MA 105 D1 Lecture 25

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Oriented surface

A surface S is called orientable if there is a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ consisting of unit normal vectors.

If S is an orientable surface and $\mathbf{F}$ is a continuous vector field of unit normal vectors, so is $-\mathbf{F}$.

An orientable surface together with a specific choice of a continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

Definition: A surface on which there exists no continuous vector field consisting of unit normal vectors is called non-orientable.

Homeomorphisms

Recall that a homeomorphism $f : U_1 \to U_2$ from one subset of $\mathbb{R}^n$ to another is a continuous, bijective map such that f^{-1} is also continuous.

The point is that many properties are preserved under homeomorphisms (which may not be preserved under a map which is only bijective and continuous). For instance we have the Invariance of Domain theorem.

Theorem: The space $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $m = n$.

The preceding theorem says that our intuitive notion of dimension can be given mathematical meaning. Incidentally, the previous theorem is closely connected to the Jordan curve theorem (and its generalisation to $\mathbb{R}^n$, the Jordan-Brouwer separation theorem).

Surfaces with boundary

The sphere S^2 clearly does not have a boundary. Neither does the torus. However, the upper hemisphere $z = \sqrt{1 - x^2 + y^2}$ obviously has the unit circle $x^2 + y^2 = 1$ as its boundary. How does one define this more precisely?

Notice that if one is at any point in the sphere, there is a small set around that point that is homeomorphic to an open disc in $\mathbb{R}^2$ (this is just a fancy way of saying that the sphere has no edge). On the other hand, on the hemisphere, if we are at one of the points $(x, \sqrt{1 - x^2}, 0)$, then this is not true any more. This allows us to make precise what we mean by the intuitive notion of boundary.

Definition: Points around which there are no sets in the surface homeomorphic to open discs are called boundary points. The set of all boundary points is called the boundary.

Closed surfaces

A closed surface in $\mathbb{R}^3$ is a surface which is bounded, whose complement is open and which has no boundary points.

For those of you who remember the word, we can define a closed surface as a compact surface that has no boundary.

Examples of closed surfaces are the sphere, the ellipsoid and the torus.

Examples of surfaces that are not closed surfaces include the surface of the paraboloid of revolution, or the one-sheeted hyperboloid (not bounded), the open unit disc in $\mathbb{R}^2$ (the complement is not open!) and the upper hemisphere (has boundary points).

The same definitions go through for higher dimensions. In that case we talk about or, hypersurfaces or, more generally, sub-manifolds without boundary.

Orienting the boundary

Let us assume that we are given an oriented surface S with a boundary that is a simple closed non-singular parametrised curve (or, more generally, a disjoint union of simple closed curves each of who is a piecewise non-singular parametrised curve). Suppose that we have chosen an orientation on S. How is the boundary oriented?

So that the surface lies to the left of an observer walking along the boundary with his head in the direction of the unit normal vector given by the choice of orientation.

Thus, the boundary of an oriented surface automatically acquires an orientation.

Stokes' Theorem

Added after class: In whatever follows, we will always assume that the complement of a surface S is open.

Theorem 42: Let S be a bounded oriented surface (more precisely, a bounded oriented non-singular $\mathcal{C}^1$ surface) and let $\mathbf{F} : D \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field, for some region D containing S. Assume further that the boundary ∂S of S is the disjoint union of simple closed curves each of which is a piecewise non-singular parametrised curve.
Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

The following special case is often of great interest: If S is a closed surface we see that the right hand side is zero, since there is no boundary.

Exercise 12.2

Exercise 12.2: Using Stokes Theorem, evaluate the line integral

$$\oint_C yz\,dx + xz\,dy + xy\,dz$$

where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

Solution: For $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$,

$$\mathrm{curl}\ (\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = 0.$$

Thus the required line integral is

$$\iint_S \mathrm{curl}\ (\mathbf{F}) \cdot \mathbf{n} dS = 0.$$

Exercise 12.4

Exercise 12.4: Compute $\oint_C \mathbf{v} \cdot d\mathbf{r}$ for $\mathbf{v} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$, where C is the circle of unit radius in the xy plane centered at the origin and oriented clockwise. Can the above line integral be computed using Stokes Theorem?

Solution: Using the parametrization $(\cos\theta, -\sin\theta)$, $0 \leq \theta < 2\pi$) (why is there a minus sign?), one has

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_C \frac{-ydx + xdy}{x^2 + y^2} = -\int_0^{2\pi} d\theta = -2\pi.$$

We note that the given vector field is not defined on the z-axis. To

apply Stokes' Theorem, one would have to work inside $U = \mathbb{R}^3 \setminus z-\text{axis}$. But there is no surface in $U = \mathbb{R}^3 \setminus z-\text{axis}$ of which C is the boundary. Remember that the complement of any such surface must be open.

The symbol $\oint$ is often used instead of just $\int$ for line integrals over closed curves (loops).

Exercise 12.6

Exercise 12.6 Calculate

$$\oint_C ydx + zdy + xdz,$$

where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

Solution: We have

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \quad \text{and} \quad \operatorname{curl} \mathbf{F} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Parametrize the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve C as

$$x\mathbf{i} + y\mathbf{j} + \frac{xy}{b}\mathbf{k}, \quad x^2 + y^2 \leq a^2.$$

The solution to Exercise 12.6

Then $\mathbf{n}dS = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

$$\iint_S \mathrm{curl}\ \mathbf{F}\cdot\mathbf{n}dS = \frac{1}{b}\iint_{x^2+y^2\leq a^2}(y+x-b)dxdy$$

$$= \frac{1}{b}\int_0^{2\pi}\int_0^a (r\sin\theta + r\cos\theta - b)rdrd\theta = -\pi a^2.$$

Fluid flow

If S is an oriented surface and $\mathbf{F}$ is the velocity field of a fluid moving in three dimensions, then

$$\iint \mathbf{F} \cdot d\mathbf{S}$$

is the net rate (units of volume/units of time) at which the fluid is crossing the surface in the outward direction (if the value of the integral is negative, this means that the net flow is inward).

Because of this interpretation of the surface integral, it is sometimes called the flux of the vector field.

Example: Find the flux of the vector field $\mathbf{j}$ across the hemisphere H defined by $x^2 + y^2 + z^2 = 1$, $x \geq 0$, oriented in the direction of increasing x.

An example: flow through a hemisphere

Without actually making the computation, we can easily see that the total net flow will be zero. This is because the amount of fluid entering the left half of the hemisphere is equal to the amount exiting the right half of the hemisphere, by symmetry.

We have already calculated the normal to the sphere for this parametrisation by spherical coordinates. It is

$$-(\sin\phi)\,\mathbf{r} = -\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

Hence,

$$-(\sin\phi)\,\mathbf{r}\cdot\mathbf{j} = -\sin^2\phi\sin\theta.$$

Note, that since we are dealing only with a hemisphere, we have $-\pi/2 \le \theta \le \pi/2$. We know that $\mathbf{\Phi}$ is orientation reversing. Hence,

$$\iint_H \mathbf{j}\cdot d\mathbf{S} = \int_0^{\pi}\int_{-\pi/2}^{\pi/2} \sin^2\phi\sin\theta d\theta d\phi = 0.$$

Fourier's Law

Let $T(x, y, z)$ denote the temperature at a point of a region V. The famous law of heat flow due to Fourier says that heat flows from regions of higher temperature to regions of lower temperature. More specifically, the heat flow vector field is proportional to the gradient field ∇T.

We write $\mathbf{F} = -k\nabla T$ for this vector field (why is there a negative sign?). Hence, if S is a surface through which heat is flowing,

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is the total rate of heat flow or flux across S.

Example: Suppose the scalar field $T(x, y, z) = x^2 + y^2 + z^2$ represents the temperaturre function at each point, and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward normal vector. Find the heat flux across the surface if $k = 1$.

Solution: The heat flow field is given by

$$\mathbf{F} = -\nabla T(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

The outward unit normal vector on S is simply given by $\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. We have

$$\mathbf{F} \cdot \hat{\mathbf{n}} = -2x^2 - 2y^2 - 2z^2 = -2$$

as the normal component of $\mathbf{F}$. Now the surface integral is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -2 \iint_S dS = -8\pi.$$

In what direction is the heat flux flowing?

Ampère's Law

We have noted that the line integral can be interpreted as the work done when a particle moves along a path in a force field.

Suppose **H** denotes the magnetic field in $\mathbb{R}^3$ and γ is a closed (oriented) curve in $\mathbb{R}^3$. Then Ampère's Law states that

$$\int_\gamma \mathbf{H} \cdot d\mathbf{s} = I,$$

where I is the current passing through the surface bounded by γ.

If **V** denotes the velocity field of a fluid, the quantity

$$\int_\gamma \mathbf{V} \cdot d\mathbf{s}$$

is an important quantity in fluid mechanics which is called the circulation of $\mathbf{V}$ around γ.

Gauss's Law

The flux of an electric field $\mathbf{E}$ over a "closed" surface is equal to the net charge Q enclosed by the surface. In terms of surface integrals we get

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q.$$

As a special case, let us consider an electric field of the form $\mathbf{E} = E\hat{\mathbf{n}}$, where E is a constant scalar. Then Gauss's law takes the form

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S E dS = Q.$$

It follows that

$$E = \frac{Q}{A(S)}.$$

Coulomb's law

In the particular case that E arises from a point charge Q_1, symmetry assures us that $\mathbf{E} = E\mathbf{n}$, where $\mathbf{n}$ is the unit normal to any sphere centered at Q_1. From this one easily computes the force on a second point charge Q_2 at a distance R from Q_1:

$$\mathbf{F} = \mathbf{E}Q_1 = EQ_1\mathbf{n} = \frac{Q_1 Q_2}{4\pi R^2}\mathbf{n}.$$

This is nothing but Coulomb's law for the force between two point charges.

(You may ask where the constant of proportionality $1/4\pi\epsilon_0$ has gone. I can give two answers: the first that I was working with cgs units where the proportionality is just 1. The other is that I am a mathematician not a physicist and I don't care about constants and always assume that the units are chosen so that the constant is 1.)

Circulation and curl

We said earlier that the curl of the velocity field of a fluid at a given point tells us whether the fluid is rotating around an axis placed at that point perpendicular to the plane of the fluid. We will see that this is an application of Stokes' Theorem.

Let C_ϵ be the circle of radius ϵ centered at a point P_0 and suppose that the enclosed disc D_ϵ has unit normal vector $\mathbf{n}$. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined in a region containing P_0. Then, we will show that

$$[(\nabla \times \mathbf{F})(P_0)] \cdot \mathbf{n} = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

We will need to combine Stokes' theorem with the mean value theorem for area integrals.

Circulation and curl

Let C_ϵ be the circle of radius ϵ centered at a point P_0 and suppose that the enclosed disc D_ϵ has unit normal vector $\mathbf{n}$. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined in a region containing P_0. Then

$$[(\nabla \times \mathbf{F})(P_0)] \cdot \mathbf{n} = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

This shows that the curl can be interpreted as the circulation per unit area in a plane with normal $\mathbf{n}$.

By Stokes' theorem,

$$\iint_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

There is a version of the mean value theorem for surface integrals (recall that the mean value theorem for double integrals says that a continuous function attains its average at some point in the region).

Using the mean value theorem

We can obtain this mean value theorem quite easily (incidentally, the same proof works for the mean value theorem for double integrals as well - after all double integrals are special cases of surface integrals).

If m (respectively M) denotes the minimum (respectively the maximum) of the (scalar) function $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ on D_ϵ, then

$$mA(D_\epsilon) \leq \iint_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \leq MA(D_\epsilon).$$

By the intermediate value theorem (which can be applied since the integrand is continuous), there is point P_ϵ in each disc D_ϵ such that

$$[(\nabla \times \mathbf{F}(P_\epsilon)) \cdot \mathbf{n} A(D_\epsilon) = \iint_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Circulation and curl - the last part of the argument

Using the mean value theorem, we see that

$$[(\nabla \times \mathbf{F}(P_\epsilon)] \cdot \mathbf{n} = \frac{1}{\pi\epsilon^2} \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

If we let $\epsilon \to 0$, we see that $P_\epsilon \to P_0$. Hence,

$$[(\nabla \times \mathbf{F}(P_0)] \cdot \mathbf{n} = \lim_{\epsilon \to 0} \frac{1}{\pi\epsilon^2} \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

Maxwell's equation

Let **E** and **H** be time-dependent electric and magnetic fields, respectively. One of Maxwell's equations is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}.$$

Let S be a surface with boundary C. Define

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \text{voltage drop around C}$$

and

$$\iint_S \mathbf{H} \cdot d\mathbf{S} = \text{magnetic flux across S.}$$

We will show that Faraday's Law can be derived from this equation of Maxwell.

Faraday's Law

Faraday's Law: The voltage (drop) around C equals the negative rate of change of magnetic flux through S.

Using Stokes' theorem

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S}.$$

Now we use Maxwell's equation to obtain

$$= \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \iint_S -\frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{S}$$

The key observation is that we can move the $\frac{\partial}{\partial t}$ across the integral sign. We can do this because the parameter t is independent of the variables dS occurring in the surface integral. This is a very useful trick called "differentiating under the integral sign".

We will not justify this step of differentiating under the integral sign (although we have all the tools necessary to do so). What we get is

$$\iint_S -\frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_S \mathbf{H} \cdot d\mathbf{s}.$$

And this is nothing but Faraday's Law.

Exercise 2: Show that differentiating under the integral sign above is justified.