**Indian Institute of Technology Bombay**

*Department of Mathematics*

MA 105: Calculus

# Quiz 10 for D1 & D2

Date: Wednesday, 01st November 2017 Max. Marks 3

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**Question:** Let $\mathbf{F}(x, y)$ be the vector field given below (for the respective codes). Compute the line integral of $\mathbf{F}$ by the first principle (that is, only by using the definition of the line integral) along the square $ABCD$ having vertices at $A = (0, 0), B = (1, 0), C = (1, 1), D = (0, 1)$ and oriented counterclockwise.

(A) $x^2y^4\mathbf{i} - x^4y^2\mathbf{j}$ (B) $x^3y^5\mathbf{i} - x^5y^3\mathbf{j}$

(C) $x^4y^6\mathbf{i} - x^6y^4\mathbf{j}$ (D) $x^2y^5\mathbf{i} - x^5y^2\mathbf{j}$.

**Solution to (A):** Let $R$ be the given square with vertices at $A, B, C, D$. The square $R$ is a piece-wise smooth curve which is union of the line segments $\mathcal{C}_1 = AB$, $\mathcal{C}_2 = BC$, $\mathcal{C}_3 = CD$ and $\mathcal{C}_4 = DA$ where $\mathcal{C}_1(t) = (t, 0)$, $\mathcal{C}_2(t) = (1, t)$, $\mathcal{C}_3(t) = (1 - t, 1)$ and $\mathcal{C}_4(t) = (0, 1 - t)$ for $t \in [0, 1]$ (note that the square is oriented counterclockwise so we define the corresponding curves this way).

Then,

$$\int_R \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{s}.$$

[1 mark]

We first compute $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s}$. Since $\mathcal{C}_1(t) = (t, 0)$ and $\mathbf{F}(x, y) = (x^2y^4, -x^4y^2)$, $\mathcal{C}_1'(t) = (1, 0)$ and $\mathbf{F}(\mathcal{C}_1(t)) = \mathbf{F}(t, 0) = (0, 0)$ and hence $\mathbf{F}(\mathcal{C}_1(t)) \cdot \mathcal{C}_1'(t) = 0$

and

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathcal{C}_1(t)) \cdot \mathcal{C}_1'(t)dt = \int_0^1 0 \ dt = 0.$$

We now compute $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$. Since $\mathcal{C}_2(t) = (1, t)$ and $\mathbf{F}(x, y) = (x^2y^4, -x^4y^2)$, $\mathcal{C}_2'(t) = (0, 1)$ and $\mathbf{F}(\mathcal{C}_2(t)) = \mathbf{F}(1, t) = (t^4, -t^2)$ and hence $\mathbf{F}(\mathcal{C}_2(t)) \cdot \mathcal{C}_2'(t) = -t^2$ and

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathcal{C}_2(t)) \cdot \mathcal{C}_2'(t)dt = \int_0^1 -t^2 \ dt = -\frac{1}{3}.$$

[1 mark]

Further, we compute $\int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s}$. Since $\mathcal{C}_3(t) = (1 - t, 1)$ and $\mathbf{F}(x, y) = (x^2y^4, -x^4y^2)$, $\mathcal{C}_3'(t) = (-1, 0)$ and $\mathbf{F}(\mathcal{C}_3(t)) = \mathbf{F}(1 - t, 1) = ((1 - t)^2, -(1 - t)^4)$ and hence $\mathbf{F}(\mathcal{C}_3(t)) \cdot \mathcal{C}_3'(t) = -(1 - t)^2$ and

$$\int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathcal{C}_3(t)) \cdot \mathcal{C}_3'(t)dt = \int_0^1 -(1 - t)^2 \ dt = -\frac{1}{3}.$$

Finally, we compute $\int_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{s}$. Since $\mathcal{C}_4(t) = (0, 1 - t)$ and $\mathbf{F}(x, y) = (x^2y^4, -x^4y^2)$, $\mathcal{C}_4'(t) = (0, -1)$ and $\mathbf{F}(\mathcal{C}_4(t)) = \mathbf{F}(0, 1 - t) = (0, 0)$ and hence $\mathbf{F}(\mathcal{C}_4(t)) \cdot \mathcal{C}_4'(t) = 0$ and

$$\int_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathcal{C}_4(t)) \cdot \mathcal{C}_4'(t)dt = \int_0^1 0 \ dt = 0.$$

Thus,

$$\int_R \mathbf{F} \cdot d\mathbf{s} = 0 - \frac{1}{3} - \frac{1}{3} + 0 = -\frac{2}{3}.$$

[1 mark]

**Solution to (B):**

$$\int_T \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{s} = 0 - \frac{1}{4} - \frac{1}{4} + 0 = -\frac{1}{2}.$$

**Solution to (C):**

$$\int_T \mathbf{F}\cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_4} \mathbf{F}\cdot d\mathbf{s} = 0 - \frac{1}{5} - \frac{1}{5} + 0 = -\frac{2}{5}.$$

**Solution to (D):**

$$\int_T \mathbf{F}\cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_4} \mathbf{F}\cdot d\mathbf{s} = 0 - \frac{1}{3} - \frac{1}{3} + 0 = -\frac{2}{3}.$$