MA-106 Linear Algebra

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D1 - Lecture 9

Random Attendance

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Linear Span: Definition

Given a collection $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V, the *linear span* of S, denoted Span(S) or Span$\{v_1, \ldots, v_n\}$,
is the set of all linear combinations of v_1, $v_2, \ldots, v_n$, i.e.,

$$\boxed{\text{Span}(S) = \{v = a_1 v_1 + \cdots + a_n v_n, \text{ for scalars } a_1, \ldots, a_n\}.}$$

Note:

1. If $v_1, \ldots, v_n$ are in $\mathbb{R}^m$, Span$\{v_1, \ldots, v_n\} = C(A)$ for $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$, an $m \times n$ matrix. Thus
v is in Span$\{v_1, \ldots, v_n\} \Leftrightarrow Ax = v$ is consistent.

2. Let $\{v_1, \ldots, v_n\}$ be n vectors in $\mathbb{R}^n$, $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.
Then A is invertible $\Leftrightarrow$ A has n pivots $\Leftrightarrow$ $Ax = v$ is consistent for every v in $\mathbb{R}^n$ $\Leftrightarrow$ Span$\{v_1, \ldots, v_n\} = \mathbb{R}^n$.

Example: Span$\{e_1, \ldots, e_n\} = \mathbb{R}^n$.

Linear Span: Examples

Examples:

1. $\text{Span}\{0\} = \{0\}$.
2. If $v \neq 0$ is a vector, $\text{Span}\{v\} = \{av, \text{ for scalars } a\}$.
   Geometrically (in $\mathbb{R}^m$): $\text{Span}\{v\}$ = the line in the direction of v passing through the origin.
3. $\text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2$.
4. If A is $m \times n$, then $\text{Span}\{A_1, \ldots, A_n\} = C(A)$.
5. If $v_1, \ldots, v_k$ are the special solutions of A,
   then $\text{Span}\ \{v_1, \ldots, v_k\}$ = $N(A)$.

Remark: All of the above are subspaces.
Exercise: $\text{Span}(S)$ is a subspace of V.

Linear Span: Examples

Example 1: Is v in Span$\{v_1, v_2, v_3, v_4\}$, where: $v = \begin{pmatrix}1 & 0 & 4\end{pmatrix}^T$, $v_1 = \begin{pmatrix}1 & 2 & 3\end{pmatrix}^T$, $v_2 = \begin{pmatrix}2 & 4 & 6\end{pmatrix}^T$, $v_3 = \begin{pmatrix}3 & 8 & 7\end{pmatrix}^T$ and $v_4 = \begin{pmatrix}5 & 12 & 13\end{pmatrix}^T$?

Let $A = \begin{pmatrix}v_1 & \cdots & v_4\end{pmatrix}$. Recall $Ax = \begin{pmatrix}1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13\end{pmatrix} x = \begin{pmatrix}b_1 \\ b_2 \\ b_3\end{pmatrix}$ is solvable $\Leftrightarrow 5b_1 - b_2 - b_3 = 0$.

$\Rightarrow$ v is not in Span$\{v_1, v_2, v_3, v_4\}$, and $w = \begin{pmatrix}1 & 0 & 5\end{pmatrix}^T = 4v_1 + (-1)v_3$ is in it.

Observe: $v_2 = 2v_1$ and $v_4 = 2v_1 + v_3$. Hence v_2, v_4 are in Span$\{v_1, v_3\}$. Therefore, Span$\{v_1, v_3\}$ = Span$\{v_1, v_2, v_3, v_4\} = C(A) =$ the plane $P : (5x - y - z = 0)$.

Q: Is the span of two vectors in $\mathbb{R}^3$ always a plane?

Linear Span: Examples

Example 2: Is $v = \begin{pmatrix} 4 & 3 & 5 \end{pmatrix}^T$ in Span$\{v_1, v_2, v_3, v_4\}$, where:

$$v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 6 \\ 7 \\ 5 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}?$$

If yes, write v as a linear combination of $\{v_1, v_2, v_3, v_4\}$.

Let $A = \begin{pmatrix} v_1 & \cdots & v_4 \end{pmatrix}$. The question can be rephrased as:

Q: Is v in $C(A)$, i.e., is $Ax = v$ solvable? If yes, find a solution.

$R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - R_1$, followed by $R_3 \mapsto R_3 + R_2$, gives

$$[A|v] = \left(\begin{array}{cccc|c} 2 & 4 & 6 & 4 & a \\ 2 & 5 & 7 & 6 & b \\ 2 & 3 & 5 & 2 & c \end{array}\right) \longrightarrow [U|w] = \left(\begin{array}{cccc|c} 2 & 4 & 6 & 4 & a \\ 0 & 1 & 1 & 2 & b-a \\ 0 & 0 & 0 & 0 & c+b-2a \end{array}\right)$$

Linear Span: Examples

$$Ax = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix} x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is solvable} \Leftrightarrow 2a - b - c = 0$$

$\Rightarrow v = \begin{pmatrix} 4 & 3 & 5 \end{pmatrix}^T$ is in Span$\{v_1, \ldots, v_4\}$,

(and, for example, $\begin{pmatrix} 4 & 3 & 4 \end{pmatrix}^T$ is not in it).

Observe: $C(A)$ is a plane!

Solve $Ax = v$: Convert U to the row reduced form R:

$$[U|w] = \begin{pmatrix} 2 & 4 & 6 & 4 & | & a \\ 0 & 1 & 1 & 2 & | & b - a \\ 0 & 0 & 0 & 0 & | & c + b - 2a \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 & 4 & | & 4 \\ 0 & 1 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & -2 & | & 4 \\ 0 & 1 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ by } R_1 \mapsto R_1 - 4R_2,\ R_1 \mapsto R_1/2.$$

Particular solution: $\begin{pmatrix} 4 & -1 & 0 & 0 \end{pmatrix}^T$ and $v = 4v_1 + (-1)v_2$.

Linear Independence: Example

With $v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 6 \\ 7 \\ 5 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$

Observe: $v_3 = v_1 + v_2$ and $v_4 = -2v_1 + 2v_2$. Hence v_3 and v_4 are in $\text{Span}\{v_1, v_2\}$. Therefore, $\text{Span}\{v_1, v_2\}$ = $\text{Span}\{v_1, v_2, v_3, v_4\} = C(A) =$ the plane $P : (2x - y - z = 0)$.

Q: Is the span of two vectors in $\mathbb{R}^3$ always a plane?
A: Not always. If v is a multiple of w, then $\text{Span}\{v, w\}$ = $\text{Span}\{w\}$, which is a line through the origin or zero.

Q: If v and w are not on the same line through the origin?
A: Yes. v, w are examples of *linearly independent vectors*.

Linear Independence: Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V, are *linearly independent* if $\boxed{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \Rightarrow a_1 = 0,\, a_2 = 0,\, \ldots,\, a_n = 0}$.

Observe: When $V = \mathbb{R}^m$, if $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$, then $v_1, v_2, \ldots, v_n$ are linearly independent $\Leftrightarrow$ $Ax = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$ has only the trivial solution $\Leftrightarrow N(A) = 0$.

The vectors $v_1, \ldots, v_n$ are *linearly dependent* if they are not linearly independent. If $V = \mathbb{R}^m$, this happens $\Leftrightarrow$

$$\boxed{Ax = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} x = 0 \text{ has non-trivial solutions.}}$$

Linear Independence: Remarks

Remarks/Examples:

1. The zero vector 0 is not linearly independent.
2. If $v \neq 0$, then it is linearly independent.
3. v, w are not linearly independent $\Leftrightarrow$ one is a multiple of the other $\Leftrightarrow$ (for $V = \mathbb{R}^m$) they lie on the same line through the origin.
4. More generally, $v_1, \ldots, v_n$ are not linearly independent $\Leftrightarrow$ one of the v_i's can be written as a linear combination of the others, i.e., v_i is in Span$\{v_j : j = 1, \ldots n, j \neq i\}$.
5. Let A be $m \times n$. Then rank$(A) = n \Leftrightarrow N(A) = 0 \Leftrightarrow A_{*1}, \cdots, A_{*n}$ are linearly independent.
   In particular, if A is $n \times n$, A is invertible $\Leftrightarrow A_1, \cdots, A_n$ are linearly independent.

Example: $e_1, \ldots, e_n$ are linearly independent vectors in $\mathbb{R}^n$.

Linear Independence: Example

Example: Are the vectors v_1, v_2, v_3, v_4 linearly independent, where
$v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 6 \\ 7 \\ 5 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$?

For $A = \begin{pmatrix} v_1 & \cdots & v_4 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

A has only 2 pivots $\Rightarrow N(A) \neq 0$, so v_1, v_2, v_3, v_4 are not independent.
A non-trivial linear combination which is zero is
$(1)v_1 + (1)v_2 + (-1)v_3 + (0)v_4$, or $(2)v_1 + (-2)v_2 + (0)v_3 + (1)v_4$.
More generally, if $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^m$, then $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is $m \times n$.
If $m < n$, then $\text{rank}(A) < n \Rightarrow N(A) \neq 0$. Thus

$\boxed{\text{In } \mathbb{R}^m\text{, any set with more than } m \text{ vectors is linearly dependent.}}$

Summary: Vector Spaces, Span and Independence

• Vector space: A triple $(V, +, *)$ which is closed under $+$ and $*$

• Subspace: A non-empty subset W of V closed under linear combinations.

• $\text{Span}\{v_1, \ldots, v_n\}$
= $\{v = a_1 v_1 + \cdots + a_n v_n$, for scalars $a_1, \ldots, a_n\}$.

Let $V = \mathbb{R}^m$, $v_1, \ldots, v_n$ be in V, and $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.

• For v in V, v is in $\text{Span}\{v_1, \ldots, v_n\} \Leftrightarrow Ax = v$ is consistent

• $v_1, \ldots, v_n$ are linearly independent
$\Leftrightarrow N(A) = 0 \Leftrightarrow \text{rank}(A) = n$.

• In particular, with $n = m$, A is invertible
$\Leftrightarrow \text{Span}\{v_1, \ldots, v_n\} = \mathbb{R}^n \Leftrightarrow v_1, \ldots, v_n$ are linearly independent
$\Leftrightarrow N(A) = 0 \Leftrightarrow \text{rank}(A) = n$.

• Any subset of $\mathbb{R}^m$ with more than m vectors is dependent.