de-Broglie Hypothesis and Wave Properties of Matter

S Uma Sankar

Department of Physics
Indian Institute of Technology Bombay
Mumbai, India

A Bit of Special Relativity

In special relativity, we need to change our notions about mass, momentum and energy.

A object has an intrinsic mass (also called rest mass) m. Such an object is said to possess a rest-mass energy $E = mc^2$.

If an object has momentum p, then its energy is given by

$$E = \sqrt{p^2c^2 + m^2c^4} = mc^2 + \frac{p^2}{2m} \text{ for } p \ll mc.$$

Momentum is related to the velocity by $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = \gamma m\vec{v}$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

All massive particles MUST have speeds less than c so γ is always real and always greater than 1.

Substituting the formula for p in E, we can derive $E = \gamma mc^2$ and $v/c = pc/E$.

Photon in Relativity

In Einstein's view light is a stream of "particles" called photons. What can one say about photons from the point of view of relativity?

In relativity we have $v/c = pc/E$. For photons, which necessarily travel with speed of light, $v = c$ and hence $E = pc$.

Comparing this with $E = \sqrt{p^2c^2 + m^2c^4}$, we get $m = 0$ for photons. That is photons are taken to be massless particles with energy E and momentum $p = E/c$. To describe the polarization of light, we also have to assume that they carry spin $= 1$ (as opposed to electrons which carry spin $1/2$).

In describing photoelectric effect, we assumed a photon collides an electron, gets absorbed and transfers its energy to the electron.

We just saw that the photon carries a momentum also. What about momentum conservation?

Compton Effect

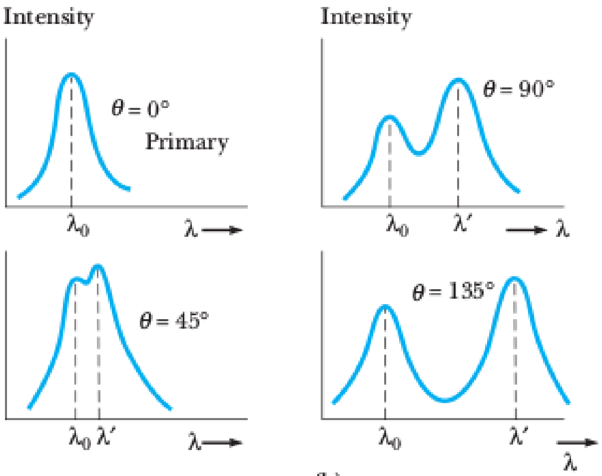
Compton observed the scattering of X-rays (energy $\sim$ 10 keV) off electrons in a thin sheet of metal. He measured the intensity of scattered X-rays as a function of the scattering angle θ.

He observed the relation

$$\lambda' - \lambda = \lambda_c (1 - \cos\theta),$$

where λ is the wavelength of the incident X-ray, λ' is the wavelength of the scattered X-ray and λ_c is a fundamental constant and can be shown to be $h/m_e c$. It is called the Compton wavelength of electron.

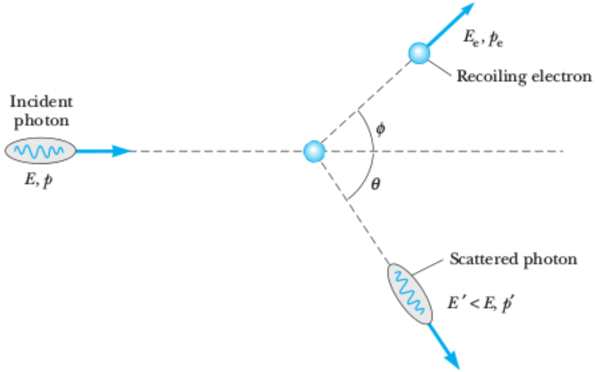
Compton Effect



(b)

Compton Effect

Compton relation can be understood by assuming the scattering to be a collision of the X-ray photon with electron, where some of the energy of the photon is transferred to the electron.



Compton Effect

Before the collision, the energy and the momentum of the photon are $(h\nu, 0, 0, h\nu/c)$ and for electron they are $(m_e c^2, 0, 0, 0)$.

After the collision, the energy and the momentum of the photon are $(h\nu', h\nu' \sin\theta/c, 0, h\nu' \cos\theta/c)$ and for electron they are $(E_e, p_e \sin\phi, 0, p_e \cos\phi)$.

Conservation of energy and momentum give the following equations:

$$\begin{aligned} h\nu + m_e c^2 &= h\nu' + E_e \\ 0 &= h\nu' \sin\theta/c + p_e \sin\phi \\ h\nu/c &= h\nu' \cos\theta/c + p_e \cos\phi. \end{aligned}$$

Compton Effect

From these equations we get

$$
\begin{aligned}
E_e^2 &= (h\nu - h\nu' - m_e c^2)^2 \\
p_e^2 c^2 &= \left(\frac{h}{c}\right)^2 \left(\nu^2 + \nu'^2 - 2\nu\nu' \cos\theta\right)
\end{aligned}
$$

The energy and momentum of the final state electron are constrained by $E_e^2 = p_e^2 c^2 + m_e^2 c^4$. Substituting for E_e and m_e and doing some algebra, we get the Compton relation $\lambda' - \lambda = \lambda_c(1 - \cos\theta)$.

Description of Waves

Mathematically, we describe a wave as a function of space coordinates x, y, z and time t.

$$A(\vec{r}, t) = A_0 \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

is a wave travelling in the direction $\vec{k}$ with speed $v = \omega/|\vec{k}|$.

$$B(x, t) = B_0 \sin(kx) \sin(\omega t)$$

is a standing wave spread along x direction.

In the above examples, A and B usually represent the displacement of some object from its equilibrium position.

In a stretched wire, the object is an infinitesimal length of wire. In a ringing bell, the object is an infinitesimal area of the metallic surface. In a water wave, the object is a tiny droplet of water. In a sound wave, the object is an air molecule.

For mechanical waves, we can actually picture massive objects oscillating about a mean position.

Description of Electromagnetic Wave

If light is an wave, then what exactly is oscillating? Thanks to Maxwell, we have the answer. The oscillating quantities are the electric and the magnetic fields associated with the radiation.

In mechanics, there is always a medium (water in the cases of water waves and air in the case of sound waves) through which the waves propagate.

What is the medium through which light (or electromagnetic waves) propagate?

There was no answer so a very complicated picture was built up. Predictions of this picture were not in agreement with the results of later experiments.

In developing special relativity, Einstein assumed that electromagnetic waves do not need a medium for propagating. They can travel through vacuum.

Relativity is consistent with all experimental data.

Radiation: Wave or particle

Classically light (in general radiation) is treated as a wave. Phenomena such as interference and diffraction could be explained only by assuming that light is described by a wave.

From the equations written earlier for waves, we must consider waves to be extended objects. They are spread over a region of space.

Mathematical derivation of formulae related to interference and diffraction depends crucially on this spread.

In photon picture of radiation, we think of it as a bundle of energy and momentum. In explaining photoelectric effect and Compton effect, we treat this bundle as if it is a particle.

But we describe the particle properties of Energy and Momentum using the language of waves! Energy $= h\nu$ and Momentum $= h\nu/c = h/\lambda$, where ν (frequency) and λ (wavelength) are quantities we use to describe waves.

How do you associate a frequency or a wavelength to a particle?

Matter should have Wave Properties!

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While people are struggling with this dual nature of radiation (particles or waves), de-Broglie jumped in and made the situation more complicated (or did he make it simple?)

He said: If waves behave like particles, then why can't particles behave like waves?

Classically, we view massive objects (such as protons and electrons) as particles and electromagnetic radiation as a wave.

In de-Broglie's point of view, both of these should have both particle and wave properties.

Picture of Matter

In classical mechanics, we do our analysis of in terms of point masses. If we have to deal with a large (or macroscopic) object, we usually picture it as a collection of point masses.

Around 1908, there was experimental proof that atoms really exist. It was realized then that the above ideal picture is quite close to the actual physical picture.

Discovery of sub-atomic particles, such as electrons and protons, did not change our picture of matter being made up of point masses.

In Classical Mechanics, these point masses are assumed to occupy a well defined position in space at any given instant. How can they have wave like properties?

I don't know if de-Broglie bothered about such a question. He just assumed that massive objects have wave like properties.

de-Broglie Hypothesis

de-Broglie combined the arguments from special relativity and Planck's hypothesis.

A light quantum has energy $E = h\nu$ and momentum $p = h\nu/c = h/\lambda$ because $\nu/c = 1/\lambda$ for light.

de-Broglie hypothesized that, with every matter particle of momentum p we should associate a wavelength $\lambda_{\mathrm{d-B}} = h/p$ and a corresponding frequency.

Why don't we see these wavelenghts? For macroscopic objects, these wavelengths are absurdly small. A ball of mass 100 g, thrown with speed of 10 m/s, has $\lambda_{\mathrm{d-B}} \sim 10^{-34}$ m. (The smallest length we can probe is about 10^{-18} m.)

Consequences of de-Broglie Hypothesis

We demand that any new hypothesis should explain a previous unexplained fact. What does de-Broglie hypothesis explain?

It provides a justification for Bohr orbits. Bohr declared that orbits satisfying angular momentum quantization rule $mvr = n\hbar$ are stable. Why?

With de-Brolgie hypothesis, Bohr quantization condition becomes $(h/\lambda_{\mathrm{d-B}})r = n\hbar$ which can be rewritten as $n\lambda_{\mathrm{d-B}} = 2\pi r$.

Orbits are stable if the perimeter contains integer multiples of de-Broglie wavelengths

Seems somewhat similar to standing waves on a stetched string. If the string is tied at both ends, then the standing waves contain integer multiples of half-wavelength.

For these **normal modes** the left moving and the right moving waves are in phase at every point. For all other modes, the phases of the two waves vary randomly so there is no sustained oscillation.