Tutorial 6

P49. Time independent SE for 1D

$$\hat{H}\psi = \hat{E}\psi$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad \hat{E} = i\hbar\frac{\partial}{\partial t}$$

$\because \phi_n(x)$ are solution of TISE with energies E_n,

$$\therefore -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\phi_n(x)) = E_n\phi_n(x)$$

Thus.

$$\hat{H}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\sum_n C_n\phi_n(x)e^{-\frac{iE_nt}{\hbar}}$$

$$= \sum_n C_n\phi_n(x)\left(i\hbar\frac{\partial}{\partial t}\left(e^{-\frac{iE_nt}{\hbar}}\right)\right).$$

$$= \sum_n C_n e^{-\frac{iE_nt}{\hbar}}(E_n\phi_n(x)).$$

Thus, $\hat{H}\psi(x,t) = \hat{E}\psi(x,t)$.

However,

$$\hat{H}\psi(x,0) = \sum_n C_nE_n\phi_n(x)$$

Thus for $\psi(x,0)$ to be a solution of TISE we must 've $\hat{H}\psi(x,0) = E\psi(x,0)$ for some real constt E i.e

$$\sum C_nE_n\phi_n(x) = E\left(\sum_n C_n\phi_n(x)\right).$$

$$\sum C_n\phi_n(E - E_n) = 0$$

$\because \hat{H} \rightarrow$ Hermitian operator

$\therefore$ the eigen values $\phi_n(x)$ must be orthogonal $\therefore$ linearly independent

$\rightarrow$ Thus, this possible only if $E = E_n\ \forall\ n$.

But $\because$ all E_ns are distinct (assuming non-degenerate levels in 1-D space), this is not possible unless $\psi(x,t)$ is not a linear combination but only a single eigen function.

P50 $\cdot -\frac{\hbar^2}{2m} \nabla^2 \psi_1(x,t) = i\hbar \frac{\partial \psi_1}{\partial t}(x,t)$ — (1)

$\frac{\hbar^2}{2m} \nabla^2 \psi_2(x,t) = i\hbar \frac{\partial \psi_2}{\partial t}(x,t)$ — (2)

Multiply $(1) \rightarrow (a)$, $(2) \rightarrow (b)$ $\}$ and add.

$\cancel{}\ a\frac{\hbar^2}{2m} \nabla^2 (a\psi_1(x,t) + b\psi_2(x,t)) = i\hbar\left[a\frac{\partial \psi_1}{\partial t}(x,t) + b\frac{\partial \psi_2}{\partial t}(x,t)\right]$

$\therefore$ Take $a\psi_1(x,t) + b\psi_2(x,t) = \psi$

$\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial}{\partial t}\psi$. ψ is a solution.

P51. $\hat{O}_1\psi = -i\frac{\partial \psi(x)}{\partial x}$ $\hat{O}_2\psi = \left(x\psi(x) - i\frac{\partial \psi}{\partial x}\right)$.

$\psi = e^{ikx}$.

$\hat{O}_1\psi = -i(ike^{ikx}) = ke^{ikx} = k\psi$ $\therefore$ Yes.

$\hat{O}_2\psi = x\psi(x) - i(ike^{ikx}) = \underbrace{(x+k)}_{\text{not constant}}\psi$ $\therefore$ No.

P52. $\hat{G} = -i\hbar\frac{d}{dx} + Ax$.

$\therefore \hat{G}f(x) = Kf(x)$ K = constant

$\therefore \left(-i\hbar\frac{d}{dx} + Ax\right) f(x) = Kf(x)$

$-i\hbar f'(x) + Axf(x) = Kf(x)$

$f'(x) = \frac{dy}{dx}$ $f(x) = y$.

$\therefore -i\hbar\frac{dy}{dx} + Axy = ky$.

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$$-i\hbar \frac{dy}{y} = ~~A0~~ \; (k - Ax)\,dx.$$

$$-i\hbar \ln y = kx - \frac{Ax^2}{2} + c.$$

$$f(x) = e^{\frac{i\left(kx - \frac{Ax^2}{2}\right) + ci}{\hbar}} = \phi(x).$$

$$\phi(a) = \phi(-a)$$

$$\therefore e^{iak} = e^{ia(-k)} \quad \therefore \boxed{k = 0}$$

eigen value.

P53. $\phi = Ae^{-\frac{x^2}{a^2}}$

$$\int_{2a}^{2a+dx} \phi^*(x)\,\phi(x)\,dx = \frac{100}{N}$$

$$\int_{0}^{a+dx} \phi^*(x)\,\phi(x)\,dx = \frac{?}{N}.$$

$$\therefore \frac{\cancel{A^2}\, e^{-2\left(\frac{2a}{a}\right)^2}}{\cancel{A^2}\, e^{-2\left(\frac{a}{a}\right)^2}} = \frac{100}{?}$$

$$? = 100\,e^{+6} = 100\,e^{6}.$$

P54 $\hat{P}\psi_1 = P_1\psi_1$ (Assume ψ_1, ψ_2 are orthonormal).

$\hat{P}\psi_2 = P_2\psi_2$.

at $t=0$. $\psi = 0.25\psi_1 + 0.75\psi_2$.

Normalize. $\psi = \dfrac{0.25\psi_1 + 0.75\psi_2}{\sqrt{0.25^2+0.75^2}}$.

$\because \psi = \beta_1\psi_1 + \beta_2\psi_2$.

probability of $\psi_i = \beta_i^2$ (in normalised).

$$\therefore \hat{P}\psi = \frac{0.25(P_1\psi_1)}{\sqrt{0.25^2+0.75^2}} + \frac{0.75}{\sqrt{0.25^2+0.75^2}} P_2\psi_2.$$

$\therefore$ Probability of observing

$$P_1 = \left(\frac{0.25}{\sqrt{0.25^2+0.75^2}}\right)^2 = \frac{1}{10}$$

P55 $\phi_1 = D(3u_1 + 4u_2)$ } Assume u_1 & u_2 orthonormal

$\phi_2 = F(4u_1 - Pu_2)$ & ϕ_1 & ϕ_2 orthonormal

$u_1, u_2 \rightarrow$ normalised } Assume.

$\phi_1, \phi_2 \rightarrow$ normalised

(a) $$9D^2\underbrace{|u_1|^2}_{1} + 4^2D^2\underbrace{|u_2|^2}_{1} + 24D^2\underbrace{u_1u_2}_{0} = 1$$

$$\therefore D^2 = \frac{1}{25} = \pm\frac{1}{5}.$$

$$F^2(16\underbrace{|u_1|^2}_{1} + P^2\underbrace{|u_2|^2}_{1} - 8P\underbrace{u_1u_2}_{0}) = 1$$

$$\therefore F^2(16+P^2) = 1$$

$\therefore \phi_1\cdot\phi_2 = 0 = 12DF - 4DFP = 0$

$\therefore \underline{P=3}$ $\therefore F = \sqrt{\dfrac{1}{25}} = \underline{\pm\dfrac{1}{5} = F}$

$\phi_1 = \frac{3}{5}u_1 + \frac{4}{5}u_2 \qquad \phi_2 = \frac{4}{5}u_1 - \frac{3}{5}u_2,$

$\hat{A}\phi_1 = \hat{A}\left(\frac{3}{5}u_1 + \frac{4}{5}u_2\right) = a_1\phi_1.$

$\hat{A}\phi_2 = \hat{A}\left(\frac{4}{5}u_1 - \frac{3}{5}u_2\right) = a_2\phi_2.$

u: $15\phi_1 = 9u_1 + 12u_2.$

$20\phi_2 = 16u_1 - 12u_2$

$\frac{15\phi_1 + 20\phi_2}{25} = u_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2.$

$= u_2 = +\frac{4}{5}\phi_1 - \frac{3}{5}\phi_2.$

$\cancel{\phi\ \frac{1}{4}\left(\phi_1 - \frac{3}{5}\left(\frac{3}{5}\phi_1 + \frac{4}{5}\phi_2\right)\right)}$

$\frac{5-9}{5}\phi_1$

$\frac{5}{5}\phi_2$

$\therefore u_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$

$u_2 = \frac{4}{5}\phi_1 - \frac{3}{5}\phi_2.$

at time $t=0$, $\left(\frac{2}{3}\phi_1 + \frac{1}{3}\phi_2\right).$

(b) $\phi = \frac{2}{3}\phi_1 + \frac{1}{3}\phi_2 \,/\, \sqrt{5}/3$

$\hat{A}\phi = \frac{2}{3}a_1\phi_1 + \frac{1}{3}a_2\phi_2.$

$\therefore a_1 \to \left(\frac{2}{\sqrt{5}}\right)^2$	$a_2 \to \left(\frac{1}{\sqrt{5}}\right)^2$
$= \frac{4}{5}$	$= \frac{1}{5}$

(c) yielded $\longrightarrow a_1$

$\therefore \hat{A}\phi = a_1 \boxed{\phi_1}$

$= \cancel{}$

$\phi_1 = \frac{3}{5}u_1 + \frac{4}{5}u_2.$

(c) $\hat{B}\phi_1 = \frac{3}{5}b_1 u_1 + \frac{4}{5}b_2 u_2.$

$\therefore b_1 \to \frac{9}{25}$ $\quad b_2 \to \frac{16}{25}$

(d) $t = 0$

$\phi = \frac{2}{3}\left(\frac{3}{5}u_1 + \frac{4}{5}u_2\right) + \frac{1}{3}\left(\frac{4}{5}u_1 - \frac{3}{5}u_2\right) \Big/ \sqrt{5}/3$

$B\phi = \left(\frac{6}{15} + \frac{4}{15}\right)b_1 u_1 + \left(\frac{8}{15} - \frac{3}{15}\right)b_2 u_2.$

$\therefore \cancel{} = \sqrt{5}/3$ $\quad \sqrt{5}/3.$

$\frac{10}{5\sqrt{5}}b_1 u_1 + \frac{5}{5\sqrt{5}}b_2 u_2$

$\frac{100}{125} \to b_1$ $\quad \frac{25}{125} \to b_2$

(e) $\therefore \phi = b_1 u_1.$

$\therefore \phi = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2.$

$A\phi = \frac{3}{5}a_1\phi_1 + \frac{4}{5}a_2\phi_2.$

$a_1 \to \frac{9}{25}$ $\quad a_2 \to \frac{16}{25}$

P56 $\psi = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$$\lim_{n\to\infty} \frac{2}{L} \int_{B}^{B+b} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$\frac{1}{L} \int_{B}^{B+b} \left[1 - \cos\frac{n\pi x}{L}\right] dx$$

$$= \frac{1}{L}\left[\left[(B+b) - \frac{L\sin n\pi(B+b)}{2\pi n \; L}\right] - \left[B - \frac{L\sin n\pi B}{2\pi n \; L}\right]\right]$$

as $n\to\infty$ $\quad 0 \quad$ as $n\to\infty$

$$\therefore \frac{b}{L}$$

P57. $\phi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$$P = \frac{1}{L} \int_{L/3}^{2L/3} \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx. \quad (n = 1 \to \text{ground state})$$

$$\frac{1}{L}\left[\frac{L}{3} - \left[\frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_{L/3}^{2L/3}\right].$$

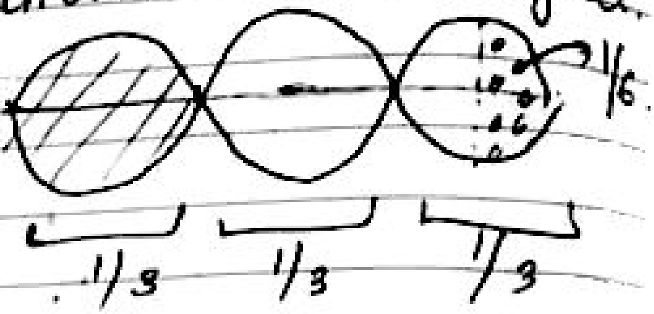
$$\frac{1}{3} - \frac{1}{2\pi}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$\boxed{\frac{1}{3} + \frac{\sqrt{3}}{2\pi}}$$

P58. $P = \frac{2}{L}\int_0^{L/6} \sin^2\left(\frac{3\pi x}{L}\right)dx$.

$P = \frac{1}{L}\int_0^{L/6}\left(1-\cos\frac{6\pi x}{L}\right)dx$.

guess = $\frac{1}{6}$ look at cos function → sin integral.



on integrating,

$\frac{1}{6} - \frac{1}{6\pi}\left[\sin\frac{6\pi x}{L}\right]_0^{L/6} \to 0$. (guess).

$= \frac{1}{6}$.

P59. $P = \frac{2}{L}\int_0^{L/4} \sin^2\frac{n\pi x}{L}\,dx$

$P = \frac{1}{L}\int_0^{L/4}\left(1-\cos\frac{2n\pi x}{L}\right)dx$.

$= \frac{1}{L}\left[\frac{L}{4} - \left[\frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/4}\right]$.

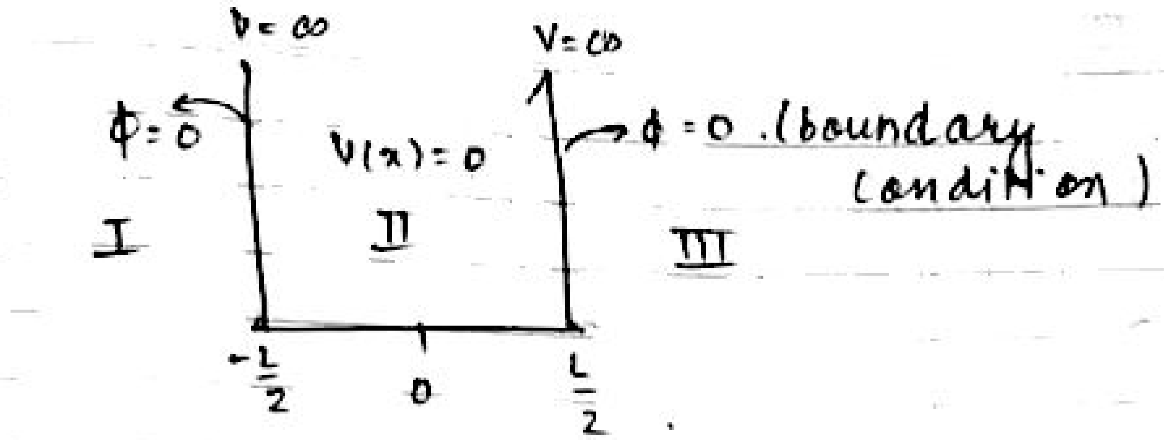
$\therefore \frac{1}{L}\left(\frac{L}{4} - \frac{L}{2\pi}\right) = \frac{1}{4} - \frac{1}{2\pi}$

$\therefore 10000\left(\frac{1}{4} - \frac{1}{2\pi}\right) \to$ particle b/w 0 & $L/4$.

immediately after 1st measurement (if particle $0 < L/4$) then probability 1.

(moment you've an observation, wave function collapses).

P60.



- $-\frac{L}{2} < x < \frac{L}{2} \rightarrow \text{II}$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} = E\phi(x)$$

$$\frac{\partial^2\phi}{\partial x^2} = -k^2\phi(x)$$

$$\phi_{II}(x) = A\cos kx + B\sin kx.$$

$$\phi_{II}\left(x=-\frac{L}{2}\right) = A\cos\frac{kL}{2} - B\sin\frac{kL}{2} = 0.$$

$$\phi_{II}\left(x=\frac{L}{2}\right) = A\cos\frac{kL}{2} + B\sin\frac{kL}{2} = 0.$$

$$\therefore A\cos\frac{kL}{2} = 0$$

$$\therefore B\sin\frac{kL}{2} = 0.$$

case 1. $A = 0, \quad \frac{kL}{2} = n\pi \qquad k = \frac{2n\pi}{L}.$

$$\phi_{II} = B\sin\frac{2n\pi x}{L}.$$

case 2 $\quad B = 0 \quad \frac{kL}{2} = n\pi - \frac{\pi}{2} \qquad k = \frac{(2n-1)\pi}{L}$

$$\therefore \phi_{II} = A\cos\frac{(2n-1)\pi x}{L}.$$

Groundstate $n = 1 \rightarrow$ there's no node.

However in case (1) $\rightarrow$ $B\sin\frac{2\pi x}{L} = 0 \rightarrow$ node. (at $x = 0$)

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