MA-108 Differential Equations I

Manoj K Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

17th April, 2018
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Constant coefficient equations with Impulses

Let us consider ODE

$$ay'' + by' + cy = f(t), \quad a, b, c \in \mathbb{R}$$

where $f(t)$ represents an impulsive force that is very large for a very short time and zero otherwise.

Example: $f(t)$ could represent force due to a hammer blow. It is difficult to find the function $f(t)$.

What is the graph of $f(t)$.

If f is an integrable function and $f(t) = 0$ outside $[t_0, t_0 + h]$, then

$$I = \int_{t_0}^{t_0+h} f(t)\, dt$$

is called the **total impulse** of f.

Idea : the solution $y(t)$ is more sensitive to total implulse I, rather that on shape of $f(t)$. So we replace f by simple rectangular function with same total impulse I.

In the idealized situation, h is so small that total impulse is assumed to be applied instantaneously at $t = t_0$.
Let $\delta(t - t_0)$ denote the unit impulse function with total impulse 1 applied at $t = t_0$.
Note $\delta(t - t_0)$ is not a function in the standard sense, since our definition implies

$$\delta(t - t_0) = 0, \quad t \neq t_0, \quad \text{and} \quad \int_{t_0}^{t_0} \delta(t - t_0)\, dt = 1$$

Let's define the meaning of solution of IVP

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \; y'(0) = 0, t_0 \geq 0$$

Theorem

Fix $t_0 \geq 0$. For each $h > 0$, let y_h be the solution of IVP

$$ay_h'' + by_h' + cy_h = f_h(t), \quad y_h(0) = 0, \; y_h'(0) = 0$$

$$f_h(t) = \begin{cases} 0, & 0 \leq t < t_0 \\ 1/h, & t_0 \leq t < t_0 + h \\ 0, & t \geq t_0 + h \end{cases}$$

Then f_h has total impulse 1. The solution of IVP

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \; y'(0) = 0, \; t_0 > 0$$

is

$$y(t) = \lim_{h \to 0+} y_h(t)$$

Proof.

Consider the IVP

$$ay_h'' + by_h' + cy_h = f_h(t), \quad y_h(0) = 0, \ y_h'(0) = 0$$

Apply Laplace transform

$$(as^2 + bs + c)Y_h(s) = F_h(s) \implies Y_h(s) = \frac{F_h(s)}{as^2 + bs + c}$$

Let $w(t) = L^{-1}\left(\frac{1}{as^2 + bs + c}\right)$. Then

$$y_h(t) = f_h(t) * w(t) = \int_0^t w(t-\tau) f_h(\tau)\, d\tau$$

$\square$

Proof.

$$y_h(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{1}{h}\int_{t_0}^{t} w(t-\tau)\,d\tau, & t_0 \le t \le t_0 + h \\ \frac{1}{h}\int_{t_0}^{t_0+h} w(t-\tau)\,d\tau, & t > t_0 + h \end{cases}$$

Therefore,

$$y(t) = \lim_{h\to 0+} y_h(t) = 0 \quad \text{if } 0 \le t \le t_0$$

We will show that

$$\lim_{h\to 0+} y_h(t) = w(t - t_0), \quad \text{if} \quad t > t_0$$

Suppose $t > t_0$ is fixed and $t - t_0 > h$. Then

$$y_h(t) - w(t - t_0) = \frac{1}{h}\int_{t_0}^{t_0+h} \left(w(t-\tau) - w(t-t_0)\right) d\tau$$

Proof.

$$|y_h(t) - w(t-t_0)| \leq \frac{1}{h}\int_{t_0}^{t_0+h} |w(t-\tau) - w(t-t_0)|\, d\tau$$

$$\leq M_h \ := \mathsf{max}_{\tau\in[t_0,t_0+h]}\, |w(t-\tau) - w(t-t_0)|$$

Since $w(t)$ is continuous,

$$\lim_{h\to 0+} M_h = 0$$

Therefore,

$$\lim_{h\to 0+} y_h(t) = w(t-t_0), \quad \text{if} \quad t > t_0$$

$\square$

So we have the following theorem.

Theorem

The solution of the IVP

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \; y'(0) = 0, \; t_0 > 0$$

is

$$y(t) = \begin{cases} 0, & 0 \le t \le t_0 \\ w(t - t_0), & t > t_0 \end{cases} = u(t - t_0)w(t - t_0)$$

where

$$w(t) = L^{-1}\left(\frac{1}{as^2 + bs + c}\right)$$

- For $t_0 > 0$, the solution $\boxed{y(t) = u(t-t_0)w(t-t_0)}$ of the IVP

$$ay'' + by' + cy = \delta(t - t_0),\ \ y(0) = 0,\ \ y'(0) = 0,\ \ t_0 > 0$$

is defined on $[0, \infty)$ and has the following properties.

1. $y(t) = 0$ for all $t < t_0$.
2. $ay'' + by' + cy = 0$ for $t \in [0, t_0) \cup (t_0, \infty)$.
3. $y'(t_0-) = 0$ and $y'(t_0+) = 1/a$.

- When $t_0 = 0$, $y'(0-)$ is not defined, so in this case

$$y(t) = u(t)w(t)$$

is a solution of

$$ay'' + by' + cy = \delta(t),\ \ y(0) = 0,\ \ y'(0) = 0$$

Example

Solve

$$y'' + 2y' + y = \delta(t-1),\ y(0) = 0,\ y'(0) = 0$$

Here

$$w(t) = L^{-1}\left(\frac{1}{s^2 + 2s + 1}\right) = L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}t$$

Therefore, the solution is given by

$$y(t) = u(t-1)w(t-1)$$

$$= u(t-1)e^{-(t-1)}(t-1)$$

Example

Solve

$$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t-1), \quad y(0) = -3, \; y'(0) = 2$$

If $y_1(t)$ is a solution of

$$y'' + 6y' + 5y = 3e^{-2t}, \quad y(0) = -3, \; y'(0) = 2$$

$$y_1(t) = \, -\frac{5}{2}e^{-t} + \frac{1}{2}e^{-5t} - e^{-2t}$$

The solution of IVP is $\quad y(t) = y_1 + y_2$,
where y_2 is a solution of

$$y'' + 6y' + 5y = 2\delta(t-1), \; y(0) = 0, \; y'(0) = 0$$

Example (continued ...)

Hence

$$w(t) = 2L^{-1}\left(\frac{1}{s^2+6s+5}\right)$$

$$= \frac{1}{2}L^{-1}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) = \frac{1}{2}(e^{-t} - e^{-5t})$$

$$y(t) = -\frac{5}{2}e^{-t} + \frac{1}{2}e^{-5t} - e^{-2t} + \frac{1}{2}u(t-1)\left(e^{-(t-1)} - e^{-5(t-1)}\right)$$

Example

Solve the IVP

$$y'' + 3y' + 2y = 6e^{2t} + 2\delta(t-1), \quad y(0) = 2, \ y'(0) = -6$$

Let $y_1(t)$ be solution of

$$y'' + 3y' + 2y = 6e^{2t}, \quad y(0) = 2, \ y'(0) = -6$$

and let $y_2(t)$ be the solution of

$$y'' + 3y' + 2y = 2\delta(t-1), \quad y(0) = 0, \ y'(0) = 0$$

Then the solution of original IVP is

$$y(t) = y_1(t) + y_2(t)$$

Example

Solve the IVP

$$y'' + y = \sin 3t + 2\delta(t - \pi/2), \qquad y(0) = 1,\ y'(0) = -1$$

Let $y_1(t)$ be solution of

$$y'' + y = \sin 3t, \quad y(0) = 1,\ y'(0) = -1$$

and let $y_2(t)$ be the solution of

$$y'' + y = 2\delta(t - \pi/2), \quad y(0) = 0,\ y'(0) = 0$$

Then the solution of original IVP is

$$y(t) = y_1(t) + y_2(t)$$

Example

Solve the IVP

$$y'' + 2y' + 2y = \delta(t-\pi) - 3\delta(t-2\pi),\ y(0) = -1,\ y'(0) = 2$$

Let $y_1(t)$ be solution of

$$y'' + 2y' + 2y = 0, \quad y(0) = -1,\ y'(0) = 2$$

and let $y_2(t)$ and $y_3(t)$ be solutions of

$$y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = 0,\ y'(0) = 0$$

$$y'' + 2y' + 2y = -3\delta(t-2\pi), \quad y(0) = 0,\ y'(0) = 0$$

Then the solution of original IVP is

$$y(t) = y_1(t) + y_2(t) + y_3(t)$$

Example

Solve the IVP

$y'' + 4y = f(t) + \delta(t - 2\pi), \quad y(0) = 0, \; y'(0) = 1$

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 2, & t \ge \pi/2 \end{cases}$$

Example

Solve the IVP

$y'' + 4y' + 4y = -\delta(t), \qquad y(0) = 1, \; y'(0+) = 5.$

Example

Find a solution, not involving unit step function which represents y on each suninterval of $[0, \infty)$ on which the forcing function is zero.

1.

$$y'' - y = \sum_{k=1}^{\infty} \delta(t-k), \qquad y(0) = 0,\ y'(0) = 1$$

2.

$$y'' - 3y' + 2y = \sum_{k=1}^{\infty} \delta(t-k),\ y(0) = 0,\ y'(0) = 1$$