CS 101: Computer Programming and Utilization

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Umesh Bellur
(cs101@cse.iitb.ac.in)

Lecture 11: Recursive Functions

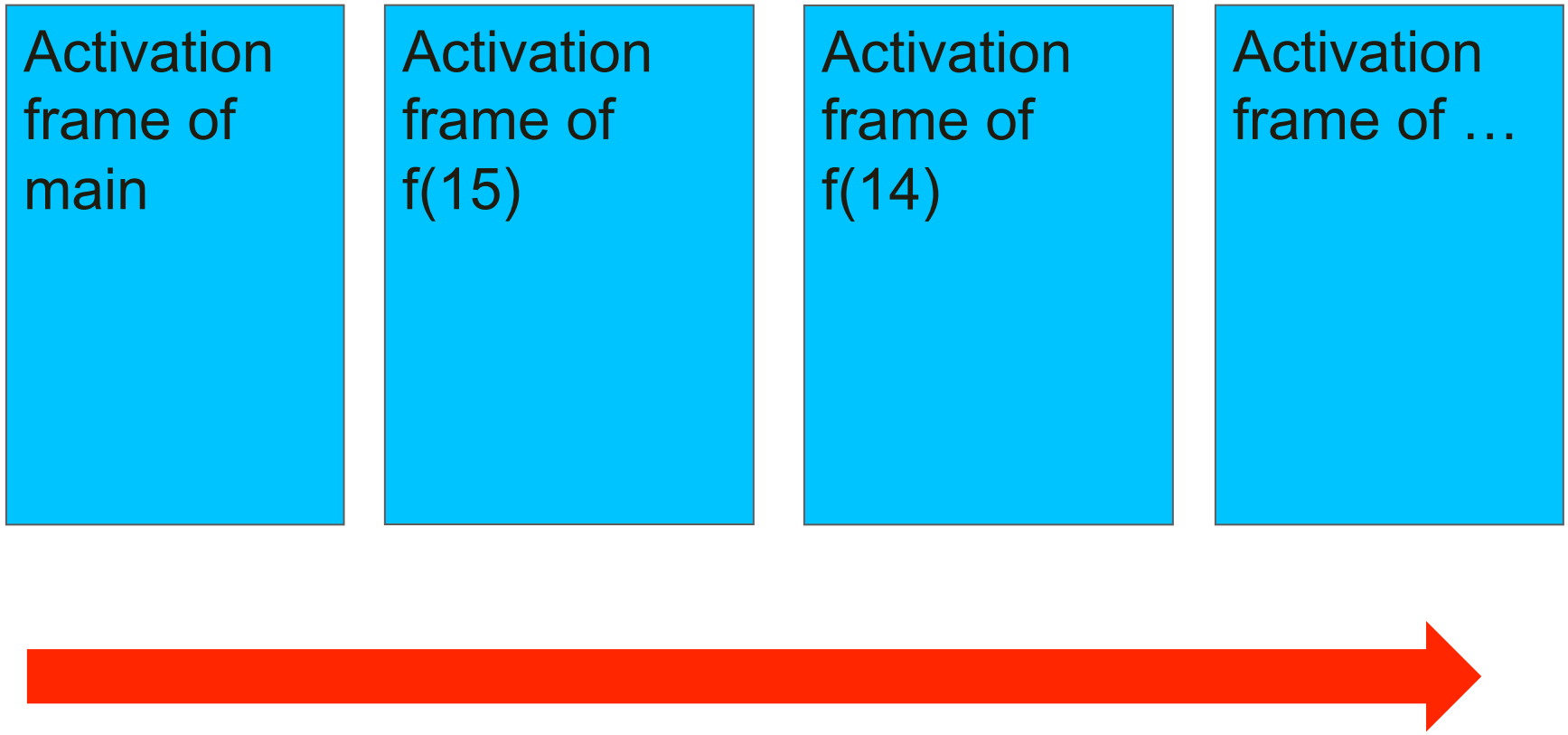
Chapter 10 & Chapter 16

Can a Function Call Itself?

```
int f(int n){
   ...
   int z = f(n-1);
   ...
}
main_program{
  int z = f(15);
}
```

- Allowed by execution mechanism
- main_program executes, calls f(15)
- Activation Frame (AF) created for f(15)
- f executes, calls f(14)
- AF created for f(14)
- Continues in this manner, with AFs created for f(13), f(12) and so on, endlessly

Activation Frames Keep Getting Created in Stack Memory



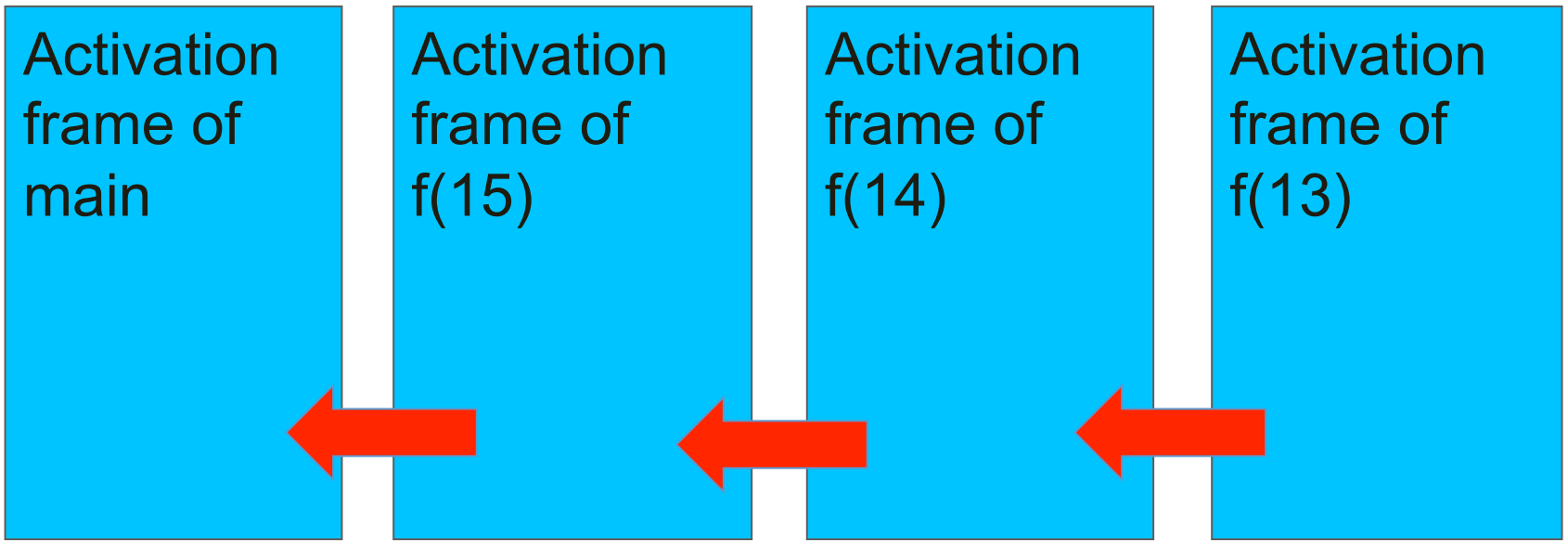
Modified version

```
int f(int n){
    …
    if(n > 13)
        z = f(n-1);
    …
}
main_program{
   int w = f(15);
}
```

- main_program executes, calls f(15)
- AF created for f(15)
- f(15) executes, calls f(14)
- AF created for f(14)
- f(14) executes, calls f(13)
- AF created for f(13)
- f(13) executes, check n>13 fails. some result returned
- Result received in f(14)
- f(14) continues and in turn returns result to f(15)
- f(15) continues, returns result to main_program
- main_program continues and finished

Activation Frames Keep Getting Created in Stack Memory

and destroyed as the functions exit



Recursion

A technique where a function called from its own body

OK if we eventually get to a call which does not call itself

Then that call will return

Previous call will return…

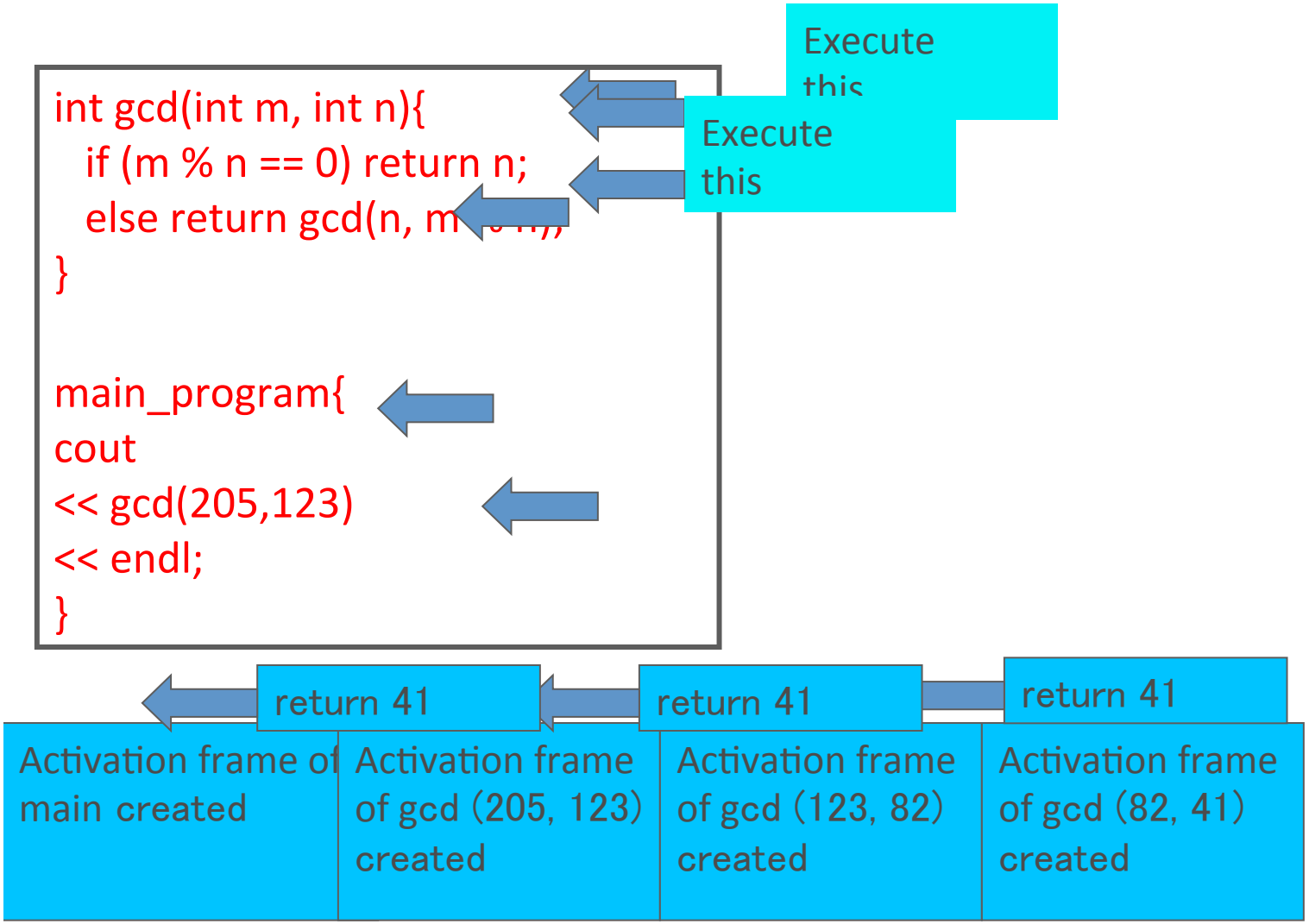
But could it be useful?

Back to Euclid's method for GCD

```
//If m % n == 0, then
//        GCD(m, n) = n,
//else GCD(m,n) = GCD(n, m % n)

int gcd(int m, int n){
    if (m % n == 0) return n;
    else return gcd(n, m % n);
}
main_program{
    cout << gcd(205,123) << endl;
}
```

Euclid's Theorem on GCD



Recursion Vs. Iteration

- **Recursion** allows multiple distinct data spaces for different executions of a **function body**
  - Data spaces are live simultaneously
  - Creation and destruction follows LIFO policy

- **Iteration** uses a single data space for different executions of a **loop body**
  - Either the same data space is shared or one data space is destroyed before the next one is created

But is the recursive GCD correct?

We prove the correctness by induction on j

For a given value of j, gcd(i,j) correctly

computes gcd(i,j) for all value of i

We prove this for all values of j by induction

- **Base case:** j=1. `gcd(i,1)` returns 1 for all i

  Obviously correct
- **Inductive hypothesis:** Assume the correctness of `gcd(i,j)` for a some j
- **Inductive step**: Show that `gcd(i, j+1)` computes the correct value

Correctness of Recursive gcd

Inductive Step: Need to show that gcd(i,j+1) computes the correct value, assuming that gcd(i,j) is correct

- ***Case***: If j+1 divides i, then the result is j+1

  Hence correct

- ***Case***: If j+1 does not divide i, then gcd(i, j+1) returns the result of calling gcd(j, i%(j+1))
  - i%(j+1) can at most be equal to j
  - By the inductive hypothesis, gcd(j, i%(j+1) computes the correct value
  - Hence gcd(i, j+1) computes the correct value

Another Example - Factorial

- **Iterative factorial function**

```
int fact(int n) {
   int res=1;
   for (int i=1; i<=n; i++)
      res = res*i;
      return res;
}
```

- **Recursive factorial function**

```
int fact(int n) {
   if (n<=0) return 1;
   else return n*fact(n-1);
}
```

Yet another example – Fibonacci Number

- **Iterative fibonacci function:**

```
int fib(int n){
    if (n <= 0) return 0;
    if (n == 1) return 1;
    int n_2 = 0, n_1 = 1,
        result = 0;
    for(int i=2;i<=n;i++){
        result = n_1 + n_2;
        n_2 = n_1;
        n_1 = result;
    }
    return result;
}
```

- **Definition**:

```
fib(0) = 0
fib(1) = 1
fib(n) = fib(n-1)
         + fib(n-2)
```

- **Recursive fibonacci function:**

```
int fib(int n){
    if (n <= 0) return 0;
    if (n == 1) return 1;
    return fib(n-1) +
           fib(n-2);
}
```

An Important Application of Recursion:
Processing Trees

Botanical trees…

Organization Tree

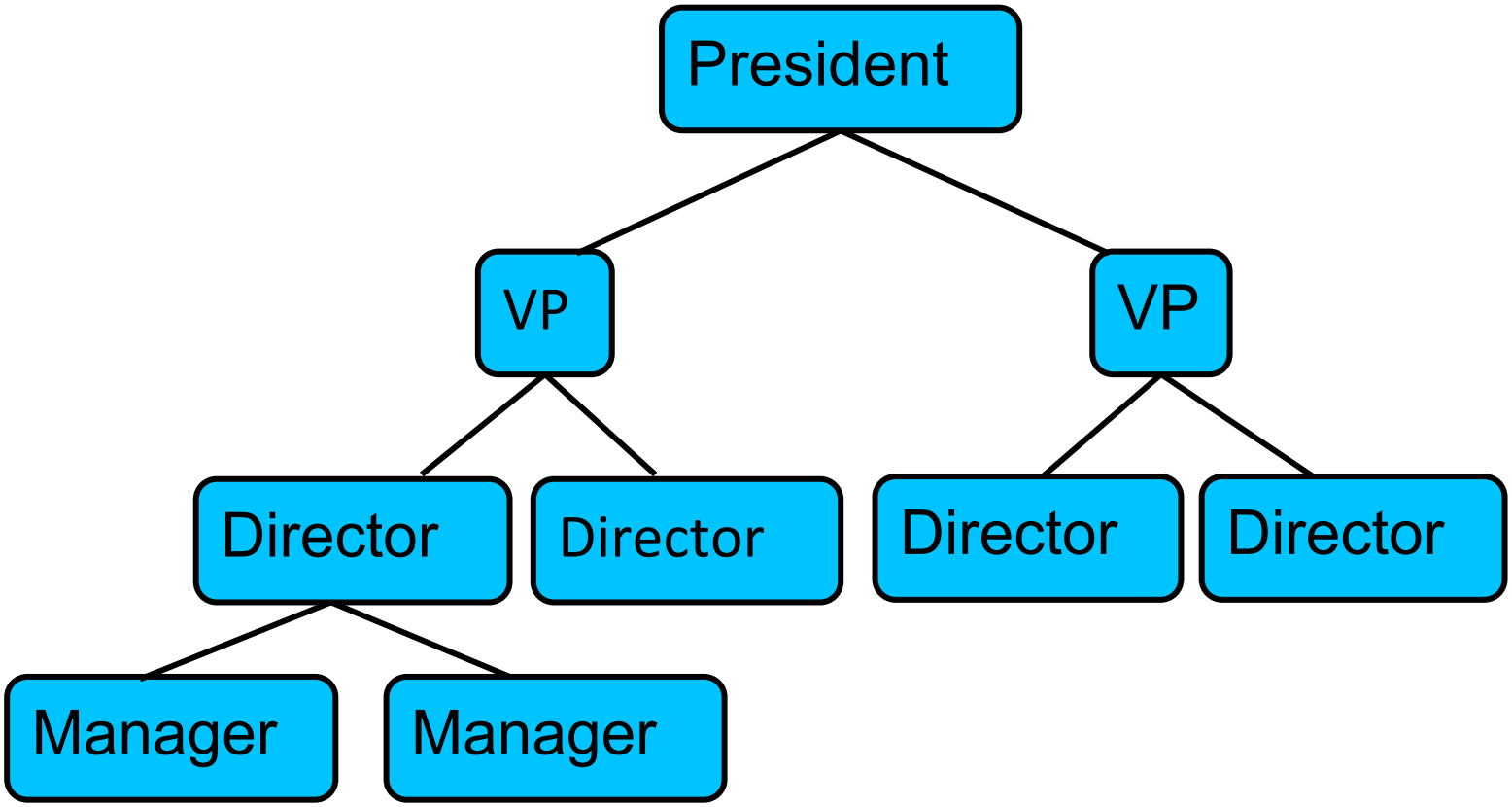
Expression Tree

Search Tree: later

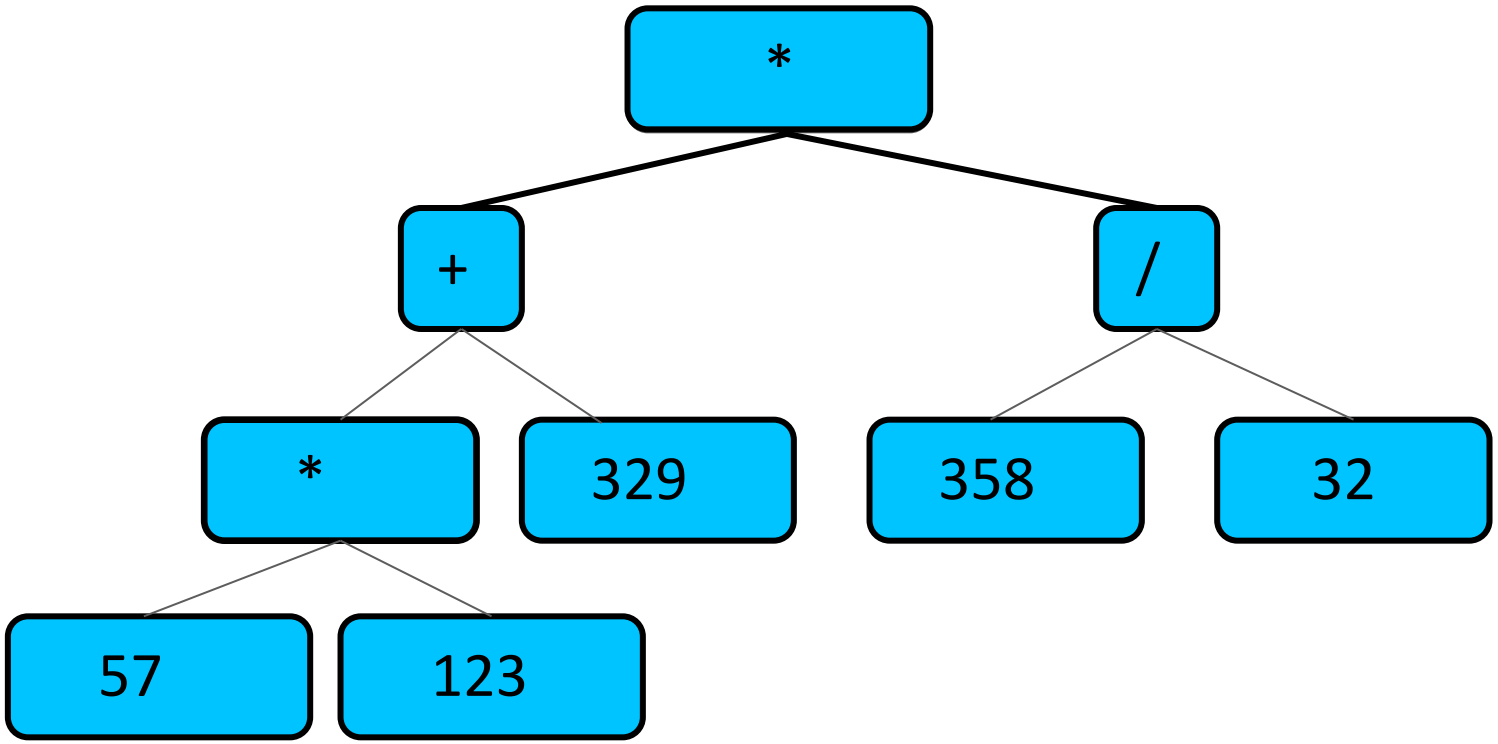
In this class we only consider how to draw trees

Must understand the ***structure of trees***

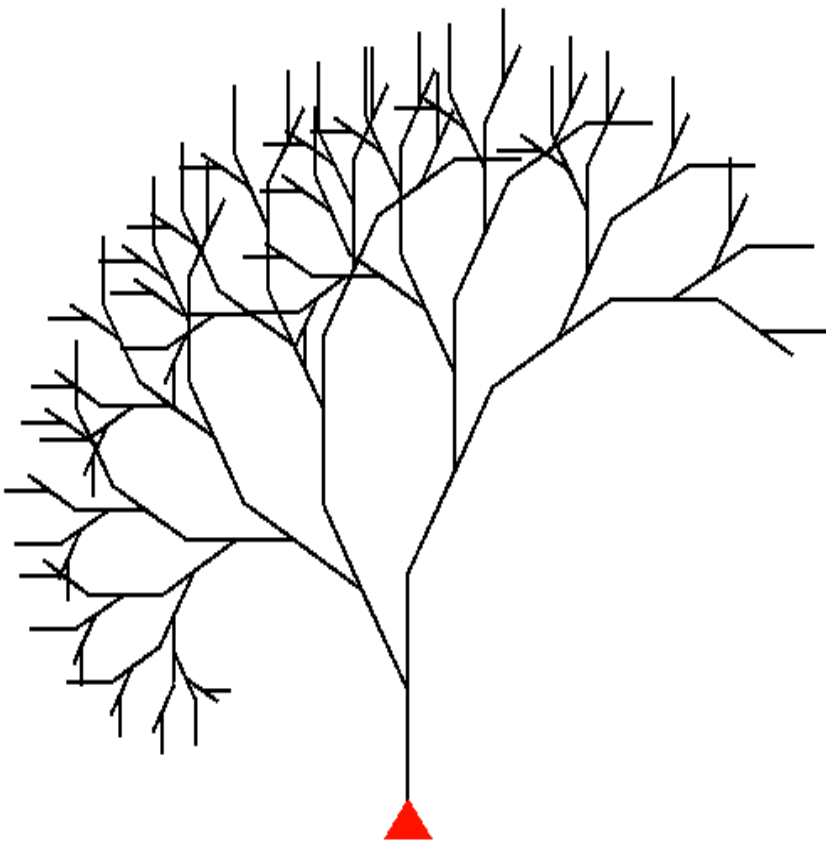
Organization Tree (Typically "grows" Downwards)

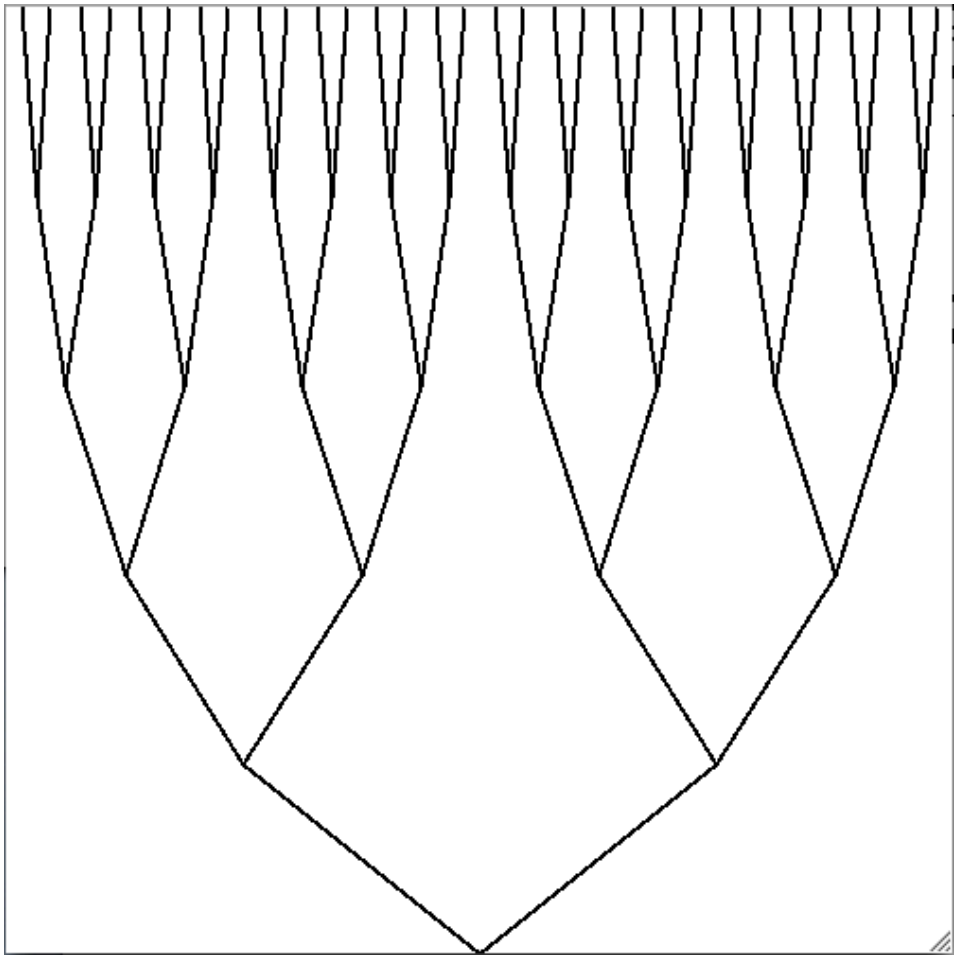


Expression Evaluation Tree
((57*123)+329)*(358/32)

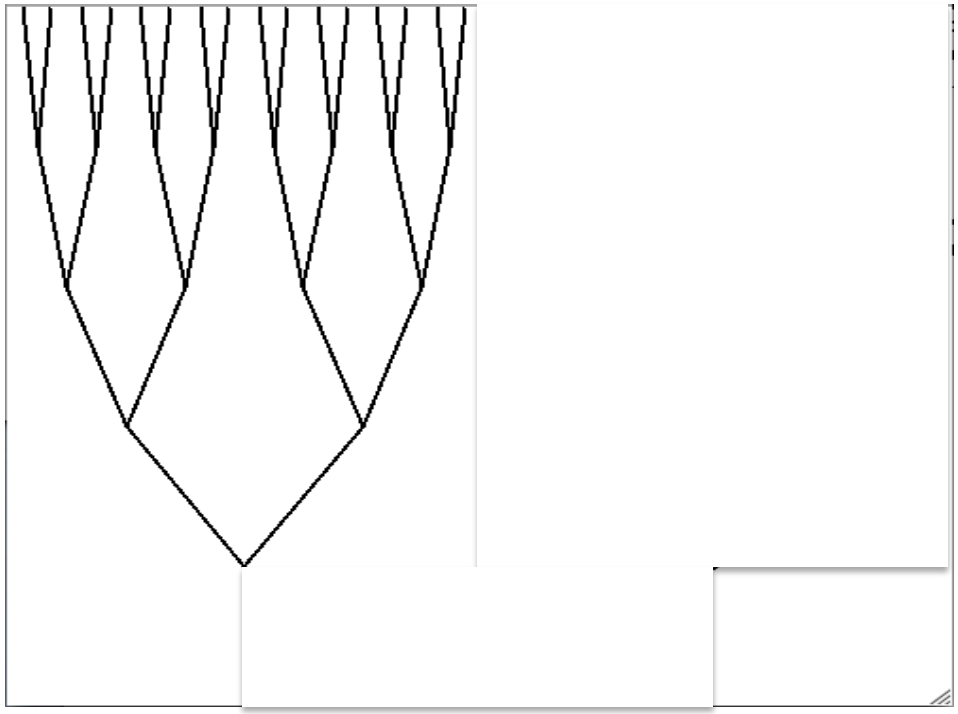


Trees drawn using Simplecpp





1 Stylized Tree = 2 Small Stylized Trees + V



When a part of an object is of the same type as the whole, the object is said to have a recursive structure.

Drawing The Stylized Tree

Parts:

Root

Left branch, Left subtree

Right branch, Right subtree

Number of levels: number of times the tree has branched going from the root to any leaf.

Number of levels in tree shown = 5

Number of levels in subtrees of tree: 4

Drawing The Stylized Tree

To draw an L level tree:

if L > 0{

Draw the left branch, and a Level L-1 on top of it.

Draw the right branch, and a Level L-1 tree on top.

}

We must give the coordinates where the lines are to be drawn

Say root is to be drawn at (rx,ry)

Total height of drawing is h.

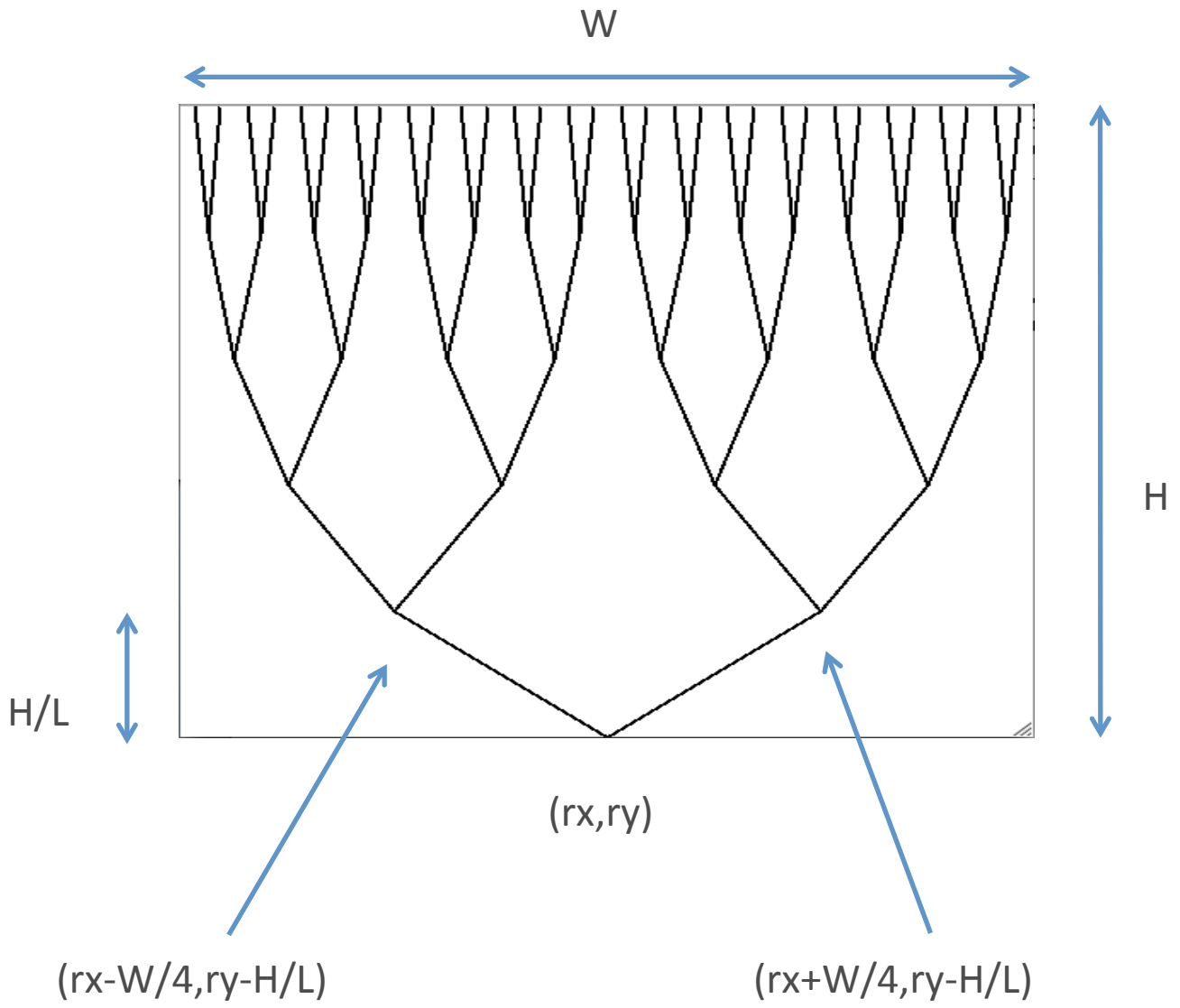
Total width of drawing is w.

We should then figure out where the roots of the subtrees will be.

Basic Primitive:

Drawing a line from (x1,y1) to (x2,y2)

Drawing The Stylized Tree



Drawing The Stylized Tree

Basic Primitive Required: Drawing a line

- Create a named shape with type Line

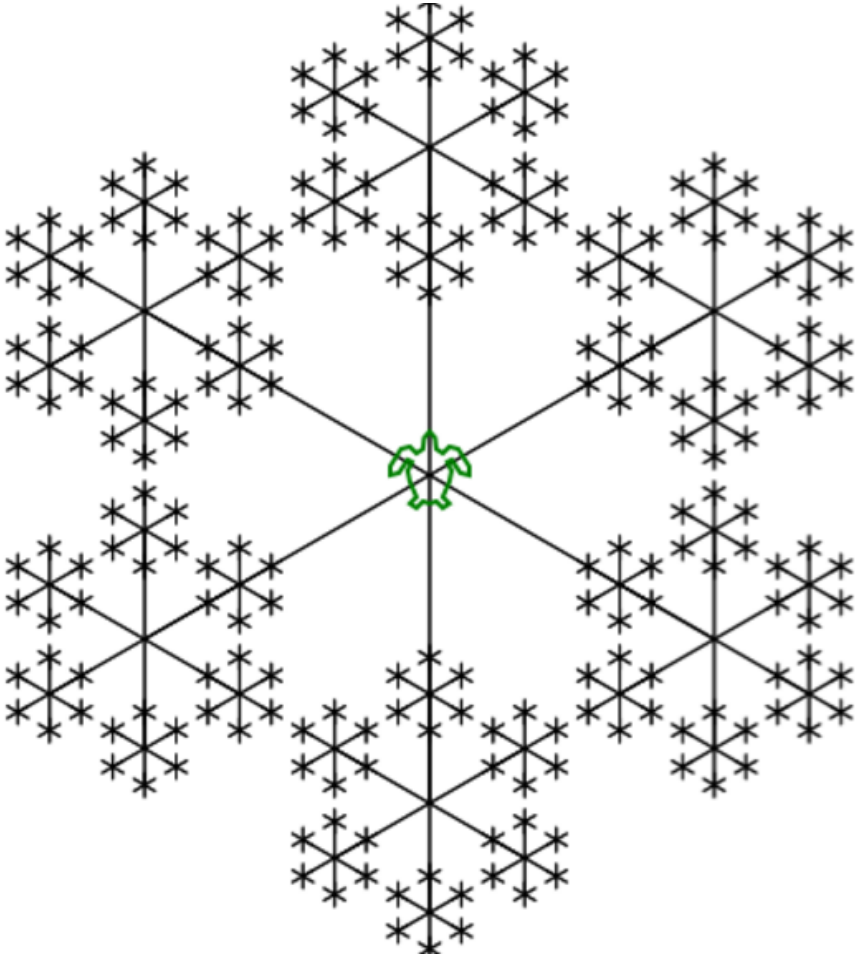
```
void draw_line(x1,y1,x2,y2);
```

Drawing The Stylized Tree

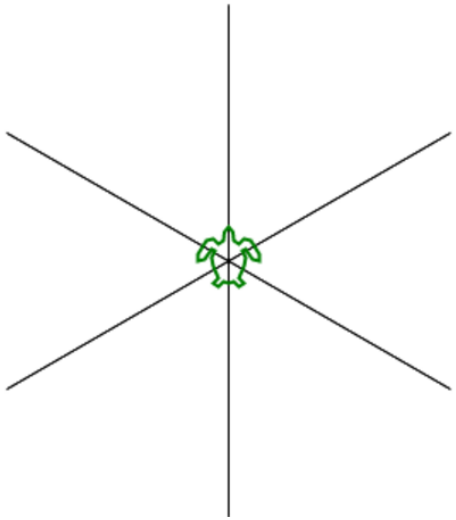
```
void tree(int L, double rx, double ry,
                 double H, double W) {
  if(L>0){
    draw_line(rx, ry, rx-W/4, ry-H/L);  // line called left
    draw_line(rx, ry, rx+W/4, ry-H/L); // line called right

    tree(L-1, rx-W/4, ry-H/L, H-H/L, W/2); // left subtree
    tree(L-1, rx+W/4, ry-H/L, H-H/L, W/2);// right subtree
  }
}
```

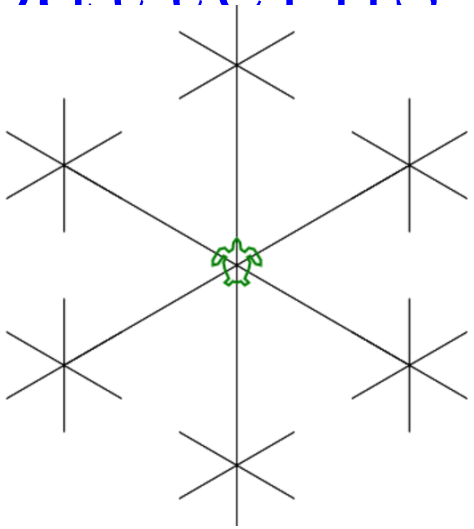
More fun drawings using recursion



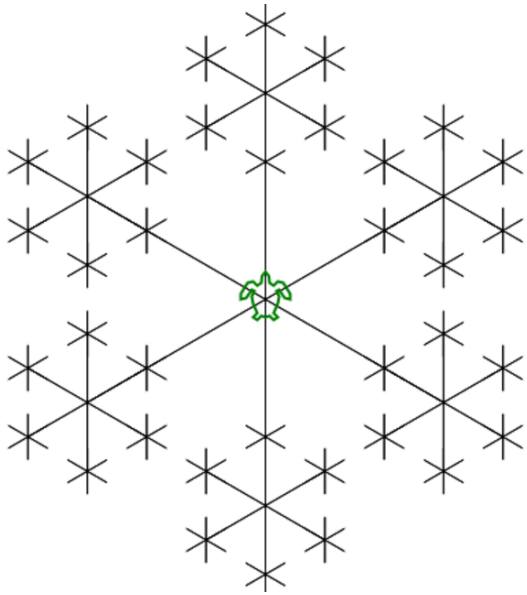
Fractals: self-similar patterns



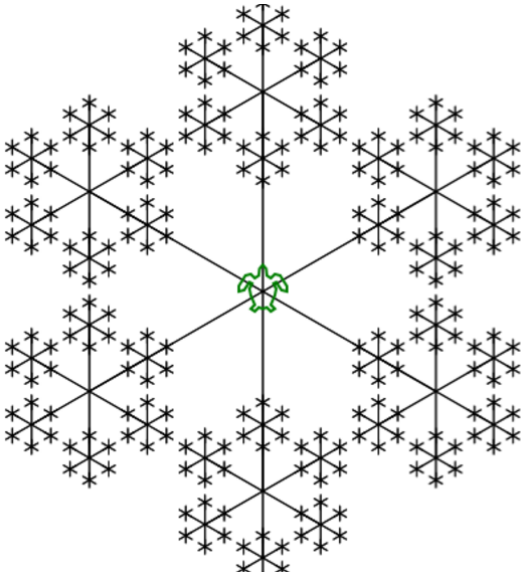
Level 1



Level 2



Level 3



Level 4

Code

```
void drawPattern(double side, int level){
     if(level>3) return;
     repeat(6){
          forward(side);
          drawPattern(side/3,level+1);
          right(180);
          forward(side);
          left(120);
      }
}
main_program{
       turtleSim();
       left(90);
       drawPattern(150,0);
}
```

Arrays and Recursion

- Recursion is very useful for designing algorithms on sequences
  - Sequences will be stored in arrays

- Topics
  - Binary Search
  - Merge Sort

Searching an array

- **Input**: An array `A` of length `n` storing numbers, number `x` (called "key")
- **Output**: true if `x` is present in `A`, `false` otherwise.
- Natural algorithm: scan through the array and return true if found.

```
for(int i=0; i<n; i++){
 if(A[i] == x) return true;
}
return false;
```

- Time consuming: we will scan through entire array if the element is not present, and on the average through half the array if it is present.
- ***Can we possibly do all this with fewer operations?***

Searching a sorted array

- **sorted array**: (non decreasing order)

  $A[0] \leq A[1] \leq \ldots \leq A[n-1]$

- **sorted array:** (non increasing order)

  $A[0] \geq A[1] \geq \ldots \geq A[n-1]$

- How do we search in a sorted array (non increasing or non decreasing)?
  - Does the sortedness help in searching?

Searching a non decreasing sorted array

- Assume array is sorted in non-decreasing order.
- Key idea for reducing the number of comparisons: First compare `x` with the "middle" element `A[n/2]` of the array.

- Suppose `x < A[n/2]`: Because `A` is sorted, `A[n/2..n-1]` will also have elements larger than `x`.
  - `x` if present will be present only in `A[0..n/2-1]`.
  - So in the rest of the algorithm we will only search first half of `A`.

- Suppose `x >= A[n/2]`:
  - `x` if present will be present in `A[n/2..n-1]`
  - `x` may be present in first half too, but if it is present it will be present in second half too.
  - So in the rest of the algorithm we will only search second half.

- How to search the "halves"?
  - Recurse.

Plan

- We will write a function `Bsearch` which will ***search a region*** of an array instead of the entire array.
- ***Region***: specified using 2 numbers: starting index `S`, length of region `L`
- When `L == 1`, we are searching a length 1 array.
  - So check if that element, `A[S] == x`.
- Otherwise, compare `x` to the "middle" element of `A[S..S+L-1]`
  - Middle element: `A[S + L/2]`
- Algorithm is called ***Binary search***, because size of the region to be searched gets halved.

The code

```
bool Bsearch(int A[], int S, int L, int x)
// Search in A[S..S+L-1]
{
   if(L == 1) return A[S] == x;
   int H = L/2;
    if (x == A[S+H]) return true;
   if(x < A[S+H]) return Bsearch(A, S,   H,   x);
   else           return Bsearch(A, S+H, L-H, x);
}

int main(){
  int A[8]={-1, 2, 2, 4, 10, 12, 30, 30};
  cout << Bsearch(A,0,8,11) << endl;
  // searches for 11.
}
```

How does the algorithm execute?

- `A = {-1, 2, 2, 4, 10, 12, 30, 30}`
- First call: `Bsearch(A, 0, 8, 11)`
  - comparison: `11 < A[0+8/2] = A[4] = 10`
  - Is `false`.
- Second call: `Bsearch(A, 4, 4, 11)`
  - comparison: `11 < A[4+4/2] = A[6] = 30`
  - Is `true`.
- Third call: `Bsearch(A, 4, 2, 11)`
  - comparison: `11 < A[4+2/2] = A[5] = 12`
  - Is true.
- Fourth call: `Bsearch(A, 4, 1, 11)`
  - Base case. Return `11 == A[4]`. So `false`.

Remarks

- If you are likely to ***search an array frequently, it is useful to first sort it.*** The time to sort the array will be be compensated by the time saved in subsequent searches.
- How do you sort an array in the first place? Next.
- Binary search can also be written without recursion. Exercise.

Sorting

- Chapter 14 discusses a simple algorithm for sorting called Selection sort.
- Selection can require n^2 comparisons to sort n keys.
- Algorithms requiring fewer comparisons are known: n log n comparisons.
- One such algorithm is Merge sort.

Mergesort idea

- Break up the sequence into two small sequences.
- Sort each small sequence. (Recurse!)
- Somehow “merge” the sorted sequences into a single long sequence.
- Hope: “merging” sorted sequences is easier than sorting the large sequence.
- Our hope is correct, as we will see soon!

Example

- Suppose we want to sort the sequence
  - 50, 29, 87, 23, 25, 7, 64
- Break it into two sequences.
  - 50, 29, 87, 23 and 25, 7, 64.
- Sort both
  - We get 23, 29, 50, 87 and 7, 25, 64.
- Merge
  - Goal is to get 7, 23, 25, 29, 50, 64, 87.

Merge sort

```
void mergesort(int S[], int n){
// Sorts sequence S of length n.
 if(n==1) return;
 int U[n/2], V[n-n/2];
 for(int i=0; i<n/2; i++) U[i]=S[i];
 for(int i=0; i<n-n/2; i++) V[i]=S[i+n/2];
 mergesort(U,n/2);
 mergesort(V,n-n/2);
//"Merge" sorted U, V into S.
 merge(U, n/2, V, n-n/2, S, n);
}
```

Merging two sorted sequences

- Think of a sorted sequence as a row of students, ordered shortest to tallest.
- We are given two such rows, U, V.
- We want to move students from both rows into a new row S, but it should still be in shortest to tallest order.

Merging

U: 23, 29, 50, 87.

V: 7, 25, 64.

S:

- The smallest overall must move into S. Smallest overall can be smaller of smallest in U and smallest in V.
- So after movement we get:

U: 23, 29, 50, 87.

V: -, 25, 64.

S: 7.

What do we do next?

U: 23, 29, 50, 87.
V: -, 25, 64.
S: 7.

- Now we need to move the second smallest into S.
- Second smallest:
  - smallest in U,V after smallest has moved out.
  - smaller of the students currently at the head of U, V.
- So we get:

U: -, 29, 50, 87.
V: -, 25, 64.
S: 7, 23.

General strategy

- While both U, V contain a student:
  - Move smallest from those at the head of U,V to the end of S.
- If only U contains students: move all to end of S.
- If only V contains students: move all to end of S.
- uf: index denoting which element of U is currently at the front.
  - U[0..uf-1] have moved out.
- vf: similarly for V.
- sb: index denoting where next element should move into S next (sb: back of S)
  - S[0..sb-1] contain elements that have moved in earlier.

Merging two sequences

```
merge(int U[], int p, int V[], int q, int S[], int n){
 for(int uf=0, vf=0, sb=0;
     sb < p + q; // while all elements havent moved
     sb++){
   if(uf<p && vf<q){  // both U,V are non empty
     if(U[uf] < V[vf]){
       S[sb] = U[uf]; uf++;
     } else{
       S[sb] = V[vf]; vf++;
     }
   } else if(uf < p){ // only U is non empty
      S[sb] = U[uf]; uf++;
   } else {              // only V is non empty
      S[sb] = V[vf]; vf++;
   }}}
```

Concluding Remarks

- Recursion allows many programs to be expressed very compactly
- The idea that the solution of a large problem can be obtained from the solution of a similar problem of the same type, is very powerful
- Euclid probably used this idea to discover his GCD algorithm
- More examples in the book