MA-106 Linear Algebra

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D1 - Lecture 4

Recall: Properties of Matrix Multiplication

If A is $m \times n$ and B is $n \times r$, then AB is $m \times r$, and

a) $(AB)_{ij} = A_{i*} \cdot B_{*j}$

b) $(AB)_{*j} = AB_{*j}$

c) $(AB)_{i*} = A_{i*}B$

Properties of Matrix Multiplication:

- (associativity) $\quad (AB)C = A(BC)$
- (distributivity) $\quad A(B + C) = AB + AC$
  $$(B + C)D = BD + CD$$
- (non-commutativity) $\quad AB \neq BA$ in general.
- (Identity) Let $I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ be $n \times n$. If A is $n \times n$, then
  $AI = A = IA$.

Inverse of a Matrix

Defn. Given A of size $n \times n$, we say B is an inverse of A if $AB = I = BA$. If this happens, we say A is *invertible*.

- An inverse may not exist. Find an example. *Hint:* $n = 1$.
- An inverse of A, if it exists, has size $n \times n$.
- If the inverse of A exists, it is unique, and is denoted A^{-1}.

Proof. Let B and C be inverses of A.

$$\begin{array}{ll}
\Rightarrow BA = I & \text{by definition of inverse.} \\
\Rightarrow (BA)C = IC & \text{multiply both sides on the right by } C. \\
\Rightarrow B(AC) = IC & \text{by associativity.} \\
\Rightarrow BI = IC & \text{since } C \text{ is an inverse of } A. \\
\Rightarrow B = C & \text{by property of the identity matrix } I.
\end{array}$$

Inverse of a Matrix

• If A and B are invertible, what about AB?
AB is invertible, with inverse $(AB)^{-1} = B^{-1}A^{-1}$.
Proof. Exercise.

• If A, B are invertible, what about $A + B$?
$A + B$ may not be invertible. Example: $I + (-I) = (0)$.

• If A is invertible, what about A^T?
A^T is invertible with inverse $(A^T)^{-1} = (A^{-1})^T$.
Proof. Use $AA^{-1} = I$. Take transpose.

• If A is symmetric and invertible, then is A^{-1} symmetric?
Yes. *Proof.* Exercise!

• (Identity) $I^{-1} = I$.

Inverses and Linear Systems

• If A is invertible then the system $Ax = b$ has a solution for every constant vector b, namely $x = A^{-1}b$. Is this unique?

• Since $x = 0$ is always a solution of $Ax = 0$, if $Ax = 0$ has a non-zero solution, then A is not invertible by the last remark.

• If A is invertible, then the Gaussian elimination of A produces n pivots.

Exercise:

1. A diagonal matrix A is invertible if and only if ____.
(Hint: When are the diagonal entries pivots?)
2. When is an upper triangular matrix invertible?

• Since $AB = \left(AB_{*1}\ AB_{*2} \cdots AB_{*n}\right)$ and $I = \left(e_1\ e_2 \cdots e_n\right)$, if $B = A^{-1}$, then B_{*j} is a solution of $Ax = e_j$ for all j.

• Strategy to find A^{-1}: Let A be an $n \times n$ invertible matrix. Solve $Ax = e_1$, $Ax = e_2$, ..., $Ax = e_n$.

Solutions to Multiple Systems

Question: Let $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$.

Is there a matrix C such that $AC = B$,
i.e., such that $AC_{*1} = b_1$, $AC_{*2} = b_2$?

<u>Rephrased question:</u> Are $Ax = b_1$ and $Ax = b_2$ both consistent?

$$[A|B] = \left(\begin{array}{ccc|cc} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|cc} \mathbf{1} & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 2 & 0 \\ 0 & \mathbf{1} & -1 & -1 & 1 \\ 0 & 2 & -2 & -2 & 2 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solutions to Multiple Systems (Contd.)

Question: Given matrices A, $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$, is there a matrix C such that $AC = B$?
STRATEGY: Solve $Ax = b_1$ and $Ax = b_2$.

$$[A|B] = \left(\begin{array}{ccc|cc} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A solution to $Ax = b_1$ is $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and to $Ax = b_2$ is $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
(Verify)! So $C = \begin{pmatrix} e_3 & e_2 \end{pmatrix}$ works! Is it unique?

Observe: In the above process, we used a *row exchange*: $R_1 \leftrightarrow R_2$ and *elimination using pivots*: $R_3 = R_3 - R_1$, $R_3 = R_3 - 2R_2$.
Both can be achieved by left multiplication by special matrices.

Random Attendance

1	170050008	Sawant Sohan Madhukar	
2	170050009	Onkar Manik Deshpande	
3	170050032	Krishna Yadav	
4	170050063	Arpit Prajapat	
5	170050073	Debabrata Mandal	
6	170050088	Sreyas Raghavan	
7	170050101	Ambati Satvik	
8	170050105	Himanshu Sheoran	Absent
9	170070052	Meruva Anjaneya Prasad	
10	17D070005	Prajwal Subhash Barapatre	
11	17D070006	Utkarsh Rajendra Bhalode	
12	17D070016	Rishikesh Shuddhodhan Meshram	

Row Operations: Elementary Matrices

Example: $$E\,x = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v - 2u \\ w \end{pmatrix}.$$

If $A = \begin{pmatrix} A_{*1} & A_{*2} & A_{*3} \end{pmatrix}$, then $EA = \begin{pmatrix} EA_{*1} & EA_{*2} & EA_{*3} \end{pmatrix}$.

Thus, EA has the same effect on A as the row operation $R_2 \mapsto R_2 + (-2)R_1$ on the matrix A.

Note: E is obtained from the identity matrix I by the row operation $R_2 \mapsto R_2 + (-2)R_1$.

Such a matrix (diagonal entries 1 and atmost one off-diagonal entry non-zero) is called an *elementary* (or elimination) matrix.

Notation: $E := E_{21}(-2)$. Similarly define $E_{ij}(\lambda)$.

Row Operations: Permutation Matrices

Example: $P\,x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ u \\ w \end{pmatrix}$

If $A = \begin{pmatrix} A_{*1} & A_{*2} & A_{*3} \end{pmatrix}$, then $PA = \begin{pmatrix} PA_{*1} & PA_{*2} & PA_{*3} \end{pmatrix}$.

Thus PA has the same effect on A as the row interchange $R_1 \leftrightarrow R_2$.

Note: We get P from the I by interchanging first and second rows. A matrix obtained from I by row exchanges is called a *permutation* (or row exchange) matrix.

Notation: $P = P_{12}$. Similarly define P_{ij}.

Remark: Row operations correspond to multiplication by elementary matrices $E_{ij}(\lambda)$ or permutation matrices P_{ij} on the left.

Elementary and Permutation Matrices: Inverses

For any $n \times n$ matrix A, observe that the row operations $R_2 \mapsto R_2 - 2R_1, R_2 \mapsto R_2 + 2R_1$ leave the matrix unchanged.

In matrix terms, $E_{21}(2)E_{21}(-2)A = IA = A$ since

$$E_{21}(-2)\ E_{21}(2) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- If $E_{21}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, what is your guess for $E_{21}(\lambda)^{-1}$? Verify.
- Let $P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_2^T \\ e_1^T \\ e_3^T \end{pmatrix}$. What is P_{12}^T? $P_{12}^T P_{12}$? P_{12}^{-1}?

Elementary and Permutation Matrices: Inverses

Notice that the row interchange $R_1 \leftrightarrow R_2$ followed by $R_1 \leftrightarrow R_2$ leaves a matrix unchanged.
In matrix terms, $P_{12}P_{12}A = IA = A$, since

$$P_{12}P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

• Let P_{ij} be obtained by interchanging the ith and jth rows of I.
Show that $P_{ij}^T = P_{ij} = P_{ij}^{-1}$.

• Let $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} e_3^T \\ e_1^T \\ e_2^T \end{pmatrix}$. Show that $P = P_{12}P_{23}$.
Hence, $P^{-1} = (P_{12}P_{23})^{-1} = P_{23}^{-1}P_{12}^{-1} = P_{23}^T P_{12}^T = P^T$.