MA105 end-semester examination

Monday, November 10, 2014 Time: 2 PM to 5 PM.

This is a composite question paper: The questions for all four codes are given together with the answers.

6. (a) True or False: Let $\gamma_n(t) = (\cos nt, \sin nt), 0 \leq t \leq 2\pi$, $n = 1, 2, 3 \ldots$ and $\mathbf{F} = (-y/(x^2 + y^2), x/(x^2 + y^2))$. Then $\int_{\gamma_5} \mathbf{F} \cdot d\mathbf{s} = 2\pi$. **(1 mark)**

   (b) If L is a line segment connecting (x_1, y_1) and (x_2, y_2), compute $\int_L xdy - ydx$ in terms of x_1, y_1, x_2 and y_2. **(1 mark)**

   (c) Use the previous part of this question to find the area of the quadrilateral with vertices $(0,0), (0,1), (1,4)$ (CODE B $(1,3)$, CODE C $(1,2)$, CODE D $(1,5)$) and $(1,0)$. **(2 marks)**

Solution: (a) False.

$$\int_{\gamma_5} \mathbf{F} \cdot d\mathbf{s} = 10\pi. \qquad \textbf{(1 mark)}$$

(b) $\int_L xdy - ydx = x_1y_2 - x_2y_1$ (**1 mark**)

If the sign is wrong above, then only 1/2 a mark was awarded.

(c) **No marks were awarded if Part (b) and Green's Theorem were not used.**

By Green's Theorem the area of the quadrilateral is given by

$$\frac{1}{2} \int_C xdy - ydx$$

where C is the boundary of the quadrilateral traversed in the counterclockwise direction. (**1 mark**). If the direction is taken to be the clockwise direction then only 1/2 a mark was awarded.

Denote the four vertices by (x_i, y_i), $i = 1 \ldots 4$, in the order in which they are given and set $(x_5, y_5) = (x_1, y_1) = (0, 0)$. We let C be the union of the successive line segments joining (x_i, y_i) and (x_{i+1}, y_{i+1}) startiting with $i = 1$ Then, the required area is

CODE A:

$$\frac{1}{2} \sum_{i=1}^{4} [x_i y_{i+1} - x_{i+1} y_i] = 5/2. \qquad \textbf{(1 mark)}.$$

CODE B: 2 (**1 mark**).

CODE C: 3/2 (**1 mark**).

CODE D: 3 (**1 mark**).

7. (a) If f satisfies the equation $\nabla^2 f = 3f$ (CODE B: $2f$, CODE C: f, CODE D: $4f$)and $\iint_D f dS = \pi$, where D is the disc of radius 2, evaluate $\int_{\partial D} f_y dx - f_x dy$, where ∂D is oriented anti-clockwise. **(1 mark)**

   (b) Let K be the surface of a right circular cone with orientation given by the outward unit normal vector. Is the parametrisation $x = u\cos v, y = u\sin v, z = u$ orientation preserving? **(1 mark)**

   (c) Find the area enclosed by one loop of the four leaved rose given by $r = 3\sin 2\theta$, $0 \le \theta \le 2\pi$. **(2 marks)**

Solution: (a) By Green's theorem

CODE A:

$$\int_{\partial D} f_y dx - f_x dy = -\iint_D f_{xx} + f_{yy} dxdy = \iint_D \nabla^2 f dxdy = -3\pi. \qquad \textbf{(1 mark)}$$

CODE B: -2π. **(1 mark)**

CODE C: $-\pi$. **(1 mark)**

CODE D: -4π. **(1 mark)**

No marks were awarded if there were any calculation mistakes (such as the wrong sign).

(b) The given unit normal has negative **k** component since the outward normal points "downward". The normal given by the parametrisation is

$$\Phi_u \times \Phi_v = (\cos v, \sin v, 1) \times (-u\sin v, u\cos v, 0) = (-u\cos v, -u\sin v, u).$$

The natural assumption is that $u \ge 0$. In this case the parametrisation is orientation reversing. **(1 mark)**

In the unlikely event that the student takes u to be negative, we get an inverted cone, so the outward normal points "upward" and has positive **k** component. But in this case the normal given by the parametrisation points "downwards". So, in either case, the parametrisation is orientation reversing.

No marks were awarded without a calculation of the normal. No marks if the normal was calculated wrongly. No marks if the conclusion was wrong even if the normal was calculated correctly.

(c) By Green's Theorem, the area in polar coordinates is given by $\frac{1}{2}\int_C r^2 d\theta$. (**1 mark**).

CODE A: Hence, the required area is

$$\frac{1}{2}\int_0^{\pi/2} (3\sin 2\theta)^2) d\theta = \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta d\theta = 9\pi/8. \qquad \textbf{(1 mark)}$$

For CODE B the curve is $r = 2\sin 2\theta$, CODE C: $r = \sin 2\theta$ and CODE D: $r = 4\sin 2\theta$.

CODE B: $\pi/2$ **(1 mark)**

CODE C: $\pi/8$ **(1 mark)**

CODE D: 2π **(1 mark)**

Some of you have calculated the area of all four leaves (so your limits are incorrect). No marks were awarded in this case. In general, no marks were awarded if there were calculation mistakes.

8. (a) True or False: If $\mathbf{F} \in C^2$ is a conservative field, then $\nabla \cdot \mathbf{F} = 0$. **(1 mark)**

   (b) Let S be the surface given by $z = 6 - (x^2 + y^2)$, $z \geq 2$ and oriented so that the unit normal vector has a positive $\mathbf{k}$ component. If $\mathbf{F} = (x^2 y, z, -2xyz)$, calculate the flux of $\mathbf{F}$ out of S. **(3 marks)**

For CODE B the surface was $z = 6 - (x^2 + y^2)$, $z \geq 1$, CODE C: $z = 6 - (x^2 + y^2)$, $z \geq -10$ and $z = 6 - (x^2 + y^2)$, $z \geq -3$

Solution: (a) False. Take $\mathbf{F} = (x, y, z)$. **1 mark**

No marks were awarded if a counter-example was not provided.

(b) Here the point is that $\nabla \cdot \mathbf{F} = 0$. This allows one to change surface from S to the surface S_1 (described below) which is just a disc. **1 mark**

(**You do not get marks above if you observed that the divergence is zero but then did not apply it properly.** In particular, those who tried to use the divergence theorem and concluded that the integral was identically zero do not get this mark.)

Hence by Gauss' Therorem,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = -\int_{S_1} \mathbf{F} \cdot \mathbf{dS},$$

CODE A where S_1 is the disc $x^2 + y^2 \leq 4$, $z = 2$, with the unit normal to S_1 given by $-\mathbf{k}$. (**1 mark**)

CODE B where S_1 is the disc $x^2 + y^2 \leq 5$, $z = 1$, with the unit normal to S_1 given by $-\mathbf{k}$. (**1 mark**)

CODE C where S_1 is the disc $x^2 + y^2 \leq 4$, $z = -3$, with the unit normal to S_1 given by $-\mathbf{k}$. (**1 mark**)

CODE D where S_1 is the disc $x^2 + y^2 \leq 9$, $z = -3$, with the unit normal to S_1 given by $-\mathbf{k}$. (**1 mark**)

Hence, we get

$$\int_{S_1} 2xy dS = 0. \qquad \textbf{(1 mark)}$$

Some students computed the flux directly by using a parametrisation.

If the integrand $\mathbf{F} \cdot \mathbf{n} dS$ was computed correctly **1/2 mark** was awarded.

For the correct limits of integration **1.5 marks** were awarded.

For an accurate calculation, **1 mark** was awarded.

9 (a) A spherical hot air balloon is expanding so that its diameter increases by 1 m./s. as the air is heated. Determine if the velocity field of the air can have the form $\mathbf{v} = (x^3y, -x^2y^2, -x^2yz)$. **(1 mark)**

(b) Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = (e^{x^2}, y^2, e^{z^2})$ and $\mathbf{c}(t) = (4t/(1+t^3), 4t^2/(1+t^3), 2t), \quad t \in [0, 1]$. **(3 marks)**

Only the curve $\mathbf{c}(t)$ changes in the CODES B, C and D.

Solution: (a) $\nabla \cdot \mathbf{v} \neq 0$. This means that the air is incompressible (neither expanding nor contracting). Hence velocity field cannot have this form. **(1 mark)**. Some students have explicitly invoked the Divergence theorem to conclude that the flux out of the surface of the balloon must be 0. They have been awarded **1 mark**. No marks were given for answers where the volume was differentiated with respect to time and various incorrect statement were made.

(b) This question was not well formulated. Although it is technically correct, the answer involves $\int_0^1 e^{t^2} dt$ or similar integrals which cannot be explicitly evaluated. We apologise for this. Because of this we have been relatively liberal with the correction of this answer. However, students who made completely incorrect statements have not been awarded marks.

The idea behind this question was to observe that $\mathbf{F}$ is a gradient field (since its curl is zero). Hence we can use any other path to evaluate this integral and we can choose a straight line, for instance. Unfortunately, the choice of $\mathbf{F}$ was poor (since one can just integrate each component to show that it is a gradient field rather than compute the curl) and, moreover, as mentioned above, the integral one gets cannot be explicitly evaluated.

For those who only wrote down the line integral as

$$\int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(t))\mathbf{c}'(t)dt :$$

1/2 mark

For those who only wrote down the line integral as

$$\int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(t))\mathbf{c}'(t)dt$$

with the wrong $\mathbf{c}'(t)$: **1/2 mark**

For those who only wrote down the line integral as

$$\int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(t))\mathbf{c}'(t)dt$$

with the correct $\mathbf{c}'(t)$: **1 mark**.

For those who wrote down the line integral as

$$\int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(t))\mathbf{c}'(t)dt$$

with the wrong $\mathbf{c}'(t)$ but the correct final answer : **2 marks**.

For those who wrote down the correct line integral as

$$\int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(t))\mathbf{c}'(t)dt,$$

got the correct $\mathbf{c}'(t)$ and final answers: **3 marks**.

Those who observed that $\nabla \times (\mathbf{F}) = 0$ and directly used Stokes' Theorem: **no marks**. This is because the given $\mathbf{c}(t)$ is not a closed curve and hence, Stokes' Theorem is not applicable to it.

Those who observed that one could change the path because of Stokes' Theorem and used another simpler path to get the correct answer: **3 marks**.

10 (a) True or False: If $f(x, y, z)$ is a C^2 function, and C is any circle in $\mathbf{R}^3$, then $\int_C (\nabla f) \cdot d\mathbf{s} = 0$. **(1 mark)**

(b) Evaluate $\iint_S xz dS$, where S is the boundary of the region given by $x^2 + y^2 = 9$ and the planes $z = 0$ and $y + z = 5$. **(3 marks)**

Solution: (a) True. One can observe that a gradient field is a conservative field and hence, that the integral along any closed path is zero. Alternatively, one can use Stokes Theorem and the curl of the gradient will be identically zero. **(1 mark)**

1/2 a mark has been cut for the failure to mention that the field is conservative.

1/2 a mark has been cut for the failure to mention that the path is closed.

(b) The surface of S involves three separate surfaces: one ellipsoidal region on the plane $y + z = 5$ (the "top" of the cylinder), one cylindrical surface and a disc in the plane $z = 0$. Each of these surface integrals is 0. You are awarded one mark for each correct surface integral, whether it was explicitly calculated or obtained by using symmetry arguments. The symmetry arguments had to be precise in order to get a mark.

For any of the three integrals, if the limits of integration and dS were computed correctly, 1/2 a mark was awarded.

Some students have argued by symmetry that the integral on the whole surface S is 0. They have received the full three marks if their arguments were clear.