MA 105 : Calculus (Autumn 2014)

Solutions to Tutorial Sheets

August 6, 2014

Solutions to Tutorial Sheet 1

(1) For a given $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq n_0$. Select $n_0 \in \mathbb{N}$ (This is possible by the *archimedean property* of $\mathbb{R}$ - but you should not probably not mention this to your students for whom this fact is surely self evident. If someone brings it up in class, you can acknowledge the comment and leave it at that.) such that

(i) $n_0 > \dfrac{10}{\epsilon}$,

(ii) $n_0 > \dfrac{5-\epsilon}{3\epsilon}$,

(iii) $n_0 > \dfrac{1}{\epsilon^3}$ as $\dfrac{1}{n^{\frac{1}{3}}} > \dfrac{n^{\frac{2}{3}}}{n+1} > |a_n|$,

(iv) $n_0 > \dfrac{2}{\epsilon}$ as $\dfrac{2}{n} > \dfrac{1}{n}\left(2 - \dfrac{1}{n+1}\right) = |a_n|$.

(2) (i) $\dfrac{n^2}{n^2+n} \leq a_n \leq \dfrac{n^2}{n^2+1} \Rightarrow \lim\limits_{n\to\infty} a_n = 1.$

(ii) $0 < \dfrac{n!}{n^n} = \left(\dfrac{n-1}{n}\right)\left(\dfrac{n-2}{n}\right)\cdots\left(\dfrac{2}{n}\right)\left(\dfrac{1}{n}\right) \leq \dfrac{1}{n} \Rightarrow \lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0.$

(iii) $0 < \dfrac{n^3+3n^2+1}{n^4+8n^2+2} = \dfrac{(1/n)+(3/n^2)+(1/n^4)}{1+(8/n^2)+(2/n^4)} < \dfrac{4}{n} \Rightarrow \lim\limits_{n\to\infty} a_n = 0.$

(iv) Let $n^{\frac{1}{n}} = 1 + h_n$. Then, for $n \geq 2$, one has

$$n = (1+h_n)^n \geq 1 + nh_n + \binom{n}{2}h_n^2 > \binom{n}{2}h_n^2.$$

Thus $0 < h_n^2 < \frac{2}{n-1}$ $(n \geq 2)$ giving $\lim_{n\to\infty} h_n = 0$. So $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(v) Since

$$0 < \left|\frac{\cos(\pi\sqrt{n})}{n^2}\right| \leq \frac{1}{n^2},$$

it follows that $\lim_{n\to\infty} a_n = 0$.

(vi) $\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} \to \dfrac{1}{2}$ as $n \to \infty$.

(3) (i) $\{\dfrac{n^2}{n+1} = (n-1) + \dfrac{1}{n+1}\}_{n\geq 1}$ is not convergent since $\dfrac{1}{n+1} \to 0$ as $n \to \infty$.

(ii) $\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}\}_{n\geq 1}$ is not convergent since $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

(4) (i) Decreasing as $a_n = 1/\left(n+\frac{1}{n}\right)$ and $\{n+\frac{1}{n}\}_{n\geq 1}$ is increasing.

(ii) Increasing as $\frac{a_{n+1}}{a_n} = \frac{6}{5} > 1$.

(iii) Increasing as $a_{n+1} - a_n = \frac{n(n-1)-1}{n^2(1+n)^2} > 0$ for $n \geq 2$.

(5) (i) By the AM-GM inequality, we see that $a_n \geq \sqrt{2}$ for all $n \geq 2$. Consequently, $a_{n+1} - a_n = \frac{2-a_n^2}{2a_n} \leq 0$ for $n \geq 2$. Thus $\{a_n\}_{n\geq 2}$ is monotonically decreasing and bounded below by $\sqrt{2}$. So $\lim_{n\to\infty} a_n = a$ (say) exists, and $a \geq \sqrt{2}$. Also $a = \frac{1}{2}\left(a+\frac{2}{a}\right)$, i.e., $a^2 = 2$. It follows that $a = \sqrt{2}$.

(ii) By induction, $\sqrt{2} \leq a_n < 2\ \forall n$. Hence $a_{n+1} - a_n = \frac{(2-a_n)(1+a_n)}{\sqrt{2+a_n}+a_n} > 0\ \forall n$. Thus $\lim_{n\to\infty} a_n = a$ (say) exists and arguing as in (i), we find $a = 2$.

(iii) By induction, $2 \leq a_n < 6\ \forall n$. Hence $a_{n+1} - a_n = \frac{6-a_n}{2} > 0\ \forall n$. Thus $\lim_{n\to\infty} a_n = a$ (say) exists and arguing as in (i), we find $a = 6$.

(6) It is clear that $\lim_{n\to\infty} a_{n+1} = L$. The inequality $||a_n| - |L|| \leq |a_n - L|$ implies that $\lim_{n\to\infty} |a_n| = |L|$.

(7) Take $\epsilon = |L|/2$. Then $\epsilon > 0$ and since $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ $\forall n \geq n_0$. Now $||a_n| - |L|| \leq |a_n - L|$ and hence $|a_n| > |L| - \epsilon = |L|/2\ \forall n \geq n_0$.

(8) Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon^2\ \forall n \geq n_0$. Hence $\left|\sqrt{a_n}\right| < \epsilon$ $\forall n \geq n_0$. [Note: For a corresponding result when $a_n \to L$, see, e.g., part (ii) of Propositions 1.9 and 2.4 of [GL-1].]

(9) Both the statements are false. Consider, for example, $a_n = 1$ and $b_n = (-1)^n$.

(10) The implication "$\Rightarrow$" is obvious. For the converse, suppose both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to ℓ. Let $\epsilon > 0$ be given. Choose $n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - \ell| < \epsilon$ for all

$n \geq n_1$ and $|a_{2n+1} - \ell| < \epsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$|a_n - \ell| < \epsilon \text{ for all } n \geq 2n_0 + 1.$$

(11) (i) The statement is **false**. For example, consider $a = -1,\ b = 1,\ c = 0$ and define $f, g : (-1, 1) \to \mathbb{R}$ by

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/x^2 & \text{if } x \neq 0. \end{cases}$$

(ii) The statement is **true** since $|g(x)| \leq M$ for all $x \in (a, b)$ implies that $0 \leq |f(x)g(x)| \leq M|f(x)|$ for all $x \in (a, b)$.

(iii) The statement is **true** since $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$.

(12) Suppose $\lim_{x \to \alpha} f(x) = L$. Then $\lim_{h \to 0} f(\alpha + h) = L$. and since

$$|f(\alpha + h) - f(\alpha - h)| \leq |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

it follows that

$$\lim_{h \to 0} |f(\alpha + h) - f(\alpha - h)| = 0.$$

The converse is **false**; e.g. consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{|x|} & \text{if } x \neq 0. \end{cases}$$

(13) (i) Continuous everywhere except at $x = 0$. To see that f is not continuous at 0, consider the sequences $\{x_n\}_{n \geq 1},\ \{y_n\}_{n \geq 1}$ where

$$x_n := \frac{1}{n\pi} \quad \text{and} \quad y_n := \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Note that both $x_n,\ y_n \to 0$, but $f(x_n) \to 0$ and $f(y_n) \to 1$.

(ii) Continuous everywhere. For ascertaining the continuity of f at $x = 0$, note that $|f(x)| \le |x|$ and $f(0) = 0$.

(iii) Continuous everywhere on $[1, 3]$ except at $x = 2$.

(14) Taking $x = 0 = y$, we get $f(0 + 0) = 2f(0)$ so that $f(0) = 0$. By the assumption of the continuity of f at 0, $\lim_{x\to 0} f(x) = 0$. Thus,

$$\lim_{h\to 0} f(c + h) = \lim_{h\to 0}[f(c) + f(h)] = f(c)$$

showing that f is continuous at $x = c$.

Optional: First verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of f to establish it for all $k \in \mathbb{R}$.

(15) Clearly, f is differentiable for all $x \neq 0$ and the derivative is

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right),\ x \neq 0.$$

Also,

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = 0.$$

Clearly, f' is continuous at any $x \neq 0$. However, $\lim\limits_{x\to 0} f'(x)$ does not exist. Indeed, for any $\delta > 0$, we can choose $n \in \mathbb{N}$ such that $x := 1/n\pi,\ y := 1/(n + 1)\pi$ are in $(-\delta, \delta)$, but $|f'(x) - f'(y)| = 2$.

(16) The inequality

$$0 \le \left|\frac{f(x + h) - f(x)}{h}\right| \le c|h|^{\alpha-1}$$

implies, by the Sandwich Theorem, that

$$\lim_{h\to 0} \left|\frac{f(x + h) - f(x)}{h}\right| = 0 \ \ \forall x \in (a, b).$$

(17) For the first part, observe that

$$\begin{aligned}\lim_{h\to 0+}\frac{f(c+h)-f(c-h)}{2h} &= \lim_{h\to 0+}\frac{1}{2}\left[\frac{f(c+h)-f(c)}{h}+\frac{f(c-h)-f(c)}{-h}\right]\\ &= \frac{1}{2}[f'(c)+f'(c)] = f'(c).\end{aligned}$$

The converse is **false**; consider, for example, $f(x) = |x|$ and $c = 0$.

(18) Since $f(x+y) = f(x)f(y)$, we obtain, in particular, $f(0) = f(0)^2$ and therefore $f(0) = 0$ or 1. If $f(0) = 0$, then

$$f(x+0) = f(x)f(0) \Rightarrow f(x) = 0 \quad \forall x.$$

Thus, trivially, f is differentiable. If $f(0) = 1$, then

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f(c)\left(\lim_{h\to 0}\frac{f(h)-f(0)}{h}\right) = f'(0)f(c).$$

(19) (i) Let $f(x) = \cos(x)$. Then $f'(x) = -\sin(x) \neq 0$ for $x \in (0,\pi)$.

Thus $g(y) = f^{-1}(y) = \cos^{-1}(y), \quad -1 < y < 1$ is differentiable

and

$$g'(y) = \frac{1}{f'(x)}, \text{ where } x \text{ is such that } f(x) = y.$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1-\cos^2(x)}} = \frac{-1}{\sqrt{1-y^2}}.$$

(ii) Note that

$$\operatorname{cosec}^{-1}(x) = \sin^{-1}\frac{1}{x} \text{ for } |x| > 1.$$

Since

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \text{ for } |x| < 1,$$

one has, by the Chain rule,

$$\frac{d}{dx}\operatorname{cosec}^{-1}(x) = \frac{1}{\sqrt{(1-\frac{1}{x^2})}}\left(\frac{-1}{x^2}\right),\ |x|>1.$$

(20) By the Chain rule,

$$\begin{aligned}\frac{dy}{dx} &= f'\left(\frac{2x-1}{x+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x+1}\right)\\ &= \sin\left(\frac{2x-1}{x+1}\right)^2\left[\frac{3}{(x+1)^2}\right] = \frac{3}{(x+1)^2}\sin\left(\frac{2x-1}{x+1}\right)^2.\end{aligned}$$

Optional exercises

(1) Note that $f(n) = nf(1)$ for all $n \in \mathbb{N}$. Show that $f(r) = rf(1)$, for every rational number r. By continuity $f(\lambda) = \lambda f(1)$ for all real λ. If we set $f(1) = k$, we are done.

(2) This can be easily done inductively, since every derivative of f satisfies the same property as f.

(3) Consider $f(x) := |x| + |1-x|$ for $x \in \mathbb{R}$.

(4) For $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to c. Then $\{f(a_n)\}_{n\geq 1}$ converges to 1 while $\{f(b_n)\}_{n\geq 1}$ converges to 0, showing that limit of f at c does not exist.

(5) Suppose $c \neq 1/2$. If $\{a_n\}_{n\geq 1}$ is a sequence of rational numbers and $\{b_n\}_{n\geq 1}$ a sequence of irrational numbers, both converging to c, then $g(a_n) = a_n \to c$, while $g(b_n) = 1 - b_n \to 1 - c$, and $c \neq 1 - c$. Thus g is not continuous at any $c \neq 1/2$. Further, if $\{a_n\}_{n\geq 1}$ is any sequence converging to $c = 1/2$, then $g(a_n) \to 1/2 = g(1/2)$. Hence, g is continuous at $c = 1/2$.

(6) Let $L = \lim_{x\to c} f(x)$. Take $\epsilon = L - \alpha$. Then $\epsilon > 0$ and so there exists a $\delta > 0$ such that

$$|f(c+h) - L| < \epsilon \text{ for } \quad 0 < |h| < \delta.$$

Consequently, $f(c+h) > L - \epsilon = \alpha$ for $0 < |h| < \delta$.

(7) (i) $\Rightarrow$ (ii): Choose $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a,b)$. Take $\alpha = f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h}, & \text{if } h \neq 0 \\ 0, & \text{if } h = 0. \end{cases}$$

(ii) $\Rightarrow$ (iii): $\displaystyle\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h\to 0} |\epsilon_1(h)| = 0$

(iii) $\Rightarrow$ (i): $\displaystyle\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \Rightarrow \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists and is equal to α.

(8) $f(0) = 0$, $f^{'}(0) = 1$ and $f'(x) = 1 + x^2$

(9) Follows easily from the definitions.

(10) Apply the Intermediate Value Theorem to the function $f(x) - x$.

Solutions to Tutorial Sheet 2

(1) $f(x) = x^3 - 6x + 3$ has stationary points at $x = \pm\sqrt{2}$.

Note that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0, f(+\sqrt{2}) = -4\sqrt{2} + 3 < 0$. Therefore f has a root in $(-\sqrt{2}, \sqrt{2})$. Also, $f \to -\infty$ as $x \to -\infty$ implying that f has a root in $(-\infty, -\sqrt{2})$. Similarly, $f \to +\infty$ as $x \to +\infty$ implying that f has a root in $(\sqrt{2}, \infty)$. Since f has at most three roots, all its root are real.

(2) For $f(x) = x^3 + px + q,\ \ p > 0,\ \ f'(x) = 3x^2 + p > 0$. Therefore f is strictly increasing and can have **at most one** real root. Since

$$\lim_{x \to \pm\infty} \left(\frac{p}{x^2} + \frac{q}{x^3} \right) = 0,$$

$$\frac{f(x)}{x^3} = 1 + \frac{p}{x^2} + \frac{q}{x^3} > 0$$

for $|x|$ very large. Thus $f(x) > 0$ if x is large positive and $f(x) < 0$ if x is large negative. By the Intermediate Value Property (IVP) f must have **at least one** real root.

(3) By the IVP, there exists **at least one** $x_0 \in (a, b)$ such that $f(x_0) = 0$. If there were another $y_0 \in (a, b)$ such that $f(y_0) = 0$, then by Rolle's theorem there would exist some c between x_0 and y_0 (and hence between a and b) with $f'(c) = 0$, leading to a contradiction.

(4) Since f has 3 distinct roots, say, $r_1 < r_2 < r_3$, by Rolle's theorem $f'(x)$ has **at least two** real roots, say, x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$. Since $f'(x) = 3x^2 + p$, this implies that $p < 0$, and $x_1 = -\sqrt{-p/3}$, $x_2 = \sqrt{-p/3}$. Now, $f''(x_1) = 6x_1 < 0 \implies f$ has a local maximum at $x = x_1$. Similarly, f has a local minimum at $x = x_2$. Since the quadratic $f'(x)$ is

negative between its roots x_1 and x_2 (so that f is decreasing over $[x_1, x_2]$) and f has a root r_2 in (x_1, x_2), we must have $f(x_1) > 0$ and $f(x_2) < 0$. Further,

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}},\ f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

so that

$$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0.$$

(5) For some c between a and b, one has

$$\left|\frac{\sin(a) - \sin(b)}{a - b}\right| = |\cos(c)| \leq 1.$$

(6) By Lagrange's Mean Value Theorem (MVT) there exists $c_1 \in \left(a, \frac{(a+b)}{2}\right)$ such that

$$\frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} = f'(c_1)$$

and there exists $c_2 \in \left(\frac{a+b}{2}, b\right)$ such that

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)} = f'(c_2).$$

Clearly one has $c_1 < c_2$, and adding the above equations one obtains

$$f'(c_1) + f'(c_2) = \frac{f(b) - f(a)}{\left(\frac{b-a}{2}\right)} = 2 \text{ (as } f(b) = b, f(a) = a).$$

(7) By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

$$f(0) - f(-a) = f'(c_1)a \text{ and } f(a) - f(0) = f'(c_2)a.$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.

Optional: Consider $g(x) = f(x) - x$, $x \in [-a, a]$. Since $g'(x) = f'(x) - 1 \leq 0$, g is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$.

(8) (i) No such function exists in view of Rolle's theorem.

(ii) $f(x) = \frac{x^2}{2} + x$

(iii) $f'' \geq 0 \Rightarrow f'$ increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence f with the required properties cannot exist.

(iv)

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if} \quad x \leq 0 \\ 1 + x + x^2 & \text{if} \quad x > 0. \end{cases}$$

(9) The points to check are the end points $x = -2$ and $x = 5$, the point of non-differentiability $x = 0$, and the stationary point $x = 2$. The values of f at these points are given by

$$f(-2) = f(2) = 13, f(0) = 1, f(5) = -14.$$

Thus, global max $= 13$ at $x = \pm 2$, and global min $= -14$ at $x = 5$.

(10) To be done in the tutorial. Asymptotes have not been discussed in class, but this curve doesn't have any, so you can skip that part.

(i) $f(x) = 2x^3 + 2x^2 - 2x - 1 \Rightarrow f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$.

Thus, $f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$ so that $f(x)$ is strictly increasing in those intervals, and $f'(x) < 0$ in $(-1, 1/3)$ so that $f(x)$ is strictly decreasing in that in

Thus, $f(x)$ has a local maximum at $x = -1$, and a local minimum at$x =$ $\frac{1}{3}$. As $f''(x) = 12x + 4$ we have that $f(x)$ is convex in $\left(-\frac{1}{3}, \infty\right)$ and concave in $\left(-\infty, -\frac{1}{3}\right.$ with a point of inflection at $x = -\dfrac{1}{3}$.

(ii) $f(x) = 1 + 12|x| - 3x^2$; f is not differentiable at $x = 0$; $f(0) = 1$. Further, $f'(x) = 0$ at $x = \pm 2$, $f'(x) < 0$ in $(-2, 0) \cup (2, 5]$, $f'(x) > 0$ in $(0, 2)$, and

$f''(x) = -6$ in $(-2,0) \cup (0,5)$. Thus f is concave in$(-2,0) \cup (0,5)$, decreasing in $(-2,0) \cup (2,5)$, and increasing in $(0,2)$; further, f has an absolute maximum at $x = \pm 2$.

(11) To be done in the tutorial.

(12) (i) x^2, (ii) $\sqrt{x}$, (iii) $\frac{1}{x}$, (iv) $\sin x$

(13) (i) True (easy).

(ii) False. Take $f(x) = g(x) = x^3$ at the point $x = 0$.

(14) This exercise involves asymptotes so you will have to introduce this idea. $y = \dfrac{x^2}{x^2+1} \quad \Rightarrow \lim\limits_{x\to\pm\infty} y = 1$
$y' = \dfrac{2x}{(x^2+1)^2} \Rightarrow y$ is increasing in $(0,\infty)$ and decreasing in $(-\infty,0)$.
Further, $y'' = -\frac{2(3x^2-1)}{(x^2+1)^3}$ implies that $y'' > 0$ if $|x| < \frac{1}{\sqrt{3}}$, and $y'' < 0$ if $|x| > \frac{1}{\sqrt{3}}$.

Therefore,

$$y \text{ is convex in } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and concave in } \mathbb{R} \setminus \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

with the points $x = \pm\dfrac{1}{\sqrt{3}}$ being the points of inflection.

Solutions to Tutorial Sheet 3

(1) The remainder terms are easy in these cases. The series are given by

(i)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} \cdot x^{2k}.$$

(ii)

$$\arctan x = x - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad \text{for } |x| < 1.$$

(2) The Taylor series is just $(x-1)^3$

(3) The Taylor series is simply

$$1729x^{1729} + 1728x^{1728} + 28x^{28} + 6x^6 + 1729.$$

Indeed, for any polynomial, the Taylor series about the point 0 just gives you the polynomial back.

(4) Let us denote the partial sums of of the given series by $s_m(x)$. We would like to show that $|s_m(x) - s_n(x)|$ can be made arbitrarily small whenever m and n are sufficiently large. Let us assume that $m > n$. We see that

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \ldots \frac{1}{2^{m-n}} \right) \leq 2 \cdot \frac{|x^n|}{n!}.$$

If N is made sufficiently large and $n > N$, the last expression can be made as small as we please.

(5) We simply integrate term by term to get

$$\log x + x + \frac{x^2}{2 \cdot 2!} + \ldots = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

(6) The solution will be provided later.

(7) Follow the hints given in the question.

(8)

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad \text{for } |x| < \frac{\pi}{2}.$$

Here B_{2n} are the *Bernoulli numbers* defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!},$$

that is, the Bernoulli numbers B_m are the numbers that appear in the Taylor series expansion for $\frac{t}{e^t-1}$.

(9) The key to the solution is the function

$$h(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Recall that this is a smooth function for which $h^{(n)}(0) = 0$ for all n. Let us construct a smooth function that is 0 outside of $[c, d]$ and 1 on $[a, b] \subset [c, d]$, for any such pairs of intervals. Let $f_1(x) = h(x - c)$ and $f_2(x) = h(a - x)$. Then

$$f(x) = \frac{f_1}{f_1 + f_2}$$

has the property that f is identically 0 to the left of c and identically 1 to the right of a. Similarly, let $g_1(x) = h(d - x)$ and $g_2(x) = h(x - b)$. Then $g = g_1/g_1 + g_2$ is identically 0 to the right of d and identically 1 to the left of b. Then $k = fg$ is the desired function.

(10) Follow the hints given in the question question.

Solutions to Tutorial Sheet 4

(1) The given fumction is integrable as it is monotone. Let P_n be the partition of $[0,2]$ into 2×2^n equal parts. Then $U(P_n, f) = 3$ and

$$L(P_n, f) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{(2^n - 1)}{2^n} \to 3$$

as $n \to \infty$. Thus, $\int_0^2 f(x)dx = 3$.

(2) (a) $f(x) \geq 0 \Rightarrow U(P,f) \geq 0,\ L(P,f) \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0$.

Suppose, moreover, f is continuous and $\int_a^b f(x)dx = 0$. Assume $f(c) > 0$ for some c in $[a,b]$. Then $f(x) > \frac{f(c)}{2}$ in a δ-nbhd of c for some $\delta > 0$. This implies that

$$U(P,f) > \delta \times \frac{f(c)}{2}$$

for any partition P, and hence, $\int_a^b f(x)dx \geq \delta f(c)/2 > 0$, a contradiction.

(b) on $[0,1]$ take $f(x) = 0$ for all $x \neq 0$ and $f(0) = 0$.

(3) (i) $S_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{\frac{3}{2}} \longrightarrow \int_0^1 (x)^{3/2}dx = \frac{2}{5}$

(ii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^2 + 1} \longrightarrow \int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}$

(iii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{\frac{i}{n}+1}} \longrightarrow \int_0^1 \frac{dx}{\sqrt{x+1}} = 2(\sqrt{2}-1)$

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{i\pi}{n} \longrightarrow \int_0^1 \cos \pi x dx = 0$

(v) $S_n \longrightarrow \int_0^1 x dx + \int_1^2 x^{3/2}dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2}-1) + \frac{19}{3}$

(4) Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

By the Chain Rule one has

$$\begin{aligned}\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x)\\ &= f(v(x))v'(x) - f(u(x))u'(x).\end{aligned}$$

(a) $\frac{dy}{dx} = \frac{1}{dx/dy} = \sqrt{1+y^2}$, $\frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} = y$.

(b) (i) $F'(x) = \cos((2x)^2)2 = 2\cos(4x^2)$.

(ii) $F'(x) = \cos(x^2)2x = 2x\cos(x^2)$

(5) Define

$$F(x) = \int_x^{x+p} f(t)dt,\ x \in \mathbb{R}.$$

Then $F'(x) = 0$ for every x.

(6) Write $\sin\lambda(x-t)$ as $\sin(\lambda x)\cos(\lambda t) - \cos(\lambda x)\sin(\lambda t)$ in the integrand, take trems in x outside the integral, evaluate $g'(x), g''(x)$, and simplify to show LHS=RHS; from the expressions for $g(x)$ and $g'(x)$ it should be clear that $g(0) = g'(0) = 0$.

The problem could also be solved by appealing to the following theorem:

Theorem A. Let $h(t,x)$ and $\frac{\partial h}{\partial x}(t,x)$ be continuous functions of t and x on the rectangle $[a,b]\times[c,d]$. Let $u(x)$ and $v(x)$ be differentiable functions of x on $[c,d]$ such that, for each x in $[c,d]$, the points $(u(x),x)$ and $(v(x),x)$ belong to $[a,b]\times[c,d]$. Then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t,x)dt = \int_{u(x)}^{v(x)} \frac{\partial h}{\partial x}(t,x)dt - u'(x)h(u(x),x) + v'(x)h(v(x),x).$$

Consider now

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Let $h(t,x) = \frac{1}{\lambda} f(t) \sin \lambda(x-t)$, $u(x) = 0$, and $v(x) = x$. Then it follows from Theorem A that

$$g'(x) = \int_0^x f(t) \cos \lambda(x-t).$$

Again applying Theorem A, we have

$$g''(x) = -\lambda \int_0^x f(t) \sin \lambda(x-t) + f(x).$$

Thus

$$g''(x) + \lambda^2 g(x) = f(x).$$

That $g(0) = g'(0) = 0$ is obvious from the expressions for $g(x)$ and $g'(x)$.

(7) (i) $\int_0^1 y\ dx = \int_0^1 (1 + x - 2\sqrt{x})\ dx = \frac{1}{6}$

(ii) $2\int_0^2 (2x^2 - (x^4 - 2x^2))\ dx = 2\int_0^2 (4x^2 - x^4)\ dx = \frac{128}{15}$

(iii) $\int_1^3 (3y - y^2 - (3-y))\ dy = \int_1^3 (4y - y^2 - 3)\ dy = \frac{4}{3}$

(8) $\int_0^{1-a} (x - x^2 - ax)\ dx = \int_0^{1-a} ((1-a)x - x^2)\ dx = 4.5$ gives $\frac{(1-a)^3}{6} = 4.5$ so that $a = -2$.

(9) Required area $= 2 \times \int_0^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2) d\theta = 4a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1) d\theta = 4\pi a^2$.

(10) (i) Length $= \displaystyle\int_0^{2\pi} \sqrt{(1-\cos(t))^2 + \sin^2(t)} dt = \int_0^{2\pi} 2|\sin(t/2)| dt = 4\int_0^{\pi} |\sin(u)| du = 8$.

(ii) Length $= \displaystyle\int_0^{\pi/4} \sqrt{1 + y'^2} dx = \int_0^{\pi/4} \sqrt{1 + \cos(2x)} dx = \sqrt{2}\int_0^{\pi/4} |\cos(x)| dx = 1$.

(11) $\dfrac{dy}{dx} = x^2 + \left(-\dfrac{1}{4x^2}\right)$.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = x^2 + \frac{1}{4x^2}.$$

Therefore,

$$\text{Length} = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

The surface area is

$$S = \int_1^3 2\pi(y+1)\frac{ds}{dx}dx = \int_1^3 2\pi\left(\frac{x^3}{3}+\frac{1}{4x}+1\right)\left(x^2+\frac{1}{4x^2}\right)dx$$

$$= 2\pi\left[\frac{x^6}{18}+\frac{x^3}{3}+\frac{x^2}{6}-\frac{1}{32x^2}-\frac{1}{4x}\right]_1^3.$$

(12) The diameter of the circle at a point x is given by

$$(8-x^2)-x^2, \;\; -2 \le x \le 2.$$

So the area of the cross-section at x is $A(x) = \pi(4-x^2)^2$. Thus

$$\text{Volume } = \int_{-2}^{2} \pi(4-x^2)^2 dx = 2\pi\int_0^2 (4-x^2)^2 dx = \frac{512\pi}{15}.$$

(13) In the first octant, the sections perpendicular to the y-axis are squares with

$$0 \le x \le \sqrt{a^2-y^2}, \;\; 0 \le z \le \sqrt{a^2-y^2}, \;\; 0 \le y \le a.$$

Since the squares have sides of length $\sqrt{a^2-y^2}$, the area of the cross-section at y is $A(y) = 4(a^2-y^2)$. Thus the required volume is

$$\int_{-a}^{a} A(y)dy = 8\int_0^a (a^2-y^2)dy = \frac{16a^3}{3}.$$

(14) Let the line be along z-axis, $0 \le z \le h$. For any fixed z, the section is a square of area r^2. Hence the required volume is $\int_0^h r^2 dz = r^2 h$.

(15) **Washer Method**

Area of washer $= \pi(1+y)^2 = \pi(1+(3-x^2))^2 = \pi(4-x^2)^2$ so that

Volume $= \int_{-2}^{2} \pi(4-x^2)^2 dx = 512\pi/15$.

(This is the same integral as in (6) above).

Shell method

Area of shell $= 2\pi(y-(-1))2x = 4\pi(1+y)\sqrt{3-y}$ so that

$$\text{Volume} = \int_{-1}^{3} 4\pi(1+y)\sqrt{3-y}dy = 512\pi/15.$$

(14) **Washer Method**

Required volume = Volume of the sphere -Volume generated by revolving the shaded region around the y-axis $= 32\pi/3-[\int_{-1}^{1}\pi x^2 dy-\pi(\sqrt{3})^2 2] = 32\pi/3-2\pi[\int_{0}^{1}(4-y^2)dy-3] = 32\pi/3-2\pi[11/3-3] = 28\pi/3$

Shell Method

Required volume = Volume of the sphere -Volume generated by revolving the shaded region around the y-axis $= 32\pi/3-\int_{\sqrt{3}}^{2} 2\pi x(2y)dx = 32\pi/3-4\pi\int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx = 32\pi/3-4\pi(1/3) = 28\pi/3$

Solutions to Tutorial Sheet 5

(1) (i) $\{(x,y) \in \mathbb{R}^2 \mid x \neq \pm y\}$

(ii) $\mathbb{R}^2 - \{(0,0)\}$

(2) (i) A level curve corresponding to any of the given values of c is the straight line $x - y = c$ in the xy-plane. A contour line corresponding to any of the given values of c is the same line shifted to the plane $z = c$ in $\mathbb{R}^3$.

(ii) Level curves do not exist for $c = -3, -2, -1$. The level curve corresponding to $c = 0$ is the point $(0,0)$. The level curves corresponding to $c = 1, 2, 3, 4$ are concentric circles centered at the origin in the xy-plane. Contour lines corresponding to $c = 1, 2, 3, 4$ are the cross-sections in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$, i.e., circles in the plane $z = c$ centered at $(0, 0, c)$.

(iii) For $c = -3, -2, -1$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the second and fourth quadrant. For $c = 1, 2, 3, 4$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the first and third quadrant. For $c = 0$, the corresponding level curve (resp. the contour line) is the union of the x-axis and the y-axis in the xy-plane (resp. in the xyz-space). A contour line corresponding to a non-zero c is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$, i.e., a rectangular hyperbola in the plane $z = c$.

(3) (i) Discontinuous at $(0,0)$. (Check $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using $y = mx^3$).

(ii) Continuous at $(0,0)$:

$$\left|xy\frac{x^2-y^2}{x^2+y^2}\right| \leq |xy|\frac{x^2+y^2}{x^2+y^2} = |xy|.$$

(iii) Continuous at $(0,0)$:

$$|f(x,y)| \leq 2(|x|+|y|) \leq 4\sqrt{x^2+y^2}.$$

(4) (i) Use the sequential definition of limit: $(x_n, y_n) \to (a,b) \implies x_n \to a$ and $y_n \to b \implies f(x_n) \to f(a)$ and $g(y_n) \to g(b) \implies f(x_n) \pm g(y_n) \to f(a) \pm g(b)$ by the continuity of f, g and limit theorems for sequences.

(ii) $(x_n, y_n) \to (a,b) \implies x_n \to a$ and $y_n \to b \implies f(x_n) \to f(a)$ and $g(y_n) \to g(b) \implies f(x_n)g(y_n) \to f(a)g(b)$ by the continuity of f, g and limit theorems for sequences.

(iii) Follows from (i) above and the following:

$$\min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \frac{|f(x)-g(y)|}{2},$$
$$\max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \frac{|f(x)-g(y)|}{2}.$$

(5) Note that limits are different along different paths: $f(x,x) = 1$ for every x and $f(x,0) = 0$.

(6) (i) $f_x(0,0) = 0 = f_y(0,0)$.

(ii)

$$f_x(0,0) = \lim_{h\to 0}\frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0}\frac{\sin^2(h)}{h|h|}$$

does not exist (Left Limit $\neq$ Right Limit). Similarly, $f_y(0,0)$ does not exist.

(7) $|f(x,y)| \leq x^2+y^2 \Rightarrow f$ is continuous at$(0,0)$.

It is easily checked that $f_x(0,0) = f_y(0,0) = 0$.

Now,

$$f_x = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,0)$, while $\dfrac{2x}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)$ is unbounded in any such disc.
(To see this, consider $(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for n a large positive integer.)

Thus f_x is unbounded in any disc around $(0,0)$.

(8) $f_x(0,0) = \lim_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\sin\dfrac{1}{h}$ does not exist. Similarly $f_y(0,0)$ does not exist. Clearly, f is continuous at $(0,0)$.

(9) (i) Let $\vec{v} = (a,b)$ be any unit vector in $\mathbb{R}^2$. We have

$$\left(D_{\vec{v}}f\right)(0,0) = \lim_{h\to 0}\frac{f(h\vec{v})}{h} = \lim_{h\to 0}\frac{f(ha,hb)}{h} = \lim_{h\to 0}\frac{h^2ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$

Therefore $\left(D_{\vec{v}}f\right)(0,0)$ exists and equals 0 for every unity vector $\vec{v}\in\mathbb{R}^2$.

For considering differentiability, note that $f_x(0,0) = \left(D_{\hat{i}}f\right)(0,0) = 0 = f_y(0,0) = \left(D_{\hat{j}}f\right)(0,0)$. We have then

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} = 0$$

since

$$0 \le \frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} \le \frac{|hk|}{\sqrt{h^2+k^2}}\frac{h^2+k^2}{h^2+k^2} \le \frac{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}.$$

Thus f is differentiable at $(0,0)$.

(ii) Note that, for any unit vector $\vec{v} = (a,b)$ in $\mathbb{R}^2$, we have

$$D_{\vec{v}}f(0,0) = \lim_{h\to 0}\frac{h^3a^3}{h(h^2(a^2+b^2))} = \lim_{h\to 0}\frac{a^3}{(a^2+b^2)} = \frac{a^3}{(a^2+b^2)}.$$

To consider differentiability, note that $f_x(0,0) = 1,\ f_y(0,0) = 0$ and

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-h\times 1-k\times 0|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|h^3/(h^2+k^2)-h|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk^2|}{(h^2+k^2)^{3/2}}$$

does not exist (consider, for example, $k = mh$). Hence f is not differentiable at $(0,0)$.

(iii) For any unit vector $\vec{v} \in \mathbb{R}^2$, one has

$$(D_{\vec{v}} f)(0,0) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left[\frac{1}{h^2(a^2+b^2)}\right]}{h} = 0.$$

Also,

$$\lim_{(h,k)\to(0,0)} \frac{\left|(h^2+k^2)\sin\left[\frac{1}{(h^2+k^2)}\right]\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0;$$

therefore f is differentiable at $(0,0)$.

(10) $f(0,0) = 0, \;\; |f(x,y)| \le \sqrt{x^2+y^2} \implies f$ is continuous at $(0,0)$.

Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$.

For $\vec{v} = (a,b)$, with $b \neq 0$, one has

$$(D_{\vec{v}})\, f(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} = \frac{(\sqrt{a^2+b^2})b}{|b|}.$$

If $\vec{v} = (a,0)$, then $(D_{\vec{v}} f)(0,0) = 0$. Hence $(D_{\vec{v}} f)(0,0)$ exists for every unit vector $\vec{v} \in \mathbb{R}^2$. Further,

$$f_x(0,0) = 0, \;\; f_y(0,0) = 1,$$

and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - 0 - h\times 0 - k\times 1|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

does not exist (consider, for example, $k = mh$) so that f is not differentiable at $(0,0)$.