MA-106 Tutorial-5

34) $\dim V = n$, $\beta = \{v_1, \ldots, v_n\}$ is a basis of V and $T: V \to V$ is a linear transformation defined by $T(v_1) = v_2$, $T(v_2) = v_3$, $\ldots$, $T(v_n) = 0$.

(a) Find $[T]_\beta^\beta = A$.

$$A = \left[\underset{\downarrow}{(T(v_1))_\beta} \quad \underset{\downarrow}{(T(v_2))_\beta} \quad \cdots \quad \underset{\downarrow}{(T(v_n))_\beta} \right]$$

$$T(v_1) = v_2 = \boxed{0} v_1 + \boxed{1} v_2 + \boxed{0} v_3 + \cdots + \boxed{0} v_n$$

$\therefore$ The (column) coordinate vector of v_1 w.r.t the ordered basis β is

$$(T(v_1))_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} = e_2$$

Similarly, $(T(v_2))_\beta = e_3$, $\ldots$, $(T(v_{n-1}))_\beta = e_n$

& $(T(v_n))_\beta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

So, $A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & 0 & & & \vdots \\ \vdots & \vdots & & \cdot & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} = [\underset{\downarrow}{e_2} \ \underset{\downarrow}{e_3} \ \cdots \ \underset{\downarrow}{e_n} \ \underset{\downarrow}{0}]$

(b) Prove that $T^n = 0$ & $T^{n-1} \neq 0$.

Note that $v_2 = T(v_1)$, $v_3 = T(v_2) = T^2(v_1)$, $\ldots$, $v_n = T(v_{n-1}) = T^{n-1}(v_1)$

So, $v_j = T^{j-1}(v_1)$ for $j = 2, 3, \cdots, n$.

Moreover, $T(v_n) = 0 \Rightarrow T(T^{n-1}(v_1)) = 0$

$$\Rightarrow T^n(v_1) = 0$$

$\therefore$ we have $T^n(v_1) = 0$ & for $j = 2, 3, \cdots, n$,

$$T^n(v_j) = T^n(T^{j-1}(v_1)) = T^{j-1}(T^n(v_1)) = T^{j-1}(0) = 0.$$

i.e. the linear transformation $T^n : V \to V$ maps each one of the basis elts to zero $\therefore$ $T^n = 0$ because any vector in V is a linear combination of elts of β.

Also, $T^{n-1}(v_1) = v_n \neq 0$ because v_n is a basis element

i.e. there exists an elt (namely, v_1) whose image under T^{n-1} is non-zero $\therefore$ $T^{n-1} \neq 0$.

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40) $S_1 : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ is defined as $S_1(A) = A + A^t$.

(i) Show that S_1 is a linear transformation.

We need to check the following -

$S_1(A+B) = S_1(A) + S_1(B)$ and $S_1(\alpha A) = \alpha S_1(A)$

for every $A, B \in \mathcal{M}_{n\times n}$ & scalar α.

$$\begin{aligned} S_1(A+B) &= (A+B) + (A+B)^t \\ &= (A+B) + (A^t + B^t) \\ &= (A + A^t) + (B + B^t) \\ &= S_1(A) + S_1(B) \end{aligned}$$

and $S_1(\alpha A) = \alpha A + (\alpha A)^t$
$= \alpha A + \alpha (A^t)$
$= \alpha (A + A^t) = \alpha S_1(A)$.

Hence $S_1 : M_{n\times n} \to M_{n\times n}$ is a linear transformation

(ii) <u>Find $N(S_1)$</u> the null space of S_1.

Let $A \in N(S_1)$. Then $S_1(A) = 0 \Rightarrow A + A^t = 0$

$\Rightarrow$ A is a skew-symmetric matrix.

Conversely, if A is skew-symmetric, then

$A + A^t = 0 \Rightarrow S_1(A) = 0 \Rightarrow A \in N(S_1)$.

$\therefore$ $N(S_1)$ = the subspace of $M_{n\times n}$ consisting of all skew-symmetric matrices.

(iii) <u>Find $C(S_1)$</u>, the image of S_1.

$C(S_1) = \{ S_1(A) \mid A \in M_{n\times n} \}$.

Let $B \in C(S_1)$. Then there exists some $A \in M_{n\times n}$ such that $B = A + A^t$

Now $B^t = (A + A^t)^t = A^t + A = B$

$\Rightarrow$ B is symmetric

$\therefore$ $C(S_1) \subseteq$ set of all symmetric matrices

Conversely, let B be a symmetric matrix.

Then $B = \frac{1}{2}(B + B) = \frac{1}{2}(B + B^t)$ $\{\because B = B^t$

$= \frac{1}{2}B + (\frac{1}{2}B)^t$

$= A + A^t$, where $A = \frac{1}{2}B$

$\Rightarrow B \in C(S_1)$

$\therefore$ $C(S_1)$ is the subspace of $M_{n\times n}$ which consists of all symmetric matrices.

42) Construct a linear map $T: P_2(\mathbb{R}) \to \mathbb{R}$ s.t. $T(1) = 1$, $T(1-x) = 2$, $T(x^2) = 3$.

Note that $\mathcal{B} = \{1, 1-x, x^2\}$ is a L.I. subset of $P_2(\mathbb{R})$. Moreover, given a polynomial $a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R})$, we can write

$$a_0 + a_1 x + a_2 x^2 = (a_0 + a_1) - a_1(1-x) + a_2 x^2$$

i-e $P_2(\mathbb{R}) \subseteq \operatorname{Span} \mathcal{B}$.

$\therefore$ $\mathcal{B}$ is a basis of $P_2(\mathbb{R})$.

Define $T: P_2(\mathbb{R}) \to \mathbb{R}$ by

$$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1 - a_1(2) + a_2(3)$$
$$= a_0 - a_1 + 3a_2.$$

Then T is a linear transformation which satisfies the given conditions.

[OR, suppose T is a linear transf^n with the required property.

Then $T(1-x) = 2 \Rightarrow T(1) - T(x) = 2$
$$\Rightarrow T(x) = T(1) - 2$$
$$= 1 - 2 = -1.$$

$\therefore$ T maps $1 \to 1$, $x \to -1$ and $x^2 \to 3$

$\therefore$ Define T by $T(a_0 + a_1 x + a_2 x^2) = a_0 - a_1 + 3a_2$]

Check that T is a lin map.

Find $N(T)$, the null space of T

Let $a_0 + a_1 x + a_2 x^2 \in N(T)$.

Then $a_0 - a_1 + 3a_2 = 0 \quad \Rightarrow a_0 = a_1 - 3a_2$

$$\therefore a_0 + a_1 x + a_2 x^2 = a_1(1+x) + a_2(-3+x^2)$$
$$\in \operatorname{Span}\{1+x, -3+x^2\} \subseteq N(T).$$

Note that $1+x$, $-3+x^2$ are L.I vectors in $N(T)$.

$\therefore$ $\{1+x, -3+x^2\}$ is a basis of $N(T)$.

dim $N(T) = 2$.

[Remark. $a_0 + a_1 x + a_2 x^2 \in N(T) \iff$ the vector $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in$ null space of the matrix $[1 \ -1 \ 3]$

& $N[1 \ -1 \ 3] = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\Rightarrow N(T) = \text{Span}\{1 + 1x, -3 + 1x^2\}$]

Find $C(T)$ the range space of T

Given $a \in \mathbb{R}$, we can write $a = T(a \cdot 1) = a \cdot T(1)$

$\Rightarrow$ T is onto $\Rightarrow$ $C(T) = \mathbb{R}$

[OR, Since $C(T)$ is a subspace of $\mathbb{R}$ & $\dim \mathbb{R} = 1$,
dim $C(T) \leq 1$, i.e dim $C(T) = 0$ or 1.
But, $C(T)$ is $\neq 0$. $\therefore$ dim $C(T)$ is $\neq 0$
i.e dim $C(T) = 1$ $\Rightarrow$ $C(T)$ must be equal to $\mathbb{R}$].

How many such maps can be constructed ?

Recall that - if V, W are two vector spaces & $\{v_1, \ldots, v_n\}$ is a basis of V, then any two linear maps $S, T : V \to W$ such that $S(v_1) = T(v_1), \ldots, S(v_n) = T(v_n)$ must be equal because every vector in V can be uniquely expressed as a lin. comb^n of $v_1, \ldots, v_n$.

Since $\beta = \{1, 1-x, x^2\}$ is a basis of $P_2(\mathbb{R})$, given $S, T: P_2(\mathbb{R}) \to \mathbb{R}$ (linear maps) such that $S(1) = T(1)$, $S(1-x) = T(1-x)$, $S(x^2) = T(x^2)$, we must have $S = T$ i.e there exists unique such map.

[Given $f \in P_2(\mathbb{R})$, there exist unique scalars a_0, a_1, a_2 such that

$$f = a_0 1 + a_1 (1-x) + a_2 x^2$$

$$\therefore S(f) = a_0 S(1) + a_1 S(1-x) + a_2 S(x^2)$$
$$= a_0 T(1) + a_1 T(1-x) + a_2 T(x^2)$$
$$= T(f)$$ & this is true for every $f \in P_2(\mathbb{R})$

$\therefore S = T$]

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37) $T: \mathbb{R}^4 \to \mathbb{R}^3$ is defined as $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix}$

Find the standard matrix of T.

Required matrix is $A = [T(e_1) \;\; T(e_2) \;\; T(e_3) \;\; T(e_4)]$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4}$$

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39) $D : P_3 \to P_2$ is the linear transformation

$$D(f) = \frac{df}{dx}.$$

(i) Find $N(D)$, $C(D)$.

$N(D) = \text{Span}\{1\}$, $\dim N(D) = 1$

$C(D) = P_2$, $\dim C(D) = 3$

Note: $\dim N(D) + \dim C(D) = \dim P_3$.

(ii) Let $\beta = \{1, x, x^2, x^3\}$ and $\beta' = \{1, x, x^2\}$ be the standard bases of P_3 & P_2 resp.

Find the matrix $[D]_{\beta'}^{\beta}$ associated with D relative to the ordered bases β & β'.

$$[D]_{\beta'}^{\beta} = A = \left[(D(1))_{\beta'} \quad (D(x))_{\beta'} \quad (D(x^2))_{\beta'} \quad (D(x^3))_{\beta'} \right]$$

$$D(1) = 0 = \boxed{0}\cdot 1 + \boxed{0}\cdot x + \boxed{0}\cdot x^2$$

$$\therefore (D(1))_{\beta'} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D(x) = 1 = \boxed{1}\cdot 1 + \boxed{0}\cdot x + \boxed{0}\cdot x^2$$

$$\therefore (D(x))_{\beta'} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$D(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$

$$D(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$