MA-108 Differential Equations I

Manoj K Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

5th April, 2018
D1

Laplace transform of Derivatives

Our goal is to apply Laplace transforms to differential equations. So we want to know the Laplace transform of derivative of a function.

Theorem

Let f be continuous on $[0, \infty)$ and of exponential order s_0.
Let f' be piecewise continuous on $[0, \infty)$.
Then the Laplace transform for f' exists for $s > s_0$ and is given by

$$L(f') = sL(f) - f(0)$$

We do not need f' to be of exponential order.

- Continuity of f is required in the equality of formula.

Proof.

Since f' is piecewise continuous, there exists $0 = t_0 < t_1 < \ldots < t_n = T$ such that f' is continuous on (t_{i-1}, t_i). Then

$$\int_0^T e^{-st} f'(t)\, dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} e^{-st} f'(t)\ dt$$

$$= \sum_{i=0}^{n-1} \left[f(t) e^{-st} \Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} (-s) e^{-st} f(t)\ dt \right]$$

$$= \ f(t_n) e^{-st_n} - e^{-st_0} f(t_0) + s \int_{t_0}^{t_n} e^{-st} f(t) dt$$

$$= f(T) e^{-sT} - f(0) + s \int_{t_0}^{T} e^{-st} f(t)\, dt$$

which goes to $\quad sL(f) - f(0) \quad$ as limit $T \to \infty$. □

Example

Let us compute $L(\cos \omega t)$ using that

$$L(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

For $f(t) = \sin \omega t$, use $L(f') = sL(f) - f(0)$.

$$L(\omega \cos \omega t) = s\frac{\omega}{s^2 + \omega^2} - 0$$

$$\omega L(\cos \omega t) = s\frac{\omega}{s^2 + \omega^2}$$

$$L(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

Example

Let us compute $L(t^n)$ for $n \geq 1$.

We have computed $L(t) = \dfrac{1}{s^2}$ for $s > 0$.

$$L((t^2)') = sL(t^2) - f(0) \implies L(t^2) = \frac{1}{s}L(2t) = \frac{2}{s^3}$$

Use induction on n. Assume

$$L(t^{n-1}) = \frac{(n-1)!}{s^n}$$

and show that

$$L(t^n) = \frac{n!}{s^{n+1}}$$

Example

Consider the IVP $\quad y' + y = 0, \quad y(0) = 5.$
We already know that the solution is given by $y = 5e^{-x}$.
Let us verify this using Laplace transform.

Let us assume that the given equation has a solution ϕ and it is of exponential order s_0 for some s_0.

$$L(\phi' + \phi) = L(0) \implies sL(\phi) - \phi(0) + L(\phi) = 0$$

$$L(\phi) = \frac{5}{s+1} \implies \phi(x) = 5e^{-x}$$

Remark. Solving IVP with Laplace transform requires initial conditions at $t = 0$.

We have have the following result about $L(f^{(n)})$.

Theorem

Assume the following.

- $f, \ f', \ \ldots, f^{(n-1)}$ *are continuous on* $[0, \infty)$.
- $f^{(n)}$ *is piecewise continuous on* $[0, \infty)$.
- $f, \ f', \ \ldots, f^{(n-1)}$ *are of exponential order* s_0 *for some* s_0.

Then Laplace transforms of $f, \ f', \ \ldots, f^{(n-1)}, \ f^{(n)}$ *exists and*

$$L(f^{(n)}) = s^n L(f) - f^{(n-1)}(0) - s f^{(n-2)}(0) - \ldots - s^{n-1} f(0).$$

Proof for $n = 2$

$$L(f'') = sL(f') - f'(0) = s[sL(f) - f(0)] - f'(0)$$

Example

Consider the IVP

$$y'' + 4y = 3\sin t,\ y(0) = 1,\ y'(0) = -1$$

We know this equation has a unique solution ϕ on $\mathbb{R}$. Assume ϕ is of exponential order $s_0 \geq 0$ and apply Laplace transform on $[0, \infty)$. We get that for all $s > s_0$

$$L(\phi'') + 4L(\phi) = \frac{3}{s^2+1}$$

$$(s^2 L(\phi) - s\phi(0) - \phi'(0)) + 4L(\phi) = \frac{3}{s^2+1}$$

$$\implies (s^2 + 4)L(\phi) - s + 1 = \frac{3}{s^2+1}$$

Example (continued ...)

$$L(\phi) = \frac{3}{(s^2+1)(s^2+4)} + \frac{s-1}{s^2+4}$$

$$= \frac{1}{s^2+1} - \frac{1}{s^2+4} + \frac{s}{s^2+4} - \frac{1}{s^2+4}$$

$$\phi(t) = L^{-1}\left(\frac{1}{s^2+1} - \frac{2}{s^2+4} + \frac{s}{s^2+4}\right)$$

$$= \sin t - \sin 2t + \cos 2t$$

Example

Solve IVP

$$y'' + 2y' + 2y = 1, \quad y(0) = -3, \; y'(0) = 1$$

The equation has a unique solution ϕ defined on all of $\mathbb{R}$. Assume ϕ is of exponential of order s_0. Then for all $s \geq s_0$,

$$L(\phi'') + 2L(\phi') + 2L(\phi) = L(1)$$

$$(s^2 L(\phi) - s\phi(0) - \phi'(0)) + 2(sL(\phi) - \phi(0)) + 2L(\phi) = \frac{1}{s}$$

$$(s^2 + 2s + 2)L(\phi) - (s+2)\phi(0) - \phi'(0) = \frac{1}{s}$$

$$((s+1)^2 + 1)L(\phi) + 3(s+2) - 1 = \frac{1}{s}$$

Example (continued ...)

$$L(\phi) = \frac{1-(3s+5)s}{((s+1)^2+1)s} := F(s)$$

We want to compute $L^{-1}(F(s))$. We use partial fractions.

$$F(s) = \frac{1-3s^2-5s}{((s+1)^2+1)s} = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}$$

$$1-3s^2-5s = A((s+1)^2+1) + (B(s+1)+C)s$$

$$s=0 \implies 1 = 2A \implies A = 1/2$$

$$s=-1 \implies 3 = A - C \implies C = -5/2$$

$$s=1 \implies -7 = 5A + 2B + C \implies B = -7/2$$

Example (continued ...)

$$L(\phi(t)) = \frac{1}{2s} - \frac{7(s+1)}{2((s+1)^2+1)} - \frac{5}{2((s+1)^2+1)}$$

$$\phi(t) = L^{-1}\left[\frac{1}{2s} - \frac{7(s+1)}{2((s+1)^2+1)} - \frac{5}{2((s+1)^2+1)}\right]$$

$$= \frac{1}{2} - e^{-t}L^{-1}\left[\frac{7s}{2(s^2+1)}\right] - e^{-t}L^{-1}\left[\frac{5}{2(s^2+1)}\right]$$

$$= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$$

Example

More generally, to solve a constant coefficient IVP

$$y'' + py' + qy = r(t), \quad y(0) = a,\ y'(0) = b, \ \ p, q \in \mathbb{R}$$

let ϕ be the unique solution, which has a Laplace transform for all $s \geq s_0$. Applying Laplace transform, we get

$$(s^2 L(\phi) - s\phi(0) - \phi'(0)) + p(sL(\phi) - \phi(0)) + qL(\phi) = L(r)$$

$$\implies (s^2 + ps + q)L(\phi) = L(r) + sa + b + pa$$

Write this as $L(\phi) = F(s)$ and compute the inverse Laplace transform of F, to get $\phi(t)$.

Unit Step Function

Let us consider IVP with constant coefficients, where the forcing function $r(t)$ is piecewise continuous.
To solve it using Laplace transform, we need to find Laplace transform of piecewise continuous functions.

Definition

The unit (or Heaviside) step function is defind as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Replacing t by $t - a$, we get

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Express the following functions in terms of unit step functions.

Example

(1) Ramp Function

$$\begin{cases} 0, & 0 < t < a \\ t - a, & t > a \end{cases} = (t-a)u(t-a)$$

(2)

$$f(t) = \begin{cases} \sin t, & 0 < t < t_0 \\ t, & t \geq t_0 \end{cases}$$

$$= \sin t + u(t - t_0)(t - \sin t)$$

Example

(3)

$$f(t) = \begin{cases} \sin t, & 0 < t < t_0 \\ \cos t, & t_0 \leq t \leq t_1 \\ t, & t > t_1 \end{cases}$$

$$= \sin t + u(t - t_0)(\cos t - \sin t) + u(t - t_1)(t - \cos t)$$

(4)

$$f(t) = \begin{cases} f_1, & 0 \leq t < t_1 \\ f_2, & t_1 \leq t < t_2 \\ . & . \\ f_n, & t_{n-1} \leq t \end{cases}$$

$$= f_1 + u(t - t_1)(f_2 - f_1) + \ldots + u(t - t_{n-1})(f_n - f_{n-1})$$

Writing a piecewise continuous function in terms of unit step functions simplifies the computation of its Laplace transform.

Theorem (Second Shifting Theorem)

Let $g(t)$ be defined for $t \geq 0$.
Assume $L(g(t+a))$ exists for $s > s_0$, where $a \geq 0$.
Then $L(u(t-a)g(t))$ exists for $s > s_0$, and

$$L(u(t-a)g(t)) = e^{-sa}L(g(t+a)).$$

$$L(u(t-a)g(t)) = \int_0^\infty e^{-st}u(t-a)g(t)\,dt$$

$$= \int_a^\infty e^{-st}g(t)\,dt = \int_0^\infty e^{-s(x+a)}g(x+a)\,dx$$

$$= e^{-sa}\,L(g(t+a))$$

Theorem (Second Shifting Theorem)

If $a \geq 0$ and $L(f)$ exists for $s > s_0$, then $L(u(t-a)f(t-a))$ exists for $s > s_0$ and

$$L(u(t-a)f(t-a)) = e^{-as}L(f(t)) = e^{-as}F(s).$$

Example

(1) $L(u(t-a)) = e^{-as}L(1) = \dfrac{e^{-as}}{s}$.

(2)

$$L(u(t-1)(t^2+1)) = e^{-s}L((t+1)^2+1)$$

$$= e^{-s}L(t^2+2t+2) = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{2}{s}\right)$$

Example

Find $L(f(t))$, where $f(t) = \begin{cases} f_1 = 1, & 0 \leq t < 2 \\ f_2 = -2t + 1, & 2 \leq t < 3 \\ f_3 = 3t, & 3 \leq t < 5 \\ f_4 = t - 1, & t \geq 5 \end{cases}$

Write $f(t)$ in terms of unit step functions as

$$f(t) = f_1 + u(t-2)(f_2 - f_1) + u(t-3)(f_3 - f_2) + u(t-5)(f_4 - f_3)$$

$$= 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1)$$

$$L(f(t)) = \frac{1}{s} - 2e^{-2s}L(t+2) + e^{-3s}L(5(t+3)-1) - e^{-5s}L(2t+11)$$

Example

$$L(f) = \frac{1}{s} - 2e^{-2s}L(t+2) + e^{-3s}L(5t+14) - e^{-5s}L(2t+11)$$

$$= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2} + \frac{11}{s}\right)$$

Example

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \frac{\pi}{2} \\ \cos t - 3\sin t, & \dfrac{\pi}{2} \leq t \end{cases}$$

$$f(t) = \sin t + u\left(t - \frac{\pi}{2}\right)(\cos t - 4\sin t)$$

$$L(f) = L(\sin t) + e^{-\pi s/2} L\left(\cos\left(t + \frac{\pi}{2}\right) - 4\sin\left(t + \frac{\pi}{2}\right)\right)$$

$$= \frac{1}{s^2+1} + e^{-\pi s/2}\left(\frac{-1-4s}{s^2+1}\right)$$

Inverse Laplace transforms

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{s}$$

Use

$$u(t-a)f(t-a) = L^{-1}\left[e^{-as}L(f(t))\right]$$

$$L^{-1}\left(\frac{e^{-2s}}{s}\right) = L^{-1}(e^{-2s}L(1)) = u(t-2)$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{s^2}$$

Here

$$F(s) = \frac{1}{s^2} \implies f(t) = t$$

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)f(t-2) = u(t-2)(t-2)$$

$$= \begin{cases} 0, & t \leq 2 \\ t-2, & t > 2 \end{cases}$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{s-3}$$

Here

$$F(s) = \frac{1}{s-3} \implies f(t) = e^{3t}$$

$$L^{-1}\left(\frac{e^{-2s}}{s-3}\right) = u(t-2)f(t-2)$$

$$= u(t-2)e^{3(t-2)} = \begin{cases} 0, & t \leq 2 \\ e^{3(t-2)}, & t > 2 \end{cases}$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{(s-3)^2}$$

Here

$$F(s) = \frac{1}{(s-3)^2} \implies f(t) = te^{3t}$$

$$L^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = u(t-2)f(t-2)$$

$$= u(t-2)(t-2)e^{3(t-2)} = \begin{cases} 0, & t \le 2 \\ (t-2)e^{3(t-2)}, & t > 2 \end{cases}$$

Example

Find inverse Laplace transform of

$$F(s) = e^{-s}\frac{1}{2s} - e^{-2s}\frac{s+1}{(s+1)^2+1}.$$

$$L^{-1}\left(\frac{1}{2s}\right) = \frac{1}{2}, \qquad L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) = e^{-t}\cos t$$

$$L^{-1}(F(s)) = \frac{1}{2}u(t-1) - u(t-2)e^{-(t-2)}\cos(t-2)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ 1/2, & 1 \le t < 2 \\ \dfrac{1}{2} - e^{-(t-2)}\cos(t-2), & t \ge 2 \end{cases}$$