Q 1. *(3 marks) Using the $\epsilon - N$ definition of the limit of a sequence, show that the sequence $a_n = 3(-1)^n$ diverges.*

Solution: Suppose that the sequence a_n converges to some limit l. We need to show that for each l there is some ϵ such that there exist infinitely many n such that $|a_n - l| \not< \epsilon$. Assume first that $l \geq 0$. Let $\epsilon = 3$ (or any value of $\epsilon < 3$ will do).

(1 mark)

Then for all $n = 2m + 1$, $m \geq 1$, we know that $a_n = -3$. Hence, for infinitely many n we see that $|(-3) - l| \not< 3$.

(1 mark)

Similarly, if $l < 0$, we see that for all $n = 2m$, $m \geq 1$, $a_n = 3$. Hence, for infinitely many n $|3 - l| \not< 3$. It follows that no such l can exist.

(1 mark)

Q 2. *Let $P_{1,n}(x)$ denote the Taylor polynomial of degree n of the function $f(x) = \frac{3e^x}{4}$ about the point 1.*

1. *(1 mark) Find $P_{1,3}(x)$*
2. *(3 marks) Determine if the following statement is true or false for all x in $(0, 2)$.*

$$|f(x) - P_{1,3}(x)| < 0.5$$

Solution: We see that $f^{(n)}(x) = \frac{3e^x}{4}$ for all n. Hence,

$$P_{1,3}(x) = \sum_{n=0}^{3} \frac{3e}{4} \cdot \frac{(x-1)^n}{n!}.$$

(1 mark)

For the remainder term $R_3 = f(x) - P_{1,3}(x)$ we have the expression

$$R_3(x) = f^{(4)}(c) \cdot \frac{(x-1)^4}{4!}.$$

for some c between x and 1.

(1 mark)

(Students may have given the general form of the remainder for an arbitrary f and arbitrary n above. You should give one mark for any form of the remainder which is correct. Note that students may write $f^{(4)}(1)$ or $f^{(4)}(x)$ in the expression above , instead of $f^{(4)}(c)$. These are both incorrect answers and should not be awarded any marks.)

The maximum value of $f^{(4)}(c)$ in the interval $(0, 2)$ is $< \frac{3}{4}e^2$ and the maximum value of $|x-1|^4$ is < 1 in the same interval.

(1 mark)

(Students must bound the relevant quantities properly above to get the mark.)

Hence,

$$|R_3(x)| < \frac{3}{4}e^2/24 < 9/24 < 0.5.$$

Hence the statement is True (we have used $e^2 < 9$ here). **(1 mark)**

Q 3. *(3 marks) Show that the tangent plane to the surface $z = x^2 - y^2$ at $(3,3,0)$ intersects the surface in two perpendicular lines.*

Solution: The tangent plane to the implicit surface $f(x,y,z) = z - x^2 + y^2 = 0$ at $(3,3,0)$ is given by

$$\frac{\partial f}{\partial x}(3,3,0)(x-3) + \frac{\partial f}{\partial y}(3,3,0)(y-3) + \frac{\partial f}{\partial z}(3,3,0)(z-0) = 0.$$

Using this formula, we obtain that the required tangent plane to be the one given by

$$-6(x-3) + 6(y-3) + z = 0.$$

(1 mark)

We now find the intersection of the tangent plane with the given surface. For, we put $z = x^2 - y^2$ in the equation of the tangent plane:

$$-6(x-3) + 6(y-3) + x^2 - y^2 = 0 \Rightarrow y^2 - 6y - x^2 + 6x = 0$$

By solving the above equation for y, we obtain that

$$y = \frac{6 \pm \sqrt{36 + 4x^2 - 24x}}{2} = \frac{6 \pm \sqrt{(2x-6)^2}}{2} = 3 \pm (x-3).$$

Hence, we obtain, the planes

$$y = x \text{ and } y = -x + 6$$

(1 mark)

Substituting these values back in the equation of the surface, we see that we obtain $z = 0$ and $12y + z - 36 = 0$ respectively. The required lines are given by

$$x - y = 0,\ z = 0 \quad \text{and} \quad x + y = 6,\ 12y + z - 36 = 0$$

(1 mark)

The normals to the respective pairs of planes are $(1,-1,0)$, $(0,0,1)$ and $(1,1,0)$, $(0,12,1)$ respectively. Taking the respective cross products we find that the first line is parallel to $(-1,-1,0)$ and and the second to $(1,-1,12)$. Taking the dot product of these two vectors we get 0, so the lines are perpendicular.

(1 bonus mark)

Explanation for the bonus mark: Many of the students have found that the intersection of the tangent plane with the surface is given by two pairs of planes but have not shown that the two lines determined by the two pairs of the planes are perpendicular. We are not deducting any mark for this but giving 1 bonus mark to those who have shown explicitly that the two lines are perpendicular.

Q 4. *(3 marks) For this question, no justification is necessary. Just provide the answer.*

1. *Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is decreasing and (strictly) concave.*
2. *Give an example of a function that is not Darboux integrable on a closed interval $[a, b]$.*
3. *Give an example of a function that is differentiable but not continuously differentiable.*

Solution:

1. $f(x) = -e^x$ **(1 mark)**
2. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

(1 mark)

3. $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. **(1 mark)**

Of course, any other correct examples have been awarded marks. A common mistake in 1. was to give a function which is convex but not striclty convex. A common mistake in the answer for 2. included giving a function that is not defined on the whole closed interval (example: $\tan x$ on $[0, \pi/2]$ is not defined at the right end point).

Q 5. *(3 marks) Let $f(x,y) = 0$ if $y = 0$ and*

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \ \text{ if } \ y \neq 0.$$

Show that the directional derivative $\nabla_u f(0,0)$ exists for every unit vector u.

Solution: For $u = (a,b)$, with $b \neq 0$, one has

$$(\nabla_u)\, f(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} \ = \frac{(\sqrt{a^2+b^2})b}{|b|} = \frac{b}{|b|}.$$

(1 mark)

If $u = (a,0)$, then $(\nabla_u f)\,(0,0) = 0$. Hence $(\nabla_u f)\,(0,0)$ exists for every unit vector $u \in \mathbb{R}^2$.

(2 marks)

(Note that some number of students will probably compute the gradient and take the dot product. This approach is incorrect.)

Q 6. *(3 marks) Find $\delta > 0$ such that*

$$\left|\frac{1}{x} - \frac{1}{5}\right| < 5^{-100}$$

whenever $|x-5| < \delta$.

Solution: Note that

$$\left|\frac{1}{x} - \frac{1}{5}\right| = \left|\frac{5-x}{5x}\right|.$$

(1 mark)

If $\delta = 5^{-100}$, we see that $x > 1$.

$$\left|\frac{1}{x} - \frac{1}{5}\right| < 5^{-100}/5x < 5^{-100}.$$

(2 marks)

A large proportion of students have proceeded **incorrectly** as follows:

$$\left|\frac{1}{x} - \frac{1}{5}\right| < 5^{-100} \implies -5^{-100} < \frac{1}{x} - \frac{1}{5} < 5^{-100}$$

$$\implies -5^{-100} < \frac{5-x}{5x} < 5^{-100} \implies -5^{-100} < \frac{\delta}{5(5-\delta)} < 5^{-100},$$

if $x = 5 - \delta$. If $x = 5 + \delta$, we get

$$\left|\frac{1}{x} - \frac{1}{5}\right| < 5^{-100} \implies -5^{-100} < \frac{-\delta}{5(5+\delta)} < 5^{-100}.$$

This yields

$$\delta < \frac{5^{-98}}{1+5^{-99}} \quad \text{or} \quad \delta < \frac{5^{-98}}{1-5^{-99}}.$$

Note that students who have done this have **not** shown what they were required to. They have **assumed** what they had to prove. Some students have gone on to pick the minimum of the two deltas above (the first one). This minimum will work. In this case they have been awarded 1 mark in spite of their faulty logic. Students who picked the larger of the two deltas above, have not been awarded any marks.

If, after finding the correct δ as above, the students have given even an incomplete argument for why it works, they have been awarded 2 marks.

Q 7. *(3 marks) Suppose f is a function from $\mathbb{R}$ to $\mathbb{R}$, for which, $f^{''}(x) = f(x)$ for all x. Suppose that $f(2) = \frac{1}{3}, f'(2) = \frac{1}{5}$. Calculate the first five terms of the Taylor series of $f(x)$ at 2.*

Solution:

Note that the Taylor series of f at a point a is given by

$$\sum_{n=0}^{\infty} a_n (x-a)^n,$$

where $a_n = \frac{f^{(n)}(a)}{n!}$ for $n \geq 1$ and $a_0 = \frac{f^{(0)}(a)}{0!} = f(a)$.

(1 mark)

It is given that $f(2) = \frac{1}{3}, f'(2) = \frac{1}{5}$ and $f^{(2)}(x) = f(x)$ for all x. Thus, we obtain that

$$f^{(2)}(2) = f(2) = \frac{1}{3},\ f^{(3)}(2) = f'(2) = \frac{1}{5},\ f^{(4)}(2) = f^{(2)}(2) = \frac{1}{3}.$$

(1 mark)

In our case $a = 2$, and hence the Taylor series of f at 2 is

$$\frac{1}{3} + \frac{1}{5}(x-2) + \frac{1}{3}\frac{(x-2)^2}{2!} + \frac{1}{5}\frac{(x-2)^3}{3!} + \frac{1}{3}\frac{(x-2)^4}{4!} + \cdots$$

(1 mark)

Q 8. *(3 marks) Let* $S_n = \frac{5}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2}\right\}$. *Evaluate* $\lim_{n\to\infty} S_n$ *by identifying it as a Riemann sum for a certain continuous function on a certain interval and with respect to a certain (tagged) partition.*

Solution: The given sum S_n is the Riemann sum for the function $f : [0,2] \to \mathbb{R}$ defined as $f(x) = 5x$ for $x \in [0,1]$ and $f(x) = 5x^{\frac{3}{2}}$ for $x \in [1,2]$, with respect to the tagged partition (P_n, t_n), where $P_n = \{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1 < \frac{n+1}{n} < \cdots < \frac{2n-1}{n} < 2\}$ and $t_n = \left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{2n-1}{n}, 2\right\}$.

(**1 mark**)

Since the function f is continuous (on the closed and bounded interval $[0,2]$), it is Riemann integrable

(**1 bonus mark**)

and

$$\int_0^2 f(x)\ dx = \left(\int_0^1 5x\ dx + \int_1^2 5x^{\frac{3}{2}}\ dx\right) = 5\left(\left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^{\frac{5}{2}}}{5/2}\right]_1^2\right)$$

$$= 5\left(\frac{1}{2} + \frac{2}{5}\left[(\sqrt{2})^5 - 1\right]\right) = \left(\frac{5}{2} + 2\left[(4\sqrt{2} - 1\right]\right).$$

(**1 mark**)

Thus, by the definition of the Riemann integral, for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|R(P,t) - \left(\frac{5}{2} + 2\left[(4\sqrt{2} - 1\right]\right)\right| < \epsilon$$

whenever (P,t) is a tagged partition of $[0,2]$ with $||(P,t)|| < \delta$.
If we take $N \in \mathbb{N}$ such that $N > \frac{1}{\delta}$, then for $n \geq N$, $||(P_n,t_n)|| = \frac{1}{n} \leq \frac{1}{N} < \delta$ and hence

$$\left|S_n - \left(\frac{5}{2} + 2\left[(4\sqrt{2} - 1\right]\right)\right| < \epsilon$$

whenever $n \geq N$. That is,

$$\lim_{n\to\infty} S_n = \left(\frac{5}{2} + 2\left[(4\sqrt{2} - 1\right]\right).$$

(**1 mark**)

Explanation for the bonus mark: Many of the students have just integrated the function f without providing any justification for this (that, why does the Riemann integral of f exist?). We are not deducting any mark for this but giving 1 bonus mark to those who have provided the correct justification.

Q 9. *(4 marks) Suppose f, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(a)g(2b) \neq 0$ for some $(a, b) \in \mathbb{R}^2$. By using the $\epsilon - \delta$ definition of the continuity, show that the function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $\phi(x, y) = f(x)g(2y)$ is continuous at (a, b).*

Solution: Let $\epsilon > 0$ be a given positive real number. It follows from the given hypothesis that $f(a) \neq 0$ and $g(2b) \neq 0$. Note that

$$f(x)g(2y) - f(a)g(2b) = f(x)g(2y) - f(a)g(2y) + f(a)g(2y) - f(a)g(2b)$$

$$= (f(x) - f(a))g(2y) + f(a)(g(2y) - g(2b)).$$

(1 mark)

Since $f(x)$ is continuous at a, there exists $\delta_f > 0$ such that

$$|x - a| < \delta_f \Rightarrow |f(x) - f(a)| < \frac{2\epsilon}{6|g(2b)|}.$$

Since $g(2y)$ is continuous at b, there exists $\delta_g > 0$ such that

$$|y - b| < \delta_g \implies |g(2y) - g(2b)| < \frac{\epsilon}{2|f(a)|}$$

and $|g(2y)| < \frac{3|g(2b)|}{2}$.

(1 mark)

Now define $\delta = \min\{\delta_f, \delta_g\}$, then it is clear that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |x - a| < \delta_f \text{ and } |y - b| < \delta_g$$

(1 mark)

and hence

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x)g(2y) - f(a)g(2b)|$$

$$\leq |(f(x) - f(a))| \cdot |g(2y)| + |f(a)| \cdot |(g(2y) - g(2b))|$$
$$< \frac{2\epsilon}{6|g(2b)|} \frac{3|g(2b)|}{2} + |f(a)| \frac{\epsilon}{2|f(a)|}$$
$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, the function $(x, y) \mapsto \phi(x, y) = f(x)g(2y)$ is continuous.

(1 mark)

Q 10. *(4 marks) Show from the definition of differentiability that the function $f(x, y) = 5x^2 + 5y^2$ from $\mathbb{R}^2 \to \mathbb{R}$ is differentiable at all points. Compute its derivative matrix.*

Solution: To show that $f(x, y)$ is differentiable at some point (x_0, y_0), we have to show that

$$\lim_{\|(h,k)\|\to 0} \frac{|f(x_0 + h, y_0 + k) - f(x_0, y_0) - hf_x(x_0, y_0) - kf_y(x_0, y_0)|}{\|(h, k)\|} = 0.$$

(1mark))

We have $f_x(x_0, y_0) = 2x_0,\ f_y(x_0, y_0) = 2y_0$. Hence, the numerator of the expression above has the form

$$5[(x_0 + h)^2(y_0 + k)^2 - x_0^2 - y_0^2 - 2x_0h - 2y_0k] = 5(h^2 + k^2).$$

(1mark))

It follows that the limit above

$$= \lim_{\|(h,k)\|\to 0} \frac{5(h^2 + k^2)}{\|(h, k)\|} = \lim_{\|(h,k)\|\to 0} 5\|(h, k)\| = 0.$$

(1mark))

The Derivative matrix is given by $(f_x(x_0, y_0)\ f_y(x_0, y_0)) = (10x_0, 10y_0)$.

(1mark))