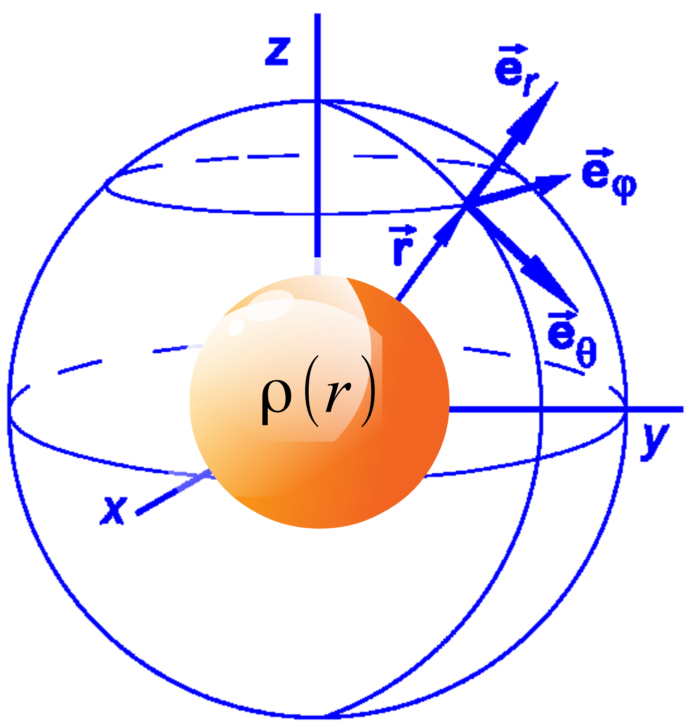
Electric field in some simple situations (symmetry + Gauss's Law)

No charge distribution is really infinite!

Very close to the surface/
Objects with very large aspect ratio

infinite approximation is useful

Spherically symmetric charge distribution



$$E_\theta = 0 \text{ and } E_\phi = 0. \; Why?$$

rotate about z-axis

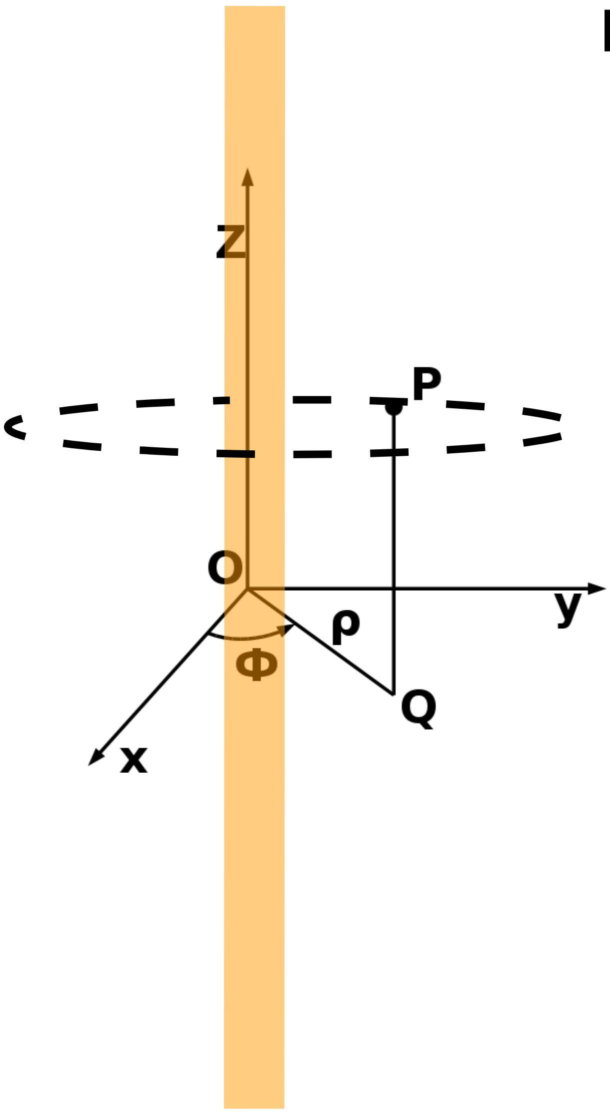
The field at the tip of $\vec{r}$, must be same at all points visited by $\vec{r}$

$$\oint \vec{E}.\vec{dl} = 0 \quad \rightarrow E_\phi = 0$$

Rotate about x-axis : show $E_\theta = 0$

$$\boxed{E_r 4\pi R^2 = \frac{1}{\epsilon_0}\int_0^R \rho(r) 4\pi r^2 \, dr}$$

Electric field in some simple situations (symmetry + Gauss's Law)



Long narrow wire type charge distribution

$$E_\phi = 0 \text{ and } E_z = 0.\ \textit{Why?}$$

rotate about z-axis

The field at the tip of ***P***, *must be same at all points visited by* ***P***

$$\oint \vec{E}\,.\,\vec{dl} = 0 \quad \rightarrow E_\phi = 0$$

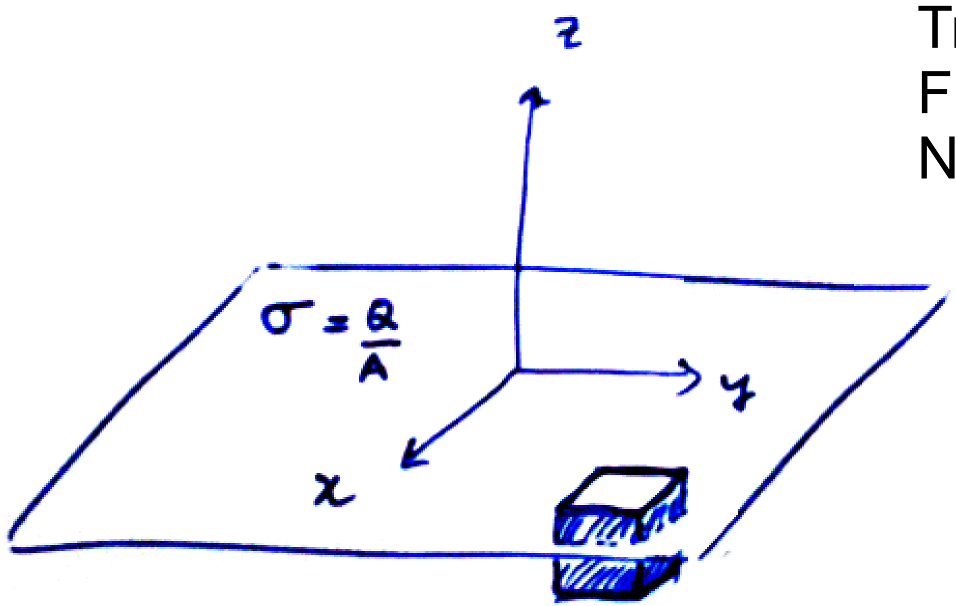
flip about z-axis : show $E_z = 0$

There is nothing to *chose z* from *-z*

$$E_\rho 2\pi\rho = \frac{1}{\epsilon_0}\lambda \qquad \lambda : \textit{charge per unit length}$$

Electric field in some simple situations (symmetry + Gauss's Law)

Infinite sheet of charge



Why cannot there be a $E_{\parallel}$ component ?

Rotate the sheet about any point
Translate sheet by any in plane vector
Field cannot change.
Not possible if there is finite component.

$$E_z = \begin{cases} \dfrac{\sigma}{2\epsilon_0} & : z > 0 \\ -\dfrac{\sigma}{2\epsilon_0} & : z < 0 \end{cases}$$

Give a symmetry argument to show that E(z) = -E(-z) must hold. Flip the "sheet“ switching "topside" and "bottom-side“. What should happen?

Easy to extend to a sheet with finite thickness .... work it out.

Superpose the field of two parallel plates, calculate capacitance

Some extra charge is placed in a conductor : why does it go to the surface?

$$\vec{\nabla} . \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{J} = \sigma \vec{E}$$

Ohm's Law

$$\frac{1}{\sigma} \vec{\nabla} . \vec{J} = \frac{\rho}{\epsilon_0}$$

Continuity equation

$$-\frac{\partial \rho}{\partial t} = \frac{\sigma}{\epsilon_0} \rho$$

$$\boxed{\rho = \rho(0) e^{-\frac{\sigma}{\epsilon_0} t}}$$

Div **J** = 0 | steady current flow (as in a wire)
OR
J=0, **E**=0 (pure electrostatics)

In either case the excess charge is NOT in bulk

But charge is conserved.
Where does the excess charge go?
From "bulk“ to surface.

How fast?

$$For\ good\ metals\ like \quad Cu, Ag, Au..$$

$$\sigma \sim 10^7 - 10^8\ S/m$$

$$\epsilon_0 = 8.85 \times 10^{-12}\ F/m$$

$$\frac{\sigma}{\epsilon_0} \sim 10^{20}$$

$$time\ constant \sim 10^{-20}\ sec$$

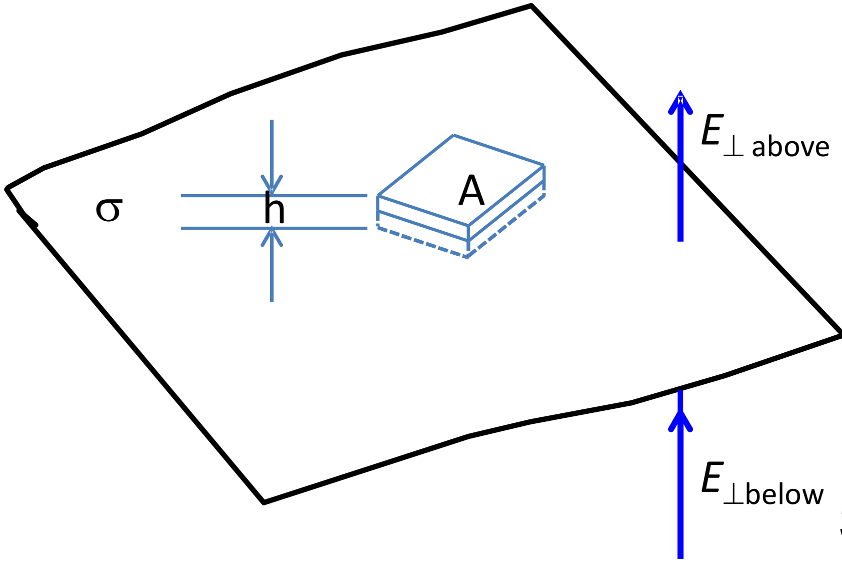
Boundary conditions and other characteristics of metals/conductors

You should now be able to justify the following :

- If no current is flowing in the conductor then E=0 inside
- All the excess charge resides on the surface of the conductor, even if there is a current flow.
- The conductor is an equipotential if J=0 (pure electrostatics)
- The electric field is normal to the surface (gradient is perpendicular to an equipotential)
- The surface charge density on a metal depends on the local radius of curvature.
- The electric field is strongest just outside sharp pointy edges.

Two spheres of radii R, r (R >>r) are in contact so at the same potential. Which one has a larger ***E*** *just outside? Why?*
Think of a cone as made of a series of successively smaller spheres...where is the electric field strongest?

Generic Electrostatic Boundary conditions (normal and tangential components)

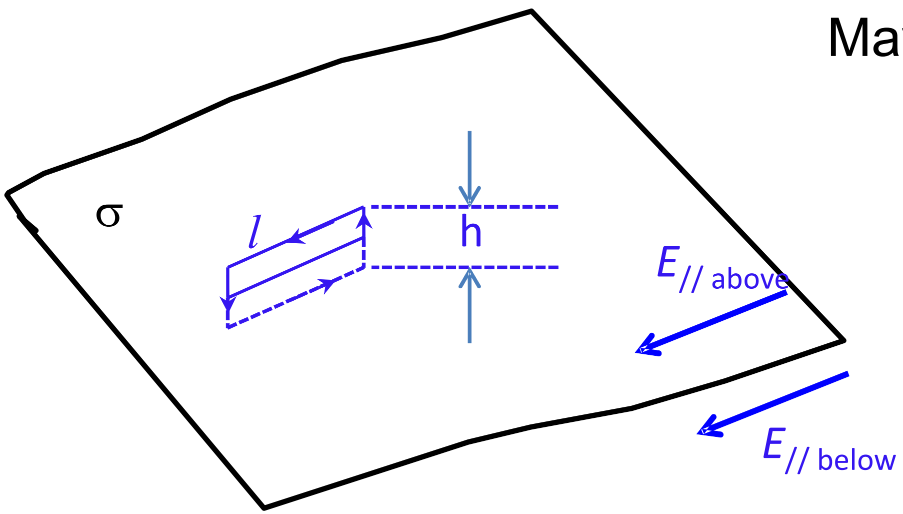


$$\oiint \vec{E}\,.\,d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

$$E_{\perp above} - E_{\perp below} = \frac{\sigma}{\epsilon_0}$$

Normal component may be discontinuous

Surfaces may be finite.
The charge density may vary from place to place.
Surface is not necessarily equipotential
May be conducting or non-conducting



$$\oint \vec{E}\,.\,d\vec{l} = 0$$

$$E_{\parallel above} - E_{\parallel below} = 0$$

Tangnetial component is continuous,
But not necessarily zero.

But electrostatic potential V is always continuous at a surface.
Reason: The discontinuity in E is finite.
So $V_2 - V_1 = -E.dl$ will go to zero as dl goes to zero

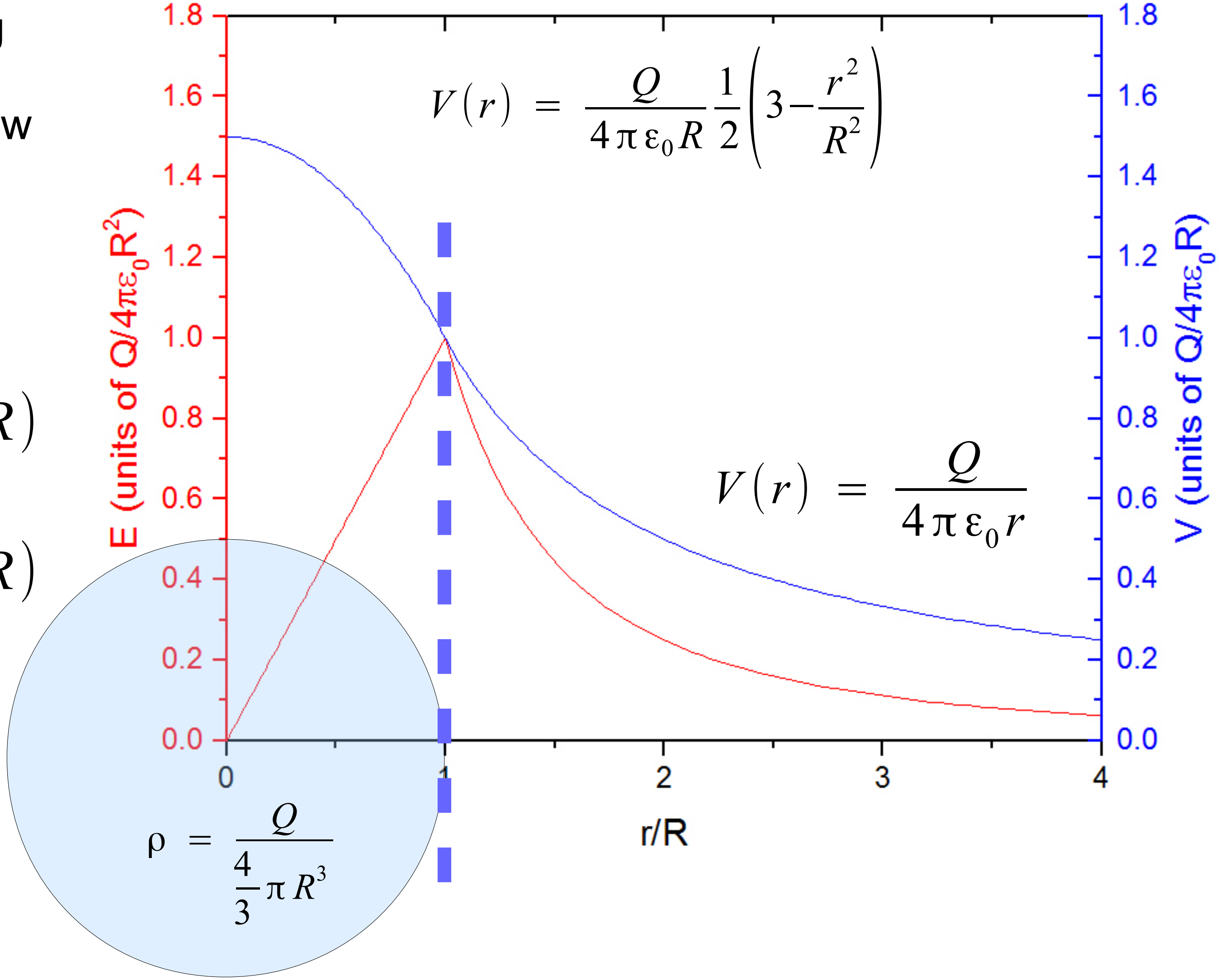
Potentials and Fields in some simple cases : charged sphere

Uniformly charged insulating sphere of radius R
Use symmetry + Gauss's Law

Solve for E(r)
Integrate to get V(r)

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r} \qquad (r<R)$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \qquad (r>R)$$

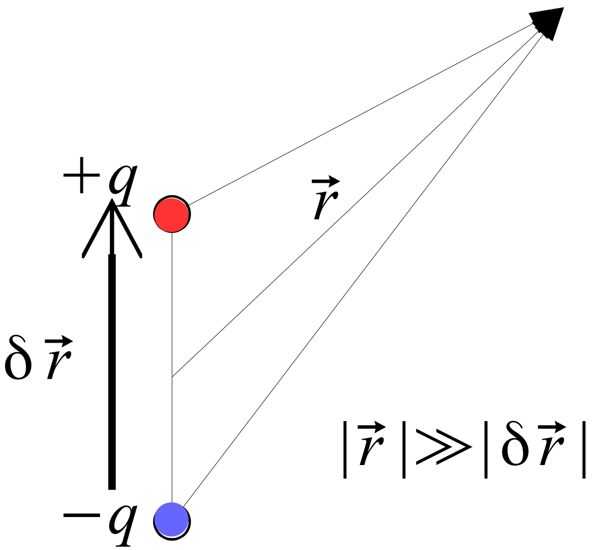


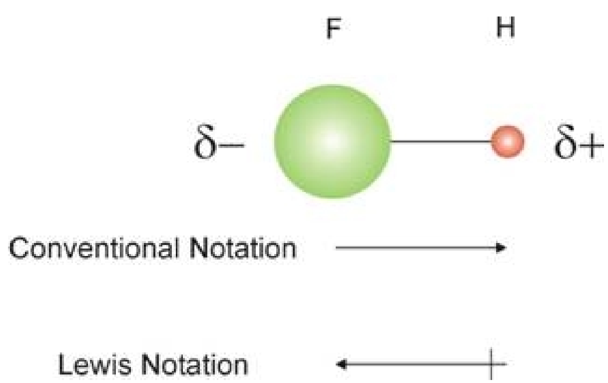
Notice the continuity at r=R. Is it consistent with the boundary conditions discussed earlier?

What if the sphere is a conducting or metallic sphere? vHollow sphere?

Use similar reasoning for a solid cylinder/wire.

Potentials and Fields in some simple cases : dipole





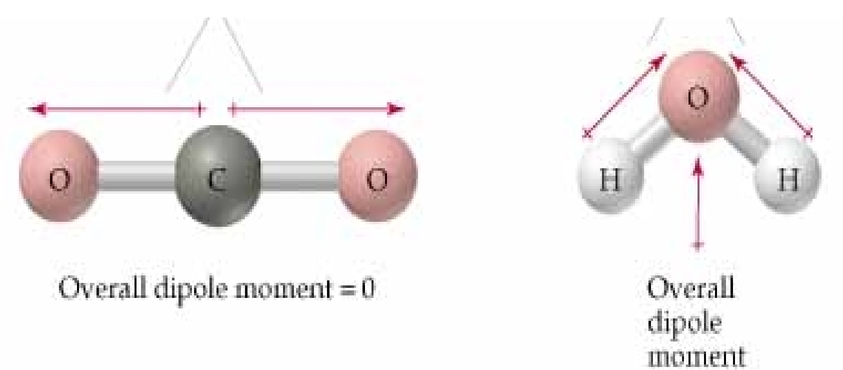
$$V(r) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left|\vec{r}-\frac{\delta\vec{r}}{2}\right|} - \frac{1}{\left|\vec{r}+\frac{\delta\vec{r}}{2}\right|}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2-\vec{r}\,.\,\delta\vec{r}+\frac{\delta\vec{r}}{2}\,.\,\frac{\delta\vec{r}}{2}}} - \frac{1}{\sqrt{r^2+\vec{r}\,.\,\delta\vec{r}+\frac{\delta\vec{r}}{2}\,.\,\frac{\delta\vec{r}}{2}}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left[\left(1-\frac{\vec{r}\,.\,\delta\vec{r}}{r^2}\right)^{-1/2} - \left(1+\frac{\vec{r}\,.\,\delta\vec{r}}{r^2}\right)^{-1/2}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left[\left(1+\frac{1}{2}\frac{\vec{r}\,.\,\delta\vec{r}}{r^2}-\ldots\right)-\left(1-\frac{1}{2}\frac{\vec{r}\,.\,\delta\vec{r}}{r^2}-\ldots\right)\right]$$

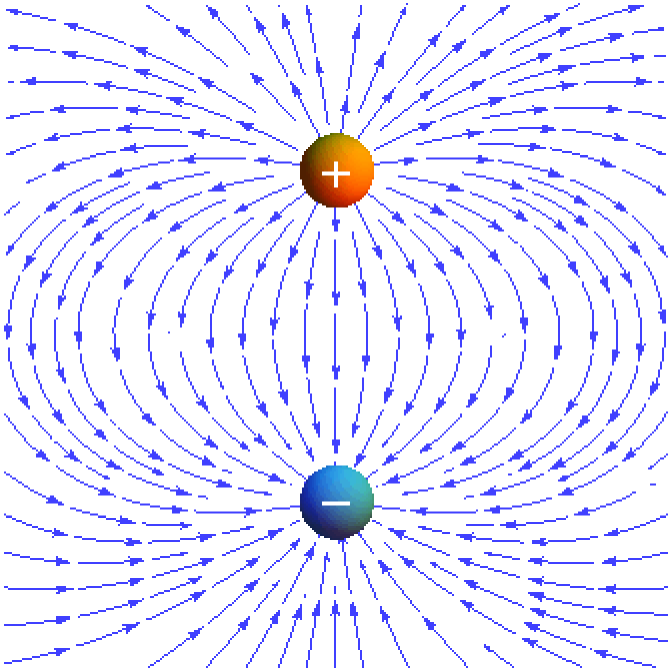
$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\,.\,\hat{r}}{r^2} \qquad \text{where} \quad (\vec{p}=q\,\delta\vec{r})$$



Monopole potential falls off as 1/r
Dipole potential falls off as $1/r^2$

Potentials and Fields in some simple cases : dipole

$$\vec{E} = -\nabla V$$
$$= \frac{p}{4\pi\varepsilon_0}\left[\hat{\epsilon}_r \frac{\partial}{\partial r}\frac{\cos\theta}{r^2} + \frac{1}{r}\hat{\epsilon}_\theta \frac{\partial}{\partial \theta}\frac{\cos\theta}{r^2}\right]$$
$$= \frac{p}{4\pi\varepsilon_0 r^3}\left[-3\cos\theta\,\hat{\epsilon}_r - \sin\theta\,\hat{\epsilon}_\theta\right]$$



Can also be written in a co-ordinate system independent way :

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}.\hat{r})\hat{r} - \vec{p}\right]$$

Solution of Laplace's equation: Average value theorem : statement

A scalar function $V(\vec{r})$ satisfies $\nabla^2 V = 0$

Consider a sphere of radius R : integrate $\nabla^2 V$ over the volume

$$\int_{vol} \vec{\nabla}.(\vec{\nabla} V) d\tau = \int_{surface} \vec{\nabla} V . d\vec{S}$$

Write the gradient in spherical polar

$$= \int [\hat{\epsilon}_r \frac{\partial V}{\partial r} + \hat{\epsilon}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\epsilon}_\phi \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}] . d\vec{S}$$

$$= \int \frac{\partial V}{\partial r} R^2 \sin\theta \, d\theta \, d\phi$$

Only the radial component survives because d**S** points radially outwards

$$0 = R^2 \frac{\partial}{\partial r} \int_{surface} V(r, \theta, \phi) \sin\theta \, d\theta \, d\phi$$

The average value $\langle V(\theta, \phi) \rangle_r$ over a sphere is independent of r.

In the limit $r \rightarrow 0$, we must have $\langle V \rangle = V(0)$

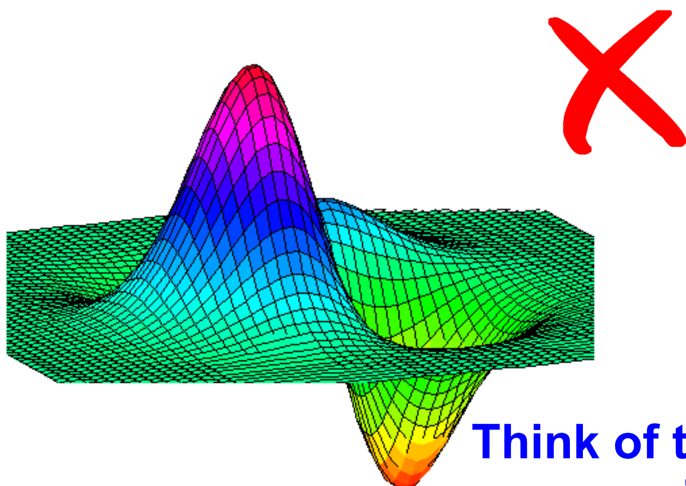
So average value over a spherical surface = value at the center

Prove the 2D (using cylindrical polar) and 1d (trivial !) cases as exercise.

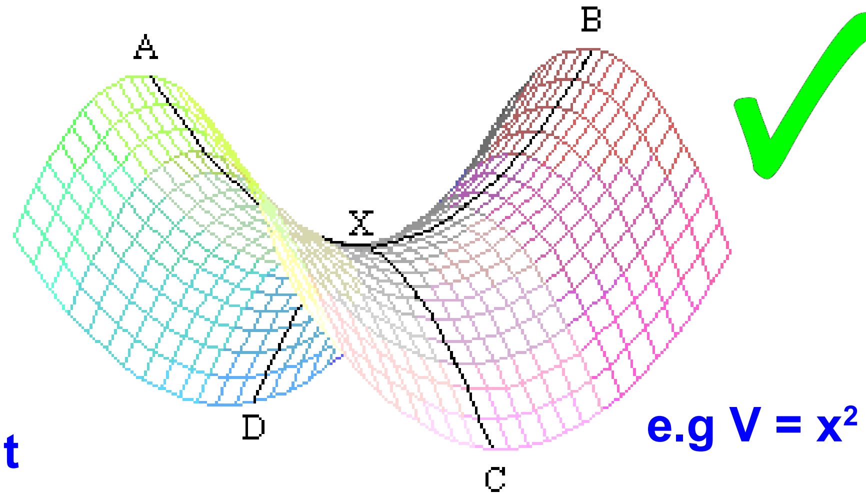
Solution of Laplace's equation: Average value theorem : consequences

There are no maxima or minima of V in a region where $\nabla^2 V = 0$

But there can be saddle points



Think of the gradient near maxima/minima



e.g $V = x^2 - y^2$

No stable equilibrium possible in purely electrostatic field $(Earnshaw)$

All extremal values must occur at the boundary

$V = const$ on ALL points on ALL boundaries $\Rightarrow V$ is constant everywhere

UNIQUENESS: There is only one possible solution of $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

consistent with a given boundary condition

Solution of Laplace's equation: Uniqueness theorem

Two functions V_1 and V_2 satisfy $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

With the same boundary conditions

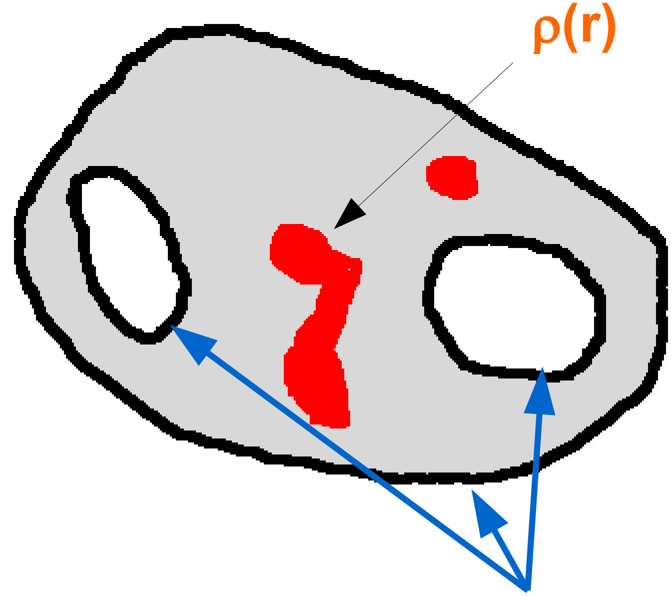
Let $\psi = V_1 - V_2$ Then

$\nabla^2 \psi = 0$ and $\psi = 0$ on ALL boundaries

Implies $\psi = 0$ everywhere

To prove this:

consider the vector function: $\psi \vec{\nabla} \psi$



$$\int_{vol} \nabla . (\psi \vec{\nabla} \psi) d\tau = \int_{surface} \psi \vec{\nabla} \psi . d\vec{S}$$

If a "guess" satisfies the boundary condition then that MUST be the solution

$$\int_{vol} \left[\psi \nabla^2 \psi + |\vec{\nabla} \psi|^2 \right] d\tau = 0$$

$$\int_{vol} |\vec{\nabla} \psi|^2 d\tau = 0$$

Possible only if ψ =constant=0 everywhere

Why is a metal cavity a "shield" ?

Arbitrary charges are outside the cavity (Q1...Qn)

Charges will be induced in the wall of the cavity.

But the wall remains an equipotential.

Inside the cavity V=0 is one possible solution satisfying the boundary conditions.

THAT IS THE UNIQUE SOLUTION.

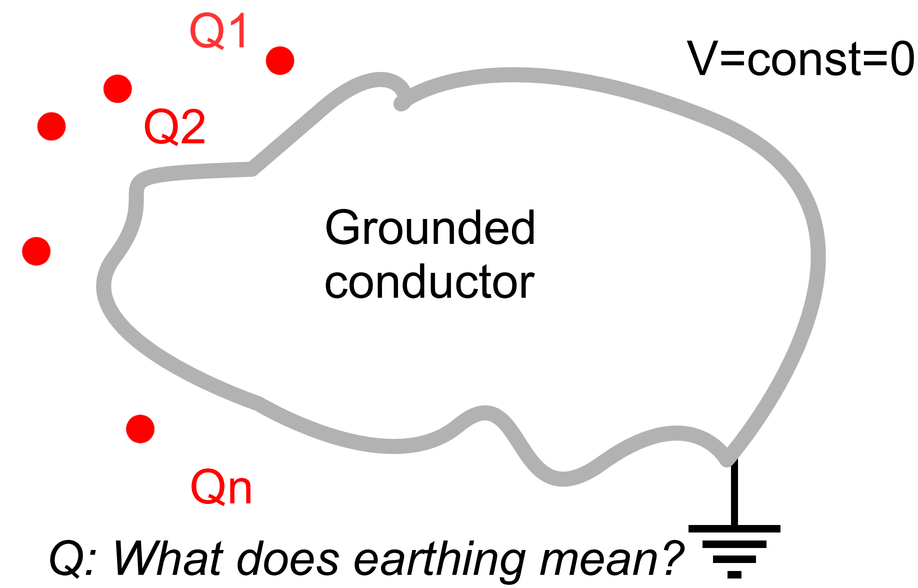
What if the wall is not fixed at V=0 (i.e. floating)?

V=constant is still correct, but the constant will depend on the charge distribution outside.

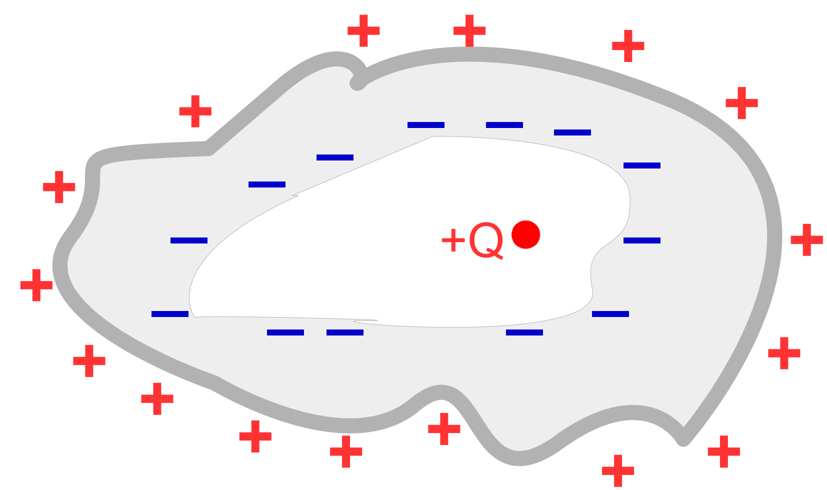
If charges are placed inside?

$$\nabla^2 V = -\frac{\rho_{in}}{\epsilon_0}$$

$V = 0$ (boundary condition) irrespective of ρ_{out}

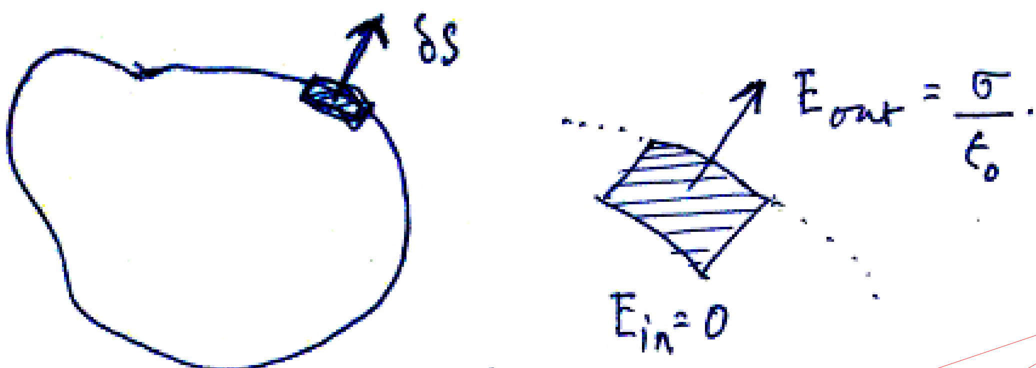


Q: What does earthing mean? Why is the potential of the earth fixed?



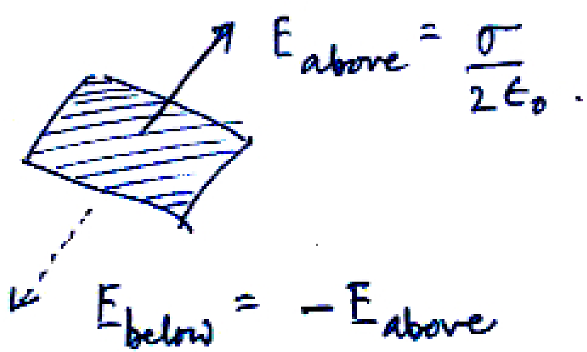
Floating conductor
Equal amounts +Q and -Q on inner and outer surfaces.

Electrostatic pressure on a conducting shell/ charged bubble



V = const. | $E_{\parallel} = 0$.

If δs was treated in isolation



Outward Pressure=Force/area
The conducting surface simplifies the calculation. For an arbitrary surface it is more complex.....

Force on the element dS is due to E field created by all the other charges.

$$\delta Q = \sigma \, \delta S \; creates$$
$$E_{\perp} = \frac{\sigma}{2\epsilon_0} \quad \text{above and below}$$
$$But \; E_{\perp} = \frac{\sigma}{\epsilon_0} \quad \text{(above)}$$
$$E_{\perp} = 0 \quad \text{(below/inside)}$$

The difference must be due to the field created by all the other charges.

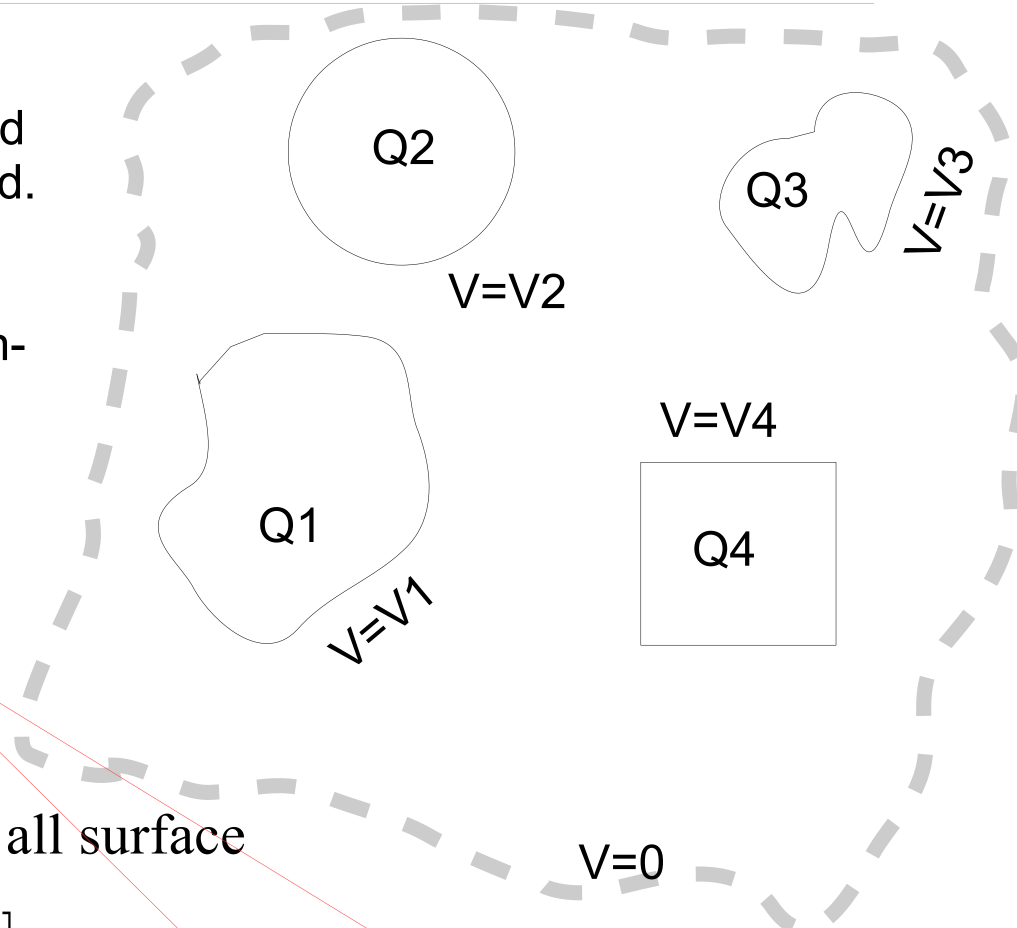
The force on dS is
Charge in dS x Field due to all charges NOT in dS

$$\delta F = (\sigma \, \delta S) \frac{\sigma}{2\epsilon_0} = \left(\frac{\sigma^2}{2\epsilon_0} \right) \delta S$$

Electrostatics of conductors : uniqueness theorem 2 & capacitance

If the charge on ALL the **conductors** is specified then the potential V(x,y,z) is uniquely determined.

Notice that we are not specifying the charge distribution, only the total charge. That's the non-trivial content.



Suppose two distinct solutions exist

$U(x,y,z), V(x,y,z)$ *: Both must satisfy*

$$\int_{surf\ i} \left(-\vec{\nabla} U\right).d\vec{S} = \int_{surf\ i} \left(-\vec{\nabla} V\right).d\vec{S} = Q_i$$

U and V must give equipotentials On each conducting surface, but we do not claim that they are the same constant to start with.

define $\psi = U - V$, *& integrate* $\psi \vec{\nabla} \psi$ over all surface

$$\sum_i \int_{surf\ i} \left(\psi \vec{\nabla} \psi\right).d\vec{S} = \int_{all\ vol} \left[\psi \nabla^2 \psi + |\nabla \psi|^2\right] d\tau$$

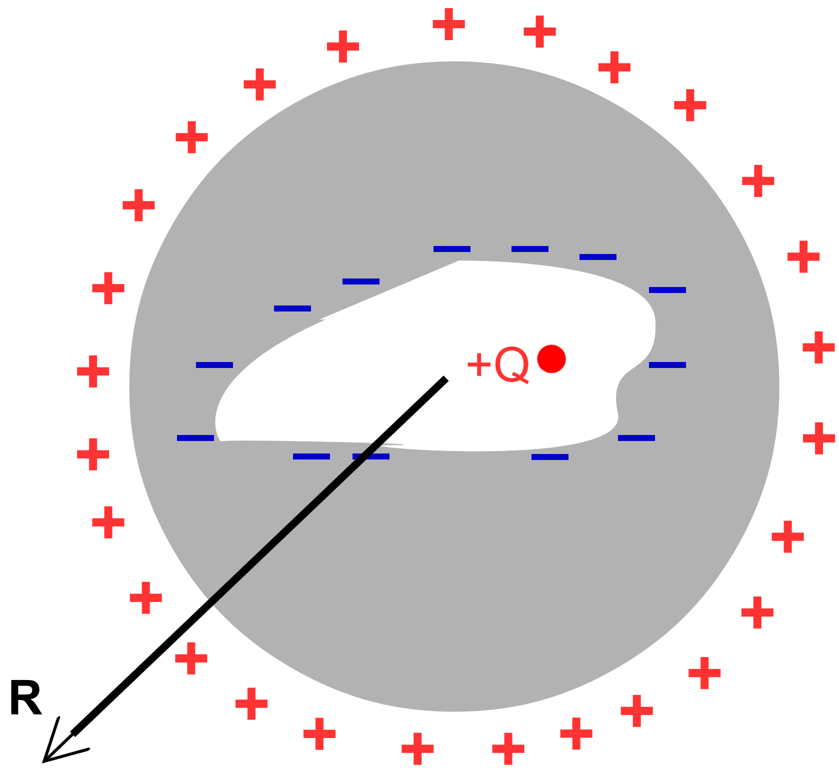
= 0

$LHS = 0 :$ *why ?*

$$So \int_{all\ vol} |\vec{\nabla} \psi|^2 d\tau = 0$$

$$Hence \quad U - V = 0$$

Why is a metal cavity a "shield" ?



Floating conductor
Equal amounts +Q and -Q on inner and outer surfaces.

Special case: irregular cavity inside a sphere.

The charge density $\sigma(\theta, \phi)$ *is uniform*

Surface is equipotential and E=0 inside. Use the boundary condition on Normal component of E and local charge density to prove the result.

$$E(R) = \frac{Q}{4\pi\epsilon_0 R^2}$$

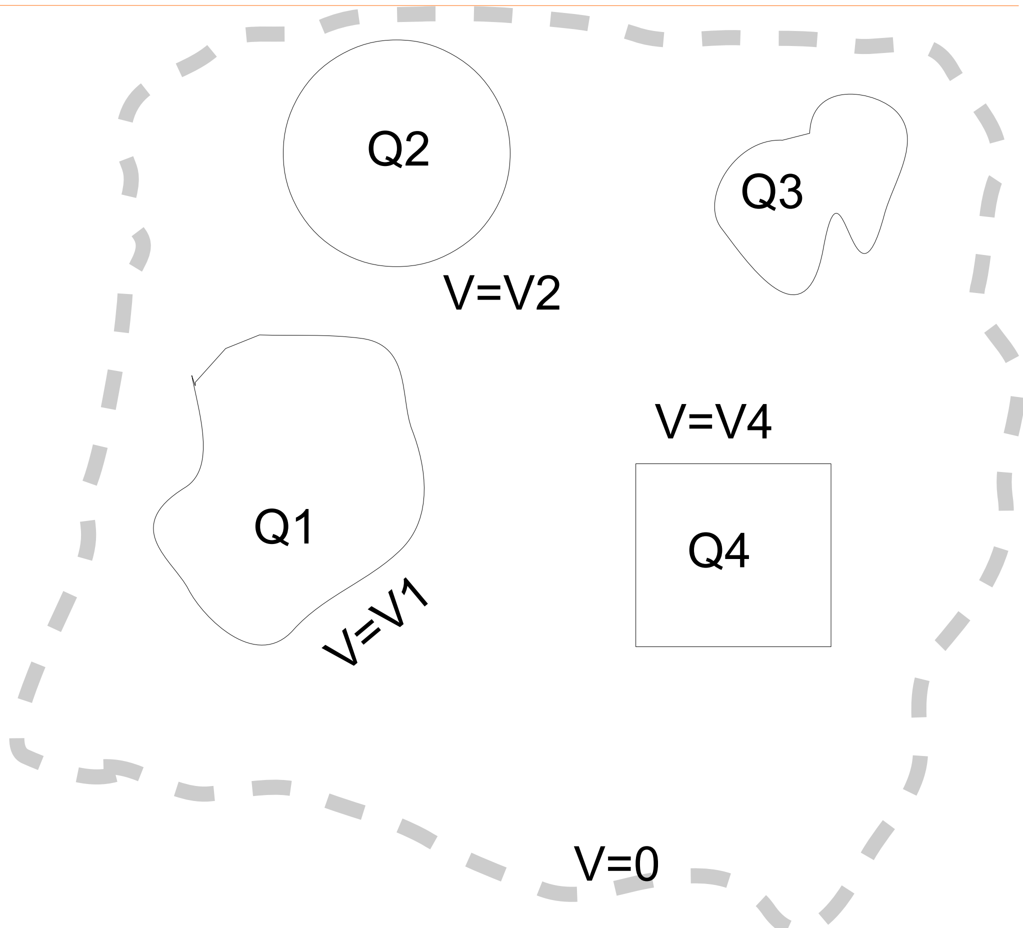
Irrespective of the location of Q inside the cavity.

Electrostatics of conductors : uniqueness theorem 2 & capacitance

You are given the charge Q1, Q2....Qn
On each conductor.

You are not told what V1,V2....Vn are.

What is the most generic statement you can make?



$$Q_i = \sum_j C_{ij} V_j$$
$$C_{ij} = C_{ji}$$

The coefficients in this LINEAR relation are the formal definition of CAPACITANCE.

For a single object it reduces to the familiar relation : **Q=CV**

For an N-condcutor system the matrix is symmetric and can be inverted.

Try writing the energy of the system in matrix form as an exercise...

Where all do we come across Laplace's equation?

flow of "DRY water", quite far from reality, still useful as a starting point

1. Fluid flow : Incompressible, "inviscid", "irrotational"

$$(\rho = const. \quad \eta = 0) \Rightarrow \nabla . v = 0$$

$$If\ \boldsymbol{\nabla} \times \boldsymbol{v} = 0 \quad then \quad \boldsymbol{v} = \boldsymbol{\nabla} \phi \quad (velocity\ potential)$$

$$\nabla^2 \phi = 0$$

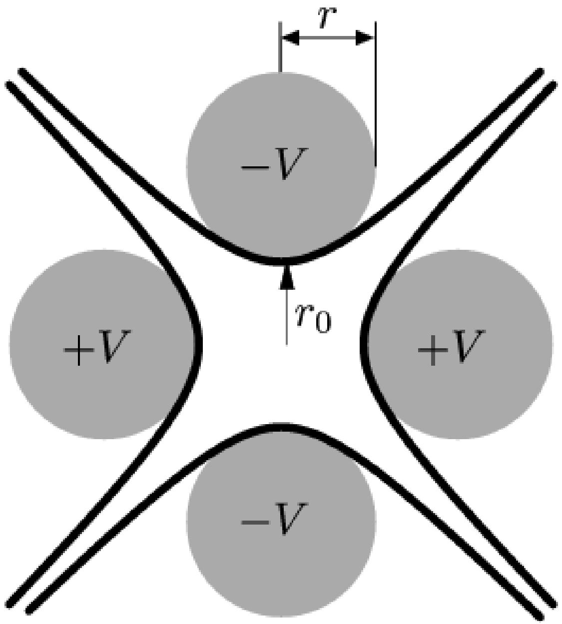
2. Heat conduction (Fourier), Diffusion equation (in steady state, time derivative =0)

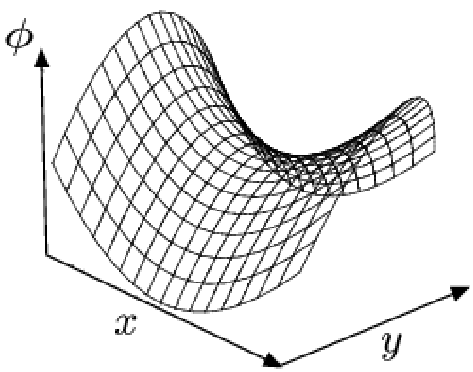
$$D \nabla^2 \theta = \frac{\partial \theta}{\partial t}$$

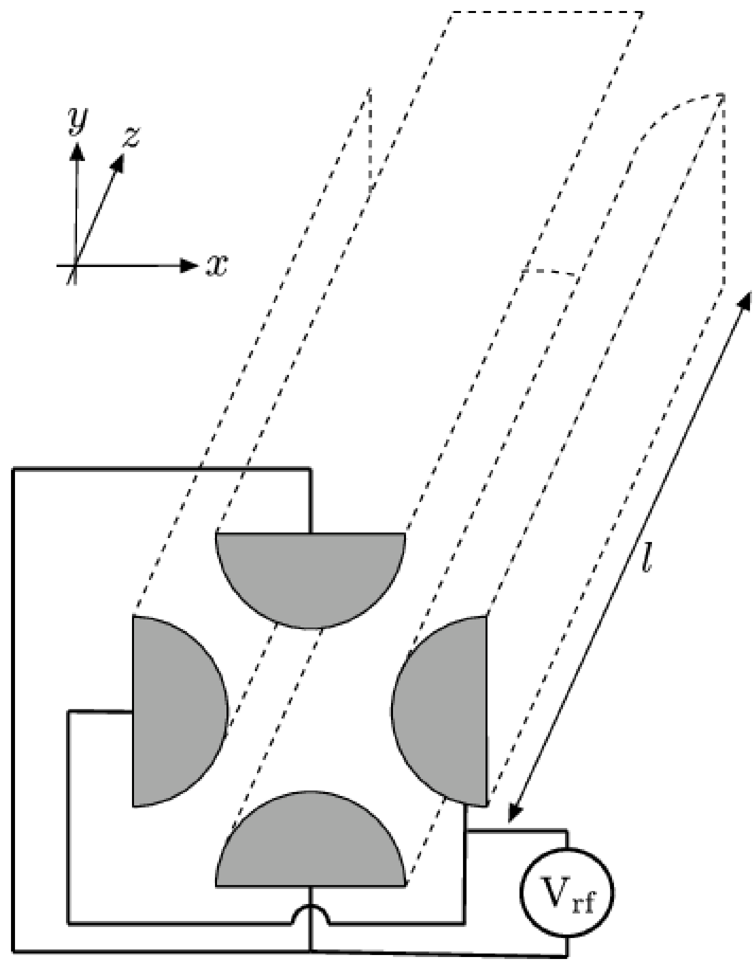
3. Electrostatic lensing :
   Electron microscope, Ion trap, particle acceleration/beam steering, mass spectrometer

   Interesting differences from optical lensing:
   Charged nature of particles,
   Not possible to have focussing from all sides

Electrostatic "quadrupole" lens : an example

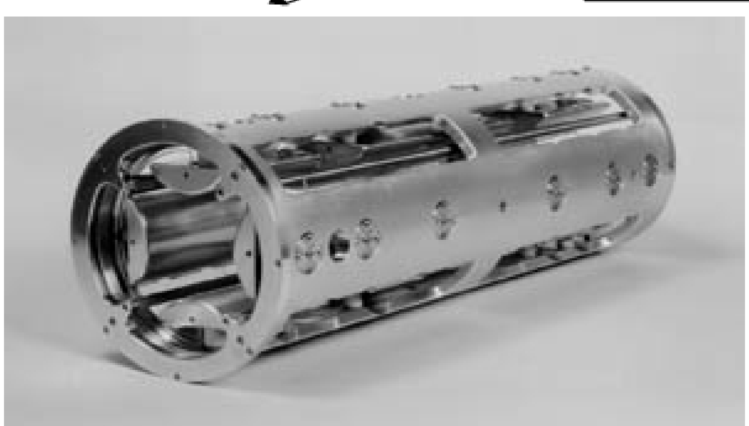






A schematic of an ideal hyperbolic electrostatic quadrupole (thick line) and the circular electrodes used to closely approximate the hyperbolic shape are shown on the top left. The saddle-shaped harmonic potential it creates is illustrated at the bottom left. The geometry formed from half-cylindrical rods and pair-wise applied a.c. potential (Vrf).

T. Brunner et al. *Nucl.Instrum.Meth.* **A676** (2012) 32-43



Notice how the problem becomes 2 dimensional
Translational symmetry of a long cylindrical geometry reduces many 3D problems to a 2D case.

Solving the Laplace equation : Image charge method

Problem: A charge distribution and some boundary conditions are given. The usual boundary conditions are fixed potentials over some surfaces.
Solve for V(r) in a certain region.

A "trick“ works for some (!! not all !!) problems.

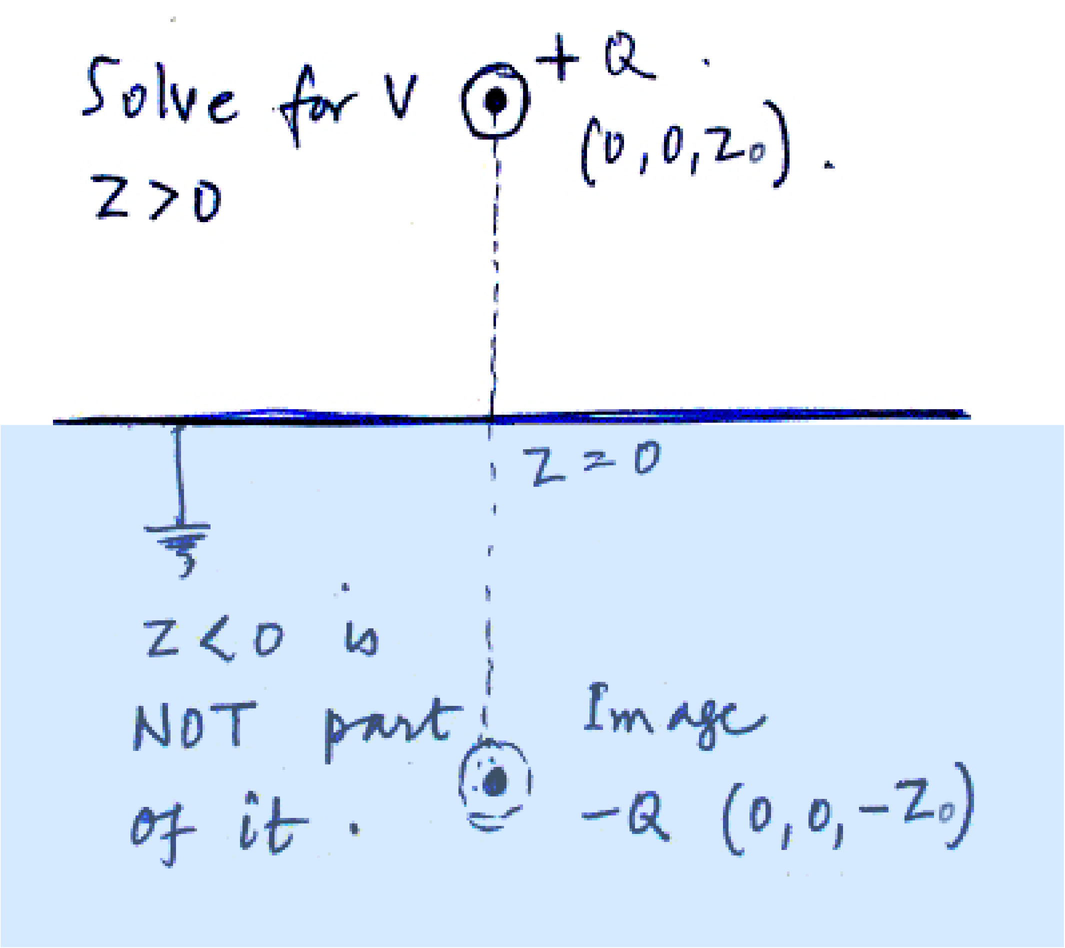
STEP 1: put some point charges in the regions NOT part of the region where you need to solve for the potential.

STEP 2: Try to arrange these external charges, so that the external + given charges together produce the desired potential at the boundaries. Forget all else!

STEP 3: Calculate the total potential in the certain (given) region using all the charges in the problem + external charges.

STEP 4: The total field/potential produced by the ALL the charges is the solution to the problem. The extra charges are called Image charges.

Image charge method : point charge near a conducting grounded plane



PROBLEM:
A point charge +Q is kept at $(0,0,z_0)$

z=0 is a grounded conducting sheet.

What is V(x,y,z) for z >0
Subject to boundary conditions:
V=0 at z=0
$V \longrightarrow 0$ as x,y,z $\longrightarrow \infty$

SOLUTION:

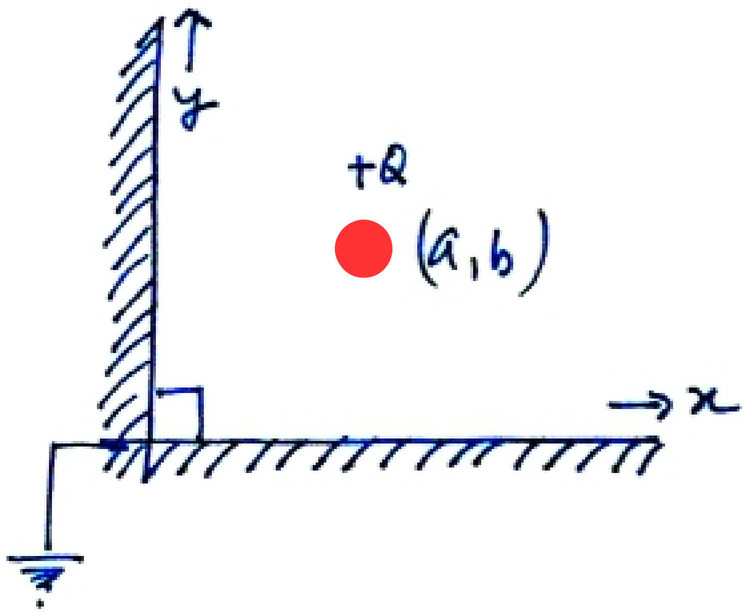
Put an extra charge -Q at $(0,0,-z_0)$

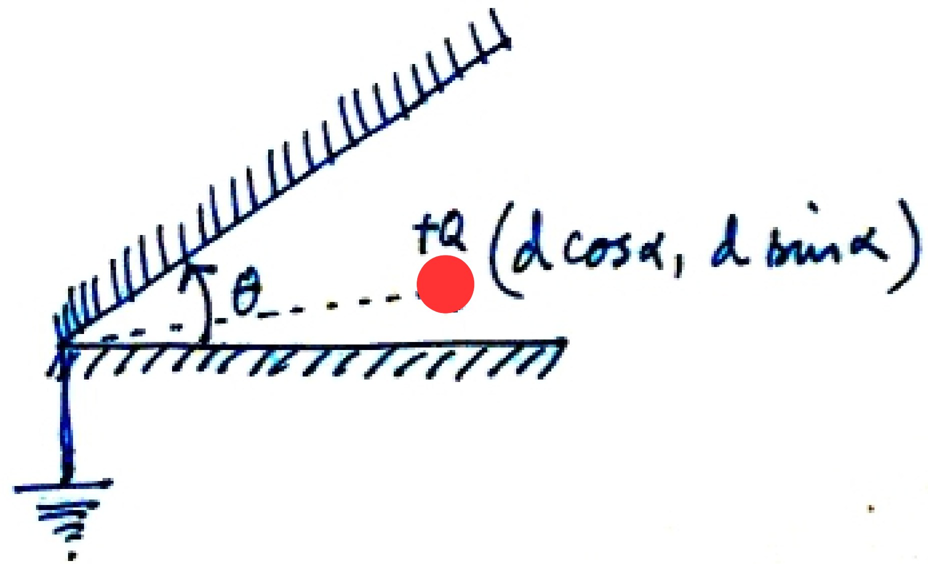
The potential due to both is

$$V(x,y,z)=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}}\right]$$

For z=0, the two terms cancel giving V=0. This must be the solution (uniqueness).
We can now calculate the force between the charge +Q, induced surface charge at every point etc.

Image charge method : point charge near a conducting grounded plane





To solve this we need three image charges:

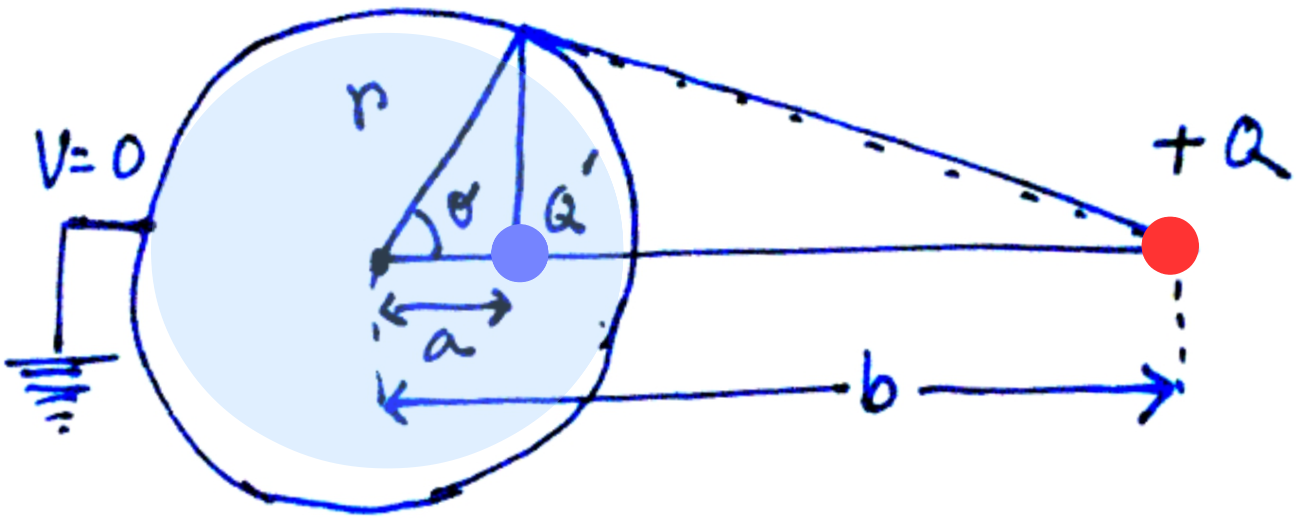
- Q at (a, -b)
+Q at (-a,-b)
- Q at (-a,b)

This problem can be solved with a finite number of images if

$$\theta \times integer = \pi$$

There is no generic method! It is a combination of guess and some calculation.....

Image charge method : point charge near a conducting grounded sphere



Solution wanted outside the sphere only

The distance of an arbitrary point on the surface from the charges Q and Q'

$$d'^2 = r^2 + a^2 - 2ar\cos\theta$$
$$d^2 = r^2 + b^2 - 2br\cos\theta$$

can we adjust a and Q' such that

$$\frac{Q}{d} - \frac{Q'}{d'} = 0 \quad \text{for all } \theta?$$

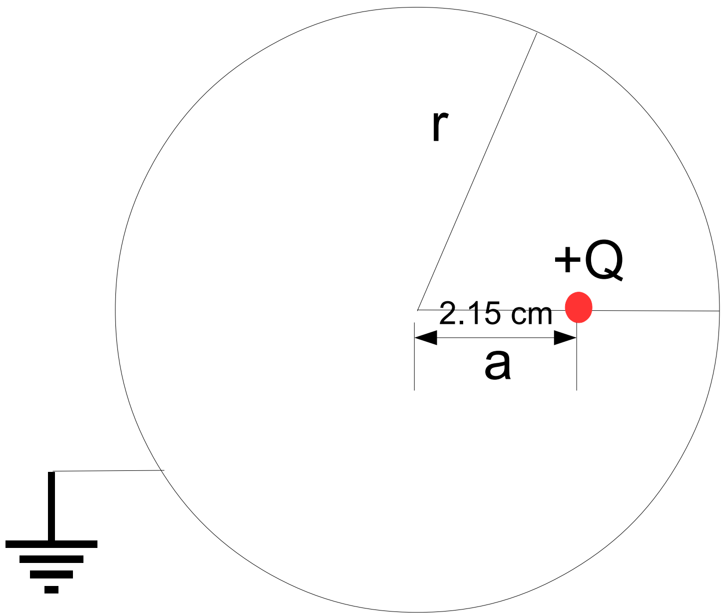
$$\frac{r^2 + a^2 - 2ar\cos\theta}{Q'^2} = \frac{r^2 + b^2 - 2br\cos\theta}{Q^2}$$

Equate the terms independent of cosθ and coefficient of cosθ on both sides

$$a = \frac{r^2}{b}$$
$$\frac{Q'}{Q} = \frac{r}{b}$$

How would you solve it if the sphere is NOT grounded?

Image charge method : off centred point charge inside grounded sphere



Hollow sphere of radius r is kept at V=0
Inside the sphere there is a charge +Q placed at a distance a from the center.

What is the potential inside the sphere?

Notice the "conjugate“ nature of this problem with the last one.

This is a characteristic of "image charge problems".

How would you adapt the image charge method for a case where the spherical surface is at a potential V≠ 0 ?

Solving Laplace's equation

$$1\mathrm{D}:\left(\textit{trivial !}\right)$$

$$\frac{d^2V}{dx^2}=0\;\Rightarrow\;V=Ax+B$$

Cannot be anything more complicated.

$$2\mathrm{D}:\left(\textit{cartesian}\right)$$

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}=0$$

In simplest (very few!) cases separation of variable will work.

$$\textit{If } V(x,y)=X(x)\,Y(y)\ \textit{ then}$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=k^2\ \ (\textit{const})$$

$$\textit{Solution}=\text{sinusoidal type}\times\text{exponential decay or growth}$$

$$V(x,y)=\sum_{\textit{allowed }k}\left(A_k\cos kx+B_k\sin kx\right)\left(C_ke^{ky}+D_ke^{-ky}\right)$$

A,B,C,D are chosen to match the given boundary conditions.

Role of x,y can be Interchanged.
X may be exponential &
Y may be sinusoidal.

Solving Laplace's equation (2D : Using complex numbers : corner)

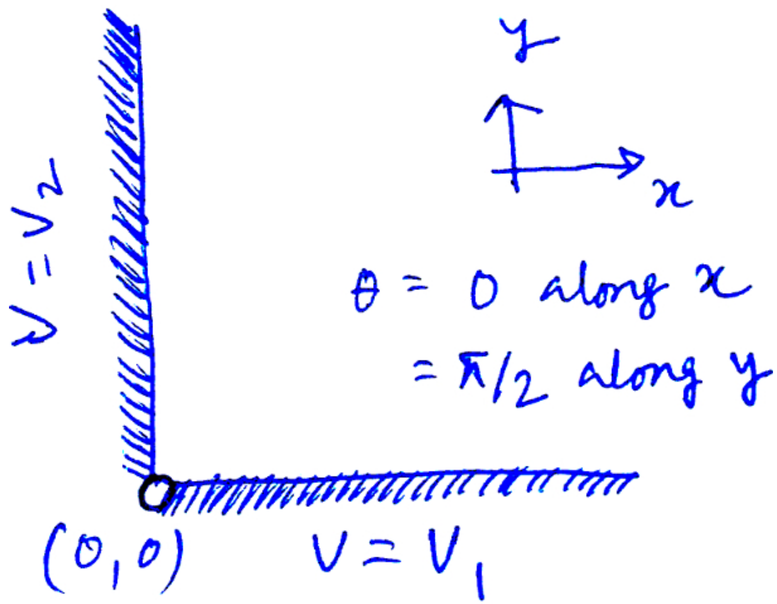
Basic idea: Take any analytic complex function (eg. z^2, sin z, e^{-z})
F(z) = u(x,y) + i v(x,y)
Both u(x,y) and v(x,y) satisfy 2D Laplace equation

By intuition/guess/imagination make u(x,y) or v(x,y) satisfy the boundary conditions only. Uniqueness theorem guarantees that the guess is THE solution.

In reality, very few problems can be solved by separation of variables.
Quite a few can be done by the complex number method – but in 2D only.

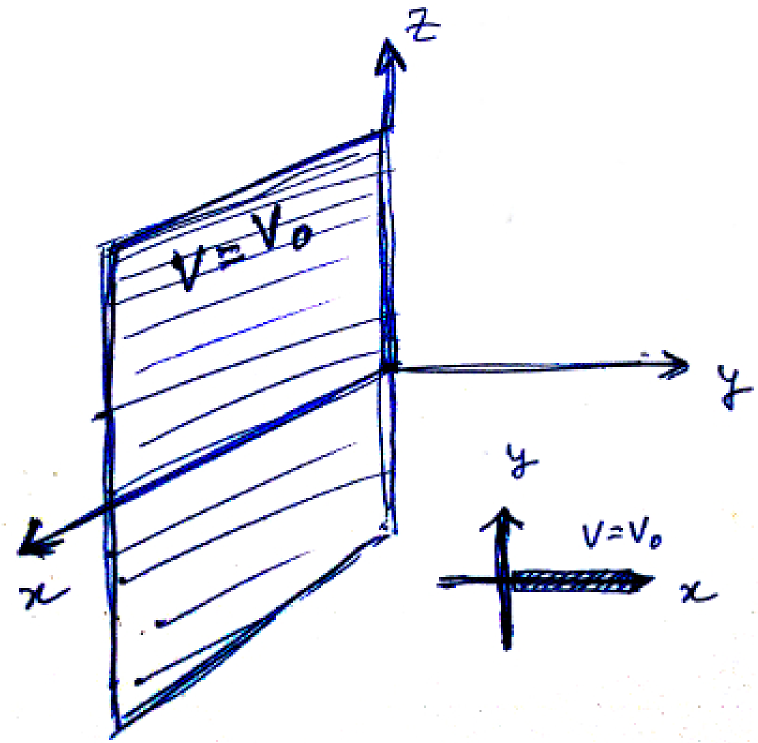
Particularly useful for solutions in near corners, slits, edges, quadrupoles.

$$F(z) = i \ln z^2$$
$$u(x,0) = 0$$
$$u(0,y) = \pi$$
Use this fact and scale the function as
$$F(z) = -(V_2 - V_1)\, i \, \ln z^2 + V_1$$
$$u(x,y) = 2\frac{V_2 - V_1}{\pi} \arctan\left(\frac{y}{x}\right) + V_1 \quad (solved\,!)$$



Notice that separation of variable doesn't work here.
How will you modify the solution if the two plates are inclined at an angle α ?

Solving Laplace's equation (2D : Using complex numbers : edge)



A semi-infinite plate occupies the region x>0 in the xy plane.

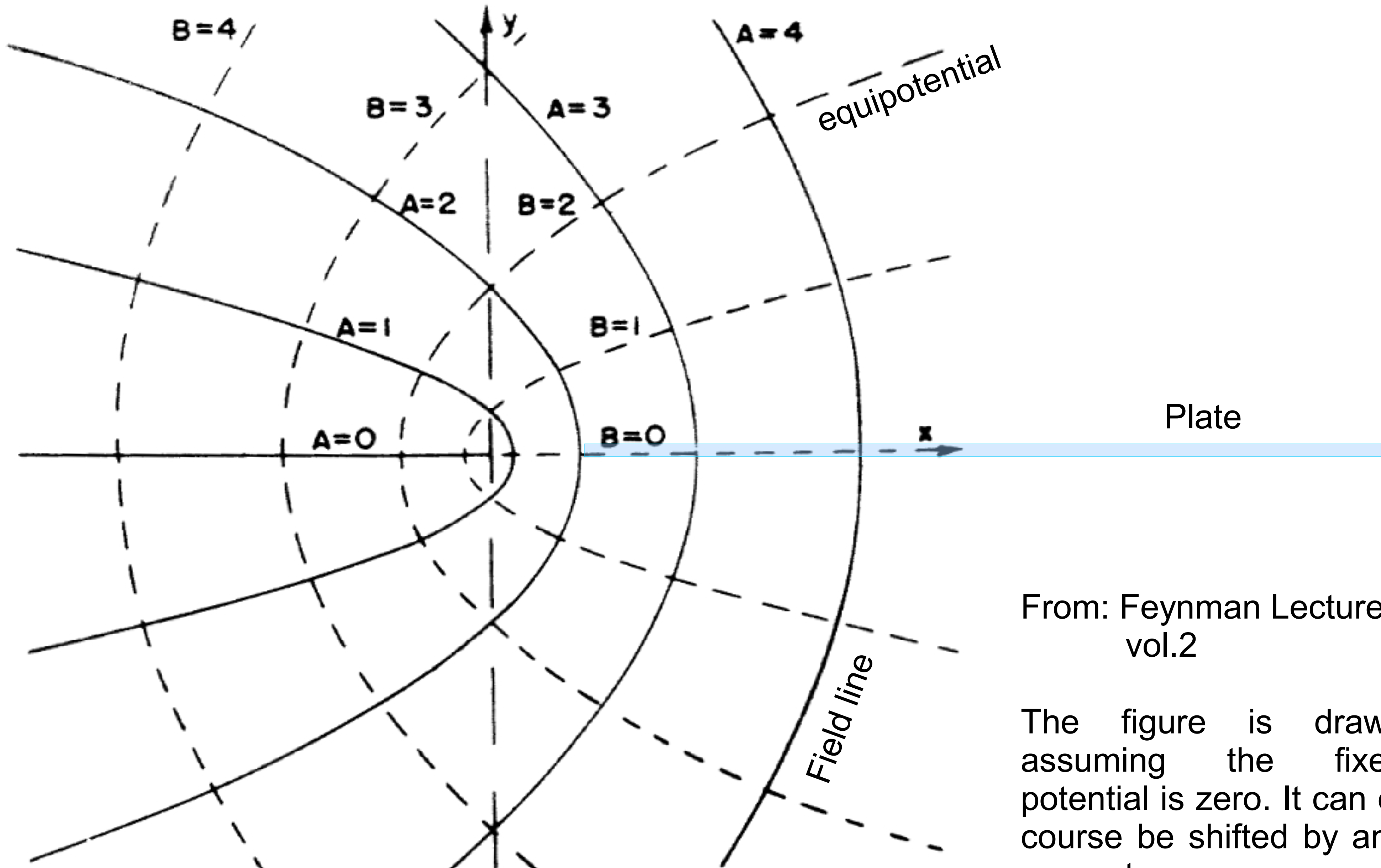
The plate is kept at V=const.

The problem shows how to handle the field near the edge of a flat thin plate.

$$F(z) = z^{1/2}$$
$$= \rho^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$
$$= \left[\frac{\left(x^2+y^2\right)^{1/2}+x}{2}\right]^{1/2} + i\left[\frac{\left(x^2+y^2\right)^{1/2}-x}{2}\right]^{1/2}$$

$$Notice \quad v(x,0) = 0 \quad if \quad x>0$$
$$v(x,y)+V_0 \quad satisfies \quad required \quad boundary \quad condition$$

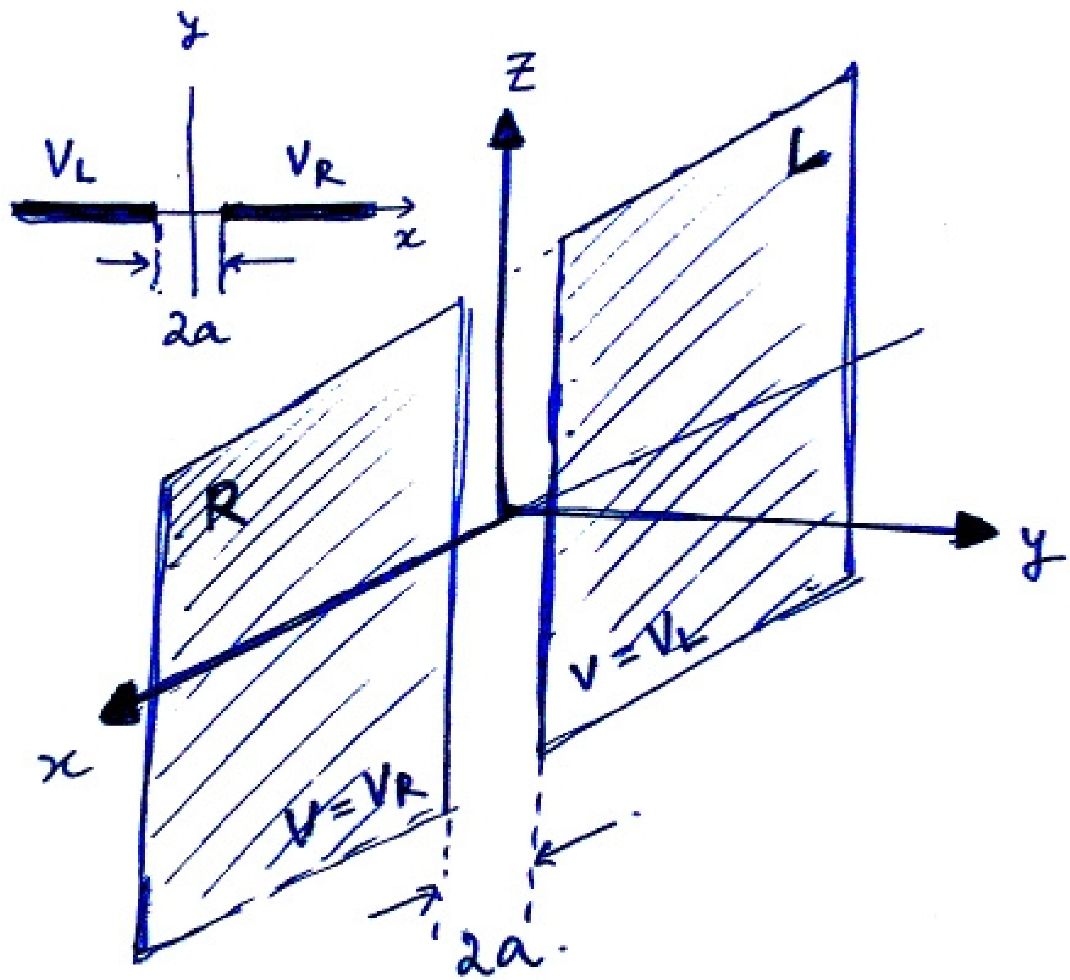
Solving Laplace's equation (2D : Using complex numbers : edge)



From: Feynman Lectures vol.2

The figure is drawn assuming the fixed potential is zero. It can of course be shifted by any amount

Solving Laplace's equation (2D : Using complex numbers : slit)



Two semi-infinite platea occupies the region [-a < x < a] in the xz plane.

The plates are kept at VL and VR

How to handle the field in a slit between equipotential plates?

The solution of this problem requires a transformation of the complex variables called "conformal transform". See: Pipes & Harvill

You cannot superpose two plates at fixed potentials to get a slit. Why?

$$v(x,y)=V_L+\frac{V_R-V_L}{\pi}\left[\arcsin\frac{1}{2}\left(\sqrt{(x/a+1)^2+(y/a)^2}-\sqrt{(x/a-1)^2+(y/a)^2}\right)+\frac{\pi}{2}\right]$$

Solving Laplace's equation (2D : Plane polar)

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

This gives:

$$r^2 \frac{d^2 R}{d r^2} + r\frac{dR}{dr} - m^2 R = 0$$

trial solution $R = Ar^n$ *gives*: $n = \pm m$, *so*

$$V(r,\theta) = \sum_m \left(A_m r^m + \frac{B_m}{r^m}\right) e^{i m\theta}$$

Try: $V = R(r) e^{i m \theta}$

Why not $e^{m\theta}$?

Why should m be
an integer?

What type of problems
can we solve with this
form?

Values given on a circle.
Solution inside should not have 1/r type solution
Solution outside (till infinity) should not have r type solution.\
Use Fourier analysis to find the coeffcients.

Solving Laplace's equation (3D : Spherical polar)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

With no ϕ dependence we try: $V(r,\theta) = R(r)\,P(\theta)$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = -\frac{1}{P}\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = l(l+1)$$

The radial solution

$$r^2 \frac{d^2 R}{dr^2} + 2\mathrm{r}\frac{dR}{dr} - l(l+1)R = 0$$

$$try \quad R = Ar^n$$

$$R = Ar^l + \frac{B}{r^{l+1}}$$

Notice the utility of wiritng the seperation constant in the l(l+1) form

Solving Laplace's equation (3D : Spherical polar)

The angular part:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+l(l+1)P=0$$

Polynomial solutions worked in the examples before this, but would not work in this case. Why?

substitute $x=\cos\theta$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+l(l+1)P=0$$

If we had kept the $e^{im\phi}$ dependence:

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left(l(l+1)-\frac{m^2}{1-x^2}\right)P=0$$

.... In atomic wavefunctions it is common

try the series: $P=\sum_0^{\infty}a_nx^n$: this gives

$$(1-x^2)\sum n(n-1)a_nx^{n-2}-2x\sum na_nx^{n-1}+l(l+1)\sum a_nx^n=0$$

$$2a_2+l(l+1)a_0=0$$

$$3.2.a_3-2a_1+l(l+1)a_1=0$$

$$a_{n+2}=-\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n$$

a0 and a1 can be arbitrarily chosen

If l is an integer, then the series will terminate at n=l

Odd and even powers do not mix in this recurrence relation

Solving Laplace's equation (3D : Spherical polar)

$$P(x)=a_0\sum(\text{even powers of } x)+a_1\sum(\text{odd powers of } x)$$

So construct each polynomial using the recurrence relation

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

Legendre Polynomials:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx=\frac{2}{2n+1}\delta_{nm}$$

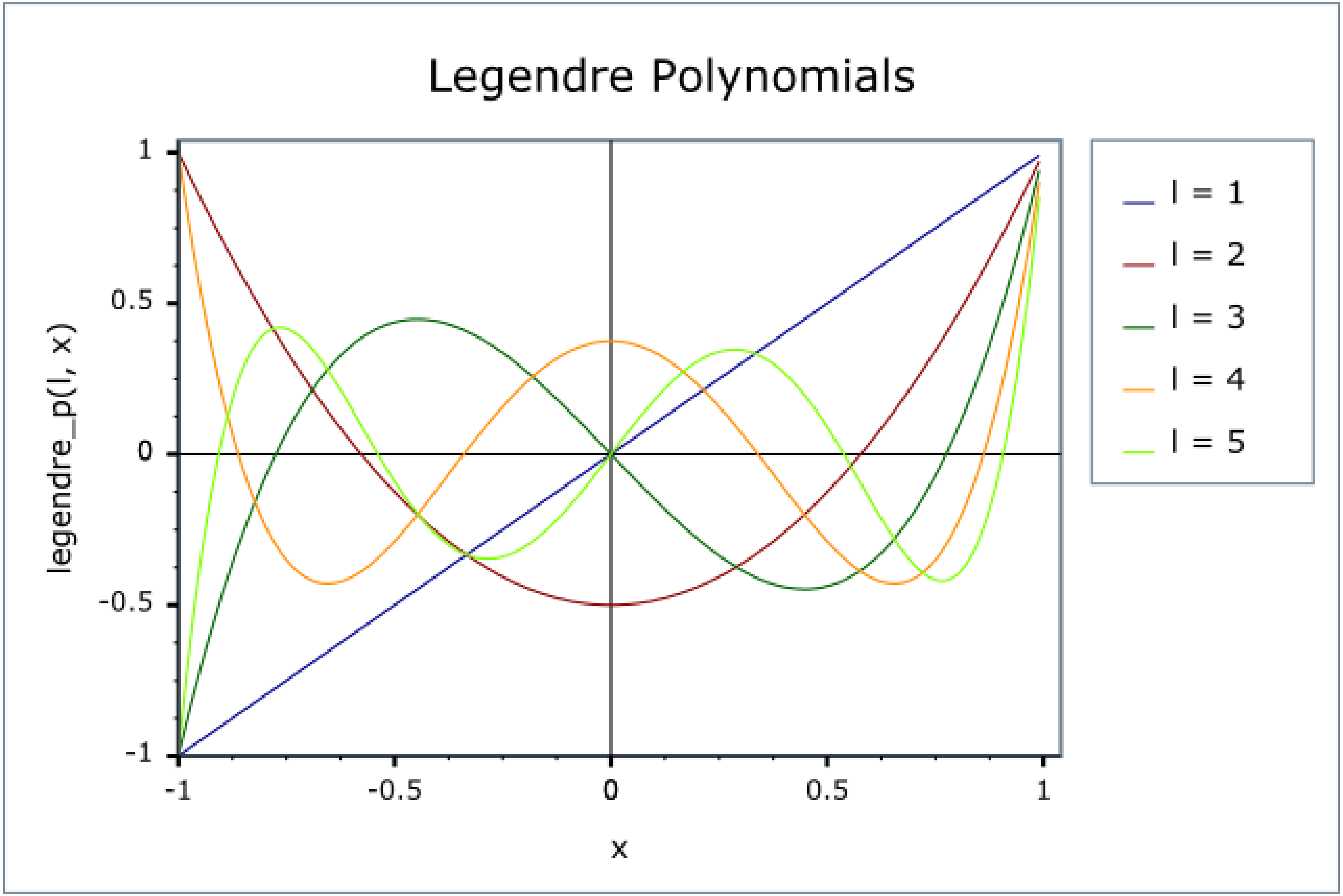
Use orthogonality
to find expansion co-effs ...

Solutions of a general class of diffn equations have this orthogonality property – called "Sturm-Liuville" diffn eqn

$$V(r,\theta)=\sum_l\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Values given on a sphere.
Solution inside should not have 1/r type solution
Solution outside (till infinity) should not have r type solution.
Use expansion in Legendre Polynomials to find the coeffcients.
See the worked examples of Griffiths...section 3.3

Solving Laplace's equation (3D : Spherical polar)



To generate the successive $P_l(x)$ use the Rodrigue's formula :

$$P_l(x) = \frac{1}{2^l\, l\,!}\frac{d^l}{dx^l}(x^2-1)^l$$

Work and Energy in electrostatics

Conservative field : Total energy of a particle is conserved.

✔KE+PE is conserved. Or equivalently

✔Work done in moving a particle very slowly from one point to another is path independent.

✔A potential energy function exists

✔The force is derivable from a scalar potential

✔Curl of the Force field is zero.

Gravitational potential:

Apple falling from a tree
Earth going round the sun...
Trajectory of a particle...

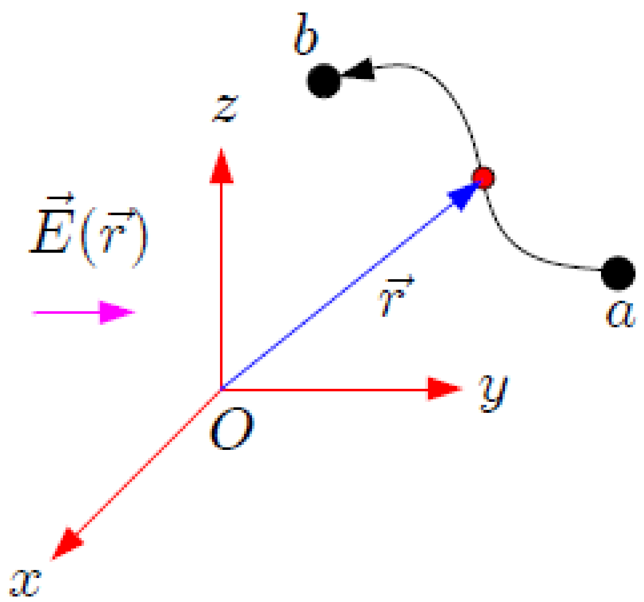
Why do we need to say more?
The answer to this is not within "electrostatics"....the need really comes
When we deal with E,B and moving charges.

Work and Energy in electrostatics

Work done on the charge

$$W = -q\int_a^b \vec{E}\,.\,d\vec{r} \quad [\text{always true}]$$

$$= q\left(V_b - V_a\right) \quad [\text{only if } \vec{\nabla}\times\vec{E}=0]$$



SI unit is Joule.
Useful unit is electron-Volt
Work needed to move one electron through 1 volt
q=-1.6 x 10^{-19}C

How do we build up a configuration of charges?

Bring the first charge : No work done

Bring the second charge from infinity to desried position : calculate work done

Bring next one. Calculate the work done due to the presence of the previous TWO

Work and Energy in electrostatics

$$W = \frac{1}{4\pi\epsilon_0}\sum_{j<i}\frac{q_i q_j}{|\vec{r}_i-\vec{r}_j|} = \frac{1}{2}\sum_i q_i \boxed{\frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j}{|\vec{r}_i-\vec{r}_j|}}$$

Potential at location of charge i

Now suppose this is a continous distribution : it means we are saying the following

$$\sum_i q_i(.....) \longrightarrow \int_{all\,space} \rho(\vec{r})\,d\tau(.....)$$

When can this breakdown?

$$\frac{1}{2}\int \rho V\,d\tau = \frac{\epsilon_0}{2}\int (\vec{\nabla}.\vec{E})V\,d\tau$$

$$= \frac{\epsilon_0}{2}\int \left[\vec{\nabla}.(\vec{E}V)-\vec{E}(-\vec{\nabla}V)\right]d\tau$$

$$= \frac{\epsilon_0}{2}\int \left[\vec{E}.\vec{E}\right]d\tau$$

Convert to surface integral
Take surface at infinity
Should go to zero

Energy of a point charge diverges!

$$\frac{\epsilon_0}{2}\int_0^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right) d\tau \qquad \text{does not converge}$$

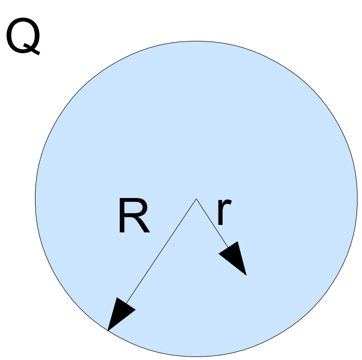
The closest we can try :

Assume that a point charge is a uniform sphere of some radius R.

The integral for field energy will then converge.

Within classical electromagnetism it is not possible to resolve this problem.

We have to accept that the concept of a point charge has some limitations



$$\rho = \frac{Q}{\frac{4}{3}\pi r^3}$$

$$E_{\text{in}} = \frac{Q}{4\pi\epsilon_0 R^2}\frac{r}{R} \qquad (r<R)$$

$$E_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r>R)$$

$$\left(\frac{\epsilon_0}{2}\right)\left[\int_0^R E_{\text{in}}^2\, d\tau + \int_R^\infty E_{\text{out}}^2\, d\tau\right] \qquad \text{will converge}$$