Wave Packets and Uncertainty Relations

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Wave or Particle or Both?

Radiation (light or X-rays), described classically by wave theory, shows particle characteristics in some experiments (photoelectric effect and Compton effect). The *particles* of radiation are required to be massless because they travel with speed of light.

Massive objects are described classically by particle mechanics. In fact, wherever possible, a massive object is treated as a point mass. But when a massive particle with no size (electron) was discovered, it was found to have wave like properties (as seen in electron diffraction and interference experiments)

We are forced to the conclusion that both matter and radiation have to be described a theory that has elements of both particle nature and wave nature.

This is quite problematic because the pictures we have built for particles and for waves have mutually exclusive properties.

Description of Particles

Particles are localized. They occupy a well defined region of space, whatever their size.

In the idealized picture, we want the particles to have no size at all, so that their position can be described by a single real number. This makes the mathematical analysis much easier.

We take this picture so seriously that when we analyze a macroscopic object, we picture it as being made of a very a large number of infintesimal objects with tiny but non-zero masses.

Such visualization has been an enormous success. A very large number of phenomena in mechanics and electromagnetism have been explained by treating matter as being made up a large number of "point masses".

Experimental proof for the existence of atoms strengthens this belief.

Dynamics (or the time evolution) of particles is described by Newton's second law, which treats the spatial coordinates of particles as functions of time.

Description of Waves

Waves are extended objects. Again, to keep the mathematics simple, we use an idealized picture, where a wave extends in space from $-\infty$ to $+\infty$.

The explanation of interference and diffraction experiments depends crucially on the fact that waves have spatial extension.

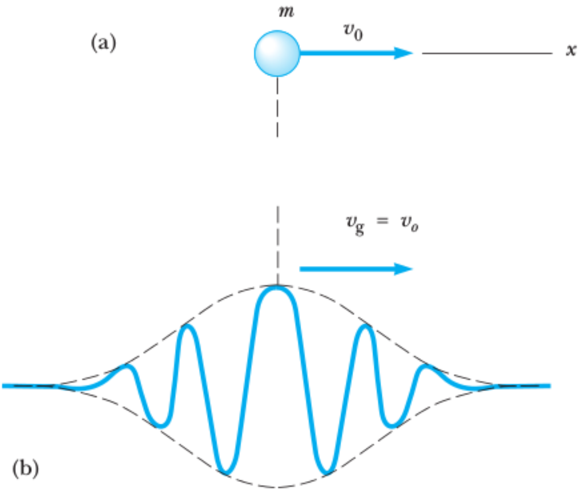
We get interference only if the wave goes through both slits **simultaneously**.
View video https://www.youtube.com/watch?v=YoQYnhHQ95U

Dynamics (evolution in space and time) is described by the wave equation, which treats the wave amplitude as a function of spatial coordinates and time.

If physical objects have both particle and wave like properties, how do we build a mathematical description, which brings together these seemingly mutually exclusive properties of particles and waves?

Wave Packet

We do it through wave packets.



Wave Packet

We make a compromise. To describe a particle of mass m and speed v, we make a **wave packet** with a finite extension and within this extension there is a wave of a few cycles.

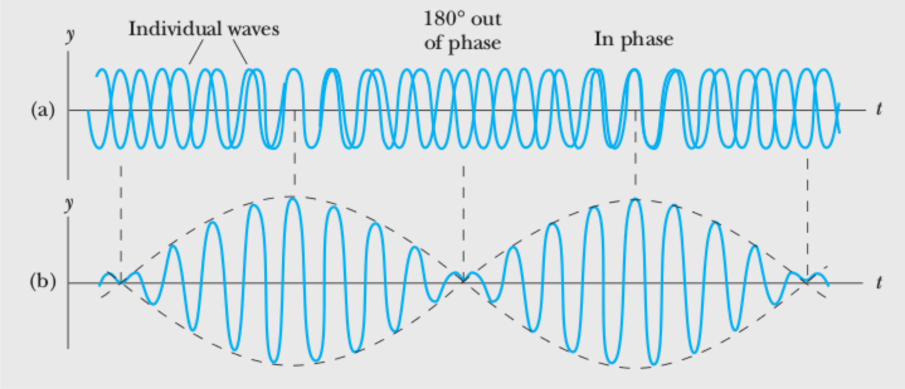
What is the actual extension and how is a cycle to be defined? De Broglie comes to our rescue. The cycle is defined in terms of the de Broglie wavelength $\lambda = h/mv$.

The "wave" in the wave packet does not look like the nice and simple sinusoidal waves we use in doing wave analysis.

The wavelength changes as a function of the spatial coordinate. This raises at least two questions?

1. How do you mathematically represent such a "wave packet"?
2. If the wavelength is a not fixed, what is the meaning of the de Broglie wavelength?

First question is fairly easy to answer. We have already have encountered wave packets in classical physics while studying **beats**.



Wave Packet

We start with two waves

$$\begin{aligned}\psi_1(x,t) &= A\sin(kx-\omega t)\\ \psi_2(x,t) &= A\sin[(k+\Delta k)x-(\omega+\Delta\omega)t]\end{aligned}$$

Both are sound waves and both travel through air with the same speed. So we must have

$$\frac{\omega}{k} = v_s \text{ and } \frac{\omega+\Delta\omega}{k+\Delta k} = v_s \implies \frac{\Delta\omega}{\Delta k} = v_s.$$

If we superpose ψ_1 and ψ_2, we get the combined wave which exhibits the phenomenon of beats.

$$\begin{aligned}\psi &= \;= \psi_1+\psi_2 = A\left(\sin(kx-\omega t)+\sin[(k+\Delta k)x-(\omega+\Delta\omega)t]\right)\\ &= A\sin[(2k+\Delta k)x-(2\omega+\Delta\omega)t]\cos(\Delta kx-\Delta\omega t)\end{aligned}$$

Now there are two oscillations: One denoted by $(2k+\Delta k)$ and the other by Δk. The wavelengths of the two are widely different. In the case of beats, both waves travel with the speed of sound.

Wave Packet

The figure of ψ looks rather similar to that of the wave packet. Thus we get the hint that we can construct a wave packet by superposing a number of different waves of different wavelength.

Thus the wave packet is characterized by the average wavelength (which will be equal to the de Broglie wavelength) and a spread (or uncertainty) in the wavelength which is related to Δk.

View video https://www.youtube.com/watch?v=NLQs7bD6w98

The maxima and the minima of the wave (and all the points in between) move with speed ω/k, called phase velocity. This is the speed with which any given point (which corresponds to a definite phase) of the wave moves.

The envelope itself moves with speed $\Delta\omega/\Delta k$, which is called the group velocity.

In general, the group velocity can be smaller than/equal to/greater than phase velocity.

Group Velocity and de Broglie Wavelength

Section 5.3 of Modern Physics:
The group velocity v_g can be related to the phase velocity v_p. v_p in general is a function of k and the angular frequency $\omega = kv_p$. Hence

$$v_g = \frac{d\omega}{dk}|_{k_0} = v_p|_{k_0} + k\frac{dv_p}{dk}|_{k_0}$$

where the quantities on the RHS are evaluated at the average wavenumber k_0 of the wave packet.

Two Examples:

1. If the phase velocity of the wave is proportional to the wave number, *i. e.* $v_p = Ak$, where A is a constant, it is easy to show that $v_g = 2v_p(k_0)$.
2. If the phase velocity of the awave is proportional to $\sqrt{\lambda}$ or $1/\sqrt{k}$, which is the case for deep water waves, then we can show $v_g = v_p(k_0)/2$.

Group Velocity of a Matter Wave

Let us calculate the group velocity for a matter wave packet of a particle of restmass m, momentum p and energy $E = \sqrt{p^2c^2 + m^2c^4}$.

From Planck hypothesis, we postulate that it has frequency $f = E/h$ and from de-Broglie hypothesis, we postulate that it has wavelength $\lambda = h/p$.

The phase velocity is given by

$$v_p = \omega/k = 2\pi f/(2\pi/\lambda) = f\lambda = E/p.$$

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Substituting for E and p, we get

$$v_p = c\sqrt{1 + \left(\frac{mc}{p}\right)^2}.$$

Momentum p can be written in terms of wave number as $p = h/\lambda = (h/2\pi)(2\pi/\lambda) = \hbar k$. Hence

$$v_p = c\sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2}.$$

Group Velocity of a Matter Wave

Substituting this in the expression for v_g, we get

$$v_g = \frac{c}{\left[1 + \left(\frac{mc}{\hbar k_0}\right)^2\right]^{1/2}} = \frac{c^2}{v_p(k_0)}.$$

The phase velocity $v_p = E/p = \gamma mc^2/\gamma mv = c^2/v$, where v is the speed of the particle and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Since v is always less than c for matter particles, the phase velocity of matter waves is always greater than the speed of light!

This is not in contradiction to relativity because the information about the particle is carried by the whole wave packet and is travelling at speed v_g.

Subsituting $v_p = c^2/v$ in the expression for v_g, we obtain $v_g = v$. The matter wave envelope moves with the same speed as the mass.

Uncertainty Relations

Section 5.4 of Modern Physics:
A wave packet is an extended object. It has non-zero spatial spread. But it is also reasonably well localized. In spatial coordinates, it is characterized by x and Δx.

If you think of the wave packet as a kind of histogram in spatial coordinate, then x_0 is the average position and Δx is its standard deviation. Δx is a measure of the spatial spread of the wave packet.

Uncertainty Relations

For a Gaussian wave packet

$$\psi(x) \sim \exp\left[-\frac{(x-x_0)^2}{2(\Delta x)^2}\right],$$

then 99.93% of the wave packet is within $x_0 \pm 3\Delta x$.

We expand any well behaved wave packet into an infinite number of sinusoidal waves, each labelled by its wave number k.

$$\psi(x) = \int_{-\infty}^{\infty} dk A(k) \sin[kx + \phi(k)],$$

where $A(k)$ is the amplitude of the wave with wave number k and $\phi(k)$ is its phase. This expansion is called Fourier Analysis.

Think of this in the following way: Given any well behaved function $f(x)$, we can obtain a McLaurin's expansion for it as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^n f}{dx^n}|_0.$$

Uncertainty Relations

That is we can expand any well behaved function in the form of an infinite power series $\sum_n a_n x^n$. The series converges and $a_n \to 0$ as $n \to \infty$.

Similarly, we can expand any well behaved wave packet in spatial coordinates as an infinite sum of sinusoidal waves.

For a well behaved wave packet, $A(k) \to 0$ as $k \to \pm\infty$. Which means $A(k)$ is a well behaved function in the variable k.

We can think of $A(k)$ also as a histogram in the variable k. It will have an average k_0 and a standard deviation Δk. Δk is a measure of the spread of frequencies which make up the wave packet.

For all wave packets, the following relation holds: $\Delta x \Delta k \sim 1$.

Uncertainty Relations

We can understand the uncertainty relation in the following manner. Suppose we want Δx very small (a very narrow pulse). The oscillations in this narrow range must necessarily be very rapid which means the average value of k is very high.

Using Fourier Analysis one can construct the histogram in the wave number, given the histogram in the spatial coordinate.

This wave number histogram will be very broad for narrow spatial pulse.

Within Fourier Analysis, the spread in spatial coordinates and the spread in wave number are **irrevocably** linked.

Fourier Analysis is based on sound mathematical principles and is found applicable in a wide range of problems.

Hence we are led to the uncertainty relations:

$$\Delta x \Delta k \sim 1 \text{ and } \Delta t \Delta \omega \sim 1.$$

These relations hold for the two wave superposition discussed earlier.

Heisenberg Uncertainty Relations

All wave packets, including matter wave packets, satisfy the uncertainty relations mentioned in the previous slide.

The value on the right hand side depends on the form of the wave packet. Using a mathematical technique called Calculus of Variations, it can shown that for Gaussian wave packets, the uncertainty is minimum: $\Delta x \Delta k = 1/2$.

Substituting it in the uncertainty relation for the Gaussian wave packet, we obtain the famous Heisenberg Uncertainty Relation

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}.$$

We can obtain a similar energy-time uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

We can get an intuitive understanding the Heisenberg uncertainty relation through the hypothetical Heisenberg Microscope.