MA 105 D1 Lecture 27

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The proof of Stokes' Theorem for a graph

Conservative fields and the curl

The Poincaré Lemma

Finding the gradient function explicitly

Examples involving the divergence theorem

Stokes' theorem for a graph

We give the proof of this theorem in the special case when the surface S is the graph of a surface $z = f(x, y)$ for a $\mathcal{C}^1$ function f. We think of the graph as a parametrised surface $(x, y, z(x, y))$, where (x, y) lies in the domain D in $\mathbb{R}^2$. Assume further that Green's theorem applies to the domain D.

How does one orient ∂S?

Since D is a planar region, it has a natural positive orientation given by the positive direction of the z axis. Now we orient the boundary ∂D as in Green's theorem, that is, in the counterclockwise direction.

Exercise 2: Once ∂D has been oriented as above, ∂S is automatically oriented. The surface S is oriented by choosing $\mathbf{\Phi}_x \times \mathbf{\Phi}_y$ as the normal with positive orientation. Verify in an example of your choosing that the region S remains to the left of an observer walking on ∂S in the direction of positive orientation.

The left hand side of Stokes' formula

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{j}$$

For the graph of a surface, the normal vector has the form

$$\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right).$$

It follows that

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D & \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) \\ & + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy\end{aligned}$$

This gives us the left hand side of Stokes' formula.

The right hand side of Stokes' formula

Let $(x(t), y(t))$, $a \leq t \leq b$, be a parametrisation of the boundary of D, oriented using the upward normal. The curve

$$\mathbf{c}(t) = (x(t), y(t), f(x(t), y(t)))$$

gives the boundary ∂S of S and is given the same orientation as ∂D.

We have

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt.$$

Now we use the chain rule to conclude that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

Substituting this in the previous expression gives us

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D} \left(F_1 + F_3 \frac{\partial z}{\partial x} \right) dx + \left(F_2 + F_3 \frac{\partial z}{\partial y} \right) dy.$$

The conclusion of the proof

Applying Green's theorem (and using the chain rule) we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3\frac{\partial^2 z}{\partial x \partial y} \right)$$
$$- \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3\frac{\partial^2 z}{\partial x \partial y} \right) dxdy.$$

Now one can compare this expression with the previous one to see that Stokes' theorem is proved in this case.

Back to conservative fields

Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined on $\mathbb{R}^3$. We have previously defined a conservative vector field to be one for which the path integrals depend only on the initial and final points of the path and not the path itself. Equivalently, the line integral of $\mathbf{F}$ around any simple closed curve should be 0.

Theorem 46: A vector field $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = 0$.

Proof: Let C be a simple closed curve in $\mathbb{R}^3$ and let S be the surface it bounds. Assume that $\nabla \times \mathbf{F} = 0$. Then using Stokes' theorem we see that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$

What about the converse?

Conservative fields and gradient fields

For the converse, since the line integral is 0 for every curve C, the surface integral must be 0 for every surface. You can easily check that this means that the integrand $(\nabla \times \mathbf{F} \cdot \mathbf{n})$ must be 0, and hence that $\nabla \times \mathbf{F} = 0$.

Corollary 47: The vector field $\mathbf{F}$ is a gradient field, that is, there exists a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

This is obvious since we have already shown that conservative fields are gradient fields when the domain is path connected. We have thus answered one of the basic questions we raised some time ago: When is a curl free vector field a gradient field?

On $\mathbb{R}^3$, always.

I should mention here that in many books, a conservative field is defined as one which is a gradient field.

Subtleties we have ignored so far

What is wrong/not clear about the proof of Theorem 46?

Is it so obvious that a simple closed curve necessarily bounds a surface in $\mathbb{R}^3$?

Let us assume instead that we are on the plane. To assert that a simple closed curve C is the boundary of a compact region in the plane we need a very non trivial result: the Jordan curve theorem. Because of this theorem, we can take the bounded component of $\mathbb{R}^2 \setminus C$ and C will be its boundary.

In $\mathbb{R}^3$ the question is even more subtle.

Back to the plane

We have just invoked the Jordan curve theorem to say that simple closed curves in the plane are necessarily boundaries of surfaces. What happens if we take $\mathbb{R}^2 \setminus \{(0,0)\}$?

It should be clear that in this case the unit circle $x^2 + y^2 = 1$ is not the boundary of a compact region in $\mathbb{R}^2 \setminus \{(0,0)\}$. The region enclosed by the unit circle is the unit disc minus the origin and this set is not closed.

This is why one can find curl free vector fields on this set which are not gradients.

It should thus be clear that the failure of a curl free vector field to be a gradient field is something that depends on the geometry of the region we are considering. In particular, we need to know (at least to apply Stokes' theorem) whether a simple closed curve bounds a compact surface or not.

Homotopy

Definition: Let $\gamma_i : I = [0,1] \to X$, $i = 1, 2$, be continuous maps into a subset X of $\mathbb{R}^n$ with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. We will say that γ_0 and γ_1 are homotopic if there exists a continuous function

$$F : I \times I \to X$$

such that $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ and $F(0,s) = P$ and $F(1,s) = Q$ for all $0 \le s \le 1$.
What this means is that one can continuously deform the curve γ_0 to γ_1 while keeping the end points fixed.

Definition: A space $X \subset \mathbb{R}^n$ in which any two paths are homotopic is called simply connected.

Fact: In a simply connected space, every simple closed curve bounds a (compact) surface.

Informally, a simply connected set in the plane is a set "without holes".

Curl free vector fields in simply connected regions

Thus, the proof of Theorem 46 is valid. It is just that we are assuming a not-so-easy fact when using Stokes' Theorem.

An alternate way to define a simply connected space is to say that every closed path (that is $\gamma(0) = \gamma(1)$) is homotopic to the constant map.

Theorem 48: Suppose that X is a connected, simply connected surface (non-singular, continuously differentiable) in $\mathbb{R}^3$ and $\mathbf{F}$ is a $\mathcal{C}^1$-vector field defined on X. Then $\nabla \times \mathbf{F} = 0$ if and only if $\mathbf{F}$ is conservative, that is, there exists a function $f : X \to R$ such that $\mathbf{F} = \nabla f$.

There is a version of this theorem for X in $\mathbb{R}^n$. It will work the moment we can make sense of the gradient and the curl in n-dimensions.

Examples

Example 1: The space $\mathbb{R}^n$ is simply connected.

Example 2: The set $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected.

Example 3: The unit circle in $\mathbb{R}^2$ is not simply connected.

Example 4: The unit sphere in $\mathbb{R}^3$ is simply connected.

Example 5: The torus is not simply connected.

Example 6: $\mathbb{R}^3 \setminus z$-axis is not simply connected.

Example 7: All convex sets are simply connected.

Example 8: All star-shaped sets are simply connected.

The divergence and curl

Similar to Corollary 47, we have the following theorem showing that divergence free vector fields necessarily arise as the curls of (other) vector fields. This time, however, the proof is straightforward and does not involve any subtleties.

Theorem 49: If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$.

Proof: In fact, we can find a vector field $\mathbf{G}$ of the special form $\mathbf{G} = G_1(x, y, z)\mathbf{i} + G_2(x, y, z)\mathbf{j}$.
The curl of a vector field $\mathbf{G}$ of this form is

$$-\frac{\partial G_2}{\partial z}\mathbf{i} + \frac{\partial G_1}{\partial z}\mathbf{j} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\mathbf{k}.$$

The Poincaré lemma

Thus, we will obtain the required **G** if we solve the equations

$$-\frac{\partial G_2}{\partial z} = F_1, \quad \frac{\partial G_1}{\partial z} = F_2 \quad \text{and} \quad \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3$$

But we can solve these equations simply by integrating:

$$G_1(x, y, z) = \int_0^z F_2(x, y, t)dt - \int_0^y F_3(x, t, 0)dt$$

and

$$G_2(x, y, z) = -\int_0^z F_1(x, y, t)dt.$$

This proves our theorem. □

As mentioned earlier one can formulate versions of Corollary 47 and Theorem 49 for $\mathbb{R}^n$ and combine to get just one statement known as the Poincaré lemma.

Theorem 50: Every closed form on $\mathbb{R}^n$ is exact.

The cross derivative test

Corollary 47 tells us that curl free vector fields on $\mathbb{R}^3$ are conservative (or gradient fields). For a vector field $\mathbf{F}$ on $\mathbb{R}^n$ we have the cross-derivative test as one of the special cases of the Poincaré lemma.

Cross derivative test A vector field $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is conservative if and only if

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = 0$$

for all $1 \leq i, j \leq n$.

Note that if $n = 3$ this is exactly the condition that the curl is zero.

Explicitly obtaining the gradient field

Let $\mathbf{F}$ be a curl free vector field. So far we have only proved the existence of a scalar function f such that $\nabla f = \mathbf{F}$ indirectly (using the fact that curl free vector fields have line integrals which are independent of the chosen path).

We can actually obtain the gradient function explicitly using the following procedure, which essentially amount to repeated integration.

By integrating we see that

$$f(x_1, \ldots, x_n) = \int F_1(x_1, \ldots, x_n) dx_1 + f_2(x_2, \ldots, x_n).$$

Now substitute this expression into the equation

$$\frac{\partial f}{\partial x_2} = F_2$$

and integrate this equation.

On integrating, the solution will now involve an arbitrary function $f_3(x_2, \ldots, x_n)$. Repeating this process we can continue until the very last equation

$$\frac{\partial f}{\partial x_n} = F_n$$

has been solved.

Exercise 2 (page 465 of Marsden, Weinstein and Tromba):
Determine whether the vector field

$$\mathbf{F} = (2x \cos y)\mathbf{i} - (x^2 \sin y)\mathbf{j}$$

is a gradient field.

Solution: Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ the cross derivative test amounts to verifying that $Q_x = P_y$. In this case we have

$$P_y = -2x \sin y = Q_x.$$

Integrating with respect to x, we get

$$f(x, y) = \int 2x \cos y dx + f_1(y) = x^2 \cos y + f_1(y),$$

where $f_1(y)$ is an arbitrary function of y. Substituting this expression in the equation $\frac{\partial f}{\partial y} = Q$, we get

$$-x^2 \sin y + f_1'(y) = -x^2 \sin y.$$

Thus, we require $f_1'(y) = 0$ or $f_1(y) = c$, for some constant c, and one can now check that this gives the required f.

Exercise 3 (page 463 of Marsden, Weinstein and Tromba:
Determine if the vector field

$$\mathbf{F} = y\mathbf{i} + (z\cos yz + x)\mathbf{j} + (y\cos yz)\mathbf{k}$$

is a gradient field and if so find a scalar function f such that $\nabla f = \mathbf{F}$.

Solution: I will leave the computation of the curl of $\mathbf{F}$ as an exercise. One can easily check that $\nabla \times \mathbf{F} = 0$.

Because of the cross-derivative test, we know that $\mathbf{F}$ is a gradient field. We proceed to determine this scalar function using the method described above.

We write the vector field $\mathbf{F}$ as

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}.$$

We first integrate the equation $\frac{\partial f}{\partial x} = F_1 = y$ to get

$$f(x, y, z) = xy + f_1(y, z).$$

Substituting this in the equation $\frac{\partial f}{\partial y} = F_2 = z \cos yz + x$ we get

$$x + \frac{\partial f_1(y, z)}{\partial y} = z \cos yz + x.$$

On integrating this equation, we obtain

$$f(x, y, z) = xy + f_1(y, z) = xy + \sin yz + f_2(z).$$

Which gives

$$f_1(y, z) = \sin yz + f_2(z).$$

We thus obtain

$$f(x, y, z) = xy + \sin yz + f_2(z).$$

Now we substitute this in the last of our equations $\frac{\partial f}{\partial z} = F_3 = y \cos yz$. Thus we get

$$y \cos yz + f_2{}'(z) = y \cos yz,$$

so $f_2{}'(z) = 0$. Thus $f_2(z) = c$ for some constant c.

Since f_2 is now determined, the general solution to our problem is

$$f(x, y, z) = xy + \sin yz + c.$$

Solutions to problems from Tutorial 13

Exercise 13.2: Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

for the region in the first octant bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution: We denote by R the region bounded by the given plane. We first evaluate the volume integral in Gauss' Theorem.

The solution to Exercise 13.2

We have $\operatorname{div}\mathbf{F} = (y + z + x)$.

$$\begin{aligned}\iiint_R (x+y+z)dV &= \iiint_R xdV + \iiint_R ydV + \iiint_R zdV \\ &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} xdxdydz + (\cdots) + (\cdots) \\ &= \frac{a^2bc}{24} + \frac{ab^2c}{24} + \frac{abc^2}{24} \\ &= \frac{abc}{24}(a+b+c).\end{aligned}$$

We now evaluate the other side of Gauss' Theorem.

The solution to Exercise 13.2, continued

The boundary of the region R consists of four triangular surfaces, three of which lie in the planes formed by the three coordinate planes. For each of these regions we can easily check that $\mathbf{F} \cdot \mathbf{n} = 0$.

For instance, we have

$$S_1 : z = 0; \frac{x}{a} + \frac{y}{b} \leq 1, x, y \geq 0$$

as one of the three boundary pieces, and along S_1

$$\mathbf{n} = -\mathbf{k}, \quad \text{so} \quad \mathbf{F} \cdot \mathbf{n} = -xz = 0 \quad (\text{as } z = 0 \text{ on } S_1),$$

and the other two triangular surfaces are treated similarly. This leaves us only the the triangular surface S_4 defined by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $x, y, z \geq 0$.

The solution to Exercise 13.2, continued

Along S_4, the outward normal (to $z = c(1 - \frac{x}{a} - \frac{y}{b}) \equiv f(x,y)$) is $(\frac{c}{a}, \frac{c}{b}, 1)$ so that

$$
\begin{aligned}
\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \iint_{\frac{x}{a}+\frac{y}{b}\leq 1; x,y\geq 0} \left(\frac{cxy}{a} + \frac{cyz}{b} + zx \right) dS \\
&= \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{cxy}{a} dxdy + (\cdots) + (\cdots) \\
&= \frac{ab^2c}{24} + \frac{abc^2}{24} + \frac{a^2bc}{24} \\
&= \frac{abc}{24}(a+b+c).
\end{aligned}
$$

This proves what we want.