Sequential continuity

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Theorem: A function $f : (a, b) \to \mathbb{R}$ is continuous at c if and only if it is sequentially continuous at c.

The theorem is often useful in its contrapositive form, that is, to show that a function is discontinuous at a point c it is enough to show that it is not sequentially continuous at the point c, i.e., that there is at least one sequence x_n such that $\lim_{n\to\infty} x_n = c$, but $\lim_{n\to\infty} f(x_n) \neq f(c)$.

Continuity implies sequential continuity

Suppose f is continuous at c. Let $\epsilon > 0$ be given. Because f is continuous at c, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta. \tag{1}$$

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Since $\lim_{n\to\infty} x_n = c$, for the $\delta > 0$ chosen above, there exisits $N \in \mathbb{N}$ such that $|x_n - c| < \delta$, whenever $n > N$.

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$N \in \mathbb{N}$ such that $|x_n - c| < \delta$, whenever $n > N$.

By equation (1) above, it follows that $|f(x_n) - c| < \epsilon$ for all $n > N$. This shows that $\lim_{n\to\infty} f(x_n) = f(c)$. $\square$

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Suppose f is sequentially continuous at c but not continuous at c.

Since f is not continuous, there exists some $\epsilon > 0$, such that for any $\delta > 0$ there is a point x such that

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - f(c)| \geq \epsilon.$$

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$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - f(c)| \geq \epsilon.$$

Fix ϵ as above. For each $n \in \mathbb{N}$, let $\delta = 1/n$. Then there exists x_n

such that $0 < |x_n - c| < 1/n$ and $|f(x_n) - f(c)| \geq \epsilon$. Clearly $\lim_{n\to\infty} x_n = c$, but $\lim_{n\to\infty} f(x_n) \neq f(c)$.

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This contradicts the sequential continuity of f at c. Hence our assumption that f is not continuous at c must have been false. $\square$

An everywhere discontinuous function

Let us return to the Exercise 3 of the Optional Exercises. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

We will show that there is no point at which f is continuous. We will use the following two facts (which are intuitively obvious).

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Fact 1: We can find a rational number between any two real numbers.

Fact 2: We can find a irrational number between any two real numbers.

We will prove both these facts a little later. For now we will assume that they are true.

Suppose that c is rational. Then, by Fact 1, in every interval $(c, c + 1/n)$, $n \in \mathbb{N}$, we can find an irrational number, say x_n. Clearly $\lim_{n\to\infty} x_n = c$. Since $f(x_n) = 1$ for all x_n, with $n \in \mathbb{N}$ and $f(c) = 0$, $|f(x_n) - f(c)| = 1$.

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Thus, if $\epsilon = 1$ we see that there is no $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. This shows that f is not continuous at c.

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Thus, if $\epsilon = 1$ we see that there is no $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. This shows that f is not continuous at c.

If c is irrational, we use Fact 2 to show (in the same way as above) that f is not continuous at c. □

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Consider the numbers of the form k/n with $k \in \mathbb{Z}$. There will be some $m \in \mathbb{Z}$ such that $m/n \leq x$ but $(m+1)/n > x$.

Clearly

$$\frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y.$$

So $(m+1)/n$ is the desired rational number.

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So $(m+1)/n$ is the desired rational number.

To prove Fact 2, replace $1/n$ in the argument above by $\sqrt{2}/n$!