

MA 105 D1 Lecture 5

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The derivative

Arnold's problem

Maxima and minima

Properties of differentiable functions

Beyond the first derivative

The definition

For now, if you did not understand the rigorous definition of the limit, forget about it. You will be able to understand what follows as long as you remember your 11th standard treatment of limits. Recall that $f : (a, b) \rightarrow \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case the value of the limit is denoted $f'(c)$ and is called the derivative of f at c . The derivative may also be denoted by $\frac{df}{dx}(c)$ or by $\frac{dy}{dx}|_c$, where $y = f(x)$.

In general, the derivative measures the rate of change of a function at a given point. Thus, if the function we are studying is the position of a particle on the x -coordinate, then $x'(t)$ is the velocity of the particle. If the function we are studying is the velocity $v(t)$ of the particle, then the derivative $v'(t)$ is the acceleration of the particle. If the function we are studying is the population of India, then the derivative measures the rate of change of the population.

The slope of the tangent

From the point of view of geometry, the derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. This becomes particularly clear if we rewrite the derivative as the following limit:

$$\lim_{y \rightarrow c} \frac{f(y) - f(c)}{y - c}.$$

The expression inside the limit obviously represents the slope of a line passing through $(c, f(c))$ and $(y, f(y))$, and as y approaches c this line obviously becomes tangent to $y = f(x)$ at the point $(c, f(c))$.

Another way of thinking of the derivative

Another way of thinking of the derivative of the function f at the point x_0 is as follows. If f is differentiable at x_0 we know that

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \rightarrow 0$$

as $h \rightarrow 0$. Since we are keeping x_0 fixed, we can treat the above quantity as a function of h . Thus we can write

$$\frac{f(x_0 + h) - f(x)}{h} - f'(x_0) = o(h)$$

for some function $o(h)$ with the property that $o(h) \rightarrow 0$ as $h \rightarrow 0$. Taking a common denominator,

$$\frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{h} = o(h) \tag{1}$$

We can use the above equality to give an equivalent definition for the derivative. A function f is said to be differentiable at the point x_0 if there exists a real number (denoted $f'(x_0)$) such that (1) holds for some function $o(h)$ such that $o(h) \rightarrow 0$ as $h \rightarrow 0$.

The derivative as a linear map

We can rewrite equation (1) as

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)h$$

Thus, the derivative of $f(x)$ at a point x_0 can be viewed as that real number (if it exists) by which you have to multiply h so that the resulting remainder goes to 0 faster than h (that is, the remainder divided by h goes to 0 as h goes to 0).

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ which has the property that $f(x + y) = f(x) + f(y)$ is called a linear function (or linear map). All such functions are given by multiplication by a real number, that is, every linear function has the form $f(x) = \lambda x$, for some real number λ . Thus the derivative may be regarded as a linear function (in the variable h). This point of view will be particularly useful in multivariable calculus.

Examples

Exercise 1.16: Let $f : (a, b) \rightarrow \mathbb{R}$ be a function such that

$$|f(x + h) - f(x)| \leq C|h|^\alpha$$

for all $x, x + h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.

Solution:

$$\left| \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \right| \leq C \lim_{h \rightarrow 0} |h|^{\alpha-1} = 0.$$

Note: Functions that satisfy the property above for some α (not necessarily greater than 1) are said to be **Lipschitz continuous with exponent α** .

Calculating derivatives

As with limits all of you are already familiar with the rule for calculating the sums, differences, products and quotients of derivatives. You should try and remember how to prove these.

You should also recall the **chain rule** for calculating the derivative of the composition of functions and try to prove it as an exercise using the $\epsilon - \delta$ definition of a limit.

Note that the proof of the chain rule given in some books involves writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}.$$

and then taking limits as $\Delta x \rightarrow 0$. This is not quite correct since Δu could be 0 even for infinitely many values of u .

Arnold's problem

Recall that you were asked to find the following limit (Exercise 3 of the previous lecture).

$$\lim_{x \rightarrow 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

The problem above was posed by the Russian mathematician Vladimir I. Arnold (see his book “Huygens and Barrow, Newton and Hooke”) as an example of a problem that seventeenth century mathematicians could solve very easily, but that modern mathematicians, even with all their extra machinery and knowledge can't. In fact, it was his habit to put up this problem while lecturing to eminent mathematicians in leading universities and challenge them to solve it within ten minutes.

Notice that the limit that we have to calculate has the form

$$\frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)}.$$

The solution to Arnold's problem

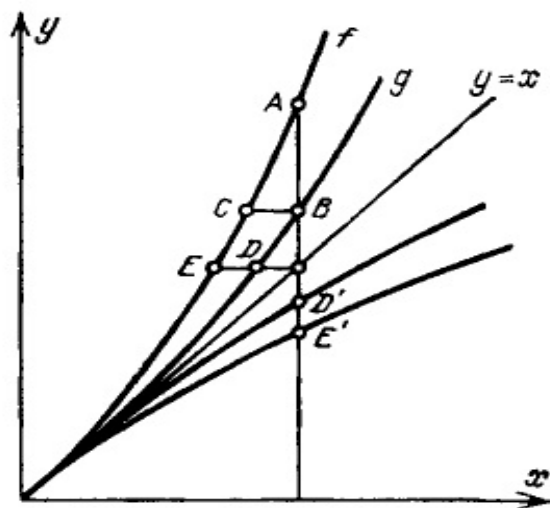


Fig. 37.

Calculation of the limit $|AB|/|D'E'|$

V. I. Arnold

The preceding picture was taken from V. I. Arnold's book
"Huygens and Barrow, Newton and Hooke" (Birkhauser 1990)



V. I. Arnold (1936-2008)
worked in geometry,
differential equations and
mathematical physics. He
was one of the most
important mathematicians
of the twentieth century.

Maxima and minima

Let $X \subset \mathbb{R}$ and let $f : X \rightarrow \mathbb{R}$ be a function (you can think of X as an open, closed or half-open interval, for instance).

Definition: The function f is said to attain a **maximum** (resp. **minimum**) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

Once again, I remind you that, in general, f may not attain a maximum or minimum at all on the set X . The standard example being $X = (0, 1)$ and $f(x) = 1/x$ (can you find an example on the closed interval $[0, 1]$?). However, if X is a closed bounded interval and f is a continuous function, Theorem 11 tells us that the maximum and minimum are actually attained. Theorem 11 is sometimes called the **Extreme Value Theorem**.

Maxima and minima and the derivative

If f has a maximum at the point x_0 and if it also differentiable at x_0 , we can reason as follows. We know that $f(x_0 + h) - f(x_0) \leq 0$ for every $h > 0$ such that $x + h \in X$. Hence, we see that (one half of the Sandwich Theorem!)

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

On the other hand, when $h < 0$, we get

$$\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Because f is assumed to be differentiable at x_0 we know that left and right hand limits must be equal. It follows that we must have $f'(x_0) = 0$. A similar argument shows that $f'(x_0) = 0$ if f has a minimum at the point x_0 .

Local maxima and minima

The preceding argument is purely **local**. Before explaining what this means, we give the following definition.

Definition: Let $f : X \rightarrow \mathbb{R}$ be a function and x_0 be in X . Suppose there is an sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a **local maximum** (resp. **local minimum**) at x_0 .

Sometimes we use the terms **global maximum** or **global minimum** instead of just maximum or minimum in order to emphasize the points are not just local maxima or minima. The argument of the previous slide actually proves the following

Theorem 13: If $f : X \rightarrow \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.

Rolle's Theorem

Theorem 13 is known as Fermat's theorem. It can be used to prove Rolle's Theorem.

Theorem 14: Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function which is differentiable in (a, b) and $f(a) = f(b)$. Then there is a point x_0 in (a, b) such that $f'(x_0) = 0$.

Proof: Since f is a continuous function on a closed bounded interval Theorem 11 tells us that f must attain its minimum and maximum somewhere in $[a, b]$. If both the minimum and maximum are attained at the end points, f must be the constant function, in which case we know that $f'(x) = 0$ for all $x \in (a, b)$. Hence, we can assume that at least one of the minimum or maximum is attained at an interior point x_0 and Theorem 13 shows that $f'(x_0) = 0$ in this case. □

One easy consequence: If $P(x)$ is a polynomial of degree n with n real roots, then all the roots of $P'(x)$ are also real. (How do we know that polynomials are differentiable?)

Problems centered around Rolle's Theorem

Exercise 2.3: Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and suppose f is differentiable on (a, b) . If $f(a)$ and $f(b)$ are of opposite signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then there is a unique point x_0 in (a, b) such that $f(x_0) = 0$.

Solution: Since the Intermediate Value Theorem guarantees the existence of a point x_0 such that $f(x_0) = 0$, the real point of this exercise is the uniqueness.

Suppose there were two points $x_1, x_2 \in (a, b)$ such that $f(x_1) = f(x_2) = 0$. Applying Rolle's Theorem, we see that there would exist $c \in (x_1, x_2)$ such that $f'(c) = 0$ contradicting our hypothesis. This proves the exercise.

Let us look at Exercise 2.8(i): Find a function f which satisfies all the given conditions, or else show that no such function exists:
 $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(0) = 1, f'(1) = 1$.

Solution: Apply Rolle's Theorem to $f'(x)$ to conclude that such a function cannot exist.

The Mean Value Theorem

Rolle's theorem is a special case of the Mean Value Theorem (MVT).

Theorem 15: Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and that f is differentiable in (a, b) . Then there is a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$



Proof: Apply Rolle's Theorem to the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

(Why does one think of the function $g(x)$?)

Applications of the MVT

Here is an application of the MVT which you have probably always taken for granted:

Theorem 16: If f satisfies the hypotheses of the MVT, and further $f'(x) = 0$ for every $x \in (a, b)$, f is a constant function.

Indeed, if $f(c) \neq f(d)$ for some two points $c < d$ in $[a, b]$,

$$0 \neq \frac{f(d) - f(c)}{d - c} = f'(x_0),$$

for some $x_0 \in (c, d)$, by the MVT. This contradicts the hypothesis.



Applications of the MVT continued

Consider Exercise 2.6.:

Let f be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = a$ and $f(b) = b$, show that there exist distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

Solution: The idea is that the function clearly has an average rate of growth equal to 1 on the interval $[a, b]$. If the derivative at some point is less than 1, there must be another point where it is greater than 1 so that the sum adds up to 2. How to use this idea?

Split the interval into two pieces - $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ - and apply the MVT to each interval.

Darboux's Theorem

Another interesting property of differentiable functions is that their derivatives have the IVP (intermediate value property). This fact is sometimes called Darboux's Theorem.

Theorem 17: Let $f : (a, b) \rightarrow \mathbb{R}$ be a differentiable function. If c, d , $c < d$ are points in (a, b) , then for every u between $f'(c)$ and $f'(d)$, there exists an x in $[c, d]$ such that $f'(x) = u$.

Proof: We can assume, without loss of generality, that $f'(c) < u < f'(d)$, otherwise we can take $x = c$ or $x = d$. Define $g(t) = ut - f(t)$. This is a continuous function on $[c, d]$, and hence, by Theorem 11, must attain its extreme values. ~~These extreme values~~ The maximum value cannot occur at c or d since $g'(c) = u - f'(c) > 0$ and $g'(d) = u - f'(d) < 0$ (~~contradicts Fermat's Theorem~~). ~~The mention of Fermat's theorem is misleading.~~ I should have been more careful with my arguments. See the next slide for my corrections.

Suppose g takes a maximum at c . Since $g'(c) > 0$, for $h > 0$ small enough, we must have $g(c + h) - g(c) > 0$, contradiction.

Suppose g takes a maximum at d . Since $g'(d) < 0$, for $h < 0$ small enough, we must have $g(d + h) - g(d) > 0$, contradiction.

It follows that there exists $x \in (c, d)$ where g takes ~~an extreme value~~ a maximum. By Fermat's Theorem $g'(x) = 0$ which yields $f'(x) = u$. □

Thanks to many of you who said you were confused by this and to Arjun who was persistent in making me go back and check.

Continuity of the first derivative

We have just seen that the derivative satisfies the IVP. Can we find a function which is differentiable but for which the derivative is not continuous?

Here is the standard example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0. \end{cases}$$

This function will be differentiable at 0 but its derivative will not be continuous at that point. In order to see this you will need to study the function in Exercise 1.13(ii). This will show that $f'(0) = 0$. On the other hand, if we use the product rule when $x \neq 0$ we get

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

which does not go to 0 as $x \rightarrow 0$.

Back to maxima and minima

We will assume that $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and that f is differentiable on (a, b) . A point x_0 in (a, b) such that $f'(x_0) = 0$ often called a **stationary point**. We will assume further that $f'(x)$ is differentiable at x_0 , that is, that the second derivative $f''(x_0)$ exists. We formulate the **Second Derivative Test** below.

Theorem 18: With the assumptions above:

1. If $f''(x_0) > 0$, the function has a local minimum at x_0 .
2. If $f''(x_0) < 0$, the function has a local maximum at x_0 .
3. If $f''(x_0) = 0$, no conclusion can be drawn.

The proof of the Second Derivative Test

Proof: The proofs are straightforward. For instance, to prove the first part we observe that

$$0 < f''(x_0) = \lim_{h \rightarrow 0} \frac{f'(x_0 + h)}{h}.$$

It follows that for $|h|$ small enough, $f'(x_0 + h) < 0$, if $h < 0$ and $f'(x_0 + h) > 0$ if $h > 0$. It follows that $f(x_0)$ is decreasing to the left of x_0 and increasing to the right of x_0 . Hence, x_0 must be a local minimum. A similar argument yields the second case. \square

If the third case of the theorem above occurs, the function may be changing from concave to convex. In this case x_0 is called a **point of inflection**. An example of this phenomenon is given by $f(x) = x^3$ at $x = 0$.