MA108, Spring 2018, Tutorial 1 Solutions

1. Solve the following initial value problems.

(a) $xy' + (1 + x\cot x)y = 0, \quad y(0) = 2.$
Ans. The IVP CANNOT BE SOLVED for the initial value prescribed at $x_0 = 0$.

(b) $y' - \frac{2x}{1+x^2}y = 0, \quad y(0) = 2.$
Ans. $y(x) = 2(1 + x^2)$.

(c) $xy' + 2y = 8x^2, \quad y(1) = 3.$
Ans. $y(x) = 2x^2 + \frac{1}{x^2}, \quad x \in (0, \infty)$.

2. Find the general solution for the following equations.

(a) $(x - 2)(x - 1)y' - (4x - 3)y = (x - 2)^3.$
Ans. $y(x) = \frac{-(x-2)^3}{2(x-1)} + C\frac{(x-2)^5}{(x-1)}$, where C is an arbitrary constant.

(b) $x^2y' + 3xy = e^x.$
Ans. $y(x) = \frac{e^x}{x_2} + \frac{C-e^x}{x^3}$, where C is an arbitrary constant.

3. In each of the following problems, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

(a) $y' + (\tan x)y = \sin x, \quad y(\pi) = 0.$
Ans. $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

(b) $(4 - x^2)y' + 2xy = 3x^2, \quad y(1) = -3.$
Ans. $(-2, 2)$.

4. Let $xy' - 2y = -1$.

(a) Find the general solutions $y_1(x)$ and $y_2(x)$ of the ODE on the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively.

(b) Show that $\lim_{x\to 0^-} y_1(x) = \lim_{x\to 0^+} y_2(x)$. This defines a continuous function $y(x)$ on $(-\infty, \infty)$. Show that $y(x)$ is differentiable on $\mathbb{R}$ and satisfies the ODE.

(c) Conclude that the IVP $xy' - 2y = -1, \quad y(0) = 1/2$ has infinitely many solutions on $\mathbb{R}$.

(d) If $x_0 > 0$ and y_0 is arbitrary, then the initial value problem $xy' - 2y = -1, \quad y(x_0) = y_0$ has a unique solution on $(0, \infty)$ by uniqueness theorem for linear ODEs. Show that this IVP has infinitely many solutions on $\mathbb{R}$. Why does this not contradict the existence and uniqueness theorem for linear ODEs?

Ans. $y_1(x) = \frac{1}{2} + C_1x^2$ on $(-\infty, 0)$ and $y_2(x) = \frac{1}{2} + C_2x^2$ on $(0, \infty)$.

1. Solve the following.

(a) $y(1 + x^3)y' = x^2$.
Ans. $y^2 = \frac{1}{3}\ln|1 + x^3| + C$, where C is an arbitrary constant.

(b) $y' = (\cos^2 x)(\cos^2 2y)$.
Ans. $\tan(2y) = x + \frac{\sin 2x}{2} + C$, where C is an arbitrary constant.

(c) $(1 + x^2)y' = x^2$.
Ans. $\frac{x^3}{3} - \frac{y^3}{3} - y = C$, where C is an arbitrary constant.

2. Show that the following equations are homogeneous of the form $y' = q(y/x)$. Solve them.

(a) $\frac{dy}{dx} = \frac{x+3y}{x-y}$.
Ans. $(x^2 + y^2)e^{\frac{2x}{x+y}} - Cx = 0$, where C is an arbitrary constant.

(b) $(x^2 + 3xy + y^2)dx - x^2dy = 0$.
Ans. $xe^{\frac{x}{x+y}} = C$, where C is an arbitrary constant.

(c) $y' = \dfrac{x^3 + y^3}{xy^2}, \quad y(1) = 3$.
Ans. $y(x) = x(3\ln x + 27)^{1/3}, \quad x \in (0, \infty)$.

3. Solve the following Bernoulli ODEs.

(a) $x^2y' + 2xy - y^3 = 0, \quad x > 0$.
Ans. $y(x) = \frac{1}{x^2}\left(\frac{2}{5x^5} + C\right)^{-\frac{1}{2}}$, where C is an arbitrary constant.

(b) $y' = \epsilon y - \sigma y^3, \quad \epsilon > 0, \sigma > 0$.
Ans. $y(x) = e^{\epsilon x}\left(\frac{\sigma}{\epsilon}e^{2\epsilon x} + C\right)^{-\frac{1}{2}}$, where C is an arbitrary constant.

(c) $x^2y' + 2y = 2e^{\frac{1}{x}}y^{\frac{1}{2}}$.
Ans. $y(x) = e^{\frac{2}{x}}\left(C - \frac{1}{x}\right)^2$, where C is an arbitrary constant.

(d) $xy' + y = x^4y^4, \quad y(1) = 1/2$.
Ans. $y(x) = \frac{1}{x}(C - 2x)^{-\frac{1}{3}}$, where C is an arbitrary constant.

4. Following may not be separable but can be made separable by substitution.

(a) $y' = \frac{-6x+y-3}{2x-y-1}$.
Ans.

$$X = C\left(\frac{Y^2}{X^2} - \frac{Y}{X} - 6\right)^{-\frac{1}{2}} e^{\frac{3}{2^{3/2}}\tan^{-1}\left(\frac{2Y-X}{2^{3/2}X}\right)},$$

where C is an arbitrary constant, $X = x - 1, Y = y - 3$.

(b) $y' = \frac{-x+3y-14}{x+y-2}$.
Ans.

$$X = C\left(\frac{X}{Y-X}\right)e^{\frac{2X}{Y-X}},$$

where C is an arbitrary constant, $X = x - 8, Y = y + 2$.

(c) $xyy' = 3x^6 + 6y^2$.
Ans. $y(x) = x^6\left(\frac{-1}{3x^6} + C\right)^{\frac{1}{2}}$, where C is an arbitrary constant.

(d) $x(\ln x)^2 y' = -4(\ln x)^2 + y\ln x + y^2$.
Ans. $y(x) = C_1\left(\ln\left(\frac{1}{\ln x} + a\right)\right)^{-1}\ln x + C_2\left(4\ln\left(\frac{1}{\ln x}\right) + b\right)\ln x$, where C_1, C_2, a, b are arbitrary constants.

5. Determine if the following equations are exact. If exact, then solve them.

(a) $(3y\cos x + 4xe^x + 2x^2e^x)\,dx + (3\sin x + 3)\,dy = 0$.
Ans. $\phi(x, y) = d$, that is, $3y(\sin x + 1) = d$, for an arbitrary constant d.

(b) $\left(\frac{1}{x} + 2x\right)dx + \left(\frac{1}{y} + 2y\right)dy = 0$.
Ans. $\phi(x, y) = \ln xy + x^2 + y^2 + C$, where C is an arbitrary constant.

(c) $(y\sin xy + xy^2\cos xy)dx + (x\sin xy + xy^2\cos xy)dy = 0$.
Ans. NOT exact.

(d) $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0$.
Ans. NOT exact.

(e) $\frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy = 0$.
Ans. $\phi(x, y) = \frac{-1}{\sqrt{x^2+y^2}} + C$, where C is an arbitrary constant.

6. Find all M such that $M(x, y)\,dx + 2xy\sin x\cos y\,dy = 0$ is exact.
Ans. $M(x, y) = 2(\sin x + x\cos x)(y\sin y + \cos y) + G(x)$, where $G(y)$ is any continuous function of x.

7. Find all N such that $(\ln xy + 2y\sin x)dx + N(x, y)dy = 0$ is exact.
Ans. $N(x, y) = \frac{x}{y} - 1 - 2\cos x + h(y)$, where $h(y)$ is any continuous function of y.