MA 105 D1 Lecture 22

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Recap: Conservative fields, Reparametrisation

Orientation and the Jordan curve theorem

Green's Theorem

Various examples

Other forms of Green's theorem

Vector fields and line integrals

In what follows we need only assume that the vector fields in question are continuous (not smooth).
We will now define the integral of a vector field along a curve. We will assume that we are given a $\mathcal{C}^1$ curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}'(t) \neq 0$ for any $t \in [a, b]$. Such a curve will be called a regular or non-singular parametrised curve.

We define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

Gradient fields

The main observation about line integrals is the following. Suppose the vector field $\mathbf{F}$ can be written as the gradient of a scalar function f, that is, $\nabla f = \mathbf{F}$, then

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = f(c(b)) - f(c(a)).$$

This shows that the value of the line integral depends only on the value of the function at the end points of the curve, not on the curve itself. Such vector fields are called conservative.

Path connectedness

Definition: A subset C of $\mathbb{R}^n$ is said to be path connected if any two points in the subset can be joined by a path (that is the image of a continuous curve) inside C.

The converse to our previous assertion is also true.

Theorem 39: Let $\mathbf{F} : D \to \mathbb{R}^3$ be a conservative vector field on a path connected open domain in $\mathbb{R}^3$. Then $\mathbf{F}$ is the gradient of a scalar function.

Curves and paths

Remember: "Curves" refers to ($\mathcal{C}^1$-)functions from $[t_1, t_2]$ to $\mathbb{R}^3$.
"Paths" refers to their images.

Let $\mathbf{c}(t) : [t_1, t_2]$ be a regular parametrised curve (this means that $\mathbf{c}'(t) \neq 0$). If $h : [u_1, u_2] \to [t_1, t_2]$ is a $\mathcal{C}^1$ diffeomorphism, $\gamma(u) = \mathbf{c}(h(u))$ is called a reparametrisation of $\mathbf{c}$.

The same path may have many different parametrisations.

Reparametrisation and orientation

There is one further piece of information that a curve carries and that the path (its image) does not. Given two points P and Q in $\mathbb{R}^3$ and a path connecting them, we can ask whether the path is traversed from P to Q or from Q to P.

Since a curve, say $\mathbf{c} : [a, b] \to \mathbb{R}^3$ comes with the information that $\mathbf{c}(a) = P$ while $\mathbf{c}(b) = Q$, (or vice-versa) it allows us to determine the direction in which the path is being traversed. This direction is called its orientation.

If the diffeomorphism h preserves the orientation of $\mathbf{c}$,

$$\int_\gamma \mathbf{F} \cdot ds = \int_{\mathbf{c}} \mathbf{F} \cdot ds.$$

If the diffeomorphism reverses the orientation,

$$\int_\gamma \mathbf{F} \cdot ds = -\int_{\mathbf{c}} \mathbf{F} \cdot ds.$$

Oriented curves

Recall that a simple closed curve $\gamma : [a, b] \to \mathbb{R}^3$ a continuous function such that $\gamma(a) = \gamma(b)$ and $\gamma(x_1) \neq \gamma(x_2)$ for all $a < x_1, x_2 < b$.

For planar curves we see that for a simple closed path lying in a plane, we get a choice of direction - clockwise or anti-clockwise. Making such a choice gives rise to an oriented curve. By convention, the positive orientation corresponds to the anti-clockwise direction and the negative orientation to the clockwise direction.

If the curve does not necessarily lie on a plane, the usual convention is the following:
We will say that the curve is positively oriented if the surface bounded by the curve always lies to the left of an observer walking along the curve in the chosen direction. Otherwise, we will say that the curve is negatively oriented. Note, that this agrees with our convention for planar curves above.

The Jordan curve theorem

What is the problem with the previous definition?

Why is it clear that a simple closed curve in $\mathbb{R}^3$ bounds a surface?

For planar curves, we have the Jordan curve theorem.

Theorem 40: The complement of a simple curve C in the plane consists of exactly two connected components, one of which is bounded and the other which is not bounded.

(A connected component consists of all points which can be joined to each other by a path which does not intersect C.)

Orienting the boundary curve

Suppose C is the boundary of a region D in the plane, but C may now consists of several components or pieces and D may have "holes".

Definition: The positive orientation of C is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the normal vector field pointing outward along the curve.

Physically, this means that if one walks along C in the direction of the positive orientation, the region D is always on one's left.

As we shall see later, if C is a closed curve in space bounding an oriented surface S, the orientation of S naturally induces an orientation on the boundary C. The above example is a special case of this.

Green's Theorem

With the preliminaries out of the way, we are now in a position to state the first major theorem of vector calculus, namely Green's Theorem.

Theorem 41 (Green's theorem): Let D be a connected open set in $\mathbb{R}^2$ with a positively oriented boundary consisting of a finite union of piecewise continuously differentiable curves. If $M : D \to \mathbb{R}$ and $N : D \to \mathbb{R}$ are $\mathcal{C}^1$ functions, then

$$\int_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a surface integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

Example 1: Let C be the circle of radius r oriented in the counterclockwise direction, and let $M(x, y) = -y$ and $N(x, y) = x$. Evaluate

$$\int_C M(x, y)dx + N(x, y)dy.$$

Solution: Let $\mathbf{F} = (-y, x, 0)$ and D denote the disc of radius r. Then $N_x - M_y = 2$. Hence, by Green's theorem

$$\int_C M(x, y)dx + N(x, y)dy = \int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D 2dxdy = 2\pi r^2.$$

Area of a region

The preceding example shows us that the area of a region enclosed can be expressed as a line integral. If C is a positively oriented curve that bounds a region D , then the area $A(D)$ is given by

$$A(D) = \frac{1}{2} \int_C x dy - y dx.$$

Example 2: Let us use the formula above to find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: We parametrise the curve C by $\mathbf{c}(t) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$. By the formula above, we get

$$\begin{aligned}
\text{Area} &= \frac{1}{2} \int_C x dy - y dx \\
&= \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) dt - (b \sin t)(-a \sin t) dt \\
&= \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab.
\end{aligned}$$

Polar coordinates

Suppose we are given a simple positively oriented closed curve $C : (r(t), \theta(t))$ in polar coordinates. Then, by the area formula above, we know that the area enclosed by C is given by

$$\frac{1}{2}\int_C xdy - ydx := \frac{1}{2}\int_C \left(x(r,\theta)\frac{dy}{dt} - y(r,\theta)\frac{dx}{dt} \right) dt$$

$$\frac{1}{2}\int_a^b x(r(t),\theta(t))\frac{\partial y}{\partial r}\frac{dr}{dt}dt + \frac{1}{2}\int_a^b x(r(t),\theta(t))\frac{\partial y}{\partial \theta}\frac{d\theta}{dt}dt$$

$$-\frac{1}{2}\int_a^b y(r(t),\theta(t))\frac{\partial x}{\partial r}\frac{dr}{dt}dt - \frac{1}{2}\int_a^b y(r(t),\theta(t))\frac{\partial x}{\partial \theta}\frac{d\theta}{dt}dt$$

$$= \frac{1}{2}\int_a^b r(t)\cos\theta(t)\sin\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r^2(t)\cos^2\theta(t)\frac{d\theta}{dt}dt$$

$$- \frac{1}{2}\int_a^b r(t)\sin\theta(t)\cos\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r(t)^2\sin^2\theta(t)\frac{d\theta}{dt}dt$$

$$= \frac{1}{2}\int_C r^2 d\theta.$$

Exercise 10.3.(i): Find the area of the cardioid $r = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$.

Solution: Using the formula we have just derived, the desired area is simply

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta = \frac{a^2}{2}\int_0^{2\pi} -2\cos\theta + \frac{\cos 2\theta}{2} + \frac{3}{2} d\theta$$
$$= \frac{3a^2\pi}{2}.$$

A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 - the rectangle being the most important such figure. Thus we will assume that D lies between $x = a$, $x = b$, $y = \phi_1(x)$ and $y = \phi_2(x)$. Similarly, we will assume the that D lies between $y = c$, $y = d$, $x = \psi_1(y)$ and $x = \psi_2(y)$.

Consider the double integral

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_D \frac{\partial N}{\partial x} dxdy - \iint_D \frac{\partial M}{\partial y} dxdy.$$

Since this is a region of type 2, the first double integral on the right hand side can be written as

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial N}{\partial x}(x, y) dxdy.$$

The proof of Green's theorem, continued

Using the Fundamental Theorem of Calculus we get

$$\int_c^d N(\psi_2(y), y) - N(\psi_1(y), y) dy = \int_c^d N(\psi_2(y), y) dy - \int_c^d N(\psi_1(y), y) dy.$$

Now we have to interpret the integrals on the right hand side as line integrals. But this is easy, since the integrand is already essentially in a parametrised form. Indeed, we can parametrise $x = \psi_2(y)$ by $\mathbf{c}(t) = (\psi_2(t), t)$, $c \leq t \leq d$. Hence, we get

$$\int_{\mathbf{c}} N dy = \int_c^d N(\psi_2(t), t) \frac{dy}{dt} dt = \int_c^d N(\psi_2(t), t) dt.$$

But if we change the name "t" to "y", we get the first of our integrals above. Similarly, the second integral becomes the line integral

$$-\int_c^d N(\psi_1(y), y) dy.$$

Completing the proof of Green's theorem

Why is there a minus sign? Because the integral is being taken in the opposite direction (that is, downward in the picture).

We need to do the line integral along the two horizontal lines $y = c$ and $y = d$. Since y is constant along these curves, we have $dy = 0$. As a result, these two line integrals are just zero. Hence, we find that

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_{\partial D} N dy.$$

Using the fact that D is a region of type 1, the integral

$$\iint_D \frac{\partial M}{\partial y} dxdy = -\int_{\partial D} M dx.$$

Where does the minus sign come from? From the fact that $y = \phi_2(x)$ is oriented in the direction of decreasing x.

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \int_{\partial D} M dx + \int_{\partial D} N dy.$$

A more general case

How does one proceed in general, that is for more general regions which may not be of both type 1 and type 2. We can try proceeding as follows:

- Break up D into smaller regions each of which is of both type 1 and type 2 but so that any two pieces meet only along the boundary.
- Apply Green's theorem to each piece.
- Observe that the line integrals along the interior boundaries cancel, leaving only the line integral around the boundary of D.

Gauss's Divergence Theorem in the plane

Theorem 41: Let D be a region as in Green's theorem and let $\mathbf{n}$ be the outward unit normal vector on the positively oriented boundary ∂D. Let $\mathbf{F} : D \to \mathbb{R}^2$ be a vector field.

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = \int \int_D \operatorname{div} \mathbf{F} dx dy.$$

Since $\mathbf{k} \times \mathbf{n} = \mathbf{T}$, we have $\mathbf{n} = -\mathbf{k} \times \mathbf{T}$, where $\mathbf{T} = \mathbf{c}' / \|\mathbf{c}'\|$ is the unit tangent vector to the curve $\mathbf{c}(t)$ which parametrises ∂D (we will assume that ∂D can be parametrised by a single curve - otherwise break up the curve into parametrisable pieces...). Hence,

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = - \int_{\partial D} \mathbf{F} \cdot (\mathbf{k} \times \mathbf{T}) ds.$$

But the integrand on the right hand side is just the scalar triple product which can be rewritten as $(\mathbf{F} \times \mathbf{k}) \cdot \mathbf{T}$, so

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = \int_{\partial D} (\mathbf{k} \times \mathbf{F}) \cdot \mathbf{T} ds = \int_{\partial D} (\mathbf{k} \times \mathbf{F}) \cdot d\mathbf{s}.$$

The proof of the divergence theorem, continued

If we write $\mathbf{F} = (A, B)$, then $\mathbf{k} \times \mathbf{F} = A\mathbf{j} - B\mathbf{i}$. If we apply Green's theorem we get

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \right) dxdy = \iint_D \operatorname{div} \mathbf{F} dxdy.$$

We can interpret the above theorem in the context of fluid flow. If $\mathbf{F}$ represents the flow of a fluid, then the left hand side of the divergence theorem represents the net flux of the fluid across the boundary ∂D. On the other hand, the right hand side represents the integral over D of the rate $\nabla \cdot \mathbf{F}$ at which fluid area is being created. In particular if the fluid is incompressible (or, more generally, if the fluid is being neither compressed nor expanded) the net flow across ∂D is zero.

Again, as we shall see next week, this theorem has a three-dimensional analogue.