Tutorial- 6 MA 106 (Linear Algebra)
Most of these problems are from reference texts for this course

1. What matrix represents $\dfrac{d^2}{dx^2}$ on $\mathcal{P}_3$ with respect to the basis $\mathcal{S} = \{1, x, x^2, x^3\}$? Find the nullspace and column space of the matrix. What do they mean in terms of polynomials? What is the matrix if the basis is $\mathcal{B} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$?

2. What 3×3 (standard) matrices represent the transformations that
   (a) project every vector onto the $x-y$ plane?
   (b) reflect every vector through the $x-y$ plane?
   (c) rotate the $x-y$ plane through $90°$, leaving the z-axis alone?
   (d) rotate the $x-y$ plane, then $x-z$, then $y-z$, through $90°$.?
   (e) carry out the same three rotations, but each one through $180°$.?

3. (a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?
   (b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?
   (c) What conditions on a, b, c and d will make the previous part impossible?

4. Prove or disprove:
   (a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det(B) = 2\det(A)$.
   (b) If $T(v)$ is known for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.
   (c) The determinant of an $n \times n$ matrix A is the product of its pivots.
   (d) If A is invertible and B is singular, then $A+B$ is invertible.
   (e) If A is invertible and B is singular, then AB is singular.
   (f) The determinant of $AB - BA$ is zero.
   (g) The eigenvectors of a 3×3 matrix A will give a basis for $\mathbb{R}^3$.

5. If every row of A adds to zero, prove that $\det(A) = 0$. If every row adds to 1, prove that $\det(A - I) = 0$. Show by an example that this does not imply $\det(A) = 1$.

6. Suppose $CD = -DC$. Find the flaw in the following argument:
   Taking determinants gives $\det(C)\det(D) = -\det(D)\det(C) \Rightarrow \det(C) = 0$ or $\det(D) = 0$.
   Thus $CD = -DC \Rightarrow C$ is singular or D is singular.

7. Find these determinants by Gaussian elimination:

$$\det\begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix}, \qquad \det\begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}$$

8. (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det(A_{3\times 3})$ will be zero?
   (b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 terms in $\det(A_{4\times 4})$ will be zero?

9. Choose the third row of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \_ & \_ & \_ \end{bmatrix}$$

so that its characteristic polynomial is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

10. Find the rank and all four eigenvalues for the matrix of ones and the chess board matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

11. Suppose A is a 3×3 matrix whose eigenvalues are $0, 3, 5$ with independent eigenvectors u, v, w.
    (a) Give a basis for the nullspace and a basis for the column space of A.
    (b) Find a particular solution to $Ax = v + w$. Find all solutions.
    (c) Show that $Ax = u$ has no solution. (Hint: If it had a solution, then $C(A) = \_\_\_\_\_\_$)

12. Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 5$. Explain why $\text{Trace}(A) = 9$, and $\det(A) = 20$. (Hint: Characteristic polynomial of A is $\_\_\_$.) Find at least three such matrices A.

13. Suppose $u, v \in \mathbb{R}^n$, and $A = uv^T$ i.e., a column times a row.
    (a) By multiplying A times u, show that u is an eigenvector. What is λ ?
    (b) What are the other eigenvalues of A (and why)? (Hint: $\text{rank}(A) = \_\_\_$.)
    (c) Compute trace(A) from the sum on the diagonal and the sum of λ s.

14. When P exchanges rows 1 and 2 and columns 1 and 2, the eigenvalues do not change. Find eigenvectors of A and PAP for $\lambda = 11$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \text{ and } PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$$

15. Let A be a 2×2 matrix satisfying $A^2 = I$.
    (a) What are the possible eigenvalues of A?
    (b) If $A \neq \pm I$, find its trace and determinant.
    (c) If the first row of A is $(3, -1)$, what is the second row?

16. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find A^{100} by diagonalising A.

17. Find all the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and write two different diagonalising matrices S.

18. Which of the following cannot be diagonalised?

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

19. Suppose A has eigenvalues $1, 2, 4$. What is the trace of A^2? What is the determinant of $(A^{-1})^T$?

20. Mark all the choices which are correct and explain why.

   (a) If the eigenvalues of A are 1, 1, 2 then,

      i. A is invertible.

      ii. A is diagonalizable.

      iii. A is not diagonalizable.

   (b) If the n columns of P (eigenvectors of A) are independent, then

      i. A is invertible.

      ii. A is diagonalizable.

      iii. P is invertible.

      iv. P is diagonalizable.

21. The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalisable because the rank of $A - 3I$ is ____. Explain.

    If you are allowed to change one entry to make A diagonalizable, which entries could you change?

22. Find Λ and P to diagonalize $A = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$.

   (a) What is the limit of Λ^k as $k \to \infty$?

   (b) What is the limit of $P\Lambda^k P^{-1}$?

   (c) $A^k = P\Lambda^k P^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____?

23. Let A be a 2×2 matrix. Let $N(A - I) = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ and $N(A - 4I) = \text{Span}\left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right\}$.

   (a) Find A. Is A diagonalizable? Explain why.

   (b) Find a diagonal matrix B such that $B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

   (c) Use previous parts to find a matrix X such that $X^2 = A$.

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