# MA-108 Differential Equations I

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## Example

Solve $x^2y' = y^2 + xy - x^2$.

$$
\begin{aligned}
y' = \frac{y^2 + xy - x^2}{x^2} &= \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1 \\
\text{Substitute } y = vx & \\
v'x + v &= v^2 + v - 1 \\
\frac{v'}{v^2 - 1} &= \frac{1}{x} \\
\frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right)v' &= \frac{1}{x} \\
\frac{1}{2}\left(\ln|v-1| - \ln|v+1|\right) &= \ln|x| + C_1 \\
\frac{v-1}{v+1} &= Cx^2 \\
v &= \frac{1 + Cx^2}{1 - Cx^2}
\end{aligned}
$$

## Example (Continued ...)

$$y = x\,\frac{1+Cx^2}{1-Cx^2} \qquad (*)$$

is a solution of

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

- **Question.** Are these all the solutions?
  Ans. No.
  Both $y = x$ and $y = -x$ are also solutions, but only $y = x$ can be obtained from the general solution.
- The solutions (*) were obtained in the intervals not containing $0$.
- **Question.** Are only solutions to the ODE, in an interval containing zero, are $y = x$ and $y = -x$?

## Example (Continued ...)

Note that

$$y = x\,\frac{1 + Cx^2}{1 - Cx^2}$$

is differentiable at $x = 0$ and satisfies the ODE

$$x^2 y' = y^2 + xy - x^2 \qquad (**)$$

at $0$ (since $y(0) = 0$).
In fact, for arbitrary $C_1, C_2 \in \mathbb{R}$, the function

$$y(x) = \begin{cases} x\,\dfrac{1 + C_1 x^2}{1 - C_1 x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1 + C_2 x^2}{1 - C_2 x^2} & \text{if } x \geq 0 \end{cases}$$

is differentiable and satisfies the ODE (**) with $y(0) = 0$.

## Example (continued ...)

Thus the IVP $\quad x^2y' = y^2 + xy - x^2, \quad y(0) = 0$
has infinitely many solutions

$$y(x) = \begin{cases} x\,\dfrac{1 + C_1x^2}{1 - C_1x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1 + C_2x^2}{1 - C_2x^2} & \text{if } x \geq 0 \end{cases}$$

one for each choice of $C_1, C_2$.

The interval of validity $I$ of $y(x)$ depends on $C_1, C_2$.

- If $C_1 \leq 0$ and $C_2 \leq 0$, then $I = \mathbb{R}$.
- If $C_1 \leq 0$ and $C_2 > 0$, then $I = (-\infty, 1/\sqrt{C_2})$.
- If $C_1 > 0$ and $C_2 \leq 0$, then $I = (-1/\sqrt{C_1}, \infty)$.
- If $C_1 > 0$ and $C_2 > 0$, then $I = \left(-1/\sqrt{C_1}, 1/\sqrt{C_2}\right)$.

## Example

Solve the IVP

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2 \qquad (*)$$

If

$$f(x,y) = \frac{y^2 + xy - x^2}{x^2},$$

then $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous in an open rectangle containing the point $(1,2) \in \mathbb{R}^2$.
By Existence and Uniqueness theorem, IVP (*) has a unique solution on sone open interval around $x_0 = 1$.
If $y \neq 0$ in an open interval, then the general solution is

$$y(x) = x\frac{1 + Cx^2}{1 - Cx^2}$$

## Example (continued ...)

$$\begin{aligned} y(1) &= \frac{1+C}{1-C} = 2 \implies C = 1/3 \\ y(x) &= x\,\frac{3+x^2}{3-x^2} \qquad (**) \end{aligned}$$

is the unique solution on some $(a,b) \subset (-\sqrt{3}, \sqrt{3})$ containing $x_0 = 1$. The interval of validity of $y(x)$ is $(-\sqrt{3}, \sqrt{3})$.

**Question.** What is the largest interval on which this solution is unique?

Note for any $x_0 \in (0, \sqrt{3})$, $y(x)$ is the unique solution of IVP

$$x^2 y' = y^2 + xy - x^2, \quad y(x_0) = x_0 \frac{3+x_0^2}{3-x_0^2}$$

on some interval in $(0, \sqrt{3})$ containing $x_0$.

## Example (continued ...)

Therefore, the largest interval $I$ containing $x_0 = 1$ on which

$$y(x) = x\,\frac{3+x^2}{3-x^2}$$

is the unique solution, contains $(0, \sqrt{3})$.
If $I$ contains $0$, then we can define another solution

$$y_1(x) = \begin{cases} x\,\dfrac{1+Cx^2}{1-Cx^2} & \text{if } a < x < 0 \\[2ex] x\,\dfrac{3+x^2}{3-x^2} & \text{if } 0 \le x < \sqrt{3} \end{cases}$$

where, $a = \dfrac{-1}{\sqrt{C}}$ if $C > 0$ and $a = -\infty$ if $C < 0$.
Thus the largest open interval, in which IVP with $y(1) = 2$ has a *unique solution*, is $(0, \sqrt{3})$.

# Examples

Describe the method to solve the following ODE.

- $y' = \dfrac{x^2 + 3x + 2}{y - 2}, \quad y(1) = 4$
  non-linear, Separable
- $(x - 2)(x - 1)y' - (4x - 3)y = (x - 2)^3$
  Linear non-homogeneous
- $(1 + x^2)y' + 2xy = \dfrac{1}{(1 + x^2)y}$
  Bernoulli Equation
- $y' = \dfrac{2x + y + 1}{x + 2y - 4}$
  Can be converted to a separable equation, use substitution $X = x + 2$, $Y = y - 3$.
- $3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0.$
  Exact equation

## Example (Exact Equation)

Solve $\quad 3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0$

Note $\quad 3x^2y^2 = \dfrac{\partial}{\partial x}(x^3y^2)$ and $2x^3y = \dfrac{\partial}{\partial y}(x^3y^2)$.

Let $\quad G(x,y) = x^3y^2$. Then

$$\begin{aligned}
&3x^2y^2 + 2x^3y\frac{dy}{dx} = 0 \\
\Longrightarrow\ &\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0 \\
\Longrightarrow\ &\frac{d}{dx}G(x,y(x)) = 0 \\
\Longrightarrow\ &G(x,y) = C
\end{aligned}$$

is an implicit solution of ODE.

## Definition

A first order ODE written in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be **exact** if there exists a function $G$ such that

$$\frac{\partial G}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial G}{\partial y} = N(x,y).$$

If the ODE is exact, then

$$G(x,y) = C$$

is an implicit solution of ODE.

# When is an ODE exact?

## Theorem

*Let* $D$ *be an open rectangle* $(a,b)\times(c,d)$. *Assume that*

$$M(x,y),\ N(x,y),\ \frac{\partial M}{\partial y},\ \frac{\partial N}{\partial x}$$

*are continuous in* $D$ *and* $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ *on* $D$.

*Then ODE*

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

*is exact on* $D$, *i.e. there exists* $G:D\to\mathbb{R}$ *s.t.*

$$\frac{\partial G}{\partial x}=M,\quad \frac{\partial G}{\partial y}=N$$

*So* $G(x,y)=C$ *is an implicit solution of ODE.*

# Exact Equations

Which of the following ODE's are exact?

1. $(2x+3)+(2y-2)y'=0$ Exact
2. $\dfrac{dy}{dx}=\dfrac{x+2y}{3x+4y}$. Not Exact
3. $(y/x+6x)dx+(\ln x-2)dy=0, \quad x,y>0.$ Exact
4. $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0.$ Not Exact

## Example

Solve $(2x+3)+(2y-2)y'=0.$

The ODE is exact, so we need to find $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = 2x+3 \text{ and } \frac{\partial \phi}{\partial y} = 2y-2$$

Integrating first equation gives

$$\begin{aligned} &\phi(x,y) = x^2+3x+h(y) \\ \Longrightarrow\ &\frac{\partial \phi}{\partial y} = \frac{dh}{dy} = 2y-2 \\ \Longrightarrow\ &h(y) = y^2-2y+c \end{aligned}$$

Therefore, an implicit solution to ODE is

$$\phi(x,y) = x^2+3x+y^2-2y = C$$

## Example

Solve $\quad (y/x + 6x)dx + (\ln x - 2)dy = 0, \quad x, y > 0.$

This is exact, so we need to find $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = \frac{y}{x} + 6x \text{ and } \frac{\partial \phi}{\partial y} = \ln x - 2$$

Integrating the first equation gives

$$\begin{aligned} \phi(x, y) &= y \ln x + 3x^2 + h(y) \\ \Longrightarrow \frac{\partial \phi}{\partial y} &= \ln x + \frac{dh}{dy} = \ln x - 2 \\ \Longrightarrow h(y) &= -2y + c \end{aligned}$$

Therefore, the solution is given by

$$\phi(x, y) = y \ln x + 3x^2 - 2y = C$$

## Example (Method of integrating factor)

Solve
$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$$

$$M = 3x^2y + 2xy + y^3, \quad N = x^2 + y^2$$
$$\frac{\partial}{\partial y}M = 3x^2 + 2x + 3y^2, \quad \frac{\partial}{\partial x}N = 2x$$

Therefore, ODE is <u>not</u> exact.

**Question.** Can it be converted to an exact equation?

The idea is to multiply the ODE by a function $\mu(x, y)$ so that it becomes exact. There is no algorithm for choosing $\mu$.

Assume
$$\mu(3x^2y + 2xy + y^3)dx + \mu(x^2 + y^2)dy = 0$$

is exact.

## Example (continued ...)

Then exactness condition gives

$$\frac{\partial}{\partial y}(\mu(3x^2y + 2xy + y^3)) = \frac{\partial}{\partial x}(\mu(x^2 + y^2)) \implies$$

$$\mu(3x^2 + 2x + 3y^2) + \frac{\partial \mu}{\partial y}(3x^2y + 2xy + y^3) = 2x\mu + \frac{\partial \mu}{\partial x}(x^2 + y^2)$$

$$\mu(3x^2 + 3y^2) + \frac{\partial \mu}{\partial y}(3x^2y + 2xy + y^3) = \frac{\partial \mu}{\partial x}(x^2 + y^2)$$

From observation, we choose $\mu$ to be independent of $y$.

Then $\partial\mu/\partial y = 0$ and above equation becomes

$$3\mu(x^2 + y^2) = \frac{d\mu}{dx}(x^2 + y^2)$$

$$\implies \frac{d\mu}{dx} = 3\mu$$

$$\implies \mu = Ce^{3x}$$

## Example (continued ...)

The ODE now becomes

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0 \quad (*)$$

Verify that this is exact. Hence there exists $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = e^{3x}(3x^2y + 2xy + y^3) \;\text{ and }\; \frac{\partial \phi}{\partial y} = e^{3x}(x^2 + y^2)$$

$$\implies \phi(x, y) = e^{3x}x^2y + \frac{1}{3}\,e^{3x}y^3 + h(y)$$

$$\implies \frac{\partial \phi}{\partial y} = e^{3x}x^2 + e^{3x}y^2 + \frac{dh}{dy} = e^{3x}(x^2 + y^2)$$

$$\implies \frac{dh}{dy} = 0 \implies h(y) = C$$

$$\implies \phi(x, y) = e^{3x}(x^2y + \frac{1}{3}\,y^3) = C \text{ : implicit solution of (*).}$$

**Question.** Is $\phi(x,y) = e^{3x}(x^2y + \frac{1}{3}\,y^3) = C$ the solution to our original ODE?
How will the solutions to the two ODE's be related?

$$\begin{aligned}
&\phi'(x,y) = 0 \\
\Longrightarrow\ &3e^{3x}(x^2y + \frac{1}{3}y^3) + e^{3x}(2xy + x^2y' + y^2y') = 0 \\
\Longrightarrow\ &(3x^2y + y^3 + 2xy) + (x^2 + y^2)y' = 0 \qquad (*)
\end{aligned}$$

since $e^{3x}$ is non-zero for all $x \in \mathbb{R}$.
Thus every $y(x)$ which is a solution to the new exact equation is a solution to the original equation $(*)$ and vice versa.

- In general, if $\mu$ is an integrating factor, then solutions to $\mu M + \mu N y' = 0$ may not be the solutions to $M + Ny' = 0$.
- If $\mu(x, y(x))$ is non vanishing for all $x$ in an open interval $I$, then the solution to exact ODE is a solution of original ODE on $I$.

## Definition (Finding the integrating factor)

We say $\mu(x, y)$ is a integrating factor of ODE

$$M(x, y) + N(x, y)y' = 0$$

if

$$\mu M + \mu N y' = 0$$

is exact, i.e.

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

or

$$\mu \left( M_y - N_x \right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M$$

# Finding the integrating factors

If the original equation

$$M(x,y) + N(x,y)y' = 0$$

was exact, then $\mu \equiv 1$ is an integrating factor.
In general, there is no clear way to determine $\mu$.

- If we assume that $\mu = \mu(x)$ is independent of $y$, then

$$\begin{aligned}
\mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M \\
\Longrightarrow \mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x} N \\
\Longrightarrow \frac{1}{\mu}\frac{d\mu}{dx} &= \frac{M_y - N_x}{N} := p(x) \\
\Longrightarrow \mu &= e^{\int p(x)\ dx}
\end{aligned}$$

  is an integrating factor if $\dfrac{M_y - N_x}{N}$ is a function of $x$ only.

# Finding the integrating factors

- If we assume that $\mu = \mu(y)$ is independent of $x$, then

$$
\begin{aligned}
\mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M \\
\implies \mu\left(M_y - N_x\right) &= -\frac{\partial \mu}{\partial y} M \\
\implies \frac{1}{\mu}\frac{d\mu}{dy} &= -\frac{M_y - N_x}{M} := q(y) \\
\implies \mu &= e^{\int q(y)\ dy}
\end{aligned}
$$

is an integrating factor if $\dfrac{M_y - N_x}{M}$ is a function of $y$ only.

# Finding the integrating factors

- If we assume that $\mu(x, y) = P(x)Q(y)$, then

$$\mu\left(M_y - N_x\right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M$$

$$\Longrightarrow P(x)Q(y)\left(M_y - N_x\right) = P'(x)Q(y)N - P(x)Q'(y)M$$

$$\Longrightarrow M_y - N_x = \frac{P'}{P}N - \frac{Q'}{Q}M$$

$$\frac{P'}{P} = p(x), \quad \frac{Q'}{Q} = q(y)$$

$$\Longrightarrow P(x) = e^{\int p(x)\,dx}, \quad Q(y) = e^{\int q(y)\,dy}$$

Thus

$$\mu(x, y) = e^{\int p(x)\,dx}\ e^{\int q(y)\,dy}$$

is an integrating factor if

$$M_y - N_x = p(x)N - q(y)M$$

## Theorem

*Consider* $\quad M(x,y) + N(x,y)y' = 0.$

*Assume that* $M$, $N$, $M_y$, $N_x$ *are continuous on an open rectangle* $R = (a,b) \times (c,d)$.

*Then* $\mu = \mu(x,y)$ *is an integrating factor of ODE, where*

1. *If* $\dfrac{M_y - N_x}{N} = p(x)$ *on* $R$, *then*
$$\mu = \mu(x) = e^{\int p(x)\,dx}$$
2. *If* $-\dfrac{M_y - N_x}{M} = q(y)$ *on* $R$, *then*
$$\mu = \mu(y) = e^{\int q(y)\,dy}$$
3. *If* $M_y - N_x = p(x)N - q(y)M$ *on* $R$, *then*
$$\mu = e^{\int p(x)\ dx}\ e^{\int q(y)\ dy}$$

## Example

Consider ODE

$$\cos x \cos y \; dx + (\sin x \cos y - \sin x \sin y + y) \; dy = 0.$$

$$\begin{aligned}
M &= \cos x \cos y \\
N &= \sin x \cos y - \sin x \sin y + y \\
M_y - N_x &= -\cos x \sin y - \cos x \cos y + \cos x \sin y \\
&= -\cos x \cos y \\
\frac{N_x - M_y}{M} &= 1
\end{aligned}$$

The integrating factor is $\mu = e^y$ and so

$$e^y \cos x \cos y \; dx + e^y(\sin x \cos y - \sin x \sin y + y) \; dy = 0$$

is exact. So there exists $\phi(x, y)$ such that

## Example (continued ...)

$$\frac{\partial \phi}{\partial x} = e^y \cos x \cos y, \quad \frac{\partial \phi}{\partial y} = e^y(\sin x \cos y - \sin x \sin y + y)$$

Integrating first equation, we get

$$\begin{aligned}
\phi(x,y) &= e^y \sin x \cos y + h(y) \\
\frac{d\phi}{\partial y} &= e^y \sin x \cos y - e^y \sin x \sin y + \frac{dh}{dy} \\
&= e^y(\sin x \cos y - \sin x \sin y + y) \\
\frac{dh}{dy} &= ye^y \\
h(y) &= e^y y + e^y + C \\
\phi(x,y) &= e^y(\sin x \cos y + y + 1) = C
\end{aligned}$$

is an implicit solution of ODE.

## Example

Solve $(3x^2y^3 - y^2 + y)dx + (-xy + 2x)dy = 0$

$$\begin{aligned}
M(x,y) &= 3x^2y^3 - y^2 + y \\
N(x,y) &= -xy + 2x \\
M_y - N_x &= 3x^2 3y^2 - 2y + 1 + y - 2 \\
&= 9x^2y^2 - y - 1 \\
\frac{-M_y + N_x}{M} &\neq q(y) \\
\frac{M_y - N_x}{N} &\neq p(x)
\end{aligned}$$

Can we write

$$\begin{aligned}
M_y - N_x &= p(x)N - q(y)M? \\
p(x) &= -2/x \\
q(y) &= -3/y
\end{aligned}$$

## Example (continued ...)

The integrating factor is then given by

$$\mu(x, y) = e^{\int -2/x \ dx} \ e^{\int -3/y \ dy} = \frac{1}{x^2 y^3}$$

We get an exact ODE

$$\begin{aligned} \frac{1}{x^2y^3}[(3x^2y^3 - y^2 + y) \ dx + (-xy + 2x) \ dy] &= 0 \\ \left(3 - \frac{1}{x^2y} + \frac{1}{x^2y^2}\right) \ dx + \left(\frac{-1}{xy^2} + \frac{2}{xy^3}\right) \ dy &= 0 \end{aligned}$$

Solve it.

**Question.** Is an integrating factor unique?
If $\mu$ is an integrating factor, then so is $c\mu$ for any constant $c \neq 0$.
What about upto constant multiple? No.

### Example

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

is not exact.
Show that

$$\mu_1(x, y) = \frac{1}{xy(2x + y)}, \quad \mu_2(x) = x$$

both are integrating factors of ODE.

However one integrating factor may give a simpler ODE than the other.

# Picard's Iteration Method

Picard's iteration method is useful in proving the existence and uniqueness theorem of the IVP

$$y' = f(x, y), \quad y(x_0) = y_0.$$

We will give an idea of the proof using this method.
Replacing $x$ by $x - x_0$ and $y$ by $y - y_0$, it is sufficient to assume that $x_0 = 0$ and $y_0 = 0$.
Suppose $y = \phi(x)$ is a solution to the IVP. Then

$$\frac{d\phi}{dx} = f(x, \phi(x)), \quad \phi(0) = 0.$$

Equivalently,

$$\phi(x) = \int_0^x f(s, \phi(s))ds, \quad \phi(0) = 0.$$

This is called an <u>integral equation</u> in the unknown function $\phi$.

Conversely, if the integral equation

$$\phi(x) = \int_0^x f(s, \phi(s))ds$$

holds, then by the Fundamental Theorem of Calculus,

$$y' = \frac{d\phi}{dx} = f(x, \phi(x)) = f(x, y).$$

Thus, solving the integral equation is equivalent to solving the IVP.

We define, iteratively, a sequence of functions $\phi_n(x)$ for $n \geq 0$

$$\begin{aligned} \phi_0(x) &\equiv 0 \\ \phi_1(x) &= \int_0^x f(s, \phi_0(s))\, ds \\ &\vdots \\ \phi_{n+1}(x) &= \int_0^x f(s, \phi_n(s))\, ds \end{aligned}$$

- Each $\phi_n$ satisfies the initial condition $\phi_n(0) = 0$.
- None of the $\phi_n$ may satisfy $y' = f(x, y)$.
- Suppose for some $n$, $\phi_{n+1} = \phi_n$. Then,

$$\begin{aligned}\phi_{n+1} = \phi_n &= \int_0^x f(s, \phi_n(s))ds \\ \implies \frac{d}{dx}(\phi_n(x)) &= f(x, \phi_n(x))\end{aligned}$$

  Thus, $y = \phi_n(x)$ is a solution of the given IVP.
- In general case, $\phi_n \neq \phi_{n+1}$ for all $n$.
- It is possible to show that, if $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ is continuous in some open rectangle (hence continuous and bounded in a smaller closed rectangle), then the sequence converges to a function

$$\phi(t) = \lim_{n \to \infty} \phi_n(t)$$

  which will be the unique solution to the given IVP.

## Example

Solve the IVP using Picard's iteration method.

$$y' = 2x(1+y);\ y(0) = 0.$$

The corresponding integral equation is

$$\phi(x) = \int_0^x 2s(1+\phi(s))ds.$$

$$\begin{aligned}
\phi_0(x) &= 0 \\
\phi_1(x) &= \int_0^x 2s\,ds = x^2, \\
\phi_2(x) &= \int_0^x 2s(1+s^2)\,ds = x^2 + \frac{x^4}{2}, \\
\phi_3(x) &= \int_0^x 2s(1+s^2+\frac{s^4}{2})\,ds = x^2 + \frac{x^4}{2} + \frac{x^6}{6}.
\end{aligned}$$

## Example (continued ...)

We claim:

$$\phi_n(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots + \frac{x^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(x) &= \int_0^x 2s(1+\phi_n(s))ds \\
&= \int_0^x 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right)ds \\
&= x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots + \frac{x^{2n}}{n!} + \frac{x^{2n+2}}{(n+1)!}.
\end{aligned}$$

## Example (continued ...)

Hence $\phi_n(x)$ is the $n$-th partial sum of the series $\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}$.

Applying the ratio test, we get:

$$\left| \frac{x^{2k+2}}{(k+1)!} \cdot \frac{k!}{x^{2k}} \right| = \frac{x^2}{k+1} \to 0$$

for all $x$ as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k!} = e^{x^2} - 1.$$

Therefore, $y(x) = e^{x^2} - 1$ is the solution of our IVP.

# Uniqueness of solution

If $f(x,y)$ is continuous in an open rectangle $R$ around $(0,0) \in \mathbb{R}^2$, then Picard's interation method shows that IVP

$$y' = f(x,y), \quad y(0) = 0$$

has atleast one solution.
Assume that $f$ and $\dfrac{\partial f}{\partial y}$ both are continuous in $R$. Then we show that IVP has a unique solution in some open interval around $0$.
Suppose $\phi$ and $\psi$ are two solutions of IVP. Then both $\phi$ and $\psi$ satisfy the integral equation.

$$\begin{aligned} \phi(x) &= \int_0^x f(x,\phi(x))\,dx \\ \psi(x) &= \int_0^x f(x,\psi(x))\,dx \end{aligned}$$

$$\begin{aligned}
\implies \phi(x) - \psi(x) &= \int_0^t (f(x,\phi(x)) - f(x,\psi(x)))\,dx \\
\implies |\phi(x) - \psi(x)| &\leq \int_0^x |f(x,\phi(x) - f(x,\psi(x)|\,dx \\
&\leq \int_0^x K|\phi(x) - \psi(x)|\,dx
\end{aligned}$$

for some constant $K$. This is because $\dfrac{\partial f}{\partial y}$ is continuous.

$$\begin{aligned}
U(x) := \int_0^x |\phi(x) - \psi(x)|\,dx \quad &\implies \quad U'(x) = |\phi(x) - \psi(x)| \\
U'(x) - KU(x) \leq 0 \quad &\implies \quad [e^{-Kx}U(x)]' \leq 0 \\
e^{-Kx}U(x) \text{ is decreasing, } U(0) = 0 \quad &\implies \quad e^{-Kx}U(x) \leq 0 \\
\implies U(x) \leq 0 \text{ for } x > 0 \text{ and } \quad U(x) &\quad \leq U(0) = 0,\ x < 0 \\
\implies 0 \leq U(x) \leq 0 \quad &\implies \quad U(x) \equiv 0 \\
\implies U(x)' \equiv 0 \quad &\implies \quad \phi(x) \equiv \psi(x)
\end{aligned}$$

# Second Order Linear ODE's

We'll consider second order linear ODE.
Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.
Though there is no formula to find all the solutions of such an ODE, we will study the existence, uniqueness and number of solutions of such an ODE.

# Second Order Linear ODE's

- If $r(x) \equiv 0$ in the equation above, i.e.,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

  then the ODE is said to be homogeneous.
  Otherwise it is called non-homogeneous.

- An IVP of a second order linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = r(x); \quad y(x_0) = a, \ y'(x_0) = b,$$

  where $p(x)$, $q(x)$ and $r(x)$ are assumed to be continuous on some open interval $I$ with $x_0 \in I$.

## Example

Solve the second order linear ODE

$$y'' + y = 0$$

Observe that $\sin x$ and $\cos x$ satisfy this equation.
Any scalar multiple of $\sin x$ and $\cos x$ is also a solution.
Any linear combination $c_1 \sin x + c_2 \cos x$ is a solution.

## Example

Solve

$$y'' - y = 0$$

It is easy to see that $e^x$ and $e^{-x}$ are solutions.
Again any linear combination $c_1 e^x + c_2 e^{-x}$ is a solution.

**Question.** Are these all the solutions?
If not, what are the other solutions, and how to find them?
If yes, why are these the only solutions?

# Solving IVP's

Let $I$ be an open interval with $x_0 \in I$.
For an integer $n \geq 0$, let $C^n(I)$ be the set of all functions $f : I \to \mathbb{R}$ such that $f$ is $n$-times differentiable on $I$ and $f^{(n)}$ is continuous on $I$.
Note that $C^n(I)$ is a vector spaces over $\mathbb{R}$ with addition and scalar multiplication defined as follows. For $x \in I$ and $k \in \mathbb{R}$,

$$\begin{aligned}(f+g)(x) &:= f(x)+g(x)\\ (k\cdot f)(x) &:= kf(x)\end{aligned}$$

Let $p, q \in C(I)$ be continuous functions on $I$. Define

$$L : C^2(I) \to C(I)$$

by

$$L(f) = f'' + p(x)f' + q(x)f.$$

# Solving IVP's

Check that $L$ is a linear transformation; i.e.

$$L(f+g) = L(f) + L(g), \quad L(cf) = cL(f)$$

for all $c \in \mathbb{R}$ and $f, g \in C^2(I)$.
The null space of $L$, denoted by $N(L)$ is

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

Thus, $N(L)$ consists of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

Therefore, solutions of (*) is a vector sub-space of $C^2(I)$.

# Existence and Uniqueness Theorem

Recall that in the first order case, existence and uniqueness was easy to prove in the linear case, whereas one need a nontrivial result in the non-linear case.
In the second order case, we need a non-trivial result in the linear case itself.

**Theorem**

*Consider the IVP*

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = a, \; y'(x_0) = b,$$

*where* $p, q \in C(I)$ *for an open interval* $I$ *with* $x_0 \in I$. *Then there is a unique solution to the IVP on* $I$.

When $p, q$ are constant, solution exists on $\mathbb{R}$.

**Example**

Find the largest open interval where the ODE

$$x^2y'' + xy' - 4y = 0$$

with initial condition $y(x_0) = y_0$ has a unique solution.

Write the ODE in standard form

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

Since $p(x) = \dfrac{1}{x}$ and $q(x) = \dfrac{-4}{x^2}$ are continuous on $(-\infty, 0) \cup (0, \infty)$, the IVP has has a unique solution on $(-\infty, 0)$ if $x_0 < 0$ and on $(0, \infty)$ if $x_0 > 0$.

- Verify that $y_1 = x^2$ is a solution of ODE on $(-\infty, \infty)$ and $y_2 = \dfrac{1}{x^2}$ is a solution on $(-\infty, 0) \cup (0, \infty)$.

## Example

Solve IVP

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \; y'(1) = 0$$

The general solution of ODE is

$$\begin{aligned} y(x) \quad &= \quad c_1x^2 + c_2\frac{1}{x^2} \\ c_1 + c_2 = 2 \quad &, \quad 2c_1 - 2c_2 = 0 \\ &\Longrightarrow \quad c_1 = 1, \quad c_2 = 1 \end{aligned}$$

Thus solution of IVP is $\quad y(x) = x^2 + \dfrac{1}{x^2}$
which is unique on the interval $(0, \infty)$.

**Exercise.** Solve
$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \; y'(-1) = 0$

## Theorem (Dimension Theorem)

*If $p(x), q(x)$ are continuous on an open interval $I$, then the set of solutions of the ODE*

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

*on $I$ is a vector space of dimension $2$.*
*Any basis $\{y_1, y_2\}$ of solutions of (*) is called a* **fundamental solutions** *of (*).*

- The theorem says that once you know that $e^x$ and $e^{-x}$ are solutions of $y'' - y = 0$, any other solution will be of the form $y(x) = c_1 e^x + c_2 e^{-x}$.
  Here $\{e^x, e^{-x}\}$ is a fundamental solutions.
- Similarly, any solution of $y'' + y = 0$ are of the form $y(x) = c_1 \sin x + c_2 \cos x$.
  Here $\{\sin x, \cos x\}$ is fundamental solutions.

## Definition

Let us consider 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

with constant coefficients $a, b, c \in \mathbb{R}$ with $a \neq 0$.
For scalar $m$, $e^{mx}$ is a solution if and only if

$$\begin{aligned} & am^2 e^{mx} + bme^{mx} + ce^{mx} = 0 \\ \implies \quad & p(m) := am^2 + bm + c = 0 \end{aligned}$$

Therefore, $e^{mx}$ is a solutioin of ODE if and only if $m$ is a root of the characteristic equation $p(m) = 0$.
The roots of the characteristic equation are given by

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider three cases:

1. Case 1: $b^2 - 4ac > 0$.
   Then characteristic equation has two distinct real roots.
2. Case 2: $b^2 - 4ac = 0$.
   Then characteristic equation has two repeated real roots.
3. Case 3: $b^2 - 4ac < 0$.
   Then characteristic equation has two distinct complex roots which are conjugates.

# Distinct real roots case

## Example

Find general solution of

$$y'' + 6y' + 5y = 0$$

The roots of characteristic equation

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5) = 0$$

are $-1$ and $-5$. Thus

$$y_1 = e^{-x}, \quad y_2 = e^{-5x}$$

are fundamental solutions of ODE. Therefore, the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

## Example

Solve IVP

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \; y'(0) = 1$$

The general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

$$\begin{aligned} y(0) = 3 \quad &\Longrightarrow \quad c_1 + c_2 = 3 \\ y'(0) = 1 \quad &\Longrightarrow \quad -c_1 - 5c_2 = 1 \end{aligned}$$

This gives $c_2 = -1$ and $c_1 = 4$.
Thus the solution to IVP is

$$y(x) = 4e^{-x} - e^{-5x}$$

# A repeated real root case

## Example

Find general solution of $y'' + 6y' + 9y = 0$
The roots of characteristic equation

$$p(m) = m^2 + 6m + 9 = (m+3)^2 = 0$$

are repeated $-3, -3$. Hence $y_1 = e^{-3x}$ is one solution.
For other solution, let $y = u(x)y_1 = u(x)e^{-3x}$. Then

$$\begin{aligned}
(u'y_1 + uy_1')' + 6(u'y_1 + uy_1') + 9(uy_1) &= 0 \\
(u''y_1 + 2u'y_1' + uy_1'') + 6(u'y_1 + uy_1') + 9(uy_1) &= 0 \\
u''y_1 + u'(2y_1' + 6y_1) + u(y_1'' + 6y_1' + 9y_1) &= 0 \\
u''y_1 + u'(2y_1' + 6y_1) &= 0
\end{aligned}$$

## Example (continued ...)

$$\begin{aligned}
\frac{u''}{u'} + \frac{2y_1'}{y_1} + 6 &= 0 \\
\ln|u'| + \ln|y_1^2| + 6x &= C \\
u'y_1^2e^{6x} &= c \\
y_1 &= e^{-3x} \\
u' &= c \\
u &= cx \\
y_2 &= xe^{-3x}
\end{aligned}$$

Therefore the general solution is

$$y(x) = e^{-3x}(c_1 + c_2x)$$

## Example

Solve IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \; y'(0) = 1$$

The general solution is

$$y(x) = e^{-3x}(c_1 + c_2 x)$$

$$y(0) = 3 \implies c_1 = 3$$

$$y'(0) = 1 = -3(3) + c_2 \implies c_2 = 10$$

Thus, the solution of IVP is

$$y(x) = e^{-3x}(3 + 10x)$$

# two distinct complex conjugate roots case

**Example**

Find general solution of

$$y'' + 4y' + 13y = 0$$

Roots of the characteristic equation

$$m^2 + 4m + 13 = (m+2)^2 + 9$$

are $-2+3i$ and $-2-3i$. So

$$\begin{aligned} e^{(-2+3i)x} &= e^{-2x}(\cos 3x + i \sin 3x) \\ e^{(-2-3i)x} &= e^{-2x}(\cos 3x - i \sin 3x) \end{aligned}$$

are complex solutions of ODE.
Taking sum and difference gives real solutions

## Example (continued ...)

$$y_1 = e^{-2x} \cos 3x, \quad y_2 = e^{-2x} \sin 3x$$

which are fundamental solutions.
Hence the general solution is

$$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$$

## Example

Solve IVP

$$y'' + 4y' + 13y = 0, \quad y(0) = 3, \; y'(0) = 1$$

$$\begin{aligned} y(0) = 3 &\quad \Longrightarrow \quad c_1 = 3, \\ y'(0) = 1 = -2(3) + 3c_2 &\quad \Longrightarrow \quad c_2 = 7/3 \end{aligned}$$

The the solution of IVP is

$$y(x) = e^{-2x}\left(3\cos 3x + \frac{7}{3}\sin 3x\right)$$

## Theorem

*Let $m_1, m_2$ be the roots of characteristic equation*

$$p(m) = am^2 + bm + c = 0 \quad \textit{of ODE}$$

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}, \ a \neq 0$$

*Then the general solution $y(x)$ is given as follows.*

1. *If $m_1 \neq m_2$ are real, then*
$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$
2. *If $m_1 = m_2 \in \mathbb{R}$,*
$$y(x) = e^{m_1 x}(c_1 + c_2 x)$$
3. *If $m_1 = \lambda + i\omega$ and $m_2 = \lambda - i\omega$, where $\omega > 0$, then*
$$y(x) = e^{\lambda x}[c_1 \cos(\omega x) + c_2 \sin(\omega x)]$$

## Theorem

*Let $p(x), q(x)$ be continuous functions on an open interval $I$. Then the set of solutions of the ODE*

$$y'' + p(x)y' + q(x)y = 0$$

*on $I$ is a vector space of dimension $2$.*

We will give a proof of this theorem using uniqueness of solution of IVP for above ODE.

## Theorem

*Consider the IVP*

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = a, \; y'(x_0) = b,$$

*where $p, q \in C(I)$ for an open interval $I$ with $x_0 \in I$. Then there is a unique solution to the IVP on $I$.*

# Proof of Dimension Theorem

If $y_1$ and $y_2$ are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

then $c_1y_1 + c_2y_2$ is also a solution of ODE.
To see this,

$$\begin{aligned} & (c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) \\ = \ & c_1[y_1'' + p(x)y_1' + q(x)y_1] + c_2[y_2'' + p(x)y_2' + q(x)y_2] \\ = \ & 0 \end{aligned}$$

Thus the solution space is a vector space. Now

1. we need to produce two linearly independent solutions, say $f$ and $g$, and
2. show that any other solution is a linear combination of $f$ and $g$.

# Existence of $f$ and $g$

Fix $x_0 \in I$. Let $y_1 = f(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 1, \; y'(x_0) = 0$$

$y_1$ exists on $I$ by uniqueness theorem.
Similarly, let $y_2 = g(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \; y'(x_0) = 1$$

We need to show that $f, g$ are linearly independent. Assume

$$af(x) + bg(x) \equiv 0 \implies af'(x) + bg'(x) \equiv 0$$

for some scalars $a$ and $b$. Evaluate at $x = x_0$, we get

$$a = 0, \quad b = 0$$

This proves $f$ and $g$ are linearly independent solutions of ODE.

# Any solution is a linear combination of $f$ and $g$

Let $h(x)$ be an arbitrary solution of the given ODE. Define

$$\widetilde{h}(x) = h(x_0)f(x) + h'(x_0)g(x)$$

Then $\widetilde{h}(x)$ and $h(x)$ both are solutions of IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = h(x_0),\ y'(x_0) = h'(x_0)$$

on $I$. By uniqueness theorem,

$$\widetilde{h} \equiv h$$

Thus any solution is a linear combination of $f$ and $g$.
Therefore, the solution space of ODE

$$y'' + p(x)y' + q(x)y = 0$$

is $2$-dimensional. $\square$

# Nonhomogeneous 2nd order linear ODE

Consider 2nd order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad (*)$$

with $p(x), q(x), r(x)$ continuous on an open interval $I$.
The solution space of its homogeneous part

$$y'' + p(x)y' + q(x)y = 0 \quad (**)$$

is a $2$-dimensional vector space.

- Suppose $y_1$ is a solution of $(*)$ and $y_2$ is a solution of $(**)$. Show that $y_1 + y_2$ is a solution of $(*)$.
- Fix a solution $y_1$ of $(*)$. If $y$ is any other solution of $(*)$, then $y - y_1 = y_2$ is a solution of $(**)$.

  Therefore, $y = y_1 + y_2$.