MA 105 D1 Lecture 26

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Gauss' Divergence Theorem

Applications

Gauss

Stokes' Theorem

Added after class: In whatever follows, we will always assume that the complement of a surface S is open.

Theorem 42: Let S be a bounded oriented surface (more precisely, a bounded oriented non-singular $\mathcal{C}^1$ surface) and let $\mathbf{F} : D \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field, for some region D containing S. Assume further that the boundary ∂S of S is the disjoint union of simple closed curves each of which is a piecewise non-singular parametrised curve.
Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

The following special case is often of great interest: If S is a closed surface we see that the right hand side is zero, since there is no boundary.

Exercise 12.4

Exercise 12.4: Compute $\oint_C \mathbf{v} \cdot d\mathbf{r}$ for $\mathbf{v} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$, where C is the circle of unit radius in the xy plane centered at the origin and oriented clockwise. Can the above line integral be computed using Stokes Theorem?

Solution: Using the parametrization $(\cos\theta, -\sin\theta)$, $0 \le \theta < 2\pi)$ (why is there a minus sign?), one has

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_C \frac{-ydx + xdy}{x^2+y^2} = -\int_0^{2\pi} d\theta = -2\pi.$$

We note that the given vector field is not defined on the z-axis. To

apply Stokes' Theorem, one would have to work inside $U = \mathbb{R}^3 \setminus z - \mathrm{axis}$. But there is no surface in $U = \mathbb{R}^3 \setminus z{-}\mathrm{axis}$ of which C is the boundary. Remember that the complement of any such surface must be open.

The symbol $\oint$ is often used instead of just $\int$ for line integrals over closed curves (loops).

Exercise 12.6

Exercise 12.6 Calculate

$$\oint_C ydx + zdy + xdz,$$

where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

Solution: We have

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \quad \text{and} \quad \mathrm{curl}\ \mathbf{F} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Parametrize the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve C as

$$x\mathbf{i} + y\mathbf{j} + \frac{xy}{b}\mathbf{k}, \quad x^2 + y^2 \leq a^2.$$

The solution to Exercise 12.6

Then $\mathbf{n}dS = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

$$\iint_S \mathrm{curl}\ \mathbf{F} \cdot \mathbf{n}dS = \frac{1}{b} \iint_{x^2+y^2 \leq a^2} (y + x - b)dxdy$$

$$= \frac{1}{b} \int_0^{2\pi} \int_0^a (r \sin\theta + r\cos\theta - b)rdrd\theta = -\pi a^2.$$

The Jordan-Brouwer Separation Theorem

Recall that a closed surface S is a compact (bounded and with open complement) surface which has no boundary. It divides $\mathbb{R}^3 \setminus S$ into two components - the bounded component or inside, and the unbounded component or outside. This last fact is not at all easy to prove. It is the analogue of the Jordan curve theorem for $\mathbb{R}^3$.

Denote by S^n the set of points $(x_1, \ldots, x_n n + 1)$ in $\mathbb{R}^{n+1}$ such that $x_1^2 + x_2^2 + \ldots x_{n+1}^2 = 1$. This set is called the unit n-sphere. When $n = 1$ we get the unit circle. When $n = 2$ the sphere in $\mathbb{R}^3$.

Theorem: Let S be homeomorphic to the sphere S^n. Then $\mathbb{R}^{n+1} \setminus S$ consists of exactly two connected components - one bounded and the other unbounded.

The theorem above is known as the Jordan-Brouwer separation theorem. When $n = 1$, it reduces to the Jordan curve theorem.

Closed surfaces and enclosed volumes

We will take for granted that a closed surface S in $\mathbb{R}^3$ has an inside (the bounded component) and outside (the unbounded component). We denote the region enclosed by S and including the points of S as well, by W. Then $\partial W = S$ and S is the boundary of W.

(The boundary of W can be defined as was done for surfaces. However, we give a different (but, of course, equivalent) definition instead. A point P in $\mathbb{R}^3$ is a boundary point of W if every open ball in $\mathbb{R}^3$ containing P intersects both W and W^c, the complement of W.)

Gauss's theorem states that the flux of a vector field out of a closed surface is equal to the divergence of that vector field over the volume enclosed by the surface.

Gauss's divergence theorem

Theorem 44: Let $S = \partial W$ be a closed oriented surface enclosing the region W with the outward normal giving the positive orientation. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined on W. Then

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

Examples

Example 1 (page 446 of Marsden, Tromba and Weinstein): Let $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and let S be the unit sphere. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: Using Gauss' theorem we see that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F}) dV,$$

where W is the unit ball bounded by the sphere. Since $\nabla \cdot \mathbf{F} = 2(1 + y + z)$ we get

$$2\iiint_W (1+y+z) dV = 2\iiint_W dV + 2\iiint_W y dV + 2\iiint_W z dV.$$

Notice that the last two integrals above are 0, by symmetry. Hence, the flux is simply

$$2\iiint_W dV = \frac{8\pi}{3}.$$

Example 2 (page 448 of Marsden, Tromba and Weinstein)
Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

Solution: Again if we use Gauss's theorem we see that we need only evaluate

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iiint_W 3(x^2 + y^2 + z^2) dxdydz,$$

where W is the unit ball.

This problem is clearly ideally suited to the use of spherical coordinates. Making a change of variables, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi \, d\rho d\phi d\theta = \frac{12\pi}{5}$$

Divergence and flux

Just as we were able to relate the circulation and curl using Stokes' theorem, we can relate the flux and the divergence using the divergence theorem.

Theorem 45: Let B_ϵ be the solid ball of radius ϵ centered at a point P in space and let S_ϵ be its boundary sphere. Let $\mathbf{F}$ be a vector field defined in an open set around P. Then

$$\nabla \cdot \mathbf{F}(P) = \lim_{\epsilon \to 0} \frac{1}{V(B_\epsilon)} \iint_{S_\epsilon} \mathbf{F} \cdot d\mathbf{S},$$

where $V(B_\epsilon)$ is the volume of B_ϵ.

Proof: Use the divergence theorem and the mean value theorem for triple integrals. The proof is virtually the same as the one relating the circulation of a vector field to its curl.

Incompressibility

Note that there is nothing special about solid balls - any class of regions for which the divergence theorem holds and whose volumes shrink to 0 will give the same result.

Theorem 45 gives us another way of showing that the divergence at a point P is the net rate per unit volume at which the fluid is flowing outwards at P.

In particular, if the divergence is 0, this means that the net outward flow of the fluid across any surface is zero. This clearly corresponds to our notion of what an incompressible fluid ought to be. Because of the theorem above, incompressible fluids are also called divergence free fluids.

A formula for the divergence in spherical coordinates

As a rather nice application of the preceding circle of ideas we can compute the formula for the divergence in spherical coordinates.

As I just said in the previous slide, we can apply Theorem 45 to any shape enclosed by a surface for which Gauss' theorem holds. In this case, we will apply it to the infinitessimal volume element W in spherical coordinates at a point (ρ, θ, ϕ). The volume of this region is given by $\rho^2 \sin \phi d\rho d\phi d\theta$.

Let us first calculate the net flux through the two faces which are orthogonal to the radial vector. If S_1 is the outer surface and S_2 in the inner surface the net flux is given by

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

We can easily calculate this net flux as follows. We denote by F_ρ the radial component of the vector field $\mathbf{F}$, and similarly define F_ρ and F_θ.

Caution: these are not to be confused with the notation for the partial derivatives of F with respect to ρ, θ and ϕ. With this notation in hand, the net flux through S_1 and S_2 becomes

$$F_\rho(\rho + d\rho, \phi, \theta)(\rho + d\rho)^2 \sin\phi d\phi d\theta - F_\rho(\rho, \phi, \theta)\rho^2 \sin\phi d\phi d\theta.$$

Using the Mean Value Theorem for the function $F_\rho \rho^2$, the above expression yields

$$\approx \frac{\partial}{\partial \rho}(F_\rho(\rho, \phi, \theta)\rho^2 \sin\phi) d\rho d\phi d\theta.$$

Dividing by the volume element we obtain

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho).$$

In exactly the same way we can compute the contribution to the net flux through the surfaces orthogonal to the ϕ and θ directions. These contributions are

$$\frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi}(\sin \phi F_\phi) \quad \text{and} \quad \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta}.$$

Putting these terms together we see that

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi}(\sin \phi F_\phi) + \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta}.$$

Of course, one could have done this computation by brute force directly, but this is perhaps a more elegant way of doing it.

Gauss' Law of Electrostatics

Gauss' Law Let W be a region in $\mathbb{R}^3$ (we will assume that W is enclosed by a closed surface). and suppose that the origin $(0,0,0)$ in not on the boundary ∂W. Then

$$\iint_{\partial W} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \begin{cases} 4\pi, & \text{if } (0,0,0) \in W \\ 0, & \text{if } (0,0,0) \notin W. \end{cases}$$

Proof: We have previously computed the divergence of the vector field $\mathbf{F}/r^3$. Indeed,

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = \frac{1}{r^3} \nabla \cdot \mathbf{r} + \nabla \left(\frac{1}{r^3} \right) \cdot \mathbf{r} = 0,$$

provided $\mathbf{r} \neq (0,0,0)$. Hence, if $(0,0,0) \notin W$, the assertion follows immediately from the divergence theorem:

$$\iint_{\partial W} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iiint_{\partial W} \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) dV = 0.$$

If $(0,0,0) \in W$, we proceed as follows. Take a solid ball centered at $(0,0,0)$ and contained in W. Consider the region $W \setminus B$. Then the origin is not contained in this region. Hence, applying Gauss's theorem to this region we see that

$$\iint_{\partial W} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS - \iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = 0$$

Why is there a minus sign in the left hand side? Because the inner boundary ∂B of $W \setminus B$ is oriented opposite to the outer boundary ∂W. But

$$\iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{\partial B} \frac{1}{r^2} dS = 4\pi.$$

Note that this calculation shows that the flux through a sphere of a vector field satisfying the inverse square law (such as the electric field given by Coulomb's Law) is independent of the radius of the sphere. In particular we can easily recover Gauss' Law in electrostatics: The flux of an electric field out of a surface is equal to the total charge inside.

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Currently, there are four students in the programme (all in the fourth year). Seven have graduated so far. Four switched from EP, one f,rom Chemistry one from EE, three from Mech., and one from Meta.

All seven who have graduated have done/are doing/are about to do a Ph.D.

Our most famous graduate is Kartik Prasanna who was a student of Andrew Wiles and is a professor in the Department of Mathematics at the University of Michigan. Three of our graduates have gone to Princeton, one to Rutgers, one to the University of Illinois (Urbana-Champaign, one to the University of Iowa.

Carl Friedrich Gauss: The prince of mathematicians (1777 C.E. - 1855 C.E.)

(http://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg)

Gauss - the early years

From “Men of Mathematics” by Eric Temple Bell:

Shortly after his seventh birthday Gauss entered his first school, a squalid relic of the Middle Ages run by a virile brute, one Büttner, whose idea of teaching the hundred or so boys in his charge was to thrash them into such a state of terrified stupidity that they forgot their own names. More of the good old days for which sentimental reactionaries long. It was in this hell-hole that Gauss found his fortune.

Nothing extraordinary happened during the first two years. Then, in his tenth year, Gauss was admitted to the class in arithmetic.

As it was the beginning class none of the boys had ever heard of an arithmetic progression. It was easy then for the heroic Büttner to give out a long problem in addition whose answer he could find by a formula in a few seconds.

The problem was of the following sort, $81297 + 81495 + 81693 + \ldots + 100899$, where the step from one number to the next is the same all along (here 198), and a given number of terms (here 100) are to be added.

It was the custom of the school for the boy who first got the answer to lay his slate on the table; the next laid his slate on top of the first, and so on.

Büttner had barely finished stating the problem when Gauss flung his slate on the table:

"There it lies," he said"Ligget se'" in his peasant dialect.

Then, for the ensuing hour, while the other boys toiled, he sat with his hands folded, favored now and then by a sarcastic glance from Büttner, who imagined the youngest pupil in the class was just another blockhead.

At the end of the period Büttner looked over the slates. On Gauss' slate there appeared but a single number.

To the end of his days Gauss loved to tell how the one number he had written down was the correct answer and how all the others were wrong.

Gauss had not been shown the trick for doing such problems rapidly. It is very ordinary once it is known, but for a boy of ten to find it instantaneously by himself is not so ordinary.

What was Gauss' trick for summing an arithmetic progression?

Philology versus mathematics

At seventeen, Gauss had to decide what he wanted to do. He was a talented student of both philology and mathematics.

What is philology? The study of languages.

At this point he made a remarkable mathematical discovery. He was so excited by this discovery that he decided to do mathematics.

What was this discovery? Gauss discovered that you can construct a regular 17-sided polygon using a straight edge (ruler) and compass only.

The four classical Greek problems

The Greek's asked how to solve the following problems using only a straight edge and compass compass.

- Trisecting an angle.
- Constructing a regular heptagon
- Doubling the cube
- Squaring the circle.