Heisenberg Uncertainty Relations

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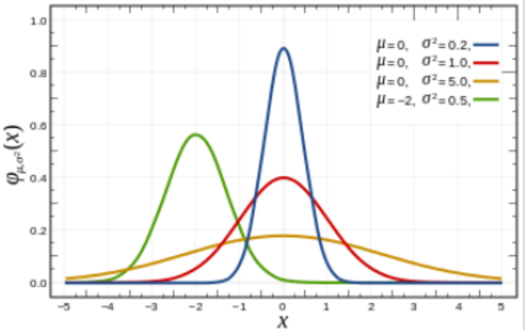
Uncertainty Relations

Section 5.4 of Modern Physics:
A wave packet is an extended object. It has non-zero spatial spread. But it is also reasonably well localized. In spatial coordinates, it is characterized by x and Δx.

If you think of the wave packet as a kind of histogram in spatial coordinate, then x_0 is the average position and Δx is its standard deviation. Δx is a measure of the spatial spread of the wave packet.



Uncertainty Relations

For a Gaussian wave packet

$$\psi(x) \sim \exp\left[-\frac{(x-x_0)^2}{2(\Delta x)^2}\right],$$

then 99.93% of the wave packet is within $x_0 \pm 3\Delta x$.

We expand any well behaved wave packet into an infinite number of sinusoidal waves, each labelled by its wave number k.

$$\psi(x) = \int_{-\infty}^{\infty} dk\,[A(k)\cos(kx) + B(k)\sin(kx)],$$

where the amplitude of the wave with wave number k is given by $\sqrt{A(k)^2 + B(k)^2}$ and its phase by $\tan^{-1}[B(k)/A(k)]$. This expansion is called Fourier Analysis.

From the above equation, we can derive the expressions for $A(k)$ and $B(k)$. They are It turns out to be

$$A(k) = \int_{-\infty}^{\infty} dx\psi(x)\cos(kx), \quad B(k) = \int_{-\infty}^{\infty} dx\psi(x)\sin(kx).$$

Uncertainty Relations

Think of this in the following way: Given any well behaved function $f(x)$, we can obtain a McLaurin's expansion for it as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^n f}{dx^n}|_0.$$

That is we can expand any well behaved function in the form of an infinite power series $\sum_n a_n x^n$. The series converges and $a_n \to 0$ as $n \to \infty$.

Similarly, we can expand any well behaved wave packet in spatial coordinates as an infinite sum of sinusoidal waves.

For a well behaved wave packet, $A(k) \to 0$ as $k \to \pm\infty$. Which means $A(k)$ is a well behaved function in the variable k.

We can think of $A(k)$ also as a histogram in the variable k. It will have an average k_0 and a standard deviation Δk. Δk is a measure of the spread of frequencies which make up the wave packet.

For all wave packets, the following relation holds: $\Delta x \Delta k \sim 1$.

Uncertainty Relations

We can understand the uncertainty relation in the following manner. Suppose we want Δx very small (a very narrow pulse). The oscillations in this narrow range must necessarily be very rapid which means the average value of k is very high.

Using Fourier Analysis one can construct the histogram in the wave number, given the histogram in the spatial coordinate.

This wave number histogram will be very broad for narrow spatial pulse.

Within Fourier Analysis, the spread in spatial coordinates and the spread in wave number are irrevocably linked.

Fourier Analysis is based on sound mathematical principles and is found applicable in a wide range of problems.

Hence we are led to the uncertainty relations:

$$\Delta x \Delta k \sim 1 \text{ and } \Delta t \Delta \omega \sim 1.$$

These relations hold for the two wave superposition discussed earlier.

Heisenberg Uncertainty Relations

All wave packets, including matter wave packets, satisfy the uncertainty relations mentioned in the previous slide.

The value on the right hand side depends on the form of the wave packet. Using a mathematical technique called Calculus of Variations, it can shown that for Gaussian wave packets, the uncertainty is minimum: $\Delta x \Delta k = 1/2$.

Substituting it in the uncertainty relation for the Gaussian wave packet, we obtain the famous Heisenberg Uncertainty Relation

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}.$$

We can obtain a similar energy-time uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

We can get an intuitive understanding the Heisenberg uncertainty relation through the hypothetical Heisenberg Microscope.

Heisenberg Uncertainty Relation

A wave packet, with a spread Δx in the co-ordinate space, necessarily has a spread Δk if it is considered as a function of the wave number k. Wave number and wavelength are related by $\lambda = 2\pi/k$.

The product $\Delta x \Delta k \sim 1$, with actual value depending on the shape of the wave packet. It can be shown that for Gaussian wave packets, the product takes the minimum value $\Delta x \Delta k = 1/2$.

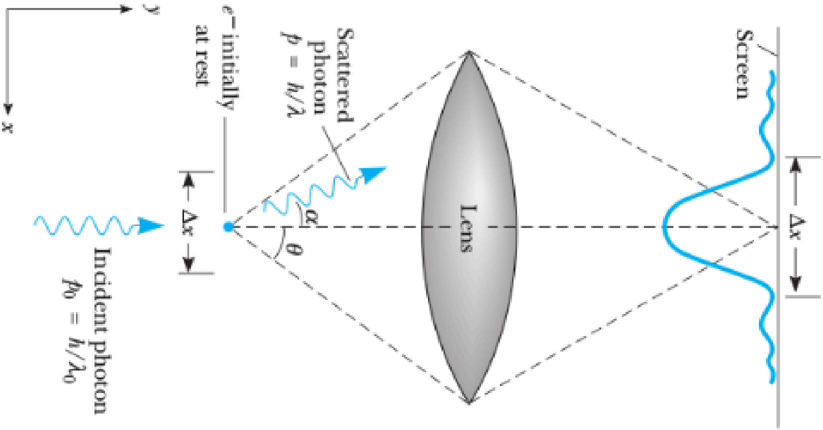
All these statements are true for a packet (or pulse) of any type. These features occur in a large number of classical physics problems, involving wave pulses. $\Delta x \Delta k \geq 1/2$ has nothing to do with any quantum idea.

The quantum idea is de Broglie hypothesis which states $\lambda = h/p$. Combining this with the $\lambda - k$ relation, we obtain $p = hk/2\pi = \hbar k$, where $\hbar = h/2\pi$.

Substituting this in the classical uncertainty relation, we get the Heisenbeg uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$.

Heisenberg Microscope

Thought microscope, imagined by Heisenberg, to illustrate his uncertainty relations.



Heisenberg Microscope

It provides one of the best illustrations of the limitations imposed on the measurement process by the Heisenberg uncertainty relations.

Key point to realize is that measurement process disturbs the object that is being observed. The state of the object changes as a result of the measurement.

In the Heisenberg microscope, let us imagine there is an electron at rest at some point (in contradiction to the uncertainty relation).

But how are we to know about it? We have to make a measurement. We do that by scattering a photon off it.

If this photon is to be collected by the lens, it has to be scattered within at an angle within the range $(-\theta, +\theta)$, as shown in the figure.

Heisenberg Microscope

The scattering imparts a momentum to the electron which is in the range $(+h\sin\theta/\lambda, -h\sin\theta/\lambda)$, where λ is the wavelength of the scattered photon.

The photon, after being collected within the lens aperture, is diffracted to the screen far away. The photon hits (strictly speaking, "most likely hits") the screen within the diffraction central maximum, whose spread is given $\lambda/(2\sin\theta)$.

This spread represents the uncertainty in the measurement of the position of the electron Δx.

The act of measurement created an uncertainty $\Delta p_x = 2h\sin\theta/\lambda$ in the momentum of the electron.

The product of the two uncertainties is

$$\Delta x \Delta p_x = \frac{2h\sin\theta}{\lambda}\frac{\lambda}{2\sin\theta} = h,$$

satisfying the Heisenberg's uncertainty principle.

Heisenberg Uncertainty Principle: Example-1

All of you have seen on TV, "*Hawky-eye*" tracking a cricket ball moving at speed of about 40 m/s or so. Does the cricket ball satisfy Heisenberg uncertainty principle?

Tracking is done using radar, which uses microwaves. To detect an object of size d, must use a probe whose sensitivity is better than d. The sensitivity of an electromagnetic wave is its wavelength.

Lowest energy microwaves have frequency $\sim 100 \ \mathrm{MHz} = 10^8 \ Hz$ implying $\lambda \sim 3$ m. Objects of similar or larger size, such as enemy planes coming to attack, could be detected by them. Radar was invented during World War II, for that purpose.

The size of a cricket ball is a few cm. One needs microwaves with wavelengths of similar size. Assume "*Hawk-eye*" uses a frequency of 100 GHz, which means the wavelength is 3 mm. This will be our Δx. Then HUP predicts

$$\Delta p \geq \hbar/\Delta x = \frac{10^{-34} \ \mathrm{J-s}}{3 \times 10^{-3} \ \mathrm{m}} \implies \Delta v \sim 10^{-31} \ \mathrm{m/s}.$$

Heisenberg Uncertainty Principle: Example-2

A claim is made that a new material is created where the inter-atomic distance is only 0.1 Angstrom or 10^{-11} m.

You want to verify this claim using electron diffraction. To what energies do you want to accelerate the electrons?

We want to measure lengths with an uncertainty less than 10^{-11} m. So we choose $\Delta x = 10^{-12}$ m.

HUP gives $\Delta p > 10^{-34}/10^{-11} = 10^{-23}$ kg-m/s. Let us take p to be similar in magnitude: $p \sim 10^{-23}$. K.E. $=$ $p^2/2m = 10^{-46}/10^{-31} = 10^{-15}\ \mathrm{J} = 10^4\ \mathrm{eV}$.

We need electrons of energy 10 keV (about a factor 100 larger than what we need to probe ordinary solids). If we do neutron diffraction, then we need a kinetic energy of only about 10 eV for the neutrons.

Heisenberg Uncertainty Principle: Example-3

Rutherford scattering experiment established that all the mass and the all the positive charge of the atom is contained in a nucleus. The same experiment allows us estimate the size of the nucleus to be 10,000 times smaller than that of the atom.

An early model of the nucleus was proposed in terms of the two known fundamental particles: proton and electron.

A nucleus of mass number A and atomic number Z contains A protons and $A - Z$ electrons (as opposed to the present picture of Z protons and $A - Z$ neutrons).

In beta decay, electrons are emitted by a nucleus. This provided a further"*proof*" of the above picture.

Can it be true?

Heisenberg Uncertainty Principle: Example-3

This above picture can refuted using Heisenberg Uncertainty Principle.

Size of the nucleus is $\sim 10^{-15}$ m. If we take this to be Δx, then $\Delta p \sim 10^{-34}/10^{-15} = 10^{-19}$ (kg-m/s).

The momentum of the electron has to be at least this much. Calculating $pc = 10^{-19} \times 10^{8} = 10^{-11}$ J, which is about 10^{8} eV, or 100 MeV. This is much larger than the rest mass energy ~ 1 MeV, of the electron.

The electrons emitted in the beta decay have energies of a few MeV. There is nothing in the above picture of nucleus, which can reduce the energy of the electron from about 100 MeV to a few MeV. So the above picture can be ruled out.

Within the present picture of nucleus, we understand beta decay as the decay of the neutron in the nucleus. And the energy of the emitted electron of the order of the mass difference between the mother nucleus and daughter nucleus.

Broadening of Spectral Lines

In the Rydberg formula we write $E_n - E_m = h\nu_{nm}$. E_n and E_m are precisely given by Bohr model, so the formula gives the impression that ν_{nm} is a single precise frequency.

Any of you who has used a spectrograph can tell that what you see is not a single line but a thick line with some width. Not a point but a small line segment.

How do we account for the width of spectral lines? The width we see actually arises due to the thermal energy of the atoms. But there is also broadening of spectral lines due to Heisenberg's uncertainty principle.

Let us consider Hydrogen atom. We can measure the decay time constant for the excited electron to decay from $2p$ state to $1s$ state. It is $\tau \approx 10^{-9}$ sec. This can be taken to be the uncertainty in the time of the decay.

There is a corresponding uncertainty in the energy of the decay given by $\Delta E = \hbar/(2\Delta t)$ and hence an uncertainty in frequency $h\Delta\nu = \hbar/(2\Delta t)$ or $\Delta\nu = 1/(4\pi \times 10^{-9} = 8 \times 10^7$ Hz.

Broadening of Spectral Lines

$\nu_{21} \approx 3 \times 10^{15} Hz$, hence $\Delta\nu/\nu \sim 10^{-8}$. Need an extremely sensitive interferometer to measure this but it has been measured.

However, a much larger broadening occurs due to thermal motion of the atoms and the resultant Doppler shift. If the atom has a speed v, the frequency emitted by it has a Doppler shift $\Delta\nu \approx (v/c)\nu$.

An atom has an average kinetic energy $(3/2)k_B T = (1/2)mv^2$. Hence the average speed of the atom is given $\sqrt{3k_B T/m}$. The spread in the speeds for Maxwell-Boltzmann distribution is roughly equal to the average speed.

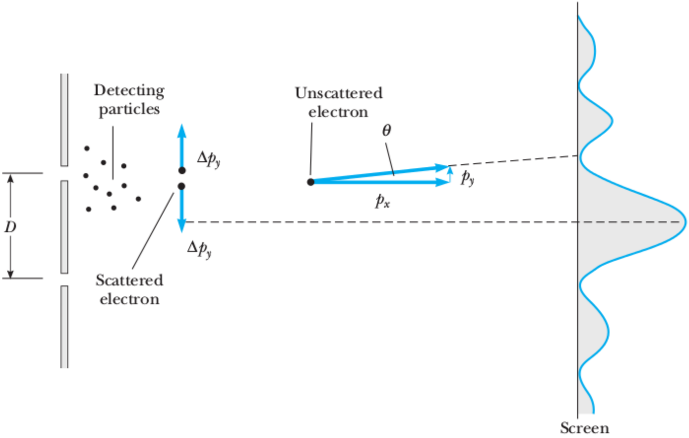
Hence

$$\Delta\nu/\nu = v/c = \sqrt{3k_B T/mc^2} = \sqrt{0.1/10^9} = 10^{-5},$$

where we assumed thermal energy of radiating atoms is about 0.1 eV and the rest mass energy of hydrogen atom is about 10^9 eV.

Doppler effect is the leading cause for the spectral broadening. There are pressure dependent effects also, which lead to larger broadening compared to quantum mechanical broadening.

Heisenberg Uncertainty Principle and Double slit experiment



Heisenberg Uncertainty Principle and Double slit experiment

Let us use the double slit experiment of the previous slide but try to measure through which slit the electron has gone. If we succeed in making this measurement, then $\Delta y \ll D$, where D is the distance between the slits. It is assumed to much larger than the slit width.

In the original interference experiment, where we do not detect electrons passing through the slits, an electron hitting the minimum position has $\tan\theta \approx \theta = p_y/p_x$.

But, in an interference experiment, based on the pathlength difference argument, we must have $\sin\theta \approx \theta = h/(2p_x D)$.

Equating the two we get $p_y \approx h/2D$. Combining this with Δy, we get $\Delta y \Delta p_y \ll h$, in violation of Heisenberg uncertainty relation.

Hence we can argue that in a single double slit experiment it is not possible to have both an interference pattern and knowledge of the slit through which the electron has gone.