MA 105 D1 Lecture 1

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About the course

Sequences

Limits of sequences

Course objectives

Welcome to IIT Bombay.

- To help the students achieve a better and more rigourous understanding of the calculus of one variable.
- To introduce the ideas and theorems in the calculus of several variable.
- To help students achieve a working knowledge of the tools and techinques of the calculus of several variables in view of the applications they are likely to encounter in the future.

For details about the syllabus, tutorials, assignments, quizes, exams and procedures for evaluation please refer to the course booklet. The course booklet can also be found on moodle:
`http://moodle.iitb.ac.in/login/index.php`
Or on my homepage:
`http://www.math.iitb.ac.in/~ravir`

The emphasis of this course will be on the underlying ideas and methods rather than very intricate problem solving involving formal manipulations (of course, there will be plenty of problems - just not many with lots of algebra tricks). The aim is to get you to think about calculus, in particular, and mathematics in general.

Ask questions! There is a good chance that if you don't understand something, many other people also do not understand it.

So, any questions before we start?

Sequences

Definition: A sequence in a set X is a function $a : \mathbb{N} \to X$, that is, a function from the natural numbers to X.

In this course X will usually be a subset of (or equal to) $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$, though we will also have occassion to consider sequences of functions sometimes. In later mathematics courses X may be the complex numbers $\mathbb{C}$ (MA 205), vector spaces (whatever those maybe) the set of continuous functions on an interval $\mathcal{C}([a, b])$ or other sets of functions (MA 106, MA 108, MA 207, MA 214).

Rather than write the value of the function at n as $a(n)$, we often write a_n for the members of the sequence. A sequence is often specified by listing the first few terms

$$a_1, a_2, a_3, \ldots$$

or, more generally by describing the n^{th} term a_n. When we want to talk about the sequence as a whole we sometimes write $\{a_n\}_{n=1}^{\infty}$, but more often we once again just write a_n.

Examples of sequences

1. $a_n = n$ (here we can take $X = \mathbb{N} \subset \mathbb{R}$ if we want, and the sequence is just the identity function. Of course, we can also take $X = \mathbb{R}$).
2. $a_n = 1/n$ (here we can take $X = \mathbb{Q} \subset \mathbb{R}$ if we want, where $\mathbb{Q}$ denotes the rational numbers, or we can take $X = \mathbb{R}$ itself).
3. $a_n = \frac{n!}{n^n}$ ($X = \mathbb{Q}$ or $X = \mathbb{R}$).
4. $a_n = n^{1/n}$ (here the values taken by a_n are irrational numbers, so it best to take $X = \mathbb{R}$).
5. $a_n = \sin\left(\frac{1}{n}\right)$ (again the values taken by a_n are irrational numbers, so it best to take $X = \mathbb{R}$).

These are all examples of sequence of real numbers.

More examples

6. $a_n = \left(n^2, \frac{1}{n}\right)$ (here $X = \mathbb{R}^2$ or $X = \mathbb{Q}^2$).

   This is a sequence in $\mathbb{R}^2$.

7. $f_n(x) = \cos(nx)$ (here X is the set of continuous functions on any interval $[a, b]$ or even on $\mathbb{R}$).

   This is a sequence of functions. More precisely, it is a sequence of continuous functions.

Series

Given a sequence a_n of real numbers, we can manufacture a new sequence, namely its sequence of partial sums:

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k.$$

8. We can take $a_n = r^n$, for some r, i.e., a geometric progression. Then $s_n = \sum_{k=0}^{n} r^k$.
9. $s_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$, or writing it out $s_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$.

   We get a sequence of polynomial functions.

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

From the examples in the previous slide, Example 1 is a monotonically increasing sequence, Example 2 is a monotonically decreasing sequence.
How about Example 3?

In Example 3 we notice that if $a_n = \frac{n!}{n^n}$,

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = a_n \times \frac{(n+1)n^n}{(n+1)^{(n+1)}} \leq a_n,$$

so the sequence is monotonically decreasing.

Eventually monotonic sequences

In Example 4 ($a_n = n^{1/n}$), we note that

$$a_1 = 1 < 2^{1/2} = a_2 < 3^{1/3} = a_3,$$

(raise both a_2 and a_3 to the sixth power to see that $2^3 < 3^2$!). However, $3^1/3 > 4^{1/4} > 5^{1/5}$. So what do you think happens as n gets larger?

In fact, $a_{n+1} \leq a_n$, for all $n \geq 3$. Prove this fact as an exercise. Such a sequence is called an eventually monotonic sequence, that is, the sequence becomes monotonic(ally decreasing) after some stage. One can similarly define eventually monotonically increasing sequences.

Let us quickly run through the other examples. Example 5 - monotonically decreasing. Example 6 - is not a sequence of real numbers. Example 7 - is a sequence of real numbers if we fix a value of x . Can it be monotonic for some x? Example 8 is monotonic for any fixed value of r and so is Example 9 for any non-negative value of x.

Limits: Preliminaries

While all of you are familiar with limits, most of you have probably not worked with a rigourous definition. We will be more interested in limits of functions of a real variable (which is what arise in the differential calculus), but limits of sequences are closely related to the former, and occur in their own right in the theory of Riemann integration.

So what does it mean for a sequence to tend to a limit? Let us look at the sequence $a_n = 1/n^2$. We wish to study the behaviour of this sequence as n gets large. Clearly as n gets larger and larger, $1/n^2$ gets smaller and smaller and seems to approach the value 0, or more precisely

the distance between $1/n^2$ and 0 becomes smaller and smaller.

In fact (and this is the key point), by choosing n large enough, we can make the distance between $1/n^2$ and 0 smaller than any prescribed quantity.

Let us examine the above statement, and then try and quantify it.

More precisely:

The distance between $1/n^2$ and 0 is given by $|1/n^2 - 0| = 1/n^2$.

Suppose I require that $1/n^2$ be less that 0.1 (that is 0.1 is my prescribed quantity). Clearly, $1/n^2 < 1/10$ for all $n > 3$.

Similarly, if I require that $1/n^2$ be less than $0.0001 (= 10^{-4})$, this will be true for all $n > 100$.

We can do this for any number, no matter how small. If $\epsilon > 0$ is any number,

$$1/n^2 < \epsilon \iff 1/\epsilon < n^2 \iff n > 1/\sqrt{\epsilon}.$$

In other words, given any $\epsilon > 0$, we can always find a natural number N (in this case any $N > 1/\sqrt{\epsilon}$) such that for all $n > N$, $|1/n^2 - 0| < \epsilon$.

The rigourous definition of a limit

Motivated by the previous example, we define the limit as follows.

Definition: A sequence a_n tends to a limit l/converges to a limit l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

This is what we mean when we write

$$\lim_{n \to \infty} a_n = l.$$

If we just want to say that the sequence has a limit without specifying what that limit is, we simply say $\{a_n\}_{n=1}^{\infty}$ converges, or that it is convergent.
A sequence that does not converge is said to diverge, or to be divergent.

Remarks on the definition

Remarks

1. Note that the N will (of course) depend on ϵ, as it did in our example, so it would have been more correct to write $N(\epsilon)$ in the definition of the limit. However, we usually omit this extra bit of notation.
2. We have already shown that $\lim_{n\to\infty} 1/n^2 = 0$. The same argument works for $\lim_{n\to\infty} 1/n^\alpha$, for any real $\alpha > 0$. We just take N to be any integer bigger than $1/\epsilon^{1/\alpha}$ for a given ϵ.
3. For a given ϵ, once one N works, any larger N will also work. In order to show that a sequence tends to a limit l we are not obliged to find the best possible N for a given ϵ, just some N that works. Thus, for the sequence $1/n^2$ and $\epsilon = 0.1$, we took $N = 3$, but we can also take $N = 10, 100, 1729$, or any other number bigger than 3.
4. Showing that a sequence converges to a limit l is not easy. One first has to guess the value l and then prove that l satisfies the definition. We will see how to get around this in various ways.

More examples of limits

Let us show that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.

For this we note that for $x \in [0, \pi/2]$, $0 \le \sin x \le x$ (try to remember why this is true).
Hence,

$$|\sin 1/n - 0| = |\sin 1/n| < 1/n.$$

Thus, given any $\epsilon > 0$, if we choose some $N > 1/\epsilon$, $n > N$ implies $1/n < 1/N < \epsilon$. It follows that $|\sin 1/n - 0| < \epsilon$.

Let us consider Exercise 1.1.(ii) of the tutorial sheet. Here we have to show that $\lim_{n\to\infty} 5/(3n+1) = 0$. Once again, we have only to note that

$$\frac{5}{3n+1} < \frac{5}{3n},$$

and if this is to be smaller than ϵ,
we must have $n > N > 5/3\epsilon$.

Formulæ for limits

If a_n and b_n are two convergent sequences then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$

Implicit in the formulæ is the fact that the limits on left hand side exist.

Note that the constant sequence $a_n = c$ has limit c, so as a special case of (2) above we have

$$\lim_{n\to\infty}(c \cdot b_n) = c \cdot \lim_{n\to\infty} b_n.$$

Using the formulæ above we can break down the limits of more complicated sequences into simpler ones and evaluate them.

The Sandwich Theorem(s)

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n.$$

A second version of the theorem is especially useful:
Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Note that we do not assume that b_n converges in this version of the theorem - we get the convergence of b_n for free . Together with the rules for sums, differences, products and quotients, this theorem allows us to handle a large number of more complicated limits.

An example using the theorems above

Consider Exercise 1.2.(iii) on the tutorial sheet. We have to show that

$$\lim_{n\to\infty} \frac{n^3+3n^2+1}{n^4+8n^2+2}$$

exists and to evaluate it.

It is clear that

$$0 < \frac{n^3+3n^2+1}{n^4+8n^2+2} \leq \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}.$$

(How do we get this?)
Note that $n^3/(n^4+8n^2+2) < n^3/n^4 = 1/n$, and the other two terms can be handled similarly.)

Hence, applying the Sandwich Theorem (Theorem 2) to the sequences

$$a_n = 0, \quad b_n = \frac{n^3+3n^2+1}{n^4+8n^2+2} \quad \text{and} \quad c_n = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

we see that the limit we want exists provided $\lim_{n\to\infty} c_n$ exists, so this is what we must concentrate on proving.

The limit $\lim_{n\to\infty} c_n$ exists provided each of the terms appearing in the sum has a limit and in that case it is equal to the sum of the limits (by the first formula). But each of these limits is quite easy to evaluate.

We already know that

$$\lim_{n\to\infty} 1/n = 0 = \lim_{n\to\infty} 1/n^4,$$

while

$$\lim_{n\to\infty} 3/n^2 = 3 \cdot \lim_{n\to\infty} 1/n^2 = 0$$

where we have used the special case of the second formula (limit of the product is the product of the limits) for the first equality in the equation above. Since all three limits converge to 0, it follows the given limit is $0 + 0 + 0 = 0$.