MA-108 Differential Equations I

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Theorem

If f is piecewise continuous and of exponential order, then

$$(i)\ \lim_{s\to\infty} F(s) = 0, \quad (ii)\ \lim_{s\to\infty} sF(s) < \infty.$$

Proof. $|f(t)| \le Me^{s_0 t}$ for $t \ge t_0$.
Further we may assume $|f(t)| \le K$ for $t \in [0, t_0]$.

$$|F(s)| = \left| \int_0^\infty f(t)e^{-st}\, dt \right| \le \int_0^\infty |f(t)|e^{-st}\, dt$$

$$\le \int_0^{t_0} Ke^{-st}\, dt + \int_{t_0}^\infty Me^{-(s-s_0)t}\, dt$$

$$= K\frac{1-e^{-st_0}}{s} + \frac{M}{s-s_0}e^{-(s-s_0)t_0}, \quad \text{for all } s > s_0$$

$$\implies \lim_{s\to\infty} F(s) = 0, \text{ and } \lim_{s\to\infty} sF(s) = K + M < \infty$$

Question. Does there exist a function $f(t)$ which is piecewise continuous and of exponential order, such that $L(f(t)) = 1$?
No. Since then $\lim_{s\to\infty} F(s) = 0$.

May be there exist some function $f(t)$ which is either not piecewise continuous or not of exponential order, and $L(f(t)) = 1$.
Yes. Dirac delta function or impluse function has this property.

Theorem

Assume f and f' both are piecewise continuous and of exponential order. Then

$$\lim_{s\to\infty} sF(s) = f(0).$$

Proof.

$$L(f'(t)) = sL(f(t)) - f(0)$$

Since f and f' both are piecewise continuous and of exponential order, we get

$$\lim_{s\to\infty} L(f'(t)) = 0, \text{ and } \lim_{s\to\infty} sF(s) < \infty$$

Therefore,

$$\lim_{s\to\infty} sF(s) = f(0)$$

Example

Let $f(t) = L^{-1}\left(\dfrac{1 - s(5+3s)}{s((s+1)^2+1)}\right)$. Find $f(0)$.

We can find $f(t)$ by partial fraction. Hence we know that f and f' are continuous and of exponential order. Therefore,

$$f(0) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} \frac{1 - s(5+3s)}{((s+1)^2+1)}$$

$$= \lim_{s\to\infty} \frac{1 - 5s - 3s^2}{s^2+2s+2} = -3$$

Theorem

If f is piecewise continuous and periodic of period T, then

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(T) e^{-st}\, dt, \;\; s > 0$$

$$L(f(t)) = \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \dots$$

$$= \int_0^T f(t)e^{-st}\,dt + \int_0^T f(t+T)e^{-s(t+T)}\,dt + \dots$$

$$= \int_0^T f(t)e^{-st}\,dt \,\left(1 + e^{-sT} + e^{-2sT} + \dots\right)$$

$$= \frac{1}{(1-e^{-sT})} \int_0^T f(t)e^{-st}\,dt\,, \;\; s > 0$$

Example

Find the Laplace transform of periodic function

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}, \quad f(t+2) = f(t)$$

$$L(f(t)) = \frac{1}{(1-e^{-2s})} \int_0^2 f(t) e^{-st}\, dt$$

$$= \frac{1}{(1-e^{-2s})} \int_0^1 t e^{-st}\, dt$$

$$= \frac{1}{(1-e^{-2s})} \left[t \frac{e^{-st}}{-s} \Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\, dt \right]$$

$$= \frac{1}{(1-e^{-2s})} \left[\frac{e^{-s}}{-s} - \frac{1}{s^2}(e^{-s} - 1) \right]$$

Additional Properties of Laplace Transform

Assume $L(f(t)) = F(s)$ is defined for $s > s_0$, then

- First shifting theorem (s-shift)
  $L\left(e^{-at} f(t)\right) = F(s+a), \quad s > s_0 + a$.
- Second shifting theorem (t-shift)
  $L(u(t-a)f(t-a)) = e^{-as}F(s), \quad s > s_0, a > 0$.
- multiplication by $1/s$
  $L\left(\int_0^t f(\tau)\, d\tau\right) = \dfrac{F(s)}{s}, \quad s > \max\{0, s_0\}$.
- differentiation w.r.t. s
  $L(tf(t)) = \; - F'(s), \quad s > s_0$.
- integration w.r.t. s
  $L\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(s')ds', \quad s > s_0$.

- f : piecewise continuous and of exponential order. Then
  - $\lim_{s\to\infty} F(s) = 0$,
  - $\lim_{s\to\infty} sF(s)$ is bounded.
- f, f' : piecewise continuous and of exponential order. Then
$$\lim_{s\to\infty} sF(s) = f(0)$$
- If f is piecewise continuous and periodic of period T, then
$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(T)e^{-st}dt, \quad s > 0$$

Exercise. Find Inverse Laplace transform of following functions $F(s)$ and varify, whether $\lim_{s\to\infty} sF(s) = f(0)$. If not, then state why it is not.

Example

$F(s) = \dfrac{s^2}{(s+1)^3}$. Find partial fractions and compute L^{-1}.

$$\frac{s^2}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

Example

$F(s) = \dfrac{s}{(s+4)^6}$. Find partial fractions and compute L^{-1}.

Example

$F(s) = \dfrac{s}{(s^2+a^2)^2}$. We can use convolution theorem for

$$F(s) = H(s)G(s), \quad H(s) = \frac{1}{s^2+a^2}, \quad G(s) = \frac{s}{s^2+a^2}$$

We can also solve it using

$$L^{-1}(H'(s)) = -tL^{-1}(H(s))$$

Example

$F(s) = \dfrac{s}{(s^2-a^2)^2}$. We can use convolution theorem for

$$F(s) = H(s)G(s), \quad H(s) = \frac{1}{s^2-a^2}, \quad G(s) = \frac{s}{s^2-a^2}$$

Example

$F(s) = \dfrac{e^{-s}}{s^5}$. Use 2nd shifting theorem.

Example

$F(s) = \dfrac{e^{-2s}}{(s+1)^2}$. Use 2nd shifting theorem.

Example

$F(s) = \dfrac{1}{\sqrt{s+1}}$.

$$L\left(\frac{1}{\sqrt{t}}\right) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-st}\,dt$$

$$= \int_0^\infty \frac{1}{x} e^{-sx^2} 2x\,dx, \quad (t = x^2)$$

$$= 2\int_0^\infty e^{-sx^2}\,dx = 2\int_0^\infty \frac{1}{\sqrt{s}} e^{-u^2}\,du, \quad (u = \sqrt{s}\,x)$$

$$= \frac{2}{\sqrt{s}} \int_0^\infty e^{-u^2}\,du = \frac{2}{\sqrt{s}}\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\frac{\pi}{s}}$$

$$L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi t}}, \quad L^{-1}\left(\frac{1}{\sqrt{s+1}}\right) = \frac{e^{-t}}{\sqrt{\pi t}}$$

Example

$F(s) = \dfrac{s}{(s-a)^{3/2}}$, $a > 0$.

$$F(s) = \frac{s-a+a}{(s-a)^{3/2}} = \frac{1}{\sqrt{s-a}} + \frac{a}{(s-a)^{3/2}}$$

We know

$$L^{-1}\left(\frac{1}{\sqrt{s+a}}\right) = \frac{e^{-at}}{\sqrt{\pi t}}$$

We can use convolution theorem for second part.
We can also use

$$L(tg(t)) = -G'(s), \quad G(s) = \frac{1}{(s-a)^{1/2}}$$

Example

$F(s) = \frac{1}{s(1-e^{-s})}$

$$F(s) = \frac{1}{s}(1 + e^{-s} + e^{-2s} + \ldots)$$

Apply 2nd shifting theorem.

Example

$F(s) = \frac{1}{s(1+e^{-s})}$

$$F(s) = \frac{1}{s}(1 - e^{-s} + e^{-2s} - \ldots)$$

Apply 2nd shifting theorem.

Example

$$F(s) = \frac{1}{(s+1)(1-e^{-2s})} = \frac{1}{s+1}(1 + e^{-2s} + e^{-4s} + \dots)$$

Apply 2nd shifting theorem.

Example

$$F(s) = \left(\frac{1}{s}\tanh s\right)$$

$$F(s) = \frac{1}{s}\left(\frac{e^{s} - e^{-s}}{e^{s} + e^{-s}}\right) = \frac{1}{s}\left(\frac{1 - e^{-2s}}{1 + e^{-2s}}\right) = \frac{1}{s}\left(1 - 2\frac{e^{-2s}}{1 + e^{-2s}}\right)$$

$$= \frac{1}{s}(1 - 2e^{-2s}(1 - e^{-2s} + e^{-4s} - \dots))$$

Apply 2nd shifting theorem.

Example

$$F(s) = \ln\left(\frac{s^2+1}{s^2+s}\right), \quad \ln\left(1+\frac{a^2}{s^2}\right), \quad \ln\left(1-\frac{a^2}{s^2}\right)$$

Compute $F'(s) = G(s)$ and use the formula

$$L\left(\frac{g(t)}{t}\right) = \int_s^\infty G(s)\,ds$$

Check that

$$\lim_{t\to 0}\frac{g(t)}{t}$$

exists.

Example

$$F(s) = \frac{1}{(s^2+1)^{1/2}} = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2} = \frac{1}{s}\left(\sum_{n\geq 0}\binom{-1/2}{n}\frac{1}{s^{2n}}\right)$$

$$= \sum_{n\geq 0}\frac{(-1/2)(-1/2-1)\dots(-1/2-n+1)}{n!}\frac{1}{s^{2n+1}}$$

$$= \sum_{n\geq 0}(-1)^n\frac{1.3.\dots.(2n-1)}{2^n n!}\frac{1}{s^{2n+1}}$$

$$L^{-1}(F(s)) = \sum_{n\geq 0}(-1)^n\frac{(2n)!}{(2^n n!)^2}L^{-1}\left(\frac{1}{s^{2n+1}}\right)$$

$$= \sum_{n\geq 0}(-1)^n\frac{1}{(2^n n!)^2}t^{2n} = \sum_{n\geq 0}\frac{(-1)^n}{(n!)^2}(t/2)^{2n}$$

Radius of convergence of this series is ∞. So the function is defined for all t.

Example

$$F(s) = \frac{1}{(s^2+1)^{3/2}} = \frac{1}{s^3}\left(1+\frac{1}{s^2}\right)^{-3/2} = \frac{1}{s^3}\left(\sum_{n\geq 0}\binom{-3/2}{n}\frac{1}{s^{3n}}\right)$$

$$= \sum_{n\geq 0}\frac{(-3/2)(-3/2-1)\dots(-3/2-n+1)}{n!}\frac{1}{s^{3n+3}}$$

$$= \sum_{n\geq 0}(-1)^n\frac{3.5\dots(2n+1)}{2^n n!}\frac{1}{s^{3n+3}}$$

$$L^{-1}(F(s)) = \sum_{n\geq 0}(-1)^n\frac{(2n+1)!}{(2^n n!)^2}L^{-1}\left(\frac{1}{s^{3n+3}}\right)$$

$$= \sum_{n\geq 0}(-1)^n\frac{(2n+1)!}{(2^n n!)^2(3n+2)!}t^{3n+2}$$

Radius of convergence of this series is ∞. So the function is defined for all t.