MA 105 D1 Lecture 10

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An example showing Darboux integrability

Let us look at Exercise 4.1. We have to show that the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1] \\ 2 \text{ if } x \in (1, 2] \end{cases}$$

is Riemann integrable from first principles.

The solution

We know from Theorem 20 that we are allowed to use any of the three definitions for the integrability, so let us show that this function is Darboux integrable.

Let $P = \{0 = x_0 < x_1, \dots x_{n-1}, x_n = 2\}$ be an arbitrary partition. The point 1 lies in only one of the partitions, say $[x_{i-1}, x_i]$. for some i. We assume that $1 \neq x_i$ and treat this case first.

$$L(f,P) = \sum_{j=1}^{i}(x_j - x_{j-1}) + \sum_{j=i+1}^{n} 2(x_j - x_{j-1}) \tag{1}$$

$$= x_i + 2(2 - x_i) = 4 - x_i, \tag{2}$$

where x_i is a point in $(1, 2]$.

If $x_i = 1$ for some i, then

$$L(f,P) = \sum_{j=1}^{i+1}(x_j - x_{j-1}) + \sum_{j=i+2}^{n} 2(x_j - x_{j-1}) \tag{3}$$
$$= (x_{i+1} - x_0) + 2(2 - x_{i+1}) = 4 - x_{i+1}, \tag{4}$$

where x_{i+1} is a point in $(1,2]$.

In either case $\sup_P L(f,P) = L(f) = 3$.

The Upper sums $U(f,P)$ can be treated in exactly the same way. In either of the cases we have treated above we get $U(f,P) = 4 - x_{i-1}$, for a point $x_{i-1} \in [0,1)$. It follows that $U(f) = \inf_P U(f,P) = 3$.

We have thus shown that $L(f) = U(f) = 3$ which shows that the function is Darboux integrable.

Properties of the Riemann integral

From the definition of the Riemann integral we can easily prove the following properties. We assume that f and g are Riemann integrable. Then

$$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dx + \int_a^b g(t)dt,$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt,$$

for any constant $c \in \mathbb{R}$, and finally if $f(t) \leq g(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$

Implicit in the properties above is that fact if f and g are Riemann

integrable, then so are $f + g$ and cf.

Properties (1) and (2) say that the Riemann integral is a linear map from the set of all Riemann integrable functions to $\mathbb{R}$.

Proving the properties of the integral

It is not hard to prove either of the properties. One needs only to use the corresponding properties for inf and sup:

$$\inf_P U(f+g,P) = \inf_P U(f,P) + \inf_P U(g,P),$$
$$\sup_P L(f+g,P) = \sup_P L(f,P) + \sup_P L(g,P),$$

and, for $c > 0$,

$$\inf_P U(cf,P) = c \inf_P U(f,P), \quad \sup_P L(cf,P) = c \sup_P L(f,P).$$

These are quite easy to do and I leave this as an exercise to the student (the third property is particularly easy).

I have modified the next slide to make it a bit clearer

Clarification on the notation

This may be a good point to talk a little more about inf and sup. ~~First, the properties we have used above are more general.~~

Suppose we have two subsets X and Y of real numbers. Then we can easily see that

$$\sup_{X,Y}(x+y) = \sup_X x + \sup_Y y \quad \text{and} \quad \sup_X cx = c \sup_X x,$$

for $c > 0$. We have similar statements for inf.

If A is a set and $f : A \to \mathbb{R}$ a function, we write $\sup_{x\in A} f(x)$ for the supremum of the set.

Sometimes we may just write $\sup_x f(x)$, when the set in which x is varying is obvious. For instance, we write

$$\sup_n (1 - 1/n) = 0,$$

because it is clear that n varies over the natural numbers in this example. Sometimes we also write just write $\sup_A f(x)$, where it is understood that x varies in A. Thus $\sup_{x\in A} f(x)$, $\sup_x f(x)$ and $\sup_A f(x)$ all mean the same thing.

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$$\sup_{X,Y}(x+y) = \sup_X x + \sup_Y y \quad \text{and} \quad \sup_X cx = c \sup_X x,$$

for $c > 0$. We have similar statements for inf.

Notice that in this case, X and Y vary over different sets of real numbers.

As I clarified in class, this is not the situation one is in when one is dealing lower and upper sums. There, one is taking the $\sup_x[f(x) + g(x)]$ as x varies in some interval. Thus $f(x)$ and $g(x)$ are not varying over different sets, so the argument there has to proceed a little differently.

The Least Upper Bound Axiom

The most interesting fact about inf and sup is, however, the following:

Theorem 22: If a set of real numbers is bounded above, it has a supremum (or least upper bound). If a set of real numbers is bounded below it has an infimum (or greatest lower bound).

Theorem 22 above is equivalent to the fact that the real numbers are complete (Theorem 4 in Lecture 4), a fact that we have not proved (remember that completeness means that every Cauchy sequence in $\mathbb{R}$ converges). It is a very useful form of the completeness and from now on we will use it whenever necessary. It is sometimes referred to as the Least Upper Bound axiom.

Tutorial Problem 4.4

Exercise 4.4 Compute

(a) $\frac{d^2y}{dx^2}$, if

$$x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$

$$(i) F(x) = \int_1^{2x} \cos(t^2) dt$$

and

$$(ii) F(x) = \int_0^{x^2} \cos(t) dt.$$

Problem 4.5

Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral

$$\int_a^{a+p} f(t)dt$$

has the same value for every real number a.

(Hint: Consider $F(a) = \int_a^{a+p} f(t)dt$.)

Problem 4.6

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x - t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and
$g(0) = 0 = g'(0)$.

The formula for arc length

Let us denote the arc length of the curve $y = f(x)$ by S. The length of any given hypotenuse in the previous slide is given by the Pythagorean Theorem: $\sqrt{\Delta x^2 + \Delta y^2}$.
Intuitively, the sum of the lengths of the n hypotenuses appears to approximate S:

$$S \sim \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i,$$

where "$\sim$" means approximately equal. We can use this idea to define the arc length as

$$S := \lim_{\Delta x_i \to 0} \sum_{i=1}^{\infty} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

provided this limit exists (in particular, we demand that the limit is a finite number).

Exercise 4.10.(ii) Find the length of the curve

$$y(x) = \int_0^x \sqrt{\cos 2t}\, dt, \; 0 \le x \le \pi/4.$$

Solution: The formula for the arc length of a curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

For the problem at hand this gives

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x} dx = \sqrt{2} \int_0^{\pi/4} \cos(x) dx = 1.$$

Rectifiable curves

Not all curves have finite arc length! Here is an example of a curve with infinite arc length.

Example: Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), \text{ if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If
http://math.stackexchange.com/questions/296397/nonrectifiable-curve
is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function. Curves for which the arc length S is finite are called rectifiable curves. You can easily check that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

Things can get even stranger

In fact, there exist space filling curves, that is curves $\gamma : [0,1] \to [0,1] \times [0,1]$ which are continuous and surjective. Obviously the graph of this curve "fills up" the entire square. Such curves are not rectifiable (can you prove this?)

The existence of such curves should make you question whether your intuitive notion of dimension actually has any mathematical basis. If a line segment can be mapped continuously onto a square, is it reasonable to say that they have different dimensions? After all, this means we can describe any point on the square using just one number.

We will answer this question (without a proof) later in this course. We will also come back to arc length of a curve when studying multivariable calculus.

The logarithm

For $x \in (0, \infty)$ we define

$$f(x) = \int_1^x \frac{1}{t} dt.$$

Then, for any y, define $g(x) = f(xy)$

Differentiating with respect to x we see that $g'(x) = f'(x)$ Hence,

$$f(x) = g(x) + C,$$

for some constant C. Set $x = 1$ to obtain $C = -f(y)$. Thus,

$$f(xy) = f(x) + f(y).$$

The logarithm and exponential functions

The function $f(x)$ is usually denoted $\ln x$. Since $f'(x) = \frac{1}{x} > 0$, whenever $x > 0$, we see that f is (strictly) monotonic increasing and concave.

By computing the Darboux lower sums associated to $\ln x$, we can easily check that $\ln x > 1$ if $x \geq 3$. By the intermediate value theorem, it follows that there exists a real number e, such that $\ln e = 1$.

It is not hard to see that f must have an inverse function. This is the exponential function sometimes denoted $\exp(x)$. Clearly $\exp(x + y) = \exp(x) \cdot \exp(y)$. Again, it requires some work to see that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

When $x = 1$ we will obtain a formula for e!

The Mean Value Theorem for Integration

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and assume that f is differentiable in (a, b). We apply the Mean Value Theorem to the function

$$F(x) = \int_a^x f(t)dt.$$

This says that there exists $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

But this is the same as saying

$$\int_a^b f(t)dt = f(c)(b - a).$$

This is the Mean Value Theorem for integration.

Convergence of Power series

There is a general test we can use to determine if a power series converges.

Theorem 25: Let $\sum_{n=0}^{\infty} a_n(x-b)^n$ be a power series about the point b. If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

for some $R \in \mathbb{R}$, the series converges in the interval $(b-R, b+R)$ to a smooth function. (if the limit is 0, the series converges on the whole real line).

Roughly speaking a_n behaves like $1/R^n$ for large n. Hence, the terms in the power series can be bounded by $(x-b)^n/R^n$, and this latter (geometric series) converges in $(b-R, b+R)$. This argument can be made precise. Proving that the series is smooth is trickier and we will not get into it.

Taylor series (or more generally "power series") can be differentiated and integrated "term by term". That is if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And similarly,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n dx.$$

We will not be proving these facts but you can use them below.
Exercise 5: Using Taylor series write down a series for the integral

$$\int \frac{e^x}{x} dx.$$

Solution: We simply integrate term by term to get

$$\log x + x + \frac{x^2}{2 \cdot 2!} + \ldots = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

As mentioned before, we can obtain series for the inverse trigonometric series in this way. Indeed we could define the function $\arcsin x$ in this way:

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt.$$

Now we can use the binomial theorem for the integrand. Note that the binomial theorem for arbitrary real exponents is an example of Taylor series:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

It is not too hard to prove that the series on the right hand side above converges for $|x| < 1$. Applying the binomial theorem for $\alpha = -1/2$ to the integrand, we get

$$\arcsin x = \int_0^x \left(1 + \frac{1}{2}t^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^4 + \dots\right) dt.$$

Integrating this term by term, you should verify that you get the series for $\arcsin x$ that you can derive directly from Taylor series.