Chapter 7 CURVILINEAR COORDINATES

TRANSFORMATION OF COORDINATES. Let the rectangular coordinates (x,y,z) of any point be expressed as functions of (u_1,u_2,u_3) so that

$$(1) \qquad x = x(u_1,u_2,u_3), \quad y = y(u_1,u_2,u_3), \quad z = z(u_1,u_2,u_3)$$

Suppose that *(1)* can be solved for u_1,u_2,u_3 in terms of x,y,z, i.e.,

$$(2) \qquad u_1 = u_1(x,y,z), \quad u_2 = u_2(x,y,z), \quad u_3 = u_3(x,y,z)$$

The functions in *(1)* and *(2)* are assumed to be single-valued and to have continuous derivatives so that the correspondence between (x,y,z) and (u_1,u_2,u_3) is unique. In practice this assumption may not apply at certain points and special consideration is required.

Given a point P with rectangular coordinates (x,y,z) we can, from *(2)* associate a unique set of coordinates (u_1,u_2,u_3) called the *curvilinear coordinates* of P. The sets of equations *(1)* or *(2)* define a *transformation of coordinates.*

ORTHOGONAL CURVILINEAR COORDINATES.

The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$, where c_1, c_2, c_3 are constants, are called *coordinate surfaces* and each pair of these surfaces intersect in curves called *coordinate curves or lines* (see Fig.1). If the coordinate surfaces intersect at right angles the curvilinear coordinate system is called *orthogonal.* The u_1, u_2 and u_3 coordinate curves of a curvilinear system are analogous to the x,y and z coordinate axes of a rectangular system.

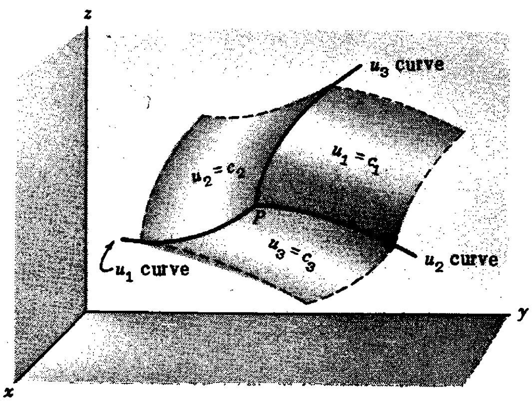


Fig. 1

UNIT VECTORS IN CURVILINEAR SYSTEMS. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P. Then *(1)* can be written $\mathbf{r} = \mathbf{r}(u_1,u_2,u_3)$. A tangent vector to the u_1 curve at P (for which u_2 and u_3 are constants) is $\dfrac{\partial \mathbf{r}}{\partial u_1}$. Then a unit tangent vector in this direction is $\mathbf{e}_1 = \dfrac{\partial \mathbf{r}}{\partial u_1} \Big/ \left|\dfrac{\partial \mathbf{r}}{\partial u_1}\right|$ so that $\dfrac{\partial \mathbf{r}}{\partial u_1} = h_1 \mathbf{e}_1$ where $h_1 = \left|\dfrac{\partial \mathbf{r}}{\partial u_1}\right|$. Similarly, if $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit tangent vectors to the u_2 and u_3 curves at P respectively, then $\dfrac{\partial \mathbf{r}}{\partial u_2} = h_2 \mathbf{e}_2$ and $\dfrac{\partial \mathbf{r}}{\partial u_3} = h_3 \mathbf{e}_3$ where $h_2 = \left|\dfrac{\partial \mathbf{r}}{\partial u_2}\right|$ and $h_3 = \left|\dfrac{\partial \mathbf{r}}{\partial u_3}\right|$. The quantities h_1, h_2, h_3 are called *scale factors.* The unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are in the directions of increasing u_1, u_2, u_3, respectively.

Since ∇u_1 is a vector at P normal to the surface $u_1 = c_1$, a unit vector in this direction is given by $\mathbf{E}_1 = \nabla u_1 / |\nabla u_1|$. Similarly, the unit vectors $\mathbf{E}_2 = \nabla u_2 / |\nabla u_2|$ and $\mathbf{E}_3 = \nabla u_3 / |\nabla u_3|$ at P are normal to the surfaces $u_2 = c_2$ and $u_3 = c_3$ respectively.

Thus at each point P of a curvilinear system there exist, in general, two sets of unit vectors, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ tangent to the coordinate curves and $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ normal to the coordinate surfaces (see Fig.2). The sets become identical if and only if the curvilinear coordinate system is orthogonal (see Problem 19). Both sets are analogous to the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors in rectangular coordinates but are unlike them in that they may change directions from point to point. It can be shown (see Problem 15) that the sets $\dfrac{\partial \mathbf{r}}{\partial u_1}, \dfrac{\partial \mathbf{r}}{\partial u_2}, \dfrac{\partial \mathbf{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ constitute reciprocal systems of vectors.

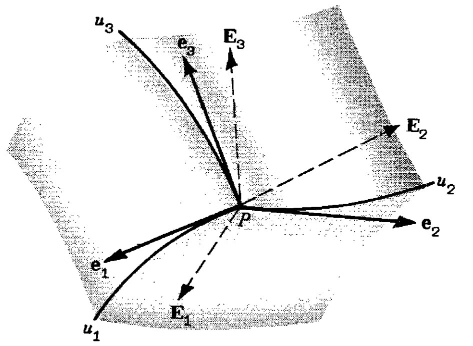


Fig. 2

A vector $\mathbf{A}$ can be represented in terms of the unit base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ or $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ in the form

$$\mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3 = a_1 \mathbf{E}_1 + a_2 \mathbf{E}_2 + a_3 \mathbf{E}_3$$

where A_1, A_2, A_3 and a_1, a_2, a_3 are the respective *components* of $\mathbf{A}$ in each system.

We can also represent $\mathbf{A}$ in terms of the base vectors $\dfrac{\partial \mathbf{r}}{\partial u_1}, \dfrac{\partial \mathbf{r}}{\partial u_2}, \dfrac{\partial \mathbf{r}}{\partial u_3}$ or $\nabla u_1, \nabla u_2, \nabla u_3$ which are called *unitary base vectors* but *are not* unit vectors in general. In this case

$$\mathbf{A} = C_1 \frac{\partial \mathbf{r}}{\partial u_1} + C_2 \frac{\partial \mathbf{r}}{\partial u_2} + C_3 \frac{\partial \mathbf{r}}{\partial u_3} = C_1 \boldsymbol{\alpha}_1 + C_2 \boldsymbol{\alpha}_2 + C_3 \boldsymbol{\alpha}_3$$

and

$$\mathbf{A} = c_1 \nabla u_1 + c_2 \nabla u_2 + c_3 \nabla u_3 = c_1 \boldsymbol{\beta}_1 + c_2 \boldsymbol{\beta}_2 + c_3 \boldsymbol{\beta}_3$$

where C_1, C_2, C_3 are called the *contravariant components* of $\mathbf{A}$ and c_1, c_2, c_3 are called the *covariant components* of $\mathbf{A}$ (see Problems 33 and 34). Note that $\boldsymbol{\alpha}_p = \dfrac{\partial \mathbf{r}}{\partial u_p}$, $\boldsymbol{\beta}_p = \nabla u_p$, $p = 1, 2, 3$.

ARC LENGTH AND VOLUME ELEMENTS. From $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ we have

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = h_1\, du_1\, \mathbf{e}_1 + h_2\, du_2\, \mathbf{e}_2 + h_3\, du_3\, \mathbf{e}_3$$

Then the differential of arc length ds is determined from $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$. For orthogonal systems, $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0$ and

$$ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$$

For non-orthogonal or general curvilinear systems see Problem 17.

Along a u_1 curve, u_2 and u_3 are constants so that $d\mathbf{r} = h_1\, du_1\, \mathbf{e}_1$. Then the differential of arc length ds_1 along u_1 at P is $h_1 du_1$. Similarly the differential arc lengths along u_2 and u_3 at P are $ds_2 = h_2 du_2$, $ds_3 = h_3 du_3$.

Referring to Fig.3 the volume element for an orthogonal curvilinear coordinate system is given by

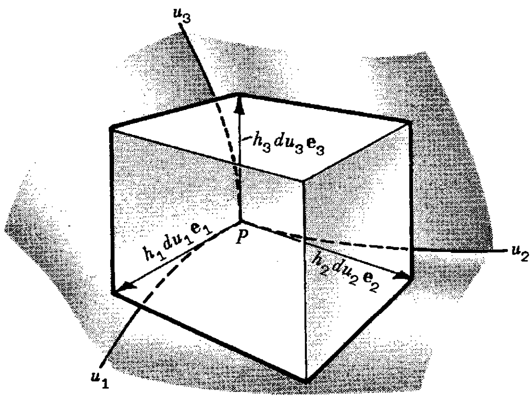


Fig. 3

$$dV = \left|(h_1 du_1 \mathbf{e}_1)\cdot(h_2 du_2 \mathbf{e}_2)\times(h_3 du_3 \mathbf{e}_3)\right| = h_1 h_2 h_3\, du_1 du_2 du_3$$

since $|\mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3| = 1$.

THE GRADIENT, DIVERGENCE AND CURL can be expressed in terms of curvilinear coordinates. If Φ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3, then the following results are valid.

1. $$\nabla\Phi = \text{grad}\,\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3$$

2. $$\nabla\cdot\mathbf{A} = \text{div}\,\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

3. $$\nabla\times\mathbf{A} = \text{curl}\,\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

4. $$\nabla^2\Phi = \text{Laplacian of }\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

If $h_1 = h_2 = h_3 = 1$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are replaced by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, these reduce to the usual expressions in rectangular coordinates where (u_1, u_2, u_3) is replaced by (x, y, z).

Extensions of the above results are achieved by a more general theory of curvilinear systems using the methods of *tensor analysis* which is considered in Chapter 8.

SPECIAL ORTHOGONAL COORDINATE SYSTEMS.

1. Cylindrical Coordinates (ρ, ϕ, z). See Fig.4 below.

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

$$\text{where} \quad \rho \geqq 0, \quad 0 \leqq \phi < 2\pi, \quad -\infty < z < \infty$$

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$$

2. Spherical Coordinates (r, θ, ϕ). See Fig.5 below.

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\text{where} \quad r \geqq 0, \quad 0 \leqq \phi < 2\pi, \quad 0 \leqq \theta \leqq \pi$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$$

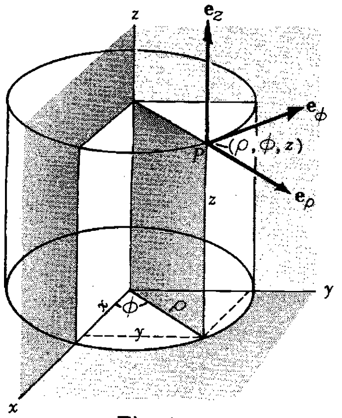


Fig. 4

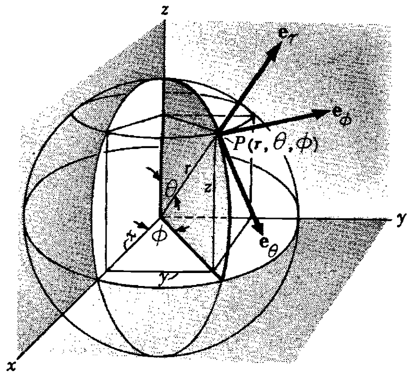


Fig. 5

3. Parabolic Cylindrical Coordinates (u, v, z). See Fig.6 below.

$$x = \tfrac{1}{2}(u^2 - v^2), \qquad y = uv, \qquad z = z$$

$$\text{where} \quad -\infty < u < \infty, \quad v \geqq 0, \quad -\infty < z < \infty$$

$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_z = 1$$

In cylindrical coordinates, $u = \sqrt{2\rho}\,\cos\dfrac{\phi}{2}, \quad v = \sqrt{2\rho}\,\sin\dfrac{\phi}{2}, \quad z = z$

The traces of the coordinate surfaces on the xy plane are shown in Fig.6 below. They are confocal parabolas with a common axis.

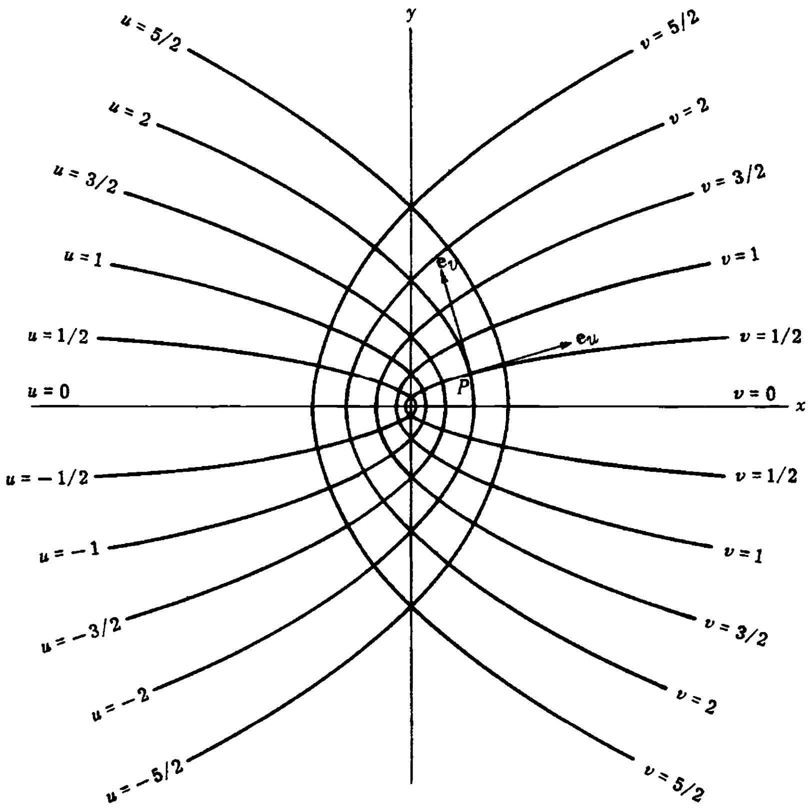


Fig. 6

4. Paraboloidal Coordinates (u, v, ϕ).

$$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \tfrac{1}{2}(u^2 - v^2)$$

$$\text{where} \quad u \geqq 0, \quad v \geqq 0, \quad 0 \leqq \phi < 2\pi$$

$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_\phi = uv$$

Two sets of coordinate surfaces are obtained by revolving the parabolas of Fig.6 above about the x axis which is relabeled the z axis. The third set of coordinate surfaces are planes passing through this axis.

5. Elliptic Cylindrical Coordinates (u, v, z). See Fig.7 below.

$$x = a\cosh u\cos v, \quad y = a\sinh u\sin v, \quad z = z$$

$$\text{where} \quad u \geqq 0, \quad 0 \leqq v < 2\pi, \quad -\infty < z < \infty$$

$$h_u = h_v = a\sqrt{\sinh^2 u + \sin^2 v}, \quad h_z = 1$$

The traces of the coordinate surfaces on the xy plane are shown in Fig.7 below. They are confocal ellipses and hyperbolas.

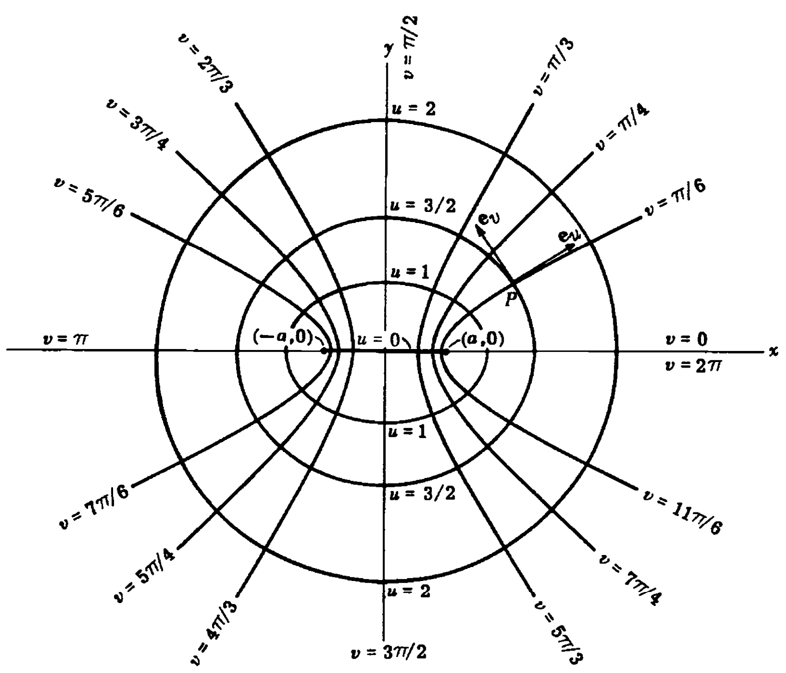


Fig. 7

6. Prolate Spheroidal Coordinates (ξ, η, ϕ).

$$x = a\sinh\xi\,\sin\eta\,\cos\phi, \quad y = a\sinh\xi\,\sin\eta\,\sin\phi, \quad z = a\cosh\xi\,\cos\eta$$

$$\text{where} \quad \xi \geqq 0, \quad 0 \leqq \eta \leqq \pi, \quad 0 \leqq \phi < 2\pi$$

$$h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}, \quad h_\phi = a\sinh\xi\,\sin\eta$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig.7 above about the x axis which is relabeled the z axis. The third set of coordinate surfaces are planes passing through this axis.

7. Oblate Spheroidal Coordinates (ξ, η, ϕ).

$$x = a\cosh\xi\,\cos\eta\,\cos\phi, \qquad y = a\cosh\xi\,\cos\eta\,\sin\phi, \qquad z = a\sinh\xi\,\sin\eta$$

$$\text{where}\quad \xi \geqq 0, \quad -\frac{\pi}{2} \leqq \eta \leqq \frac{\pi}{2}, \quad 0 \leqq \phi < 2\pi$$

$$h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}, \quad h_\phi = a\cosh\xi\,\cos\eta$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig.7 above about the y axis which is relabeled the z axis. The third set of coordinate surfaces are planes passing through this axis.

8. Ellipsoidal Coordinates (λ, μ, ν).

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1, \qquad \lambda < c^2 < b^2 < a^2$$

$$\frac{x^2}{a^2-\mu} + \frac{y^2}{b^2-\mu} + \frac{z^2}{c^2-\mu} = 1, \qquad c^2 < \mu < b^2 < a^2$$

$$\frac{x^2}{a^2-\nu} + \frac{y^2}{b^2-\nu} + \frac{z^2}{c^2-\nu} = 1, \qquad c^2 < b^2 < \nu < a^2$$

$$h_\lambda = \frac{1}{2}\sqrt{\frac{(\mu-\lambda)(\nu-\lambda)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}, \qquad h_\mu = \frac{1}{2}\sqrt{\frac{(\nu-\mu)(\lambda-\mu)}{(a^2-\mu)(b^2-\mu)(c^2-\mu)}}$$

$$h_\nu = \frac{1}{2}\sqrt{\frac{(\lambda-\nu)(\mu-\nu)}{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}$$

9. Bipolar Coordinates (u, v, z). See Fig.8 below.

$$x^2 + (y - a\cot u)^2 = a^2\csc^2 u, \qquad (x - a\coth v)^2 + y^2 = a^2\operatorname{csch}^2 v, \qquad z = z$$

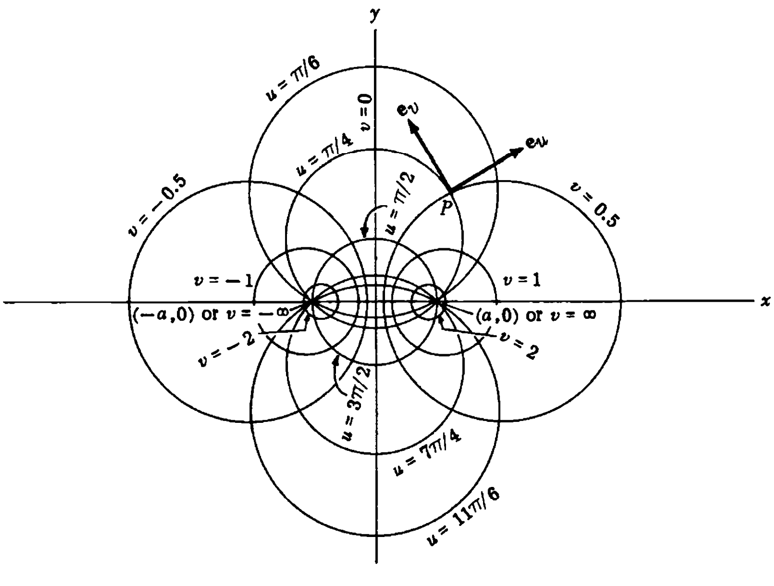


Fig. 8

or $$x = \frac{a \sinh v}{\cosh v - \cos u}, \quad y = \frac{a \sin u}{\cosh v - \cos u}, \quad z = z$$

where $0 \leqq u < 2\pi, \quad -\infty < v < \infty, \quad -\infty < z < \infty$

$$h_u = h_v = \frac{a}{\cosh v - \cos u}, \quad h_z = 1$$

The traces of the coordinate surfaces on the xy plane are shown in Fig.8 above. By revolving the curves of Fig.8 about the y axis and relabeling this the z axis a *toroidal coordinate system* is obtained.

SOLVED PROBLEMS

1. Describe the coordinate surfaces and coordinate curves for (*a*) cylindrical and (*b*) spherical coordinates.

(*a*) The coordinate surfaces (or level surfaces) are:

$\rho = c_1$ cylinders coaxial with the z axis (or z axis if $c_1 = 0$).
$\phi = c_2$ planes through the z axis.
$z = c_3$ planes perpendicular to the z axis.

The coordinate curves are:

Intersection of $\rho = c_1$ and $\phi = c_2$ (z curve) is a straight line.
Intersection of $\rho = c_1$ and $z = c_3$ (ϕ curve) is a circle (or point).
Intersection of $\phi = c_2$ and $z = c_3$ (ρ curve) is a straight line.

(*b*) The coordinate surfaces are:

$r = c_1$ spheres having center at the origin (or origin if $c_1 = 0$).
$\theta = c_2$ cones having vertex at the origin (lines if $c_2 = 0$ or π, and the xy plane if $c_2 = \pi/2$).
$\phi = c_3$ planes through the z axis.

The coordinate curves are:

Intersection of $r = c_1$ and $\theta = c_2$ (ϕ curve) is a circle (or point).
Intersection of $r = c_1$ and $\phi = c_3$ (θ curve) is a semi-circle ($c_1 \neq 0$).
Intersection of $\theta = c_2$ and $\phi = c_3$ (r curve) is a line.

2. Determine the transformation from cylindrical to rectangular coordinates.

The equations defining the transformation from rectangular to cylindrical coordinates are

$$(1)\ x = \rho \cos\phi, \qquad (2)\ y = \rho \sin\phi, \qquad (3)\ z = z$$

Squaring (*1*) and (*2*) and adding, $\rho^2(\cos^2\phi + \sin^2\phi) = x^2 + y^2$ or
$\rho = \sqrt{x^2+y^2}$, since $\cos^2\phi + \sin^2\phi = 1$ and ρ is positive.

Dividing equation (*2*) by (*1*), $\dfrac{y}{x} = \dfrac{\rho \sin\phi}{\rho\cos\phi} = \tan\phi$ or $\phi = \arctan\dfrac{y}{x}$.

Then the required transformation is $(4)\ \rho = \sqrt{x^2+y^2}$, $(5)\ \phi = \arctan\frac{y}{x}$, $(6)\ z = z$.

For points on the z axis ($x = 0, y = 0$), note that ϕ is indeterminate. Such points are called *singular points* of the transformation.

3. Prove that a cylindrical coordinate system is orthogonal.

The position vector of any point in cylindrical coordinates is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\,\mathbf{k}$$

The tangent vectors to the ρ, ϕ and z curves are given respectively by $\frac{\partial\mathbf{r}}{\partial\rho}$, $\frac{\partial\mathbf{r}}{\partial\phi}$ and $\frac{\partial\mathbf{r}}{\partial z}$ where

$$\frac{\partial\mathbf{r}}{\partial\rho} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \qquad \frac{\partial\mathbf{r}}{\partial\phi} = -\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}, \qquad \frac{\partial\mathbf{r}}{\partial z} = \mathbf{k}$$

The unit vectors in these directions are

$$\mathbf{e}_1 = \mathbf{e}_\rho = \frac{\partial\mathbf{r}/\partial\rho}{|\partial\mathbf{r}/\partial\rho|} = \frac{\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \frac{\partial\mathbf{r}/\partial\phi}{|\partial\mathbf{r}/\partial\phi|} = \frac{-\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$

$$\mathbf{e}_3 = \mathbf{e}_z = \frac{\partial\mathbf{r}/\partial z}{|\partial\mathbf{r}/\partial z|} = \mathbf{k}$$

Then

$$\mathbf{e}_1\cdot\mathbf{e}_2 = (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j})\cdot(-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) = 0$$

$$\mathbf{e}_1\cdot\mathbf{e}_3 = (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j})\cdot(\mathbf{k}) = 0$$

$$\mathbf{e}_2\cdot\mathbf{e}_3 = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\cdot(\mathbf{k}) = 0$$

and so $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are mutually perpendicular and the coordinate system is orthogonal.

4. Represent the vector $\mathbf{A} = z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}$ in cylindrical coordinates. Thus determine A_ρ, A_ϕ and A_z.

From Problem 3,

$$(1)\ \mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \qquad (2)\ \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \qquad (3)\ \mathbf{e}_z = \mathbf{k}$$

Solving (1) and (2) simultaneously,

$$\mathbf{i} = \cos\phi\,\mathbf{e}_\rho - \sin\phi\,\mathbf{e}_\phi, \qquad \mathbf{j} = \sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi$$

Then

$$\begin{aligned}\mathbf{A} &= z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}\\ &= z(\cos\phi\,\mathbf{e}_\rho - \sin\phi\,\mathbf{e}_\phi) - 2\rho\cos\phi(\sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi) + \rho\sin\phi\,\mathbf{e}_z\\ &= (z\cos\phi - 2\rho\cos\phi\sin\phi)\mathbf{e}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\mathbf{e}_\phi + \rho\sin\phi\,\mathbf{e}_z\end{aligned}$$

and $A_\rho = z\cos\phi - 2\rho\cos\phi\sin\phi$, $A_\phi = -z\sin\phi - 2\rho\cos^2\phi$, $A_z = \rho\sin\phi$.

5. Prove $\frac{d}{dt}\mathbf{e}_\rho = \dot\phi\,\mathbf{e}_\phi$, $\frac{d}{dt}\mathbf{e}_\phi = -\dot\phi\,\mathbf{e}_\rho$ where dots denote differentiation with respect to time t.

From Problem 3,

$$\mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \qquad \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$

Then

$$\frac{d}{dt}\mathbf{e}_\rho = -(\sin\phi)\dot\phi\,\mathbf{i} + (\cos\phi)\dot\phi\,\mathbf{j} = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\dot\phi = \dot\phi\,\mathbf{e}_\phi$$

$$\frac{d}{dt}\mathbf{e}_\phi = -(\cos\phi)\dot\phi\,\mathbf{i} - (\sin\phi)\dot\phi\,\mathbf{j} = -(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j})\dot\phi = -\dot\phi\,\mathbf{e}_\rho$$

6. Express the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a particle in cylindrical coordinates.

In rectangular coordinates the position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the velocity and acceleration vectors are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad \text{and} \quad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$$

In cylindrical coordinates, using Problem 4,

$$\begin{aligned}\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} &= (\rho\cos\phi)(\cos\phi\,\mathbf{e}_\rho - \sin\phi\,\mathbf{e}_\phi) \\ &\quad + (\rho\sin\phi)(\sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi) + z\,\mathbf{e}_z \\ &= \rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z\end{aligned}$$

Then $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\rho}{dt}\mathbf{e}_\rho + \rho\frac{d\mathbf{e}_\rho}{dt} + \frac{dz}{dt}\mathbf{e}_z = \dot{\rho}\,\mathbf{e}_\rho + \rho\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\,\mathbf{e}_z$$

using Problem 5. Differentiating again,

$$\begin{aligned}\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{\rho}\,\mathbf{e}_\rho + \rho\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\,\mathbf{e}_z) \\ &= \dot{\rho}\frac{d\mathbf{e}_\rho}{dt} + \ddot{\rho}\,\mathbf{e}_\rho + \rho\dot{\phi}\frac{d\mathbf{e}_\phi}{dt} + \rho\ddot{\phi}\,\mathbf{e}_\phi + \dot{\rho}\dot{\phi}\,\mathbf{e}_\phi + \ddot{z}\,\mathbf{e}_z \\ &= \dot{\rho}\dot{\phi}\,\mathbf{e}_\phi + \ddot{\rho}\,\mathbf{e}_\rho + \rho\dot{\phi}(-\dot{\phi}\,\mathbf{e}_\rho) + \rho\ddot{\phi}\,\mathbf{e}_\phi + \dot{\rho}\dot{\phi}\,\mathbf{e}_\phi + \ddot{z}\,\mathbf{e}_z \\ &= (\ddot{\rho} - \rho\dot{\phi}^2)\,\mathbf{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\,\mathbf{e}_\phi + \ddot{z}\,\mathbf{e}_z\end{aligned}$$

using Problem 5.

7. Find the square of the element of arc length in cylindrical coordinates and determine the corresponding scale factors.

First Method.

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

$$dx = -\rho\sin\phi\,d\phi + \cos\phi\,d\rho, \quad dy = \rho\cos\phi\,d\phi + \sin\phi\,d\rho, \quad dz = dz$$

Then $$\begin{aligned}ds^2 = dx^2 + dy^2 + dz^2 &= (-\rho\sin\phi\,d\phi + \cos\phi\,d\rho)^2 + (\rho\cos\phi\,d\phi + \sin\phi\,d\rho)^2 + (dz)^2 \\ &= (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2 = h_1^2(d\rho)^2 + h_2^2(d\phi)^2 + h_3^2(dz)^2\end{aligned}$$

and $h_1 = h_\rho = 1$, $h_2 = h_\phi = \rho$, $h_3 = h_z = 1$ are the scale factors.

Second Method. The position vector is $\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\mathbf{k}$. Then

$$\begin{aligned}d\mathbf{r} &= \frac{\partial\mathbf{r}}{\partial\rho}d\rho + \frac{\partial\mathbf{r}}{\partial\phi}d\phi + \frac{\partial\mathbf{r}}{\partial z}dz \\ &= (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j})\,d\rho + (-\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j})\,d\phi + \mathbf{k}\,dz \\ &= (\cos\phi\,d\rho - \rho\sin\phi\,d\phi)\,\mathbf{i} + (\sin\phi\,d\rho + \rho\cos\phi\,d\phi)\,\mathbf{j} + \mathbf{k}\,dz\end{aligned}$$

Thus $$\begin{aligned}ds^2 = d\mathbf{r}\cdot d\mathbf{r} &= (\cos\phi\,d\rho - \rho\sin\phi\,d\phi)^2 + (\sin\phi\,d\rho + \rho\cos\phi\,d\phi)^2 + (dz)^2 \\ &= (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2\end{aligned}$$

8. Work Problem 7 for (*a*) spherical and (*b*) parabolic cylindrical coordinates.

(*a*) $$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Then
$$dx = -r\sin\theta\sin\phi\,d\phi + r\cos\theta\cos\phi\,d\theta + \sin\theta\cos\phi\,dr$$
$$dy = r\sin\theta\cos\phi\,d\phi + r\cos\theta\sin\phi\,d\theta + \sin\theta\sin\phi\,dr$$
$$dz = -r\sin\theta\,d\theta + \cos\theta\,dr$$

and
$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

The scale factors are $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$.

(*b*) $$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

Then $$dx = u\,du - v\,dv, \quad dy = u\,dv + v\,du, \quad dz = dz$$

and $$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (u^2+v^2)(du)^2 + (u^2+v^2)(dv)^2 + (dz)^2$$

The scale factors are $h_1 = h_u = \sqrt{u^2+v^2}$, $h_2 = h_v = \sqrt{u^2+v^2}$, $h_3 = h_z = 1$.

9. Sketch a volume element in (*a*) cylindrical and (*b*) spherical coordinates giving the magnitudes of its edges.

(*a*) The edges of the volume element in cylindrical coordinates (Fig.(*a*) below) have magnitudes $\rho\,d\phi$, $d\rho$ and dz. This could also be seen from the fact that the edges are given by

$$ds_1 = h_1 du_1 = (1)(d\rho) = d\rho, \quad ds_2 = h_2 du_2 = \rho\,d\phi, \quad ds_3 = (1)(dz) = dz$$

using the scale factors obtained from Problem 7.

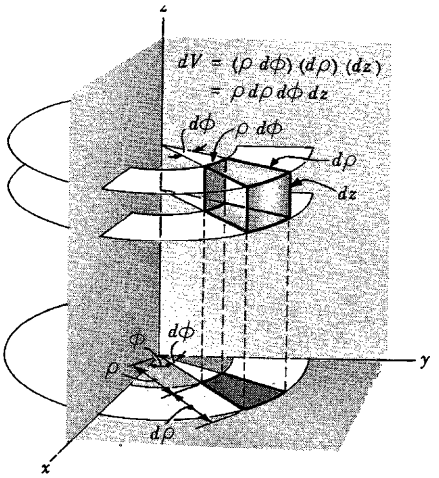


Fig.(*a*) **Volume element in cylindrical coordinates.**

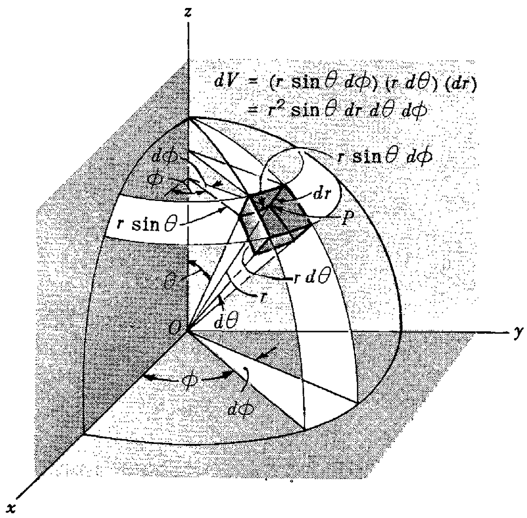


Fig.(*b*) **Volume element in spherical coordinates.**

(*b*) The edges of the volume element in spherical coordinates (Fig.(*b*) above) have magnitudes dr, $r\,d\theta$ and $r\sin\theta\,d\phi$. This could also be seen from the fact that the edges are given by

$$ds_1 = h_1 du_1 = (1)(dr) = dr, \quad ds_2 = h_2 du_2 = r\,d\theta, \quad ds_3 = h_3 du_3 = r\sin\theta\,d\phi$$

using the scale factors obtained from Problem 8(*a*).

10. Find the volume element dV in (*a*) cylindrical, (*b*) spherical and (*c*) parabolic cylindrical coordinates.

The volume element in orthogonal curvilinear coordinates u_1, u_2, u_3 is

$$dV = h_1 h_2 h_3 \, du_1 du_2 du_3$$

(*a*) In cylindrical coordinates $u_1 = \rho$, $u_2 = \phi$, $u_3 = z$, $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$ (see Problem 7). Then

$$dV = (1)(\rho)(1)\, d\rho\, d\phi\, dz = \rho\, d\rho\, d\phi\, dz$$

This can also be observed directly from Fig. (*a*) of Problem 9.

(*b*) In spherical coordinates $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, $h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$ (see Problem 8(*a*)). Then

$$dV = (1)(r)(r \sin\theta)\, dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

This can also be observed directly from Fig. (*b*) of Problem 9.

(*c*) In parabolic cylindrical coordinates $u_1 = u$, $u_2 = v$, $u_3 = z$, $h_1 = \sqrt{u^2 + v^2}$, $h_2 = \sqrt{u^2 + v^2}$, $h_3 = 1$ (see Problem 8(*b*)). Then

$$dV = (\sqrt{u^2+v^2})(\sqrt{u^2+v^2})(1)\, du\, dv\, dz = (u^2 + v^2)\, du\, dv\, dz$$

11. Find (*a*) the scale factors and (*b*) the volume element dV in oblate spheroidal coordinates.

(*a*) $x = a \cosh\xi \cos\eta \cos\phi, \quad y = a \cosh\xi \cos\eta \sin\phi, \quad z = a \sinh\xi \sin\eta$

$$dx = -a\cosh\xi\cos\eta\sin\phi\, d\phi - a\cosh\xi\sin\eta\cos\phi\, d\eta + a\sinh\xi\cos\eta\cos\phi\, d\xi$$
$$dy = a\cosh\xi\cos\eta\cos\phi\, d\phi - a\cosh\xi\sin\eta\sin\phi\, d\eta + a\sinh\xi\cos\eta\sin\phi\, d\xi$$
$$dz = a\sinh\xi\cos\eta\, d\eta + a\cosh\xi\sin\eta\, d\xi$$

Then
$$\begin{aligned}(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 &= a^2(\sinh^2\xi + \sin^2\eta)(d\xi)^2 \\ &\quad + a^2(\sinh^2\xi + \sin^2\eta)(d\eta)^2 \\ &\quad + a^2\cosh^2\xi\cos^2\eta\,(d\phi)^2\end{aligned}$$

and $h_1 = h_\xi = a\sqrt{\sinh^2\xi + \sin^2\eta}$, $h_2 = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}$, $h_3 = h_\phi = a\cosh\xi\cos\eta$.

(*b*)
$$\begin{aligned}dV &= (a\sqrt{\sinh^2\xi + \sin^2\eta})(a\sqrt{\sinh^2\xi + \sin^2\eta})(a\cosh\xi\cos\eta)\, d\xi\, d\eta\, d\phi \\ &= a^3(\sinh^2\xi + \sin^2\eta)\cosh\xi\cos\eta\, d\xi\, d\eta\, d\phi\end{aligned}$$

12. Find expressions for the elements of area in orthogonal curvilinear coordinates.

Referring to Figure 3, p.136, the area elements are given by

$$dA_1 = |(h_2\, du_2\, \mathbf{e}_2) \times (h_3\, du_3\, \mathbf{e}_3)| = h_2 h_3 |\mathbf{e}_2 \times \mathbf{e}_3|\, du_2\, du_3 = h_2 h_3\, du_2\, du_3$$

since $|\mathbf{e}_2 \times \mathbf{e}_3| = |\mathbf{e}_1| = 1$. Similarly

$$dA_2 = |(h_1\, du_1\, \mathbf{e}_1) \times (h_3\, du_3\, \mathbf{e}_3)| = h_1 h_3\, du_1\, du_3$$

$$dA_3 = |(h_1\, du_1\, \mathbf{e}_1) \times (h_2\, du_2\, \mathbf{e}_2)| = h_1 h_2\, du_1\, du_2$$

13. If u_1, u_2, u_3 are orthogonal curvilinear coordinates, show that the Jacobian of x,y,z with respect to u_1, u_2, u_3 is

$$J\left(\frac{x,y,z}{u_1,u_2,u_3}\right) = \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} = h_1 h_2 h_3$$

By Problem 38 of Chapter 2, the given determinant equals

$$\left(\frac{\partial x}{\partial u_1}\mathbf{i} + \frac{\partial y}{\partial u_1}\mathbf{j} + \frac{\partial z}{\partial u_1}\mathbf{k}\right) \cdot \left(\frac{\partial x}{\partial u_2}\mathbf{i} + \frac{\partial y}{\partial u_2}\mathbf{j} + \frac{\partial z}{\partial u_2}\mathbf{k}\right) \times \left(\frac{\partial x}{\partial u_3}\mathbf{i} + \frac{\partial y}{\partial u_3}\mathbf{j} + \frac{\partial z}{\partial u_3}\mathbf{k}\right)$$

$$= \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} = h_1\mathbf{e}_1 \cdot h_2\mathbf{e}_2 \times h_3\mathbf{e}_3$$

$$= h_1 h_2 h_3\, \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = h_1 h_2 h_3$$

If the Jacobian equals zero identically then $\frac{\partial \mathbf{r}}{\partial u_1}, \frac{\partial \mathbf{r}}{\partial u_2}, \frac{\partial \mathbf{r}}{\partial u_3}$ are coplanar vectors and the curvilinear coordinate transformation breaks down, i.e. there is a relation between x,y,z having the form $F(x,y,z) = 0$. We shall therefore require the Jacobian to be different from zero.

14. Evaluate $\iiint_V (x^2+y^2+z^2)\, dx\, dy\, dz$ where V is a sphere having center at the origin and radius equal to a.

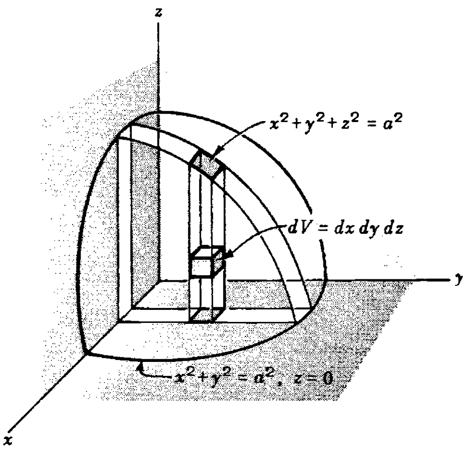


Fig. (a)

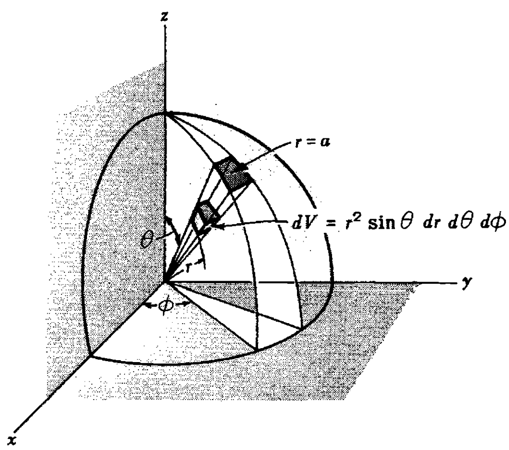


Fig. (b)

The required integral is equal to eight times the integral evaluated over that part of the sphere contained in the first octant (see Fig. (a) above).

Then in rectangular coordinates the integral equals

$$8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$$

but the evaluation, although possible, is tedious. It is easier to use spherical coordinates for the evaluation. In changing to spherical coordinates, the integrand $x^2+y^2+z^2$ is replaced by its equivalent r^2 while the volume element $dx\,dy\,dz$ is replaced by the volume element $r^2 \sin\theta \; dr\,d\theta\,d\phi$ (see Problem 10(*b*)). To cover the required region in the first octant, fix θ and ϕ (see Fig. (*b*) above) and integrate from $r=0$ to $r=a$; then keep ϕ constant and integrate from $\theta=0$ to $\pi/2$; finally integrate with respect to ϕ from $\phi=0$ to $\phi=\pi/2$. Here we have performed the integration in the order r,θ,ϕ although any order can be used. The result is

$$8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}(r^2)(r^2\sin\theta\; dr\,d\theta\,d\phi) \;=\; 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^4\sin\theta\; dr\,d\theta\,d\phi$$

$$= \;8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\frac{r^5}{5}\sin\theta\,\Big|_{r=0}^{a}d\theta\,d\phi \;=\; \frac{8a^5}{5}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin\theta\; d\theta\,d\phi$$

$$= \;\frac{8a^5}{5}\int_{\phi=0}^{\pi/2} -\cos\theta\,\Big|_{\theta=0}^{\pi/2} d\phi \;=\; \frac{8a^5}{5}\int_{\phi=0}^{\pi/2} d\phi \;=\; \frac{4\pi a^5}{5}$$

Physically the integral represents the moment of inertia of the sphere with respect to the origin, i.e. the polar moment of inertia, if the sphere has unit density.

In general, when transforming multiple integrals from rectangular to orthogonal curvilinear coordinates the volume element $dx\,dy\,dz$ is replaced by $h_1h_2h_3\,du_1du_2du_3$ or the equivalent $J(\frac{x,y,z}{u_1,u_2,u_3})\,du_1du_2du_3$ where J is the Jacobian of the transformation from x,y,z to u_1,u_2,u_3 (see Problem 13).

15. If u_1,u_2,u_3 are general coordinates, show that $\dfrac{\partial\mathbf{r}}{\partial u_1}, \dfrac{\partial\mathbf{r}}{\partial u_2}, \dfrac{\partial\mathbf{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocal systems of vectors.

We must show that $\dfrac{\partial\mathbf{r}}{\partial u_p}\cdot\nabla u_q = \begin{cases}1 & \text{if } p=q\\ 0 & \text{if } p\neq q\end{cases}$ where p and q can have any of the values 1,2,3. We have

$$d\mathbf{r} \;=\; \frac{\partial\mathbf{r}}{\partial u_1}du_1 + \frac{\partial\mathbf{r}}{\partial u_2}du_2 + \frac{\partial\mathbf{r}}{\partial u_3}du_3$$

Multiply by $\nabla u_1\;\cdot$. Then

$$\nabla u_1\cdot d\mathbf{r} \;=\; du_1 \;=\; (\nabla u_1\cdot\frac{\partial\mathbf{r}}{\partial u_1})\,du_1 + (\nabla u_1\cdot\frac{\partial\mathbf{r}}{\partial u_2})\,du_2 + (\nabla u_1\cdot\frac{\partial\mathbf{r}}{\partial u_3})\,du_3$$

or
$$\nabla u_1\cdot\frac{\partial\mathbf{r}}{\partial u_1} = 1, \qquad \nabla u_1\cdot\frac{\partial\mathbf{r}}{\partial u_2} = 0, \qquad \nabla u_1\cdot\frac{\partial\mathbf{r}}{\partial u_3} = 0$$

Similarly, upon multiplying by $\nabla u_2\;\cdot$ and $\nabla u_3\;\cdot$ the remaining relations are proved.

16. Prove $\left\{\dfrac{\partial\mathbf{r}}{\partial u_1}\cdot\dfrac{\partial\mathbf{r}}{\partial u_2}\times\dfrac{\partial\mathbf{r}}{\partial u_3}\right\}\left\{\nabla u_1\cdot\nabla u_2\times\nabla u_3\right\} \;=\; 1.$

From Problem 15, $\dfrac{\partial\mathbf{r}}{\partial u_1}, \dfrac{\partial\mathbf{r}}{\partial u_2}, \dfrac{\partial\mathbf{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocal systems of vectors. Then the required result follows from Problem 53(*c*) of Chapter 2.

The result is equivalent to a theorem on Jacobians for

$$\nabla u_1 \cdot \nabla u_2 \times \nabla u_3 = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{vmatrix} = J(\frac{u_1,u_2,u_3}{x,y,z})$$

and so $J(\frac{x,y,z}{u_1,u_2,u_3})\, J(\frac{u_1,u_2,u_3}{x,y,z}) = 1$ using Problem 13.

17. Show that the square of the element of arc length in general curvilinear coordinates can be expressed by

$$ds^2 = \sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq}\, du_p\, du_q$$

We have

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = \boldsymbol{\alpha}_1\, du_1 + \boldsymbol{\alpha}_2\, du_2 + \boldsymbol{\alpha}_3\, du_3$$

Then

$$\begin{aligned} ds^2 = d\mathbf{r}\cdot d\mathbf{r} &= \boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1\, du_1^2 + \boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2\, du_1\, du_2 + \boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_3\, du_1\, du_3 \\ &\quad + \boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_1\, du_2\, du_1 + \boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2\, du_2^2 + \boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_3\, du_2\, du_3 \\ &\quad + \boldsymbol{\alpha}_3\cdot\boldsymbol{\alpha}_1\, du_3\, du_1 + \boldsymbol{\alpha}_3\cdot\boldsymbol{\alpha}_2\, du_3\, du_2 + \boldsymbol{\alpha}_3\cdot\boldsymbol{\alpha}_3\, du_3^2 \\ &= \sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq}\, du_p\, du_q \quad \text{where } g_{pq} = \boldsymbol{\alpha}_p\cdot\boldsymbol{\alpha}_q \end{aligned}$$

This is called the *fundamental quadratic form* or *metric form*. The quantities g_{pq} are called *metric coefficients* and are symmetric, i.e. $g_{pq} = g_{qp}$. If $g_{pq} = 0$, $p \neq q$, then the coordinate system is orthogonal. In this case $g_{11} = h_1^2$, $g_{22} = h_2^2$, $g_{33} = h_3^2$. The metric form extended to higher dimensional space is of fundamental importance in the theory of relativity (see Chapter 8).

GRADIENT, DIVERGENCE AND CURL IN ORTHOGONAL COORDINATES.

18. Derive an expresssion for $\nabla\Phi$ in orthogonal curvilinear coordinates.

Let $\nabla\Phi = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$ where f_1, f_2, f_3 are to be determined.

Since

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \\ &= h_1\mathbf{e}_1\, du_1 + h_2\mathbf{e}_2\, du_2 + h_3\mathbf{e}_3\, du_3 \end{aligned}$$

we have

$$(1)\quad d\Phi = \nabla\Phi\cdot d\mathbf{r} = h_1 f_1\, du_1 + h_2 f_2\, du_2 + h_3 f_3\, du_3$$

But

$$(2)\quad d\Phi = \frac{\partial\Phi}{\partial u_1} du_1 + \frac{\partial\Phi}{\partial u_2} du_2 + \frac{\partial\Phi}{\partial u_3} du_3$$

Equating (*1*) and (*2*), $f_1 = \dfrac{1}{h_1}\dfrac{\partial\Phi}{\partial u_1}, \quad f_2 = \dfrac{1}{h_2}\dfrac{\partial\Phi}{\partial u_2}, \quad f_3 = \dfrac{1}{h_3}\dfrac{\partial\Phi}{\partial u_3}.$

Then
$$\nabla\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

This indicates the operator equivalence
$$\nabla \equiv \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

which reduces to the usual expression for the operator ∇ in rectangular coordinates.

19. Let u_1, u_2, u_3 be orthogonal coordinates. (*a*) Prove that $|\nabla u_p| = h_p^{-1}$, $p = 1,2,3$.
(*b*) Show that $\mathbf{e}_p = \mathbf{E}_p$.

(*a*) Let $\Phi = u_1$ in Problem 18. Then $\nabla u_1 = \dfrac{\mathbf{e}_1}{h_1}$ and so $|\nabla u_1| = |\mathbf{e}_1|/h_1 = h_1^{-1}$, since $|\mathbf{e}_1| = 1$. Similarly by letting $\Phi = u_2$ and u_3, $|\nabla u_2| = h_2^{-1}$ and $|\nabla u_3| = h_3^{-1}$.

(*b*) By definition $\mathbf{E}_p = \dfrac{\nabla u_p}{|\nabla u_p|}$. From part (*a*), this can be written $\mathbf{E}_p = h_p \nabla u_p = \mathbf{e}_p$ and the result is proved.

20. Prove $\mathbf{e}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3$ with similar equations for $\mathbf{e}_2$ and $\mathbf{e}_3$, where u_1, u_2, u_3 are orthogonal coordinates.

From Problem 19, $\nabla u_1 = \dfrac{\mathbf{e}_1}{h_1}, \ \nabla u_2 = \dfrac{\mathbf{e}_2}{h_2}, \ \nabla u_3 = \dfrac{\mathbf{e}_3}{h_3}.$

Then $\nabla u_2 \times \nabla u_3 = \dfrac{\mathbf{e}_2 \times \mathbf{e}_3}{h_2 h_3} = \dfrac{\mathbf{e}_1}{h_2 h_3}$ and $\mathbf{e}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3$.

Similarly $\mathbf{e}_2 = h_3 h_1 \nabla u_3 \times \nabla u_1$ and $\mathbf{e}_3 = h_1 h_2 \nabla u_1 \times \nabla u_2$.

21. Show that in orthogonal coordinates

$$(a) \quad \nabla \cdot (A_1 \mathbf{e}_1) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(A_1 h_2 h_3)$$
$$(b) \quad \nabla \times (A_1 \mathbf{e}_1) = \frac{\mathbf{e}_2}{h_3 h_1}\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\mathbf{e}_3}{h_1 h_2}\frac{\partial}{\partial u_2}(A_1 h_1)$$

with similar results for vectors $A_2\mathbf{e}_2$ and $A_3\mathbf{e}_3$.

(*a*) From Problem 20,

$$\begin{aligned}
\nabla \cdot (A_1 \mathbf{e}_1) &= \nabla \cdot (A_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \\
&= \nabla(A_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + A_1 h_2 h_3 \nabla \cdot (\nabla u_2 \times \nabla u_3) \\
&= \nabla(A_1 h_2 h_3) \cdot \frac{\mathbf{e}_2}{h_2} \times \frac{\mathbf{e}_3}{h_3} + 0 = \nabla(A_1 h_2 h_3) \cdot \frac{\mathbf{e}_1}{h_2 h_3} \\
&= \left[\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2}(A_1 h_2 h_3) + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}(A_1 h_2 h_3) \right] \cdot \frac{\mathbf{e}_1}{h_2 h_3} \\
&= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(A_1 h_2 h_3)
\end{aligned}$$

(*b*)
$$\begin{aligned}\nabla\times(A_1\mathbf{e}_1) &= \nabla\times(A_1h_1\nabla u_1)\\ &= \nabla(A_1h_1)\times\nabla u_1 + A_1h_1\nabla\times\nabla u_1\\ &= \nabla(A_1h_1)\times\frac{\mathbf{e}_1}{h_1} + \mathbf{0}\\ &= \left[\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1h_1) + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2}(A_1h_1) + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}(A_1h_1)\right]\times\frac{\mathbf{e}_1}{h_1}\\ &= \frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)\end{aligned}$$

22. Express $\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A}$ in orthogonal coordinates.

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \nabla\cdot(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3) = \nabla\cdot(A_1\mathbf{e}_1) + \nabla\cdot(A_2\mathbf{e}_2) + \nabla\cdot(A_3\mathbf{e}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]\end{aligned}$$

using Problem 21(*a*).

23. Express $\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A}$ in orthogonal coordinates.

$$\begin{aligned}\nabla\times\mathbf{A} &= \nabla\times(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3) = \nabla\times(A_1\mathbf{e}_1) + \nabla\times(A_2\mathbf{e}_2) + \nabla\times(A_3\mathbf{e}_3)\\ &= \frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)\\ &\quad + \frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial u_3}(A_2h_2)\\ &\quad + \frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u_1}(A_3h_3)\\ &= \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right] + \frac{\mathbf{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\partial}{\partial u_1}(A_3h_3)\right]\\ &\quad + \frac{\mathbf{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\partial}{\partial u_2}(A_1h_1)\right]\end{aligned}$$

using Problem 21(*b*). This can be written

$$\nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ A_1h_1 & A_2h_2 & A_3h_3\end{vmatrix}$$

24. Express $\nabla^2\psi$ in orthogonal curvilinear coordinates.

From Problem 18, $$\nabla\psi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\psi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\psi}{\partial u_3}.$$

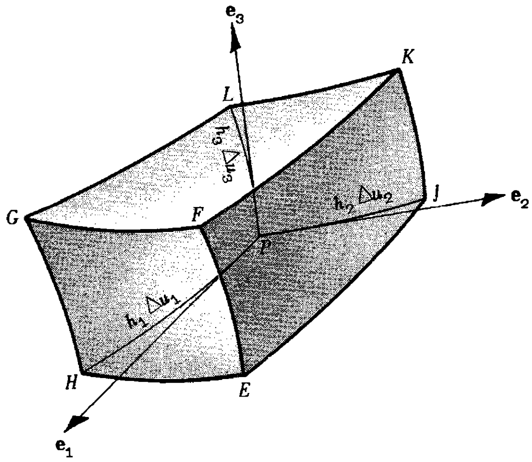
If $\mathbf{A} = \nabla\psi$, then $\quad A_1 = \dfrac{1}{h_1}\dfrac{\partial\psi}{\partial u_1},\quad A_2 = \dfrac{1}{h_2}\dfrac{\partial\psi}{\partial u_2},\quad A_3 = \dfrac{1}{h_3}\dfrac{\partial\psi}{\partial u_3}\quad$ and by Problem 22,

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \nabla\cdot\nabla\psi = \nabla^2\psi \\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]\end{aligned}$$

25. Use the integral definition

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \lim_{\Delta V\to 0}\frac{\displaystyle\iint_{\Delta S}\mathbf{A}\cdot\mathbf{n}\,dS}{\Delta V}$$

(see Problem 19, Chapter 6) to express $\nabla\cdot\mathbf{A}$ in orthogonal curvilinear coordinates.



Consider the volume element ΔV (see adjacent figure) having edges $h_1\Delta u_1$, $h_2\Delta u_2$, $h_3\Delta u_3$.

Let $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ and let $\mathbf{n}$ be the outward drawn unit normal to the surface ΔS of ΔV. On face $JKLP$, $\mathbf{n} = -\mathbf{e}_1$. Then we have approximately,

$$\begin{aligned}\iint_{JKLP}\mathbf{A}\cdot\mathbf{n}\,dS &= (\mathbf{A}\cdot\mathbf{n}\text{ at point }P)\,(\text{Area of }JKLP) \\ &= [(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3)\cdot(-\mathbf{e}_1)]\,(h_2h_3\Delta u_2\Delta u_3) \\ &= -A_1h_2h_3\,\Delta u_2\Delta u_3\end{aligned}$$

On face $EFGH$, the surface integral is

$$A_1h_2h_3\,\Delta u_2\Delta u_3 + \frac{\partial}{\partial u_1}(A_1h_2h_3\,\Delta u_2\Delta u_3)\,\Delta u_1$$

apart from infinitesimals of order higher than $\Delta u_1\Delta u_2\Delta u_3$. Then the net contribution to the surface integral from these two faces is

$$\frac{\partial}{\partial u_1}(A_1h_2h_3\,\Delta u_2\Delta u_3)\,\Delta u_1 = \frac{\partial}{\partial u_1}(A_1h_2h_3)\,\Delta u_1\Delta u_2\Delta u_3$$

The contribution from all six faces of ΔV is

$$\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]\Delta u_1\Delta u_2\Delta u_3$$

Dividing this by the volume $h_1h_2h_3\,\Delta u_1\Delta u_2\Delta u_3$ and taking the limit as $\Delta u_1, \Delta u_2, \Delta u_3$ approach zero, we find

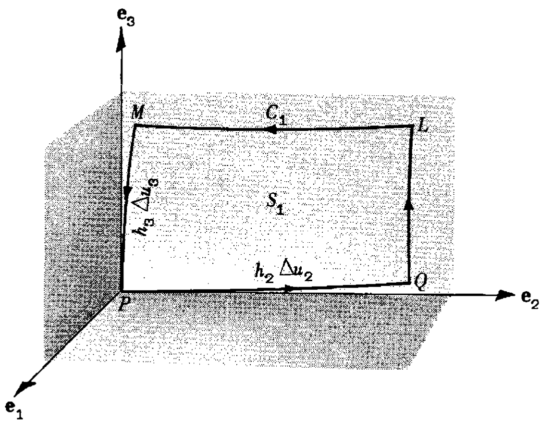
$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

Note that the same result would be obtained had we chosen the volume element ΔV such that P is at its center. In this case the calculation would proceed in a manner analogous to that of Problem 21, Chapter 4.

26. Use the integral definition

$$(\text{curl}\,\mathbf{A})\cdot\mathbf{n} = (\nabla\times\mathbf{A})\cdot\mathbf{n} = \lim_{\Delta S\to 0}\frac{\oint_C \mathbf{A}\cdot d\mathbf{r}}{\Delta S}$$

(see Problem 35, Chapter 6) to express $\nabla\times\mathbf{A}$ in orthogonal curvilinear coordinates.



Let us first calculate $(\text{curl}\,\mathbf{A})\cdot\mathbf{e}_1$. To do this consider the surface S_1 normal to $\mathbf{e}_1$ at P, as shown in the adjoining figure. Denote the boundary of S_1 by C_1. Let $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$. We have

$$\oint_{C_1}\mathbf{A}\cdot d\mathbf{r} = \int_{PQ}\mathbf{A}\cdot d\mathbf{r} + \int_{QL}\mathbf{A}\cdot d\mathbf{r} + \int_{LM}\mathbf{A}\cdot d\mathbf{r} + \int_{MP}\mathbf{A}\cdot d\mathbf{r}$$

The following approximations hold

$$(1)\qquad \int_{PQ}\mathbf{A}\cdot d\mathbf{r} = (\mathbf{A}\text{ at }P)\cdot(h_2\,\Delta u_2\,\mathbf{e}_2)$$
$$= (A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3)\cdot(h_2\,\Delta u_2\,\mathbf{e}_2) = A_2\,h_2\,\Delta u_2$$

Then

$$\int_{ML}\mathbf{A}\cdot d\mathbf{r} = A_2\,h_2\,\Delta u_2 + \frac{\partial}{\partial u_3}(A_2\,h_2\,\Delta u_2)\,\Delta u_3$$

or

$$(2)\qquad \int_{LM}\mathbf{A}\cdot d\mathbf{r} = -A_2\,h_2\,\Delta u_2 - \frac{\partial}{\partial u_3}(A_2\,h_2\,\Delta u_2)\,\Delta u_3$$

Similarly,

$$\int_{PM}\mathbf{A}\cdot d\mathbf{r} = (\mathbf{A}\text{ at }P)\cdot(h_3\,\Delta u_3\,\mathbf{e}_3) = A_3\,h_3\,\Delta u_3$$

or

$$(3)\qquad \int_{MP}\mathbf{A}\cdot d\mathbf{r} = -A_3\,h_3\,\Delta u_3$$

and

$$(4)\qquad \int_{QL}\mathbf{A}\cdot d\mathbf{r} = A_3\,h_3\,\Delta u_3 + \frac{\partial}{\partial u_2}(A_3\,h_3\,\Delta u_3)\,\Delta u_2$$

Adding *(1)*, *(2)*, *(3)*, *(4)* we have

$$\oint_{C_1}\mathbf{A}\cdot d\mathbf{r} = \frac{\partial}{\partial u_2}(A_3\,h_3\,\Delta u_3)\,\Delta u_2 - \frac{\partial}{\partial u_3}(A_2\,h_2\,\Delta u_2)\,\Delta u_3$$
$$= \left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right]\Delta u_2\,\Delta u_3$$

apart from infinitesimals of order higher than $\Delta u_2\,\Delta u_3$.

Dividing by the area of S_1 equal to $h_2h_3\Delta u_2\Delta u_3$ and taking the limit as Δu_2 and Δu_3 approach zero,

$$(\text{curl}\,\mathbf{A})\cdot\mathbf{e}_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right]$$

Similarly, by choosing areas S_2 and S_3 perpendicular to $\mathbf{e}_2$ and $\mathbf{e}_3$ at P respectively, we find $(\text{curl}\,\mathbf{A})\cdot\mathbf{e}_2$ and $(\text{curl}\,\mathbf{A})\cdot\mathbf{e}_3$. This leads to the required result

$$\begin{aligned}\text{curl}\,\mathbf{A} &= \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right]\\ &\quad + \frac{\mathbf{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\partial}{\partial u_1}(A_3h_3)\right]\\ &\quad + \frac{\mathbf{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\partial}{\partial u_2}(A_1h_1)\right] \;=\; \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}\end{aligned}$$

The result could also have been derived by choosing P as the center of area S_1; the calculation would then proceed as in Problem 36, Chapter 6.

27. Express in cylindrical coordinates the quantities (a) $\nabla\Phi$, (b) $\nabla\cdot\mathbf{A}$, (c) $\nabla\times\mathbf{A}$, (d) $\nabla^2\Phi$.

For cylindrical coordinates (ρ,ϕ,z),

$$u_1=\rho,\ u_2=\phi,\ u_3=z\ ;\quad \mathbf{e}_1=\mathbf{e}_\rho,\ \mathbf{e}_2=\mathbf{e}_\phi,\ \mathbf{e}_3=\mathbf{e}_z\ ;$$

and

$$h_1=h_\rho=1,\quad h_2=h_\phi=\rho,\quad h_3=h_z=1$$

(a)
$$\begin{aligned}\nabla\Phi &= \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3\\ &= \frac{1}{1}\frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{1}{1}\frac{\partial\Phi}{\partial z}\mathbf{e}_z\\ &= \frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z\end{aligned}$$

(b)
$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]\\ &= \frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial\rho}\Big((\rho)(1)A_\rho\Big) + \frac{\partial}{\partial\phi}\Big((1)(1)A_\phi\Big) + \frac{\partial}{\partial z}\Big((1)(\rho)A_z\Big)\right]\\ &= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{\partial A_\phi}{\partial\phi} + \frac{\partial}{\partial z}(\rho A_z)\right]\end{aligned}$$

where $\mathbf{A} = A_\rho\mathbf{e}_1 + A_\phi\mathbf{e}_2 + A_z\mathbf{e}_3$, i.e. $A_1=A_\rho$, $A_2=A_\phi$, $A_3=A_z$.

(c)
$$\nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z\\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

$$= \frac{1}{\rho}\left[\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial}{\partial z}(\rho A_\phi)\right)\mathbf{e}_\rho + \left(\rho\frac{\partial A_\rho}{\partial z} - \rho\frac{\partial A_z}{\partial \rho}\right)\mathbf{e}_\phi + \left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)\mathbf{e}_z\right]$$

(d) $$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

$$= \frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial \rho}\left(\frac{(\rho)(1)}{(1)}\frac{\partial\Phi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{(1)(1)}{\rho}\frac{\partial\Phi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\frac{(1)(\rho)}{(1)}\frac{\partial\Phi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

28. Express (a) $\nabla\times\mathbf{A}$ and (b) $\nabla^2\psi$ in spherical coordinates.

Here $u_1 = r,\ u_2 = \theta,\ u_3 = \phi$; $\mathbf{e}_1 = \mathbf{e}_r,\ \mathbf{e}_2 = \mathbf{e}_\theta,\ \mathbf{e}_3 = \mathbf{e}_\phi$; $h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_\phi = r\sin\theta$.

(a) $$\nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{(1)(r)(r\sin\theta)}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left\{\frac{\partial}{\partial\theta}(r\sin\theta\,A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)\right\}\mathbf{e}_r\right.$$

$$\left. + \left\{\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r\sin\theta\,A_\phi)\right\} r\,\mathbf{e}_\theta + \left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\} r\sin\theta\,\mathbf{e}_\phi\right]$$

(b) $$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$= \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{(r\sin\theta)(1)}{r}\frac{\partial\psi}{\partial\theta}\right)\right.$$

$$\left. + \frac{\partial}{\partial\phi}\left(\frac{(1)(r)}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

29. Write Laplace's equation in parabolic cylindrical coordinates.

From Problem 8(b),

$$u_1 = u,\ u_2 = v,\ u_3 = z\ ;\quad h_1 = \sqrt{u^2+v^2},\ h_2 = \sqrt{u^2+v^2},\ h_3 = 1$$

Then $$\nabla^2\psi = \frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\partial\psi}{\partial v}\right) + \frac{\partial}{\partial z}\left((u^2+v^2)\frac{\partial\psi}{\partial z}\right)\right]$$
$$= \frac{1}{u^2+v^2}\left(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right) + \frac{\partial^2\psi}{\partial z^2}$$

and Laplace's equation is $\nabla^2\psi = 0$ or
$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} + (u^2+v^2)\frac{\partial^2\psi}{\partial z^2} = 0$$

30. Express the heat conduction equation $\dfrac{\partial U}{\partial t} = \kappa\nabla^2 U$ in elliptic cylindrical coordinates.

Here $u_1 = u,\ u_2 = v,\ u_3 = z\ ;\quad h_1 = h_2 = a\sqrt{\sinh^2 u + \sin^2 v},\ h_3 = 1$. Then

$$\nabla^2 U = \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left[\frac{\partial}{\partial u}\left(\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial z}\left(a^2(\sinh^2 u + \sin^2 v)\frac{\partial U}{\partial z}\right)\right]$$
$$= \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left[\frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2}\right] + \frac{\partial^2 U}{\partial z^2}$$

and the heat conduction equation is
$$\frac{\partial U}{\partial t} = \kappa\left\{\frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left[\frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2}\right] + \frac{\partial^2 U}{\partial z^2}\right\}$$

SURFACE CURVILINEAR COORDINATES

31. Show that the square of the element of arc length on the surface $\mathbf{r} = \mathbf{r}(u,v)$ can be written
$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

We have $$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial u}du + \frac{\partial\mathbf{r}}{\partial v}dv$$

Then $$ds^2 = d\mathbf{r}\cdot d\mathbf{r}$$
$$= \frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial u}du^2 + 2\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}du\,dv + \frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial v}dv^2$$
$$= E\,du^2 + 2F\,du\,dv + G\,dv^2$$

32. Show that the element of surface area of the surface $\mathbf{r} = \mathbf{r}(u,v)$ is given by
$$dS = \sqrt{EG - F^2}\,du\,dv$$

The element of area is given by
$$dS = \left|\left(\frac{\partial\mathbf{r}}{\partial u}du\right)\times\left(\frac{\partial\mathbf{r}}{\partial v}dv\right)\right| = \left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right|du\,dv = \sqrt{\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)}\,du\,dv$$

The quantity under the square root sign is equal to (see Problem 48, Chapter 2)
$$\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial v}\right) - \left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right) = EG - F^2$$ and the result follows.