$\boxed{\text{Code A}}$

(1) $(D-1)^3$ annihilates $e^x x^2$

$(D-1)$ " e^x

$((D+1)^2+1)^2$ annihilates $xe^{-x}\sin x$

$(D^2+1)^2$ annihilates $x\cos x - 2\sin x$

D annihilates 3

Thus, $\boxed{D(D-1)^3(D^2+1)^2((D+1)^2+1)^2}$ is the required operator.

(2) L is an operator of 9.

$$L = (D^2+9)^2 (D-2)\left((D-1)^2+1\right)^2$$

Thus, a fundamental set of solutions is

$$\{ \sin 3x, \cos 3x, x\sin 3x, x\cos 3x, e^{2x}, e^x \sin x, e^x \cos x,$$
$$x e^x \sin x, x e^x \cos x \}$$

(3) Let $h(x) = y(x-1)$

$$\Rightarrow \frac{dh}{dx}(x) = \frac{dy}{dx}(x-1) = \cancel{\frac{2x+y+1}{x+2(y+1)}}$$

$$= \frac{2x + y(x-1) + 1}{x + 2(y(x-1)+1)}$$

$$= \frac{2x + h(x) + 1}{x + 2(h(x)+1)}$$

Let $g(x) = h(x) + 1$

$$\Rightarrow \frac{dg}{dx}(x) = \frac{dh}{dx}(x) = \frac{2x + g(x)}{x + 2g(x)}$$

Let us solve the ODE

$$g' = \frac{2x+g}{x+2g}$$

Put $g = xv \Rightarrow xv' + v = \frac{2+v}{1+2v}$

$$\Rightarrow xv' = \frac{2(1-v^2)}{1+2v}$$

$$\Rightarrow \frac{v'}{2}\left[\frac{1+2v}{1-v^2}\right] = \frac{1}{x}$$

$$\frac{1+2v}{1-v^2} = \frac{A}{1-v} + \frac{B}{1+v} \Rightarrow A = \frac{3}{2}, \ B = -\frac{1}{2}$$

$$\Rightarrow \frac{3v'}{1-v} - \frac{v'}{1+v} = \frac{4}{x}$$

$$\Rightarrow -\ln|1-v|^3 - \ln|1+v| = \ln x^4$$

$$\Rightarrow \frac{1}{(1-v)^3(1+v)} = cx^4$$

$$\Rightarrow \frac{x^4}{(x-y)^3(x+y)} = cx^4$$

$$\Rightarrow \frac{\cancel{x^4}}{(x-y)^3(x+y)} = c$$

Now put $y(x) = h(x)+1 = y(x-1)+1$

$$\Rightarrow \frac{\cancel{x^4}}{(x-1-y(x-1))^3(x+1+y(x-1))} = c$$

Put $x=2$ to get

$$\frac{4}{(1-3)^3(3+3)} = c = \frac{4}{-8\cdot 6} = -\frac{1}{12\cdot 4}$$

Thus, $\cancel{\frac{(x+1)^2}{(x-y)^3(x+y+2)} = -\frac{1}{12}}$

$$\frac{(x+1)^2}{(x-y)^3(x+y+2)} = \cancel{-\frac{1}{12}} \quad -\frac{1}{48}$$

(After putting $x-1 = X$)

④ $y' = f(x,y)$; $y(x_0) = y_0$

Theorem : (a) If (x_0, y_0) is such that $\exists$ $R > 0$ such that ~~and~~ f is continuous on $\{(x,y) \mid |x-x_0| < R$ and $|y-y_0| < R\}$

Then the IVP has a solution in a small nbd around x_0.

(b) If (x_0, y_0) is such that $\exists$ $R > 0$ such that both f and $\frac{\partial f}{\partial y}$ are continuous on

$\{(x,y) \mid |x-x_0| < R$ and $|y-y_0| < R\}$

Then the IVP has a unique solution in a small nbd around x_0.

$$f(x,y) = \cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}$$

$$\frac{\partial f}{\partial y}(x,y) = -1 - \frac{\cos(x+y)^2}{\sin(x+y)^2}$$

Let $Z \subset \mathbb{R}^2$ be the set

$$\{(x,y) \mid x+y = n\pi \text{ for some } n \in \mathbb{Z}\}$$

Then for every $(x_0, y_0) \in \mathbb{R}^2 \setminus Z$ the IVP has a unique solution in some nbd around x_0.

(5) (a) $M = y(x\cos x + 2\sin x)$

$N = x(y+1)\sin x$

$M_y = x\cos x + 2\sin x$

$N_x = (y+1)[\sin x + x\cos x]$

Since $M_y \neq N_x \Rightarrow$ ODE is not exact.

(b) $M_y - N_x = p(x)N - q(y)M$

$$M_y - N_x = x\cos x + 2\sin x - (y+1)[\sin x + x\cos x]$$

$$= x\cos x + 2\sin x - [y\sin x + yx\cos x + \sin x + x\cos x]$$

$$= \sin x - y\sin x - yx\cos x$$

Let $q(y) = 1$, then ~~$M_y - N_x + q(y)M$~~

$$M_y - N_x + q(y)M = M_y - N_x + M$$

$$= \sin x - y\sin x - \cancel{yx\cos x} + \cancel{yx\cos x} + 2y\sin x$$

$$= \sin x + y\sin x = (y+1)\sin x$$

$$= \frac{1}{x}N$$

Thus, $M_y - N_x = \frac{1}{x}N - M$

$$\Rightarrow \mu(x,y) = e^{\int \frac{1}{x}dx} e^{\int 1 \cdot dy} = xe^y$$

(c) After multiplying with $\mu(x,y)$ the ODE becomes

$$x e^y y\,(x\cos x + 2\sin x) + x^2 e^y (y+1)\sin x \; y' = 0$$

$$M = x e^y y\,(x\cos x + 2\sin x)$$

$$N = x^2 e^y (y+1)\sin x$$

$$M_y = \left(e^y + y e^y\right) x\,(x\cos x + 2\sin x)$$

$$N_x = e^y (y+1)\left[2x\sin x + x^2\cos x\right]$$

Thus, $M_y = N_x$ and so the ODE is exact.

(6) $M = e^x x^4 y^2 + e^x 4x^3 y^2 + e^x$

$N = 2x^4 y e^x + 2y$

$\frac{d\phi}{dx} = M \Rightarrow \phi(x,y) = e^x x^4 y^2 + e^x + h(y)$

$\frac{d\phi}{dy} = 2e^x x^4 y + h'(y) = 2x^4 y e^x + 2y$

$\Rightarrow h'(y) = 2y \Rightarrow h(y) = y^2$

Thus, $\phi(x,y) = e^x x^4 y^2 + e^x + y^2$

Thus, the implicit ~~soul~~ solution is $\phi(x,y) = C$

$e^x x^4 y^2 + e^x + y^2 = C$.

(7) Let us first solve

$$x^2 y'' - 3xy' + 3y = 0$$

Consider the polynomial $X^2 - 4X + 3 = 0$

Its roots are $X = 3, 1$.

Thus, the two solutions are $\{e^{3\ln x}, e^{\ln x}\}$

$$= \{x^3, x\}$$

Let $y_1 = x$, $y_2 = x^3$

$$W(y_1, y_2; x) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$$

$$\cancel{y_p = x^3 \int \frac{x\, x^4}{2x^3}\, dx - x}$$

~~First~~, First write ODE in standard form.

$$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = x^2$$

$$y_p = x^3 \int \frac{x\, x^2}{2x^3}\, dx - x \int \frac{x^3 \cdot x^2}{2x^3}\, dx$$

$$= \frac{x^4}{2} - \frac{x^4}{6} = \frac{x^4}{3}$$

The general solution is $\frac{x^4}{3} + c_1 x + c_2 x^3$

$y(1) = 0 \implies \frac{1}{3} + c_1 + c_2 = 0 \qquad y'(1) = 0 \implies \frac{4}{3} + c_1 + 3c_2 = 0$

$\Rightarrow \quad 1 + 2c_2 = 0 \quad \Rightarrow \quad c_2 = -1/2$

$\Rightarrow \quad \frac{1}{3} + c_1 - \frac{1}{2} = 0 \quad \Rightarrow \quad c_1 = \frac{1}{6}$

Thus, $y(x) = \frac{x^4}{3} + \frac{x}{6} - \frac{x^3}{2}$

(8) The given integral is

$$(f*g)(t) \quad \text{where} \quad f(t) = t^8 \quad g(t) = t^{13}$$

Thus, taking Laplace transform we get

$$L(f*g) = L(f)L(g)$$

$$= \frac{8!}{s^9}\frac{13!}{s^{14}} = \frac{8!\,13!}{22!}\frac{22!}{s^{23}}$$

$$\Rightarrow (f*g)(t) = \frac{8!\,13!}{22!}\,t^{22}$$

(9) $L(y') = sL(y) - 1$

$L(y'') = sL(y') - 0$

$= s^2 L(y) - s$

$L(y''') = sL(y'') - 2$

$= s^3 L(y) - s^2 - 2$

Thus, $s^3 L(y) - s^2 - 2 - L(y) = 0$

$\Rightarrow L(y) = \dfrac{s^2+2}{s^3-1}$

$$\frac{s^2+2}{s^3-1} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$$

$$\Rightarrow A = \left.\frac{s^2+2}{s^2+s+1}\right|_{s=1} = \frac{3}{3} = 1$$

$$\Rightarrow \frac{s^2+2}{s^3-1} - \frac{s^2+s+1}{s^3-1} = \frac{Bs+C}{s^2+s+1}$$

$$= -\frac{s-1}{s^3-1} = \frac{-1}{s^2+s+1}$$

$\Rightarrow B = 0, \quad C = -1$

$$\frac{s^2+2}{s^3-1} = \frac{1}{s-1} - \frac{1}{s^2+s+1}$$

$$\Rightarrow y(t) = e^t - L^{-1}\left(\frac{1}{(s+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= e^{t} - \frac{e^{-t/2}}{\sqrt{3}/2} L^{-1}\left(\frac{1 \cdot \sqrt{3}/2}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right)$$

$$= e^{t} - \frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t$$

(10) $F'(s) = \dfrac{3s^2}{s^3+1} - \dfrac{1}{s+2} - \dfrac{2}{s} = G(s)$

$$\frac{3s^2}{s^3+1} = \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$\Rightarrow A = \left.\frac{3s^2}{s^2-s+1}\right|_{s=-1} = \frac{3}{3} = 1$$

$$\frac{Bs+C}{s^2-s+1} = \frac{3s^2}{s^3+1} - \frac{s^2-s+1}{s^3+1}$$

$$= \frac{2s^2+s-1}{s^3+1} = \frac{(2s-1)\cancel{(s+1)}}{\cancel{s^3+1}\ s^2-s+1}$$

$$= \frac{2s-1}{s^2-s+1} \qquad B=2, \ C=-1$$

$$\frac{3s^2}{s^3+1} = \frac{1}{s+1} + \frac{2s-1}{s^2-s+1}$$

$$= \frac{1}{s+1} + 2\frac{(s-1/2)}{(s-1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow L^{-1}\left(\frac{3s^2}{s^3+1}\right) = e^{-t} + 2e^{t/2}\cos\frac{\sqrt{3}}{2}t$$

$$\Rightarrow L^{-1}(F'(s)) = e^{-t} + 2e^{t/2}\cos\frac{\sqrt{3}}{2}t - e^{-2t} - 2 = g(t)$$

$$\lim_{t\to 0}\frac{g(t)}{t} = \frac{e^{-t} - e^{-2t} - 2 + 2e^{t/2}\cos\frac{\sqrt{3}}{2}t}{t}$$

exists by L'Hospital Rule.

Thus, $L^{-1}(F(s)) = \dfrac{e^{-t} - e^{-2t} - 2 + 2e^{t/2}\cos\frac{\sqrt{3}}{2}t}{t}$

(11) (a) $f(t) = 4e^{t}\left(u(t) - u(t-1)\right)$
$\qquad + e^{-t}u(t-1)$

(b) $$L(f(t)) = 4L(e^{t}u(t)) - 4L(e^{t}u(t-1)) + L(e^{-t}u(t-1))$$

$$= 4\frac{1}{s-1} - 4ee^{-s}\frac{1}{s-1} + e^{-1}L\left(e^{-(t-1)}u(t-1)\right)$$

$$= \frac{4}{s-1} - 4e\frac{e^{-s}}{s-1} + \frac{e^{-1}e^{-s}}{s+1}$$

(c) $$(s+1)^2 L(y) = \frac{4}{s-1} - \frac{4e^{-(s-1)}}{(s-1)} + \frac{e^{-(s+1)}}{s+1}$$

$$\frac{1}{(s-1)(s+1)^2} = \frac{1}{2}\frac{1}{s+1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] - \frac{1}{2(s+1)^2}$$

$$\Rightarrow \frac{4}{(s-1)(s+1)^2} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}$$

$$\frac{4e^{-(s-1)}}{(s-1)(s+1)^2} = \frac{e^{-(s-1)}}{s-1} - \frac{e^{-(s-1)}}{s+1} - \frac{2e^{-(s-1)}}{(s+1)^2}$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$- L^{-1}\left(\frac{e^{-(s-1)}}{s-1}\right) + L^{-1}\left(\frac{e^{-(s-1)}}{s+1}\right) + 2L^{-1}\left(\frac{e^{-(s-1)}}{(s+1)^2}\right)$$

$$+ L^{-1}\left(\frac{e^{-(s+1)}}{(s+1)^3}\right)$$

$$= e^{t} - e^{-t} - 2te^{-t} - e^{t}u(t-1)$$

$$+ e^{2}e^{-t}u(t-1) + 2e^{2}e^{-t}(t-1)u(t-1)$$

$$+ \frac{e^{-t}(t-1)^2 u(t-1)}{2}$$

$\boxed{\text{CODE - B}}$

(1) $f(x) = e^{-x}x^2 + e^{-x} - x e^{x}\sin x + x\cos x - 2\sin x + 3$

Annihilator of

$$e^{-x}x^2 \quad - \quad (D+1)^3$$

$$e^{-x} \quad - \quad (D+1)$$

$$x e^{x}\sin x \quad - \quad \left((D-1)^2+1\right)^2$$

$$x\cos x \quad - \quad (D^2+1)^2$$

$$\sin x \quad - \quad (D^2+1)$$

$$3 \quad - \quad D$$

Hence least order annihilator of $f(x)$

= lcm of each one

$$= D(D+1)^3\left((D-1)^2+1\right)^2(D^2+1)^2.$$

② $L = (D^2+9)^2(D-2)(D^2+2D+2)^2$

Basis for solution space of $Ly=0$ is

$\{\sin 3x,\ \cos 3x,\ x\sin 3x,\ x\cos 3x,\ e^{2x},\ e^{-x}\cos x,$
$e^{-x}\sin x,\ xe^{-x}\cos x,\ xe^{-x}\sin x\}.$

③ $y' = \dfrac{2x+y+1}{x+2y-1}, \quad y(2)=4$

Put $x = X+h,\ y = Y+k,$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{2X+Y}{X+2Y}$$

where $2h+k+1=0$
$h+2k-1=0$
$\Rightarrow -3k+3=0 \Rightarrow k=1$
$\Rightarrow h=-1.$

Put $\frac{Y}{X}=v$, we get

$$\frac{dY}{dX} = v + Xv' = \frac{2+v}{1+2v}$$

$$\Rightarrow Xv' = \frac{2+v}{1+2v} - v$$

$$= \frac{2-2v^2}{1+2v}$$

$$\Rightarrow \frac{1+2v}{2-2v^2}v' = \frac{1}{X}$$

$$\frac{1+2v}{2-2v^2} = \frac{1+2v}{2(1+v)(1-v)}$$

$$= -\frac{1}{4(1+v)} + \frac{3}{4(1-v)}$$

$$v'\left[\frac{3}{4(1-v)} - \frac{1}{4(1+v)}\right] = \frac{1}{X}$$

Integrate. wrt X.

$$\frac{3}{4}\frac{\ln(1-v)}{-1} - \frac{1}{4}\ln(1+v) = \ln X + \ln C$$

$$\ln\frac{1}{(1-v)^3(1+v)} = \ln CX^4$$

$$\Rightarrow \ln\frac{X^4}{(X-Y)^3(X+Y)} = \ln CX^4$$

$$\Rightarrow \cancel{X^4} = C\cancel{X^4}(X-Y)^3(X+Y)$$

$$\Rightarrow C(x+1-(y-1))^3(x+1+(y-1)) = 1$$

$$\Rightarrow \boxed{C(x-y+2)^3(x+y)=1}$$

$y(2)=5$

$C(2-5+2)^3(2+5)=1$

$\boxed{C=-1/7}$

(2) $L = (D^2+9)^2 (D-2) (D^2+2D+2)^2$

$= (D^2+9)^2 (D-2) ((D+1)^2+1)^2$

$\{ \sin 3x, \cos 3x, x\sin 3x, x\cos 3x, e^{2x}, e^{-x}\sin x, e^{-x}\cos x,$

$x e^{-x}\sin x, x e^{-x}\cos x \}$

(3) $y' = \dfrac{2x+y+1}{x+2y-1}$, $y(2)=5$.

$x = X+h$, $y = Y+k$,

$2h+k+1=0$, $h+2k-1=0$ $\Rightarrow$ $h=-1$, $k=1$

$$\frac{dY}{dX} = \frac{2+Y/X}{1+2Y/X}.$$

Put $Y = vX$,

$$v'X + v = \frac{2+v}{1+2v}$$

$$v'X = \frac{2-2v^2}{1+2v}$$

$$\left(\frac{1+2v}{2-2v^2}\right) v' = \frac{1}{X}$$

$$\left(\frac{3}{4(1-v)} - \frac{1}{4(1+v)}\right) v' = \frac{1}{X}$$

Integrate. $\dfrac{3}{4} \ln \dfrac{1}{1-v} + \dfrac{1}{4} \ln\left(\dfrac{1}{1+v}\right) = \ln X + \ln \tilde{C}$

$$\ln \frac{1}{(1-v)^3(1+v)} = \ln C X^4$$

$$C X^4 (1-v)^3 (1+v) = 1$$

$$C(X-Y)^3(X+Y) = 1$$

$x = X-1$

$y = Y+1$

$$C\left(x+1-(y-1)\right)^3 (x+1+y-1) = 1$$

$$\boxed{C(x-y+2)^3(x+y) = 1}$$

$y(2)=5$ $\Rightarrow$ $C(2-5+2)^3(2+5)=1$ $\Rightarrow$ $\boxed{C = -1/7}$

(4) $y' = \tan(x+y), \quad y(x_0) = y_0.$

<u>Statement</u>: $y' = f(x,y), \quad y(x_0) = y_0.$

If $R = (a,b) \times (c,d)$ is an open rectangle,

$f(x,y)$ is continuous on R and

$\dfrac{\partial f}{\partial y}$ is continuous on R

and $(x_0, y_0) \in R$.

Then $\exists$ an open interval I s.t.

$x_0 \in I \subset (a,b)$ such that

IVP $y' = f(x,y), \quad y(x_0) = y_0$

has a unique solution on I.

- Consider $y' = \tan(x+y), \quad y(x_0) = y_0$

If $x_0 + y_0 \neq n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}$,

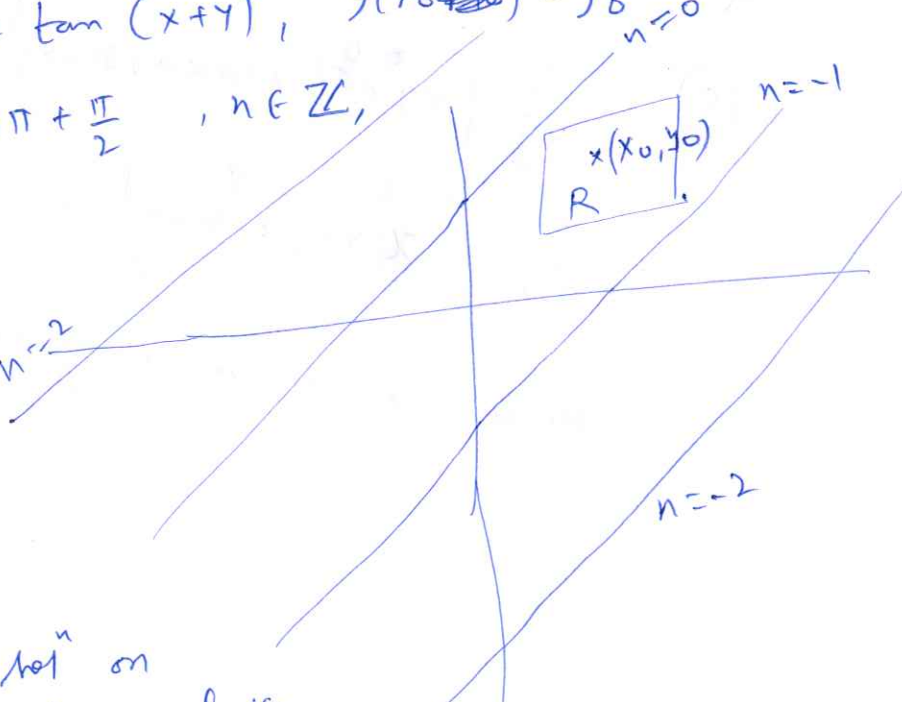
Then we can find

a rectangle

R st.

f and $\dfrac{\partial f}{\partial y}$ is

Continuous on R.



Hence $\exists$ a unique sol^n on

some open interval around x_0.

$\therefore \ (x_0, y_0) \in \mathbb{R}^2 - \bigcup_{n \in \mathbb{Z}} \left\{ (x,y) \,\middle|\, x+y = n\pi + \frac{\pi}{2} \right\}.$

(5) (a). $\underbrace{y(x\cos x + 2\sin x)}_{M} + \underbrace{x(y+1)\sin x}_{N} \cdot y' = 0$

$$\frac{\partial M}{\partial y} = x\cos x + 2\sin x, \qquad \frac{\partial N}{\partial x} = (y+1)(\sin x + x\cos x)$$

Hence ODE is not exact.

(b) $$M_y - N_x = (x\cos x + 2\sin x) - (y+1)(\sin x + x\cos x)$$
$$= \sin x - y(\sin x + x\cos x)$$

$$N\,p(x) - M\,q(y) = (x(y+1)\sin x)\left(\frac{1}{x}\right) - (yx\cos x + 2y\sin x)(1)$$

$$= y\sin x + \sin x - y(x\cos x + 2\sin x)$$
$$= \sin x - y(x\cos x + \sin x)$$

$$\therefore \mu(x,y) = e^{\int \frac{1}{x}dx} \cdot e^{\int 1\,dy}$$

$$\boxed{\mu = \boxed{x e^{y}}}$$

(c) $$\frac{\partial}{\partial y}\left(x y e^{y}(x\cos x + 2\sin x)\right) = x(x\cos x + 2\sin x)e^{y}(y+1)$$
$$\parallel$$
$$\frac{\partial}{\partial x}\left(x^2(y+1)e^{y}\sin x\right) = e^{y}(y+1)(2x\sin x + x^2\cos x)$$

Hence μ is an integrating factor.

(6) $e^x(x^4y^2+4x^3y^2+1)+(2x^4ye^x+2y)y'=0.$

Given that the ODE is exact.

Hence $\exists$ $\phi(x,y)$ st. $\phi(x,y)=c$ is an implicit solⁿ.

$$\frac{\partial\phi}{\partial x}=e^x(x^4y^2+4x^3y^2+1)$$

$$\frac{\partial\phi}{\partial y}=2x^4ye^x+2y$$

Integrating 2nd eqⁿ.

$$\phi(x,y)=x^4y^2e^x+y^2+h(x).$$

$$\frac{\partial\phi}{\partial x}=y^2e^x(x^4+4x^3)+h'(x)$$

$$=e^x(x^4y^2+4x^3y^2+1)$$

$$\Rightarrow h'(x)=e^x \quad \Rightarrow h(x)=e^x.$$

Hence the solⁿ is

$$\boxed{x^4y^2e^x+y^2+e^x=C}$$

(7) $x^2y'' - 3xy' + 3y = x^5, \quad y(1)=0, \quad y'(1)=0.$

homogeneous eqn:

$$x^2y'' - 3xy' + 3y = 0 \quad - \text{ Cauchy Euler}$$

$$\lambda^2 + (-3-1)\lambda + 3 = 0 \quad \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda = 1, 3.$$

Hence $y_1 = x$, $y_2 = x^3$ are solutions of homogeneous ODE.

To find a particular soln –

$\boxed{\text{method - 1}}$ observe cx^5 is form of a particular soln.

$$c\left(x^2(20 \cdot x^3) - 3x(5x^4) + 3 \cdot x^5\right) = x^5$$

$$\Rightarrow cx^5(20 - 15 + 3) = x^5$$

$$\Rightarrow c = 1/8$$

$$\Rightarrow y_p = \frac{1}{8}x^5.$$

$\boxed{\text{method. 2}}$ Apply variation of parameter method

For this, write ODE in std form:

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = x^3.$$

$$W(x, x^3) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3$$

$$y_p = y_2\int \frac{y_1 r}{W} - y_1 \int \frac{y_2 r}{W}$$

$$= x^3 \int \frac{x \cdot x^3}{2x^3} - x\int \frac{x^3 \cdot x^3}{2x^3} = x^3\int \frac{x}{2} - x\int \frac{x^3}{2} = \frac{x^5}{4} - \frac{x^5}{8} = \frac{1}{8}x^5.$$

$$\boxed{y = c_1 x + c_2 x^3 + \frac{1}{8}x^5}$$

put $y(1) = 0$, $y'(1) = 0$

$$\boxed{c_1 = \frac{1}{8}, \quad c_2 = -\frac{1}{4}}$$

(8) $$L\left(\int_0^t x^7 (t-x)^{14}\, dx\right)$$

$$= L\left(t^7 * t^{14}\right) = \frac{7!}{s^8} \cdot \frac{(14)!}{s^{15}}$$

$$= \frac{(7!)\,(14)!}{s^{23}}$$

$$\int_0^t x^7 (t-x)^{14}\, dx = L^{-1}\left(\frac{(7)!\,(14)!}{s^{23}}\right)$$

$$= \frac{(7)!\,(14)!}{(22)!}\, t^{22}.$$

(9) $y''' + y = 0$, $y(0)=1$, $y'(0)=0$, $y''(0)=2$

Apply Laplace transform.

$$s^3\mathcal{L}(y) - 2 - s\cdot 0 - s^2\cdot 1 + \mathcal{L}(y) = 0$$

$$\Rightarrow \mathcal{L}(y) = \frac{2+s^2}{s^3+1} = F(s)$$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$A = F(s)\cdot(s+1)\Big|_{s=-1} = \frac{2+1}{1+1+1} = \frac{3}{3} = 1$$

$$2+s^2 = (s^2-s+1) + (s+1)(Bs+C)$$

$\underline{s=0}$: $2 = 1 + C \Rightarrow C = 1$.

$\underline{s=1}$: $3 = 1 + 2(B+1)$

$\Rightarrow 1 = B+1 \Rightarrow B = 0$

$$\therefore \mathcal{L}(y) = \frac{1}{s+1} + \frac{1}{s^2-s+1}$$

check:

$$= \frac{s^2-s+1+s+1}{s^3+1} = \frac{s^2+2}{s^3+1}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1} + \frac{1}{(s-\frac{1}{2})^2+\frac{3}{4}}\right)$$

$$\boxed{y(t) = e^{-t} + e^{\frac{1}{2}t}\frac{2}{\sqrt{3}}\cdot\sin\frac{\sqrt{3}}{2}t}$$

(10) $F(s) = \ln\left(\frac{s^3-1}{s^3+2s^2}\right)$

$\lim_{s\to\infty} F(s) = 0.$

$F'(s) = \frac{3s^2}{s^3-1} - \frac{3s^2+4s}{s^3+2s^2}$

$\frac{3s^2}{s^3-1} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$

$A \doteq \frac{3}{1+1+1} = 1$

$3s^2 = (s^2+s+1) + (Bs+C)(s-1)$

$\underline{s=0}$: $0 = 1-C \Rightarrow C=1.$

$\underline{s=-1.}$ $3 = 1 + (-B+1)(-2)$

$-B+1 = -1 \Rightarrow B = 2.$

$$\boxed{\frac{3s^2}{s^3-1} = \frac{1}{s-1} + \frac{2s+1}{s^2+s+1}}$$

Check:

$s^2+s+1 + (2s+1)(s-1)$

$= 3s^2$

$\frac{3s^2+4s}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$

$C = \frac{3(4)-8}{4} = \frac{4}{4} = 1.$

$3s^2+4s = As(s+2) + B(s+2) + \cancel{C}s^2$

$\underline{s=0}$ $0 = 2B \Rightarrow B=0$

$\underline{s=1.}$ $3+4 = A\cdot 3 + 1 \Rightarrow 3A=6 \Rightarrow A=2.$

$$\boxed{\frac{3s^2+4s}{s^2(s+2)} = \frac{2}{s} + \frac{1}{s+2}}$$

$$G(s) = F'(s) = \left(\frac{1}{s-1} + \frac{2s+1}{s^2+s+1}\right) - \left(\frac{2}{s} + \frac{1}{s+2}\right)$$

$$g(t) = \mathcal{L}^{-1}(G(s)) = e^{t} + \mathcal{L}^{-1}\left(\frac{2s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right) - 2 - e^{-2t}$$

$$= e^{t} + e^{-\frac{1}{2}t}\,\mathcal{L}^{-1}\left(\frac{2(s-1/2)+1}{s^2+3/4}\right) - 2 - e^{-2t}$$

$$\boxed{g(t) = e^{t} + e^{-t/2}\left(2\cdot \cos\frac{\sqrt{3}}{2}t\right) - 2 - e^{-2t}}$$

<u>Note</u> - $\lim\limits_{t\to 0} \frac{g(t)}{t} = \frac{1+2-2-1}{0} = \frac{0}{0}$

$$= \lim_{t\to 0}\left(e^{t} + e^{-t/2}\,2\cos\frac{\sqrt{3}}{2}t - 2 - e^{-2t}\right)'$$

(L'Hôpital rule)

= exists.

Hence $\mathcal{L}^{-1}\left(\int_s^\infty F'(s)\,ds\right) = \frac{g(t)}{t}$

i.e. $\mathcal{L}^{-1}\left(-F(s)\right) = \frac{g(t)}{t}$ (Note $\lim\limits_{s\to\infty} F(s) = 0$)

$$\therefore \boxed{\mathcal{L}^{-1}\left(F(s)\right) = -\frac{g(t)}{t}}$$

(11) $f(t) = \begin{cases} 4e^{t}, & 0 \le t < 3 \\ e^{-t}, & 3 \le t \end{cases}$

(a) $f(t) = 4e^{t} + u(t-3)\left(e^{-t} - 4e^{t}\right)$

(b) $\mathcal{L}(f(t)) = \dfrac{4}{s-1} + e^{-3s}\mathcal{L}\left(e^{-(t+3)} - 4e^{(t+3)}\right)$

$$= \frac{4}{s-1} + \frac{e^{-3(s+1)}}{s+1} - \frac{4e^{-3(s-1)}}{s-1}$$

(c) $y'' + 2y' + y = f(t)$,

$y(0) = 0, \quad y'(0) = 0.$

Taking Laplace Transform,

$$(s^2+2s+1)\,\mathcal{L}(y) = \frac{4}{s-1} + \frac{e^{-3(s+1)}}{s+1} - \frac{4e^{-3(s-1)}}{s-1}$$

$$\boxed{\mathcal{L}(y) = \frac{4\left(1-e^{-3(s-1)}\right)}{(s-1)(s+1)^2} + \frac{e^{-3(s+1)}}{(s+1)^3}}$$

$$\frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$A = \frac{1}{4}, \quad C = -\frac{1}{2}$

$\frac{1}{4}(s+1)^2 + B(s+1)(s-1) + C(s-1) = 1$

$\underline{s=0.} \quad \frac{1}{4} - B + \frac{1}{2} = 1, \quad B = -1 + \frac{1}{4} + \frac{1}{2} = -\frac{1}{4}$

$$\frac{1}{(s-1)(s+1)^2} = \frac{1}{4(s-1)} - \frac{1}{4(s+1)} - \frac{1}{2(s+1)^2}$$

check -

$(s+1)^2 - (s^2-1) - 2(s-1)$

$= \cancel{s^2} + 1 + \cancel{2s} - \cancel{s^2} + 1 - \cancel{2s} + 2$

$= 4.$

$$\mathcal{L}(y) = \left(1 - e^{-3(s-1)}\right)\left(\frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}\right)$$

$$- \frac{e^{-3(s+1)}}{(s+1)^3}$$

$$y(t) = \left(e^{t} - e^{-t} - 2te^{-t}\right)$$

$$- e^{t} u(t-3) + e^{6} e^{-t} u(t-3)$$

$$+ 2e^{6} e^{-t} (t-3) u(t-3)$$

$$- \frac{e^{-t}}{2} (t-3)^2 u(t-3)$$