MA-106 Linear Algebra

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D1 - Lecture 12

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6	170070031	Jayesh Songara	
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8	17D070013	Paras Vijay Bodake	Absent
9	17D070026	Anubhav Agarwal	
10	17D070031	Bhavesh Garg	
11	17D070037	Shreyas Goenka	
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13	170050003	Makwana Jigar	
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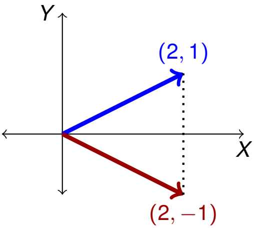
Matrices as Transformations: Examples

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. Let $\mathbf{x} = (2,1)^T$.

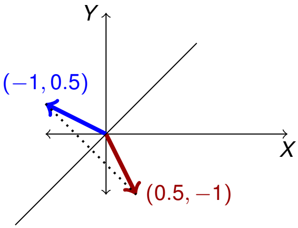
What is $\mathbf{A}x$? How does A transform x?

A reflects vectors across the X-axis.



Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$. If $\mathbf{x} = (-1, 0.5)^T$, then $\mathbf{B}x = (0.5, -1)^T$. How does B transform x?

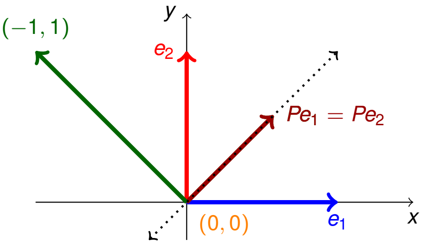
B reflects vectors across the line $x_1 = x_2$.



Q: Do reflections preserve scalar multiples? Sums of vectors?

Matrices as Transformations: Examples

- $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ transforms $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to $Px = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$.



$Pe_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = Pe_2$.

P transforms the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to the origin.

Q: Geometrically, how is P transforming the vectors?

A: Projects onto the line $x_1 = x_2$.

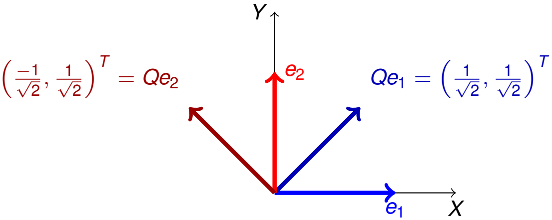
Q: What happens to sums of vectors when you project them? What about scalar multiples?

Exercise: Understand the effect of $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ on e_1 and e_2 and interpret what P represents geometrically.

Matrices as transformations: Examples

Let $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}$.

How does Q transform the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$?



Q: What does the transformation $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto Qx$ represent geometrically?

Again note that rotations map sum of vectors to sum of their images and scalar multiple of a vector to scalar multiple of its image.

Matrices as Transformations

• An $m \times n$ matrix A transforms a vector x in $\mathbb{R}^n$ into the vector Ax in $\mathbb{R}^m$. Thus $T(x) = Ax$ defines a function $T : \mathbb{R}^n \to \mathbb{R}^m$.

• The domain of T is ___. The codomain of T is ___.

• Let $b \in \mathbb{R}^m$. Then b is in $C(A) \Leftrightarrow Ax = b$ is consistent $\Leftrightarrow T(x) = b$, i.e., b is in the image (or range) of T.
Hence, the range of T is ___.

Example: Let $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$. Then $T(x) = Ax$ is a function with domain $\mathbb{R}^4$, codomain $\mathbb{R}^3$, and range equal to
$C(A) = \{(a, b, c)^T \mid 2a - b - c = 0\} \subseteq \mathbb{R}^3$.

Q: How does T transform sums and scalar multiples of vectors?
A. Nicely! For scalars a and b, and vectors x and y,
$T(ax + by) = A(ax + by) = aAx + bAy = aT(x) + bT(y)$. Thus

$$\boxed{T \text{ takes linear combinations to linear combinations.}}$$

Linear Transformations

Defn. Let V and W be vector spaces. A *linear transformation* from V to W is a function $T : V \to W$ which takes linear combinations of vectors in V to the linear combinations of their images, i.e., for $x, y \in V$, scalars a and b,

$$\boxed{T(ax + by) = aT(x) + bT(y)}$$

• The image (or range) of T is defined to be $C(T) = \{y \in W \mid T(x) = y \text{ for some } x \in V\}$.

• The kernel (or null space) of T is defined as $N(T) = \{x \in V \mid T(x) = 0\}$.

Main Example:

Let A be an $m \times n$ matrix. Define $T(x) = Ax$.

• This defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

• The image of T is the column space of A, i.e., $C(T) = C(A)$.

• The kernel of T is the null space of A, i.e., $N(T) = N(A)$.

Linear Transformations: Examples

Which of the following functions are linear transformations?

• $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $g(x_1, x_2, x_3)^T = (x_1, x_2, 0)^T$

$$ag(x) + bg(y) = ag\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + bg\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ 0 \end{pmatrix} + \begin{pmatrix} by_1 \\ by_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ 0 \end{pmatrix} = g(ax + by) \text{ is a linear transformation.}$$

Exercise: Find $N(g)$ and $C(g)$.

• $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $h(x_1, x_2, x_3)^T = (x_1, x_2, 5)^T$.

Note: $h(0 + 0) \neq h(0) + h(0)$.

Observe: A linear transformation must map $0 \in V$ to $0 \in W$.

Linear Transformations: Examples

• $f : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $f(x_1, x_2)^T = (x_1, 0, x_2, x_2^2)^T$.

Note: f transforms the Y-axis in $\mathbb{R}^2$ to $\{(0, 0, y, y^2)^T \mid y \in \mathbb{R}\}$.

Observe: A linear transformation must transform a subspace of V into a subspace of W.

• $S : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ defined by $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$is a linear transformation.

Observe: The function S is onto $\Rightarrow C(S) = \mathbb{R}^4$,
and $S(A) = S(B) \Rightarrow A = B$, i.e., the function S is one-one. In particular, $N(S) = \{0\}$.

Linear Transformations: Examples

Show that the following functions are linear transformations.

$T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $T(x_1, x_2, \ldots) = (x_1 + x_2, x_2 + x_3, \ldots,)$.
Exercise: What is $N(T)$?

$S : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.
Exercise: Find $C(S)$, and a basis of $N(S)$.

Let $T : \mathcal{P}_2 \to \mathcal{P}_1$ be $S(a_0 + a_1 x + a_2 x^2) = a_1 + 4a_2 x$.

Exercise: Show that dim $(N(T)) = 1$, and find $C(T)$.

Let $D : \mathcal{C}^\infty([0, 1]) \to \mathcal{C}^\infty([0, 1])$ defined as $Df = \frac{df}{dx}$.

Exercise: Is $D^2 = D \circ D$ linear? What about D^3?

Exercise: What is $N(D)$? $N(D^2)$? $N(D^k)$?

Question: Is integration linear?

Observe: Images and null spaces are subspaces!
Of which vector space?

Properties of Linear transformations

Let $\mathcal{B} = \{v_1, \ldots, v_n\} \subseteq V$, $T : V \to W$ be linear. Then:

• T takes linear combinations to linear combinations.
In particular, $T(0) = 0$.

• $N(T)$ is a subspace of V. Why? $C(T)$ is a subspace of W. Why?

• If $\text{Span}(\mathcal{B}) = V$, is $\text{Span}\{T(v_1), \ldots, T(v_n)\} = W$?
Observe: $\text{Span}\{T(v_1), \ldots, T(v_n)\} = C(T)$. Why?
Conclusion: (i) If $\dim(V) = n$, then $\dim(C(T)) \leq n$.
(ii) T is onto $\Leftrightarrow \text{Span}\{T(v_1), \ldots, T(v_n)\} = C(T) = W$.

• $T(u) = T(v) \Leftrightarrow u - v \in N(T)$.
Conclusion: T is one-one $\Leftrightarrow N(T) = 0$.

• If $\mathcal{B} \subseteq V$ is linearly independent, is $\{T(v_1), \ldots, T(v_n)\} \subseteq W$ linearly independent?
HINT: $a_1 T(v_1) + \cdots a_n T(v_n) = 0 \Rightarrow a_1 v_1 + \cdots a_n v_n \in N(T)$.

• If $S : U \to V$, $T : V \to W$ are linear, then the composition $T \circ S : U \to W$ is linear. **Exercise:** Show that $N(S) \subseteq N(T \circ S)$. How are $C(T \circ S)$ and $C(T)$ related?