MA-108 Differential Equations I

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S1 - Lecture 1

Class Information

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- Reference Text : Elementary Differential Equations by William Trench available at ramanujan.math.trinity.edu/wtrench/texts/index.shtml
- Two short quiz of 5 marks each on 21st March and 18th April in the tutorial classes during 3:00-3:10 PM.
- Main quiz of 30 marks on 4th April from 8:15-9:15 AM.
- End Semester exam of 60 marks.
- Minimum passing marks is 30.
- Be Honest. Cheating in exams will give you <u>atleast</u> an FR grade in the course.

Definition

Let $y = y(x)$ be an unknown function of x.
An Ordinary differential equation (ODE) is an equation involving atleast one derivative of y.
The order of an ODE is the highest order of derivative of y occuring in the ODE.

Example

(1) $y' = x^2y^2 + x$ is a 1st order ODE.
(2) $y'' + 2xy' + y = \sin x$ is a 2nd order (linear) ODE.

Definition

An ODE of order n is called linear if it can be written as

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x),$$

If $a < b$ are real numbers, then

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

is an open interval.
$\mathbb{R} = (-\infty, \infty)$ is also an open interval.
$\mathbb{R} - \{0\}$ is not an open interval. It is union of two open intervals.

$$\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$$

Definition

An (explicit) solution of an ODE is a function $y = f(x)$ which satisfies the ODE on some open interval.

First simple example of an ODE

Consider the linear (homogeneous) ODE $y' + ay = 0$, $a \in \mathbb{R}$.
Note that $y \equiv 0$ is the (trivial) solution.
Let $y = y(x)$ be a non-trivial solution, i.e. $y(x) \neq 0$.
Since y is a continuous function, there exists an open interval, say I in $\mathbb{R}$ such that y does not take 0 value on I.
Let us solve the ODE on I.

$$
\begin{aligned}
y' + ay = 0 \implies & \frac{y'}{y} = -a \\
\implies & \frac{d}{dx} \ln |y| = -a \\
\implies & \ln |y| = -ax + c \\
\implies & |y| = e^c e^{-ax} \\
\implies & y(x) = Ce^{-ax},
\end{aligned}
$$

is a solution of $y' + ay = 0$ on $I = (-\infty, \infty)$, where $C = e^c$ when $y(x) > 0$ and $C = -e^c$ when $y(x) < 0$ on I.

1st order linear homogeneous ODE

Consider the ODE with $a(x)$ continuous on an open interval I,

$$y' + a(x)y = 0 \qquad (1)$$

Let $y = y(x)$ be a non-trivial solution, i.e. $y(x) \neq 0$.
Since y is a continuous function, there exists an open interval, say $J \subset I$ such that $y(x)$ does not take 0 value on J.

$$\begin{aligned} y' + a(x)y = 0 &\implies y'/y = -a(x) \\ &\implies \ln|y| = -\int a(x)\,dx + c \\ &\implies |y| = e^c e^{-\int a(x)\,dx} \\ &\implies y(x) = Ce^{-\int a(x)\,dx} \end{aligned}$$

$C = e^c$ when $y(x) > 0$ and $C = -e^c$ when $y(x) < 0$ on J.
Thus, $y(x) = Ce^{-\int a(x)\,dx}$ is a solution of (1) on $J = I$.

Theorem

Let $p(x)$ be a continuous function on an open interval (a, b). Then the general solution of

$$y' + p(x)y = 0 \qquad (1)$$

on the interval (a, b) is $\boxed{y(x) = Ce^{-P(x)}}$,
where $P(x)$ is any anti-derivative of $p(x)$ on (a, b), i.e.

$$P'(x) = p(x), \quad x \in (a, b)$$

- *General solution means $y(x) = Ce^{-P(x)}$ is a solution of (1) for all choices of $C \in \mathbb{R}$.*
- *Further, any solution of (1) can be obtained from the general solution for some choice of C.*
- *This may not be true for non-linear ODEs.*

Second simple example of an ODE

Consider the linear (non-homogeneous) ODE

$$y' + ay = f(x) \qquad (1)$$

where $f(x)$ is continuous on some open interval I.
The solution of $y' + ay = 0$ is $y_1(x) = e^{-ax}$ on $\mathbb{R}$.
Let us try to look for a solution of (1) of the type $y = u(x)e^{-ax}$.

Substituting into the differential equation (1), we get on I

$$\begin{aligned} & u'e^{-ax} - aue^{-ax} + aue^{-ax} = f(x) \\ \implies & u' = f(x)e^{ax} \\ \implies & u(x) = \int f(x)e^{ax}\,dx + C \end{aligned}$$

Thus

$$y(x) = e^{-ax}\Big(\int f(x)e^{ax}\,dx + C\Big)$$

is a solution of (1) on the (open) interval I.

1st order Linear non-homogeneous ODE

Let $p(x)$ and $f(x)$ be continuous on (a, b). Let us solve

$$y' + p(x)y = f(x) \qquad (1)$$

$y' + p(x)y = 0$ is the Complementary equation of (1).
Let $y_1(x) = e^{-\int p(x)\,dx}$ be a solution of C.E.
Substitute $y(x) = u(x)y_1$ into ODE, we get

$$\begin{aligned}
& u'y_1 + uy_1' + p(x)uy_1 = f(x) \\
\implies & u'y_1 = f(x) \\
\implies & u(x) = \int f(x)e^{\int p(x)dx} + C
\end{aligned}$$

$$\implies y(x) = e^{-\int p(x)dx}\left(\int f(x)e^{\int p(x)dx} + C\right)$$

is the general solution of (1) on (a, b).

Theorem (Existence Theorem)

Let $p(x)$ and $f(x)$ be continuous functions on an open interval (a,b). Then the general solution of

$$y' + p(x)y = f(x) \qquad (1)$$

on the interval (a,b) is

$$y(x) = e^{-\int p(x)} \left(\int f(x) e^{\int p(x)dx}\, dx + C \right) \qquad (2)$$

- *General solution means $y(x)$ in (2) is a solution of (1) for all choices of $C \in \mathbb{R}$.*
- *Further, any solution of (1) can be obtained from the general solution for some choice of C.*
- *This may not be true for non-linear ODEs.*

Example

Solve $\quad y' + 2y = x^3 e^{-2x}. \qquad$ (1)
C.E. $\quad y' + 2y = 0$ has a solution $y_1(x) = e^{-2x}$.
The solution of (1) is $y = uy_1$

$$\begin{aligned} u' y_1 &= x^3 e^{-2x} \\ \Longrightarrow u' &= x^3 \\ \Longrightarrow u(x) &= x^4/4 + C \end{aligned}$$

Therefore,

$$y(x) = e^{-2x}(x^4/4 + C)$$

is a solution of ODE on $\mathbb{R}$.

Example

(1) Solve $y' - 2xy = 1$.
C.E. $y' - 2xy = 0$ has a solution $y_1(x) = e^{\int 2x\,dx} = e^{x^2}$.
The solution of ODE is $y = uy_1$, where

$$\begin{aligned} u'y_1 &= 1 \\ \implies u(x) &= \int e^{-x^2} dx + C \\ \implies y(x) &= e^{x^2}\left(\int e^{-x^2} dx + C\right) \end{aligned}$$

(2) Solve the IVP $\boxed{y' - 2xy = 1,\ y(0) = y_0}$.
Write the solution of ODE as

$$\begin{aligned} y(x) &= e^{x^2}\left(\int_0^x e^{-x^2} dx + C\right) \\ y(0) = y_0 &\implies C = y_0 \end{aligned}$$

Definition

An Initial value problem (IVP) for 1st order ODE is

$$y' = F(x, y), \quad y(x_0) = y_0.$$

A function $y = y(x)$ defined on some open interval (a, b) containing x_0 is a solution of the IVP if y satisfies the ODE on (a, b) and $y(x_0) = y_0$.

Theorem (Existence and Uniqueness Theorem for IVP)

Let $p(x)$ and $f(x)$ be continuous functions on an interval (a, b). Let $x_0 \in (a, b)$ and $y_0 \in \mathbb{R}$. Then the IVP

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a unique solution on (a, b).

Definition

Let $y(x)$ be an explicit solution of IVP

$$y' = F(x, y), \quad y(x_0) = y_0$$

on some open interval containing x_0.
The interval of validity of $y(x)$ is the largest open interval containing x_0 where $y(x)$ is a solution of IVP.

Example

The function

$$y = (x^2/3) + (1/x)$$

satisfies

$$xy' + y = x^2$$

on $(-\infty, 0) \cup (0, \infty)$.

- For IVP

  $$xy' + y = x^2, \quad y(1) = 4/3$$

  the interval of validity of $y(x)$ is $(0, \infty)$.

- For IVP

  $$xy' + y = x^2, \quad y(-1) = -2/3$$

  the interval of validity of $y(x)$ is $(-\infty, 0)$.

Definition

- An explicit solution of an ODE is a function $y = y(x)$ which satisfies the ODE on some open interval (a, b).
- A <u>solution curve</u> of an ODE is the graph of an explicit solution of the ODE.
- An <u>implicit solution</u> of an ODE is an equation $g(x, y) = 0$ that gives an explicit solution of the ODE on some open interval.
- A curve C is an <u>integral curve</u> of an ODE if the following holds: If the graph of a function $y = f(x)$ is a portion of the curve C, then $y = f(x)$ is a solution of the ODE.
- An integral curve C of an ODE is the curve defined by an implicit solution of the ODE.

Note that a solution curve is also an integral curve, but an integral curve may not be a solution curve, since an integral curve C may not be the graph of a single function.

Example

Circle C defined by $x^2 + y^2 = 1$ is an integral curve of

$$y' = -x/y$$

Only functions whose graph is a segment of C are

$$y_1 = \sqrt{1 - x^2}, \quad y_2 = -\sqrt{1 - x^2}$$

on $(-1, 1)$.
So graphs of y_1 and y_2 are solution curves.
But C is not a solution curve as C is not the graph of a function.

Separation of variable method : 1st order ODE

Assume that the ODE can be written in the form

$$h(y)y' = g(x)$$

Let $H(y)$ and $G(x)$ be antiderivatives of $h(y)$ and $g(x)$ respectively. Then

$$\frac{d}{dy}H(y) = H'(y) = h(y), \quad G'(x) = g(x)$$

Then our ODE is

$$\frac{d}{dx}H(y) = H'(y)y' = \frac{d}{dx}G(x)$$

Integrating, we get

$$H(y) = G(x) + C$$

This is an implicit solution of ODE.

Separable ODE's

Example

Solve $\qquad y' = 2xy^2$.

Assume $y \neq 0$. Rewrite ODE as

$$\frac{1}{y^2} y' = 2x$$

Integrating, we get

$$\begin{aligned} \frac{-1}{y} &= x^2 + C \\ \implies y &= \frac{-1}{x^2 + C} \end{aligned}$$

The solution $y \equiv 0$ cannot be obtained for any choice of C.

Example

Solve IVP

$$y' = 2xy^2, \quad y(0) = y_0$$

and find the interval of validity.

The solution is

$$y = \frac{-1}{x^2 + C}$$

- If $y_0 = 0$, the solution is $y \equiv 0$ and the interval of validity is $\mathbb{R}$.
- If $y_0 \neq 0$, then $C = -\dfrac{1}{y_0}$. Hence $y = \dfrac{-y_0}{y_0 x^2 - 1}$.
- If $y_0 < 0$, the solution is defined for all x. Hence the interval of validity is $\mathbb{R}$.
- If $y_0 > 0$, the solution is valid when $x \in \mathbb{R} - \{\pm 1/\sqrt{y_0}\}$. Hence the interval of validity is $\left(\dfrac{-1}{\sqrt{y_0}}, \dfrac{1}{\sqrt{y_0}}\right)$.

Example

Solve IVP

$$y' = \frac{y\cos x}{1+2y^2}; \quad y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\begin{array}{rcl} \frac{1+2y^2}{y} y' & = & \cos x \\ \ln|y| + y^2 & = & \sin x + c \\ y(0) = 1 & \Longrightarrow & c = 1 \\ \ln|y| + y^2 & = & \sin x + 1 \end{array}$$

is an implicit solution of IVP.
Note: $y \equiv 0$ is a solution to the ODE, but it is not a solution to the given IVP.

Linear vs Non-Linear ODE

Theorem (Existence and Uniqueness of solution : $y' = f(x,y)$)

Let $D = (a,b) \times (c,d)$ *be an open rectangle containing the point* (x_0, y_0) *and consider the IVP*

$$y' = f(x,y), \quad y(x_0) = y_0$$

- *(Existence) Assume* $f(x,y)$ *is continuous on* D*. Then IVP has at least one solution on some interval* $(a_1, b_1) \subset (a,b)$ *containing* x_0*.*
- *(Uniqueness) If both* $f(x,y)$ *and* $\dfrac{\partial f}{\partial y}$ *are continuous on* D*, then IVP has a unique solution on some interval* $(a', b') \subset (a,b)$ *containing* x_0*.*

Linear vs Non-Linear ODE

- For the solution of a non-linear ODE, the interval where the solution exists, depends on the choice of our initial condition.
- The general solution of a non-linear ODE involving an arbitrary constant, may not give all solutions.
- For example, for non-linear ODE $y' = 2xy^2$, our solution $y = -1/(x^2 + C)$ does not give the solution $y \equiv 0$ for any value of C.
- In an implicit solution of a non-linear ODE, not every value of C will give an actual solution.

Example

The circle $x^2 + y^2 = C$ is an implicit solution of $yy' = -x$. For $C = -1$, it does not give any solution to ODE, since the curve $x^2 + y^2 = -1$ is empty.

Example

Consider the IVP

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0 \quad (*)$$

$$\begin{aligned} f(x,y) &= \frac{x^2 - y^2}{1 + x^2 + y^2}, \\ \frac{\partial f}{\partial y} &= \frac{-2y}{1 + x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(1 + x^2 + y^2)^2} \\ &= \frac{-2y(1 + 2x^2)}{(1 + x^2 + y^2)^2} \end{aligned}$$

Since $f(x,y)$ and $\partial f/\partial y$ are continuous for all $(x,y) \in \mathbb{R}^2$,

by existence and uniqueness theorem, for any $(x_0, y_0) \in \mathbb{R}^2$, IVP has a unique solution on some open interval containing x_0.

Example

Consider the IVP $\quad y' = f(x,y), \quad y(x_0) = y_0 \quad (*)$

$$
\begin{aligned}
f(x,y) &= \frac{x^2 - y^2}{x^2 + y^2} \\
\frac{\partial f}{\partial y} &= \frac{-2y}{x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(x^2 + y^2)^2} \\
&= \frac{-4x^2 y}{(x^2 + y^2)^2}
\end{aligned}
$$

f and $\partial f / \partial y$ are continuous for all $(x,y) \in \mathbb{R}^2 \setminus (0,0)$.

Assume $(x_0, y_0) \neq (0,0)$.

- There is an open rectangle R containing (x_0, y_0) but not containing $(0,0)$.
- $f(x,y)$ and $\partial f / \partial y$ are continuous on R.
- By existence and uniqueness theorem, $(*)$ has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{x+y}{x-y}, \quad y(x_0) = y_0 \qquad (*)$$

If

$$f(x,y) = \frac{x+y}{x-y}, \quad \text{then} \quad \frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}$$

Here $f(x,y)$ and $\partial f/\partial y$ are continuous everywhere except on the line $y = x$.

Assume $x_0 \neq y_0$.

- There is an open rectangle R containing (x_0, y_0) that does not intersect with the line $y = x$.
- $f(x,y)$ and $\partial f/\partial y$ are continuous on R.
- By existence and uniqueness theorem, $(*)$ has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{10}{3}\,xy^{2/5}, \quad y(x_0) = y_0 \quad (*)$$

$$f(x,y) = \frac{10}{3}\,xy^{2/5} \;\text{ and }\; \frac{\partial f}{\partial y} = \frac{4}{3}\,xy^{-3/5}$$

- Since $f(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$, IVP $(*)$ has atleast one solution for all $(x_0, y_0) \in \mathbb{R}^2$.
- If $y \neq 0$, then $f(x,y)$ and $\partial f/\partial y$ both are continuous for all $(x,y) \in \mathbb{R}^2$.
- If $y_0 \neq 0$, there is an open rectangle R containing (x_0, y_0) s.t. f and $\partial f/\partial y$ are continuous on R.
  Hence IVP $(*)$ has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{10}{3}\, x y^{2/5}, \quad y(0) = 0 \qquad (*)$$

Since $\dfrac{\partial f}{\partial y} = \dfrac{4}{3}\, x y^{-3/5}$ is not continuous if $y = 0$,
(*) may have more than one solution on every open interval containing $x_0 = 0$.
$y \equiv 0$ is one solution of IVP $(*)$.

Let y be a non-zero solution of ODE.

$$\begin{aligned} \frac{y'}{y^{2/5}} &= (10/3)\, x \\ (5/3)\, y^{3/5} &= (5/3)\,(x^2 + C) \\ y(x) &= (x^2 + C)^{5/3} \end{aligned}$$

Example (continued ...)

Note that $y(x) = (x^2 + C)^{5/3}$ is defined for all (x, y) and

$$y' = \frac{5}{3}\,(x^2 + C)^{2/3}(2x) = \frac{10}{3}\,xy^{2/5}, \qquad \forall\, x \in (-\infty, \infty)$$

Thus $y(x)$ is a solution on $\mathbb{R}$ for all C.

$$y(0) = 0 \implies C = 0$$

Thus, the IVP

$$y' = \frac{10}{3}y^{2/5}, \quad y(0) = 0 \qquad\qquad (*)$$

has two solutions, $y_1 \equiv 0$ and $y_2(x) = x^{10/3}$.

We can construct two more solutions of IVP (*). How?

Example

Consider the IVP

$$y' = \frac{10}{3}\, xy^{2/5}, \quad y(0) = -1 \qquad (*)$$

$$f(x,y) = \frac{10}{3}\, xy^{2/5}, \quad \frac{\partial f}{\partial y} = \frac{4}{3} xy^{-3/5}$$

are continuous in an open rectangle containing $(0,-1)$. Hence the IVP has a unique solution on some open interval containing $x_0 = 0$.

Question. Find the unique solution and its interval of validity.

Let $y \neq 0$ be the solution of $y' = (10/3)\, xy^{2/5}$. Then

$$y(x) = (x^2 + C)^{5/3}$$

$$y(0) = -1 \implies C = -1$$

$$\implies y(x) = (x^2 - 1)^{5/3}$$

Example (continued ...)

- $y(x) = (x^2 - 1)^{5/3}$ is a solution on $(-\infty, \infty)$ of IVP

$$y' = (10/3)\, xy^{2/5}, \quad y(0) = -1$$

  Hence interval of validity of this solution is $\mathbb{R}$.

- We have seen that if $y_0 \neq 0$, then the IVP

$$y' = (10/3)\, xy^{2/5}, \quad y(x_0) = y_0$$

  has a unique solution on some open interval around x_0.

- $y(x) = (x^2 - 1)^{5/3}$ is non-zero on $(-1, 1)$. Therefore, $y(x)$ is the unique solution on $(-1, 1)$.

  To see this, If $w(x)$ is another solution on $(-1, 1)$. Then $w(x) \equiv y(x)$ on some interval (ϵ', ϵ) containing 0.
  We need to show that $\epsilon = 1$ and $\epsilon' = -1$.

Example (continued ...)

- If $\epsilon \neq 1$, then $w(\epsilon) = y(\epsilon) = c \neq 0$ as w and y are continuous. Hence there exists a unique solution of ODE with IV $y(\epsilon) = c \neq 0$. Hence $w \equiv y$ on an open interval around ϵ. Thus $\epsilon = 1$. Similarly, $\epsilon' = -1$.
- $(-1, 1)$ is the largest interval on which the ODE with IV $y(0) = -1$ has a **unique** solution.
  To see this, we can define another solution

$$y_1(x) = \begin{cases} (x^2 - 1)^{5/3} & , \quad -1 \leq x \leq 1 \\ 0 & , \quad |x| > 1 \end{cases}$$

Exercise. Find largest interval where the IVP

$$y' = \frac{10}{3}\, xy^{2/5}, \quad y(0) = 1$$

has a unique solution.

Transforming Non-Linear into Separable ODE

A non-linear differential equation

$$y' + p(x)y = f(x)y^r$$

where $r \in \mathbb{R} - \{0, 1\}$ is said to be a **Bernoulli Equation**.
For $r = 0, 1$, it is linear.
If $y_1 = e^{-\int p(x)\,dx}$ is a non-zero solution of $y' + p(x)y = 0$, then putting $y = u(x)y_1$ in ODE, we get

$$\begin{aligned}
& u'y_1 + uy_1' + puy_1 = fu^r y_1^r \\
\implies & u'y_1 = fu^r y_1^r \\
\implies & \frac{u'}{u^r} = f(x)(y_1(x))^{r-1} \\
\implies & \frac{u^{-r+1}}{-r+1} = \int f(x)(y_1(x))^{r-1}\,dx + C
\end{aligned}$$

Example (Bernoulli Equation)

Consider

$$y' + y = xy^2$$

Set $y = u(x)e^{-x}$, where $y_1 = e^{-x}$ is solution of homogeneous part.

$$\begin{aligned}
&u'e^{-x} - ue^{-x} + ue^{-x} = u^2e^{-2x}x \\
\implies &u'e^{-x} = u^2e^{-2x}x \\
\implies &\frac{u'}{u^2} = xe^{-x} \\
\implies &\frac{-1}{u} = -(1+x)e^{-x} + C \\
\implies &u = \frac{1}{(1+x)e^{-x} - C} \\
\implies &y = \frac{e^{-x}}{(1+x)e^{-x} - C} = \frac{1}{1 + x - Ce^{x}}
\end{aligned}$$

Example

Consider Bernoulli equation

$$xy' - 2y = \frac{x^2}{y^6} \implies y' - \frac{2}{x}y = \frac{x}{y^6}$$

The solution to homogeneous part is $y_1 = x^2$.
Set $y = u(x)y_1$,

$$\begin{aligned} u'y_1 &= x(uy_1)^{-6} \\ u^6u' &= x(x^2)^5 = x^{11} \\ \frac{1}{7}u^7 &= \frac{1}{12}x^{12} + C \\ (1/7)y^7 &= \left[(1/12)x^{12} + C\right] y_1^7 \\ y^7 &= \left[(7/12)x^{12} + 7C\right] x^{14} \end{aligned}$$

is an implicit solution.
We do not have an explicit solution.

Homogeneous Non-Linear Equations

Definition

An ODE

$$y' = f(x, y)$$

is said to be **homogeneous** if it can be written as

$$y' = q(y/x)$$

Substitute $y = v(x)x$ in homogeneous ODE, we get

$$v'x + v = q(v)$$

This is a separable ODE.

Example

Solve

$$xy' = y + x$$

Rewrite it as

$$y' = \frac{y}{x} + 1$$

This is homogeneous ODE.

Substitute $y = vx$. We get

$$\begin{aligned} & v'x + v = v + 1 \\ \implies & v'x = 1 \\ \implies & v' = 1/x \\ \implies & v(x) = \ln|x| + C \\ \implies & y = x(\ln|x| + C) \end{aligned}$$

Example

Solve $\quad x^2y' = y^2 + xy - x^2$.

$$\begin{aligned}
y' = \frac{y^2 + xy - x^2}{x^2} \quad &= \quad \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1 \\
\text{Substitute} \quad y = vx & \\
v'x + v \quad &= \quad v^2 + v - 1 \\
\frac{v'}{v^2 - 1} \quad &= \quad \frac{1}{x} \\
\frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right)v' \quad &= \quad \frac{1}{x} \\
\frac{1}{2}\left(\ln|v-1| - \ln|v+1|\right) \quad &= \quad \ln|x| + C_1 \\
\frac{v-1}{v+1} \quad &= \quad Cx^2 \\
v \quad &= \quad \frac{1 + Cx^2}{1 - Cx^2}
\end{aligned}$$

Example (Continued ...)

$$y = x\,\frac{1+Cx^2}{1-Cx^2} \qquad (*)$$

is a solution of

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

- **Question.** Are these all the solutions?
  Ans. No.
  Both $y = x$ and $y = -x$ are also solutions, but only $y = x$ can be obtained from the general solution.
- The solutions (*) were obtained in the intervals not containing 0.
- **Question.** Are only solutions to the ODE, in an interval containing zero, are $y = x$ and $y = -x$?

Example (Continued ...)

Note that

$$y = x\,\frac{1+Cx^2}{1-Cx^2}$$

is differentiable at $x = 0$ and satisfies the ODE

$$x^2y' = y^2 + xy - x^2 \qquad (**)$$

at 0 (since $y(0) = 0$).
In fact, for arbitrary $C_1, C_2 \in \mathbb{R}$, the function

$$y(x) = \begin{cases} x\,\dfrac{1+C_1x^2}{1-C_1x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1+C_2x^2}{1-C_2x^2} & \text{if } x \geq 0 \end{cases}$$

is differentiable and satisfies the ODE (**) with $y(0) = 0$.

Example (continued ...)

Thus the IVP $\quad x^2y' = y^2 + xy - x^2, \quad y(0) = 0$
has infinitely many solutions

$$y(x) = \begin{cases} x\,\dfrac{1 + C_1x^2}{1 - C_1x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1 + C_2x^2}{1 - C_2x^2} & \text{if } x \geq 0 \end{cases}$$

one for each choice of C_1, C_2.

The interval of validity I of $y(x)$ depends on C_1, C_2.

- If $C_1 \leq 0$ and $C_2 \leq 0$, then $I = \mathbb{R}$.
- If $C_1 \leq 0$ and $C_2 > 0$, then $I = (-\infty, 1/\sqrt{C_2})$.
- If $C_1 > 0$ and $C_2 \leq 0$, then $I = (-1/\sqrt{C_1}, \infty)$.
- If $C_1 > 0$ and $C_2 > 0$, then $I = (-\alpha, \alpha)$, where $\alpha = \min\{1/\sqrt{C_1}, 1/\sqrt{C_2}\}$.

Example

Solve the IVP

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2 \qquad (*)$$

If

$$f(x,y) = \frac{y^2 + xy - x^2}{x^2},$$

then $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous in an open rectangle containing the point $(1,2) \in \mathbb{R}^2$.

By Existence and Uniqueness theorem, IVP (*) has a unique solution on sone open interval around $x_0 = 1$.

If $y \neq 0$ in an open interval, then the general solution is

$$y(x) = \frac{1 + Cx^2}{1 - Cx^2}$$

Example (continued ...)

$$\begin{aligned} y(1) &= \frac{1+C}{1-C} = 2 \implies C = 1/3 \\ y(x) &= x\,\frac{3+x^2}{3-x^2} \qquad (**) \end{aligned}$$

is the unique solution on some $(a,b) \subset (-\sqrt{3}, \sqrt{3})$ containing $x_0 = 1$. The interval of validity of $y(x)$ is $(-\sqrt{3}, \sqrt{3})$.

Question. What is the largest interval on which this solution is unique?

Note for any $x_0 \in (0, \sqrt{3})$, $y(x)$ is the unique solution of IVP

$$x^2 y' = y^2 + xy - x^2, \quad y(x_0) = x_0 \frac{3+x_0^2}{3-x_0^2}$$

on some interval in $(0, \sqrt{3})$ containing x_0.

Example (continued ...)

Therefore, the largest interval I containing $x_0 = 1$ on which

$$y(x) = x\,\frac{3+x^2}{3-x^2}$$

is the unique solution, contains $(0, \sqrt{3})$.
If I contains 0, then we can define another solution

$$y_1(x) = \begin{cases} x\,\dfrac{1+Cx^2}{1-Cx^2} & \text{if } a < x < 0 \\[2ex] x\,\dfrac{3+x^2}{3-x^2} & \text{if } 0 \le x < \sqrt{3} \end{cases}$$

where, $a = \dfrac{-1}{\sqrt{C}}$ if $C > 0$ and $a = -\infty$ if $C < 0$.
Thus the largest open interval, in which IVP with $y(1) = 2$ has a *unique solution*, is $(0, \sqrt{3})$.

Examples

Describe the method to solve the following ODE.

- $$y' = \frac{x^2+3x+2}{y-2}, \quad y(1) = 4$$
  non-linear, Separable
- $$(x-2)(x-1)y' - (4x-3)y = (x-2)^3$$
  Linear non-homogeneous
- $$(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y}$$
  Bernoulli Equation
- $$y' = \frac{2x+y+1}{x+2y-4}$$
  Can be converted to a separable equation, use substitution $X = x+2$, $Y = y-3$.
- $$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0.$$
  Exact equation

Example (Exact Equation)

Solve $\quad 3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0$

Note $\quad 3x^2y^2 = \dfrac{\partial}{\partial x}(x^3y^2)$ and $2x^3y = \dfrac{\partial}{\partial y}(x^3y^2)$.

Let $\quad G(x, y) = x^3y^2$. Then

$$
\begin{aligned}
& 3x^2y^2 + 2x^3y\frac{dy}{dx} = 0 \\
\Longrightarrow\ & \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0 \\
\Longrightarrow\ & \frac{d}{dx}G(x, y(x)) = 0 \\
\Longrightarrow\ & G(x, y) = C
\end{aligned}
$$

is an implicit solution of ODE.

Definition

A first order ODE written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is said to be **exact** if there exists a function G such that

$$\frac{\partial G}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial G}{\partial y} = N(x, y).$$

If the ODE is exact, then

$$G(x, y) = C$$

is an implicit solution of ODE.

When is an ODE exact?

Theorem

Let D *be an open rectangle* $(a,b)\times(c,d)$. *Assume that*

$$M(x,y),\ N(x,y),\ \frac{\partial M}{\partial y},\ \frac{\partial N}{\partial x}$$

are continuous in D *and* $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ *on* D.

Then ODE

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

is exact on D, *i.e. there exists* $G:D\to\mathbb{R}$ *s.t.*

$$\frac{\partial G}{\partial x}=M,\quad \frac{\partial G}{\partial y}=N$$

So $G(x,y)=C$ *is an implicit solution of ODE.*

Exact Equations

Which of the following ODE's are exact?

1. $(2x+3)+(2y-2)y'=0$ Exact
2. $\dfrac{dy}{dx}=\dfrac{x+2y}{3x+4y}$. Not Exact
3. $(y/x+6x)dx+(\ln x-2)dy=0,\quad x,y>0.$ Exact
4. $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0.$ Not Exact

Example

Solve $\quad (2x+3)+(2y-2)y'=0.$

The ODE is exact, so we need to find $\phi(x,y)$ such that

$$\frac{\partial\phi}{\partial x}=2x+3 \text{ and } \frac{\partial\phi}{\partial y}=2y-2$$

Integrating first equation gives

$$\begin{aligned}\phi(x,y)&=x^2+3x+h(y)\\ \Longrightarrow \frac{\partial\phi}{\partial y}&=\frac{dh}{dy}=2y-2\\ \Longrightarrow h(y)&=y^2-2y+c\end{aligned}$$

Therefore, an implicit solution to ODE is

$$\phi(x,y)=x^2+3x+y^2-2y=C$$

Example

Solve $\quad (y/x + 6x)dx + (\ln x - 2)dy = 0, \quad x, y > 0.$

This is exact, so we need to find $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = \frac{y}{x} + 6x \text{ and } \frac{\partial \phi}{\partial y} = \ln x - 2$$

Integrating the first equation gives

$$\begin{aligned} \phi(x, y) &= y \ln x + 3x^2 + h(y) \\ \implies \frac{\partial \phi}{\partial y} &= \ln x + \frac{dh}{dy} = \ln x - 2 \\ \implies h(y) &= -2y + c \end{aligned}$$

Therefore, the solution is given by

$$\phi(x, y) = y \ln x + 3x^2 - 2y = C$$

Example (Method of integrating factor)

Solve

$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$$

$$M = 3x^2y + 2xy + y^3, \quad N = x^2 + y^2$$

$$\frac{\partial}{\partial y}M = 3x^2 + 2x + 3y^2, \quad \frac{\partial}{\partial x}N = 2x$$

Therefore, ODE is <u>not</u> exact.

Question. Can it be converted to an exact equation?

The idea is to multiply the ODE by a function $\mu(x, y)$ so that it becomes exact. There is no algorithm for choosing μ.

Assume

$$\mu(3x^2y + 2xy + y^3)dx + \mu(x^2 + y^2)dy = 0$$

is exact.

Example (continued ...)

Then exactness condition gives

$$\frac{\partial}{\partial y}(\mu(3x^2y + 2xy + y^3)) = \frac{\partial}{\partial x}(\mu(x^2 + y^2)) \implies$$

$$\mu(3x^2 + 2x + 3y^2) + \frac{\partial \mu}{\partial y}(3x^2y + 2xy + y^3) = 2x\mu + \frac{\partial \mu}{\partial x}\,(x^2 + y^2)$$

$$\mu(3x^2 + 3y^2) + \frac{\partial \mu}{\partial y}(3x^2y + 2xy + y^3) = \frac{\partial \mu}{\partial x}\,(x^2 + y^2)$$

From observation, we choose μ to be independent of y.

Then $\partial\mu/\partial y = 0$ and above equation becomes

$$3\mu(x^2 + y^2) = \frac{d\mu}{dx}(x^2 + y^2)$$

$$\implies \frac{d\mu}{dx} = 3\mu$$

$$\implies \mu = Ce^{3x}$$

Example (continued ...)

The ODE now becomes

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0 \quad (*)$$

Verify that this is exact. Hence there exists $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = e^{3x}(3x^2y + 2xy + y^3) \text{ and } \frac{\partial \phi}{\partial y} = e^{3x}(x^2 + y^2)$$

$$\implies \phi(x, y) = e^{3x}x^2y + \frac{1}{3}\,e^{3x}y^3 + h(y)$$

$$\implies \frac{\partial \phi}{\partial y} = e^{3x}x^2 + e^{3x}y^2 + \frac{dh}{dy} = e^{3x}(x^2 + y^2)$$

$$\implies \frac{dh}{dy} = 0 \implies h(y) = C$$

$\implies \phi(x, y) = e^{3x}(x^2y + \frac{1}{3}\,y^3) = C$: implicit solution of (*).

Question. Is $\phi(x, y) = e^{3x}(x^2 y + \frac{1}{3} y^3) = C$ the solution to our original ODE?
How will the solutions to the two ODE's be related?

$$\begin{aligned}
&\phi'(x, y) = 0 \\
\Longrightarrow\ & 3e^{3x}(x^2 y + \frac{1}{3} y^3) + e^{3x}(2xy + x^2 y' + y^2 y') = 0 \\
\Longrightarrow\ & (3x^2 y + y^3 + 2xy) + (x^2 + y^2) y' = 0 \qquad (*)
\end{aligned}$$

since e^{3x} is non-zero for all $x \in \mathbb{R}$.
Thus every $y(x)$ which is a solution to the new exact equation is a solution to the original equation $(*)$ and vice versa.

- In general, if μ is an integrating factor, then solutions to $\mu M + \mu N y' = 0$ may not be the solutions to $M + N y' = 0$.
- If $\mu(x, y(x))$ is non vanishing for all x in an open interval I, then the solution to exact ODE is a solution of original ODE on I.

Definition (Finding the integrating factor)

We say $\mu(x,y)$ is a integrating factor of ODE

$$M(x,y) + N(x,y)y' = 0$$

if

$$\mu M + \mu N y' = 0$$

is exact, i.e.

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

or

$$\mu \left(M_y - N_x\right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M$$

Finding the integrating factors

If the original equation

$$M(x,y) + N(x,y)y' = 0$$

was exact, then $\mu \equiv 1$ is an integrating factor.
In general, there is no clear way to determine μ.

- If we assume that $\mu = \mu(x)$ is independent of y, then

$$\begin{aligned}
\mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M \\
\Longrightarrow \mu\left(M_y - N_x\right) &= \frac{\partial \mu}{\partial x} N \\
\Longrightarrow \frac{1}{\mu}\frac{d\mu}{dx} &= \frac{M_y - N_x}{N} := p(x) \\
\Longrightarrow \mu &= e^{\int p(x)\ dx}
\end{aligned}$$

is an integrating factor if $\dfrac{M_y - N_x}{N}$ is a function of x only.

Finding the integrating factors

- If we assume that $\mu = \mu(y)$ is independent of x, then

$$
\begin{aligned}
&\mu\left(M_y - N_x\right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M \\
\Longrightarrow\ &\mu\left(M_y - N_x\right) = -\frac{\partial \mu}{\partial y} M \\
\Longrightarrow\ &\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{M_y - N_x}{M} := q(y) \\
\Longrightarrow\ &\mu = e^{\int q(y)\ dy}
\end{aligned}
$$

is an integrating factor if $\dfrac{M_y - N_x}{M}$ is a function of y only.

Finding the integrating factors

- If we assume that $\mu(x,y) = P(x)Q(y)$, then

$$\mu\left(M_y - N_x\right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M$$
$$\implies P(x)Q(y)\left(M_y - N_x\right) = P'(x)Q(y)N - P(x)Q'(y)M$$
$$\implies M_y - N_x = \frac{P'}{P}N - \frac{Q'}{Q}M$$
$$\frac{P'}{P} = p(x), \quad \frac{Q'}{Q} = q(y)$$
$$\implies P(x) = e^{\int p(x)\,dx}, \quad Q(y) = e^{\int q(y)\,dy}$$

Thus

$$\mu(x,y) = e^{\int p(x)\,dx}\; e^{\int q(y)\,dy}$$

is an integrating factor if

$$M_y - N_x = p(x)N - q(y)M$$

Theorem

Consider $\quad M(x,y) + N(x,y)y' = 0.$

Assume that M, N, M_y, N_x *are continuous on an open rectangle* $R = (a,b) \times (c,d)$.

Then $\mu = \mu(x,y)$ *is an integrating factor of ODE, where*

1. *If* $\dfrac{M_y - N_x}{N} = p(x)$ *on* R, *then*
$$\mu = \mu(x) = e^{\int p(x)\,dx}$$
2. *If* $-\dfrac{M_y - N_x}{M} = q(y)$ *on* R, *then*
$$\mu = \mu(y) = e^{\int q(y)\,dy}$$
3. *If* $M_y - N_x = p(x)N - q(y)M$ *on* R, *then*
$$\mu = e^{\int p(x)\ dx}\ e^{\int q(y)\ dy}$$

Example

Consider ODE

$$\cos x \cos y \; dx + (\sin x \cos y - \sin x \sin y + y) \; dy = 0.$$

$$\begin{aligned} M &= \cos x \cos y \\ N &= \sin x \cos y - \sin x \sin y + y \\ M_y - N_x &= -\cos x \sin y - \cos x \cos y + \cos x \sin y \\ &= -\cos x \cos y \\ \frac{N_x - M_y}{M} &= 1 \end{aligned}$$

The integrating factor is $\mu = e^y$ and so

$$e^y \cos x \cos y \; dx + e^y(\sin x \cos y - \sin x \sin y + y) \; dy = 0$$

is exact. So there exists $\phi(x, y)$ such that

Example (continued ...)

$$\frac{\partial \phi}{\partial x} = e^y \cos x \cos y, \quad \frac{\partial \phi}{\partial y} = e^y(\sin x \cos y - \sin x \sin y + y)$$

Integrating first equation, we get

$$\begin{aligned}
\phi(x, y) &= e^y \sin x \cos y + h(y) \\
\frac{d\phi}{\partial y} &= e^y \sin x \cos y - e^y \sin x \sin y + \frac{dh}{dy} \\
&= e^y(\sin x \cos y - \sin x \sin y + y) \\
\frac{dh}{dy} &= ye^y \\
h(y) &= e^y y + e^y + C \\
\phi(x, y) &= e^y(\sin x \cos y + y + 1) = C
\end{aligned}$$

is an implicit solution of ODE.

Example

Solve $(3x^2y^3 - y^2 + y)dx + (-xy + 2x)dy = 0$

$$
\begin{aligned}
M(x,y) &= 3x^2y^3 - y^2 + y \\
N(x,y) &= -xy + 2x \\
M_y - N_x &= 3x^2 3y^2 - 2y + 1 + y - 2 \\
&= 9x^2y^2 - y - 1 \\
\frac{-M_y + N_x}{M} &\neq q(y) \\
\frac{M_y - N_x}{N} &\neq p(x)
\end{aligned}
$$

Can we write

$$
\begin{aligned}
M_y - N_x &= p(x)N - q(y)M? \\
p(x) &= -2/x \\
q(y) &= -3/y
\end{aligned}
$$

Example (continued ...)

The integrating factor is then given by

$$\mu(x, y) = e^{\int -2/x \; dx} \; e^{\int -3/y \; dy} = \frac{1}{x^2 y^3}$$

We get an exact ODE

$$\begin{aligned} \frac{1}{x^2y^3}[(3x^2y^3 - y^2 + y)\; dx + (-xy + 2x)\; dy] &= 0 \\ \left(3 - \frac{1}{x^2y} + \frac{1}{x^2y^2}\right)\; dx + \left(\frac{-1}{xy^2} + \frac{2}{xy^3}\right)\; dy &= 0 \end{aligned}$$

Solve it.

Question. Is an integrating factor unique?
If μ is an integrating factor, then so is $c\mu$ for any constant $c \neq 0$.
What about upto constant multiple? No.

Example

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

is not exact.
Show that

$$\mu_1(x, y) = \frac{1}{xy(2x + y)}, \quad \mu_2(x) = x$$

both are integrating factors of ODE.

However one integrating factor may give a simpler ODE than the other.

Picard's Iteration Method

Picard's iteration method is useful in proving the existence and uniqueness theorem of the IVP

$$y' = f(x, y), \quad y(x_0) = y_0.$$

We will give an idea of the proof using this method.
Replacing x by $x - x_0$ and y by $y - y_0$, it is sufficient to assume that $x_0 = 0$ and $y_0 = 0$.
Suppose $y = \phi(x)$ is a solution to the IVP. Then

$$\frac{d\phi}{dx} = f(x, \phi(x)), \quad \phi(0) = 0.$$

Equivalently,

$$\phi(x) = \int_0^x f(s, \phi(s))ds, \quad \phi(0) = 0.$$

This is called an <u>integral equation</u> in the unknown function ϕ.

Conversely, if the integral equation

$$\phi(x) = \int_0^x f(s, \phi(s))ds$$

holds, then by the Fundamental Theorem of Calculus,

$$y' = \frac{d\phi}{dx} = f(x, \phi(x)) = f(x, y).$$

Thus, solving the integral equation is equivalent to solving the IVP.

We define, iteratively, a sequence of functions $\phi_n(x)$ for $n \geq 0$

$$\begin{aligned} \phi_0(x) &\equiv 0 \\ \phi_1(x) &= \int_0^x f(s, \phi_0(s))\, ds \\ &\vdots \\ \phi_{n+1}(x) &= \int_0^x f(s, \phi_n(s))\, ds \end{aligned}$$

- Each ϕ_n satisfies the initial condition $\phi_n(0) = 0$.
- None of the ϕ_n may satisfy $y' = f(x, y)$.
- Suppose for some n, $\phi_{n+1} = \phi_n$. Then,

$$\begin{aligned} \phi_{n+1} = \phi_n &= \int_0^x f(s, \phi_n(s))ds \\ \Longrightarrow \frac{d}{dx}(\phi_n(x)) &= f(x, \phi_n(x)) \end{aligned}$$

  Thus, $y = \phi_n(x)$ is a solution of the given IVP.
- In general case, $\phi_n \neq \phi_{n+1}$ for all n.
- It is possible to show that, if $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ is

  continuous in some open rectangle (hence continuous and bounded in a smaller closed rectangle), then the sequence converges to a function

$$\phi(t) = \lim_{n \to \infty} \phi_n(t)$$

  which will be the unique solution to the given IVP.

Example

Solve the IVP using Picard's iteration method.

$$y' = 2x(1+y);\ y(0) = 0.$$

The corresponding integral equation is

$$\phi(x) = \int_0^x 2s(1+\phi(s))ds.$$

$$\begin{aligned}
\phi_0(x) &= 0 \\
\phi_1(x) &= \int_0^x 2s\,ds = x^2, \\
\phi_2(x) &= \int_0^x 2s(1+s^2)\,ds = x^2 + \frac{x^4}{2}, \\
\phi_3(x) &= \int_0^x 2s(1+s^2+\frac{s^4}{2})\,ds = x^2 + \frac{x^4}{2} + \frac{x^6}{6}.
\end{aligned}$$

Example (continued ...)

We claim:

$$\phi_n(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots + \frac{x^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(x) &= \int_0^x 2s(1+\phi_n(s))ds \\
&= \int_0^x 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots + \frac{x^{2n}}{n!} + \frac{x^{2n+2}}{(n+1)!}.
\end{aligned}$$

Example (continued ...)

Hence $\phi_n(x)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}$.

Applying the ratio test, we get:

$$\left| \frac{x^{2k+2}}{(k+1)!} \cdot \frac{k!}{x^{2k}} \right| = \frac{x^2}{k+1} \to 0$$

for all x as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k!} = e^{x^2} - 1.$$

Therefore, $y(x) = e^{x^2} - 1$ is the solution of our IVP.

Uniqueness of solution

If $f(x,y)$ is continuous in an open rectangle R around $(0,0) \in \mathbb{R}^2$, then Picard's interation method shows that IVP

$$y' = f(x,y), \quad y(0) = 0$$

has atleast one solution.

Assume that f and $\dfrac{\partial f}{\partial y}$ both are continuous in R. Then we show that IVP has a unique solution in some open interval around 0.

Suppose ϕ and ψ are two solutions of IVP. Then both ϕ and ψ satisfy the integral equation.

$$\begin{aligned}
\phi(x) &= \int_0^x f(x, \phi(x))\, dx \\
\psi(x) &= \int_0^x f(x, \psi(x))\, dx
\end{aligned}$$

$$
\begin{aligned}
\implies \phi(x) - \psi(x) &= \int_0^t (f(x,\phi(x)) - f(x,\psi(x)))\,dx \\
\implies |\phi(x) - \psi(x)| &\leq \int_0^x |f(x,\phi(x) - f(x,\psi(x)|\,dx \\
&\leq \int_0^x K|\phi(x) - \psi(x)|\,dx
\end{aligned}
$$

for some constant K. This is because $\dfrac{\partial f}{\partial y}$ is continuous.

$$
\begin{aligned}
U(x) := \int_0^x |\phi(x) - \psi(x)|\,dx &\implies U'(x) = |\phi(x) - \psi(x)| \\
U'(x) - KU(x) \leq 0 &\implies [e^{-Kx}U(x)]' \leq 0 \\
\implies 0 \leq U(x) \leq 0 &\implies U(x) \equiv 0 \\
\implies U(x)' \equiv 0 &\implies \phi(x) \equiv \psi(x)
\end{aligned}
$$