MA 105 D1 Lecture 6

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Maxima and Minima

1. The Extreme Value Theorem
2. Fermat's theorem
3. The second derivative test

Local and Global extrema.

The continuity of the first derivative

1. The IVP property and Darboux's theorem.

2. Differentiable but not continuously differentiable functions.
Here is the standard example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0. \end{cases}$$

Concavity and convexity

Let I denote an interval (open or closed or half-open).
Definition: A function $f : I \to \mathbb{R}$ is said to be concave (or sometimes concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

for all x_1 and x_2 in I and $t \in [0,1]$. Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

By replacing the $\geq$ and $\leq$ signs above by strict inequalities we can define strictly concave and strictly convex functions.

For various reasons, convex functions are more important in mathematics than concave functions and for this reason we will concentrate on the former rather than the latter. On the other hand, note that if $f(x)$ is a concave function, $-f(x)$ is a convex function, so it is really enough to study one class or the other.

Examples of concave and convex functions

Here are some examples of convex functions.

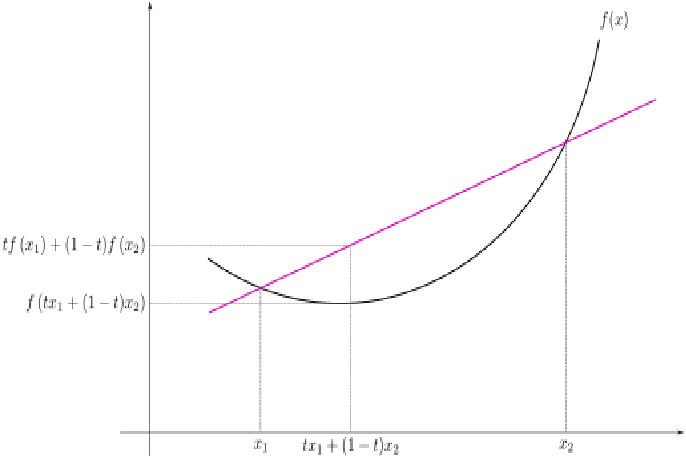
1. $f(x) = x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $[0, \infty)$.
3. $f(x) = e^x$ on $\mathbb{R}$.

Examples of concave functions include

1. $f(x) = -x^2$
2. $f(x) = x^3$ on $(-\infty, 0]$
3. $f(x) = \log x$ on $(0, \infty)$.

For a convex function f and point $c \in (x_1, x_2)$, the point $(c, f(c))$ always lies below the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Convexity illustrated graphically



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Propeties of Convex functions

Convex functions have many nice properties. For instance, it is easy to show that convex functions are continuous (do this!). More is true.

Exercise 1. Every convex function is Lipschitz continuous (a function is Lipschitz continuous if it satisfies the inequality given in Exercise 1.16 but with $\alpha = 1$). In fact, much more is true. A convex function is actually differentiable at all but at most countably many points.

A differentiable function is convex if and only if its derivative is monotonically increasing. Moreover, if a function is both differentiable and convex, it is continuously differentiable, that is, its derivative is continuous (feel free to try proving these facts).

Convexity and the second derivative

A twice differentiable function on an interval will be convex if its second derivative is everywhere non-negative. If the second derivative is positive, the function will be strictly convex.

However, the converse of the second statement above is not true. Can you give a counter-example to the converse of the second statement?

How about $f(x) = x^4$?

Definition: A point of inflection x_0 for a function f is a point where the function changes its behavior from concave to convex (or vice-versa). At such a point, if the function is twice differentiable, $f''(x_0) = 0$, but this is only a necessary, not a sufficient condition.(Why?) If further, we also assume that the lowest order (≥ 2) non-zero derivative is odd, then we get a sufficient condition.

We will now introduce some notation. The space $\mathcal{C}^k(I)$, will denote the space of k times continuously differentiable functions on an (open) interval I, for some fixed $k \in \mathbb{N}$, that is, the space of functions for which k derivatives exist and such that the k-th derivative is a continuous functions.

The space $\mathcal{C}^\infty(I)$ will consist of functions that lie in $\mathcal{C}^k(I)$ for every $k \in \mathbb{N}$. Such functions are called smooth or infinitely differentiable functions.

From now on we will denote the k-th derivative of a function $f(x)$ by $f^{(k)}(x)$.

Our aim will be to enlarge the class of functions we understand using the polynomials as stepping stones.

The Taylor polynomials

Given a function $f(x)$ which is n times differentiable at some point x_0 in an interval I, we can associate to it a family of polynomials $P_0(x), P_1(x), \ldots P_n(x)$ called the Taylor polynomials of degrees $0, 1, \ldots n$ at x_0 as follows.

We let $P_0(x) = f(x_0)$,

$$P_1(x) = f(x_0) + f^{(1)}(x_0)(x - x_0),$$

$$P_2(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2$$

We can continue in this way to define

$$P_n(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's Theorem

The Taylor polynomials are rigged exactly so that the degree n Talyor polynomial has the same first n derivatives at the point x_0 as the function $f(x)$ has, that is, $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $0 \leq k \leq n$, where $f^{(0)} = f(x)$ by convention.

Taylor's Theorem says that we can recover a lot of information about the function from the Taylor polynomials.

Warning: The version of Taylor's theorem I stated in class was slightly imprecise. I have corrected the mistakes below and added a couple of steps for clarity.

Theorem 19: Let I be an open interval and suppose that $[a, b] \subset I$. Suppose that $f \in \mathcal{C}^n(I)$ $(n \geq 0)$ and suppose that $f^{(n)}$ is differentiable on I. Then there exists $c \in (a, b)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where $P_n(x)$ denotes the Taylor polynomial of degree n at a.

The proof of Taylor's theorem

Proof: From the definition, we see that

$$P_n(b) = f(a) + f^{(1)}(a)(b-a) + \cdots \frac{f^{(n}(a)}{n!}.$$

Consider the function

$$F(x) = f(b) - f(x) - f^{(1)}(x)(b-x) - \frac{f^{(2)}(x)}{2!}(b-x)^2 - \ldots - \frac{f^{(n)}(x)}{n!}(b-x)^n.$$

Clearly $F(b) = 0$, and

$$F^{(1)}(x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!}. \tag{1}$$

We would like to apply Rolle's Theorem here, but $F(a) \neq 0$. So consider

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^{n+1} F(a)$$

(this is similar to the method by which we reduced the MVT to Rolle's Theorem), and we see that $g(a) = 0$. Applying Rolle's Theorem we see that there is a $c \in (a, b)$ such that $g'(c) = 0$.

This yields

$$F^{(1)}(c) = -(n+1)\left[\frac{(b-c)^n}{(b-a)^{n+1}}\right] F(a). \tag{2}$$

We can eliminate $F^{(1)}(c)$ using (1). This gives

$$-(n+1)\left[\frac{(b-c)^n}{(b-a)^{n+1}}\right] F(a) = -\frac{f^{(n+1)}(c)(b-c)^n}{n!},$$

from which we obtain

$$F(a) = \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

This proves what we want. $\square$

Remarks on Taylor's Theorem and some examples

Remark 1: When $n = 0$ in Taylor's Theorem we get the MVT. When $n = 1$, Taylor's Theorem is called the Extended Mean Value Theorem.

Remark 2: The Taylor polynomials are nothing but the partial sums of the Taylor Series associated to a $\mathcal{C}^\infty$ function about (or at) the point a:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(b-a)^k.$$

We can show that this series converges provided we know that the difference $f(x) - P_n(x) = R_n(x)$ can be made less than any $\epsilon > 0$ when n is sufficiently large. We will see how to do this for certain simple functions like e^x or $\sin x$.

The Taylor series for e^x

Let us show that the Taylor series for the function e^x about the point 0 is a convergent series for any value of $x = b \geq 0$ and that it converges to the value e^b (a similar proof works for $b < 0$).

In this case, at any point a, $f^{(n)}(a) = e^a$, so at $a = 0$ we obtain $f^{(n)}(0) = 1$. Hence the series about 0 is

$$\sum_{k=0}^{\infty} \frac{b^k}{k!}.$$

If we look at $R_n(b) = e^b - s_n(b)$ we obtain

$$|R_n(b)| = \frac{e^c b^{n+1}}{(n+1)!} \leq \frac{e^b b^{n+1}}{(n+1)!},$$

since $c \leq b$. As $n \to \infty$ this clearly goes to 0. This shows that the Taylor series of e^b converges to the value of the function at each real number b.