MA 105 D1 Lecture 23

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Parametrised surfaces

The tangent plane

Surface area

Orientability

The definition of a parametrised surface

A curve is a "one-dimensional" object. Intuitively, this means that if we want to describe a curve, it should be possible to do so using just one variable or parameter. To do line integration, we further required some extra properties of the curve - that it should be $\mathcal{C}^1$ and non-singular. In this lecture we are going to develop the analogous theory for surfaces.

In order to describe a surface, which is a two-dimensional object, we clearly need two parameters. And, in order to do this we will need some analogue of a non-singular curve.

Definition: Let D be a domain in $\mathbb{R}^2$. A parametrised surface is a function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

Geometric parametrised surfaces

In all examples, D will be a connected open set in $\mathbb{R}^2$. As with curves and paths we will distinguish between the surface $\mathbf{\Phi}$ and its image. Thus the image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

(In fact, it is customary to call paths geometric curves.)

Note that for a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$. Each of the coordinates of the vector depends on u and v. Hence we write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on D.

Examples

Example 1: Graphs of real valued functions are parametrised surfaces.

Indeed, let $f(x, y)$ be a scalar function and let $z = f(x, y)$. If D is the domain of f in $\mathbb{R}^2$, we can define the parametrised surface $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)).$$

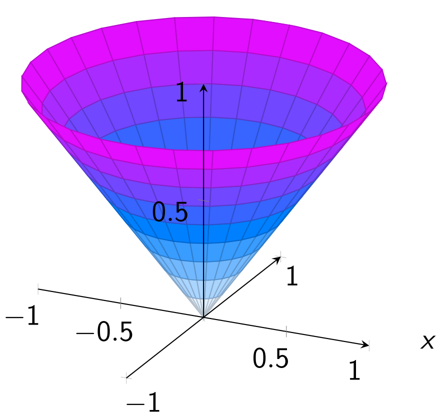
More specifically, we have $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$.

Example 2: Consider $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$x = u \cos v, \quad y = u \sin v, \quad \text{and} \quad z = u,$$

$u \geq 0$. What geometric surface does it describe?

A right-circular cone.



The graph of $z = \sqrt{x^2 + y^2}$, also known as the parametrised surface

$$x = u\cos v, \quad y = u\sin v, \quad \text{and} \quad z = u, \quad u \geq 0.$$

A hyperboloid with one sheet

Example 3: Find a parametrisation for the hyperboloid of one sheet: $x^2 + y^2 - z^2 = 1$

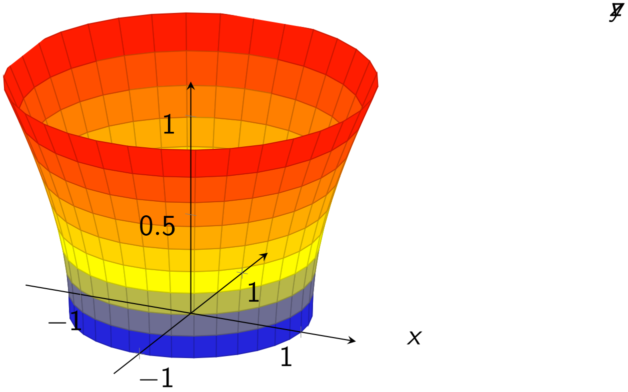
Solution: Note that the figure is obviously invariant under rotations about the z axis, since this would leave the quantities $x^2 + y^2$ and z^2 both unchanged. Whenever one has such invariance, it is logical to use polar coordinates to parametrise x and y. Thus, we set

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

The equation of the hyperboloid now becomes $r^2 - z^2 = 1$, so it is reasonable to parametrise r and z by hyperbolic functions: $r = \cosh u$, $z = \sinh u$. Hence, we obtain

$$x(u,\theta) = \cosh u\cos\theta,\ y(u,\theta) = \cosh u\sin\theta \quad \text{and} \quad z = \sinh u$$

as a parametrisation. Here $0 \leq \theta \leq 2\pi$ and $-\infty < u < \infty$.



The graph of $x^2 + y^2 - z^2 = 1$, also known as the parametrised surface

$$x = \cosh u \cos\theta, \quad y = \cosh u \sin\theta, \quad \text{and} \quad z = \sinh u,$$

Surfaces of revolution around the x-axis

Example 4: The preceding example involved rotations around the z-axis. If we have the graph of a function $y = f(x)$ which we rotate around x-axis, we can parametrise it as follows:

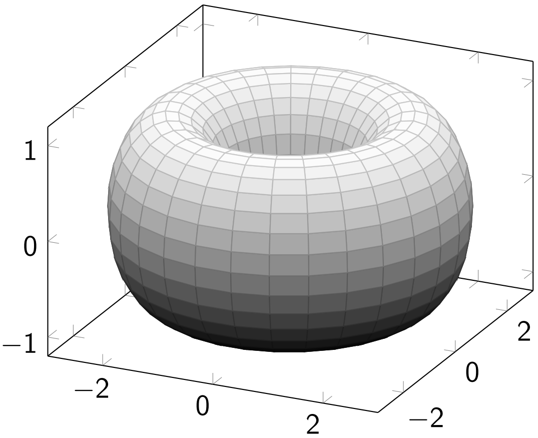
$$x = u, \quad y = f(u)\cos v, \quad \text{and} \quad z = f(u)\sin v$$

Here $a \leq u \leq b$ if $[a, b]$ is the domain of f, and $0 \leq v \leq 2\pi$.

Exercise 1: Find a parametrisation for a torus - that is the outer surface of a doughnut or cycle tire tube.

This shows that parametrised surfaces are more general than graphs of functions.

A picture of the torus



Tangent vectors for a parametrised surface

Let $\mathbf{\Phi}(u, v)$ be a parametrised surface. If we fix the variable v, say $v = v_0$, we obtain a curve $\mathbf{c}(u, v_0)$ that lies on the surface. Thus

$$\mathbf{c}(u) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

What is the tangent vector to the curve at the point u_0? Clearly, it is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

So far we have talked only about partial derivatives for scalar valued functions. However, we can *define* the partial derivative of a vector valued function by

$$\mathbf{\Phi}_u(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing u and varying v we obtain a curve $\mathbf{d}(u_0, v)$ and we can set

$$\mathbf{\Phi}_v(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent plane

The two tangent vectors $\mathbf{\Phi}_u$ and $\mathbf{\Phi}_v$ will, in general define a plane. Indeed, the normal to this plane $\mathbf{n}$ is simply given by $\mathbf{n} = \mathbf{\Phi}_u \times \mathbf{\Phi}_v$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

More explicitly, if $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the equation of the plane is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Let us find the equation of the tangent plane at points on the various parametrised surfaces we have already looked at.

Example 1: This was the case when we were looking at the graph of a function $z = f(x, y)$, so $\mathbf{\Phi}(u, v) = (u, v, f(u, v))$. In this case

$$\mathbf{\Phi}_u = \mathbf{i} + \frac{\partial f}{\partial u}(u_0, v_0)\mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v = \mathbf{j} + \frac{\partial f}{\partial v}(u_0, v_0)\mathbf{k}.$$

Hence,

$$\mathbf{n} = \mathbf{\Phi}_u \times \mathbf{\Phi}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).$$

Thus the equation of the tangent plane is

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) = 0;$$

which yields,

$$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0).$$

Example 2: This was the example of the right circular cone. The parametric surface was given by

$$\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u),$$

with $u \geq 0$. In this case we get

$$\mathbf{\Phi}_u = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v = -u \sin v\mathbf{i} + u \cos v\mathbf{j},$$

whence

$$\mathbf{n} = (-u \cos v, -u \sin v, u).$$

Note that if $(u, v) = (0, 0)$, then $\mathbf{n} = 0$, so the tangent plane is not defined at the origin. However, it is defined at any other point.

Non-singular surfaces

In analogy with the situation for curves, we will call $\mathbf{\Phi}$ a regular of non-singular parametrised surface if $\mathbf{\Phi}_u \times \mathbf{\Phi}_v \neq 0$ at all points. Moreover, we will assume that $\mathbf{\Phi}$ is a $\mathcal{C}^1$ function, again, except possibly along a finite union of curves on the surface (where it should still be assumed to be continuous).

As we just saw, the right circular cone is not a regular parametrised surface. However, as with curves, integration of bounded functions will not pose a problem if we omit the origin. Indeed, we are free to omit the a finite union of graphs of curves where the surface fails to have a non-zero normal vector.

Exercise 2: Find the tangent plane at an arbitrary point of the one sheeted hyperboloid and the torus.

Parametrisation and coordinate change

As before, let S be the geometric surface corresponding to a parametrised surface $\mathbf{\Phi}(u, v)$. Given a rectangle R with corners (u, v), $(u + \Delta u, v)$, $(u + \Delta u, v + \Delta v)$ and $(u, v + \Delta v)$, we would like to compute the area of the "area element" on S bounded by the four points $\mathbf{\Phi}(u, v)$, $\mathbf{\Phi}(u + \Delta u, v)$, $\mathbf{\Phi}(u + \Delta u, v + \Delta v)$ and $\mathbf{\Phi}(u, v + \Delta v)$.

Actually, we have already done this! This is exactly what we did when computing the formula for the change of variables, except that in that case, the coordinate change took an area in $\mathbb{R}^2$ back to an area in $\mathbb{R}^2$ (recall the situation for polar coordinates). The only difference now is that $\mathbf{\Phi}(R)$ no longer lies in the plane. This doesn't really change anything.

We do have to be a little careful: we must make sure that $\mathbf{\Phi}$ is bijective and that it is non-singular. In fact, the inverse function theorem guarantees us that if $\mathbf{\Phi}$ is non-singular it is automatically bijective in a small enough neighborhood on the surface.

The area vector of an infinitesimal surface element

We see that $\mathbf{\Phi}$ takes the small rectangle R to the parallelogram given by the vectors $\mathbf{\Phi}_u \Delta u$ and $\mathbf{\Phi}_v \Delta v$.

It follows that the area $\Delta\mathbf{S}$ of this parallelogram is

$$\Delta\mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v)\Delta u \Delta v.$$

Thus the surface area is to be thought of as a vector pointing in the direction of the normal to the surface.

The magnitude of the surface area is given by

$$dS = \|d\mathbf{S}\| = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\Delta u \Delta v$$

and we can write

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

The magnitude of the area vector

It remains to compute the magnitude dS. To do this we must find $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$. Writing this out in terms of x, y and z, we see that

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

Hence,

$$dS = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The surface area integral

Because of the calculations we have just made, the surface area is given by the double integral

$$\iint_S dS = \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The area is nothing but the integral of the constant function 1 on the surface S. We can likewise integrate any scalar function $f : S \to \mathbb{R}$:

$$\iint_S f dS = \iint_D f(x,y,z) \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

If Σ is a union of parametrised surfaces S_i that intersect only along their boundary curves, then we can define

$$\iint_\Sigma f dS = \sum_i \iint_{S_i} f dS.$$

The surface integral of a vector field

It is just as easy to integrate vector fields. Indeed, we proceed almost exactly like we did for line integrals:

Let $\mathbf{F}$ be a vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : D \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_D \mathbf{F}(\mathbf{\Phi}(u,v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv.$$

This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

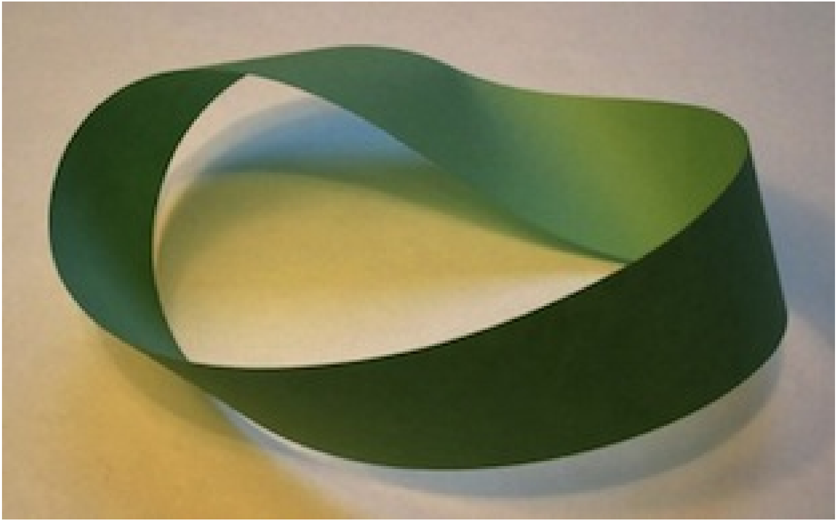
Oriented surfaces - a first attempt

In what follows we will assume that any parametrised surface $\mathbf{\Phi}$ is $\mathcal{C}^1$ and non-singular. Recall that we made the following definition.
Definition: An oriented surface S is a two-sided surface with one side specified as the outside (or positive side) and the other side as the inside (or negative side).

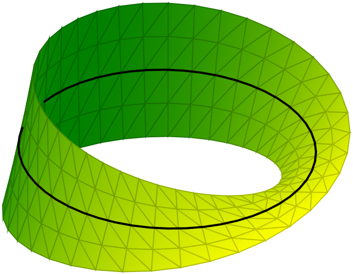
What is the problem with this definition?

There are surfaces which have only one side! The easiest such surface is the Möbius strip, named after its discoverer, a famous Swedish mathematician of the eighteenth century.

The Möbius strip



Another Möbius strip



Orientable surfaces - a second attempt

What the preceding example shows us is that we need a definition that helps us make sense of terms like "inside" and "outside". Let us try a second definition.

Recall that for us a vector field on a surface S is a function $\mathbf{F} : S \to \mathbb{R}^3$.

Definition: A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

At each point of S there are two possible directions for the normal vector to S. The question is whether the normal vector field be can be chosen so that the resulting vector field is continuous.

Examples of orientable surfaces

Example: For the unit sphere in $\mathbb{R}^3$ we can choose an orientation by selecting the unit vector $\hat{\mathbf{n}}(x, y, z) = \hat{\mathbf{r}}$, where $\mathbf{r}$ points outwards from the surface of the sphere.

More explicitly, we define

$$\mathbf{F}(x, y, z) = (x, y, z).$$

This obviously defines a continuous vector field on S. Hence, we see that the unit sphere in $\mathbb{R}^3$ is orientable.

Notice, that we can also define a vector field $\mathbf{G}(x, y, z) = -(x, y, z)$. The vector field $\mathbf{G} = -\mathbf{F}$ is also obviously continuous. This is not specific to the example of a the sphere.

Choosing an orientation

As we have just seen in the preceding example, if S is an orientable surface and $\mathbf{F}$ is a continuous vector field of unit normal vectors, so is $-\mathbf{F}$.

An orientable surface together with a specific choice of continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

Once one has chosen a particular vector field of normal vectors it makes sense to talk about the "outside" or "positive side" of the surface: usually, it is the side given by the direction of the unit normal vector. The other side is then called the "inside" or "negative side". However, which side one calls "positive" or "negative" is a matter of choice.

The preceding paragraph shows why the second (and correct) definition of orientation is related to the first.