Quantum Mechanical Problems in Higher Dimensions

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Quantum Mechanics in Two Dimensions

- To keep the algebra simple, so far we have been discussing only one-dimensional problems. The reason for this is to emphasize the principles and the techniques of quantum mechanics without being distracted by the additional details which arise when the motion occurs in more than one dimension.
- Hoping that all of you have digested these principles and techniques very well, we free ourselves from the groove in which we were struck and venture out into the plane.
- Life immediately becomes complicated! Which way to go? Previously one could move in only two directions: forward or backward. Now there are an infinite number of directions to choose from!
- Previously bound motion is necessarily oscillatory. Now another type of bound motion becomes possible. Rotational motion!
- I briefly mentioned about quantization of oscillatory motion. Now we have to take care of rotational motion also.

Quantum Mechanics in Two Dimensions

- You can ask, "Why are you so flustered by this notion of two dimensions? We live in Euclidean space. The motion in x-direction is completely independent of the motion in y-direction".
- True, if you have a problem which neatly separates itself into two sub-problems: one in x-direction and one in y-direction.
- That happens only when the potential can be written as the sum of two terms, one which depends only on x and the other only on y.
- If the potential is a general function of x and y, we must consider both the equations of motion simultaneously.
- Very difficult to do if the potential is an arbitrary function.
- We focus our attention of potentials which are invariant under some symmetry. Using the symmetry, we can simplify the problem.
- There are a number of natural cases, where the potentials do have symmetries inherent in them.

Particle in a Rectangular Potential Well

The simplest problem in two dimensions is that of a particle in a two dimensional infinite potential well, given by

$$\begin{aligned} V &= 0 \text{ for } 0 < \mathrm{x} < \mathrm{L_x} \text{ and } 0 < \mathrm{y} < \mathrm{L_y} \\ &= \infty \text{ everywhere else.} \end{aligned}$$

The time independent Schroedinger equation is

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2} + V(x,y)\psi(x,y)\right] = E\psi(x,y).$$

Since the potential is specified in terms of cartesian coordinates, we can try our old trick of separation of variables:

$$\psi(x,y) = \psi_x(x)\psi_y(y).$$

Substituting this in Schroedinger's quation, and dividing by $\psi(x,y)$, we get

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\psi_x(x)}\frac{d^2\psi_x(x)}{dx^2} + \frac{1}{\psi_y(y)}\frac{d^2\psi_y(y)}{dy^2}\right] = (E - V).$$

Particle in a Rectangular Potential Well

Since RHS is a constant, each term in LHS also has to be a constant. So we write them as

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{1}{\psi_x(x)}\frac{d^2\psi_x(x)}{dx^2} &= E_x \\
-\frac{\hbar^2}{2m}\frac{1}{\psi_y(y)}\frac{d^2\psi_y(y)}{dy^2} &= E_y
\end{aligned}
$$

Each of these equations is like the Schroedinger's equation for a one dimensional infinite potential well. So we have

$$E_x = \frac{n_x^2\pi^2\hbar^2}{2mL_x^2} \text{ and } E_y = \frac{n_y^2\pi^2\hbar^2}{2mL_y^2}.$$

The total energy of the particle is

$$E = E_x + E_y = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right).$$

The energy eigenvalue E now depends on two integers n_x and n_y.