MA-106 Linear Algebra

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D1 - Lecture 1

Some Class Policies

ATTENDANCE:

- Attendance in the first week of classes is mandatory.
- Random attendance bonus for everyone: 2 marks.
- Absence during each random in-class attendance: -1 mark.

EVALUATION: 100 marks are waiting to be earned:

In-Tutorial Quizzes (2 best out of 3)	2×15 marks
Final	70 marks
Random Attendance*	2 marks
Total	100 marks

*Will not be counted if total exceeds 100 marks.

ACADEMIC HONESTY: Be honest. Do not to violate the academic integrity of the Institute. Any form of academic dishonesty will invite severe penalties.

What is Linear Algebra?

Twitter version: It is the theory of solving simultaneous linear equations in a finite number of unknowns.

Example:

Q: Suppose you have 10, 3, 6 and 1 currency notes respectively of denominations Rs. 100, Rs. 200, Rs. 500 and Rs. 1000. You buy food worth Rs. 2600 at the Gulmohar cafeteria. How many notes of each denomination will you need to pay?

A: Give names to unknowns: Let t, u, v and w be number of notes of denominations Rs. 100, Rs. 200, Rs. 500 and Rs. 1000 respectively that you will pay.

Want to solve: $\boxed{100t + 200u + 500v + 1000w = 2600.}$

What is Linear Algebra?

Want to solve: $\boxed{100t + 200u + 500v + 1000w = 2600.}$

A possible solution: $(t, u, v, w) = (1, 0, 5, 0)$.
Is this unique? No. e.g., $(10, 3, 2, 0)$ or $(1, 0, 3, 1)$(?)

Are the following solutions permissible:
$(1, 5, 3, 0)$? No. We need $u \leq 3$.
or $(-1, 1, 5, 0)$? Only if the shopkeeper gives back change.
or $(0, 0, 5.2, 0)$? No. We need integer values.

Key note:
In general, we are looking for all possible solutions to the given system, i.e., <u>without any constraints</u>, unlike the introductory example.

An Example

Solve the system: (1) $x + 2y = 3$,   (2) $3x + y = 4$.

Elimination of variables:

Eliminate x by $(2) - 3 \times (1)$ to get $y = 1$.

Cramer's Rule (determinant): $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \dfrac{4-9}{1-6} = 1$

In either case, back substitution gives $x = 1$

We could also solve for x first and use back substitution for y.

Comparison: For a large system, say 100 equations in 100 variables, elimination method is preferred, since computing 101 determinants of size 100×100 is time-consuming.

Geometry of linear equations

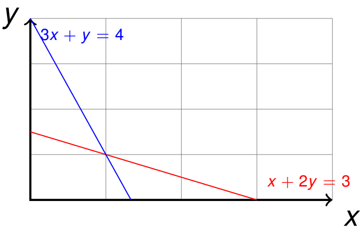
Row method:

$$\boxed{x+2y=3}$$

and

$$\boxed{3x+y=4}$$

represent lines in $\mathbb{R}^2$ passing through $\boxed{(0,3/2) \quad \text{and} \quad (3,0)}$ and through $\boxed{(0,4) \quad \text{and} \quad (4/3,0)}$ respectively.



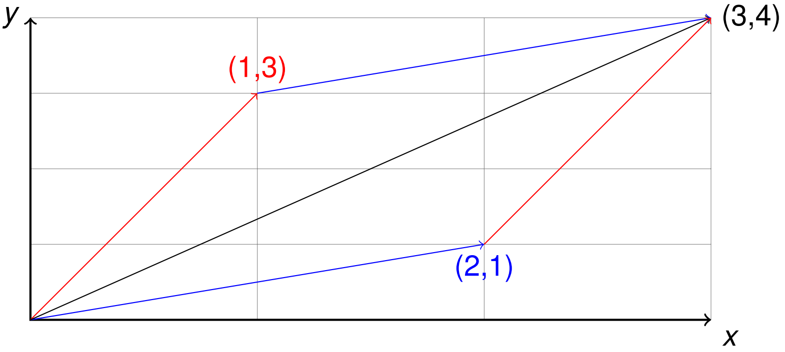
The intersection of the two lines is the unique point $(1,1)$. Hence $(x,y) = (1,1)$ is the (only) solution of above system of linear equations.

Geometry of linear equations

Column method: The system is $x\begin{pmatrix}1\\3\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$.

We need to find a *linear combination* of the column vectors on LHS to produce the column vector on RHS.

Geometrically this is same as completing the parallelogram with given directions and diagonal.



Here $x = 1$, $y = 1$ will work.

Equations in 3 variables

Row method: A linear equation in 3 variables represents a plane in a 3 dimensional space $\mathbb{R}^3$.

Example: (1)

$$\boxed{\text{x+2y+3z=6}}$$

represents a plane passing through: $(0,0,2)$, $(0,3,0)$, $(6,0,0)$.

Example: (2)

$$\boxed{\text{x+2y+3z=0}}$$

represents a plane passing through: $(0,0,0)$, $(-3,0,1)$, $(-2,1,0)$.

In Example (2) we are looking for (x,y,z) such that $(x,y,z)\cdot(1,2,3)=0$, i.e., plane (2) is the set of all vectors perpendicular to the vector (1,2,3).

Equations in 3 variables: Examples

Example 1: (1) $x + 2y + 3z = 6$ (2) $x + 2y + 3z = 0$.

The two equations represent planes with normal vector (1,2,3) and are parallel to each other.

How many solutions can we find? There are *no solutions*.

Example 2: (1) $x + 2y + 3z = 0$ (2) $-x + 2y + z = 0$

The two equations represent planes passing through (0,0,0).

The intersection is non-empty, i.e., the system has <u>at least</u> one solution.
In fact, the *solution set* is a line passing through the origin.

Exercise: Find all the solutions in the second example.

3 equations in 3 variables

- Solving 3 by 3 system by the **row method** means finding an intersection of three planes, say P_1, P_2, P_3.

  This is same as the intersection of a line L (intersection of P_1 and P_2) with the plane P_3.

- If the line L does not intersect the plane P_3, then the linear system has no solution, i.e., the system is *inconsistent*.

- If the line L is contained in the plane P_3, then the system has infinitely many solutions.

  In this case, every point of L is a solution.

- Workout some examples.

Linear Combinations (or Weighted Sum)

Column method: Consider the 3×3 system:
x+2y+3z = 0, -2x+3y = 1, -x+5y+2z = 2. Equivalently,

$$x \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

We want a *linear combination* (or *weighted sum*) of the column vectors on LHS which is equal to RHS.
Observe: 1. $x = 1, y = 1, z = -1$ is a solution. **Q:** Is it unique?
2. Since each column represents a vector in $\mathbb{R}^3$ from origin, we can find the solution geometrically, as in the 2×2 case.
Q: Can we do the same when number of variables is > 3?

Solve the system by other techniques to answer such questions.

Gaussian Elimination

Example: $2x + y + z = 5, \;\; 4x - 6y = -2, \;\; -2x + 7y + 2z = 9.$

Algorithm: Eliminate x from last 2 equations by
$(2) - 2 \times (1)$, and $(3) + (1)$ to get the *equivalent system:*

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \;\; 8y + 3z = 14$$

The coefficient used for eliminating a variable is called a *pivot*.

The first pivot is 2, second pivot is -8. Eliminate y from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \;\; z = 2$$

Solve this triangular system by *back substitution*, to get the *unique solution*

$$z = 2, \;\; y = 1, \;\; x = 1.$$