A simple physical derivation of Child–Langmuir space-charge-limited emission using vacuum capacitance

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The fundamental Child–Langmuir limit on the maximum current density in a vacuum between two infinite parallel electrodes is one of the most well known and often applied rules of plasma physics. We develop a simple model using vacuum capacitance, conservation of energy, and conservation of charge to derive the Child–Langmuir space-charge-limited emission. This capacitive model provides physical insight into the origins of the well known (voltage)$^{3/2}$/(gap distance)^2 scaling of the classical current density and does not require the solution of the nonlinear differential equation normally associated with the Child–Langmuir formulation. In addition, the full space-charge-limited solution is reproduced without imposing the condition that the electric field be driven to zero at the cathode surface.
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I. INTRODUCTION

In a variety of electron devices, the charge densities are sufficiently large that the self-consistent effects of the charges on themselves as well as the applied fields must be considered. One of the most often encountered self-interaction effects is the fundamental limit on the current density that can be drawn between two infinite parallel-plane electrodes. Child and Langmuir[1,2] first derived classical space-charge-limited emission for two such electrodes in a vacuum separated by a distance D and a potential difference V. The traditional solution to this emission involves solving a second-order nonlinear differential equation, as we will see in Sec. II. In Sec. III, we will present a novel physical picture of space-charge-limited emission that gives an approximate solution whose scaling matches the exact solution. Section IV discusses how we can easily find the exact solution once the proper scaling is determined, and Sec. V refines the suppositions of the approximate solution so that the exact solution can be obtained using this simple technique. A method for not solving this second-order nonlinear differential equation, and yet still arriving at the exact solution is discussed in Secs. III–V.[3] Such a technique often is overlooked in favor of attacking a differential equation via brute force.

II. SOLVING THE DIFFERENTIAL EQUATION FOR SPACE-CHARGE-LIMITED EMISSION

We first briefly describe the traditional solution to space-charge-limited emission.[4,5] The question we wish to answer is the following: If we have an infinite supply of free electrons at rest (at the grounded cathode) and apply a voltage V to an anode a distance D from the cathode, how much current will be drawn across this gap in the steady state? Because both the anode and cathode are parallel planes of infinite extent, we will look for the limiting current density rather than a total current:

$$J(z)=\rho(z)v(z)=-J_{\mathrm{SCL}}, \tag{1}$$

where ρ and v are the density and velocity of the electrons, respectively, in the gap as a function of the position z between the two plates. Because electrons are born only at the cathode, current will only flow when the anode is biased positively with respect to the cathode (no current flows when the anode is negative with respect to the cathode). This diode is assumed to be of infinite extent in the x and y directions, so these components are ignored and all vectors are treated as scalars. By charge conservation, J_{SCL} cannot vary with z and is thus equal to the fixed value we seek. Because we are interested in solely the electron current density carried in the positive z direction (from cathode to anode), we introduce a minus sign to be consistent with the conventional definition of current. Now $v(z)$ can be found from conservation of energy:

$$\frac{1}{2}mv^2(z)-e\phi(z)=\text{constant}=0, \tag{2}$$

where m and e are the electron mass and charge, respectively, $\phi(z)$ is the potential field in the gap, and for simplicity, the total energy has been set to zero (for electrons initially at rest at the grounded cathode). The potential $\phi(z)$ must obey Poisson's equation

$$\nabla^2\phi(z)=\frac{-\rho(z)}{\varepsilon_0}, \tag{3}$$

where ε_0 is the permittivity of free space. If we eliminate $\rho(z)$ and $v(z)$ using Eqs. (1) and (2), Eq. (3) becomes a second-order nonlinear differential equation for the potential:

$$\frac{d^2\phi}{dz^2}=\frac{J_{\mathrm{SCL}}}{\varepsilon_0}\sqrt{\frac{m}{2e}}\frac{1}{\sqrt{\phi(z)}}. \tag{4}$$

The solution to Eq. (4) is not readily apparent, but can be accomplished by not too much manipulation.

A first integration of Eq. (4) can be performed after first multiplying both sides by $d\phi/dz$:

$$\begin{aligned}\left(\frac{d\phi}{dz}\right)^2&=\frac{4J_{\mathrm{SCL}}}{\varepsilon_0}\sqrt{\frac{2e}{m}}\sqrt{\phi(z)}+K\\&=\frac{4J_{\mathrm{SCL}}}{\varepsilon_0}\sqrt{\frac{2e}{m}}\sqrt{\phi(z)}+\left(\left.\frac{d\phi}{dz}\right|_{z=0}\right)^2,\end{aligned} \tag{5}$$

where the constant of integration K is found by applying the boundary condition of the potential at the cathode, $\phi(0)=0$. Recall that the magnitude of the electric field E is simply the negative of the derivative of the potential ($E=-d\phi/dz$ for this one-dimensional case), so the constant K is the square of the magnitude of the electric field at the cathode surface.

If there were no electrons in the anode–cathode gap region, the magnitude of the electric field at the cathode would simply be V/D, with a direction such that the electrons would be accelerated from the cathode toward the anode. As electrons enter the gap, they modify the vacuum electric field due to their own charge; eventually so many electrons enter the gap region that the applied electric field is exactly canceled at the cathode surface, and no additional electrons are accelerated into the gap. This assumption of zero-electric field at the cathode surface is the condition for space-charge-limited emission—any additional electron injection would cause the field to reverse direction and hence drive electrons back toward the cathode. In the steady state, electrons leave the gap region by being collected at the anode so that more electrons can enter via the cathode; this process gives rise to the space-charge-limited current density J_{SCL}.

If we set the electric field at the cathode to zero, we can take the square root of Eq. (5) and then integrate to find the potential in this space-charge-limited gap:

$$\phi(z)=\left(\frac{3}{2}\right)^{4/3}\left(\frac{J_{\mathrm{SCL}}}{\varepsilon_0}\right)^{2/3}\left(\frac{m}{2e}\right)^{1/3}z^{4/3}+K, \tag{6}$$

where, again, the constant of integration is set to zero due to the boundary condition $\phi(0)=0$. If we apply the boundary condition of the fixed potential anode, $\phi(D)=V$, we can solve Eq. (6) for the space-charge-limited current density:

$$J_{\mathrm{SCL}}=\frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{D^2}. \tag{7}$$

Since the derivation of this fundamental emission limit, many important and useful variations on the classical Child–Langmuir law in Eq. (7) have been investigated to account for effects such as cylindrical[6–8] and spherical[9] geometries, relativistic electron energies,[10] Maxwellian,[11] and uniform[12] initial electron velocities, quantum mechanical effects,[13,14] and a nonzero-electric field at the cathode surface.[15,16] Recently, efforts have been made to determine the seemingly simple space-charge-limited emission[17–22] from an infinite planar cathode whose emission region is now finite in size.

Given the wide interest in and application of space-charge-limited emission, it is instructive to have a simple physical picture on which to base a straightforward derivation of this fundamental limit. We next present a derivation of the Child–Langmuir law that does not require solving the nonlinear differential equation in Eq. (4). During the course of the derivation, new physical insight is gained on space-charge-limited emission scaling.

III. ESTIMATING SPACE-CHARGE-LIMITED EMISSION DENSITY USING VACUUM CAPACITANCE

One of us has recently developed a simple method for modeling classical space-charge-limited emission in the Child–Langmuir diode described previously by considering the vacuum capacitance of the gap.[23] The magnitude of the bound surface charge Q_b on the cathode of a capacitor is $Q_b=CV$, where the capacitance C for the parallel plate, with a vacuum gap (no space charge) of area A is $C=\varepsilon_0 A/D$. Let us suppose that the negative bound charge is liberated from the cathode at zero-initial velocity and is allowed to traverse the gap as negative free charge Q_f. As a first-order approximation, we will assume that the free charge entering the gap is equal to the bound charge previously stored. Thus,

$$Q_b=\varepsilon_0 AV/D\approx Q_f. \tag{8}$$

The validity of this assumption will be addressed in Sec. IV but, for now, it is reasonable to posit that the total emitted free charge should be proportional to the applied voltage and be of the same magnitude as the previously bound charge. The charge Q_f will completely traverse the gap during a characteristic transit time τ given by:

$$\tau=D/v_{\mathrm{ave}}, \tag{9}$$

where v_{ave} is the time-averaged velocity of the charge during the transit. We assume that the charge leaves the cathode with negligible initial velocity, and, from energy conservation, we know that upon reaching the anode, the final velocity of the charge $v_{\mathrm{max}}=\sqrt{2eV/m}$. Again, to first-order only, we may approximate v_{ave} by taking the simple arithmetic mean of the initial and final velocities:

$$v_{\mathrm{ave}}\approx\frac{(0+v_{\mathrm{max}})}{2}=\frac{1}{2}\sqrt{\frac{2eV}{m}}. \tag{10}$$

The accuracy of this approximation also will be addressed in Sec. IV, but it is certain that v_{ave} must lie between zero and v_{max}. Now, the total electron current I traversing the gap is simply:

$$I\approx\frac{Q_f}{\tau}=\frac{\varepsilon_0 AV}{D}\frac{1}{2D}\sqrt{\frac{2eV}{m}}, \tag{11}$$

so the estimated electron current density J is then

$$J=\frac{1}{2}\varepsilon_0\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{D^2}, \tag{12}$$

which is remarkably close to the result for J_{SCL} derived in Eq. (7).

Although this simple model is not in exact agreement with the solution of the Eq. (4), it provides physical insight into the familiar $V^{3/2}/D^2$ scaling of J_{SCL}. Because the bound charge Q_b, and hence the free charge Q_f, are proportional to

V and the transit time is proportional to $V^{-1/2}$, the emission density must be proportional to $V^{3/2}$. That is, as the voltage is increased, not only is there more charge in the gap, but the charge also moves more quickly. (Note that this well known scaling argument holds not just for infinite parallel plates, but also for any arbitrary geometry.) In addition, because the bound charge Q_b, and hence the free charge Q_f, are inversely proportional to D and the transit time is proportional to D, the emission density must be proportional to D^{-2}. That is, as the gap distance increases, the vacuum capacitance (and, hence, the induced free charge) decreases linearly, while the time needed to traverse the gap increases linearly. (This scaling argument does not remain the same for all geometries, because the vacuum capacitance is no longer inversely proportional to the electrode separation.) In contrast to the conditions imposed by Child and Langmuir to reach their solution, recall that here the only required information for finding an approximate J is the vacuum capacitance and the arithmetic mean of the initial and final charge velocities.

We next discuss how we can use this vacuum capacitance model to solve for $\phi(z)$ and more accurately determine J.

IV. FINDING THE EXACT $\phi(z)$ AND J_{SCL}

For this infinite parallel-plate diode, charge conservation requires that $J=\rho(z)v(z)$, which is a constant across the entire gap from $z=0$ to $z=D$; $\rho(z)$ and $v(z)$ are the electron charge density and velocity, respectively, in the gap. From Eq. (12), we know that the approximate J follows

$$J=\frac{1}{2}\varepsilon_0\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{D^2}\propto\frac{\phi^{3/2}(z)}{z^2}. \tag{13}$$

Because J is a constant for all z, from Eq. (13) we must have $\phi(z)\propto z^{4/3}$. If we now apply the physical condition imposed by the anode electrode, $\phi(D)=V$, we conclude

$$\phi(z)=V\left(\frac{z}{D}\right)^{4/3}, \tag{14}$$

so that

$$\phi'(z)=\frac{4}{3}\frac{V}{D}\left(\frac{z}{D}\right)^{1/3}, \tag{15}$$

and

$$\phi''(z)=\frac{4}{9}\frac{V}{D^2}\left(\frac{z}{D}\right)^{-2/3}. \tag{16}$$

In this manner, we have completely solved for the potential inside the Child–Langmuir diode without having to solve the nonlinear second-order differential equation presented by Poisson's equation. Poisson's equation dictates that the electron charge density be given by

$$\rho(z)=\frac{4}{9}\varepsilon_0\frac{V}{D^2}\left(\frac{z}{D}\right)^{-2/3}. \tag{17}$$

We recall that energy conservation gives $v=\sqrt{2e\phi/m}$ and arrive at the current density

$$J=\rho(z)v(z)=\frac{4}{9}\varepsilon_0\frac{V}{D^2}\left(\frac{z}{D}\right)^{-2/3}\sqrt{\frac{2e\phi(z)}{m}}, \tag{18}$$

which, when evaluated at the anode, gives

$$J=\frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{D^2}. \tag{19}$$

In this way, our capacitive model now exactly agrees with the Child–Langmuir result, J_{SCL}. Surprisingly, we have even arrived at the full space-charge-limited solution without requiring that the electric field be driven to zero at the cathode surface. [Of course, any $\phi(z)\propto z^{1+\delta}$ will result in a vanishing cathode electric field for all positive δ.] The capacitive model and energy conservation alone give the scaling of the potential, while the application of charge conservation and the anode boundary potential gives the exact result.

V. MODIFICATIONS TO INITIAL ESTIMATES

Armed with the results of Sec. III, we now address the validity of the assumptions made in Sec. II about the magnitude of the free charge and the average velocity of the charge. To determine the true total electron charge in the gap, we recognize from Eq. (14) that the electric field $E(z)$ is given by

$$E(z)=-\phi'(z)=-\frac{4}{3}\frac{V}{D}\left(\frac{z}{D}\right)^{1/3}, \tag{20}$$

so $E(0)=0$ and $E(D)=-4V/3D$. Thus, the total bound negative surface charge on the cathode is zero, and the total bound positive surface charge on the anode is

$$Q_b=\sigma_b A=\varepsilon_0|E(D)|A=\frac{4}{3}\frac{\varepsilon_0 AV}{D}, \tag{21}$$

which, for this charge-neutral system, must be equal in magnitude to the total free negative charge. Hence, instead of the approximation used in Eq. (8), we now have

$$Q_f=\frac{4}{3}\frac{\varepsilon_0 AV}{D}, \tag{22}$$

for the total electron charge traversing the gap. To determine the time-averaged velocity of this charge, we make use of charge conservation $J=\rho(z)v(z)$ and the solution for $\rho(z)$, Eq. (17), to find that the velocity must scale as $z^{2/3}$. Thus,

$$v(z)=v_{\max}\left(\frac{z}{D}\right)^{2/3}=\sqrt{\frac{2eV}{m}}\left(\frac{z}{D}\right)^{2/3}=\frac{dz}{dt}, \tag{23}$$

where $v(0)=0$ for electrons initially at rest at the cathode. Equation (23) can be readily integrated to find the position as a function of time $z(t)$:

$$3z^{1/3}=v_{\max}D^{-2/3}t+K, \tag{24}$$

where the constant of integration $K=0$, which is found by looking at the cathode boundary, that is, $z=0$ at $t=0$. The transit time τ is then found from Eq. (24) by looking at the anode boundary, that is, $z=D$ at $t=\tau$:

$$3D^{1/3}=v_{\max}D^{-2/3}\tau, \tag{25}$$

or

$$\tau=3D/v_{\max}. \tag{26}$$

Hence, instead of the approximate v_{ave} used in Eq. (10), it is seen that the actual time-averaged velocity must be

$$v_{\mathrm{ave}}=\frac{v_{\max}}{3}=\frac{1}{3}\sqrt{\frac{2eV}{m}}. \tag{27}$$

If we use the exact values for the free charge and average velocity in Eqs. (22) and (27), we find that our capacitance-based derivation for J in Eq. (12) is now in exact agreement with the result of Child and Langmuir for J_{SCL}.

VI. CONCLUSIONS

The capacitive method of deriving classical space-charge-limited emission not only avoids the need to solve a nonlinear differential equation, but also provides additional physical insight into the $V^{3/2}/D^2$ scaling of the Child–Langmuir law. In the Child–Langmuir solution, Poisson's equation for $\phi(z)$ must be completely solved so that J_{SCL} can be determined. In contrast, we first arrive at an approximate J that provides the appropriate scaling for $\phi(z)$, so that the exact solution can then be readily determined by simply applying the boundary potentials at the cathode and anode. The physical condition of the electric field being driven to zero at the cathode is not required in this derivation; nevertheless, this condition is still satisfied by our solution for $\phi(z)$.

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