MA-106 Linear Algebra

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D1 - Lecture 16

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Recall: Formula for Determinant

- **Using Permutations:** For $n \times n$ matrix $A = (a_{ij})$,

$$\det(A) = \sum_{\text{all permutations } P} (a_{1\alpha_1} \ldots a_{n\alpha_n}) \det(P).$$

Here a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ corresponds to the permutation matrix $P = \begin{bmatrix} e_{i_1}^T \\ \vdots \\ e_{i_n}^T \end{bmatrix}$.

- **Using Cofactors:** Let C_{1j} be the coefficient of a_{1j} in the expansion

$$\det(A) = \sum_{\text{all permutations } P} (a_{1\alpha_1} \ldots a_{n\alpha_n}) \det(P)$$

- Then $\boxed{\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}.}$

Note: $C_{1j} = (-1)^{1+j}\det(M_{1j})$, where M_{1j} is obtained from A by deleting the 1$^{\text{st}}$ row and j^{th} column.

Recall: Formula for Determinant

- **Expansion along ith row:** If C_{ij} is the coefficient of a_{ij} in the formula of $\det(A)$, then $\boxed{\det(A) \;=\; a_{i1}\,C_{i1} + \ldots + a_{in}\,C_{in}}$.

Note: $C_{ij} = (-1)^{i+j}\det(M_{ij})$, where M_{ij} is obtained from A by deleting i-th row and j-th column.
Idea: Using $i-1$ row exchanges, make A_{i*} the first row.

- **Expansion along jth column:** $C_{ij}(A^T) = C_{ji}(A)$ and $\det(A) = \det(A^T) \Rightarrow \quad \boxed{\det(A) = a_{1j}\,C_{1j}(A) + \ldots + a_{nj}\,C_{nj}(A).}$

Computing determinants: Exercise

Example: Let $F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & -1 \\ & & & 1 & 1 \end{vmatrix}$ be a $(1,1,-1)$ tri-diagonal $n \times n$ matrix. Expanding along the first row, we get

$$F_n = F_{n-1} + (-1)^{1+2}(-1) \begin{vmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & 1 & 1 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & -1 \\ & & & & 1 & 1 \end{vmatrix} = F_{n-1} + F_{n-2},$$

by expanding along first column.

Since $F_1 = \_\_$, $F_2 = \_\_$, the sequence F_n is $\_\_, \_\_, \_\_, \ldots$.

Computing determinants: Examples

Find the determinants for the following examples

- $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. Expand along 1st column.
- $B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Expand along 2nd row.
- $E = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$. Find the cofactor matrix C. Compute EC^T.

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad EC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note: $\det(E) = 4$ and $EC^T = 4I \Rightarrow E^{-1} = \dfrac{1}{\det(E)} C^T$.

Applications: 1. Computing A^{-1}

If $C = (C_{ij})$: cofactor matrix of A, then $\boxed{A\,C^T = \det(A)\,I}$ i.e.,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \dots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \det(A) \end{bmatrix}$$

Proof. We have seen that $a_{i1}C_{i1} + \ldots + a_{in}C_{in} = \det(A)$. Now

$$a_{11}C_{21} + a_{12}C_{22} + \ldots + a_{1n}C_{2n} = \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{11} & \dots & a_{1n} \\ a_{31} & \dots & a_{3n} \\ . & & . \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = 0.$$

Similarly, if $i \neq j$, then $a_{i1}C_{j1} + a_{i2}C_{j2} + \ldots + a_{in}C_{jn} = 0$. □

Remark. If A is invertible, then $A^{-1} = \dfrac{1}{\det(A)} C^T$.

For $n \geq 4$, this is *not* a good formula to find A^{-1}.
Use elimination to find A^{-1} for $n \geq 4$.

Reading Slide - Exercise

Find the inverse of $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} & \mathbf{1} \end{bmatrix}$.

$$\det(A) = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} + (-1)^{1+2}(2) \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

$C_{11} = 3$, $C_{12} = 0$, $C_{13} = 0$

$C_{21} = -2$, $C_{22} = 1$, $C_{23} = -4$

$C_{31} = 0$, $C_{32} = 0$, $C_{33} = 3$.

Hence $A^{-1} = \dfrac{1}{\det(A)} C^T = \dfrac{1}{3} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}$

Check. $AA^{-1} = I$.

Applications: 2. Solving $Ax = b$

Cramer's rule: If A is invertible, the $Ax = b$ has a unique solution.

$$x = A^{-1}b = \frac{1}{\det(A)}\, C^T b = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Hence $x_j = \dfrac{1}{\det(A)}(b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}) = \dfrac{1}{\det(A)}\det(B_j)$,

where B_j is obtained by replacing j^{th} column of A by b, and $\det(B_j)$ is computed along the j^{th} column.

Extra Reading - Applications: 3. A Formula for Pivots

Observation: If row exchanges are not required, then the first k pivots are determined by the top-left $k \times k$ submatrices $\widetilde{A}_k$ of A.

Example. If $A = [a_{ij}]_{3\times3}$, then $\widetilde{A}_1 = (a_{11})$, $\widetilde{A}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and

$\widetilde{A}_3 = A$.

Assume the pivots are $d_1, \ldots, d_n$, obtained without row exchange. Then

- $\det(\widetilde{A}_1) = a_{11} = d_1$
- $\det(\widetilde{A}_2) = d_1\, d_2 = \det(A_1)\, d_2$
- $\det(\widetilde{A}_3) = d_1\, d_2\, d_3 = \det(A_2)\, d_3$ *etc.*,
- If $\det(\widetilde{A}_k) = 0$, then we need a row exchange in elimination.
- Otherwise the k-th pivot is $\boxed{d_k = \det(\widetilde{A}_k)/\det(\widetilde{A}_{k-1})}$

Extra Reading - Example

Example. Find if row exchange is required in Gauss Elimination of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 9 \end{pmatrix}$ and find the pivots.

Solution. $\widetilde{A}_1 = (1),\ \widetilde{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix},\ \widetilde{A}_3 = A$

$$\det(\widetilde{A}_1) = 1, \quad \det(\widetilde{A}_2) = 3$$

$$\det(\widetilde{A}_3) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 3 & 8 \end{vmatrix} = 15$$

Therefore row exchange is not required in elimination and the pivots are $d_1 = 1,\ d_2 = 3/1 = 3,\ d_3 = 15/3 = 5.$
Verify this directly by elimination!

Summary: Determinants

Let A and B $n \times n$, and t a scalar.

- $\det(A+B) \neq \det(A) + \det(B)$, and $\det(tA) = t^n \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(A) = \det(A^T)$.
- If A is orthogonal, i.e., $AA^T = I$, then $\det(A) =$
- If $A = [a_{ij}]$ is triangular, then $\det(A) =$
- A is invertible $\Leftrightarrow \det(A) \neq 0$. If this happens, then $\det(A^{-1}) =$
- If A and B are similar, i.e., $B = S^{-1}AS$ for an invertible matrix S, then $\det(B) =$
- If A is invertible, and $d_1, \ldots, d_n$ are the pivots of A, then $\det(A) =$

Eigenvalues and Eigenvectors: Motivation

• Solve for the differential equation for u: $\quad du/dt = 3u$.

The solution is $\quad u(t) = c\,e^{3t}, \quad c \in \mathbb{R}$
With initial condition $u(0) = 2$, the solution is $u(t) = 2e^{3t}$.

• Consider the system of linear 1$^{\text{st}}$ order differential equations (ODE) with constant coefficients:

$$\begin{aligned} du_1/dt &= 4u_1 - 5u_2, \\ du_2/dt &= 2u_1 - 3u_2, \end{aligned}$$

How does one find the solution?
• Write the system in matrix form $\quad \boxed{du/dt = Au,}$

where $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$.

• Assuming the solution is $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = e^{\lambda t}\,v = \begin{pmatrix} e^{\lambda t}\,x \\ e^{\lambda t}\,y \end{pmatrix}$, where

$v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we need to find λ and v.

Eigenvalues and Eigenvectors: Definition

We have $u_1' = 4u_1 - 5u_2$, $u_2' = 2u_1 - 3u_2$, where
$u_1(t) = e^{\lambda t}x, u_2(t) = e^{\lambda t}y$

$$\lambda\, e^{\lambda t}x = 4e^{\lambda t}x - 5e^{\lambda t}y,$$
$$\lambda\, e^{\lambda t}y = 2e^{\lambda t}x - 3e^{\lambda t}y.$$

Cancelling $e^{\lambda t}$, we get

Eigenvalue problem: Find λ and $v = (x, y)^T$ satisfying

$$4x - 5y = \lambda x,$$

$$2x - 3y = \lambda y.$$

In the matrix form, it is $\boxed{Av = \lambda v}$.
This equation has two unknowns, λ and v.

If there exists a λ such that $Av = \lambda v$ has a non-zero solution v, then λ is called an eigenvalue of A and all *nonzero* v satisfying $Av = \lambda v$ are called eigenvectors of A associated to λ.

Q: Given A $n \times n$, how does one find its eigenvalues and eigenvectors?

Eigenvalues and Eigenvectors: Solving $Ax = \lambda x$

- Write $Av = \lambda v$ as $\boxed{(A - \lambda I)v = 0.}$
- λ is an eigenvalue of A

  $\Leftrightarrow$ there is a nonzero v in the nullspace of $A - \lambda I$

  $\Leftrightarrow N(A - \lambda I) \neq 0$, i.e., dim $(N(A - \lambda I)) \geq 1$,

  $\Leftrightarrow A - \lambda I$ is singular

  $\Leftrightarrow \det(A - \lambda I) = 0$.

- $\det(A - \lambda I)$ is a polynomial in the variable λ of degree n. Hence it has at most n roots $\Rightarrow A$ has atmost n eigenvalues.
- $\det(A - \lambda I)$ is called the characteristic polynomial of A.
- If λ is an eigenvalue of A, then the nullspace of $A - \lambda I$ is called the eigenspace of A associated to eigenvalue λ.
- $\boxed{\lambda = 0 \text{ is an eigenvalue of } A \Leftrightarrow \det(A) = 0 \Leftrightarrow A \text{ is singular.}}$

Eigenvalues and Eigenvectors: Example

To summarise: An eigenvalue of A is a root of its characteristic polynomial, and any non-zero vector
in the corresponding eigenspace is an associated eigenvector.

Recall: The ODE system we want to solve is

$$u_1' = 4u_1 - 5u_2, \quad u_1(0) = 8, \qquad u_2' = 2u_1 - 3u_2, \quad u_2(0) = 5.$$

The solutions are $u_1(t) = e^{\lambda t}x, u_2(t) = e^{\lambda t}y$, where $(x, y)^T$ is a solution of:

$$\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \qquad (Av = \lambda v)$$

The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix} \\ &= (4-\lambda)(-3-\lambda) + 10 \\ &= \lambda^2 - \lambda - 2 \quad = (\lambda+1)(\lambda-2) \end{aligned}$$

The eigenvalues of A are $\boxed{\lambda_1 = -1, \lambda_2 = 2.}$

Eigenvalues and Eigenvectors: Example

Eigenvectors v_1 and v_2 associated to $\lambda_1 = -1$ and $\lambda_2 = 2$ respectively, are in:
$N(A - \lambda_1 I) = N(A + I)$, and $N(A - \lambda_2 I) = N(A - 2I)$.
Solving $(A + I)v = 0$, i.e., $\begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$, we get
$N(A + I) = \left\{ \begin{pmatrix} y \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}$ and $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to $\lambda_1 = -1$.

Similarly, solving $(A - 2I)v = 0$ gives $N(A - 2I) = \left\{ \begin{pmatrix} \frac{5y}{2} \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}$.

In particular, $v_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is an eigenvector associated to $\lambda_2 = 2$.

Thus, the system $du/dt = Au$ has two special solutions

$$e^{-t} v_1 \text{ and } e^{2t} v_2.$$

Reading Slide - Complete Solution to ODE

Note: When two functions satisfy $du/dt = Au$, then so do their linear combinations.

Complete solution: $u(t) = c_1 e^{-t} v_1 + c_2 e^{2t} v_2$,

$$\text{i.e. } \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

$$\text{i.e. } u_1(t) = c_1 e^{-t} + 5c_2 e^{2t}, \quad u_2(t) = c_1 e^{-t} + 2c_2 e^{2t}.$$

If we put initial conditions (IC) $u_1(0) = 8$ and $u_2(0) = 5$, then

$$c_1 + 5c_2 = 8, \quad c_1 + 2c_2 = 5 \Rightarrow c_1 = 3,\ c_2 = 1.$$

Hence the solution of the original ODE system with the given IC is

$$u_1(t) = 3e^{-t} + 5e^{2t}, \quad u_2(t) = 3e^{-t} + 2e^{2t}.$$

Examples

In some cases it is easy to find the eigenvalues.

Example: $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ is diagonal. Characteristic polynomial $(3-\lambda)(2-\lambda)$.

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$.
Eigenvectors: $(A - 3I)v_1 = 0 \Rightarrow Av_1 = 3v_1 \Rightarrow v_1 = e_1$
Similarly, an eigenvector associated to λ_2 is $v_2 = e_2$

Further, $\mathbb{R}^2$ has a basis consisting of eigenvectors of A: $\{e_1, e_2\}$.

General case: If A is a diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then

Eigenvalues: $\lambda_1, \cdots, \lambda_n$ Eigenvectors: $e_1, \cdots, e_n$,

which form a basis for $\mathbb{R}^n$.

Examples

Example: Projection onto the line $x = y$: $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

$v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ projects onto itself $\Rightarrow \lambda_1 = 1$ with eigenvector v_1.

$v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T \mapsto 0 \Rightarrow \lambda_2 = 0$ with eigenvector v_2.

Further, $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$.

Q: Do a collection of eigenvectors always form a basis of $\mathbb{R}^n$?