A Note on Spherical Co-ordinates

In cartesian co-ordinates the volume element $d^3r = dx\ dy\ dz$. In spherical coordinates (r, θ, ϕ), $d^3r = r^2 dr \sin\theta d\theta d\phi$. The volume element is the dot product of an area element and the displacement in a perpendicular direction. The area element in $y-z$ plane is defined by $d\vec{A}_{yz} = dy\hat{y} \times dz\hat{z}$. The volume element is $dx\hat{x} \cdot d\vec{A}_{yz} = dx\ dy\ dz$.

The spherical coordinates are defined by the radial distance r from the center, polar angle θ (the angle $\vec{r}$ makes with z axis) and the azimuthal angle ϕ (the angle the projection of $\vec{r}$ on $x-y$ plane makes with x-axis). With these definitions it is straight forward to work out the cartesian coordinates to be

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta. \end{aligned}$$

We consider three mutually perpendicular displacements. The radial displacement dr occurs when we hold θ and ϕ fixed. This is along the radial unit vector $\hat{r}$, whose direction changes from point to point (unlike the cartesian unit vectors). The displacement along θ occurs when we hold r and ϕ fixed. This is given by $rd\theta\hat{\theta}$. On the surface of the earth, this is equivalent to moving along a longitude. The corresponding unit vector $\hat{\theta}$ is tangential to the longitude. The displacement along ϕ occurs when we hold r and θ fixed. This is given by $r \sin\theta d\phi\hat{\phi}$. On the surface of the earth, this is equivalent to moving along a latitude, whose radius is $r \sin\theta$. The unit vector $\hat{\phi}$ is tangential to the latitude. In spherical co-cordinates, the position vector is $\vec{r} = r\hat{r}$ but the most general displacement is

$$d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}.$$

When the position vector changes, in general, both its magnitude r and its direction $\hat{r}$ change. The change in $\hat{r}$ is what leads to the additional terms in $d\vec{r}$.

The three unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ form a right-handed triad of unit vectors, just as $\hat{x}$, $\hat{y}$ and $\hat{z}$ do. Based on the above definitions, you should be able to derive

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$
\begin{aligned}
\hat{\theta} &= \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \\
\hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y}.
\end{aligned}
$$

The area element on the surface of a sphere is given by the cross product of the two independent displacements on it,

$$d\vec{A}_{\theta-\phi} = rd\theta\hat{\theta} \times r\sin\theta d\phi\hat{\phi} = r^2\sin\theta d\theta d\phi\hat{r}.$$

This quantity is sometimes written as $r^2 d\Omega$, where $d\Omega$ is called the **solid angle** subtended by the area element $dA_{\theta-\phi}$ at the origin. The volume element is given by

$$d^3r = dr\hat{r} \cdot d\vec{A}_{\theta-\phi} = r^2 dr\sin\theta d\theta d\phi.$$

If we want to cover the whole space, the range of r is from 0 (origin) to ∞. The range of θ is from 0 (along positive z-axis) to π (along negative z-axis). The range of ϕ is from 0 to 2π. Quite often, we use the variable $\cos\theta$ rather than θ. Then the volume element is written as

$$d^3r = r^2 dr d(\cos\theta) d\phi,$$

with the range of $\cos\theta$ going from -1 to $+1$. It is, of course, true that $\sin\theta d\theta$ is $-d(\cos\theta)$ but the negative sign is taken care of by flipping the limits of integration. The rule of thumb is that if we use θ as the variable of integration, then we integrate from lower value of θ to higher value of θ. If we use $\cos\theta$ as the variable of integration then we integrate from lower value of $\cos\theta$ to higher value of $\cos\theta$. This automatically takes care of the negative sign.

From the notion of displacement vector

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z},$$

we can define the differential operator

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

In spherical co-ordinates we use exactly the same idea. From

$$d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi},$$

we define

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

In general relativity (which uses the language of differential geometry) $d\vec{r}$ is called *contra-variant vector* and $\nabla\phi$ is called a *co-variant vector* where ϕ is a scalar function. From the above definition of ∇, we can compute

$$\nabla^2 = \nabla\cdot\nabla = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right).$$

A similar definition in the cartesian case gives us only three terms because the unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ do not change in direction as we move from point to point. That is not true in the case of the unit vectors in spherical polar coordinates. $\hat{r}$ and $\hat{\theta}$ are functions of both θ and ϕ whereas $\hat{\phi}$ is a function of ϕ. This dependence has to be kept in mind while taking the derivatives. If we do the calculation carefully, we will get

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}.$$