MA-106 Linear Algebra

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Properties of Linear transformations

Let $\mathcal{B} = \{v_1, \dots, v_n\} \subseteq V$, $T : V \to W$ be linear. Then:

• T takes linear combinations to linear combinations.
In particular, $T(0) = 0$.

• $N(T)$ is a subspace of V. Why? $C(T)$ is a subspace of W. Why?

• If $\mathrm{Span}(\mathcal{B}) = V$, is $\mathrm{Span}\{T(v_1), \dots, T(v_n)\} = W$?
Observe: $\mathrm{Span}\{T(v_1), \dots, T(v_n)\} = C(T)$. Why?
Conclusion: (i) If $\dim(V) = n$, then $\dim(C(T)) \leq n$.
(ii) T is onto $\Leftrightarrow \mathrm{Span}\{T(v_1), \dots, T(v_n)\} = C(T) = W$.

• $T(u) = T(v) \Leftrightarrow u - v \in N(T)$.
Conclusion: T is one-one $\Leftrightarrow N(T) = 0$.

• If $\mathcal{B} \subseteq V$ is linearly independent, is $\{T(v_1), \dots, T(v_n)\} \subseteq W$ linearly independent?
HINT: $a_1 T(v_1) + \cdots a_n T(v_n) = 0 \Rightarrow a_1 v_1 + \cdots a_n v_n \in N(T)$.

• If $S : U \to V$, $T : V \to W$ are linear, then the composition $T \circ S : U \to W$ is linear. **Exercise:** Show that $N(S) \subseteq N(T \circ S)$. How are $C(T \circ S)$ and $C(T)$ related?

Isomorphism of vector spaces

A linear map $T : V \to W$ is an *isomorphism* if T is one-one and onto, i.e., T is a linear bijection. **Notation:** $V \simeq W$.

Q: If $T : V \to W$ is an isomorphism, is $T^{-1} : W \to V$ linear?

Recall: T is one-one $\Leftrightarrow N(T) = 0$ and T is onto $\Leftrightarrow C(T) = W$.

Thus $\boxed{T \text{ is an isomorphism} \Leftrightarrow N(T) = 0 \text{ and } C(T) = W.}$

Example: If V is the subspace of convergent sequences in $\mathbb{R}^\infty$, then $L : V \to \mathbb{R}$ given by $L(x_1, x_2, \ldots) = \lim\limits_{n\to\infty}(x_n)$ is linear.

What is $N(L)$? $C(L)$? Is L one-one or onto?

Exercise: Given $A \in \mathcal{M}_{m\times n}$, let $T(x) = Ax$ for $x \in \mathbb{R}^n$.
Then T is an isomorphism $\Leftrightarrow m = n$ and A is invertible.

Exercise: In the previous examples, identify linear maps which are one-one, and those which are onto.

Example: $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$ is an isomorphism since $N(S) = 0$ and $C(S) = \mathbb{R}^4$. Thus $\mathcal{M}_{2\times 2} \simeq \mathbb{R}^4$. What is S^{-1}?

Linear Maps and Basis

$\bullet$ Consider $S : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ given by $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$.
Recall that $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ is a basis of $\mathcal{M}_{2\times 2}$ such that
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}$.
Observe that $S(e_{11}) = e_1, S(e_{12}) = e_2, S(e_{21}) = e_3$, $S(e_{22}) = e_4$.
Thus, $S(A) = aS(e_{11}) + bS(e_{12}) + cS(e_{21}) + dS(e_{22})$
$= ae_1 + be_2 + ce_3 + de_4 = (a, b, c, d)^T$.

General case:

If $\{v_1, \ldots, v_n\}$ is a basis of V, $T : V \to W$ is linear, $v \in V$, then
$v = a_1 v_1 + \cdots a_n v_n \Rightarrow T(v) = a_1 T(v_1) + \cdots a_n T(v_n)$. Why? Thus,

$\boxed{T \text{ is determined by its action on a basis.}}$

Finite-dimensional Vector Spaces

Important Observation:

Let dim $(V) = n$, and $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V.
Define $T : V \to \mathbb{R}^n$ by $T(v_i) = e_i$.

e.g., If $v = v_1 + v_n$, then $T(v) = ?$
If $v = 3v_2 - 5v_3$, then $T(v) = ?$

If $v = a_1 v_1 + \cdots + a_n v_n$, then $T(v) = ?$
Thus $T(v) = [v]_{\mathcal{B}}$.

What is $N(T)$? What is $C(T)$?

Conclusion: $\boxed{\text{If dim (V) = n, then } V \simeq \mathbb{R}^n.}$

What is the isomorphism?
How many such isomorphisms can you construct?

Exercise: Find 3 isomorphisms each from $\mathcal{P}_3$ and $\mathcal{M}_{2\times 2}$ to $\mathbb{R}^4$.

Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Example: $T(e_1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ T(e_2) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},\ T(e_3) = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

defines a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$.

If $x = (x_1, x_2, x_3)^T$, then $T(x) = T(x_1 e_1 + x_2 e_2 + x_3 e_3) =$

$x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \end{pmatrix}$, i.e.,

$T(x) = Ax$, where $A = \begin{pmatrix} 3 & 2 & -5 \\ 1 & -1 & 0 \end{pmatrix}$. **Question:** $A_{*j} = T(e_j)$.

General case: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

for $x = (x_1, \ldots, x_n)^T$ in $\mathbb{R}^n$,

$$T(x) = x_1 T(e_1) + \cdots x_n T(e_n) = Ax,$$

where $A = \begin{pmatrix} T(e_1) & \cdots & T(e_n) \end{pmatrix} \in \mathcal{M}_{m \times n}$, i.e., $A_{*j} = T(e_j)$.

Defn. A is called the *standard matrix* of T. Thus

$$\boxed{\text{Linear transformations from } \mathbb{R}^n \text{ to } \mathbb{R}^m}$$

$$\boxed{\text{are in one-one correspondence with } m \times n \text{ matrices.}}$$

Q: Can you imitate this if V and W are not $\mathbb{R}^n$ and $\mathbb{R}^m$? THINK!

Matrix Associated to a Linear Map: Example

$S : \mathcal{P}_2 \to \mathcal{P}_1$ given by $S(a_0 + a_1x + a_2x^2) = a_1 + 4a_2x$ is linear.

Question: Is there a matrix associated to S?
Expected size: 2×3. Why?

IDEA: Construct an associated linear map $\mathbb{R}^3 \to \mathbb{R}^2$.
Use coordinate vectors! Fix bases $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$, and $\mathcal{C} = \{1, x\}$ of $\mathcal{P}_1$ to do this.

Identify $f = a_0 + a_1x + a_2x^2 \in \mathcal{P}_2$ with $[f]_\mathcal{B} = (a_0, a_1, a_2)^T \in \mathbb{R}^3$,

and $S(f) \in \mathcal{P}_1$ with $[S(f)]_\mathcal{C} = (a_1, 4a_2)^T \in \mathbb{R}^2$.

The associated linear map $S' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by
$S'(a_0, a_1, a_2)^T = (a_1, 4a_2)^T$, i.e., $S'([f]_\mathcal{B}) = [S(f)]_\mathcal{C}$, i.e.,
S' is defined by $S'(e_1) = (0, 0)^T$, $S'(e_2) = (1, 0)^T$, $S'(e_3) = (0, 4)^T \Rightarrow$
the standard matrix of S' is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Q: How is A related to S?

Observe: $A_{*1} = [S(1)]_\mathcal{C}$, $A_{*2} = [S(x)]_\mathcal{C}$, $A_{*3} = [S(x^2)]_\mathcal{C}$.

Matrix Associated to a Linear Map

Example: The matrix of $S(a_0 + a_1 x + a_2 x^2) = a_1 + 4a_2 x$, w.r.t. the bases $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$ and $\mathcal{C} = \{1, x\}$ of $\mathcal{P}_1$ is $A =$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $\boxed{A_{*1} = [S(1)]_{\mathcal{C}}, A_{*2} = [S(x)]_{\mathcal{C}}, A_{*3} = [S(x^2)]_{\mathcal{C}}.}$

General Case: If $T : V \to W$ is linear, then the matrix of T w.r.t. the ordered bases $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V, and $\mathcal{C} = \{w_1, \ldots, w_m\}$ of W, denoted $[T]_{\mathcal{C}}^{\mathcal{B}}$, is

$A = \left([T(v_1)]_{\mathcal{C}} \ \cdots \ [T(v_n)]_{\mathcal{C}}\right) \in \mathcal{M}_{m \times n}$.

Example: Projection onto the line $x_1 = x_2$

$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$ has standard matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.
This is the matrix of P w.r.t. the standard basis.

Q: What is $[P]_{\mathcal{B}}^{\mathcal{B}}$ where $\mathcal{B} = \{(1,1)^T, (-1,1)^T\}$?

Conclusion: The matrix of a transformation depends on the chosen basis. Some are better than others!