Tutorial 8

P66. The expectation value can be calculated as

$$\langle E\rangle = \langle \phi^* | \hat{H} | \phi\rangle = \int_0^L \phi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}(Ax(l-x))\right)dx$$

$$= \int_0^L Ax(l-x)\left(\frac{-\hbar^2}{2m}\right)(-2A)\,dx$$

$$\frac{A^2\hbar^2}{m}\int_0^L x(l-x)\,dx = \frac{A^2\hbar^2}{m}\left[\frac{x^2 l}{2} - \frac{x^3}{3}\right]_0^L$$

$$= \frac{A^2\hbar^2 L^3}{6m}$$

For A, normalize $\phi(x)$:

$$\langle\phi^*|\phi\rangle = \int_0^L A^2x^2(L-x)^2\,dx = 1$$

$$A^2\int_0^L (x^2L^2 - 2x^3L + L^2x^4)\,dx = 1$$

$$\therefore A^2\left[\frac{x^3L^2}{3} - \frac{2x^4L}{4} + \frac{L^2x^5}{5}\right]_0^L = 1.$$

$$\therefore A^2 = \frac{30}{L^5} \qquad A^2 = \frac{30}{L^5}$$

$$\langle E\rangle = \frac{5\hbar^2}{mL^2}$$

for particle in an infinite well, eigen values of energy function are given as

$$\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

$\therefore$ probability of finding particle in gs

$$\langle\psi_1|\phi\rangle^2 = \left(\int_0^L \left[\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\right)\left(Ax(L-x)\right)\right]dx\right)^2$$

$$= \left[\sqrt{60}\times\frac{4}{\pi^3}\right]^2 = \frac{960}{\pi^6} = \frac{960}{\pi^6} = 0.9985$$

P67. $\psi(x,0) = A(\phi_1 + \phi_2 + \phi_4)$

Given: $\langle \phi_i | \phi_j \rangle = 1$ for $i = j$ normalized

$\langle \phi_i | \phi_j \rangle = 0$ for $i \neq j$ orthonormal.

$$\langle \psi^* | \psi \rangle = 1 = A^2 \langle \phi_1 + \phi_2 + \phi_4 | \phi_1 + \phi_2 + \phi_4 \rangle$$

(a) $A^2 = \frac{1}{3}$ $\quad A = \frac{1}{\sqrt{3}}$

(b) $\psi(x,t) = \frac{1}{\sqrt{3}} (\phi_1' + \phi_2' + \phi_4')$

where $\phi_1' = \phi_1 e^{-iE_1 t/\hbar}$

$\phi_2' = \phi_2 e^{-iE_2 t/\hbar}$

$\phi_3' = \phi_3 e^{-iE_4 t/\hbar}$

(c) $E = \frac{1}{3}(E_1 + E_2 + E_4)$

P68.

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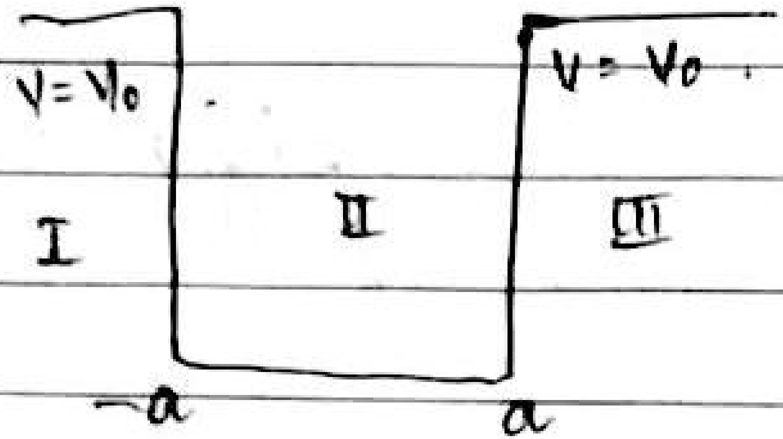


$$\psi_1 = Ae^{\alpha x}$$

$$\psi_2 = C\sin kx + D\cos kx$$

$$\psi_3 = Fe^{-\alpha x}.$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \cancel{\alpha = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}}.$$

$$\alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

∵ Well is symmetric, solutions will be of odd or even parity.

Even ~~odd~~ parity.

$\psi_1 = Ae^{\alpha x}$ $\quad \psi_2 = D\cos kx$ $\quad \psi_3 = Fe^{-\alpha x}$.

$Ae^{-\alpha a} = D\cos ka = Fe^{-\alpha a}$

$\therefore F = A$.

$-\alpha A e^{-\alpha a} = -Dk\sin ka$.

$+\alpha = +k\tan ka$

$\therefore \tan ka = \dfrac{\alpha}{k}$ ——— (I)

~~Even~~ odd parity.

$\psi_1 = Ae^{\alpha x}$ $\quad \psi_2 = C\sin kx$ $\quad \psi_3 = Fe^{-\alpha x}$.

$Ae^{-\alpha a} = -C\sin ka$

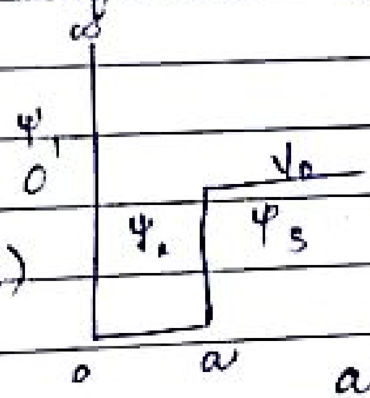
$C\sin ka = Fe^{-\alpha a}$.

$A\alpha e^{\alpha(-a)} = Ck\cos ka = -\alpha F e^{-\alpha a}$.

$Ck\cos ka = \alpha C\sin ka$

$-\dfrac{k}{\alpha} = \tan ka$ ——— (II)

② . infinite semi-potential well.



$\psi_1 = 0$

$\psi_2 = A\cos kx + B\sin kx$

$\psi_3 = Ce^{-\alpha x}$

$k = \sqrt{\dfrac{2mE}{\hbar^2}}$ $\quad \alpha = \sqrt{\dfrac{2m(V_0-E)}{\hbar^2}}$

at $x = 0$.

$A = 0$.

at $x = a$

$B\sin ka = Ce^{-\alpha a}$

$Bk\cos ka = C(-\alpha)e^{-\alpha a}$.

$k\cot ka = (-\alpha)$

$-\dfrac{k}{\alpha} = \tan ka$ ——— (III)

To be ~~ne~~ this is half (a) of the given finite well (2a)

Also, $n_{max} \propto L$

$n_{max} \to$ max no. of levels for a given potential.

$\therefore$ ② will have half the no. of energy levels as compared to ①

from (I) (II) (III),
we can see that
solutions for semi-infinite = odd parity
solutions for finite.

$$\tan ka = -\frac{k}{\alpha} \therefore \left(-\sqrt{\frac{E}{V_0 - E}}\right) = \tan\left(\sqrt{\frac{2mE}{\hbar^2}}\,a\right)$$

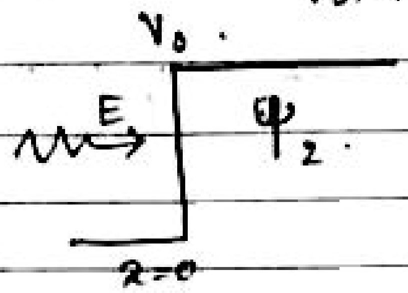
substitute & check
which values of E given
satisfy this equation.

Date / /

Tutorial 8

$V_0 > E$

69.



$\psi_2 = Ae^{-kx}$

$$k = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

(a). $\frac{|\psi(0)|^2}{|\psi(x_0)|^2} = e$. $\quad \frac{|\psi(0)|^2}{e} = |\psi(x_0)|^2$.

$$\therefore \frac{A^2}{e} = A^2 e^{-2kx_0}.$$

$$1 = 2kx_0.$$

$$x_0 = \frac{1}{2k} = \frac{\hbar}{\sqrt{8m(V_0-E)}}$$

$$x_0 = \frac{\hbar}{\sqrt{8m(V_0-E)}}$$

(b) $\Delta x = x_0 = \frac{\hbar}{\sqrt{8m(V_0-E)}}$

$\Delta p \cdot \Delta x = \frac{\hbar}{2} \quad \Delta p = \frac{\hbar}{2\Delta x} = \sqrt{2m(V_0-E)}$

$E = \frac{p^2}{2m} = \cancel{(\ldots)}$. $\quad \Delta E = \frac{p\Delta p}{m}$

~~Now $E' = E \pm \Delta E$~~

$$\therefore \Delta E = p\sqrt{\frac{2(V_0-E)}{m}} = \sqrt{4(V_0-E)\frac{p^2}{2m}}$$

$\because \frac{p^2}{2m} = V_0 - E$

$$\therefore \Delta E = 2(V_0-E)$$

$E' = E \pm \Delta E \Rightarrow E + \Delta E = 2V_0 - E.$

$E - \Delta E = 3E - 2V_0$

$\because$ ~~....~~, $\therefore$ E may exceed V_0 due to uncertainty in energy. This explains why the particle is able to penetrate the potential barrier even though it's classically forbidden.

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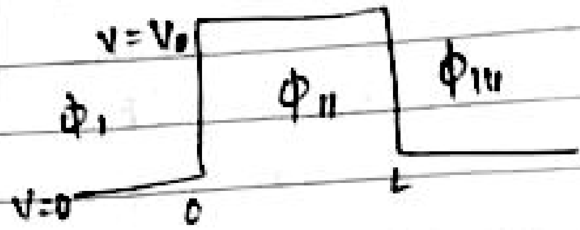
P70*



$\phi_1 = Ae^{ikx} + Be^{-ikx}$

$\phi_2 = Cx + D$

$\phi_3 = Fe^{ikx}$. (no reflection)

At $x = 0$. $A + B = D$ $(\phi_1 = \phi_2)$.

$ik(A-B) = C$ $(\phi_1' = \phi_2')$

At $x = L$. $CL + D = Fe^{ikL}$ $(\phi_2 = \phi_3)$

$C = ikFe^{ikL}$ $(\phi_2' = \phi_3')$

On solving, $\frac{F}{A} = \frac{2e^{-ikL}}{2 - ikL}$

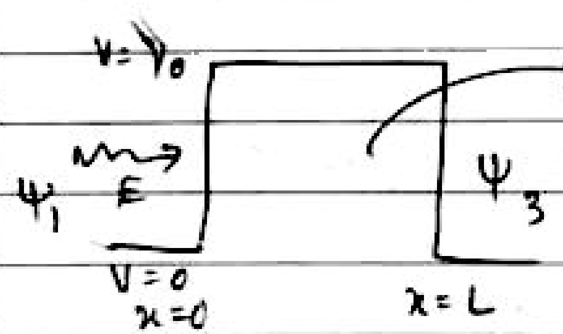
Transmission coefficient $= \left|\frac{F}{A}\right|^2 = \left(\frac{2e^{-ikL}}{2 - ikL}\right)\left(\frac{2e^{ikL}}{2 + ikL}\right)$

(a) $T = \frac{4}{4 + k^2L^2}$ $= \frac{4}{4 + k^2L^2}$

For $T = 0.5$, we get $kL = 2$ $\therefore L = \frac{2}{k} = \frac{\lambda}{\pi}$

(b) $L = \frac{\lambda}{\pi}$

P71.



$\psi_2 : Ae^{-\alpha x} + Be^{\alpha x}$.

$\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

$\psi_1 = Ce^{ik_0x} + De^{-ikx}$

$\psi_3 : Fe^{ikx}$

$\psi_1 = \psi_2$ $A + B = C + D$

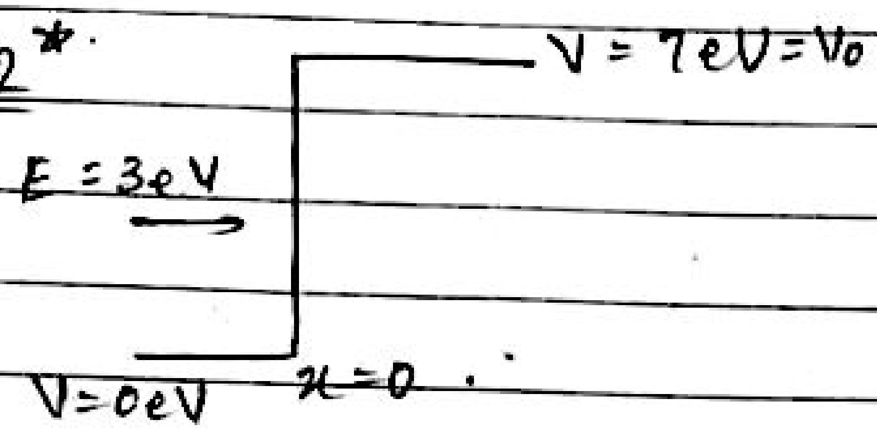
$\psi_1' = \psi_2'$ $ik(C - D) = -\alpha(A - B)$

$\psi_2 = \psi_3$ $Ce^{-ikL} + De^{ikL} = Fe^{ikL}$

$\psi_2' = \psi_3'$ $-ik(Ce^{-ikL} - De^{ikL}) = ik(Fe^{ikL})$

Solve

P72*



$$\psi_2 = Ae^{-Kx} \qquad K = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$V_0 = 7eV$

$E = 3eV$.

Similar concept as P69.

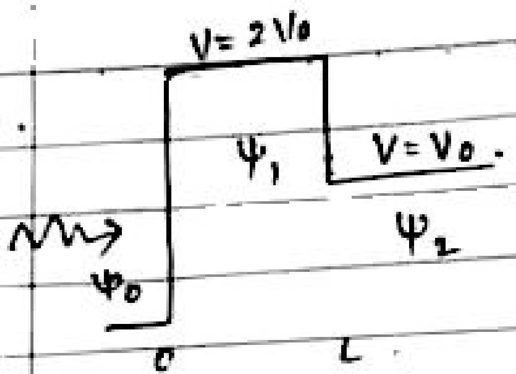
$$\frac{|\psi(0)|^2}{2} = |\psi(x_0)|^2 \qquad x_0 > x$$

$$\frac{\cancel{A^2}}{2} = \cancel{A^2} e^{-2Kx_0} \qquad \frac{1}{2K}\ln 2 = x_0$$

$$K = \sqrt{\frac{2 \times 9.1 \times 10^{-31} \times (7-3) \times 1.6 \times 10^{-19} \times 4 \times \pi^2}{(6.626 \times 10^{-34})^2}} = 1.02 \times 10^{10}$$

$x = 0.33864$ Å. $\qquad x_0 = 0.34$ Å

P73.



$\psi_0 = Ae^{ik_0x} + Be^{-ik_0x}$

$\psi_1 = Ce^{-k_1x} + De^{k_1x}$

$\psi_2 = Fe^{-k_2x}$ (no reflection)

At $x=0$: $\psi_0 = \psi_1$ $\quad A + B = C + D$

$\psi_0' = \psi_1'$ $\quad (B-A)k = -ik_1(C-D)$ — ①

At $x = L$: $\psi_1 = \psi_2$ $\quad Ce^{-k_1L} + De^{k_1L} = Fe^{-k_2L}$

$\psi_1' = \psi_2'$ $\quad -k_1(Ce^{-k_1L} - De^{k_1L}) = -k_2 Fe^{-k_2L}$ — ②

To show: $D = 0$.

Let's assume $D=0$. By ① & ②.

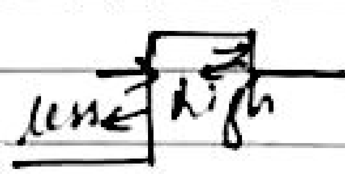
$$-k_1(Ce^{-k_1L}) = -k_2(Ce^{-k_1L}).$$

$$k_1 = k_2 \longrightarrow \times$$

(proof by contradiction)

$\therefore D \neq 0$.

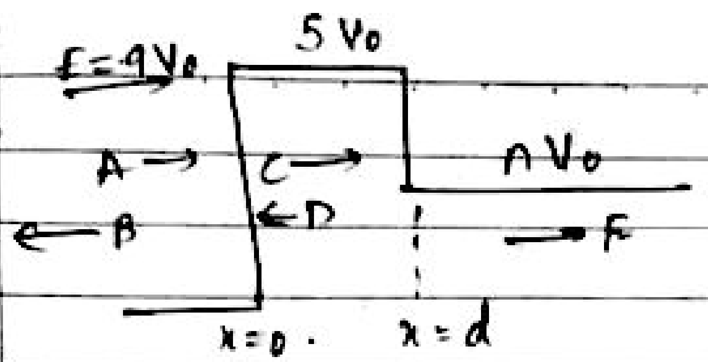
$\therefore$ claim is incorrect.

When going from  one to other potential ~~high potential~~ there must be both reflection and transmission at each barrier.

(unless $V \to \infty \to$ complete reflection

$V \to 0 \to$ complete transmission)

P74.



$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x} \qquad k_1 = \sqrt{\frac{2m(E)}{\hbar^2}} = \sqrt{\frac{2m(9V_0)}{\hbar^2}}$$

$$\psi_2 = Ce^{ik_2x} + De^{-ik_2x} \qquad k_2 = \sqrt{\frac{2m(4V_0)}{\hbar^2}}$$

$$\psi_3 = Fe^{ik_3x} \qquad k_3 = \sqrt{\frac{2m(9-n)V_0}{\hbar^2}}.$$

Boundary

at $x=0$ $\quad \psi_1 = \psi_2 : \left[A+B = C+D\right]$

$\psi_1' = \psi_2' : \left[ik_1(A-B) = ik_2(C-D)\right]$

at $x=d$ $\quad \psi_2 = \psi_3 : \left[Ce^{ik_2d} + De^{-ik_2d} = Fe^{ik_3d}\right]$

$\psi_2' = \psi_3' : \left[ik_2(Ce^{ik_2d} - De^{-ik_2d}) = ik_3 Fe^{ik_3d}\right]$

On solving

$$\frac{k_3}{k_2} = \frac{C-D}{C+D}.$$

Transmission $= \frac{3}{4}$

Reflection $= \frac{1}{4}$.

$$\frac{2}{3}\,\frac{A+B}{A-B} = \frac{C+D}{C-D} = \frac{k_2}{k_3}.$$

$$\frac{A+B}{A-B} = \frac{3}{\sqrt{9-n}}$$

$$\therefore \left|\frac{B}{A}\right|^2 = \frac{1}{4}$$

$$\therefore \left|\frac{B}{A}\right| = \frac{1}{2}.$$

$$\frac{1+\frac{B}{A}}{1-\frac{B}{A}} = \frac{3}{\sqrt{9-n}}$$

$$\therefore \frac{B}{A} = \pm\frac{1}{2}$$

$$\frac{B}{A} = \frac{1}{2} \therefore \frac{\frac{3}{2}}{\frac{1}{2}} = \frac{3}{\sqrt{9-n}} \qquad \underline{n=8}$$

$$\frac{B}{A} = -\frac{1}{2} \therefore \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{3}{\sqrt{9-n}}$$

$$\underline{n = -72}$$

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(b) $\frac{C+D}{C-D} = \frac{2}{1}$ $\quad \frac{B}{A} = \frac{1}{2}$ $\quad B = \frac{A}{2}$.

(for $n=8$.) $C = \frac{9}{8}A$, $D = \frac{3A}{8}$.

(c) yes, no.

$$B = A\frac{(k_1 - k_3)}{(k_1 + k_3)}.$$

phase change.

$$\mathrm{Im}\left(\frac{k_1 - k_3}{k_1 + k_3}\right)$$

$$\mathrm{Im}\left(\frac{3 - \sqrt{9-n}}{3 + \sqrt{9-n}}\right),$$

$n = 9$, $\mathrm{Im}(1) = 0$. $\therefore$ phase change $= 0$.

~~a=72~~. ~~809~~