Indian Institute of Technology Bombay

Department of Mathematics

MA 105: Calculus

Quiz 9 for D1 & D2

Date: Wednesday, 25th October 2017 Max. Marks 3

Question: Determine if the vector field $\mathbf{F}(x,y,z)$ (defined below for the respective codes) is conservative. Also, compute the gradient of $\nabla \cdot \mathbf{F}$ at $(0,0,0)$.

(A) $\sin(x^2)\mathbf{i} + 2xy\mathbf{j} + 2xz\mathbf{k}$ (B) $2xy\mathbf{i} + \sin(y^2)\mathbf{j} + 2yz\mathbf{k}$

(C) $2xz\mathbf{i} + 2yz\mathbf{j} + \sin(z^2)\mathbf{k}$ (D) $\cos(x^2)\mathbf{i} + 2xy\mathbf{j} + 2xz\mathbf{k}$.

Solution to (A): If we write $\mathbf{F}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$, then

$$\nabla\times\mathbf{F} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$

and hence

$$\nabla \times \mathbf{F} = 0\mathbf{i} - 2z\mathbf{j} + 2y\mathbf{k} \neq (0,0,0).$$

Thus, $\mathbf{F}$ is not conservative. [1 mark]

And,

$$\nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

and hence

$$\nabla \cdot \mathbf{F}(x,y,z) = 2x\cos(x^2) + 4x.$$

[1 mark]

Thus, $\nabla(\nabla \cdot \mathbf{F}(x,y,z)) = (-4x^2\sin(x^2) + 2\cos(x^2) + 4, 0, 0)$

$\implies \nabla(\nabla \cdot \mathbf{F})(0,0,0) = (6,0,0).$ [1 mark]

Solution to (B), (C) and (D): None of the vector fields are conservative because of the same reason as that in (A). In (B), $\nabla(\nabla\cdot\mathbf{F})(0,0,0) = (0,6,0)$. In (C), $\nabla(\nabla \cdot \mathbf{F})(0,0,0) = (0,0,6)$. In (D), $\nabla(\nabla \cdot \mathbf{F})(0,0,0) = (4,0,0)$.