MA 105 D1 Lecture 18

Ravi Raghunathan

Department of Mathematics

September 28, 2017

Defining the double integral over a region

Assume that $f(x, y)$ is defined in some bounded region D bounded by a finite union of graphs of C^1-functions. Since D is bounded, we may assume that $D \subset R$, for some rectangle R.

Define a function

$$g(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \end{cases}$$

on R. If g is integrable on R, we will say that f is integrable on D and we define

$$\int\int_D f(x, y) dxdy := \int\int_R g(x, y) dxdy.$$

Elementary regions

We will call the two simple types of regions that we are going to describe elementary regions.

Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions. (Since our criterion for Riemann integrability is that the region D should have a boundary which consists of piecewise C^1 curves, it is probably best to assume that h_1 and h_2 and k_1, k_2 below are C^1 functions, although, in practice, we may need only continuity in many examples.) Consider the set of points

$$D_1 = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}.$$

Such a region is said to be of type 1.

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions, the set of points

$$D_2 = \{(x, y) \mid c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of type 2.
If D is a union of regions of types 1 and 2, we call it a region of type 3.

Evaluating integrals on regions of type 1

Let D be a region of type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let g be the corresponding function on R (obtained by extending f by zero).

$$\int\int_D f(x,y)dxdy = \int\int_R g(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta g(x,y)dy\right] dx,$$

where the second equality follows because of Theorem 34. In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} g(x,y)dy\right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $g(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$. Similarly for type 2 regions we have

$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Examples

Example 1 (page 298 of Marsden, Tromba and Weinstein but slightly modified): Evaluate the integral

$$\int\int_D \sqrt{1-y^2}dxdy,$$

where D is the part of the unit disc that lies in the first quadrant.

Solution: Clearly the region D can be viewed as either a type 1 region or a type 2 region. Let us write down the iterated integrals in each case.

Type 1: Let $h_1(x) = 0$ and $h_2(x) = \sqrt{1-x^2}$. We may view our region as being contained in the square $[0,1] \times [0,1]$. Then, our integral becomes

$$\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}dy\right] dx.$$

Type 2: Let $k_1(y) = 0$ and $k_2(y) = \sqrt{1 - y^2}$ and R be $[0, 1] \times [0, 1]$. Then our integral becomes

$$\int_0^1 \left[\int_0^{\sqrt{1-y^2}} \sqrt{1 - y^2} dx \right] dy.$$

One of the iterated integrals above is clearly easier to evaluate than the other. Which one?

Clearly the second one. Let us evaluate it. We get

$$\int_0^1 \left[\sqrt{1-y^2}x\right]_0^{\sqrt{1-y^2}} dy = \int_0^1 (1-y^2)dy = \left(y - \frac{y^3}{3}\right)\Big|_0^1 = \frac{2}{3}.$$

The point of the above exercise is to show you that it is often easier to evaluate one of the iterated integrals. In particular, if a given iterated integral looks difficult, you can switch the order of integration and try to do the other iterated integral.

The mean value theorem for double integrals

Let us first state the mean value theorem for functions of one variable.

Theorem 35: Suppose that f is a continuous function on $[a, b]$. There exists a point x_0 in $[a, b]$ such that

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x)dx.$$

How does one interpret the above statement geometrically?

Theorem 36: If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exist (x_0, y_0) in D such that

$$f(x_0, y_0) = \frac{1}{A(D)} \int \int_D f(x, y)dA.$$

Changing variables

When computing the volume of a solid we have seen that there are several ways to divide up the volume and integrate - one can approximate by cuboids, take slices or cylindrical shells and so on.

For calculating double integrals, it may be convenient to describe the area elements in the given region D by other coordinate systems, such as coordinates, and the integrals themselves might become simpler to evaluate. This is the analogue of the substitution rule in one variable integration.

We will see what happens to the double integral when changing coordinate systems in general. The most important and useful coordinate systems (other than the usual cartesian coordinate system) are the polar, spherical and cylindrical coordinate systems.

Polar Coordinates

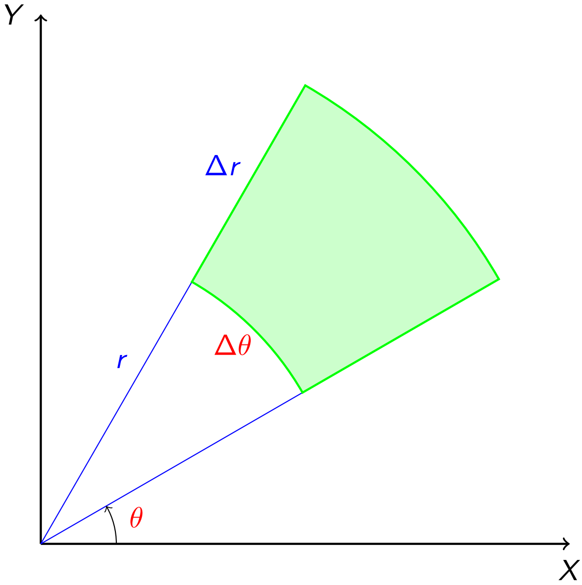
Instead of starting with a completely arbitrary coordinate system, let us see what happens when we use polar coordinates. If we have a function $f : D \to \mathbb{R}$, as a function of the usual (cartesian) $x - y$ coordinates, we can get a function g of the polar coordinates r and θ by composition:

$$g(r, \theta) = f(r \cos\theta, r \sin\theta).$$

To integrate the function g on a domain D^* we need to cut up D^* into small rectangles, but these will be rectangles in the $r - \theta$ coordinate system.

What shape does a rectangle $[r, r + \Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the $x - y$ plane?

An area element in polar coordinates



The integral in polar coordinates

Clearly a part of a sector of a circle. What is the area of this part of a sector? It is

$$\frac{1}{2} \cdot [(r + \Delta r)^2 \Delta\theta - r^2 \Delta\theta] \sim r \Delta r \Delta\theta.$$

It follows that the integral we want is approximated by a sum of the form

$$\sum_{i,j} g(t_{ij}) r_i \Delta r_i \Delta\theta_j,$$

where $t = \{t_{ij}\}$ is a tag for the partition of the "rectangle" in polar coordinates. Taking limits, we see that

$$\int\int_D f(x, y) dx dy = \int\int_{D^*} f(r \cos\theta, r \sin\theta) r dr d\theta,$$

where D is the image of the region D^*. This is the change of variable formula for polar coordinates.

The integral of the Gaussian

We would like to evaluate the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

What does this integral mean? - so far we have only looked at Riemann integrals inside closed bounded intervals, so the end points were always finite numbers a and b. An integral like the one

above is called an improper integral. We can assign it a meaning as follows. It is defined as

$$\lim_{T \to \infty} \int_{-T}^{T} e^{-x^2} dx,$$

provided, of course, this limit exists. We will see how to evaluate this.

The most amazing trick ever

Consider

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy.$$

We view this product as an iterated integral!

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

But this iterated integral can be viewed as a double integral on the whole plane. Now under polar coordinates, the plane is sent to the plane. Hence, we can write this as

$$\int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} rdr \right] d\theta.$$

But we can now evaluate the inner integral. Hence, we get

$$\int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

The answer

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function (next semester) and in many other contexts.

There are many other ways of evaluating the integral I, but the method above is easily the cleverest.

Joseph Liouville (1809 - 1892)

The definition of a mathematician

The preceding trick for evaluating the integral I, was discovered by Joseph Liouville (1809 -1892), a great mathematician with contributions in complex analysis, differential equations and number theory.

Liouville inspired William Thomson, Lord Kelvin, the famous Scottish physicist, to one of the most satisfying definitions of a mathematician that has ever been given, “Do you know what a mathematician is?” Kelvin once asked a class. He stepped to the board and wrote:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Putting his finger on what he had written, he turned to the class and said:

“A mathematician is one to whom that is as obvious as that twice two makes four is to you. Liouville was a mathematician.”

The area element for a change of coordinates

Let us suppose that we have a general change of coordinates given by $x = \phi(u, v)$ and $y = \psi(u, v)$. Can we compute the the area of the image of a rectangle in the $u - v$ plane, and hence, give a change of variables formula as we did for polar coordinates?

Using the chain rule for functions of two variables we see that

$$\Delta x \sim \frac{\partial \phi}{\partial u}\Delta u + \frac{\partial \phi}{\partial v}\Delta v$$

and

$$\Delta y \sim \frac{\partial \psi}{\partial u}\Delta u + \frac{\partial \psi}{\partial v}\Delta v.$$

The Jacobian

You may recognize the matrix

$$J = \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix}$$

It is the derivative matrix for the function $h = (\phi, \psi) : \mathbb{R}^2 \to \mathbb{R}^2$. It is not too hard to see that the area of the image is given (upto higher order terms) by

$$\left| \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix} \right| \Delta u \Delta v,$$

where the first term on the right hand side is the determinant of the matrix J, which is called the Jacobian determinant. It is customary to write $x = x(u, v)$ and $y = y(u, v)$ rather than using additional letters ϕ and ψ (just like we sometimes write $y = y(x)$ rather than $y = f(x)$). In this case we use the notation

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

for the Jacobian determinant.

The change of variables formula

Let D be a region in the xy plane and D^* a region in the uv plane such that $h(D^*) = D$. Then

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Let us see what we get in the familiar case of polar coordinates. We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

which is what we obtained previously.