P90. This question can have two solutions

Solution 1

Assumption → Each energy level is two-fold spin degenerate i.e. we've 2 states with $E=0$, 8 states with $E=2E$, 16 states with $E=3E$. Each level can accomodate at most 1 electron.

(a) $(1,3,2)$ $\quad {}^2C_1\,{}^8C_3\,{}^{16}C_2 = 2\times 56\times 120 = 13440$

(b) $(2,0,4)$ $\quad {}^2C_2\,{}^8C_0\,{}^{16}C_4 = 1\times 1\times 1820 = 1820$

$\therefore$ Ratio $= \dfrac{1820}{13440} = 0.135$

Solution 2

Assumption → Each energy level is not two-fold spin degenerate i.e. we've 1 state with $E=0$, 4 states with $E=2E$ and 8 states with $E=3E$. Each level can accommodate at most 1 electron.

(a) $(1,3,2)$ $\quad {}^1C_1\,{}^4C_3\,{}^8C_2 = 1\times 4\times 28 = 112$

(b) $(2,0,4)$ $\quad 0\times {}^4C_0\,{}^8C_4 = 0$

0 as we cannot place 2 fermions in the same quantum state

$\therefore$ Ratio $= 0$.

P7.1(a) $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \Rightarrow n^2 = \frac{2mE_nL^2}{\pi^2\hbar^2} = \frac{2(mc^2)E_nL^2}{\pi^2\hbar^2c^2}$

$$n^2 = \frac{2\times 0.5\times 10^6 \times 0.1\times 1 \ (\text{eV-}\mu\text{m})^2}{\pi^2 \times (0.2)^2 \ (\text{eV-}\mu\text{m})^2} \approx \frac{10^5}{10\times 0.04}$$

$\Rightarrow n = 500$

Each level contains two electrons.

Number of electrons $= 1000$

b) For a 1-D potential well-

$g(n) = \frac{2}{L}$ (one state per 'n' and 2 electrons per state)

$$g(n)\,dn = g(E)\,dE \Rightarrow g(E) = \frac{g(n)}{\frac{dE}{dn}}$$

$$\frac{dE}{dn} = \frac{\pi^2\hbar^2}{2mL^2}\cdot 2n = \frac{\pi^2\hbar^2c^2}{(mc^2)L^2}\,n$$

$$g(E)\Big|_{E_F} = \frac{2}{L}\,\frac{mc^2L^2}{\pi^2\hbar^2c^2}\cdot\frac{1}{n(E_F)} = \frac{2\times 0.5\times 10^6\times 1 \ (\text{eV-}\mu\text{m})}{10\times(0.2)^2\times 500 \ (\text{eV-}\mu\text{m})^2}$$

$$= 5000 \ (\text{eV-}\mu\text{m})^{-1}$$

12. $E = (n_x^2 + n_y^2)\dfrac{h^2}{8mL^2}$

$n_x^2 + n_y^2 = n^2$

$E = \dfrac{n^2h^2}{8mL^2}$

One unit square inside the given area $\longleftrightarrow$ One value of (n_x, n_y)

To count no. of degenerate states, consider 'n' to be continuous.

$g(n)\,dn = \dfrac{1}{4} \times 2\pi n\,dn \times (2s+1) = \dfrac{\pi n}{2}(2s+1)\,dn$

($s = |spin|$)
electron
$2s+1$
$= 2\times\frac{1}{2}+1$
$= 2$
$\therefore$ spin deg $= 2$

Transforming the variable to E

$n = \sqrt{\dfrac{8mL^2}{h^2}E}$

$dn = \sqrt{\dfrac{8mL^2}{h^2}} \times \dfrac{1}{2\sqrt{E}}\,dE$

$\therefore g(E)\,dE = \dfrac{\pi}{2} \times \dfrac{8mL^2}{h^2} \times \dfrac{1}{2} \times (2s+1)\,dE$

$= \dfrac{2m\pi L^2 (2s+1)\,dE}{h^2}$ $\quad (L^2 = A)$.

At $T = 0$, $f(E) = 1$

$\therefore N = \displaystyle\int_0^{E_F} \frac{2\pi m L^2 (2s+1)\,dE}{h^2}$ $\therefore N = \dfrac{2\pi m A (2s+1) E_F}{h^2}$.

$\therefore E_F = \dfrac{h^2 N}{2\pi m A (2s+1)}$.

$\langle E \rangle = \dfrac{1}{N}\displaystyle\int_0^{E_F} \frac{(2s+1)\,2\pi m L^2}{h^2} E\,dE$

$= \dfrac{1}{N} \times \dfrac{N}{E_F} \times \dfrac{E_F^2}{2} = \dfrac{E_F}{2}$.

P 93 (a) $E = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega + (n_z + \frac{1}{2})(2\omega)$.

$= (n_x + n_y + 2n_z + 2)\hbar\omega$

$E_{GS} = 2\hbar\omega$.

(b) $E = (n_x + n_y + 2n_z + 2)\hbar\omega = 7\hbar\omega$.

$\Rightarrow n_x + n_y + 2n_z = 5$

$n_z = 0 \Rightarrow n_x + n_y = 5 \rightarrow$ gives degeneracy of 6. (0,5) (1,4) (2,3) (3,2) (4,1) (5,0)

$n_z = 1 \Rightarrow n_x + n_y = 3 \rightarrow$ gives degeneracy of 4 (0,3), (1,2) (2,1) (3,0)

$n_z = 2 \Rightarrow n_x + n_y = 1 \rightarrow$ gives degeneracy of 2 (0,1) (1,0)

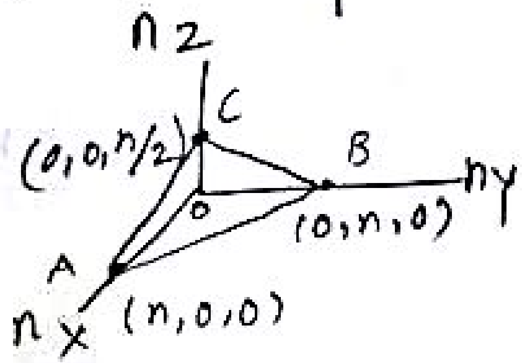
Total degeneracy = 6 + 4 + 2 = 12

(c) for $E = n\hbar\omega$ $(n >> 1)$, calculate $g(n)$

$$\Rightarrow \quad (n_x + n_y + 2n_z + 2)\hbar\omega = n\hbar\omega.$$

$\therefore n = n_x + n_y + 2n_z + 2$ $\therefore$ $n \approx n_x + n_y + 2n_z$. defines a plane.

Need to calculate the no. of points between the planes



$ABC = [(n,0,0), (0,n,0), (0,0,\frac{n}{2})]$ &

$A'B'C' = [(n+dn,0,0), (0,n+dn,0), (0,0,\frac{n+dn}{2})]$.

Volume of tetrahedron OABC $= \frac{1}{6}\left(n \cdot n \cdot \frac{n}{2}\right) = \frac{n^3}{12} = V(n)$

change in volume $dV = V(n+dn) - V(n)$

$$= \frac{1}{12}\left[(n+dn)^3 - n^3\right] \approx \frac{1}{12} 3n^2 dn = \frac{n^2}{4} dn$$

since there's one state per unit volume, the density of states $g(n) = \frac{n^2}{4}$

p94.

$$g(n)\,dn \approx \frac{n^2}{2}\,dn$$

$$g(E)\,dE = \frac{E^2}{2h^2f^2}\,\frac{dE}{hf}$$

$$= \frac{1}{2}\,\frac{E^2}{(hf)^3}\,dE$$

P95 a) Sun consists mostly of hydrogen

$$N_{sun} = \frac{2 \times 10^{30}}{1.67 \times 10^{-27}} = 1.2 \times 10^{57}.$$

b) $V = \frac{4}{3}\pi R^3 = 32 \times 10^{21} m^3$

$$\frac{N}{V} = \frac{1.2 \times 10^{57}}{32 \times 10^{21}} \approx 3.75 \times 10^{34} / m^3$$

If spin is included, $E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} \approx 4.1 \times 10^4 eV$

If spin is ignored, $E_F = \frac{\hbar^2}{2m}\left(6\pi^2 \frac{N}{V}\right)^{2/3} \approx 6.5 \times 10^4 eV.$

(c) $E = pc = \hbar k c = \hbar \cdot \frac{n\pi}{L} \cdot c$

Assuming non-spin case,

$$g(n)\,dn = \frac{n\pi^2}{2}\,dn = \frac{\pi}{2}\left(\frac{L}{\pi\hbar c}\right)^3 E^2\,dE$$

$$N = \int_0^{E_F} g(E)\,dE = \frac{\pi}{2}\,\frac{V}{(\hbar\pi c)^3}\int_0^{E_F} E^2\,dE$$

$$\frac{N}{V} = \frac{\pi}{6(\hbar\pi c)^3}\cdot E_F^{\,3} \Rightarrow \boxed{E_F = \hbar c\left(6\pi^2\frac{N}{V}\right)^{1/3}}$$

P96 Assume a cubic BBR cavity

$E = (\hbar k)c$

The boundary conditions are similar

$\Rightarrow k_x L = n_x \pi$, $k_y L = n_y \pi$, $k_z L = n_z \pi$

$E = \hbar c\sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\hbar c \pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\hbar c \pi n}{L}$

This expression is different from particle in a box
However, density of state in 'n' space is same.

$g(n)\,dn = 2 \times \frac{1}{8} \times 4\pi n^2 dn = \pi n^2 dn$

(two possible polarizations)

$\therefore g(n)\,dn = \pi n^2 dn$

$E = \frac{\hbar c \pi}{L} \times n \Rightarrow dE = \frac{\hbar c \pi}{L} dn$ changing variable

$g(E)\,dE = \pi\left(\frac{L}{\hbar c \pi}\right)^3 E^2 dE.$

(Take $\alpha = 0$, as no. of particles aren't fixed)

To put in standard

$\hbar = \frac{h}{2\pi}$, $E = \frac{hc}{\lambda}$, $dE = \frac{hc}{\lambda^2}|-d\lambda|$

$N(\lambda)\,d\lambda = \frac{\pi V\left(\frac{2\pi}{hc}\right)^3\left(\frac{hc}{\lambda}\right)^2\left(\frac{hc}{\lambda^2}\right)d\lambda}{e^{hc/\lambda kT} - 1} = \frac{8\pi V}{\lambda^4}\frac{d\lambda}{e^{hc/\lambda kT} - 1}$

for energy density, multiply by hc/λ

further multiply by $\frac{c}{4}$ to convert to $e(\lambda)d\lambda$

** (See this derivation in appendix of Richtmyer and Kennard).

p97 $$\langle E\rangle = \frac{\int_0^{E_F} E\,N(E)\,dE}{\int_0^{E_F} N(E)\,dE} = \frac{\int_0^{E_F} E_F^{3/2}\,dE}{\int_0^{E_F} E_F^{1/2}\,dE} = \frac{3}{5}E_F.$$

for N electrons, multiply by N.

P98. (a) $f(E) = \dfrac{1}{e^{(E-E_F)/K_BT}+1}$

$f(E_F + 0.1K_BT) = \dfrac{1}{e^{0.1}+1}$

$f(E_F+10K_BT) = \dfrac{1}{e^{10}+1}$ → can be neglected!

$f(E_F + K_BT) = \dfrac{1}{e+1}$

$f(E_F+2K_BT) = \dfrac{1}{e^{2}+1}$

(b) $\dfrac{1}{4} = \dfrac{1}{e^{\delta/K_BT}+1}$

$e^{\delta/K_BT} = 3$

$\dfrac{3}{4} = \dfrac{1}{e^{\delta/K_BT}+1}$

$e^{\delta/K_BT} = \dfrac{1}{3}$.

Solve.

c) Probability of occupation of a state with energy higher than E_F by $\Delta E = \dfrac{1}{e^{\Delta E/K_BT}+1}$

Probability ~~of~~ that a state is not occupied which has energy $E_F - \Delta E$ $= 1 - \dfrac{1}{e^{-\Delta E/K_BT}+1} = 1 - \dfrac{e^{\Delta E/K_BT}}{1+e^{\Delta E/K_BT}}$

$= \dfrac{1}{e^{\Delta E/K_BT}+1}$

P9g $E = 7.04\,eV$

$V = \sqrt{\frac{2E}{m}}$ $\lambda = \sqrt{\frac{h^2}{2mE}}$. ~orders of 4.6 Å (please calculate completely).

Diffraction depends on size of crystal, if comparable, can diffract.

Lattice constant of crystal $\sim 10^{-10}$ (order of)

$\therefore \lambda \approx 10^{-10}$ (comparable) diffraction takes place.

P100 Rough estimation of free electron contribution to specific heat assuming that only electrons below KT of E_F get excited and gain energy $\sim \frac{3}{2} kT$.

$$\text{Fraction} = \frac{\int_{E_F - KT}^{E_F} E^{1/2}\, dE}{\int_0^{E_F} E^{1/2}\, dE} = \frac{E_F^{3/2} - (E_F - KT)^{3/2}}{E_F^{3/2}}$$

$$= \left[1 - \left(1 - \frac{KT}{E_F}\right)^{3/2}\right] \approx \frac{3}{2} \frac{kT}{E_F}$$

↳ (expand).

For Cu at 300K this is $\sim 0.5\%$.

at 1360 K this is $\sim 2.5\%$.

Specific heat contribution

$$= \frac{d}{dT}\left[\frac{3}{2} KT \times \frac{3}{2} \frac{KT}{E_F}\right] N_A = \frac{9}{2} \frac{KTR}{E_F}$$

Actual contribution. $\frac{\pi^2}{2} \frac{KT}{E_F} R.$