MA105 Final Examination-Code A

9:30 - 12:30 , Friday, November 17, 2017

Solutions with the marking schemes

Q. 1A: (a) [1 mark] Let $f(x,y) = x\sin y$. Find all critical points and classify them.
(b) [3 marks] Let $f(x,y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$ where $A > 0$ and $AC - B^2 > 0$.
(i) Show that there is a unique critical point of $f(x,y)$, say (x_1, y_1).
(ii) Show that $f(x,y)$ has a relative minimum at (x_1, y_1).

Solution: (a) $\frac{\partial f}{\partial x} = \sin y, \frac{\partial f}{\partial y} = x\cos y$. Hence the critical points are $(0, n\pi), n \in \mathbb{Z}$. At these critical points $\Delta = -\cos^2 y < 0$. Thus all critical points are saddle points. [1]
(b) (i) The critical points are solutions to $f_x = f_y = 0$. Hence
$f_x = 2Ax + 2By + 2D = 0 = f_y = 2Bx + 2Cy + 2E$. [1]
Write these equations as

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -D \\ -E \end{bmatrix}$$

Since the determinant of the coefficient matrix $AC - B^2 > 0$, there is a unique solution say (a, b) of these equations. [1]
(b) (ii) Note that $f_{xx} = 2A, f_{xy} = f_{yx} = 2B$ and $f_{yy} = 2C$. Hence $f_{xx}f_{yy} - f_{xy}^2 = 4(AC - B^2) > 0$ and $A > 0$. Hence $f(x,y)$ has a local minimum at (a, b). [1]
(a) Common errors

- Critical points are neither maximum nor minimum.
- Critical points are saddle points but did not write what are the critical points.
- Critical points and determinant of Hessian less than zero, but no conclusion about the classification of critical points.

(b) Common errors

- Two linear equations of two variables $\Rightarrow$ unique solution.
- Error in calculation of partial derivatives .
- Calculation mistake in solving linear system and wrong solution.
- Quadratic equation, A>0 and C>0, $\Rightarrow$ unique solution.

(b(ii)) Common errors.

- determinant of Hessian greater than zero but nothing written about sign of f_{xx}.
- f_{xx}>0 but nothing written about determinant of Hessian.

Q. 2A: (a) [2 marks] Evaluate

$$\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dxdydz$$

where D is the region defined by $1 \le x^2 + y^2 + z^2 \le 2$ and $z \ge 0$.
(b) [2 marks] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \frac{xy^2}{x^2+y^4}$ if $x \neq 0$ and $f(x,y) = 0$ otherwise. Find the directional derivative of f at the origin in the direction (a, b) where $a \neq 0$. Is $f(x,y)$ continuous at the origin ?

Solution: (a) Use spherical coordinates

$$\iiint_D e^{(x^2+y^2+z^2)^{3/2}} dxdydz = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi}\int_{\rho=1}^{\sqrt{2}} e^{\rho^3}\rho^2 \sin\phi \, d\rho d\theta d\phi \quad [1]$$
$$= 2\pi \int_{\phi=0}^{\pi/2}\int_{\rho=1}^{\sqrt{2}} e^{\rho^3}\rho^2 d\rho \sin\phi d\phi$$
$$= \frac{2\pi}{3} e^{\rho^3}\Big|_1^{\sqrt{2}} (-\cos\phi)\Big|_0^{\pi/2}$$
$$= \frac{2\pi}{3}(e^{\sqrt{8}} - e). \quad [1]$$

(b) Let $\mathbf{u} = (a,b), a \neq 0$. Then

$$\frac{f(\mathbf{0}+h\mathbf{u}) - f(\mathbf{0})}{h} = \frac{ab^2}{a^2+h^2b^4}.$$

Hence $D_{\mathbf{u}}(f)(0,0) = b^2/a$ since $a \neq 0$. [1]
Note that for $x \neq 0$, $f(x^2,x) = 1/2$. Hence f is not continuous at the origin. [1]

Common errors: (a) The domain for z given in the problem is $z \geq 0$. This means that the limits for ϕ are from 0 to $\pi/2$. Many students took the limit from 0 to π.
(b) The definition of directional derivative does not require that the vector given is a unit vector. Students who used $u/||u||$ have been given full marks. We have also accepted the definition given in Wikipedia where one divides by $h||u||$ and then takes the limit.

Q. 3A: (a) [1 mark] Find the limits in the double integral after Interchanging the order of integration:

$$\int_0^1\int_{x^2}^{x} dydx.$$

(b) [3 marks] Calculate the double integral

$$\int\int_D (x-y)^2\sin^2(x+y)dxdy$$

where D is the parallelogram with vertices $(\pi,0), (2\pi,\pi), (\pi,2\pi)$ and $(0,\pi)$.
Solution: (a) The domain of integration is the region between the curves $y = x^2$ and $y = x$. The double integral after reversing the order of integration is $\displaystyle\int_0^1\int_y^{\sqrt{y}} dxdy$. [1]
(b) Put $x = \frac{u-v}{2}, y = \frac{u+v}{2}$. Then the rectangle $R = \{\pi \leq u \leq 3\pi, \ -\pi \leq v \leq \pi\}$ in the uv-plane gets mapped to D. [1]
Further,

$$J = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$$

and then

$$\int\int_D (x-y)^2\sin^2(x+y)dxdy = \int\int_R v^2\sin^2(u)\frac{1}{2}dudv \quad [1]$$

$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} v^2 dv\right)\left(\int_{\pi}^{3\pi} \sin^2(u) du\right) = \frac{1}{2}\left(2 \times \frac{\pi^3}{3}\right)(\pi) = \frac{\pi^4}{3}. \quad [1]$$

Common errors: (a) Many students interchanged the limits in the integral for example instead of y to y^n they did y^n to y in which case they got 0 mark out of 1.

(b) There are several types of mistake in this problem. Some of them miscalculated the image rectangle and as a consequence they got wrong answer with correct remaining part. They lost 1 mark. Also there are students with wrong jacobian due to wrong formula. They lost 1 mark. Some students used wrong change of variable formula, e.g. not using Jacobian, lost 2 marks.

Question 4 A.

(a) (1 mark) True or False: The vector field $\mathbf{F}(x, y, z) = (e^x \cos y, e^x \sin y, 0)$ can be written as curl of some vector field on $\mathbb{R}^3$.

(b) (3 marks) Show that the vector field $\mathbf{F}(x, y, z) = (y \cos x + y^2, \sin x + 2xy - y, 0)$ is irrotational and determine a scalar function $f(x, y, z)$ for which $\mathbf{F} = \nabla f$.

Solution.

(a) False. Since $\nabla \cdot (\nabla \times \mathbf{G}) = 0$ for any C^2 vector field $\mathbf{G}$ but $\nabla \cdot \mathbf{F} = 2e^x \cos y \neq 0$, $\mathbf{F}$ cannot be written as curl of any vector field on $\mathbb{R}^3$.

[**1 mark**]

(b) By the cross-ratio test, we obtain that $\nabla \times \mathbf{F} = (0, 0, \cos x + 2y - \cos x - 2y) = (0, 0, 0)$, hence $\mathbf{F}$ is irrotational. [**1 mark**]

Let $f(x, y, z)$ be a function for which $\mathbf{F} = \nabla f$. This shows that

$$\frac{\partial f}{\partial x} = y \cos x + y^2, \quad \frac{\partial f}{\partial y} = \sin x + 2xy - y, \quad \text{and} \quad \frac{\partial f}{\partial z} = 0.$$

Integrating the first equation with respect to x, we obtain that

$$f(x, y, z) = y \sin x + y^2 x + g(y)$$

for some function $g(y)$ (since the function $f(x, y, z)$ is not depending on z). [**1 mark**]

Those who have written a constant C in place of $g(y)$ above, have been awarded 0 marks for the above step.

Now by putting this value of $f(x, y, z)$ in the second equation, we obtain that

$$\sin x + 2yx + g'(y) = \sin x + 2xy - y.$$

Thus, $g'(y) = -y$ and by integrating this, we find that

$$g(y) = -\frac{y^2}{2} + c$$

for some constant c. Hence

$$f(x, y, z) = y \sin x + y^2 x - \frac{y^2}{2} + c$$

for some constant c. [**1 mark**]

Those who have directly written $f(x, y, z) = y \sin x + y^2 x - \frac{y^2}{2} + c$ without showing the intermediate calculations, have been awarded 0 mark for the last two steps.

Question 5 A.

(a) (1 mark) Evaluate $\int_C x\ ds$ where C is the line segment from $(1,2)$ to $(3,4)$.

(b) (3 marks) Show that

$$\iint_R \left(\phi\nabla^2\phi + \nabla\phi\cdot\nabla\phi\right)\ dxdy = \oint_{\partial R} \phi\nabla\phi\cdot\mathbf{n}\ ds$$

where ϕ is a $\mathcal{C}^2$ function defined on a region $R\subset\mathbb{R}^2$ for which ∂R is a positively oriented simple closed curve and $\mathbf{n}$ is the outward unit normal to ∂R.

Solution.

(a) The line segment can be parametrised as $C(t) = (1,2)+t[(3,4)-(1,2)] = (1+2t, 2+2t)$ for $t\in[0,1]$. Also, $C'(t) = (2,2)$. By definition of the line integral, it follows that

$$\int_C x\ ds = \int_0^1 (1+2t)\sqrt{8}dt = \sqrt{8}\left[t+\frac{2t^2}{2}\right]_0^1 = 4\sqrt{2}.$$

[1 mark]

(b) By the Gauss' divergence theorem in the plane, we obtain that

$$\oint_{\partial R} \phi\nabla\phi\cdot\mathbf{n}\ ds = \iint_R \nabla\cdot(\phi\nabla\phi)dxdy$$

[1 mark]

and if we write

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

then

$$\nabla\cdot(\phi\nabla\phi) = \frac{\partial}{\partial x}\left(\phi\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\phi}{\partial y}\right).$$

Note that

$$\frac{\partial}{\partial x}\left(\phi\frac{\partial\phi}{\partial x}\right) = \left(\frac{\partial\phi}{\partial x}\right)^2 + \phi\frac{\partial^2\phi}{\partial x^2}$$

and

$$\frac{\partial}{\partial y}\left(\phi\frac{\partial\phi}{\partial y}\right) = \left(\frac{\partial\phi}{\partial y}\right)^2 + \phi\frac{\partial^2\phi}{\partial y^2}.$$

Thus,

$$\nabla\cdot(\phi\nabla\phi) = \left(\frac{\partial\phi}{\partial x}\right)^2 + \phi\frac{\partial^2\phi}{\partial x^2} + \left(\frac{\partial\phi}{\partial y}\right)^2 + \phi\frac{\partial^2\phi}{\partial y^2} = \phi\nabla^2\phi + \nabla\phi\cdot\nabla\phi.$$

[1 mark]

Those who have neither computed $\nabla\cdot(\phi\nabla\phi)$ nor mentioned the product rule for the divergence of the vector field $\phi\nabla\phi$ have been awarded 0 mark for the above step.

Hence

$$\oint_{\partial R} \phi\nabla\phi\cdot\mathbf{n}\ ds = \iint_R \nabla\cdot(\phi\nabla\phi)dxdy = \iint_R \left(\phi\nabla^2\phi + \nabla\phi\cdot\nabla\phi\right).$$

[1 mark]

Those who have used the same notation for the gradient (∇f) and the divergence ($\nabla\cdot\mathbf{F}$) should always remember that the gradient operator is applied to a scalar function f and the resulting function ($\nabla f(x,y,z)$) is a vector field while the divergence operator is applied to a vector field $\mathbf{F}$ and the resulting function $\nabla\cdot\mathbf{F}(x,y,z)$ is a scalar function.

Question 6 A.

(a) (1 mark) What is the derivative of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x, y) = (2x + y, x + 2y)$?

(b) (3 marks) Let the scalar functions $f = f(x, y)$ and $g = g(x, y)$ have continuous partial derivatives on the unit disc $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. If $f(x, y) = 1$ and $g(x, y) = y$ for all (x, y) on the boundary circle $C := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and

$$\mathbf{u} = f^2\mathbf{i} + g^2\mathbf{j};\ \mathbf{v} = (f_x - f_y)\mathbf{i} + (g_x - g_y)\mathbf{j},$$

find

$$\iint_D \mathbf{u} \cdot \mathbf{v}\, dxdy.$$

Solution.

(a) The derivative of the given function at a point $(x, y) \in \mathbb{R}^2$ is the 2×2 matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. [**1 mark**]

Since the derivative of a multi-variable function is defined as a matrix, those who have not written the derivative in matrix form have been awarded 0 mark.

(b) Note that $\mathbf{u} \cdot \mathbf{v} = f^2 f_x - f^2 f_y + g^2 g_x - g^2 g_y = (g^2 g_x - f^2 f_y) + (f^2 f_x - g^2 g_y)$ and hence

$$\iint_D \mathbf{u} \cdot \mathbf{v}\, dxdy = \iint_D \left[(g^2 g_x - f^2 f_y) + (f^2 f_x - g^2 g_y)\right] dxdy$$
$$= \iint_D (g^2 g_x - f^2 f_y) dxdy + \iint_D (f^2 f_x - g^2 g_y) dxdy.$$

[**1 mark**]

For the given function, we have information only about its values at the points on the boundary circle of the given disk. Those who have considered $f(x, y) = 1$ and $g(x, y) = y$ for all $(x, y) \in D$ have been awarded 0 mark.

We now compute both the integrals by using the Green's theorem. We now define the vector fields $\mathbf{F} = \frac{1}{3}(f^3, g^3), \mathbf{G} = \frac{1}{3}(g^3, f^3)$. Then by Green's theorem, it follows that

$$\iint_D (g^2 g_x - f^2 f_y) dxdy = \int_C \mathbf{F} \cdot d\mathbf{s}$$

and

$$\iint_D (f^2 f_x - g^2 g_y) dxdy = \int_C \mathbf{G} \cdot d\mathbf{s}.$$

[**1 mark**]

We now parametrise the boundary circle C as $C(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ so that the circle is positively oriented. Then, $\mathbf{F}(C(t)) = \frac{1}{3}(1, \sin^3 t)$ and $\mathbf{G}(C(t)) = \frac{1}{3}(\sin^3 t, 1)$. Also, $C'(t) = (-\sin t, \cos t)$. Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \frac{1}{3} \int_0^{2\pi} (-\sin t + \sin^3 t \cos t) dt = 0$$

and

$$\int_C \mathbf{G} \cdot d\mathbf{s} = \frac{1}{3} \int_0^{2\pi} (-\sin^4 t + \cos t) dt = -\frac{\pi}{4}.$$

Hence

$$\iint_D \mathbf{u} \cdot \mathbf{v}\, dxdy = 0 - \frac{\pi}{4} = -\frac{\pi}{4}.$$

[**1 mark**]

If the calculation in the last step or the final answer is not correct, only 0 mark has been awarded to this step.

Question 7 A

(a) (1 mark) True or False: Consider the boundary of the solid cylinder $x^2 + y^2 = 4, -1 \leq z \leq 1$, oriented with the outward normal vector. The parametrisation of the surface S at the top of the cylinder by cylindrical coordinates is orientation reversing.

(b) (3 marks) Find the closed surface S for which the flux of the vector field $\mathbf{F} = (x^2y - xy^2, -yz^2 - xy^2, z - zx^2)$ through S is a maximum.

Solution: (a) False. The natural orientation for the boundary is the outward normal. The normal given the by parametrisation for S is

$$\Phi_r \times \Phi_\theta = (\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0) = (0, 0, r).$$

Since $r > 0$ we see that this normal is pointing in the direction of the positive z-axis, i.e., in the direction outward from the cylinder. Thus the parametrisation is orientation preserving. (**1 mark**)
(Note: This solution is independent of the code.)
Many students have incorrectly parametrised the top surface of the cylinder. In this case no marks have been awarded.

(b) Let S be a closed surface. We may apply Gauss' theorem to obtain

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} dV. \iiint_W (1 - x^2 - y^2 - z^2) dV$$

Where W is the solid region enclosed by S. (**1 mark**)

Now $\nabla \cdot \mathbf{F} = 1 - x^2 - y^2 - z^2$. Hence the integral becomes

$$\iiint_W (1 - x^2 - y^2 - z^2) dV.$$

The integrand will be nonnegative if $x^2 + y^2 + z^2 \leq 1$. Outside of this sphere it will be negative, so integrating the function on any region outside will give a negative contribution to the integral.

(**1 mark**)

(More precisely, let Y be any region in $\mathbb{R}^3$. Let $Y_1 = Y \cap W$ and $Y_2 = Y \cap W^c$, where W^c is the complement of W. Then

$$\iiint_Y (1-x^2-y^2-z^2) dV = \iiint_{Y_1} (1-x^2-y^2-z^2) dV + \iiint_{Y_2} (1-x^2-y^2-z^2) dV.$$

The first integral on the right is positive, while the second is negative. Thus the left-hand side will be maximised if the first integral is maximised and the second minimised. This will happen if $Y = W$.)

Thus the maximum value of the integral will be attained for the largest region on which the integrand remains positive, which is the sphere $x^2 + y^2 + z^2 = 1$.
(**1 mark**)

Many students have simply asserted that the desired surface must satisfy $\nabla \cdot \mathbf{F} = 0$. This is incorrect and no marks have been awarded for this.

Question 8 A

(a) (1 mark) True or False: If $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^3$.

(b) (3 marks) Let C be a simple closed curve lying on the plane $2x+y+z=1$ and enclosing an area π. If $\mathbf{F} = (2z, -2x, 3y)$, calculate the value of the circulation of $\mathbf{F}$ around C. Make sure you specify the relevant orientations clearly.

Solution: (a) False. $\mathbf{F} = (y, x, z)$ has divergence 0, but its curl is nonzero. Hence, it cannot be conservative. (**1 mark**).
(Note: A two-dimensional example, e.g. $\mathbf{F} = (-y, x)$ is acceptable.)
Students who have not given a counterexample have not been awarded marks.

(b) Let S be the surface on the plane enclosed by C. We choose the normal $\mathbf{n} = \frac{1}{\sqrt{6}}(2,1,1)$ to orient S. With this choice or orientation, C automatically acquires the anti clockwise orientation when seen from point high on the positive z-axis. (**1 mark**)
(Note: It is important that students specify the orientation on the surface first. This automatically induces an orientation on the boundary, and it is for these two orientations that Stokes' theorem is valid. Students who did not give a proper choice of orientation were not awarded marks. Specifically just saying C has "anticlockwise" is insufficient, since you have to say where you are looking at the curve from.)

Using Stokes' theorem we get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

(**1 mark**)

We see that $\nabla \times \mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \sqrt{6}$. It follows that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S \sqrt{6} dS = \sqrt{6}\pi.$$

(**1 mark**)

Minor mistakes in calculation at the end should be excused.

Question 9 A

(a) (1 mark) True or False: If $\mathbf{F}$ is a conservative vector field defined in the region D in the plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, the line integral of $\mathbf{F}$ along any closed loop is 0.

(b) (3 marks) Consider the following "light bulb" shaped surface S with the outward unit normal as the chosen orientation (that is out of the bulb). It is the union of the surfaces $S_1 : x^2 + y^2 = 1, 0 \le z \le 1$ and $S_2 : x^2 + y^2 + (z-3)^2 = 5, z \ge 1$. Calculate the flux of the vector field $\nabla \times \mathbf{F}$, where $\mathbf{F} = (e^{z^2-2z}x, \sin(xyz) + y + 1, e^{z^2}\sin(z^2))$.

Solution: (a) True. Since $\mathbf{F}$ is conservative, $\mathbf{F} = \nabla f$ for some scalar function f. In this case the line integral of $\mathbf{F}$ along any path depends only on the difference of the values of f at the final and initial end points. Since these are the same the integral must be zero. (**1 mark**)
(Note: The fact that the region D is not simply connected is irrelevant. The region D is path connected and that is all that is needed for the work done by conservative fields to be path independent.)
Many students have applied Stokes' theorem to the boundary - this has not been awarded any marks, since this proves the desired result only for one curve - not all curves!

(b) Since S is oriented with the outward normal, the boundary C which is the circle $x^2 + y^2 = 1$ in the xy-plane acquires the anti clockwise orientation when viewed from above the xy-plane.

(**1 mark**)

Using Stokes' theorem we see that the flux

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s},$$

and C is the circle $x^2 + y^2 = 1$ in the xy-plane. (**1 mark**)

Parametrising the circle as $\mathbf{r}(t) = (\cos t, \sin t, 0)$ we have

$$\mathbf{F}(\mathbf{r}(t)) = (\cos t, \sin t + 1, 0) \quad \text{and} \quad \mathbf{r}'(t) = (-\sin t, \cos t, 0).$$

Thus $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \cos t$ and the required integral is just

$$\int_0^{2\pi} \cos t dt = 0.$$

(**1 mark**)

Alternate solution: Take S_1 and S_2 and apply Stokes' theorem separately to both. There are three boundaries to orient. Orientation was done correctly for all three boundaries.

(**2 marks**)

Note: Some students have oriented only some of the boundaries correctly in which case they have got at most 1 mark at this stage and no further marks were awarded.
Noticing that two line of the line integrals cancel out and evaluating the line integral in the xy plane ($z = 0$) gives the answer.

(1 mark)

Alternate solution: Use Stokes' theorem twice. Specify orientation of the boundary C as above.

(1 mark)

Then use Stokes' theorem to convert the flux into a line integral, as above.

(1 mark)

Then use Stokes' theorem again to convert the line integral along C to a surface integral on the unit disc in the xy plane. Observe that the curl of $\mathbf{F}$ on the unit disc in the xy plane is 0. Hence conclude that the desired integral is 0.

(1 mark)

Alternate Solution: Use Gauss' theorem on the closed surface $S \cup D$ where D is the closed unit disc in the xy-plane.

Since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, we see that

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iiint_W \nabla \cdot (\nabla \times \mathbf{F}) dV$$

(1 mark)

Since the divergence of curl is zero, it follows that

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = -\iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

(1 mark)

Finally, since curl of $\mathbf{F}$ vanishes on D, the last integral must be 0.

(1 mark)

Minor calculation mistakes when the calculating the curl were excused, but major mistakes were not, even if the final answere was correct.

Alternate solution: Use the Gauss divergence theorem 3 times! Only one student did this and was awarded full marks.

Question 10 A

(a) (1 mark) True or false: A gas is being compressed by a piston in a cylinder. Then its velocity field $\mathbf{F}$ cannot have the form $(3xy^2, e^{yx^2}, x^2e^z)$.

(b) (3 marks) Compute the area of that portion of the paraboloid $x^2+z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.

Solution:
(a) **This is an updated version of the marking scheme for this question. The earlier version was incorrect and was wrongly uploaded.**
True. Note that $\nabla \cdot \mathbf{F} = 3y^2 + x^2e^{yx^2} + x^2e^z$. This is always positive which means that the gas is expanding at every point, and hence cannot be undergoing compression. Thus the velocity field cannot have this form.

(**1 mark**)

Note: Many students wrote "False" and saying that the gradients is always positive and that the gas cannot therefore be undergoing compression. They were awarded the mark.

It is a question of which of the two sentences in the question one is asserting is true or false. Technically, it should be the second sentence that one should be declaring true or false, since it is given that the gas is being compressed (thus, this is not up for debate). However, as long students have given a proper justification for their answer, both answers above have been awarded marks.

Students who said only that $\nabla \cdot \mathbf{F} \neq 0$ have not been awarded any marks, since one cannot conclude anything from this.

(b) The area of the paraboloid $x^2 + z^2 = 2ay$ between $y = 0$ and $y = b$ is given by

$$S = \iint_T \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dxdz$$

where T is the region $\{z^2 + x^2 \leq 2a^2\}$ in the zx-plane. (**1 mark**)

Hence,

$$S = \iint_T \sqrt{1 + \frac{x^2}{a^2} + \frac{z^2}{a^2}} dxdz$$

(**1 mark**)

This can be evaluated as

$$= \int_0^{2\pi} \int_0^{a\sqrt{2}} \sqrt{1 + \frac{r^2}{a^2}}\ rdrd\theta.$$

$$= \frac{2\pi}{3}(3\sqrt{3} - 1)a^2.$$

(**1 mark**)

Note: Some students may have solved this question by recognising the paraboloid as a surface of revolution. Minor mistakes in calculation at the end should be excused.