SUFFICIENT STATISTICS FOR NONLINEAR TAX SYSTEMS WITH GENERAL ACROSS-INCOME HETEROGENEITY

Antoine Ferey, Benjamin B. Lockwood, and Dmitry Taubinsky*

April 2022

Abstract

This paper provides general and empirically implementable sufficient statistics formulas for optimal nonlinear tax systems in the presence of across-income heterogeneity in preferences, inheritances, income-shifting capabilities, and other sources. We study unrestricted tax systems on income and savings (or other commodities) that implement the optimal direct-revelation mechanism, as well as simpler tax systems that impose common restrictions like separability between earnings and savings taxes. We characterize the optimum using familiar elasticity concepts and a sufficient statistic for general across-income heterogeneity: the difference between the cross-sectional variation of savings with income, and the causal effect of income on savings. The Atkinson-Stiglitz Theorem is a knife-edge case corresponding to zero difference, and a number of other key results in optimal tax theory are subsumed as special cases. We provide tractable extensions of these results that include multidimensional heterogeneity, additional efficiency rationales for taxing heterogeneous returns, and corrective motives to encourage more saving. Applying these formulas in a calibrated model

*We are grateful to Afras Sial for excellent research assistance. This project was supported through a Quartet Pilot Research award and was funded by the Center for Health Incentives and Behavioral Economics at the University of Pennsylvania, by The Research Council of Norway 315765, and by the Alfred P. Sloan Foundation. Ferey gratefully acknowledges the financial support of Labex ECODEC at CREST, and of the Deutsche Forschungsgemeinschaft through CRC TRR 190 at LMU Munich. The content is solely the responsibility of the authors and does not necessarily represent the official views of the University of Pennsylvania. We thank Pierre Boyer, Philippe Choné, Ashley Craig, Nathan Hendren, Laurence Jacquet, Claus Kreiner, Dirk Krueger, Etienne Lehmann, Jonas Loebbing, Jean-Baptiste Michau, Andreas Peichl, Dominik Sachs, Emmanuel Saez, Florian Scheuer, Stefanie Stantcheva, Joseph Stiglitz, Aleh Tsyvinski, Nicolas Werquin, Eric Zwick, participants at NBER Macro Public Finance, NBER Public Economics, IIPF, LAGV, NTA, and audiences at UC Berkeley, CREST-Ecole Polytechnique, University of Michigan, and LMU Munich for their valuable comments. Ferey: LMU Munich & CESIFO (antoine.ferey@econ.lmu.de). Lockwood: Wharton & NBER (ben.lockwood@wharton.upenn.edu). Taubinsky: UC Berkeley & NBER (dmitry.taubinsky@berkeley.edu). Supplementary material available at `https://benlockwood.com/papers/FLT_Supplement.pdf`.

of the U.S. economy, we find that the optimal savings tax is positive and progressive.
JEL Codes: D61, H21, H24

1 Introduction

Taxes on capital income, estates, inheritances, and certain categories of consumption are a widespread feature of modern tax systems. Yet there is considerable debate, both among economists and in policy circles, about their optimal design. The celebrated theorem of Atkinson and Stiglitz (1976) is sometimes interpreted to suggest that such taxes should be eliminated. The theorem states that if preferences are homogeneous and weakly separable in consumption and labor, then differential taxes on commodities—including on future consumption in the form of savings—are suboptimal, and welfare is maximized when redistribution is carried out solely through an income tax. However, as was appreciated by contemporaneous work (Mirrlees, 1976) and emphasized by the authors themselves (Stiglitz, 2018), the assumptions underpinning the Atkinson-Stiglitz Theorem are strong, and the theorem does not apply in settings where earnings ability covaries with preferences for commodities or with other attributes such as heterogeneous inheritances, rates of return, or income-shifting abilities.

As a result, an active literature has developed to demonstrate that non-zero commodity and savings taxes may be optimal when the Atkinson-Stiglitz assumptions are relaxed. Yet general, elasticity-based "sufficient statistics" formulas for optimal nonlinear commodity and savings taxes, of the kind common in the modern optimal income tax literature (e.g., Saez, 2001; Rothschild and Scheuer, 2013; Sachs, Tsyvinski, and Werquin, 2020; Hendren, 2020; Bierbrauer, Boyer, and Hansen, 2020), have remained elusive. Existing results have instead studied settings with restrictions to a small number of discrete "types" or restrictions on the form of utility functions or tax functions, or they have focused on qualitative insights.[1]

In this paper, we derive sufficient statistics formulas for optimal linear and nonlinear commodity taxes in a general setting where attributes other than ability—such as preferences, inheritances, or rates of return—vary across the income distribution. We study a general version of standard models where consumers with heterogeneous earning abilities and tastes choose labor supply and a consumption and savings bundle that exhausts their after-tax income.[2] Our formulas nest prior results in this setting, as well as the Atkinson-Stiglitz Theorem itself, as special cases. For concreteness in what follows, we describe

[1]Of particular note, Saez (2002) used a model like the one in this paper to answer the qualitative question of when a "small" *linear* commodity (savings) tax can increase welfare in the presence of preference heterogeneity, but left to future work the task of deriving an expression for the optimal tax, writing "It would of course be extremely useful to obtain optimal commodity tax formulas" in such a framework.

[2]See, e.g., Atkinson and Stiglitz (1976); Saez (2002); Farhi and Werning (2010); Diamond and Spinnewijn (2011); Golosov et al. (2013); Piketty and Saez (2013); Scheuer and Wolitzky (2016); Saez and Stantcheva (2018); Allcott et al. (2019); Gaubert et al. (2021)

results in terms of taxes on savings, although they also apply to other commodities.

We organize the paper around the following key contributions.

The first is a set of results about the optimal unrestricted, nonlinear tax system on earnings and savings. We begin with the question of implementation: Can the optimal allocation be implemented by a smooth (i.e., differentiable) tax on earnings and savings? Unlike an optimal mechanism, a smooth tax system cannot disallow *double deviations*, where individuals can jointly alter their earnings and savings to reach bundles not chosen by any other type. The broader forms of across-income heterogeneity we consider thus introduce a complication not present in the standard income taxation model of Mirrlees (1971), nor in Atkinson and Stiglitz (1976), as double deviations are generally not attractive to individuals in these settings. Nevertheless, we show that under an extended Spence-Mirrlees condition and under modest regularity conditions, it is possible to construct a smooth tax system that implements the optimal direct-revelation mechanism.

We then present new elasticity-based formulas for the optimal nonlinear tax on savings and earnings. We show that these formulas can be written entirely in terms of welfare weights and empirically measurable statistics, including a key sufficient statistic for across-income heterogeneity justifying taxes on s: the difference between the cross-sectional variation of savings s with earnings z, denoted $s'(z)$, and the causal effect of income changes on savings, which we denote $s'_{inc}(z)$. The residual, $s'_{het}(z) := s'(z) - s'_{inc}(z)$, is a sufficient statistic for (local) across-income heterogeneity.[3] Intuitively, the total derivative of s with respect to z is the sum of two partial derivatives: (i) the causal income effect s'_{inc}, holding all else constant and (ii) the degree to which higher-ability types prefer, or are able to obtain, more s, holding earnings constant. The second component is captured by s'_{het}, which we show can be estimated from existing data on the correlational and causal associations with earnings, avoiding the need to explicitly measure or model the relationship between earnings ability and unobserved factors, like preferences.

The formula for optimal savings tax rates is a product of s'_{het} and a term that resembles the optimal income tax formula in Saez (2001), with the elasticity of earnings replaced by the elasticity of savings with respect to the savings tax rate. This result provides an immediate generalization of the Atkinson-Stiglitz Theorem, as it implies that the optimal savings tax rate is everywhere zero when $s'_{het}(z) = 0$ for all earnings levels z. We also present Pareto-efficiency conditions that use the same sufficient statistics and that can be used to test for (and address) inefficiencies in existing tax systems. These Pareto efficiency conditions inform whether there is a set of joint perturbations to the income and savings tax that decrease distortions from taxation without decreasing revenue, such that all individuals

[3] To our knowledge, this statistic was first employed in Allcott et al. (2019), in a setting restricted to a separable linear commodity tax, which of course cannot implement the optimal mechanism.

are at least weakly better off.

We show that these formulas apply in a variety of settings that depart from the Atkinson-Stiglitz assumptions, including heterogeneous endowments or inheritances, differential rates of return on investments, human capital investments that enhance productivity, deviations from weak separability, and the ability to engage in income shifting. In each case, the difference between the cross-sectional profile of savings and the causal income effect on savings, $s'(z) - s'_{inc}(z)$, is the key sufficient statistic for across-income heterogeneity. Consequently, these formulas can be viewed both as a synthesis of prior work that qualitatively studied these extensions in isolation, and as a method for quantifying optimal tax rates when several of these forces are at play simultaneously.

Our second contribution is a characterization of what we call "simple tax systems." Across a large number of countries, the tax system consists of a nonlinear tax on income, accompanied by taxes on savings vehicles that can be classified as one of three types: (i) a separable linear (SL) savings tax; (ii) a separable nonlinear (SN) savings tax; or (iii) a system with a linear earnings-dependent (LED) savings tax, which allows, for example, lower-income people to have their savings taxed at a lower linear rate, as is the case for long-term capital gains in the U.S. We show that the optimal tax policy within each of these classes of simple systems can be expressed using the same sufficient statistics that appear in our formulas for the optimal smooth unrestricted tax system. We also provide sufficient conditions for the SN and LED systems to implement the optimal mechanism. Importantly, when focusing on simple tax systems, we extend our results to multidimensional heterogeneity and to a potentially suboptimal income tax. In these cases, the causal effect of income on savings, together with the joint distribution of savings and income, remain sufficient statistics for characterizing the optimal savings tax.

We generalize our results to several other key applications. First, we consider many dimensions of consumption and savings. For example, different categories of savings might be taxed differently. In this case, the additional necessary sufficient statistics are cross-price elasticities, which allow us to compute *tax diversion ratios*—the fiscal spillovers to taxes collected on goods $j \neq i$ relative to the reduction in taxes collected on good i, when the price of good i is increased. The optimal tax rate on good s_i is the sum of the formula in our baseline result and the tax diversion ratios.

Second, we consider situations where the government wants to alter or correct individual behavior. Our model generalizes the setup of Farhi and Werning (2010), in which the government puts more weight on future generations than the parents, to allow for heterogeneous preferences. Our results also cover the case where individuals under-save due to behavioral biases such as myopia or lack of self control, as in Moser and Olea de Souza e Silva (2019).

Third, we study settings in which there is an additional efficiency rationale for taxing savings, because the government can collect savings taxes either before or after returns are earned, and therefore can arbitrage heterogeneous private rates of return by shifting tax collections onto post-returns savings for high earners. This extension relates to independent work by Gerritsen et al. (2020), who study the special case where all across-income heterogeneity is from differences in rates of return, characterizing and quantifying the optimal separable nonlinear savings tax in terms of model primitives.

In the final part of the paper, we apply our sufficient statistics formulas to study the optimal tax treatment of savings in the U.S. We calibrate the distribution of savings across the income distribution using the Distributional National Accounts micro-files of Piketty et al. (2018). This evidence suggests that savings are approximately constant at low incomes but increase convexly at higher incomes, so that the cross-sectional slope $s'(z)$ is increasing with income. To calibrate the causal income effect on savings, we draw on two sources. The first is Fagereng et al. (2021), which estimates the medium-run marginal propensity to save out of windfall income using lottery prizes. The second is a new probability-based survey representing the U.S. adult population, conducted on the AmeriSpeak panel, which asked respondents about their savings behavior in response to a possible raise. The two sources are consistent in suggesting similar magnitudes for $s'_{inc}(z)$, with little variation across incomes. Together, these findings yield a positive and increasing value of the residual $s'(z) - s'_{inc}(z) = s'_{het}(z)$, our sufficient statistic for heterogeneity, across most of the income distribution. Incorporated into our formulas, this implies a (mostly) positive and progressive optimal tax on savings. Our baseline estimates of optimal savings tax rates are somewhat higher than those currently in place in the U.S. across much of the income distribution, although as in other work, these results are sensitive to the elasticity of savings with respect to tax rates, about which there is still substantial uncertainty.

Our paper contributes to a number of literatures. The first is the literature studying optimal commodity and savings taxation in the presence of preference heterogeneity. Saez (2002) considers the special case of a separable linear commodity tax and derives conditions under which its optimal value is non-zero, but does not provide a formula for the magnitude. Golosov et al. (2013) derive conditions characterizing the optimal mechanism in a model like the one we study, but formulate their results in terms of first-order conditions on structural primitives rather than empirically estimable sufficient statistics. Their empirical estimates suggest substantially less across-income heterogeneity than ours do, resulting in much lower optimal savings tax rates. This difference could be because they study heterogeneity in time discounting only, rather than the broader set of forces that can contribute to $s'_{het}(z)$ and that we allow in our general characterization.[4] Saez and Stantcheva (2018) study nonlinear capital taxation in a setting without income effects, which corresponds to the special case of our model where $s'_{inc}(z) = 0$ and $s'_{het}(z) = s'(z)$. They consider multidimensional heterogeneity when tax systems are restricted to be either separable linear or separable nonlinear, so their results can be viewed as a special case of our extension characterizing optimal simple tax systems with multidimensional heterogeneity. Allcott et al. (2019) derive a sufficient statistics formula for the optimal separable linear commodity tax in the presence of preference heterogeneity across incomes.[5] Our results build on these insights by developing methods to characterize and implement the optimal mechanism using an unrestricted smooth nonlinear tax system, by studying other more restricted but still nonlinear tax systems that are commonly used in practice, and by incorporating forms of across-income heterogeneity that are not just preference-based.

Second, we contribute to the literature studying optimal taxation in settings with other sources of across-income heterogeneity. Gahvari and Micheletto (2016) and Gerritsen et al. (2020) study heterogeneous rates of return, Boadway et al. (2000) and Cremer et al. (2003) study heterogeneous endowments, Christiansen and Tuomala (2008) study income shifting, and Bovenberg and Jacobs (2005) and Bovenberg and Jacobs (2011) study human capital investments.[6] Our methods can be viewed as providing a unified treatment of these different sources of across-income heterogeneity, as well as a unified approach that can address—using the same set of sufficient statistics—both the growing literature on simpler tax systems with multidimensional heterogeneity and the smaller literature on optimal mechanisms with unidimensional heterogeneity.

Third, this paper complements the literature on dynamic taxation (see overviews by Golosov and Tsyvinski, 2006; Stantcheva, 2020), which typically assumes homogeneous preferences, but derives a theoretically robust role for capital taxation via the inverse Euler equation (e.g., Golosov et al., 2003; Farhi and Werning, 2013). Our work is complementary

[4]The lower measured heterogeneity in Golosov et al. (2013) could also be driven by attenuation bias. They measure preference heterogeneity by regressing a structural estimate of time preferences on a plausibly noisy proxy of earnings ability (performance on the Armed Forces Qualification Test), which may be biased toward zero due to a noisy right-hand-side variable.

[5]The application of separable *linear* savings taxes in the presence of multidimensional heterogeneity is also considered in Piketty and Saez (2013), Diamond and Spinnewijn (2011), and Gauthier and Henriet (2018). Piketty and Saez (2013) derive sufficient statistics formulas but make the additional restriction of a linear income tax. Diamond and Spinnewijn (2011) and Gauthier and Henriet (2018) allow for a nonlinear income tax but assume a finite number of possible earnings levels, and derive results in terms of model primitives. Jacquet and Lehmann (2021a) provide a generalization to a separable sum of many one-dimensional nonlinear tax schedules.

[6]See Stantcheva (2017) for an analysis of human capital policies in a dynamic setting. Our framework spans the static models in Bovenberg and Jacobs (2005) and Bovenberg and Jacobs (2011) but not more dynamic models.

in relaxing the assumption of homogeneous and weakly separable preferences, but using a static (two-period) framework. Quantitatively, the dynamic taxation literature tends to find optimal savings "wedges" of only several percentage points (see, e.g., Golosov and Tsyvinski, 2015; Golosov et al., 2016; Farhi and Werning, 2013)—substantially lower than those suggested by our baseline calibrations at the same assumed values of elasticities. This suggests that across-income heterogeneity may play a quantitatively larger role in determining optimal savings tax policy than do the social insurance motives analyzed in the dynamic taxation literature, and it motivates future research incorporating our method of measuring and incorporating across-income heterogeneity into fully dynamic models.

The rest of this paper proceeds as follows. Section 2 presents our model and assumptions. Section 3 shows that smooth tax systems can implement the optimal mechanism, and provides sufficient statistics for optimal smooth tax systems. Section 5 studies simple tax systems. Section 6 presents extensions to our results. Section 7 applies our formulas to quantify optimal savings tax rates in the United States. Section 8 concludes. All proofs are gathered in the Appendix.

2 Baseline Model and Assumptions

For concreteness, we begin by discussing preference heterogeneity, and then generalize the analysis to broader forms of across-income heterogeneity in Section 4.

Setting. There is a population of heterogeneous individuals indexed by their type $\theta \in \Theta$, where Θ is compact. They differ in their disutility from generating earnings z and in their preferences for a consumption bundle (c, s) embodied in their utility function $U(c, s, z; \theta)$. We begin with the common assumption that heterogeneity is unidimensional, $\Theta \subset \mathbb{R}$, in which case we interpret type θ as reflecting earnings ability; Section 5.2 considers multi-dimensional heterogeneity. We assume that θ has a continuously differentiable cumulative distribution function $F(\theta)$.

One application is where c is period-1 consumption and s is the realized savings in period 2, as in Saez (2002), Golosov et al. (2013), and many others. A second application is where c is period-1 consumption by the parents, while s is the wealth bequeathed to their children and consumed in period 2, as in Farhi and Werning (2010). A third application is where c is numeraire consumption and s is another dimension of commodity consumption that could be taxed nonlinearly, such as energy or housing (Gaubert et al., 2021).

Throughout the paper, we assume the following regularity conditions for the utility function:

Assumption 1. *$U(c,s,z;\theta)$ is twice continuously differentiable, increasing and weakly concave in c and s, and decreasing and strictly concave in z. The first derivatives U_c' and U_s' are bounded.*

We also assume a linear production technology with marginal rate of transformation p between s and c. In the savings and inheritance interpretations of the model, $p = 1/R$, where R is the gross rate of return in a linear savings technology between the two periods.

Preference heterogeneity. To introduce preference heterogeneity, consider the following example. In applications where s represents savings, a frequently used functional form (e.g., Saez, 2002; Golosov et al., 2013) involves additively separability and heterogeneity in individuals' productivity w and discount factor δ:

$$U(c,s,z;\theta) = u(c) + \delta(\theta)u(s) - k(z/w(\theta)), \tag{1}$$

with $u(.)$ the utility from consumption and $k(z/w)$ the disutility from work. There is preference heterogeneity for s across income-earning ability when the discount factor $\delta(\theta)$ covaries with productivity $w(\theta)$.

More generally, we say that there is across-ability preference heterogeneity for consumption bundles when the marginal rate of substitution between c and s varies with earnings ability. We denote this marginal rate of substitution by $\mathcal{S}(c,s,z;\theta) := \frac{U_s'(c,s,z;\theta)}{U_c'(c,s,z;\theta)}$ and similarly let $\mathcal{Z}(c,s,z;\theta) := \frac{U_z'(c,s,z;\theta)}{U_c'(c,s,z;\theta)}$ be the marginal rate of substitution between consumption c and earnings z. Using the shorthand $\mathcal{S}_\theta'(c,s,z;\theta_0) := \frac{\partial}{\partial\theta}\mathcal{S}(c,s,z;\theta)|_{\theta=\theta_0}$, we formally define preference heterogeneity as follows:

Definition 1. *There is across-ability preference heterogeneity for consumption bundles if some individuals prefer different (c,s) bundles conditional on having the same earnings z; i.e.,*

$$\exists\theta_0, \forall (c,s,z), \ \mathcal{S}_\theta'(c,s,z;\theta_0) \neq 0. \tag{2}$$

For instance, in example (1), $\mathcal{S}_\theta'(c,s,z;\theta) > 0$ whenever $\delta(\theta)$ covaries positively with $w(\theta)$. Such across-ability preference heterogeneity in consumption bundles is the focus of our results to follow, and for the rest of the paper we will refer to it simply as "preference heterogeneity."

Government. The government seeks to maximize a weighted sum of utilities:

$$\max \int_\theta \alpha(\theta)\, U(c(\theta), s(\theta), z(\theta);\theta)\, dF(\theta), \tag{3}$$

where $\alpha(\theta)$ represents some set of Pareto weights across types. Selecting a particular set of weights requires normative assumptions, which we discuss when introducing social marginal welfare weights in Section 3.2.2.

Type θ is private information and cannot be observed by the government; only the distribution of types, $F(\theta)$, is known. Therefore the government designs a tax and transfer function $\mathcal{T}$ that depends on the observable variables c, s, and z, which can be written without loss of generality as a tax on s and z only.[7] The government anticipates that individuals choose these variables to maximize their utility subject to their individual budget constraints, $c + ps \leq z - \mathcal{T}(s, z)$. Thus, an optimal tax system maximizes (3) subject to individual optimization, and subject to a resource constraint, $\int_\theta \mathcal{T}(s(\theta), z(\theta))dF(\theta) \geq E$, where E is an exogenous revenue requirement.

If the tax system $\mathcal{T}(s, z)$ is unrestricted, this problem is equivalent to the problem of selecting an optimal allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ to maximize the objective in (3) subject to the resource constraint

$$\int_\theta [z(\theta) - ps(\theta) - c(\theta)]\, dF(\theta) \geq E, \tag{4}$$

and subject to incentive compatibility constraints

$$\forall\ (\theta, \theta') \in \Theta^2,\ U(c(\theta), s(\theta), z(\theta); \theta) \geq U(c(\theta'), s(\theta'), z(\theta'); \theta). \tag{5}$$

We refer to an allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ that maximizes (3) subject to (4) and (5) as an *optimal incentive-compatible allocation*.

3 Optimal Smooth Tax Systems

In this section, we provide two key results about *smooth tax systems*, by which we mean twice continuously differentiable tax functions $\mathcal{T}(s, z)$. First, we show that an optimal incentive-compatible allocation can be implemented by a *smooth* tax system. Second, we derive a sufficient statistics characterization of optimal smooth tax systems.

[7] Expressing the tax more generally as $\mathcal{T}(c, s, z)$ is redundant. Given such a tax function, any choice of s and z implies a consumption value given by $\mathcal{C}(s, z) := \max\{c | c = z - s - \mathcal{T}(c, s, z)\}$; thus, one can re-express the tax as a function of only savings and earnings: $\tilde{\mathcal{T}}(s, z) = \mathcal{T}(\mathcal{C}(s, z), s, z)$.

3.1 Implementability with Smooth Tax Systems

Our implementation result relies on regularity conditions and on an extended Spence-Mirrlees condition about the optimal incentive-compatible allocation.

Assumption 2. *In the optimal incentive-compatible allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$, we assume that: (i) c, s, and z are smooth functions of θ, (ii) any type θ strictly prefers its allocation $(c(\theta), s(\theta), z(\theta))$ to the allocation $(c(\theta'), s(\theta'), z(\theta'))$ of another type $\theta' \neq \theta$, and (iii) the set of types for which $z'(\theta) = 0$ is of measure zero, and, when $z'(\theta) \neq 0$, the following extended Spence-Mirrlees condition holds for any type θ':*

$$\mathcal{S}'_\theta \left(c(\theta), s(\theta), z(\theta); \theta'\right) \frac{s'(\theta)}{z'(\theta)} + \mathcal{Z}'_\theta \left(c(\theta), s(\theta), z(\theta); \theta'\right) > 0. \tag{6}$$

Assumptions (i) and (ii) are standard assumptions required to apply optimal control methods to characterize the optimal allocation. The main component of Assumption 2 is an extension of the Spence-Mirrlees condition that $\mathcal{Z}'_\theta(c, s, z; \theta) > 0$. Intuitively, the standard Spence-Mirrlees condition states that higher types are more willing to trade labor for consumption. Our condition extends this to our more general setting, stating higher types are more willing to trade labor for the joint change in c and s that can be obtained along the allocation path. Paralleling Mirrlees (1971), our conditions ensure that $z(\theta)$ is strictly increasing in θ (Appendix Lemma B.1), allowing us to define the function $\vartheta(z)$, which maps each earnings level z to the type to which it is assigned in the optimal incentive-compatible allocation.

Definition 2. *We say that an allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ is implementable by a tax system $\mathcal{T}$ if*

1. *$\mathcal{T}$ satisfies individual feasibility: $c(\theta) + ps(\theta) + \mathcal{T}(s(\theta), z(\theta)) = z(\theta)$ for all $\theta \in \Theta$, and*

2. *$\mathcal{T}$ satisfies individual optimization: $(c(\theta), s(\theta), z(\theta))$ maximizes $U(c, s, z; \theta)$ subject to the constraint $c + ps + \mathcal{T}(s, z) \leq z$, for all $\theta \in \Theta$.*

Our first result shows that an optimal incentive-compatible allocation is implementable by some smooth tax system.

Theorem 1. *Under Assumptions 1 and 2, an optimal incentive-compatible allocation is implementable by a smooth tax system. In this smooth tax system, individuals' choices are interior (first-order conditions hold), and their local optima are strict (strict second-order conditions).*

Although it is clear that the optimal incentive-compatible allocation $\{(c(\theta), s(\theta), z(\theta))\}_\theta$ can always be implemented by *some* two-dimensional tax system—for example, by defining $\mathcal{T}(s(\theta), z(\theta)) = z(\theta) - c(\theta) - s(\theta)$ for $\theta \in \Theta$ and letting $\mathcal{T}(s, z) \to \infty$ for $(c, s, z) \notin \{(c(\theta), s(\theta), z(\theta))\}_\theta$—such a tax system is not guaranteed to be smooth. A smooth tax system allows individuals to locally adjust s and z to points not chosen by any other type in the optimal allocation, and thus the set of possible deviations is much larger than when the optimal mechanism can simply disallow certain allocations.

Starting from any given allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$, a smooth tax system can implement the allocation only by satisfying each type's θ first-order conditions:

$$\mathcal{T}_s'\left(s\left(\theta\right), z\left(\theta\right)\right) = \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta\right) - p \tag{7}$$

$$\mathcal{T}_z'\left(s\left(\theta\right), z\left(\theta\right)\right) = \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta\right) + 1. \tag{8}$$

In the presence of preference heterogeneity, individuals' temptation to deviate from their assigned allocation $(c(\theta), s(\theta), z(\theta))$ are stronger under a smooth tax system than under a direct mechanism. For example, suppose that higher types θ have a stronger relative preference for s. If they deviate downward to some other earnings level $z(\theta') < z(\theta)$, then under a direct mechanism they will be forced to choose $s(\theta')$. Under a smooth tax system, however, the deviating type θ will choose $s' > s(\theta')$ at earnings level $z(\theta')$, making this *double deviation* more appealing.

Tax implementation results that involve multidimensional consumption bundles and multidimensional tax systems typically avoid the difficulties associated with double deviations by ruling out the type of preference heterogeneity that we consider here.[8] Thus, to our knowledge, our proof of Theorem 1 is different from typical implementation proofs in the optimal tax literature. The proof, contained in Appendix C.2, proceeds in three steps. The first step is to construct a sequence of tax systems $\mathcal{T}_k$ such that each element in the sequence satisfies type-specific feasibility and the first-order conditions above. The sequence is ordered such that successive elements are increasingly convex around the bundles $(s(\theta), z(\theta))$ offered in the optimal mechanism.

In the second step of the proof, we show that for each type θ there exists a k sufficiently large such that this type's second-order conditions hold at $(c(\theta), s(\theta), z(\theta))$. In other words, for each type there is a sufficiently large k such that $(c(\theta), s(\theta), z(\theta))$ is a *local* optimum under the tax system $\mathcal{T}_k$. This step requires Appendix Lemmas C.1 and C.2, which char-

[8] As pointed out by Kocherlakota (2005), Werning (2010), and others, smooth tax systems can also generate double deviations in dynamic settings where there is a discrete set of types. Werning (2010) provides a general implementation proof for a dynamic setting where productivity is smoothly distributed. The setting studied by Werning (2010), and the proof technique, is distinct from ours because time preferences, and thus preferences for period-2 consumption, are assumed homogeneous.

acterize individuals' budget constraints and second derivatives of indirect utility functions for any tax system $\mathcal{T}$ that preserves individuals' first-order conditions.

In the third step, we show that there exists a sufficiently large k such that for *all* types θ, $(c(\theta), s(\theta), z(\theta))$ is a *global* optimum under $\mathcal{T}_k$. We complete this step via a proof by contradiction. Under the assumption that such a k does not exist, there exists an infinite sequence of values k and types θ_k such that type θ_k prefers to deviate from $(c(\theta_k), s(\theta_k), z(\theta_k))$ under $\mathcal{T}_k$. Because the type space is compact, the Bolzano-Weierstrass Theorem allows us to extract a convergent subsequence of types θ_j who all prefer to deviate from the allocation assigned to them under the optimal mechanism. We show that this implies a contradiction because the limit type of this sequence, $\hat{\theta}$, must then prefer to deviate from $(c(\hat{\theta}), s(\hat{\theta}), z(\hat{\theta}))$ to some other allocation $(c(\theta'), s(\theta'), z(\theta'))$ offered in the optimal mechanism.

Theorem 1 is an existence result, and our proof of the theorem does not offer insight into the structure of an optimal tax system. However, because individuals' choices are shown to satisfy first-order and second-order conditions in a smooth tax system, we can use variational methods to characterize optimal tax systems. We now proceed by deriving optimal tax formulas expressed in terms of empirically estimable sufficient statistics that transparently highlight the key economic forces governing the optimal tax system.

3.2 Sufficient Statistics for Smooth Tax Systems

3.2.1 Definitions

Assumptions. To define our sufficient statistics, it is helpful to write individuals' optimization problem under a tax system $\mathcal{T}(s, z)$ as

$$\max_{z} \left\{ \max_{c,s} U(c, s, z; \theta) \text{ s.t. } c \leq z - ps - \mathcal{T}(s, z) \right\}, \tag{9}$$

where the inner problem represents the optimal choices of $c(z; \theta)$ and $s(z; \theta)$ for a given earnings level z, and the outer problem represents the optimal choice of earnings $z(\theta)$ taking into account endogenous choices of c and s.

When Assumptions 1 and 2 hold, our implementation result for smooth tax systems $\mathcal{T}(s, z)$ holds (Theorem 1) and we do not need to impose any other requirements. In cases where these assumptions fail, or when we study simpler tax systems for which this implementation result may not hold, our optimal tax formulas remain valid under the following assumption:

Assumption 3. *The tax systems under consideration are such that at the optimum: (i) these tax systems are smooth, (ii) $z(\theta)$ is smooth and strictly increasing, $c(z;\theta)$ and $s(z;\theta)$ are smooth functions of z and θ, and (iii) individuals' optima are unique and their first-order and second-order conditions strictly hold.*

Elasticities for z-choices. Earnings responses to tax reforms are captured through ζ_z^c, the compensated elasticity of labor income with respect to the marginal labor income tax rate, and η_z, the income effect parameter. Formally, for each level of earnings $z(\theta)$ chosen by a type θ, we define

$$\zeta_z^c\left(z(\theta)\right) := -\frac{1-\mathcal{T}_z'\left(s(\theta),z(\theta)\right)}{z\left(\theta\right)}\frac{\partial z\left(\theta\right)}{\partial\mathcal{T}_z'\left(s(\theta),z(\theta)\right)} \tag{10}$$

$$\eta_z\left(z(\theta)\right) := -(1-\mathcal{T}_z'\left(s(\theta),z(\theta)\right))\frac{\partial z\left(\theta\right)}{\partial\mathcal{T}\left(s(\theta),z(\theta)\right)} \tag{11}$$

where $\mathcal{T}\left(s(\theta),z(\theta)\right)$ is the tax liability and $\mathcal{T}_z'\left(s(\theta),z(\theta)\right)$ is the marginal labor income tax rate. Since earnings choices take into account endogenous choices of c and s, these elasticity concepts take into account the full sequence of adjustments due to changes in choices of c and s, as well as those due to any nonlinearities in the tax system.[9]

Elasticities for s-choices. Changes in s in response to tax reforms are captured through $\zeta_{s|z}^c$, the compensated elasticity of s with respect to the marginal tax rate on s, $\eta_{s|z}$, the income effect parameter, and s_{inc}', the causal effect on consumption of s from a marginal change in gross pre-tax income z. These are formally defined as follows:

$$\zeta_{s|z}^c\left(z(\theta)\right) := -\frac{1+\mathcal{T}_s'\left(s\left(z;\theta\right),z\right)}{s\left(z;\theta\right)}\frac{\partial s\left(z;\theta\right)}{\partial\mathcal{T}_s'\left(s\left(z;\theta\right),z\right)}\Big|_{z=z(\theta)} \tag{12}$$

$$\eta_{s|z}\left(z(\theta)\right) := -\left(1+\mathcal{T}_s'\left(s\left(z;\theta\right),z\right)\right)\frac{\partial s\left(z;\theta\right)}{\partial\mathcal{T}\left(s\left(z;\theta\right),z\right)}\Big|_{z=z(\theta)} \tag{13}$$

$$s_{inc}'(z(\theta)) := \frac{\partial s\left(z;\theta\right)}{\partial z}\Big|_{z=z(\theta)} \tag{14}$$

where $\mathcal{T}_s'\left(s\left(z;\theta\right),z\right)$ is the marginal tax rate on s of a type θ who earns labor income z. Elasticity concepts $\zeta_{s|z}^c$ and $\eta_{s|z}$ are conditional on z. They measure responses to tax reforms accounting for nonlinearities in the tax system, holding labor income fixed at $z(\theta)$.

Note that we define the elasticity of s with respect to $1+\mathcal{T}_s'$, rather than with respect to $p+\mathcal{T}_s'$. This choice is irrelevant when $p\equiv 1$—a normalization that can be adopted without

[9] This corresponds to the type of circular adjustments described in, e.g., Jacquet and Lehmann (2021b).

loss of generality, as shown in our discussion of generalized budget constraints in section 4.1 below. However, defining the elasticity with respect to $p + \mathcal{T}_s'$ may be more natural in applications where s is a commodity sold at an after-tax price of $q = p + \mathcal{T}_s'$.[10] For all elasticity concepts, we use the "bar" notation, as in $\overline{\zeta^c_{s|z}}$, to denote a population elasticity.

Preference heterogeneity. To quantify preference heterogeneity, we decompose the cross-sectional profile of $s(z) := s(z; \vartheta(z))$. Intuitively, $s'(z)$, the cross-sectional change in s with respect to z, comprises both the causal income effect and the degree to which preferences vary across earnings z. We thus define our measure of local across-income preference heterogeneity, $s'_{het}(z)$, as the difference between the cross-sectional change in s along the earnings distribution and the causal income effect $s'_{inc}(z)$:

$$s'_{het}(z(\theta)) := s'(z(\theta)) - s'_{inc}(z(\theta)) \tag{15}$$

Formally, $s'(z)$ is a total derivative equal to the sum of two partial derivatives:

$$\underbrace{\frac{ds(z, \vartheta(z))}{dz}}_{s'(z)} = \underbrace{\frac{\partial s(z'; \vartheta(z))}{\partial z'}\bigg|_{z'=z}}_{s'_{inc}(z)} + \underbrace{\frac{\partial s(z; \vartheta(z'))}{\partial z'}\bigg|_{z'=z}}_{s'_{het}(z)} \tag{16}$$

The first term on the right-hand side measures how a change in z affects s consumption, holding the type θ constant. The second term, $s'_{het}(z)$, measures how a change in type affects s consumption, holding earnings z constant. For instance, in example (1) above, $s'_{het}(z)$ would be directly proportional to the local change in the discount factor $\delta(\vartheta(z))$.

Violations of weak separability. The sufficient statistic s'_{het} also captures failures of weak separability between labor and consumption as in, e.g., Corlett and Hague (1953). For example, suppose that higher types get a higher hourly wage rate, and that consumption of s and leisure are complements. Then, because higher types θ obtain more leisure at a fixed level of earnings z, higher types will have a stronger preference for s holding z constant. In our setup, this amounts to $s'_{het} > 0$.

[10] It is straightforward to convert our results between these elasticity definitions: in this case, the appropriate formulas can be obtained from Theorem 2 and Proposition 3 by multiplying $\zeta^c_{s|z}$ by $(p + \mathcal{T}_s')/(1 + \mathcal{T}_s')$. The only change in Theorem 2 is that the left-hand-side in equation (20) becomes $\frac{\mathcal{T}_s'(s(z),z)}{p+\mathcal{T}_s'(s(z),z)}$, and analogously for Proposition 3.

3.2.2 Social Marginal Welfare Weights

To encode the policymaker's redistributive tastes, we follow the literature in defining social marginal welfare weights as the marginal social welfare derived from an increase in consumption for an individual of type θ at points $s(\theta)$ and $z(\theta)$, normalized by the marginal value of public funds λ:

$$g(s(\theta), z(\theta)) := \frac{\alpha(\theta)}{\lambda} U_c'\big(z(\theta) - \mathcal{T}(s(\theta), z(\theta)) - ps(\theta),\ s(\theta),\ z(\theta)\,;\,\theta\big). \tag{17}$$

We define $\hat{g}(s, z)$ as the social marginal welfare weights augmented with the fiscal impact of income effects. This represents the full social value of marginally increasing the disposable income of individuals at points s and z. Formally,

$$\hat{g}(s,z) := g(s,z) + \mathcal{T}_z'(s,z)\frac{\eta_z(z)}{1-\mathcal{T}_z'(s,z)} + \mathcal{T}_s'(s,z)\left(\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'(s,z)} + s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'(s,z)}\right), \tag{18}$$

where the last term comes from the fact that income effects on earnings, proportional to $\eta_z(z)$, induce changes in s consumption proportional to $s_{inc}'(z)$, affecting tax revenues.

Social marginal welfare weights embed judgments about interpersonal utility comparisons. These are usually treated as normative assumptions, although some research has utilized survey data to estimate these weights (see Appendix C of Saez and Stantcheva, 2016) or estimated them from existing policies via an "inverse optimum" procedure (e.g., Bourguignon and Spadaro 2012; Lockwood and Weinzierl 2016). Such normative assumptions are particularly strong when there is preference heterogeneity, because individuals prefer different bundles—and face different tax burdens—even when they have identical budget sets. For example, a savings tax might be viewed as an unfair tax on those with relatively high discount factors. Lockwood and Weinzierl (2015) show that this difficulty arises even in the standard Mirrlees (1971) model; because there is no formal distinction between heterogeneous earnings ability and heterogeneous preferences for exerting labor effort—both manifest as a revealed preference for a lower level of earnings.

We write our theoretical results in terms of flexible welfare weights that span the degree of heterogeneity in individuals' types, so that optimal policy can be computed using whatever welfare weights the policymaker prefers.[11] For results in the case of unidimensional heterogeneity, welfare weights are written as a function only of earnings, $g(z(\theta))$, without loss of generality. For results involving multidimensional heterogeneity, in which savings are heterogeneous conditional on income, we write social marginal welfare weights as a

[11] Our empirical application in Section 7 employs a version of the inverse optimum approach, estimating optimal savings taxes consistent with the current U.S. taxes on labor income.

function of both s and z, $g(s(\theta), z(\theta))$.

3.3 Characterization of Optimal Smooth Tax Systems

A key result used to derive our sufficient statistics formula is an equivalence result for tax reforms affecting marginal tax rates on s versus z. This result is a generalization of Lemma 1 in Saez (2002) to nonlinear smooth tax systems.

Lemma 1. *A small increase $d\tau$ in the marginal tax rate on s faced by an individual earning z induces the same earnings change as a small increase $s'_{inc}(z)\, d\tau$ in the marginal tax rate on z.*

Lemma 1 relates the labor supply distortions induced by increasing taxes on s to the labor supply distortions induced by increasing taxes on earnings z. Intuitively, if the marginal tax rate on earnings z increases by $d\tau_z$, an individual realizes they must now pay an additional $d\tau_z$ on each marginal dollar of earnings, so they earn less in response. Alternatively, if the marginal tax rate on commodity s increases by $d\tau_s$, and the individual adjusts s by s'_{inc} for every dollar adjustment in earnings, then the individual realizes they must now effectively pay an additional $s'_{inc} d\tau_s$ more for each marginal dollar of earnings, accounting for the way in which they will also adjust s. If $d\tau_z = s'_{inc} d\tau_s$, then the induced earnings changes will be the same for both reforms.

We are now in a position to write formulas characterizing necessary conditions for the optimal smooth tax system in terms of sufficient statistics. In the results that follow, we use $H(s, z)$ and $h(s, z)$ to denote the cumulative and density functions over (s, z), with h_s and h_z denoting the marginal density over s and z, respectively.

Theorem 2. *Under the assumptions of Theorem 1 or under Assumption 3, at each bundle (c, s, z) chosen by a type θ, an optimal smooth tax system satisfies the following conditions on marginal tax rates on z and s, respectively:*

$$\frac{\mathcal{T}'_z(s,z)}{1-\mathcal{T}'_z(s,z)} = \frac{1}{z\,\zeta^c_z(z)} \frac{1}{h_z(z)} \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) - s'_{inc}(z) \frac{\mathcal{T}'_s(s,z)}{1-\mathcal{T}'_z(s,z)} \qquad (19)$$

$$\frac{\mathcal{T}'_s(s,z)}{1+\mathcal{T}'_s(s,z)} = s'_{het}(z) \frac{1}{s\,\zeta^c_{s|z}(z)} \frac{1}{h_z(z)} \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x). \qquad (20)$$

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}'_s(s,z)}{1+\mathcal{T}'_s(s,z)} = s'_{het}(z) \frac{z\,\zeta^c_z(z)}{s\,\zeta^c_{s|z}(z)} \frac{\mathcal{T}'_z(s,z) + s'_{inc}(z)\mathcal{T}'_s(s,z)}{1-\mathcal{T}'_z(s,z)}. \qquad (21)$$

Formula (19) constitutes a familiar "ABC" condition analogous to Saez (2001), with one modification: When tax rates on s are non-zero, the formula also accounts for how changes in earnings affect consumption of s, and therefore tax revenue. To see this, consider a small increase in the earnings marginal tax rate, $d\tau_z$, in a small bandwidth of earnings, dz, around z. This reform triggers a lump-sum tax increase, $d\tau_z dz$, for all individuals with higher earnings that changes tax revenue and welfare by $d\tau_z dz \int_{x\geq z}(1-\hat{g}(x))\, dH_z(x)$. Moreover, it induces the mass of individuals in the bandwidth, $h_z(z)dz$, to reduce their earnings and thus tax revenue by $\mathcal{T}'_z(s,z)\, \frac{z}{1-\mathcal{T}'_z(s,z)}\zeta^c_z(z)d\tau_z$. This earnings reduction also leads to a reduction in the consumption of s by an amount proportional to s'_{inc}, which reduces tax revenue by $\mathcal{T}'_s(s,z)\, s'_{inc}(z)\frac{z}{1-\mathcal{T}'_z(s,z)}\zeta^c_z(z)d\tau_z$. Summing these effects and characterizing the optimum as a situation where they cancel each other yields (19).

Formula (20) is one of our key results: Optimal marginal tax rates on s satisfy a condition that is remarkably similar to the standard "ABC" formula, and that provides a transparent generalization of the Atkinson-Stiglitz Theorem. When the sufficient statistic s'_{het} is equal to zero, the condition implies that the optimal tax on s must equal zero as well. When $s'_{het} > 0$, implying that higher earners have a stronger relative preference for s, the condition implies that the optimal tax rate on s must be positive and that its magnitude is decreasing in the elasticity of s with respect to the tax rate, increasing in the strength of redistributive motives, and decreasing in the density of individuals at point $s(z)$.

To obtain intuition for the result, combine the earnings tax *increase* discussed above with a small *decrease* in the marginal tax rate on s, $d\tau_s$, in a small bandwidth ds around $s(z)$. Set $ds = s'(z)dz$ such that the two bandwidths coincide, $h_s(s(z))ds = h_z(z)dz$.[12] By Lemma (1), the $d\tau_s$ decrease in the marginal tax rate on s induces individuals in this bandwidth to increase their earnings by $\frac{z}{1-\mathcal{T}'_z(s,z)}\zeta^c_z(z)s'_{inc}d\tau_s$. Hence, setting $d\tau_z = s'_{inc}(z)d\tau_s$ implies that earnings changes induced by these two reforms cancel out, leaving the earnings of individuals in the bandwidth unchanged. As a result, the joint reform (i) induces individuals in the bandwidth to increase their consumption of s and thus tax revenue by $\mathcal{T}'_s(s,z)\, \frac{s(z)}{1-\mathcal{T}'_s(s,z)}\zeta^c_{s|z}(z)d\tau_s$ and (ii) generates a lump-sum tax decrease, $dT = d\tau_s ds - d\tau_z dz = (s'(z) - s'_{inc}(z))\, d\tau dz$, for all individuals with earnings higher than z, which induces a $dT \int_{x\geq z}(1-\hat{g}(x))\, dH_z(x)$ change in tax revenue. Setting the sum of these effects to zero yields (20).

We can combine conditions (19) and (20) to derive the Pareto-efficiency condition in (21).[13] Because the condition in (21) does not feature social marginal welfare weights,

[12]Here we use the fact that $h_s(s(z)) = \frac{h_z(z)}{s'(z)}$, which highlights that this heuristic proof strategy relies on the mapping $s(z)$ being strictly monotonic. Our formal appendix proof follows a more sophisticated proof strategy that does not require strict monotonicity of $s(z)$.

[13]This corresponds to combining tax reforms with $d\tau_z = s'(z)d\tau_s$ to cancel out lump-sum tax changes

it is an efficiency condition that must hold for any tax system that is not Pareto dominated. Intuitively, it quantifies the efficient balance between taxing s and taxing z, given the measure s'_{het} of how relative tastes for s covary with earnings ability. The stronger the association between relative preferences for s and earnings ability, the more efficient it is to tax s instead of z. An important implication of this Pareto-efficiency condition is that in the absence of preference heterogeneity, positive tax rates on s are Pareto dominated. On the other hand, any Pareto-efficient tax system must feature non-zero tax rates on s in the presence of preference heterogeneity.

4 Across-Income Heterogeneity and its Determinants

4.1 Budget Heterogeneity and Auxiliary Choices

So far, we have considered economies with type-specific preferences $U(c, s, z; \theta)$ but type-independent budget constraints $c \leq B(s, z) - \mathcal{T}(s, z)$, with budget domain $B(s, z) := z - ps$. In this environment, across-income heterogeneity captured by the sufficient statistic $s'_{het}(z)$ originates from preference heterogeneity only.

However, this approach readily extends to other sources of across-income heterogeneity. For example, across-income heterogeneity in prices of s (Gahvari and Micheletto, 2016; Gerritsen et al., 2020), in income shifting (Slemrod, 1995; Christiansen and Tuomala, 2008), and in endowments (Boadway et al., 2000; Cremer et al., 2003) may all contribute to differences between the cross-sectional profile $s'(z)$ and the causal income effect $s'_{inc}(z)$. A key feature of our sufficient statistics approach is that the model can be reinterpreted so that $s'_{het}(z)$ represents these alternative sources of heterogeneity—or a combination of them—and the characterization of optimal tax schedules remains intact.

To formalize this idea, consider the following modifications to our baseline model. First, assume that individuals might face type-dependent budget constraints given by $c \leq B(s, z, \chi; \theta) - \mathcal{T}(s, z)$, where χ represents a vector of auxiliary choices made by individuals that may affect their budget domain. Second, assume that individuals may manipulate their taxable earnings z and taxable savings s, or alter their ability to produce taxable earnings z by making productivity-enhancing investments, resulting in general type-dependent preferences $U(c, \phi_s(z, s, \chi; \theta), \phi_z(z, s, \chi; \theta), \chi; \theta)$ affected by auxiliary choices χ.

We establish the following equivalence result between economies:

above earnings z. This result builds on Konishi (1995), Laroque (2005), and Kaplow (2006), who derive Pareto-optimality conditions under the more restrictive assumptions of the Atkinson-Stiglitz theorem.

Lemma 2. *For any given tax system $\mathcal{T}(s,z)$, individuals make identical choices in (1) an economy in which they have type-dependent preferences $U(c, \phi_s(z,s,\chi;\theta), \phi_z(z,s,\chi;\theta), \chi;\theta)$ and type-dependent budget domains $B(s,z,\chi;\theta)$ that are potentially affected by auxiliary choices χ, and in (2) an economy in which individuals have type-independent budget domains $B(s,z)$, no auxiliary choices, and type-dependent preferences*

$$\tilde{U}(c,s,z;\theta) := \max_{\chi} U(c+B(s,z,\chi;\theta)-B(s,z), \phi_s(z,s,\chi;\theta), \phi_z(z,s,\chi;\theta), \chi;\theta). \quad (22)$$

Lemma 2 shows that an economy that features preference heterogeneity and budget heterogeneity as well as auxiliary choices is equivalent to an economy with preference heterogeneity only, provided that preferences are suitably defined. The intuition is that under a given tax system $\mathcal{T}(s,z)$, individuals' utility maximization problems are equivalent in both economies. This means that all individuals make identical choices and attain the same level of utility, and the government collects the same tax revenue.[14] Since this equivalence holds for any tax system $\mathcal{T}(s,z)$, it immediately follows that these equivalent economies share the same optimal tax system, leading to the following proposition:

Proposition 1. *In an economy with preference heterogeneity, budget heterogeneity, and auxiliary choices, an optimal smooth tax system remains characterized by Theorem 2, as long as the equivalent economy with preference heterogeneity only satisfies the assumptions of the Theorem.*

This result highlights the generality of our sufficient statistics characterization of optimal taxes, where the key sufficient statistic s'_{het} captures all relevant dimensions of across-income heterogeneity that justify taxing s. While these different sources of across-income heterogeneity have previously been studied in isolation to qualitatively assess the robustness of the Atkinson-Stiglitz Theorem, our sufficient statistics techniques can be applied to account for them in a quantitative and general manner. To emphasize this point, we provide in Appendix B.3 a structural characterization of s'_{inc} and s'_{het} in economies with preference heterogeneity, budget heterogeneity, and auxiliary choices.

The intuition for this generality stems from the logic of Feldstein (1999), which shows that the elasticity of taxable income is a sufficient statistic for efficiency losses irrespective

[14]This holds because the tax $\mathcal{T}(s,z)$ is measured in dollars and paid out of earnings. If instead the tax had a two-part structure where individuals pay T_1 in units of c (e.g., dollars) and T_2 in units of s (e.g., liters of soda), then budget heterogeneity would affect tax revenue. Such a system is relatively common for savings, where taxes are often paid in units of "period 2" dollars, after returns have been realized. This creates an additional arbitrage motive to tax individuals with higher returns more heavily in units of s. We explore such arbitrage motives in Section 6.3.

of whether it is due to real labor supply responses or costly avoidance behavior.[15] We now discuss some examples of budget heterogeneity and auxiliary choices, focusing on those that are relevant in the context of savings taxation.

Heterogeneous Prices. One widespread form of budget heterogeneity is price heterogeneity. Suppose that individuals face prices $p(s, z, \chi; \theta)$ for s that depend on their level of s, earnings z, effort χ to seek out lower prices, or types θ, such that their budget domain is $B(s, z, \chi; \theta) = z - p(s, z, \chi; \theta)s$ and their preferences are $U(c, s, z, \chi; \theta)$, allowing for monetary or psychic costs of effort χ. For instance, in the context of savings taxation, higher savings might allow individuals to lock in better interest rates, higher income might generate beneficial network effects that expose individuals to better opportunities (both are examples of "scale dependence"), and higher ability types may be better at finding lower prices or higher returns on investments (an example of "type dependence").

Lemma 2 shows that this economy is equivalent to an economy with budget domain $B(s, z) := z - s$, with the price normalized to $p \equiv 1$, where individuals' utility function is $\tilde{U}(c, s, z; \theta) := \max_\chi U\left(c + (1 - p(s, z, \chi; \theta))s, s, z, \chi; \theta\right)$. This also demonstrates that in our baseline model, the price p can be normalized to 1 without loss of generality—a feature we employ in some appendix proofs. Intuitively, with a price of $p = p'$ instead of $p = 1$, individuals enjoy $(1 - p')s$ more consumption of c at a given choice s.

An insight from this reinterpretation is that some sources of across-income price heterogeneity justify taxing s, while others do not, and the decomposition of the cross-sectional profile $s'(z)$ into $s'_{inc}(z)$ and $s'_{het}(z)$ correctly distinguishes between them. In particular, *type*-dependent heterogeneity in prices will generally lead to $s'_{het}(z) \neq 0$ and thus to deviations from the Atkinson-Stiglitz result, whereas *scale*-dependent heterogeneity in prices would contribute to $s'_{inc}(z)$ and would thus preserve the Atkinson-Stiglitz result.

Heterogeneous Endowments. Another widespread form of budget heterogeneity is heterogeneity in initial endowments. Suppose that individuals have endowments $y_0(\theta)$, from inheritance or other sources, such that their budget domain is $B(s, z, \chi; \theta) = z + y_0(\theta) - s$. In the presence of income effects, endowments would influence individuals' decisions, and *type*-dependent endowments would therefore enter s'_{het}. For instance, this could represent a situation in which higher productivity individuals have higher savings in part because they have larger inheritances, resulting in $s'_{het} > 0$ and justifying taxes on s.

[15]Chetty (2009) suggests limitations to Feldstein's (1999) results due to some avoidance behaviors generating new types of fiscal externalities, or due to behavioral biases. Variations of these considerations are relevant in our setting as well, as explored in Sections 6.2 and 6.3, respectively.

Income Shifting. Suppose that some auxiliary actions allow individuals to shift some of their labor income to capital income. Let $\tilde{z}$ and $\tilde{s}$ be the individuals' true labor income and savings, which are unobserved by the tax authority. Let χ denote the amount of labor income $\tilde{z}$ that individuals shift to be realized as taxable savings (including capital income), and let $m(s, z, \chi; \theta)$ denote any financial costs involved in income shifting. Individuals' taxable labor income is thus $z = \tilde{z} - \chi$ and their taxable savings are $s = \tilde{s} + \chi$. This describes an economy with budget domain $B(s, z, \chi; \theta) = z - s + 2\chi - m(s, z, \chi; \theta)$ and preferences $U(c, s - \chi, z + \chi, \chi; \theta)$, where χ might directly influence utility because of effort or psychic costs. In this example, $\phi_s(s, \chi) = s - \chi$ and $\phi_z(z, \chi) = z + \chi$. If individuals with higher earnings ability are better able to engage in income shifting, this type dependence will effectively translate in $s'_{het} > 0$. In contrast, any scale dependence in income shifting would affect s'_{inc}, but not s'_{het}.

Human Capital Investments. Our framework can also be related to models studying human capital investments. Suppose that s represents human capital investments such as education, which affects productivity during working life (Bovenberg and Jacobs, 2005; Stantcheva, 2017). Following the static framework in Bovenberg and Jacobs (2005), the productivity-enhancing effect of human capital may be directly captured in preferences through, e.g., $\phi_z(z, s; \theta)$. Using Lemma 2, this economy is equivalent to one with preferences $\tilde{U}(c, s, z; \theta) := U(c, s, \phi_z(z, s; \theta); \theta)$. The functional form assumptions in Bovenberg and Jacobs (2005) imply that $s'_{het}(z) < 0$ and thus that human capital investments are subsidized at the optimum; Bovenberg and Jacobs (2011) consider more general settings in which these assumptions are relaxed.

4.2 Measuring s'_{het}

Because the key sufficient statistic $s'_{het}(z)$ is equal to the difference $s'(z) - s'_{inc}(z)$, it can be measured as a residual using familiar estimation strategies. The term $s'(z)$ represents the cross-sectional variation of s across the income distribution, which can be directly measured using standard data sources. The statistic $s'_{inc}(z)$ can be measured using a variety of strategies. Here we present three different methods of measuring $s'_{inc}(z)$, which rely on different types of quasi-experimental variation.

Proposition 2. *Define $\xi^s_w(z)$ as the elasticity of s with respect to the wage rate w, $\xi^h_w(z)$ as the elasticity of hours with respect to the wage rate, and $\chi^c_s(z)$ as the elasticity of s with respect to the marginal net of tax rate on labor income. The sufficient statistic $s'_{inc}(z)$ can be measured as follows:*

M1. *If preferences are weakly separable and the tax system is separable,* $s'_{inc}(z) = \frac{1-T'_z(z)}{1+T'_s(s(z))}\eta_{s|z}(z)$.

M2. *If wage rates* w *and hours* h *are observable,* $s'_{inc}(z) = s(z)\frac{\xi^s_w(z)}{w(z)+h(z)\xi^h_w(z)}$.

M3. *If responses to tax reforms are measurable,* $s'_{inc}(z) = \frac{s(z)}{z}\frac{\chi^c_s(z)}{\zeta^c_z(z)}$.

If individuals' preferences are weakly separable as in example (1) above, and if the tax system is separable in s and z, then $s'_{inc}(z)$ is proportional to the income effect parameter for s. If individuals' preferences are not weakly separable but wage rates w and hours h are observable, $s'_{inc}(z)$ can be related to the elasticity of s with respect to the wage rate and to the elasticity of hours with respect to the wage rate. If the elasticities of both s and z with respect to the marginal tax rate on z are observable, $s'_{inc}(z)$ can be recovered from these elasticities.

A key question for empirical implementation is the time horizon over which the statistics must be measured. Interpreting our static model to represent a steady-state economy, $s'_{inc}(z)$ corresponds to the causal effect of a change in steady-state labor income on steady-state consumption of s.[16] Under the weak separability assumption, it is therefore necessary to measure the *long-run* marginal propensity to consume s. In the case of savings, this is the long-run marginal propensity to save, as estimated by Fagereng et al. (2021) for example, in response to a change in unearned income.[17]

5 Simple Tax Systems and Multidimensional Heterogeneity

In practice, tax systems must be defined by policymakers and implemented by institutions, which may impose constraints on the degree of complexity in the tax function. In this section, we apply our sufficient statistics methods to characterize the optimal policy for a few

[16]A natural question is whether the effect of income received earlier in life—e.g., family income in childhood—should be used to measure the long-run income effect $s'_{inc}(z)$. It should not. As shown by Lemma 1 above, the role of $s'_{inc}(z)$ is to quantify the distortion in work-life income induced by a change in the steady-state tax on s, and this distortion depends on the causal effect of earnings *during work life* on s. To the extent that income earlier in life affects s consumption differently from income during work-life, the former behaves like a component of preference heterogeneity.

[17]Fagereng et al. (2021) use lottery winnings as a source of exogenous variation in unearned income. However, if individuals respond differently to a one-time change in unearned income than to a persistent change of equal present value, then $s'_{inc}(z)$ should be measured based on the latter. We discuss this issue, and an alternative measure of $s'_{inc}(z)$ based on survey data, in Section 7.

There is some evidence that mental accounting and other behavioral frictions affect people's propensity to consume and save out of windfalls. For example, Thakral and To (2021) show that people save more out of long-anticipated windfalls. Since steady-state changes in earnings correspond to anticipated changes in earnings, unanticipated windfalls could lead to an under-estimate of s'_{inc} when s corresponds to savings, and an over-estimate when s corresponds to immediate consumption.

classes of tax systems with restrictions such as separability or linearity. Focusing on more restricted systems also allows us to characterize optimal policy with multiple dimensions of heterogeneity. As is well-known, in settings with multidimensional heterogeneity, optimal fully flexible mechanisms tend to feature bunching, randomization, and other irregularities, which are sensitive to model assumptions. A common approach is thus to characterize the optimal policy within a restricted class of tax systems using conventional perturbation methods.[18] This section extends this literature by considering a more varied set of simple tax systems, and by expressing all results in terms of the sufficient statistics discussed in the previous section.

5.1 A Taxonomy of Common Simple Savings Tax Systems

Many governments tax both labor income and savings (or capital interest income). While these tax systems take a variety of forms, the details of which depend on specifics such as timing, many of these tax policies can be interpreted as a function of earnings and savings, analogous to our function $\mathcal{T}(s, z)$. Table I presents three classes of simple tax systems: separable linear (SL), separable nonlinear (SN), and linear earnings-dependent (LED).

Table I: Types of simple tax systems

Type of simple tax system	$\mathcal{T}(s,z)$	$\mathcal{T}'_s(s,z)$	$\mathcal{T}'_z(s,z)$
SL: separable linear	$\tau_s\, s + T_z(z)$	τ_s	$T'_z(z)$
SN: separable nonlinear	$T_s(s) + T_z(z)$	$T'_s(s)$	$T'_z(z)$
LED: linear earnings-dependent	$\tau_s(z)\, s + T_z(z)$	$\tau_s(z)$	$T'_z(z) + \tau'_s(z)\, s$

Table A1, provided as Supplementary Material, categorizes the tax policies on each of five classes of savings-related tax bases—wealth, capital gains, property taxes, pensions, and inheritances—for 21 countries, showing that most of these taxes can be understood as fitting into one of the three simple tax system classes from Table I. In cases where there is ambiguity, we provide supplementary information.[19]

In the United States, for example, most property taxes, levied at the state and local level, take the form of a separable linear tax, with a flat tax rate, independent of one's

[18] See Piketty and Saez (2013), Diamond and Spinnewijn (2011), and Gauthier and Henriet (2018) for examples restricting to a linear tax on s, and Saez and Stantcheva (2018) for a nonlinear separable tax on s and z.

[19] We impose several simplifications for our interpretation of the tax codes. First, we treat ordinary income as consisting primarily of labor income (earnings), written as z in our notation. Second, we separately consider taxes on five broad categories of savings vehicles: wealth, capital gains, real property, private pensions, and inheritances. These categories may overlap—real property is a component of wealth, for example—but we use these groups to reflect the tax instruments that many governments use in practice.

labor earnings, applied to the assessed value of the total property. The estate tax takes the form of a separable nonlinear tax: Tax rates rise progressively with the value of the estate, but they do not vary with labor income of the donor or the recipient. Taxes on long-term capital gains and qualified dividends take the form of linear earnings-dependent taxes.[20] In 2020, for example, an individual with \$50,000 in labor earnings faced a tax rate of 15% on long-term capital gains, whereas an individual earning \$500,000 faced a tax rate of 20%. Finally, although we focus on savings tax policies, these classes of simple tax systems are also relevant for other classes of commodities. Separable linear commodity taxes are ubiquitous (e.g., on lodging, airfare, and sales taxes that apply to specific classes of consumption); while separable nonlinear and linear income-dependent tax structures are often used for subsidies on goods like energy and education.[21]

5.2 Optimal Simple Tax Systems

We now present optimality conditions for SL, SN and LED tax systems. We focus on marginal tax rates on s in the body of the paper, and present conditions for marginal tax rates on z in Appendix B.5. The preference heterogeneity statistic s'_{het} remains the key sufficient statistic for the marginal tax rate on s. For SL systems, it is convenient to introduce the term $s_{het}(z) := \int_{x=z_{min}}^{z} s'_{het}(x)dx$, which integrates local preference heterogeneity across incomes to obtain a measure of total preference heterogeneity up to earnings level z.

Proposition 3. *Suppose that the optimal SL, SN, and LED systems satisfy Assumption 3, and that in the SN system s is strictly monotonic (increasing or decreasing) in z. Then, at each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions:*

[20] This is a slight approximation since the linear capital gains tax rate in the U.S. is a function of total income (including capital gains) rather than solely labor income.

[21] One practical distinction between taxes on savings and on other commodities involves the measurement of the tax base. In our baseline model, the argument of the tax function s represents the amount of the commodity s consumed. This is natural for many commodities, but in the setting of savings, it is common for the tax system to be written as a function of gross savings before taxes, e.g., a tax $T_1(z)$ in period 1, and a tax $T_2(s_g, z)$ on gross pre-tax savings $s_g = (1+r)(z - T_1(z) - c)$ in period 2, so that period-2 consumption is given by $s = s_g - T_s(s_g, z)$, where r is the compounded rate of return. In this formulation, a SL structure is one where $T_2(s_g, z) = \tau_s s_g$, a SN structure is one where $T_2(s_g, z)$ is a function of s_g only, and a LED structure is one where $T_2(s_g, z) = \tau_s(z) s_g$. Fortunately, there is an equivalence between these formulations of two-period tax systems and the type of "static" tax function $\mathcal{T}$ considered in our baseline model, allowing us to translate results between them. Appendix B.7 shows the nature of this equivalence in two steps. First, the set of allocations implementable by these systems is identical, as there is a simple and natural translation between the "static" tax function $\mathcal{T}$ we consider and the two-period function. Second, if $T_1(z) + T_2(s_g, z)$ implements the same allocation as $\mathcal{T}(s, z)$, then $T_1(z) + T_2(s_g, z)$ will be SL/SN/LED if and only if $\mathcal{T}(s, z)$ is SL/SN/LED.

$$SL: \frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s \zeta^c_{s|z}}} \int_z s'_{het}(z) \left[\int_{x \geq z} (1-\hat{g}(x)) dH_z(x) \right] dz \tag{23}$$

$$= -\frac{1}{\overline{s \zeta^c_{s|z}}} Cov[s_{het}(z), \hat{g}(z)] \tag{24}$$

$$SN: \frac{T'_s(s)}{1+T'_s(s)} = \frac{1}{s\, \zeta^c_{s|z}(z)} \frac{1}{h_z(z)} s'_{het}(z) \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) \tag{25}$$

$$LED: \frac{\tau_s(z)}{1+\tau_s(z)} = \frac{1}{s\, \zeta^c_{s|z}(z)} \frac{1}{h_z(z)} s'_{het}(z) \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) \tag{26}$$

Moreover, if a SL/SN/LED tax system is not Pareto dominated by another SL/SN/LED system, then it must satisfy the following conditions:

$$SL: \frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s \zeta^c_{s|z}}} \int_z s'_{het}(z)\; z\zeta^c_z(z)\; \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} \, dH_z(z) \tag{27}$$

$$SN: \frac{T'_s(s)}{1+T'_s(s)} = s'_{het}(z)\; \frac{z\zeta^c_z(z)}{s\zeta^c_{s|z}(z)} \; \frac{T'_z(z) + s'_{inc}(z) T'_s(s)}{1-T'_z(z)} \tag{28}$$

$$LED: \frac{\tau_s(z)}{1+\tau_s(z)} = s'_{het}(z)\; \frac{z\zeta^c_z(z)}{s\zeta^c_{s|z}(z)} \; \frac{T'_z(z) + \tau'_s(z)\, s + s'_{inc}(z)\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\, s} \tag{29}$$

The optimal tax formulas and the Pareto-efficiency conditions for SN and LED systems are analogous to the conditions for $\mathcal{T}'_s$ in the fully flexible smooth tax systems derived in Theorem 2. Appendix B.2 derives sufficiency conditions under which SN and LED systems can implement the optimal allocation. In the case of SN systems, these conditions are relatively weak, although they do require that s is weakly increasing with z in the optimal allocation. In contrast, the sufficiency conditions for LED systems are more restrictive, loosely requiring that the local preference for s must not increase "too quickly" across incomes, or else the second-order condition for earnings may be violated. Yet LED systems do not require a non-decreasing schedule of $s(z)$ in the optimal allocation. Thus these results illustrate that SN and LED systems have distinct advantages in different settings.

The SL system is the most restrictive and generically cannot implement the same allocation as the optimal smooth tax system. This is because the optimal smooth tax system does not generally feature constant marginal tax rates on s. Correspondingly, the optimal tax formula for the SL systems takes a different form. As shown in expression (23), the constant marginal tax rate τ_s for SL systems is in a certain sense an average of the z-earner specific marginal tax rates in expressions (25) and (26). Intuitively, the constant marginal tax rate is a population aggregate of the tax rates that would be optimal for individuals with different earnings levels. This is mirrored in the Pareto-efficiency condition (27). Expression (24) expresses this optimality condition in an alternative way, which was first derived by Allcott et al. (2019). This formulation has a familiar form resembling the Diamond (1975) "many-person Ramsey tax rule." The expression is identical to the Diamond (1975) formula when $s_{het}(z) = s(z)$; i.e., when there are no income effects and thus all consumption differences are driven by preference heterogeneity.[22] In contrast, when all consumption differences are driven by heterogeneity in income such that $s_{het}(z) \equiv 0$, it reduces to the original statement of the Atkinson-Stiglitz Theorem. More generally, even for arbitrarily nonlinear taxes on s, the optimal tax rate is always inversely proportional to the elasticity $\zeta^c_{s|z}(z)$, consistent with standard Ramsey principles, as long as $s'_{het}(z) \neq 0$.

5.3 Multidimensional Heterogeneity, Suboptimal Earnings Taxes

Our next result generalizes Proposition 3 in two key ways. First, it allows for multidimensional heterogeneity, where types θ belong to a subset of $\mathbb{R}^n$ for $n \geq 2$, so that the support of the distribution of (s, z) can be two-dimensional. Second, it characterizes optimal taxes on s even when the earnings tax $T_z(z)$ is not necessarily optimal.[23] Proposition B.6 in the Appendix characterizes optimal earnings tax schedules. The combination of Proposition B.6 and the result below provides a complete characterization of optimal simple tax schedules with multidimensional heterogeneity.

In settings with multidimensional heterogeneity, the relevant sufficient statistics may vary across the joint distribution of s and z. We use $\zeta^c_z(s, z)$ and $s'_{inc}(s, z)$ to denote the compensated elasticity of s and the causal income effect on s for an individual choosing the bundle (s, z). The formulas below also allow social marginal welfare weights g to be functions of both s and z.

Proposition 4. *Consider smooth simple tax systems with (potentially suboptimal) earnings tax schedules $T_z(z)$ and optimally set marginal tax rates on s. Assume that individuals' first- and second-order conditions hold in these tax systems, and that there is no bunching. Then, at each bundle (c^0, s^0, z^0) chosen by some type θ^0, the marginal tax rates on s in SL/SN/LED systems must satisfy the following optimality conditions:*

[22] In this special case of no income effects, the SN optimal tax formula (25) also nests a previous result: The optimal nonlinear tax formula of Saez and Stantcheva (2018), in which wealth choices originate entirely from preferences for wealth. Use $h_s(s(z)) = \frac{h_z(z)}{s'(z)}$ to recover the equivalence with formula (11) in Saez and Stantcheva (2018).

[23] For reference, we provide a characterization of optimal taxes on s assuming unidimensional heterogeneity and a given (potentially suboptimal) earnings tax $T_z(z)$, in Appendix B.4.

$$SL: \frac{\tau_s}{1+\tau_s}\int_z \left\{ \mathbb{E}\left[s\zeta^c_{s|z}(s,z) \middle| z \right] \right\} dH_z(z) = \int_z \left\{ \mathbb{E}\left[(1-\hat{g}(s,z))\, s \middle| z \right] \right.$$

$$\left. - \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z)\, s'_{inc}(s,z) \middle| z \right] \right\} dH_z(z) \tag{30}$$

$$SN: \frac{T'_s(s^0)}{1+T'_s(s^0)}\int_z \left\{ s^0\zeta^c_{s|z}(s^0,z) \right\} h\left(s^0,z\right) dz = \int_z \left\{ \mathbb{E}\left[1-\hat{g}(s,z) \middle| z, s \geq s^0 \right] \right\} dH_z(z)$$

$$- \int_z \left\{ \frac{T'_z(z) + s'_{inc}\left(s^0,z\right) T'_s\left(s^0\right)}{1-T'_z(z)} z\zeta^c_z(s^0,z) s'_{inc}\left(s^0,z\right) \right\} h\left(s^0,z\right) dz \tag{31}$$

$$LED: \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)\, s + s'_{inc}(s,z)\,\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\, s} z\zeta^c_z(s,z) s \middle| z^0 \right] h_z\left(z^0\right) = \int_{z \geq z^0} \mathbb{E}\left[(1-\hat{g}(s,z))\, s \middle| z \right] dH_z(z)$$

$$- \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta^c_{s|z}(s,z) \middle| z \right] + \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)\, s + s'_{inc}(s,z)\,\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\, s} z\zeta^c_z(s,z) s'_{inc}(s,z) \middle| z \right] \right\} dH_z(z) \tag{32}$$

Because Lemma 1 still holds, the s'_{inc} statistic, together with standard elasticity concepts, allow us to characterize optimal taxes on s in terms of observables. If all terms inside the expectation operators $\mathbb{E}[\cdot|z]$ in Proposition 4 are independent of each other, then the expectation can be applied to each statistic separately, and thus the unidimensional formulas are similar to the multidimensional formulas provided that all statistics are interpreted as averages conditional on z. For example, the first term in the integral in expression (30) can be written as $\left(1-\overline{\hat{g}(z)}\right)\overline{s(z)}$, where the "bar" notation denotes income-conditional averages. The second term in the integral can be written as

$$\frac{T'_z(z)\,\overline{s'_{inc}(z)} + \tau_s \overline{s'_{inc}(s,z)^2}}{1-T'_z(z)} z\overline{\zeta^c_z(z)}. \tag{33}$$

The main new effect is the square of s'_{inc} inside the integral. Because $\int (s'_{inc})^2 dH > \left(\int s'_{inc} dH\right)^2$ and because the square enters into the optimal tax expression negatively, this implies that ignoring multidimensional heterogeneity can lead to over-estimates of optimal marginal tax rates on s. Formulas (31) and (32) also involve squares of s'_{inc}, implying that multidimensional heterogeneity can similarly lower the optimal tax rate on s through $(s'_{inc})^2$. We quantify the importance of this insight in our empirical application in Section 7. More generally, positive covariances between pairs of statistics inside the expectation operator will tend to lower the optimal tax rate on s, while negative covariances will tend to increase it.

6 Extensions

In this section we provide three key extensions. First, we generalize our results to more than two dimensions of consumptions. This allows us to cover settings where, for example, individuals have access to multiple saving vehicles that are taxed differentially. Second, we allow for the possibility that the government wants people to save more than their perceived private optima, either because of a misalignment between private and social intergenerational preferences or because of individuals' behavioral biases. Third, we consider the case where taxes can be collected both in units of c and in units of s, as is often the case for savings taxes. These extensions highlight that s'_{het} remains a key sufficient statistic for optimal taxes, and that our previous formulas readily extend to these settings.

6.1 Multiple Goods

Suppose individuals consume $n+1$ goods, c and $\boldsymbol{s} = (s_1, s_2, ..., s_n)$. For example, $\boldsymbol{s}$ might correspond to different categories of saving, which the government might choose to tax in different ways. We consider a tax system $\mathcal{T}(\boldsymbol{s}, z) = \mathcal{T}(s_1, s_2, ..., s_n, z)$, where we retain the normalization that the numeraire c is untaxed. We normalize $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ to measure consumption in units of the numeraire. An individual of type θ then maximizes $U(c, \boldsymbol{s}, z; \theta)$ subject to the budget constraint $c + \sum_{i=1}^{n} s_i \leq z - \mathcal{T}(\boldsymbol{s}, z)$.

We denote own-price elasticities of goods by $\zeta^c_{s_i|z}(z)$, and we define cross-substitution elasticities by $\xi^c_{s_{j,i}|z}(z) := -\frac{\mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta),z)}{s_j(z;\theta)} \frac{\partial s_j(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta),z)}\Big|_{\theta=\vartheta(z)}$, where $s_j(z;\theta)$ denotes type θ consumption of good j when earning labor income z. We denote causal income effects on good s_j by $s'_{j,inc}(z) := \frac{\partial s_j(z;\theta)}{\partial z}\Big|_{\theta=\vartheta(z)}$. We continue using $\hat{g}(z)$ to denote the social marginal welfare effect of increasing a z-earner's consumption of c by one unit.[24]

Proposition 5. *Under Assumption 3, for each taxed good $i = 1, \ldots, n$ and at each bundle $(c, \boldsymbol{s}, z)$ chosen by a type θ, an optimal smooth tax system satisfies*

$$\frac{\mathcal{T}'_{s_i}(\boldsymbol{s}, z)}{1 + \mathcal{T}'_{s_i}(\boldsymbol{s}, z)} = s'_{i,het}(z) \frac{1}{s_i \zeta^c_{s_i|z}(z)} \frac{1}{h_z(z)} \int_{x \geq z} \Big[1 - \hat{g}(x)\Big] dH_z(x) + \underbrace{\sum_{j \neq i} \frac{\mathcal{T}'_{s_j}(\boldsymbol{s}, z)}{\mathcal{T}'_{s_i}(\boldsymbol{s}, z)} \frac{s_j \xi^c_{s_{j,i}|z}(z)}{s_i \zeta^c_{s_i|z}(z)}}_{\text{Tax diversion ratio}}. \tag{34}$$

[24]The formula for $\hat{g}(z)$ in this more general setting is in Appendix C.10.

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}'_{s_i}(\boldsymbol{s},z)}{1+\mathcal{T}'_{s_i}(\boldsymbol{s},z)} = s'_{i,het}(z)\frac{z\zeta^c_z(z)}{s_i\zeta^c_{s_i|z}(z)}\frac{\mathcal{T}'_z(\boldsymbol{s},z)+\sum_{j=1}^n s'_{j,inc}(z)\mathcal{T}'_{s_j}(\boldsymbol{s},z)}{1-\mathcal{T}'_z(\boldsymbol{s},z)} + \underbrace{\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}(\boldsymbol{s},z)}{\mathcal{T}'_{s_i}(\boldsymbol{s},z)}\frac{s_j\xi^c_{s_{j,i}|z}(z)}{s_i\zeta^c_{s_i|z}(z)}}_{\text{Tax diversion ratio}}. \tag{35}$$

Proposition 5 features all of the familiar terms of Theorem 2, and includes a novel term that captures the tax implications of substitution effects between the different goods. Intuitively, substituting from s_i to higher-taxed goods generates positive fiscal externalities that motivate higher marginal tax rates on s_i, while substitution to lower-taxed goods generates negative fiscal externalities that motivate lower marginal tax rates on s_i. These effects are summarized by what we call the tax diversion ratio—the impact on taxes collected on goods $j \neq i$ relative to the impact on taxes collected on good i, when the price of good i is increased. The higher the diversion ratio, the more favorable are the fiscal externalities associated with substitution away from good i, and thus the higher is the optimal tax rate on good i.

6.2 Optimal Taxation with Corrective Motives

Our framework can be interpreted as a bequest model in which parents work and consume in the first period, and leave a bequest to their heirs in the second period. Under this interpretation, our baseline model makes the implicit assumption that the government values bequests in the same way as parents. Farhi and Werning (2010) consider a model where the weight that parents attach to the well-being of future generations is too low from a normative perspective. This misalignment introduces a motive to encourage bequests, which we consider in this extension.

Following Farhi and Werning (2010), we assume additively separable preferences given by

$$U\big(c,s,z;\theta\big) = u\left(c;\theta\right) - k\left(z;\theta\right) + \beta v\left(s;\theta\right), \tag{36}$$

where $u\left(c;\theta\right)$ is the utility parents derive from consumption c, $k\left(z;\theta\right)$ is the disutility parents incur to obtain earnings z, $v\left(s;\theta\right)$ is the utility heirs derive from a bequest s, and β is the weight parents attach to the well-being of their heirs. As in Farhi and Werning (2010), the government maximizes

$$\int_\theta \big[U\left(c\left(\theta\right),s\left(\theta\right),z\left(\theta\right);\theta\right) + \nu v\left(s\left(\theta\right);\theta\right)\big]dF\left(\theta\right), \tag{37}$$

where ν parametrizes the degree of misalignment. Farhi and Werning (2010) microfound ν as the Lagrange multiplier associated with a constraint that the future generation attains a required level of well-being $\int_\theta v\left(s\left(\theta\right);\theta\right)dF\left(\theta\right)\geq \underline{V}$.

The formal model above can be interpreted more generally beyond the bequest application and can be used to analyze behavioral biases as a motivation for encouraging savings. In particular, suppose that $v(s;\theta)=\delta(\theta)u(s;\theta)$, where δ is the "exponential discount factor" and β is "present focus," as in Laibson (1997). If the government utilizes the "long-run criterion" for welfare, then the degree of misalignment is given by $\nu=(1-\beta)$.[25] More generally, β may be heterogeneous, so that misalignment is type-dependent and given by $\nu(\theta)=(1-\beta(\theta))$. For example, Lockwood (2020) summarizes evidence suggesting that individuals with higher earnings ability have lower degrees of present focus.

Below, we characterize optimal taxation with heterogeneous misalignment, where $\beta(z)$ and $\nu(z)$ denote the parameters corresponding to a z-earner. This generalizes the result in Farhi and Werning (2010) by (i) allowing heterogeneity in preferences for s, and by (ii) allowing heterogeneity in the misalignment parameter ν.

Proposition 6. *Under Assumption 3, at each bundle (c,s,z) chosen by a type θ, an optimal smooth tax system satisfies the following marginal tax rate condition*

$$\frac{\mathcal{T}_s'\left(s,z\right)}{1+\mathcal{T}_s'\left(s,z\right)}=s_{het}'\left(z\right)\frac{1}{s\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left[1-\hat{g}(x)\right]dH_z(x)-\frac{\nu(z)}{\beta(z)}g(z). \tag{38}$$

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}_s'\left(s,z\right)}{1+\mathcal{T}_s'\left(s,z\right)}=s_{het}'\left(z\right)\frac{z\zeta_z^c(z)}{s\zeta_{s|z}^c(z)}\left[\frac{\mathcal{T}_z'\left(s,z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s,z\right)}{1-\mathcal{T}_z'\left(s,z\right)}+s_{inc}'(z)\frac{\nu(z)}{\beta(z)}g(z)\right]-\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z). \tag{39}$$

This is an intuitive generalization of Theorem 2, where the key new term is a form of Pigovian correction, given by $\frac{\nu(z)}{\beta(z)}g(z)$. As equation (38) shows, the presence of misalignment motivates the government to lower the tax rate on s. The degree by which the government lowers the tax rate depends on the degree of misalignment (relative to the discount factor β), and on the social marginal welfare weight. Because social marginal welfare weights decline with z, equation (38) gives the "progressive estate taxation" result of Farhi and Werning (2010)—i.e., savings subsidies that decline with income—under the special assumptions that (i) $s_{het}'\equiv 0$ and (ii) $\beta(z)\equiv\beta\in\mathbb{R}$, $\nu(z)\equiv\nu\in\mathbb{R}$. This core result of

[25] See Bernheim and Taubinsky (2018) for a detailed discussion of such a criterion, as well as alternative normative approaches to studying the implications of present focus.

Farhi and Werning (2010) extends the standard Pigovian taxation logic to optimal screening of distortions with a nonlinear tax.

More generally, Proposition 6 provides a simple formula for balancing the "corrective motives" studied by Farhi and Werning (2010) with the additional motives to tax s in the presence of preference heterogeneity studied in this paper. This extends the Allcott et al. (2019) results for linear commodity taxes with biased consumers to study optimal screening of biases with a nonlinear tax. If $s'_{het}(z) > 0$ and $\nu(z)/\beta(z)$ and $g(z)$ are decreasing with z, Proposition 6 suggests a progressive tax on s that can feature subsidies at low incomes and taxes at high incomes.

6.3 Tax Arbitrage with Heterogeneous Prices

Thus far we have considered tax functions where the tax is always paid in units of the numeraire commodity c. In some applications it is also natural to consider tax systems with *multidimensional range*, which include taxes collected in units of c and also in units of s. This is natural, for example, if c and s correspond to period 1 and period 2 consumption, respectively, and taxes must be paid in both periods. The additional richness in the range does not alter the optimal tax implications when the rates of transformation p are homogeneous; in equilibrium, the government's rate of transformation is the same as the homogenous rate for individuals, and it does not matter what portion of the total tax bill is collected in units of s. However, we show that when prices are heterogeneous, there is an additional efficiency rationale for differentially taxing s. Heterogeneity in prices motivates a form of "tax arbitrage," where the government collects relatively more taxes in units of s from individuals who can obtain s at a low price or, in the case of savings, it imposes relatively higher savings taxes (and lower earnings taxes) on individuals with high rates of returns. This extension provides a generalization both of our baseline results and the independent work of Gerritsen et al. (2020), which also studies the role of such efficiency effects.

Formally, suppose that the government uses a two-part tax structure, where individuals pay a tax $T_1(z)$ in units of c and a tax $T_2(s, z)$ in units of s. For instance, in a two-period model where s is savings, T_1 represents the earnings tax levied in period 1 and paid in period-1 dollars, while T_2 represents the savings tax levied in period 2 and paid in period-2 dollars, and $p = 1/(1 + r)$ is a function of the rate of return r. For concreteness, we refer to T_1 as period-1 taxes and to T_2 as period-2 taxes, though we emphasize that the presence of efficiency effects is not about dynamics per se, but rather that T_2 is collected in units of s.

Following Gahvari and Micheletto (2016), we consider heterogeneous prices $p(z, \theta)$

that are a function of gross earnings and type. For example, wealthier individuals may have access to better rates of return on savings or prices of commodities. Alternatively, higher earnings ability may be associated with a better ability to obtain high rates of return or to find better prices.

Individuals maximize utility $U(c, s, z; \theta)$ subject to the budget constraint $c+p(z,\theta)s \leq z-T_1(z)-p(z,\theta)T_2(s,z)$. Denoting by $\vartheta(z)$ the type θ of individuals who choose earnings z, we slightly abuse notation to define $p(z) := p(z, \vartheta(z))$. The government, as before, maximizes a weighted average of utilities,

$$\int_z \left\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z), z)\big), s(z), z; \vartheta(z)\Big) \right\} dH_z(z), \tag{40}$$

subject to the constraints

$$\int_z T_1(z) dH_z(z) \geq E_1 \quad \text{and} \quad \int_z T_2(s(z), z) dH_z(z) \geq E_2, \tag{41}$$

which generate marginal values of public funds λ_1 and λ_2 . We continue using $\hat{g}(z)$ to denote the social marginal welfare effect of increasing a z-earner's consumption of c by one unit, normalized by the marginal value of public funds λ_1.[26]

Heterogeneity in p generates efficiency effects through two channels, which are represented in Proposition 7 below. First, for individuals with relatively low $p(z)$, it is efficient for the government to decrease T_1 and increase T_2. This efficiency effect is present irrespective of the mechanism for the cross-sectional variation of p with z and leads to a deviation from the Atkinson-Stiglitz Theorem.

Second, lump-sum changes in T_2 trigger novel substitution effects. This is because a lump-sum increase dT in T_2 has the same effect on an individual's utility as a $p(z)dT$ increase in T_1, and thus changes behavior as much as a marginal tax rate change of $\frac{\partial p}{\partial z} dT$ in T_1. We denote by $\varphi(z) := -\left(T_1'(z) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(z)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{z\zeta_z^c(z)}{1-T_1'(z)}\frac{\partial p}{\partial z}$ the fiscal impacts of this substitution effect at earnings z. The impact of a uniform lump-sum change in T_2 is then $\overline{\hat{g}p - \varphi}$, where the "bar" notation is used to denote a population average across all earnings levels. Thus, $\lambda_2/\lambda_1 = \overline{\hat{g}p - \varphi}$, as we formally show in Appendix C.12.3.

Proposition 7. *Under Assumption 3 and under the assumption that for SN systems s is strictly monotonic (increasing or decreasing) in z, at each bundle (c, s, z) chosen by a type*

[26] The formula for $\hat{g}(z)$ in this more general setting is in Appendix C.12.3.

θ, an optimal SN two-part tax system $\{T_1(z), T_2(s)\}$ satisfies

$$\frac{\lambda_2/\lambda_1 T_2'(s)}{1+p(z)T_2'(s)} = \frac{1}{s\zeta^c_{s|z}(z)}\frac{1}{h_z(z)}\left\{s'_{het}(z)\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x)+\frac{s'(z)}{p(z)}\Big(\Psi(z)+\int_{x\geq z}\Big[\varphi(x)-\overline{\varphi}\Big]dH_z(x)\Big)\right. \tag{42}$$

where

$$\Psi(z) := \Big(1-H_z(z)\Big)\int_{x\leq z}\hat{g}(x)\Big(p(x)-p(z)\Big)dH_z(x)+H_z(z)\int_{x\geq z}\hat{g}(x)\Big(p(z)-p(x)\Big)dH_z(x). \tag{43}$$

An optimal LED two-part tax system $\{T_1(z), \tau_s(z)s\}$ satisfies

$$\begin{aligned}\frac{\lambda_2/\lambda_1 \tau_s(z)}{1+p(z)\tau_s(z)} &= \frac{1}{s(z)\zeta^c_{s|z}(z)}\frac{1}{h_z(z)}\left\{s'_{het}(z)\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x)+\frac{p'(z)}{p(z)}s(z)\int_{x\geq z}\Big[1-\hat{g}(x)\Big]dH_z(x)\right\}\\ &\quad+\frac{1}{\zeta^c_{s|z}(z)}\frac{1}{p(z)}\Big\{\overline{\hat{g}p}-\overline{\hat{g}}p(z)+\varphi(z)-\overline{\varphi}\Big\}.\end{aligned} \tag{44}$$

Proposition 7 shows that the sufficient statistic $s'_{het}(z)$ continues to play a central role.[27] On the left-hand side of (42) and (44), the presence of $p(z)$ in the denominator is because an individual's marginal tax rate on s, translated to units of c, is $p(z)\frac{\partial T_2}{\partial s}$. The presence of λ_2/λ_1 in the numerator of the left-hand side is because fiscal externalities generated by substitution away from s must be weighted by the "period-2" marginal value of public funds.

Proposition 7 also introduces novel efficiency terms that to lead to taxes on s, even when $s'_{het} \equiv 0$. In the SN formula, there are two additional efficiency effects. These terms are both positive and thus push toward taxing s when higher earners (i) face lower prices p (e.g., higher rates of returns on savings) and choose higher levels of s, and (ii) exhibit larger substitution effects φ. The first term, proportional to $\Psi(z)$, captures the efficiency effects of increasing period-2 taxes. This term is unambiguously positive when p decreases cross-sectionally with z and captures the intuition that with a SN system, increasing marginal tax rates on s at any point $z > z_{\min}$ increases period-2 taxes on individuals with below-average p. The second term, proportional to $\int_{x\geq z}\Big[\varphi(x)-\overline{\varphi}\Big]dH_z(x)$, captures the fact that increasing marginal tax rates on s motivates individuals to increase labor supply in order to get lower prices p when $\frac{\partial p}{\partial z} < 0$. The SN formula generalizes the result in Gerritsen et al. (2020) to incorporate other forms of across-income heterogeneity and makes transparent the sign of these terms in a formula employing measurable sufficient statistics.

The implications for LED tax systems are somewhat different. Assume again that p de-

[27] A characterization of the optimal earnings tax schedule $T_1(z)$ is in Appendix C.12.4.

clines cross-sectionally with z (i.e., $p'(z) < 0$). The first novel term in equation (44), proportional to $p'(z)/p(z)$, reflects the fact that higher earners are less responsive to marginal changes in T_2 when $p(z)$ declines with income, since period-2 consumption is "cheaper" for them than period-1 consumption. The second term, proportional to $\overline{\hat{g}p} - \bar{\hat{g}}p(z) + \varphi(z) - \overline{\varphi}$, is also negative for sufficiently low values of z, as in this case both $\overline{\hat{g}p} - \bar{\hat{g}}p(z)$ and $\varphi(z) - \overline{\varphi}$ are negative. However, this term is positive for sufficiently high values of z. Thus, when $s'_{het}(z) \equiv 0$, the optimal LED system features subsidies on s for lower-income individuals and taxes on s for higher-income individuals.

The contrast in implications for SN versus LED tax systems—everywhere-positive tax rates in the former, subsidies followed by taxes in the latter—highlights that the new efficiency considerations from heterogeneous rates of return depend on the types of restrictions imposed on the tax system. The reason for this dependence is because positive tax rates on s are a consequence of a missing instrument problem. In a fully flexible tax system, the efficiency gains of taxing a person in period 2 instead of period 1 could be obtained by shifting each individual's total tax burden onto their lowest-cost tax base up to the point that heterogeneous prices are arbitraged away, without the distortion of increasing marginal tax rates on s. But less flexible tax systems can only generate this shifting of the tax burden by altering marginal tax rates on s, and the optimal means of doing this depend on the nature of the restricted tax system.

7 Empirical Application

We apply our formulas to the question of savings taxes in the United States. We first calibrate the relevant sufficient statistics from micro data and empirical studies. In section 7.2, we devote particular attention to the calibration of the sufficient statistic $s'_{het}(z)$. We then use the Pareto-efficiency conditions derived in Proposition 3 to compute the SL, SN and LED savings tax schedules that would be consistent with the status quo income tax schedule. This allows us to study the welfare-improving reforms that could be made to the existing tax system, taking as given the distributional preferences already embedded in the existing income tax. As is typical for calculations based on sufficient statistics formulas, these results are approximations, as they do not account for changes in the underlying distributions and sufficient statistics that might arise if the savings tax were reformed. These results suggest that across-income heterogeneity leads to a (mostly) positive and progressive schedule of savings tax rates, which range from approximately 0% at the bottom of the income distribution up to between 15% and 20% at higher incomes in our baseline calibration.

7.1 Calibration

We calibrate a model of the U.S. economy that can be interpreted through the lens of our model with a joint savings and income tax function $\mathcal{T}(s, z)$, expressed in terms of the three simple tax systems described in Table I. Appendix D discusses details of this calibration; here, we summarize the key steps. We calibrate a two-period model economy with a fine grid of incomes, where the first period corresponds to work life and the second to retirement. We assume a constant (and, in our baseline, homogeneous) annual rate of return of 3.8% before taxes, drawing from Fagereng et al. (2020). We calibrate a version of the economy with unidimensional heterogeneity (i.e., a single level of savings at each income) and a version with multidimensional heterogeneity, reporting results for each below.

Because our model builds on standard models of commodity taxation, it implicitly assumes that z and $\mathcal{T}(s, z)$ are measured in the same units as consumption, which in a dynamic setting corresponds to "period-1" dollars. In practice, savings taxes are typically levied after returns, and they are thus measured in "period-2" dollars. We accordingly translate all tax rates into units of period-2 dollars when reporting results, so that a marginal savings tax rate of 10% indicates that if an individual's total wealth at retirement increases by \$1, then they must pay an additional \$0.10 in taxes when they retire. Appendix D.1 discusses details of our calibration and this conversion.

To calibrate the earnings and savings distributions—and thus the across-income savings profile $s(z)$—we use the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65). Discretizing the income distribution into percentiles by age group, our measure of annualized earnings during the working life z at the nth percentile is constructed by averaging earnings at the nth percentile across those aged 20 to 44 and those aged 45 to 64. Our measure of s is the average value of net personal wealth, *hweal*, projected forward to age 65 based on the value observed at each income percentile in the 45-64 age bucket. This measure of wealth includes housing assets, business assets, and financial assets, net of liabilities, as well as defined-contribution pension and life insurance assets.[28] This provides us with a population of representative individuals at each percentile of the income distribution, for whom period 1 represents their working life, with a representative age of 45, and period 2 represents retirement, which occurs 20 years later at age 65. We normalize both labor earnings and retirement savings by the number of years worked.

[28] The ongoing methodological discussion regarding the different ways to measure wealth (See e.g. Saez and Zucman, 2020; Smith et al., 2021) has important implications for estimates of wealth in the top 1% but has little impact on the wealth distribution of the rest of the population that we are using here.

Figure A1 plots our estimate of gross (i.e., after-returns and before-tax) discretionary savings per year worked, across the income distribution. This does not include Social Security, which we model as lump-sum forced savings that are received during retirement. The figure shows that the stock of savings upon retirement are approximately zero at low incomes, but increase substantially with income. We convert this to net-of-tax savings using a calibration of savings tax rates across the earnings distribution in the U.S., derived by computing the weighted average of savings tax rates using the asset composition of savings portfolios reported in Bricker et al. (2019); see Appendix D.1.2 for details. The convex shape of the savings profile, which persists after accounting for taxes, indicates that the cross-sectional slope $s'(z)$ rises with income, as shown by the solid blue line in Figure I.

We assume a constant compensated earnings elasticity of $\bar{\zeta}_z^c = 0.33$, drawn from the meta-analysis of Chetty (2012). The value of the savings elasticity $\zeta_{s|z}^c$ is related to the elasticity of taxable wealth (e.g., Jakobsen et al., 2020) and to the elasticity of capital gains realizations with respect to the capital gains tax (e.g., Agersnap and Zidar, 2021). However, studies that measure elasticities from responses to tax reforms are likely inflated by cross-base responses, as taxpayers re-optimize their savings portfolio towards savings vehicles that are relatively less taxed after the reform.[29] We report results for a broad range of values spanning $\zeta_{s|z}^c = 0.7$ to $\zeta_{s|z}^c = 3$, with a baseline of $\zeta_{s|z}^c = 1$, which approximately aligns with the baseline calibration considered in Golosov et al. (2013), in which the intertemporal elasticity of substitution is set to 1. Appendix D.1.4 discusses this conversion.

7.2 Estimation of s'_{het}

To estimate our measure of local preference heterogeneity $s'_{het}(z) = s'(z) - s'_{inc}(z)$, we employ two complementary strategies.

The first estimation strategy is motivated by the Proposition 2 result that when preferences are separable in s and z, the causal income effect of windfall income on s consumption identifies $s'_{inc}(z)$. To the extent that separability is plausible, we can exploit exogenous shocks to unearned income in order to estimate $s'_{inc}(z)$. To implement this strategy, we draw from Fagereng et al. (2021), who estimate the marginal propensity to consume (MPC) out of windfall income across the earnings distribution using information on lottery prizes linked with administrative data in Norway.[30] Lottery consumption is widespread in Nor-

[29]Our extension to many goods (Section 6.1) shows how the inclusion of cross-base responses affect optimal savings tax formulas. It could be used to compute the optimal savings tax on different savings vehicles, if there was a larger body of empirical evidence on savings elasticities and cross-base responses.

[30]Two other recent studies point to the promise of estimating such causal marginal propensities in a variety of settings. Golosov et al. (2021) study the response to lottery prize winnings in the U.S., although the

way—over 70% of adults from all income groups participated in 2012—and administrative records of asset and wealth holdings allow for direct measures of savings and consumption responses to lottery winnings. Fagereng et al. (2021) find that individuals' consumption peaks during the winning year and then gradually reverts to the the pre-win baseline. Over a 5-year horizon, they estimate that winners consume close to 90% of the prize (see their Figure 2, "aggregate consumption response") which translates into a long-run MPC of 0.9, and a marginal propensity to save of 0.1. They do not find significant heterogeneity across incomes in this MPC. We convert this homogeneous MPC into a response of net retirement savings to changes in pre-tax labor income using our calibrated schedules of income and savings tax rates—which introduces a small amount of heterogeneity across incomes—in order to arrive at $s'_{inc}(z)$. The resulting profile is plotted as the dashed red line in Figure I. The difference between this estimated schedule of $s'_{inc}(z)$ and the cross-sectional profile $s'(z)$, also plotted in Figure I, is $s'_{het}(z)$.

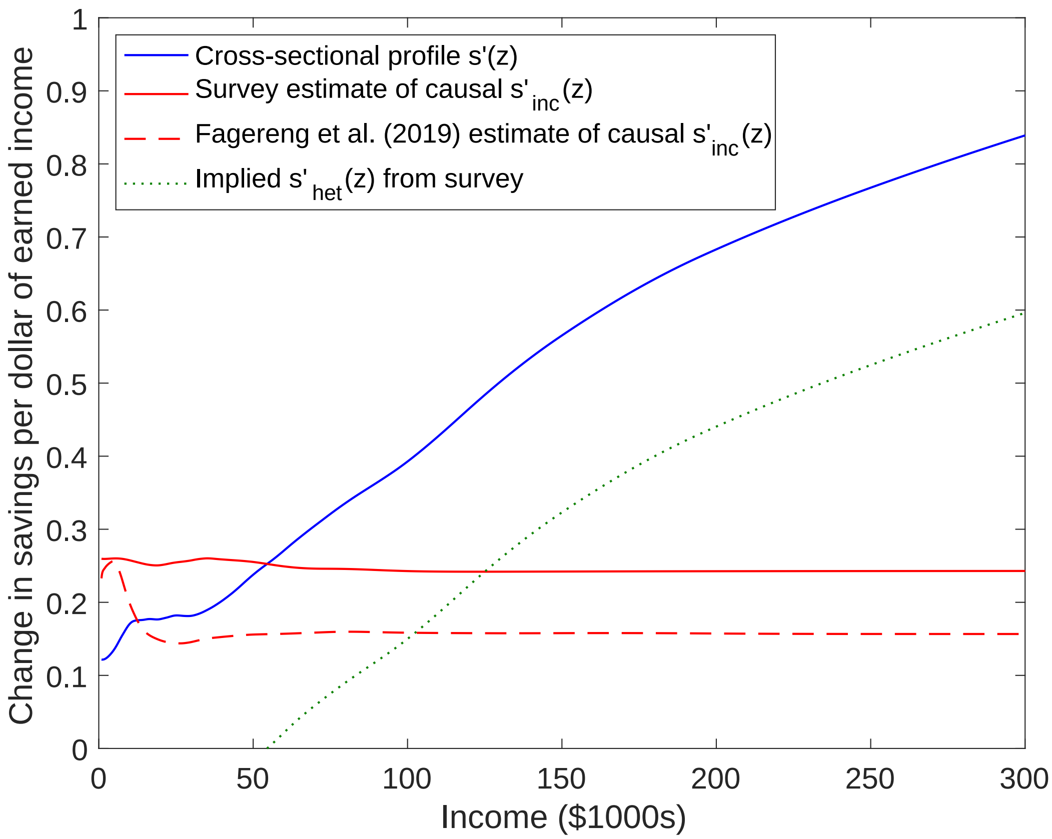
Our second strategy for estimating $s'_{het}(z)$ utilizes a new probability-based survey representing the U.S. adult population, conducted on the AmeriSpeak panel in the spring of 2021. We asked respondents to report how much more they would save each year if they received a hypothetical raise that increased their household's annual income by \$1000 in the coming years. This strategy has the advantage of being based on the U.S. population, and of asking directly about a modest, *persistent* change in *pre-tax earned* income, rather than a large one-time windfall, so that it does not require a weak separability assumption. The survey results suggest a short-run marginal propensity to save of 0.60, close to that reported in Fagereng et al. (2021), with little variation across incomes. We translate this into a long-run MPC using the response profile of Fagereng et al. (2021), which we in turn convert to the across-income schedule of $s'_{inc}(z)$ displayed as the solid red line in Figure I. See Appendix D.1.3 for additional details about the survey results.

The two estimation strategies provide remarkably consistent estimates of $s'_{inc}(z)$, as shown in Figure I. There is a substantial difference between the cross-income profile $s'(z)$ and the causal income effect $s'_{inc}(z)$, suggesting that $s'_{het}(z)$ is positive across most of the income distribution and rises with income.[31]We use the survey-based estimate as the baseline for our numerical results.

absence of third-party administrative reporting of wealth in the U.S. complicates the measurement of marginal propensities to save. Straub (2018) estimates the propensity to save out of permanent income, although the absence of quasi-experimental variation in earnings makes it difficult to separate causal income effects from across-income heterogeneity.

[31]Our measure of $s'_{het}(z)$ appears to be slightly negative at low incomes, which in our simulations gives rise to slight savings subsidies at low incomes. However, we note that this could be driven by limitations in our ability to measure $s'_{inc}(z)$ at low incomes. This emphasizes the value of additional empirical research on this statistic.

Figure I: Decomposition of Cross-Sectional Savings Profile



Notes: This figure reports the slope of the cross-sectional profile of savings $s'(z)$ (blue), as well as our calibrations of $s'_{inc}(z)$ based on causal income effects, derived from Fagereng et al. (2021) and from a new nationally representative survey. See Section 7 and Appendix D.1 for details.

By way of comparison, Golosov et al. (2013) estimate preference heterogeneity by estimating differences in discount factors across ability levels. They infer discount factors from a simple parametric model of savings choice applied to survey data on individuals' household income paths and net worth, and they use survey respondents' results to the Armed Forces Qualification Test (AFQT) as a proxy for ability. In contrast to our findings, their estimation strategy finds very little measured preference heterogeneity, amounting to less than 1% of the cross-sectional variation in savings (see Appendix D.1.3). This discrepancy could be driven by attenuation bias due to measurement error in their proxy for ability—an issue we avoid by computing preference heterogeneity directly as a difference of two statistics rather than from regression analysis. It could also be driven by their use of a narrower measure of across-income heterogeneity based only on time preferences, as opposed to all of the possible forms of across-income heterogeneity that our statistic comprises.

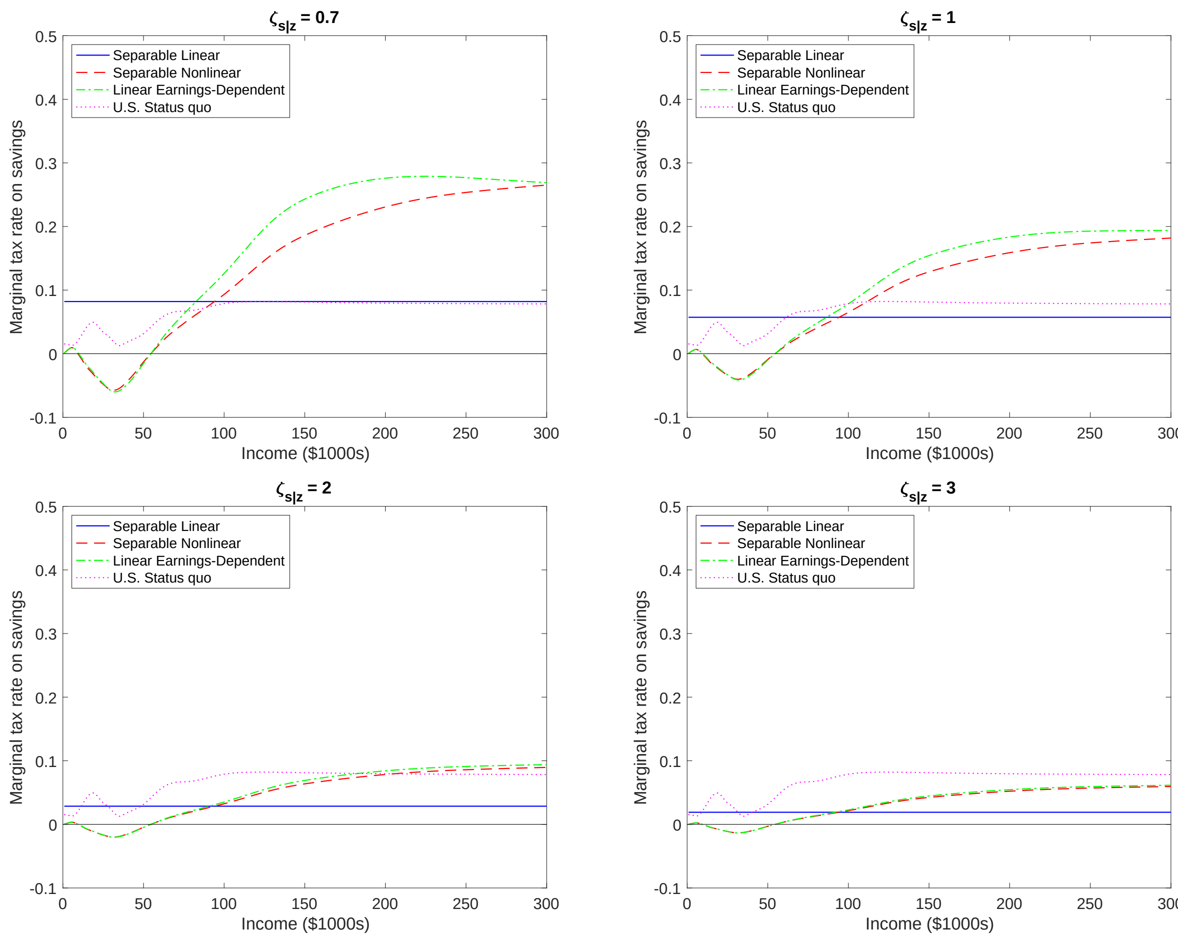
7.3 Results

Figure II reports the schedule of marginal tax rates for SL, SN and LED tax systems that satisfy the Pareto efficiency formulas in Proposition 3, taking the existing U.S. income tax schedule and income distribution as given. In each case, we translate the tax into a marginal tax rate on gross savings at retirement, measured in period-2 dollars. Each panel reports results for a different value of the savings elasticity. For SL tax systems, the linear savings tax rate τ_s is by definition constant across earnings levels. For LED tax systems, the linear savings tax rate $\tau_s(z)$ is earnings dependent and we thus report the linear savings tax rate at each earnings level. For SN tax systems, the nonlinear savings tax schedule $T_s(s)$ depends on the value of savings s, and not on earnings z. But to make the SN system visually comparable to the other systems, we plot the marginal savings tax rate faced at the margin by each earner, given their level of saving (represented on Figure A1).

In each panel, marginal savings tax rates are mostly positive, and the nonlinear tax schedules are progressive, with marginal rates increasing with income. The magnitudes depend on the value of the savings elasticity parameter. In the baseline case of $\zeta^c_{s|z} = 1$, savings tax rates in SN and LED tax systems average approximately 0% for annual incomes below $\$50,000$, then steadily increase up to nearly 20% for annual incomes around $\$200,000$, remaining stable thereafter. The savings tax rate in a SL tax system is approximately 6%. Changing the savings elasticity parameter scales the efficient savings tax rates without affecting the overall pattern: across-income heterogeneity calls for (mostly) positive and progressive savings tax rates. At the lower elasticity values, our estimates of optimal tax rates are substantially higher than the prevailing savings tax rates in the U.S., which are also shown in Figure II.

Figure III considers two key extensions to these results: multidimensional heterogeneity, as in Section 5.3, and heterogeneous rates of return with “tax arbitrage” efficiency effects, as in Section 6.3. For comparability with our baseline results, all other parameters, including elasticity parameters and income-dependent welfare weights, are held fixed at the values from our baseline calibration. These results are computed using our baseline savings elasticity of $\zeta^c_{s|z} = 1$. We plot both types of nonlinear tax schedules, LED and SN, omitting the separable linear plots for legibility.

Figure II: Savings Tax Rates Implied by Pareto-Efficiency Formulas

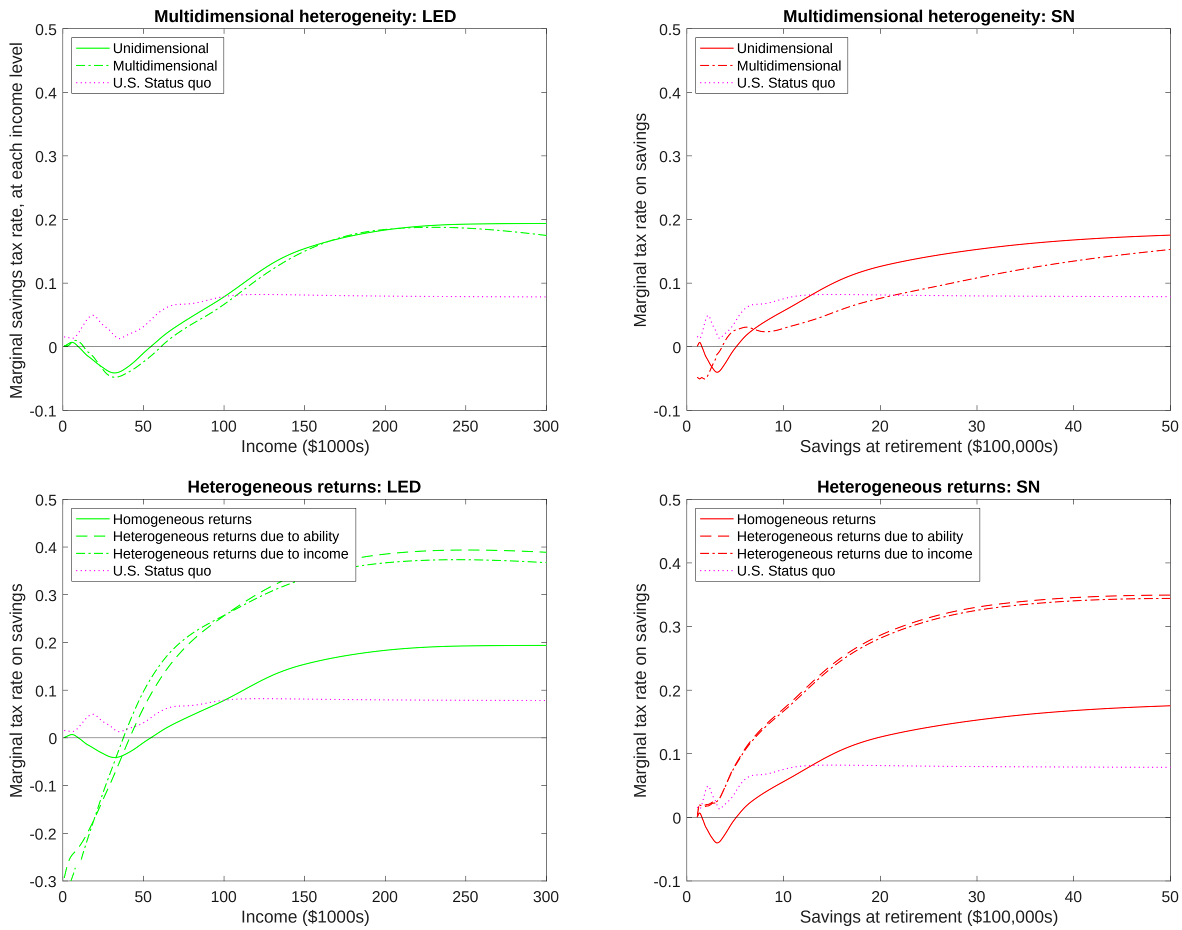


Notes: This figure presents the marginal savings tax rates values that satisfy the Pareto-efficiency formulas in Proposition 3, plotted against the earnings level to which they apply. We plot these schedules for four different values of the savings elasticity $\zeta_{s|z}$, with $\zeta_{s|z} = 1$ representing our baseline case.

In the case of multidimensional heterogeneity, we use the same measure of gross savings, but rather than compute average savings at each income, we partition the population into four levels of savings at each level of income, representing quartiles of the income-conditional savings distribution.

In the case of heterogeneous rates of return, we follow Gerritsen et al. (2020) who, relying on empirical work by Fagereng et al. (2020), assume that rates of return rise by 1.4 percentage points from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return.

Figure III: Effects of Multidimensional Heterogeneity and Heterogeneous Returns



Notes: This figure plots the marginal savings tax rate schedules which are optimal, according to the first-order condition formulas presented in the text, for two extensions discussed in Section 6: multidimensional heterogeneity (top row), and heterogeneous returns (bottom row). All plots also reproduce the Pareto-efficient savings schedules from Figure II for comparison, as well as the status quo U.S. savings taxes. These plots use the same set of social welfare weights, calibrated to rationalize the status quo income tax in the unidimensional model. The Linear Earnings-Dependent (LED) schedules, in the left column, are plotted across earnings during work life. The Separable Nonlinear (SN) schedules, in the right column, cannot be plotted this way, because individuals with a given income have different levels of savings and are thus subject to different savings taxes. We therefore plot them over total savings at the time of retirement. See Section 7 and Appendices D.2 and D.3 for details.

Consistent with the intuition described in Section 5.3, the top two panels of Figure III show that incorporating multidimensional heterogeneity reduces the magnitude of optimal tax rates in LED systems (top left panel) and in SN systems (top right panel). The effect is particularly pronounced for SN systems, where savings tax rates are plotted as a function of total savings at the time of retirement, since individuals with the same income save different amounts and thus face different savings tax rates. In this extension, marginal savings tax rates are still progressive, and are above status quo savings tax rates across high incomes in our baseline specification.

The bottom two panels show that the presence of heterogeneous rates of returns tends to significantly raise optimal savings tax rates, reflecting the efficiency effects of tax arbitrage highlighted in Proposition 7.[32] The bottom right panel shows that tax rates in the SN system are higher at all levels of income, consistent with our discussion of the formula for SN systems in Proposition 7. On the other hand, recall that the formula for LED systems implied lower savings tax rates at low incomes and higher tax rates at higher incomes. Consistent with this, the bottom left panel shows that relative to the baseline, the optimal savings tax rates with heterogeneous rates of return are even more progressive. For example, substantial savings subsidies are optimal for incomes below about $\$40,000$, whereas savings taxes are substantially higher at higher incomes.

Taken together, our empirical results show a robust role for progressive savings taxes, stemming from across-income heterogeneity captured by the s'_{het} statistic. This highlights the importance of this statistic and motivates additional empirical work estimating the long-run marginal propensity to save out of earned income, as well as across-income consumption profiles and causal income effects in other applications.

8 Conclusion

This paper characterizes optimal tax systems on earnings and savings (or other dimensions of consumption) in the presence of general across-income heterogeneity. We first prove that with unidimensional heterogeneity, the optimal allocation can be implemented by a smooth tax on earnings and savings. We then derive formulas that characterize the optimal smooth tax system through familiar empirical statistics, as well as a key sufficient statistic for across-income heterogeneity, $s'_{het}(z)$. This statistic is empirically estimable and captures heterogeneity in preferences, heterogeneous rates of return, endowments, or income-shifting abilities. We then use the same sufficient statistics to characterize a set of "simple" separable tax systems that are widely used in practice, considering both unidimensional and multidimensional heterogeneity. We also provide tractable extensions to multiple goods, corrective motives, and heterogeneous prices with "tax arbitrage" efficiency effects. Finally, we apply our theoretical formulas to the setting of savings taxes in the U.S.

[32]Consistent with the tax arbitrage interpretation, these efficiency effects are (almost) unaffected by whether return heterogeneity is driven by income (scale dependence) or by ability (type dependence).

Results suggest that the savings tax rates that would be consistent with the existing income tax are progressive and (mostly) positive. Together, these results provide a practical and general method for quantifying optimal tax systems for savings, inheritances, and other commodities in the presence of general across-income heterogeneity.

References

AGERSNAP, O. AND O. M. ZIDAR (2021): "The Tax Elasticity of Capital Gains and Revenue-Maximizing Rates," *American Economic Review: Insights*, 3, 399–416.

ALLCOTT, H., B. B. LOCKWOOD, AND D. TAUBINSKY (2019): "Regressive Sin Taxes, with an Application to the Optimal Soda Tax," *Quarterly Journal of Economics*, 134, 1557–1626.

ATKINSON, A. B. AND J. E. STIGLITZ (1976): "The design of tax structure: direct versus indirect taxation," *Journal of Public Economics*, 6, 55–75.

BERNHEIM, B. D. AND D. TAUBINSKY (2018): "Behavioral Public Economics," in *Handbook of Behavioral Economics*, ed. by B. D. Bernheim, S. DellaVigna, and D. Laibson, New York: Elsevier, vol. 1, 381–516.

BIERBRAUER, F., P. BOYER, AND E. HANSEN (2020): "Pareto-improving tax reforms and the Earned Income Tax Credit," *Working Paper no. 8358, CESifo*.

BOADWAY, R., M. MARCHAND, AND P. PESTIEAU (2000): "Redistribution with Unobservable Bequests: A Case for Taxing Capital Income," *The Scandinavian Journal of Economics*, 102, 253–267.

BOURGUIGNON, F. AND A. SPADARO (2012): "Tax–benefit revealed social preferences," *The Journal of Economic Inequality*, 10, 75–108.

BOVENBERG, A. L. AND B. JACOBS (2005): "Redistribution and education subsidies are Siamese twins," *Journal of Public Economics*, 89.

——— (2011): "Optimal taxation of human capital and the earnings function," *Journal of Public Economic Theory*, 13, 957–971.

BRICKER, J., K. B. MOORE, AND J. P. THOMPSON (2019): "Trends in Household Portfolio Composition," *Finance and Economics Discussion Series 2019-069. Washington: Board of Governors of the Federal Reserve System*.

CHETTY, R. (2009): "Is the Taxable Income Elasticity Sufficient to Calculate Deadweight Loss? The Implications of Evasion and Avoidance," *American Economic Journal: Economic Policy*, 1, 31–52.

——— (2012): "Bounds on elasticities with optimization frictions: A synthesis of micro and macro evidence on labor supply," *Econometrica*, 80, 969–1018.

CHRISTIANSEN, V. AND M. TUOMALA (2008): "On taxing capital income with income shifting," *International Tax and Public Finance*, 15, 527–545.

CORLETT, W. J. AND D. C. HAGUE (1953): "Complementarity and the excess burden of taxation," *The Review of Economic Studies*, 21, 21–30.

CREMER, H., P. PESTIEAU, AND J.-C. ROCHET (2003): “Capital income taxation when inherited wealth is not observable,” *Journal of Public Economics*, 87, 2475–2490.

DIAMOND, P. (1975): “A Many-Person Ramsey Tax Rule,” *Journal of Public Economics*, 4, 335–342.

DIAMOND, P. AND J. SPINNEWIJN (2011): “Capital income taxes with heterogeneous discount rates,” *American Economic Journal: Economic Policy*, 3, 52–76.

FAGERENG, A., L. GUISO, D. MALACRINO, AND L. PISTAFERRI (2020): “Heterogeneity and persistence in returns to wealth,” *Econometrica*, 88, 115–170.

FAGERENG, A., M. B. HOLM, AND G. J. NATVIK (2021): “MPC heterogeneity and household balance sheets,” *American Economic Journal: Macroeconomics*, 13, 1–54.

FARHI, E. AND I. WERNING (2010): “Progressive Estate Taxation,” *Quarterly Journal of Economics*, 125, 635–673.

——— (2013): “Insurance and Taxation over the Life Cycle,” *Review of Economic Studies*, 80, 596–635.

FELDSTEIN, M. S. (1999): “Tax Avoidance and the Deadweight Loss of the Income Tax,” *Review of Economics and Statistics*, 81, 674–680.

GAHVARI, F. AND L. MICHELETTO (2016): “Capital income taxation and the Atkinson–Stiglitz theorem,” *Economics Letters*, 147, 86–89.

GAUBERT, C., P. KLINE, AND D. YAGAN (2021): “Place-Based Redistribution,” *Working Paper no. 28337, National Bureau of Economic Research*.

GAUTHIER, S. AND F. HENRIET (2018): “Commodity taxes and taste heterogeneity,” *European Economic Review*, 101, 284–296.

GERRITSEN, A., B. JACOBS, A. V. RUSU, AND K. SPIRITUS (2020): “Optimal Taxation of Capital Income with Heterogeneous Rates of Return,” *Working Paper no. 8395, CESifo*.

GOLOSOV, M., M. GRABER, M. MOGSTAD, AND D. NOVGORODSKY (2021): “How Americans Respond to Idiosyncratic and Exogenous Changes in Household Wealth and Unearned Income,” *Working Paper no. 29000, National Bureau of Economic Research*.

GOLOSOV, M., N. KOCHERLAKOTA, AND A. TSYVINSKI (2003): “Optimal Indirect and Capital Taxation,” *Review of Economic Studies*, 70, 569–588.

GOLOSOV, M., M. TROSHKIN, AND A. TSYVINSKI (2016): “Redistribution and Social Insurance,” *American Economic Review*, 106, 359–386.

GOLOSOV, M., M. TROSHKIN, A. TSYVINSKI, AND M. WEINZIERL (2013): “Preference heterogeneity and optimal capital income taxation,” *Journal of Public Economics*, 97, 160–175.

GOLOSOV, M. AND A. TSYVINSKI (2006): “New Dynamic Public Finance: a User’s Guide,” *NBER Macroeconomics Annual*, 317-379.

——— (2015): “Policy Implications of Dynamic Public Finance,” *Annual Reviews of Economics*, 7, 147–171.

HENDREN, N. (2020): “Measuring economic efficiency using inverse-optimum weights,” *Journal of Public Economics*, 187, 1–13.

JACQUET, L. AND E. LEHMANN (2021a): “How to Tax Different Incomes?” *Discussion*

Paper no. 14739, IZA Institute of Labor Economics.

——— (2021b): "Optimal income taxation with composition effects," *Journal of the European Economic Association*, 19, 1299–1341.

JAKOBSEN, K., K. JAKOBSEN, H. KLEVEN, AND G. ZUCMAN (2020): "Wealth taxation and wealth accumulation: Theory and evidence from Denmark," *The Quarterly Journal of Economics*, 135, 329–388.

KAPLOW, L. (2006): "On the undesirability of commodity taxation even when income taxation is not optimal," *Journal of Public Economics*, 90, 1235–1250.

KOCHERLAKOTA, N. R. (2005): "Zero expected wealth taxes: A mirrlees approach to dynamic optimal taxation," *Econometrica*, 73, 1587–1621.

KONISHI, H. (1995): "A Pareto-improving commodity tax reform under a smooth nonlinear income tax," *Journal of Public Economics*, 56, 413–446.

LAIBSON, D. (1997): "Golden Eggs and Hyperbolic Discounting," *The Quarterly Journal of Economics*, 112, 443–478.

LAROQUE, G. R. (2005): "Indirect taxation is superfluous under separability and taste homogeneity: A simple proof," *Economics Letters*, 87, 141–144.

LOCKWOOD, B. B. (2020): "Optimal income taxation with present bias," *American Economic Journal: Economic Policy*, 12, 298–327.

LOCKWOOD, B. B. AND M. WEINZIERL (2015): "De Gustibus non est Taxandum: Heterogeneity in preferences and optimal redistribution," *Journal of Public Economics*, 124, 74–80.

——— (2016): "Positive and normative judgments implicit in U.S. tax policy, and the costs of unequal growth and recessions," *Journal of Monetary Economics*, 77, 30–47.

MILGROM, P. AND I. SEGAL (2002): "Envelope Theorems for Arbitrary Choice sets," *Econometrica*, 70, 583–601.

MIRRLEES, J. A. (1971): "An Exploration in the Theory of Optimum Income Taxation," *The Review of Economic Studies*, 38, 175–208.

——— (1976): "Optimal tax theory: a synthesis," *Journal of Public Economics*, 6, 327–358.

MOSER, C. AND P. OLEA DE SOUZA E SILVA (2019): "Optimal paternalistic savings policies," *Institute Working Paper 17, Federal Reserve Bank of Minneapolis*.

PIKETTY, T. AND E. SAEZ (2013): "A theory of optimal inheritance taxation," *Econometrica*, 81, 1851–1886.

PIKETTY, T., E. SAEZ, AND G. ZUCMAN (2018): "Distributional national accounts: methods and estimates for the United States," *The Quarterly Journal of Economics*, 133, 553–609.

ROTHSCHILD, C. AND F. SCHEUER (2013): "Redistributive Taxation in the Roy Model," *Quarterly Journal of Economics*, 128, 623–668.

SACHS, D., A. TSYVINSKI, AND N. WERQUIN (2020): "Nonlinear tax incidence and optimal taxation in general equilibrium," *Econometrica*, 88, 469–493.

SAEZ, E. (2001): "Using elasticities to derive optimal income tax rates," *The Review of Economic Studies*, 68, 205–229.

——— (2002): "The desirability of commodity taxation under non-linear income taxation and heterogeneous tastes," *Journal of Public Economics*, 83, 217–230.

SAEZ, E. AND S. STANTCHEVA (2016): "Generalized social marginal welfare weights for optimal tax theory," *American Economic Review*, 106, 24–45.

——— (2018): "A simpler theory of optimal capital taxation," *Journal of Public Economics*, 162, 120–142.

SAEZ, E. AND G. ZUCMAN (2020): "Trends in US Income and Wealth Inequality: Revising After the Revisionists," *Working Paper no. 27921, National Bureau of Economic Research*.

SCHEUER, F. AND A. WOLITZKY (2016): "Capital Taxation under Political Constraints," *American Economic Review*, 106, 2304–2328.

SLEMROD, J. (1995): "Income Creation or Income Shifting? Behavioral Responses to the Tax Reform Act of 1986," *The American Economic Review*, 85, 175–180.

SMITH, M., O. M. ZIDAR, AND E. ZWICK (2021): "Top wealth in America: New estimates and implications for taxing the rich," *Working Paper no. 29374, National Bureau of Economic Research*.

STANTCHEVA, S. (2017): "Optimal taxation and human capital policies over the life cycle," *Journal of Political Economy*, 125, 1931–1990.

——— (2020): "Dynamic taxation," *Annual Review of Economics*, 12, 801–831.

STIGLITZ, J. E. (2018): "Pareto efficient taxation and expenditures: pre-and redistribution," *Journal of Public Economics*, 162, 101–119.

STRAUB, L. (2018): "Consumption, savings, and the distribution of permanent income," *Unpublished manuscript, Harvard University*.

THAKRAL, N. AND L. TO (2021): "Anticipation and Consumption," *working paper*.

WERNING, I. (2010): "Nonlinear Capital Taxation," *working paper*.

Online Appendix

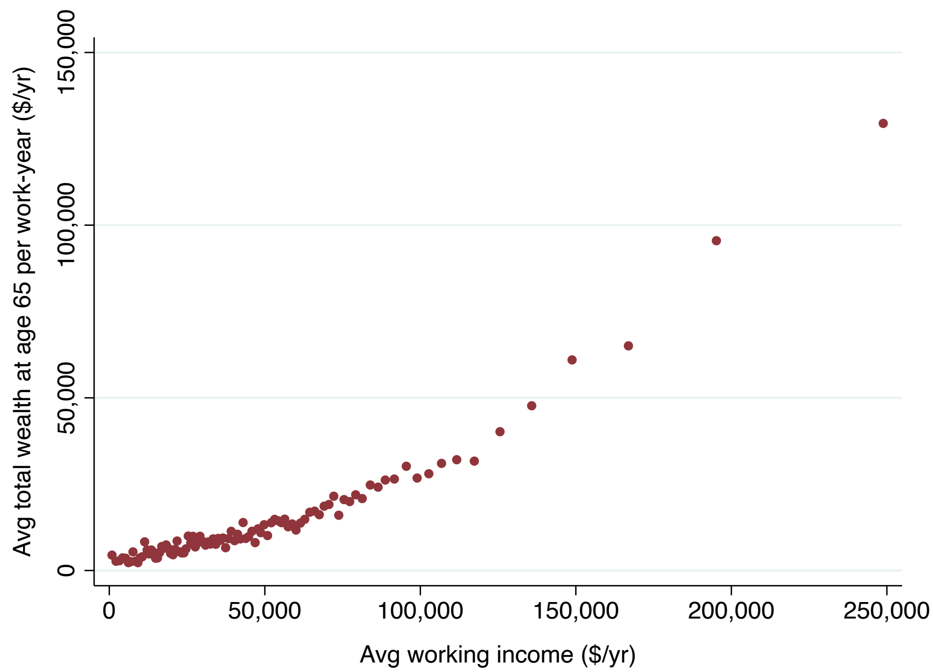
Sufficient Statistics for Nonlinear Tax Systems With Preference Heterogeneity

Antoine Ferey, Benjamin B. Lockwood, and Dmitry Taubinsky

Table of Contents

A Supplementary Tables & Figures

Figure A1: Savings Across Incomes in the United States



Notes: The earnings and savings distribution in the U.S. is calibrated based on the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65), to impute gross savings at the time of retirement, which we normalize by the number of work years. See Appendix D.1 for further details.

B Supplementary Theoretical Results

B.1 Monotonicity with Preference Heterogeneity

Lemma B.1. *Under Assumption 1 and 2, earnings z are strictly increasing with type θ in the optimal incentive-compatible allocation $\mathcal{A}$.*

B.2 Implementation Results for Simple Tax Systems

We proceed in three steps to provide sufficient conditions under which some SN and LED tax systems decentralize the optimal incentive-compatible allocation $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$.

First, we define candidate SN and LED tax systems that satisfy type-specific feasibility and individuals' first-order conditions at the optimal allocation. Second, in Proposition B.1, we present sufficient conditions under which these SN and LED tax systems also satisfy individuals' second-order conditions at the optimal allocation, implying local optimality. Third, in Proposition B.2, we present sufficient conditions under which local optima are ensured to be global optima, implying that the candidate SN and LED systems are indeed implementing the optimal allocation.

There are interesting differences between SN and LED tax systems in their ability to implement the optimal allocation. Under our baseline assumptions, we have shown that $z^*(\theta)$ is strictly increasing with type (Lemma B.1). However, $s^*(\theta)$ may not be monotonic. When the optimal incentive-compatible allocation $\mathcal{A}$ features a monotonic $s^*(\theta)$, we show that implementation by a SN tax system requires relatively weaker conditions than implementation by a LED tax system. However, when the optimal incentive-compatible allocation $\mathcal{A}$ features non-monotonicity in $s^*(\theta)$, we show that a LED tax system may be able to implement the optimal allocation, whereas a SN tax system cannot – the candidate SN tax system is not even well defined. Hence, all implementation results for SN tax systems are made under the following assumption:

Assumption 4. *When the SN system is studied, $s^*(\theta)$ is assumed strictly monotonic (increasing or decreasing) in type θ.*

Step 1: Defining candidate tax systems. We first define a candidate SN tax system $\mathcal{T}(s,z) = T_s(s)+T_z(z)$, with the nonlinear functions T_s and T_z defined across all savings and earnings bundles of the optimal allocation $\mathcal{A} = (c^*(\theta), s^*(\theta), z^*(\theta))_\theta$ as follows:

$$T_s(s^*(\theta)) := \int_{\vartheta=\theta_{min}}^{\theta} \left(U_s'(\vartheta)/U_c'(\vartheta) - 1\right) s^{*\prime}(\vartheta) d\vartheta, \tag{45}$$

$$T_z(z^*(\theta)) := z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) + \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta. \tag{46}$$

where θ_{min} denotes the lowest earning type of the compact type space Θ, and the derivatives are evaluated at the bundle assigned in the optimal allocation (e.g., $U_s'(\vartheta) = U_s'(c^*(\vartheta), s^*(\vartheta), z^*(\vartheta); \vartheta)$). Under this tax system, the optimal allocation satisfies by definition each type's first-order conditions for individual optimization given in Equations (7) and (8). By Lemma C.1, this tax system thus satisfies type-specific feasibility.

We similarly define a candidate LED tax system $\mathcal{T}(s,z) = \tau_s(z) \cdot s + T_z(z)$ as follows:

$$\tau_s(z^*(\theta)) := U_s'(\theta)/U_c'(\theta) - 1, \tag{47}$$

$$\begin{aligned} T_z(z^*(\theta)) :=& z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) \\ &+ \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta - s^*(z) \cdot \left(\tau_s(z) - \tau_s(z^*(\theta_{min}))\right). \end{aligned} \tag{48}$$

This tax system also satisfies local first-order conditions for individual optimization and type-specific feasibility.

Step 2: Local maxima. We can now derive sufficient conditions under which the above candidate SN and LED tax systems satisfy the second-order conditions for individual optimization, implying that under these conditions assigned bundles are local optima. These conditions can be simply stated in terms of the marginal rates of substitution between consumption and, respectively, savings $\mathcal{S}(c,s,z;\theta)$ and earnings $\mathcal{Z}(c,s,z;\theta)$. These marginal rates of substitutions are smooth functions of c, s, z, and θ by the smoothness of the allocation and the utility function, and sufficient conditions for local second-order conditions are given by the following proposition.

Proposition B.1. *Suppose that an allocation satisfies the conditions in Theorem 1. Under the SN tax system defined by Equations* (45) *and* (46)*, each individual's assigned choice of savings and earnings is a local optimum if the following conditions hold at each point in the allocation:*

$$\mathcal{S}_c' \geq 0,\ \mathcal{S}_z' \geq 0,\ \mathcal{S}_\theta' \geq 0 \tag{49}$$

and

$$\mathcal{Z}_c' \leq 0,\ \mathcal{Z}_s' \geq 0,\ \mathcal{Z}_\theta' \geq 0. \tag{50}$$

Under the LED tax system defined by Equations (47) *and* (48)*, each individual's assigned choice of savings and earnings is a local optimum if the utility function is additively separable in consumption, savings, and earnings ($U_{cs}'' = 0$, $U_{cz}'' = 0$, and $U_{sz}'' = 0$), and additionally the following conditions hold at each point in the allocation:*

$$\mathcal{S}_\theta' \geq 0,\ \mathcal{S}_\theta' \leq \frac{z^{*\prime}(\theta)}{s^{*\prime}(\theta)} \mathcal{Z}_\theta',\ \mathcal{S}_\theta' \leq s^{*\prime}(\theta) \left(\mathcal{S} \cdot \mathcal{S}_c' - \mathcal{S}_s' \right). \tag{51}$$

The sufficiency conditions (49) and (50) are quite weak; they are satisfied under many common utility functions used in calibrations of savings and income taxation models, including the simple example function in equation (1). Conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{Z}_\theta' \geq 0$ are single crossing conditions for savings and earnings, while other conditions intuitively relate to the concavity of preferences.

The sufficiency conditions for LED systems are more restrictive. Beyond the single-crossing conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{Z}_\theta' \geq 0$, they place a constraint on the extent of local preference heterogeneity for savings, as compared with preference heterogeneity in earnings. In words, the preference for savings must not increase "too quickly" across types, or else the second-order condition for earnings may be violated. The intuition for this result can be seen from the definition of the potentially optimal LED system. If the marginal rate of substitution for saving, $\mathcal{S}$, increases very quickly with income at some point in the allocation, then the savings tax rate $\tau_s(z)$ must rise very quickly with z at that point, by equation (47). Since the savings tax rate $\tau_s(z)$ applies to total savings (including inframarginal savings), this increase in $\tau_s(z)$ must be offset by a sharp decrease in $T_z(z)$ at the same point in the distribution, by equation (48). Yet a sufficiently steep decrease in $T_z(z)$ will cause the second-order condition for earnings choice—holding fixed savings choice—to be violated.

Step 3: Global maxima. Having presented conditions under which the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ assigned to type θ is a local optimum under the candidate SN and LED tax systems, we now present a set of regularity conditions ensuring that these local optima are also global optima.

Proposition B.2. *Assume single-crossing conditions for earnings and savings ($\mathcal{Z}_\theta' \geq 0$ and $\mathcal{S}_\theta' \geq 0$), that preferences are weakly separable ($U_{cz}'' = 0$ and $U_{sz}'' = 0$), and that commodities c and s are weak complements ($U_{cs}'' \geq 0$). If $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, local optima correspond to global optima when:*

1. *$\mathcal{T}$ is a SN system, and we have that for all $s > s^*(\theta)$ and θ, $\frac{-U_{ss}''(c(s,\theta),s,z^*(\theta);\theta)}{U_s'(c(s,\theta),s,z^*(\theta);\theta)} > \frac{-T_{ss}''(s)}{1+T_s'(s)}$.*
2. *$\mathcal{T}$ is a LED system, and we have that*

(a) *for all* $s < s^*(\theta)$ *and* θ, $\frac{-U''_{cc}(c(s,\theta),s,z^*(\theta);\theta)}{U'_c(c(s,\theta),s,z^*(\theta);\theta)} > \frac{1}{1+\tau_s(z^*(\theta))} \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$,

(b) *for all* $s > s^*(\theta)$ *and* θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$,

where $c(s,\theta) := z^*(\theta) - s - \mathcal{T}(s, z^*(\theta))$

In essence, global optimality is ensured under the following assumptions. First, higher types θ derive higher gains from working and allocating those gains to consumption or savings — generalized single-crossing conditions. Second, additive separability of consumption and savings from labor. Third, the utility function U is sufficiently concave in consumption and savings.

For the case of SN tax systems, condition 1 imposes a particular concavity requirement with respect to savings. For the case of LED tax systems, condition 2 imposes particular concavity requirements with respect to both consumption and savings. Notably, these concavity conditions need only be checked when earnings are fixed at each type's assigned earnings level $z^*(\theta)$.

We can naturally apply this result to the candidate SN tax system defined in equations (45) and (46), and to the candidate LED tax system defined in equations (47) and (48). Because these candidate tax systems are defined in terms of individuals' preferences and optimal allocations, we can then express conditions 1 and 2 fully in terms of individuals' preferences and optimal allocations.

B.3 Structural characterization of s'_{inc} and s'_{het}

In economies with preference heterogeneity, budget heterogeneity, and auxiliary choices, individuals solve

$$\max_{c,s,z,\chi} U(c, \phi_s(s,z,\chi;\theta), \phi_z(s,z,\chi;\theta), \chi;\theta) \ s.t. \ c \leq B(s,z,\chi;\theta) - \mathcal{T}(s,z)$$
$$\iff \max_z \left\{ \max_s \left[\max_\chi U(B(s,z,\chi;\theta) - \mathcal{T}(s,z), \phi_s(s,z,\chi;\theta), \phi_z(s,z,\chi;\theta), \chi;\theta) \right] \right\}. \quad (52)$$

We denote $\chi(s,z;\theta)$ the solution to the inner problem, $s(z;\theta)$ the solution to the intermediate problem, and $z(\theta)$ the solution to the outer problem. We assume that $\chi(s,z;\theta)$ and $s(z;\theta)$ are interior solutions that satisfy the first-order conditions of these problems, and we maintain the assumption that $z(\theta)$ is strictly increasing to denote $\vartheta(z)$ the type that chooses earnings z.

In this setting, we decompose across-income heterogeneity in $s(z) := s(z;\vartheta(z))$ between $s'_{inc}(z) := \frac{\partial s(z;\vartheta(z'))}{\partial z}|_{z'=z}$ and $s'_{het}(z) := \frac{\partial s(z';\vartheta(z))}{\partial z}|_{z'=z}$ as follows:

Proposition B.3. *In economies with preference heterogeneity, budget heterogeneity, and auxiliary choices, sufficient statistics* $s'_{inc}(z)$ *and* $s'_{het}(z)$ *are given by*

$$s'_{inc}(z) = -\frac{\mathcal{N}^1_{inc}(z) + \mathcal{N}^2_{inc}(z)}{\mathcal{D}^1(z) + \mathcal{D}^2(z)} \quad (53)$$

$$s'_{het}(z) = -\frac{\mathcal{N}^1_{het}(z) + \mathcal{N}^2_{het}(z)}{\mathcal{D}^1(z) + \mathcal{D}^2(z)} \quad (54)$$

where terms in the numerators and denominators are

$$\mathcal{N}^1_{inc} := \mathcal{K}_c\Big[B'_z + B'_\chi\frac{\partial\chi}{\partial z} - \mathcal{T}'_z\Big] + \mathcal{K}_s\Big[\frac{\partial\phi_s}{\partial z} + \frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial z}\Big] + \mathcal{K}_z\Big[\frac{\partial\phi_z}{\partial z} + \frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial z}\Big] + \mathcal{K}_\chi\frac{\partial\chi}{\partial z} \tag{55}$$

$$\mathcal{N}^2_{inc} := U'_c\Big[B''_{sz} + B''_{s\chi}\frac{\partial\chi}{\partial z} - \mathcal{T}''_{sz}\Big] + U'_s\Big[\frac{\partial^2\phi_s}{\partial s\partial z} + \frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial z}\Big] + U'_z\Big[\frac{\partial^2\phi_z}{\partial s\partial z} + \frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial z}\Big] \tag{56}$$

$$\mathcal{N}^1_{het} := \mathcal{K}_c B'_\chi\frac{\partial\chi}{\partial\theta} + \mathcal{K}_s\Big[\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial\theta} + \frac{\partial\phi_s}{\partial\theta}\Big] + \mathcal{K}_z\Big[\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial\theta} + \frac{\partial\phi_z}{\partial\theta}\Big] + \mathcal{K}_\chi\frac{\partial\chi}{\partial\theta} + \mathcal{K}_\theta \tag{57}$$

$$\mathcal{N}^2_{het} := U'_c B''_{s\chi}\frac{\partial\chi}{\partial\theta} + U'_s\Big[\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial\theta} + \frac{\partial^2\phi_s}{\partial s\partial\theta}\Big] + U'_z\Big[\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial\theta} + \frac{\partial^2\phi_z}{\partial s\partial\theta}\Big] \tag{58}$$

$$\mathcal{D}^1 := \mathcal{K}_c\Big[B'_s + B'_\chi\frac{\partial\chi}{\partial s} - \mathcal{T}'_s\Big] + \mathcal{K}_s\Big[\frac{\partial\phi_s}{\partial s} + \frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial s}\Big] + \mathcal{K}_z\Big[\frac{\partial\phi_z}{\partial s} + \frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial s}\Big] + \mathcal{K}_\chi\frac{\partial\chi}{\partial s} \tag{59}$$

$$\mathcal{D}^2 := U'_c\Big[B''_{ss} + B''_{s\chi}\frac{\partial\chi}{\partial s} - \mathcal{T}''_{ss}\Big] + U'_s\Big[\frac{\partial^2\phi_s}{(\partial s)^2} + \frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\Big] + U'_z\Big[\frac{\partial^2\phi_z}{(\partial s)^2} + \frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\Big] \tag{60}$$

with all quantities being evaluated at z, $s(z)$, $\vartheta(z)$, $\chi(z) := \chi(s(z)), z; \vartheta(z))$, as well as $c(z) := B(s(z), z, \chi(z); \vartheta(z)) - \mathcal{T}(s(z), z)$, and where

$$\mathcal{K}_c := U''_{cc}\left(B'_s - \mathcal{T}'_s\right) + U''_{cs}\frac{\partial\phi_s}{\partial s} + U''_{cz}\frac{\partial\phi_z}{\partial s} \tag{61}$$

$$\mathcal{K}_s := U''_{cs}\left(B'_s - \mathcal{T}'_s\right) + U''_{ss}\frac{\partial\phi_s}{\partial s} + U''_{sz}\frac{\partial\phi_z}{\partial s} \tag{62}$$

$$\mathcal{K}_z := U''_{cz}\left(B'_s - \mathcal{T}'_s\right) + U''_{sz}\frac{\partial\phi_s}{\partial s} + U''_{zz}\frac{\partial\phi_z}{\partial s} \tag{63}$$

$$\mathcal{K}_\chi := U''_{c\chi}\left(B'_s - \mathcal{T}'_s\right) + U''_{s\chi}\frac{\partial\phi_s}{\partial s} + U''_{z\chi}\frac{\partial\phi_z}{\partial s} \tag{64}$$

$$\mathcal{K}_\theta := U''_{c\theta}\left(B'_s - \mathcal{T}'_s\right) + U''_{s\theta}\frac{\partial\phi_s}{\partial s} + U''_{z\theta}\frac{\partial\phi_z}{\partial s}. \tag{65}$$

Numerators of $s'_{inc}(z)$ and $s'_{het}(z)$ are different as they capture direct changes in s, coming from either a change in z or a change in $\vartheta(z)$. Denominators are the same because they capture processes of circular adjustments induced by direct changes in s.

In a simple setting like example (1) with additively separable utility, a separable tax system and preference heterogeneity for s only, the only non-zero term in the numerator of s'_{inc} would be proportional to $\mathcal{K}_c$ capturing changes in the marginal utility of c from changes in z, and the only non-zero term in the numerator of s'_{inc} would be proportional to $\mathcal{K}_\theta$ capturing changes in marginal utility of s from changes in θ.

B.4 Optimal Taxes on s in Simple Tax Systems

We present optimal savings tax formulas for simple tax systems, which characterize the optimal savings tax schedule for *any* given earnings tax schedule—including a potentially suboptimal one. These formulas are derived assuming unidimensional heterogeneity and are similar to those presented in Proposition 4, where heterogeneity is allowed to be multidimensional.

Proposition B.4. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$, suppose that SL, SN and LED systems verify Assumption 3, and suppose that in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions for taxes on s:*

$$SL: \frac{\tau_s}{1+\tau_s}\int_{x=z_{min}}^{z_{max}} s(x)\zeta_{s|z}^c(x)dH_z(x) = \int_{x=z_{min}}^{z_{max}}\left\{s\left(x\right)\left(1-\hat{g}(x)\right) - \frac{T_z'(x)+s_{inc}'(x)\,\tau_s}{1-T_z'\left(x\right)}x\zeta_z^c(x)\,s_{inc}'(x)\right\}dH_z(x) \tag{66}$$

$$SN: \frac{T_s'(s)}{1+T_s'(s)}s\zeta_{s|z}^c(z)h_z(z) = s'\left(z\right)\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - \frac{T_z'(z)+s_{inc}'(z)T_s'(s)}{1-T_z'(z)}z\zeta_z^c(z)s_{inc}'(z)h_z(z) \tag{67}$$

$$LED: \frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'(z)\tau_s(z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(z)\,s\,h_z(z) + \int_{x\geq z}\frac{\tau_s(x)}{1+\tau_s(x)}s(x)\zeta_{s|z}^c(x)\,dH_z(x) \tag{68}$$

$$= \int_{x\geq z}\left\{\left(1-\hat{g}(x)\right)s\left(x\right) - \frac{T_z'\left(x\right)+\tau_s'\left(x\right)s\left(x\right)+s_{inc}'(x)\tau_s(x)}{1-T_z'\left(x\right)-\tau_s'\left(x\right)s\left(x\right)}x\zeta_z^c(x)s_{inc}'(x)\right\}\,dH_z(x).$$

These optimal savings tax formulas are all different, reflecting differences between the savings tax instruments that we consider, yet they share common elements. First, the preference heterogeneity term $s_{het}'\left(z\right)$ no longer appears in the formulas. The intuition is that outside of the full optimum, it may still be desirable to tax savings in the absence of preference heterogeneity, implying that optimality may clash with Pareto efficiency when the earnings tax is suboptimal. Second, $s_{inc}'\left(z\right)$ is a key sufficient statistic for optimal savings tax schedules. Indeed, by Lemma 1, a larger $s_{inc}'\left(z\right)$ means that savings tax reforms impose higher distortions on earnings and thus generally calls for lower savings tax rate.

B.5 Optimal Taxes on z in Simple Tax Systems

We now present optimal earnings tax formulas for simple tax systems.

Proposition B.5. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$ and suppose that SL, SN and LED systems verify Assumption 3, and suppose that in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions for taxes on z:*

$$SL: \frac{T_z'\left(z\right)}{1-T_z'\left(z\right)} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - s_{inc}'(z)\frac{\tau_s}{1-T_z'\left(z\right)} \tag{69}$$

$$SN: \frac{T_z'\left(z\right)}{1-T_z'\left(z\right)} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - s_{inc}'(z)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)} \tag{70}$$

$$LED: \frac{T_z'\left(z\right)+\tau_s'\left(z\right)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - s_{inc}'(z)\frac{\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}. \tag{71}$$

These conditions pinning down the optimal schedule of marginal earnings tax rates are a direct application of equation (19) presented in Theorem 2 for smooth tax systems. While formulas for SL and SN tax systems look almost identical to the general condition, the formula for LED tax system looks a bit different. This difference only reflects the fact that, for a LED tax system, the marginal earnings tax rate is given by $\mathcal{T}'_z(s,z) = T'_z(z) + \tau'_s(z)\, s(z)$, accounting for the earnings-dependent nature of savings taxes.

B.6 Optimal Taxes on z with Multidimensional Heterogeneity

Proposition B.6. *Consider given (and potentially suboptimal) SL, SN, and LED savings tax schedule, and assume that under each simple tax system individuals first-order and second-order conditions strictly hold. Then, at each bundle (c^0, s^0, z^0) chosen by a type θ^0, marginal tax rates on z in SL/SN/LED systems must satisfy the following optimality conditions:*

$$SL: \frac{T'_z(z^0)}{1-T'_z(z^0)}\mathbb{E}\left[\zeta^c_z(s,z)\Big|z^0\right] = \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\}dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z)\frac{\tau_s}{1-T'_z(z)}\zeta^c_z(s,z)\Big|z^0\right] \tag{72}$$

$$SN: \frac{T'_z(z^0)}{1-T'_z(z^0)}\mathbb{E}\left[\zeta^c_z(s,z)\Big|z^0\right] = \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\}dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z)\frac{T'_s(s)}{1-T'_z(z)}\zeta^c_z(s,z)\Big|z^0\right] \tag{73}$$

$$LED: \mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)\,s}{1-T'_z(z)-\tau'_s(z)\,s}\zeta^c_z(s,z)\Big|z^0\right] = \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\}dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z)\frac{\tau_s(z)}{1-T'_z(z)-\tau'_s(z)\,s}\zeta^c_z(s,z)\Big|z^0\right] \tag{74}$$

These conditions are similar to those presented above for optimal marginal earnings tax rates under unidimensional heterogeneity (Proposition B.5). Indeed, Lemma 1 still applies such that the previous derivations carry over when adding an expectation with respect to savings. Proofs are thus omitted.

B.7 Equivalences with Tax Systems Involving Gross Period-2 Savings

Suppose that there are two periods, and set $1+r = 1/p$. In period 1 the individual earns z, consumes c, and pays income taxes $T_1(z)$. In period 2 the individual realizes *gross pre-tax savings* $s_g = (z-c-T_1(z))(1+r)$ and pays income taxes $T_2(s_g, z)$. The realized savings s are given by $s_g - T_2(s_g, z)$. The total tax paid in "period-1 dollars" is given by $T_1(z) + T_2(s_g, z)/(1+r)$. The individual maximizes $U(c,s,z)$ subject to the constraint

$$\begin{aligned} s &\leq (z-c-T_1(z))(1+r) - T_2(s_g,z) \\ \Leftrightarrow c + \frac{s}{1+r} &\leq z - T_1(z) - \frac{T_2((z-c-T_1(z))(1+r),z)}{1+r}. \end{aligned}$$

In our baseline formulation with period-1 tax function $\mathcal{T}(s,z)$, individuals choose s and z to maximize $U(z-s-\mathcal{T}(s,z),s,z;\theta)$. To convert from the formulation with period-2 taxes to our baseline formulation, define a function $\tilde{s}_g(s,z)$ implicitly to satisfy the equation

$$\tilde{s}_g - T_2(\tilde{s}_g, z) = s$$

Note that $\tilde{s}_g$ is generally uniquely defined if we have a system with monotonic realized savings s. Then, the equivalence in tax schedules is given by

$$\mathcal{T}_s'(s,z) = \frac{1}{1+r}\frac{\partial}{\partial s_g}T_2(s_g,z)|_{s_g=\tilde{s}_g}\frac{\partial}{\partial s}\tilde{s}_g \tag{75}$$

and $\mathcal{T}_z' = T_z'$. equation (75) simply computes how a marginal change in s changes the tax burden in terms of period-1 units of money, and the division by $1+r$ is to convert to period-1 units. Now differentiating the definition of $\tilde{s}_g$ gives

$$\frac{\partial}{\partial s}\tilde{s}_g - \frac{\partial}{\partial s_g}T_2(s_g,z)\frac{\partial}{\partial s}\tilde{s}_g = 1$$

and thus

$$\frac{\partial}{\partial s}\tilde{s}_g = \frac{1}{1-\frac{\partial}{\partial s_g}T_2(s_g,z)}$$

from which it follows that

$$\mathcal{T}_s'(s,z) = \frac{1}{1+r}\frac{\frac{\partial}{\partial s_g}T_2(s_g,z)|_{s_g=\tilde{s}_g}}{1-\frac{\partial}{\partial s_g}T_2(s_g,z)|_{s_g=\tilde{s}_g}}. \tag{76}$$

We can also start with a schedule $\mathcal{T}$ and converts it to the two-period tax schedule. Now if s is the realized savings, we can define gross savings in period 2 as $s_g = s + \mathcal{T}(z,s)(1+r) - \mathcal{T}(z,0)$, and we define the function $\tilde{s}(s_g,z)$ to satisfy

$$s_g = \tilde{s} + (1+r)\left(\mathcal{T}(\tilde{s},z) - \mathcal{T}(0,z)\right).$$

Then,

$$\begin{aligned}\frac{\partial}{\partial s_g}T_2(s_g,z) &= (1+r)\mathcal{T}_s'(\tilde{s},z)\frac{\partial}{\partial s_g}\tilde{s} \\ &= \frac{(1+r)\mathcal{T}_s'(\tilde{s},z)}{1+(1+r)\mathcal{T}_s'(\tilde{s},z)}\end{aligned} \tag{77}$$

B.7.1 Separable tax systems (SN).

Now if T_2 is a function of s_g only (a separable tax system), then s_g will be a function of s only, and thus $\mathcal{T}_s'$ will only depend on s. Conversely, note that if $\mathcal{T}$ is a separable system, so that $\mathcal{T}_s'$ does not depend on z, then (77) implies that $\frac{\partial}{\partial s_g}T_2(s_g,z)$ does not depend on z either. Thus, separability is a property preserved under these transformations.

Now if we start with a separable $\mathcal{T}$, then T_2 is given by

$$T_2'(s_g) = (1+r)\frac{\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}{1+\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}$$

where $\tilde{s}$ is the value that solves $s_g = \tilde{s} + \mathcal{T}(\tilde{s})$.

B.7.2 Linear tax systems (LED and SL).

If $T_2 = s_g\tau(z)$, a linear earnings-dependent system, then $s = s_g(1-\tau(z))$ and $s_g = \frac{s}{1-\tau(z)}$. Moreover, $\frac{\partial}{\partial s}s_g = \frac{1}{1-\tau(z)}$, and thus we have that

$$\mathcal{T}_s' = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$$

which again implies that we have a linear earnings-dependent system with rate $\tilde{\tau}(z) = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$.

Conversely, if we start with a LED system $\mathcal{T}$ with $\mathcal{T}_s' = \tau(z)$, then

$$\frac{\partial}{\partial s_g}T_2(s_g, z) = (1+r)\frac{\tau(z)}{1+\tau(z)}.$$

When the tax rates τ are not functions of z, the calculations above show that the conversions preserve not just linearity, but also independence of z.

C Proofs

C.1 Proof of Lemma B.1 (Monotonicity with Preference Heterogeneity)

We show by contradiction that the extended Spence-Mirrlees condition implies that type $\theta_2 > \theta_1$ chooses earnings $z(\theta_2) > z(\theta_1)$. Note that, by Assumption 2, $c(\theta)$, $s(\theta)$, and $z(\theta)$ are smooth functions of θ in the optimal incentive-compatible allocation, and that by Assumption 1 utility U is twice continuously differentiable.

Assume (without loss of generality) that there is an open set $(\theta_1, \theta_2) \in \Theta$ where $z(\theta)$ is decreasing with θ such that $z(\theta_2) < z(\theta_1)$.[33]Then,

[33]Since we assume no bunching in income, meaning that types θ for which $z'(\theta) = 0$ are of measure zero, this inequality has to be strict with $\theta_2 > \theta_1$.

$$\begin{aligned}
&U\left(c(\theta_2), s(\theta_2), z(\theta_2); \theta_2\right) - U\left(c(\theta_1), s(\theta_1), z(\theta_1); \theta_2\right)\\
&= \int_{\theta=\theta_1}^{\theta_2} \left[\frac{dU\left(c(\theta), s(\theta), z(\theta); \theta_2\right)}{d\theta}\right] d\theta\\
&= \int_{\theta=\theta_1}^{\theta_2} \left[U_c'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) c'(\theta) + U_s'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) s'(\theta) + U_z'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) z'(\theta)\right] d\theta\\
&= \int_{\theta=\theta_1}^{\theta_2} U_c'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) \Big[c'(\theta) + \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) s'(\theta) + \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) z'(\theta)\Big] d\theta
\end{aligned} \tag{78}$$

Now, for each $\theta \in (\theta_1, \theta_2)$ the first-order condition implied by incentive compatibility implies that, at point $(c(\theta), s(\theta), z(\theta))$,

$$\begin{aligned}
&U_c'\left(c(\theta), s(\theta), z(\theta); \theta\right) c'(\theta) + U_s'\left(c(\theta), s(\theta), z(\theta); \theta\right) s'(\theta) + U_z'\left(c(\theta), s(\theta), z(\theta); \theta\right) z'(\theta) = 0\\
&\iff c'(\theta) + \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta\right) s'(\theta) + \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta\right) z'(\theta) = 0.
\end{aligned} \tag{79}$$

When $z'(\theta) \neq 0$, the extended Spence-Mirrlees condition states that for any θ',

$$\begin{aligned}
&\mathcal{S}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) \frac{s'(\theta)}{z'(\theta)} + \mathcal{Z}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) > 0\\
&\iff \mathcal{S}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) s'(\theta) + \mathcal{Z}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) z'(\theta) < 0
\end{aligned} \tag{80}$$

where the last inequality is reversed because $z(\theta)$ is decreasing in $\theta \in (\theta_1, \theta_2)$, meaning $z'(\theta) < 0$. This implies that with $\theta_2 > \theta$

$$c'(\theta) + \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) s'(\theta) + \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) z'(\theta) < 0. \tag{81}$$

Since $U_c' > 0$, and since we assume no bunching on income (meaning that types θ for which $z'(\theta) = 0$ are of measure zero), this means that the integral above is negative, and thus that

$$U\left(c(\theta_2), s(\theta_2), z(\theta_2); \theta_2\right) < U\left(c(\theta_1), s(\theta_1), z(\theta_1); \theta_2\right). \tag{82}$$

This is a contradiction with the fact that type θ_2 (strictly) prefers its allocation $(c(\theta_2), s(\theta_2), z(\theta_2))$ in the optimal incentive-compatible allocation, which concludes the proof.

C.2 Proof of Theorem 1 (Implementation with a Smooth Tax System)

In the appendix, we adopt the notation that individual's allocations in the optimal mechanism are labeled with a "star"; i.e., $(c^*(\theta), s^*(\theta), z^*(\theta))$. We construct a smooth tax system that implements the optimal incentive-compatible allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ by introducing penalties for deviations away from these allocations. This proof relies on Lemma C.1 and Lemma C.2, which we derive at the end of this subsection. Throughout, we adopt $p \equiv 1$ to economize on notations.

With unidimensional heterogeneity, type θ belongs to the compact space $\Theta = [\theta_{min}, \theta_{max}]$. Moreover, there is always a mapping $s^*(z)$ that denotes the savings level associated with earnings level $z = z^*(\theta)$ at the optimal incentive-compatible allocation. We consider without loss of generality the case in which $s(z)$ is strictly increasing; the proof can be adapted to cases with non-monotonic $s(z)$.

Let $z_{max} := z^*(\theta_{max})$ and $z_{min} := z^*(\theta_{min})$ denote the maximal and minimal, respectively, earnings levels in the allocation. Let $s_{max} := \max_z s^*(z)$ and $s_{min} := \min_z s^*(z)$ denote the maximal and minimal savings levels.

Step 1: Defining the smooth tax system. We start from a separable and smooth tax system $T_s(s) + T_z(z)$ that satisfies type-specific feasibility and individuals' first-order conditions at the optimal incentive-compatible allocation. We then add quadratic penalty terms parametrized by k for deviations from this allocation. For a given deviation, this allows to make the penalty arbitrarily large and enables us to make the individuals' optimization problems locally concave around the optimal incentive-compatible allocation. The other terms that we add are there to guarantee the smoothness of the penalized tax system $\mathcal{T}(s, z; k)$ at the boundaries of the set of optimal allocations.

Formally, $\mathcal{T}_k = \mathcal{T}(s, z; k)$ is defined by:

1. $T_s(s_{min}) = 0$ and $T_z(z_{min}) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\forall z \in [\, z_{min}\, ;\, z_{max}\,]\, , T_z'(z) = \mathcal{Z}(c^*(\theta_z), s^*(\theta_z), z^*(\theta_z); \theta_z) + 1$ with θ_z such that $z = z^*(\theta_z)$
3. $\forall s \in [\, s_{min}\, ;\, s_{max}\,]\, , T_s'(s) = \mathcal{S}(c^*(\theta_s), s^*(\theta_s), z^*(\theta_s); \theta_s) - 1$ with θ_s such that $s = s^*(\theta_s)$
4. $$\mathcal{T}(s, z; k) = \begin{cases} T_s(s) + T_z(z) + k(s - s^*(z))^2 & \text{if } z_{min} \leq z \leq z_{max}, \\ & s_{min} \leq s \leq s_{max} \\ T_s(s_{min}) + T_z(z) + k(s - s^*(z))^2 + T_s'(s_{min})(s - s_{min}) & \text{if } z_{min} \leq z \leq z_{max}, s < s_{min} \\ T_s(s_{max}) + T_z(z) + k(s - s^*(z))^2 + T_s'(s_{max})(s - s_{max}) & \text{if } z_{min} \leq z \leq z_{max}, s > s_{max} \\ T_s(s) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 & \text{if } z < z_{min}, s_{min} \leq s \leq s_{max} \\ \quad + T_z'(z_{min})(z - z_{min}) & \\ T_s(s_{min}) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 & \text{if } z < z_{min}, s < s_{min} \\ \quad + T_z'(z_{min})(z - z_{min}) + T_s'(s_{min})(s - s_{min}) & \\ T_s(s_{max}) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 & \text{if } z < z_{min}, s > s_{max} \\ \quad + T_z'(z_{min})(z - z_{min}) + T_s'(s_{max})(s - s_{max}) & \\ T_s(s) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 & \text{if } z > z_{max}, s_{min} \leq s \leq s_{max} \\ \quad + T_z'(z_{max})(z - z_{max}) & \\ T_s(s_{max}) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 & \text{if } z > z_{max}, s > s_{max} \\ \quad + T_z'(z_{max})(z - z_{max}) + T_s'(s_{max})(s - s_{max}) & \\ T_s(s_{min}) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 & \text{if } z > z_{max}, s < s_{min} \\ \quad + T_z'(z_{max})(z - z_{max}) + T_s'(s_{min})(s - s_{min}) & \end{cases}$$

Assumptions 1 and 2 guarantee that the separable tax system $T_s(s) + T_z(z)$ is smooth, i.e., a twice continuously differentiable function. Our construction of the penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$ guarantees that it inherits this smoothness property.

Step 2: Local maxima for sufficiently large k. For a given type θ, we show that the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum under the tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$ for sufficiently large k. To do so, we first establish that type-specific feasibility is satisfied together with the first-order conditions of type θ's maximization problem. We then show that for sufficiently large k, second-order conditions are also satisfied implying that the intended bundle is a local maximum.

The previous definition of the tax system implies

$$\mathcal{T}'_z(s^*(\theta), z^*(\theta); k) = T'_z(z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$$
$$\mathcal{T}'_s(s^*(\theta), z^*(\theta); k) = T'_s(s^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$$

meaning type-specific feasibility is satisfied by Lemma C.1 (see below).

Now, defining

$$V(s, z; \theta, k) := U(z - s - \mathcal{T}(s, z; k), s, z; \theta), \tag{83}$$

the first-order conditions for type θ's choice of savings s and earnings z are

$$V'_s(s, z; \theta, k) = -(1 + \mathcal{T}'_s(s, z; k))U'_c(z - s - \mathcal{T}(s, z; k), s, z; \theta) + U'_s(z - s - \mathcal{T}(s, z; k), s, z; \theta) = 0$$
$$V'_z(s, z; \theta, k) = (1 - \mathcal{T}'_z(s, z; k))U'_c(z - s - \mathcal{T}(s, z; k), s, z; \theta) + U'_z(z - s - \mathcal{T}(s, z; k), s, z; \theta) = 0$$

and they are by construction satisfied at $(s^*(\theta), z^*(\theta))$ for each type θ.

Using Lemma C.2 (see below), second-order conditions at $(s^*(\theta), z^*(\theta))$ are

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \leq 0 \tag{84}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \leq 0 \tag{85}$$

$$\begin{aligned}\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg] \leq 0\end{aligned} \tag{86}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Here, U, $\mathcal{S}$, and $\mathcal{Z}$ are smooth functions, implying that their derivatives are continuous functions over compact spaces and are thus bounded. Now, by definition of $\mathcal{T}_k = \mathcal{T}(s, z; k)$, we have $\mathcal{T}''_{sz} = -2ks^{*\prime}(z)$ which is negative for any $k \geq 0$ and increasing in magnitude with k.

Noting $U'_c > 0$ and $s^{*\prime}(z) > 0$, this implies that V''_{ss} and V''_{zz} must be negative for sufficiently large k, thanks to the last term, since the other terms are bounded and do not depend on k. By the same reasoning, under the extended Spence-Mirrlees single-crossing assumption that $\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta > 0$, we also have that $(V''_{sz})^2 - V''_{ss}V''_{zz}$ must be negative for sufficiently large k.

This shows that for a given type θ, there exists a k_0 such that for all $k \geq k_0$ the allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum to type θ's maximization problem under the smooth penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$.

Step 3: Global maxima for sufficiently large k. Let $s_{\mathcal{T}_k}(\theta)$ and $z_{\mathcal{T}_k}(\theta)$ denote the level of savings and earnings, respectively, that a type θ chooses given a smooth penalized tax system $\mathcal{T}_k$. To prove implementability of optimal incentive-compatible allocations, we show that there exists a sufficiently large k such that for all θ, $s_{\mathcal{T}_k}(\theta) = s^*(\theta)$ and $z_{\mathcal{T}_k}(\theta) = z^*(\theta)$.

Let's proceed by contradiction, and suppose that it is not the case. Then, there exists an infinite sequence of types θ_k, choosing savings $s_{\mathcal{T}_k}(\theta_k) \neq s^*(\theta_k)$ and earnings $z_{\mathcal{T}_k}(\theta_k) \neq z^*(\theta_k)$ under tax system $\mathcal{T}_k$, and enjoying utility gains from this "deviation" to allocation $(s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$.

First, the fact that we impose quadratic penalties for earnings choices outside of $[\, z_{min}\,;\, z_{max}\,]$ guarantees that for any $\varepsilon > 0$, there exists k_1, such that for all $k \geq k_1$ and for all types θ, individuals' earnings choices belong to the compact set $[\, z_{min} - \varepsilon\,;\, z_{max} + \varepsilon\,]$. Indeed, starting from a given earnings level $z > z_{max} + \varepsilon$, the utility change associated with an earnings change is $[(1 - \mathcal{T}_z')\, U_c' + U_z']\, dz$. By construction, the marginal earnings tax rate in this region is $\mathcal{T}_z' = 2k(z - z_{max}) + T_z'(z_{max})$. Noting that $U_c' > 0$, $U_z' < 0$, and that the type space is compact, we can make for all individuals the utility change from a decrease in earnings arbitrarily positive for sufficiently large k. This shows that all individuals choose earnings $z \leq z_{max} + \varepsilon$ for sufficiently large k. Symmetrically, we can show that all individuals choose earnings $z \geq z_{min} - \varepsilon$ for sufficiently large k.

Second, since earnings shape individuals' disposable incomes, the fact that earnings belong to a compact set for sufficiently large k implies that individuals' allocation choices also belong to a compact set. Indeed, for sufficiently large k, individuals' allocation choices must belong to the set of (c, s, z) such that $c \geq 0$, $s \geq 0$, $z \in [\, z_{min} - \varepsilon\,;\, z_{max} + \varepsilon\,]$, and $c + s \leq z - \mathcal{T}(s, z; k)$ where the tax function is smooth and finite. These constraints make the space of allocations compact.

As a result, the infinite sequence $(\theta_k, s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$ belongs to a compact space of allocations and types, it is thus bounded. By the Bolzano-Weierstrass theorem, this means that there exists a convergent subsequence $\left(\theta_j, s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{\theta}, \hat{s}, \hat{z})$. Since all types θ_j belong to $[\, \theta_{min}\,;\, \theta_{max}\,]$, we know that the limit type $\hat{\theta}$ must belong to $[\, \theta_{min}\,;\, \theta_{max}\,]$. Now, from the previous paragraph, individuals' earnings choices have to be arbitrarily close to $[\, z_{min}\,;\, z_{max}\,]$ as the penalty grows. This implies that the limit $\hat{z}$ must belong to $[\, z_{min}\,;\, z_{max}\,]$.

Next, we establish that the limit $\hat{s}$ must be such that $\hat{s} = s^*(\hat{z})$. First fix an earnings level $z \in [\, z_{min}\,;\, z_{max}\,]$. Then, starting from a savings level $s \neq s^*(z)$, the utility change associated with a savings change is $[-(1 + \mathcal{T}_s')\, U_c' + U_s']\, ds$. Assuming without loss of generality that s belongs to $[\, s_{min}\,;\, s_{max}\,]$, the marginal savings tax rate in this region is $\mathcal{T}_s' = T_s'(s) + 2k(s - s^*(z))$. Noting that $U_c' > 0$, and that U_s' is bounded, we can make the utility gains from a savings change towards $s^*(z)$ arbitrarily large for sufficiently large k. As a result, for any $\varepsilon > 0$, there exists k_2 such that for all $k \geq k_2$, type $\hat{\theta}$ chooses savings $s \in [\, s^*(z) - \varepsilon\,;\, s^*(z) + \varepsilon\,]$ for a fixed z.[34] Since type $\hat{\theta}$'s savings choice can be made arbitrarily close to $s^*(z)$ for sufficiently large k, we must have at the limit $s = s^*(z)$. Now, because earnings z converge to $\hat{z}$ and the function $s^*(z)$ is by assumption continuous, we must have at the limit $\hat{s} = s^*(\hat{z})$.

We have thus established that the limit $(\hat{\theta}, \hat{s}, \hat{z})$ is such that $\hat{\theta} \in [\, \theta_{min}\,;\, \theta_{max}\,]$, $\hat{z} \in [\, z_{min}\,;\, z_{max}\,]$, and $\hat{s} = s^*(\hat{z})$. This means that the limit allocation $(\hat{c}, \hat{s}, \hat{z})$ belongs to the set of optimal incentive-compatible allocations. Given our continuity and monotonicity assumptions, this implies that it is the optimal allocation of some type θ and it has to be by definition that of type $\hat{\theta}$. Hence, $(\hat{c}, \hat{s}, \hat{z}) = (c^*(\hat{\theta}), s^*(\hat{\theta}), z^*(\hat{\theta}))$.

To complete the proof and find a contradiction, fix a value $k^\dagger$ that is large enough such that

[34] A way to see this is that the marginal rate of substitution between consumption and savings $\mathcal{S}$ is continuous on a compact space and thus bounded, whereas the marginal tax rate parametrized by k can be made arbitrarily large. As a result, individuals' first-order conditions can never hold for sufficiently large k, while we can similarly exclude corner solutions for sufficiently large k.

second-order conditions hold for type $\hat{\theta}$ at allocation $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ under tax system $\mathcal{T}_{k^\dagger}$ – this $k^\dagger$ exists by step 2. This implies that there exists an open set N containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ such that $V(s, z; \hat{\theta}, k^\dagger)$ is strictly concave over $(s, z) \in N$.

Now, consider types θ^j with $j \geq k^\dagger$. Since these individuals belong to the previously defined subsequence, they prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_j$. Because penalties are increasingly large and $j \geq k^\dagger$, this implies that types θ^j also prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_{k^\dagger}$.

Yet, by continuity, the function $V\left(s, z; \theta_j, k^\dagger\right)$ gets arbitrarily close to the function $V(s, z; \hat{\theta}, k^\dagger)$ for sufficiently large j since $\theta_j \to \hat{\theta}$. For the same reason, $(s^*(\theta_j), z^*(\theta_j)) \to (s^*(\hat{\theta}), z^*(\hat{\theta}))$. Moreover, by definition $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{s}, \hat{z})$. As a result, for any open set $N' \subsetneq N$ containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$, there exists a $j^\dagger \geq k^\dagger$ such that $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ and such that both $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ and $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ belong to the set N'.

Since $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$, it has a unique optimum on N'. By definition of $\mathcal{T}_{k^\dagger}$, type $\theta_{j^\dagger}$'s first-order conditions are satisfied at $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$. Hence, $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ is type $\theta_{j^\dagger}$'s maximum under the tax system $\mathcal{T}_{k^\dagger}$. This contradicts the fact established above that type $\theta_{j^\dagger}$ prefers $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ to allocation $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ under tax system $\mathcal{T}_{k^\dagger}$, which completes the proof.

Lemma for type-specific feasibility.

Lemma C.1. *A smooth tax system $\mathcal{T}$ satisfies type-specific feasibility over the compact type space $[\,\theta_{min}\,;\,\theta_{max}\,]$ if it satisfies the following conditions:*

1. $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\mathcal{T}_z'(s^*(\theta), z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$
3. $\mathcal{T}_s'(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$

Proof. Consider the type-specific feasible tax system $T_\theta^*(\theta) = z^*(\theta) - s^*(\theta) - c^*(\theta)$, and note that the lemma amounts to showing that $T_\theta^*(\theta) = \mathcal{T}(s^*(\theta), z^*(\theta))$ for all θ. To that end, note that the first-order condition for truthful reporting of θ under the optimal mechanism implies

$$U_c' \cdot (z'(\theta) - s'(\theta) - T_\theta^{*\prime}(\theta)) + U_s' \cdot s'(\theta) + U_z' \cdot z'(\theta) = 0,$$

with derivatives evaluated at the optimal allocation. This can be rearranged as

$$\begin{aligned} T_\theta^{*\prime}(\theta) &= \left(\frac{U_s'}{U_c'} - 1\right) s'(\theta) + \left(\frac{U_z'}{U_c'} + 1\right) z'(\theta) \\ &= \mathcal{T}_s'(s^*(\theta)) s^{*\prime}(\theta) + \mathcal{T}_z'(z^*(\theta)) z^{*\prime}(\theta). \end{aligned}$$

It thus follows that

$$\begin{aligned} \mathcal{T}(s^*(\theta), z^*(\theta)) - \mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) &= \int_{\vartheta=\theta_{min}}^{\vartheta=\theta} \left(\mathcal{T}_s'(s^*(\vartheta)) s^{*\prime}(\vartheta) + \mathcal{T}_z'(z^*(\vartheta)) z^{*\prime}(\vartheta)\right) d\vartheta \\ &= T_\theta^*(\theta) - T_\theta^*(\theta_{min}). \end{aligned}$$

Since $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = T^*_\theta(\theta_{min})$, this implies that $\mathcal{T}(s^*(\theta), z^*(\theta)) = T^*_\theta(\theta)$ for all θ. The smooth tax system $\mathcal{T}$ therefore satisfies type-specific feasibility. $\square$

Lemma on second-order conditions.

Lemma C.2. *Consider a smooth tax system $\mathcal{T}$ satisfying the conditions in Lemma C.1 and define*

$$V(s,z;\theta) := U(z - s - \mathcal{T}(s,z), s, z; \theta). \tag{87}$$

When evaluated at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$, we show that

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \tag{88}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \tag{89}$$

$$\begin{aligned}\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg]\end{aligned} \tag{90}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Proof. The first-order derivatives are

$$\begin{aligned}V'_s(s,z;\theta) &= -(1+\mathcal{T}'_s(s,z))U'_c(z-s-\mathcal{T}(s,z),s,z;\theta) + U'_s(z-s-\mathcal{T}(s,z),s,z;\theta)\\ V'_z(s,z;\theta) &= (1-\mathcal{T}'_z(s,z))U'_c(z-s-\mathcal{T}(s,z),s,z;\theta) + U'_z(z-s-\mathcal{T}(s,z),s,z;\theta).\end{aligned}$$

The second-order derivatives are

$$V''_{ss}(s,z;\theta) = -\mathcal{T}''_{ss}U'_c - (1+\mathcal{T}'_s)\left(-(1+\mathcal{T}'_s)U''_{cc} + U''_{cs}\right) - (1+\mathcal{T}'_s)U''_{cs} + U''_{ss} \tag{91}$$

$$V''_{zz}(s,z;\theta) = -\mathcal{T}''_{zz}U'_c + (1-\mathcal{T}'_z)\left((1-\mathcal{T}'_z)U''_{cc} + U''_{cz}\right) + (1-\mathcal{T}'_z)U''_{cz} + U''_{zz}. \tag{92}$$

To obtain the first result of the Lemma, we compute $\mathcal{T}''_{ss}$ by differentiating both sides of $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta);\theta) - 1$ with respect to θ:

$$\begin{aligned}\mathcal{T}''_{ss}s^{*\prime}(\theta) + \mathcal{T}''_{sz}z^{*\prime}(\theta) &= \frac{d}{d\theta}\mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta);\theta)\\ &= \mathcal{S}'_c c^{*\prime}(\theta) + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta,\end{aligned}$$

plugging in $c^{*\prime}(\theta) = (1-\mathcal{T}'_z)\,z^{*\prime}(\theta) - (1+\mathcal{T}'_s)\,s^{*\prime}(\theta)$ and denoting $s^{*\prime}(z^*) := s^{*\prime}(\theta)/z^{*\prime}(\theta)$. The previous expression can be rearranged as

$$\mathcal{T}''_{ss} = \mathcal{S}'_c\frac{1-\mathcal{T}'_z}{s^{*\prime}(z^*)} - \mathcal{S}'_c(1+\mathcal{T}'_s) + \mathcal{S}'_s + \frac{\mathcal{S}'_z}{s^{*\prime}(z^*)} + \frac{\mathcal{S}'_\theta}{s^{*\prime}(\theta)} - \frac{\mathcal{T}''_{sz}}{s^{*\prime}(z^*)}. \tag{93}$$

Moreover, we differentiate the definition $\mathcal{S} := \frac{U_s'}{U_c'}$ to express the derivative of $\mathcal{S}$ with respect to c as

$$\begin{aligned}\mathcal{S}_c'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) &= \frac{U_c' U_{sc}'' - U_s' U_{cc}''}{(U_c')^2} \\ &= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'} U_{cc}'' + U_{sc}''\right) \\ &= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s')U_{cc}'' + U_{sc}''\right) \end{aligned} \tag{94}$$

and the derivative of $\mathcal{S}$ with respect to s as

$$\begin{aligned}\mathcal{S}_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) &= \frac{U_c' U_{ss}'' - U_s' U_{cs}''}{(U_c')^2} \\ &= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'} U_{cs}'' + U_{ss}''\right) \\ &= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s')U_{cs}'' + U_{ss}''\right). \end{aligned} \tag{95}$$

Substituting equations (93), (94) and (95) into (91), we have

$$\begin{aligned} V_{ss}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) &= -U_c' \cdot \left(\mathcal{S}_c' \frac{1-\mathcal{T}_z'}{s^{*\prime}(z)} - \mathcal{S}_c'(1+\mathcal{T}_s') + \mathcal{S}_s' + \frac{\mathcal{S}_z'}{s^{*\prime}(z)} + \frac{\mathcal{S}_\theta'}{s^{*\prime}(\theta)} - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) - (1+\mathcal{T}_s')U_s'\mathcal{S}_c' + U_c'\mathcal{S}_s' \\ &= -U_c' \cdot \left(\frac{1-T_z'}{s^{*\prime}(z)}\mathcal{S}_c' + \frac{1}{s^{*\prime}(z)}\mathcal{S}_z' + \frac{1}{s^{*\prime}(\theta)}\mathcal{S}_\theta' - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) \\ &= \frac{U_z'}{s^{*\prime}(z)}\mathcal{S}_c' - \frac{U_c'}{s^{*\prime}(z^*)}\mathcal{S}_z' - \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{S}_\theta' + \frac{U_c'}{s^{*\prime}(z^*)}\mathcal{T}_{sz}'' \end{aligned} \tag{96}$$

where we have used $U_z' = -U_c'\left(1-\mathcal{T}_z'\right)$ in the last line.

Similarly, we can obtain the second result of the Lemma by writing $\mathcal{T}_{zz}''$ as

$$\mathcal{T}_{zz}'' = \mathcal{Z}_c'\left(1-\mathcal{T}_z'\right) - \mathcal{Z}_c'\left(1+\mathcal{T}_s'\right)s^{*\prime}(z^*) + \mathcal{Z}_s' s^{*\prime}(z^*) + \mathcal{Z}_z' + \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} - \mathcal{T}_{sz}'' s^{*\prime}(z^*). \tag{97}$$

Using

$$\mathcal{Z}_c' = \frac{1}{U_c'}\left(U_{cz}'' + \left(1-\mathcal{T}_z'\right)U_{cc}''\right)$$

as well as

$$\mathcal{Z}_z' = \frac{1}{U_c'}\left(U_{zz}'' + \left(1-\mathcal{T}_z'\right)U_{cz}''\right)$$

we get

$$V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = U_s' s^{*\prime}(z^*)\mathcal{Z}_c' - U_c' s^{*\prime}(z^*)\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} + U_c'\mathcal{T}_{sz}'' s^{*\prime}(z^*). \tag{98}$$

Finally, to obtain the third result of the Lemma, we must compute $\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}''$. Note that the first-order condition $V_s'(s^*(\theta), z^*(\theta); \theta) = 0$ holds at every θ by construction. Differentiating with respect to θ we get

$$\frac{d}{d\theta}V'_s(s^*(\theta), z^*(\theta); \theta) = V''_{ss}s^{*\prime}(\theta) + V''_{sz}z^{*\prime}(\theta) + V''_{s\theta} = 0 \tag{99}$$

which we can rearrange as

$$-V''_{sz} = V''_{ss}s^{*\prime}(z^*) + \frac{V''_{s\theta}}{z^{*\prime}(\theta)}. \tag{100}$$

Similarly, by totally differentiating the first-order condition $V'_z(s^*(\theta), z^*(\theta); \theta) = 0$ and rearranging we find

$$-V''_{sz} = \frac{V''_{zz}}{s^{*\prime}(z^*)} + \frac{V''_{z\theta}}{s^{*\prime}(\theta)}. \tag{101}$$

Writing $(V''_{sz})^2$ as the product of the right-hand sides of equations (100) and (101) yields

$$\begin{aligned}\left(V''_{sz}\right)^2 =& \left(V''_{ss}s^{*\prime}(z) + \frac{V''_{s\theta}}{z^{*\prime}(\theta)}\right)\left(\frac{V''_{zz}}{s^{*\prime}(z)} + \frac{V''_{z\theta}}{s^{*\prime}(\theta)}\right) \\ =& V''_{ss}V''_{zz} + \frac{1}{z^{*\prime}(\theta)}V''_{ss}V''_{z\theta} + \frac{1}{s^{*\prime}(\theta)}V''_{zz}V''_{s\theta} + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}V''_{s\theta}V''_{z\theta}.\end{aligned} \tag{102}$$

Now from the definition $V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta)$, we can compute

$$\begin{aligned}V''_{s\theta}(s, z; \theta) &= -\left(1 + \mathcal{T}'_s(s, z)\right)U''_{c\theta} + U''_{s\theta} \\ V''_{z\theta}(s, z; \theta) &= \left(1 - \mathcal{T}'_z(s, z)\right)U''_{c\theta} + U''_{z\theta}\end{aligned}$$

and use the fact that at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ we have

$$\begin{aligned}\mathcal{S}'_\theta &= \frac{1}{U'_c}\left(U''_{s\theta} - \left(1 + \mathcal{T}'_s\right)U''_{c\theta}\right) \\ \mathcal{Z}'_\theta &= \frac{1}{U'_c}\left(U''_{z\theta} + \left(1 - \mathcal{T}'_z\right)U''_{c\theta}\right)\end{aligned}$$

to obtain

$$V''_{s\theta}(s^*(\theta), z^*(\theta); \theta) = U'_c\mathcal{S}'_\theta \tag{103}$$

$$V''_{z\theta}(s^*(\theta), z^*(\theta); \theta) = U'_c\mathcal{Z}'_\theta. \tag{104}$$

Substituting these into equation (102) and rearranging, we have

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{1}{z^{*\prime}(\theta)}V''_{ss}U'_c\mathcal{Z}'_\theta + \frac{1}{s^{*\prime}(\theta)}V''_{zz}U'_c\mathcal{S}'_\theta + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\left(U'_c\right)^2\mathcal{S}'_\theta\mathcal{Z}'_\theta. \tag{105}$$

Expanding V''_{ss} from equation (96), and V''_{zz} from equation (98) yields after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta$$
$$+\left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg],$$

which gives the third result of the Lemma above. □

C.3 Proof of Proposition B.1 & B.2 (Implementation with Simple Tax Systems)

C.3.1 Proof of Proposition B.1

SN tax system. The sufficient conditions for local optimality under the candidate SN tax system follow directly from Lemma C.2 which computes individuals' second-order conditions (SOCs) at the optimal incentive-compatible allocation under a general tax system $\mathcal{T}(s,z)$. Indeed, individuals' SOCs are satisfied if equations (88), (89), and (90) are negative under the SN tax system. Since the cross-partial derivative $\mathcal{T}''_{sz}$ is zero for a SN tax system, it is easy to verify that conditions (49) and (50) on the derivatives of $\mathcal{S}$ and $\mathcal{Z}$, combined with monotonicity ($s^{*\prime}(\theta) > 0$, $s^{*\prime}(z) > 0$) and Assumption 1 on the derivatives of U, jointly imply that each of these three equations is the sum of negative terms. This implies that individuals' SOCs are satisfied at the optimal incentive-compatible allocation under the candidate SN tax system.

LED tax system. To derive sufficient conditions for local optimality under the candidate LED tax system, we begin from results obtained in the derivations of Lemma C.2 which computes individuals' SOCs at the optimal incentive-compatible allocation. We consider the requirements $V''_{ss} < 0$, $V''_{zz} < 0$, and $V''_{ss}V''_{zz} > (V''_{sz})^2$ in turn.

First, to show that V''_{ss} is negative, note that under a LED tax system, $\mathcal{T}''_{ss} = 0$. Therefore, using the fact that under the candidate LED tax system we have $1 + \mathcal{T}'_s = \frac{U'_s}{U'_c}$ at the optimal incentive-compatible allocation, the general expression for V''_{ss} given in equation (91) reduces to

$$V''_{ss}(s^*(\theta), z^*(\theta); \theta) = \left(\frac{U'_s}{U'_c}\right)^2 U''_{cc} - 2\frac{U'_s}{U'_c}U''_{cs} + U''_{ss}.$$

Therefore when utility is additively separable in c and s (implying $U''_{cs} = 0$), the concavity of preferences ($U''_{cc} \leq 0$ and $U''_{ss} \leq 0$) guarantees that this expression is negative.

Second, to show that V''_{zz} is negative, note that under the candidate LED tax system defined in equations (47) and (48) we have

$$\mathcal{T}''_{sz}(s,z) = \tau'_s(z).$$

We can thus find an expression for $\tau'_s(z)$ at any point in the allocation in question by totally differentiating equation (47) with respect to θ:

$$\begin{aligned}\tau_s'\left(z^*(\theta)\right)z^{*\prime}(\theta) &= \frac{d}{d\theta}\Big[\mathcal{S}\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\Big]\\ &= \frac{d}{d\theta}\Big[\mathcal{S}\left(z^*(\theta)-s^*(\theta)-\mathcal{T}\left(s^*(\theta), z^*(\theta)\right), s^*(\theta), z^*(\theta);\theta\right)\Big]\\ &= \mathcal{S}_c'\cdot\left[(1-\mathcal{T}_z')z^{*\prime}(\theta)-\left(1+\mathcal{T}_s'\right)s^{*\prime}(\theta)\right]+\mathcal{S}_s's^{*\prime}(\theta)+\mathcal{S}_z'z^{*\prime}(\theta)+\mathcal{S}_\theta',\end{aligned}$$

which yields

$$\tau_s'\left(z^*(\theta)\right)=\mathcal{S}_c'\cdot\left(1-\mathcal{T}_z'\right)-\mathcal{S}_c'\cdot\left(1+\mathcal{T}_s'\right)s^{*\prime}(z^*)+\mathcal{S}_s'\cdot s^{*\prime}(z^*)+\mathcal{S}_z'+\frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}.$$

Substituting this into the expression for V_{zz}'' in (98), we have

$$\begin{aligned}V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right);\theta\right) &= U_s's^{*\prime}(z^*)\mathcal{Z}_c'-U_c's^{*\prime}(z^*)\mathcal{Z}_s'-U_c'\frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)}\\ &\quad+U_c's^{*\prime}(z^*)\left[\mathcal{S}_c'\cdot\left(1-\mathcal{T}_z'\right)-\mathcal{S}_c'\cdot\left(1+\mathcal{T}_s'\right)s^{*\prime}(z^*)+\mathcal{S}_s'\cdot s^{*\prime}(z^*)+\mathcal{S}_z'+\frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}\right].\end{aligned} \tag{106}$$

Now employing the assumption that utility is separable in c, s, and z, (implying both $U_{cz}''=0$ and $U_{cs}''=0$) we have

$$\begin{aligned}U_s'\mathcal{Z}_c'+U_c'\mathcal{S}_c'(1-\mathcal{T}_z') &= U_s'\mathcal{Z}_c'-U_z'\mathcal{S}_c'\\ &= U_s'\frac{U_c'U_{cz}''-U_z'U_{cc}''}{(U_c')^2}-U_z'\frac{U_c'U_{cs}''-U_s'U_{cc}''}{(U_c')^2}\\ &= 0.\end{aligned}$$

Substituting this result into equation (106), and noting that $\mathcal{Z}_s'=\mathcal{S}_z'=0$ by separability, yields

$$V_{zz}''(s^*(\theta), z^*(\theta);\theta)=\left(s^{*\prime}(z^*)\right)^2\left[U_c'\mathcal{S}_s'-U_s'\mathcal{S}_c'\cdot\right]-\frac{U_c'}{z^{*\prime}(\theta)}\left[\mathcal{Z}_\theta'-s^{*\prime}(z^*)\mathcal{S}_\theta'\right]. \tag{107}$$

Again employing separability, we have

$$U_c'\mathcal{S}_s'-U_s'\mathcal{S}_c'=U_c'\frac{U_c'U_{ss}''-U_s'U_{cs}''}{(U_c')^2}-U_s'\frac{U_c'U_{cs}''-U_s'U_{cc}''}{(U_c')^2}=U_{ss}''+\left(\frac{U_s'}{U_c'}\right)^2U_{cc}''\leq 0,$$

implying that the first term on the right-hand side of equation (107) is negative. The condition $\mathcal{Z}_\theta'-s^{*\prime}(z^*)\mathcal{S}_\theta'\geq 0$ from (51) in the Proposition then implies equation (107) (and thus V_{zz}'') is negative.

Third, to show $V''_{ss}V''_{zz} > (V''_{sz})^2$, we proceed from equation (90) in Lemma C.2:

$$
\begin{aligned}
&\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} \\
&= \frac{U'_c}{s^{*\prime}(\theta)}\left[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta + \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\right] \\
&= \left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta \\
&+ \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta U'_c\mathcal{T}''_{sz} + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta\left(U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{z^{*\prime}(\theta)} + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz}\right).
\end{aligned}
$$

Recognizing that the last bracket term is exactly the expression for V''_{zz} given in Lemma C.2, this gives

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta U'_c\mathcal{T}''_{sz} + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}.$$

Using the previous expression derived for $\mathcal{T}''_{sz} = \tau'_s$, and the fact that separability ensures $\mathcal{S}'_z = 0$, we obtain after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = -\frac{(U'_c)^2}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\mathcal{Z}'_\theta\left[s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right) - \mathcal{S}'_\theta\right] + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}.$$

We have already shown that V''_{zz} is negative. Thus the conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{S}'_\theta \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right)$ from (51) in the Proposition imply that both terms on the right-hand side are negative, implying that all second-order conditions hold.

C.3.2 Proof of Proposition B.2

We begin with a more general statement, and then derive Proposition B.2 as a corollary. For a fixed type θ, let $c(z,\theta)$ and $s(z,\theta)$ be its preferred consumption and savings choices at earnings z, given the budget constraint induced by $\mathcal{T}(s,z)$.

Lemma C.3. *Suppose that $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, where $z^*(\theta)$ is increasing. Individuals' local optima correspond to their global optima when*

1. $\mathcal{Z} = \frac{U'_z(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ *and* $\mathcal{S} = \frac{U'_s(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ *are strictly increasing in θ for all (c,s,z).*
2. *For any two types θ and θ', we cannot have both*

$$
\begin{aligned}
&U'_c\Big(c^*(\theta), s^*(\theta), z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U'_z\Big(c^*(\theta), s^*(\theta), z;\theta\Big) \\
&< U'_c\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\sigma_c\Big(s\left(z^*(\theta),\theta'\right), z^*(\theta)\Big) \qquad (108) \\
&+ U'_z\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)
\end{aligned}
$$

and

$$U_s'\Big(c^*(\theta), s^*(\theta), z^*(\theta); \theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U_z'\Big(c^*(\theta), s^*(\theta), z; \theta\Big)\Big)$$
$$< U_s'\Big(c\left(z^*(\theta), \theta'\right), s\left(z^*(\theta), \theta'\right), z^*(\theta); \theta\Big)\sigma_s\Big(s\left(z^*(\theta), \theta'\right), z^*(\theta)\Big) \tag{109}$$
$$+ U_z'\Big(c\left(z^*(\theta), \theta'\right), s\left(z^*(\theta), \theta'\right), z^*(\theta); \theta$$

where $\sigma_c(s,z) := 1 - \mathcal{T}_z'(s,z)$ *and* $\sigma_s(s,z) := \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Condition 1 corresponds to single-crossing assumptions for earnings and savings. Condition 2 is a requirement that if type θ preserves its assigned earnings level $z^*(\theta)$, but chooses some other consumption level s (corresponding to a level that some other type θ' would choose if forced to choose earnings level $z^*(\theta)$), then at this alternative consumption bundle, type θ cannot have both higher marginal utility from increasing its savings through one more unit of work *and* increasing its consumption through one more unit of work. Generally, this condition will hold as long as U is sufficiently concave in consumption and savings when type θ chooses earnings level $z^*(\theta)$.

Proof. To prove that individuals' local optima are global optima, we want to show that for any given type θ^*, utility decreases when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$.

The first step is to compute type θ^*'s utility change. The envelope theorem applied to savings choices $s(z,\theta^*)$ implies

$$\begin{aligned}&\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\\ &= U_c'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\sigma_c\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\end{aligned}$$

where $\sigma_c(s,z) = 1 - \mathcal{T}_z'(s,z)$. Note that, as established by Milgrom and Segal (2002), these equalities hold as long as U is differentiable in z (holding s and c fixed)—differentiability of $c(z,\theta^*)$ or $s(z,\theta^*)$ is actually not required.

Similarly, the envelope theorem applied to consumption choices $c(z,\theta^*)$ implies

$$\begin{aligned}&\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\\ &= U_s'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\sigma_s\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\end{aligned} \tag{110}$$

where $\sigma_s(s,z) = \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Therefore, type θ^*'s utility change when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$ is

$$\begin{aligned}&U\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right) - U\left(c(z^*(\theta^*),\theta^*), s(z^*(\theta^*),\theta^*), z^*(\theta^*); \theta^*\right)\\ &= \int_{x=z^*(\theta^*)}^{x=z}\Big[\min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)\sigma_s\left(s(x,\theta^*), x\right)\right\}\\ &\qquad + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)\Big]dx\end{aligned} \tag{111}$$

where the $\min$ operator is introduced without loss of generality given that both terms are equal.

The second step is to show that under our assumptions, type θ^*'s utility change in equation (111) is negative. To do so, let θ_x be the type that chooses earnings x. Then, by definition, type θ_x's utility is maximal at earnings x, implying both

$$U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0$$
$$U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0$$

such that

$$\begin{aligned}&\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right)\right\}\\ &+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0.\end{aligned} \tag{112}$$

Now, by condition 2, we either have[35]

$$\begin{aligned}&U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\\ &\geq U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\end{aligned}$$

or

$$\begin{aligned}&U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\\ &\geq U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\end{aligned}$$

implying that

$$\begin{aligned}&\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right)\right\}\\ &+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\\ &\geq \min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right)\right\}\\ &+ U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right).\end{aligned} \tag{113}$$

But since the maximum is zero, this minimum has to be negative. Hence, we have either

$$\begin{aligned}&U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0\\ \iff\ &\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)\end{aligned}$$

or

$$\begin{aligned}&U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0\\ \iff\ &\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right).\end{aligned}$$

Suppose that $z > z^*(\theta^*)$ such that $x > z^*(\theta^*)$; the case $z < z^*(\theta^*)$ follows identically. For

[35] Not having $\{a < c \quad and \quad b < c\}$ means having $\{a \geq c \quad \text{or} \quad b \geq d\}$, which implies $\max(a,b) \geq \min(c,d)$.

any $x > z^*(\theta^*)$, the monotonicity of the earnings function means that $\theta_x > \theta^*$. Then, by the single-crossing conditions for $\mathcal{Z} = \frac{U'_z}{U'_c}$ and $\mathcal{S} = \frac{U'_s}{U'_c}$, this means that we have either[36]

$$\frac{U'_z\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)}{U'_c\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)} \le -\sigma_c\left(s(x,\theta^*), x\right)$$

or

$$\frac{U'_z\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)}{U'_s\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)} \le -\sigma_c\left(s(x,\theta^*), x\right)$$

implying that for any $x > z^*(\theta^*)$,

$$\begin{aligned}&\min\left\{U'_c\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_c\left(s(x,\theta^*), x\right), U'_s\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_s\left(s(x,\theta^*), x\right)\right\}\\ &+ U'_z\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right) \le 0.\end{aligned} \tag{114}$$

As a result, the right hand-side of equation (111) is an integral of negative terms, which shows that

$$U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right) - U\left(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*);\theta^*\right) \le 0. \tag{115}$$

The case with $z < z^*(\theta^*)$ follows identically, proving Lemma C.3. □

Proof of Proposition B.2

We now derive Proposition B.2 as a consequence of Lemma C.3 by deriving assumptions under which condition 2 is met for SN and LED tax systems.

SN systems. First, suppose that $s < s^*(\theta)$, then $c > c^*(\theta)$. Noting that $\sigma_c = 1 - T'_z\left(z^*\left(\theta\right)\right)$ is not a function of s, we can use $U''_{cc} \le 0$ and $U''_{cs} \ge 0$ to obtain

$$U'_c\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) \ge U'_c\left(c, s, z^*(\theta);\theta\right)\sigma_c\left(s, z^*(\theta)\right).$$

Further relying on the fact that $U''_{cz} = 0$ and $U''_{sz} = 0$, we obtain

$$\begin{aligned}&U'_c\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) + U'_z\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\\ &\ge U'_c\left(c, s, z^*(\theta);\theta\right)\sigma_c\left(s, z^*(\theta)\right) + U'_z\left(c, s, z^*(\theta);\theta\right).\end{aligned}$$

Conversely, suppose that $s > s^*(\theta)$, then $c < c^*(\theta)$. We have

$$\begin{aligned}&\frac{d}{ds}\left[\frac{U'_s(z - T_z\left(z\right) - s - T_s\left(s\right), s, z^*(\theta);\theta)}{1 + T'_s(s)}\right]\\ &= -U''_{cs} + \frac{1}{\left(1 + T'_s(s)\right)}\left[U''_{ss} - U'_s\frac{T''_{ss}(s)}{1 + T'_s(s)}\right].\end{aligned}$$

The condition that $\frac{U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} < \frac{T''_{ss}(s)}{1+T'_s(s)}$, together with $U''_{cs} > 0$, implies that $\frac{U'_s(c(s,\theta),s,z^*(\theta);\theta)}{1+T'_s(s)}$

[36]Note that having both $\mathcal{Z}$ and $\mathcal{S}$ increasing in θ also implies that $\frac{\mathcal{Z}}{\mathcal{S}} = \frac{U'_z}{U'_s}$ is increasing in θ.

is decreasing in s and thus that

$$\frac{U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)}{1 + T_s'(s^*(\theta))} \geq \frac{U_s'(c, s, z^*(\theta); \theta)}{1 + T_s'(s)}.$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, and that $\mathcal{T}_s' = T_z'(z)$ is independent of s, we obtain

$$\begin{aligned} &U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\, \sigma_s(s^*(\theta), z^*(\theta)) + U_z'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \\ &\geq U_s'(c, s, z^*(\theta); \theta)\, \sigma_s(s, z^*(\theta)) + U_z'(c, s, z^*(\theta); \theta). \end{aligned}$$

LED systems. First, consider a type θ' choosing earnings $z = z^*(\theta) > z^*(\theta')$. We have

$$\begin{aligned} &\frac{d}{ds}\left[U_c'(z - s - \tau_s(z^*(\theta))\, s - T_z(z^*(\theta)), s, z^*(\theta); \theta)\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))\, s\right)\right] \\ &= U_{cs}''\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))\, s\right) - U_{cc}''(1 + \tau_s(z^*(\theta)))\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))\, s\right) - U_c'\tau_s'(z^*(\theta)). \end{aligned}$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z^*(\theta) > z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') < s^*(\theta)$. In this case, condition *(2a)* of the proposition implies that the remaining terms are negative such that

$$U_c'(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta)\, \sigma_c(s, z^*(\theta))$$

is increasing in s for $s < s^*(\theta)$, where $\sigma_c(s, z^*(\theta)) = 1 - T_z'(z) - \tau_s'(z)s$. As a result,

$$\begin{aligned} &U_c'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\, \sigma_c(s^*(\theta), z^*(\theta)) \\ &\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right) \sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) \end{aligned}$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$\begin{aligned} &U_c'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\, \sigma_c(s^*(\theta), z^*(\theta)) + U_z'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \\ &\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right) \sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right). \end{aligned}$$

Second, consider a type θ' choosing $z = z^*(\theta) < z^*(\theta')$. We have

$$\begin{aligned} &\frac{d}{ds}\left[U_s'(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta)\, \frac{1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))\, s}{1 + \tau_s(z)}\right] \\ &= -U_{cs}''\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))\, s\right) + U_{ss}''\frac{1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))\, s}{1 + \tau_s(z)} + U_s'\frac{\tau_s'(z^*(\theta))}{1 + \tau_s(z)}. \end{aligned}$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z = z^*(\theta) < z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') > s^*(\theta)$. Hence, condition *(2b)* of the proposition implies that the remaining terms are negative such that

$$U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\sigma_s\left(s, z^*(\theta)\right)$$

is decreasing in s for $s > s^*(z)$, where $\sigma_s\left(s, z^*(\theta)\right) = \frac{1 - T_z'(z) - \tau_s'(z)s}{1 + \tau_s(z)}$. This ensures that

$$\begin{aligned}&U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right)\\ &\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right)\end{aligned}$$

and thus, relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$\begin{aligned}&U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\\ &\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right).\end{aligned}$$

C.4 Proof of Proposition 2 (Measurement of Causal Income Effects)

Here, we derive that the different expressions of the sufficient statistic $s_{inc}'(z)$ can be measured empirically.

Case 1. If individuals' preferences are weakly separable between the utility of consumption $u(.)$ and the disutility to work $k(.)$, type θ's problem is written as

$$\max_{c,s,z} u(c, s; \theta) - k(z/w(\theta)) \ s.t.\ c \leq z - ps - \mathcal{T}(s, z),$$

meaning that conditional on earnings z, savings $s(z; \theta)$ is defined as the solution to

$$-\left(p + \mathcal{T}_s'(s, z)\right)u_c'\left(z - ps - \mathcal{T}(s, z), s; \theta\right) + u_s'\left(z - ps - \mathcal{T}(s, z), s; \theta\right) = 0.$$

Differentiating this equation with respect to savings s and earnings z yields

$$\frac{\partial s}{\partial z} = -\frac{\left[-\mathcal{T}_{sz}''u_c' - (p + \mathcal{T}_s')(1 - \mathcal{T}_z')u_{cc}'' + (1 - \mathcal{T}_z')u_{cs}''\right]}{\left[-\mathcal{T}_{ss}''u_c' + (p + \mathcal{T}_s')^2 u_{cc}'' - 2\left(p + \mathcal{T}_s'(s, z)\right)u_{cs}'' + u_{ss}''\right]}.$$

Differentiating this equation with respect to savings s and disposable income y yields

$$\frac{\partial s}{\partial y} = -\frac{\left[-(p + \mathcal{T}_s')u_{cc}'' + u_{cs}''\right]}{\left[-\mathcal{T}_{ss}''u_c' + (p + \mathcal{T}_s')^2 u_{cc}'' - 2\left(p + \mathcal{T}_s'(s, z)\right)u_{cs}'' + u_{ss}''\right]}.$$

Hence, if $\mathcal{T}_{sz}'' = 0$, we get

$$s_{inc}'(z) := \frac{\partial s(z; \theta)}{\partial z} = \left(1 - \mathcal{T}_z'\right)\frac{\partial s}{\partial y} = \left(1 - \mathcal{T}_z'\right)\frac{\eta_{s|z}(z(\theta))}{1 + \mathcal{T}_s'},$$

where the last equality follows from the definition of $\eta_{s|z}(z(\theta))$. The intuition behind this result is that with separable preferences, savings s depend on earnings z only through disposable income

$y = z - ps - \mathcal{T}(s, z)$.

Case 2. If individuals' wage rates w and hours h are observable, and earnings z are given by $z = w \cdot h$, we can infer s'_{inc} from changes in wages through

$$\begin{aligned}
\frac{\partial s}{\partial w} &= \frac{\partial s(w \cdot h; \theta)}{\partial w} = \frac{\partial s(z; \theta)}{\partial z}\left(1 + \frac{\partial h}{\partial w}\right) \\
\iff \frac{\partial s(z; \theta)}{\partial z} &= \frac{\frac{\partial s}{\partial w}}{1 + \frac{\partial h}{\partial w}} = s\frac{\frac{w}{s}\frac{\partial s}{\partial w}}{w + h\frac{w}{h}\frac{\partial h}{\partial w}} \\
\iff s'_{inc}(z) &= s(z)\frac{\xi_w^s(z)}{w(z) + h(z)\,\xi_w^h(z)}
\end{aligned}$$

where $\xi_w^s(z) \equiv \frac{w(z)}{s(z)}\frac{\partial s(z)}{\partial w(z)}$ is individuals' elasticity of savings with respect to their wage rate, and $\xi_w^h(z) \equiv \frac{w(z)}{h(z)}\frac{\partial h(z)}{\partial w(z)}$ is individuals' elasticity of hours with respect to their wage rate.

Case 3. Otherwise, if we can measure the elasticity of savings s and earnings z upon a compensated change in the marginal earnings tax rate $\mathcal{T}'_z$, respectively denoted $\chi_s^c := -\frac{1-\mathcal{T}'_z}{s}\frac{\partial s}{\partial \mathcal{T}'_z}$ and $\zeta_z^c := -\frac{1-\mathcal{T}'_z}{z}\frac{\partial z}{\partial \mathcal{T}'_z}$, we then have

$$\begin{aligned}
\frac{\partial s}{\partial \mathcal{T}'_z} &= \frac{\partial s(z; \theta)}{\partial z}\frac{\partial z}{\mathcal{T}'_z} \\
\iff \left(-\frac{s}{1-\mathcal{T}'_z}\chi_s^c\right) &= s'_{inc}(z)\left(-\frac{z}{1-\mathcal{T}'_z}\zeta_z^c\right) \\
\iff s'_{inc}(z) &= \frac{s(z)}{z}\frac{\chi_s^c(z)}{\zeta_z^c(z)}.
\end{aligned}$$

C.5 Proof of Lemma 1 (Earnings Responses to Taxes on s)

Throughout the paper, we characterize earnings responses to (different) savings tax reforms using generalizations of Lemma 1 in Saez (2002). The robust insight in all cases is that a $\Delta\tau$ increase in the marginal tax rate on s induces the same earnings changes (through substitution effects) as a $s'_{inc}(z)\Delta\tau$ increase in earnings tax rate. This is what appears in the body of the text as Lemma 1.

In our proofs we use a version that pertains to reforms that have an LED, SL, or SN structure. For example, a reform with LED structure adds a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0 + \Delta z\right]$. Note that the reform itself has an LED structure, but it can applied to any nonlinear tax system, not just one with an LED structure. The results below allow for multidimensional heterogeneity.

Let

$$V(\mathcal{T}(., z), z; \theta) = \max_s\ U(z - ps - \mathcal{T}(s, z), s, z; \theta)$$

be type θ's indirect utility function at earnings z.

LED reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, that is:[37]

$$\Delta\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_s\left(z-z^0\right)s & if\ z \in [z^0, z^0+\Delta z] \\ \Delta\tau_s\,\Delta z\, s & if\ z \geq z^0+\Delta z \end{cases}$$

We now construct for each type θ a tax reform $\Delta\mathcal{T}_z^\theta$ that affects marginal earnings tax rates, and induces the same earnings response as the initial perturbation $\Delta\mathcal{T}_s$. We define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z)+\Delta\mathcal{T}_s(.,z),z;\theta) = V(\mathcal{T}(.,z)+\Delta\mathcal{T}_z^\theta(.,z),z;\theta).$$

Then, by construction, the perturbation $\Delta\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $\Delta\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with earnings $z(\theta) \in [z^0, z^0+\Delta z]$, this implies

$$\begin{aligned} &U_c'\,\Delta\tau_s\left(z-z^0\right)s\left(z;\theta\right) = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\left(z-z^0\right)s\left(z;\theta\right). \end{aligned}$$

Differentiating both sides with respect to z and letting $\Delta z \to 0$, this implies that in the phase-in region, the reform induces the same earnings change as a small increase $s_{inc}'\left(z\right)\Delta\tau_s$ in the marginal earnings tax rate.

For types θ with earnings $z(\theta) \geq z^0+\Delta z$, this implies

$$\begin{aligned} &U_c'\,\Delta\tau_s\,\Delta z\, s\left(z;\theta\right) = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\,\Delta z\, s\left(z;\theta\right). \end{aligned}$$

That is, above the phase-in region, the reform induces the same earnings changes as a $\Delta\tau_s\Delta z\, s\left(z\right)$ increase in tax liability combined with a $\Delta\tau_s\Delta z\, s_{inc}'\left(z\right)$ increase in the marginal earnings tax rate.

SL reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s$ on s for all individuals. This is a special case of a LED reform. As a result, we directly obtain that this reform induces the same earnings changes as a $\Delta\tau_s\, s\left(z\right)$ increase in tax liability combined with a $\Delta\tau_s\, s_{inc}'\left(z\right)$ increase in the marginal earnings tax rate.

[37] This reform, which is natural to consider for LED tax systems, allows us to derive a sufficient statistics characterization of the optimal smooth tax system (Theorem 2) without the requirement that $s(z)$ is monotonic. In contrast, if we were to rely on an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we would need further assumptions.

SN reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in a small increase $\Delta\tau_s$ in the marginal tax rate on s in a bandwidth $\left[s^0, s^0+\Delta s\right]$, with $\Delta\tau_s$ much smaller than Δs:

$$\Delta\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ s \le s^0 \\ \Delta\tau_s(s-s^0) & if\ s\in[s^0, s^0+\Delta s] \\ \Delta\tau_s\Delta s & if\ s \ge s^0+\Delta s \end{cases}$$

We now construct for each type θ a perturbation of the earnings tax $\Delta\mathcal{T}_z^\theta$ that induces the same earnings response as the initial perturbation $\Delta\mathcal{T}_s$. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z)+\Delta\mathcal{T}_s(.,z),z;\theta) = V(\mathcal{T}(.,z)+\Delta\mathcal{T}_z^\theta(.,z),z;\theta).$$

Then, by construction, the perturbation $\Delta\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $\Delta\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with $s(z;\theta)\in[s^0, s^0+\Delta s]$, this implies

$$\begin{aligned} &U_c'\Delta\tau_s\left(s\left(z;\theta\right)-s^0\right) = U_c'\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \left(s\left(z;\theta\right)-s^0\right)\Delta\tau_s. \end{aligned}$$

Differentiating both sides with respect to z and letting $\Delta s\to 0$, this implies that a small increase $\Delta\tau_s$ in the marginal tax rate on s induces the same earnings change as a small increase $s_{inc}'\left(z\right)\Delta\tau_s$ in the marginal earnings tax rate.

For types θ with $s(z;\theta)\ge s^0+\Delta s$, this implies

$$\begin{aligned} &U_c'\,\Delta\tau_s\,\Delta s = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\,\Delta s. \end{aligned}$$

Thus, a $\Delta\tau_s\,\Delta s$ lump-sum (savings) tax increase induces the same earnings change as a $\Delta\tau_s\,\Delta s$ lump-sum (earnings) tax increase.

C.6 Proof of Theorem 2 (Optimal Smooth Tax Systems)

When $z\left(\theta\right)$ is a strictly increasing function, we can define its inverse by $\vartheta\left(z\right)$. This allows us to define consumption of good c as $c(z) := c(z;\vartheta\left(z\right))$, consumption of good s as $s\left(z\right) := s\left(z;\vartheta\left(z\right)\right)$, and the planner's weights as $\alpha(z) := \alpha(\vartheta\left(z\right))$.

In this notation, the problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z\Big[\alpha(z)\,U\Big(c(z), s(z), z;\vartheta(z)\Big) + \lambda\Big(\mathcal{T}\left(s(z),z\right)-E\Big)\Big]\,dH_z(z), \tag{116}$$

where λ is the social marginal value of public funds, and the tax function implicitly enters individuals' utility through $c(z) = z - s(z) - \mathcal{T}\left(s(z),z\right)$.

C.6.1 Optimality Condition for Marginal Tax Rates on z

Reform. We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal tax rate on earnings in a small bandwidth Δz. Formally,

$$\Delta\mathcal{T}(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_z(z-z^0) & if\ z \in [z^0, z^0+\Delta z] \\ \Delta\tau_z\Delta z & if\ z \geq z^0+\Delta z \end{cases}$$

We characterize the impact of this reform on the government's objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*:

$$\int_{z\geq z^0}\Big(1-\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right)\Big)\,\Delta\tau_z\Delta z\,dH_z(z)$$

- *behavioral effects from changes in z:*[38]

$$-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\,\Delta z h_z(z^0)$$
$$-\int_{z\geq z^0}\mathcal{T}_z'\left(s\left(z\right),z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\Delta\tau_z\Delta z\,dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}_s'\left(s(z^0),z^0\right)s_{inc}'(z^0)\left[\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\right]\Delta z h_z(z^0)$$
$$-\int_{z\geq z^0}\mathcal{T}_s'\left(s(z),z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\right]\Delta\tau_z\Delta z\,dH_z(z)$$

Summing over these different effects yields the total impact of the reform

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta z} = &\int_{z\geq z^0}\left(1-\hat{g}(z)\right)\,\Delta\tau_z dH_z(z) \\ &-\Big(\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s(z^0),z^0\right)\Big)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\,h_z(z^0)\end{aligned} \tag{117}$$

[38]Note that by definition elasticity concepts include all circularities and adjustments induced by tax reforms such that changes in z and s are given by

$$\begin{cases} dz = -\frac{z}{1-\mathcal{T}_z'}\zeta_z^c(z)\Delta\mathcal{T}_z'(s,z)-\frac{\eta_z(z)}{1-\mathcal{T}_z'}\Delta\mathcal{T}(s,z) \\ ds = -\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'}\Delta\mathcal{T}(s,z)+s_{inc}'(z)dz \end{cases}$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$\hat{g}(z) = \frac{\alpha(z)}{\lambda} U_c'(c(z), s(z), z; \vartheta(z)) - \frac{\mathcal{T}_z'(s(z), z) + s_{inc}'(z)\mathcal{T}_s'(s(z), z)}{1 - \mathcal{T}_z'(s(z), z)} \eta_z(z) - \frac{\mathcal{T}_s'(s(z), z)}{1 + \mathcal{T}_s'(s(z), z)} \eta_{s|z}(z).$$

Optimality. A direct implication is a sufficient statistics characterization of the optimal schedule of marginal tax rates on z. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L} = 0$, meaning that at each earnings z^0 the optimal marginal earnings tax rate satisfies

$$\frac{\mathcal{T}_z'\left(s\left(z^0\right), z^0\right)}{1 - \mathcal{T}_z'\left(s\left(z^0\right), z^0\right)} = \frac{1}{\zeta_z^c(z^0)} \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z))\, dH_z(z) - s_{inc}'(z^0) \frac{\mathcal{T}_s'\left(s(z^0), z^0\right)}{1 - \mathcal{T}_z'\left(s\left(z^0\right), z^0\right)} \tag{118}$$

which is the optimality condition in equation (19) presented in Theorem 2.

C.6.2 Optimality Condition for Marginal Tax Rates on s

Reform. We consider a small reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s \Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0 + \Delta z\right]$, that is:[39]

$$\Delta\mathcal{T}_s(s, z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_s \left(z - z^0\right) s & if\ z \in [z^0, z^0 + \Delta z] \\ \Delta\tau_s\, \Delta z\, s & if\ z \geq z^0 + \Delta z \end{cases}$$

Let $s^0 = s(z^0)$. We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z \geq z^0} \left(1 - \frac{\alpha(z)}{\lambda} U_c'(c(z), s(z), z; \theta(z))\right) \Delta\tau_s \Delta z\, s(z)\, dH_z(z) \tag{119}$$

[39]We use this reform to derive a sufficient statistics characterization of the optimal smooth tax system, without the requirement that $s(z)$ is monotonic. If we instead consider an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we need this extra assumption.

- *behavioral effects from changes in z:*[40]

$$
\begin{aligned}
& -\mathcal{T}_z'\left(s^0,z^0\right)\left[\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\,\Delta\tau_s s^0\right]h_z(z^0)\Delta z \\
& -\int_{z\geq z^0}\mathcal{T}_z'\left(s\left(z\right),z\right)\left[\frac{z\zeta_z^c(z)s_{inc}'(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}+\frac{\eta_z(z)s(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\right]\Delta\tau_s\Delta z\,dH_z(z) \qquad (120)
\end{aligned}
$$

- *behavioral effects from changes in s:*

$$
\begin{aligned}
& -\mathcal{T}_s'\left(s^0,z^0\right)s_{inc}'(z^0)\left[\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\,\Delta\tau_s\,s^0\right]h_z(z^0)\Delta z \\
& -\int_{z\geq z^0}\mathcal{T}_s'\left(s(z),z\right)\left[\frac{\zeta_{s|z}^c(z)+\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\,s(z)+s_{inc}'(z)\left[\frac{z\zeta_z^c(z)s_{inc}'(z)}{1-\mathcal{T}_z'\left(s(z),z\right)}+\frac{\eta_z(z)s(z)}{1-\mathcal{T}_z'\left(s(z),z\right)}\right]\right]\Delta\tau_s\Delta z\,dH_z(z) \qquad (121)
\end{aligned}
$$

Summing over these different effects yields the total impact of the reform

$$
\begin{aligned}
&\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s\Delta z} \\
&=-\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s^0,z^0\right)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}z^0\zeta_z^c(z^0)\,s^0\,h_z(z^0) \qquad (122)\\
&+\int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z)
\end{aligned}
$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$
\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\eta_z(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}\eta_{s|z}(z).
$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, which implies that at each $s^0=s\left(z^0\right)$ and earnings z^0, the optimal marginal tax rate on s satisfies

$$
\begin{aligned}
&\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_z'\left(s^0,z^0\right)}{1-\mathcal{T}_s'\left(s^0,z^0\right)}z^0\zeta_z^c(z^0)\,s^0\,h_z(z^0) \qquad (123)\\
&=\int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\frac{\mathcal{T}_z'\left(s(z),z\right)+s_{inc}'(z)\mathcal{T}_z'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s(z),z\right)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z)
\end{aligned}
$$

[40] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$
\begin{cases}dz=-\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'}\,\Delta\tau_s\,s^0\\ ds=s_{inc}'(z^0)dz\end{cases}
\quad\text{and}\quad
\begin{cases}dz=-\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}\,\Delta\tau_s\Delta z\,s_{inc}'(z)-\frac{\eta_z(z)}{1-\mathcal{T}_z'}\,\Delta\tau_s\Delta z\,s\left(z\right)\\ ds=-\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'}\,\Delta\tau_s\Delta z-\frac{\eta_s(z)}{1+\mathcal{T}_s'}\Delta\tau_s\Delta z\;s\left(z\right)+s_{inc}'(z)dz\end{cases}
$$

Using the formula for the optimal schedule of marginal earnings tax rates in equation (118) to replace the term on the left-hand side, this formula can be rearranged as

$$s(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z) \tag{124}$$
$$=\int_{z\geq z^0}\left\{(1-\hat{g}(z))\,s(z)-\frac{\mathcal{T}_z'\left(s(z),z\right)+s_{inc}'(z)\mathcal{T}_z'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s(z),z\right)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z)$$

Differentiating both sides with respect to z^0 yields

$$s'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)-s^0(1-\hat{g}(z^0))h_z(z^0)-\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)$$
$$=-\left(1-\hat{g}(z^0)\right)s^0h_z(z^0)+s_{inc}'(z^0)\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s^0,z^0\right)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)z^0h_z(z^0)$$

where both $s^0(1-\hat{g}(z^0))h_z(z^0)$ terms cancel out. Using equation (118) again, the last term is equal to $s_{inc}'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)$ such that we finally obtain

$$\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s(z^0)\zeta_{s|z}^c(z^0)h_z(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{het}'(z^0)}\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z), \tag{125}$$

which is the optimality condition in equation (20) presented in Theorem 2.

C.6.3 Pareto Efficiency Condition

We can combine formulas for optimal marginal tax rates on z and on s to obtain a characterization of Pareto efficiency. Indeed, leveraging the above optimal formula for marginal tax rates on s written in terms of $s_{het}'(z^0)$, and replacing the integral term by its value from the optimal formula for marginal earnings tax rates in equation (118) yields

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+T_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0)=s_{het}'(z^0)\frac{\mathcal{T}_z'(z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)$$

which is the Pareto-efficiency condition in equation (21) presented in Theorem (2).

C.7 Proof of Proposition B.3 (Structural characterization of s_{inc}' and s_{het}')

In economies with preference heterogeneity, budget heterogeneity, and auxiliary choices, $s(z;\theta)$ solves

$$\max_s U\Big(B\big(s,z,\chi(s,z;\theta);\theta\big)-\mathcal{T}(s,z),\phi_s\big(z,s,\chi(s,z;\theta);\theta\big),\phi_z\big(z,s,\chi(s,z;\theta);\theta\big),\chi(s,z;\theta);\theta\Big) \tag{126}$$

where $\chi(s,z;\theta)$ denotes utility-maximizing auxiliary choices. As a result, applying the envelope theorem to changes in χ, $s(z;\theta)$ is defined by the following first-order condition

$$U_c'(.)\big[B_s'(s(z;\theta),z,\chi(s(z;\theta),z;\theta);\theta)-\mathcal{T}_s'(s(z;\theta),z)\big] \\ +U_s'(.)\frac{\partial\phi_s(z,s,\chi(s,z;\theta);\theta)}{\partial s}\Big|_{s=s(z;\theta)}+U_z'(.)\frac{\partial\phi_z(z,s,\chi(s,z;\theta);\theta)}{\partial s}\Big|_{s=s(z;\theta)}=0. \quad (127)$$

Now, to compute $s_{inc}'=\frac{\partial s(z;\theta)}{\partial z}$, we differentiate this first-order condition with respect to z while holding θ fixed:

$$\begin{aligned}&\underbrace{\Big[U_{cc}''(.)(B_s'-\mathcal{T}_s')+U_{cs}''(.)\frac{\partial\phi_s}{\partial s}+U_{cz}''(.)\frac{\partial\phi_z}{\partial s}\Big]}_{\mathcal{K}_c}\Big[B_s'\frac{\partial s(z;\theta)}{\partial z}+B_z'+B_\chi'\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)-\mathcal{T}_s'\frac{\partial s(z;\theta)}{\partial z}-\mathcal{T}_z'\Big]\\
&+\underbrace{\Big[U_{cs}''(.)(B_s'-\mathcal{T}_s')+U_{ss}''(.)\frac{\partial\phi_s}{\partial s}+U_{sz}''(.)\frac{\partial\phi_z}{\partial s}\Big]}_{\mathcal{K}_s}\Big[\frac{\partial\phi_s}{\partial z}+\frac{\partial\phi_s}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\phi_s}{\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]\\
&+\underbrace{\Big[U_{cz}''(.)(B_s'-\mathcal{T}_s')+U_{sz}''(.)\frac{\partial\phi_s}{\partial s}+U_{zz}''(.)\frac{\partial\phi_z}{\partial s}\Big]}_{\mathcal{K}_z}\Big[\frac{\partial\phi_z}{\partial z}+\frac{\partial\phi_z}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\phi_z}{\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]\\
&+\underbrace{\Big[U_{c\chi}''(.)(B_s'-\mathcal{T}_s')+U_{s\chi}''(.)\frac{\partial\phi_s}{\partial s}+U_{z\chi}''(.)\frac{\partial\phi_z}{\partial s}\Big]}_{\mathcal{K}_\chi}\Big[\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big]\\
&+U_c'\Big[B_{ss}''\frac{\partial s(z;\theta)}{\partial z}+B_{sz}''+B_{s\chi}''\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)-\mathcal{T}_{ss}''\frac{\partial s(z;\theta)}{\partial z}-\mathcal{T}_{sz}''\Big]\\
&+U_s'\Big[\frac{\partial^2\phi_s}{\partial s\partial z}+\frac{\partial^2\phi_s}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]\\
&+U_z'\Big[\frac{\partial^2\phi_z}{\partial s\partial z}+\frac{\partial^2\phi_z}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]=0. \end{aligned} \quad (128)$$

Rearranging terms yields

$$\frac{\partial s(z;\theta)}{\partial z}=-\frac{\mathcal{K}_c\left[B_z'+B_\chi'\frac{\partial\chi}{\partial z}-\mathcal{T}_z'\right]+\mathcal{K}_s\left[\frac{\partial\phi_s}{\partial z}+\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial z}\right]+\mathcal{K}_z\left[\frac{\partial\phi_z}{\partial z}+\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial z}\right]+\mathcal{K}_\chi\left[\frac{\partial\chi}{\partial z}\right]+...}{\mathcal{K}_c\left[B_s'+B_\chi'\frac{\partial\chi}{\partial s}-\mathcal{T}_s'\right]+\mathcal{K}_s\left[\frac{\partial\phi_s}{\partial s}+\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial s}\right]+\mathcal{K}_z\left[\frac{\partial\phi_z}{\partial s}+\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial s}\right]+\mathcal{K}_\chi\left[\frac{\partial\chi}{\partial s}\right]+...} \\ \frac{...+U_c'\left[B_{sz}''+B_{s\chi}''\frac{\partial\chi}{\partial z}-\mathcal{T}_{sz}''\right]+U_s'\left[\frac{\partial^2\phi_s}{\partial s\partial z}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial z}\right]+U_z'\left[\frac{\partial^2\phi_z}{\partial s\partial z}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial z}\right]}{...+U_c'\left[B_{ss}''+B_{s\chi}''\frac{\partial\chi}{\partial s}-\mathcal{T}_{ss}''\right]+U_s'\left[\frac{\partial^2\phi_s}{(\partial s)^2}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\right]+U_z'\left[\frac{\partial^2\phi_z}{(\partial s)^2}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\right]}. \quad (129)$$

Similarly, to compute $s_{het}'=\frac{\partial s(z;\theta)}{\partial\theta}$, we differentiate the first-order condition for $s(z;\theta)$ with respect to θ while holding z fixed:

$$\underbrace{\left[U''_{cc}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{cs}(.)\frac{\partial \phi_s}{\partial s}+U''_{cz}(.)\frac{\partial \phi_z}{\partial s}\right]}_{\mathcal{K}_c}\left[B'_s\frac{\partial s(z;\theta)}{\partial \theta}+B'_\chi\left(\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right)-\mathcal{T}'_s\frac{\partial s(z;\theta)}{\partial \theta}\right]$$
$$+\underbrace{\left[U''_{cs}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{ss}(.)\frac{\partial \phi_s}{\partial s}+U''_{sz}(.)\frac{\partial \phi_z}{\partial s}\right]}_{\mathcal{K}_s}\left[\frac{\partial \phi_s}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \phi_s}{\partial \chi}\left(\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right)+\frac{\partial \phi_s}{\partial \theta}\right]$$
$$+\underbrace{\left[U''_{cz}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{sz}(.)\frac{\partial \phi_s}{\partial s}+U''_{zz}(.)\frac{\partial \phi_z}{\partial s}\right]}_{\mathcal{K}_z}\left[\frac{\partial \phi_z}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \phi_z}{\partial \chi}\left(\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right)+\frac{\partial \phi_z}{\partial \theta}\right]$$
$$+\underbrace{\left[U''_{c\chi}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{s\chi}(.)\frac{\partial \phi_s}{\partial s}+U''_{z\chi}(.)\frac{\partial \phi_z}{\partial s}\right]}_{\mathcal{K}_\chi}\left[\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right]$$
$$+\underbrace{\left[U''_{c\theta}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{s\theta}(.)\frac{\partial \phi_s}{\partial s}+U''_{z\theta}(.)\frac{\partial \phi_z}{\partial s}\right]}_{\mathcal{K}_\theta}$$
$$+U'_c\left[B''_{ss}\frac{\partial s(z;\theta)}{\partial \theta}+B''_{s\chi}\left(\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right)-\mathcal{T}''_{ss}\frac{\partial s(z;\theta)}{\partial \theta}\right]$$
$$+U'_s\left[\frac{\partial^2 \phi_s}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial^2 \phi_s}{\partial s\partial \chi}\left(\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right)+\frac{\partial^2 \phi_s}{\partial s\partial \theta}\right]$$
$$+U'_z\left[\frac{\partial^2 \phi_z}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial^2 \phi_z}{\partial s\partial \chi}\left(\frac{\partial \chi}{\partial s}\frac{\partial s(z;\theta)}{\partial \theta}+\frac{\partial \chi}{\partial \theta}\right)+\frac{\partial^2 \phi_z}{\partial s\partial \theta}\right]=0.$$

Rearranging terms yields

$$\frac{\partial s(z;\theta)}{\partial \theta}=-\frac{\mathcal{K}_c\left[B'_\chi\frac{\partial \chi}{\partial \theta}\right]+\mathcal{K}_s\left[\frac{\partial \phi_s}{\partial \chi}\frac{\partial \chi}{\partial \theta}+\frac{\partial \phi_s}{\partial \theta}\right]+\mathcal{K}_z\left[\frac{\partial \phi_z}{\partial \chi}\frac{\partial \chi}{\partial \theta}+\frac{\partial \phi_z}{\partial \theta}\right]+\mathcal{K}_\chi\left[\frac{\partial \chi}{\partial \theta}\right]+\mathcal{K}_\theta+...}{\mathcal{K}_c\left[B'_s+B'_\chi\frac{\partial \chi}{\partial s}-\mathcal{T}'_s\right]+\mathcal{K}_s\left[\frac{\partial \phi_s}{\partial s}+\frac{\partial \phi_s}{\partial \chi}\frac{\partial \chi}{\partial s}\right]+\mathcal{K}_z\left[\frac{\partial \phi_z}{\partial s}+\frac{\partial \phi_z}{\partial \chi}\frac{\partial \chi}{\partial s}\right]+\mathcal{K}_\chi\left[\frac{\partial \chi}{\partial s}\right]+...}$$
$$\frac{...+U'_c\left[B''_{s\chi}\frac{\partial \chi}{\partial \theta}\right]+U'_s\left[\frac{\partial^2 \phi_s}{\partial s\partial \chi}\frac{\partial \chi}{\partial \theta}+\frac{\partial^2 \phi_s}{\partial s\partial \theta}\right]+U'_z\left[\frac{\partial^2 \phi_z}{\partial s\partial \chi}\frac{\partial \chi}{\partial \theta}+\frac{\partial^2 \phi_z}{\partial s\partial \theta}\right]}{...+U'_c\left[B''_{ss}+B''_{s\chi}\frac{\partial \chi}{\partial s}-\mathcal{T}''_{ss}\right]+U'_s\left[\frac{\partial^2 \phi_s}{(\partial s)^2}+\frac{\partial^2 \phi_s}{\partial s\partial \chi}\frac{\partial \chi}{\partial s}\right]+U'_z\left[\frac{\partial^2 \phi_z}{(\partial s)^2}+\frac{\partial^2 \phi_z}{\partial s\partial \chi}\frac{\partial \chi}{\partial s}\right]}. \quad (130)$$

C.8 Proof of Propositions 3, B.4, and B.5 (Optimal Simple Tax Systems)

The derivation of optimal earnings tax formulas for simple tax systems parallels that of general smooth tax systems and the optimal formula for marginal earnings tax rates formula, equation (19), continues to hold. This proves Proposition B.5.

Moreover, the particular linear reforms considered in the sufficient statistics characterization of optimal marginal tax rates on s for general smooth tax systems $\mathcal{T}(s,z)$ are also available for LED tax systems. As a result, the derivation of optimal marginal tax rates on s in LED tax systems is identical to the derivation for general smooth tax systems, and the optimality formula in equation

(20) continues to hold. This, in turn, implies that the Pareto-efficiency condition in equation (21) also holds, thereby proving all sufficient statistics characterizations for LED tax systems.

In contrast, LED reforms of tax rates on s are not available under SL and SN tax systems, and we derive below sufficient statistics characterizations of optimal tax rates on s and Pareto-efficiency conditions in SL and SN tax systems.

C.8.1 SL tax system

SL tax reform. When the government uses a linear tax on s such that $\mathcal{T}(s,z) = \tau_s\, s + T_z(z)$, we consider a small reform of the linear tax rate τ_s that consists in a small increase $\Delta\tau_s$. For an individual with earnings z, this reform increases tax liability by $\Delta\tau_s\, s(z)$ and increases the marginal tax rate on s by $\Delta\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_z \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_s\, s\left(z\right)\, dH_z(z) \tag{131}$$

- *behavioral effects from changes in z:*[41]

$$-\int_z T_z'(z)\left[\frac{z\zeta_z^c(z)}{1-T_z'\left(z\right)}\,\Delta\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1-T_z'\left(z\right)}\,\Delta\tau_s\, s\left(z\right)\right] dH_z(z) \tag{133}$$

- *behavioral effects from changes in s:*

$$-\int_z \tau_s\left[\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\Delta\tau_s + \frac{\eta_{s|z}(z)}{1+\tau_s}\Delta\tau_s\, s\left(z\right)\right] dH_z(z)$$
$$-\int_z \tau_s s_{inc}'(z)\left[\frac{z\zeta_z^c(z)}{1-T_z'}\,\Delta\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1-T_z'}\,\Delta\tau_s\, s\left(z\right)\right] dH_z(z) \tag{134}$$

Summing over these different effects yields the total impact of the reform

$$\frac{d\mathcal{L}}{\lambda} = \int_z \left\{\left(1-\hat{g}(z)\right)s\left(z\right) - \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)} z\zeta_z^c(z)\, s_{inc}'(z) - \frac{\tau_s}{1+\tau_s} s(z)\zeta_{s|z}^c(z)\right\}\Delta\tau_s\, dH_z(z), \tag{135}$$

with social marginal welfare weights augmented with the fiscal impact of income effects given by

$$\hat{g}(z) = \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \theta(z)\right) + \frac{T_z'(z)}{1-T_z'\left(z\right)}\eta_z(z) + \tau_s\left[\frac{\eta_{s|z}(z)}{1+\tau_s} + s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'}\right].$$

[41] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z\zeta_z^c(z)}{1-T_z'(z)}\,\Delta\tau_s\, s_{inc}'(z) - \frac{\eta_z(z)}{1-T_z'(z)}\,\Delta\tau_s\, s\left(z\right) \\ ds = -\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\,\Delta\tau_s - \frac{\eta_{s|z}(z)}{1+\tau_s}\Delta\tau_s\, s\left(z\right) + s_{inc}'(z)dz \end{cases} \tag{132}$$

Optimal linear tax rate on s. A direct implication of this result is a sufficient statistics characterization of the optimal linear tax rate τ_s. Indeed, at the optimum, the reform should have a zero impact on the government objective, meaning that the optimal τ_s satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta_{s|z}^c(z)dH_z(z)=\int_z\left\{(1-\hat{g}(z))\ s(z)-\frac{T_z'(z)+\tau_s\,s_{inc}'(z)}{1-T_z'(z)}z\zeta_z^c(z)\,s_{inc}'(z)\right\}dH_z(z). \tag{136}$$

This is equation (69) in Proposition B.4, and it holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z(z)$.

Now, assume that the earnings tax schedule is optimal. Equation (118) applied to SL tax systems then implies that at each earnings z,

$$\frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)}=\frac{1}{\zeta_z^c(z)}\frac{1}{zh_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) \tag{137}$$

such that plugging in this expression to replace the last term, we obtain

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta_{s|z}^c(z)dH_z(z)=\int_z\left\{s(z)(1-\hat{g}(z))\right\}dH_z(z)-\int_z\left\{s_{inc}'(z)\int_{x\geq z}(1-\hat{g}(x))\,h_z(x)dx\right\}dz. \tag{138}$$

Defining $s_{inc}(z)\equiv\int_{x=0}^z s_{inc}'(x)\,dx$, we can integrate by parts the last term to re-express it as[42]

$$\int_z\left\{s_{inc}'(z)\int_{x\geq z}(1-\hat{g}(x))\,h_z(x)dx\right\}dz=\int_z\left\{s_{inc}(z)(1-\hat{g}(z))\,h_z(z)\right\}dz \tag{139}$$

to obtain

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta_{s|z}^c(z)dH_z(z)=\int_z\left\{[s(z)-s_{inc}(z)]\,(1-\hat{g}(z))\right\}dH_z(z). \tag{140}$$

Note that here $\int_z\frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z)dH_z(z)$ is the aggregate population response to a change in τ_s. Defining $\overline{\zeta_{s|z}^c}$ as the aggregate elasticity of $\bar{s}:=\int_z s(z)dH_z(z)$, we can rewrite this term as $\frac{\bar{s}}{1+\tau_s}\overline{\zeta_{s|z}^c}$ such that

$$\frac{\tau_s}{1+\tau_s}=\frac{1}{\bar{s}\overline{\zeta_{s|z}^c}}\int_z s_{het}(z)\,(1-\hat{g}(z))\,dH_z(z). \tag{141}$$

This is equation (23) in Proposition 3. Integrating by part the right-hand side, this formula is also equivalent to

$$\frac{\tau_s}{1+\tau_s}=\frac{1}{\bar{s}\overline{\zeta_{s|z}^c}}\int_z\left[s_{het}'(z)\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x)\right]dz. \tag{142}$$

[42]Define $\phi(z)=\int_{x=0}^z s_{inc}'(x)\,dx$ such that $\phi'(z)=s_{inc}'(z)$ and $\psi(z)=\int_{x=z}^{z_{max}}(1-\hat{g}(x))\,h_z(x)dx$ such that $\psi'(z)=-(1-\hat{g}(z))\,h_z(z)$, and apply

$$\int_{x=z}^{z_{max}}\left[\phi'(x)\,\psi(x)\right]dx=\left[\phi(z_{max})\,\psi(z_{max})-\phi(z)\,\psi(z)\right]-\int_{x=z}^{z_{max}}\left[\phi(x)\,\psi'(x)\right]dx.$$

Pareto efficiency for SL tax systems. To characterize Pareto efficiency, we combine tax reforms in a way that neutralizes all lump-sum changes in tax liability, thereby offsetting all utility changes.

We start with a small reform of the linear tax rate τ_s that consists in small increase $\Delta\tau_s$. At the bottom of the earnings distribution ($z = z_{min}$), the mechanical effect of the reform is an increase in tax liability by $s\left(z_{min}\right)\Delta\tau_s$. We thus adjust the earnings tax liability through a downward lump-sum shift by $s\left(z_{min}\right)\Delta\tau_s$ at all earnings levels. This joint reform has the following impact on the government objective

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}}^{z_{max}} \left\{ [1-\hat{g}(z)]\ \left[s\left(z\right) - s\left(z_{min}\right)\right] - \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \tag{143}$$

meaning that the lump-sum change in tax liability is nil at earnings $z = z_{min}$, but not at earnings $z \geq z_{min}$.

To cancel out lump-sum changes in tax liability at all earnings levels, we construct a sequence of earnings tax reforms. We discretize the range of earnings $[z_{min}, z_{max}]$ into N bins and consider reforms in the small earnings bandwidths $\Delta z = \frac{z_{max}-z_{min}}{N}$. We proceed by induction to derive a general formula:

- First, consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min}\right)\Delta\tau_s$ over the bandwidth $[z_{min}, z_{min} + \Delta z]$. In this bandwidth, this additional reform (i) cancels out lump-sum changes in tax liability to a first-order approximation since $\left[s\left(z_{min} + \Delta z\right) - s\left(z_{min}\right)\right]\Delta\tau_s \approx s'\left(z_{min}\right)\Delta z \Delta\tau_s$, and (ii) induces earnings responses through the change in marginal tax rates. Moreover, it also decreases the lump-sum tax liability on all individuals with earnings $z \geq z_{min} + \Delta z$ by $s'\left(z_{min}\right)\Delta z\Delta\tau_s$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} &= \int_{z=z_{min}+\Delta z}^{z_{max}} \left\{ [1-\hat{g}(z)]\ \left[s\left(z\right) - s\left(z_{min}\right) - s'\left(z_{min}\right)\Delta z\right] \right\} \Delta\tau_s\, dH_z(z) \\ &\quad - \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \\ &\qquad + \int_{z=z_{min}}^{z_{min}+\Delta z} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)} z\zeta_z^c(z) \left(s'\left(z_{min}\right)\Delta\tau_s\right)\ \Delta H_z(z).\end{aligned} \tag{144}$$

- Second, consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min} + \Delta z\right)\Delta\tau_s$ over the bandwidth $[z_{min}+\Delta z, z_{min}+2\Delta z]$. This again cancels out lump-sum changes in this bandwidth up to a first-order approximation since $\left[s\left(z_{min} + 2\Delta z\right) - s\left(z_{min}\right) - s'\left(z_{min}\right)\Delta z\right] \approx s'\left(z_{min} + \Delta z\right)\Delta z$.

The total impact of this sequence of reforms is then

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}+2\Delta z}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - s'(z_{min})\Delta z - s'(z_{min}+\Delta z)\Delta z \right] \right\} \Delta\tau_s \, dH_z(z)$$
$$- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T'_z(z) + \tau_s \, s'_{inc}(z)}{1-T'_z} z\zeta^c_z(z) \, s'_{inc}(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta^c_{s|z}(z) \right\} \Delta\tau_s \, dH_z(z)$$
$$+ \int_{z=z_{min}}^{z_{min}+\Delta z} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) \left(s'(z_{min}) \Delta\tau_s \right) \, dH_z(z)$$
$$+ \int_{z=z_{min}+\Delta z}^{z_{min}+2\Delta z} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) \left(s'(z_{min}+\Delta z) \Delta\tau_s \right) \, dH_z(z). \quad (145)$$

- Iterating over to step k, in which we consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min} + (k-1)\frac{\Delta z}{N}\right)\Delta\tau_s$ over the bandwidth $[z_{min} + (k-1)\frac{\Delta z}{N}, z_{min} + k\frac{\Delta z}{N}]$. The total impact of this sequence of reforms is then

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}+k\frac{\Delta z}{N}}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - \frac{\Delta z}{N} \left[\sum_{p=0}^{k-1} s'\left(z_{min} + p\frac{\Delta z}{N} \right) \right] \right] \right\} \Delta\tau_s \, dH_z(z)$$
$$- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T'_z(z) + \tau_s \, s'_{inc}(z)}{1-T'_z} z\zeta^c_z(z) \, s'_{inc}(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta^c_{s|z}(z) \right\} \Delta\tau_s \, dH_z(z)$$
$$+ \sum_{p=0}^{k-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) \Delta\tau_s \, dH_z(z). \quad (146)$$

- Pushing the iteration forward until $k = N$, the first integral disappears (integration over an empty set) such that the total impact of this sequence of reforms is given by

$$\frac{d\mathcal{L}}{\lambda} = - \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T'_z(z) + \tau_s \, s'_{inc}(z)}{1-T'_z} z\zeta^c_z(z) \, s'_{inc}(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta^c_{s|z}(z) \right\} \Delta\tau_s \, dH_z(z)$$
$$+ \sum_{p=0}^{N-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) \Delta\tau_s \, dH_z(z). \quad (147)$$

Let's now compute the last term at the limit $N \to \infty$. Denoting $z^p := z_{min} + p\frac{\Delta z}{N}$, we have

$$\sum_{p=0}^{N-1} \int_{z=z^p}^{z^p+\frac{\Delta z}{N}} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) s'(z^p) \Delta\tau_s \, dH_z(z)$$
$$\approx \sum_{p=0}^{N-1} \frac{T'_z(z^p) + s'_{inc}(z^p)\tau_s}{1-T'_z(z^p)} (z^p) \zeta^c_z(z^p) s'(z^p) \Delta\tau_s h_z(z^p) \frac{\Delta z}{N}$$
$$\underset{N\to\infty}{\longrightarrow} \int_{z=z_{min}}^{z_{max}} \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta^c_z(z) s'(z) \Delta\tau_s h_z(z) \, dz \quad (148)$$

where the last line follows from the (Riemann) definition of the integral in terms of Riemann sums. Hence, the total impact of this sequence of reforms is at the limit given by

$$\frac{d\mathcal{L}}{\lambda} = -\int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1 - T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} \Delta\tau_s\, h_z(z) dz$$
$$+ \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1 - T_z'(z)} z\zeta_z^c(z)\, s'(z)\, \Delta\tau_s\, h_z(z)\; dz. \quad (149)$$

By construction, the sequence of reforms we have constructed does not affect individuals' utility, and only affects tax revenue through the expression above. When the impact of this reform is non-zero, the type of sequence of reforms we consider delivers a Pareto improvement over the existing tax system. Characterizing a Pareto-efficient SL tax system as one that cannot be reformed in a Pareto-improving way yields the following Pareto-efficiency formula

$$\frac{\tau_s}{1+\tau_s} \int_z s(z)\zeta_{s|z}^c(z) h_z(z) dz = \int_z \underbrace{\left[s'(z) - s_{inc}'(z)\right]}_{s_{het}'(z)} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1 - T_z'(z)} z\zeta_z^c(z)\, h_z(z)\, dz, \quad (150)$$

which is equation (27) in Proposition 3.

C.8.2 SN tax systems

SN tax reform. When the government uses a SN tax system such that $\mathcal{T}(s,z) = T_s(s) + T_z(z)$, we consider a small reform of the tax on s at $s^0 = s(\theta^0)$ that consists in a small increase $\Delta\tau_s$ of the marginal tax rate on s in a small bandwidth Δs. Formally,

$$\Delta\mathcal{T}(s,z) = \begin{cases} 0 & if\ s \le s^0 \\ \Delta\tau_s(s - s^0) & if\ s \in [s^0, s^0 + \Delta s] \\ \Delta\tau_s \Delta s & if\ s \ge s^0 + \Delta s \end{cases}$$

Assuming there exists a strictly increasing mapping between z and s, we denote z^0 the earnings level such that $s^0 = s(z^0)$.[43] We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta s \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z \ge z^0} \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_s \Delta s\, dH_z(z)$$

[43] Our sufficient statistic characterization of optimal SN tax systems fundamentally relies on monotonicity of the function $s(z)$. Hence, it is also valid if we assume a strictly decreasing mapping $s(z)$. Moreover, it can be extended to weakly monotonic $s(z)$ (i.e., non-decreasing or non-increasing) with slight modifications.

- *behavioral effects from changes in z:*[44]

$$-\mathcal{T}_z'(s^0,z^0)\left[\frac{z^0}{1-\mathcal{T}_z'(s^0,z^0)}\zeta_z^c(z^0)\,s_{inc}'(z)\Delta\tau_s\right]\Delta s\,\frac{h_z(z^0)}{s'(z^0)}-\int_{z\geq z^0}\mathcal{T}_z'\left(s,z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s,z\right)}\Delta\tau_s\Delta s\,dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}_s'\left(s^0,z^0\right)\left[\frac{s^0}{1+\mathcal{T}_s'\left(s^0,z^0\right)}\zeta_{s|z}^c(z^0)\Delta\tau_s+s_{inc}'(z^0)\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\,s_{inc}'(z^0)\Delta\tau_s\right]\Delta s\,\frac{h_z(z^0)}{s'(z^0)}$$
$$-\int_{z\geq z^0}\mathcal{T}_s'\left(s,z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s,z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s,z\right)}\right]\Delta\tau_s\Delta s\,dH_z(z).$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta s}=s'\left(z^0\right)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)\Delta\tau_s\,dH_z(z)-\left\{\mathcal{T}_s'\left(s^0,z^0\right)\frac{s^0}{1+\mathcal{T}_s'\left(s^0,z^0\right)}\zeta_{s|z}^c(z^0)\right.$$
$$\left.+\left[\mathcal{T}_z'(s^0,z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)s_{inc}'(z^0)\right\}\Delta\tau_s\,h_z(z^0). \quad (152)$$

Optimal nonlinear tax rate on s. A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, which implies that at each $s^0=s\left(z^0\right)$ the optimal marginal tax rate on s satisfies

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+\mathcal{T}_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)$$
$$=s'\left(z^0\right)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^0)\frac{\mathcal{T}_z'(s^0,z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)h_z(z^0) \quad (153)$$

which is equation (67) in Proposition B.4, recognizing that $\mathcal{T}_z'(s,z)=T_z'(z)$ and $\mathcal{T}_s'(s,z)=T_s'(s)$. This characterization holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z(z)$.

Now, further assume that the earnings tax schedule is optimal. Equation (118) applied to SN

[44] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz=-\frac{z}{1-T_z'}\zeta_z^c(z)\Delta T_z^{\theta\prime}-\frac{\eta_z(z)}{1-T_z'}\Delta T_z^\theta \\ ds=-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\Delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\Delta T_s+s_{inc}'(z)dz \end{cases} \quad (151)$$

where T_z^θ is a $s_{inc}'(z)\Delta\tau_s$ increase in the marginal earnings tax rate when $s\in[s^0,s^0+\Delta s]$, and a $\Delta\tau_s\Delta s$ increase in tax liability when $s\geq s^0+\Delta s$. Moreover, the mass of individuals in the bandwidth is $\Delta s\,h_s(s(z^0))=\Delta s\,\frac{h_z(z^0)}{s'(z^0)}$.

tax systems then implies that at each earnings z^0,

$$\frac{T_z'(z^0) + s_{inc}'(z^0)T_s'(s(z^0))}{1 - T_z'(z^0)} = \frac{1}{\zeta_z^c(z^0)} \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z))\, dH_z(z). \tag{154}$$

Using this expression to substitute the last term in the formula for optimal marginal tax rates on s yields

$$\frac{T_s'(s^0)}{1 + T_s'(s^0)} s^0 \zeta_{s|z}^c(z^0) h_z(z^0) = \underbrace{\left[s'(z^0) - s_{inc}'(z^0)\right]}_{s_{het}'(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z))\, dH_z(z)$$

which is equation (25) in Proposition 3.

Pareto efficiency for SN tax systems. We can combine formulas for optimal marginal tax rates on s and z to obtain a characterization of Pareto efficiency. Indeed, leveraging the previous optimal formula for marginal tax rates on s written in terms of $s_{het}'(z^0)$, and replacing the integral term by its value given from the optimal formula for marginal earnings tax rates yields

$$\frac{\mathcal{T}_s'(s^0, z^0)}{1 + T_s'(s^0, z^0)} s^0 \zeta_{s|z}^c(z^0) = s_{het}'(z^0) \frac{\mathcal{T}_z'(z^0) + s_{inc}'(z^0)\mathcal{T}_s'(s^0, z^0)}{1 - \mathcal{T}_z'(s^0, z^0)} z^0 \zeta_z^c(z^0)$$

which is the Pareto-efficiency condition (28) presented in Proposition 3, recognizing that $\mathcal{T}_z'(s, z) = T_z'(z)$ and $\mathcal{T}_s'(s, z) = T_s'(s)$.

C.9 Proof of Proposition 4 (Multidimensional Heterogeneity)

We characterize in Proposition 4 optimal tax rates on s for each type of simple tax system in the presence of multidimensional heterogeneity. These formulas take the actual earnings tax schedule as given, be they optimally set or not, and extend the results derived in the unidimensional case. Crucially, we are able to provide similar characterizations because Lemma 1 still holds in the presence of multidimensional heterogeneity.

C.9.1 Separable linear (SL) tax system

Consider a reform that consists in a $\Delta\tau_s$ increase in the linear tax rate τ_s. For all individuals, this triggers an increase in tax liability by $s\,\Delta\tau_s$ and an increase in the marginal tax rate on s by $\Delta\tau_s$, which by Lemma 1 produces earnings responses equivalent to an increase in the marginal earnings tax rate by $s_{inc}'\Delta\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\begin{aligned} &\int_z \int_s \Big[(1 - g(s, z))\, s\, \Delta\tau_s\Big] h(s, z)\, ds dz \\ &= \int_z \mathbb{E}\left[(1 - g(s, z))\, s | z\right] \Delta\tau_s h_z(z)\, dz \end{aligned} \tag{155}$$

- *behavioral effects from changes in* z[45]

$$\int_z T_z'(z)\left\{\int_s\left(-\frac{z}{1-T_z'(z)}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\Delta\tau_s-\frac{\eta_z(s,z)}{1-T_z'(z)}s\,\Delta\tau_s\right)h(s,z)\,ds\right\}dz$$
$$=-\int_z\frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)+\eta_z(s,z)s|z\right]\right\}\Delta\tau_s h_z(z)\,dz \tag{157}$$

- *behavioral effects from changes in* s

$$\begin{aligned}&\tau_s\int_z\int_s\Big\{-\frac{s}{1+\tau_s}\zeta_{s|z}^c(s,z)\,\Delta\tau_s-\frac{\eta_{s|z}(s,z)}{1+\tau_s}\,s\,\Delta\tau_s\\&\qquad+s_{inc}'(s,z)\left(-\frac{z}{1-T_z'(z)}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\Delta\tau_s-\frac{\eta_z(s,z)}{1-T_z'(z)}s\,\Delta\tau_s\right)\Big\}h(s,z)\,dsdz\\&=-\tau_s\int_z\Big\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)+\eta_{s|z}(s,z)\,s|z\right]\\&\qquad+\frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2+\eta_z(s,z)\,s\,s_{inc}'(s,z)\,|z\right]\right)\Big\}\Delta\tau_s h_z(z)\,dz\end{aligned} \tag{158}$$

such that the total impact of the reform on the government objective is

$$\begin{aligned}\frac{d\mathcal{L}}{\Delta\tau_s}&=\int_z\mathbb{E}\left[(1-g(s,z))\,s|z\right]\,h_z(z)\,dz\\&-\int_z\frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)+\eta_z(s,z)\,s|z\right]\right\}h_z(z)\,dz\\&-\tau_s\int_z\Big\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)+\eta_{s|z}(s,z)\,s|z\right]\\&+\frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2+\eta_z(s,z)\,s\,s_{inc}'(s,z)\,|z\right]\right)\Big\}h_z(z)\,dz.\end{aligned} \tag{159}$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z)=g(s,z)+\frac{T_z'(z)}{1-T_z'(z)}\eta_z(s,z)+\frac{\tau_s}{1+\tau_s}\eta_{s|z}(s,z)+\frac{\tau_s}{1-T_z'(z)}\eta_z(s,z)s_{inc}'(s,z) \tag{160}$$

[45]Applying Lemma 1, changes in z and s are here given by

$$\begin{cases}dz=-\frac{z\zeta_z^c(s,z)}{1-T_z'(z)}\,\Delta\tau_s\,s_{inc}'(s,z)-\frac{\eta_z(s,z)}{1-T_z'(z)}\,\Delta\tau_s\,s\\ds=-\frac{s(z)\zeta_{s|z}^c(s,z)}{1+\tau_s}\,\Delta\tau_s-\frac{\eta_{s|z}(,sz)}{1+\tau_s}\Delta\tau_s\,s+s_{inc}'(s,z)dz\end{cases} \tag{156}$$

we finally get

$$\frac{d\mathcal{L}}{\Delta\tau_s} = \int_z \mathbb{E}\left[(1-\hat{g}(s,z))\, s|z\right] \, h_z(z)\, dz - \int_z \frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)\,|z\right]\right\}h_z(z)\, dz \tag{161}$$

$$-\tau_s \int_z \left\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right] + \frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2|z\right]\right)\right\}h_z(z)\, dz.$$

Characterizing the optimal linear tax rate τ_s through $d\mathcal{L}=0$, it satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z \left\{\mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right]\right\}dH_z(z) \tag{162}$$

$$= \int_z \left\{\mathbb{E}\left[(1-\hat{g}(s,z))\, s|z\right] - \mathbb{E}\left[\frac{T_z'(z)+s_{inc}'(s,z)\,\tau_s}{1-T_z'(z)} z\zeta_z^c(s,z)s_{inc}'(s,z)\,|z\right]\right\}dH_z(z)$$

which is equation (30) in Proposition 4.

C.9.2 Separable nonlinear (SN) tax system

Consider a reform that consists in a small $\delta\tau_s$ increase in the marginal tax rate on s across the bandwidth $\left[s^0, s^0+\Delta s\right]$. For all individuals with savings above s^0, this triggers a $\Delta s\,\Delta\tau_s$ increase in tax liability. For individuals at s^0, this triggers a $\Delta\tau_s$ increase in the marginal tax rate on s – which by Lemma 1 produces earnings responses equivalent to a $s_{inc}'\Delta\tau_s$ increase in the marginal earnings tax rate.

We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta s \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\int_{s\geq s^0}\int_z \left\{(1-g(s,z))\,\Delta s\Delta\tau_s\right\}h(s,z)\, dsdz \tag{163}$$

$$= \int_z \left\{\mathbb{E}\left[1-g(s,z)\,|z, s\geq s^0\right]\right\}\Delta s\Delta\tau_s h_z(z)\, dz$$

- *behavioral effects from changes in z*[46]

$$
\begin{aligned}
&-\int_z T_z'(z)\left\{\frac{z}{1-T_z'}\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)\Delta\tau_s\right\}\Delta s\, h\left(s^0,z\right)dz \qquad (165)\\
&-\int_{s\geq s^0}\int_z T_z'(z)\left\{\frac{\eta_z(s,z)}{1-T_z'(z)}\Delta\tau_s\Delta s\right\}h(s,z)\,dsdz\\
&=-\int_z\frac{T_z'(z)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)s|z,s\geq s^0\right]\right\}\Delta\tau_s\Delta s\, h_z(z)\,dz
\end{aligned}
$$

- *behavioral effects from changes in s*

$$
-T_s'\left(s^0\right)\int_z\left\{\frac{s^0}{1+T_s'\left(s^0\right)}\zeta_{s|z}^c(s^0,z)\,\Delta\tau_s+s_{inc}'\left(s^0,z\right)\frac{z}{1-T_z'(z)}\zeta_z^c(s^0,z)\,s_{inc}'\left(s^0,z\right)\Delta\tau_s\right\}\Delta s\, h\left(s^0,z\right)dz \tag{166}
$$

$$
\begin{aligned}
&-\int_{s\geq s^0}\int_z\left\{T_s'(s)\left(\frac{\eta_{s|z}(s,z)}{1+T_s'(s)}\Delta s\Delta\tau_s+s_{inc}'(s,z)\frac{\eta_z(s,z)}{1-T_z'(z)}\Delta s\Delta\tau_s\right)\right\}h(s,z)\,dsdz\\
&=-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'(z)}s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\}\Delta\tau_s\Delta s\, h_z(z)\,dz \qquad (167)\\
&-\int_z\left\{\mathbb{E}\left[\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}\Delta s\Delta\tau_s h_z(z)\,dz
\end{aligned}
$$

such that the total impact of the reform on the government objective is

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s\Delta\tau_s}=&\int_z\left\{\mathbb{E}\left[1-g(s,z)\,|z,s\geq s^0\right]\right\}dH_z(z) \qquad (168)\\
&-\int_z\frac{T_z'(z)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)|z,s\geq s^0\right]\right\}dH_z(z)\\
&-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'(z)}s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\}dH_z(z)\\
&-\int_z\left\{\mathbb{E}\left[\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}dH_z(z)\,.
\end{aligned}
$$

[46]Applying Lemma 1, changes in z and s are here given by

$$
\begin{cases}dz=-\frac{z}{1-T_z'}\zeta_z^c(z)\delta T_z^{\theta\prime}-\frac{\eta_z(z)}{1-T_z'}\delta T_z^{\theta}\\ ds=-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\delta T_s+s_{inc}'(z)dz\end{cases} \tag{164}
$$

where the reform δT_z^{θ} is a $s_{inc}'(s,z)\delta\tau_s$ increase in the marginal earnings tax rate when $s\in[s^0,s^0+ds]$, and a $\delta\tau_s\delta s$ increase in tax liability when $s\geq s^0+\delta s$.

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z) = g(s,z) + \frac{T_z'(z)}{1-T_z'(z)}\eta_z(s,z) + \frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z) + s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z) \quad (169)$$

we finally get

$$\frac{d\mathcal{L}}{\Delta s \Delta \tau_s} = \int_z \left\{ \mathbb{E}\left[1-\hat{g}(s,z)\,|z, s\geq s^0\right]\right\} dH_z(z) - \int_z \left\{ \frac{T_z'(z)}{1-T_z'} z\zeta_z^c\left(s^0,z\right) s_{inc}'\left(s^0,z\right)\right\} dH_z(z)$$
$$- \int_z \left\{ \frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} s^0 \zeta_{s|z}^c(s^0,z) + \frac{T_s'\left(s^0\right)}{1-T_z'(z)} s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\} dH_z(z)\,. \quad (170)$$

Characterizing the optimal marginal tax rate on s, through $\frac{d\mathcal{L}}{\Delta s\Delta\tau_s} = 0$, it satisfies at each savings s^0,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} \int_z \left\{ s^0\zeta_{s|z}^c(s^0,z)\right\} dH_z(z) \quad (171)$$
$$= \int_z \left\{ \mathbb{E}\left[1-\hat{g}(s,z)\,|z,s\geq s^0\right]\right\} dH_z(z) - \int_z \left\{ \frac{T_z'(z) + s_{inc}'\left(s^0,z\right) T_s'\left(s^0\right)}{1-T_z'(z)} z\zeta_z^c\left(s^0,z\right) s_{inc}'\left(s^0,z\right)\right\} dH_z(z)$$

which is equation (73) in Proposition B.5.

C.9.3 Linear earnings-dependent (LED) tax system

Consider a reform that consists in a $\Delta\tau_s\Delta z$ increase in $\tau_s(z)$, the linear earnings-dependent tax rate on s, phased-in across the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$.[47]

For all individuals with earnings above $z^0+\Delta z$, this triggers an increase in the linear tax rate by $\Delta\tau_s\Delta z$ meaning that the marginal tax rate on s increases by the same magnitude, which—by Lemma 1—triggers earnings responses equivalent to those induced by a $s_{inc}'\Delta\tau_s\Delta z$ increase in the marginal earnings tax rate, and so individuals' tax liability increases by $s\,\Delta\tau_s\Delta z$.

For individuals in the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, the only direct effect of the reform is to induce earnings responses which by Lemma 1 are equivalent to an increase in the marginal earnings tax rate given by $s\,\Delta\tau_s$.

We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\int_{z\geq z^0}\int_s \left\{ (1-g(s,z))\,\Delta z\Delta\tau_s s\right\} h(s,z)\,ds dz$$
$$= \int_{z\geq z^0} \left\{ E_s\left[(1-g(s,z))\,s\middle|z\right]\Delta z\Delta\tau_s\right\} h_z(z)\,dz \quad (172)$$

[47] To avoid any ambiguity, we here use d for integration and δ for attributes of the reform we consider.

- *behavioral effects from changes in* z[48]

$$
\begin{aligned}
&-\int_s \left(T_z'\left(z^0\right)+\tau_s'\left(z^0\right)s\right)\left\{\frac{z^0\zeta_z^c(s,z^0)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s}\,s\,\Delta\tau_s\right\}\Delta z\, h\left(s,z^0\right)ds \\
&-\int_{z\geq z^0}\int_s \left(T_z'\left(z\right)+\tau_s'\left(z\right)s\right)\left\{\frac{z\zeta_z^c(s,z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}s_{inc}'\Delta z\Delta\tau_s+\frac{\eta_z(s,z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\,s\,\Delta z\Delta\tau_s\right\}h\left(s,z\right)ds \\
&=-\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s\middle|z=z^0\right]\Delta z\Delta\tau_s\, h_z\left(z^0\right) \qquad (173)\\
&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\left(z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)+s\,\eta_z(s,z)\right)\middle|z\right]\right\}\Delta z\Delta\tau_s h_z\left(z\right)dz
\end{aligned}
$$

- *behavioral effects from changes in* s

$$
\begin{aligned}
&-\tau_s\left(z^0\right)\int_s s_{inc}'\left(s,z^0\right)\left\{\frac{z^0\zeta_z^c(s,z^0)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s}\,s\,\Delta\tau_s\right\}\Delta z\, h\left(s,z^0\right)ds \\
&-\int_{z\geq z^0}\int_s \tau_s\left(z\right)\left\{\frac{s\zeta_{s|z}^c(s,z)}{1+\tau_s\left(z\right)}\Delta z\Delta\tau_s+s_{inc}'\left(s,z\right)\frac{z\zeta_z^c(s,z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}s_{inc}'\left(s,z\right)\Delta z\Delta\tau_s\right\}h\left(s,z\right)dz \\
&-\int_{z\geq z^0}\int_s\left\{\tau_s\left(z\right)\left(\frac{\eta_{s|z}(s,z)}{1+\tau_s\left(z\right)}\,s\,\Delta z\Delta\tau_s+s_{inc}'\left(s,z\right)\frac{\eta_z(s,z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\,s\,\Delta z\Delta\tau_s\right)\right\}h\left(s,z\right)dsdz \\
&=-\tau_s\left(z^0\right)\mathbb{E}\left[s_{inc}'\left(s,z\right)\frac{z\zeta_z^c(s,z)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\middle|z=z^0\right]\Delta z\Delta\tau_s\, h_z\left(z^0\right) \qquad (174)\\
&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\middle|z\right]+\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)^2\right]\right\}\Delta z\Delta\tau_s\, h_z\left(z\right)dz \\
&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\eta_{s|z}(s,z)\middle|z\right]+\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}s\eta_z(s,z)s_{inc}'\left(s,z\right)\middle|z\right]\right\}\Delta z\Delta\tau_s\, h_z\left(z\right)dz
\end{aligned}
$$

such that the total impact of the reform on the government objective is

[48] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$
\begin{cases} dz=-\frac{z^0\zeta_z^c(s,z^0)}{1-T_z'}\,\Delta\tau_s\, s \\ ds=s_{inc}'(s,z^0)dz \end{cases}
\quad \text{and} \quad
\begin{cases} dz=-\frac{z\zeta_z^c(s,z)}{1-T_z'}\,\Delta\tau_s\Delta z\, s_{inc}'(s,z)-\frac{\eta_z(s,z)}{1-T_z'}\,\Delta\tau_s\Delta z\, s \\ ds=-\frac{s\zeta_{s|z}^c(s,z)}{1+T_s'}\,\Delta\tau_s\Delta z-\frac{\eta_s(s,z)}{1+T_s'}\Delta\tau_s\Delta z\, s+s_{inc}'(s,z)dz \end{cases}
$$

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s\Delta\tau_s} = & \int_{z\geq z^0}\left\{E_s\left[(1-g(s,z))\,s\middle|z\right]\right\}h_z(z)\,dz \\
& -\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^0\right]h_z\left(z^0\right) \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}\left(z\zeta_z^c(s,z)s_{inc}'(s,z)+s\,\eta_z(s,z)\right)\middle|z\right]\right\}h_z(z)\,dz \\
& -\mathbb{E}\left[\frac{s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^0\right]\delta z\delta\tau_s\,h_z\left(z^0\right) \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]+\mathbb{E}\left[\frac{s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\middle|z\right]\right\}h_z(z)\,dz \\
& -\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\eta_{s|z}(s,z)\middle|z\right]+\mathbb{E}\left[\frac{s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}s\eta_z(s,z)\middle|z\right]\right\}h_z(z)\,dz.
\end{aligned} \tag{175}
$$

Redefining augmented social marginal welfare weights as

$$
\hat{g}(s,z)=g(s,z)+\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}\eta_z(s,z)+\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(s,z) \tag{176}
$$

we finally get

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s\Delta\tau_s} = & \int_{z\geq z^0}\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right]h_z(z)\,dz \\
& -\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^0\right]h_z\left(z^0\right) \\
& -\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\Big)\middle|z\right]h_z(z)\,dz \\
& -\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]h_z(z)\,dz
\end{aligned} \tag{177}
$$

Characterizing the optimal linear earnings-dependent tax rate $\tau_s(.)$ through $d\mathcal{L}=0$, it satisfies at each earnings z^0

$$
\begin{aligned}
& \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]h_z(z)\,dz \\
& =\int_{z\geq z^0}\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right]h_z(z)\,dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\middle|z\right]h_z(z)\,d
\end{aligned} \tag{178}
$$

which is equation (74) in Proposition B.5.

C.10 Proof of Proposition 5 (Multiple Goods)

C.10.1 Setting and definitions

The problem of the government is to maximize the following Lagrangian

$$\mathcal{L} = \int_z \left\{ \alpha(z)\, U\Big(z - \mathcal{T}(\boldsymbol{s}(z), z) - \sum_{i=1}^{n} s_i(z), \boldsymbol{s}(z), z; \vartheta(z)\Big) + \lambda \mathcal{T}(\boldsymbol{s}(z), z) - E \right\} dH_z(z) \quad (179)$$

where we use the fact that $z(\theta)$ is a bijective mapping to denote $\vartheta(z)$ its inverse, and to define Pareto weights $\alpha(z) := \alpha(\vartheta(z))$ and the vector of n consumption goods as $\boldsymbol{s}(z) := \boldsymbol{s}(z; \vartheta(z))$.

In this setting, we express optimal tax formulas in terms of the following elasticity concepts that measure consumption responses of s_i and s_j to changes in $\mathcal{T}'_{s_i}$:

$$\zeta^c_{s_i|z}(z(\theta)) := -\frac{1 + \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}{s_i(z;\theta)} \frac{\partial s_i(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}\Big|_{z=z(\theta)} \quad (180)$$

$$\xi^c_{s_{j,i}|z}(z(\theta)) := \frac{\mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}{s_j(z;\theta)} \frac{\partial s_j(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}\Big|_{z=z(\theta)} \quad (181)$$

and in terms of the following statistics,

$$s'_{i,inc}(z(\theta)) := \frac{\partial s_i(z;\theta)}{\partial z}\Big|_{z=z(\theta)} \quad (182)$$

$$\hat{g}(z(\theta)) := \left[\alpha(z)\frac{U'_c(z)}{\lambda} - \left(\mathcal{T}'_z(\boldsymbol{s}(z), z) + \sum_{i=1}^{n} s'_{i,inc}(z)\mathcal{T}'_{s_i}(\boldsymbol{s}(z), z)\right)\frac{\partial z(.)}{\partial \mathcal{T}} - \sum_{i=1}^{n} \mathcal{T}'_{s_i}(\boldsymbol{s}(z), z)\frac{\partial s_i(.)}{\partial \mathcal{T}}\right]\Big|_{z=z(\theta)}. \quad (183)$$

C.10.2 Optimal marginal tax rates on earnings z

We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate $\mathcal{T}'_z$ in a small bandwidth Δz. The impact of this reform on the Lagrangian as $\Delta z \to 0$ is

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_z \Delta z} = \int_{x \geq z^0} \Big(1 - \alpha(x)\frac{U'_c(x)}{\lambda}\Big) dH_z(x) \quad (184)$$

$$+ \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z^0) + \int_{x \geq z_0} \mathcal{T}'_z(\boldsymbol{s}(x), x)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} dH_z(x)$$

$$+ \sum_{i=1}^{n} \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right) s'_{i,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z^0)$$

$$+ \int_{x \geq z_0} \sum_{i=1}^{n} \mathcal{T}'_{s_i}(\boldsymbol{s}(x), x)\left[\frac{\partial s_i(.)}{\partial \mathcal{T}}\Big|_{z=x} + s'_{i,inc}(x)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x}\right] dH_z(x). \quad (185)$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(z)$, we obtain

$$-\left[\mathcal{T}_z'\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{i=1}^{n} \mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0), z^0\right) s_{i,inc}'(z^0)\right] \frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} h_z(z_0) = \int_{x \geq z_0} \Big(1 - \hat{g}(x)\Big) dH_z(x). \tag{186}$$

C.10.3 Optimal marginal tax rates on good i

We consider a small reform at earnings level z^0 that consists in adding a linear tax rate $\Delta\tau_s \Delta z$ on s_i for all individuals with earnings z above z^0, phased-in over the earnings bandwidth $\left[z^0, z^0 + \Delta z\right]$. In the bandwidth $\left[z^0, z^0 + \Delta z\right]$, this reform induces labor supply distortions on earnings z. At earnings $z \geq z^0 + \Delta z$, this reform induces (a) substitution effects away from s_i, (b) labor supply distortions on earnings z, and, new to this setting, (c) cross-effects on the consumption of goods s_{-i}.[49]

The impact of this reform on the Lagrangian as $\Delta z \to 0$ is

$$\begin{aligned}
\frac{1}{\lambda} \frac{d\mathcal{L}}{\Delta\tau_s \Delta z} = & \int_{x=z^0}^{z_{max}} \Big(1 - \alpha(x) \frac{U_c'(x)}{\lambda}\Big) s_i(x) dH_z(x) \\
& + \mathcal{T}_z'\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_i(z^0) h_z(z^0) + \int_{x=z^0}^{z_{max}} \mathcal{T}_z'\left(\boldsymbol{s}(x), x\right) \left[\frac{\partial z(.)}{\partial \mathcal{T}_z'} s_{i,inc}'(x) + \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x)\right] dH_z(z) \\
& + \sum_{j=1}^{n} \mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0), z^0\right) \left[s_{j,inc}'(z^0) \frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_i(z^0)\right] h_z(z^0) \\
& + \int_{x=z^0}^{z_{max}} \sum_{j=1}^{n} \mathcal{T}_{s_j}'\left(\boldsymbol{s}(x), x\right) \left\{\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=x} + \frac{\partial s_j(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x) + s_{j,inc}'(x) \left[\frac{\partial z(.)}{\partial \mathcal{T}_z'} s_{i,inc}'(x) + \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x)\right]\right\} dH_z(z)
\end{aligned} \tag{187}$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}$ capture cross-effects for all $j \neq i$.

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(x)$, we obtain

$$\begin{aligned}
& -\left[\mathcal{T}_z'\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{j=1}^{n} \mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0), z^0\right) s_{j,inc}'(z^0)\right] \frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0} s_i(z^0) h_z(z^0) = \int_{x=z^0}^{z_{max}} \Big(1 - \hat{g}(x)\Big) s_i(x) dH_z(z) \\
& + \int_{x=z^0}^{z_{max}} \left\{\left[\mathcal{T}_z'\left(\boldsymbol{s}(x), x\right) + \sum_{j=1}^{n} \mathcal{T}_{s_j}'\left(\boldsymbol{s}(x), x\right) s_{j,inc}'(x)\right] \frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=x} s_{i,inc}'(x) + \sum_{j=1}^{n} \mathcal{T}_{s_j}'\left(\boldsymbol{s}(x), x\right) \frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=x}\right\} dH_z(z).
\end{aligned} \tag{188}$$

[49] Applying Lemma 1, which still holds in this setting, changes in z and s_j at earnings z^0 and above earnings z^0 are respectively

$$\begin{cases} dz = \frac{\partial z(.)}{\partial \mathcal{T}_z'} \Delta\tau_s \, s_i(z^0) \\ ds_j = s_{j,inc}'(z^0) dz \end{cases} \quad \text{and} \quad \begin{cases} dz = \frac{\partial z(.)}{\partial \mathcal{T}_z'} \Delta\tau_s \Delta z \, s_{i,inc}'(z) + \frac{\partial z(.)}{\partial \mathcal{T}} \Delta\tau_s \Delta z \, s_i(z) \\ ds_j = \frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'} \Delta\tau_s \Delta z + \frac{\partial s_j(.)}{\partial \mathcal{T}} \Delta\tau_s \Delta z \, s_i(z) + s_{j,inc}'(z) dz \end{cases}$$

C.10.4 Deriving Proposition 5

For any good i, we combine the optimality condition for marginal tax rates on earnings z with the one for marginal tax rates on good i to obtain

$$s_i(z^0)\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x)=\int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)s_i(x)dH_z(z)$$
$$+\int_{x=z^0}^{z_{max}}\left[\left[\mathcal{T}_z'\left(\boldsymbol{s}(x),x\right)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)s_{j,inc}'(x)\right]\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=x}s_{i,inc}'(x)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=x}\right]dH_z(z) \tag{189}$$

such that differentiating with respect to earnings z^0 gives after simplification

$$s_i'(z^0)\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x) \tag{190}$$
$$=-\left[\left[\mathcal{T}_z'\left(\boldsymbol{s}(z^0),z^0\right)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)s_{j,inc}'(z^0)\right]\frac{\partial z(.)}{\partial \mathcal{T}_z'}\Big|_{z=z^0}s_{i,inc}'(z^0)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=z^0}\right]h_z(z^0)$$

Making use of the optimality condition for marginal earnings tax rates, we can substitute the first term on the right-hand side to obtain

$$-\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=z^0}=\underbrace{\left[s_i'(z^0)-s_{i,inc}'(z^0)\right]}_{s_{i,het}(z^0)}\frac{1}{h_z(z^0)}\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x). \tag{191}$$

Isolating the term relative to $\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)$ on the left-hand side yields the following optimal tax formula in terms of $s_{i,het}'$

$$-\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_i(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=z^0}=\frac{1}{h_z(z^0)}s_{i,het}'(z^0)\int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x)+\sum_{j\neq i}\mathcal{T}_{s_j}'\left(z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}\Big|_{z=z^0} \tag{192}$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}_{s_i}'}$ capture cross-effects for all $j\neq i$.

We can rewrite this optimality condition in terms of the compensated elasticity $\zeta_{s_i|z}^c$ and the cross elasticity $\xi_{s_{j,i}|z}^c$ to finally obtain

$$\frac{\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)}{1+\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)}=s_{i,het}'(z^0)\frac{1}{s_i(z^0)\zeta_{s_i|z}^c(z^0)}\frac{1}{h_z(z^0)}\int_{z=z^0}^{z_{max}}\Big(1-\hat{g}(z)\Big)dH_z(z)+\sum_{j\neq i}\frac{\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)}{\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)}\frac{s_j(z^0)\xi_{s_{j,i}|z}^c(z^0)}{s_i(z^0)\zeta_{s_i|z}^c(z^0)} \tag{193}$$

which is the first condition stated in Proposition 5.

To derive the second condition stated in Proposition 5, we substitute the first term on the right-hand side using the optimality condition for marginal tax rates on earnings z to directly obtain

$$\frac{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0),z^0\right)}{1+\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0),z^0\right)}=s'_{i,het}(z^0)\frac{\mathcal{T}'_z\left(\boldsymbol{s}(z^0),z^0\right)+\sum_{j=1}^n\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0),z^0\right)s'_{j,inc}(z^0)}{1-\mathcal{T}'_z\left(\boldsymbol{s}(z^0),z^0\right)}\frac{z^0\zeta_z^c(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}$$
$$+\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0),z^0\right)}{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0),z^0\right)}\frac{s_j(z^0)\xi^c_{s_{j,i}|z}(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}. \tag{194}$$

This completes the proof of Proposition 5.

C.11 Proof of Proposition 6 (Bequest Taxation and Behavioral Biases)

C.11.1 Setting

We here provide a sufficient statistics characterization of a smooth tax system $\mathcal{T}(s,z)$ under the following additively separable representation of individuals' preferences

$$U\big(c,s,z;\theta\big)=u\left(c;\theta\right)-k\left(z;\theta\right)+\beta\left(\theta\right)v\left(s;\theta\right),$$

and for a utilitarian government that maximizes

$$\int_\theta\big[U\left(c\left(\theta\right),s\left(\theta\right),z\left(\theta\right);\theta\right)+\nu\left(\theta\right)v\left(s\left(\theta\right);\theta\right)\big]dF\left(\theta\right), \tag{195}$$

where $\nu(\theta)$ parametrizes the degree of misalignment on the valuation of s.

Using the mapping between types θ and earnings z, the Lagrangian of the problem is written as

$$\mathcal{L}=\int_z\Big[U\left(c\left(z\right),s\left(z\right),z;\vartheta\left(z\right)\right)+\nu\left(z\right)v\left(s\left(z\right);\vartheta\left(z\right)\right)+\lambda\left(\mathcal{T}\left(s,z\right)-E\right)\Big]dH_z\left(z\right). \tag{196}$$

As before, we derive optimal tax formulas by considering reforms of marginal tax rates on z and s. Thanks to the additively separable representation of preferences, there are no income effects on labor supply choices. As a result, the only substantial change is that savings changes now lead to changes in social welfare proportional to the degree of misalignment.

C.11.2 Optimal marginal tax rates on z.

A small reform at earnings z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate in a small bandwidth Δz has the following effect as $\Delta z\to 0$,

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_z\Delta z}=&\int_{z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z)\\&-\left(\mathcal{T}'_z\left(s\left(z^0\right),z^0\right)+s'_{inc}(z^0)\Big(\mathcal{T}'_s\left(s(z^0),z^0\right)+\nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\Big)\right)\frac{z^0}{1-\mathcal{T}'_z\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\,h_z(z^0).\end{aligned}$$

In this context, social marginal welfare weights augmented with income effects $\hat{g}(z)$ are equal to

$$\hat{g}(z)=\frac{u'\left(c\left(z\right)\right)}{\lambda}+\Big(\mathcal{T}_s'\left(s(z),z\right)+\nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}\Big)\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}$$

and we can use individuals' first-order condition for s, $\left(1+\mathcal{T}_s'\right)u'\left(c\right)=\beta v'\left(s\right)$, to express the misalignment wedge in terms of the social marginal welfare weights $g\left(z\right):=\frac{u'(c(z))}{\lambda}$ as

$$\nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}=\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\right).$$

The optimal schedule of marginal earnings tax rates is thus characterized by

$$\begin{aligned}\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}=&\frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z=z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z)\\&-s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}-s_{inc}'(z^0)\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\frac{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}.\end{aligned}$$

C.11.3 Optimal marginal tax rates on s.

A small reform at earnings level z^0 that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, phased-in over the earnings bandwidth $\left[z^0,z^0+\Delta z\right]$, has the following effect as $\Delta s\to 0$,

$$\begin{aligned}&\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s\Delta z}\\&=-\left[\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s^0,z^0\right)+\nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\Big)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\,s^0\,h_z(z^0)\\&+\int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\left[\mathcal{T}_s'\left(s(z),z\right)+\nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\}dH_z(z)\\&-\int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\Big(\mathcal{T}_s'\left(s(z),z\right)+\nu\left(z\right)\frac{v'\left(s\left(z\right)\right)}{\lambda}\Big)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}s_{inc}'(z)\right\}dH_z(z).\end{aligned}$$

We characterize optimal taxes through $d\mathcal{L}=0$. Replacing the misalignment wedge by its expression in terms of social marginal welfare weights $g\left(z\right)$, we obtain that the optimal schedule of marginal tax rates on s is characterized by

$$\begin{aligned}&\left[\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s^0,z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z)\left(1+\mathcal{T}_s'\left(s^0,z^0\right)\right)\Big)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\,s^0\,h_z(z^0)\\&=\int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\left[\mathcal{T}_s'\left(s(z),z\right)+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\}dH_z(z)\\&-\int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\Big(\mathcal{T}_s'\left(s(z),z\right)+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\Big)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}s_{inc}'(z)\right\}dH_z(z).\end{aligned}$$

C.11.4 Deriving Proposition 6

Combining optimality conditions for marginal tax rates on z and s yields

$$s(z^0)\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z)=\int_{z\geq z^0}(1-\hat{g}(z))\,s\,(z)\,dH_z(z) \tag{197}$$
$$-\int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\left(\mathcal{T}_s'\left(s(z),z\right)+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}s_{inc}'(z)\right\}dH_z(z)$$
$$-\int_{z\geq z^0}\left\{\left[\mathcal{T}_s'\left(s(z),z\right)+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\}dH_z(z).$$

Differentiating with respect to z^0, we obtain after simplification

$$s'(z^0)\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z) \tag{198}$$
$$=\left\{\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\Big)\right]\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'}s_{inc}'(z^0)\right\}h_z(z^0)$$
$$+\left\{\left[\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\right]\frac{s(z^0)\zeta_{s|z}^c(z^0)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}\right\}h_z(z^0).$$

Substituting the first term on the right-hand side by its expression from the optimality condition for marginal tax rates on z, and rearranging we obtain

$$\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{het}'(z^0)}\frac{1}{s(z^0)\zeta_{s|z}^c(z^0)}\frac{1}{h_z(z^0)}\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z) \tag{199}$$

which is the first optimality condition in Proposition 6.

Conversely, substituting the term on the left-hand side by its expression from the optimality condition for marginal tax rates on z, and rearranging we obtain

$$\left[\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\right]\frac{s(z^0)\zeta_{s|z}^c(z^0)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)} \tag{200}$$
$$=s_{het}'(z^0)\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\Big)\right]\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}$$

which is the second optimality condition in Proposition 6.

C.12 Proof of Proposition 7 (Multidimensional Range with Heterogeneous Prices)

C.12.1 Setting

We consider heterogeneous marginal rates of transformation or "prices" $p(z,\theta)$ between c and s, and a two-part tax structure, where a person must pay a tax $T_1(z)$ in units of c and a tax $T_2(s,z)$ in units of s. In particular, we consider simple tax systems of the SN type, where the tax on s is nonlinear but independent of earnings z such that $T_2(s,z) = T_2(s)$, and of the LED type, where the tax on s is linear but earnings-dependent such that $T_2(s,z) = \tau_s(z)\, s$.

In this setting, we can write type θ's problem as

$$\max_{c,s,z} U(c,s,z;\theta) \text{ s.t. } c + p(z,\theta)s \leq z - T_1(z) - p(z,\theta)T_2(s,z) \tag{201}$$

$$\iff \max_z \left\{ \max_s U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z; \theta\Big)\right\} \tag{202}$$

where the inner problem leads to consumption choices $c\,(z;\theta)$ and $s\,(z;\theta)$, and the outer problem leads to an earnings choice $z\,(\theta)$. Assuming $z\,(\theta)$ continues to be a bijective mapping, we again denote $\vartheta\,(z)$ its inverse. This allows us to define $s\,(z) := s\,(z;\vartheta\,(z))$, $p(z) := p(z(\vartheta(z));\vartheta(z))$ and to formulate the problem in terms of observable earnings z.[50]

Let λ_1 and λ_2 be the marginal values of public funds associated with the resource constraints

$$\int_z T_1(z)dH_z(z) \geq E_1 \tag{203}$$

$$\int_z T_2(s(z),z)dH_z(z) \geq E_2. \tag{204}$$

The problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z \Bigg\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z),z)\big), s(z), z; \vartheta(z)\Big) + \lambda_1 T_1(z) + \lambda_2 T_2(s(z),z) - E_1 - E_2 \Bigg\} dH_z(z). \tag{205}$$

C.12.2 Adapting Lemma 1

Lemma C.4. *For an type $\theta = \vartheta(z)$, we have that:*
(1a) *a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.*
(1b) *a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s'_{inc}(z)\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.*
(2) *a small increase ΔT in the T_2 tax liability faced by type $\theta = \vartheta(z)$ generates the same earnings change as a small increase $p(z)\Delta T$ in the T_1 tax liability.*

[50] When taking derivatives, the presence of these two arguments is implicit. For instance, a total derivative corresponds to $\frac{dp}{dz} := \frac{\partial p}{\partial z} + \frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial z}$, whereas a partial derivative $\frac{\partial p}{\partial z}$ represents variation in only the first argument.

Proof. We first derive an abstract characterization that we then apply to different tax reforms.

Let type θ indirect utility function at earnings z be

$$V\left(T_1(z), T_2(., z), z; \theta\right) := \max_s U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z; \theta\Big). \tag{206}$$

Consider a small reform $\Delta T_2(s,z)$ of T_2, and construct for each type θ a perturbation $\Delta T_1^\theta(z)$ of T_1 that induces the same earnings response as the initial perturbation. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(T_1(z) + \Delta T_1^\theta(z), T_2(., z), z; \theta) = V(T_1(z), T_2(., z) + \Delta T_2(., z), z; \theta). \tag{207}$$

Then, by construction, the perturbation $\Delta T_1^\theta(z)$ induces the same earnings response dz as the initial perturbation $\Delta T_2(., z)$. Moreover, both tax reforms must induce the same utility change for type θ. Applying the envelope theorem yields

$$-U_c'(z;\theta) \cdot \Delta T_1^\theta(z) = -U_c'(z;\theta)\, p(z,\theta) \cdot \Delta T_2\left(s\left(z;\theta\right), z\right) \tag{208}$$

such that finally, the perturbation $\Delta T_1^\theta(z)$ is

$$\Delta T_1^\theta(z) = p(z,\theta) \cdot \Delta T_2\left(s\left(z;\theta\right), z\right). \tag{209}$$

and we can now apply this abstract characterization to different tax reforms.

(1a) Consider a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ over a small bandwidth of income $\left[z^0, z^0 + \Delta z\right]$. Then, for any type θ such that $z(\theta) \in \left[z^0, z^0 + \Delta z\right]$, we have $\Delta T_2\left(s\left(z;\theta\right), z\right) = \Delta\tau_z\left(z - z^0\right)$ such that $\Delta T_1^\theta(z) = p(z,\theta)\Delta\tau_z\left(z - z^0\right)$ and differentiating with respect to z we get

$$\Big(\Delta T_1^\theta(z)\Big)' = \frac{\partial p(z,\theta)}{\partial z}\Delta\tau_z\left(z - z^0\right) + p(z,\theta)\Delta\tau_z. \tag{210}$$

At the limit $\Delta z \to 0$ such that $z \to z^0$, a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)\Delta\tau_z$ in the marginal tax rate $T_1'(z)$.

(1b) Consider a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ over a small bandwidth of savings $\left[s^0, s^0 + \Delta s\right]$. Then, for any type θ such that $s(\theta) \in \left[s^0, s^0 + \Delta s\right]$, we have $\Delta T_2\left(s\left(z;\theta\right), z\right) = \Delta\tau_s\left(s\left(z;\theta\right) - s^0\right)$ such that $\Delta T_1^\theta(z) = p(z,\theta)\Delta\tau_z\left(s\left(z;\theta\right) - s^0\right)$ and differentiating with respect to z we get

$$\Big(\Delta T_1^\theta(z)\Big)' = \frac{\partial p(z,\theta)}{\partial z}\Delta\tau_z\left(s\left(z;\theta\right) - s^0\right) + p(z,\theta)\Delta\tau_z s_{inc}'(z). \tag{211}$$

At the limit $\Delta s \to 0$ such that $s \to s^0$, a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s_{inc}'(z)\Delta\tau_z$ in the marginal tax rate $T_1'(z)$.

(2) Consider a small lump-sum increase ΔT in the T_2 tax liability for a type θ who earns z, we then have $\Delta T_1^\theta(z) = p(z,\theta)\Delta T$ such that the equivalent reform is no longer a lump-sum increase. Hence, a small increase ΔT in the T_2 tax liability faced by a type $\vartheta(z)$ generates the same earnings change as a small increase $p(z)\Delta T$ in the T_1 tax liability. □

C.12.3 Marginal values of public funds

An important prerequisite to derive optimality conditions is to pin down the marginal values of public funds λ_1 and λ_2. At the optimum, λ_1 and λ_2 are pinned down by optimally setting the tax level T_1 and T_2. Characterizing the impact of lump-sum changes in tax liabilities yields the following two equations that can be solved for λ_1 and λ_2:

$$\int_{x=z_{min}}^{z_{max}} \left\{ -\alpha(x)U_c'(x) + \lambda_1 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'(x)\lambda_2 \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} + \lambda_2 \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_1} \right\} dH_z(x) = 0 \tag{212}$$

$$\int_{x=z_{min}}^{z_{max}} \left\{ -\alpha(x)p(x)U_c'(x) + \lambda_2 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'(x)\lambda_2 \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_2} + \lambda_2 \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_2} \right\} dH_z(x) = 0 \tag{213}$$

where $z(.)$ and $s(.)$ denote, with a slight abuse of notation, the earnings and savings choices, and all partial derivatives are evaluated at earnings x.

Renormalizing these equations by λ_1, we can use the fact that by Lemma C.4, $\frac{\partial z(.)}{\partial T_2} = \frac{\partial z(.)}{\partial T_1} p(z) + \frac{\partial z(.)}{\partial T_1'} \frac{\partial p}{\partial z}$ and that $\frac{\partial s(.)}{\partial T_2} = \frac{\partial s(.)}{\partial T_1} p(z)$ to obtain

$$\int_{x=z_{min}}^{z_{max}} \left\{ 1 - \left[\alpha(x) \frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z} + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_1} \right] \right\} dH_z(z) = 0 \tag{214}$$

$$\begin{aligned} \int_{x=z_{min}}^{z_{max}} \Bigg\{ \frac{\lambda_2}{\lambda_1} - p(x) \left[\alpha(x) \frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z} + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_1} \right] \quad (215) \\ + \left(T_1'(z) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z} + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1'} \frac{\partial p}{\partial z} \Bigg\} dH_z(x) = 0. \end{aligned}$$

At any given earnings x, defining social marginal welfare weights augmented with the fiscal impact of income effects $\hat{g}(x)$ and the fiscal impacts of the novel substitution effects $\varphi(x)$ as respectively

$$\hat{g}(x) := \alpha(x) \frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z} + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_1} \tag{216}$$

$$\varphi(x) := \left(T_1'(x) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z} + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1'} \frac{\partial p}{\partial z} \tag{217}$$

where all partial derivatives are evaluated at x, we finally obtain

$$\overline{\hat{g}} := \int_{x=z_{min}}^{z_{max}} \hat{g}(x)\, dH_z(x) = 1 \tag{218}$$

$$\overline{\hat{g}p - \varphi} := \int_{x=z_{min}}^{z_{max}} \Big(\hat{g}(x)p(x) - \varphi(x) \Big)\, dH_z(x) = \frac{\lambda_2}{\lambda_1}. \tag{219}$$

C.12.4 Optimal tax rates on z

We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate $T_1'(z)$ in a small bandwidth Δz. The impact on the Lagrangian is as $\Delta z \to 0$,

$$\begin{aligned}\frac{d\mathcal{L}}{\Delta\tau_z \Delta z} &= \int_{x \geq z^0} \Big(\lambda_1 - \alpha(x) U_c'(x)\Big) dH_z(x) \\ &+ \left[\lambda_1 T_1'\left(z^0\right) + \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} h_z(z^0) \\ &+ \int_{x \geq z^0} \left[\lambda_1 T_1'\left(x\right) + \lambda_2 \frac{\partial T_2}{\partial z}\right] \frac{\partial z(.)}{\partial T_1} dH_z(x) \\ &+ \lambda_2 \frac{\partial T_2}{\partial s}\Big|_{z=z^0} s_{inc}'(z^0) \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} h_z(z^0) \\ &+ \int_{x \geq z^0} \lambda_2 \frac{\partial T_2}{\partial s} \left[\frac{\partial s(.)}{\partial T_1} + s_{inc}'(x) \frac{\partial z(.)}{\partial T_1}\right] dH_z(x).\end{aligned} \tag{220}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing everything by λ_1, plugging in social marginal welfare weights augmented with income effects $\hat{g}(x)$, we obtain the following optimality condition for marginal earnings tax rates at each earnings z^0

$$-\left[T_1'(z^0) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z}\Big|_{z^0} + s_{inc}'(z^0) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s}\Big|_{z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} = \frac{1}{h_z(z^0)} \int_{x \geq z^0} \Big[1 - \hat{g}(x)\Big] dH_z(x). \tag{221}$$

C.12.5 Optimal tax rates on s

SN tax system. We consider a small reform at $s^0 = s(z^0)$ that consists in a small increase $\Delta\tau_s$ of $\frac{\partial T_2}{\partial s}$, the marginal tax rate on s, in a small bandwidth Δs. Using Lemma 2, we characterize the impact of the reform on the Lagrangian as $\Delta s \to 0$

$$\begin{aligned}\frac{d\mathcal{L}}{\Delta\tau_s \Delta s} &= \int_{x \geq z^0} \Big(\lambda_2 - \alpha(x) p(x) U_c'(x)\Big) dH_z(x) \\ &+ \left[\lambda_1 T_1'(z^0) + \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} s_{inc}'(z^0) p(z^0) \frac{h_z(z^0)}{s'(z^0)} \\ &+ \int_{x \geq z^0} \left[\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z}\right] \left(\frac{\partial z(.)}{\partial T_1} p(x) + \frac{\partial z(.)}{\partial T_1'\left(x\right)} \frac{\partial p}{\partial z}\right) dH_z(x) \\ &+ \lambda_2 \frac{\partial T_2}{\partial s}\Big|_{z=z^0} \left[\frac{\partial s(.)}{\partial \left(\frac{\partial T_2}{\partial s}\Big|_{z=z^0}\right)} + s_{inc}'(z^0) \frac{\partial z(.)}{\partial T_1'(z^0)} s_{inc}'(z^0) p(z^0)\right] \frac{h_z(z^0)}{s'(z^0)} \\ &+ \int_{x \geq z^0} \lambda_2 \frac{\partial T_2}{\partial s} \left[\frac{\partial s(.)}{\partial T_2} + s_{inc}'(x) \left(\frac{\partial z(.)}{\partial T_1} p(x) + \frac{\partial z(.)}{\partial T_1'\left(x\right)} \frac{\partial p}{\partial z}\right)\right] dH_z(x).\end{aligned} \tag{222}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing by λ_1 and using $\frac{\partial s(.)}{\partial T_2} = \frac{\partial s(.)}{\partial T_1} p(x)$, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for marginal tax rates on s at each savings $s^0 = s(z^0)$:

$$- \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s}\Big|_{z^0} \frac{\partial s(.)}{\partial \left(\frac{\partial T_2}{\partial s}\Big|_{z^0} \right)} h_z(z^0) = s'(z^0) \int_{x \geq z^0} \left\{ \frac{\lambda_2}{\lambda_1} - \hat{g}(x) p(x) + \varphi(x) \right\} dH_z(x) \tag{223}$$
$$+ \left[T_1'(z^0) + \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial z}\Big|_{z^0} + s'_{inc}(z^0) \frac{\lambda_2}{\lambda_1} \frac{\partial T_2}{\partial s}\Big|_{z^0} \right] \frac{\partial z(.)}{\partial T_1'(z^0)} s'_{inc}(z^0) p(z^0) h_z(z^0)$$

LED tax system. We consider a small reform at $s^0 = s(z^0)$ that consists in a small increase $\Delta \tau_s$ of the linear savings tax rate $\tau_s(z)$ phased in over the earnings bandwidth $[z^0, z^0 + \Delta z]$. Using Lemma (2), we characterize the impact of the reform on the Lagrangian as $\Delta z \to 0$

$$\frac{d\mathcal{L}}{\Delta \tau_s \Delta z} = \int_{x \geq z^0} \Big(\lambda_2 - \alpha(x) p(x) U_c'(x) \Big) s(x) \, dH_z(x) \tag{224}$$
$$+ \left(\lambda_1 T_1'(z^0) + \lambda_2 \tau_s'(z^0) s(z^0) \right) \frac{\partial z(.)}{\partial T_1'(z^0)} p(z^0) s(z^0) \, h_z(z^0)$$
$$+ \int_{x \geq z^0} \left(\lambda_1 T_1'(x) + \lambda_2 \tau_s'(z^0) s(z^0) \right) \left[\frac{\partial z(.)}{\partial T_1} p(x) s(x) + \frac{\partial z(.)}{\partial T_1'(x)} \left(\frac{\partial p}{\partial z} s(x) + p(x) s'_{inc}(x) \right) \right] dH_z(x)$$
$$+ \lambda_2 \tau_s(z^0) s'_{inc}(z^0) \left[\frac{\partial z(.)}{\partial T_1'(z^0)} p(z^0) s(z^0) \right] h_z(z^0)$$
$$+ \int_{x \geq z^0} \lambda_2 \tau_s(x) \left[\frac{\partial s(.)}{\partial \left(\frac{\partial T_2}{\partial s}\Big|_x \right)} + \frac{\partial s(.)}{\partial T_1} p(x) s(x) + s'_{inc}(x) \left[\frac{\partial z(.)}{\partial T_1} p(x) s(x) + \frac{\partial z(.)}{\partial T_1'(x)} \left(\frac{\partial p}{\partial z} s(x) + p(x) s'_{inc}(x) \right) \right] \right] dH_z$$

since the reform triggers for individuals at z^0 changes in earnings z equivalent to those induced by a $p(z) \, \Delta \tau_s \, s(z)$ increase in $T_1'(z^0)$, and for individuals above z^0 an increase in tax liability equivalent to a $p(z) \, \Delta \tau_s \, \Delta z \, s(z)$ increase in T_1 and a change in marginal earnings tax rates equivalent to a $\left(\frac{\partial p}{\partial z} \, s(z) + p(z) \, s'_{inc}(z) \right) \Delta \tau_s \, \Delta z$ increase in $T_1'(z)$, in addition to the $\Delta \tau_s \Delta z$ increase in the linear tax rate on s.

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing by λ_1, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for linear earnings-dependent tax rates on s at each earnings z^0

$$- \left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1} \tau_s'(z^0) s(z^0) + \frac{\lambda_2}{\lambda_1} s'_{inc}(z^0) \tau_s(z^0) \right) \frac{\partial z(.)}{\partial T_1'(z^0)} p(z^0) s(z^0) \, h_z(z^0)$$
$$= \int_{x \geq z^0} \left\{ \Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(x) p(x) + \varphi(x) \Big) s(x) + \frac{\lambda_2}{\lambda_1} \tau_s(x) \frac{\partial s(.)}{\partial \left(\frac{\partial T_2}{\partial s}\Big|_x \right)} \right\} dH_z(x) \tag{225}$$
$$+ \int_{x \geq z^0} \left(T_1'(x) + \frac{\lambda_2}{\lambda_1} \tau_s'(x) s(x) + \frac{\lambda_2}{\lambda_1} s'_{inc}(x) \tau_s(x) \right) \frac{\partial z(.)}{\partial T_1'(x)} p(x) s'_{inc}(x) \, dH_z(x)$$

C.12.6 Deriving Proposition 7

SN tax system. A two-part SN tax system $\{T_1(z), T_2(s)\}$ thus satisfies two optimality conditions: the optimality condition in equation (221) for $T_1'(z)$ and the optimality condition in equation (223) for $T_2'(s)$. Combining these two conditions, we get that at each earnings z^0, the optimal SN tax system satisfies

$$-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} = \frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1}-\hat{g}(x)p(x)+\varphi(x)\right\}dH_z(x)-p(z^0)\frac{s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x) \tag{226}$$

Adding and subtracting $p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x=z^0}^{z_{max}}\Big[1-\hat{g}(x)\Big]\,dH_z(x)$ yields

$$\begin{aligned}-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} &= p(z^0)\frac{s'(z^0)-s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x) \qquad (227)\\ &+\frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1}-\hat{g}(x)p(x)+\varphi(x)\right\}dH_z(x)-p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x).\end{aligned}$$

Defining $\zeta^c_{s|z}(z) = -\frac{1+p\frac{\partial T_2}{\partial s}\Big|_{z^0}}{s}\frac{\partial s(.)}{p\,\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}$ such that $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} = -\frac{p\,s}{1+p\frac{\partial T_2}{\partial s}\Big|_{z^0}}\zeta^c_{s|z}(z)$, we get[51]

$$\begin{aligned}&\frac{\overline{\hat{g}p-\varphi}\frac{\partial T_2}{\partial s}\Big|_{z^0}}{1+p(z^0)\frac{\partial T_2}{\partial s}\Big|_{z^0}} \qquad (228)\\ &=\frac{1}{s(z^0)\zeta^c_{s|z}(z^0)}\frac{1}{h_z(z^0)}\left\{s'_{het}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+\frac{s'(z^0)}{p(z^0)}\Big[\Psi(z^0)+\Phi(z^0)\Big]\right\}\end{aligned}$$

[51]With homogeneous p, a SN savings tax levied in period-1 dollars $T_s(s)$ is simply equal to $T_s(s) = pT_2(s)$. As a result, this elasticity definition ensures that $\zeta^c_{s|z}(z)$ coincides with the elasticity concept introduced before:

$$\zeta^c_{s|z}(z) = -\frac{1+T_s'(s)}{s}\frac{\partial s(.)}{\partial T_s'(s)} = -\frac{1+pT_2'(s)}{s}\frac{\partial s(.)}{p\partial T_2'(s)}.$$

where we use $\overline{\hat{g}p - \varphi} = \frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}(x)} = 1$ to obtain the additional terms

$$\begin{aligned}
\Psi(z^0) &:= \int_{z \geq z^0} \left[\overline{\hat{g}p} - \hat{g}(z)p(z)\right] dH_z(z) - p(z^0) \int_{z=z^0}^{z_{max}} \left[\overline{\hat{g}} - \hat{g}(z)\right] dH_z(z) \qquad (229)\\
&= \int_{z \geq z^0} \left[\int_{x=z_{min}}^{z_{max}} \hat{g}(x)p(x)dH_z(x) - \hat{g}(z)p(z)\right] dH_z(z) - p(z^0) \int_{z \geq z^0} \left[\int_{x=z_{min}}^{z_{max}} \hat{g}(x)dH_z(x) - \hat{g}(z)\right] dH_z(z)\\
&= \left(1 - H_z(z^0)\right) \int_{x=z_{min}}^{z_{max}} \hat{g}(x)p(x)dH_z(x) - \int_{z \geq z^0} \hat{g}(z)p(z)dH_z(z)\\
&\qquad - p(z^0)\left(1 - H_z(z^0)\right) \int_{x=z_{min}}^{z_{max}} \hat{g}(x)dH_z(x) - p(z^0) \int_{z \geq z^0} \hat{g}(z)dH_z(z)\\
&= \left(1 - H_z(z^0)\right) \int_{x=z_{min}}^{z_{max}} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(x) - \int_{x \geq z^0} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(x)\\
&= \Big(1 - H_z(z^0)\Big) \int_{x \leq z^0} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(z) + H_z(z^0) \int_{x \geq z^0} \hat{g}(x)\Big(p(z^0) - p(x)\Big)dH_z(x) \qquad (230)
\end{aligned}$$

$$\Phi(z^0) := \int_{x \geq z^0} \left[\varphi(x) - \overline{\varphi(x)}\right] dH_z(x) \tag{231}$$

which proves the optimal formula for SN tax systems in Proposition 7.

LED tax system. A two-part LED tax system $\{T_1(z), \tau_s(z)s\}$ thus satisfies two optimality conditions: the optimality condition in equation (221) for $T_1'(z)$ and the optimality condition in equation (225) for $\tau_s(z)$. Combining these two conditions, we get that at each earnings z^0 the optimal LED tax system satisfies

$$\begin{aligned}
&p(z^0)s(z^0) \int_{x \geq z^0} \left[1 - \hat{g}(x)\right] dH_z(x)\\
&= \int_{x \geq z^0} \left\{ \Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\Big)s(x) + \frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)} \right\} dH_z(x)\\
&+ \int_{x \geq z^0} \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x) + \frac{\lambda_2}{\lambda_1}s_{inc}'(x)\tau_s(x)\right) \frac{\partial z(.)}{\partial T_1'(x)} p(x)s_{inc}'(x)\, dH_z(x).
\end{aligned}$$

Differentiating with respect to z^0 yields

$$\begin{aligned}
&\Big(p'(z^0)s(z^0) + p(z^0)s'(z^0)\Big) \int_{x \geq z^0} \left[1 - \hat{g}(x)\right] dH_z(x) - p(z^0)s(z^0)\left[1 - \hat{g}(z^0)\right] h_z(z^0)\\
&= -\left\{ \Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(z^0)p(z^0) + \varphi(z^0)\Big)s(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} \right\} h_z(z^0)\\
&- \left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0) + \frac{\lambda_2}{\lambda_1}s_{inc}'(z^0)\tau_s(z^0)\right) \frac{\partial z(.)}{\partial T_1'(z^0)} p(z^0)s_{inc}'(z^0)\, h_z(z^0).
\end{aligned}$$

Using the optimality condition in equation (221) for $T_1'(z)$, the last term is equal to $p(z^0)s'_{inc}(z^0)\int_{x\geq z^0}[1-\hat{g}(x)]\,dH_z(x$ at the optimum such that

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}h_z(z^0)\\
&=p(z^0)s'_{het}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+p'(z^0)s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\\
&+\left\{\frac{\lambda_2}{\lambda_1}-\Big(\hat{g}(z^0)p(z^0)-\varphi(z^0)\Big)-p(z^0)\Big[1-\hat{g}(z^0)\Big]\right\}s(z^0)h_z(z^0).
\end{aligned}
$$

We can now plug in the elasticity $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)}=-\frac{p(z^0)s(z^0)}{1+p(z^0)\frac{\partial T_2}{\partial s}\big|_{z^0}}\zeta^c_{s|z}(z^0)$ with $\frac{\partial T_2}{\partial s}\big|_{z^0}=\tau_s(z^0)$ and use the fact that $\overline{\hat{g}p-\varphi}=\frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}}=1$ to obtain

$$
\begin{aligned}
&\overline{\hat{g}p-\varphi}\frac{\tau_s(z^0)}{1+p(z^0)\tau_s(z^0)} \qquad (232)\\
&=\frac{1}{s(z^0)\zeta^c_{s|z}(z^0)}\frac{1}{h_z(z^0)}\left\{s'_{het}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+\frac{p'(z^0)}{p(z^0)}s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\right\}\\
&+\frac{1}{p(z^0)}\frac{1}{\zeta^c_{s|z}(z^0)}\left\{\left[\overline{\hat{g}p}-p(z^0)\overline{\hat{g}}\right]-\left[\overline{\varphi}-\varphi(z^0)\right]\right\}
\end{aligned}
$$

which proves the optimal formula for LED tax systems in Proposition 7.

D Details on the Empirical Application

This appendix describes the details underlying the numerical results presented in Section 7. In Section D.1, we describe how we calibrate a baseline two-period, unidimensional model of the U.S. economy, which we use to compute the simple savings tax schedules that are consistent with the prevailing income tax, i.e., that satisfy the Pareto-efficiency formulas in Proposition 3. These are reported in Figure II. In Section D.2, we describe how we extend this exercise to calibrate the optimal simple savings tax systems in the presence of multidimensional heterogeneity as in Proposition 4, assuming that redistributive preferences and other sufficient statistics are the same as in the baseline calibration. In Section D.3, we describe how we instead extend the baseline exercise to allow for heterogeneous rates of return, with an efficiency-based rationale for taxing those with access to high returns, as in Proposition 7. Results for these extensions are reported in Figure III.Throughout this exercise, we make two assumptions for tractability: We assume that preferences are weakly separable as described in Proposition 2, so that the income effect on savings, $\eta_{s|z}(z)$ can be identified from $s'_{inc}(z)$, and we assume that income effects on labor supply ($\eta_z(z)$) are negligible.

For comparability with the literature on wealth taxation, we express all savings tax rates in terms of "period-2" taxes on gross savings, so that a marginal savings tax rate of 0.1 indicates that if an individual's total wealth at retirement increases by \$1, then they must pay an additional \$0.10 in taxes when they retire.[52]

[52]Notationally, we write this translation as in Appendix B.7, with s_1 and s_g denoting gross savings before

The LaTeX source code underlying this document—which can be viewed in the accompanying replication files—uses equation labels that match those in the Matlab simulation code.

D.1 Baseline Calibration with Unidimensional Heterogeneity

We first calibrate a simplified version of the U.S. economy with unidimensional heterogeneity. This calibration has two periods, with the first period corresponding to working life and the second to retirement. We assume these periods are separated by 20 years, with a risk-free annual rate of return of 3.8% per year between period 1 and period 2 (see Fagereng et al. (2020), Table 3).

D.1.1 Joint Distribution of Earnings and Savings, and the Status Quo Income Tax

We calibrate the joint distribution of earnings and savings using the Distributional National Accounts micro-files of Piketty et al. (2018), henceforth PSZ. We use individual measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) as well as the age category (20 to 44 years old, 45 to 64, and above 65) and household information. We discretize the income distribution into percentiles by age group, and we partition the top percentile into the top 0.01%, the top 0.1% (excluding the top 0.01%), and the rest of the top percentile. Our measure of annualized earnings during work life z at the n-th percentile is constructed by averaging *plinc* at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. For married households, we use the average earnings of the couple and assign both members of the couple to the same percentile of income. For households with one member above 65 years old, we keep only the younger spouse in the sample. We drop the bottom 2% of observations with non-positive labor income; these individuals have positive average income from other sources, suggesting they are not representative of the zero-ability types which would correspond to $z = 0$ in our model.

Our measure of gross retirement savings per year worked, which we denote s_g in the notation of Appendix B.7, at each labor income quantile is constructed by projecting forward to age 65 the average wealth we observe in the 45 to 65 age category. We project forward by assuming that individuals within each percentile save the same share of post-tax income while young and middle-aged.[53] For married households, we take household wealth to be the average wealth of its members. We then normalize the total wealth at retirement by the number of working years ($65 - 25 = 40$) so that z and s_g are in comparable units measured per working year. This yields a monotonic profile of savings across earnings z, and pins down the cross-sectional variation in gross savings $s_g'(z)$.

taxes, measured in period-1 and period-2 dollars, respectively, and $T_2(s_g, z)$ denoting the savings tax function in period 2. Appendix B.7 demonstrates that the simplicity structure of a tax system (SL, SN, and LED) is preserved when translating between $\mathcal{T}(s, z)$ and $\{T_1(z), T_2(s_g, z)\}$. In the accompanying code replication files, all savings taxes are computed in terms of $\mathcal{T}(s, z)$, but marginal tax rates are converted into $\frac{\partial T_2(s_g,z)}{\partial s_g}$ when plotted in figures.

[53] Specifically, we construct a representative working agent for each income percentile in each age category: a "young" agent of age 35 (in the 20 to 44 age category, where we assume work begins at age 25), and a "middle-aged" agent of age 55 (in the 45 to 64 age category). We assume wealth at middle age is the result of the sum of 20 years' worth of savings while young, with returns compounded for an average of $55 - 35 = 20$ years, and 10 years of saving during middle age, compounded for an average of 5 years, with a constant share of post-tax income saved in the age range.

We convert this discrete distribution of labor income and savings into a smooth distribution with 1000 gridpoints with equal log-spacing, to ensure a smooth marginal tax function that converges to a fixed point when we iterate using the first-order conditions from our propositions. This conversion is performed using the smoothing spline fit in Matlab, with a smoothing parameter of 0.9 and the scale normalization setting set to "on." Measures of savings are noisy at low incomes, which also have outlier values of $\ln(z)$ after the logarithmic transformation used for our savings fit. To avoid having those percentiles generate a strong pull on the fit, we fit the log of savings to $\ln(z+k)$, where a larger k reduces the extent to which the low incomes are outliers. Our baseline uses $k = \$20,000$.

We construct the status quo income tax function by comparing gross income to the PSZ measure *diinc* ("extended disposable income") of post-tax income $z - T_1(z)$. We use the median value within each pre-tax income percentile, constructing a smoothed profile of disposable income y by fitting log *diinc* to log *plinc*, with the same setting described above. In the DINA files, total disposable income *diinc* exceeds total labor income *plinc*, reflecting non-labor factors of production in the economy and the taxes on them. For internal consistency, we apply a lump-sum adjustment so that total y and z are equal, although our results are not sensitive to this adjustment. We then calibrate the smooth marginal income tax rate schedule as $1 - \frac{dy}{dz}$. We treat Social Security as a fixed amount of forced savings, which are added to net-of-tax disposable savings to arrive at our total measure of net savings s. [54]

D.1.2 Status Quo Savings Tax Rates in the United States

We are interested in comparing our results to the profile of status quo effective tax rates on savings in the U.S. Constructing such a schedule presents several difficulties, however. There are many different types of taxes which apply to savings in the U.S., including capital gains taxes (which differ depending on the length of asset ownership), ordinary income taxes, and property taxes. Moreover, effective tax rates depend on assumptions about incidence, about which there is substantial disagreement.

We use a simple approach to construct an approximation of the U.S. savings tax based on the composition of savings portfolios across the income distribution. Bricker et al. (2019) use the Survey of Consumer Finances to construct a decomposition of saving types by asset ownership percentile; we summarize the analogous decomposition by income percentile in Figure A2 below. We then construct a savings tax rate at each income level based on the asset-weighted average of the tax rates that apply to each asset class.

For comparison to our results, the savings tax rate of interest is the distortion between work-life consumption and savings. Therefore savings which are subject to labor income taxes but no further taxes, such as a Roth IRA, should be understood as being subject to zero savings tax. We similarly classify traditional IRAs and pension plans as being subject to zero taxes, since they are also subject only to ordinary income taxes. We therefore treat assets in the "Financial (retirement)" category as subject to zero savings tax. We assume "Financial (transaction)" assets, which include checking and savings accounts, represent liquidity needs and similarly do not count toward taxed savings. We

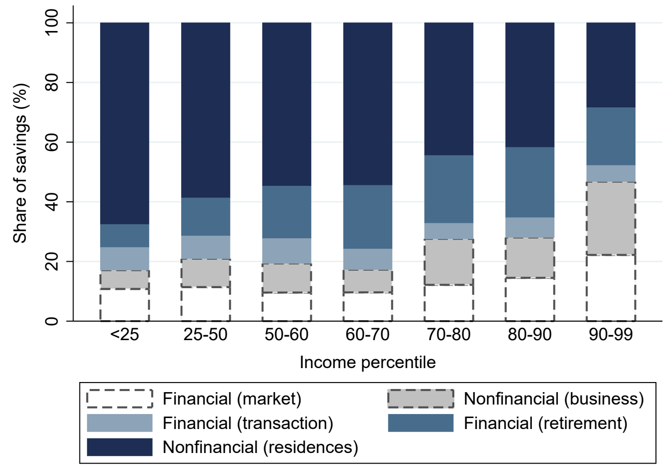
[54] The amount is computed as follows, using the SSA Fact Sheet: Retired workers receive on average \$1,514 per month from social security, which is $12 \times 1,514 = \$18,168$ annually. Through the lens of our two-period model, these benefits are received over an average retirement length of 20 years, and stem from contributions paid over 40 working years. We therefore approximate this as forced savings at the time of retirement of \$9000 per working year.

view property taxes on "Nonfinancial (residences)" savings as a tax that is incident on renters, and thus a component of imputed rent, which is paid regardless of whether the asset is owned by the user, so we also assume the tax rate on these savings is 0%. Therefore we view only the dotted-outline asset classes "Financial (market)" and "Nonfinancial (business)" as subject to savings taxes, in the form of capital gains. We do not know what share of these holdings represent gains, as opposed to the original contributions. To be conservative, we treat the entire asset classes as though they were subject to capital gains taxes at the time of retirement.

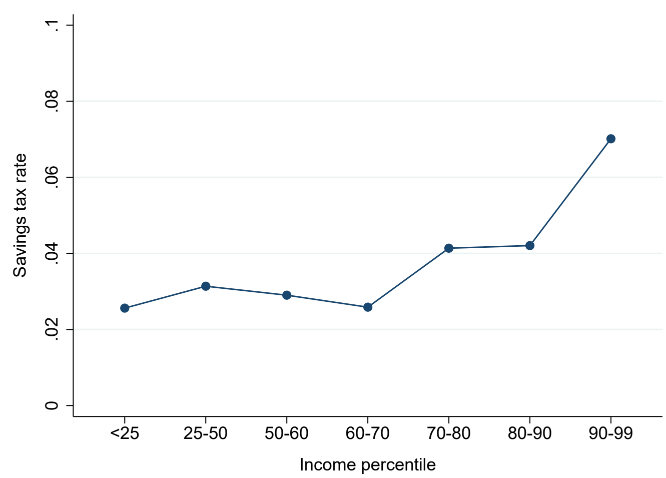
We treat this savings tax rate profile as a schedule of *average* tax rates on one's savings portfolio at each point in the income distribution. We smooth this schedule of average rates using the spline fit procedure described above, and apply that average tax rate to the calibrated level of gross savings at each point in the income distribution to reach a calibrated schedule of total savings taxes paid. We then compute the schedule of marginal rates that would give rise to that nonlinear profile of average tax rates; this schedule is plotted as the "U.S. Status quo" savings tax, e.g., in Figure II.

Figure A2: Calibration of Savings Tax Rates Across Incomes in the U.S.

(a) Decomposition of Savings Types: Bricker et al. (2019)



(b) Calibrated Savings Tax Rates in the United States, by Income Percentile



Notes: This figure illustrates the calibration of savings tax rates in the U.S. across the income distribution. Panel (a) plots the composition of asset types in individuals' portfolios across the income distribution, reported by Bricker et al. (2019). Panel (b) plots the implied weighted average savings tax rate in each bin. See Appendix D.1.2 for details.

D.1.3 Measures of s'_{inc}

A key input for our sufficient statistics is the marginal propensity to save out of earned income, $s'_{inc}(z) := \frac{\partial s(z)}{\partial z}\big|_{\theta=\theta(z)}$, which relates changes in the amount of net-of-tax savings at the time of retirement to changes in the amount of pre-tax earnings z. We draw from two sources of empirical data to calibrate our marginal propensities to consume (or save), translated into measures of $s'_{inc}(z)$. These results are plotted in Figure I.

Norwegian estimates from Fagereng et al. (2021). Fagereng et al. (2021) estimate marginal propensities to consume (MPC) across the earnings distribution using information on lottery prizes linked with administrative data in Norway. They find that individuals' consumption peaks during the year in which the prize is won, before gradually reverting to their previous consumption level. Over a 5-year horizon, they estimate winners consume close to 90% of the tax-exempt lottery prize; see the "consumption" panels in Fagereng et al. (2021) Figure 2. This translates into an MPC of 0.9, and thus a marginal propensity to save of 0.1. Under the assumption that preferences are weakly separable with respect to the disutility of labor supply, this is also the marginal propensity to save out of net earned income from labor supply. (See Proposition 2.)

They find little evidence of variation in MPCs across income levels which implies

$$\frac{\partial c(z)}{\partial(z - T_1(z))} = 0.9$$

and recognizing that individuals' budget constraint is $s_1(z) = z - T_1(z) - c(z)$, we get

$$\frac{\partial s_1(z)}{\partial(z - T_1(z))} = 1 - \frac{\partial c(z)}{\partial(z - T_1(z))} = 0.1.$$

The identity $s = (s_1 - T_s(s))(1+r)$ implies that $\frac{\partial s}{\partial s_1} = \frac{1}{\frac{1}{1+r} + T'_s(s)}$, and thus that the local causal effect of *pre-tax* income z on *net* savings s satisfies

$$\begin{aligned} s'_{inc}(z) &= \frac{\partial s_1(z)}{\partial(z - T_1(z))} \cdot \frac{\partial s}{\partial s_1} \cdot \frac{\partial(z - T_1(z))}{\partial z} \\ &= 0.1 \cdot \frac{1 - T'_1(z)}{\frac{1}{1+r} + T'_s(s(z))}. \end{aligned} \tag{233}$$

We can then use our calibrated U.S. tax schedule to obtain a profile of $s'_{inc}(z)$, under the key assumption that U.S. households have similar MPCs as Norwegian households. This profile is plotted in Figure I.

U.S. estimates from a new AmeriSpeak survey. We conducted a probability-based survey of the American population in the spring of 2021, which asked the following question:

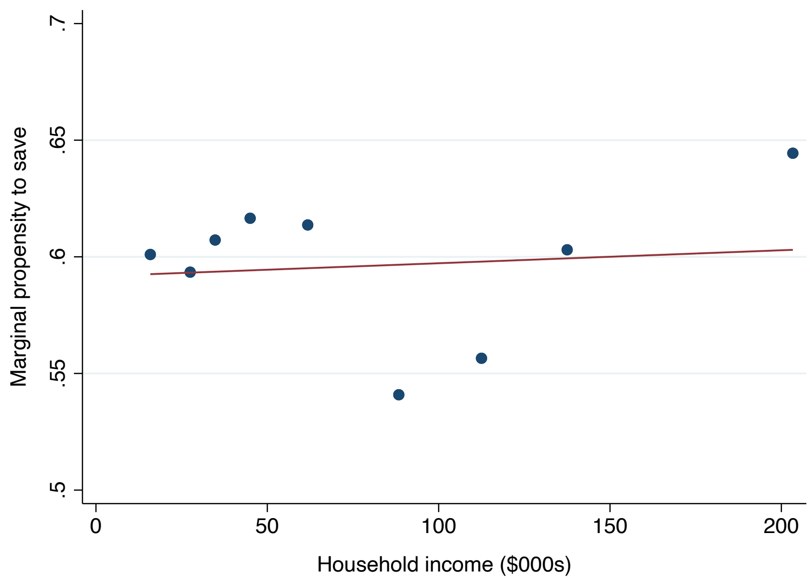
Imagine that you or someone else in your household gets a raise such that over the next five years, your household's income is \$1,000 higher each year than what you expected. How much of this would your household spend, and how much would your household save over each of the next five years? (For purposes of this question, consider paying off debt, such as reducing your mortgage, a form of saving.) If no one in your household is going to be employed for most of the next five years, please write "N/A."

Spend an extra \$ ______ per year

Save an extra \$ ______ per year

Answers to this question provide information about individuals' reported marginal propensity to consume (MPC) and marginal propensity to save (MPS) out of a small and persistent change in *earned* income – in contrast to empirical estimates based on lottery winnings which measure MPC and MPS out of a one-time *windfall* income gain. Our survey sample consisted of $1{,}703$ respondents who reported an average marginal propensity to save of 0.60 in the year of the raise.[55] We also requested information on household income in the survey, so we can observe marginal propensity to save across earnings levels, plotted in Figure A3. Marginal propensities to save appear quite stable across income levels, a finding that is consistent with the results of Fagereng et al. (2021).

Figure A3: Marginal propensity to save across household income (own survey)



Notes: Marginal propensities to save are computed from the answers to our survey question. They are computed as the ratio between the amount respondents report they would save and the amount of the raise.

Since our survey question asks about consumption and spending within each year, we interpret these estimates as short-run responses. Fagereng et al. (2021) show that positive income shocks are followed by consumption responses that can last up to 5 years. We use their impulse-response profile to convert these 1-year MPS into a 5-year MPS, which we interpret as a total effect on savings before returns. To do so, we use the fact that they report a 1-year MPC of 0.52 and a 5-year MPC of 0.90; we therefore compute our long run marginal propensity to save as

$$MPS_{5y} = MPS_{1y} \cdot \frac{1 - 0.90}{1 - 0.52} = 0.60 \cdot 0.208 = 0.125.$$

[55] This average is computed using the sample weights provided AmeriSpeak; the unweighted average is 0.59.

Because our survey question asked about a change in pre-tax income, we do not need to multiply by $1 - T_1'(z)$ as in equation (233); we just divide by $\frac{1}{1+r} + T_s'(s(z))$ to reach our measure of $s_{inc}'(z)$. This results in an estimate somewhat higher than that obtained by Fagereng et al. (2021) for Norway, plotted in Figure I. We use this as the baseline measure of $s_{inc}'(z)$ for our simulations, and the difference between the cross-sectional slope $s'(z)$ and $s_{inc}'(z)$ provides our estimate of the key statistic for preference heterogeneity, $s_{het}'(z)$, which is also plotted in Figure I.

Comparison to Golosov et al. (2013). Golosov et al. (2013) also study preference heterogeneity, providing a useful point of comparison. In their baseline calibration, they assume individuals' preferences are Constant-Relative-Risk-Aversion

$$U(c, s, l) = \frac{\alpha(w)}{1 + \alpha(w)} \ln c + \frac{1}{1 + \alpha(w)} \ln s - \frac{1}{\sigma} (l)^{\sigma},$$

where l is the labor supply of an individual with hourly wage w such that earnings are given by $z = wl$. The risk-aversion parameter is set to $\gamma = 1$, the isoelastic disulity from labor effort is such that $\sigma = 3$, and the taste parameter is given by

$$\alpha(w) = 1.0526 (w)^{-0.0036}.$$

In other words, the taste parameter varies from 1.0433 for individuals in the bottom quintile of the earnings distribution (mean hourly wage of \$12.35, in 1992 dollars) to 1.0406 for individuals in the top quintile of the earnings distribution (mean hourly wage of \$25.39, in 1992 dollars). This means that this taste parameter is almost constant with income around an average of $\overline{\alpha} = 1.042$.

To illustrate how little preference heterogeneity this implies, we compute the s_{inc}' and s_{het}' implied by their calibration. Individuals' savings choices follow from maximizing $U(c, s, \frac{z}{w})$ subject to the budget constraint $c \leq z - \frac{1}{R}s - \mathcal{T}(s, z)$. This implies

$$s = \frac{z - \mathcal{T}(s, z)}{1/R + \alpha(1/R + \mathcal{T}_s')}$$

such that, neglecting the (potential) curvature of the tax function $\mathcal{T}'' \approx 0$, we can decompose the variation of savings s across earnings z as

$$\underbrace{\frac{ds}{dz}}_{s'(z)} = \underbrace{\frac{1 - \mathcal{T}_z'}{1/R + \alpha(1/R + \mathcal{T}_s') + \mathcal{T}_s'}}_{s_{inc}'(z)} + \underbrace{\frac{-(1/R + \mathcal{T}_s')}{1/R + \alpha(1/R + \mathcal{T}_s') + \mathcal{T}_s'} \frac{d\alpha}{dz} s}_{s_{het}'(z)}.$$

To obtain an approximation of $s_{het}'(z)$ in their setting, we use the fact that Golosov et al. (2013) report in their simulation results that individuals with an annual income $z = \$100,000$ have an hourly wage $w = \$40$ while those with an annual income $z = \$150,000$ have an hourly wage $w = \$62.5$. We can thus approximate $\frac{d\alpha}{dz} = \frac{\alpha(62.5) - \alpha(40)}{150{,}000 - 100{,}000} = \frac{1.0370 - 1.0387}{50{,}000} = -34 * 10^{-9}$. For $\mathcal{T}_z'$, we assume a linear income tax rate $\tau_z = 0.3$, for $R = 2.1$ we use our real interest rate of 3.8% compounded over 20 years), and for $\mathcal{T}_s'$ we assume a linear income tax rate $\tau_s = 0.01$ which we show below (see equation (235)) to be consistent with a linear tax of 4% on capital gains (the approximate average in Figure A2b).

This gives a constant $s'_{inc} = \frac{1-0.3}{1/2.1+1.042*(1/2.1+0.01)} = 0.71$, which is much higher than our estimate. Leveraging the fact that s'_{inc} is constant, we can also infer that at an annual income of $\$125,000$, the annual amount of savings available for consumption in period 2 (including compounded interest) is approximately equal to $s = s'_{inc} * \$125,000 = 0.71 * 125,000 = \$88,750$. Thus, $s'_{het} = \frac{1/2.1+0.02}{1/2.1+1.042*(1/2.1+0.02)+0.02} * (34 * 10^{-9}) * 88,750 = 0.0015$.[56]

These values for s'_{inc} and s'_{het} imply that in the calibration of Golosov et al. (2013), preference heterogeneity is substantially smaller than our estimate of across-income heterogeneity, as it only explains $\frac{s'_{het}}{s'_{het}+s'_{inc}} = \frac{0.0015}{0.71+0.0015} = 0.2\%$ of the variation in savings between individuals earning $\$100,000$ annually and those earning $\$150,000$.

D.1.4 Savings elasticity

For purposes of calibration, we assume that the income-conditional compensated elasticity of savings is constant across earnings, $\zeta^c_{s|z}(z) = \bar{\zeta}^c_{s|z}$. We follow Golosov et al. (2013) in drawing on the literature estimating the intertemporal elasticity of substitution (IES), and reporting results for a range of values. To motivate these values, we describe here how we can translate from the IES to a compensated elasticity $\zeta^c_{s|z}$ in the case of a representative agent.

The IES is defined as the elasticity of the growth rate of consumption with respect to the net price of consumption. We assume consumption is smoothed during retirement, so that retirement consumption is proportional to the net stock of savings s, and thus the elasticity of the growth rate of consumption (with respect to a tax change) is the same as the elasticity of the ratio of s to work-life consumption c. We consider a change in the price of retirement consumption induced by a small reform to a SL system like the one described in Table I with a constant linear tax rate τ_s, in which case the net-of-tax price of retirement savings is $\frac{R}{1+R\tau_s}$. (This can be found using the relationship $(s_1 - \tau_s s)R = s$ and solving for $\frac{ds}{ds_1} = -\frac{ds}{dc}$.) We can therefore write

$$\begin{aligned}
IES &= \frac{d\ln(s/c)}{d\ln(\frac{R}{1+R\tau_s})} \\
&= -\frac{d\ln(s/c)}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{d\ln c}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{dc}{d\ln(1+R\tau_s)}\frac{1}{c} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{ds}{d\ln(1+R\tau_s)}\frac{dc}{ds}\frac{1}{c}.
\end{aligned}$$

[56] More specifically, we postulate $s'_{het} \ll s'_{inc}$ to infer $s(z) = s'_{inc} \cdot z$ and then compute s'_{het}. Since we obtain a value that verifies $s'_{het} \ll s'_{inc}$, this reasoning is consistent and proves that $s'_{het} \ll s'_{inc}$. Put differently, even if we assume $s'_{het} \approx s'_{inc}$ which implies that $s(z) = 2s'_{inc} \cdot z$, we still obtain $s'_{het} \ll s'_{inc}$.

Substituting for $\frac{dc}{ds} = \frac{1+R\tau_s}{R}$, we then obtain

$$\begin{aligned}
IES &= -\frac{d\ln s}{d\ln(1+R\tau_s)} - \frac{d\ln s}{d\ln(1+R\tau_s)}\frac{1+R\tau_s}{R}\frac{s}{c} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln s}{d\ln(1+R\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln(1+\tau_s)}{d\ln(1+R\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\left(\frac{d(1+R\tau_s)}{d\tau_s}\right)^{-1}\frac{1+R\tau_s}{1+\tau_s}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
\implies \quad \frac{d\ln s}{d\ln(1+\tau_s)} &= -\frac{IES}{\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}}. \qquad (234)
\end{aligned}$$

Using a value of $s/c = 0.67$ (the population average in our calibrated two-period economy), and using the values $R = 2.1$ (from our real interest rate of 3.8% compounded over 20 years) and $\tau_s = 0.01$ (corresponding to a linear tax of 4% on capital gains, the approximate average in Figure A2b), we find[57]

$$\frac{d\ln s}{d\ln(1+\tau_s)} = -\frac{IES}{0.64}.$$

Treating this as the population estimate of $\frac{d\ln \bar{s}}{d\ln(1+\tau_s)}$, we can then compute the value of the elasticity $\bar{\zeta}^c_{s|z}$ that is consistent with this estimate. From the proof of the optimal SL tax system (see Appendix C.8.1, equation (134)), the response of aggregate savings $\bar{s}$ to a change in the separable

[57] A linear tax rate τ^{cg} on capital gains $(R-1)\, s_1$ leads to net savings $s = s_1(1+(R-1)(1-\tau^{cg}))$. Similarly, a period-1 linear tax τ_s on net savings s leads to net savings $s = (s_1 - \tau_s s)\, R \iff s = \frac{s_1 R}{1+\tau_s R}$. As a result,

$$\begin{aligned}
s_1(1+(R-1)(1-\tau^{cg})) &= \frac{s_1 R}{1+\tau_s R} \\
\iff 1+\tau_s R &= \frac{R}{1+(R-1)(1-\tau^{cg})} \\
\iff \tau_s &= \frac{1}{1+(R-1)(1-\tau^{cg})} - \frac{1}{R}. \qquad (235)
\end{aligned}$$

linear tax rate τ_s (measured in period-1 dollars, as distinct from $\tau_{s,2}$) is:

$$\frac{d\bar{s}}{d\tau_s} = -\int_z \left\{ \frac{1}{1+\tau_s}\left(s(z)\bar{\zeta}^c_{s|z} + \eta_{s|z}(z)s(z)\right) + \frac{s'_{inc}(z)}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z)\right)\right\} dH_z(z)$$

$$\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{1} = -\bar{\zeta}^c_{s|z}\bar{s} - \int_z \left\{ \eta_{s|z}(z)s(z) + s'_{inc}(z)\frac{1+\tau_s}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z)\right)\right\} dH_z(z)$$

$$\underbrace{\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{\bar{s}}}_{\frac{d\ln\bar{s}}{d\ln(1+\tau_s)}} = -\bar{\zeta}^c_{s|z} - \int_z \left\{ \eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}}\frac{1+\tau_s}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + s(z)\,\eta_z(z)\right)\right\} dH_z(z)$$

$$\bar{\zeta}^c_{s|z} = -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}}\frac{1+\tau_s}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z)\right)\right]$$

This could be computed directly if we had an independent estimate of the income-conditional income effect $\eta_{s|z}$. We instead invoke our assumptions of weak separability and a separable tax system, implying $\eta_{s|z}(z) = s'_{inc}(z)\frac{1+T'_s(s(z))}{1-T'_z(z)}$ (see Proposition 2), and negligible income effects on earnings, to write

$$\begin{aligned}\bar{\zeta}^c_{s|z} &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)}\frac{s'_{inc}(z)}{\bar{s}}\left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z)\right)\right] \\ &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \frac{1}{\bar{s}}\cdot\mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)}s'_{inc}(z)\left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z)\right)\right]. \end{aligned} \tag{236}$$

In our calibration, the value of the second term is 0.38, suggesting a translation of $\bar{\zeta}^c_{s|z} \approx IES/0.64 - 0.38$. Thus a value of $IES = 1$, the baseline in Golosov et al. (2013), suggests an elasticity of $\bar{\zeta}^c_{s|z} = 1.2$. We use a baseline value of $\bar{\zeta}^c_{s|z} = 1$. IES values of 0.5 and 2 (the "low" and "high" values considered in Golosov et al. (2013)) suggest savings elasticities of $\bar{\zeta}^c_{s|z} = 0.4$ and $\bar{\zeta}^c_{s|z} = 2.7$. This is a wide range; values of savings elasticities below $\bar{\zeta}^c_{s|z} = 0.6$ in particular suggest that consistency with the status quo income tax requires a savings tax that is extreme or non-convergent.[58] We report results for alternative values of $\bar{\zeta}^c_{s|z} = 0.7$, $\bar{\zeta}^c_{s|z} = 2$, and $\bar{\zeta}^c_{s|z} = 3$.

D.2 Simulations of Optimal Savings Taxes with Multidimensional Heterogeneity

We now extend the above calibration to accommodate multidimensional heterogeneity, which we use to apply the formulas derived in Proposition 4. In the multidimensional setting, we do not have Pareto-efficiency formulas like those for unidimensional setting, because in the presence of income-conditional savings heterogeneity, Pareto-improving reforms are not generally available. Therefore, we use the formulas in Proposition 4 to compute the *optimal* schedule of savings tax rates for each type of simple tax system. In order to isolate and illustrate the effects of multidimensional

[58]Intuitively, as the savings elasticity becomes low, one's level of savings becomes a reliable signal of underlying ability, and more of the total redistribution in the tax system should be carried out through the savings tax, rather than the income tax. Thus for sufficiently low $\bar{\zeta}^c_{s|z}$, the status quo income tax cannot be Pareto efficient.

heterogeneity, we hold fixed the sufficient statistics used in the unidimensional setting. We also hold fixed the distributional preferences of the policy maker. The Pareto-efficiency computations above are equivalent to computing the optimal tax under "inverse optimum" welfare weights that would rationalize the status quo income tax. We compute these welfare weights explicitly, as described below, assuming that they vary with earnings, but not with savings conditional on earnings. We then use those inverse optimum weights for the optimal tax calculations.

D.2.1 Inverse Optimum Approach

We assume that income effects on labor supply are negligible, so that $\eta_z \approx 0$, which simplifies the computation of $\hat{g}(z)$ from equation (18) to

$$\hat{g}(z) = g(z) + \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right)\eta_{s|z}(z). \tag{237}$$

We also assume that preferences are weakly separable so that, as shown in Proposition 2, we have

$$\eta_{s|z}(z) = s_{inc}'(z)\,\frac{1+T_s'(s(z))}{1-T_z'(z)}. \tag{238}$$

Because equations (237) and (238) depend on the marginal savings tax rates $\mathcal{T}_s'$, we must impose an assumption about how they adjust when we recompute the savings tax. We assume that the welfare weights $g(z)$ remain proportional to those calibrated using the inverse optimum procedure described above but we rescale to preserve the normalization $\int_z \hat{g}(z)dH_z(z) = 1$.[59] In equation (238), after computing $\eta_{s|z}(z)$ from the baseline calibration of $s_{inc}'(z)$, we assume that $\eta_{s|z}(z)$ remains stable when savings taxes are recomputed.

The inverse optimum computes the social marginal welfare weights (SMWW) consistent with existing tax policy (Bourguignon and Spadaro, 2012; Lockwood and Weinzierl, 2016). This exercise is typically performed using labor income taxes. Our setting presents a complication, as we have both a status quo income tax and savings tax, which need not produce a consistent set of weights. We compute weights assuming that the status quo schedule of earnings tax rates is optimal, for consistency with the Pareto-efficiency formulas above. Since the status quo savings tax rates also appear in this calculation, we must choose whether to use the status quo rates, or the rates that would counterfactually be optimal. In practice, results are insensitive to this latter issue; for consistency with the "inverse optimum" motivation, we use the Pareto-efficient set of SN tax rates.

Under these assumptions, we can compute the inverse optimum social marginal welfare weights

[59] Specifically, letting $g^0(z)$ denote our baseline welfare weights, we set $g(z) = \kappa g^0(z)$, where

$$\kappa = \frac{1 - \int_z \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right)\eta_{s|z}(z)dH_z(z)}{\int_z g^0(z)dH_z(z)}. \tag{239}$$

at each earnings z by inverting the optimal tax rate condition,

$$\frac{T_z'(z)}{1-T_z'(z)} = \frac{1}{\zeta_z^c(z)\, z}\frac{1}{h_z(z)}\int_{x=z}^{z_{max}}(1-\hat{g}(x))\, dH_z(x) - s_{inc}'(z)\frac{T_s'(s(z))}{1-T_z'(z)} \tag{240}$$

$$\iff \int_{x=z}^{z_{max}}(1-\hat{g}(x))\, dH_z(x) = \zeta_z^c(z)\, z\, h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}, \tag{241}$$

where the right-hand side term can be identified from the data. Differentiating with respect to z yields the expression we use to implement this computation numerically,

$$\hat{g}(z) = 1 + \frac{1}{h_z(z)}\cdot\frac{d}{dz}\left[\zeta_z^c(z)\, z\, h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}\right]. \tag{242}$$

Using the fact that augmented social marginal welfare weights are defined as

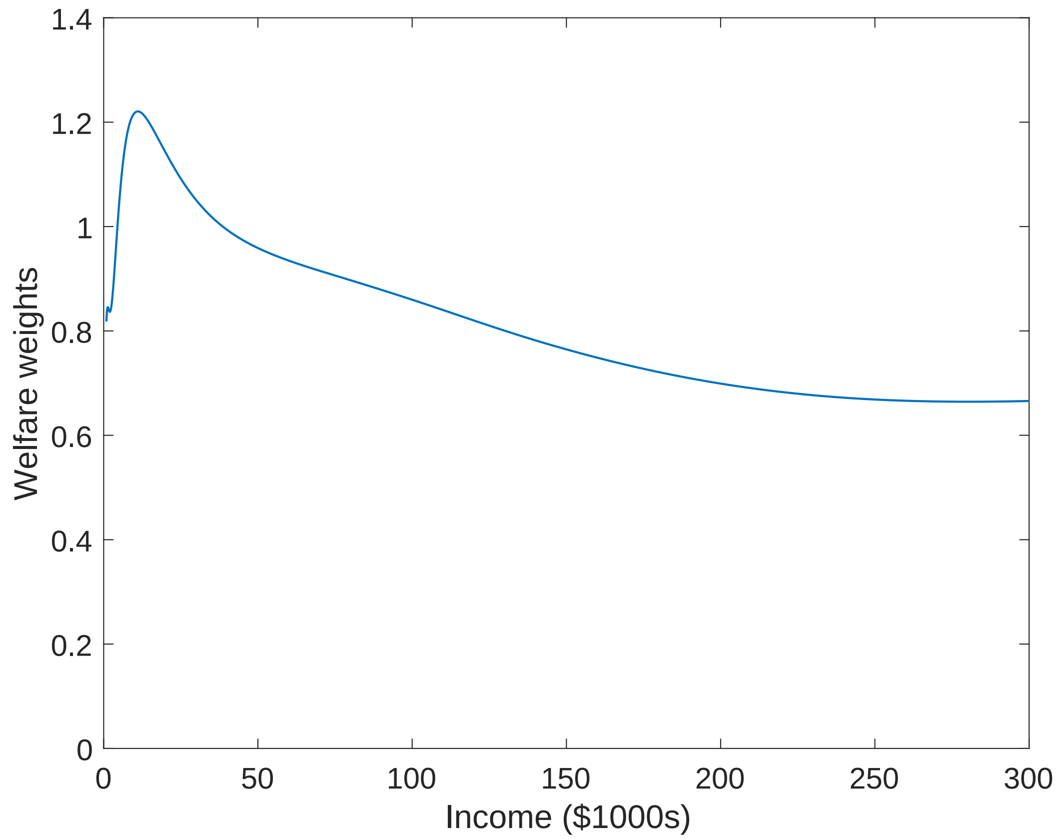
$$\hat{g}(z) := g(z) + T_z'(z)\frac{\eta_z(z)}{1-T_z'(z)} + T_s'(s(z))\left(\frac{\eta_{s|z}(z)}{1+T_s'(s(z))} + s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'(z)}\right), \tag{243}$$

and assuming preferences are weakly separable, such that by Proposition 2 we have $s_{inc}'(z) = \frac{1-T_z'(z)}{1+T_s'(s(z))}\eta_{s|z}(z)$, inverse optimum weights $g(z)$ are obtained from $\hat{g}(z)$ as follows:

$$g(z) = \hat{g}(z) - s_{inc}'(z)\left(\frac{T_s'(s(z))}{1-T_z'(z)}\right). \tag{244}$$

Figure A4 plots our estimated profile of inverse optimum weights.

Figure A4: Schedule of Inverse Optimum Social Welfare Weights in the U.S.



Notes: This figure plots the schedule of inverse optimum welfare weights that would rationalize the U.S. income tax schedule. These weights are computed under the assumption that the savings tax is the Pareto-efficient SN schedule reported in Figure II.

D.2.2 Calibration Details

To extend our calibrated two-period model economy to a multidimensional setting, we retain the same discretized grid of incomes as in the unidimensional case, using the calibration described in Appendix D.1. At each income, we now allow for heterogeneous levels of savings. Specifically, using the same measure of gross savings described in Appendix D.1, we now use a calibration with four different levels of savings at each level of income, each representing a quartile of the income-conditional savings distribution. Across the income distribution, we assume savings within each quartile are a constant ratio of the income-conditional average level of saving. These ratios are 15%, 40%, 70% and 280% of the income-conditional average savings level; they are calibrated to reflect the average ratios across percentiles 50 to 100 in the PSZ data. We calibrate these ratios excluding the bottom portion of the distribution because the average level of saving is very low in the bottom half, resulting in noisily measured ratios.

To calibrate the savings income effect $\eta_{s|z}(s, z)$, we assume that the income elasticity of savings is constant within earnings and equal to its unidimensional counterpart, implying that $\eta_{s|z}(s,z) = \frac{s}{\overline{s(z)}}\eta_{s|z}(z)$, where $\overline{s(z)} := \mathbb{E}\left[s\middle|z\right]$ denotes the average savings level at earnings z, and that similarly, $s'_{inc}(s,z) = \frac{s}{\overline{s(z)}}s'_{inc}(z)$. Using these expressions, we can adapt equation (239)—the scaling factor necessary to ensure that $\hat{g}(s,z)$ integrates to one when recomputing savings taxes—to this setting:

$$\kappa = \frac{1 - \int_z \int_s \left(\frac{\mathcal{T}'_s}{1+\mathcal{T}'_s}\right)\eta_{s|z}(s,z)\,h(s,z)dsdz}{\int_z g^0(z)dH_z(z)}.$$

Letting $\mathcal{T}'_s = \tau$ in the SL case and $\mathcal{T}'_s = \tau_s(z)$ in the LED case, we have

$$\kappa = \frac{1 - \int_z \frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\,dH_z(z)}{\int_z g^0(z)dH_z(z)}, \tag{245}$$

and with $\mathcal{T}'_s = T'_s(s)$ in the SN case such that

$$\kappa = \frac{1 - \int_s \frac{T'_s(s)}{1+T'_s(s)}\int_z \eta_{s|z}(s,z)\,h(s,z)dzds}{\int_z g^0(z)dH_z(z)}. \tag{246}$$

D.2.3 Separable linear (SL) tax system

The optimal savings tax formula with multidimensional heterogeneity (Proposition 4) is

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\int_z\left\{\mathbb{E}\left[s\zeta^c_{s|z}(s,z)\middle|z\right]\right\}dH_z(z) \qquad (247)\\ &= \int_z\left\{\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right] - \mathbb{E}\left[\frac{T'_z(z)+s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)}z\zeta^c_z(s,z)\,s'_{inc}(s,z)\middle|z\right]\right\}dH_z(z).\end{aligned}$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$, replacing $s'_{inc}(s,z)$ and $\eta_{s|z}(s,z)$ by their values, and assuming η_z is negligible gives

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\int_z\left\{\overline{s(z)}\zeta^c_{s|z}\right\}dH_z(z) \qquad (248)\\ &= \int_z\left\{\mathbb{E}\left[\left(1-g(z)-\tau_s\frac{\eta_{s|z}(z)}{1+\tau_s}\frac{s}{\overline{s(z)}}\right)s\middle|z\right] - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[T'_z(z)\frac{s}{\overline{s(z)}} + s'_{inc}(z)\,\tau_s\left(\frac{s}{\overline{s(z)}}\right)^2\middle|z\right]\right\}dH_z(z)\end{aligned}$$

which after rearranging yields

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z\overline{s(z)}dH_z(z) \qquad (249)\\ &= \int_z\left\{(1-g(z))\,\overline{s(z)} - \frac{\tau_s}{1+\tau_s}\frac{\eta_{s|z}(z)}{\overline{s(z)}}\mathbb{E}\left[s^2\middle|z\right] - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[\frac{T'_z(z)}{\overline{s(z)}}s + \frac{s'_{inc}(z)\,\tau_s}{\overline{s(z)}^2}s^2\middle|z\right]\right\}dH_z(z).\end{aligned}$$

We can now use $\mathbb{E}\left[s^2\middle|z\right] = \mathbb{E}\left[\left(s - \overline{s(z)}\right)^2 + 2s\overline{s(z)} - \overline{s(z)}^2\middle|z\right] = \mathbb{V}(s|z) + \overline{s(z)}^2$ to obtain

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z \overline{s(z)}dH_z(z) \qquad (250)\\ &= \int_z \left\{\left[1 - g(z) - \frac{\tau_s\eta_{s|z}(z)}{1+\tau_s}\left(1 + \frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}\right)\right]\overline{s(z)} - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\left[T'_z(z) + s'_{inc}(z)\tau_s\left(1 + \frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}\right)\right]\right\}dH_z(z)\end{aligned}$$

which we can finally rewrite as

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\int_z \overline{s(z)}\zeta^c_{s|z}dH_z(z) \qquad (251)\\ &= \int_z \left\{\left(1 - g(z) - \frac{\tau_s}{1+\tau_s}\eta_{s|z}(z)\right)\overline{s(z)} - \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)}z\zeta^c_z s'_{inc}(z)\right\}dH_z(z)\\ &- \int_z \left\{\underbrace{\frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}}_{\geq 0}\underbrace{\left(\frac{\tau_s}{1+\tau_s}\eta_{s|z}(z)\overline{s(z)} + \frac{s'_{inc}(z)\tau_s}{1-T'_z(z)}z\zeta^c_z s'_{inc}(z)\right)}_{\geq 0}\right\}dH_z(z).\end{aligned}$$

The first two lines correspond to the optimal savings tax formula under unidimensional heterogeneity (Proposition B.4) and the last line captures the effect of multidimensional heterogeneity through $\mathbb{V}(s|z)$. Multidimensional heterogeneity adds a corrective term which is unambiguously negative and thus prescribes a lower linear savings tax rate.

D.2.4 Separable nonlinear (SN) tax system

At any given savings level s^0, the optimal savings tax formula with multidimensional heterogeneity (Proposition 4) is

$$\begin{aligned}\frac{T'_s(s^0)}{1+T'_s(s^0)}\int_z \left\{s^0\zeta^c_{s|z}(s^0,z)\right\}h(s^0,z)\,dz &= \int_z \left\{\mathbb{E}\left[1 - \hat{g}(s,z)\middle|z, s \geq s^0\right]\right\}h_z(z)\,dz \qquad (252)\\ &- \int_z \left\{\frac{T'_z(z) + s'_{inc}(s^0,z)T'_s(s^0)}{1-T'_z(z)}z\zeta^c_z(s^0,z)s'_{inc}(s^0,z)\right\}h(s^0,z)\,dz.\end{aligned}$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$ and assuming η_z is negligible gives

$$\begin{aligned}\frac{T'_s(s^0)}{1+T'_s(s^0)}s^0\zeta^c_{s|z}\int_z h(s^0,z)\,dz &= \int_{s\geq s^0}\left\{\int_z\left[1 - g(z) - T'_s(s)\frac{\eta_{s|z}(s,z)}{1+T'_s(s)}\right]h(s,z)\,dz\right\}ds \qquad (253)\\ &- \int_z \left[\frac{T'_z(z) + s'_{inc}(s^0,z)T'_s(s^0)}{1-T'_z(z)}z\zeta^c_z s'_{inc}(s^0,z)\right]h(s^0,z)\,dz\end{aligned}$$

or equivalently, expressing this as a function of the savings density $h_s(s) = \int_z h(s,z)\,dz$,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c h_s\left(s^0\right)=\int_{s\geq s^0}\left\{\mathbb{E}\left[1-g(z)-T_s'\left(s\right)\frac{\eta_{s|z}\left(s,z\right)}{1+T_s'\left(s\right)}\Big|s\right]\right\}h_s\left(s\right)ds \\ -\mathbb{E}\left[\frac{T_z'\left(z\right)+s_{inc}'\left(s,z\right)T_s'\left(s\right)}{1-T_z'\left(z\right)}z\zeta_z^c s_{inc}'\left(s,z\right)\Big|s=s^0\right]h_s\left(s^0\right) \tag{254}$$

where the expectations operator denotes integration with respect to earnings conditional on savings.

For implementation, we assume that at each point in the income continuum, there are M different equal-sized saver bins (e.g., bottom-, middle-, and top-third of savers), indexed by $m=1,\ldots,M$. Thus we can write $s_m(z)$ as the savings map for saver bin m at each income, with $s_m'(z)$ the cross-sectional savings profile within each saver-bin. Then the income density in each saver-bin is $h_{z,m}(z)=h(z)/M$, since the bins are equally sized conditional on income. The savings density among saver-bin m is therefore $h_{s,m}(s)=h_{z,m}(z)/s_m'(z)$, and we have $H(s)=\sum_{m=1}^M\int_{s=0}^\infty h_{s,m}(s)ds$, and $h_s(s)=\sum_{m=1}^M h_{s,m}(s)$. And the savings-conditional average of some $x(s,z)$ is $\mathbb{E}[x(s,z)|s]=\frac{\sum_{m=1}^M x(s_m,z)h_{s,m}(s)}{h_s(s)}$.

To better picture the link with the unidimensional formula in equation (67), let us also rewrite the latter as a function of the savings density $h_s\left(s\right)$—implicitly defining $z\left(s\right)$ as the earnings level of individuals with savings s—this yields

$$\frac{T_s'(s^0)}{1+T_s'(s^0)}s^0\zeta_{s|z}^c h_s\left(s^0\right)=\int_{s\geq s^0}\left\{1-g\left(z(s)\right)-\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}\left(z\left(s\right)\right)\right\}h_s\left(s\right)ds \\ -\frac{T_z'(z\left(s^0\right))+s_{inc}'(z\left(s^0\right))T_s'(s^0)}{1-T_z'(z\left(s^0\right))}z\left(s^0\right)\zeta_z^c s_{inc}'\left(z(s^0)\right)h_s\left(s^0\right). \tag{255}$$

While it is clear that the multidimensional formula extends the unidimensional formula, determining the impact of multidimensional heterogeneity on tax rates is more analytically difficult and we thus rely on numerical simulations.

D.2.5 Linear earnings dependent (LED) tax system

At earnings z^0, the optimal LED savings tax formula in the presence of multidimensional heterogeneity (Proposition 4) is

$$\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s\Big|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\Big|z\right]h_z\left(z\right)dz \\ =\int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z\Big]h_z\left(z\right)dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)\Big|z\right]h_z\left(z\right)d \tag{256}$$

which proves particularly cumbersome to use in numerical simulations, even under the aforementioned assumptions. To obtain an expression that is more easily implementable numerically, we further assume that the earnings tax is optimal (see Proposition B.6) such that

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)\Big|z^0\right]h_z\left(z^0\right)=\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z\right]\right\}h_z\left(z\right)dz. \tag{257}$$

Now, observing that $s=\overline{s\left(z^0\right)}+s-\overline{s\left(z^0\right)}$, we can rewrite the first term of the optimal savings tax formula as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s\Big|z^0\right]=\overline{s\left(z^0\right)}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\Big|z^0\right] \tag{258}$$
$$+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right].$$

Plugging this back into the optimal savings tax formula and using the optimal earnings tax formula, this implies that

$$\begin{aligned}&\overline{s\left(z^0\right)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z\right]\right\}h_z\left(z\right)dz \\ &+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\Big|z\right]h_z\left(z\right)dz \\ &=\int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z\Big]h_z\left(z\right)dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s_{inc}'\left(s,z\right)\Big|z\right]h_z\left(z\right)dz.\end{aligned} \tag{259}$$

Differentiating with respect to z^0 then yields

$$\begin{aligned}&\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z\right]\right\}h_z\left(z\right)dz-\overline{s\left(z^0\right)}\,\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z^0\right]h_z\left(z^0\right) \\ &+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]h_z\left(z^0\right)\right)-\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\Big|z^0\right]h_z\left(z^0\right) \\ &=-\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z^0\Big]h_z\left(z^0\right)+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s_{inc}'\left(s,z\right)\Big|z^0\right]h_z\left(z^0\right).\end{aligned} \tag{260}$$

Rearranging gives

$$\begin{aligned}&\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c+\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s_{inc}'\left(s,z\right)\Big|z^0\right]h_z\left(z^0\right) \\ &=\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)|z\right]\right\}h_z\left(z\right)dz-\mathbb{E}\Big[\left(\hat{g}\left(s,z\right)\right)\left(s-\overline{s\left(z^0\right)}\right)\Big|z^0\Big]h_z\left(z^0\right) \\ &+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]h_z\left(z^0\right)\right).\end{aligned} \tag{261}$$

Now, with $s'_{inc}(s,z) = \frac{s}{s(z)} s'_{inc}(z)$ as well as $\eta_{s|z}(s,z) = \frac{s}{s(z)} \eta_{s|z}(z)$ and $\hat{g}(s,z) = g(z) + \frac{\tau_s(z)}{1+\tau_s(z)} \frac{s}{s(z)} \eta_{s|z}(z)$, we get

$$\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta^c_{s|z} + \frac{T'_z(z) + \tau'_s(z)s + \frac{s}{s(z)} s'_{inc}(z)\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)s} z\zeta^c_z \frac{s}{s(z)} s'_{inc}(z) \Big| z^0\right] h_z(z^0) \tag{262}$$

$$= \frac{d\left(\overline{s(z^0)}\right)}{dz^0} \int_{z \geq z^0} \left\{\mathbb{E}\left[1 - g(z) - \frac{\tau_s(z)}{1+\tau_s(z)} \frac{s}{s(z)} \eta_{s|z}(z) \Big| z\right]\right\} h_z(z)\, dz$$

$$- \mathbb{E}\left[\left(g(z) + \frac{\tau_s(z)}{1+\tau_s(z)} \frac{s}{s(z)} \eta_{s|z}(z)\right)\left(s - \overline{s(z^*)}\right) \Big| z^0\right] h_z(z^0)$$

$$+ \frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{s(z)} s'_{inc}(z)\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)s} z\zeta^c_z \left(s - \overline{s(z^0)}\right) \Big| z = z^0\right] h_z(z^0)\right)$$

which simplifies to the following exact formula

$$\frac{\tau_s(z^0)}{1+\tau_s(z^0)} \overline{s(z^0)} \zeta^c_{s|z} h_z(z^0) + z^0 \zeta^c_z \frac{s'_{inc}(z^0)}{s(z^0)} \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{s(z)} s'_{inc}(z)\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)s} s \Big| z^0\right] h_z(z^0) \tag{263}$$

$$= \frac{d\left(\overline{s(z^0)}\right)}{dz^0} \int_{z \geq z^0} \left\{1 - g(z) - \frac{\tau_s(z)}{1+\tau_s(z)} \eta_{s|z}(z)\right\} h_z(z)\, dz - \frac{\tau_s(z^0)}{1+\tau_s(z^0)} \eta_{s|z}(z^0) \frac{\mathbb{V}\left[s \Big| z^0\right]}{\overline{s(z^0)}} h_z(z^0)$$

$$+ \frac{d}{dz^0}\left(z^0 \zeta^c_z \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{s(z)} s'_{inc}(z)\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)s} \left(s - \overline{s(z^0)}\right) \Big| z = z^0\right] h_z(z^0)\right)$$

using $\mathbb{E}\left[s\left(s - \overline{s(z^0)}\right) \Big| z^0\right] = \mathbb{E}\left[s^2 \Big| z^0\right] - \left(\overline{s(z^0)}\right)^2 = \mathbb{V}\left[s \Big| z^0\right]$.

Since the marginal tax rate on earnings $T'_z(z) + \tau'_s(z)s$ features savings s, it is hard to further simplify this formula while retaining an exact characterization. To further simplify this expression, we disregard this dependence by setting $s = \overline{s(z^0)}$ in marginal earnings tax rates. We believe that these formulas are informative in that they converge to exact expressions as the linear earnings dependent savings tax rate tends to a simple linear savings tax rate—that is $\tau'_s(z) = 0$ for all z. Moreover, although these approximations are not unbiased in that they provide an upper bound on the linear-earnings dependent savings tax rate, these upper bounds are tight as the approximation only amounts to assuming $\tau'_s(z^0)\, \mathbb{V}(s|z^0)$ is negligible.

We thus use

$$\mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)s + \frac{s}{s(z)} s'_{inc}(z)\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)s} s \Big| z^0\right]$$

$$\approx \overline{s(z^0)}\left[\frac{T'_z(z^0) + \tau'_s(z^0)\overline{s(z^0)}}{1 - T'_z(z^0) - \tau'_s(z^0)\overline{s(z^0)}} + \frac{s'_{inc}(z^0)\tau_s(z^0)}{1 - T'_z(z^0) - \tau'_s(z^0)\overline{s(z^0)}}\left(1 + \frac{\mathbb{V}(s|z^0)}{\overline{s(z^0)}^2}\right)\right]$$

as well as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+\frac{s}{s(z)}s_{inc}'(z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]$$
$$\approx\overline{s\left(z^0\right)}\left[\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}\right]$$

to finally obtain

$$\begin{aligned}&\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\overline{s\left(z^0\right)}\zeta_{s|z}^c h_z\left(z^0\right)+s_{inc}'\left(z^0\right)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}+s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}z^0\zeta_z^c h_z\left(z^0\right)\\
&=\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\}h_z(z)\,dz \qquad (264)\\
&-z^0\zeta_z^c s_{inc}'\left(z^0\right)\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}h_z\left(z^0\right)-\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\eta_{s|z}\left(z^0\right)\frac{\mathbb{V}\left(s\middle|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\\
&+\frac{d}{dz^0}\left(z^0\zeta_z^c\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\right).\end{aligned}$$

As an element of comparison, a similar derivation under unidimensional heterogeneity combining the optimal LED savings tax formula (Proposition B.4) and the optimal earnings tax formula (Proposition B.5) yields the following unidimensional analogue

$$\begin{aligned}&\frac{\tau_s(z^0)}{1+\tau_s(z^0)}s(z^0)\zeta_{s|z}^c(z^0)\,h_z(z^0)+z^0\zeta_z^c(z^0)s_{inc}'(z^0)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)s\left(z^0\right)+s_{inc}'(z^0)\tau_s(z^0)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s\left(z^0\right)}h_z(z^0) \qquad (265)\\
&=s'(z^0)\int_{z\geq z^0}\left(1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right)dH_z(z).\end{aligned}$$

Multidimensional heterogeneity thus adds new terms related to $\mathbb{V}\left(s|z^0\right)$ that naturally wash out under unidimensional heterogeneity. The two terms on the third line of equation (264) are clearly negative and push for lower savings tax rates in the presence of multidimensional heterogeneity. The term on the fourth line cannot be signed unambiguously. In our calibration, it appears to be negative at low earnings but positive at high earnings. However, its order magnitude is so small (around 10^{-4}) that it does not meaningfully affects the optimal LED savings tax rate and can thus be neglected. As a result, we also get in this case that taking multidimensional heterogeneity into account calls for lower tax rates.

D.3 Simulations of Optimal Savings Taxes with Heterogeneous Returns

For the extension to the case with efficiency arbitrage effects, considered in Section 6.3, we now compute the optimal savings tax rates using the formulas derived in Proposition 7, again using the same set of inverse optimum welfare weights derived above.

These results are reported in the bottom two panels of Figure III, which display schedules of LED and SN savings tax rates computed under the assumption that (i) individuals with different income levels differ in their private rates of return, and that (ii) the savings tax is levied in period-2 dollars. We compute the tax schedules that satisfy the equations for the optimal tax conditions in Proposition 7. As in the case of multidimensional heterogeneity, we hold fixed the schedule of marginal social welfare weights $g(z)$ proportional to those which rationalize the status quo income tax in our baseline inverse optimum calculation. Building on the findings of Fagereng et al. (2020), we follow Gerritsen et al. (2020) in assuming that rates of return rise by 1.4% from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return.

Maintaining our assumptions of negligible labor supply income effects and weakly separable preferences, equation (216) simplifies to

$$\hat{g}(x) := g(x) + \frac{\lambda_2}{\lambda_1}\frac{T_2'(s)}{1+pT_2'(s)}\eta_{s|z}(z) \tag{266}$$

for an SN system. To ensure that $\hat{g}(z)$ still integrates to one, the rescaling factor in equation (239) now becomes

$$\kappa = \frac{1-\int_z\left(\frac{\lambda_2}{\lambda_1}\frac{T_2'(s)}{1+pT_2'(s)}\right)\eta_{s|z}(z)dH_z(z)}{\int_z g^0(z)dH_z(z)}. \tag{267}$$

Similarly, equation (217) simplifies to

$$\varphi(x) = -\left(T_1'(x) + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}T_2'(s)\right)\left(\zeta_z^c(x)\frac{x}{1-T_1'(x)}\right)\frac{\partial p}{\partial z}. \tag{268}$$

For an LED system we can replace $T_2'(s)$ with $\tau_s(z)$ in the previous formulas.