MA-106 Linear Algebra

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D1 - Lecture 11

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Recall: Basis and Dimension

• A basis of a vector space V is a linearly independent subset $\mathcal{B}$ which spans V.

• A basis is a maximal linearly independent subset of V
$\Rightarrow$ any linearly independent subset in V can be extended to a basis of V.

• A basis is a minimal spanning set of V
$\Rightarrow$ every spanning set of V contains a basis.

• The number of elements in each basis is the same,
and the dimension of V,
dim(V) = number of elements in a basis of V.

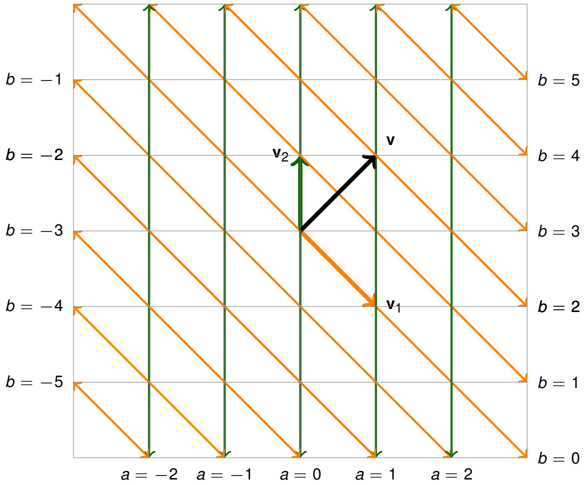
• $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for $V \Leftrightarrow$ every $v \in V$ can be uniquely written as a linear combination of $\{v_1, \ldots, v_n\}$.

• dim $(\mathbb{R}^n) = n$, and the set $\mathcal{B} = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n \Leftrightarrow$ $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is invertible.

Example: A basis for $\mathbb{R}^2$

Pick $\mathbf{v}_1 \neq 0$. Choose $\mathbf{v}_2$, not a multiple of $\mathbf{v}_1$. For any $\mathbf{v}$ in $\mathbb{R}^2$, there are unique scalars a and b such that $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$.

e.g., pick $\mathbf{v}_1 = (1, -1)^T$, $\mathbf{v}_2 = (0, 1)^T$, and let $\mathbf{v} = (1, 1)^T$.



Thus the lines $a = 0$ and $b = 0$ give a set of axes for $\mathbb{R}^2$, and $\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2$.

With this basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, the coordinates of $\mathbf{v}$ will be $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Basis and Coordinates

A basis for $\mathcal{M}_{2\times 2}$, the vector space of 2×2 matrices is $\mathcal{B} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$, where

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Check this!) Hence $\dim(\mathcal{M}_{2\times 2}) = 4$.

Every 2×2 matrix $A = (a_{ij})$ can be written as

$$A = a_{11}e_{11} + a_{12}e_{12} + a_{21}e_{21} + a_{22}e_{22}.$$

For this fixed basis $\mathcal{B}$, the *coordinate vector* of A with respect to $\mathcal{B}$, denoted

$$[A]_{\mathcal{B}} = (a_{11}, a_{12}, a_{21}, a_{22})^T$$

completely determines the matrix A.

Since $\dim(\mathcal{M}_{2\times 2}) = 4$, once we fix a basis, we will need 4 coordinates to describe each matrix.

Exercise: Find two bases and the dimension of $\mathcal{M}_{m\times n}$, the vector space of $m \times n$ matrices.

Coordinate Vectors

1. Consider the basis $\mathcal{B} = \{v_1 = (1, -1)^T, v_2 = (0, 1)^T\}$ of $\mathbb{R}^2$, and $v = (1, 1)^T$. Note that $v = 1v_1 + 2v_2$. Hence, the coordinate vector of v w.r.t. $\mathcal{B}$ is $[v]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
2. **Exercise:** Show that $\mathcal{B}_1 = \{1, x, x^2\}$ is a basis of $\mathcal{P}_2$.
   The coordinate vector of $v = 2x^2 - 3x + 1$ w.r.t. $\mathcal{B}$ is
   $[v]_{\mathcal{B}} = (1, -3, 2)^T$.
3. **Exercise:** Show that $\mathcal{B}' = \{1, (x-1), (x-1)^2, (x-1)^3\}$ is a basis of $\mathcal{P}_3$. HINT: Taylor expansion.
   Then $[x^3]_{\mathcal{B}'} = (\_\_, \_\_, \_\_, \_\_)^T$.

Observe: To write the coordinates, we have a to fix a basis $\mathcal{B}$, with a fixed *order* of elements in it!

The Four Fundamental Subspaces

Let A be an $m \times n$ matrix. Associated to A, we have four fundamental subspaces:

- The column space of A: $C(A) = \{v : Ax = v \text{ is consistent}\} \subseteq \mathbb{R}^m$.
- The null space of A: $N(A) = \{x : Ax = 0\} \subseteq \mathbb{R}^n$.
- The row space of A = $\text{Span}\{A_{1*}, \ldots, A_{m*}\} = C(A^T) \subseteq \mathbb{R}^n$.
- The left null space of A = $\{x : x^T A = 0\} = N(A^T) \subseteq \mathbb{R}^m$.

Q: Why are the row space and the left null space subspaces?

Let U be the echelon form of A, and R its reduced form.

- Recall, $N(A) = N(U) = N(R)$.

Observe: The rows of U(and R) are linear combinations of the rows of A, and vice versa $\Rightarrow$ their row spaces are same, i.e.,

- $C(A^T) = C(U^T) = C(R^T)$.

We now compute bases and dimensions for these special subspaces.

The Big Four: An Example

Let $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$. Find the four fundamental subspaces of A, their bases and dimensions.

Recall:

The reduced form of A is $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

• The 1st and 2nd are pivot columns $\Rightarrow$ rank(A) = 2.

• $v = \begin{pmatrix} a & b & c \end{pmatrix}^T$ is in $C(A) \Leftrightarrow Ax = v$ is solvable $\Leftrightarrow 2a - b - c = 0$.

• We can compute special solutions to $Ax = 0$. The number of special solutions to $Ax = 0$ is the number of free variables.

The Big Four: $N(A)$

For $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$$N(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -c + 2d \\ -c - 2d \\ c \\ d \end{pmatrix} = c \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

= Span$\left\{ w_1 = \begin{pmatrix} -1 & -1 & 1 & 0 \end{pmatrix}^T, w_2 = \begin{pmatrix} 2 & -2 & 0 & 1 \end{pmatrix}^T \right\}$.

w_1, w_2 are linearly independent (Why?)

$\Rightarrow \mathcal{B} = \{w_1, w_2\}$ forms a basis for $N(A) \Rightarrow \dim(N(A)) = 2$.

A basis for $N(A)$ is the set of special solutions.

$\dim(N(A)) =$ no. of free variables = no. of variables - rank(A)

Exercise: Show that $w = (-3, -7, 5, 1)^T$ in $N(A)$. What is $[w]_{\mathcal{B}}$?

The Big Four: $C(A)$

For $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Write $A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$ and $R = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}$.

Recall: Relations between the column vectors of A are the same as the relations between column vectors of R.

$\Rightarrow Ax = v_3$ has a solution has the same solution as $Rx = w_3$, and $Ax = v_4$ has a same solution as $Rx = w_4$.

Particular solutions are $(1, 1, 0, 0)^T$ and $(-2, 2, 0, 0)^T$ respectively $\Rightarrow$ $v_3 = v_1 + v_2$, $v_4 = -2v_1 + 2v_2$.

Observe:

- v_1 and v_2 correspond to the pivot columns of A.
- $\{v_1, v_2\}$ are linearly independent. Why?
- $C(A) = \text{Span}\{v_1, \ldots, v_4\} = \text{Span}\{v_1, v_2\}$.

Thus $\mathcal{B} = \{v_1, v_2\}$ is a basis of $C(A)$. **Q:** What is $[v_i]_{\mathcal{B}}$?

The Big Four: Rank-Nullity Theorem

More generally, for an $m \times n$ matrix A,

• Let $\text{rank}(A) = r$. The r pivot columns are linearly independent since their reduced form contains an $r \times r$ identity matrix.

• For each non-pivot column A_{*j} of A, find particular solution of $Ax = A_{*j}$. Use this to write A_{*j} as a linear combination of the pivots columns. Thus

$$\boxed{\text{A basis for } C(A) \text{ is given by the pivot columns of } A.}$$

$$\boxed{\dim(C(A)) = \text{no. of pivot variables} = \text{rank}(A).}$$

Rank-Nullity Theorem: Let A be an $m \times n$ matrix. Then

$$\dim(C(A)) + \dim(N(A)) = \text{no. of variables} = n$$

The Big Four: $C(A^T)$

Recall: If $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, then $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Recall: R is obtained from A by taking non-zero scalar multiples of rows and their sums $\Rightarrow C(R^T) = C(A^T)$.

Observe: The non-zero rows of R will span $C(A^T)$, and they contain an identity submatrix $\Rightarrow$ they are linearly independent.

Thus, the non-zero rows of R form a basis for $C(R^T) = C(A^T)$.

Exercise: Give two different basis for $C(A^T)$.

Since the number of non-zero rows of R = number of pivots of A, we have:

$$\boxed{\dim C(A^T) = \text{no. of pivots of } A = \text{rank}(A).}$$

- Recall that $\dim C(A^T) = \text{rank}(A^T)$. Thus,

$$\boxed{\text{rank}(A^T) = \dim\,(C(A^T)) = \text{rank}(A)}$$

The Big Four: $N(A^T)$

The no. of columns of A^T is m.
By Rank-Nullity Theorem, $\text{rank}(A^T) + \dim(N(A^T)) = m$.
Hence:

$$\boxed{\dim(N(A^T)) = m - \text{rank}(A).}$$

Exercise: Complete the example by finding a basis for $N(A^T)$.

$A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Q. Can you use R to compute the basis for $N(A^T)$? Why not?

A. Need the reduced form of A^T which is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.