MA 105 D1 Lecture 19

Ravi Raghunathan

Department of Mathematics

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A linear change of coordinates

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First, let us write down the linear map in more compact notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

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Clearly, a rectangle $[1, 0] \times [0, 1]$ in the $u - v$ plane is mapped to a parallelogram in the $x - y$ plane. The sides of the parallelogram are given by $(a + t_1, c + t_2)$ and $(b + t_1, d + t_2)$.

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How does one compute the area of this parallelogram?

The area element for a change of coordinates

It is obviously given by the cross product of the vectors

$$(a, c, 0) \times (b, d, 0) = (ad - bc) \cdot \mathbf{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \mathbf{k}.$$

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Using the chain rule for functions of two variables we see that

$$\Delta x \sim \frac{\partial \phi}{\partial u} \Delta u + \frac{\partial \phi}{\partial v} \Delta v$$

and

$$\Delta y \sim \frac{\partial \psi}{\partial u} \Delta u + \frac{\partial \psi}{\partial v} \Delta v.$$

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Using our previous notation, we can write

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

The Jacobian

You may recognize the matrix

$$J = \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix}$$

that appears in the preceding formula. It is the derivative matrix for the function $h = (\phi, \psi) : \mathbb{R}^2 \to \mathbb{R}^2$.

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We know that the derivative matrix is the linear approximation to the function h, at least in a neighborhood of a point, say (u_0, v_0). Or, to say it slightly differently, in a neighborhood of the point (u_0, v_0), the function h and the function J, behave very similarly (that is, they are the same upto the first order terms - use Taylor's theorem!).

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In particular, it is easy to see how the area of a small rectangle changes under h, since we have already done so in the case of a linear map. It simply changes by the (absolute value of) determinant of J! This is how we get the change of variable formulæ.

The change of variables formula

Let D be a region in the xy plane and D^* a region in the uv plane such that $h(D^*) = D$. Then

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

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Let us see what we get in the familiar case of polar coordinates. We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left|\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r,$$

which is what we have already obtained in this case.

Exercise 7.4: Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y)\, dxdy$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

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Solution: Put

$$x = \frac{u-v}{2}, y = \frac{u+v}{2},$$

Then the rectangle

$$R = \{\pi \le u \le 3\pi,\ \ -\pi \le v \le \pi\}$$

in the uv-plane gets mapped to D, a parallelogram in the xy-plane. Further,

$$J = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

$$\int\int_D (x-y)^2 \sin^2(x+y)dxdy = \int\int_R v^2 \sin^2(u)\frac{1}{2}dudv$$

$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} v^2 dv\right)\left(\int_{\pi}^{3\pi} \sin^2(u)du\right)$$

$$= \frac{1}{2}\left(2 \times \frac{\pi^3}{3}\right)(\pi) = \frac{\pi^4}{3}.$$

Exercise 7.5: Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Find

$$\iint_D dA$$

by transforming it to

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where $x = \frac{u}{v}, y = uv, v > 0$.

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Solution: Put

$$x = \frac{u}{v},\ \ y = uv.$$

Then the rectangle $R = \{1 \leq u \leq 3, 1 \leq v \leq 2\}$ in the uv-plane gets mapped to D in the xy-plane. Further,

$$J = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{2u}{v}$$

Hence,

$$\int\int_D dA = \mathrm{Area}(D) = \int\int_R \frac{2u}{v} dudv$$

$$= \left(\int_1^3 2udu\right)\left(\int_1^2 \frac{dv}{v}\right) = 8\ln 2.$$

Triple integrals in a box

If we have a function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$ we can integrate it over this rectangular parallelepiped. As in the one and two variable cases, we divide the parallelepiped into smaller ones B_{ijk}, making sure that the length, breadth and height of the small parallelepiped are all small. In particular, we can use the regular partition of order n to obtain the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

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As before we say that f integrable $\lim_{n\to\infty} S(f, P_n, t)$ exists for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

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First, if f is bounded and continuous in B, except possibly on (a finite union of) graphs of $\mathcal{C}^1$ functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

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Once we have defined the triple integral in this way, it remains to evaluate it.

Evaluating triple integrals

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Thus, if f integrable on the box B we have

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx.$$

There are, in fact, five other possibilities for the iterated integrals and each of these exists and is equal to the value of the triple integral.

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There are, in fact, five other possibilities for the iterated integrals and each of these exists and is equal to the value of the triple integral.

The triple integrals that are easiest to evaluate are those for which the region P in space can be described by bounding z between the graphs of two functions in x and y. This is the analogue of an elementary region in the plane and we will call such regions also elementary regions (but in space). In general, we may be able to express more complicated domains as unions of elementary domains.

Evaluating triple integrals continued

In this case we proceed as follows. Suppose that the region P lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of P on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then

$$\iiint_P f(x, y, z)dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)dzdydx.$$

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Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region P, where P is the unit sphere.

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Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region P, where P is the unit sphere.

The sphere can be described as the region lying between $z = -\sqrt{1 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

The volume of the unit sphere

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1-x^2}{2} dx = \frac{4}{3}\pi.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

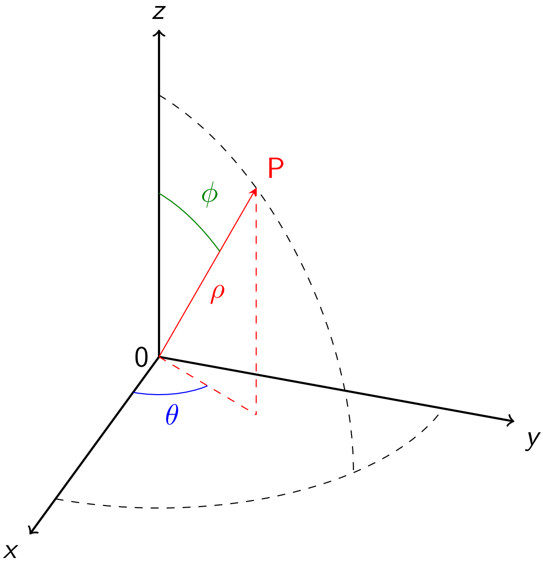
$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (\phi, \psi, \rho)$, the function g is defined as $g = f(\phi, \psi, \rho)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

is just the Jacobian determinant for a function of three variables.

Spherical Coordinates - the big picture



Spherical coordinates in formulæ

The relation between the spherical and cartesian coordinates is

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta \quad \text{and} \quad z = \rho\cos\phi$$

for $\rho \geq 0$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$.

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Exercise 1: Show that

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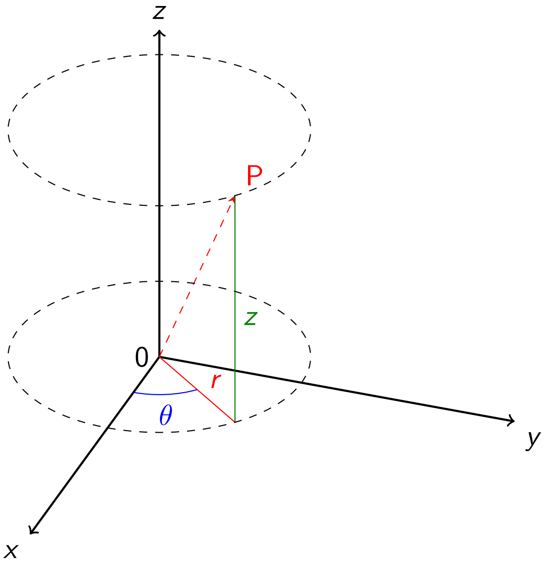
Note that my convention is different from Wikipedia's.

Cylindrical coordinates: the picture



Cylindrical coordinates in formulae

If $P = (x, y, z) \in \mathbb{R}^3$, its cylindrical coordinates (r, θ, z) are defined by

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

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It is very easy to see that

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The good thing about our convention is that θ means the same thing in both the cylindrical and spherical coordinate systems as well as in the (two-dimensional) polar coordinate system, and r means the same thing in both the cylindrical and (two-dimensional) polar coordinate systems.

Centre of Mass: the definition

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Suppose we have solid in a region W with a continuous mass density given by a function $\rho(x, y, z)$. The centre of mass is the point with the coordinates

$$\bar{x} = \frac{\iiint_W x\rho(x, y, z)dxdydz}{\iiint \rho(x, y, z)dxdydz}, \quad \bar{y} = \frac{\iiint_W y\rho(x, y, z)dxdydz}{\iiint \rho(x, y, z)dxdydz}$$

and $$\bar{z} = \frac{\iiint_W z\rho(x, y, z)dxdydz}{\iiint \rho(x, y, z)dxdydz}.$$

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The importance of the centre of mass is the following. Outside a spherically symmetric planet whose density need not be uniform, the gravitational potential is given by $V = GMm/R$, where R is the distance of the smaller body from the centre of mass of the bigger body. Thus, a spherically symmetric body attracts as if all its mass were concentrated at the centre.

Determining the centre of mass - planes of symmetry

A body in the region W is said to be symmetric with respect to a plane if for every particle located on one side of the plane there is a particle of equal mass located at its mirror image.

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A body in the region W is said to be symmetric with respect to a plane if for every particle located on one side of the plane there is a particle of equal mass located at its mirror image.

Assume that W is symmetric with respect to the xy-plane. In terms of the density function this means that $\rho(x,y,z) = \rho(x,y,-z)$ Let W^+ and W^- denote the portions of W above and below the plane, respectively. Let us calculate the z-coordinate of the centre of mass of W. We have

$$
\begin{aligned}
\bar{z} &= \frac{\iiint_W z\rho(x,y,z)dxdydz}{\iiint \rho(x,y,z)dxdydz} \\
&= \frac{\iiint_{W^+} z\rho(x,y,z)dxdydz}{\iiint \rho(x,y,z)dxdydz} + \frac{\iiint_{W^-} z\rho(x,y,z)dxdydz}{\iiint \rho(x,y,z)dxdydz} \\
&= \frac{\iiint_{W^+} z\rho(x,y,z)dxdydz}{\iiint \rho(x,y,z)dxdydz} + \frac{\iiint_{W^+} -z\rho(x,y,-z)dxdydz}{\iiint \rho(x,y,z)dxdydz} = 0.
\end{aligned}
$$

Archimedes's discovery

From the preceding discussion we see that the centre of mass must lie on the xy plane. There was nothing special about the xy plane. If our body had been symmetric about some other plane, we could simply have rotated it and translated it so that the xy plane became the plane of symmetry. This shows that if a body is symmetric with respect to a plane, its centre of mass must lie on the plane.

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In particular we see that the following law of mechanics holds: If a body is symmetric with respect to two planes, then its centre of mass lies on their line of intersection.

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For a sphere, we can take the xy, yz and zx planes as planes of symmetry to see that the centre of mass is nothing but the centre of the sphere.

The validity of the change of variables formula

We have pushed many things under the carpet in our hurry to develop a change of variables formula. We must perhaps ask ourselves when and under what conditions the formula might be valid.

The validity of the change of variables formula

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For instance, should we allow the following transformation $h(u, v) = (1, 2)$ (a constant map)?

Clearly not, since all information about the region D is lost when we make such a transformation, considering such functions will not lead to anything useful.

Properties to look for in coordinate changes

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If we remember the method of substitution in one variable, only substitution changes that were bijective were allowed. We should clearly require this of our function h, not just locally (which is guaranteed if we assume $|J| \neq 0$), but for the whole domain.

Diffeomorphisms

Finally, another requirement comes from the final formula we derived. This includes $|J|$, which, if we want some kind of integrability, forces h to be a $\mathcal{C}^1$ function.

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The final condition we must impose is that h must be a diffeomorphism, that is, the inverse function h^{-1} (which exists, since h is assumed bijective) must also be a $\mathcal{C}^1$ function. This will ensure that the image $h(D)$ of a small disc D in the $u - v$ plane will contain a small disc in the $x - y$ plane.

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Again, this is automatic locally, if we assume that $|J| \neq 0$. This fact is called the Inverse function theorem.

Coordinate changes

Let us summarize our discussion above. A change of coordinates $h : D \to D' \subset \mathbb{R}^2$ is a function $h(u, v) = (\phi(u, v), \psi(u, v))$ which is $\mathcal{C}^1$ on D, which is a bijection from D to D' and such that h^{-1} is a $\mathcal{C}^1$ function.

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There is only one thing we have still been vague about. We have not specified what kind of regions D and D' should be. For our purposes we will assume that D and D' are the interiors of regions of type 3 (the interior of a region of type 3 is simply the part of the region that is not on the boundary - for instance, the open disc is the interior of the closed disc). More generally, we can assume that D and D' are open sets (see next slide).

Open sets

Definition: A subset U of $\mathbb{R}^n$ is called open if around every point $x \in U$ and some $r > 0$, there is a disc $D_r(x)$ of radius r around x contained inside U.

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Exercise 1: Conversely, prove that any open set in $\mathbb{R}$ is necessarily the (countable) disjoint union of open intervals.

Open discs and rectangles are examples of open sets in $\mathbb{R}^2$ as are unions (finite or infinite) of theses. The n-dimensional open discs and rectangles are open sets in $\mathbb{R}^n$.

The Inverse Function Theorem

Theorem 38: If U is an open set in $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ function such that the Jacobian determinant of F at p, $|J_F(p)| \neq 0$, for some point $p \in U$, then the inverse function F^{-1} exists for some open set V containing $F(p)$ and F^{-1} is a $\mathcal{C}^1$ function.

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Another way of thinking of the Inverse Function Theorem is the following. If x and y are functions of u and v, the Inverse Function Theorem tells us that (when $|J| \neq 0$) we can write u and v as functions of x and y, at least in some small open set V.

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As I mentioned above, in order to check whether the function h gives a coordinate change, it is enough by the Inverse Function Theorem to check that it is bijective and that its Jacobian determinant is everywhere non-zero.

Flow lines for vector fields

Recall that a vector field was just a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. For the moment let us suppose that all vector fields under consideration are continuous.

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If we write $\mathbf{c}(t) = (x(t), y(t), z(t))$, and $F = (F_1, F_2, F_3)$, we see that finding a flow line is equivalent to solving the following system of equations.

$$\begin{aligned}
x'(t) &= F_1(x(t), y(t), z(t)) \\
y'(t) &= F_2(x(t), y(t), z(t)) \\
z'(t) &= F_3(x(t), y(t), z(t))
\end{aligned}$$

The del operator on functions

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One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.

The del operator on vector fields

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$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

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It is useful to represent it as a determinant:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Angular velocity

Recall that if a particle P is moving in the three-dimensional space, its position vector $\mathbf{r}$ and velocity vector $\mathbf{v}$ together define a plane at any given instant in time. This plane is called the plane of rotation.

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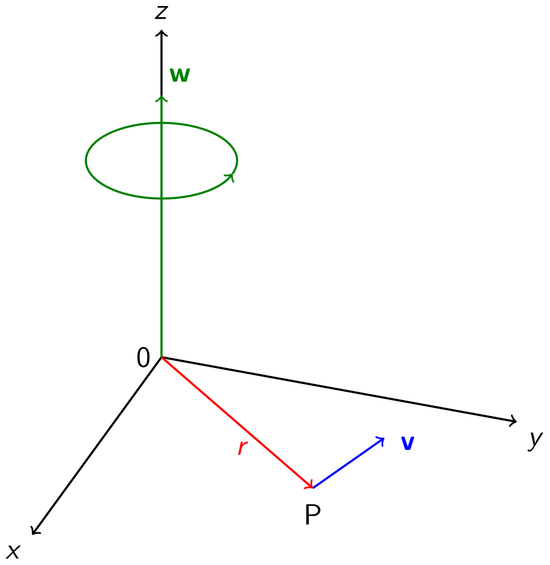
The angular velocity vector which measures the rate of change of angular displacement is defined as

$$\mathbf{w} = \frac{\mathbf{r} \times \mathbf{v}}{r^2}.$$

Clearly, the direction of $\mathbf{w}$ is the direction of the axis of rotation. In the picture in the next slide we assume that the point P is moving in a plane. In this case we can assume that the axis of rotation is the z-axis, since we can always rotate our picture so that this happens.

Angular velocity for a single particle



The curl and angular velocity

Since the particle moves in the $x - y$ plane, we can write $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. From the definition it follows that

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = -wy\mathbf{i} + wx\mathbf{j},$$

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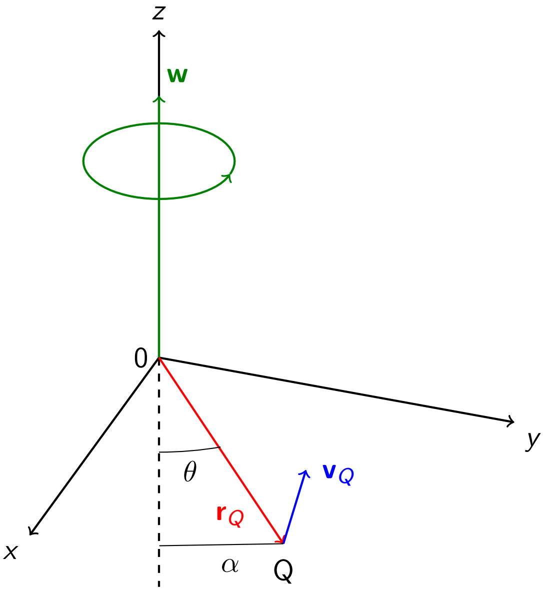
Suppose we assume that P is part of a rigid body and that Q is another point in the body. In the picture that follows, $\mathbf{r}_Q$ is the position vector of Q and θ is the angle made by $\mathbf{r}_Q$ with the axis of rotation (which we continue to assume is the z axis).

Angular velocity for another particle in the same rigid body



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We can easily determine the tangential velocity $\mathbf{v}_T$ of Q. It is directed counterclockwise in along the tangent to a circle parallel to the xy plane with radius α (as in the picture). It follows that

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$$\|\mathbf{v}_T\| = w\alpha = wr_Q \sin\theta,$$

whence, we see that $\mathbf{v}_T = \mathbf{w} \times \mathbf{r}_Q$. Now the same computation as before shows that $\nabla \times \mathbf{v}_T = 2\mathbf{w}$.

The angular velocity for a rigid body

What we can conclude from our previous calculations is that for a rigid body, the curl of the velocity vector field is a constant vector field. Its direction at each point is simply the direction of the axis of rotation, while the magnitude is twice the angular speed.

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Remark: It is more conventional to denote angular velocity by the letter ω. However, I have used **w** because I am unable to get a good boldface font for ω.

Irrotational flow

Instead of looking at the velocity field of a rigid body we can look at the velocity field $\mathbf{F}$ of the flow of a fluid. In this case, what does it mean if $\nabla \times \mathbf{F} = 0$ at a point P?

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It means that the fluid is free from rigid rotations at that point. In physical terms it means that if you put a small paddle wheel face down in the fluid, it will move with the fluid but will not rotate around its axis. In terms of the fluid itself, this translates into there being no whirlpools centred at the point P. In this case, the vector field is called irrotational.

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For instance, water draining into sink produces an irrotational field, except at the very centre of the circular drain.

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

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$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}.$$

Clearly, if f is $\mathcal{C}^2$ (which we will assume), $\nabla \times \mathbf{F} = 0$. In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. If its curl is not zero at some point, it cannot arise as a gradient.

Is the condition $\nabla \times F = 0$ sufficient?

Recall that we have previously looked at the vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2} \cdot \mathbf{i} + \frac{-x}{x^2 + y^2} \cdot \mathbf{j},$$

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Finally, as a special case of the curl, we can define the scalar curl. If $\mathbf{F} = (M(x, y), N(x, y), 0)$ is a vector field in the plane, then

$$\nabla \times \mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

The function $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is called the scalar curl of $\mathbf{F}$.

The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

Definition: Let $\mathbf{F} = (F_1, F_2, F_2)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

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If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Examples

Let us consider the divergence of different vector fields.

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$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

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$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fact the vectors get smaller in size which means that the fluid is getting compressed.

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Example 2: $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. In this case the fluid is moving countercolockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The change in area in a flow

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The change of variable formula tells us how a unit area changes. One has to simply multiply by the Jacobian determinant of the transformation $\phi(x, y) = (X, Y)$. In this case we have

$$J = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix}.$$

The rate of change of area in a flow

We would like to compute the rate at which the unit area is changing. This is simply given by $\frac{\partial J}{\partial t}$. We first write out the function J as

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$$= \frac{\partial}{\partial x}\left(\frac{\partial X}{\partial t}\right)\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial Y}{\partial t}\right)$$
$$-\frac{\partial}{\partial y}\left(\frac{\partial X}{\partial t}\right)\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right).$$

We keep going by brute force

Now we observe that $(X(t), Y(t))$ describes a flow line for the velocity field (F_1, F_2). Hence, the tangent vector of the curve is the same as the value of $\mathbf{F}$ at any point. Hence,

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This is true because (X, Y) describes a flow line of $\mathbf{F}$ as t varies. Hence, the tangent vector of the curve is the same as the value of $\mathbf{F}$ at any point.

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This shows that if $\nabla \cdot \mathbf{F} = 0$, then $\frac{\partial J}{\partial t} = 0$. Hence, $J(x, y, t)$ is a constant function. But $J(x, y, 0) = 1$, so J must be the constant function 1, that is, there is no change of volume along the flow. This shows that if the divergence is zero the fluid is incompressible.

More exercises

Exercise 2: Try doing the above calculation in three dimensions. By this I mean, take a divergence free vector field in the plane and show that the rate of expansion of volue by unit volume in is 0 along flow lines. This is the same calculation as before but there are now eighteen terms in the Jacobian! Not fun.

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Exercise 3: In Lecture 18 we have represented three different vector fields in pictures. Calculate their curls and divergences.

The Laplace operator

Just as the curl of a gradient was 0, we similarly have the divergence of any curl is zero. In other words, if $\mathbf{F}$ is a $\mathcal{C}^2$ vector field,

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Finally, the composition of the gradient followed by the divergence gives one of the most important operators in mathematics and physics. The Laplace operator denoted ∇^2 is defined by

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One can easily check that the function $f(x, y, z) = \frac{1}{\|\mathbf{r}\|}$ satisfies $\nabla^2 f = 0$.