MA-108 Differential Equations I

Manoj K Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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D1

Laplace Transforms

- $L(e^{at}) = \dfrac{1}{s-a}, \qquad s > a, \quad a \in \mathbb{R}.$
- $L(te^{at}) = \dfrac{1}{(s-a)^2}, \qquad s > a.$
- $L(t^n) = \dfrac{n!}{s^{n+1}}, \qquad s > 0, \quad n \geq 1$
- $L(\sin \omega t) = \dfrac{\omega}{s^2 + \omega^2}, \quad s > 0, \ \omega \in \mathbb{R}$

  $L(\cos \omega t) = \dfrac{s}{s^2 + \omega^2}, \quad s > 0$

Theorem (Linearity property)

Suppose $L(f_i)$ is defined for $s > s_i$ for $1 \leq i \leq n$. Let s_0 be maximum of s_i's and $c_i \in \mathbb{R}$. Then

$$L(c_1 f_1 + \ldots + c_n f_n) = c_1 L(f_1) + \ldots + c_n L(f_n), \quad s > s_0$$

Example

$L(e^{at}) = \dfrac{1}{s-a},\ s > a$. Then for $b \neq 0$.

$$L(\cosh bt) = L\left(\frac{e^{bt}+e^{-bt}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right)$$

$$= \frac{s}{s^2-b^2}, \quad s > \max\{b, -b\} = |b|$$

Example

$$L(\sinh bt) = L\left(\frac{e^{bt} - e^{-bt}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right)$$

$$= \frac{b}{s^2 - b^2}, \quad s > |b|$$

Theorem (First Shifting Theorem)

If $F(s) = L(f(t))$ *for* $s > s_0$, *then*

$$L(e^{at}f(t)) = F(s-a) \quad \text{for } s > s_0 + a$$

Proof.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad s > s_0$$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt, \quad s-a > s_0$$

$$= L(e^{at}f(t)), \quad s > a + s_0$$

$\square$

Example

1. $$F(1) = \frac{1}{s},\ s > 0 \implies F(e^{at}) = \frac{1}{s-a},\ s > a.$$
2. $$F(t^n) = \frac{n!}{s^{n+1}},\ s > 0 \implies$$
   $$F(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}},\ s > a.$$
3. $$F(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}, \quad F(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \quad s > 0$$
   $$F(e^{at} \sin \omega t) = \frac{\omega}{(s-a)^2 + \omega^2},\ s > a.$$
   $$F(e^{at} \cos \omega t) = \frac{s-a}{(s-a)^2 + \omega^2},\ s > a.$$
4. $$L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2},\ s > a + |b|.$$
5. $$L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2},\ s > a + |b|.$$

Example

Find Laplace transform of piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ e^{-t}, & t \geq 1 \end{cases}$$

$$\begin{aligned} L(f) &= F(s) = \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^1 e^{-st} dt + \int_1^\infty e^{-st} e^{-t} dt \\ &= -\frac{1}{s} e^{-st}\Big|_0^1 + \frac{-1}{s+1} e^{-(s+1)t}\Big|_1^\infty \\ &= \begin{cases} \dfrac{1-e^{-s}}{s} + \dfrac{e^{-(s+1)}}{s+1} & , \quad s > -1, s \neq 0 \\ 1 + \dfrac{1}{e} & , \quad s = 0 \end{cases} \end{aligned}$$

Inverse Laplace Transform

If $L(f(t)) = F(s)$ is the Laplace transform of f, then we say f is an **inverse Laplace transform** of F, and write

$$f = L^{-1}(F)$$

To solve an IVP using Laplace transform, we need to find inverse Laplace transforms.
We will use the table of Laplace transform to find inverse Laplace transform.

Theorem (Linearity Property)

If $F_1(s), \ldots, F_r(s)$ are Laplace transforms of $f_1(t), \ldots, f_r(t)$, i.e. $L^{-1}(F_i) = f_i$, then for $c_i \in \mathbb{R}$,

$$L^{-1}(c_1 F_1 + \ldots + c_r F_r) = c_1 L^{-1}(F_1) + \ldots + c_r L^{-1}(F_r).$$

Example

- $L^{-1}\left(\dfrac{1}{s^2-1}\right) = \sinh t,$
- $L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t.$
- $L(f) = F \implies L\left(e^{at}f(t)\right) = F(s-a)$

  Equivalently, $L^{-1}(F(s-a)) = e^{at}f(t) = e^{at}L^{-1}(F(s))$.
- $f(t) = L^{-1}\left(\dfrac{8}{s+5} + \dfrac{7}{s^2+3}\right)$

$$= L^{-1}\left(\frac{8}{s+5}\right) + L^{-1}\left(\frac{7}{s^2+3}\right)$$

$$= 8e^{-5t} + \frac{7}{\sqrt{3}}\sin\left(\sqrt{3}t\right)$$

Example

$$f(t) = L^{-1}\left(\frac{3s+8}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{3(s+1)+5}{(s+1)^2+4}\right) = e^{-t}L^{-1}\left(\frac{3s+5}{s^2+4}\right)$$

$$= e^{-t}L^{-1}\left(\frac{3s}{s^2+4}\right) + e^{-t}L^{-1}\left(\frac{5}{s^2+4}\right)$$

$$= e^{-t}\left[3\cos 2t + \frac{5}{2}\sin 2t\right]$$

- If P, Q are polynomials with deg $P <$ deg Q, then L^{-1} of $P(s)/Q(s)$ is found, by finding partial fractions.

Example

Find $L^{-1}(F(s))$, where

$$F(s) = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-1)(s-2)(s+1)}$$

The partial fraction of $F(s)$ is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}$$

$$A = F(s)s\Big|_{s=0} = \frac{6 + (s+1)(s^2 - 5s + 11)}{(s-1)(s-2)(s+1)}\Big|_{s=0} = \frac{17}{2}$$

$$B = F(s)(s-1)\Big|_{s=1} = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-2)(s+1)}\Big|_{s=1}$$

$$= \frac{6 + 2.7}{-2} = -10$$

Example (continued ...)

$$C = F(s)(s-2)\Big|_{s=2} = \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s+1)}\Big|_{s=2}$$

$$= \frac{6+3.5}{6} = \frac{7}{2}$$

$$D = F(s)(s+1)\Big|_{s=-1} = \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)}\Big|_{s=-1}$$

$$= \frac{6}{-6} = -1$$

$$L^{-1}\left(F(s)\right) = L^{-1}\left(\frac{17}{2s} - \frac{10}{s-1} + \frac{7}{2(s-2)} - \frac{1}{s+1}\right)$$

$$= 17/2 - 10e^{t} + (7/2)e^{2t} - e^{-t}$$

Example

Let $F(s) = \dfrac{s^2 - 5s + 7}{(s+2)^3}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

To find A, B, C, expand the numerator of $F(s)$ in powers of $(s+2)$.

$$s^2 - 5s + 7 = ((s+2) - 2)^2 - 5((s+2) - 2) + 7$$

$$= (s+2)^2 - 9(s+2) + 21$$

Therefore

$$A = 1,\ B = -9,\ C = 21$$

Example (continued ...)

$$F(s) = \left(\frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3} \right)$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3} \right)$$

$$= e^{-2t} L^{-1}\left(\frac{1}{s} - \frac{9}{s^2} + \frac{21}{s^3} \right)$$

$$= e^{-2t}\left(1 - 9t + \frac{21}{2} t^2 \right)$$

Example

Let $F(s) = \dfrac{8+3s}{(s^2+1)(s^2+4)}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$\frac{8+3s}{(s^2+1)(s^2+4)} = \frac{A+Bs}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$8+3s = (A+Bs)(s^2+4) + (C+Ds)(s^2+1)$$

$$= s^3(B+D) + s^2(A+C) + s(4B+D) + (4A+C)$$

Equate the powers of s and solve to get A, B, C, D.

We have a simpler method in this particular case.
Here denominator of $F(s)$ is a polynomial in s^2, put $x = s^2$ and use

Example (continued ...)

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right)$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right)$$

$$L^{-1}(F(s)) = L^{-1}\left[\frac{8}{3(s^2+1)} + \frac{s}{s^2+1} - \frac{8}{3(s^2+4)} - \frac{s}{s^2+4}\right]$$

$$= \left(\frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t\right)$$