MA 105 D1 Lecture 24

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Integrating scalar functions on a surface

Orientability

Orientation and parametrisation

The surface area integral

Because of the calculations we have just made, the surface area is given by the double integral

$$\iint_S dS = \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The area is nothing but the integral of the constant function 1 on the surface S. We can likewise integrate any scalar function $f : S \to \mathbb{R}$:

$$\iint_S fdS = \iint_D f(x,y,z) \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

If Σ is a union of parametrised surfaces S_i that intersect only along their boundary curves, then we can define

$$\iint_\Sigma fdS = \sum_i \iint_{S_i} fdS.$$

Exercise 11.4

Exercise 11.4: A parametric surface S is described by the vector equation

$$\Phi(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k},$$

where $0 \leq u \leq 4$ and $0 \leq v \leq 2\pi$.

(i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.

(ii) Compute the vector $\Phi_u \times \Phi_v$ in terms of u and v.

(iii) The area of S is $\frac{\pi}{n}\left[65\sqrt{65} - 1\right]$ where n is an integer. Compute the value of n.

Solution: (i) A point (x, y, z) on the surface satisfies $z = x^2 + y^2$. The surface is thus the portion of a paraboloid of revolution about the z-axis between $z = 0$ and $z = 16$. Fixing $u = c$ gives a horizontal circle, while $v = c$ gives a curve which is a portion of a half parabola.

The solution to Exercise 11.4

(ii) We need to compute $\Phi_u \times \Phi_v$. We see that

$$\Phi_u = (\cos v, \sin v, 2u) \quad \text{and} \quad \Phi_v = (-u \sin v, u \cos v, 0).$$

It follows that

$$\Phi_u \times \Phi_v = -2u^2(\cos v\mathbf{i} + \sin v\mathbf{j}) + u\mathbf{k}.$$

(iii) We have

$$S = \int_{v=0}^{2\pi} \int_{u=0}^{4} \|\Phi_u \times \Phi_v\| du dv = 2\pi \int_0^4 u\sqrt{4u^2+1}\, du = \frac{\pi}{6}\left[65\sqrt{65} - 1\right]$$

Therefore, $n = 6$.

The surface integral of a vector field

It is just as easy to integrate vector fields. Indeed, we proceed almost exactly like we did for line integrals:

Let $\mathbf{F}$ be a vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : D \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_D \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv.$$

This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

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(i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.

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Orientable surfaces - a second attempt

What the example of the Möbius strip shows us is that we need a definition that helps us make sense of terms like "inside" and "outside". Let us try a second definition.

Recall that for us a vector field on a surface S is a function $\mathbf{F} : S \to \mathbb{R}^3$.

Definition: A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

At each point of S there are two possible directions for the normal vector to S. The question is whether the normal vector field be can be chosen so that the resulting vector field is continuous.

Examples of orientable surfaces

Example: For the unit sphere in $\mathbb{R}^3$ we can choose an orientation by selecting the unit vector $\hat{\mathbf{n}}(x, y, z) = \hat{\mathbf{r}}$, where $\mathbf{r}$ points outwards from the surface of the sphere.

More explicitly, we define

$$\mathbf{F}(x, y, z) = (x, y, z).$$

This obviously defines a continuous vector field on S. Hence, we see that the unit sphere in $\mathbb{R}^3$ is orientable.

Notice, that we can also define a vector field $\mathbf{G}(x, y, z) = -(x, y, z)$. The vector field $\mathbf{G} = -\mathbf{F}$ is also obviously continuous. This is not specific to the example of a the sphere.

Choosing an orientation

As we have just seen in the preceding example, if S is an orientable surface and $\mathbf{F}$ is a continuous vector field of unit normal vectors, so is $-\mathbf{F}$.

An orientable surface together with a specific choice of a continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

Once one has chosen a particular vector field of normal vectors it makes sense to talk about the "outside" or "positive side" of the surface: usually, it is the side given by the direction of the unit normal vector. The other side is then called the "inside" or "negative side". However, which side one calls "positive" or "negative" is a matter of choice.

The preceding paragraph shows why the second (and correct) definition of orientation is related to the first.

Non-orientable surfaces

Definition: A surface on which there exists no continuous vector field consisting of unit normal vectors is called non-orientable.

Exercise 1: Make a Möbius strip out of a piece of paper. Starting at the top draw a series of stick figures, head to toe, and label their left and right hands. When the stick figure comes back to the top (on the underside) compare the left and right hands of the two stick figures at the top.

Exercise 2: Give a parametrisation for the Möbius strip.

Exercise 3: The Möbius strip is an example of a non-orientable surface. Can you prove this? (Hard.)

The orientation of parametrised surfaces

Let us suppose that we are given an oriented geometric surface S that is described as a $\mathcal{C}^1$ non-singular parametrised surface $\mathbf{\Phi}(u, v)$.

Notice that a parametrised surface $\mathbf{\Phi}$ comes equipped with a natural vector field of unit normal vectors:

$$\hat{\mathbf{n}} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}.$$

Definiition: If the unit normal vector $\hat{\mathbf{n}}$ agrees with the given orientation of S we say that the parametrisation $\mathbf{\Phi}$ is orientation preserving. Otherwise we say that $\mathbf{\Phi}$ is orientation reversing.

Clarification

Two slides ago I asked you to find a parametrisation for a Möbius strip which is not orientable. In the previous slide I declare that the "parametrised surface Φ" comes equipped with an orientation"? How can both statements be true?

In slide 27 I am assuming that the the surface S is oriented. Only then am I claiming that the parametrisation Φ (of the surface S) produces a continuous vector field of unit normals. Thus my statement in the previous slide is not for any parametrised surface but only for those already assumed to be orientable.

The sphere

As an example, let us consider the surface S of the unit sphere in $\mathbb{R}^3$ with the outward radial unit vector as the chosen orientation. Specifically:

$$\hat{\mathbf{n}}(x, y, z) = (x, y, z) = \mathbf{r},$$

for any point (x, y, z) on S. Now consider the parametrisation of this sphere given by the spherical coordinates:

$$x = \cos\theta \sin\phi, \quad y = \sin\theta \sin\phi, \quad z = \cos\phi.$$

Let us check whether this parametrisation is orientation preserving or reversing: we calculate $\mathbf{\Phi}_\theta \times \mathbf{\Phi}_\phi / \|\mathbf{\Phi}_\theta \times \mathbf{\Phi}_\phi\|$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix} = -\mathbf{r}\sin\phi.$$

Since $0 \le \phi \le \pi$, $\sin\phi \ge 0$, $-\mathbf{r}\sin\phi$ is a normal vector in the direction opposite to $\mathbf{r}$. Hence, this parametrisation is orientation reversing.

Independence of parametrisation

Let S be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of S and let $\mathbf{F}$ be a continuous vector field on S.

- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For an oriented surface, the notation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

is unambiguous.

Exercise 11.7

Exercise 11.7: Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and let $\mathbf{F} = (x, y, z)$. Let $\hat{\mathbf{n}}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ using

(i) The vector representation $\Phi(u, v) = (u+v)\mathbf{i} + (u-v)\mathbf{j} + (1-2u)\mathbf{k}$.

(ii) An explicit representation of the form $z = f(x, y)$.

Solution: (i) Note that $\Phi_u \times \Phi_v = -2(\mathbf{i} + \mathbf{j} + \mathbf{k})$ has negative z-component. This parametrisation is thus orientation reversing. Thus

$$\hat{\mathbf{n}}\, dS = -(\Phi_u \times \Phi_v) dudv.$$

The solution to Exercise 11.7 (i)

Hence,

$$\mathbf{F} \cdot \hat{\mathbf{n}} dS = 2(x + y + z) du dv$$

Note that on the surface S, $x + y + z = 1$. Hence,

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = 2\text{Area}(S^*),$$

where S^* is the region in the uv-plane such that $\Phi(S^*) = S$. The region S^* is the triangle with vertices $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{2}, -\frac{1}{2})$, so the area of S^* is $\frac{1}{4}$. Hence,

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{1}{2}.$$

The solution to Exercise 11.7 (i)

(ii) The surface S satisfies $z = 1 - x - y \geq 0,\ x \geq 0,\ y \geq 0$. Thus, using

$$\hat{\mathbf{n}} dS = (-z_x, -z_y, 1) dxdy$$

we get

$$\mathbf{F} \cdot \hat{\mathbf{n}} dS = (x, y, z) \cdot (1, 1, 1) dxdy = (x + y + z) dxdy = dxdy.$$

Given $S_1^* = \{x + y \leq 1, x \geq 0, y \geq 0\}$ as the parametrizing region, one has

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_1^*} dxdy = \mathrm{Area}(S_1^*) = \frac{1}{2}.$$

Exercise 11.9

Exercise 11.9: A fluid flow has flux density vector

$$\mathbf{F}(x, y, z) \;=\; x\mathbf{i} - (2x + y)\mathbf{j} + z\mathbf{k}.$$

Let S denote the hemisphere $x^2 + y^2 + z^2 \;=\; 1$, $z \geq 0$, and let $\hat{\mathbf{n}}$ denote the unit outward normal. Calculate the mass of the fluid flowing through S in unit time in the direction of $\hat{\mathbf{n}}$.

Solution: The hemisphere is given by the graph

$$z = \sqrt{1 - x^2 - y^2},\; x^2 + y^2 \leq 1.$$

We have

$$\hat{\mathbf{n}} dS = (-z_x, -z_y, 1) dxdy = \left(\frac{x}{z}, \frac{y}{z}, 1\right) dxdy.$$

The solution to Exercise 11.9

Hence,

$$\mathbf{F}\cdot\hat{\mathbf{n}}dS = (x, -2x-y, z)\cdot\left(\frac{x}{z},\frac{y}{z},1\right)dxdy$$

$$= \frac{(1-2xy-2y^2)}{z}dxdy.$$

The hemisphere lies over the region $T=\{(x,y): x^2+y^2\le 1\}$.

Using polar coordinates, we see that

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \iint_T \frac{(1-2xy-2y^2)}{\sqrt{1-x^2-y^2}}dxdy$$

$$= \int_0^{2\pi}\int_0^1 \frac{(1-r^2\sin 2\theta-2r^2\sin^2\theta)rdrd\theta}{\sqrt{1-r^2}} = \frac{2\pi}{3}.$$

Homeomorphisms

Recall that a homeomorphism $f : U_1 \to U_2$ from one subset of $\mathbb{R}^n$ to another is a continuous, bijective map such that f^{-1} is also continuous.

The point is that many properties are preserved under homeomorphisms (which may not be preserved under a map which is only bijective and continuous). For instance we have the Invariance of Domain theorem.

Theorem: The space $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $m = n$.

The preceding theorem says that our intuitive notion of dimension can be given mathematical meaning. Incidentally, the previous theorem is closely connected to the Jordan curve theorem (and its generalisation to $\mathbb{R}^n$, the Jordan-Brouwer separation theorem).

Surfaces with boundary

The sphere S^2 clearly does not have a boundary. Neither does the torus. However, the upper hemisphere $z = \sqrt{1 - x^2 + y^2}$ obviously has the unit circle $x^2 + y^2 = 1$ as its boundary. How does one define this more precisely?

Notice that if one is at any point in the sphere, there is a small set around that point that is homeomorphic to an open disc in $\mathbb{R}^2$ (this is just a fancy way of saying that the sphere has no edge). On the other hand, on the hemisphere, if we are at one of the points $(x, \sqrt{1 - x^2}, 0)$, then this is not true any more. This allows us to make precise what we mean by the intuitive notion of boundary.

Definition: Points around which there are no sets in the surface homeomorphic to open discs are called boundary points. The set of all boundary points is called the boundary.