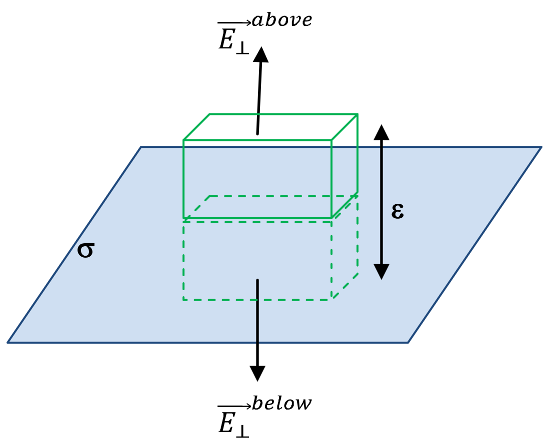
Potential & Potential Energy

Lecture 10: Electromagnetic Theory
Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Electrostatic Boundary Conditions :

We had seen that electric field has a discontinuity at a charged boundary between two media. Let us examine this a little more carefully.



Let us consider the interface between two media which has a charge density σ. (This is not necessarily the infinite sheet discussed earlier, it could be a surface of any shape and size).

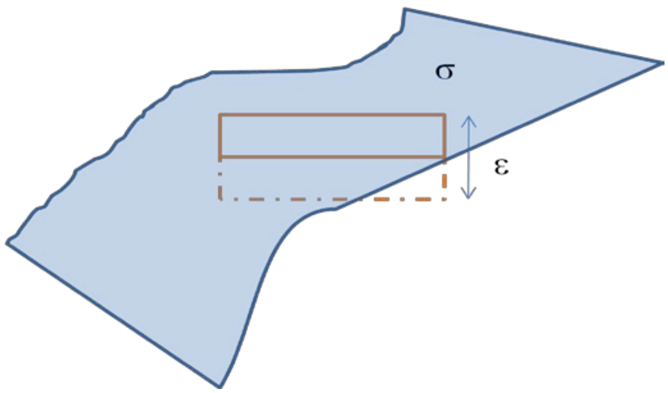
Consider a Gaussian pillbox in the shape of a rectangular parallelepiped of cross section A and height ϵ half of which is above the plane and half below. Using Gauss's law, the flux out of the parallelepiped is due the flux from the top and the bottom surfaces as well as from the side surfaces. Let us look at the normal components of the electric field denoted by $E_\perp$. As the height of the parallelepiped ϵ becomes infinitesimally small, the contribution to the flux from the sides become vanishingly small and only the top and the bottom surfaces contribute. If $E_\perp^{above}$ and $E_\perp^{above}$ are the normal components of the electric field on the top and the bottom faces, we have

$$\Phi = E_\perp^{above} A - E_\perp^{above} A = \frac{\sigma A}{\epsilon_0}$$

Thus the normal component of the electric field has a discontinuity, given by,

$$E_\perp^{above} - E_\perp^{above} = \frac{\sigma}{\epsilon_0}$$

What about the tangential component?



Consider a rectangular loop of length $l$and heighte ϵ which is infinitesimally small, whose plane is perpendicular to the interface and half of which is above the interface and half below. As the electric field is conservative, we have $\oint \vec{E}\cdot d\vec{l} = 0$. Since the contribution to the line integral from the perpendicular sections become infinitely small, we have $(E_{\parallel}^{above} - E_{\parallel}^{below})l = 0$, which gives,

$$E_{\parallel}^{above} - E_{\parallel}^{below} = 0$$

Thus, though the normal component of the electric field has a discontinuity, the tangential component is continuous.

Potential Energy of a charge distribution :

We had seen that the potential could be interpreted as the potential energy associated with a unit test charge at a point. We will now consider a collection of discrete charges and calculate the potential energy of such a charge distribution.

Suppose the final configuration of charges is to have charges $\{q_i\}$ at $\{\vec{r_i}\}$. We proceed to assemble the charge distribution as follows. Initially let us assume that all charges are at infinity separated by infinite distance from one another. The potential energy of this configuration is zero.

We now take the charge q_1 and bring it from infinity to its final place $\vec{r_1}$. This involves no work as the region through which it moves is a field free region and hence it experiences no force while being brought in. Let us now bring a second charge, say, $q_2$and place it at $\vec{r_1}$. This charge is not being moved in a field free region because the first charge is already in place at $\vec{r_1}$ and has its electric field already established in space. The potential at $\vec{r_2}$ due to this charge is $\varphi(\vec{r_2}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r_2}-\vec{r_1}|}$. The work done in bringing the charge q_2 from infinity to this point is $q_2\varphi(\vec{r_2}) = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{|\vec{r_2}-\vec{r_1}|}$.

Let us now bring in the third charge q_3 and place it at $\overrightarrow{r_3}$. This charge now experiences a force due to superposition of fields established because of two charges q_1 at $\overrightarrow{r_1}$ and q_2 at $\overrightarrow{r_2}$. The work done in bringing the charge q_3 is given by q_3 times the potential at $\overrightarrow{r_3}$, i.e.

$$\frac{1}{4\pi\epsilon_0}q_3\left(\frac{q_1}{|\overrightarrow{r_3}-\overrightarrow{r_1}|}+\frac{q_2}{|\overrightarrow{r_3}-\overrightarrow{r_2}|}\right)$$

Continuing like this, we get the net work done in bringing a charge q_j to the position $\vec{r_j}$ is

$$\frac{1}{4\pi\epsilon_0}q_j\sum_i\frac{q_i}{|\vec{r_i}-\vec{r_j}|}$$

The total work done would then be obtained by summing over this over the index j. However, there is a restriction $j \neq i$ because a charge does not exert force ion itself. Further we need an additional factor of half to avoid double counting, for there is only one interaction term between a pair. Thus the net work done, which gets stored as the potential energy of the system is given by

$$W = \frac{1}{2}\cdot\frac{1}{4\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}\frac{q_iq_j}{|\vec{r_i}-\vec{r_j}|}$$
$$= \frac{1}{2}\sum_i q_i\varphi(\vec{r_i})$$

Energy of a continuous charge distribution :

The expression for the energy obtained above for a discrete charge distribution is readily extended to a continuous charge distribution by taking a small volume element dv at the position $\vec{r}$. If the charge density at that position is $\rho(\vec{r})$, the element of volume contains $\rho(\vec{r})dv$ of charge. The potential energy is then obtained by multiplying the charge element with the potential at that position and integrating over the volume containing the charge distribution,

$$W = \frac{1}{2}\int\rho(\vec{r})\varphi(\vec{r})d^3r \qquad (1)$$

where we have written d^3r in place of dv to emphasize that the volume element is in three dimension. We can convert eqn. (1) to other convenient forms by observing that $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, using which we can write

$$W = \frac{\epsilon_0}{2}\int\left(\vec{\nabla}\cdot\vec{E}\right)\varphi(\vec{r})d^3r$$
$$= -\frac{\epsilon_0}{2}\int\varphi(\vec{r})\nabla^2\varphi(\vec{r})d^3r \qquad (2)$$

where we have used $\vec{E} = -\nabla\varphi$.

We will simplify this expression a little more by using a vector identity,

$$\nabla\cdot(\varphi\nabla\varphi) = \varphi\nabla^2\varphi + |\nabla\varphi|^2$$

Which enables us to rewrite (2) as

$$W = -\frac{\epsilon_0}{2}\int [\nabla\cdot(\varphi\nabla\varphi) - |\nabla\varphi|^2] d^3r$$
$$= -\frac{\epsilon_0}{2}\int_S \varphi\overrightarrow{\nabla\varphi}\cdot d\vec{S} + \frac{\epsilon_0}{2}\int_V |\nabla\varphi|^2 d^3r \qquad (3)$$

Where, in the first term we have used the divergence theorem.

This equation can be handled in two different ways. Firstly, observe that the volume that we take can be any volume as long as it encloses all the charges in the distribution. Looking at eqn. (1) the integrand has $\rho(\vec{r})$ as a factor. Even if we are to take a much bigger volume than the original volume, outside the physical volume the charge density is zero and hence it does not contribute to the integral. Thus in principle, we can take the volume to be infinite. If we do that, the first term in eqn. (3) becomes zero because the surface being at infinity, the potential is zero over such a surface. Thus (3) becomes,

$$W = \frac{\epsilon_0}{2}\int_{All\ Space} |E|^2 d^3r \qquad (4)$$

Where we have used $\vec{E} = -\nabla\varphi$. The integration in this case is over all space because we have taken the surface term to be zero.

Self Energy Problem :

Equation (4) says that the potential energy of a charge distribution is positive definite. We can define a positive energy density $\frac{\epsilon_0}{2}|E|^2$ associated with the field. This causes some contradiction with the situation that exists for a system of discrete charges which may be negative. For instance, a pair of opposite charges have a negative interaction energy. The apparently anomaly is resolved if you recognize that in obtaining the interaction energy of discrete charges, we had assumed that the discrete charges already existed and no energy was spent in creating them. To be more specific, the self interaction of the electric field of a charge with itself was excluded. When we consider the work done in assembling the charges themselves, we would get the self energy effect and when added with the interaction energy, the resulting sum would be positive.

The self energy of a point charge is infinite as can be seen by integrating the square of the electric field due to a point charge over all space using eqn. (4), for

$$W_{self} = \int \frac{q^2}{16\pi^2\epsilon_0^2 r^4} d^3r = \frac{q^2}{16\pi^2\epsilon_0^2}\int_0^\infty \frac{4\pi}{r^2} dr$$

which diverges at the lower limit. We could have anticipated this because the potential at the position of the charge is itself infinite. This problem is known as the self energy divergence problem a discussion of which is beyond our scope. In our calculation of the potential energy of a charge distribution we assume that this divergent self energy is excluded from calculation.

Potential Energy of a uniformly charged sphere :

We will illustrate the method of calculating the potential energy of a continuous charge distribution by taking the example of a uniformly charged sphere and calculate the energy by four different methods.

Method 1 : Using basic definition as given in eqn. (1) :

If the total charge is Q, the charge density is constant and is given by $\rho = \frac{Q}{\frac{4\pi R^3}{3}} = \frac{3Q}{4\pi R^3}$. The potential can be obtained from the expression for the electric field that we have calculated earlier, viz.,

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & for \quad r > R \\ \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r} & for \quad r < R \end{cases}$$

The potential at the position $\vec{r}$ is obtained by evaluating the integral $-\int \vec{E}\cdot d\vec{r}$ from infinity to the position $\vec{r}$.

For r>R, the potential is given by $\varphi(\vec{r} > R) = -\int_{\infty}^{R} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0 r}$. The potential at points r < R is obtained by adding to the value of the potential given by this expression at r=R, the line integral of the electric field inside the sphere from r = R to r. For this case, the appropriate expression to be used for the electric field is the second expression given above,

$$\begin{aligned}\varphi(r) &= \frac{Q}{4\pi\epsilon_0 R} - \int_R^r \frac{Qr}{4\pi\epsilon_0 R^3} dr \\ &= \frac{Q}{4\pi\epsilon_0 R} - \frac{Q}{8\pi\epsilon_0 R^3}(r^2 - R^2) \\ &= \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\end{aligned}$$

In the first method we need to calculate the volume integral

$$W = \frac{1}{2}\int \rho(\vec{r})\varphi(\vec{r}) d^3r$$

Since the charge density is zero outside the physical sphere, the expression for the potential that need to use is the second expression which is appropriate to r < R,

$$\begin{aligned}W &= \frac{1}{2}\frac{3Q}{4\pi R^3}\int_0^R \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) 4\pi r^2 dr \\ &= \frac{3Q^2}{20\pi\epsilon_0 R}\end{aligned}$$

Method 2 : In this method we will use eqn. (4) which uses the infinite space for volume integration, throwing out the surface term. The electric field is given inside and outside the physical region by the expressions given earlier.

$$W = \frac{\epsilon_0}{2}\int_{All\ Space} |E|^2 d^3r$$
$$= \frac{\epsilon_0}{2}\left[4\pi\int_0^R \frac{q^2}{16\pi^2\epsilon_0^2}\frac{r^2}{R^6}r^2dr + 4\pi\int_R^\infty \frac{q^2}{16\pi^2\epsilon_0^2}\frac{1}{r^4}r^2dr\ \right]$$
$$= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{R^6}R^5 + \frac{1}{R}\right] = \frac{3Q^2}{20\pi\epsilon_0 R}$$

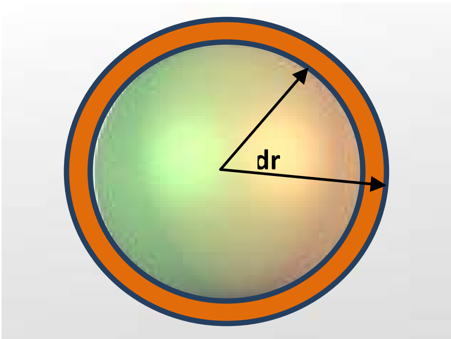
Method 3 : In this method we use eqn. (3) but consider the physical volume and surface. As the surface is in finite space, we cannot ignore the surface contribution. The fields and potential that contribute to this case are the ones corresponding to r <R.

$$W = -\frac{\epsilon_0}{2}\int_S \varphi\overrightarrow{\nabla\varphi}\cdot d\vec{S} + \frac{\epsilon_0}{2}\int_V |\nabla\varphi|^2 d^3r$$
$$= \frac{\epsilon_0}{2}\int_S \varphi(R)\vec{E}\cdot\hat{n}dS + \frac{\epsilon_0}{2}\int_V |E|^2 d^3r$$
$$= \frac{\epsilon_0}{2}\varphi(R)|E(R)|4\pi R^2 + \frac{\epsilon_0}{2}\int_0^R\left(\frac{Q^2}{16\pi^2\epsilon_0^2}\right)\frac{r^2}{R^6}4\pi r^2 dr\mathrm{s}$$
$$= \frac{\epsilon_0}{2}\frac{Q}{4\pi\epsilon_0 R}\frac{Q}{4\pi\epsilon_0 R^2}4\pi R^2 + \frac{\epsilon_0}{2}\frac{Q^2}{4\pi\epsilon_0^2}\frac{1}{R^6}\frac{R^5}{5}$$
$$= \frac{3Q^2}{20\pi\epsilon_0 R}$$

In the surface terms above we have used the value of the potential and the field on the surface of the sphere.

Method 4 : Calculating from the first principle, building up the sphere layer by layer :

In this method we assume that at certain instant we have a charged sphere of radius r having a charge q. We add to this sphere an additional charge dq which we spread uniformly over it, increasing the radius from r to r+dr. If the final charge is to be Q and the radius of the sphere is to be R, at the instant when the radius is r, the charge on the sphere is $q(r) = \frac{Qr^3}{R^3}$.



The additional charge dq is contained in a spherical shell between radii r and r+dr. Thus

$$dq = 4\pi r^2 dr\rho = \frac{3Q}{R^3} r^2 dr$$

The work done in bringing the charge dq and spreading it over the existing sphere is

$$dW = \varphi(r)dq = \frac{1}{4\pi\epsilon_o}\frac{q(r)dq}{r}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{Qr^3}{R^3}\frac{3Q}{R^3} rdr = \frac{3Q^2}{4\pi\epsilon_0 R^6} r^4 dr$$

The potential energy, which is the total work done in building the sphere starting with zero radius to the final radius is obtained by integrating this expression from r=0 to r=R,

$$W = \frac{3Q^2}{4\pi\epsilon_0 R^6}\int_0^R r^4 dr = \frac{3Q^2}{20\pi\epsilon_0 R}$$

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Tutorial Assignment :

1. A sphere of radius R has a spherically symmetric charge density $\rho(r) = \rho_0\left(1 - \frac{r}{R}\right)$, where r is the distance from the centre. Calculate the energy stored in the whole space as well as the energy stored within the volume of the sphere.
2. In the text we had calculated the energy of a uniformly charged sphere by assembling charges layer by layer (see method 4). This method fails while calculating energy of a spherical conductor because charges on a conductor only reside on its surface. Try the following variation. Start with an

uncharged sphere of radius *a*. Calculate the work done in bringing charge q from infinity to the surface of this sphere and spreading it uniformly in a shell of thickness *b-a*. Take the limit $b \to a$.

Solutions to Tutorial Problems :

1. We first calculate the total charge $Q = \int \rho dv = \int_0^R \rho_0 \left(1 - \frac{r}{R}\right) 4\pi r^2 dr = \frac{\rho_0 \pi R^3}{3}$. Consider a shell of radius r > R and width dr. Since r >R, the electric field at r is due to the charge Q concentrated at origin and is given by $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$. The energy stored in the surrounding space is $W = \left(\frac{1}{2}\right)\epsilon_0 \int |E|^2 dv = \frac{\epsilon_0}{2}\frac{Q^2}{16\pi^2\epsilon_0^2}\int_R^\infty \frac{1}{r^4} 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon_0}\int_R^\infty \frac{1}{r^2} dr = \frac{Q^2}{8\pi\epsilon_0 R} = \frac{\pi\rho_0^2 R^5}{72\epsilon_0}$.
The energy stored within the volume of the sphere can be obtained in a similar way. However, one has to observe that when r <R, the electric field at the position r is given by the amount of charge enclosed within the sphere of radius r. The charge within such a sphere of radius r is
$$Q(r) = \int_0^r \rho_0 \left(1 - \frac{r}{R}\right) 4\pi r^2 dr = 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) = \frac{\pi\rho_0 r^2}{3R}(4rR - 3r^2)$$
The electric field at the position r (r<R) is given by $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q(r)}{r^2} = \frac{\rho_0}{12\epsilon_0 R}(4rR - 3r^2)$. The energy is obtained by integrating the square of this expression within the volume,
$W = \left(\frac{1}{2}\right)\epsilon_0 \int |E|^2 dv = \frac{\epsilon_0}{2}\frac{\rho_0^2 4\pi}{144R^2\epsilon_0^2}\int_0^R (4rR - 3r^2)^2 r^2 dr = \frac{17}{2520}\frac{\rho_0^2 \pi R^5}{\epsilon_0}$.

2. Suppose we start with a neutral sphere of radius *a* and deposit charge Q uniformly over a shell of thickness (*b-a*), I.e charge is contained within a spherical shell of radii *a* and *b*. The charge density is $\rho = \frac{3Q}{4\pi(b^3 - a^3)}$. If at a certain instant the charges are deposited over a shell of radius *(r-a),* the amount of charge contained in the shell is$q(r) = \frac{Q(r^3 - a^3)}{b^3 - a^3}$. Let us bring an amount of charge *dq* from infinity and spread it uniformly over this shell so that this charge lies in the shell between *r* and *r+dr*. The charge dq is given by the volume of the shell between r and r+dr multiplied by the charge density, i.e.,
$$dq = 4\pi r^2 dr \times \frac{3Q}{4\pi(b^3 - a^3)} = \frac{3Q}{b^3 - a^3} r^2 dr$$
Since the potential on the existing sphere of radius r is
$$\varphi(r) = \frac{q(r)}{4\pi\epsilon_0 r} = \frac{Q(r^3 - a^3)}{4\pi\epsilon_0 r(b^3 - a^{3)}}$$
, the work done is $\varphi(r)dq$. Thus the total work done in creating a uniformly charged shell of radii *a* and *b* is
$$W = \frac{3Q^2}{4\pi\epsilon_0 (b^3 - a^3)^2}\int_a^b (r^3 - a^3)\, r dr$$
$$= \frac{3Q^2}{4\pi\epsilon_0 (b^3 - a^3)^2}\left[\frac{b^5 - a^5}{5} - \frac{a^3(b^2 - a^2)}{2}\right]$$

We would obtain the desired result if we let $b \to a$. To see this, let $b = a + \delta$. We would expand the above expression in terms of powers of δ and retain leading order of terms and finally let $\delta \to 0$.

Note that

$$\begin{aligned}
\frac{b^5 - a^5}{5} - \frac{a^3(b^2 - a^2)}{2} &= \frac{(a+\delta)^5 - a^5}{5} - \frac{a^3[(a+\delta)^2 - a^2]}{2} \\
&= \left(\frac{1}{10}\right)\left[2a^5\left(! + \ 5\frac{\delta}{a} + 10\frac{\delta^2}{a^2} + \cdots - 1\right) - 5a^5\left(2\frac{\delta}{a} + \delta^2\right)\right] \\
&= \frac{3}{2}\delta^2 a^3 \\
(b^3 - a^3)^2 &= [(a+\delta)^3 - a^3]^2 \\
&= (3a^2\delta + 3a\delta^2)^2 \approx 9a^4\delta^2
\end{aligned}$$

Substituting these in the expression for W, we get,

$$W = \frac{3Q^2}{4\pi\epsilon_0}\frac{3\delta^2 a^3}{18a^4\delta^2} = \frac{Q^2}{8\pi\epsilon_0 a}$$

[The limit can be taken in a simpler way if you treat *b* as a variable and use L'Hospital's rule for taking the limit $b \to a$.]

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Self Assessment Quiz

1. A sphere of radius R has a spherically symmetric charge density $\rho(r) = kr$, where r is the distance from the centre. Calculate the energy stored in the whole space as well as the energy stored within the volume of the sphere. (Do this problem using all the four methods discussed in the lecture.)
2. Find the energy per unit length of a uniformly charged cylinder of radius R having a charge of λ Per unit length.

Solutions to Self Assessment Quiz

1. For this problem we need to calculate the electric field, the potential and the charge. Since $\rho = kr$, the charge enclosed within a radius r is $Q(r) = 4\pi \int_0^r kr^3 dr = \pi k r^4$ and the total charge is $Q = \pi k R^4$. The field both inside and outside can be calculated using this and Gauss's law. We get

$$\vec{E} = \begin{cases} \dfrac{kr^2}{4\epsilon_0}\hat{r} & \text{for } r < R \\ \dfrac{kR^4}{4\epsilon_0 r^2}\hat{r} & \text{for } r > R \end{cases}$$

The potential outside the volume is $\varphi(r > R) = \frac{kR^4}{4\epsilon_0 r}$ with $\varphi(\infty) = 0$. The potential inside is calculated by taking the line integral from the surface (where the potential is known from the above) to the point r,

$$\varphi(r < R) = \frac{kR^4}{4\epsilon_0 r} - \int_R^r \frac{kr^2}{4\epsilon_0} dr$$
$$= \frac{kR^3}{3\epsilon_0} - \frac{kr^3}{12\epsilon_0}$$

With these expressions, we will calculate the self energy in four different ways.

Method 1 :

In this method we use the formula $W = \frac{1}{2}\int \rho(\vec{r})\varphi(\vec{r})d^3r$. We have since the charge density outside the given sphere is zero,

$$W = \frac{1}{2}\int_0^R kr\left(\frac{kR^3}{3\epsilon_0} - \frac{kr^3}{12\epsilon_0}\right)4\pi r^2 dr$$
$$= \frac{2\pi k^2}{\epsilon_0}\left(\frac{R^7}{12} - \frac{R^7}{84}\right) = \frac{\pi k^2 R^7}{7\epsilon_0}$$

Method 2 :

In this method we extend the integrals to infinite space, dropping the surface term. We have

$$W = \frac{\epsilon_0}{2}\int_{All\ Space} |E|^2 d^3r$$
$$= \frac{\epsilon_0}{2}\int_0^R \frac{k^2 r^4}{16\epsilon_0^2}4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty \frac{k^2 R^8}{16\epsilon_0^2}\frac{1}{r^4}4\pi r^2 dr$$
$$= \frac{\pi k^2 R^7}{56\epsilon_0} + \frac{\pi k^2 R^7}{8\epsilon_0} = \frac{\pi k^2 R^7}{7\epsilon_0}$$

Method 3 :

In this method we restrict to the volume of the sphere, thereby we need to keep both the volume and the surface terms. The volume term is given by the first term of method 2. The surface term is

$$W_{surface} = \frac{\epsilon_0}{2}\varphi(R)\int_S \vec{E}\cdot\hat{n}dS = \frac{1}{2}\varphi(R)Q_{total} = \frac{1}{2}\frac{kR^3}{4\epsilon_0}\ \pi k R^4 = \frac{\pi k^2 R^7}{8}$$

Since this term exactly equals the volume contribution to energy from outside the sphere, this will also give $W = \frac{\pi k^2 R^7}{7\epsilon_0}$

Method 4 :

Method 4 consists of building up of the sphere layer by layer. We have calculated the charge enclosed in a sphere of radius r to be $Q(r) = \pi k r^4$. Suppose at certain instant the radius of the sphere is r so that it has enclosed charge $Q(r)$. We bring an additional amount of charge $dq(r) = 4\pi r^2 dr\,\rho(r) = 4\pi k\ r^3 dr$ and spread it over the existing sphere so that the new radius is r + dr. The potential on the surface of the sphere of radius r is

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q(r)}{r} = \frac{kr^3}{4\epsilon_0}$$

The work done in bringing this charge to the surface of the sphere is $\varphi(r)dq(r)$, so that the total work done is given by

$$W = \int \varphi(r) dq(r) = \int_0^R \frac{kr^3}{4\epsilon_0} 4\pi k\, r^3 dr = \frac{\pi k^2 R^7}{7\epsilon_0}$$

2. The electric field due to the cylinder (ignore edge effect) can be easily calculated using Gauss's law. It gives,

$$\vec{E} = \begin{cases} \frac{\lambda\rho}{2\pi R^2 \epsilon_0}\hat{\rho} & \text{for } \rho < R \\ \frac{\lambda}{2\pi\epsilon_0 r}\hat{\rho} & \text{for } \rho < R \end{cases}$$

Where ρ is the distance from the axis of the cylinder of radius R and $\hat{\rho}$ is the radial direction of the cylindrical coordinates. The potential inside the cylinder is directly calculated from the above expression,

$$\varphi(r < R) = -\frac{\lambda}{4\pi\epsilon_0 R^2}\rho^2$$

The potential outside the cylinder can be obtained from the expression for the electric field above, with the constant being fixed by continuity of the potential at $\rho = R$,

$$\varphi(r > R) = -\frac{\lambda}{2\pi\epsilon_0}\left(\ln\frac{\rho}{R} + \frac{1}{2}\right)$$

(This is messy, but fortunately we can do with only the charge density and potential inside the cylinder by method 1. Taking infinite surface is tricky here because the log does not behave nicely) Thus, energy per unit length

$$\begin{aligned} W = W &= \frac{1}{2l}\int \rho(\vec{r})\varphi(\vec{r}) d^3r \\ &= \frac{1}{2l}\int_0^R \left(\frac{\lambda}{\pi R^2}\right)\left(-\frac{\lambda}{4\pi\epsilon_0 R^2}\rho^2\right) 2\pi\rho l d\rho \\ &= -\frac{\lambda^2 l}{4\pi\epsilon_0 R^4}\int_0^R \rho^3 d\rho = -\frac{\lambda^2}{16\pi\epsilon_0} \end{aligned}$$