MA 105 D1 Lecture 16

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Lagrange multipliers

Integral quadratic forms

The Riemann integral in several variables

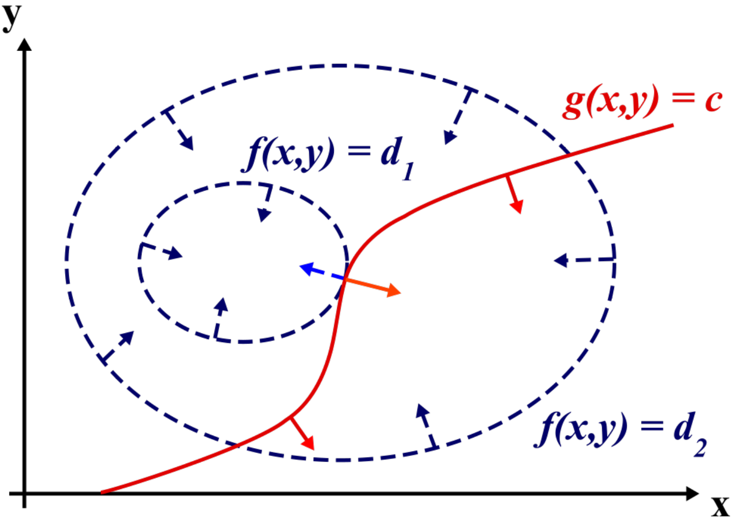
The two variable problem

Suppose we are given a function $f(x, y)$ in two variables. We would like maximize or minimize it subject to the constraint that $g(x, y) = c$. In geometric terms, we want to find the maximum or minimum values of f while staying on the curve $g(x, y) = c$.

Where might we find these maxima or minima?

One way of doing it would be to walk along the curve $g(x, y) = c$ meeting the level curves $f(x, y) = d$ as d increases. The smallest value of d (if it exists!) such that $g(x, y) = c$ and $f(x, y) = d$ meet will surely be the minimum value for the function $f(x, y)$ on the curve $g(x, y) = c$. What special property holds at this point?

In pictures



http://en.wikipedia.org/wiki/Lagrange_multiplier#mediaviewer/
File:LagrangeMultipliers2D.svg

The condition on the normals

From the picture one might guess that at the value d_1 the curves are tangent to each other, or, equivalently, their normals are parallel to each other. Why is this?

Suppose that this were not the case. Recall that the normals to the level curves of f are the directions in which f is decreasing on increasing the fastest. If this normal is not perpendicular to the curve $g(x, y) = c$, then it will have a component tangent to this curve.

Hence, the function $f(x, y)$ will be either increasing or decreasing at the point of intersection of the two curves as one goes along $g(x, y) = c$. This shows that the point of intersection cannot be an extreme point.

The gradient condition

From the discussion in the previous slide, it follows that the normals of the curves $f(x, y) = d$ and $g(x, y) = c$ must be parallel to each other at the extrema. We are thus looking for points (x_0, y_0) such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$

subject to the constraint condition.

$$g(x_0, y_0) = c.$$

The λ that occurs above is called the Lagrange multiplier. The points (x_0, y_0) satisfying the above conditions are like the critical points we encounter in the unconstrained problem. We have to actually check whether they are extrema or not.

Checking for the extrema

To check for extrema all we have to do is to evaluate the function $f(x, y)$ at each of points (x_0, y_0) that we find after solving the equation above. This will enable us to decide whether a given point is a maximum or minimum.

Exercise: Find the maximum and minimum values of the function $f(x, y) = x^2 + 2y^2$ that lie on the circle $x^2 + y^2 = 1$.

Solution: We know that we must solve the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad x^2 + y^2 = 1,$$

where $g(x, y) = x^2 + y^2$. The first equation yields the pair of equations

$$2x = \lambda \cdot 2x \quad \text{and} \quad 4y = \lambda \cdot 2y.$$

The first equation above yields $x = 0$ or $\lambda = 1$. If $x = 0$, we must have $y = \pm 1$. If $\lambda = 1$, then $4y = 2y$ in the second equation, so $y = 0$ and $x = \pm 1$.

Thus we must evaluate $f(x, y)$ at the points $(\pm 1, 0)$ and $(0, \pm 1)$.
We obtain $f(1, 0) = 1,\ f(-1, 0) = 1,\ f(0, 1) = 2,\ f(0, -1) = 2$.
Thus the maximum value is 2 and the minimum 1.

The three variable problem

The same reasoning as before applies to the three variable constrained problem, that is, to find the maxima and minima of a function $f(x, y, z)$ subject to $g(x, y, z) = c$. If we are moving on the surface of g, this means that at any point, we are in the tangent plane of g. On the other hand, the directions in which f is stationary, are those perpendicular to its gradient.

It follows that we must once again solve the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = c,$$

to obtain the extreme points, and then evaluate the function $f(x, y, z)$ at these points.

Exercise: Find the dimensions of the box with the largest volume having a surface area of $150cm^2$.

Solution: The volume of the box is given by $f(x, y, z) = xyz$. This has to be maximized subject to the constraint that $g(x, y, z) = 2xy + 2yz + 2zx = 150$.

The condition $\nabla f = \lambda \nabla g$ yields

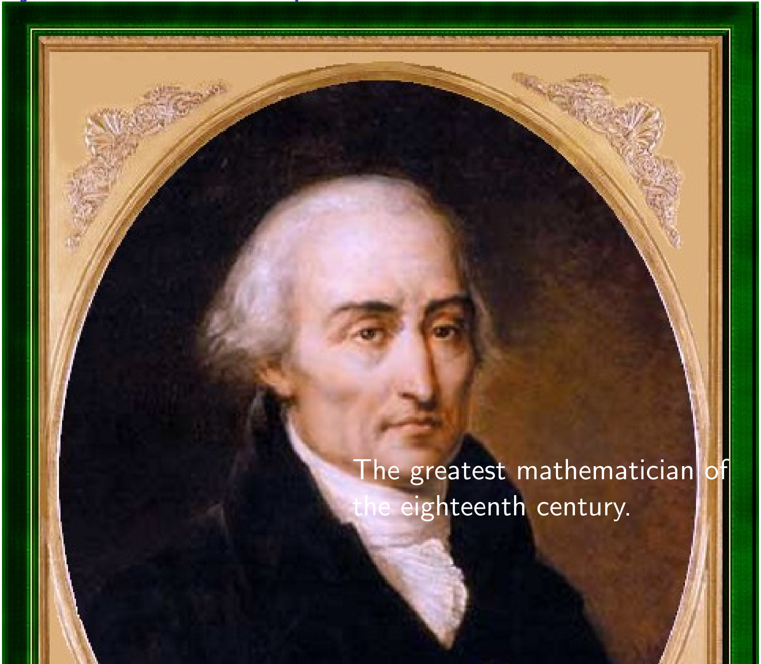
$$yz = \lambda(2y + 2z), \ zx = \lambda(2z + 2x), \ xy = \lambda(2x + 2y).$$

while the constraint condition give $xy + yx + zx = 75$. Multiplying the previous equations by x, y and z respectively we get

$$xyz = 2\lambda x(y + z), \ xyz = 2\lambda y(z + x), \ xyz = 2\lambda z(x + y).$$

Equating the first two gives $\lambda = 0$ or $x = y$. But if $\lambda = 0$ we get $y = 0$ or $z = 0$, which is not possible. The equations above are symmetric in x, y and z, so if $x = y$ we must have $x = y = z$. This yields $3x^2 = 75$ or $x = y = z = 5$.

Identify the man in the picture



The greatest mathematician of the eighteenth century.

The four squares theorem

We will be studying the method of Lagrange multipliers next in the context of finding constrained optima. But for the moment, I would like to tell (remind?) you of another theorem of Lagrange.

Theorem: Every positive integer can be written as a sum of four squares.

Let us restate the above theorem. Given a natural number n, if there exist integers a_1, a_2, a_3 and a_4 (note that some of these may take the value 0!) such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = n,$$

we say that the function $x_1^2 + x_2^2 + x_3^2 + x_4^2$ in four variables represents the number n. So the theorem above says:

Theorem: The function $x_1^2 + x_2^2 + x_3^2 + x_4^2$ represents every natural number.

Quadratic forms

An *n-ary quadratic form* over the real numbers is a function from $\mathbb{R}^n$ or $\mathbb{Z}^n$ to $\mathbb{R}$ of the form

$$q(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i,j \leq n} q_{ij} x_i x_j, \quad a_{ij} \in \mathbb{R}.$$

The example $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is an example of a quartenary quadratic form. It is a diagonal form, that is, only square terms appear. Forms need not be diagonal - for instance $q_1(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$ is a non-diagonal form in two variables. Another example (in three variables) is $q_2(x_1, x_2, x_3) = \sqrt{2} x_1^2 + 5x_2^2 - 7x_3^2 + 2x_1 x_3$.

We will be interested only in integral quadratic forms, that is quadratic forms which take $\mathbb{Z}^n$ to $\mathbb{Z}$. Lagrange's quadratic form and q_1 above are integral, but q_2 above is not.

The Bhargava-Hanke Theorem

A quadratic form is called positive definite if $q(x_1, \ldots, x_n) > 0$ for all $(x_1, \ldots, x_n)$ in $\mathbb{R}^n \setminus \{(0, 0, \ldots, 0)\}$.

In the examples above, the first two forms are positive definite, while the third is not (easy to check).

Theorem: If a positive definite (integral) quadratic form represents every number $n \leq 290$ it represents natural numbers.

This is an absolutely remarkable theorem (guessed at by Conway, a very well-known mathematician at Princeton). It says that you only need to check finitely manly values in order to prove something for infinitely many values!

Manjul Bhargava

Rectangles

Any rectangle R in the plane can be described as the set of points in the cartesian product $[a, b] \times [c, d]$ of two closed intervals.

We can define the notion of a partition of a rectangle. How can this be done?

The easiest way is to take a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$ and take the product of the two partitions. Thus if

$$P_1 = \{a = x_0, x_1 \ldots, x_m = b)\} \quad \text{and} \quad P_2 = \{c = y_0, y_1, \ldots, y_n = d\},$$

we take the collection of points

$$P = \{(x_i, y_j) \,|\, 1 \leq i \leq m,\, 1 \leq j \leq n\}.$$

The point (x_i, y_j) is the left bottom corner of the rectangle $R_{ij} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$. As i and j vary, we get a family of rectangles R_{ij}, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$. By identifying each rectangle with its left bottom corner we can think of P as the collection of these rectangles R_{ij}. Clearly, $R = \cup_{i,j} R_{ij}$, and the collection of rectangles P is called a partition of R.

Partitions of rectangles and the integral

You may recall that we have several equivalent definitions of Riemann integration in one variable. We can model the notion of Riemann integration in two variables on any one of those. Clearly, the notion of upper and lower sums associated to a partition makes sense so we could define the Darboux integral as before.

We set $\Delta_{ij} = (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, the area of the rectangle R_{ij}.

If $f : R \to \mathbb{R}$ is a function of two variables, define

$$m_{ij} = \inf_{(x,y) \in R_{ij}} f(x, y)$$

and the lower sum of f associated to the partition P by.

$$L(f, P) = \sum_{i,j}^{m-1,n-1} m_{ij} \Delta_{ij}.$$

We can similarly define the upper sums, and now it is clear that the lower and upper integrals, and hence also the Darboux integral can be defined.

The Riemann integral

A second approach arises by defining the norm of a partition of a rectangle. Again, this is easy. In the one variable case the norm of a partition was simply the length of the largest of the sub-intervals. What is the analogue for rectangles? Clearly, we could define

$$\|P\|_1 = \max_{0 \leq i \leq m-1, 0 \leq j \leq n-1} (x_{i+1} - x_i) \times (y_{j+1} - y_j).$$

As before, we can define a tagged partition (P, t), where $t = \{t_{ij}\}_{i,j}$ is a collection of points such that $t_{ij} \in R_{ij}$.

The Riemann sum S associated to (P, t) is defined by

$$S(f, P, t) = \sum_{i,j}^{m-1,n-1} f(t_{ij}) \Delta_{ij}.$$

Now it should be clear how to define the Riemann integral: it is a number S such that for any $\epsilon > 0$, there is a δ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) with the property that $\|P\|_1 \leq \delta$.

One problem with the previous definition

There is one problem with the definition made in the last slide. The norm that we have selected for partitions is not such a good one.

For instance, one could take $P_1 = \{0 < 1/2 < 1\}$ and $P_{2,n} = \{0 < 1/n, 2/n, \ldots < 1\}$ and take $P_n = P_1 \times P_{2,n}$ as a partition of the unit square. In this case, clearly $\|P\| = 1/2n$ goes to zero as $n \to \infty$.

Exercise 1: Find a function $f(x, y)$ on the unit square and an $\epsilon > 0$ such that for all n

$$|S(f, P_n, t) - V| > \epsilon,$$

for a tag t_n of P_n, where V is the volume of the solid region lying above the unit square and below the graph of $z = f(x, y)$ (note, there are very simple functions with the property).

Remedy

How to fix the problem? The problem is that the rectangles in our partition may be very thin in one direction but remain fat in the other one.

In order to avoid such situations, we have to change the definition of the norm that we have given. We have to select a norm so that when the norm of P is small, both sides of our rectangles are guaranteed to be small.

We define $\|P\| = \max_{i,j}\{(x_{i+1} - x_i), (y_{j+1} - y_j)\}$. Clearly this norm has the desired property and is the correct analogue of the norm of a partition in one variable.

Thus, to get the correct definition of a Riemann integral for a two variable function one must replace $\|P\|_1$ in the definition given above by $\|P\|$.

Regular Partitions

We will not use either of the above approaches, preferring instead the third approach. Recall that in Definition 2 of the one variable integral we saw that it was enough to restrict our attention to a fixed family of partitions. This is what we will do, taking a particularly simple family of partitions.

The regular partition of R of order n is is partition defined inductively by $x_0 = a$ and $y_0 = c$ and

$$x_{i+1} = x_i + \frac{b-a}{n} \quad \text{and} \quad y_{j+1} = y_j + \frac{d-c}{n},$$

$1 \leq i, j \leq n-1$. We take $t = \{t_{ij} \in R_{ij}\}$ to be an arbitrary tag.

Definition: We say that the function $f : R \to \mathbb{R}$ is Riemann integrable if the Riemann sum

$$S(f, P_n, t) = \sum_{i,j=0}^{n-1} f(t_{ij})\Delta_{ij}$$

tends to a limit S for any choice of tag t.

The Riemann integral continued

This limit value is usually denoted as

$$\int\int_R f, \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

The preceding definition is sometimes rewritten as

$$\lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(t_{ij})\Delta_{ij} = \int\int_R f.$$

If $f(x,y) \geq 0$ for all values of x and y, then the Riemann integral has a geometric interpretation. It is obviously the volume of the region under the graph of the function $z = f(x,y)$ and above the rectangle R in xy-plane.

The integral may also be interpreted as mass in some physical situations; for example, if we have a rectangular plate and $f(x,y)$ represents the density of the plate at a given point, then the integral above gives the mass of the whole plate.

The main theorem

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable. The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In two variables the geometry of the set of points of discontinuity can be more complicated. Still, what are the analogues of points in this case? In other words what sets have "zero area"?

Theorem 33: If a function f is bounded and continuous on R except possibly along a finite number of graphs of $\mathcal{C}^1$ functions, then f is integrable on R.