$. Hence, $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ is agent $\theta_{j^\dagger}$ maximum under the tax system $\mathcal{T}_{k^\dagger}$. This contradicts the fact established above that agent $\theta_{j^\dagger}$ prefers $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ to allocation $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ under tax system $\mathcal{T}_{k^\dagger}$, which completes the proof.

Lemma for type-specific feasibility.

Lemma B.1. *A smooth tax system $\mathcal{T}$ satisfies type-specific feasibility over the compact type space $[\theta_{min}\,;\theta_{max}]$ if it satisfies the following conditions:*

1. $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\mathcal{T}'_z(s^*(\theta), z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$
3. $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$

Proof. Consider the type-specific feasible tax system $T^*_\theta(\theta) = z^*(\theta) - s^*(\theta) - c^*(\theta)$, and note that the lemma amounts to showing that $T^*_\theta(\theta) = \mathcal{T}(s^*(\theta), z^*(\theta))$ for all θ. To that end, note that the first-order condition for truthful reporting of θ under the optimal mechanism implies

$$U'_c \cdot (z'(\theta) - s'(\theta) - T^{*\prime}_\theta(\theta)) + U'_s \cdot s'(\theta) + U'_z \cdot z'(\theta) = 0,$$

with derivatives evaluated at the optimal allocation. This can be rearranged as

$$\begin{aligned} T^{*\prime}_\theta(\theta) &= \left(\frac{U'_s}{U'_c} - 1\right) s'(\theta) + \left(\frac{U'_z}{U'_c} + 1\right) z'(\theta) \\ &= \mathcal{T}'_s(s^*(\theta)) s^{*\prime}(\theta) + \mathcal{T}'_z(z^*(\theta)) z^{*\prime}(\theta). \end{aligned}$$

It thus follows that

$$\begin{aligned} \mathcal{T}(s^*(\theta), z^*(\theta)) - \mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) &= \int_{\vartheta=\theta_{min}}^{\vartheta=\theta} \left(\mathcal{T}'_s(s^*(\vartheta)) s^{*\prime}(\vartheta) + \mathcal{T}'_z(z^*(\vartheta)) z^{*\prime}(\vartheta)\right) d\vartheta \\ &= T^*_\theta(\theta) - T^*_\theta(\theta_{min}). \end{aligned}$$

Since $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = T^*_\theta(\theta_{min})$, this implies that $\mathcal{T}(s^*(\theta), z^*(\theta)) = T^*_\theta(\theta)$ for all θ. The smooth tax system $\mathcal{T}$ therefore satisfies type-specific feasibility. □

Lemma on second-order conditions.

Lemma B.2. *Consider a smooth tax system $\mathcal{T}$ satisfying the conditions in Lemma B.1 and define*

$$V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta). \tag{78}$$

When evaluated at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$*, we show that*

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \tag{79}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \tag{80}$$

$$\begin{aligned}\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg]\end{aligned} \tag{81}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Proof. First-order derivatives are

$$\begin{aligned}V'_s(s,z;\theta) &= -(1+\mathcal{T}'_s(s,z))U'_c(z-s-\mathcal{T}(s,z),s,z;\theta) + U'_s(z-s-\mathcal{T}(s,z),s,z;\theta) \\ V'_z(s,z;\theta) &= (1-\mathcal{T}'_z(s,z))U'_c(z-s-\mathcal{T}(s,z),s,z;\theta) + U'_z(z-s-\mathcal{T}(s,z),s,z;\theta).\end{aligned}$$

Second-order derivatives are

$$V''_{ss}(s,z;\theta) = -\mathcal{T}''_{ss}U'_c - (1+\mathcal{T}'_s)\left(-(1+\mathcal{T}'_s)U''_{cc} + U''_{cs}\right) - (1+\mathcal{T}'_s)U''_{cs} + U''_{ss} \tag{82}$$

$$V''_{zz}(s,z;\theta) = -\mathcal{T}''_{zz}U'_c + (1-\mathcal{T}'_z)\left((1-\mathcal{T}'_z)U''_{cc} + U''_{cz}\right) + (1-\mathcal{T}'_z)U''_{cz} + U''_{zz}. \tag{83}$$

To obtain the first result of the Lemma, we compute $\mathcal{T}''_{ss}$ by differentiating both sides of $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta);\theta) - 1$ with respect to θ

$$\begin{aligned}\mathcal{T}''_{ss}s^{*\prime}(\theta) + \mathcal{T}''_{sz}z^{*\prime}(\theta) &= \frac{d}{d\theta}\mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta);\theta) \\ &= \mathcal{S}'_c c^{*\prime}(\theta) + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta,\end{aligned}$$

plugging in $c^{*\prime}(\theta) = (1-\mathcal{T}'_z)\,z^{*\prime}(\theta) - (1+\mathcal{T}'_s)\,s^{*\prime}(\theta)$ and denoting $s^{*\prime}(z^*) := s^{*\prime}(\theta)/z^{*\prime}(\theta)$. The previous expression can be rearranged as

$$\mathcal{T}''_{ss} = \mathcal{S}'_c\frac{1-\mathcal{T}'_z}{s^{*\prime}(z^*)} - \mathcal{S}'_c(1+\mathcal{T}'_s) + \mathcal{S}'_s + \frac{\mathcal{S}'_z}{s^{*\prime}(z^*)} + \frac{\mathcal{S}'_\theta}{s^{*\prime}(\theta)} - \frac{\mathcal{T}''_{sz}}{s^{*\prime}(z^*)}. \tag{84}$$

Moreover, we differentiate Equation (2) to express the derivative of $\mathcal{S}$ with respect to c as

$$\begin{aligned}\mathcal{S}'_c(c^*(\theta), s^*(\theta), z^*(\theta);\theta) &= \frac{U'_cU''_{sc} - U'_sU''_{cc}}{\left(U'_c\right)^2} \\ &= \frac{1}{U'_c}\left(-\frac{U'_s}{U'_c}U''_{cc} + U''_{sc}\right) \\ &= \frac{1}{U'_c}\left(-(1+\mathcal{T}'_s)U''_{cc} + U''_{sc}\right)\end{aligned} \tag{85}$$

and the derivative of $\mathcal{S}$ with respect to s as

$$\begin{aligned}
\mathcal{S}'_s(c^*(\theta), s^*(\theta), z^*(\theta); \theta) &= \frac{U'_c U''_{ss} - U'_s U''_{cs}}{(U'_c)^2} \\
&= \frac{1}{U'_c}\left(-\frac{U'_s}{U'_c} U''_{cs} + U''_{ss}\right) \\
&= \frac{1}{U'_c}\left(-(1+\mathcal{T}'_s) U''_{cs} + U''_{ss}\right). \qquad (86)
\end{aligned}$$

Substituting equations (84), (85) and (86) into (82), we have

$$\begin{aligned}
V''_{ss}\left(s^*(\theta), z^*(\theta); \theta\right) &= -U'_c \cdot \left(\mathcal{S}'_c \frac{1-\mathcal{T}'_z}{s^{*\prime}(z)} - \mathcal{S}'_c(1+\mathcal{T}'_s) + \mathcal{S}'_s + \frac{\mathcal{S}'_z}{s^{*\prime}(z)} + \frac{\mathcal{S}'_\theta}{s^{*\prime}(\theta)} - \frac{\mathcal{T}''_{sz}}{s^{*\prime}(z)}\right) - (1+\mathcal{T}'_s) U'_s \mathcal{S}'_c + U'_c \mathcal{S}'_s \\
&= -U'_c \cdot \left(\frac{1-T'_z}{s^{*\prime}(z)} \mathcal{S}'_c + \frac{1}{s^{*\prime}(z)} \mathcal{S}'_z + \frac{1}{s^{*\prime}(\theta)} \mathcal{S}'_\theta - \frac{\mathcal{T}''_{sz}}{s^{*\prime}(z)}\right) \\
&= \frac{U'_z}{s^{*\prime}(z)} \mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)} \mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)} \mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)} \mathcal{T}''_{sz} \qquad (87)
\end{aligned}$$

where we have used $U'_z = -U'_c\left(1-\mathcal{T}'_z\right)$ in the last line.

Similarly, we can obtain the second result of the Lemma by writing $\mathcal{T}''_{zz}$ as

$$\mathcal{T}''_{zz} = \mathcal{Z}'_c\left(1-\mathcal{T}'_z\right) - \mathcal{Z}'_c\left(1+\mathcal{T}'_s\right) s^{*\prime}(z^*) + \mathcal{Z}'_s s^{*\prime}(z^*) + \mathcal{Z}'_z + \frac{\mathcal{Z}'_\theta}{z^{*\prime}(\theta)} - \mathcal{T}''_{sz} s^{*\prime}(z^*). \qquad (88)$$

Using

$$\mathcal{Z}'_c = \frac{1}{U'_c}\left(U''_{cz} + \left(1-\mathcal{T}'_z\right) U''_{cc}\right)$$

as well as

$$\mathcal{Z}'_z = \frac{1}{U'_c}\left(U''_{zz} + \left(1-\mathcal{T}'_z\right) U''_{cz}\right)$$

we get

$$V''_{zz}\left(s^*(\theta), z^*(\theta); \theta\right) = U'_s s^{*\prime}(z^*) \mathcal{Z}'_c - U'_c s^{*\prime}(z^*) \mathcal{Z}'_s - U'_c \frac{\mathcal{Z}'_\theta}{z^{*\prime}(\theta)} + U'_c \mathcal{T}''_{sz} s^{*\prime}(z^*). \qquad (89)$$

Finally, to obtain the third result of the Lemma, we must compute $\left(V''_{sz}\right)^2 - V''_{ss} V''_{zz}$. Note that the first-order condition $V'_s(s^*(\theta), z^*(\theta); \theta) = 0$ holds at every θ by construction. Differentiating with respect to θ we get

$$\frac{d}{d\theta} V'_s(s^*(\theta), z^*(\theta); \theta) = V''_{ss} s^{*\prime}(\theta) + V''_{sz} z^{*\prime}(\theta) + V''_{s\theta} = 0 \qquad (90)$$

which we can rearrange as

$$-V''_{sz} = V''_{ss} s^{*\prime}(z^*) + \frac{V''_{s\theta}}{z^{*\prime}(\theta)}. \qquad (91)$$

Similarly, by totally differentiating the first-order condition $V_z'(s^*(\theta), z^*(\theta); \theta) = 0$ and rearranging we find

$$-V_{sz}'' = \frac{V_{zz}''}{s^{*\prime}(z^*)} + \frac{V_{z\theta}''}{s^{*\prime}(\theta)}. \tag{92}$$

Writing $(V_{sz}'')^2$ as the product of the right-hand sides of Equations (91) and (92) yields

$$\begin{aligned}\left(V_{sz}''\right)^2 =& \left(V_{ss}''s^{*\prime}(z) + \frac{V_{s\theta}''}{z^{*\prime}(\theta)}\right)\left(\frac{V_{zz}''}{s^{*\prime}(z)} + \frac{V_{z\theta}''}{s^{*\prime}(\theta)}\right) \\ =& V_{ss}''V_{zz}'' + \frac{1}{z^{*\prime}(\theta)}V_{ss}''V_{z\theta}'' + \frac{1}{s^{*\prime}(\theta)}V_{zz}''V_{s\theta}'' + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}V_{s\theta}''V_{z\theta}''.\end{aligned} \tag{93}$$

Now from the definition $V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta)$, we can compute

$$\begin{aligned}V_{s\theta}''(s, z; \theta) &= -\left(1 + \mathcal{T}_s'(s, z)\right)U_{c\theta}'' + U_{s\theta}'' \\ V_{z\theta}''(s, z; \theta) &= \left(1 - \mathcal{T}_z'(s, z)\right)U_{c\theta}'' + U_{z\theta}''\end{aligned}$$

and use the fact that at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ we have

$$\begin{aligned}\mathcal{S}_\theta' &= \frac{1}{U_c'}\left(U_{s\theta}'' - \left(1 + \mathcal{T}_s'\right)U_{c\theta}''\right) \\ \mathcal{Z}_\theta' &= \frac{1}{U_c'}\left(U_{z\theta}'' + \left(1 - \mathcal{T}_z'\right)U_{c\theta}''\right)\end{aligned}$$

to obtain

$$V_{s\theta}''\left(s^*(\theta), z^*(\theta); \theta\right) = U_c'\mathcal{S}_\theta' \tag{94}$$

$$V_{z\theta}''\left(s^*(\theta), z^*(\theta); \theta\right) = U_c'\mathcal{Z}_\theta'. \tag{95}$$

Substituting these into Equation (93) and rearranging, we have

$$\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' = \frac{1}{z^{*\prime}(\theta)}V_{ss}''U_c'\mathcal{Z}_\theta' + \frac{1}{s^{*\prime}(\theta)}V_{zz}''U_c'\mathcal{S}_\theta' + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\left(U_c'\right)^2\mathcal{S}_\theta'\mathcal{Z}_\theta'. \tag{96}$$

Expanding V_{ss}'' from Equation (87), and V_{zz}'' from Equation (89) yields after simplification

$$\begin{aligned}\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' = \frac{U_c'}{s^{*\prime}(\theta)}\Bigg[&\left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\mathcal{Z}_\theta' + \left(U_s'\mathcal{Z}_c' - U_c'\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}_\theta' \\ &+ \left(\mathcal{Z}_\theta' + s^{*\prime}(z^*)\mathcal{S}_\theta'\right)U_c'\mathcal{T}_{sz}''\Bigg],\end{aligned}$$

which gives the third result of the Lemma above. □

B.3 Proof of Proposition A.1 & A.2 (Implementation with Simple Tax Systems)

B.3.1 Proof of Proposition A.1

SN tax system. The sufficient conditions for local optimality under the candidate SN tax system follow directly from Lemma B.2 which computes agents' SOCs at the optimal incentive-compatible allocation under a general tax system $\mathcal{T}(s,z)$. Indeed, agents' SOCs are satisfied if Equations (79), (80), and (81) are negative under the SN tax system. Since the cross-partial derivative $\mathcal{T}''_{sz}$ is zero for a SN tax system, it is easy to verify that conditions (59) and (60) on the derivatives of $\mathcal{S}$ and $\mathcal{Z}$, combined with monotonicity ($s^{*\prime}(\theta) > 0$, $s^{*\prime}(z) > 0$) and Assumption 1 on the derivatives of U, jointly imply that each of these three equations is the sum of negative terms. This implies that agents' SOCs are satisfied at the optimal incentive-compatible allocation under the candidate SN tax system.

LED tax system. To derive sufficient conditions for local optimality under the candidate LED tax system, we begin from results obtained in the derivations of Lemma B.2 which computes agents' SOCs at the optimal incentive-compatible allocation. We consider the requirements $V''_{ss} < 0$, $V''_{zz} < 0$, and $V''_{ss}V''_{zz} > (V''_{sz})^2$ in turn.

First, to show that V''_{ss} is negative, note that under a LED tax system, $\mathcal{T}''_{ss} = 0$. Therefore, using the fact that under the candidate LED tax system we have $1 + \mathcal{T}'_s = \frac{U'_s}{U'_c}$ at the optimal incentive-compatible allocation, the general expression for V''_{ss} given in Equation (82) reduces to

$$V''_{ss}(s^*(\theta), z^*(\theta); \theta) = \left(\frac{U'_s}{U'_c}\right)^2 U''_{cc} - 2\frac{U'_s}{U'_c}U''_{cs} + U''_{ss}.$$

Therefore when utility is additively separable in c and s (implying $U''_{cs} = 0$), the concavity of preferences ($U''_{cc} \leq 0$ and $U''_{ss} \leq 0$) guarantees that this expression is negative.

Second, to show that V''_{zz} is negative, note that under the candidate LED tax system defined in Equations (57) and (58) we have

$$\mathcal{T}''_{sz}(s,z) = \tau'_s(z).$$

We can thus find an expression for $\tau'_s(z)$ at any point in the allocation in question by totally differentiating Equation (57) with respect to θ:

$$\begin{aligned}
\tau'_s(z^*(\theta))\, z^{*\prime}(\theta) &= \frac{d}{d\theta}\Big[\mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\Big] \\
&= \frac{d}{d\theta}\Big[\mathcal{S}(z^*(\theta) - s^*(\theta) - \mathcal{T}(s^*(\theta), z^*(\theta)), s^*(\theta), z^*(\theta); \theta)\Big] \\
&= \mathcal{S}'_c \cdot \left[(1-\mathcal{T}'_z)z^{*\prime}(\theta) - \left(1+\mathcal{T}'_s\right)s^{*\prime}(\theta)\right] + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta,
\end{aligned}$$

which yields

$$\tau_s'\left(z^*(\theta)\right) = \mathcal{S}_c' \cdot \left(1 - \mathcal{T}_z'\right) - \mathcal{S}_c' \cdot \left(1 + \mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{S}_s' \cdot s^{*\prime}(z^*) + \mathcal{S}_z' + \frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}.$$

Substituting this into the expression for V_{zz}'' in (89), we have

$$\begin{aligned} V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = {} & U_s' s^{*\prime}(z^*) \mathcal{Z}_c' - U_c' s^{*\prime}(z^*) \mathcal{Z}_s' - U_c' \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} \\ & + U_c' s^{*\prime}(z^*) \left[\mathcal{S}_c' \cdot \left(1 - \mathcal{T}_z'\right) - \mathcal{S}_c' \cdot \left(1 + \mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{S}_s' \cdot s^{*\prime}(z^*) + \mathcal{S}_z' + \frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}\right]. \end{aligned} \tag{97}$$

Now employing the assumption that utility is separable in c, s, and z, (implying both $U_{cz}'' = 0$ and $U_{cs}'' = 0$) we have

$$\begin{aligned} U_s'\mathcal{Z}_c' + U_c'\mathcal{S}_c'(1 - \mathcal{T}_z') &= U_s'\mathcal{Z}_c' - U_z'\mathcal{S}_c' \\ &= U_s' \frac{U_c'U_{cz}'' - U_z'U_{cc}''}{(U_c')^2} - U_z' \frac{U_c'U_{cs}'' - U_s'U_{cc}''}{(U_c')^2} \\ &= 0. \end{aligned}$$

Substituting this result into Equation (97), and noting that $\mathcal{Z}_s' = \mathcal{S}_z' = 0$ by separability, yields

$$V_{zz}''(s^*(\theta), z^*(\theta); \theta) = \left(s^{*\prime}(z^*)\right)^2 \left[U_c'\mathcal{S}_s' - U_s'\mathcal{S}_c'\cdot\right] - \frac{U_c'}{z^{*\prime}(\theta)} \left[\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta'\right]. \tag{98}$$

Again employing separability, we have

$$U_c'\mathcal{S}_s' - U_s'\mathcal{S}_c' = U_c' \frac{U_c'U_{ss}'' - U_s'U_{cs}''}{(U_c')^2} - U_s' \frac{U_c'U_{cs}'' - U_s'U_{cc}''}{(U_c')^2} = U_{ss}'' + \left(\frac{U_s'}{U_c'}\right)^2 U_{cc}'' \leq 0,$$

implying that the first term on the right-hand side of Equation (98) is negative. The condition $\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta' \geq 0$ from (61) in the Proposition then implies Equation (98) (and thus V_{zz}'') is negative.

Third, to show $V_{ss}''V_{zz}'' > (V_{sz}'')^2$, we proceed from Equation (81) in Lemma B.2:

$$\begin{aligned} & \left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' \\ & = \frac{U_c'}{s^{*\prime}(\theta)} \left[\left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\mathcal{Z}_\theta' + \left(U_s'\mathcal{Z}_c' - U_c'\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{s^{*\prime}(\theta)}\right) s^{*\prime}(z^*)\mathcal{S}_\theta' + \left(\mathcal{Z}_\theta' + s^{*\prime}(z^*)\mathcal{S}_\theta'\right) U_c'\mathcal{T}_{sz}''\right] \\ & = \left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right) \frac{U_c'}{s^{*\prime}(\theta)} \mathcal{Z}_\theta' + \frac{U_c'}{s^{*\prime}(\theta)} \mathcal{Z}_\theta' U_c' \mathcal{T}_{sz}'' + \frac{U_c'}{s^{*\prime}(\theta)} \mathcal{S}_\theta' \left(U_s' s^{*\prime}(z^*)\mathcal{Z}_c' - U_c' s^{*\prime}(z^*)\mathcal{Z}_s' - U_c' \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} + U_c' s^{*\prime}(z^*)\mathcal{T}_{sz}''\right). \end{aligned}$$

Recognizing that the last bracket term is exactly the expression for V_{zz}'' given in Lemma B.2, this gives

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta U'_c\mathcal{T}''_{sz} + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}$$

using the previous expression derived for $\mathcal{T}''_{sz} = \tau'_s$, and the fact that separability ensures $\mathcal{S}'_z = 0$, we obtain after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = -\frac{\left(U'_c\right)^2}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\mathcal{Z}'_\theta\left[s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right) - \mathcal{S}'_\theta\right] + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}.$$

We have already shown that V''_{zz} is negative. Thus the conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{S}'_\theta \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right)$ from (61) in the Proposition imply that both terms on the right-hand side are negative, implying that all second-order conditions hold.

B.3.2 Proof of Proposition A.2

We begin with a more general statement, and then derive Proposition A.2 as a corollary. For a fixed type θ, let $c(z,\theta)$ and $s(z,\theta)$ be its preferred consumption and savings choices at earnings z, given the budget constraint induced by $\mathcal{T}(s,z)$.

Lemma B.3. *Suppose that $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, where $z^*(\theta)$ is increasing. Individuals' local optima correspond to their global optima when*

1. *$\mathcal{Z} = \frac{U'_z(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ and $\mathcal{S} = \frac{U'_s(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ are strictly increasing in θ for all (c,s,z).*
2. *For any two types θ and θ', we cannot have both*

$$\begin{aligned}&U'_c\Big(c^*(\theta), s^*(\theta), z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U'_z\Big(c^*(\theta), s^*(\theta), z;\theta\Big)\\ &< U'_c\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\sigma_c\Big(s\left(z^*(\theta),\theta'\right), z^*(\theta)\Big) + U'_z\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\end{aligned} \tag{99}$$

and

$$\begin{aligned}&U'_s\Big(c^*(\theta), s^*(\theta), z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U'_z\Big(c^*(\theta), s^*(\theta), z;\theta\Big)\Big)\\ &< U'_s\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\sigma_s\Big(s\left(z^*(\theta),\theta'\right), z^*(\theta)\Big) + U'_z\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\end{aligned} \tag{100}$$

where $\sigma_c(s,z) := 1 - \mathcal{T}'_z(s,z)$ and $\sigma_s(s,z) := \frac{1-\mathcal{T}'_z(s,z)}{1+\mathcal{T}'_s(s,z)}$.

Condition 1 corresponds to single-crossing assumptions for earnings and savings. Condition 2 is a requirement that if type θ preserves its assigned earnings level $z^*(\theta)$, but chooses some other consumption level s (corresponding to a level that some other type θ' would choose if forced to choose earnings level $z^*(\theta)$), then at this alternative consumption bundle agent θ cannot have both higher marginal utility from increasing its savings through one more unit of work *and* increasing its consumption through one more unit of work. Generally, this condition will hold as long as U is sufficiently concave in consumption and savings when type θ chooses earnings level $z^*(\theta)$.

Proof. To prove agents' local optima are global optima, we want to show that for any given agent θ^*, utility decreases when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$.

The first step is to compute agent θ^* utility change. The envelope theorem applied to savings choices $s(z,\theta^*)$ implies

$$\begin{aligned}
&\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right) \\
&= U_c'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\sigma_c\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)
\end{aligned}$$

where $\sigma_c(s,z) = 1 - \mathcal{T}_z'(s,z)$. Note that, as established by Milgrom and Segal (2002), these equalities hold as long as U is differentiable in z (holding s and c fixed)—differentiability of $c(z,\theta^*)$ or $s(z,\theta^*)$ is actually not required.

Similarly, the envelope theorem applied to consumption choices $c(z,\theta^*)$ implies

$$\begin{aligned}
&\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right) \\
&= U_s'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\sigma_s\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)
\end{aligned} \tag{101}$$

where $\sigma_s(s,z) = \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Therefore, agent's θ^* utility change when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$ is

$$\begin{aligned}
&U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right) - U\left(c(z^*(\theta^*),\theta^*), s(z^*(\theta^*),\theta^*), z^*(\theta^*);\theta^*\right) \\
&= \int_{x=z^*(\theta^*)}^{x=z}\Big[\min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_c\left(s(x,\theta^*),x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_s\left(s(x,\theta^*),x\right)\right\} \\
&\qquad + U_z'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\Big]dx
\end{aligned} \tag{102}$$

where the min operator is introduced without loss of generality given the fact that both terms are equal.

The second step is to show that under our assumptions, agent θ^* utility change (102) is negative. To do so, let θ_x be the type that chooses earnings x. Then, by definition, agent θ_x utility is maximal at earnings x implying both

$$\begin{aligned}
U_c'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right)\sigma_c\left(s^*(\theta_x),x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right) = 0 \\
U_s'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right)\sigma_s\left(s^*(\theta_x),x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right) = 0
\end{aligned}$$

such that

$$\begin{aligned}
&\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right)\sigma_c\left(s^*(\theta_x),x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right)\sigma_s\left(s^*(\theta_x),x\right)\right\} \\
&+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right) = 0.
\end{aligned} \tag{103}$$

Now, by condition 2, we either have[33]

$$\begin{aligned}&U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\\ &\geq U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\end{aligned}$$

or

$$\begin{aligned}&U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\\ &\geq U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\end{aligned}$$

implying that

$$\begin{aligned}&\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right)\right\}\\ &+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\\ &\geq \min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right)\right\}\\ &+ U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right).\end{aligned} \tag{104}$$

But since the maximum is zero, this minimum has to be negative. Hence, we have either

$$\begin{aligned}&U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0\\ \iff &\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)\end{aligned}$$

or

$$\begin{aligned}&U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0\\ \iff &\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)\end{aligned}$$

Suppose that $z > z^*(\theta^*)$ such that $x > z^*(\theta^*)$; the case $z < z^*(\theta^*)$ follows identically. For any $x > z^*(\theta^*)$, the monotonicity of the earnings function means that $\theta_x > \theta^*$. Then, by the single-crossing conditions for $\mathcal{Z} = \frac{U_z'}{U_c'}$ and $\mathcal{S} = \frac{U_s'}{U_c'}$, this means that we have either[34]

$$\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)}{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)$$

or

$$\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)}{U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)$$

[33]Not having $\{a < c \quad and \quad b < c\}$ means having $\{a \geq c \quad \text{or} \quad b \geq d\}$ which implies $\max(a,b) \geq \min(c,d)$

[34]Note that having both $\mathcal{Z}$ and $\mathcal{S}$ increasing in θ also implies that $\frac{\mathcal{Z}}{\mathcal{S}} = \frac{U_z'}{U_s'}$ is increasing in θ.

implying that for any $x > z^*(\theta^*)$,

$$\begin{aligned}&\min\left\{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_c\left(s(x,\theta^*),x\right),U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_s\left(s(x,\theta^*),x\right)\right\}\\&+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\leq 0.\end{aligned} \tag{105}$$

As a result, the right hand-side of Equation (102) is an integral of negative terms, which shows that

$$U\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)-U\left(c^*(\theta^*),s^*(\theta^*),z^*(\theta^*);\theta^*\right)\leq 0. \tag{106}$$

The case with $z < z^*(\theta^*)$ follows identically, proving Lemma B.3. □

Proof of Proposition A.2

We now derive Proposition A.2 as a consequence of Lemma B.3 by deriving assumptions under which condition 2 is met for SN and LED tax systems.

SN systems. First, suppose that $s < s^*(\theta)$, then $c > c^*(\theta)$. Noting that $\sigma_c = 1 - T_z'\left(z^*\left(\theta\right)\right)$ is not a function of s, we can use $U_{cc}'' \leq 0$ and $U_{cs}'' \geq 0$ to obtain

$$U_c'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right)\geq U_c'\left(c,s,z^*(\theta);\theta\right)\sigma_c\left(s,z^*(\theta)\right).$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we obtain

$$\begin{aligned}&U_c'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right)+U_z'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\\&\geq U_c'\left(c,s,z^*(\theta);\theta\right)\sigma_c\left(s,z^*(\theta)\right)+U_z'\left(c,s,z^*(\theta);\theta\right).\end{aligned}$$

Conversely, suppose that $s > s^*(\theta)$, then $c < c^*(\theta)$. We have

$$\begin{aligned}&\frac{d}{ds}\left[\frac{U_s'(z-T_z\left(z\right)-s-T_s\left(s\right),s,z^*(\theta);\theta)}{1+T_s'(s)}\right]\\&=-U_{cs}''+\frac{1}{(1+T_s'(s))}\left[U_{ss}''-U_s'\frac{T_{ss}''(s)}{1+T_s'(s)}\right].\end{aligned}$$

The condition that $\frac{U_{ss}''(c(s,\theta),s,z^*(\theta);\theta)}{U_s'(c(s,\theta),s,z^*(\theta);\theta)} < \frac{T_{ss}''(s)}{1+T_s'(s)}$, together with $U_{cs}'' > 0$, implies that $\frac{U_s'(c(s,\theta),s,z^*(\theta);\theta)}{1+T_s'(s)}$ is decreasing in s and thus that

$$\frac{U_s'(c^*(\theta),s^*(\theta),z^*(\theta);\theta)}{1+T_s'(s^*(\theta))}\geq\frac{U_s'(c,s,z^*(\theta);\theta)}{1+T_s'(s)}.$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, and that $\mathcal{T}_s' = T_z'\left(z\right)$ is independent of s, we obtain

$$\begin{aligned}&U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_s\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\\ &\geq U_s'\left(c, s, z^*(\theta); \theta\right)\sigma_s\left(s, z^*(\theta)\right) + U_z'\left(c, s, z^*(\theta); \theta\right).\end{aligned}$$

LED systems. First, consider a type θ' choosing earnings $z = z^*(\theta) > z^*(\theta')$. We have

$$\begin{aligned}&\frac{d}{ds}\left[U_c'\left(z - s - \tau_s\left(z^*(\theta)\right)s - T_z\left(z^*(\theta)\right), s, z^*(\theta); \theta\right)\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right)\right]\\ &= U_{cs}''\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) - U_{cc}''\left(1 + \tau_s(z^*(\theta))\right)\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) - U_c'\tau_s'\left(z^*(\theta)\right).\end{aligned}$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z^*(\theta) > z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') < s^*(\theta)$. In this case, condition *(a)* of the proposition implies that the remaining terms are negative such that

$$U_c'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\sigma_c\left(s, z^*(\theta)\right)$$

is increasing in s for $s < s^*(\theta)$, where $\sigma_c\left(s, z^*(\theta)\right) = 1 - T_z'(z) - \tau_s'(z)s$. As a result,

$$\begin{aligned}&U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right)\\ &\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right)\end{aligned}$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$\begin{aligned}&U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\\ &\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right).\end{aligned}$$

Second consider a type θ' choosing $z = z^*(\theta) < z^*(\theta')$. We have

$$\begin{aligned}&\frac{d}{ds}\left[U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1 + \tau_s(z)}\right]\\ &= -U_{cs}''\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) + U_{ss}''\frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1 + \tau_s(z)} + U_s'\frac{\tau_s'\left(z^*(\theta)\right)}{1 + \tau_s(z)}\end{aligned}$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z = z^*(\theta) < z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') > s^*(\theta)$. Hence, condition *(b)* of the proposition implies that the remaining terms are negative such that

$$U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\sigma_s\left(s, z^*(\theta)\right)$$

is decreasing in s for $s > s^*(z)$, where $\sigma_s(s, z^*(\theta)) = \frac{1-T_z'(z)-\tau_s'(z)s}{1+\tau_s(z)}$. This ensures that

$$\begin{aligned} &U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\, \sigma_c(s^*(\theta), z^*(\theta)) \\ &\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right) \sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) \end{aligned}$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$\begin{aligned} &U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\, \sigma_c(s^*(\theta), z^*(\theta)) + U_z'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \\ &\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right) \sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right). \end{aligned}$$

B.4 Proof of Proposition 1 (Measurement of Causal Income Effects)

Here, we derive that the different expressions of the sufficient statistic $s_{inc}'(z)$ can be measured empirically.

Case 1. If agents preferences are weakly separable between the utility of consumption $u(.)$ and the disutility to work $k(.)$, agent θ problem writes

$$\max_{c,s,z}\; u(c, s; \theta) - k(z/w(\theta)) \;\; s.t.\; c \leq z - s - \mathcal{T}(s, z)$$

meaning that conditional on earnings z, savings $s(z; \theta)$ is defined as the solution to

$$-\left(1 + \mathcal{T}_s'(s, z)\right) u_c'(z - s - \mathcal{T}(s, z), s; \theta) + u_s'(z - s - \mathcal{T}(s, z), s; \theta) = 0.$$

Differentiating this equation with respect to savings s and earnings z yields

$$\frac{\partial s}{\partial z} = -\frac{\left[-\mathcal{T}_{sz}'' u_c' - (1 + \mathcal{T}_s')(1 - \mathcal{T}_z') u_{cc}'' + (1 - \mathcal{T}_z') u_{cs}''\right]}{\left[-\mathcal{T}_{ss}'' u_c' + (1 + \mathcal{T}_s')^2 u_{cc}'' - 2(1 + \mathcal{T}_s'(s, z)) u_{cs}'' + u_{ss}''\right]}.$$

Differentiating this equation with respect to savings s and disposable income y yields

$$\frac{\partial s}{\partial y} = -\frac{\left[-(1 + \mathcal{T}_s') u_{cc}'' + u_{cs}''\right]}{\left[-\mathcal{T}_{ss}'' u_c' + (1 + \mathcal{T}_s')^2 u_{cc}'' - 2(1 + \mathcal{T}_s'(s, z)) u_{cs}'' + u_{ss}''\right]}.$$

Hence, if $\mathcal{T}_{sz}'' = 0$, we get

$$s_{inc}'(z) := \frac{\partial s(z; \theta)}{\partial z} = \left(1 - \mathcal{T}_z'\right) \frac{\partial s}{\partial y} = \left(1 - \mathcal{T}_z'\right) \frac{\eta_{s|z}(z(\theta))}{1 + \mathcal{T}_s'},$$

where the last equality follows from the definition of $\eta_{s|z}(z(\theta))$. The intuition behind this result is that with separable preferences, savings s depend on earnings z only through disposable income $y = z - s - \mathcal{T}(s, z)$.

Case 2. If agents wage rates w and hours h are observable, and earnings z are given by $z = w \cdot h$, we can infer s'_{inc} from changes in wages through

$$\begin{aligned}
\frac{\partial s}{\partial w} &= \frac{\partial s\left(w \cdot h;\theta\right)}{\partial w} = \frac{\partial s\left(z;\theta\right)}{\partial z}\left(1+\frac{\partial h}{\partial w}\right) \\
\iff \quad \frac{\partial s\left(z;\theta\right)}{\partial z} &= \frac{\frac{\partial s}{\partial w}}{1+\frac{\partial h}{\partial w}} = s\frac{\frac{w}{s}\frac{\partial s}{\partial w}}{w+h\frac{w}{h}\frac{\partial h}{\partial w}} \\
\iff \quad s'_{inc}\left(z\right) &= s\left(z\right)\frac{\xi_w^s\left(z\right)}{w\left(z\right)+h\left(z\right)\xi_w^h\left(z\right)}
\end{aligned}$$

where $\xi_w^s\left(z\right) \equiv \frac{w(z)}{s(z)}\frac{\partial s(z)}{\partial w(z)}$ is individuals' elasticity of savings with respect to their wage rate, and $\xi_w^h\left(z\right) \equiv \frac{w(z)}{h(z)}\frac{\partial h(z)}{\partial w(z)}$ is individuals' elasticity of hours with respect to their wage rate.

Case 3. Otherwise, if we can measure the elasticity of savings s and earnings z upon a compensated change in the marginal earnings tax rate $\mathcal{T}'_z$, respectively denoted $\chi_s^c := -\frac{1-\mathcal{T}'_z}{s}\frac{\partial s}{\partial \mathcal{T}'_z}$ and $\zeta_z^c := -\frac{1-\mathcal{T}'_z}{z}\frac{\partial z}{\partial \mathcal{T}'_z}$, we then have

$$\begin{aligned}
\frac{\partial s}{\partial \mathcal{T}'_z} &= \frac{\partial s\left(z;\theta\right)}{\partial z}\frac{\partial z}{\mathcal{T}'_z} \\
\iff \quad \left(-\frac{s}{1-\mathcal{T}'_z}\chi_s^c\right) &= s'_{inc}\left(z\right)\left(-\frac{z}{1-\mathcal{T}'_z}\zeta_z^c\right) \\
\iff \quad s'_{inc}\left(z\right) &= \frac{s\left(z\right)}{z}\frac{\chi_s^c\left(z\right)}{\zeta_z^c\left(z\right)}.
\end{aligned}$$

B.5 Proof of Lemma 1 (Earnings Responses to Taxes on s)

Throughout the paper, we characterize earnings responses to (different) savings tax reforms using generalizations of Lemma 1 in Saez (2002). The robust insight in all cases is that a $\Delta\tau$ increase in the marginal tax rate on s induces the same earnings changes (through substitution effects) as a $s'_{inc}(z)\Delta\tau$ increase in earnings tax rate. This is what appears in the body of the text as Lemma 1.

In our proofs we use a version that pertains to reforms that have an LED, SL, or SN structure. For example, a reform with LED structure adds a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$. Note that the reform itself has an LED structure, but it can applied to any nonlinear tax system, not just one with an LED structure. The results below allow for multidimensional heterogeneity.

Let

$$V\left(\mathcal{T}(.,z),z;\theta\right) = \max_s \; U\left(z-s-\mathcal{T}(s,z),s,z;\theta\right)$$

be agent θ's indirect utility function at earnings z.

LED reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, that is:[35]

$$\Delta \mathcal{T}_s(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_s\,(z - z^0)\, s & if\ z \in [z^0, z^0 + \Delta z] \\ \Delta\tau_s\,\Delta z\, s & if\ z \geq z^0 + \Delta z \end{cases}$$

We now construct for each type θ a tax reform $\Delta\mathcal{T}_z^\theta$ that affects marginal earnings tax rates, and induces the same earnings response as the initial perturbation $\Delta\mathcal{T}_s$. We define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z) + \Delta\mathcal{T}_s(.,z), z;\theta) = V(\mathcal{T}(.,z) + \Delta\mathcal{T}_z^\theta(.,z), z;\theta).$$

Then, by construction, the perturbation $\Delta\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $\Delta\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with earnings $z(\theta) \in [z^0, z^0 + \Delta z]$, this implies

$$\begin{aligned} U_c'\,\Delta\tau_s\left(z - z^0\right) s\,(z;\theta) &= U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ \Delta\mathcal{T}_z^\theta(z) &= \Delta\tau_s\left(z - z^0\right) s\,(z;\theta) \end{aligned}$$

Differentiating both sides with respect to z and letting $\Delta z \to 0$, this implies that in the phase-in region, the reform induces the same earnings change as a small increase $s_{inc}'(z)\,\Delta\tau_s$ in the marginal earnings tax rate.

For types θ with earnings $z(\theta) \geq z^0 + \Delta z$, this implies

$$\begin{aligned} U_c'\,\Delta\tau_s\,\Delta z\, s\,(z;\theta) &= U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ \Delta\mathcal{T}_z^\theta(z) &= \Delta\tau_s\,\Delta z\, s\,(z;\theta)\,. \end{aligned}$$

That is, above the phase-in region, the reform induces the same earnings changes as a $\Delta\tau_s\Delta z\, s\,(z)$ increase in tax liability combined with a $\Delta\tau_s\Delta z\, s_{inc}'(z)$ increase in the marginal earnings tax rate.

SL reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s$ on s for all individuals. This is a special case of a LED reform. As a result, we directly obtain that this reform induces the same earnings changes as a $\Delta\tau_s\, s\,(z)$ increase in tax liability combined with a $\Delta\tau_s\, s_{inc}'(z)$ increase in the marginal earnings tax rate.

[35] This reform, which is natural to consider for LED tax systems, allows us to derive a sufficient statistics characterization of the optimal smooth tax system (Theorem 2) without the requirement that $s(z)$ is monotonic. In contrast, if we were to rely on an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we would need further assumptions.

SN reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in a small increase $\Delta\tau_s$ in the marginal tax rate on s in a bandwidth $\left[s^0, s^0+\Delta s\right]$, with $\Delta\tau_s$ much smaller than Δs:

$$\Delta\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ s \leq s^0 \\ \Delta\tau_s(s-s^0) & if\ s \in [s^0, s^0+\Delta s] \\ \Delta\tau_s \Delta s & if\ s \geq s^0+\Delta s \end{cases}$$

We now construct for each type θ a perturbation of the earnings tax $\Delta\mathcal{T}_z^\theta$ that induces the same earnings response as the initial perturbation $\Delta\mathcal{T}_s$. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z)+\Delta\mathcal{T}_s(.,z),z;\theta) = V(\mathcal{T}(.,z)+\Delta\mathcal{T}_z^\theta(.,z),z;\theta).$$

Then, by construction, the perturbation $\Delta\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $\Delta\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with $s(z;\theta) \in [s^0, s^0+\Delta s]$, this implies

$$\begin{aligned} & U_c'\Delta\tau_s\left(s\left(z;\theta\right)-s^0\right) = U_c'\Delta\mathcal{T}_z^\theta(z) \\ \iff\ & \Delta\mathcal{T}_z^\theta(z) = \left(s\left(z;\theta\right)-s^0\right)\Delta\tau_s. \end{aligned}$$

Differentiating both sides with respect to z and letting $\Delta s \to 0$, this implies that a small increase $\Delta\tau_s$ in the marginal tax rate on s induces the same earnings change as a small increase $s_{inc}'\left(z\right)\Delta\tau_s$ in the marginal earnings tax rate.

For types θ with $s(z;\theta) \geq s^0+\Delta s$, this implies

$$\begin{aligned} & U_c'\,\Delta\tau_s\,\Delta s = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ & \Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\,\Delta s. \end{aligned}$$

Thus, a $\Delta\tau_s\,\Delta s$ lump-sum (savings) tax increase induces the same earnings change as a $\Delta\tau_s\,\Delta s$ lump-sum (earnings) tax increase.

B.6 Proof of Theorem 2 (Optimal Smooth Tax Systems)

When $z\left(\theta\right)$ is a strictly increasing function, we can define its inverse by $\vartheta\left(z\right)$. This allows us to define consumption of good c as $c(z) := c(z;\vartheta\left(z\right))$, consumption of good s as $s\left(z\right) := s\left(z;\vartheta\left(z\right)\right)$, and the planner's weights as $\alpha(z) := \alpha(\vartheta\left(z\right))$.

In this notation, the problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z \Big[\alpha(z)\,U\Big(c(z),s(z),z;\vartheta(z)\Big) + \lambda\Big(\mathcal{T}\left(s(z),z\right)-E\Big)\Big]\,dH_z(z), \tag{107}$$

where λ is the social marginal value of public funds, and the tax function implicitly enters agents' utility through $c(z) = z - s(z) - \mathcal{T}(s(z), z)$.

B.6.1 Optimality Condition for Marginal Tax Rates on z

Reform. We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal tax rate on earnings in a small bandwidth Δz. Formally,

$$\Delta\mathcal{T}(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_z(z - z^0) & if\ z \in [z^0, z^0 + \Delta z] \\ \Delta\tau_z \Delta z & if\ z \geq z^0 + \Delta z \end{cases}$$

We characterize the impact of this reform on the government's objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*:

$$\int_{z \geq z^0} \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_z \Delta z\, dH_z(z)$$

- *behavioral effects from changes in z:*[36]

$$-\mathcal{T}_z'\left(s\left(z^0\right), z^0\right) \frac{z^0}{1 - \mathcal{T}_z'\left(s\left(z^0\right), z^0\right)} \zeta_z^c(z^0) \Delta\tau_z\, \Delta z h_z(z^0)$$
$$- \int_{z \geq z^0} \mathcal{T}_z'\left(s\left(z\right), z\right) \frac{\eta_z(z)}{1 - \mathcal{T}_z'\left(s\left(z\right), z\right)} \Delta\tau_z \Delta z\, dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}_s'\left(s(z^0), z^0\right) s_{inc}'(z^0) \left[\frac{z^0}{1 - \mathcal{T}_z'\left(s\left(z^0\right), z^0\right)} \zeta_z^c(z^0) \Delta\tau_z\right] \Delta z h_z(z^0)$$
$$- \int_{z \geq z^0} \mathcal{T}_s'\left(s(z), z\right) \left[\frac{\eta_{s|z}(z)}{1 + \mathcal{T}_s'\left(s(z), z\right)} + s_{inc}'(z) \frac{\eta_z(z)}{1 - \mathcal{T}_z'\left(s\left(z\right), z\right)}\right] \Delta\tau_z \Delta z\, dH_z(z)$$

[36] Note that by definition elasticity concepts include all circularities and adjustments induced by tax reforms such that changes in z and s are given by

$$\begin{cases} dz = -\frac{z}{1-\mathcal{T}_z'} \zeta_z^c(z) \Delta\mathcal{T}_z'(s,z) - \frac{\eta_z(z)}{1-\mathcal{T}_z'} \Delta\mathcal{T}(s,z) \\ ds = -\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'} \Delta\mathcal{T}(s,z) + s_{inc}'(z) dz \end{cases}$$

Summing over these different effects yields the total impact of the reform

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta z} = &\int_{z\geq z^0}(1-\hat{g}(z))\ \Delta\tau_z dH_z(z)\\ &-\left(\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\, h_z(z^0)\end{aligned} \tag{108}$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$\hat{g}(z) = \frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right) - \frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\eta_z(z) - \frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}\eta_{s|z}(z).$$

Optimality. A direct implication is a sufficient statistics characterization of the optimal schedule of marginal tax rates on z. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, meaning that at each earnings z^0 the optimal marginal earnings tax rate satisfies

$$\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} = \frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z) - s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} \tag{109}$$

which is the optimality condition (23) presented in Theorem 2.

B.6.2 Optimality Condition for Marginal Tax Rates on s

Reform. We consider a small reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0,z^0+\Delta z\right]$, that is:[37]

$$\Delta\mathcal{T}_s(s,z) = \begin{cases}0 & if\ z\leq z^0\\ \Delta\tau_s\left(z-z^0\right)s & if\ z\in[z^0,z^0+\Delta z]\\ \Delta\tau_s\,\Delta z\, s & if\ z\geq z^0+\Delta z\end{cases}$$

Let $s^0=s(z^0)$. We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta z\to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z\geq z^0}\left(1-\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\theta(z)\right)\right)\ \Delta\tau_s\Delta z\, s\left(z\right)\, dH_z(z) \tag{110}$$

[37] We use this reform to derive a sufficient statistics characterization of the optimal smooth tax system, without the requirement that $s(z)$ is monotonic. If we instead consider an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we need this extra assumption.

- *behavioral effects from changes in z*:[38]

$$
\begin{aligned}
&- \mathcal{T}_z'\left(s^0, z^0\right) \left[\frac{z^0 \zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s^0, z^0\right)} \Delta\tau_s s^0\right] h_z(z^0)\Delta z \\
&- \int_{z \geq z^0} \mathcal{T}_z'\left(s\left(z\right), z\right) \left[\frac{z\zeta_z^c(z) s_{inc}'(z)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)} + \frac{\eta_z(z)s(z)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)}\right] \Delta\tau_s \Delta z \, dH_z(z)
\end{aligned}
\tag{111}
$$

- *behavioral effects from changes in s*:

$$
\begin{aligned}
&- \mathcal{T}_s'\left(s^0, z^0\right) s_{inc}'(z^0) \left[\frac{z^0 \zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s^0, z^0\right)} \Delta\tau_s \, s^0\right] h_z(z^0)\Delta z \\
&- \int_{z \geq z^0} \mathcal{T}_s'\left(s(z), z\right) \left[\frac{\zeta_{s|z}^c(z) + \eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z), z\right)} s(z) + s_{inc}'(z) \left[\frac{z\zeta_z^c(z) s_{inc}'(z)}{1-\mathcal{T}_z'\left(s(z), z\right)} + \frac{\eta_z(z)s(z)}{1-\mathcal{T}_z'\left(s(z), z\right)}\right]\right] \Delta\tau_s \Delta z \, dH_z(z).
\end{aligned}
\tag{112}
$$

Summing over these different effects yields the total impact of the reform

$$
\begin{aligned}
&\frac{1}{\lambda} \frac{d\mathcal{L}}{\Delta\tau_s \Delta z} \\
&= -\frac{\mathcal{T}_z'\left(s^0, z^0\right) + s_{inc}'(z^0)\mathcal{T}_s'\left(s^0, z^0\right)}{1-\mathcal{T}_z'\left(s^0, z^0\right)} z^0 \zeta_z^c(z^0) \, s^0 \, h_z(z^0) \\
&+ \int_{z \geq z^0} \left\{ \left(1-\hat{g}(z)\right) s\left(z\right) - \frac{\mathcal{T}_z'\left(s\left(z\right), z\right) + s_{inc}'(z)\mathcal{T}_s'\left(s(z), z\right)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\mathcal{T}_s'\left(s(z), z\right)}{1+\mathcal{T}_s'\left(s(z), z\right)} s(z) \zeta_{s|z}^c(z) \right\} dH_z(z)
\end{aligned}
\tag{113}
$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$
\hat{g}(z) = \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right) - \frac{\mathcal{T}_z'\left(s\left(z\right), z\right) + s_{inc}'(z)\mathcal{T}_s'\left(s(z), z\right)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)} \eta_z(z) - \frac{\mathcal{T}_s'\left(s(z), z\right)}{1+\mathcal{T}_s'\left(s(z), z\right)} \eta_{s|z}(z).
$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L} = 0$, which implies that at each $s^0 = s\left(z^0\right)$ and earnings z^0, the optimal marginal tax rate on s satisfies

$$
\begin{aligned}
&\frac{\mathcal{T}_z'\left(s^0, z^0\right) + s_{inc}'(z^0)\mathcal{T}_z'\left(s^0, z^0\right)}{1-\mathcal{T}_s'\left(s^0, z^0\right)} z^0 \zeta_z^c(z^0) \, s^0 \, h_z(z^0) \\
&= \int_{z \geq z^0} \left\{ \left(1-\hat{g}(z)\right) s\left(z\right) - \frac{\mathcal{T}_z'\left(s(z), z\right) + s_{inc}'(z)\mathcal{T}_z'\left(s(z), z\right)}{1-\mathcal{T}_z'\left(s(z), z\right)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\mathcal{T}_s'\left(s(z), z\right)}{1+\mathcal{T}_s'\left(s(z), z\right)} s(z) \zeta_{s|z}^c(z) \right\} dH_z(z).
\end{aligned}
\tag{114}
$$

Using the formula for the optimal schedule of marginal earnings tax rates (109) to replace the

[38] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$
\begin{cases} dz = -\frac{z^0 \zeta_z^c(z^0)}{1-\mathcal{T}_z'} \Delta\tau_s \, s^0 \\ ds = s_{inc}'(z^0) dz \end{cases}
\quad \text{and} \quad
\begin{cases} dz = -\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'} \Delta\tau_s \Delta z \, s_{inc}'(z) - \frac{\eta_z(z)}{1-\mathcal{T}_z'} \Delta\tau_s \Delta z \, s\left(z\right) \\ ds = -\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'} \Delta\tau_s \Delta z - \frac{\eta_s(z)}{1+\mathcal{T}_s'} \Delta\tau_s \Delta z \; s\left(z\right) + s_{inc}'(z) dz \end{cases}
$$

term on the left-hand side, this formula can be rearranged as

$$s(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z) \tag{115}$$
$$=\int_{z\geq z^0}\left\{(1-\hat{g}(z))\,s(z)-\frac{\mathcal{T}_z'\,(s(z),z)+s_{inc}'(z)\mathcal{T}_z'\,(s(z),z)}{1-\mathcal{T}_z'\,(s(z),z)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\,(s(z),z)}{1+\mathcal{T}_s'\,(s(z),z)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z).$$

Differentiating both sides with respect to z^0 yields

$$s'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)-s^0(1-\hat{g}(z^0))h_z(z^0)-\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)$$
$$=-\left(1-\hat{g}(z^0)\right)s^0h_z(z^0)+s_{inc}'(z^0)\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s^0,z^0\right)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)z^0h_z(z^0)$$

where both $s^0(1-\hat{g}(z^0))h_z(z^0)$ terms cancel out. Using (109) again, the last term is equal to $s_{inc}'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)$ such that we finally obtain

$$\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s(z^0)\zeta_{s|z}^c(z^0)h_z(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{pref}'(z^0)}\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z), \tag{116}$$

which is the optimality condition (24) presented in Theorem 2.

B.6.3 Pareto-efficiency Condition

We can combine formulas for optimal marginal tax rates on z and on s to obtain a characterization of Pareto efficiency. Indeed, leveraging the above optimal formula for marginal tax rates on s written in terms of $s_{pref}'(z^0)$, and replacing the integral term by its value from the optimal formula for marginal earnings tax rates (109) yields

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+T_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0)=s_{pref}'(z^0)\frac{\mathcal{T}_z'(z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)$$

which is the Pareto-efficiency condition (25) presented in Theorem (2).

B.7 Proof of Propositions 2, A.3, and A.4 (Optimal Simple Tax Systems)

The derivation of optimal earnings tax formulas for simple tax systems parallels that of general smooth tax systems and the optimal formula for marginal earnings tax rates formula, (23), continues to hold. This proves Proposition A.4.

Moreover, the particular linear reforms considered in the sufficient statistics characterization of optimal marginal tax rates on s for general smooth tax systems $\mathcal{T}(s,z)$ are also available for LED tax systems. As a result, the derivation of optimal marginal tax rates on s in LED tax systems is identical to the derivation for general smooth tax systems, and the optimality formula

(24) continues to hold. This, in turn, implies that the Pareto-efficiency condition (25) also holds, thereby proving all sufficient statistics characterizations for LED tax systems.

In contrast, LED reforms of tax rates on s are not available under SL and SN tax systems, and we below derive sufficient statistics characterizations of optimal tax rates on s and Pareto-efficiency conditions in SL and SN tax systems.

B.7.1 SL tax system

SL tax reform. When the government uses a linear tax on s such that $\mathcal{T}(s,z) = \tau_s\, s + T_z(z)$, we consider a small reform of the linear tax rate τ_s that consists in a small increase $\Delta\tau_s$. For an individual with earnings z, this reform increases tax liability by $\Delta\tau_s\, s(z)$ and increases the marginal tax rate on s by $\Delta\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_z \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_s\, s\left(z\right)\, dH_z(z) \tag{117}$$

- *behavioral effects from changes in z:*[39]

$$-\int_z T_z'(z) \left[\frac{z\zeta_z^c(z)}{1-T_z'\left(z\right)}\, \Delta\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1-T_z'\left(z\right)}\, \Delta\tau_s\, s\left(z\right)\right] dH_z(z) \tag{119}$$

- *behavioral effects from changes in s:*

$$\begin{aligned} &-\int_z \tau_s \left[\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s} \Delta\tau_s + \frac{\eta_{s|z}(z)}{1+\tau_s} \Delta\tau_s\, s\left(z\right)\right] dH_z(z) \\ &\qquad - \int_z \tau_s s_{inc}'(z) \left[\frac{z\zeta_z^c(z)}{1-T_z'}\, \Delta\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1-T_z'}\, \Delta\tau_s\, s\left(z\right)\right] dH_z(z). \end{aligned} \tag{120}$$

Summing over these different effects yields the total impact of the reform

$$\frac{d\mathcal{L}}{\lambda} = \int_z \left\{\left(1-\hat{g}(z)\right) s\left(z\right) - \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)} z\zeta_z^c(z)\, s_{inc}'(z) - \frac{\tau_s}{1+\tau_s} s(z)\zeta_{s|z}^c(z)\right\} \Delta\tau_s\, dH_z(z), \tag{121}$$

[39] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z\zeta_z^c(z)}{1-T_z'(z)}\, \Delta\tau_s\, s_{inc}'(z) - \frac{\eta_z(z)}{1-T_z'(z)}\, \Delta\tau_s\, s\left(z\right) \\ ds = -\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\, \Delta\tau_s - \frac{\eta_{s|z}(z)}{1+\tau_s} \Delta\tau_s\ s\left(z\right) + s_{inc}'(z) dz \end{cases} \tag{118}$$

with social marginal welfare weights augmented with the fiscal impact of income effects given by

$$\hat{g}(z) = \frac{\alpha(z)}{\lambda} U_c' \left(c(z), s(z), z; \theta(z)\right) + \frac{T_z'(z)}{1 - T_z'(z)} \eta_z(z) + \tau_s \left[\frac{\eta_{s|z}(z)}{1 + \tau_s} + s_{inc}'(z) \frac{\eta_z(z)}{1 - T_z'} \right].$$

Optimal linear tax rate on s. A direct implication of this result is a sufficient statistics characterization of the optimal linear tax rate τ_s. Indeed, at the optimum, the reform should have a zero impact on the government objective meaning that the optimal τ_s satisfies

$$\frac{\tau_s}{1 + \tau_s} \int_z s(z) \zeta_{s|z}^c(z) dH_z(z) = \int_z \left\{ (1 - \hat{g}(z)) \; s(z) - \frac{T_z'(z) + \tau_s \, s_{inc}'(z)}{1 - T_z'(z)} z \zeta_z^c(z) \, s_{inc}'(z) \right\} dH_z(z). \tag{122}$$

This is Equation (65) in Proposition A.3, and it holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z(z)$.

Now, let assume that the earnings tax schedule is optimal. Equation (109) applied to SL tax systems then implies that at each earnings z,

$$\frac{T_z'(z) + s_{inc}'(z) \tau_s}{1 - T_z'(z)} = \frac{1}{\zeta_z^c(z)} \frac{1}{z h_z(z)} \int_{x \geq z} (1 - \hat{g}(x)) \, dH_z(x) \tag{123}$$

such that plugging in this expression to replace the last term, we obtain

$$\frac{\tau_s}{1 + \tau_s} \int_z s(z) \zeta_{s|z}^c(z) dH_z(z) = \int_z \left\{ s(z) \left(1 - \hat{g}(z)\right) \right\} dH_z(z) - \int_z \left\{ s_{inc}'(z) \int_{x \geq z} (1 - \hat{g}(x)) \, h_z(x) dx \right\} dz \tag{124}$$

Defining $s_{inc}(z) \equiv \int_{x=0}^{z} s_{inc}'(x) \, dx$, we can integrate by part the last term to re-express it as[40]

$$\int_z \left\{ s_{inc}'(z) \int_{x \geq z} (1 - \hat{g}(x)) \, h_z(x) dx \right\} dz = \int_z \left\{ s_{inc}(z) (1 - \hat{g}(z)) \, h_z(z) \right\} dz \tag{125}$$

to obtain

$$\frac{\tau_s}{1 + \tau_s} \int_z s(z) \zeta_{s|z}^c(z) dH_z(z) = \int_z \left\{ \left[s(z) - s_{inc}(z) \right] (1 - \hat{g}(z)) \right\} dH_z(z). \tag{126}$$

Note that here $\int_z \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) dH_z(z)$ is the aggregate population response to a change in τ_s. Defining $\overline{\zeta_{s|z}^c}$ as the aggregate elasticity of $\overline{s} := \int_z s(z) dH_z(z)$, we can rewrite this term as $\frac{\overline{s}}{1+\tau_s} \overline{\zeta_{s|z}^c}$ such that

$$\frac{\tau_s}{1 + \tau_s} = \frac{1}{\overline{s} \overline{\zeta_{s|z}^c}} \int_z s_{pref}(z) \left(1 - \hat{g}(z)\right) dH_z(z). \tag{127}$$

This is Equation (32) in Proposition 2. Integrating by part the right-hand side, this formula is also

[40] Define $\phi(z) = \int_{x=0}^{z} s_{inc}'(x) \, dx$ such that $\phi'(z) = s_{inc}'(z)$ and $\psi(z) = \int_{x=z}^{z_{max}} (1 - \hat{g}(x)) \, h_z(x) dx$ such that $\psi'(z) = -(1 - \hat{g}(z)) \, h_z(z)$, and apply

$$\int_{x=z}^{z_{max}} \left[\phi'(x) \, \psi(x) \right] dx = \left[\phi(z_{max}) \, \psi(z_{max}) - \phi(z) \, \psi(z) \right] - \int_{x=z}^{z_{max}} \left[\phi(x) \, \psi'(x) \right] dx.$$

equivalent to

$$\frac{\tau_s}{1+\tau_s} = \frac{1}{\bar{s}\overline{\zeta^c_{s|z}}} \int_z \left[s'_{pref}(z) \int_{x\geq z} (1-\hat{g}(x))\, dH_z(x) \right] dz. \tag{128}$$

Pareto efficiency for SL tax systems. To characterize Pareto-efficiency, we combine tax reforms in a way that anihilates all lump-sum changes in tax liability, thereby offsetting all utility changes.

We start with a small reform of the linear tax rate τ_s that consists in small increase $\Delta\tau_s$. At the bottom of the earnings distribution ($z = z_{min}$), the mechanical effect of the reform is an increase in tax liability by $s(z_{min})\Delta\tau_s$. We thus adjust the earnings tax liability through a downward lump-sum shift by $s(z_{min})\Delta\tau_s$ at all earnings levels. This joint reform has the following impact on the government objective

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}}^{z_{max}} \left\{ [1-\hat{g}(z)]\, [s(z) - s(z_{min})] - \frac{T'_z(z) + \tau_s\, s'_{inc}(z)}{1-T'_z} z\zeta^c_z(z)\, s'_{inc}(z) - \tau_s \frac{s(z)}{1+T'_s}\zeta^c_{s|z}(z) \right\} \Delta\tau_s\, dH_z(z) \tag{129}$$

meaning that the lump-sum change in tax liability is nil at earnings $z = z_{min}$, but not at earnings $z \geq z_{min}$.

To cancel out lump-sum changes in tax liability at all earnings levels, we construct a sequence of earnings tax reforms. We discretize the range of earnings $[z_{min}, z_{max}]$ into N bins and consider reforms in the small earnings bandwidths $\Delta z = \frac{z_{max}-z_{min}}{N}$. We proceed by induction to derive a general formula:

- First, consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'(z_{min})\Delta\tau_s$ over the bandwidth $[z_{min}, z_{min}+\Delta z]$. In this bandwidth, this additional reform (i) cancels out lump-sum changes in tax liability to a first-order approximation since $[s(z_{min}+\Delta z) - s(z_{min})]\Delta\tau_s \approx s'(z_{min})\Delta z\Delta\tau_s$, and (ii) induces earnings responses through the change in marginal tax rates. Moreover, it also decreases the lump-sum tax liability on all individuals with earnings $z \geq z_{min}+\Delta z$ by $s'(z_{min})\Delta z\Delta\tau_s$. The total impact of this sequence of reforms is then

$$\begin{aligned} \frac{d\mathcal{L}}{\lambda} = & \int_{z=z_{min}+\Delta z}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - s'(z_{min})\Delta z \right] \right\} \Delta\tau_s\, dH_z(z) \\ & - \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T'_z(z) + \tau_s\, s'_{inc}(z)}{1-T'_z} z\zeta^c_z(z)\, s'_{inc}(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta^c_{s|z}(z) \right\} \Delta\tau_s\, dH_z(z) \\ & + \int_{z=z_{min}}^{z_{min}+\Delta z} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) \left(s'(z_{min})\Delta\tau_s \right) \Delta H_z(z). \end{aligned} \tag{130}$$

- Second, consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'(z_{min}+\Delta z)\Delta\tau_s$ over the bandwidth $[z_{min}+\Delta z, z_{min}+2\Delta z]$. This again cancels out lump-sum changes in this bandwidth up to a first-order approximation since $[s(z_{min}+2\Delta z) - s(z_{min}) - s'(z_{min})\Delta z] \approx$

$s'(z_{min} + \Delta z)\,\Delta z$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} = &\int_{z=z_{min}+2\Delta z}^{z_{max}} \left\{ [1-\hat{g}(z)]\, \left[s(z) - s(z_{min}) - s'(z_{min})\,\Delta z - s'(z_{min}+\Delta z)\,\Delta z \right] \right\} \Delta\tau_s\, dH_z(z) \\ &- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \\ &+ \int_{z=z_{min}}^{z_{min}+\Delta z} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min})\,\Delta\tau_s \right)\, dH_z(z) \\ &+ \int_{z=z_{min}+\Delta z}^{z_{min}+2\Delta z} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min}+\Delta z)\,\Delta\tau_s \right)\, dH_z(z). \end{aligned} \tag{131}$$

- Iterating over to step k, in which we consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min} + (k-1)\frac{\Delta z}{N}\right)\Delta\tau_s$ over the bandwidth $[z_{min} + (k-1)\frac{\Delta z}{N}, z_{min} + k\frac{\Delta z}{N}]$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} = &\int_{z=z_{min}+k\frac{\Delta z}{N}}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - \frac{\Delta z}{N}\left[\sum_{p=0}^{k-1} s'\left(z_{min} + p\frac{\Delta z}{N} \right) \right] \right] \right\} \Delta\tau_s\, dH_z(z) \\ &- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \\ &+ \sum_{p=0}^{k-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) \Delta\tau_s\, dH_z(z). \end{aligned} \tag{132}$$

- Pushing the iteration forward until $k = N$, the first integral disappears (integration over an empty set) such that the total impact of this sequence of reforms is given by

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} = &- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \\ &+ \sum_{p=0}^{N-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) \Delta\tau_s\, dH_z(z). \end{aligned} \tag{133}$$

Let's now compute the last term at the limit $N \to \infty$. Denoting $z^p := z_{min} + p\frac{\Delta z}{N}$, we have

$$\begin{aligned} &\sum_{p=0}^{N-1} \int_{z=z^p}^{z^p+\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'(z^p)\,\Delta\tau_s\, dH_z(z) \\ &\approx \sum_{p=0}^{N-1} \frac{T_z'(z^p) + s_{inc}'(z^p)\tau_s}{1-T_z'(z^p)} (z^p)\, \zeta_z^c(z^p)\, s'(z^p)\,\Delta\tau_s h_z(z^p)\, \frac{\Delta z}{N} \\ &\underset{N\to\infty}{\longrightarrow} \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\,\Delta\tau_s\, h_z(z)\, dz \end{aligned} \tag{134}$$

where the last line follows from the (Riemann) definition of the integral in terms of Riemann sums.

Hence, the total impact of this sequence of reforms is at the limit given by

$$\frac{d\mathcal{L}}{\lambda} = -\int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, h_z(z)dz$$
$$+ \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\, \Delta\tau_s\, h_z(z)\, dz. \quad (135)$$

By construction, the sequence of reforms we have constructed does not affect agents' utility, and only affects tax revenue through the expression above. When the impact of this reform is non-zero, the type of sequence of reforms we consider delivers a Pareto improvement over the existing tax system. Characterizing a Pareto-efficient SL tax system as one that cannot be reformed in a Pareto-improving way yields the following Pareto-efficiency formula

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta_{s|z}^c(z)h_z(z)dz = \int_z \underbrace{\left[s'(z) - s_{inc}'(z)\right]}_{s_{pref}'(z)} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, h_z(z)\, dz, \quad (136)$$

which is Equation (36) in Proposition 2.

B.7.2 SN tax systems

SN tax reform. When the government uses a SN tax system such that $\mathcal{T}(s,z) = T_s(s) + T_z(z)$, we consider a small reform of the tax on s at $s^0 = s(\theta^0)$ that consists in a small increase $\Delta\tau_s$ of the marginal tax rate on s in a small bandwidth Δs. Formally,

$$\Delta\mathcal{T}(s,z) = \begin{cases} 0 & if\ s \le s^0 \\ \Delta\tau_s(s-s^0) & if\ s \in [s^0, s^0+\Delta s] \\ \Delta\tau_s\Delta s & if\ s \ge s^0+\Delta s \end{cases}$$

Assuming there exists a strictly increasing mapping between z and s, we denote z^0 the earnings level such that $s^0 = s(z^0)$.[41] We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta s \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z\ge z^0} \left(1 - \frac{\alpha(z)}{\lambda}U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_s\Delta s\, dH_z(z)$$

[41] Our sufficient statistic characterization of optimal SN tax systems fundamentally relies on monotonicity of the function $s(z)$. Hence, it is also valid if we assume a strictly decreasing mapping $s(z)$. Moreover, it can be extended to weakly monotonic $s(z)$ (i.e., non-decreasing or non-increasing) with slight modifications.

- *behavioral effects from changes in* z:[42]

$$-\mathcal{T}_z'(s^0,z^0)\left[\frac{z^0}{1-\mathcal{T}_z'(s^0,z^0)}\zeta_z^c(z^0)\,s_{inc}'(z)\Delta\tau_s\right]\Delta s\,\frac{h_z(z^0)}{s'(z^0)}-\int_{z\geq z^0}\mathcal{T}_z'\left(s,z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s,z\right)}\Delta\tau_s\Delta s\,dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}_s'\left(s^0,z^0\right)\left[\frac{s^0}{1+\mathcal{T}_s'\left(s^0,z^0\right)}\zeta_{s|z}^c(z^0)\Delta\tau_s+s_{inc}'(z^0)\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\,s_{inc}'(z^0)\Delta\tau_s\right]\Delta s\,\frac{h_z(z^0)}{s'(z^0)}$$
$$-\int_{z\geq z^0}\mathcal{T}_s'\left(s,z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s,z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s,z\right)}\right]\Delta\tau_s\Delta s\,dH_z(z).$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta s}=s'\left(z^0\right)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)\Delta\tau_s\,dH_z(z)-\left\{\mathcal{T}_s'\left(s^0,z^0\right)\frac{s^0}{1+\mathcal{T}_s'\left(s^0,z^0\right)}\zeta_{s|z}^c(z^0)\right.$$
$$\left.+\left[\mathcal{T}_z'(s^0,z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)s_{inc}'(z^0)\right\}\Delta\tau_s\,h_z(z^0).\qquad(138)$$

Optimal nonlinear tax rate on s**.** A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, which implies that at each $s^0=s\left(z^0\right)$ the optimal marginal tax rate on s satisfies

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+\mathcal{T}_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)$$
$$=s'\left(z^0\right)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^0)\frac{\mathcal{T}_z'(s^0,z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)h_z(z^0)\qquad(139)$$

which is Equation (63) in Proposition A.3, recognizing that $\mathcal{T}_z'(s,z)=T_z'(z)$ and $\mathcal{T}_s'(s,z)=T_s'(s)$. This characterization holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z(z)$.

Now, let further assume that the earnings tax schedule is optimal. Equation (109) applied to SN tax systems then implies that at each earnings z^0,

$$\frac{T_z'(z^0)+s_{inc}'(z^0)T_s'(s(z^0))}{1-T_z'(z^0)}=\frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z).\qquad(140)$$

[42] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases}dz=-\frac{z}{1-T_z'}\zeta_z^c(z)\Delta T_z^{\theta\prime}-\frac{\eta_z(z)}{1-T_z'}\Delta T_z^{\theta}\\ ds=-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\Delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\Delta T_s+s_{inc}'(z)dz\end{cases}\qquad(137)$$

where T_z^{θ} is a $s_{inc}'(z)\Delta\tau_s$ increase in the marginal earnings tax rate when $s\in[s^0,s^0+\Delta s]$, and a $\Delta\tau_s\Delta s$ increase in tax liability when $s\geq s^0+\Delta s$. Moreover, the mass of individuals in the bandwitdh is $\Delta s\,h_s(s(z^0))=\Delta s\,\frac{h_z(z^0)}{s'(z^0)}$.

Using this expression to substitute the last term in the formula for optimal marginal tax rates on s yields

$$\frac{T_s'(s^0)}{1+T_s'(s^0)}s^0\zeta_{s|z}^c(z^0)h_z(z^0) = \underbrace{\left[s'(z^0) - s_{inc}'(z^0)\right]}_{s_{pref}'(z^0)}\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)$$

which is Equation (34) in Proposition 2.

Pareto efficiency for SN tax systems. We can combine formulas for optimal marginal tax rates on s and z to obtain a characterization of Pareto efficiency. Indeed, leveraging the previous optimal formula for marginal tax rates on s written in terms of $s_{pref}'(z^0)$, and replacing the integral term by its value given from the optimal formula for marginal earnings tax rates yields

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+T_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0) = s_{pref}'(z^0)\frac{\mathcal{T}_z'(z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)$$

which is the Pareto-efficiency condition (37) presented in Proposition 2, recognizing that $\mathcal{T}_z'(s,z) = T_z'(z)$ and $\mathcal{T}_s'(s,z) = T_s'(s)$.

B.8 Proof of Proposition 3 (Simple Tax systems and Multidimensional Heterogeneity)

We characterize in Proposition 3 optimal tax rates on s for each type of simple tax system in the presence of multidimensional heterogeneity. These formulas take the actual earnings tax schedule as given, be they optimally set or not, and extend the results derived in the unidimensional case. Crucially, we are able to provide similar characterizations because Lemma 1 still holds in the presence of multidimensional heterogeneity.

B.8.1 Separable linear (SL) tax system

Consider a reform that consists in a $\Delta\tau_s$ increase in the linear tax rate τ_s. For all agents, this triggers an increase in tax liability by $s\,\Delta\tau_s$ and an increase in the marginal tax rate on s by $\Delta\tau_s$, which by Lemma 1 produces earnings responses equivalent to an increase in the marginal earnings tax rate by $s_{inc}'\Delta\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\begin{aligned}&\int_z\int_s\Big[(1-g\left(s,z\right))\,s\,\Delta\tau_s\Big]h\left(s,z\right)dsdz\\ &=\int_z \mathbb{E}\left[(1-g\left(s,z\right))\,s|z\right]\,\Delta\tau_s h_z\left(z\right)dz \qquad (141)\end{aligned}$$

- *behavioral effects from changes in* z[43]

$$\begin{aligned}
&\int_z T_z'(z)\left\{\int_s \left(-\frac{z}{1-T_z'(z)}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\Delta\tau_s - \frac{\eta_z(s,z)}{1-T_z'(z)}s\,\Delta\tau_s\right)h(s,z)\,ds\right\}dz \\
&= -\int_z \frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)+\eta_z(s,z)s|z\right]\right\}\Delta\tau_s h_z(z)\,dz \qquad (143)
\end{aligned}$$

- *behavioral effects from changes in* s

$$\begin{aligned}
&\tau_s\int_z\int_s\Bigg\{-\frac{s}{1+\tau_s}\zeta_{s|z}^c(s,z)\,\Delta\tau_s - \frac{\eta_{s|z}(s,z)}{1+\tau_s}\,s\,\Delta\tau_s \\
&\qquad + s_{inc}'(s,z)\left(-\frac{z}{1-T_z'(z)}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\Delta\tau_s - \frac{\eta_z(s,z)}{1-T_z'(z)}s\,\Delta\tau_s\right)\Bigg\}h(s,z)\,dsdz \\
&= -\tau_s\int_z\Bigg\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)+\eta_{s|z}(s,z)\,s|z\right] \qquad (144) \\
&\qquad + \frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2+\eta_z(s,z)\,s\,s_{inc}'(s,z)\,|z\right]\right)\Bigg\}\Delta\tau_s h_z(z)\,dz
\end{aligned}$$

such that the total impact of the reform on the government objective is

$$\begin{aligned}
\frac{d\mathcal{L}}{\Delta\tau_s} &= \int_z \mathbb{E}\left[(1-g(s,z))\,s|z\right]\,h_z(z)\,dz \qquad (145) \\
&- \int_z \frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)+\eta_z(s,z)\,s|z\right]\right\}h_z(z)\,dz \\
&- \tau_s\int_z\Bigg\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)+\eta_{s|z}(s,z)\,s|z\right] \\
&+ \frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2+\eta_z(s,z)\,s\,s_{inc}'(s,z)\,|z\right]\right)\Bigg\}h_z(z)\,dz.
\end{aligned}$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z) = g(s,z) + \frac{T_z'(z)}{1-T_z'(z)}\eta_z(s,z) + \frac{\tau_s}{1+\tau_s}\eta_{s|z}(s,z) + \frac{\tau_s}{1-T_z'(z)}\eta_z(s,z)s_{inc}'(s,z) \qquad (146)$$

we finally get

$$\begin{aligned}
\frac{d\mathcal{L}}{\Delta\tau_s} &= \int_z \mathbb{E}\left[(1-\hat{g}(s,z))\,s|z\right]\,h_z(z)\,dz - \int_z \frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)\,|z\right]\right\}h_z(z)\,dz \qquad (147) \\
&- \tau_s\int_z\left\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right] + \frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2|z\right]\right)\right\}h_z(z)\,dz.
\end{aligned}$$

[43] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z\zeta_z^c(s,z)}{1-T_z'(z)}\,\Delta\tau_s\,s_{inc}'(s,z) - \frac{\eta_z(s,z)}{1-T_z'(z)}\,\Delta\tau_s\,s \\ ds = -\frac{s(z)\zeta_{s|z}^c(s,z)}{1+\tau_s}\,\Delta\tau_s - \frac{\eta_{s|z}(,sz)}{1+\tau_s}\Delta\tau_s\;s + s_{inc}'(s,z)dz \end{cases} \qquad (142)$$

Characterizing the optimal linear tax rate τ_s through $d\mathcal{L} = 0$, it satisfies

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\int_z \left\{ \mathbb{E}\left[s\zeta^c_{s|z}(s,z)|z \right] \right\} dH_z(z) \\
&= \int_z \left\{ \mathbb{E}\left[(1-\hat{g}(s,z))\, s|z \right] - \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z) s'_{inc}(s,z)\,|z \right] \right\} dH_z(z)
\end{aligned} \tag{148}$$

which is Equation (39) in Proposition 3.

B.8.2 Separable nonlinear (SN) tax system

Consider a reform that consists in a small $\delta\tau_s$ increase in the marginal tax rate on s across the bandwidth $\left[s^0, s^0 + \Delta s\right]$. For all agents with savings above s^0, this triggers a $\Delta s\, \Delta\tau_s$ increase in tax liability. For agents at s^0, this triggers a $\Delta\tau_s$ increase in the marginal tax rate on s – which by Lemma 1 produces earnings responses equivalent to a $s'_{inc}\Delta\tau_s$ increase in the marginal earnings tax rate.

We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta s \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\begin{aligned}
&\int_{s\geq s^0}\int_z \left\{ (1-g(s,z))\, \Delta s \Delta\tau_s \right\} h(s,z)\, ds dz \\
&= \int_z \left\{ \mathbb{E}\left[1-g(s,z)\,|z, s\geq s^0 \right] \right\} \Delta s \Delta\tau_s h_z(z)\, dz
\end{aligned} \tag{149}$$

- *behavioral effects from changes in z*[44]

$$\begin{aligned}
&-\int_z T'_z(z) \left\{ \frac{z}{1-T'_z} \zeta^c_z\left(s^0,z\right) s'_{inc}\left(s^0,z\right) \Delta\tau_s \right\} \Delta s\, h\left(s^0,z\right) dz \\
&-\int_{s\geq s^0}\int_z T'_z(z) \left\{ \frac{\eta_z(s,z)}{1-T'_z(z)}\, \Delta\tau_s \Delta s \right\} h(s,z)\, ds dz \\
&= -\int_z \frac{T'_z(z)}{1-T'_z} \left\{ z\zeta^c_z\left(s^0,z\right) s'_{inc}\left(s^0,z\right) + \mathbb{E}\left[\eta_z(s,z) s|z, s\geq s^0 \right] \right\} \Delta\tau_s \Delta s\, h_z(z)\, dz
\end{aligned} \tag{151}$$

[44] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z}{1-T'_z}\zeta^c_z(z)\delta T^{\theta\prime}_z - \frac{\eta_z(z)}{1-T'_z}\delta T^{\theta}_z \\ ds = -\frac{s(z)}{1+T'_s}\zeta^c_{s|z}(z)\delta T'_s - \frac{\eta_{s|z}(z)}{1+T'_s}\delta T_s + s'_{inc}(z) dz \end{cases} \tag{150}$$

where the reform δT^{θ}_z is a $s'_{inc}(s,z)\delta\tau_s$ increase in the marginal earnings tax rate when $s \in [s^0, s^0 + ds]$, and a $\delta\tau_s\delta s$ increase in tax liability when $s \geq s^0 + \delta s$.

- *behavioral effects from changes in s*

$$-T_s'\left(s^0\right)\int_z\left\{\frac{s^0}{1+T_s'\left(s^0\right)}\zeta_{s|z}^c(s^0,z)\,\Delta\tau_s+s_{inc}'\left(s^0,z\right)\frac{z}{1-T_z'\left(z\right)}\zeta_z^c(s^0,z)\,s_{inc}'\left(s^0,z\right)\Delta\tau_s\right\}\Delta s\,h\left(s^0,z\right)dz \tag{152}$$

$$-\int_{s\geq s^0}\int_z\left\{T_s'\left(s\right)\left(\frac{\eta_{s|z}(s,z)}{1+T_s'\left(s\right)}\Delta s\Delta\tau_s+s_{inc}'\left(s,z\right)\frac{\eta_z(s,z)}{1-T_z'\left(z\right)}\Delta s\Delta\tau_s\right)\right\}h\left(s,z\right)dsdz$$

$$=-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'\left(z\right)}s_{inc}'\left(s^0,z\right)^2z\zeta_z^c(s^0,z)\right\}\Delta\tau_s\Delta s\,h_z\left(z\right)dz \tag{153}$$

$$-\int_z\left\{\mathbb{E}\left[\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'\left(s,z\right)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}\Delta s\Delta\tau_sh_z\left(z\right)dz$$

such that the total impact of the reform on the government objective is

$$\frac{d\mathcal{L}}{\Delta s\Delta\tau_s}=\int_z\left\{\mathbb{E}\left[1-g\left(s,z\right)|z,s\geq s^0\right]\right\}dH_z\left(z\right) \tag{154}$$

$$-\int_z\frac{T_z'\left(z\right)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)|z,s\geq s^0\right]\right\}dH_z\left(z\right)$$

$$-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'\left(z\right)}s_{inc}'\left(s^0,z\right)^2z\zeta_z^c(s^0,z)\right\}dH_z\left(z\right)$$

$$-\int_z\left\{\mathbb{E}\left[\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'\left(s,z\right)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}dH_z\left(z\right).$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z)=g(s,z)+\frac{T_z'\left(z\right)}{1-T_z'\left(z\right)}\eta_z(s,z)+\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}(s,z)+s_{inc}'(s,z)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)}\eta_z(s,z) \tag{155}$$

we finally get

$$\frac{d\mathcal{L}}{\Delta s\Delta\tau_s}=\int_z\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z,s\geq s^0\right]\right\}dH_z\left(z\right)-\int_z\left\{\frac{T_z'\left(z\right)}{1-T_z'}z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)\right\}dH_z\left(z\right) \tag{156}$$

$$-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'\left(z\right)}s_{inc}'\left(s^0,z\right)^2z\zeta_z^c(s^0,z)\right\}dH_z\left(z\right).$$

Characterizing the optimal marginal tax rate on s, through $\frac{d\mathcal{L}}{\Delta s\Delta\tau_s}=0$, it satisfies at each savings s^0,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}\int_z\left\{s^0\zeta_{s|z}^c(s^0,z)\right\}dH_z\left(z\right) \tag{157}$$

$$=\int_z\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z,s\geq s^0\right]\right\}dH_z\left(z\right)-\int_z\left\{\frac{T_z'\left(z\right)+s_{inc}'\left(s^0,z\right)T_s'\left(s^0\right)}{1-T_z'\left(z\right)}z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)\right\}dH_z\left(z\right)$$

which is Equation (69) in Proposition (A.4).

B.8.3 Linear earnings-dependent (LED) tax system

Consider a reform that consists in a $\Delta\tau_s\Delta z$ increase in $\tau_s(z)$, the linear earnings-dependent tax rate on s, phased-in across the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$.[45]

For all individuals with earnings above $z^0+\Delta z$, this triggers an increase in the linear tax rate by $\Delta\tau_s\Delta z$ meaning that the marginal tax rate on s increases by the same magnitude – which triggers by Lemma 1 earnings responses equivalent to those induced by a $s'_{inc}\Delta\tau_s\Delta z$ increase in the marginal earnings tax rate – and that agents' tax liability increases by $s\,\Delta\tau_s\Delta z$.

For individuals in the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, the only direct effect of the reform is to induce earnings responses which by Lemma 1 are equivalent to an increase in the marginal earnings tax rate given by $s\,\Delta\tau_s$.

We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\begin{aligned}&\int_{z\geq z^0}\int_s\Big\{(1-g(s,z))\,\Delta z\Delta\tau_s s\Big\}h(s,z)\,dsdz\\&=\int_{z\geq z^0}\Big\{E_s\Big[(1-g(s,z))\,s\Big|z\Big]\Delta z\Delta\tau_s\Big\}h_z(z)\,dz\end{aligned}\tag{158}$$

- *behavioral effects from changes in z*[46]

$$\begin{aligned}&-\int_s\left(T_z'\left(z^0\right)+\tau_s'\left(z^0\right)s\right)\left\{\frac{z^0\zeta_z^c(s,z^0)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s}\,s\,\Delta\tau_s\right\}\Delta z\,h\left(s,z^0\right)ds\\&-\int_{z\geq z^0}\int_s\left(T_z'(z)+\tau_s'(z)\,s\right)\left\{\frac{z\zeta_z^c(s,z)}{1-T_z'(z)-\tau_s'(z)\,s}s_{inc}'\Delta z\Delta\tau_s+\frac{\eta_z(s,z)}{1-T_z'(z)-\tau_s'(z)\,s}\,s\,\Delta z\Delta\tau_s\right\}h(s,z)\,dsdz\\&=-\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^0\right]\Delta z\Delta\tau_s\,h_z\left(z^0\right)\\&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}\Big(z\zeta_z^c(s,z)s_{inc}'(s,z)+s\,\eta_z(s,z)\Big)\middle|z\right]\right\}\Delta z\Delta\tau_s h_z(z)\,dz\end{aligned}\tag{159}$$

[45] To avoid any ambiguity, we here use d for integration and δ for attributes of the reform we consider.

[46] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$\begin{cases}dz=-\frac{z^0\zeta_z^c(s,z^0)}{1-T_z'}\,\Delta\tau_s\,s\\ ds=s_{inc}'(s,z^0)dz\end{cases}\quad\text{and}\quad\begin{cases}dz=-\frac{z\zeta_z^c(s,z)}{1-T_z'}\,\Delta\tau_s\Delta z\,s_{inc}'(s,z)-\frac{\eta_z(s,z)}{1-T_z'}\,\Delta\tau_s\Delta z\,s\\ ds=-\frac{s\zeta_{s|z}^c(s,z)}{1+T_s'}\,\Delta\tau_s\Delta z-\frac{\eta_s(s,z)}{1+T_s'}\Delta\tau_s\Delta z\,s+s_{inc}'(s,z)dz\end{cases}$$

- *behavioral effects from changes in s*

$$
\begin{aligned}
& -\tau_s\left(z^0\right)\int_s s'_{inc}\left(s,z^0\right)\left\{\frac{z^0\zeta_z^c(s,z^0)}{1-T'_z\left(z^0\right)-\tau'_s\left(z^0\right)s}\,s\,\Delta\tau_s\right\}\Delta z\,h\left(s,z^0\right)ds \\
& -\int_{z\geq z^0}\int_s \tau_s\left(z\right)\left\{\frac{s\zeta_{s|z}^c(s,z)}{1+\tau_s\left(z\right)}\Delta z\Delta\tau_s + s'_{inc}\left(s,z\right)\frac{z\zeta_z^c(s,z)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}\,s'_{inc}\left(s,z\right)\Delta z\Delta\tau_s\right\}h\left(s,z\right)dz \\
& -\int_{z\geq z^0}\int_s\left\{\tau_s\left(z\right)\left(\frac{\eta_{s|z}(s,z)}{1+\tau_s\left(z\right)}\,s\,\Delta z\Delta\tau_s + s'_{inc}\left(s,z\right)\frac{\eta_z(s,z)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}\,s\,\Delta z\Delta\tau_s\right)\right\}h\left(s,z\right)dsdz \\
& = -\tau_s\left(z^0\right)\mathbb{E}\left[s'_{inc}\left(s,z\right)\frac{z\zeta_z^c(s,z)s}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}\middle|z=z^0\right]\Delta z\Delta\tau_s\,h_z\left(z^0\right) \qquad (160) \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\middle|z\right]+\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}z\zeta_z^c(s,z)s'_{inc}\left(s,z\right)^2\right]\right\}\Delta z\Delta\tau_s\,h_z\left(z\right)dz \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\eta_{s|z}(s,z)\middle|z\right]+\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}s\eta_z(s,z)s'_{inc}\left(s,z\right)\middle|z\right]\right\}\Delta z\Delta\tau_s\,h_z\left(z\right)dz
\end{aligned}
$$

such that the total impact of the reform on the government objective is

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s\Delta\tau_s} = & \int_{z\geq z^0}\left\{E_s\left[\left(1-g\left(s,z\right)\right)s\middle|z\right]\right\}h_z\left(z\right)dz \qquad (161) \\
& -\mathbb{E}\left[\frac{T'_z\left(z\right)+\tau'_s\left(z\right)s}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}z\zeta_z^c(s,z)s\middle|z=z^0\right]h_z\left(z^0\right) \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{T'_z\left(z\right)+\tau'_s\left(z\right)s}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}\left(z\zeta_z^c(s,z)s'_{inc}\left(s,z\right)+s\,\eta_z(s,z)\right)\middle|z\right]\right\}h_z\left(z\right)dz \\
& -\mathbb{E}\left[\frac{s'_{inc}\left(s,z\right)\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}z\zeta_z^c(s,z)s\middle|z=z^0\right]\delta z\delta\tau_s\,h_z\left(z^0\right) \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\middle|z\right]+\mathbb{E}\left[\frac{s'_{inc}\left(s,z\right)\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}z\zeta_z^c(s,z)s'_{inc}\left(s,z\right)\middle|z\right]\right\}h_z\left(z\right)dz \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\eta_{s|z}(s,z)\middle|z\right]+\mathbb{E}\left[\frac{s'_{inc}\left(s,z\right)\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}s\eta_z(s,z)\middle|z\right]\right\}h_z\left(z\right)dz.
\end{aligned}
$$

Redefining augmented social marginal welfare weights as

$$
\hat{g}(s,z) = g(s,z) + \frac{T'_z\left(z\right)+\tau'_s\left(z\right)s+s'_{inc}(s,z)\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}\eta_z(s,z) + \frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}\eta_{s|z}(s,z) \qquad (162)
$$

we finally get

$$\begin{aligned}\frac{d\mathcal{L}}{\Delta s\Delta\tau_s} =& \int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z\Big]h_z\left(z\right)dz \\ &-\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s\Big|z=z^0\right]h_z\left(z^0\right) \\ &-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)\Big)\Big|z\right]h_z\left(z\right)dz \\ &-\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\Big|z\right]h_z\left(z\right)dz\end{aligned} \tag{163}$$

Characterizing the optimal linear earnings-dependent tax rate $\tau_s(.)$ through $d\mathcal{L}=0$, it satisfies at each earnings z^0

$$\begin{aligned}&\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s\Big|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\Big|z\right]h_z\left(z\right)dz \\ &=\int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z\Big]h_z\left(z\right)dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)\Big|z\right]h_z\left(z\right)dz\end{aligned} \tag{164}$$

which is Equation (70) in Proposition (A.4).

B.9 Proof of Proposition 4 (Many Goods)

B.9.1 Setting and definitions

The problem of the government is to maximize the following Lagrangian

$$\mathcal{L}=\int_z\left\{\alpha(z)\,U\Big(z-\mathcal{T}(\boldsymbol{s}(z),z)-\sum_{i=1}^{n}s_i(z),\boldsymbol{s}(z),z;\vartheta\left(z\right)\Big)+\lambda\mathcal{T}(\boldsymbol{s}(z),z)-E\right\}dH_z(z) \tag{165}$$

where we use the fact that $z\left(\theta\right)$ is a bijective mapping to denote $\vartheta\left(z\right)$ its inverse, and to define Pareto weights $\alpha(z):=\alpha(\vartheta(z))$ and the vector of n consumption goods as $\boldsymbol{s}\left(z\right):=\boldsymbol{s}\left(z;\vartheta\left(z\right)\right)$.

In this setting, we express optimal tax formulas in terms of the following elasticity concepts that measure consumption responses of s_i and s_j to changes in $\mathcal{T}_{s_i}'$

$$\zeta_{s_i|z}^c\left(z(\theta)\right):=-\frac{1+\mathcal{T}_{s_i}'\left(\boldsymbol{s}\left(z;\theta\right),z\right)}{s_i\left(z;\theta\right)}\frac{\partial s_i\left(z;\theta\right)}{\partial\mathcal{T}_{s_i}'\left(\boldsymbol{s}\left(z;\theta\right),z\right)}\Big|_{z=z(\theta)} \tag{166}$$

$$\xi_{s_{j,i}|z}^c\left(z(\theta)\right):=\frac{\mathcal{T}_{s_i}'\left(\boldsymbol{s}\left(z;\theta\right),z\right)}{s_j\left(z;\theta\right)}\frac{\partial s_j\left(z;\theta\right)}{\partial\mathcal{T}_{s_i}'\left(\boldsymbol{s}\left(z;\theta\right),z\right)}\Big|_{z=z(\theta)} \tag{167}$$

and in terms of the following statistics

$$s'_{i,inc}(z(\theta)) := \frac{\partial s_i(z;\theta)}{\partial z}\Big|_{z=z(\theta)} \tag{168}$$

$$\hat{g}(z(\theta)) := \left[\alpha(z)\frac{U'_c(z)}{\lambda} - \left(\mathcal{T}'_z\left(\boldsymbol{s}(z), z\right) + \sum_{i=1}^{n} s'_{i,inc}(z)\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z), z\right)\right)\frac{\partial z(.)}{\partial \mathcal{T}} - \sum_{i=1}^{n} \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z), z\right)\frac{\partial s_i(.)}{\partial \mathcal{T}}\right]\Big|_{z=z(\theta)}. \tag{169}$$

B.9.2 Optimal marginal tax rates on earnings z

We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate $\mathcal{T}'_z$ in a small bandwidth Δz. The impact of this reform on the Lagrangian as $\Delta z \to 0$ is

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_z\Delta z} &= \int_{x\geq z^0}\Big(1-\alpha(x)\frac{U'_c(x)}{\lambda}\Big)dH_z(x) \\ &+ \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0}h_z(z^0) + \int_{x\geq z_0}\mathcal{T}'_z\left(\boldsymbol{s}(x), x\right)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x}dH_z(x) \\ &+ \sum_{i=1}^{n}\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)s'_{i,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0}h_z(z^0) + \int_{x\geq z_0}\sum_{i=1}^{n}\mathcal{T}'_{s_i}\left(\boldsymbol{s}(x), x\right)\left[\frac{\partial s_i(.)}{\partial \mathcal{T}}\Big|_{z=x} + s'_{i,inc}(x)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x}\right]dH_z(x).\end{aligned} \tag{170}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(z)$, we obtain

$$-\left[\mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{i=1}^{n}\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)s'_{i,inc}(z^0)\right]\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0}h_z(z_0) = \int_{x\geq z_0}\Big(1-\hat{g}(x)\Big)dH_z(x). \tag{171}$$

B.9.3 Optimal marginal tax rates on good i

We consider a small reform at earnings level z^0 that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s_i for all individuals with earnings z above z^0, phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$. In the bandwidth $\left[z^0, z^0+\Delta z\right]$, this reform induces labor supply distortions on earnings z. At earnings $z \geq z^0 + \Delta z$, this reform induces (a) substitution effects away from s_i, (b) labor supply distortions on earnings z, and, new to this setting, (c) cross-effects on the consumption of goods s_{-i}.[47]

[47] Applying Lemma 1, which still holds in this setting, changes in z and s_j at earnings z^0 and above earnings z^0 are respectively

$$\begin{cases} dz = \frac{\partial z(.)}{\partial \mathcal{T}'_z}\,\Delta\tau_s\,s_i(z^0) \\ ds_j = s'_{j,inc}(z^0)dz \end{cases} \quad \text{and} \quad \begin{cases} dz = \frac{\partial z(.)}{\partial \mathcal{T}'_z}\,\Delta\tau_s\Delta z\,s'_{i,inc}(z) + \frac{\partial z(.)}{\partial \mathcal{T}}\,\Delta\tau_s\Delta z\,s_i\left(z\right) \\ ds_j = \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\,\Delta\tau_s\Delta z + \frac{\partial s_j(.)}{\partial \mathcal{T}}\Delta\tau_s\Delta z\,s_i\left(z\right) + s'_{j,inc}(z)dz \end{cases}$$

The impact of this reform on the Lagrangian as $\Delta z \to 0$ is

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s \Delta z} = \int_{x=z^0}^{z_{max}} \Big(1-\alpha(x)\frac{U_c'(x)}{\lambda}\Big)s_i(x)dH_z(x) \tag{172}$$
$$+ \mathcal{T}_z'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_i(z^0)h_z(z^0) + \int_{x=z^0}^{z_{max}} \mathcal{T}_z'\left(\boldsymbol{s}(x),x\right)\left[\frac{\partial z(.)}{\partial \mathcal{T}_z'}s_{i,inc}'(x) + \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x)\right]dH_z(z)$$
$$+ \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)\left[s_{j,inc}'(z^0)\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_i(z^0)\right]h_z(z^0)$$
$$+ \int_{x=z^0}^{z_{max}}\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)\left\{\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=x} + \frac{\partial s_j(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x) + s_{j,inc}'(x)\left[\frac{\partial z(.)}{\partial \mathcal{T}_z'}s_{i,inc}'(x) + \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x)\right]\right\}dH_z(z)$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}$ capture cross-effects for all $j \neq i$.

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(x)$, we obtain

$$-\left[\mathcal{T}_z'\left(\boldsymbol{s}(z^0),z^0\right) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)s_{j,inc}'(z^0)\right]\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_i(z^0)h_z(z^0) = \int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)s_i(x)dH_z(z)$$
$$+ \int_{x=z^0}^{z_{max}}\left\{\left[\mathcal{T}_z'\left(\boldsymbol{s}(x),x\right) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)s_{j,inc}'(x)\right]\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=x} s_{i,inc}'(x) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=x}\right\}dH_z(z). \tag{173}$$

B.9.4 Deriving Proposition 4

For any good i, we combine the optimality condition for marginal tax rates on earnings z with the one for marginal tax rates on good i to obtain

$$s_i(z^0)\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x) = \int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)s_i(x)dH_z(z)$$
$$+ \int_{x=z^0}^{z_{max}}\left[\left[\mathcal{T}_z'\left(\boldsymbol{s}(x),x\right) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)s_{j,inc}'(x)\right]\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=x} s_{i,inc}'(x) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=x}\right]dH_z(z) \tag{174}$$

such that differentiating with respect to earnings z^0 gives after simplification

$$s_i'(z^0)\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x) \tag{175}$$
$$= -\left[\left[\mathcal{T}_z'\left(\boldsymbol{s}(z^0),z^0\right) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)s_{j,inc}'(z^0)\right]\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_{i,inc}'(z^0) + \sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=z^0}\right]h_z(z^0).$$

Making use of the optimality condition for marginal earnings tax rates, we can substitute the first term on the right-hand side to obtain

$$-\sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} = \underbrace{\left[s'_i(z^0) - s'_{i,inc}(z^0)\right]}_{s_{i,pref}(z^0)} \frac{1}{h_z(z^0)} \int_{x=z_0}^{z_{max}} \Big(1 - \hat{g}(x)\Big) dH_z(x). \quad (176)$$

Isolating the term relative to $\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)$ on the left-hand side yields the following optimal tax formula in terms of $s'_{i,pref}$

$$-\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial s_i(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} = \frac{1}{h_z(z^0)} s'_{i,pref}(z^0) \int_{x=z^0}^{z_{max}} \Big(1 - \hat{g}(x)\Big) dH_z(x) + \sum_{j \neq i} \mathcal{T}'_{s_j}\left(z^0\right) \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} \quad (177)$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}$ capture cross-effects for all $j \neq i$.

We can rewrite this optimality condition in terms of the compensated elasticity $\zeta^c_{s_i|z}$ and the cross elasticity $\xi^c_{s_{j,i}|z}$ to finally obtain

$$\begin{aligned} \frac{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)}{1 + \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)} &= s'_{i,pref}(z^0) \frac{1}{s_i(z^0)\zeta^c_{s_i|z}(z^0)} \frac{1}{h_z(z^0)} \int_{z=z^0}^{z_{max}} \Big(1 - \hat{g}(z)\Big) dH_z(z) \\ &\quad + \sum_{j \neq i} \frac{\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right)}{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)} \frac{s_j(z^0)\xi^c_{s_{j,i}|z}(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)} \end{aligned} \quad (178)$$

which is the first condition stated in Proposition 4.

To derive the second condition stated in Proposition 4, we substitute the first term on the right-hand side using the optimality condition for marginal tax rates on earnings z to directly obtain

$$\begin{aligned} \frac{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)}{1 + \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)} &= s'_{i,pref}(z^0) \frac{\mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{j=1}^n \mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right) s'_{j,inc}(z^0)}{1 - \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right)} \frac{z^0 \zeta^c_z(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)} \\ &\quad + \sum_{j \neq i} \frac{\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right)}{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right)} \frac{s_j(z^0)\xi^c_{s_{j,i}|z}(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}. \end{aligned} \quad (179)$$

This completes the proof of Proposition 4.

B.10 Proof of Proposition 5 (Bequest Taxation and Behavioral Agents)

B.10.1 Setting

We here provide a sufficient statistics characterization of a smooth tax system $\mathcal{T}(s, z)$ under the following additively separable representation of agents' preferences

$$U\big(c, s, z; \theta\big) = u(c; \theta) - k(z; \theta) + \beta(\theta)\, v(s; \theta),$$

and for a utilitarian government that maximizes

$$\int_\theta \Big[U\left(c\left(\theta\right),s\left(\theta\right),z\left(\theta\right);\theta\right)+\nu\left(\theta\right)v\left(s\left(\theta\right);\theta\right)\Big]dF\left(\theta\right), \tag{180}$$

where $\nu\left(\theta\right)$ parametrizes the degree of misalignment on the valuation of s.

Using the mapping between types θ and earnings z, the Lagrangian of the problem writes

$$\mathcal{L}=\int_z \Big[U\left(c\left(z\right),s\left(z\right),z;\vartheta\left(z\right)\right)+\nu\left(z\right)v\left(s\left(z\right);\vartheta\left(z\right)\right)+\lambda\left(\mathcal{T}\left(s,z\right)-E\right)\Big]dH_z\left(z\right), \tag{181}$$

As before, we derive optimal tax formulas by considering reforms of marginal tax rates on z and s. Thanks to the additively separable representation of preferences, there are no income effects on labor supply choices. As a result, the only substantial change is that savings changes now lead to changes in social welfare proportional to the degree of misalignment.

B.10.2 Optimal marginal tax rates on z.

A small reform at earnings z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate in a small bandwidth Δz has the following effect as $\Delta z\to 0$,

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_z\Delta z}=&\int_{z^0}^{z_{max}}\left(1-\hat{g}(z)\right)\,dH_z(z)\\&-\left(\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\Big)\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\,h_z(z^0).\end{aligned}$$

In this context, social marginal welfare weights augmented with income effects $\hat{g}(z)$ are equal to

$$\hat{g}(z)=\frac{u'\left(c\left(z\right)\right)}{\lambda}+\Big(\mathcal{T}_s'\left(s(z),z\right)+\nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}\Big)\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}$$

and we can use agents' first-order condition for s, $\left(1+\mathcal{T}_s'\right)u'\left(c\right)=\beta v'\left(s\right)$, to express the misalignment wedge in terms of the social marginal welfare weights $g\left(z\right):=\frac{u'(c(z))}{\lambda}$ as

$$\nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}=\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\right).$$

The optimal schedule of marginal earnings tax rates is thus characterized by

$$\begin{aligned}\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}=&\frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z=z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z)\\&-s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}-s_{inc}'(z^0)\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\frac{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}.\end{aligned}$$

B.10.3 Optimal marginal tax rates on s.

A small reform at earnings level z^0 that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, has the following effect as $\Delta s \to 0$,

$$
\begin{aligned}
&\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s\Delta z} \\
&= -\left[\mathcal{T}_z'\left(s^0,z^0\right) + s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s^0,z^0\right) + \nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\Big)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\, s^0\, h_z(z^0) \\
&+ \int_{z\geq z^0}\left\{(1-\hat{g}(z))\, s\left(z\right) - \left[\mathcal{T}_s'\left(s(z),z\right) + \nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\} dH_z(z) \\
&- \int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right) + s_{inc}'(z)\Big(\mathcal{T}_s'\left(s(z),z\right) + \nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}\Big)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}s_{inc}'(z)\right\} dH_z(z).
\end{aligned}
$$

We characterize optimal taxes through $d\mathcal{L}=0$. Replacing the misalignment wedge by its expression in terms of social marginal welfare weights $g\left(z\right)$, we obtain that the optimal schedule of marginal tax rates on s is characterized by

$$
\begin{aligned}
&\left[\mathcal{T}_z'\left(s^0,z^0\right) + s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s^0,z^0\right) + \frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z)\left(1+\mathcal{T}_s'\left(s^0,z^0\right)\right)\Big)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\, s^0\, h_z(z^0) \\
&= \int_{z\geq z^0}\left\{(1-\hat{g}(z))\, s\left(z\right) - \left[\mathcal{T}_s'\left(s(z),z\right) + \frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\} dH_z(z) \\
&- \int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right) + s_{inc}'(z)\Big(\mathcal{T}_s'\left(s(z),z\right) + \frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\Big)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}s_{inc}'(z)\right\} dH_z(z).
\end{aligned}
$$

B.10.4 Deriving Proposition 5

Combining optimality conditions for marginal tax rates on z and s yields

$$
\begin{aligned}
&s(z^0)\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\, dH_z(z) = \int_{z\geq z^0}(1-\hat{g}(z))\, s\left(z\right) dH_z(z) \qquad (182)\\
&- \int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right) + s_{inc}'(z)\left(\mathcal{T}_s'\left(s(z),z\right) + \frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}s_{inc}'(z)\right\} dH_z(z) \\
&- \int_{z\geq z^0}\left\{\left[\mathcal{T}_s'\left(s(z),z\right) + \frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\} dH_z(z).
\end{aligned}
$$

Differentiating with respect to z^0, we obtain after simplification

$$
\begin{aligned}
& s'(z^0)\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z) \qquad (183)\\
&=\left\{\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\Big)\right]\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'}s_{inc}'(z^0)\right\}h_z(z^0)\\
&+\left\{\left[\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\right]\frac{s(z^0)\zeta_{s|z}^c(z^0)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}\right\}h_z(z^0).
\end{aligned}
$$

Substituting the first term on the right-hand side by its expression from the optimality condition for marginal tax rates on z, and rearranging we obtain

$$
\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{pref}'(z^0)}\frac{1}{s(z^0)\zeta_{s|z}^c(z^0)}\frac{1}{h_z(z^0)}\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z) \qquad (184)
$$

which is the first optimality condition in Proposition 5.

Conversely, substituting the term on the left-hand side by its expression from the optimality condition for marginal tax rates on z, and rearranging we obtain

$$
\begin{aligned}
&\left[\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\right]\frac{s(z^0)\zeta_{s|z}^c(z^0)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)} \qquad (185)\\
&=s_{pref}'(z^0)\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\Big)\right]\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}
\end{aligned}
$$

which is the second optimality condition in Proposition 5.

B.11 Proof of Proposition 6 (Multi-Dimensional Tax Range with Heterogeneous Prices)

B.11.1 Setting

We consider heterogeneous marginal rates of transformation or "prices" $p(z,\theta)$ between c and s, and a two-part tax structure, where a person must pay a tax $T_1(z)$ in units of c and a tax $T_2(s,z)$ in units of s. In particular, we consider simple tax systems of the SN type, where the tax on s is nonlinear but independent of earnings z such that $T_2(s,z)=T_2(s)$, and of the LED type, where the tax on s is linear but earnings-dependent such that $T_2(s,z)=\tau_s(z)\,s$.

In this setting, we can write agent θ problem as

$$
\max_{c,s,z}\; U(c,s,z;\theta) \text{ s.t. } c+p(z,\theta)s\le z-T_1(z)-p(z,\theta)T_2(s,z) \qquad (186)
$$

$$
\iff \max_z\left\{\max_s\; U\Big(z-T_1(z)-p(z,\theta)\left(s+T_2(s,z)\right),s,z;\theta\Big)\right\} \qquad (187)
$$

where the inner problem leads to consumption choices $c(z;\theta)$ and $s(z;\theta)$, and the outer problem leads to an earnings choice $z(\theta)$. Assuming $z(\theta)$ continues to be a bijective mapping, we again denote $\vartheta(z)$ its inverse. This allows us to define $s(z) := s(z;\vartheta(z))$, $p(z) := p(z(\vartheta(z));\vartheta(z))$ and to formulate the problem in terms of observable earnings z.[48]

Let λ_1 and λ_2 be the marginal values of public funds associated with the resource constraints

$$\int_z T_1(z)dH_z(z) \geq E_1 \tag{188}$$

$$\int_z T_2(s(z),z)dH_z(z) \geq E_2. \tag{189}$$

The problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z \Big\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z),z)\big), s(z), z; \vartheta(z)\Big) \\ + \lambda_1 T_1(z) + \lambda_2 T_2(s(z),z) - E_1 - E_2 \Big\}\, dH_z(z) \tag{190}$$

B.11.2 Adapting Lemma 1

Lemma B.4. *For an agent $\theta = \vartheta(z)$, we have that:*
(1a) a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.
(1b) a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s'_{inc}(z)\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.
(2) a small increase ΔT in the T_2 tax liability faced by agent $\theta = \vartheta(z)$ generates the same earnings change as a small increase $p(z)\Delta T$ in the T_1 tax liability.

Proof. We first derive an abstract characterization that we then apply to different tax reforms.

Let type θ indirect utility function at earnings z be

$$V\left(T_1(z), T_2(.,z), z;\theta\right) := \max_s\ U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z;\theta\Big). \tag{191}$$

Consider a small reform $\Delta T_2(s,z)$ of T_2, and let construct for each type θ a perturbation $\Delta T_1^\theta(z)$ of T_1 that induces the same earnings response as the initial perturbation. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(T_1(z) + \Delta T_1^\theta(z), T_2(.,z), z;\theta) = V(T_1(z), T_2(.,z) + \Delta T_2(.,z), z;\theta). \tag{192}$$

Then, by construction, the perturbation $\Delta T_1^\theta(z)$ induces the same earnings response dz as the initial perturbation $\Delta T_2(.,z)$. Moreover, both tax reforms must induce the same utility change for

[48] When taking derivatives, the presence of these two arguments is implicit. For instance, a total derivative corresponds to $\frac{dp}{dz} := \frac{\partial p}{\partial z} + \frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial z}$, whereas a partial derivative $\frac{\partial p}{\partial z}$ represents variation in only the first argument.

agent θ. Applying the envelope theorem yields

$$-U_c'(z;\theta)\cdot\Delta T_1^\theta(z) = -U_c'(z;\theta)\, p(z,\theta)\cdot\Delta T_2(s(z;\theta),z) \tag{193}$$

such that finally, the perturbation $\Delta T_1^\theta(z)$ is

$$\Delta T_1^\theta(z) = p(z,\theta)\cdot\Delta T_2(s(z;\theta),z). \tag{194}$$

and we can now apply this abstract characterization to different tax reforms.

(1a) Consider a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ over a small bandwidth of income $\left[z^0, z^0+\Delta z\right]$. Then, for any agent θ such that $z(\theta)\in\left[z^0,z^0+\Delta z\right]$, we have $\Delta T_2(s(z;\theta),z) = \Delta\tau_z\left(z-z^0\right)$ such that $\Delta T_1^\theta(z) = p(z,\theta)\Delta\tau_z\left(z-z^0\right)$ and differentiating with respect to z we get

$$\left(\Delta T_1^\theta(z)\right)' = \frac{\partial p(z,\theta)}{\partial z}\Delta\tau_z\left(z-z^0\right) + p(z,\theta)\Delta\tau_z. \tag{195}$$

At the limit $\Delta z\to 0$ such that $z\to z^0$, a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)\Delta\tau_z$ in the marginal tax rate $T_1'(z)$.

(1b) Consider a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ over a small bandwidth of savings $\left[s^0,s^0+\Delta s\right]$. Then, for any agent θ such that $s(\theta)\in\left[s^0,s^0+\Delta s\right]$, we have $\Delta T_2(s(z;\theta),z) = \Delta\tau_s\left(s(z;\theta)-s^0\right)$ such that $\Delta T_1^\theta(z) = p(z,\theta)\Delta\tau_z\left(s(z;\theta)-s^0\right)$ and differentiating with respect to z we get

$$\left(\Delta T_1^\theta(z)\right)' = \frac{\partial p(z,\theta)}{\partial z}\Delta\tau_z\left(s(z;\theta)-s^0\right) + p(z,\theta)\Delta\tau_z s_{inc}'(z). \tag{196}$$

At the limit $\Delta s\to 0$ such that $s\to s^0$, a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s_{inc}'(z)\Delta\tau_z$ in the marginal tax rate $T_1'(z)$.

(2) Consider a small lump-sum increase ΔT in the T_2 tax liability for an agent θ who earns z, we then have $\Delta T_1^\theta(z) = p(z,\theta)\Delta T$ such that the equivalent reform is no longer a lump-sum increase. Hence, a small increase ΔT in the T_2 tax liability faced by an agent $\vartheta(z)$ generates the same earnings change as a small increase $p(z)\Delta T$ in the T_1 tax liability. □

B.11.3 Marginal values of public funds

An important prerequisite to derive optimality conditions is to pin down the marginal values of public funds λ_1 and λ_2. At the optimum, λ_1 and λ_2 are pinned down by optimally setting the tax level T_1 and T_2. Characterizing the impact of lump-sum changes in tax liabilities yields the following two equations that can be solved for λ_1 and λ_2:

$$\int_{x=z_{min}}^{z_{max}} \left\{-\alpha(x)U_c'(x) + \lambda_1 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'(x)\lambda_2 \frac{\partial T_2}{\partial s}\right) \frac{\partial z(.)}{\partial T_1} + \lambda_2 \frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1}\right\} dH_z(x) = 0 \tag{197}$$

$$\int_{x=z_{min}}^{z_{max}} \left\{-\alpha(x)p(x)U_c'(x) + \lambda_2 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'(x)\lambda_2 \frac{\partial T_2}{\partial s}\right) \frac{\partial z(.)}{\partial T_2} + \lambda_2 \frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_2}\right\} dH_z(x) = 0 \tag{198}$$

where $z(.)$ and $s(.)$ denote, with a slight abuse of notation, the earnings and savings choices, and all partial derivatives are evaluated at earnings x.

Renormalizing these equations by λ_1, we can use the fact that by Lemma B.4, $\frac{\partial z(.)}{\partial T_2} = \frac{\partial z(.)}{\partial T_1}p(z) + \frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z}$ and that $\frac{\partial s(.)}{\partial T_2} = \frac{\partial s(.)}{\partial T_1}p(z)$ to obtain

$$\int_{x=z_{min}}^{z_{max}} \left\{1 - \left[\alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1}\right]\right\} dH_z(z) = 0 \tag{199}$$

$$\begin{aligned}\int_{x=z_{min}}^{z_{max}} \Bigg\{\frac{\lambda_2}{\lambda_1} - p(x)\left[\alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1}\right] \\ + \left(T_1'(z) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z}\Bigg\} dH_z(x) = 0.\end{aligned} \tag{200}$$

At any given earnings x, defining social marginal welfare weights augmented with the fiscal impact of income effects $\hat{g}(x)$ and the fiscal impacts of the novel substitution effects $\varphi(x)$ as respectively

$$\hat{g}(x) := \alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1} \tag{201}$$

$$\varphi(x) := \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z} \tag{202}$$

where all partial derivatives are evaluated at x, we finally obtain

$$\overline{\hat{g}} := \int_{x=z_{min}}^{z_{max}} \hat{g}(x)\, dH_z(x) = 1 \tag{203}$$

$$\overline{\hat{g}p - \varphi} := \int_{x=z_{min}}^{z_{max}} \left(\hat{g}(x)p(x) - \varphi(x)\right) dH_z(x) = \frac{\lambda_2}{\lambda_1}. \tag{204}$$

B.11.4 Optimal tax rates on z

We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate $T_1'(z)$ in a small bandwidth Δz. The impact on the Lagrangian is as $\Delta z \to 0$,

$$\begin{aligned}
\frac{d\mathcal{L}}{\Delta\tau_z\Delta z} = & \int_{x\geq z^0}\Big(\lambda_1-\alpha(x)U_c'(x)\Big)dH_z(x) \\
& + \left[\lambda_1 T_1'\left(z^0\right)+\lambda_2\frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right]\frac{\partial z(.)}{\partial T_1'\left(z^0\right)}h_z(z^0) \\
& + \int_{x\geq z^0}\left[\lambda_1 T_1'\left(x\right)+\lambda_2\frac{\partial T_2}{\partial z}\right]\frac{\partial z(.)}{\partial T_1}dH_z(x) \\
& + \lambda_2\frac{\partial T_2}{\partial s}\Big|_{z=z^0}s_{inc}'(z^0)\frac{\partial z(.)}{\partial T_1'\left(z^0\right)}h_z(z^0) \\
& + \int_{x\geq z^0}\lambda_2\frac{\partial T_2}{\partial s}\left[\frac{\partial s(.)}{\partial T_1}+s_{inc}'(x)\frac{\partial z(.)}{\partial T_1}\right]dH_z(x).
\end{aligned} \tag{205}$$

We characterize optimal taxes through $d\mathcal{L}=0$. Renormalizing everything by λ_1, plugging in social marginal welfare weights augmented with income effects $\hat{g}(x)$, we obtain the following optimality condition for marginal earnings tax rates at each earnings z^0

$$-\left[T_1'(z^0)+\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z}\Big|_{z^0}+s_{inc}'(z^0)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\right]\frac{\partial z(.)}{\partial T_1'\left(z^0\right)}=\frac{1}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x). \tag{206}$$

B.11.5 Optimal tax rates on s

SN tax system. We consider a small reform at $s^0=s(z^0)$ that consists in a small increase $\Delta\tau_s$ of $\frac{\partial T_2}{\partial s}$, the marginal tax rate on s, in a small bandwidth Δs. Using Lemma 2, we characterize the impact of the reform on the Lagrangian as $\Delta s\to 0$

$$\begin{aligned}
\frac{d\mathcal{L}}{\Delta\tau_s\Delta s} = & \int_{x\geq z^0}\Big(\lambda_2-\alpha(x)p(x)U_c'(x)\Big)dH_z(x) \\
& + \left[\lambda_1 T_1'(z^0)+\lambda_2\frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right]\frac{\partial z(.)}{\partial T_1'\left(z^0\right)}s_{inc}'(z^0)p(z^0)\frac{h_z(z^0)}{s'(z^0)} \\
& + \int_{x\geq z^0}\left[\lambda_1 T_1'(x)+\lambda_2\frac{\partial T_2}{\partial z}\right]\left(\frac{\partial z(.)}{\partial T_1}p(x)+\frac{\partial z(.)}{\partial T_1'\left(x\right)}\frac{\partial p}{\partial z}\right)dH_z(x) \\
& + \lambda_2\frac{\partial T_2}{\partial s}\Big|_{z=z^0}\left[\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z=z^0}\right)}+s_{inc}'(z^0)\frac{\partial z(.)}{\partial T_1'(z^0)}s_{inc}'(z^0)p(z^0)\right]\frac{h_z(z^0)}{s'(z^0)} \\
& + \int_{x\geq z^0}\lambda_2\frac{\partial T_2}{\partial s}\left[\frac{\partial s(.)}{\partial T_2}+s_{inc}'(x)\left(\frac{\partial z(.)}{\partial T_1}p(x)+\frac{\partial z(.)}{\partial T_1'\left(x\right)}\frac{\partial p}{\partial z}\right)\right]dH_z(x)
\end{aligned} \tag{207}$$

We characterize optimal taxes through $d\mathcal{L}=0$. Renormalizing by λ_1 and using $\frac{\partial s(.)}{\partial T_2}=\frac{\partial s(.)}{\partial T_1}p(x)$, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for marginal tax rates on s at each savings $s^0 = s(z^0)$:

$$-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}h_z(z^0) = s'(z^0)\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right\}dH_z(x) \quad (208)$$
$$+\left[T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z}\Big|_{z^0} + s'_{inc}(z^0)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\right]\frac{\partial z(.)}{\partial T_1'(z^0)}s'_{inc}(z^0)p(z^0)h_z(z^0)$$

LED tax system. We consider a small reform at $s^0 = s(z^0)$ that consists in a small increase $\Delta\tau_s$ of the linear savings tax rate $\tau_s(z)$ phased in over the earnings bandwidth $[z^0, z^0 + \Delta z]$. Using Lemma 2, we characterize the impact of the reform on the Lagrangian as $\Delta z \to 0$

$$\frac{d\mathcal{L}}{\Delta\tau_s\Delta z} = \int_{x\geq z^0}\Big(\lambda_2 - \alpha(x)p(x)U_c'(x)\Big)s(x)\,dH_z(x) \quad (209)$$
$$+\left(\lambda_1 T_1'(z^0) + \lambda_2\tau_s'(z^0)s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\,h_z(z^0)$$
$$+\int_{x\geq z^0}\left(\lambda_1 T_1'(x) + \lambda_2\tau_s'(z^0)s(z^0)\right)\left[\frac{\partial z(.)}{\partial T_1}p(x)s(x) + \frac{\partial z(.)}{\partial T_1'(x)}\left(\frac{\partial p}{\partial z}s(x) + p(x)s'_{inc}(x)\right)\right]dH_z(x)$$
$$+\lambda_2\tau_s(z^0)s'_{inc}(z^0)\left[\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\right]h_z(z^0)$$
$$+\int_{x\geq z^0}\lambda_2\tau_s(x)\left[\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)} + \frac{\partial s(.)}{\partial T_1}p(x)s(x) + s'_{inc}(x)\left[\frac{\partial z(.)}{\partial T_1}p(x)s(x) + \frac{\partial z(.)}{\partial T_1'(x)}\left(\frac{\partial p}{\partial z}s(x) + p(x)s'_{inc}(x)\right)\right]\right]dH_z(x)$$

since the reform triggers for individuals at z^0 changes in earnings z equivalent to those induced by a $p(z)\,\Delta\tau_s\,s(z)$ increase in $T_1'(z^0)$, and for individuals above z^0 an increase in tax liability equivalent to a $p(z)\,\Delta\tau_s\,\Delta z\,s(z)$ increase in T_1 and a change in marginal earnings tax rates equivalent to a $\left(\frac{\partial p}{\partial z}\,s(z) + p(z)\,s'_{inc}(z)\right)\Delta\tau_s\,\Delta z$ increase in $T_1'(z)$, in addition to the $\Delta\tau_s\Delta z$ increase in the linear tax rate on s.

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing by λ_1, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for linear earnings-dependent tax rates on s at each earnings z^0

$$-\left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0) + \frac{\lambda_2}{\lambda_1}s'_{inc}(z^0)\tau_s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\,h_z(z^0)$$
$$= \int_{x\geq z^0}\left\{\left(\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right)s(x) + \frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)}\right\}dH_z(x) \quad (210)$$
$$+\int_{x\geq z^0}\left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x) + \frac{\lambda_2}{\lambda_1}s'_{inc}(x)\tau_s(x)\right)\frac{\partial z(.)}{\partial T_1'(x)}p(x)s'_{inc}(x)\,dH_z(x)$$

B.11.6 Deriving Proposition 6

SN tax system. A two-part SN tax system $\{T_1(z), T_2(s)\}$ thus satisfies two optimality conditions: the optimality condition (206) for $T_1'(z)$ and the optimality condition (208) for $T_2'(s)$. Combining these two conditions, we get that at each earnings z^0 the optimal SN tax system satisfies

$$-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} = \frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1}-\hat{g}(x)p(x)+\varphi(x)\right\}dH_z(x)-p(z^0)\frac{s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x) \tag{211}$$

Adding and subtracting $p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x=z^0}^{z_{max}}\Big[1-\hat{g}(x)\Big]dH_z(x)$ yields

$$\begin{aligned}-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} &= p(z^0)\frac{s'(z^0)-s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x) \\ &+\frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1}-\hat{g}(x)p(x)+\varphi(x)\right\}dH_z(x)-p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x).\end{aligned} \tag{212}$$

Defining $\zeta^c_{s|z}(z) = -\frac{1+p\frac{\partial T_2}{\partial s}\big|_{z^0}}{s}\frac{\partial s(.)}{p\,\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)}$ such that $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)} = -\frac{p\,s}{1+p\frac{\partial T_2}{\partial s}\big|_{z^0}}\zeta^c_{s|z}(z)$, we get[49]

$$\begin{aligned}&\frac{\overline{\hat{g}p-\varphi}\frac{\partial T_2}{\partial s}\Big|_{z^0}}{1+p(z^0)\frac{\partial T_2}{\partial s}\Big|_{z^0}} \\ &=\frac{1}{s(z^0)\zeta^c_{s|z}(z^0)}\frac{1}{h_z(z^0)}\left\{s'_{pref}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x)+\frac{s'(z^0)}{p(z^0)}\Big[\Psi(z^0)+\Phi(z^0)\Big]\right\}\end{aligned} \tag{213}$$

[49]With homogeneous p, a SN savings tax levied in period 1 dollar $T_s(s)$ is simply equal to $T_s(s) = pT_2(s)$. As a result, this elasticity definition ensures that $\zeta^c_{s|z}(z)$ coincides with the elasticity concept introduced before:

$$\zeta^c_{s|z}(z) = -\frac{1+T_s'(s)}{s}\frac{\partial s(.)}{\partial T_s'(s)} = -\frac{1+pT_2'(s)}{s}\frac{\partial s(.)}{p\partial T_2'(s)}$$

where we use $\overline{\hat{g}p - \varphi} = \frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}(x)} = 1$ to obtain the additional terms

$$\begin{aligned}
\Psi(z^0) &:= \int_{z \geq z^0} \left[\overline{\hat{g}p} - \hat{g}(z)p(z)\right] dH_z(z) - p(z^0) \int_{z=z^0}^{z_{max}} \left[\overline{\hat{g}} - \hat{g}(z)\right] dH_z(z) \qquad (214)\\
&= \int_{z \geq z^0} \left[\int_{x=z_{min}}^{z_{max}} \hat{g}(x)p(x)dH_z(x) - \hat{g}(z)p(z)\right] dH_z(z) - p(z^0) \int_{z \geq z^0} \left[\int_{x=z_{min}}^{z_{max}} \hat{g}(x)dH_z(x) - \hat{g}(z)\right] dH_z(z)\\
&= \left(1 - H_z(z^0)\right) \int_{x=z_{min}}^{z_{max}} \hat{g}(x)p(x)dH_z(x) - \int_{z \geq z^0} \hat{g}(z)p(z)dH_z(z)\\
&\qquad - p(z^0)\left(1 - H_z(z^0)\right) \int_{x=z_{min}}^{z_{max}} \hat{g}(x)dH_z(x) - p(z^0) \int_{z \geq z^0} \hat{g}(z)dH_z(z)\\
&= \left(1 - H_z(z^0)\right) \int_{x=z_{min}}^{z_{max}} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(x) - \int_{x \geq z^0} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(x)\\
&= \Big(1 - H_z(z^0)\Big) \int_{x \leq z^0} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(z) + H_z(z^0) \int_{x \geq z^0} \hat{g}(x)\Big(p(z^0) - p(x)\Big)dH_z(x)
\end{aligned} \tag{215}$$

$$\Phi(z^0) := \int_{x \geq z^0} \left[\varphi(x) - \overline{\varphi(x)}\right] dH_z(x) \tag{216}$$

which proves the optimal formula for SN tax systems in Proposition 6.

LED tax system. A two-part LED tax system $\{T_1(z), \tau_s(z)s\}$ thus satisfies two optimality conditions: the optimality condition (206) for $T_1'(z)$ and the optimality condition (210) for $\tau_s(z)$. Combining these two conditions, we get that at each earnings z^0 the optimal LED tax system satisfies

$$\begin{aligned}
&p(z^0)s(z^0) \int_{x \geq z^0} \left[1 - \hat{g}(x)\right] dH_z(x)\\
&= \int_{x \geq z^0} \left\{ \Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\Big)s(x) + \frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)} \right\} dH_z(x)\\
&+ \int_{x \geq z^0} \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x) + \frac{\lambda_2}{\lambda_1}s_{inc}'(x)\tau_s(x)\right) \frac{\partial z(.)}{\partial T_1'(x)} p(x)s_{inc}'(x)\, dH_z(x).
\end{aligned}$$

Differentiating with respect to z^0 yields

$$\begin{aligned}
&\Big(p'(z^0)s(z^0) + p(z^0)s'(z^0)\Big) \int_{x \geq z^0} \left[1 - \hat{g}(x)\right] dH_z(x) - p(z^0)s(z^0)\Big[1 - \hat{g}(z^0)\Big] h_z(z^0)\\
&= -\left\{ \Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(z^0)p(z^0) + \varphi(z^0)\Big)s(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} \right\} h_z(z^0)\\
&- \left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0) + \frac{\lambda_2}{\lambda_1}s_{inc}'(z^0)\tau_s(z^0)\right) \frac{\partial z(.)}{\partial T_1'(z^0)} p(z^0)s_{inc}'(z^0)\, h_z(z^0).
\end{aligned}$$

Using optimality condition (206) for $T_1'(z)$, the last term is equal to $p(z^0)s'_{inc}(z^0)\int_{x\geq z^0}[1-\hat{g}(x)]\,dH_z(x)$ at the optimum such that

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}h_z(z^0)\\
&=p(z^0)s'_{pref}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+p'(z^0)s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\\
&+\left\{\frac{\lambda_2}{\lambda_1}-\Big(\hat{g}(z^0)p(z^0)-\varphi(z^0)\Big)-p(z^0)\Big[1-\hat{g}(z^0)\Big]\right\}s(z^0)h_z(z^0).
\end{aligned}
$$

We can now plug in the elasticity $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)}=-\frac{p(z^0)s(z^0)}{1+p(z^0)\frac{\partial T_2}{\partial s}\big|_{z^0}}\zeta^c_{s|z}(z^0)$ with $\frac{\partial T_2}{\partial s}\big|_{z^0}=\tau_s(z^0)$ and use the fact that $\overline{\hat{g}p-\varphi}=\frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}}=1$ to obtain

$$
\begin{aligned}
&\overline{\hat{g}p-\varphi}\frac{\tau_s(z^0)}{1+p(z^0)\tau_s(z^0)} \qquad (217)\\
&=\frac{1}{s(z^0)\zeta^c_{s|z}(z^0)}\frac{1}{h_z(z^0)}\left\{s'_{pref}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+\frac{p'(z^0)}{p(z^0)}s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\right\}\\
&+\frac{1}{p(z^0)}\frac{1}{\zeta^c_{s|z}(z^0)}\left\{\left[\overline{\hat{g}p}-p(z^0)\overline{\hat{g}}\right]-\left[\overline{\varphi}-\varphi(z^0)\right]\right\}
\end{aligned}
$$

which proves the optimal formula for LED tax systems in Proposition 6.

C Details on the Empirical Application

This appendix describes the details underlying the numerical results presented in Section 6. In Section C.1, we describe how we calibrate a baseline two-period, unidimensional model of the U.S. economy, which we use to compute the simple savings tax schedules that are consistent with the prevailing income tax, i.e., that satisfy the Pareto-efficiency formulas in Proposition 2. These are reported in Figure III. In Section C.2, we describe how we extend this exercise to calibrate the optimal simple savings tax systems in the presence of multidimensional heterogeneity as in Proposition 3, assuming that redistributive preferences and other sufficient statistics are the same as in the baseline calibration. In Section C.3, we describe how we instead extend the baseline exercise to allow for heterogeneous rates of return, with an efficiency-based rationale for taxing those with access to high returns, as in Proposition 6. Results for these extensions are reported in Figure IV.Throughout this exercise, we make two assumptions for tractability: we assume that preferences are weakly separable as described in Proposition 1, so that the income effect on savings, $\eta_{s|z}(z)$ can be identified from $s'_{inc}(z)$, and we assume that income effects on labor supply ($\eta_z(z)$) are negligible.

For comparability with the literature on wealth taxation, we express all savings tax rates in terms of "period 2" taxes on gross savings, so that a marginal savings tax rate of 0.1 indicates that if an individual's total wealth at retirement increases by \$1, then they must pay an additional \$0.10 in taxes when they retire.[50]

The LaTeX source code underlying this document—which can be viewed in the accompanying replication files—uses equation labels that match those in the Matlab simulation code.

C.1 Baseline Calibration with Unidimensional Heterogeneity

We first calibrate a simplified version of the U.S. economy with unidimensional heterogeneity. This calibration has two periods, with the first period corresponding to working life and the second to retirement. We assume these periods are separated by 20 years, with a risk-free annual rate of return of 3.8% per year between period 1 and period 2 (see Fagereng et al. (2020), Table 3).

C.1.1 Joint Distribution of Earnings and Savings, and the Status Quo Income Tax

We calibrate the joint distribution of earnings and savings using the Distributional National Accounts micro-files of Piketty et al. (2018), henceforth PSZ. We use individual measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) as well as the age category (20 to 44 years old, 45 to 64, and above 65) and household information. We discretize the income distribution into percentiles by age group, and we partition the top percentile into the top 0.01%, the top 0.1% (excluding the top 0.01%), and the rest of the top percentile. Our measure of annualized earnings during work life z at the n-th percentile is constructed by averaging *plinc* at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. For married households, we use the average earnings of the couple and assign both members of the couple to the same percentile of income. For households with one member above 65 years old, we keep only the younger spouse in the sample. We drop the bottom 2% of observations with non-positive labor income; these individuals have positive average income from other sources, suggesting they are not representative of the zero-ability types which would correspond to $z = 0$ in our model.

Our measure of gross retirement savings per year worked, which we denote s_g in the notation of Appendix A.6, at each labor income quantile is constructed by projecting forward to age 65 the average wealth we observe in the 45 to 65 age category. We project forward by assuming that individuals within each percentile save the same share of post-tax income while young and middle-aged.[51] For married households, we take household wealth to be the average wealth of its members. We then normalize the total wealth at retirement by the number of working years $(65 - 25 = 40)$ so

[50] Notationally, we write this translation as in Appendix A.6, with s_1 and s_g denoting gross savings before taxes, measured in period-1 and period-2 dollars, respectively, and $T_2(s_g, z)$ denoting the savings tax function in period 2. Appendix A.6 demonstrates that the simplicity structure of a tax system (SL, SN, and LED) is preserved when translating between $\mathcal{T}(s, z)$ and $\{T_1(z), T_2(s_g, z)\}$. In the accompanying code replication files, all savings taxes are computed in terms of $\mathcal{T}(s, z)$, but marginal tax rates are converted into $\frac{\partial T_2(s_g,z)}{\partial s_g}$ when plotted in figures.

[51] Specifically, we construct a representative working agent for each income percentile in each age category: a "young" agent of age 35 (in the 20 to 44 age category, where we assume work begins at age 25), and a "middle-aged" agent of age 55 (in the 45 to 64 age category). We assume wealth at middle-aged is the result of the sum of 20 years' worth of savings while young, with returns compounded for an average of $55 - 35 = 20$ years, and 10 years of saving while middle-aged, compounded for an average of 5 years, with a constant share of post-tax income saved in age range.

that z and s_g are in comparable units measured per working year. This yields a monotonic profile of savings across earnings z, and pins down the cross-sectional variation in gross savings $s'_g(z)$.

We convert this discrete distribution of labor income and savings into a smooth distribution with 1000 gridpoints with equal log-spacing, to ensure a smooth marginal tax function that converges to a fixed point when we iterate using the first-order conditions from our propositions. This conversion is performed using the smoothing spline fit in Matlab, with a smoothing parameter of 0.9 and the scale normalization setting set to "on." Measures of savings are noisy at low incomes, which also have outlier values of $\ln(z)$ after the logarithmic transformation used for our savings fit. To avoid having those percentiles generate a strong pull on the fit, we fit the log of savings to $\ln(z+k)$, where a larger k reduces the extent to which the low incomes are outliers. Our baseline uses $k = \$20,000$.

We construct the status quo income tax function by comparing gross income to the PSZ measure *diinc* ("extended disposable income") of post-tax income $z - T_1(z)$. We use the median value within each pre-tax income percentile, constructing a smoothed profile of disposable income y by fitting log *diinc* to log *plinc*, with the same setting described above. In the DINA files, total disposable income *diinc* exceeds total labor income *plinc*, reflecting non-labor factors of production in the economy and the taxes on them. For internal consistency, we apply a lump-sum adjustment so that total y and z are equal, although our results are not sensitive to this adjustment. We then calibrate the smooth marginal income tax rate schedule as $1 - \frac{dy}{dz}$. We treat Social Security as a fixed amount of forced savings, which are added to net-of-tax disposable savings to arrive at our total measure of net savings s. [52]

C.1.2 Status Quo Savings Tax Rates in the United States

We are interested in comparing our results to the profile of status quo effective tax rates on savings in the U.S. Constructing such a schedule presents several difficulties, however. There are many different types of taxes which apply to savings in the U.S., including capital gains taxes (which differ depending on the length of asset ownership), ordinary income taxes, property taxes, and more. Moreover, effective tax rates depend on assumptions about incidence, about which there is substantial disagreement.

We use a simple approach to construct an approximation of the U.S. savings tax based on the composition of savings portfolios across the income distribution. Bricker et al. (2019) use the Survey of Consumer Finances to construct a decomposition of saving types by asset ownership percentile; we summarize the analogous decomposition by income percentile in Figure A1 below. We then construct a savings tax rate at each income level based on the asset-weighted average of the tax rates that apply to each asset class.

For comparison to our results, the savings tax rate of interest is the distortion between work-life consumption and savings. Therefore savings which are subject to labor income taxes, but no

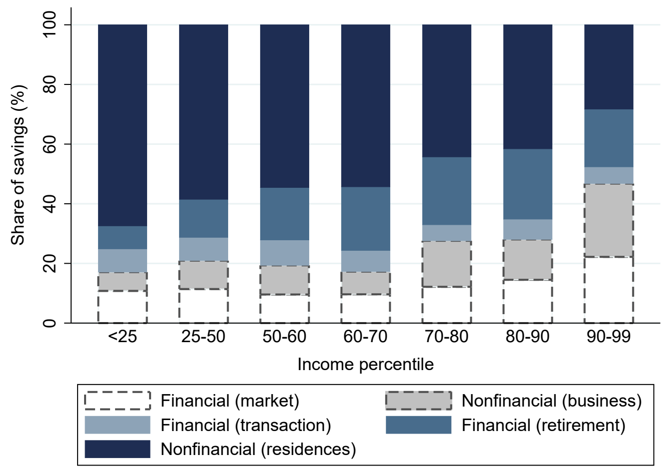
[52] The amount is computed as follows, using the SSA Fact Sheet: Retired workers receive on average \$1,514 per month from social security, meaning $12 * 1,514 = \$18,168$ annually. Through the lens of our two-period model, these benefits are received over an average retirement length of 20 years, and stem from contributions paid over 40 working years. We therefore approximate this as forced savings at the time of retirement of \$9000 per working year.

further taxes, such as a Roth IRA, should be understood as being subject to zero savings tax. We similarly classify traditional IRAs and pension plans as being subject to zero taxes, since they are also subject only to ordinary income taxes. We therefore treat assets in the "Financial (retirement)" category as subject to zero savings tax. We assume "Financial (transaction)" assets, which include checking and savings accounts, represent liquidity needs and similarly do not count toward taxed savings. We view property taxes on "Nonfinancial (residences)" savings as a tax that is incident on renters, and thus a component of imputed rent, which is paid regardless of whether the asset is owned by the user, so we also assume the tax rate on these savings is 0%. Therefore we view only the dotted-outline asset classes "Financial (market)" and "Nonfinancial (business)" as subject to savings taxes, in the form of capital gains. We do not know what share of these holdings represent gains, as opposed to the original contributions. To be conservative, we treat the entire asset classes as though they were subject to capital gains taxes at the time of retirement.

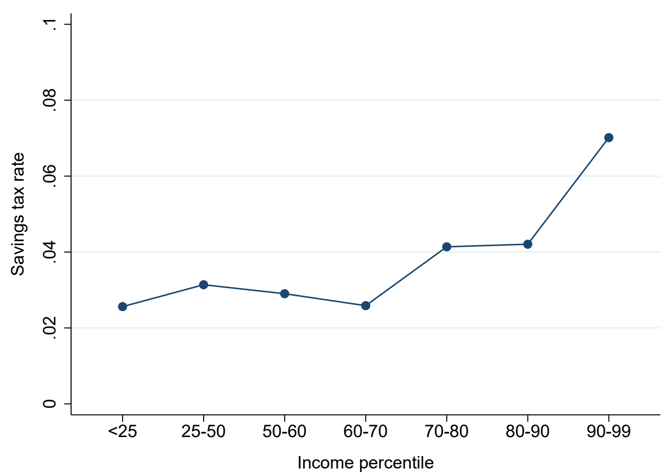
We treat this savings tax rate profile as a schedule of *average* tax rates on one's savings portfolio at each point in the income distribution. We smooth this schedule of average rates using the spline fit procedure described above, and apply that average tax rate to the calibrated level of gross savings at each point in the income distribution to reach a calibrated schedule of total savings taxes paid. We then compute the schedule of marginal rates that would give rise to that nonlinear profile of average tax rates; this schedule is plotted as the "U.S. Status Quo" savings tax, e.g., in Figure III.

Figure A1: Calibration of Savings Tax Rates Across Incomes in the U.S.

(a) Decomposition of Savings Types: Bricker et al. (2019)



(b) Calibrated Savings Tax Rates in the United States, by Income Percentile



Notes: This figure illustrates the calibration of savings tax rates in the U.S. across the income distribution. Panel (a) plots the composition of asset types in individuals' portfolios across the income distribution, reported by Bricker et al. (2019). Panel (b) plots the implied weighted average savings tax rate in each bin. See Appendix C.1.2 for details.

C.1.3 Measures of s'_{inc}

A key input for our sufficient statistics is the marginal propensity to save out of earned income, $s'_{inc}(z) := \frac{\partial s(z)}{\partial z}\big|_{\theta=\theta(z)}$, which relates changes in the amount of net-of-tax savings at the time of retirement to changes in the amount of pre-tax earnings z. We draw from two sources of empirical data to calibrate our marginal propensities to consume (or save), translated into measures of $s'_{inc}(z)$. These results are plotted in Figure II.

Norwegian estimates from Fagereng et al. (2021). Fagereng et al. (2021) estimate marginal propensities to consume (MPC) across the earnings distribution using information on lottery prizes linked with administrative data in Norway. They find that individuals' consumption peaks during the year in which the prize is won, before gradually reverting to their previous consumption level. Over a 5-year horizon, they estimate winners consume close to 90% of the tax-exempt lottery prize; see the "consumption" panels in their Figure 2. This translates into an MPC of 0.9, and thus a marginal propensity to save of 0.1. Under the assumption that preferences are weakly separable with respect to the disutility of labor supply, this is also the marginal propensity to save out of net earned income from labor supply. (See Proposition 1.)

They find little evidence of variation in MPCs across income levels which implies

$$\frac{\partial c\left(z\right)}{\partial\left(z-T_{1}\left(z\right)\right)}=0.9$$

and recognizing that individuals' budget constraint is $s_1(z) = z - T_1\left(z\right) - c(z)$, we get

$$\frac{\partial s_{1}\left(z\right)}{\partial\left(z-T_{1}\left(z\right)\right)}=1-\frac{\partial c\left(z\right)}{\partial\left(z-T_{1}\left(z\right)\right)}=0.1.$$

The identity $s = (s_1 - T_s(s))(1+r)$ implies that $\frac{\partial s}{\partial s_1} = \frac{1}{\frac{1}{1+r}+T_s'(s)}$, and thus that the local causal effect of *pre-tax* income z on *net* savings s satisfies

$$\begin{aligned} s_{inc}'(z) &= \frac{\partial s_{1}\left(z\right)}{\partial\left(z-T_{1}\left(z\right)\right)}\cdot\frac{\partial s}{\partial s_{1}}\cdot\frac{\partial(z-T_{1}(z))}{\partial z} \\ &= 0.1\cdot\frac{1-T_{1}'\left(z\right)}{\frac{1}{1+r}+T_{s}'(s(z))}. \end{aligned} \tag{218}$$

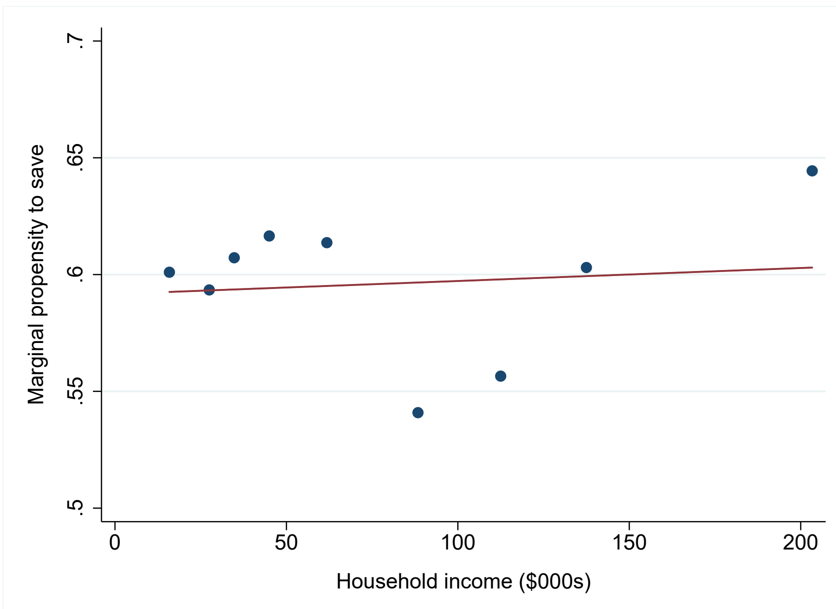
We can then use our calibrated U.S. tax schedule to obtain a profile of $s_{inc}'\left(z\right)$, under the key assumption that U.S. households have similar MPCs as Norwegian households. This profile is plotted in Figure II.

U.S. estimates from a new AmeriSpeak survey. In a survey experiment run on a representative sample of U.S. households, we ask the following question:

> Imagine that you or someone else in your household gets a raise such that over the next five years, your household's income is $1,000 higher each year than what you expected. How much of this would your household spend, and how much would your household save over each of the next five years? (For purposes of this question, consider paying off debt, such as reducing your mortgage, a form of saving.) If no one in your household is going to be employed for most of the next five years, please write "N/A."
>
> Spend an extra $ ______ per year
>
> Save an extra $ ______ per year

Answers to this question provide information about individuals' reported marginal propensity to consume (MPC) and marginal propensity to save (MPS) out of a small and persistent change in *earned* income – in contrast to empirical estimates based on lottery winnings which measure MPC and MPS out of a one-time *windfall* income gain. Since we also collect information on household income in the survey, we can observe marginal propensity to save across earnings levels, plotted in Figure A2. Respondents report on average a marginal propensity to save of 0.6 in the year of the raise. Moreover, marginal propensities to save appear strikingly constant across income levels which is consistent with the results of Fagereng et al. (2021).

Figure A2: Marginal propensity to save across household income (own survey)



Notes: Marginal propensities to save are computed from the answers to our survey question. They are computed as the ratio between the amount respondents report they would save and the amount of the raise.

Since our survey question asks about consumption and spending within each year, we interpret these estimates as short-run responses. Fagereng et al. (2021) show that positive income shocks are followed by consumption responses that can last up to 5 years. We use their impulse-response profile to convert these 1-year MPS into a 5-year MPS, which we interpret as a total effect on savings before returns. To do so, we use the fact that they report a 1-year MPC of 0.52 and a 5-year MPC of 0.90; we therefore compute our long run marginal propensity to save as

$$MPS_{5y} = MPS_{1y} \cdot \frac{1-0.90}{1-0.52} = 0.6 \cdot 0.208 = 0.125.$$

Because our survey question asked about a change in pre-tax income, we do not need to multiply by $1 - T_1'(z)$ as in equation (218); we just divide by $\frac{1}{1+r} + T_s'(s(z))$ to reach our measure of $s_{inc}'(z)$. This results in an estimate somewhat higher than that obtained by Fagereng et al. (2021) for Norway, plotted in Figure II. We use this as the baseline measure of $s_{inc}'(z)$ for our simulations, and the difference between the cross-sectional slope $s'(z)$ and $s'_{inc}(z)$ provides our estimate of the key statistic for preference heterogeneity, $s'_{pref}(z)$, which is also plotted in Figure II.

Comparison to Golosov et al. (2013). Golosov et al. (2013) also study preference heterogeneity, providing a useful point of comparison. In their baseline calibration, they assume individuals preferences are CRRA

$$U(c,s,l) = \frac{\alpha(w)}{1+\alpha(w)} \ln c + \frac{1}{1+\alpha(w)} \ln s - \frac{1}{\sigma}(l)^{\sigma}$$

where l is the labor supply of an individual with hourly wage w such that earnings are given by $z = wl$. The risk aversion parameter is set to $\gamma = 1$, the isoelastic disulity from labor effort is such that $\sigma = 3$, and the taste parameter is given by

$$\alpha(w) = 1.0526\,(w)^{-0.0036}.$$

In other words, the taste parameter varies from 1.0433 for individuals in the bottom quintile of the earnings distribution (mean hourly wage of \$12.35, in 1992 dollars) to 1.0406 for individuals in the top quintile of the earnings distribution (mean hourly wage of \$25.39, in 1992 dollars). This means that this taste parameter is almost constant with income around an average of $\overline{\alpha} = 1.042$.

To illustrate how little preference heterogeneity this implies, we compute the s'_{inc} and s'_{pref} implied by their calibration. Individuals savings choices follow from maximizing $U(c, s, \frac{z}{w})$ subject to the budget constraint $c \leq z - \frac{1}{R}s - \mathcal{T}(s,z)$. This implies

$$s = \frac{z - \mathcal{T}(s,z)}{1/R + \alpha\,(1/R + \mathcal{T}'_s)}$$

such that, neglecting the (potential) curvature of the tax function $\mathcal{T}'' \approx 0$, we can decompose the variation of savings s across earnings z as

$$\underbrace{\frac{ds}{dz}}_{s'(z)} = \underbrace{\frac{1-\mathcal{T}'_z}{1/R + \alpha\,(1/R + \mathcal{T}'_s) + \mathcal{T}'_s}}_{s'_{inc}(z)} + \underbrace{\frac{-(1/R + \mathcal{T}'_s)}{1/R + \alpha\,(1/R + \mathcal{T}'_s) + \mathcal{T}'_s} \frac{d\alpha}{dz} s}_{s'_{pref}(z)}.$$

To obtain an approximation of $s'_{pref}(z)$ in their setting, we use the fact that Golosov et al. (2013) report in their simulation results that individuals with an annual income $z = \$100,000$ have an hourly wage $w = \$40$ while those with an annual income $z = \$150,000$ have an hourly wage $w = \$62.5$. We can thus approximate $\frac{d\alpha}{dz} = \frac{\alpha(62.5)-\alpha(40)}{150{,}000-100{,}000} = \frac{1.0370-1.0387}{50{,}000} = -34 * 10^{-9}$. For $\mathcal{T}'_z$, we assume a linear income tax rate $\tau_z = 0.3$, for $R = 2.1$ we use our real interest rate of 3.8% compounded over 20 years) and for $\mathcal{T}'_s$ we assume a linear income tax rate $\tau_s = 0.01$ which we show below (see (220)) to be consistent with a linear tax of 4% on capital gains (the approximate average in Figure A1b).

This gives a constant $s'_{inc} = \frac{1-0.3}{1/2.1+1.042*(1/2.1+0.01)} = 0.71$, which is much higher than our estimate. Leveraging the fact that s'_{inc} is constant, we can also infer that at an annual income of \$125,000, the annual amount of savings available for consumption in period 2 (including coumpounded interest) is approximately equal to $s = s'_{inc} * \$125,000 = 0.71 * 125,000 = \$88,750$. Thus, $s'_{pref} = \frac{1/2.1+0.02}{1/2.1+1.042*(1/2.1+0.02)+0.02} * (34 * 10^{-9}) * 88,750 = 0.0015$.[53]

These values for s'_{inc} and s'_{pref} imply that in the calibration of Golosov et al. (2013), preference heterogeneity is substantially smaller than our estimate of across-income heterogeneity, as it only explains $\frac{s'_{pref}}{s'_{pref}+s'_{inc}} = \frac{0.0015}{0.71+0.0015} = 0.2\%$ of the variation in savings between individuals earning \$100,000 annually and those earning \$150,000.

C.1.4 Savings elasticity

For purposes of calibration, we assume that the income-conditional compensated elasticity of savings is constant across earnings, $\zeta^c_{s|z}(z) = \bar{\zeta}^c_{s|z}$. We follow Golosov et al. (2013) in drawing on the literature estimating the intertemporal elasticity of substitution (IES), and reporting results for a range of values. To motivate these values, we describe here how we can translate from the IES to a compensated elasticity $\zeta^c_{s|z}$ in the case of a representative agent.

The IES is defined as the elasticity of the growth rate of consumption with respect to the net price of consumption. We assume consumption is smoothed during retirement, so that retirement consumption is proportional to the net stock of savings s, and thus the elasticity of the growth rate of consumption (with respect to a tax change) is the same as the elasticity of the ratio of s to work-life consumption c. We consider a change in the price of retirement consumption induced by a small reform to a SL system like the one described in Table I with a constant linear tax rate τ_s, in which case the net-of-tax price of retirement savings is $\frac{R}{1+R\tau_s}$. (This can be found using the relationship $(s_1 - \tau_s s)R = s$ and solving for $\frac{ds}{ds_1} = -\frac{ds}{dc}$.) We can therefore write

$$\begin{aligned}
IES &= \frac{d\ln(s/c)}{d\ln(\frac{R}{1+R\tau_s})} \\
&= -\frac{d\ln(s/c)}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{d\ln c}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{dc}{d\ln(1+R\tau_s)}\frac{1}{c} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{ds}{d\ln(1+R\tau_s)}\frac{dc}{ds}\frac{1}{c}.
\end{aligned}$$

[53]More specifically, we postulate $s'_{pref} \ll s'_{inc}$ to infer $s(z) = s'_{inc} \cdot z$ and then compute s'_{pref}. Since we obtain a value that verifies $s'_{pref} \ll s'_{inc}$, this reasoning is consistent and proves that $s'_{pref} \ll s'_{inc}$. Put differently, even if we assume $s'_{pref} \approx s'_{inc}$ which implies that $s(z) = 2s'_{inc} \cdot z$, we still obtain $s'_{pref} \ll s'_{inc}$.

Substituting for $\frac{dc}{ds} = \frac{1+R\tau_s}{R}$, we then obtain

$$\begin{aligned}
IES &= -\frac{d\ln s}{d\ln(1+R\tau_s)} - \frac{d\ln s}{d\ln(1+R\tau_s)}\frac{1+R\tau_s}{R}\frac{s}{c} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln s}{d\ln(1+R\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln(1+\tau_s)}{d\ln(1+R\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\left(\frac{d(1+R\tau_s)}{d\tau_s}\right)^{-1}\frac{1+R\tau_s}{1+\tau_s}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
\implies \quad \frac{d\ln s}{d\ln(1+\tau_s)} &= -\frac{IES}{\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}}. \qquad (219)
\end{aligned}$$

Using a value of $s/c = 0.67$ (the population average in our calibrated two-period economy), and using the values $R = 2.1$ (from our real interest rate of 3.8% compounded over 20 years) and $\tau_s = 0.01$ (corresponding to a linear tax of 4% on capital gains, the approximate average in Figure A1b), we find[54]

$$\frac{d\ln s}{d\ln(1+\tau_s)} = -\frac{IES}{0.64}.$$

Treating this as the population estimate of $\frac{d\ln \bar{s}}{d\ln(1+\tau_s)}$, we can then compute the value of the elasticity $\bar{\zeta}^c_{s|z}$ that is consistent with this estimate. From the proof of the optimal SL tax system (see Appendix B.7.1, Equation (120)), the response of aggregate savings $\bar{s}$ to a change in the

[54] A linear tax rate τ^{cg} on capital gains $(R-1)\,s_1$ leads to net savings $s = s_1(1+(R-1)(1-\tau^{cg}))$. Similarly, a period-1 linear tax τ_s on net savings s leads to net savings $s = (s_1 - \tau_s s)\,R \iff s = \frac{s_1 R}{1+\tau_s R}$. As a result,

$$\begin{aligned}
s_1(1+(R-1)(1-\tau^{cg})) &= \frac{s_1 R}{1+\tau_s R} \\
\iff 1+\tau_s R &= \frac{R}{1+(R-1)(1-\tau^{cg})} \\
\iff \tau_s &= \frac{1}{1+(R-1)(1-\tau^{cg})} - \frac{1}{R}. \qquad (220)
\end{aligned}$$

separable linear tax rate τ_s (measured in period-1 dollars, as distinct from $\tau_{s,2}$) is:

$$\frac{d\bar{s}}{d\tau_s} = -\int_z \left\{ \frac{1}{1+\tau_s} \left(s(z)\bar{\zeta}^c_{s|z} + \eta_{s|z}(z)s(z) \right) + \frac{s'_{inc}(z)}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z) \right) \right\} dH_z(z)$$

$$\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{1} = -\bar{\zeta}^c_{s|z}\bar{s} - \int_z \left\{ \eta_{s|z}(z)s(z) + s'_{inc}(z)\frac{1+\tau_s}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z) \right) \right\} dH_z(z)$$

$$\underbrace{\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{\bar{s}}}_{\frac{d\ln\bar{s}}{d\ln(1+\tau_s)}} = -\bar{\zeta}^c_{s|z} - \int_z \left\{ \eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}}\frac{1+\tau_s}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + s(z)\,\eta_z(z) \right) \right\} dH_z(z)$$

$$\bar{\zeta}^c_{s|z} = -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}}\frac{1+\tau_s}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z) \right) \right]$$

This could be computed directly if we had an independent estimate of the income-conditional income effect $\eta_{s|z}$. We instead invoke our assumptions of weak separability and a separable tax system, implying $\eta_{s|z}(z) = s'_{inc}(z)\frac{1+T'_s(s(z))}{1-T'_z(z)}$ (see Proposition 1), and negligible income effects on earnings, to write

$$\begin{aligned}\bar{\zeta}^c_{s|z} &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)}\frac{s'_{inc}(z)}{\bar{s}} \left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z) \right) \right] \\ &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \frac{1}{\bar{s}}\cdot\mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)} s'_{inc}(z) \left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z) \right) \right] \end{aligned} \quad (221)$$

In our calibration, the value of the second term is 0.38, suggesting a translation of $\bar{\zeta}^c_{s|z} \approx IES/0.64 - 0.38$. Thus a value of $IES = 1$, the baseline in Golosov et al. (2013), suggests an elasticity of $\bar{\zeta}^c_{s|z} = 1.2$. We use a baseline value of $\bar{\zeta}^c_{s|z} = 1$. IES Values of 0.5 and 2 (the "low" and "high" values considered in Golosov et al. (2013)) suggest savings elasticities of $\bar{\zeta}^c_{s|z} = 0.4$ and $\bar{\zeta}^c_{s|z} = 2.7$. This is a wide range, and in particular values of savings elasticities below $\bar{\zeta}^c_{s|z} = 0.6$ suggest that consistency with the status quo income tax requires a savings tax that is extreme or non-convergent.[55] We report results for alternative values of $\bar{\zeta}^c_{s|z} = 0.7$, $\bar{\zeta}^c_{s|z} = 2$, and $\bar{\zeta}^c_{s|z} = 3$.

C.2 Simulations of Optimal Savings Taxes with Multidimensional Heterogeneity

We now extend the above calibration to accommodate multidimensional heterogeneity, which we use to apply the formulas derived in Proposition 3. In the multidimensional setting, we do not have Pareto-efficiency formulas like those for unidimensional setting, because in the presence of income-conditional savings heterogeneity, Pareto-improving reforms are not generally available. Therefore, we use the formulas in Proposition 3 to compute the *optimal* schedule of savings tax rates for each type of simple tax system. In order to isolate and illustrate the effects of multidimensional heterogeneity, we hold fixed the sufficient statistics used in the unidimensional setting. We also hold

[55] Intuitively, as the savings elasticity becomes low, one's level of savings becomes a reliable signal of underyling ability, and more of the total redistribution in the tax system should be carried out through the savings tax, rather than the income tax. Thus for sufficiently low $\bar{\zeta}^c_{s|z}$, the status quo income tax cannot be Pareto efficient.

fixed the distributional preferences of the policy maker. The Pareto-efficiency computations above are equivalent to computing the optimal tax under "inverse optimum" welfare weights that would rationalize the status quo income tax. We compute these welfare weights explicitly, as described below, assuming that they vary with earnings, but not with savings conditional on earnings. We then use those inverse optimum weights for the optimal tax calculations.

C.2.1 Inverse Optimum Approach

We assume that income effects on labor supply are negligible, so that $\eta_z \approx 0$, which simplifies the computation of $\hat{g}(z)$ from Equation (22) to

$$\hat{g}(z) = g(z) + \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right)\eta_{s|z}(z). \tag{222}$$

We also assume that preferences are weakly separable so that as shown in Proposition 1, we have

$$\eta_{s|z}(z) = s_{inc}'(z)\,\frac{1+T_s'(s(z))}{1-T_z'(z)}. \tag{223}$$

Because Equations (222) and (223) depend on the marginal savings tax rates $\mathcal{T}_s'$, we must impose an assumption about how they adjust when we recompute the savings tax. We assume that the welfare weights $g(z)$ remain proportional to those calibrated using the inverse optimum procedure described above, but rescaled to preserve the normalization $\int_z \hat{g}(z)dH_z(z) = 1$.[56] In Equation (223), after computing $\eta_{s|z}(z)$ from the baseline calibration of $s_{inc}'(z)$, we assume that $\eta_{s|z}(z)$ remains stable when savings taxes are recomputed.

The inverse optimum computes the social marginal welfare weights (SMWW) consistent with existing tax policy (Bourguignon and Spadaro, 2012; Lockwood and Weinzierl, 2016). This exercise is typically performed using labor income taxes. Our setting presents a complication, as we have both a status quo income tax and savings tax, which need not produce a consistent set of weights. We compute weights assuming that the status quo schedule of earnings tax rates is optimal, for consistency with the Pareto-efficiency formulas above. Since the status quo savings tax rates also appear in this calculation, in principle we must choose whether to use the status quo rates, or those that would counterfactually be optimal. In practice, results are insensitive to this latter issue; for consistency with the "inverse optimum" motivation, we use the Pareto-efficient set of SN tax rates.

Under these assumptions, we can compute the inverse optimum social marginal welfare weights

[56] Specifically, letting $g^0(z)$ denote our baseline welfare weights, we set $g(z) = \kappa g^0(z)$, where

$$\kappa = \frac{1-\int_z \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right)\eta_{s|z}(z)dH_z(z)}{\int_z g^0(z)dH_z(z)}. \tag{224}$$

at each earnings z by inverting the optimal tax rate condition,

$$\frac{T_z'(z)}{1-T_z'(z)} = \frac{1}{\zeta_z^c(z)\, z}\frac{1}{h_z(z)}\int_{x=z}^{z_{max}}(1-\hat{g}(x))\, dH_z(x) - s_{inc}'(z)\frac{T_s'(s(z))}{1-T_z'(z)} \tag{225}$$

$$\iff \int_{x=z}^{z_{max}}(1-\hat{g}(x))\, dH_z(x) = \zeta_z^c(z)\, z\, h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}, \tag{226}$$

where the right-hand side term can be identified from the data. Differentiating with respect to z yields the expression we use to implement this computation numerically,

$$\hat{g}(z) = 1 + \frac{1}{h_z(z)}\cdot\frac{d}{dz}\left[\zeta_z^c(z)\, z\, h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}\right]. \tag{227}$$

Using the fact that augmented social marginal welfare weights are defined as

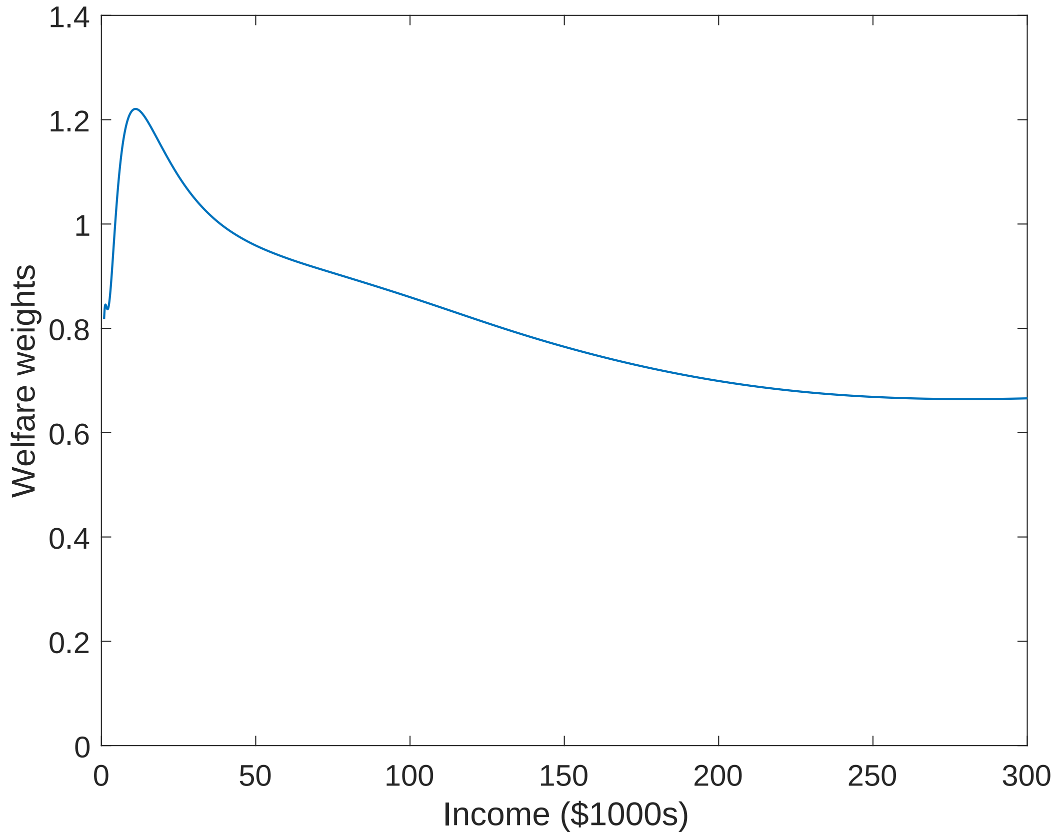
$$\hat{g}(z) := g(z) + T_z'(z)\frac{\eta_z(z)}{1-T_z'(z)} + T_s'(s(z))\left(\frac{\eta_{s|z}(z)}{1+T_s'(s(z))} + s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'(z)}\right), \tag{228}$$

and assuming preferences are weakly separable, such that by Proposition 1 we have $s_{inc}'(z) = \frac{1-T_z'(z)}{1+T_s'(s(z))}\eta_{s|z}(z)$, inverse optimum weights $g(z)$ are obtained from $\hat{g}(z)$ as follows:

$$g(z) = \hat{g}(z) - s_{inc}'(z)\left(\frac{T_s'(s(z))}{1-T_z'(z)}\right). \tag{229}$$

Figure A3 plots our estimated profile of inverse optimum weights.

Figure A3: Schedule of Inverse Optimum Social Welfare Weights in the U.S.



Notes: This figure plots the schedule of inverse optimum welfare weights that would rationalize the U.S. income tax schedule. These weights are computed under the assumption that the savings tax is the Pareto-efficient SN schedule reported in Figure III.

C.2.2 Calibration Details

To extend our calibrated two-period model economy to a multidimensional setting, we retain the same discretized grid of incomes as in the unidimensional case, using the calibration described in Appendix C.1. At each income, we now allow for heterogeneous levels of savings. Specifically, using the same measure of gross savings described in Appendix C.1, we now use a calibration with four different levels of savings at each level of income, each representing a quartile of the income-conditional savings distribution. Across the income distribution, we assume savings within each quartile are a constant ratio of the income-conditional average level of saving. These ratios are 15%, 40%, 70%, and 280% of the income-conditional average savings level; they are calibrated to reflect the average ratios across percentiles 50 to 100 in the PSZ data. We calibrate these ratios excluding the bottom portion of the distribution, because the average level of saving is very low in the bottom half, resulting in noisily measured ratios.

To calibrate the savings income effect $\eta_{s|z}(s, z)$, we assume that the income elasticity of savings is constant within earnings and equal to its unidimensional counterpart, implying that $\eta_{s|z}(s,z) = \frac{s}{\overline{s(z)}}\eta_{s|z}(z)$, where $\overline{s(z)} := \mathbb{E}\left[s\middle|z\right]$ denotes the average savings level at earnings z, and that similarly, $s'_{inc}(s,z) = \frac{s}{\overline{s(z)}}s'_{inc}(z)$. Using these expressions, we can adapt Equation (224)—the scaling factor necessary to ensure that $\hat{g}(s,z)$ integrates to one when recomputing savings taxes—to this setting:

$$\kappa = \frac{1 - \int_z \int_s \left(\frac{\mathcal{T}'_s}{1+\mathcal{T}'_s}\right)\eta_{s|z}(s,z)\,h(s,z)dsdz}{\int_z g^0(z)dH_z(z)}$$

Letting $\mathcal{T}'_s = \tau$ in the SL case and $\mathcal{T}'_s = \tau_s(z)$ in the LED case, we have

$$\kappa = \frac{1 - \int_z \frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\,dH_z(z)}{\int_z g^0(z)dH_z(z)}, \tag{230}$$

and with $\mathcal{T}'_s = T'_s(s)$ in the SN case such that

$$\kappa = \frac{1 - \int_s \frac{T'_s(s)}{1+T'_s(s)} \int_z \eta_{s|z}(s,z)\,h(s,z)dzds}{\int_z g^0(z)dH_z(z)}. \tag{231}$$

C.2.3 Separable linear (SL) tax system

The optimal savings tax formula with multidimensional heterogeneity (Proposition 3) is

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\int_z \left\{\mathbb{E}\left[s\zeta^c_{s|z}(s,z)\middle|z\right]\right\}dH_z(z) \qquad (232)\\
&= \int_z \left\{\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right] - \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z)\,s'_{inc}(s,z)\middle|z\right]\right\}dH_z(z).
\end{aligned}$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$, replacing $s'_{inc}(s,z)$ and $\eta_{s|z}(s,z)$ by their values, and assuming η_z is negligible gives

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\int_z \left\{\overline{s(z)}\zeta^c_{s|z}\right\}dH_z(z) \qquad (233)\\
&= \int_z \left\{\mathbb{E}\left[\left(1 - g(z) - \tau_s\frac{\eta_{s|z}(z)}{1+\tau_s}\frac{s}{\overline{s(z)}}\right)s\middle|z\right] - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[T'_z(z)\frac{s}{\overline{s(z)}} + s'_{inc}(z)\,\tau_s\left(\frac{s}{\overline{s(z)}}\right)^2\middle|z\right]\right\}dH_z(z)
\end{aligned}$$

which after rearranging yields

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z \overline{s(z)}dH_z(z) \qquad (234)\\
&= \int_z \left\{(1-g(z))\,\overline{s(z)} - \frac{\tau_s}{1+\tau_s}\frac{\eta_{s|z}(z)}{\overline{s(z)}}\mathbb{E}\left[s^2\middle|z\right] - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[\frac{T'_z(z)}{\overline{s(z)}}s + \frac{s'_{inc}(z)\,\tau_s}{\overline{s(z)}^2}s^2\middle|z\right]\right\}dH_z(z).
\end{aligned}$$

We can now use $\mathbb{E}\left[s^2\middle|z\right] = \mathbb{E}\left[\left(s - \overline{s(z)}\right)^2 + 2s\overline{s(z)} - \overline{s(z)}^2\middle|z\right] = \mathbb{V}(s|z) + \overline{s(z)}^2$ to obtain

$$\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z \overline{s(z)}dH_z(z) \tag{235}$$
$$= \int_z \left\{\left[1 - g(z) - \frac{\tau_s\eta_{s|z}(z)}{1+\tau_s}\left(1 + \frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}\right)\right]\overline{s(z)} - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\left[T'_z(z) + s'_{inc}(z)\tau_s\left(1 + \frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}\right)\right]\right\}dH_z(z)$$

which we can finally rewrite as

$$\frac{\tau_s}{1+\tau_s}\int_z \overline{s(z)}\zeta^c_{s|z}dH_z(z) \tag{236}$$
$$= \int_z \left\{\left(1 - g(z) - \frac{\tau_s}{1+\tau_s}\eta_{s|z}(z)\right)\overline{s(z)} - \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)}z\zeta^c_z s'_{inc}(z)\right\}dH_z(z)$$
$$- \int_z \left\{\underbrace{\frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}}_{\geq 0}\underbrace{\left(\frac{\tau_s}{1+\tau_s}\eta_{s|z}(z)\overline{s(z)} + \frac{s'_{inc}(z)\tau_s}{1-T'_z(z)}z\zeta^c_z s'_{inc}(z)\right)}_{\geq 0}\right\}dH_z(z).$$

The first two lines correspond to the optimal savings tax formula under unidimensional heterogeneity (Proposition A.3) and the last line captures the effect of multidimensional heterogeneity through $\mathbb{V}(s|z)$. Multidimensional heterogeneity adds a corrective term which is unambiguously negative, it thus prescribes a lower linear savings tax rate.

C.2.4 Separable nonlinear (SN) tax system

At any given savings level s^0, the optimal savings tax formula with multidimensional heterogeneity (Proposition 3) is

$$\frac{T'_s(s^0)}{1+T'_s(s^0)}\int_z \left\{s^0\zeta^c_{s|z}(s^0,z)\right\}h(s^0,z)\,dz = \int_z \left\{\mathbb{E}\left[1 - \hat{g}(s,z)\middle|z, s \geq s^0\right]\right\}h_z(z)\,dz \tag{237}$$
$$- \int_z \left\{\frac{T'_z(z) + s'_{inc}(s^0,z)T'_s(s^0)}{1-T'_z(z)}z\zeta^c_z(s^0,z)s'_{inc}(s^0,z)\right\}h(s^0,z)\,dz.$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$ and assuming η_z is negligible gives

$$\frac{T'_s(s^0)}{1+T'_s(s^0)}s^0\zeta^c_{s|z}\int_z h(s^0,z)\,dz = \int_{s\geq s^0}\left\{\int_z \left[1 - g(z) - T'_s(s)\frac{\eta_{s|z}(s,z)}{1+T'_s(s)}\right]h(s,z)\,dz\right\}ds \tag{238}$$
$$- \int_z \left[\frac{T'_z(z) + s'_{inc}(s^0,z)T'_s(s^0)}{1-T'_z(z)}z\zeta^c_z s'_{inc}(s^0,z)\right]h(s^0,z)\,dz$$

or equivalently, expressing this as a function of the savings density $h_s(s) = \int_z h(s,z)\,dz$,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} s^0 \zeta_{s|z}^c h_s\left(s^0\right) = \int_{s\geq s^0} \left\{\mathbb{E}\left[1-g(z)-T_s'\left(s\right)\frac{\eta_{s|z}\left(s,z\right)}{1+T_s'\left(s\right)}\Big|s\right]\right\} h_s\left(s\right)ds \\ -\mathbb{E}\left[\frac{T_z'\left(z\right)+s_{inc}'\left(s,z\right)T_s'\left(s\right)}{1-T_z'\left(z\right)} z\zeta_z^c s_{inc}'\left(s,z\right)\Big|s=s^0\right] h_s\left(s^0\right) \tag{239}$$

where the expectations operator denotes integration with respect to earnings conditional on savings.

For implementation, we assume that at each point in the income continuum, there are M different equal-sized saver bins (e.g., bottom-, middle-, and top-third savers), indexed by $m = 1, \ldots, M$. Thus we can write $s_m(z)$ as the savings map for saver bin m at each income, with $s_m'(z)$ the cross-sectional savings profile within each saver-bin. Then the income density in each saver-bin is $h_{z,m}(z) = h(z)/M$, since the bins are equally sized conditional on income. The savings density among saver-bin m is therefore $h_{s,m}(s) = h_{z,m}(z)/s_m'(z)$, and we have $H(s) = \sum_{m=1}^{M}\int_{s=0}^{\infty} h_{s,m}(s)ds$, and $h_s(s) = \sum_{m=1}^{M} h_{s,m}(s)$. And the savings-conditional average of some $x(s,z)$ is $\mathbb{E}[x(s,z)|s] = \frac{\sum_{m=1}^{M} x(s_m,z)h_{s,m}(s)}{h_s(s)}$.

To better picture the link with the unidimensional formula (63), let also rewrite the latter as a function of the savings density $h_s\left(s\right)$ – implicitly defining $z\left(s\right)$ as the earnings level of individuals with savings s – this yields

$$\frac{T_s'(s^0)}{1+T_s'(s^0)} s^0 \zeta_{s|z}^c h_s\left(s^0\right) = \int_{s\geq s^0} \left\{1-g\left(z(s)\right)-\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}\left(z\left(s\right)\right)\right\} h_s\left(s\right)ds \\ -\frac{T_z'(z\left(s^0\right))+s_{inc}'(z\left(s^0\right))T_s'(s^0)}{1-T_z'(z\left(s^0\right))} z\left(s^0\right)\zeta_z^c s_{inc}'\left(z(s^0)\right)h_s\left(s^0\right). \tag{240}$$

While it is clear that the multidimensional formula extends the unidimensional formula, determining the impact of multidimensional heterogeneity on tax rates is analytically more difficult and we thus rely on numerical simulations.

C.2.5 Linear earnings dependent (LED) tax system

At earnings z^0, the optimal LED savings tax formula in the presence of multidimensional heterogeneity (Proposition 3) is

$$\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s} z\zeta_z^c(s,z)s\Big|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\Big|z\right]h_z\left(z\right)dz \\ =\int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z\Big]h_z\left(z\right)dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)\Big|z\right]h_z\left(z\right)dz \tag{241}$$

which proves particularly cumbersome to use in numerical simulations, even under the aforementioned assumptions. To obtain an expression that is more easily implementable numerically, we further assume that the earnings tax is optimal (see Proposition A.5) such that

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c(s,z)\Big|z^0\right]h_z\left(z^0\right)=\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz. \tag{242}$$

Now, observing that $s=\overline{s(z^0)}+s-\overline{s(z^0)}$, we can rewrite the first term of the optimal savings tax formula as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s\Big|z^0\right]=\overline{s(z^0)}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\Big|z^0\right] \tag{243}$$
$$+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right].$$

Plugging this back into the optimal savings tax formula and using the optimal earnings tax formula, this implies that

$$\begin{aligned}
&\overline{s(z^0)}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz \qquad (244)\\
&+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\Big|z\right]h_z(z)\,dz\\
&=\int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}(s,z)\right)s\Big|z\Big]h_z(z)\,dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s_{inc}'(s,z)\Big|z\right]h_z(z)\,dz.
\end{aligned}$$

Differentiating with respect to z^0 then yields

$$\begin{aligned}
&\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz-\overline{s(z^0)}\,\mathbb{E}\left[1-\hat{g}(s,z)|z^0\right]h_z\left(z^0\right) \qquad (245)\\
&+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z\left(z^0\right)\right)-\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\Big|z^0\right]h_z\left(z^0\right)\\
&=-\mathbb{E}\Big[\left(1-\hat{g}(s,z)\right)s\Big|z^0\Big]h_z\left(z^0\right)+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s_{inc}'(s,z)\Big|z^0\right]h_z\left(z^0\right).
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
&\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c+\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s_{inc}'(s,z)\Big|z^0\right]h_z\left(z^0\right) \qquad (246)\\
&=\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz-\mathbb{E}\Big[\left(\hat{g}(s,z)\right)\left(s-\overline{s(z^0)}\right)\Big|z^0\Big]h_z\left(z^0\right)\\
&+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z\left(z^0\right)\right).
\end{aligned}$$

Now, with $s'_{inc}(s,z) = \frac{s}{\overline{s(z)}} s'_{inc}(z)$ as well as $\eta_{s|z}(s,z) = \frac{s}{\overline{s(z)}}\eta_{s|z}(z)$ and $\hat{g}(s,z) = g(z) + \frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{\overline{s(z)}}\eta_{s|z}(z)$, we get

$$\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta^c_{s|z} + \frac{T'_z(z) + \tau'_s(z)s + \frac{s}{\overline{s(z)}}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} z\zeta^c_z \frac{s}{\overline{s(z)}} s'_{inc}(z) \Big| z^0\right] h_z\left(z^0\right) \tag{247}$$

$$= \frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-g(z) - \frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{\overline{s(z)}}\eta_{s|z}(z)\Big| z\right]\right\} h_z(z)\,dz$$

$$-\mathbb{E}\left[\left(g(z) + \frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{\overline{s(z)}}\eta_{s|z}(z)\right)\left(s - \overline{s(z^*)}\right)\Big| z^0\right] h_z\left(z^0\right)$$

$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{\overline{s(z)}}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} z\zeta^c_z\left(s - \overline{s(z^0)}\right)\Big| z = z^0\right] h_z\left(z^0\right)\right)$$

which simplifies to the following exact formula

$$\frac{\tau_s(z^0)}{1+\tau_s(z^0)}\overline{s(z^0)}\zeta^c_{s|z} h_z\left(z^0\right) + z^0\zeta^c_z \frac{s'_{inc}(z^0)}{\overline{s(z^0)}}\mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{\overline{s(z)}}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} s\Big| z^0\right] h_z\left(z^0\right) \tag{248}$$

$$= \frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z) - \frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\} h_z(z)\,dz - \frac{\tau_s(z^0)}{1+\tau_s(z^0)}\eta_{s|z}\left(z^0\right)\frac{\mathbb{V}\left[s\big| z^0\right]}{\overline{s(z^0)}} h_z\left(z^0\right)$$

$$+\frac{d}{dz^0}\left(z^0\zeta^c_z\mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{\overline{s(z)}}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}\left(s - \overline{s(z^0)}\right)\Big| z = z^0\right] h_z\left(z^0\right)\right)$$

using $\mathbb{E}\left[s\left(s-\overline{s(z^0)}\right)\Big| z^0\right] = \mathbb{E}\left[s^2\big| z^0\right] - \left(\overline{s(z^0)}\right)^2 = \mathbb{V}\left[s\big| z^0\right]$.

Since the marginal tax rate on earnings $T'_z(z) + \tau'_s(z)s$ features savings s, it is hard to further simplify this formula while retaining an exact characterization. To further simplify this expression, we disregard this dependence by setting $s = \overline{s(z^0)}$ in marginal earnings tax rates. We believe that these formulas are informative in that they converge to exact expressions as the linear earnings dependent savings tax rate tends to a simple linear savings tax rate—that is $\tau'_s(z) = 0$ for all z. Moreover, although these approximations are not unbiased in that they provide an upper bound on the linear-earnings dependent savings tax rate, these upper bounds are tight as the approximation only amounts to assuming $\tau'_s\left(z^0\right)\mathbb{V}\left(s|z^0\right)$ is negligible.

We thus use

$$\mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{\overline{s(z)}}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} s\Big| z^0\right]$$

$$\approx \overline{s(z^0)}\left[\frac{T'_z(z^0) + \tau'_s(z^0)\overline{s(z^0)}}{1-T'_z(z^0)-\tau'_s(z^0)\overline{s(z^0)}} + \frac{s'_{inc}(z^0)\tau_s(z^0)}{1-T'_z(z^0)-\tau'_s(z^0)\overline{s(z^0)}}\left(1 + \frac{\mathbb{V}\left(s|z^0\right)}{\overline{s(z^0)}^2}\right)\right]$$

as well as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+\frac{s}{\overline{s(z)}}s_{inc}'(z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]$$
$$\approx\overline{s\left(z^0\right)}\left[\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}\right]$$

to finally obtain

$$\begin{aligned}
&\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\overline{s\left(z^0\right)}\zeta_{s|z}^c h_z\left(z^0\right)+s_{inc}'\left(z^0\right)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}+s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}z^0\zeta_z^c h_z\left(z^0\right)\\
&=\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\}h_z(z)\,dz \qquad (249)\\
&-z^0\zeta_z^c s_{inc}'\left(z^0\right)\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}h_z\left(z^0\right)-\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\eta_{s|z}\left(z^0\right)\frac{\mathbb{V}\left(s\middle|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\\
&+\frac{d}{dz^0}\left(z^0\zeta_z^c\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\right).
\end{aligned}$$

As an element of comparison, a similar derivation under unidimensional heterogeneity combining the optimal LED savings tax formula (Proposition A.3) and the optimal earnings tax formula (Proposition A.4) yields the following unidimensional analogue

$$\begin{aligned}
&\frac{\tau_s(z^0)}{1+\tau_s(z^0)}s(z^0)\zeta_{s|z}^c(z^0)\,h_z(z^0)+z^0\zeta_z^c(z^0)s_{inc}'(z^0)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)s\left(z^0\right)+s_{inc}'(z^0)\tau_s(z^0)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s\left(z^0\right)}h_z(z^0) \qquad (250)\\
&=s'(z^0)\int_{z\geq z^0}\left(1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right)dH_z(z).
\end{aligned}$$

Multidimensional heterogeneity thus add new terms related to $\mathbb{V}\left(s|z^0\right)$ that naturally wash out under unidimensional heterogeneity. The two terms on the third line of (249) are clearly negative and push for lower savings tax rates in the presence of multidimensional heterogeneity. The term on the fourth line cannot be signed unambiguously. In our calibration, it appears to be negative at low earnings but positive at high earnings. However, its order magnitude is so small (around 10^{-4}) that it does not meaningfully affects the optimal LED savings tax rate and can thus be neglected. As a result, we also get in this case that taking multidimensional heterogeneity into account calls for lower tax rates.

C.3 Simulations of Optimal Savings Taxes with Heterogeneous Returns

For the extension to the case with efficiency arbitrage effects, considered in Section 5.3, we now compute the optimal savings tax rates using the formulas derived in Proposition 6, again using the same set of inverse optimum welfare weights derived above.

These results are reported in the bottom two panels of Figure IV, which display schedules of LED and SN savings tax rates computed under the assumption that (i) individuals with different income levels differ in their private rates of return, and that (ii) the savings tax is levied in period-2 dollars. We compute the tax schedules that satisfy the equations for the optimal tax conditions in Proposition 6. As in the case of multidimensional heterogeneity, we hold fixed the schedule of marginal social welfare weights $g(z)$ proportional to those which rationalize the status quo income tax in our baseline inverse optimum calculation. Building on the findings of Fagereng et al. (2020), we follow Gerritsen et al. (2020) in assuming that rates of return rise by 1.4% from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return.

Maintaining our assumptions of negligible labor supply income effects and weakly separable preferences, Equation (201) simplifies to

$$\hat{g}(x) := g(x) + \frac{\lambda_2}{\lambda_1} \frac{T_2'(s)}{1 + pT_2'(s)} \eta_{s|z}(z) \tag{251}$$

for an SN system. To ensure that $\hat{g}(z)$ still integrates to one, the rescaling factor in Equation (224) now becomes

$$\kappa = \frac{1 - \int_z \left(\frac{\lambda_2}{\lambda_1} \frac{T_2'(s)}{1+pT_2'(s)} \right) \eta_{s|z}(z) dH_z(z)}{\int_z g^0(z) dH_z(z)}. \tag{252}$$

Similarly, Equation (202) simplifies to

$$\varphi(x) = -\left(T_1'(x) + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} T_2'(s) \right) \left(\zeta_z^c(x) \frac{x}{1 - T_1'(x)} \right) \frac{\partial p}{\partial z}. \tag{253}$$

For an LED system we can replace $T_2'(s)$ with $\tau_s(z)$ in the previous formulas.

D Details of Tax Systems by Country

In Table II, we consider five categories of savings subject to various taxation regimes in different countries: (i) wealth, (ii) capital gains, (iii) property, (iv) pensions, and (v) inheritance, which are typically defined in tax codes as follows. First, wealth, which is free from taxation in most advanced economies, is defined as the aggregate value of certain classes of assets, such as real estate, stocks, and bank deposits. Next, capital gains consist of realized gains from financial and real estate investments, and include interest and dividend payments. Third, property consists of real estate holdings, such as land, private residences, and commercial properties. Fourth, for our purposes, pensions are defined as private retirement savings in dedicated accounts, excluding government transfers to retired individuals, such as Social Security in the United States. Lastly, inheritances—also known as estates—are the collections of assets bequeathed by deceased individuals to living individuals, often relatives.

For each country, we label the tax system applied to each category of savings with the types described in Table I or "Other," which encompasses all other tax systems. An additional common simple tax structure is a "composite" tax, in which savings and labor income are not distinguished for the purposes of taxation. Composite taxes are often applied to classes of income for which it is unclear whether the income should be considered capital income or labor income. For example, in a majority of the countries in Table II, rental income—which requires some active participation from the recipient of the income—is subject to composite taxation.

In the subsections below, we have included additional details about the tax system in each country in Table II. Note that we characterize tax systems that feature a flat tax on savings above an exempt amount as having a separable nonlinear tax system. In addition, when benefits are withdrawn from pension accounts, they are often subject to the same progressive tax rates as labor income. We characterize these tax systems as separable nonlinear rather than composite since benefits are generally received after retirement from the labor force when the taxpayer's income is primarily composed of savings.

Australia

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a composite tax applies. Gains from certain assets are exempt or discounted.
- **Property:** At the state level, land tax rates are progressive; primary residence land is typically exempt. At the local level, generally flat taxes are assessed on property but the taxes can be nonlinear as well, depending on the locality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account. A component of pension benefits may be subject to taxation when withdrawn, in which case the lesser of a flax tax or the same progressive tax rates as apply to labor income is assessed.
- **Inheritance:** No inheritance tax.

Austria

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed, with the rate depending on the type of asse; taxpayers with lower labor income can opt to apply their labor income tax rate instead. Gains from certain classes of assets are exempt.
- **Property:** Either flat or progressive tax rates are assessed on property, depending on its intended use. Rates vary by municipality.

- **Pensions:** Generally no tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, with discounts applicable to certain types of withdrawals.
- **Inheritance:** No inheritance tax.

Canada

- **Wealth:** No wealth tax.
- **Capital gains:** For most capital gains, a discount is first applied to the gain and then the discounted gain is added to labor income and taxed progressively. For certain gains, such as interest income, no discount is applied. Lifetime exemptions up to a limit apply to gains from certain classes of assets.
- **Property:** Generally a flat tax is assessed on property, with rates varying by province and locality.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, with exemptions applicable to certain types of withdrawals.
- **Inheritance:** No separate inheritance tax. A final year tax return is prepared for the deceased, including income for that year, that treats all assets as if they have just been sold and applies the relevant taxes (e.g., labor income and capital gains taxes) accordingly.

Denmark

- **Wealth:** No wealth tax.
- **Capital gains:** Progressive taxation with two tax brackets. Gains from certain classes of assets are exempt.
- **Property:** At the national level, property is subject to progressive taxation with two tax brackets. Pensioners under an income threshold can receive tax relief. Land taxes—assessed at the local level—are flat taxes, with rates varying by municipality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income (excluding a labor market surtax), a flat tax, or are exempt from taxation, depending on the type of pension.
- **Inheritance:** Generally a flat tax is assessed on the inheritance above an exemption, with a higher tax rate for more distant relatives. Transfers to spouses and charities are exempt. Inheritances above a certain value are subject to additional taxes.

France

- **Wealth:** No wealth tax.
- **Capital gains:** Different rates—progressive and flat—apply to gains from different classes of assets. Certain low-income individuals are either exempt from taxes or can opt to apply their labor income tax rate, depending on the type of asset. High-income individuals are subject to a surtax. Gains from certain assets are exempt or discounted.
- **Property:** Residence taxes are assessed on property users, while property taxes on developed and undeveloped properties are assessed on owners. Rates are set at the local level and apply to the estimated rental value of the property. Exemptions, reductions, and surcharges may apply depending on the taxpayer's reference income and household composition, certain events, and property characteristics. Surcharges may also apply to higher-value properties. An additional property wealth tax applies at the national level; rates are progressive above an exemption.
- **Pensions:** Generally no tax on capital gains made within the pension account. Pension benefits beyond an exemption are generally subject to the same progressive tax rates as labor income. A flat tax is assessed on certain types of withdrawals, and special rules apply to certain types of accounts.
- **Inheritance:** Either a flat tax or progressive tax rates are assessed on the inheritance above an exemption, with rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Transfers to spouses/civil partners are exempt. Certain shares are required to pass to the deceased's children.

Germany

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on gains above an exemption, but taxpayers with lower labor income can opt to apply their labor income tax rate instead. Gains from certain classes of assets are exempt or subject to special rules.
- **Property:** A flat tax is assessed on property, with rates depending on the class of property and subject to a multiplier, which varies by locality.
- **Pensions:** No tax on capital gains made within the pension account. A portion of pension benefits, which depends on the type of account, is subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions both depending on the relation of the recipient to the deceased. Pension entitlements are exempt.

Ireland

- **Wealth:** No wealth tax.
- **Capital gains:** A flat tax is assessed on gains above an exemption, with the rate depending on the type of asset. Certain classes of individuals, such as farmers and entrepreneurs, qualify for lower rates and additional exemptions.
- **Property:** Progressive tax rates are assessed on residential properties, with local authorities able to vary the rates to a certain extent. A flat tax is assessed on commercial properties, with rates varying by locality.
- **Pensions:** No tax on capital gains made within the pension account. Depending on the type of withdrawal, pension benefits are either subject to the same progressive tax rates as labor income or different progressive tax rates beyond an exemption. A surtax is assessed on high-value accounts.
- **Inheritance:** A flat tax is assessed on inheritances above an exemption. Exemptions are associated with the recipient and apply to the sum of all inheritances bequeathed to the recipient from certain classes of relatives.

Israel

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on real gains (i.e., the inflationary component of gains is exempt). High-income individuals are subject to a surtax.
- **Property:** Generally the tax increases in the area of the property, with amounts depending on property characteristics and varying by municipality. Tax relief may apply to certain taxpayers, such as new immigrants and low-income individuals, depending on the municipality.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income; certain taxpayers qualify for exemptions.
- **Inheritance:** No inheritance tax.

Italy

- **Wealth:** A flat tax is assessed on bank deposits and financial investments held abroad, with exemptions on bank deposits if the average annual account balance is below a certain threshold.

- **Capital gains:** Generally a flat tax is assessed on financial capital gains. For certain real estate capital gains, individuals can choose between separable or composite taxation, either applying a flat tax or their labor income tax rate.
- **Property:** Generally a flat tax is assessed on property, with rates depending on property characteristics and varying by municipality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account, with the rate depending on the type of asset. Pension benefits are also subject to flat taxes, with rates varying with the duration of the contribution period.
- **Inheritance:** A flat tax is assessed on inheritances, with higher rates for more distant relatives. Different amounts of the inheritance are exempt from taxation for certain close relatives.

Japan

- **Wealth:** No wealth tax.
- **Capital gains:** A flat tax is assessed on gains from certain classes of assets, such as securities and real estate, with the rate depending on the type of asset. Progressive tax rates, composite taxation, exemptions, and discounts apply to gains from different classes of assets.
- **Property:** A flat tax is assessed on property above an exemption, with a lower rate or reduction applicable to certain types of property.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to progressive tax rates, with the rates depending on the type of withdrawal.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above a general exemption and an exemption that depends on the relation of the recipient to the deceased and their disability status. A surtax applies to more distant relatives. Certain shares are required to pass to certain relatives.

Netherlands

- **Wealth:** A progressive, fictitious estimated return from net assets not intended for daily use is taxed at a flat rate depending on the amount above the exemption.
- **Capital gains:** Gains from a company in which an individual has a substantial stake are subject to a flat tax. Most other capital gains are not subject to taxation.
- **Property:** At the municipal level, a flat tax is assessed on property, with rates depending on property characteristics and varying by municipality. At the national level, progressive tax rates are assessed on the fictitious estimated rental values of primary residences, with

substantial deductions applicable to the portion of the tax exceeding the mortgage interest deduction.

- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, though certain accounts with taxed contributions allow tax-free withdrawals.

- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Additional exemptions apply to certain classes of assets.

New Zealand

- **Wealth:** No wealth tax.

- **Capital gains:** Capital gains from financial assets are generally either subject to composite taxation or are exempt from taxation, depending on the type of gain. Special rules apply to certain classes of assets. Capital gains from real estate are generally subject to composite taxation. Depending on transaction characteristics, gains from the sale of commercial property may be subject to an additional tax, while gains from the sale of residential property may be exempt from taxation.

- **Property:** Generally a fixed fee plus a flat tax is assessed on property, with rates set at the municipal level. Low-income individuals qualify for rebates for owner-occupied residential property.

- **Pensions:** A flat tax is assessed on capital gains made within the pension account, with the rate depending on the type of account; for certain accounts, the rate depends on the taxpayer's labor income in prior years. Pension benefits are generally exempt from taxation.

- **Inheritance:** No inheritance tax.

Norway

- **Wealth:** A flat tax is assessed on wealth above an exemption, with the value of certain classes of assets, such as primary and secondary residences, discounted.

- **Capital gains:** A flat tax is assessed on gains from financial assets above the "risk-free" return (i.e., the counterfactual return on treasury bills of the same value). Gains from certain financial assets, such as dividends, are multiplied by a factor before the tax is assessed. A flat tax is assessed on real estate gains, with exemptions for certain types of property.

- **Property:** A flat tax is assessed on discounted property values, with rates varying by municipality and discounts varying by property type.

- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to a lower tax rate than labor income, and taxpayers with smaller benefits qualify for larger tax deductions.
- **Inheritance:** No inheritance tax.

Portugal

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on gains from financial assets, but for certain types of gains, such as interest, low-income individuals can opt to apply their labor income tax rate. For real estate capital gains, a discount is first applied to the gain and then the discounted gain is added to labor income and taxed progressively. Certain classes of real estate are exempt.
- **Property:** Progressive tax rates are assessed on property, with exemptions for certain taxpayers. Rates and exemptions vary based on property characteristics, and an additional exemption applies to low-income individuals.
- **Pensions:** No tax on capital gains made within the pension account, except for dividends, which are generally subject to a flat tax. For different types of withdrawals above an exemption, capital gains are either subject to a flat tax or the same progressive tax rates as labor income when withdrawn. Depending on how contributions were initially taxed and the type of withdrawal, the non-capital gains component of benefits is exempt from taxation, or subject to a flat tax or the same progressive tax rates as labor income on the amount above an exemption.
- **Inheritance:** A flat tax is assessed on the inheritance, with a higher rate for real estate transfers. Transfers to spouses/civil partners, ascendants, and descendants are exempt (except for real estate transfers, which are subject to a low flat tax).

Singapore

- **Wealth:** No wealth tax.
- **Capital gains:** Most capital gains are not subject to taxation. Depending on transaction characteristics, composite taxation may apply.
- **Property:** Progressive tax rates are assessed on the estimated rental value of the property, with rates varying by property type and occupancy status.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income; benefits from contributions made before a certain year are exempt from taxation.

- **Inheritance:** No inheritance tax.

South Korea

- **Wealth:** No wealth tax.
- **Capital gains:** Various flat and progressive tax rates are assessed on gains above an exemption; rates and exemptions depend on the type of asset. Gains from certain classes of assets are entirely exempt. Dividends and interest are subject to flat taxation below a certain limit and composite taxation above that limit.
- **Property:** Progressive tax rates are assessed on property, with rates varying by property type.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond a progressive exemption (i.e., greater portions are exempt at smaller benefit levels) are generally subject to the same progressive tax rates as labor income; the exempt amount may also depend on the type of withdrawal and taxpayer characteristics.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above either a lump-sum or itemized deduction, which depends on the composition of the inheritance and relation of the recipient to the deceased. Transfers to spouses are exempt. The top tax rate increases for controlling shares in a company.

Spain

- **Wealth:** Progressive tax rates are assessed on net assets above an exemption, with an additional exemption for residences.
- **Capital gains:** Progressive tax rates are generally assessed on gains, with exemptions for elderly individuals under certain conditions and for certain real estate gains.
- **Property:** Generally a flat tax is assessed on property, with rates depending on the property type and varying by locality. Exemptions or discounts may apply depending on taxpayer and property characteristics, including taxpayer income.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Certain classes of assets, such as family businesses and art collections, are eligible for additional exemptions.

Switzerland

- **Wealth:** A flat tax is assessed on the net value of certain classes of assets and liabilities, with tax rates and exemptions varying by canton.
- **Capital gains:** Progressive tax rates are assessed on gains from real estate, with rates varying by canton. Most capital gains from financial assets are not subject to taxation. Dividends and interest are subject to composite taxation.
- **Property:** Generally a flat tax is imposed on property, with rates varying by canton; a minimum amount per property may apply. For owner-occupied properties not rented out, an estimated rental value is subject to composite taxation.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are subject to either the same progressive tax rates as labor income or lower progressive tax rates, depending on the type of withdrawal.
- **Inheritance:** In most cantons, progressive tax rates are assessed on the inheritance and depend on the relation of the recipient to the deceased. Transfers to spouses and children are exempt in most cantons.

Taiwan

- **Wealth:** No wealth tax.
- **Capital gains:** Most capital gains from financial assets are subject to composite taxation; taxpayers can opt for a flat tax to be assessed on dividends, and certain gains are exempt from taxation. A flat tax is assessed on gains from real estate, with the rate depending on the type of asset, and an exemption for primary residences.
- **Property:** Flat or progressive tax rates are assessed on land, depending on its intended use. A flat tax is generally assessed on buildings, with rates depending on their intended use. Certain classes of land and buildings are exempt or subject to reduced rates.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond an exemption—which depends on the duration of the contribution period—are subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, which depends on the relation of the recipient to the deceased, their disability status, and their age.

United Kingdom

- **Wealth:** No wealth tax.

- **Capital gains:** Either flat or progressive tax rates are assessed on gains, with rates depending on the taxpayer's labor income tax bracket; higher rates generally apply to taxpayers in higher labor income tax brackets. Exemptions for part or all of the gain apply to certain types of assets, such as dividends and primary residences.
- **Property:** Progressive tax rates are assessed on property, with rates varying by locality. Exemptions or discounts may apply to certain taxpayers depending on characteristics, such as age.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond an exemption are subject to the same progressive tax rates as labor income. An additional flat tax may be imposed on accounts with a value exceeding a lifetime limit, with the tax rate depending on the type of withdrawal.
- **Inheritance:** A flat tax is assessed on the inheritance above an exemption, with larger exemptions for transfers to children. Transfers to spouses/civil partners, charities, and amateur sports clubs are exempt. The tax rate is reduced if a certain share is transferred to charity.

United States

- **Wealth:** No wealth tax.
- **Capital gains:** Gains from "short-term" assets (held for less than a year) are subject to composite taxation. Gains from "long-term" assets are subject to a flat tax, with higher rates for higher-income individuals. Dividends are also subject to either composite taxation or flat taxes that increase with labor income, depending on their source.
- **Property:** Generally a flat tax is assessed on property, with rates varying by state, county, and municipality.
- **Pensions:** No tax on capital gains made within the pension account. Depending on the type of account, benefits are generally either exempt from taxation or subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption. Transfers to spouses are generally exempt.