Indian Institute of Technology Bombay

Department of Mathematics

MA 105: Calculus

Quiz 7 for D1 D2

Date: Wednesday, 4 October 2017 Max. Marks 3

Question: Find the critical points, points of local minima, maxima and saddle points for the function $f(x, y)$.

(A) $f(x, y) = x^3 - 3xy^2$ (B) $f(x, y) = x^5 - 5xy^2$

(C) $f(x, y) = x^7 - 7xy^2$ (D) $f(x, y) = x^9 - 9xy^2$

Solution to (A): $f_x = 3x^2 - 3y^2, f_y = -6xy, f_{xx} = 6x, f_{xy} = -6y, f_{yy} = -6x$. The only critical point is $(0, 0)$. [1mark]

At this point all these partial derivatives vanish. Hence the second derivative test cannot be used since $D = 0$. [1mark]

However, for any $a > 0$, $f(a, 0) = a^3 > 0$. For $a < 0$, $f(a, 0) = a^3 < 0$. Hence f has a saddle point at the origin. [1mark]

Note: The last step can be given 2 marks even if the second step is missing. The second step is really not necessary. But it can be given 1 mark with an overall ceiling of 3 marks for the answer.

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Department of Mathematics

MA 105: Calculus

Quiz 7 for D3 D4

Date: Monday, 9 October 2017 Max. Marks 3

Question: Using the method of Lagrange multipliers, find the absolute minimum value of $f(x,y) = x^2 + y^2$ on the line $y = mx + 1$ where

(A) $m = 1$ (B) $m = 2$

(C) $m = 3$ (D) $m = 4$

Solution to (A): Write the constraint as $g(x,y) = y - x - 1$. Find the solutions to $\nabla f = \lambda \nabla g$ and $y = x + 1$. [1mark]

These lead to the equations $2x = -\lambda, 2y = \lambda, y = x + 1$. [1mark]

Solving these we get only one critical point, namely, $x = -1/2, y = 1/2$. Thus the absolute minimum value of $f(x,y)$ is $1/2$. [1mark]

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Department of Mathematics

MA 105: Calculus : Quiz 6

Date: 27 September 2017 Max. Marks 3

Question: Examine $f(x,y)$ for continuity and partial derivatives at $(0,0)$. Let $f(0,0)=0$.

(A) $f(x,y)=\frac{\sin^2(x+y)}{x+y}$ (B) $f(x,y)=\frac{\sin^2(3x+y)}{3x+y}$

(C) $f(x,y)=\frac{\sin^2(5x+y)}{5x+y}$ (D) $f(x,y)=\frac{\sin^2(7x+y)}{7x+y}$

Solution:(A) For continuity we need to check for the limit of $f(x,y)$ as $(x,y)\to(0,0)$. But this is $\lim_{(x,y)\to(0,0)}\sin(x+y)\frac{\sin(x+y)}{x+y}=0.1=0=f(0,0)$. Hence $f(x,y)$ is continuous at $(0,0)$. [1mark]

The partial derivative $f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)}{h}=\lim_{h\to 0}\frac{\sin^2 h}{h^2}=1$. [1mark]

Similarly $f_y(0,0)=1$. [1mark]

Remark: For the above argument to work for continuity one may define $f(x,y)=0$ on the line $x+y=0$. If students prove that teh function is continuous without noticing lack of definition of $f(x,y)$ on $x+y=0$, you may award one mark. If some students say that since $f(x,y)$ is undefined in every neghborhood of the origin, the function is not continuous, one mark may be awarded.

Indian Institute of Technology Bombay

Department of Mathematics

MA 105: Calculus : Quiz 6

Date: 25 September 2017 Max. Marks 3

Question: Find the unit vectors $u = (a, b)$ for which the directional derivative $D_u(f)$ of the function $f(x, y)$ exist at $(0, 0)$. Prove or disprove: $f(x, y)$ is differentiable at $(0, 0)$. Take $f(0, 0) = 0$.

(A) $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$ (B) $f(x, y) = \frac{x^3}{x^2+y^2}$

(C) $f(x, y) = (x^2 + y^2)\sin\left(\frac{1}{x^2+y^2}\right)$ (D) $f(x, y) = (x^2 + y^2)\cos\left(\frac{1}{x^2+y^2}\right)$.

Solution: (A) Let $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$ and $v = (a, b)$ be a unit vector. We have

$$(D_v f)(0, 0) = \lim_{h\to 0} \frac{f(hv)}{h} = \lim_{h\to 0} \frac{f(ha, hb)}{h} = \lim_{h\to 0} \frac{h^2 ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$

Hence $(D_v f)(0, 0)$ exists and equals 0 for every unity vector $v \in \mathbb{R}^2$.[1mark]

For considering differentiability, note that $f_x(0, 0) = \left(D_{\hat{i}} f\right)(0, 0) = 0 = f_y(0, 0) = \left(D_{\hat{j}} f\right)(0, 0)$. We have then

$$\lim_{(h,k)\to(0,0)} \frac{|f(h, k) - f(0, 0) - hf_x(0, 0) - kf_y(0, 0)|}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk(h^2 - k^2)|}{(h^2 + k^2)^{3/2}} = 0$$

[1mark for setting up derivative] since

$$0 \le \frac{|hk(h^2 - k^2)|}{(h^2 + k^2)^{3/2}} \le \frac{|hk|}{\sqrt{h^2 + k^2}} \frac{h^2 + k^2}{h^2 + k^2} \le \frac{\sqrt{h^2 + k^2}\sqrt{h^2 + k^2}}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2}.$$

Thus f is differentiable at $(0, 0)$. [1mark for right limit]

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Department of Mathematics

MA 105: Calculus : Quiz 5

Date: 6 September 2017 Max. Marks 3

Question A: Find the area of the region between the curve $y = 4 - x^2$, the x-axis and $0 \leq x \leq 3$.

Solution: The x-intercept of the curve partitions $[0, 3]$ into subintervals on which $f(x) = 4 - x^2$ has the same sign. To find the required area we integrate f over each subinterval and add the absolute values of the results. [1]

Integral over $[0, 2]$: $\int_0^2 (4 - x^2)dx = 16/3$. [1]

Integral over $[2, 3]$ is $\int_2^3 (4 - x^2)dx = -7/3$. [1]

Therefore the required area is 23/3.

Question B: Find the area of the region between the graph of $y = f(x) = x^2 - 6x + 8$, the x-axis over the interval $[0, 3]$. (**Answer 22/3**)

Question C: Find the area of the region between the graph of $y = f(x) = 2x - x^2$, the x-axis over the interval $[0, 3]$. (**Answer 8/3**)

Question D: Find the area of the region between the graph of $y = f(x) = 3x - x^2$, the x-axis over the interval $[0, 4]$. (**Answer 19/3**)

Indian Institute of Technology Bombay

Department of Mathematics

MA 105: Calculus : Quiz 5

Date: 4 September 2017 Max. Marks 3

Question A: Find the length of the path $c(t) = (t - \sin t, 1 - \cos t)$ for $t \in [0, 2\pi]$.

Solution: The speed is $||c'(t)|| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$. [1]

Therefore the arc length is $\int_0^{2\pi} \sqrt{2 - 2\cos t}dt$ [1]

Thus the arc lenghth is $2\int_0^{2\pi} \sqrt{\frac{1-\cos t}{2}}dt = 2\int_0^{2\pi} \sin(t/2)dt = 8$. [1]

Question B: Find the arc length of the curve given by $y = \int_0^x \sqrt{\cos 2t}dt$ where $0 \le x \le \pi/4$. **Solution:** Note that $y' = \cos 2x$. Hence The arc length is [1]

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1+(y')^2}dx &= \int_0^{\pi/4} \sqrt{1+\cos(2x)}dx & [1]\\ &= \sqrt{2}\int_0^{\pi/4} |\cos x|dx = 1. & [1]\end{aligned}$$

Question C: Find the arc length of the curve $c(t) = (e^t \cos t, e^t \sin t)$ for $t \in [0, 2]$.

Solution: The required length is

$$\int_0^2 \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2}dt = \sqrt{2}[e^2 - 1].$$

Question D: Find the arc length of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ for $x \in [1, 3]$.

Solution: $\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$. Therefore the required length is

$$\int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right)dx = \frac{53}{6}.$$

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Department of Mathematics

MA 105: Calculus : Quiz 4

Date: 30 August 2017 Max. Marks 3

Question : Find a formula for the Taylor polynomial $\sum_{m=0}^{n} a_m (x-a)^m$ where

$$(A)\ f(x) = 2^x \text{ and } a = 0 \qquad (B)\ f(x) = \sqrt{x} \text{ and } a = 4$$

$$(C)\ f(x) = \sqrt{x+1} \text{ and } a = 3 \qquad (D)\ f(x) = 3^x \text{ and } a = 0$$

Solution :

Let $f(x) = 2^x$. Then $\ln f(x) = x \ln 2$. By chain rule, $f'(x) = 2^x \ln 2$. We claim that $f^{(n)}(x) = 2^x (\ln 2)^n$. [1mark]

This is clearly true for $n = 1$. Assume this for n. Differentiate $f^{(n)}(x) = 2^x (\ln 2)^n$ to get $f^{(n+1)}(x) = 2^x (\ln 2)^{n+1}$. [1mark]

Hence The Taylor polynomial at $a = 0$ is $\sum_{m=0}^{n} \frac{(\ln 2)^m}{m!} x^m$. [1mark]

Indian Institute of Technology Bombay

Department of Mathematics

MA 105: Calculus : Quiz 4

Date: 28 August 2017 Max. Marks 3

Question : Find a formula for the Taylor polynomial $\sum_{m=0}^{n} a_m(x-a)^m$ where

$$(A)\ f(x) = \frac{1}{x} \text{ and } a = 2 \qquad (B)\ f(x) = \frac{1}{x^2} \text{ and } a = 1$$
$$(C)\ f(x) = \frac{1}{x+1} \text{ and } a = 1 \qquad (D)\ f(x) = \frac{1}{(x+1)^2} \text{ and } a = 0$$

Solution: Let $f(x) = \frac{1}{x}$ and $a = 2$. Then $f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$. [1mark] It is true for $n = 1$ since $f'(x) = -x^{-2}$. Suppose that the formula is true for n. Differentiate the formula for $f^{(n)}(x)$ to get $f^{(n+1)}(x) = (-1)^{n+1}(n+1)!x^{-(n+2)}$. [1 mark]
Therefore $\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$. Hence the Taylor polynomial is $\sum_{m=0}^{n} \frac{(-1)^m}{2^{m+1}}(x-2)^m$. [1mark]

Remark: Students may guess the answer by calculating a few derivatives and not mention use of mathematical induction. Full marks could be given to correct solutions which have some justification.