MA 105 D1 Lecture 8

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The Darboux integral

Riemann integration

Partitions

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Definition: A partition $P' = \{a = x_0' < x_1' < \ldots < x_m' = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x_j' \in P'$ such that $x_i = x_j'$.

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Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals. Any two partitions have a common refinement.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{b-1} < x_n = b\}$ and a function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n$$

Defintion: We define the Lower sum as

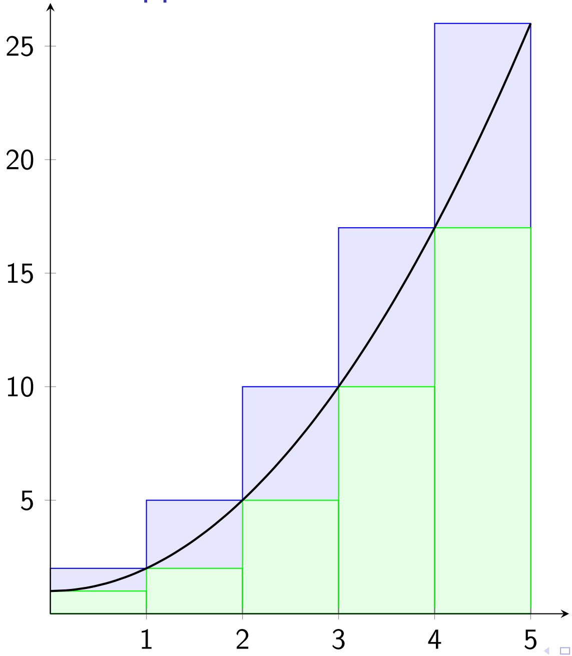
$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

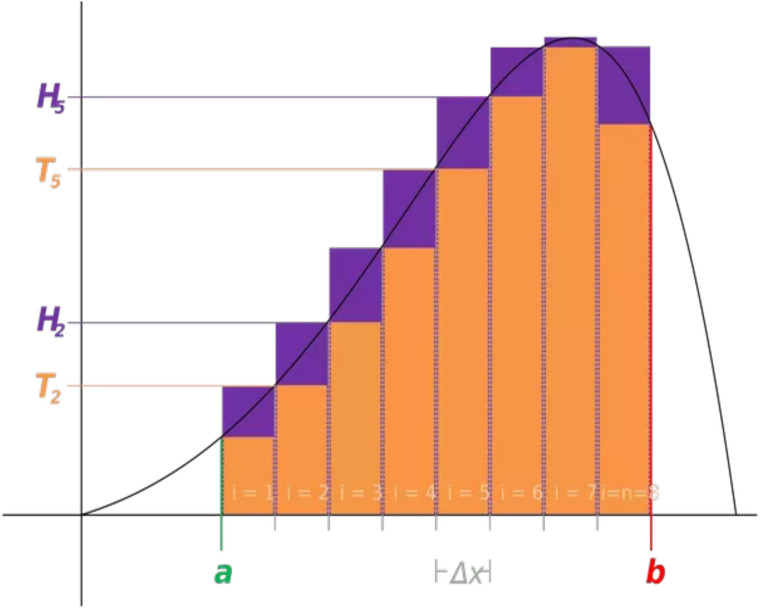
$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words "infimum" and "supremum" bother you, you can think "minimum" and "maximum most of time since we will usually be dealing with continuous functions on $[a, b]$.

Lower and Upper Darboux sums



A picture for a non-monotonic function



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Clearly, the minimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the maximum $M_j = \frac{j}{n}$ at $\frac{j}{n}$. And finally, $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$, for all $1 \leq j \leq n$.

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An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P)\colon P \text{ is a partition of } [a, b]\},$$

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If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\,dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

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Can we conclude that the Darboux integral is $1/2$ by letting $n \to \infty$? Unfortunately, no.

Useful properties of the Darboux sums

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One of the most useful properties of the Darboux sums is the following. If P' is a refinement of P then obviously

$$L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P).$$

Riemann Sums

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Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

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When the size of the partition is small, it means that every interval in the partition is small.

The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

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Notice, that as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

Also notice that if P' is a refinement of P, then $\|P'\| \leq \|P\|$.

The Riemann integral continued

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Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P such that for every tagged refinement of (P', t') of P with $\|P'\| \leq \delta$,

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The nice thing about the above definition is that one only has to check that $|R(f, P', t') - R|$ is small for for refinements of a fixed partition, and not for all partitions.

Note: the slide in class had P in one place, and (P, t) in another. I have corrected this.

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Let $\epsilon > 0$ be arbitrary. For our fixed partition we take $P = P_n$ where $n > \frac{1}{2\epsilon}$ is some fixed number. The statements between the parentheses are true but not logically essential for our proof. [The Riemann sum is trapped between the upper and lower sums:

$$L(f, P_n) = \frac{1}{2} - \frac{1}{2n} \leq R(f, P_n, t) \leq U(f, P_n) = \frac{1}{2} + \frac{1}{2n},$$

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Before going any further we will formally state what we have already been referring to for several slides.

Comparison with the Darboux integral

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With this theorem in hand, we see that the function $f(x) = x$ is also Darboux integrable.

How does one prove Theorem 20? It is not too hard but it takes some work and is roundabout.

The easiest way is to proceed as follows. It is clear that if f is Riemann integrable in the sense of Definition 1, it is Riemann integrable in the sense of Definition 2. Next, one shows that if f is Riemann integrable in the sense of Definition 2, then it is Darboux integrable. And finally, one can show that if the Darboux integral exists, then the Riemann integral exists in the sense of Definition 1. An interested student can try this as an exercise.

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In fact, one can allow even countably many discontinuities and the Theorem will remain true.
Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 21 and the extension to countably many discontinuities (Warning: there is one crucial fact about continuous functions that we have not covered that you will have to discover for yourself).