PH108 (Division 1) Lectures on
TUESDAY SLOT 1B (9:30 -10:25 AM)
THURSDAY SLOT 1C (10:35-11:30 AM) **LA 101**

Instructor (D1): Kantimay Das Gupta : kdasgupta@phy

Reference texts:

D J Griffiths : Introduction to electromagnetism
Feynman Lectures: vol 2

Vector Analysis (Schaum series) M Spiegel
Mathematical methods : Pipes & Harvil

Several other classic texts:
Panofsky and Philips
J D Jackson

ATTENDANCE : 80% REQUIRED
EVALUATION Quiz1=15 : Midsem=30 : Quiz2=15 : Endsem=40 (typical)

Some key aspects of Classical Electromagnetic Theory

Basic principles known for ~150 years.

Mature subject with a well defined structure.

Regime of validity well undesrtood.

Great success: explaining propagation & generation of electromagnetic radiation,
Forces of adhesion and cohesion....examples?

First example of a classical field theory.
It showed that particles and fields both carry energy and momentum
….What is meant by a "field theory" / " action-at-a-distance-theory" ?

It played a remarkable role in the discovery of special relativity.

Fails when we go to atomic / nuclear scale

Gravity and electromagnetism are markedly different too, though both
are "inverse square force".

What is a field ? What are the typical questions one asks?

A quantity defined or measured over a certain area/volume of space.

Scalar field Temperature defined over a region $T(x,y,z)$

Vector field Electric, Magnetic field : $\mathbf{E}(x,y,z)$ $\mathbf{B}(x,y,z)$
velocity of water $\mathbf{v}(x,y,z)$ in a pipe, river, ocean

Matrix/Tensor field Stress, Strain inside a material like a concrete beam.
With every point a matrix like object is associated.

A field is also like an object with a large number of degrees of freedom.

How is the field created? What is the "source" ?
How does the field affect particles in it (Interaction of field with matter)?

A systematic way of handling co-ordinate systems : Part 1

Many types of co-ordinates are used, so that we can use the natural symmetry of a problem.

Equations would have the simplest form and minimum number of free variables if the co-ordinate system is chosen intelligently.

How to define a co-ordinate system?

Few typical systems:

Plane Polar
Spherical Polar
Cylindrical

How to define your own if you need?

How to write arc-length, area element & volume element?

Co-ordinate transformation and the "Jacobean", connection to the "metric"

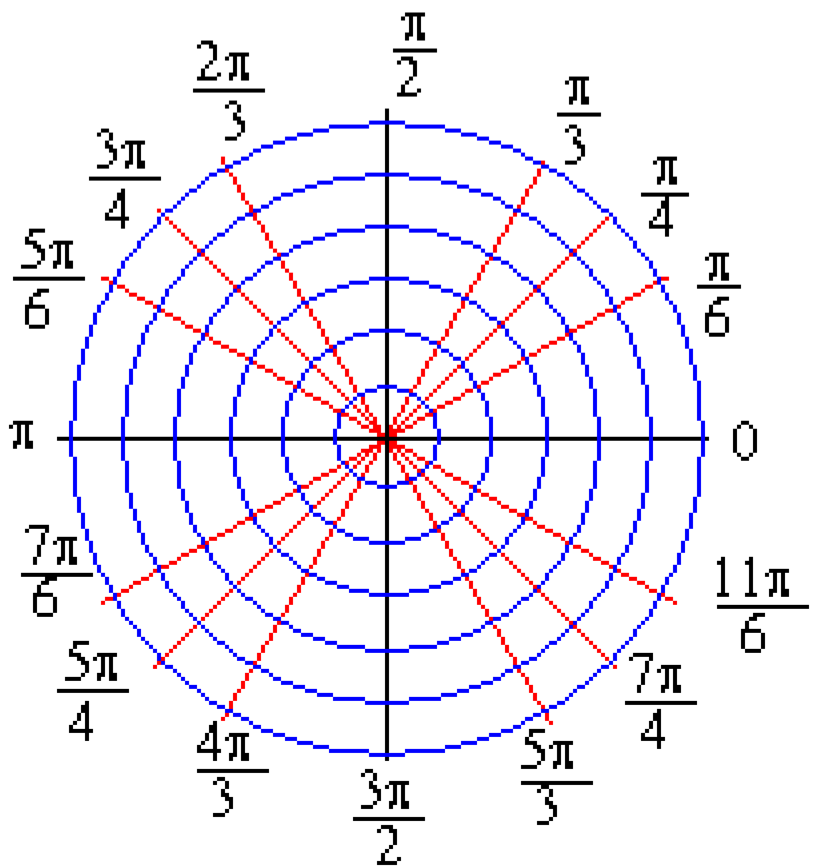
See Spiegel's book for a nice list of systems....

Plane Polar (r,θ) in detail

STEP 1: Write down the relation with (x,y) co-ordinates

$$x = r \cos\theta$$
$$y = r \sin\theta$$

STEP 2: Draw the co-ordinate grid



How do r=constant lines look?

How do θ= constant lines look?

Plane Polar (r,θ) in detail

STEP 3: What happens when the independent variables are changed infinitesimally

$$\begin{aligned}\delta x &= \cos\theta\,\delta r - r\sin\theta\,\delta\theta\\ \delta y &= \sin\theta\,\delta r + r\cos\theta\,\delta\theta\end{aligned}$$

STEP 4: Which direction would we move, if only one variable was changed?

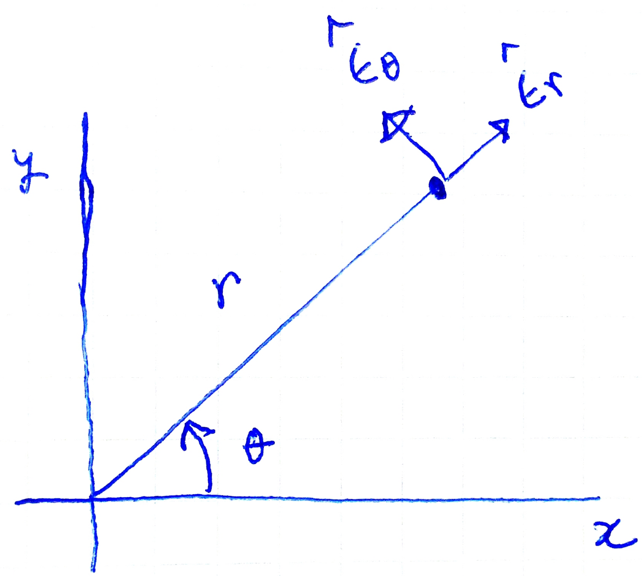
$$\delta\theta=0$$

$$\begin{aligned}\boldsymbol{i}\,\delta x+\boldsymbol{j}\,\delta y &= (\boldsymbol{i}\cos\theta+\boldsymbol{j}\sin\theta)\,\delta r\\ &= \hat{\boldsymbol{\epsilon}}_r\,\delta r\end{aligned}$$

$$\delta r=0$$

$$\begin{aligned}\boldsymbol{i}\,\delta x+\boldsymbol{j}\,\delta y &= (-\boldsymbol{i}\sin\theta+\boldsymbol{j}\cos\theta)\,r\,\delta\theta\\ &= \hat{\boldsymbol{\epsilon}}_\theta\,r\,\delta\theta\end{aligned}$$

$$\delta\vec{r} = \hat{\boldsymbol{\epsilon}}_r\,\delta r+\hat{\boldsymbol{\epsilon}}_\theta\,r\,\delta\theta$$



$$\hat{\boldsymbol{\epsilon}}_r\cdot\hat{\boldsymbol{\epsilon}}_\theta=0$$

Curvilinear but still orthogonal

Plane Polar (r,θ) in detail

STEP 5: What happens to the element of area?

i.e take a small step in the direction and a small step in the direction

What is the "infinitesimal" area enclosed by these two perpendicular vectors?

$$
\begin{aligned}
dA &= \left| \boldsymbol{\epsilon}_r \delta r \times \boldsymbol{\epsilon}_\theta \delta \theta \right| \\
&= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \delta\theta\,\delta r \\
&= r\,\delta\theta\,\delta r
\end{aligned}
$$

STEP 6: What happens to the element of distance or arclength?

$$
\begin{aligned}
ds^2 &= \delta\vec{r}\,.\,\delta\vec{r} \\
&= dr^2 + r^2 d\,\theta^2
\end{aligned}
$$

In orthogonal co-ordinates there will be no cross terms in the arclength expression.

Plane Polar (r,θ) in detail

STEP 7: Now suppose a SCALAR function of co-ordinates is defined (like Temperature over a region), T(x,y,z)

We change our position by a small VECTOR $\delta \boldsymbol{r}$, and ask dT = ?

We want a function such that :

$$\begin{aligned}
\delta T &= \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta \\
&= [some \quad fn].\delta\vec{r} \\
&= [some \quad fn].\left(\hat{\boldsymbol{\epsilon}}_r \delta r + \hat{\boldsymbol{\epsilon}}_\theta r\,\delta\theta\right)
\end{aligned}$$

$$[some \quad fn] = \hat{\boldsymbol{\epsilon}}_r \frac{\partial T}{\partial r} + \hat{\boldsymbol{\epsilon}}_\theta \frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$The \quad combination \quad is \quad called \quad gradient$$

$$\vec{\nabla} = \hat{\boldsymbol{\epsilon}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{\epsilon}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}$$

Gradient is the generalisation of the derivative in 1 dimension

Prove :

grad T is perpendicular to surfaces of constant T

What form would grad T take in cartesian co-ordinates?

Plane Polar (r,θ) in detail

STEP 8: What are velocity and acceleration components, when a particle's motion
Is described using polar co-ordinates?

$$\vec{v} = \frac{\delta \vec{r}}{\delta t}$$

$$= \frac{\left(\hat{\boldsymbol{\epsilon}}_r \delta r + \hat{\boldsymbol{\epsilon}}_\theta r \delta \theta\right)}{\delta t}$$

$$= \hat{\boldsymbol{\epsilon}}_r \frac{dr}{dt} + \hat{\boldsymbol{\epsilon}}_\theta r \frac{d\theta}{dt}$$

$$\vec{a} = \frac{d}{dt}\left(\hat{\boldsymbol{\epsilon}}_r \frac{dr}{dt} + \hat{\boldsymbol{\epsilon}}_\theta r \frac{d\theta}{dt}\right)$$

Using our result from STEP 4...

$$\begin{pmatrix} \hat{\boldsymbol{\epsilon}}_r \\ \hat{\boldsymbol{\epsilon}}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \end{pmatrix}$$

hence

$$\begin{pmatrix} \dot{\hat{\boldsymbol{\epsilon}}}_r \\ \dot{\hat{\boldsymbol{\epsilon}}}_\theta \end{pmatrix} = \dot{\theta} \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \end{pmatrix}$$

Unlike cartesian unit vectors the unit vectors here are not constant and must be differentiated themselves.

$$\dot{\hat{\boldsymbol{\epsilon}}}_r = ?$$

$$\dot{\hat{\boldsymbol{\epsilon}}}_\theta = ?$$

Plane Polar (r,θ) in detail

Using the last two results:

$$\begin{pmatrix} \dot{\hat{\epsilon}}_r \\ \dot{\hat{\epsilon}}_\theta \end{pmatrix} = \dot{\theta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_r \\ \hat{\epsilon}_\theta \end{pmatrix} \qquad \begin{cases} \dot{\hat{\epsilon}}_r = \dot{\theta}\,\hat{\epsilon}_\theta \\ \dot{\hat{\epsilon}}_\theta = -\dot{\theta}\,\hat{\epsilon}_r \end{cases}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\hat{\epsilon}_r \dot{r} + \hat{\epsilon}_\theta r \dot{\theta}\right)$$

$$= \hat{\epsilon}_\theta \dot{\theta} \dot{r} + \hat{\epsilon}_r \ddot{r} - \hat{\epsilon}_r \dot{\theta} r \dot{\theta} + \hat{\epsilon}_\theta \dot{r} \dot{\theta} + \hat{\epsilon}_\theta r \ddot{\theta}$$

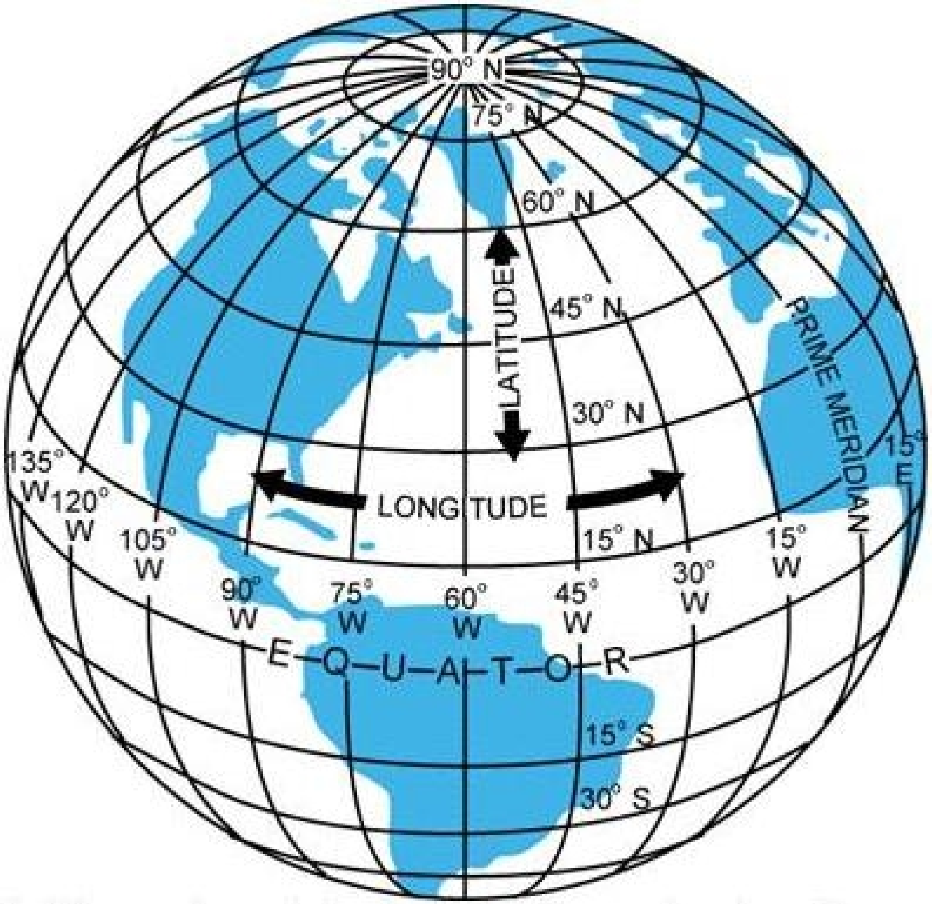
$$= \left(\ddot{r} - \dot{\theta}^2 r\right)\hat{\epsilon}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\epsilon}_\theta$$

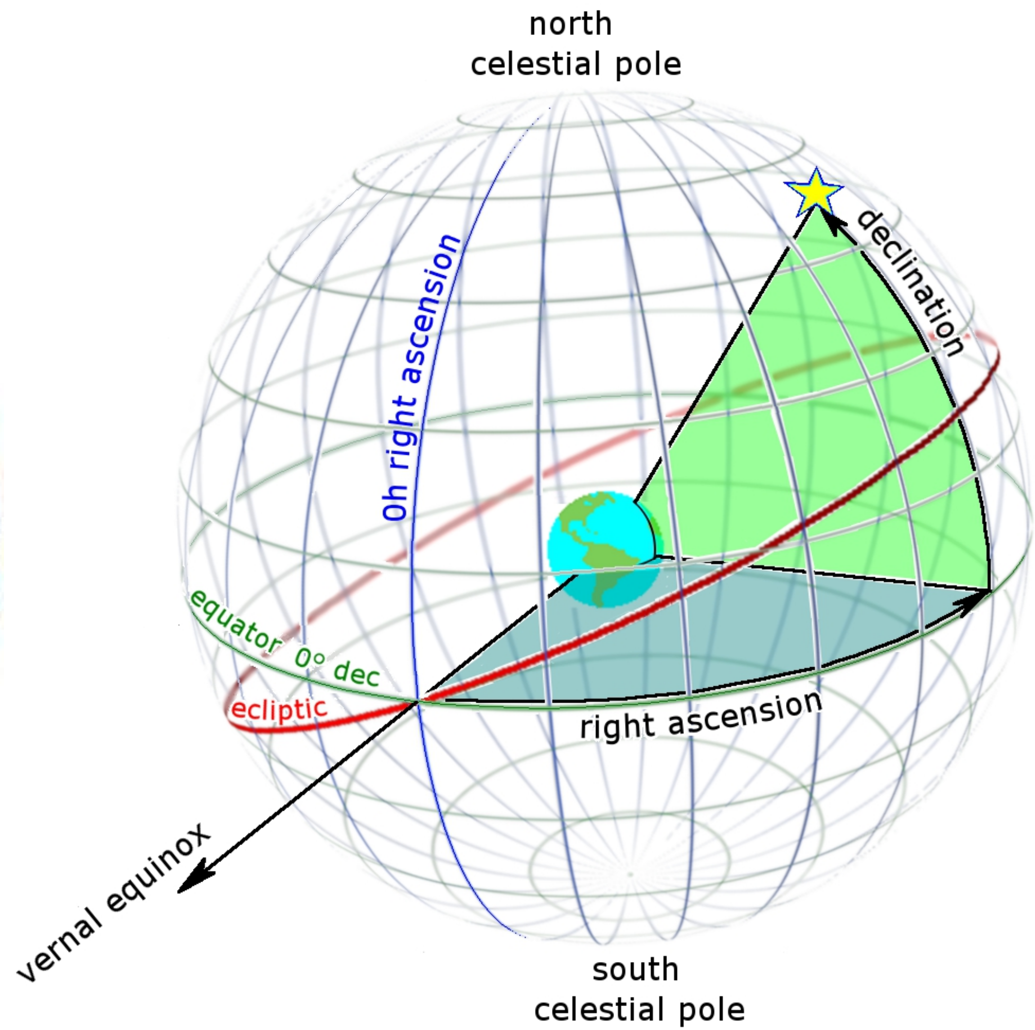
What are the physical meanings of the various terms in the result for accelaration?

If the force on the particle is "central", then which quantity is conserved?

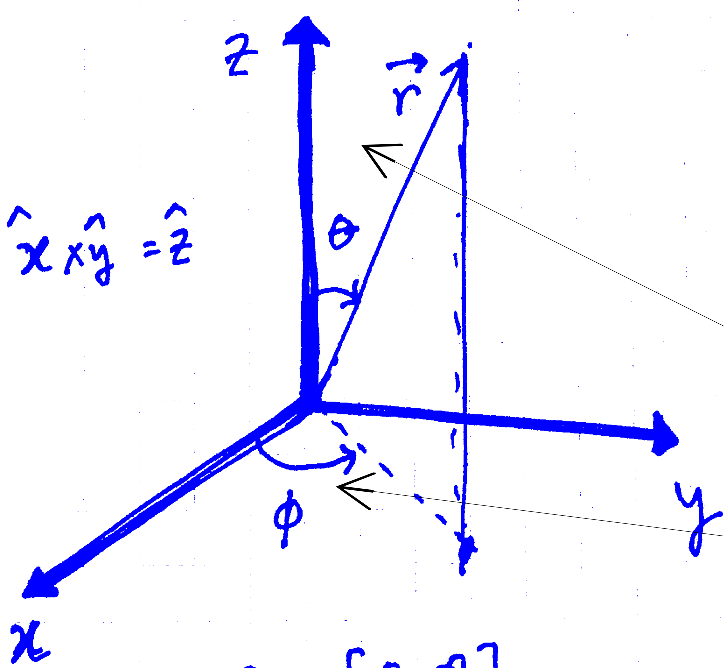
You are given an arbitrary vector in cartesian $(\hat{\boldsymbol{i}} F_x + \hat{\boldsymbol{j}} F_y)$.
How will you go over to $(\hat{\epsilon}_r F_r + \hat{\epsilon}_\theta F_\theta)$?
What can you say about the matrix connecting the two sets and the inverse relation ?

Spherical Polar (r,θ,ϕ) : two obvious examples





Spherical Polar (r,θ,φ)



$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$r = [0, \infty]$

$\theta = [0, \pi]$

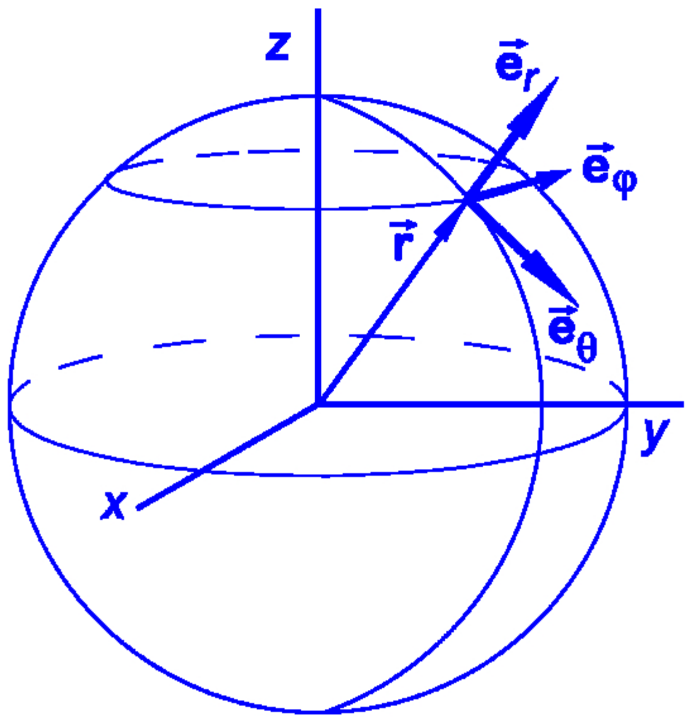
$\phi = [0, 2\pi]$

$$
\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} \sin\theta\ \cos\phi & r\ \cos\theta\ \cos\phi & -r\ \sin\theta\ \sin\phi \\ \sin\theta\ \sin\phi & r\ \cos\theta\ \sin\phi & r\ \sin\theta\ \cos\phi \\ \cos\theta & -r\ \sin\theta & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta\phi \end{pmatrix}
$$

Spherical Polar (r,θ,ϕ) : unit vectors, volume element, arc length

$$
\begin{aligned}
\hat{\epsilon}_r &= \sin\theta\cos\phi\,\hat{\boldsymbol{i}} + \sin\theta\sin\phi\,\hat{\boldsymbol{j}} + \cos\theta\,\hat{\boldsymbol{k}}\\
\hat{\epsilon}_\theta &= \cos\theta\cos\phi\,\hat{\boldsymbol{i}} + \cos\theta\sin\phi\,\hat{\boldsymbol{j}} + -\sin\theta\,\hat{\boldsymbol{k}}\\
\hat{\epsilon}_\phi &= -\sin\phi\,\hat{\boldsymbol{i}} + \cos\phi\,\hat{\boldsymbol{j}}
\end{aligned}
$$

Can you invert this set of equations? It is easy!



$$
\begin{aligned}
\delta\vec{r} &= \hat{\epsilon}_r\,\delta r + \hat{\epsilon}_\theta\, r\,\delta\theta + \hat{\epsilon}_\phi\, r\sin\theta\,\delta\phi\\
ds^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2
\end{aligned}
$$

$$
\begin{aligned}
dV &= \left|\, \hat{\epsilon}_r . (\hat{\epsilon}_\theta r)\times(\hat{\epsilon}_\phi r\sin\theta)\,\right| dr\; d\theta\; d\phi\\
&= r^2\sin\theta\, dr\; d\theta\; d\phi
\end{aligned}
$$

Spherical Polar (r,θ,ϕ) : the area element

$$If \quad r = constant \quad (surface \quad of \quad a \quad sphere) \quad \delta r = 0$$

$$\begin{aligned} dA &= \left| r\,\hat{\boldsymbol{\epsilon}}_\theta \times r\,\sin\theta\,\hat{\boldsymbol{\epsilon}}_\phi \right| d\theta\, d\phi \\ &= r^2 \sin\theta\, d\theta\, d\phi \end{aligned}$$

$$If \quad \theta = constant \quad \delta\theta = 0$$

$$\begin{aligned} dA &= \left| \hat{\boldsymbol{\epsilon}}_r \times r\,\sin\theta\,\hat{\boldsymbol{\epsilon}}_\phi \right| d r\, d\phi \\ &= r \sin\theta\, d r\, d\phi \end{aligned}$$

$$If \quad \phi = constant \quad (plane \quad polar \quad \text{in} \quad a \quad vertical \quad plane) \quad \delta\phi = 0$$

$$\begin{aligned} dA &= \left| \hat{\boldsymbol{\epsilon}}_r \times r\,\hat{\boldsymbol{\epsilon}}_\theta \right| d r\, d\theta \\ &= r\, d r\, d\theta \end{aligned}$$

Q:
Suppose you were confined on the surface of a sphere – but you were not told that. Would you be able to figure out?

Spherical Polar (r,θ,ϕ)

We still need to express the derivatives $\left(\dot{\hat{\boldsymbol{\epsilon}}}_r, \dot{\hat{\boldsymbol{\epsilon}}}_\theta, \dot{\hat{\boldsymbol{\epsilon}}}_\phi\right)$ in *terms of* $\left(\hat{\boldsymbol{\epsilon}}_r, \hat{\boldsymbol{\epsilon}}_\theta, \hat{\boldsymbol{\epsilon}}_\phi\right)$

$$\begin{pmatrix} \dot{\hat{\boldsymbol{\epsilon}}}_r \\ \dot{\hat{\boldsymbol{\epsilon}}}_\theta \\ \dot{\hat{\boldsymbol{\epsilon}}}_\phi \end{pmatrix} = \dot{\boldsymbol{M}}\,\boldsymbol{M}^T \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_r \\ \hat{\boldsymbol{\epsilon}}_\theta \\ \hat{\boldsymbol{\epsilon}}_\phi \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \quad \boldsymbol{M}^T = \begin{pmatrix} \sin\theta\,\cos\phi & \cos\theta\,\cos\phi & -\sin\phi \\ \sin\theta\,\sin\phi & \cos\theta\,\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\dot{\boldsymbol{M}} = \begin{pmatrix} \cos\theta\,\cos\phi\,\dot{\theta} - \sin\theta\,\sin\phi\,\dot{\phi} & \cos\theta\,\sin\phi\,\dot{\theta} + \sin\theta\,\cos\phi\,\dot{\phi} & -\sin\theta\,\dot{\theta} \\ -\sin\theta\,\cos\phi\,\dot{\theta} - \cos\theta\,\sin\phi\,\dot{\phi} & -\sin\theta\,\sin\phi\,\dot{\theta} + \cos\theta\,\cos\phi\,\dot{\phi} & -\cos\theta\,\dot{\theta} \\ -\cos\phi\,\dot{\phi} & -\sin\phi\,\dot{\phi} & 0 \end{pmatrix}$$

Spherical Polar (r,θ,ϕ)

This appears very messy! But if you work through the matrix multiplication then:

$$\begin{pmatrix} \dot{\hat{\boldsymbol{\epsilon}}}_r \\ \dot{\hat{\boldsymbol{\epsilon}}}_\theta \\ \dot{\hat{\boldsymbol{\epsilon}}}_\phi \end{pmatrix} = \dot{\boldsymbol{M}}\,\boldsymbol{M}^T \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_r \\ \hat{\boldsymbol{\epsilon}}_\theta \\ \hat{\boldsymbol{\epsilon}}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dot{\theta} & \sin\theta\,\dot{\phi} \\ -\dot{\theta} & 0 & \cos\theta\,\dot{\phi} \\ -\sin\theta\,\dot{\phi} & -\cos\theta\,\dot{\phi} & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_r \\ \hat{\boldsymbol{\epsilon}}_\theta \\ \hat{\boldsymbol{\epsilon}}_\phi \end{pmatrix}$$

The result is remarkably simple.

Why are the diagonal terms zero? Can you see the physical implication?

Notice that the matrix connecting the two vectors is anti-symmetric.

This was also the case in the plane polar co-ordinates. But we didn't mention it there.

The problem for velocity and acceleration components can now be completed...

Spherical Polar (r,θ,ϕ) : the gradient

$$\textit{If we have a function } T(r,\theta,\phi) \textit{ then we want}$$

$$\delta T = \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta + \frac{\partial T}{\partial \phi}\delta\phi$$

$$= \vec{\nabla} T . \delta\vec{r}$$

since

$$\delta\vec{r} = \hat{\boldsymbol{\epsilon}}_r \delta r + \hat{\boldsymbol{\epsilon}}_\theta r\delta\theta + \hat{\boldsymbol{\epsilon}}_\phi r\sin\theta\delta\phi$$

$$\textit{we must have}$$

$$\vec{\nabla} T = \hat{\boldsymbol{\epsilon}}_r \frac{\partial T}{\partial r} + \hat{\boldsymbol{\epsilon}}_\theta \frac{1}{r}\frac{\partial T}{\partial \theta} + \hat{\boldsymbol{\epsilon}}_\phi \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

Spherical Polar (r,θ,ϕ) : velocity & acceleration

You should be able to show the following now:

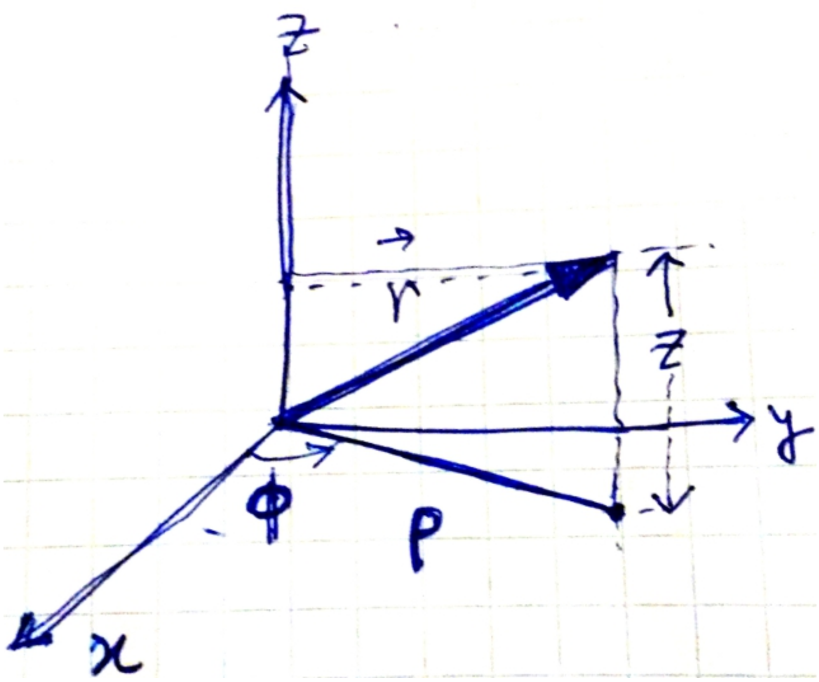
$$\vec{v} = \hat{\epsilon}_r \, \dot{r} + \hat{\epsilon}_\theta \, r \, \dot{\theta} + \hat{\epsilon}_\phi \, r \sin\theta \, \dot{\phi}$$

$$\begin{aligned} \vec{a} = \; & \hat{\epsilon}_r \left(\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\phi}^2 \sin^2\theta \right) + \\ & \hat{\epsilon}_\theta \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} - r\,\dot{\phi}^2 \sin\theta\cos\theta \right) + \\ & \hat{\epsilon}_\phi \left(r\,\ddot{\phi}\sin\theta + 2\,\dot{r}\,\dot{\phi}\sin\theta + 2r\,\dot{\theta}\,\dot{\phi}\cos\theta \right) \end{aligned}$$

We now have all the necessary bits to solve dynamical problems in this co-ordinate

Key step: differentiation of the unit vectors and writing the result in terms of the unit vectors themselves.

Cylindrical polar (ρ,θ,z) : length, area and volume elements



$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

Wires, co-axial cables,
Pipes etc.

$$\hat{\boldsymbol{\epsilon}}_\rho = \cos\phi\,\hat{\boldsymbol{i}} + \sin\phi\,\hat{\boldsymbol{j}}$$
$$\hat{\boldsymbol{\epsilon}}_\phi = -\sin\phi\,\hat{\boldsymbol{i}} + \cos\phi\,\hat{\boldsymbol{j}}$$
$$\hat{\boldsymbol{\epsilon}}_z = \hat{\boldsymbol{k}}$$

$$\delta\vec{r} = \hat{\boldsymbol{\epsilon}}_\rho\,\delta\rho + \hat{\boldsymbol{\epsilon}}_\phi\,\rho\,\delta\phi + \hat{\boldsymbol{\epsilon}}_z\,\delta z$$

Cylindrical polar (ρ,θ,z) : length, area and volume elements

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$dA = \rho\, d\rho\, d\phi \qquad z = constant$$

$$dA = \rho\, d\phi\, dz \qquad \rho = constant$$

$$dA = d\rho\, dz \qquad \phi = constant$$

$volume$

$$dV = \rho\, d\rho\, d\phi\, dz$$

$gradient$

$$\vec{\nabla} = \hat{\epsilon}_\rho \frac{\partial}{\partial \rho} + \hat{\epsilon}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\epsilon}_z \frac{\partial}{\partial z}$$

Follow exactly the same process as we did for spherical polar...

Generic orthogonal curvilinear co-ordinates : unit vector, scale factor

Writing the basic information about orthogonal co-ordinates....

$$d\vec{r} = \hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 du_1 + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 du_2 + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 du_3$$

$$ds^2 = ?$$

$$dV = ?$$

A shorthand compact way of writing co-ordinates

$$d\vec{r} = \sum \hat{\boldsymbol{\epsilon}}_{\boldsymbol{i}} h_i d\, u_i$$

Summation convention :

REPEATED INDEX IMPLIES SUMMATION

$$d\vec{r} = \hat{\boldsymbol{\epsilon}}_{\boldsymbol{i}} h_i d\, u_i$$

Flux and circulation

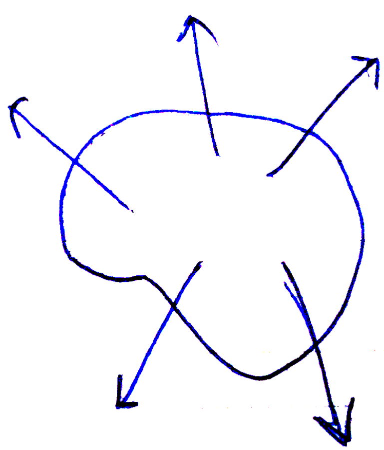
The volume of water flowing out through the SURFACE per unit time

$$\oiint \vec{v} \, . \, d\vec{S}$$

The shape of the surface can be arbitrary

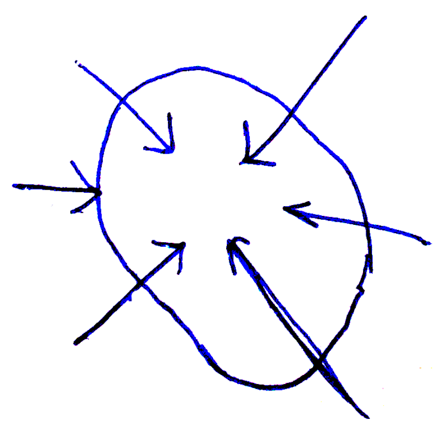
dS points OUTWARD

This has a unique meaning only if the surface is closed.



something flowing out

$$\oiint \boldsymbol{v} \, . \, d\boldsymbol{S} \; > \; 0$$



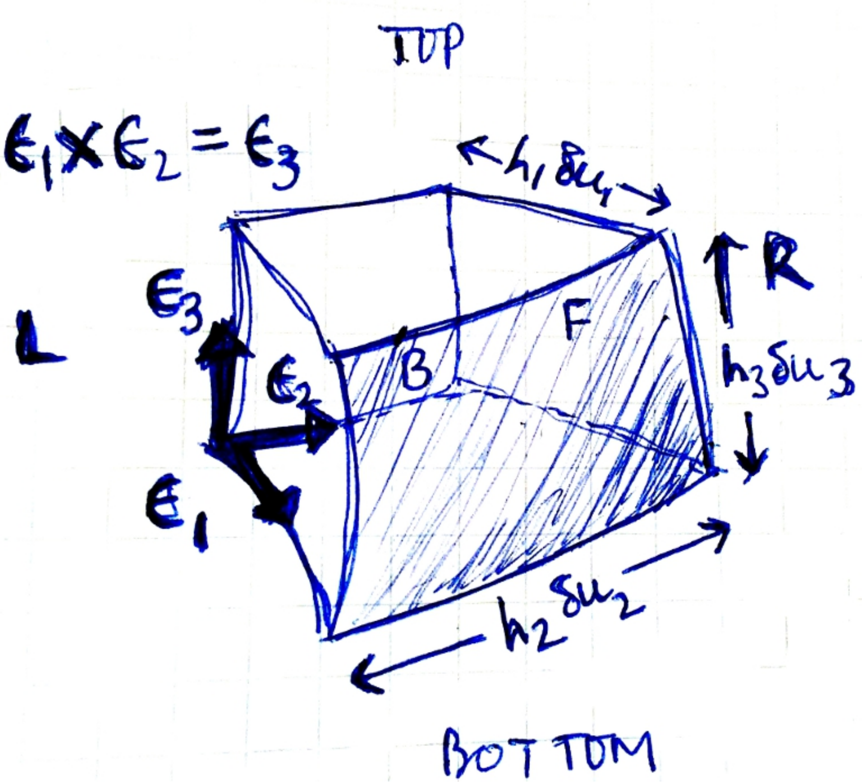
something flowing in

$$\oiint \boldsymbol{v} \, . \, d\boldsymbol{S} \; < \; 0$$

Consider a vector $\vec{F}$

Is it possible to have a function $X(\vec{F})$ such that

$$X(\vec{F})dV = \vec{F}.d\vec{S}$$



Flux through BACK

$$f_B = -F_1 h_2 \delta u_2 h_3 \delta u_3$$

Flux through FRONT

$$f_F = F_1 h_2 \delta u_2 h_3 \delta u_3 + \frac{\partial}{\partial u_1}\left(F_1 h_2 \delta u_2 h_3 \delta u_3\right)\delta u_1$$

$$f_B + f_F = \left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right)\right]\delta u_1 \delta u_2 \delta u_3$$

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Flux and circulation

The LEFT + RIGHT pair gives

$$f_L + f_R = \left[\frac{\partial}{\partial u_2}\left(F_2 h_1 h_3\right)\right]\delta u_1 \delta u_2 \delta u_3$$

The BOTTOM + TOP pair gives

$$f_{Bottom} + f_{Top} = \left[\frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]\delta u_1 \delta u_2 \delta u_3$$

$$f_{TOTAL} = \left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(F_2 h_1 h_3\right) + \frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]\delta u_1 \delta u_2 \delta u_3$$

$$\frac{\vec{F}\,.\,\delta \vec{S}}{\delta V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(F_2 h_3 h_1\right) + \frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]$$

Now break a finite volume into small volume elements

Flux from neighbouring walls of two infinitesimal volume elements will cancel

Only faces which form the part of the boundary of the volume will not cancel

This function is called DIVERGENCE, denoted by $\vec{\nabla}\,.\,\vec{F}$

$$\oiiint \vec{\nabla}\,.\,\vec{F}\,dV = \oiint \vec{F}\,.\,d\vec{S}$$

Called Gauss's theorem

Divergence of a vector is a scalar quantity

"divergence" should convey a visual picture of the Vector field.... What is it?

How should a vector field look around points of stable/unstable equilibrium ?

In Cartesian:

$$\vec{\nabla}\,.\,\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

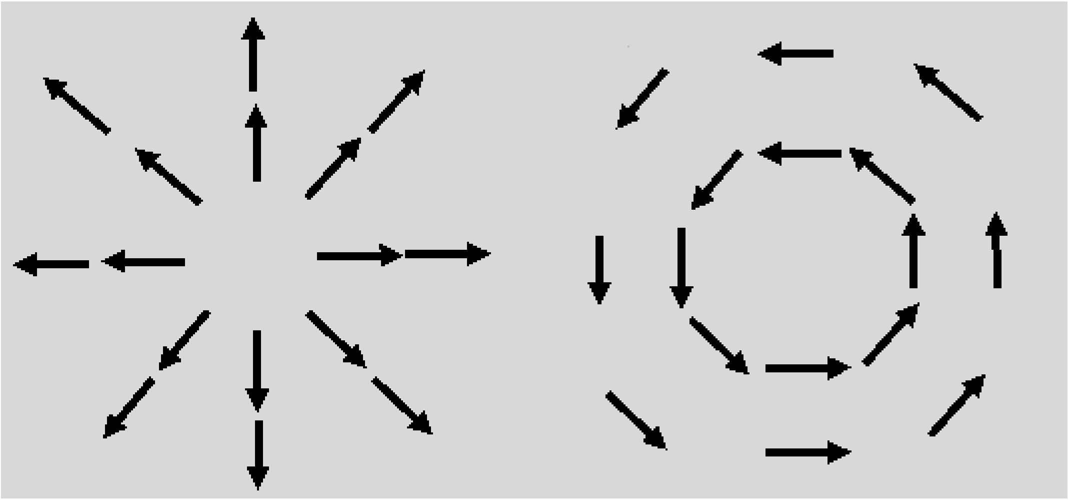
In Spherical polar:

$$\vec{\nabla}\,.\,\vec{F} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\, F_r\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta\, F_\theta\right) + \frac{\partial}{\partial \phi}\left(rF_\phi\right)\right]$$

In cylindrical polar

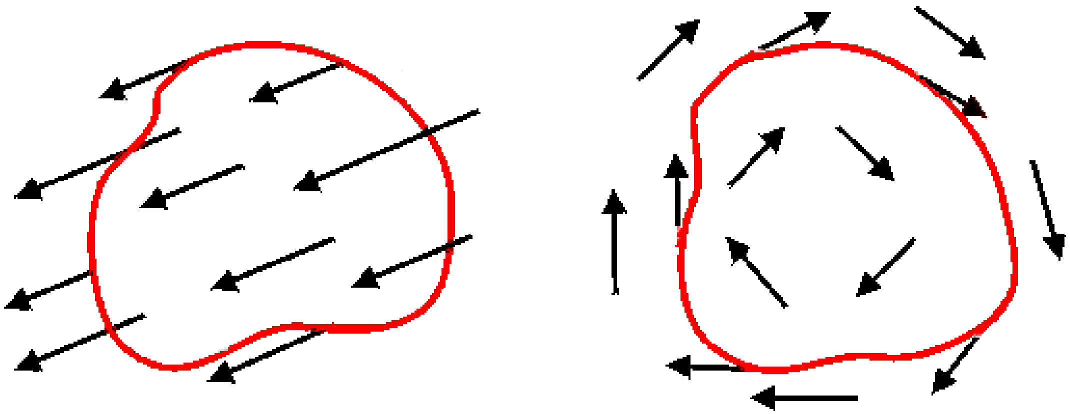
$$\vec{\nabla}\,.\,\vec{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{\partial}{\partial \phi}\left(F_\phi\right) + \frac{\partial}{\partial z}\left(\rho F_z\right)\right]$$

Divergence and continuity equation....



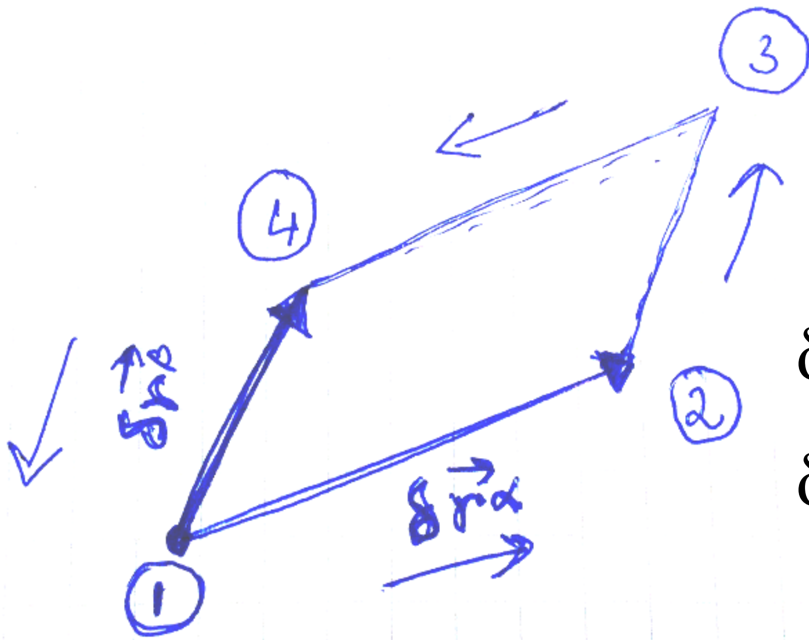
$\oiint \vec{F}\,.\,d\,\vec{S}$ *identifies a distinctive field pattern.*

Another possible one is a circulating pattern.



When will $\oint \vec{F}\,.\,d\,\vec{l}$ *be nonzero ?*

Consider two arbitray infinitesimal displacements



$$\delta \vec{r}^{\alpha} = \hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 \delta u_1^{\alpha} + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 \delta u_2^{\alpha} + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 \delta u_3^{\alpha}$$

$$\delta \vec{r}^{\beta} = \hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 \delta u_1^{\beta} + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 \delta u_2^{\beta} + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 \delta u_3^{\beta}$$

The vector field is $\vec{F}$.

Is it possible to have a function $X(\vec{F})$ such that

$$X(\vec{F}).\,\delta \vec{S} = \sum_{\substack{peri-\\meter}} \vec{F}.\,\delta \vec{l}$$

If possible then this function will connect some characteristics of inside points with the boundary

$$d\vec{S} = \delta\vec{r}^{\alpha} \times \delta\vec{r}^{\beta} = \begin{vmatrix} \hat{\boldsymbol{\epsilon}}_1 & \hat{\boldsymbol{\epsilon}}_2 & \hat{\boldsymbol{\epsilon}}_3 \\ h_1 \delta u_1^{\alpha} & h_2 \delta u_2^{\alpha} & h_3 \delta u_3^{\alpha} \\ h_1 \delta u_1^{\beta} & h_2 \delta u_2^{\beta} & h_3 \delta u_3^{\beta} \end{vmatrix}$$

$$\begin{aligned} X(\vec{F}).d\vec{S} = \; & X_1 h_2 h_3 [\delta u_2^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_2^{\beta}] \\ & - X_2 h_1 h_3 [\delta u_1^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_1^{\beta}] \\ & + X_3 h_1 h_2 [\delta u_1^{\alpha} \delta u_2^{\beta} - \delta u_2^{\alpha} \delta u_1^{\beta}] \end{aligned}$$

Try writing RHS in this form and compare.
The co-efficients of the arbitrary displacments must agree

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Flux and circulation

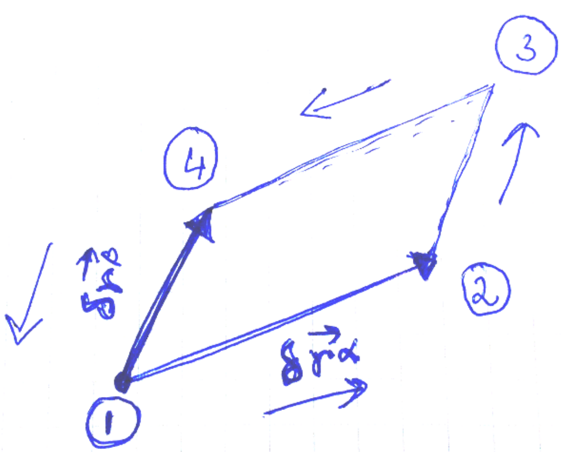
Consider the pair of paths $(1 \to 2)$ and $(3 \to 4)$

$$\vec{F}.\delta\vec{l}_{|1\to 2} = F_1 h_1 \delta u_1^{\alpha} + F_2 h_2 \delta u_2^{\alpha} + F_3 h_3 \delta u_3^{\alpha}$$

$$\vec{F}.\delta\vec{l}_{|3\to 4} = \left[F_i h_i + (\nabla F_i h_i).\delta\vec{r}^{\beta}\right](-\delta u_i^{\alpha}) \qquad (i=1,2,3)$$

Write contributions from $\vec{F}.\delta\vec{l}_{|2\to 3}$ & $\vec{F}.\delta\vec{l}_{|4\to 1}$ similarly.

Full path gives: $\left(\nabla\vec{F}.\delta\vec{r}^{\beta}\right).\delta\vec{r}^{\alpha} - \left(\nabla\vec{F}.\delta\vec{r}^{\alpha}\right).\delta\vec{r}^{\beta}$



$$= \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^{\beta}\right] h_k \delta u_k^{\alpha}$$

$$-\sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^{\alpha}\right] h_k \delta u_k^{\beta}$$

$$= \sum_{k,i}\left[\frac{\partial F_i h_i}{\partial u_k} - \frac{\partial F_k h_k}{\partial u_i}\right]\delta u_i^{\beta}\delta u_k^{\alpha}$$

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Now compare the co-efficient of $\delta u_2^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_2^{\beta}$

We need to put $i=3, k=2$ and then $i=2, k=3$

this gives $X_1 h_2 h_3 = \left[\frac{\partial F_3 h_3}{\partial u_2} - \frac{\partial F_2 h_2}{\partial u_3} \right]$

So $$X(\vec{F}) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\boldsymbol{\epsilon}}_1 & h_2 \hat{\boldsymbol{\epsilon}}_2 & h_3 \hat{\boldsymbol{\epsilon}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \equiv \begin{cases} \nabla \times \vec{F} \\ curl\ \vec{F} \\ rot\ \vec{F} \end{cases}$$

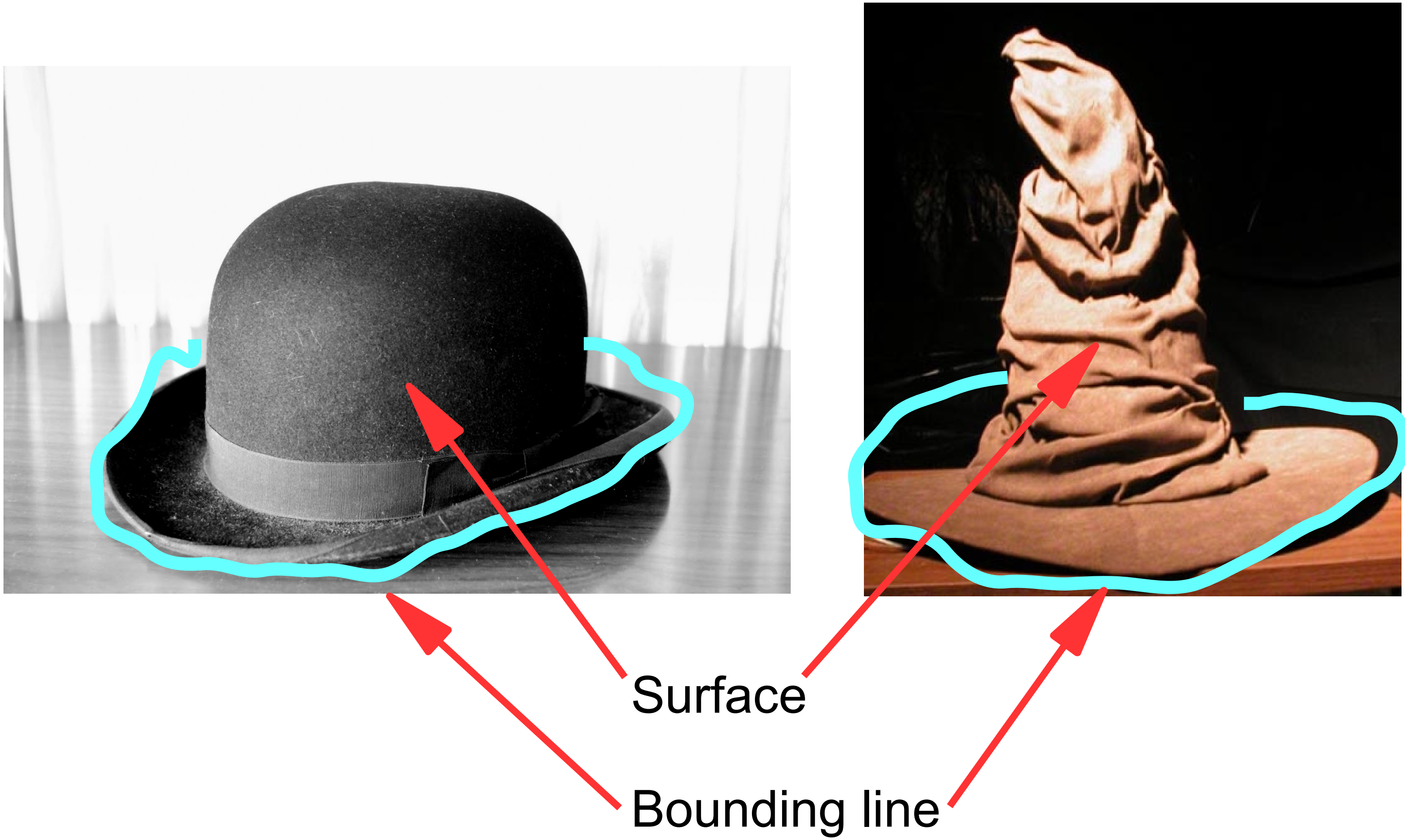
We have $\iint \nabla \times \vec{F} . d\vec{S} = \oint \vec{F} . d\vec{l}$ $\quad$ $(called\ Stoke's\ theorem)$

Now break a finite surface into small area elements

Line integral from neighbouring perimeters of two infinitesimal area elements will cancel

Only line segments which form the part of the perimeter will not cancel

Flux and circulation : which surface ?



Any surface with the same bounding edge will work.

Curl F over any closed surface should be zero. WHY?

Divergence of a curl = ?

Curl of a gradient = ?

Multiple vector products : ε–δ notation (Levi-Civita)

Write the dot product as $\vec{A}.\vec{B} = \delta_{ij} A_i B_j \quad where \quad \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$

Write the cross product as

$$\vec{A} \times \vec{B}|_i = \epsilon_{ijk} A_j B_k \quad where \quad \epsilon_{ijk} = \begin{cases} 1 & if \quad ijk \quad is \quad an \quad even \quad permutation \quad of \quad 123 \\ -1 & if \quad ijk \quad is \quad an \quad odd \quad permutation \quad of \quad 123 \\ 0 & otherwise \end{cases}$$

Convince yourself that $\epsilon_{ijk} = \hat{\boldsymbol{\epsilon}}_{\boldsymbol{i}} . \hat{\boldsymbol{\epsilon}}_{\boldsymbol{j}} \times \hat{\boldsymbol{\epsilon}}_{\boldsymbol{k}}$

This works with operators also : $with \quad x_i \quad for \quad x, y, z$

$$\nabla . \vec{A} = \frac{\partial A_i}{\partial x_i}$$

$$\nabla \times \vec{A}\Big|_i = \epsilon_{ijk} \frac{\partial A_j}{\partial x_k}$$

Notice how the summation convention on repeated indices have been used.

Q: How does it help?

Multiple vector products : ε–δ notation (Levi-Civita)

Consider a vector triple product $\boldsymbol{A} \times \left(\boldsymbol{B} \times \boldsymbol{C}\right)$

$$\begin{aligned} \vec{A} \times \left(\vec{B} \times \vec{C}\right)\Big|_i &= \epsilon_{ijk} A_j \left(\vec{B} \times \vec{C}\right)_k \\ &= \epsilon_{ijk} A_j \left(\epsilon_{kpq} B_p C_q\right) \\ &= \epsilon_{kij} \epsilon_{kpq} A_j B_p C_q \end{aligned}$$

What to do with a product like $\epsilon_{ijk} \epsilon_{kpq}$?

This is either -1 or 0 or 1

kpq itself is some permutation of the sequence lmn, otherwise the product will vanish.

WITHOUT summation,
we can write for a generic product term: $(why\,?)$

$$\begin{aligned} \epsilon_{lmn} \epsilon_{kpq} = \quad & \delta_{lk} \delta_{mp} \delta_{nq} + \delta_{lp} \delta_{mq} \delta_{nk} + \delta_{lq} \delta_{mk} \delta_{np} \\ & - \delta_{lk} \delta_{mq} \delta_{np} - \delta_{lp} \delta_{mk} \delta_{nq} - \delta_{lq} \delta_{mp} \delta_{nk} \end{aligned}$$

odd/even permutations

kpq	$+$
kqp	$-$
pkq	$-$
pqk	$+$
qkp	$+$
qpk	$-$

Multiple vector products : ε–δ notation (Levi-Civita)

$$\begin{aligned}\epsilon_{kij}\epsilon_{kpq} &= \delta_{kk}\delta_{ip}\delta_{jq} + \delta_{kp}\delta_{iq}\delta_{jk} + \delta_{kq}\delta_{ik}\delta_{jp} \\ &\quad -\delta_{kk}\delta_{iq}\delta_{jp} - \delta_{kp}\delta_{ik}\delta_{jq} - \delta_{kq}\delta_{ip}\delta_{jk} \\ &= \delta_{kk}\left(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\right)+\delta_{kp}\left(\delta_{iq}\delta_{jk}-\delta_{ik}\delta_{jq}\right)+\delta_{kq}\left(\delta_{ik}\delta_{jp}-\delta_{ip}\delta_{jk}\right) \\ &= 3\left(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\right)+\left(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}\right)+\left(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}\right) \\ &= \delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\end{aligned}$$

sum over k

$$Using\ the\ last\ result\left(with\quad i=p\right)$$

$$\begin{aligned}\epsilon_{kij}\epsilon_{kiq} &= \delta_{ii}\delta_{jq}-\delta_{iq}\delta_{ji} \\ &= 3\delta_{jq}-\delta_{jq} \\ &= 2\delta_{jq}\end{aligned}$$

sum over k and i

$$Using\ the\ last\ result\left(with\quad j=q\right)$$

$$\begin{aligned}\epsilon_{kij}\epsilon_{kij} &= 2\delta_{jj} \\ &= 6\end{aligned}$$

sum over k,i and j

Successive summation over indices.

The first sum is most frequently encountered. It allows you to write a cross product in terms of dot product like terms

Multiple vector products : ε–δ notation (Levi-Civita)

Should be able to prove all of these easily....
(list from the last page of Griffith's book)

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$