Tunnelling

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Scattering in One dimension

- We studied the problem of a steady stream of particles scattering off a step potential of height V_0 which extends from $x = 0$ to $x \to \infty$.
- We considered two cases:
  1. $E < V_0$: The particles do not have energy to over the potential barrier. All of them are reflected back, leading to the reflection coefficient $R = 1$. This is similar to the classical situation.

     Unlike in the classical case, we find that the particle/wave penetrates the potential barrier to a depth

     $$d \sim \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2mE}}.$$

  2. $E > V_0$: The particles are energetic enough to cross te barrier. Classically, all of them should cross the barrier.

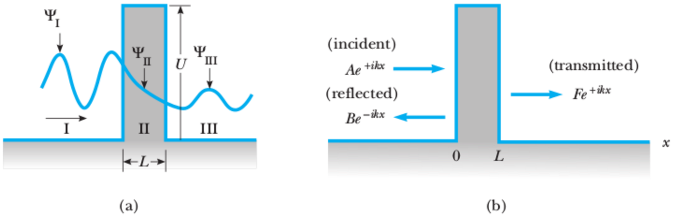
     But quantum mechanics says that they should have wave like properties. Applying Schroedinger's equation, we find that some particles are reflected and some are transmitted.

     The reflection and the transmission occur such that the reflection coefficient R and the transmission coefficient T satisfy $R + T = 1$.

Tunnelling

Let us now consider a potential step of finite width.

$$\begin{aligned} V(x) &= 0 \text{ for } x \leq 0 \\ &= V_0 \text{ for } 0 < x < L, \text{ where } V_0 > 0 \\ &= 0 \text{ for } x \geq L \end{aligned}$$



As before, we will consider two cases: First the case $E < V_0$ and later $E > V_0$.

Tunnelling

- Outside the barrier, wave number of the particle is $k = \sqrt{2mE}/\hbar$. For $E < V_0$, the particle has imaginary wave number $k' = i\kappa$ ($\kappa = \sqrt{2m(V_0 - E)}/\hbar$) inside the barrier.
- The allowed forms of the wave function are

$$
\begin{aligned}
\psi_{\mathrm{I}}(x) &= Ae^{ikx} + Be^{-ikx} \\
\psi_{\mathrm{II}}(x) &= Ce^{\kappa x} + De^{-\kappa x} \\
\psi_{\mathrm{III}}(x) &= Fe^{ikx} + Ge^{-ikx}
\end{aligned}
$$

- As in the second problem $G = 0$ because the boundary condtions do not allow it. However C here is non-zero, unlike in the first problem, because the rising exponential extends only upto a finite distance L.

Tunnelling

- We impose the continuity of the wave function and its first derivative at $x = 0$ and at $x = L$. We get the equations

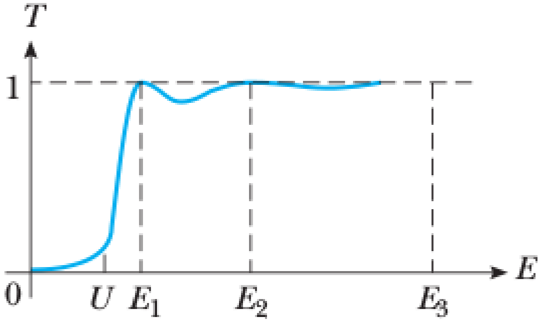
$$\begin{aligned} A + B &= C + D \\ ik(A - B) &= \kappa(C - D), \\ Ce^{\kappa L} + De^{-\kappa L} &= Fe^{ikL} \\ \kappa(Ce^{\kappa L} - De^{-\kappa L}) &= ikFe^{ikL} \end{aligned}$$

- Solving these four equations, we can obtain the expressions for the four ratios B/A, C/A, D/A and F/A.
- The reflection coefficient $R = |B/A|^2$ and the transmission coefficient $T = |F/A|^2$ are functions of k and hence E, the energy of the projectile. They satisfy $R(E) + T(E) = 1$.
- In regions I and III the wave numbers are the same. So no additional factor in the definition of T.

The expression for T is quite complicated

$$T(E) = \left[1 + \frac{1}{4}\left(\frac{V_0^2}{V_0(V_0 - E)}\right)\sinh^2 \kappa L\right]^{-1},$$

$\sinh(\kappa L) = [\exp(\kappa L) - \exp(-\kappa L)]/2.$



For $E \ll V_0$ $T \ll 1$, whereas it is a few percent for $E \simeq V_0$.

Example 7.1 from Serway:

Two conducting copper wires are separated by an insulating layer of copper-oxide. We model the oxide layer as a rectangular barrier of height 10 eV. Calculate the transmission coefficient for penetration by 7 eV electrons, if the layer thickness is (a) 5 nm and (b) 1 nm.

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} == \sqrt{2 \times 511000 \times 3}1973 = 0.9(\mathrm{Angstrom})^{-1}.$$

For $L \gg 1$, $\kappa L \approx 45 \implies \sinh(\kappa L) \approx \exp(\kappa L)/2$.

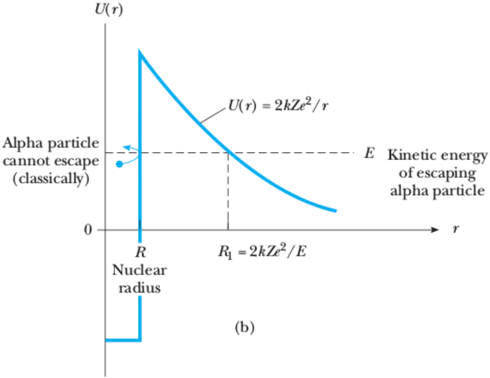
Thus we $T \approx 4\exp(-2\kappa L)$ leading to

$$\frac{T(L=50)}{T(L=10)} = 4\exp(-2 \times 0.9 \times 40) \approx 10^{-31}.$$

Because of the exponential factor, small changes of the barrier height or width lead to large changes in the tunnelling probability.

Application of Tunnelling: α Decay

George Gamow pictured α particle as being pre-formed within a heavy nucleus and being loosely bound to it, in a potential shown below.



The attractive part of the potential arises due to the strong nuclear force and the repulsive part due to the Coulomb force.

Application of Tunnelling: α Decay

Example 7.4 of Serway

- Thorium undergoes α decay with a lifetime of $\sim 10^{10}$ years with $E_\alpha = 4$ MeV. Polonium undergoes α decay with a lifetime of $\sim 10^{-7}$ seconds with $E_\alpha = 9$ MeV.
- Gamow's calculation, based on the concept of tunnelling through the Coulomb potential barrier, could account for the factor of 10^{24} difference in the lifetimes, eventhough the difference in the energies of the two α particles is only a factor of 2.
- In the calculation of the α decay rate, the there are two terms in the exponential with opposite sign.
- In the case of Polonium, there is a near cancellation of these two terms which results in a small negative exponential and a large decay rate.
- In the case of Thorium, the negative term in the exponential is much larger the positive term, leading to a large negative exponential and a small decay rate.

Transmission Resonances

- Now let us consider the case of $E > V_0$ in the case of a potential barrier of finite width. Example 7.3 of Serway.
- Outside the barrier, wave number of the particle is $k = \sqrt{2mE}/\hbar$. In the barrier, the wave number is $k' = \sqrt{2m(E - V_0)}/\hbar < k$.
- The allowed forms of the wave function are

$$\begin{aligned}
\psi_{\mathrm{I}}(x) &= Ae^{ikx} + Be^{-ikx} \\
\psi_{\mathrm{II}}(x) &= Ce^{ik'x} + De^{-ik'x} \\
\psi_{\mathrm{III}}(x) &= Fe^{ikx} + Ge^{-ikx}
\end{aligned}$$

- Here again we have $G = 0$ and $C \neq 0$.

Transmission Resonances

- We impose the continuity of the wave function and its first derivative at $x = 0$ and at $x = L$. We get the equations

$$
\begin{aligned}
A + B &= C + D \\
k(A - B) &= k'(C - D), \\
Ce^{ik'L} + De^{-ik'L} &= Fe^{ikL} \\
k'(Ce^{ik'L} - De^{-ik'L}) &= kFe^{ikL}
\end{aligned}
$$

- Solving these four equations, we can obtain the expressions for the four ratios B/A, C/A, D/A and F/A.
- Once again, the reflection coefficient $R = |B/A|^2$ and the transmission coefficient $T = |F/A|^2$ can be obtained as functions of k and hence E.
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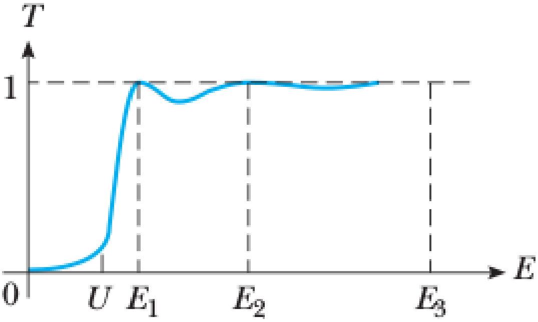
Transmission Resonances

The expression for T is

$$T(E) = \left[1 + \frac{1}{4}\left(\frac{V_0^2}{E(E-V_0)}\right)\sin^2(k'L)\right]^{-1},$$



We see that $T = 1$ when $k'L = n\pi$. That is we get perfect transmission for certain special values of k'.

Transmission Resonances

- The reflected wave can be thought of as the superposition of reflections from the leading and the trailing edges of the barrier.
- If the two waves interfere destructively, then it is possible to have no reflection at all.
- The condition for this destructive interference is $2k'L + \pi = (2n+1)\pi \implies k'L = n\pi$.
- This is a well known phenomenon in optics which is utilised in designing anti-reflection coating on camera lenses.
- You can easily verify that the formula for T holds for scattering off a potential well ($V = -V_1$ for $0 < x < L$) with the replacement $V_0 \to -V_1$.
- When the condition $k'L = n\pi$ is satisfied, the wave function between $x = 0$ and $x = L$ is a standing wave and the particle is almost bound to the potential well, eventhough it has $E > 0$.