Code B, D

① Solve the IVP

$$y' = \frac{3x+6y}{2x-y} \quad ; \quad y(1)=0$$

Solution: Let $y = xv$.

Then ~~the~~ $y' = v + xv'$.

The above ODE becomes

$$v + xv' = \frac{3+6v}{2-v}$$

$$\Rightarrow \quad xv' = \frac{3+6v}{2-v} - v = \frac{3+4v+v^2}{2-v}$$

$$\Rightarrow \quad \boxed{v'\left(\frac{2-v}{3+4v+v^2}\right) = \frac{1}{x}} \quad \text{——} \quad (1)$$

$$3+4v+v^2 = (v+3)(v+1)$$

Let $\dfrac{\alpha}{v+3} + \dfrac{\beta}{v+1} = \dfrac{2-v}{3+4v+v^2}$

$$\Rightarrow \quad \alpha v + \alpha + \beta v + 3\beta = 2 - v$$

$$\Rightarrow \quad \begin{aligned} \alpha + \beta &= -1 \\ \alpha + 3\beta &= 2 \end{aligned} \quad \Rightarrow \quad 2\beta = -3 \quad \Rightarrow \quad \beta = -\frac{3}{2}$$

$$\Rightarrow \quad \alpha = \frac{1}{2}$$

Thus, we get

$$\frac{1}{2}\frac{v'}{v+3} - \frac{3}{2}\frac{v'}{v+1} = \frac{1}{x}$$

$$\Rightarrow \ln|v+3|^{1/2} - \ln|v+1|^{3/2} = \ln|x| + c$$

$$\Rightarrow \frac{|v+3|^{1/2}}{|v+1|^{3/2}} = cx$$

$$\Rightarrow \boxed{\frac{(y+3x)}{(y+x)^3} = c^2 x^2} \quad 2$$

To find c, ~~we~~ put $x=1$ and $y(1)=0$

$$\Rightarrow \frac{3}{1} = c^2 .$$

Thus, the solution to the IVP is given by

$$\boxed{\frac{(y+3x)}{(y+x)^3} = 3x^2} \quad 2$$

(2) Let ϕ_i denote the i^{th} iterate in Picard's iteration method. Find ϕ_1 and ϕ_2 for the following IVP

$$y' = x^2 + y^2 + 1 \quad ; \quad y(0) = 0$$

<u>Solution</u>: Let $\phi_0 \equiv 0$. Let $f(s,t) = s^2 + t^2 + 1$.

Then $\phi_1(t) = \int_0^t f(s, \phi_0)\, ds = \int_0^t (s^2 + 1)\, ds$

$$\boxed{\phi_1(t) = \cancel{\frac{t^3}{3} + t} \;\; \frac{t^3}{3} + t} \quad \text{— } 1$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds = \int_0^t \left(s^2 + \phi_1(s)^2 + 1\right) ds$$

$$= \int_0^t \left(s^2 + \frac{s^6}{9} + \frac{2s^3}{3} + s^2 + 1\right) ds$$

$$= \int_0^t \left(2s^2 + \frac{s^6}{9} + \frac{2s^3}{3} + 1\right) ds$$

$$= \cancel{\frac{2t^3}{3} + \frac{t^7}{63} + \frac{t^4}{6} + t}$$

$$\boxed{\phi_2(t) = \frac{2t^3}{3} + \frac{t^7}{63} + \frac{t^4}{6} + t} \quad \text{— } 1$$

③ Find the integrating factor and solve the IVP

$$(12x^3y + 24x^2y^2) + (9x^4 + 32x^3y + 4y)y' = 0 \quad ; \quad y(0) = 1$$

Solution :

$$M = 12x^3y + 24x^2y^2$$
$$N = 9x^4 + 32x^3y + 4y$$

$$\Rightarrow \quad M_y = 12x^3 + 48x^2y$$
$$N_x = 36x^3 + 96x^2y$$

$$M_y - N_x = -24x^3 - 48x^2y$$

$$\Rightarrow \quad \frac{M_y - N_x}{-M} = \frac{24x^3 + 48x^2y}{12x^3y + 24x^2y^2} = \frac{2}{y}$$

Thus, we can take $\boxed{\mu = e^{\int \frac{2}{y}\,dy} = y^2.}$ — 2

Multiplying the ODE by y^2 we get

$$(12x^3y^3 + 24x^2y^4) + (9x^4y^2 + 32x^3y^3 + 4y^3)y' = 0$$

Let us find f such that

$$\frac{df}{dx} = 12x^3y^3 + 24x^2y^4 \quad \text{— (a)}$$

$$\frac{df}{dy} = 9x^4y^2 + 32x^3y^3 + 4y^3 \quad \text{— (b)}$$

$$\text{(a)} \Rightarrow \quad f(x,y) = 3x^4y^3 + 8x^3y^4 + h(y)$$

$$\Rightarrow 9x^4y^2 + 32x^3y^3 + h'(y) = 9x^4y^2 + 32x^3y^3 + 4y^3$$

$$\Rightarrow h'(y) = 4y^3 \Rightarrow h(y) = y^4 + c$$

Thus, the solution is

$$\boxed{3x^4y^3 + 8x^3y^4 + y^4 = c} \quad \text{— } 2$$

To compute c, let $x = 0$ and $y(0) = 1$

$$\Rightarrow c = 1.$$

Thus, the solution to the IVP is given by

$$\boxed{3x^4y^3 + 8x^3y^4 + y^4 = 1} \quad \text{— } 1$$

4. For $i=1,2$ let $p_i(x)$ and $q_i(x)$ be continuous functions on $(0,1)$. Assume that the two differential equations

$$y'' + p_i(x)y' + q_i(x)y = 0 \qquad i=1,2$$

have the same set of solutions. Show that $p_1 = p_2$ and $q_1 = q_2$ on $(0,1)$.

Solution: $\boxed{\text{For solution using IVP method, see code } A.}$

Let $\{f, g\}$ be a basis for the solution space of

$$y'' + p_1(x)y' + q_1(x)y = 0 \quad \text{———} \quad (a)$$

Then $\{f, g\}$ is also a basis for the solution space of

$$y'' + p_2(x)y' + q_2(x)y = 0 \quad \text{———} \quad (b)$$

From equation (a), we know that the Wronskian $W(f,g;x)$ satisfies

$$\frac{W'(f,g;x)}{W(f,g;x)} = -p_1(x)$$

From equation (b), we know that the Wronskian satisfies

$$\frac{W'(f,g;x)}{W(f,g;x)} = -p_2(x)$$

$$\Rightarrow \boxed{p_1 = p_2} \quad \text{—} \quad 3$$

Let $\alpha \in (0,1)$. Let $y_\alpha(x)$ be the unique solution to the IVP

$$y'' + p_1(x) y' + q_1(x) y = 0 \quad ; \quad y_\alpha(\alpha) = 1 \, , \quad y_\alpha'(\alpha) = 0$$

Then y_α is also a solution to

$$y'' + p_2(x) y' + q_2(x) y = 0$$

Thus, y_α satisfies (by subtracting the two)

$$\cancel{p_1(x)} \Big(q_1(x) - q_2(x) \Big) y_\alpha(x) = 0$$

Let $x = \alpha$. Then we get

$$\big(q_1(\alpha) - q_2(\alpha) \big) y_\alpha(\alpha) = 0$$

Since $y_\alpha(\alpha) = 1$, $\Rightarrow q_1(\alpha) - q_2(\alpha) = 0$

Thus, $\boxed{q_1 = q_2 \text{ on } (0,1).}$ — 2

5. Use variation of parameters to find the general solution of

$$y'' - 6y' + 9y = \frac{e^{3x}}{x} \qquad x > 0$$

<u>Solution</u>: The homogeneous part is

$$y'' - 6y' + 9y = (D^2 - 6D + 9)y = 0$$

$$D^2 - 6D + 9 = (D-3)^2$$

$\Rightarrow$ The two solutions to the homogeneous part are e^{3x} and xe^{3x}.

The Wronskian is

$$W(e^{3x}, xe^{3x}; x) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

2

The particular solution is given by

$$y_p = y_2 \int \frac{y_1 r(s)}{W} ds - y_1 \int \frac{y_2 r(s)}{W} ds$$

$$= xe^{3x} \int \frac{e^{3s}\, e^{3s}}{s\, e^{6s}} ds - e^{3x} \int \frac{s e^{3s}}{e^{6s}} \frac{e^{3s}}{s} ds$$

$$= xe^{3x} \int \frac{1}{s} ds - e^{3x} \int 1\, ds$$

$$\boxed{y_p = xe^{3x} \ln x - xe^{3x}}$$

2

The general solution is given by

$$\boxed{y = xe^{3x}\ln x - xe^{3x} + c_1 e^{3x} + c_2 x e^{3x}} \text{ — } 1$$

6 Consider the IVP

$$y' = 6x\,y^{2/3}\ ; \qquad y(0) = -8$$

(a) Find one solution of IVP on $\mathbb{R}$.

(b) What is the largest interval I containing 0, on which $y_1(x)$ is the unique solution of IVP?

(c) If $I \neq \mathbb{R}$, find another solution of IVP on $\mathbb{R}$.

Solution (a)

$$\frac{y'}{y^{2/3}} = 6x$$

$$\Rightarrow 3\frac{d}{dx}\left(y^{1/3}\right) = 3\frac{d}{dx}x^2$$

$$\Rightarrow y^{1/3} = x^2 + c$$

$$\Rightarrow \boxed{y = (x^2+c)^3} \text{ — } 2$$

To find c, put $x = 0$, $y(0) = -8$

$$\Rightarrow -8 = c^3 \Rightarrow \boxed{c = -2.} \text{ — } 1$$

Thus, $y(x) = (x^2 - 2)^3$.

This is a solution on all of $\mathbb{R}$.

(b) The given ODE is

$$y' = f(x,y), \quad \text{where } f(x,y) = 6xy^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{4x}{y^{1/3}}$$

$$\Rightarrow \frac{\partial f}{\partial y}(x, y(x)) = \frac{4x}{x^2 - 2}$$

Thus, there is a problem when $x = \pm\sqrt{2}$.

Here $\frac{\partial f}{\partial y}$ is not defined.

$\therefore$ The interval $\boxed{I = (-\sqrt{2}, \sqrt{2})}$ — 2

(c) $y(\pm\sqrt{2}) = 0$. ~~Thus, define~~

Also observe that the constant function $y = 0$ satisfies the ODE (but not the IVP)

Define a new function as

$$y(x) = \begin{cases} 0 & |x| \geq \sqrt{2} \\ \ldots \end{cases}$$ — 2

$\boxed{y_1(x) \text{ is clearly continous.}}$

Let us check $y_1(x)$ is differentiable at $\pm\sqrt{2}$.

~~$\lim_{x \to \sqrt{2}^+} y_1'(x) = \lim_{x \to \sqrt{2}^+}$~~

On $(-\sqrt{2}, \sqrt{2})$, $y_1'(x) = 3(x^2-2)^2 \cdot (2x)$

~~Clearly, $y_1'(\pm\sqrt{2})$~~

On $(-\infty, -\sqrt{2})$, $y_1'(x) = 0$

On $(\sqrt{2}, \infty)$, $y_1'(x) = 0$

$\boxed{\text{Clearly } y_1'(x) \text{ is also continuous at } \pm\sqrt{2}.}$ 1

Thus, $y_1(x)$ is another solution on $\mathbb{R}$.

1

(A, C)

$$y' = \frac{x+6y}{3x-2y}, \quad y(1)=0$$

put $y/x = v$ or $y = vx$,

$$v'x + v = \frac{1+6v}{3-2v}$$

$$v'x = \frac{1+6v - v(3-2v)}{3-2v} = \frac{2v^2+3v+1}{3-2v}$$

$$\frac{3-2v}{2v^2+3v+1} v' = \frac{1}{x}$$

$$\boxed{\frac{3-2v}{(v+1)(2v+1)} v' = \frac{1}{x}}$$ 1

$$\frac{A}{v+1} + \frac{B}{2v+1} = \frac{3-2v}{(v+1)(2v+1)} \Rightarrow$$

$(2v+1)A + (v+1)B = 3-2v$

$2A + B = 3$

$A + B = -2$

$A = 5, \quad B = -7$

$$\left[\frac{5}{v+1} - \frac{7}{2v+1}\right] v' = \frac{1}{x}$$

Integrate wrt x,

$$5\ln|v+1| - \frac{7}{2}\ln|2v+1| = \ln|x| + C_1$$

$$\frac{(v+1)^5}{(2v+1)^{7/2}} = Cx$$ 2

$$\boxed{\frac{(y+x)^5}{(2y+x)^{7/2}} = C x^{5/2}}$$

$y(1) = 0$

$\Rightarrow C = 1$

$$\boxed{\frac{(y+x)^5}{(2y+x)^{7/2}} = x^{5/2}}$$ 2

is the implicit solⁿ of IVP.

(2)

(A, C)

$$y' = x^2+y^2+1 = f(x,y), \quad y(0)=0$$

$$\phi_0(x) = 0$$

$$\phi_1(x) = \int_0^x f(x,\phi_0)\,dx$$

$$= \int_0^x (x^2+1)\,dx$$

$$\boxed{\phi_1 = \frac{x^3}{3} + x} \quad (1)$$

$$\phi_2(x) = \int_0^x f(x, \phi_1(x))\,dx$$

$$= \int_0^x \left[x^2 + \left(\frac{x^3}{3}+x\right)^2 + 1\right] dx$$

$$= \int_0^x \left[x^2 + \frac{x^6}{9} + x^2 + \frac{2x^4}{3} + 1\right] dx$$

$$\boxed{\phi_2 = \frac{2x^3}{3} + \frac{x^7}{63} + \frac{2x^5}{15} + x} \quad (1)$$

(3) (A.C). $\overbrace{(6xy^2+2y)}^{M} + \overbrace{(12xy^2+6x+3)}^{N} y' = 0, \quad y(0)=1$

$M_y = 12xy+2$

$N_x = 24xy+6$

$M_y - N_x = -(12xy+4)$

$$\frac{M_y - N_x}{-M} = \frac{-(12xy+4)}{-y(6xy+2)} = \frac{2}{y}$$

Hence integrating factor $\mu = \mu(y)$ is

$$\mu(y) = e^{\int \frac{2}{y} dy} = e^{\ln y^2} = y^2.$$

$\boxed{\mu(y) = y^2}$ 2

Now $y^2(6xy^2+2y) + y^2(12xy^2+6x+3)y' = 0$

is exact.

$\left(M_y = 24xy^3 + 6y^2, \quad N_x = 24xy^3+6y^2 \right)$

Hence $\exists\ \phi(x,y)$ s.t.

$$\frac{\partial \phi}{\partial x} = y^2(6xy^2+2y), \quad \frac{\partial \phi}{\partial y} = y^2(12xy^2+6x+3)$$

$\phi(x,y) = 3x^2y^4 + 2xy^3 + h(y)$

$$\frac{\partial \phi}{\partial y} = 12x^2y^3 + 6xy^2 + h'(y)$$

$$= 12x^2y^3 + 6xy^2 + 3y^2$$

$\Rightarrow \quad h'(y) = 3y^2 \quad \Rightarrow h(y) = y^3.$

Hence the sol^n of ODE is

$\boxed{\phi(x,y) = 3x^2y^4 + 2xy^3 + y^3 = c}$ 2

$y(0)=1 \quad \Rightarrow \quad c = 1.$

$\therefore \quad \boxed{3x^2y^4 + 2xy^3 + y^3 = 1}$ 1

4 (A, C)

$y'' + p_1(x)y' + q_1(x)y = 0$ have same set of sol^{n}s. By dimension theorem, solution space is 2-dimensional, say V.

All fns in V satisfy both eqn^{n}s, hence their difference $(p_1 - p_2)(x)\, y' + (q_1 - q_2)(x)\, y = 0$.

If $(p_1 - p_2)(x_0) \neq 0$ for some $x_0 \in (0,1)$, then $\exists$ a sol^n $y_1 \in V$ st $y_1(x_0) = 0$, $y_1'(x_0) = 1$.

(Such a y_1 exists and is unique by Existence and uniqueness of sol^n of IVP.)

Then $(p_1 - p_2)(x_0)\, y'(x_0) = (p_1 - p_2)(x_0) = 0$ #.

a contradiction, hence $\boxed{p_1 - p_2 \equiv 0.}$ 3

Similarly, if $(q_1 - q_2)(x_0) \neq 0$ for some $x_0 \in (0,1)$, we can consider an IVP with $y(x_0) = 1$, $y'(x_0) = 0$.

This gives a unique $y_2 \in V$.

Now $(p_1 - p_2)(x_0)\, y_2'(x_0) + (q_1 - q_2)(x_0)\, y_2(x_0) = 0$

$\Rightarrow (q_1 - q_2)(x_0) = 0$

a contradiction,

hence $\boxed{q_1 - q_2 \equiv 0}$ 2

(5) (A,C) $y'' - 4y' + 4y = \frac{e^{2x}}{x}$, $x > 0$

$D^2 - 4D + 4 = (D-2)^2$.

Hence basis of homog eqn is e^{2x}, xe^{2x}.

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$\boxed{W = e^{4x}(1 + 2x - 2x) = e^{4x}.}$$ (2)

By variation of parameter method, a particular solution y_p is given by

$$y_p = y_2 \int \frac{y_1 r}{W} dx - y_1 \int \frac{y_2 r}{W} dx$$

$$= xe^{2x} \int \frac{e^{2x} \cdot e^{2x}/x}{e^{4x}} - e^{2x} \int \frac{xe^{2x} \cdot e^{2x}/x}{e^{4x}} dx$$

$$= xe^{2x} \int \frac{1}{x} dx - e^{2x} \int 1 \, dx$$

$$\boxed{y_p = e^{2x}(x \ln x - x)}$$ (2)

General soln is

$$\boxed{y = e^{2x}(x \ln x - x) + c_1 e^{2x} + c_2 xe^{2x}}$$ (1)

6 (A, C).

$y' = 6xy^{2/3}$, $y(0) = -1$

(a) $\frac{y'}{y^{2/3}} = 6x \Rightarrow 3y^{1/3} = 3x^2 + 3C$

$\Rightarrow \boxed{y^{1/3} = x^2 + C}$ 2

$y(0) = -1 \Rightarrow C = -1$.

$y^{1/3} = x^2 - 1 \Rightarrow \boxed{y = (x^2-1)^3}$ 1

This is one solⁿ of IVP on $\mathbb{R}$.

(b) The largest interval I on which IVP has a unique solⁿ is $(-1, 1)$.

$\boxed{I = (-1, 1)}$ 2

(c) Another solⁿ of IVP on $\mathbb{R}$ is

$$y_2(x) = \begin{cases} (x^2-1)^3 & , x \in (-1,1) \\ 0 & , |x| \geq 1 \end{cases}$$ 2

Note that $y \equiv 0$ is a solⁿ of differential equation.

We need to show that y_2 is differential at $x = \pm 1$.

$\lim_{x \to \pm 1} y_2(x) = 0 = y_2(\pm 1)$, hence y_2 is continuous at ± 1.

Further,

$$\lim_{x \to 0} \frac{y_2(\pm 1) - y_2(\pm 1 + x)}{x}$$

$$= \lim_{x \to 0} \frac{0}{x} \left(\text{or } \frac{((\pm 1 + x)^2 - 1)^3}{x} \right)$$

$$= 0$$

Hence y_2 is differential at ± 1.

$\therefore$ $y_2(x)$ is another sol^n of IVP on $\mathbb{R}$.

B