Cauchy sequences.

Definition: A sequence (x_n) is said to be a Cauchy sequence if given any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } n, m \geq K.$$

Thus, a sequence is not a Cauchy sequence if there exists $\varepsilon > 0$ and a subsequence $(x_{n_k} : k \in \mathbb{N})$ with

$$|x_{n_k} - x_{n_{k+1}}| \geq \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

3.5.5 A sequence converges if and only if it is a Cauchy sequence.

Proof. If a sequence converges, it is easy to see that it must be a Cauchy sequence (exercise). Next, if it is a Cauchy sequence, then it is bounded (why) and hence it has a convergent subsequence say (x_{n_k}) by the Bolzano-Weierstrass theorem. Suppose $\lim(x_{n_k} : k \in \mathbb{N}) = x$, it is easy to check that $\lim(x_n) = x$ (exercise).

Example If (s_n) is a sequence such that $|s_n - s_{n+1}| \leq r^n$ for all $n \in \mathbb{N}$, $0 \leq r < 1$, then the sequence converges. In particular, if $s_n = x_1 + x_2 + \cdots + x_n$ where $|x_n| \leq r^n$, then (s_n) is a Cauchy sequence and hence converges.

3.5.6 (a) $x_1 = 1,\ x_2 = 2$ and $x_{n+2} = (x_{n+1} + x_n)/2$ for all n.

(b) $s_n = -1 + \dfrac{1}{2!} + \cdots + \dfrac{(-1)^n}{n!}$.

(c) $|x_{n+1} - x_n| \leq r|x_n - x_{n-1}|$ for all $n \geq 2$.

Example 3.5.10. Solution of $x^3 - 7x + 2 = 0$.

Define $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$ for suitable choice of x_1.

Indeed, one can also define $u = \sup\{x : x^3 - 7x + 2 < 0\}$.

Construction of Real numbers by Cauchy Sequence.

Recall that we have constructed the set of rational numbers. However, we have yet shown that there exists a set of real numbers, i.e., a complete ordered field. We will now construct this set using rational numbers.

Since we have yet to define real numbers. We will say (x_n) is a Cauchy sequence of rational numbers if given any positive rational number r, there exists $K \in \mathbb{N}$ such that $|x_n - x_m| < r$ for all $n, m \geq K$.

Let S be the collection of Cauchy sequences of rational numbers. We say $(x_n) \sim (y_n)$ if given any positive rational number ε, there exists $K \in \mathbb{N}$ such that $|x_n - y_n| < \varepsilon$ for all $n \in K$. We then show that this defines an equivalence relation on S.

Exercise: $(x_n) \sim (x_{n+K} : n \in \mathbb{N})$.

We will now define $\mathbb{R}$ as the set of equivalence classes in S and let us identify any rational number r with the equivalence class that contains the sequence (r). Note that if $(a) \sim (b)$ where $a, b \in \mathbb{Q}$, then $a = b$; see Homework/Exercise.

We will now show that $\mathbb{R}$ is a field.

The only more difficult part is to show that each nonzero equivalence class has a multiplicative inverse.

First, we will need a simple observation.

Fact: If (x_n) is not a Cauchy sequence that is equivalence to 0, then there exists $M, K \in \mathbb{N}$ such that $|x_n| > 1/M$ for all $n \geq K$. It is then clear that (x_n) is equivalence to a sequence (y_n) of rational numbers with absolute value more than $1/M$.

Using the above sequence, we now claim that the sequence $(1/y_n)$ is Cauchy and clearly it is the multiplicative inverse of (x_n).

We next show that $\mathbb{R}$ is an ordered field.

Similar to the previous argument, we will take the positive real numbers to be the set of equivalence classes (y_n) with $y_n > r$ for all n for some positive rational number r.

Finally, we show that it is complete.

In general, let $((y(k)_n : n \in \mathbb{N}) : k \in \mathbb{N})$ be a Cauchy sequence of Cauchy sequences of rational numbers. Then note that $\lim(y(k) : k \in \mathbb{N}) = (z_n : n \in \mathbb{N})$ for some Cauchy sequence of rational numbers (z_n). Hence every Cauchy sequence converges. For more details, see the Appendix at the end of the note.

Now suppose S is a nonempty set of equivalence classes of Cauchy sequences that is bounded above. For each fixed $n \in \mathbb{N}$, consider the set (recall that we have identified constant sequences (r) with rational numbers r)

$$T_n = \{k \in \mathbb{Z} : k/2^n \text{ is an upper bound of } S\}.$$

First, let us show that it is nonempty. Let X be an upper bound of S. We may assume that $X > 0$. Now, note that $1/X > 0$ and there exists rational number $r = m/K$, $m, K \in \mathbb{N}$ such that $0 < m/K < 1/X$. Clearly $1/X > 1/K$ and hence $K2^n \in T_n$.
We next show that T_n has a least number. For simplicity, let us first assume that the set has a nonnegative real number (an equivalence class of Cauchy sequences of rational numbers). Then it has a least element say n_0 by a consequence of well ordering property of natural number. Note that $i_n - 1 \notin T_n$ and hence $(i_n - 1)/2^n$ is not an upper bound while $i_n/2^n$ is an upper bound. (Note that we have identified the number $i_n/2^n$ to the constant sequence (r) with $r = i_n/2^n$. Define $a_n = i_n/2^n$ and check that (a_n) is a Cauchy sequence of rational numbers since $|a_n - a_m| \leq 1/2^n$ for any $m \geq n$. Finally, we need to show that $[(i_n/2^n)]$ is the least upper bound of S. Note that $((i_n - 1)/2^n) \in [(i_n/2^n)]$.

In general if X is a set with a "distance function", we say it is complete if and only if every Cauchy sequence in the set converges.

$\mathbb{R}^n$ **is complete.** It is not an ordered field for $n > 1$. But it is complete, i.e., every Cauchy sequences converges.

Homework/Exercise:

A. Show that multiplication of "real numbers" is well defined.

B. Let $a, b \in \mathbb{Q}$. Prove that the constant sequences $(a) \sim (b)$ if and only if $a = b$.

C. Show that if we define the positive real numbers as above, then the set of "real numbers" satisfies properties 2.1.5.

Sec 3.5: all questions except 1,3 and 7. (4th ed is the same as 3rd ed)

Additional questions for HW4: Sec 3.5: 6,8,12.

Sec 3.6: all questions except 1,6,10. (4th ed is the same as 3rd ed)

Homework5: 5,7,8 in the homework on limsup and liminf below, Questions B and Sec 3.6: 1, 6, 10.

Properly divergent sequences

We say $\lim(x_n) = \infty$ if given any real number $\alpha \in \mathbb{R}$, there exists $K \in \mathbb{N}$ such that

$$x_n > \alpha \quad \text{for all } n \geq K.$$

Note that then the sequence diverges. Similarly, we could define the meaning of $\lim(x_n) = -\infty$.

Example : $\lim(n) = \infty$ and hence $\lim(n^k), \lim(n!), \lim(r^n) = \infty$ (if $r > 1$.)

Moreover, if $\lim(x_{n+1}/x_n) = L > 1$ where (x_n) is a sequence of positive real numbers, then $\lim(x_n) = \infty$.

A simple fact (3.6.4)

Suppose $x_n \leq y_n$ for all n. Then

$\lim(x_n) = \infty \implies \lim(y_n) = \infty$.

Operation with infinite limits:

Suppose $\lim(x_n) = x$ and $\lim(y_n) = l$, l could be infinite while $x \in \mathbb{R}$. Then

(i) $\lim(x_n + y_n) = x + l$ ($x + \infty = \infty$ and $x - \infty = -\infty$.)

(ii) $\lim(x_n y_n) = lx$ if $x \neq 0$ ($a \times \infty = \infty$ if $a > 0$.)

(iii) $\lim(x_n/y_n) = 0$ if l is infinite and $y_n \neq 0$ for all n.

Theorem 3.6.5 Let $(x_n), (y_n)$ be sequences of positive real numbers and $\lim(x_n/y_n) = L > 0$ ($L \neq \infty$). Then $\lim(x_n) = \infty$ if and only if $\lim(y_n) = \infty$.

Limit superior and limit inferior

Definition Let $\{a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We define its set of cluster points as follows:

$S(a_n) = \{c \in \mathbb{R}$: there exists a subsequence $(a_{n_k} : k \in \mathbb{N})$ of (a_n) that converges to $c\}$.

Examples

(i) $S((-1)^n) = \{1, -1\}$.

(ii) $S(\frac{1}{2}, 1, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{1}{5}, \cdots) = [0, 1]$.

(iii) $S(\sin n) =?$.

Definition Let (a_n) be a bounded sequence. We define

$$\overline{\lim}(a_n) = \overline{\lim_{n\to\infty}}\, a_n = \limsup_{n\to\infty} a_n = \sup S(a_n)$$

$$\underline{\lim}(a_n) = \underline{\lim_{n\to\infty}}\, a_n = \liminf_{n\to\infty} a_n = \inf S(a_n)$$

Useful Fact $\overline{\lim}(a_n) = l$ if and only if

(i) there exists a subsequence of (a_n) that converges to l and

(ii) $\lim(a_{n_k}) \leq l$ for any convergent subsequence (a_{n_k}) of (a_n).

Proof. It is clear that $l = \overline{\lim}_{n\to\infty} a_n$ if (i) and (ii) hold. Conversely, let $\overline{\lim}_{n\to\infty} a_n = l$. Then (ii) follows immediately from the definition. To prove (i), by the definition of the supremum, for each $k \in \mathbb{N}$ there exists an $u_k \in S(a_n)$ such that $l \geq u_k > l - \frac{1}{2k}$. But there exists a subsequence (a_{k_m}) of (a_n) that converges to u_k. By choosing a subsequence of the subsequence, we may assume that $|a_{k_m} - u_k| < 1/2m$ for each $m \in \mathbb{N}$. Moreover, we may assume that $k_m < (k+1)_m$ for $k, m \in \mathbb{N}$ (again by choosing subsequences). We now note that (a_{k_k}) is a subsequence of (a_n) that converges to l.

Remark By modifying the above argument, we can also show that the set of cluster points is closed, i.e., if there exists (u_k), $u_k \in S(a_n)$ for all k such that $u_k \to u$, then $u \in S(a_n)$.

Examples

(1) $\overline{\lim}_{n\to\infty} \sin(n\pi/4) = 1$.

Proof. It is clear that $\sin(n\pi/4) \leq 1$ for all $n \in \mathbb{N}$ and $\lim(\sin(8n+2)\pi/4) = 1$.

(2) Let $a_n = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is odd} \\ \frac{7}{n} - 2 & \text{if } n \text{ is even.} \end{cases}$ Then $\overline{\lim}(a_n) = 1$ and $\underline{\lim}(a_n) = -2$.

Proof. Let (a_{n_k}) be a convergent subsequence of (a_n). It is clear that $\min\{1 + \frac{1}{n_k}, \frac{7}{n_k} - 2\} \leq a_{n_k} \leq \max\{1 + \frac{1}{n_k}, \frac{7}{n_k} - 2\}$ and hence $-2 \leq \lim(a_{n_k}) \leq 1$. Finally, since $\lim(a_{2k}) = -2$ and $\lim(a_{2k+1}) = 1$, the answer follows.

Theorem Let (a_n) be a sequence of real numbers. If it is bounded above we let

$$U_k = \sup\{a_n : n \geq k\}.$$

And if (a_n) is bounded below we let

$$L_k = \inf\{a_n : n \geq k\}.$$

If (a_n) is bounded, then $\overline{\lim}(a_n) = \lim(U_k)$ and $\underline{\lim}(a_n) = \lim(L_k)$.

Useful properties of limit superior and limit inferior

(i) $\overline{\lim}(a_n) \leq u$ if and only if given any $\varepsilon > 0$, there exists an $N > 0$ such that

$$a_n < u + \varepsilon \quad \text{for} \quad n > N \qquad (*)$$

Proof: $(\Longrightarrow)$ done in class using $U_k = \sup\{a_n : n \geq k\}$.

$(\Leftarrow)$ Suppose on the contrary that $\overline{\lim}(a_n) = a > u$. Then (a_n) has a subsequence (a_{n_k}) that converges to a. Take $\varepsilon = (a-u)/2$. Then there exists $N' \in \mathbb{N}$ such that $|a_{n_k} - a| < \varepsilon$ for all $k \geq N'$. In particular, we have

$$u + \varepsilon > a_{n_k} > a - \varepsilon \quad \text{if } k \geq \max\{N, N'\} \quad \text{where } N \text{ is given in } (*).$$

which is impossible as $u + \varepsilon = (a+u)/2 = a - \varepsilon$.

(ii) (Comparison of limit superior) Let $a_n \leq b_n$ for $n \in \mathbb{N}$. Then $\overline{\lim}(a_n) \leq \overline{\lim}(b_n)$.

(iii)

$$\underline{\lim}(a_n)+\underline{\lim}(b_n) \leq \underline{\lim}(a_n+b_n) \leq \underline{\lim}(a_n)+\overline{\lim}(b_n) \leq \overline{\lim}(a_n+b_n) \leq \overline{\lim}(a_n)+\overline{\lim}(b_n).$$

(iv) $\overline{\lim}(-a_n) = -\underline{\lim}(a_n)$.

(v) If $\overline{\lim}_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = l$, then (a_n) converges to l.

Remark

(i) If a set S is not bounded above then we write $\sup S = \infty$, similarly we write $\inf S = -\infty$ if S is not bounded below. Clearly, if $\emptyset \neq A \subset B$, then

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

We will usually write $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

(ii) We can extend the definition of limit superior to unbounded sequences as follows: if (a_n) is not bounded, it is still possible to define limit superior and limit inferior. However, we will have to define it by $\sup S(a_n)$ and $\inf S(a_n)$ and include ∞ and $-\infty$ as possible limits.

Homework on limit superior and inferior

(1) Given $\lim(a_n) \geq 0$ and (b_n) is bounded, show that

$$\overline{\lim_{n\to\infty}}\, a_n b_n = \lim_{n\to\infty} a_n \overline{\lim_{n\to\infty}}\, b_n.$$

(2) Show that a convergent sequence has at least a maximum or minimum element.

(3) Let $a_n > 0$ for $n \in \mathbb{N}$. If $\overline{\lim_{n\to\infty}}\, a_{n+1}/a_n = r$, show that $\{a_n\}_{n=1}^{\infty}$ converges if $r < 1$. What if $r \geq 1$?

(4) Show that

$$\lim_{n\to\infty} \left(\int_0^{\pi/6} \sin^n x dx \right)^{1/n} = \frac{1}{2}.$$

(5) Given $\underline{\lim}_{n\to\infty} a_n \geq 5$ with $a_n > -3$ for $n \in \mathbb{N}$, show that

$$\underline{\lim_{n\to\infty}} \frac{4a_n}{3 + a_n} \geq 5/2.$$

(6) Show that the set of cluster points of the sequence $((-1)^n)$ is $\{-1, 1\}$.

(7) Given $\underline{\lim}_{n\to\infty} a_n \geq a$, show by using the definition that

$$\underline{\lim}_{n\to\infty} \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \geq a.$$

(8) Show that

$$\underline{\lim}(a_n)+\underline{\lim}(b_n) \leq \underline{\lim}(a_n+b_n) \leq \underline{\lim}(a_n)+\overline{\lim}(b_n) \leq \overline{\lim}(a_n+b_n) \leq \overline{\lim}(a_n)+\overline{\lim}(b_n).$$

Appendix We will give a sketch to show that any Cauchy sequence of equivalence classes of Cauchy sequence of rational numbers converges to a (an equivalence class of) Cauchy sequence of rational numbers.

Indeed, first we will need to choose a better representative from each equivalence class, we will choose them (by taking subsequences) such that

$$|y(k)_n - y(k)_m| < \frac{1}{2^{k+n}} \text{ if } m \geq n.$$

Moreover, we can choose a subsequence $(y(m_k))$ of the Cauchy sequence such that

$$|y(m_k) - y(m_l)| < 1/2^{k+1} \text{ if } l \geq k.$$

First note we have for all $l \geq k$,

$$|y(m_k)_n - y(m_l)_n| < 1/2^k \text{ for all } n \geq 2.$$

Indeed, for each $l \geq k$, there exists $N \in \mathbb{N}$ such that $|y(m_k)_N - y(m_l)N| < 1/2^{k+1}$. Hence

$$|y(m_k)_n - y(m_l)_n| \leq |y(m_k)_n - y(m_k)_N| + |y(m_k)_N - y(m_l)_N| + |y(m_l)_N - y(m_l)_n| < 1/2^k$$

for all $n \geq 2$.

Claim: we may take $z_n = y(m_n)_n$.

It is easy to see that it is a Cauchy sequence and $\lim([y(m_k)]) = [(z_n)]$ and hence $\lim([y(k)]) = [(z_n)]$.