Proofs of Stokes' and Gauss' Theorem

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The proof of Stokes' theorem for parametrised surfaces

The proof of Gauss'theorem

The basic assumptions

We give a proof of Stokes's theorem for parametrised surfaces (which include spheres, tori and the outer surface of a cuboid/parallelepiped).

The general statement of Stokes' theorem will then follow by decomposing the given surface into a union of parametrised surfaces which intersect only along curves (more specifically, any two parametrised surfaces should intersect in a finite union of graphs of curves).

We will assume that S is a bounded geometric surface in $\mathbb{R}^3$ given by a non-singular $\mathcal{C}^1$ parametrisation $\mathbf{\Phi} : D \to \mathbb{R}^3$, where D is a bounded subset of $\mathbb{R}^2$.

We will further assume that the map $\mathbf{\Phi} : D \to S$ is a bijective map so $\mathbf{\Phi}(D) = S$ and that ∂D consists of a finite disjoint union of simple closed curves each of which is a piecewise $\mathcal{C}^1$ non-singular parametrised curve.

(Note that because of the inverse function theorem $\mathbf{\Phi} : D \to S$ is actually a diffeomorphism, though we will not be using this in the proof).

Orientation of the boundary of the surface

Exercise 1: Check that $\mathbf{\Phi}$ takes the boundary ∂D of D to the boundary ∂S of S.

How does one orient ∂S?

Since D is a planar region, it has a natural positive orientation given by the positive direction of the z axis. Now we orient the boundary ∂D as in Green's theorem, that is, in the counterclockwise direction.

Exercise 2: Once ∂D has been oriented as above, ∂S is automatically oriented. The surface S is oriented by choosing $\mathbf{\Phi}_u \times \mathbf{\Phi}_v$ as the normal with positive orientation. Verify in an example of your choosing that the region S remains to the left of an observer walking on ∂S in the direction of positive orientation.

Reduction of the surface integral to a double integral

Now that we have specified clearly how various regions have been oriented, we can start the proof. We will assume that $\mathbf{F} : W \to \mathbb{R}^3$ is a $\mathcal{C}^1$-vector field on a subset W of $\mathbb{R}^3$ which contains S.

Let us first compute the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

We will write $\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$, where F_1 and F_2 and F_3 are the three scalar components of $\mathbf{F}$. We will also write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are the three scalar components of $\mathbf{\Phi}$. With this notation, we have previously seen that

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D [(\nabla \times \mathbf{F})(\mathbf{\Phi}(u, v))] \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv.$$

Calculating the curl and the normal vector

Once can easily show that

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

One can also easily calculate the normal vector to the surface:

$$\mathbf{\Phi}_u \times \mathbf{\Phi}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\mathbf{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\mathbf{k}$$

Calculating the dot product of the curl and the normal

We now calculate the dot product

$$(\nabla \times \mathbf{F}) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v).$$

It is

$$\frac{\partial F_3}{\partial y}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial z}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}$$
$$\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial z}\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial F_3}{\partial x}\frac{\partial x}{\partial u}\frac{\partial z}{\partial v}$$
$$\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial x}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

We thus have an explicit expression for the double integral

$$\iint_D [(\nabla \times \mathbf{F})(\mathbf{\Phi}(u,v))] \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) dudv.$$

Parametrising the boundary of S

Let us assume that ∂D is given by a pieciewise $\mathcal{C}^1$ curve. To prove the more general theorem we have stated when the boundary is in several pieces, we simply parametrise each piece and take the resulting sum of the line integrals.

Let $(u(t), v(t))$, $a \leq t \leq b$, be a parametrisation of ∂D. Then

$$\mathbf{c}(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))),$$

$a \leq t \leq b$ is a parametrisation of ∂S. Hence, the tangent vector is given by

$$\mathbf{c}'(t) = \left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}, \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}, \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} \right).$$

Our aim is to evaluate the line integral that appears in Stokes's theorem

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Using the parametrisation we have given we can write this integral out as

Reduction to a line integral in the plane

$$\int_a^b F_1(x(u(t),v(t)),y(u(t),v(t)),z(u(t),v(t)))\left(\frac{\partial x}{\partial u}\frac{du}{dt}+\frac{\partial x}{\partial v}\frac{dv}{dt}\right)dt$$
$$+F_2(x(u(t),v(t)),y(u(t),v(t)),z(u(t),v(t)))\left(\frac{\partial y}{\partial u}\frac{du}{dt}+\frac{\partial y}{\partial v}\frac{dv}{dt}\right)dt$$
$$+F_3(x(u(t),v(t)),y(u(t),v(t)),z(u(t),v(t)))\left(\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\right)dt.$$

After changing variables this line integral has the less cumbersome form:

$$\int_{\partial D} F_1(x(u,v),y(u,v),z(u,v))\left[\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right]$$
$$+F_2(x(u,v),y(u,v),z(u,v))\left[\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right]$$
$$+F_3(x(u,v),y(u,v),z(u,v))\left[\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right].$$

Using Green's theorem

The previous integral can be rewritten as

$$\int_{\partial D} \left(F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u} \right) du + \left(F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v} \right) dv.$$

This integral is of the form

$$\int_{\partial D} M(u, v)du + N(u, v)dv,$$

so we may apply Green's Theorem to it. Thus, we obtain

$$\int_{\partial D} M(u, v)du + N(u, v)dv = \iint_D (N_u(u, v) - M_v(u, v))dudv.$$

Recall that we have already expressed the surface integral in Stokes' theorem as a double integral over D. So all that is required now is to show that the integrand in the integral above is the same as the one we obtained before.

Evaluating the integrand

We proceed to calculate N_v and M_u.

$$M_v = \frac{\partial}{\partial v}\left(F_1(x,y,z)\frac{\partial x}{\partial u} + F_2(x,y,z)\frac{\partial y}{\partial u} + F_3(x,y,z)\frac{\partial z}{\partial u}\right)$$

$$= \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} + F_1(x,y,z)\frac{\partial^2 x}{\partial v \partial u} + \dots$$

Similarly

$$N_u = \frac{\partial}{\partial u}\left(F_1(x,y,z)\frac{\partial x}{\partial v} + F_2(x,y,z)\frac{\partial y}{\partial v} + F_3(x,y,z)\frac{\partial z}{\partial v}\right)$$

$$= \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + F_1(x,y,z)\frac{\partial^2 x}{\partial u \partial v} + \dots$$

The end of the proof

Recall that we need to calculate $N_u - M_v$. By looking at the two expressions in the previous slide, we see that first and fourth terms cancel, leaving four terms behind. There will similarly be four surviving terms coming from the terms involving F_2 and F_3 giving a total of twelve such terms. One can then easily see that these are exactly the twelve terms that occur in the double integral that came from the surface integral. This completes the proof.

As I mentioned earlier, the proof above works when ∂D has just one component. If there are several we need to parametrise each one and write down the corresponding line integrals. Green's theorem will then say that the sum of these line integrals is the double integral that we obtained above.

The proof of Gauss' theorem

Before embarking on the proof, I must add that I am following Marsden, Weinstein and Tromba's exposition very closely. I would also like to thank my colleague Professor Garge for lending me some of his slides which I have used with only slight modifications.

We will prove Gauss' theorem for regions that are of all three types, that is, they can be realized as the region between two graphs no matter which coordinate we single out.

The most important examples of such regions are the parallelepipeds (or cuboids). Other examples include tetrahedra, pyramids, more generally, convex polyhedra and also spheres.

Recall that we have a $\mathcal{C}^1$-vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ defined and on W. We will further assume that surfaces that bound W (that is, the graphs that appear in the description of W as a region of one type or another) are defined by $\mathcal{C}^1$ functions.

The left hand side

We start with the left hand side (the volume integral) of the theorem. Since,

$$\mathrm{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

the volume integral becomes

$$\iiint_W \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

which breaks up as a sum

$$\iiint_W \frac{\partial P}{\partial x} dV + \iiint_W \frac{\partial Q}{\partial y} dV + \iiint_W \frac{\partial R}{\partial z} dV.$$

The right hand side

Similarly the right hand side breaks up as a sum as follows:

$$\begin{aligned}\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\partial W} (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot d\mathbf{S} \\ &= \iint_{\partial W} P\mathbf{i} \cdot d\mathbf{S} + \iint_{\partial W} Q\mathbf{j} \cdot d\mathbf{S} + \iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S}\end{aligned}$$

The theorem will follow if we establish the equalities:

$$\iint_{\partial W} P\mathbf{i} \cdot d\mathbf{S} = \iiint_W \frac{\partial P}{\partial x} dV, \quad \iint_{\partial W} Q\mathbf{j} \cdot d\mathbf{S} = \iiint_W \frac{\partial Q}{\partial y} dV$$

and

$$\iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S} = \iiint_W \frac{\partial R}{\partial z} dV.$$

Reduction to a single equality

It is clearly enough to prove the theorem for vector fields with only one non-zero component, since any vector field can be written as a sum of three such vector fields as above.

Thus, we can (and will) assume that the vector field $\mathbf{F}$ has only one nonzero component, and we take it to be in the direction of the unit vector $\mathbf{k}$, $\mathbf{F} = R\mathbf{k}$. We will show that the third of the equalities above holds. So we need

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S},$$

where the solid W is given by a region D in the xy-plane and functions $f_1, f_2 : D \to \mathbb{R}$:

$$W = \{(x, y, z) : (x, y) \in D, f_1(x, y) \leq z \leq f_2(x, y)\},$$

since the region is of type 1.

Reduction to a double integral

Using Fubini's theorem, we write the triple integral as an iterated integral:

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_D \left(\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z} dz \right) dxdy.$$

Using the fundamental theorem of calculus, we get

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_D \left[R\left(x, y, f_2(x, y)\right) - R\left(x, y, f_1(x, y)\right) \right] dxdy.$$

We will not simplify the left hand side any further, but will turn our attention to the right hand side instead.

Computing the integral over the boundary

The boundary of W is a union of six faces: the graph of f_1 is the bottom face S_1, the graph of f_2 is the top face S_2 and the other four faces are S_3, S_4, S_5, S_6 (in any order).

Not all of these four faces may actually occur. If W happens to be the solid cylinder it is possible to use just two faces to describe ∂W. A similar situation presents itself for a solid sphere. In any event we can write

$$\iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S} = \iint_{S_1} R\mathbf{k} \cdot d\mathbf{S} + \iint_{S_2} R\mathbf{k} \cdot d\mathbf{S} + \sum_{i=3}^{6} \iint_{S_i} R\mathbf{k} \cdot d\mathbf{S}.$$

Reduction to surface integrals over two surfaces

Note that the normals to the surfaces S_3, S_4, S_5 and S_6 are normal to the z-axis, because at every point on these surfaces the z-axis is contained in the tangent plane. It follows that

$$\iint_{S_i} R\mathbf{k} \cdot d\mathbf{S} = \iint_{S_i} (R\mathbf{k} \cdot \mathbf{n_i}) dS = 0 \quad 3 \leq i \leq 6.$$

Hence,

$$\iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S} = \iint_{S_1} R\mathbf{k} \cdot d\mathbf{S} + \iint_{S_2} R\mathbf{k} \cdot d\mathbf{S}.$$

It remains to compute the two surface integrals above explicitly. We will use the fact that they are both parametrised surfaces (being graphs).

Computing the surface integrals over the remaining surfaces

The surfaces S_1 and S_2 are graphs of functions f_1 and f_2 respectively, so we parametrise them by $\Phi : D \to \mathbb{R}^3$ and $\Psi : D \to \mathbb{R}^3$ given by

$$\Phi(x, y) = (x, y, f_1(x, y)) \quad \text{and} \quad \Psi(x, y) = (x, y, f_2(x, y)).$$

To compute the surface integrals we must compute the normals. They are

$$\begin{aligned} \Phi_x \times \Phi_y &= \left(\mathbf{i} + \frac{\partial f_1}{\partial x}\mathbf{k}\right) \times \left(\mathbf{j} + \frac{\partial f_1}{\partial y}\mathbf{k}\right) \\ &= -\frac{\partial f_1}{\partial x}\mathbf{i} - \frac{\partial f_1}{\partial y}\mathbf{j} + \mathbf{k} \end{aligned}$$

and

$$\Psi_x \times \Psi_y = -\frac{\partial f_2}{\partial x}\mathbf{i} - \frac{\partial f_2}{\partial y}\mathbf{j} + \mathbf{k}.$$

Note that the normal given by our parametrisation of S_1 is "upward" in the direction of $\mathbf{k}$, that is, it goes inside the solid W, so while computing the surface integral we multiply by -1:

$$\begin{aligned}\iint_{S_1} R\mathbf{k}\cdot d\mathbf{S} &= -\iint_D R\big(\Phi(x,y)\big)\mathbf{k}\cdot(\Phi_x\times\Phi_y)dxdy\\ &= \iint_D R\big(\Phi(x,y)\big)\mathbf{k}\cdot\left(\frac{\partial f_1}{\partial x}\mathbf{i}+\frac{\partial f_1}{\partial y}\mathbf{j}-\mathbf{k}\right)dxdy\\ &= -\iint_D R\big(x,y,f_1(x,y)\big)dxdy.\end{aligned}$$

and similarly we get

$$\iint_{S_2} R\mathbf{k}\cdot d\mathbf{S} = \iint_D R\big(x,y,f_2(x,y)\big)dxdy.$$

Adding the two integrals above, we see that Gauss' theorem follows immediately in this case.

Gauss' divergence theorem for more general regions

While we have proved Gauss' theorem only for regions which are simultaneously of types 1, 2 and 3, we stated the theorem for a much more general class of regions - those bounded by closed surfaces.

It turns out that any region bounded by a closed surface can be approximated from within by parallelepipeds which intersect only along planar surfaces. As in the proof of Stokes' theorem for general surfaces the surface integrals on the common boundary of two parallelepipeds cancels because they are oppositely oriented.

It follows that only the surface integral on the outer boundary of the union of two parallelepipeds survives. By decreasing the size of the parallelepipeds we can eventually approximate the region more and more efficiently and prove the general case.