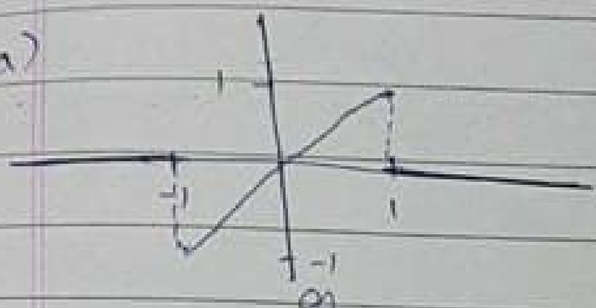
Tutorial 5

P39. $f(x) = x \qquad -1 \le x \le 1 \qquad f(x) = 0$ elsewhere.

a)



b) $g(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx = \int_{-1}^{1} x e^{ikx} dx$

$$\left[\frac{x e^{ikx}}{ik} - \frac{e^{ikx}}{(ik)^2} \right]_{-1}^{1}$$

$$\left[\frac{e^{ik}}{ik} - \frac{e^{ik}}{(ik)^2} \right] - \left[\frac{-e^{-ik}}{ik} - \frac{e^{-ik}}{(ik)^2} \right]$$

$$\frac{e^{ik} + e^{-ik}}{ik} + \frac{e^{-ik} - e^{ik}}{(ik)^2} \qquad \frac{e^{ik} - e^{-ik}}{k^2}$$

$$\frac{-i \times 2}{k} (\cos k) + \frac{2i \sin k}{k^2}$$

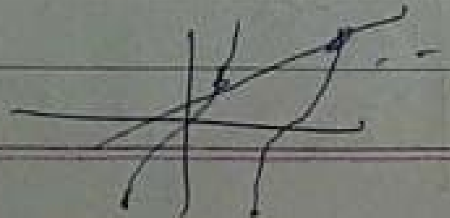
$$g(k) = \frac{2i}{k} \left(\frac{\sin k}{k} - \cos k \right).$$

c) $|g(k)| = \left| \frac{2}{k} \left(\frac{\sin k}{k} - \cos k \right) \right|.$

$k \to 0 \quad |g(k)| \to \infty$. Let's find zeroes

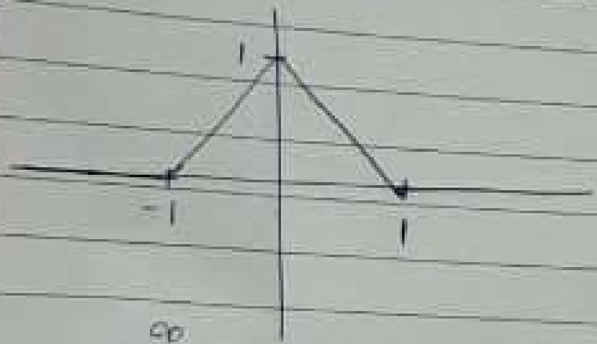
$$\frac{\sin k}{k} - \cos k = 0.$$

$\tan k = k$ for some $k_0, k_1, \ldots$



P40*

(a) $f(x) = 1 - |x|$. $\quad -1 \le x \le 1 \quad f(x) = 0.$



(b) $g(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$

$$= \int_{-1}^{0} (1+x) e^{ikx}\, dx + \int_{0}^{1} (1-x) e^{ikx}\, dx$$

$$\int_{-1}^{1} e^{ikx}\, dx + \int_{-1}^{0} x e^{ikx}\, dx - \int_{0}^{1} x e^{ikx}\, dx$$

$$\left[\frac{e^{ikx}}{ik}\right]_{-1}^{1} + \left[\frac{x e^{ikx}}{ik} - \frac{e^{ikx}}{(ik)^2}\right]_{-1}^{0} + \left[\frac{x e^{ikx}}{ik} - \frac{e^{ikx}}{(ik)^2}\right]_{0}^{1}$$

$$\left[\frac{e^{ik} - e^{-ik}}{ik}\right] \checkmark + \left\{\left[\frac{-1}{(ik)^2}\right] \checkmark + \left[\frac{+e^{-ik}}{ik} - \frac{e^{-ik}}{(ik)^2}\right]\right\} + \left\{\left[\frac{-1}{(ik)^2}\right] \checkmark - \left[\frac{e^{ik}}{ik} + \frac{e^{ik}}{(ik)^2}\right]\right\}$$

$$\frac{\cancel{2\sin k}}{k} + \frac{2}{k^2} - \frac{e^{ik}}{k^2} - \frac{e^{ik}}{ik} + \frac{e^{-ik}}{ik} - \frac{e^{-ik}}{k^2}$$

$$-\frac{2\cos k}{k^2} - \frac{\cancel{2\sin k}}{k}$$

$$= \frac{2-2\cos k}{k^2} = \frac{2}{k^2}(1-\cos k)$$

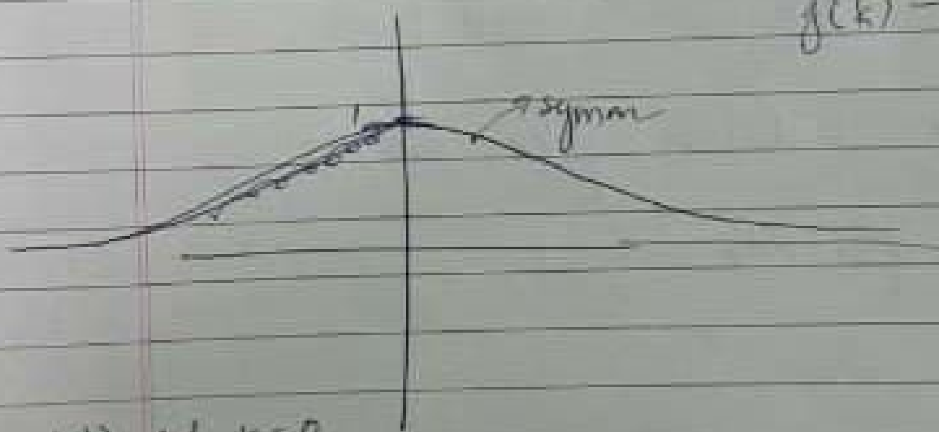
$$= \frac{2\cdot 2\sin^2 k/2}{k^2}$$

$$= \boxed{\frac{4\sin^2\frac{k}{2}}{k^2}}$$

(c) $k \to 0$ $\left(\frac{\sin^2\frac{k}{2}}{\left(\frac{k}{2}\right)^2}\right) = 1$

$k \to \infty$

$f(k) \to 0$



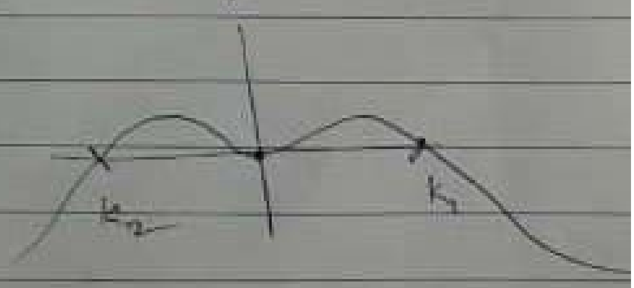
(d) At $k=0$

(e) $\frac{1}{2} = \frac{4\sin^2 k/2}{k^2}$

$\frac{k^2}{8} = \sin^2\frac{k}{2}$

$\frac{k}{2\sqrt{2}} = \sin\frac{k}{2}$

$\sin^2\frac{k}{2} - \frac{k^2}{8}$



(f)

P41 $\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$.

$\therefore \Delta p_x \geq 5.27 \times 10^{-25}$

% error = $\frac{\Delta p_x}{p_x} \times 100 = 3.08\%$

(Assumption: all momentum is along x-axis to get max error $p_x = 1.7 \times 10^{-23}$).

P42 Since X and Y axes are independent, can't find along Y

P42 check. e^- falls from a height of 10 m.

$v = \sqrt{2gh} = 10\sqrt{2}$ m/s

DeBroglie $\lambda = \frac{h}{mv} = 5.1 \times 10^{-3}$ cm.

Dimension of slit = ~~[illegible]~~ 2 cm (10^{2})

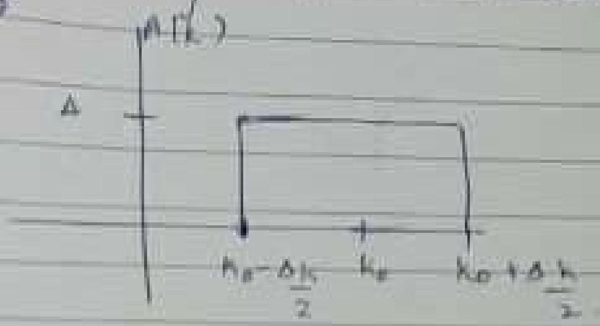
All matter behave like waves

wavelength << slit

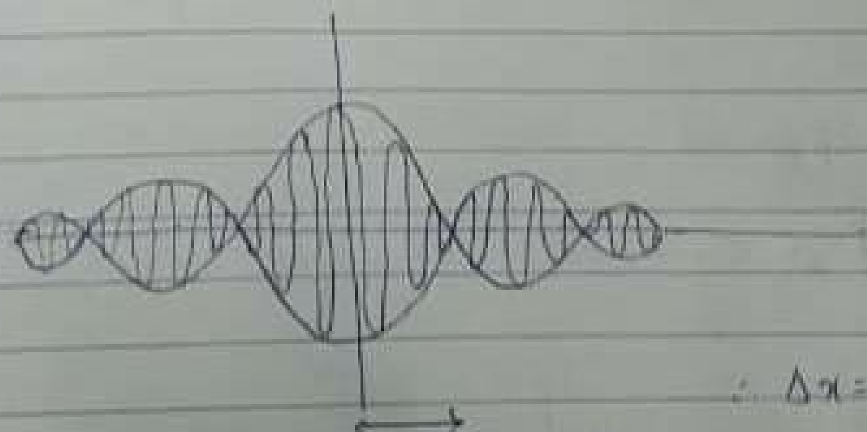
$\therefore$ particle nature.

Generated by CamScanner

P45*. $y(x,t) = \int A(k) \cos(kx - \omega t)\, dk$



$$y(x,t) = \underbrace{\frac{2A}{x} \sin\left(\frac{\Delta k x}{2}\right)}_{\text{Envelope}} \underbrace{\cos\left(k_0 x - \omega t\right)}_{\text{oscillatory term}}.$$



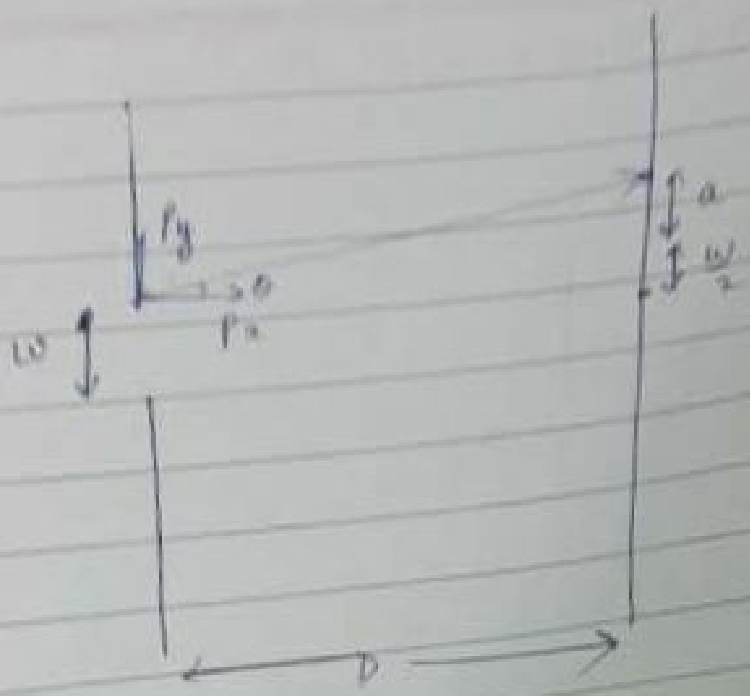
$\therefore \Delta x = \frac{2\pi}{\Delta k}$ — ①

$\therefore \Delta x = \frac{2\pi}{\Delta k}$

$\Delta x \cdot \Delta k = 2\pi > \frac{1}{2}$ ✓

$$\left(\begin{array}{l} \Delta x\, \Delta(\hbar k) \geq \frac{\hbar}{2} \\ \Delta x \cdot \Delta k \geq \frac{1}{2} \end{array} \right)$$

P 94*



$$\tan\theta = \frac{a}{D} = \frac{p_y}{p_x} \approx \frac{p_y}{p} \qquad a = \frac{D p_y}{p}$$

$$\Delta p_y \cdot \Delta y \geq \frac{\hbar}{2}$$

$$\Delta p_y \cdot w \geq \frac{\hbar}{2w}$$

minimize condition $\quad \Delta p_y = \frac{\hbar}{2w}$

$p_y + \Delta p_y$, $p_y - \Delta p_y$

$$2p_y = \Delta p_y$$

Answer $\Rightarrow$ spot size :

$$R = \frac{w}{2} + a$$

$$R = \frac{w}{2} + \frac{D p_y}{p}$$

$$= \frac{w}{2} + \frac{D \Delta p_y}{2p}$$

$$R = \frac{w}{2} + \frac{D\hbar}{4wp}$$

minimize this

45(a) $\Delta x \cdot \Delta p = \hbar$

$$\Delta p = \frac{6.626 \times 10^{-34}}{2 \times 3.14 \times 10^{-14}}$$

$$\Delta p = 1.05 \times 10^{-20}.$$

$\because \Delta p$ is uncertainty in momentum, momentum of electron should atleast ie $p = 1.05 \times 10^{-20}$.

$\Longrightarrow$ high momentum, $\therefore$ velocity comparable to speed of light

$\therefore$ calculate energy relativistically

$$\boxed{E = \sqrt{m_i^2 c^4 + p^2 c^2}}$$

$$\therefore E = \sqrt{(9.1)^2 \times 10^{-62} \times (3 \times 10^8)^4 + (1.05 \times 10^{-20})^2 \times (3 \times 10^8)^2}$$

$$E = 3.15 \times 10^{-12} \text{ J}$$

$$\boxed{E = 19.6 \text{ MeV}}$$

$\therefore$ If electron exists inside the nucleus, it should've energy $= 19.6$ MeV (to the order of).

However, β particles ejected from nucleus have energies ≈ 3 MeV.

Also, experimental results show that no electron / particle in an atom possess energy greater than 4 MeV.

$\therefore$ Electrons do not exist inside nucleus.

P45 (b) Total energy $= KE_{av} + PE_{av}$

$$= \frac{\langle p_x^2 \rangle}{2m} + \frac{1}{2} k x^2_{av}$$

Average PE $= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} k x_i^2$

$$= \frac{k}{2} \sum_{i=1}^{n} \langle x^2 \rangle$$

$$(\Delta x)^2 = \sigma^2 = \langle x^2 \rangle - (\langle x \rangle)^2$$

you've to assume $\Delta x = 0$ to get right answer

$$= \langle x^2 \rangle - 0$$

$$\therefore (\Delta x)^2 = \langle x^2 \rangle .$$

Total energy $= \frac{(\Delta p_x)^2}{2m} + \frac{k}{2} (\Delta x)^2 .$

$\because \Delta x \cdot \Delta p \geq \frac{\hbar}{2}$ $\therefore \Delta p = \frac{\hbar}{2 \Delta x}$.

Total energy $= \left(\frac{\hbar}{2 \Delta x} \right)^2 \frac{1}{2m} + \frac{k}{2} (\Delta x)^2$

Differentiate wrt Δx.

$\frac{dE}{d(\Delta x)} = 0$, get value of Δx, substitute

$(\text{Total energy})_{\text{minimum}} = \frac{1}{2} \hbar \omega$ $\left(\omega = \sqrt{\frac{k}{m}} \right.$

(c) $PE = -\frac{Ke^2}{r}$ $KE = \frac{(\Delta p_x)^2}{2m}$

$\Delta x = r$

$\Delta p_x = \frac{\hbar}{2r}$ $\left(\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2} \right)$.

Total energy $= -\frac{Ke^2}{r} + \frac{\hbar^2}{8mr^2}$

Differentiate wrt r.

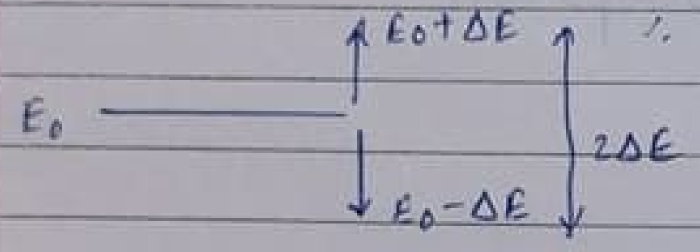
Min total energy is ground state energy.

P4) $\frac{E}{c} = mv$

Recoil energy $= \frac{1}{2}mv^2$
$= \frac{E^2}{2mc^2}$

$\Delta e \cdot \Delta t \geq \frac{\hbar}{2}$.

$\Delta E = 3.3 \times 10^{-8}\,eV$.



$\therefore$ Width $= 2\Delta E = 6.6 \times 10^{-8}\,eV$.

$\frac{E^2}{2mc^2} = 6.6 \times 10^{-8}\,eV$.

$m = 100 \times 1.6 \times 10^{-27}\,kg$

$\therefore E \approx 111\,eV$.

(For those interested!)

Resonant absorbtion theory:

When we've an electron absorbing a photon and undergoing transition of energy E_1, the energy ~~~~ of the photon would've to be greater than E_1 in order to compensate for the KE of the atom. Similarly, if a photon is released from a transition ~~that~~ then its energy is less than E_1.

To conserve E, we need recoil energy to be of the order of natural line width. Now for emitted photon to be absorbed by another atom, it must belong to some transition energy. But since the other atom is also moving with recoil energy, it'll see a doppler shifted photon

which it won't be able to absorb.
The only resolution is if all three are of the same order, in which case, absorption is possible.
(reason for cosmic microwave background to be present everywhere → resonant absorption)
→ Basically the math behind doppler shift: recoil energy being the same as the photon energy enable the absorption.

P48. Energy conservation

$$0 + \frac{3}{2}k_BT + E_2 = E_{10} + E_{recoil} \quad \text{— (1)}$$

Momentum conservation

$$-\frac{E_2}{c} + \underbrace{\sqrt{3kTm}}_{\sqrt{2m\underbrace{E}_{\frac{3}{2}k_BT}}} = -p_{recoil}$$

$$E_2 = E_{10} + \frac{E_2^2}{2mc^2} - \frac{E_2}{m}\sqrt{3kTm}$$

$E_{10} = 10.2\,eV \rightarrow$ Known

$E_2 = 10.19990886\,eV$

$E_1 = 10.2\,eV$.