MA 105 D1 Lecture 20

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The change of variables formula

What is a coordinate change?

Vector fields revisited

The del operator

Curl

Divergence

Defining the line integral

Exercise 7.5: Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Find

$$\iint_D dA$$

by transforming it to

$$\iint_E dudv,$$

where $x = \frac{u}{v}, y = uv, v > 0$.

Solution: Put

$$x = \frac{u}{v},\ \ y = uv.$$

Then the rectangle $R = \{1 \leq u \leq 3, 1 \leq v \leq 2\}$ in the uv-plane gets mapped to D in the xy-plane. Further,

$$J = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{2u}{v}$$

Hence,

$$\int\int_D dA = \mathrm{Area}(D) = \int\int_R \frac{2u}{v}\,du\,dv$$

$$= \left(\int_1^3 2u\,du\right)\left(\int_1^2 \frac{dv}{v}\right) = 8\ln 2.$$

How does one explain how we thought of this particular change of coordinates? Ideally, one wants a rectangle in the uv plane to map to the given area in the xy plane.

Presumably the boundary should go to the boundary. Thus, we want a map that sends (vertical) straight lines to hyperbolas and (horizontal) straight lines to slanting lines.

Finding the coordinate change

Vertical straight lines are given by $u = a$ and horizontal lines by $v = b$ for constants a and b.

Notice that on the vertical lines, $x(u, v)$ and $y(u, v)$ are functions of only v. How would one get a hyperbola?

Clearly, $x = c(u)/v, y = c(u)v$ will give a hyperbola $xy = c(u)^2$ for any function $c(u)$.

On the other hand, x and y will be functions only of u on the horizontal lines. We also require the horizontal lines to go to either the line $y = x$ or $y = 4x$.

This will happen if the functions x and y have the form $x = d(v)u, y = u/d(v)$ for some function $d(v)$.

Equating these two forms, we get

$$c(u)/v = d(v)u,$$

so $c(u) = u$ and $d(v) = 1/v$, which gives $x = u/v$ and $y = uv$.

Since we require $xy = 1$ and $xy = 9$, we see (since $u^2 = 1, 9$) that $u = 1$ and $u = 3$ will be the vertical lines.

Since we require $y = x$ and $y = 4x$, we must have (since $v^2 = 1, 4$) $v = 1$ and $v = 2$.

Note that we are also given $v > 0$. This rules out the possibilities $u = -1, -3$ and $v = -1, -2$. If you do not impose this condition, you can use one of the other rectangles.

Exercise 7.6 revisited

Exercise 7.6: Find

$$\lim_{r\to\infty} \iint_{A(R)} e^{-(x^2+y^2)} d(x,y),$$

where $A(R)$ is one of the following regions:

(i) $\{(x,y) : x^2+y^2 \le R^2\}$.

(ii) $\{(x,y) : x^2+y^2 \le R^2,\ x \ge 0,\ y \ge 0\}$.

(iii) $\{(x,y) : |x| \le R,\ |y| \le R\}$.

(iv) $\{(x,y) : 0 \le x \le R,\ 0 \le y \le R\}$.

We call the region in (i) $D(R)$ and the region in (iii) $S(R)$.

The solution to Exercise 7.6

Solution:

(i) We use polar coordinates:

$$x = r\cos\theta, y = r\sin\theta, \quad 0 \le r \le R, 0 \le \theta \le 2\pi.$$

Recalling that the Jacobian $J = r \ge 0$, we get

$$\iint_{D(R)} e^{-(x^2+y^2)} d(x,y) = \int_0^{2\pi} \int_0^R e^{-r^2} r dr d\theta = \pi(1 - e^{-R^2}).$$

Hence, letting $R \to \infty$, we obtain π as the limit.

(ii) By symmetry, the required limit is

$$\lim_{r\to\infty} \frac{\pi}{4}\left(1 - e^{-R^2}\right) = \frac{\pi}{4}.$$

The solution to Exercise 7.6 continued

(iii) Let

$$S(R) = \{|x| \leq R, |y| \leq R\}.$$

$$\iint_{D(R)} e^{-(x^2+y^2)} d(x,y) \leq \iint_{S(R)} e^{-(x^2+y^2)} d(x,y)$$

$$\leq \iint_{D(R\sqrt{2})} e^{-(x^2+y^2)} d(x,y).$$

Therefore, letting $R \to \infty$ we see, using the Sandwich theorem, that the limit is π.

(iv) The required integral, being one-fourth of the integral in (iii), is $\frac{\pi}{4}$.

Revisiting $\int_{-\infty}^{\infty} e^{-x^2} dx$

Recall that we defined $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ as follows. We set

$$I(T) = \lim_{T \to \infty} \int_{-T}^{T} e^{-x^2} dx.$$

and $I = \lim_{T \to \infty} I(T)$.
Now we note that

$$I^2(T) = \int_{-T}^{T} e^{-x^2} dx \cdot \int_{-T}^{T} e^{-y^2} dy$$

$$= \iint_{S(T)} e^{-(x^2+y^2)} d(x,y),$$

where $S(T)$ is as defined in the previous slide. It follows that $I^2 = \pi$ and $I = \sqrt{\pi}$.

The validity of the change of variables formula

We have pushed many things under the carpet in our hurry to develop a change of variables formula. We must perhaps ask ourselves when and under what conditions the formula might be valid.

The first question to ask is "What does changing coordinates mean?". What kinds of functions h is it reasonable to consider?

For instance, should we allow the following transformation $h(u, v) = (1, 2)$ (a constant map)?

Clearly not, since all information about the region D^* is lost when we make such a transformation, considering such functions will not lead to anything useful.

Properties to look for in coordinate changes

One of the problems with the preceding example is that it "destroys area", that is, a rectangle with non-zero area is taken to a point with zero area.

It should not be hard to convince yourself that transformations h that do this should not be "allowed". So the function $h(u, v) = (u, 0)$ or the function $h(u, v) = (0, v^2)$ is "disqualified". More generally, when is "area destroyed" by h?

Clearly when $|J| = 0$. So the non-vanishing of $|J|$ is clearly necessary. The remarkable fact is that, locally at least, this guarantees that the function is bijective (just like in the once variable case)!

If we remember the method of substitution in one variable, only substitution changes that were bijective were allowed. We should clearly require this of our function h, not just locally (which is guaranteed if we assume $|J| \neq 0$), but for the whole domain.

Diffeomorphisms

Finally, another requirement comes from the final formula we derived. This includes $|J|$, which, if we want some kind of integrability, forces h to be a $\mathcal{C}^1$ function.

The final condition we must impose is that h must be a diffeomorphism, that is, the inverse function h^{-1} (which exists, since h is assumed bijective) must also be a $\mathcal{C}^1$ function. This will ensure that the image $h(D^*)$ of a small disc D^* in the uv plane will contain a small disc in the xy plane.

Again, this is automatic locally, if we assume that $|J| \neq 0$. This fact is called the Inverse function theorem.

Coordinate changes

Let us summarize our discussion above. A change of coordinates $h : D^* \to D \subset \mathbb{R}^2$ is a function $h(u, v) = (\phi(u, v), \psi(u, v))$ which is $\mathcal{C}^1$ on D^*, which is a bijection from D^* to D and such that h^{-1} is a $\mathcal{C}^1$ function on D.

There is only one thing we have still been vague about. We have not specified what kind of regions D^* and D should be. For our purposes we will assume that D^* and D are the interiors of regions of type 3 (the interior of a region of type 3 is simply the part of the region that is not on the boundary - for instance, the open disc is the interior of the closed disc). More generally, we can assume that D^* and D are open sets (see next slide).

Open sets

Definition: A subset U of $\mathbb{R}^n$ is called open if around every point $x \in U$ there is a disc $D_r(x)$ of radius r around x contained inside U.

An open interval in $\mathbb{R}$ is an example of an open set in $\mathbb{R}$ ($n = 1$). Finite and countable unions of open intervals are also open sets in $\mathbb{R}$.

Exercise 1: Conversely, prove that any open set in $\mathbb{R}$ is necessarily the (countable) disjoint union of open intervals.

Open discs and rectangles are examples of open sets in $\mathbb{R}^2$ as are unions (finite or infinite) of theses. The n-dimensional open discs and rectangles are open sets in $\mathbb{R}^n$.

The Inverse Function Theorem

Theorem 38: If U is an open set in $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ function such that the Jacobian determinant of F at p, $|J_F(p)| \neq 0$, for some point $p \in U$, then the inverse function F^{-1} exists for some open set V containing $F(p)$ and F^{-1} is a $\mathcal{C}^1$ function.

Another way of thinking of the Inverse Function Theorem is the following. If x and y are functions of u and v, the Inverse Function Theorem tells us that (when $|J| \neq 0$) we can write u and v as functions of x and y, at least in some small open set V.

As I mentioned above, in order to check whether the function h gives a coordinate change, it is enough by the Inverse Function Theorem to check that it is bijective and that its Jacobian determinant is everywhere non-zero.

Flow lines for vector fields

Recall that a vector field was just a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. For the moment let us suppose that all vector fields under consideration are continuous.

Definition: If $\mathbf{F}$ is a vector field, a flow line or integral curve for $\mathbf{F}$ is a curve $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

Basically, one is trying to fit a curve so that the tangent vector of the curve at any point is the same as the vector given by the vector field at that point.

If we write $\mathbf{c}(t) = (x(t), y(t), z(t))$, and $F = (F_1, F_2, F_3)$, we see that finding a flow line is equivalent to solving the following system of equations.

$$\begin{aligned} x'(t) &= F_1(x(t), y(t), z(t)) \\ y'(t) &= F_2(x(t), y(t), z(t)) \\ z'(t) &= F_3(x(t), y(t), z(t)) \end{aligned}$$

The del operator on functions

We will assume from now on that our vector fields are smooth wherever they are defined.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.

The del operator on vector fields

The del operator can be made to operate on vector fields as follows. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

It is useful to represent it as a determinant:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Irrotational flow

Instead of looking at the velocity field of a rigid body we can look at the velocity field $\mathbf{F}$ of the flow of a fluid. In this case, what does it mean if $\nabla \times \mathbf{F} = 0$ at a point P?

It means that the fluid is free from rigid rotations at that point. In physical terms it means that if you put a small paddle wheel face down in the fluid, it will move with the fluid but will not rotate around its axis. In terms of the fluid itself, this translates into there being no whirlpools centred at the point P. In this case, the vector field is called irrotational.

For instance, water draining into sink produces an irrotational field, except at the very centre of the circular drain.

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}.$$

Clearly, if f is $\mathcal{C}^2$ (which we will assume), $\nabla \times \mathbf{F} = 0$. In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. If its curl is not zero at some point, it cannot arise as a gradient.

Is the condition $\nabla \times F = 0$ sufficient?

Recall that we have previously looked at the vector field

$$\mathbf{F} = \frac{y}{x^2+y^2} \cdot \mathbf{i} + \frac{-x}{x^2+y^2} \cdot \mathbf{j},$$

Exercise 1: Check that $\nabla \times \mathbf{F} = 0$. Can you express $\mathbf{F}$ as the gradient of a suitable scalar function?

Finally, as a special case of the curl, we can define the scalar curl. If $\mathbf{F} = (M(x,y), N(x,y), 0)$ is a vector field in the plane, then

$$\nabla \times \mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

The function $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is called the scalar curl of $\mathbf{F}$.

The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

Definition: Let $\mathbf{F} = (F_1, F_2, F_2)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

$$\operatorname{div} \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Examples

Let us consider the divergence of different vector fields.
Example 1: $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

The flow lines of this vector field point radially outward from the origin, so it is clear that the fluid is expanding as it flows. This is reflected in the fact that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fact the vectors get smaller in size which means that the fluid is getting compressed.

Example 2: $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. In this case the fluid is moving countercolockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The change in area in a flow

As before let us assume that our vector field $\mathbf{F} = (F_1, F_2)$ represents the velocity field of a fluid, but this time in just two dimensions. Let us compute the rate of change of a unit area of the fluid as it flows along the integral curve.

We assume that we start at time $t = 0$ at a point $P = (x, y)$ in $\mathbb{R}^2$. We let the point evolve under the flow to a point (X, Y) at time t. Explicitly, we see that

$$X = X(x, y, t) \quad \text{and} \quad Y = Y(x, y, t)$$

The change of variable formula tells us how a unit area changes. One has to simply multiply by the Jacobian determinant of the transformation $\phi(x, y) = (X, Y)$. In this case we have

$$J = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix}.$$

The rate of change of area in a flow

We would like to compute the rate at which the unit area is changing. This is simply given by $\frac{\partial J}{\partial t}$. We first write out the function J as

$$J = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}.$$

It follows that

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right] - \frac{\partial}{\partial t}\left[\frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right]$$

$$\begin{aligned} &= \frac{\partial}{\partial x}\left(\frac{\partial X}{\partial t}\right)\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial Y}{\partial t}\right) \\ &- \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial t}\right)\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right). \end{aligned}$$

We keep going by brute force

Now we observe that $(X(t), Y(t))$ describes a flow line for the velocity field (F_1, F_2). Hence, the tangent vector of the curve is the same as the value of $\mathbf{F}$ at any point. Hence,

$$\frac{\partial X}{\partial t} = F_1, \quad \text{and} \quad \frac{\partial Y}{\partial t} = F_2.$$

This is true because (X, Y) describes a flow line of $\mathbf{F}$ as t varies. Hence, the tangent vector of the curve is the same as the value of $\mathbf{F}$ at any point.

Using these equations we see that

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{F})J.$$

This shows that if $\nabla \cdot \mathbf{F} = 0$, then $\frac{\partial J}{\partial t} = 0$. Hence, $J(x, y, t)$ is a constant function. But $J(x, y, 0) = 1$, so J must be the constant function 1, that is, there is no change of volume along the flow. This shows that if the divergence is zero the fluid is incompressible.

More exercises

Exercise 2: Try doing the above calculation in three dimensions. By this I mean, take a divergence free vector field in $\mathbb{R}^3$ and show that the rate of expansion of volume by unit volume is 0 along flow lines. This is the same calculation as before but there are now eighteen terms in the Jacobian! Not fun.
Moral: There has got to be an easier way.

Exercise 3: In Lecture 18 we have represented three different vector fields in pictures. Calculate their curls and divergences.

The Laplace operator

Just as the curl of a gradient was 0, we similarly have the divergence of any curl is zero. In other words, if $\mathbf{F}$ is a $\mathcal{C}^2$ vector field,

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Exercise 4: If $\nabla \cdot \mathbf{F} = 0$, does it imply that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$?

Finally, the composition of the gradient followed by the divergence gives one of the most important operators in mathematics and physics. The Laplace operator denoted ∇^2 is defined by

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

One can easily check that the function $f(x, y, z) = \frac{1}{\|\mathbf{r}\|}$ satisfies $\nabla^2 f = 0$.

Vector fields and line integrals

In what follows we only assume that the vector field in question are continuous (not smooth).

We will now define the integral of a vector field along a curve. We will assume that we are given a $\mathcal{C}^1$ curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}'(t) \neq 0$ for any $t \in [a, b]$. Such a curve will be called a regular or non-singular parametrised curve.

We define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

If we write $\mathbf{c}(t)$ in vector notation, that is

$$\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

we see that

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$