Magnetostatics

Magnetostatics: Field due to steady currents

Magnetic fields are created by:

1. Charges in motion
2. Intrinsic spin (dipole) moments of some elementary particles.

The field created by a single moving charge is not very easy to write down!
Historically steady currents were understood first.

Maxwell's equations tell us the field created by currents (moving charges).
The response of a particle to the fields (E and B) is an independent input.

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

Lorentz Force Law is not derivable from Maxwell's equation

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

= 0 for now

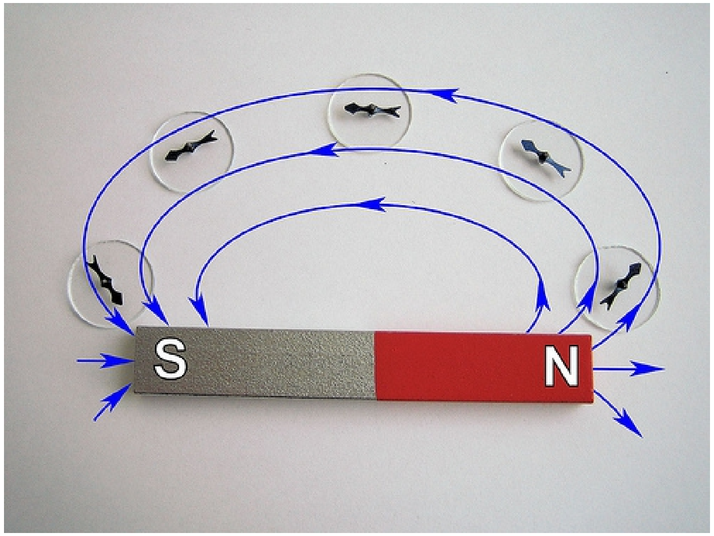
Zero divergence means :
No analogue of charge as in electrostatics.

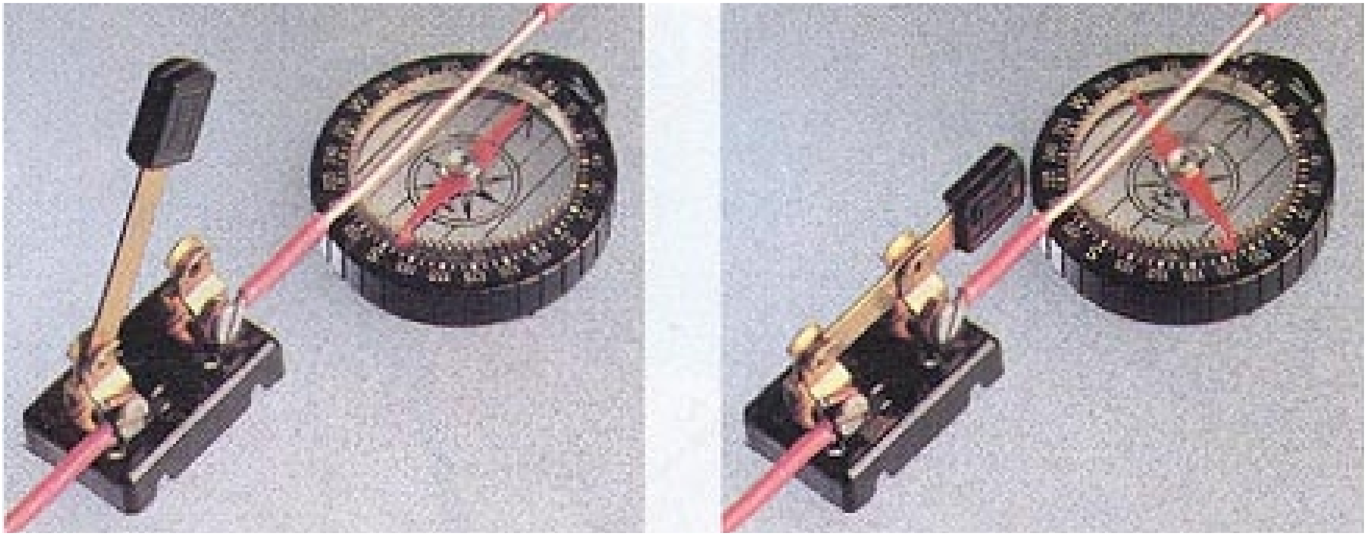
$$\mu_0 = 4\pi\times 10^{-7}\, Henry.m^{-1}$$

Historically these equations were written later.

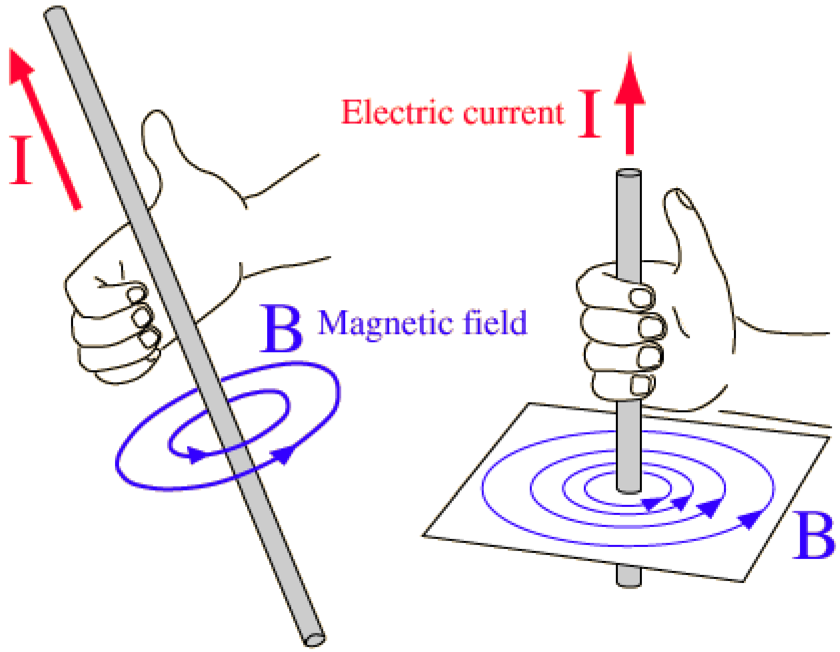
Recall that curl and div specifies a vector field fully if suitable boundary conditions are also given.

Histroical observations, Biot Savart Law





Magnets deflect a "compass" and so does a nearby current carrying wire. So a current carrying wire must be creating a magnetic field. (Oersted, Ampere)

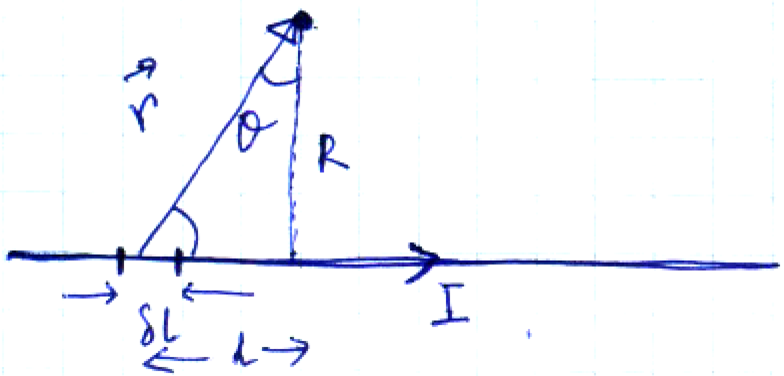


$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{r}}{r^3}$$

Then integrate over the wire to find the full field.

Common geometries.....
Straight lines, coils etc.

Field from a wire segment, loops, coils, toroids etc



$$R = r\cos\theta$$

$$l = R\tan\theta$$

$$\delta l = R\sec^2\theta\,\delta\theta$$

$$\delta B = \frac{\mu_0}{4\pi} I \frac{\vec{\delta l}\times\vec{r}}{r^3}$$

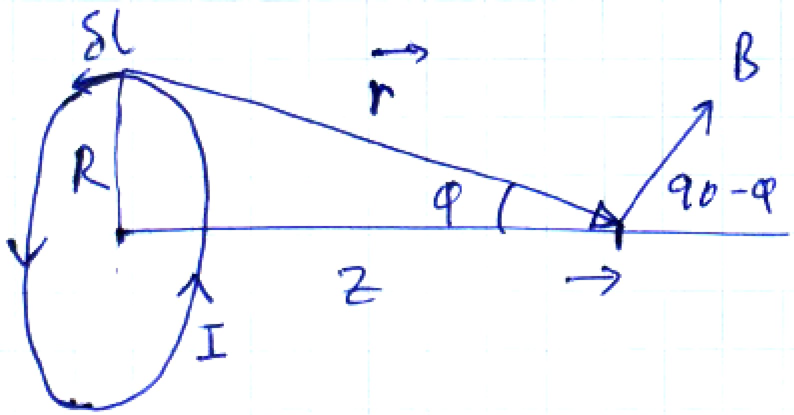
$$= \frac{\mu_0 I}{4\pi} \cdot \frac{R\sec^2\theta\,\delta\theta \cdot \sin(\pi/2-\theta) \cdot R\sec\theta}{R^3\sec^3\theta}$$

$$B = \frac{\mu_0 I}{4\pi R} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I}{4\pi} (\sin\theta_2 - \sin\theta_1)$$

Infinite wire $\begin{cases} \theta_1 = -\pi/2 \\ \theta_2 = \pi/2 . \end{cases}$

$$B = \frac{\mu_0 I}{2\pi R}$$ (Can use symmetry arguments)

Field from a wire segment, loops, coils, toroids etc



$$r^2 = R^2 + z^2$$

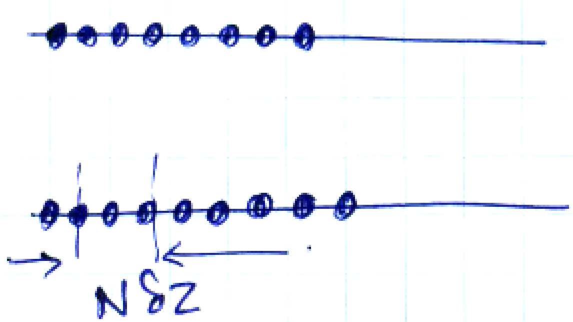
$$\sin\varphi = R/r$$

$$\delta l = R\,\delta\theta$$

$$\delta B = \frac{\mu_0}{4\pi} \cdot I(R\delta\theta) \cdot \frac{1}{R^2+z^2} \cdot \frac{R}{r}$$

$$B = \frac{\mu_0 I}{2} \cdot \frac{R^2}{(R^2+z^2)^{3/2}}$$

Integrate over z for a finite coil.

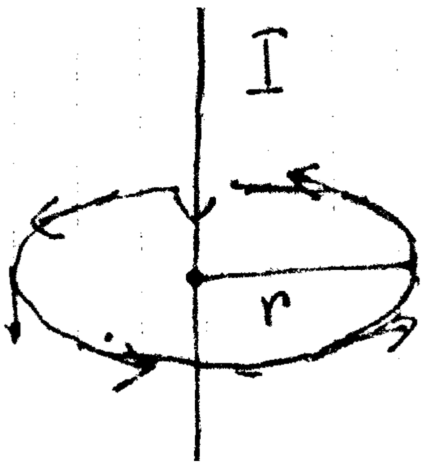


$$\delta B = \frac{\mu_0 I}{2} \frac{R^2 N \delta z}{(R^2+z^2)^{3/2}}$$

Using the integral form in symmetrical cases: long wire, coil, full toroid

$$\oint \vec{B}\,.\,d\vec{l} = \mu_0 I_{enc} \quad \text{same as} \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

Infinitely long wire



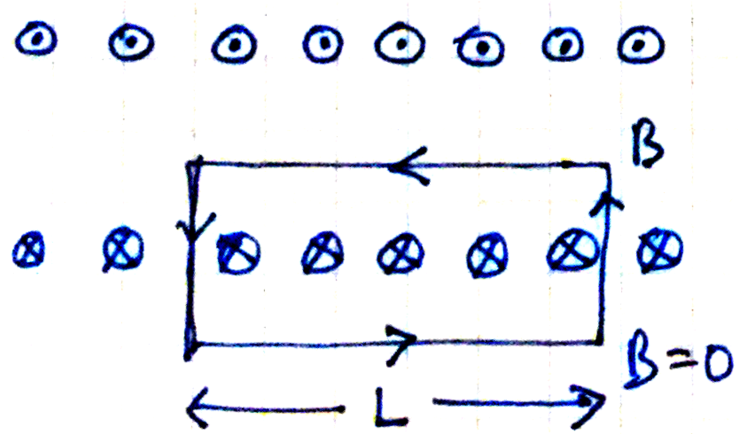
$$B_\varphi 2\pi r = \mu_0 I$$

$$B_\varphi = \frac{\mu_0 I}{2\pi r}$$

$$B_r = 0 \quad (\nabla \cdot B = 0)$$

$$B_z = 0$$

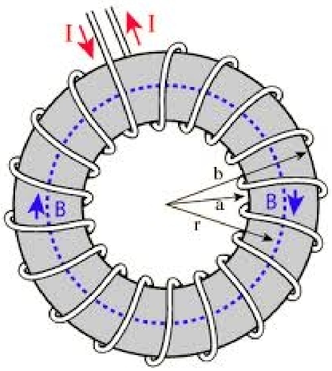
Infinitely long coil

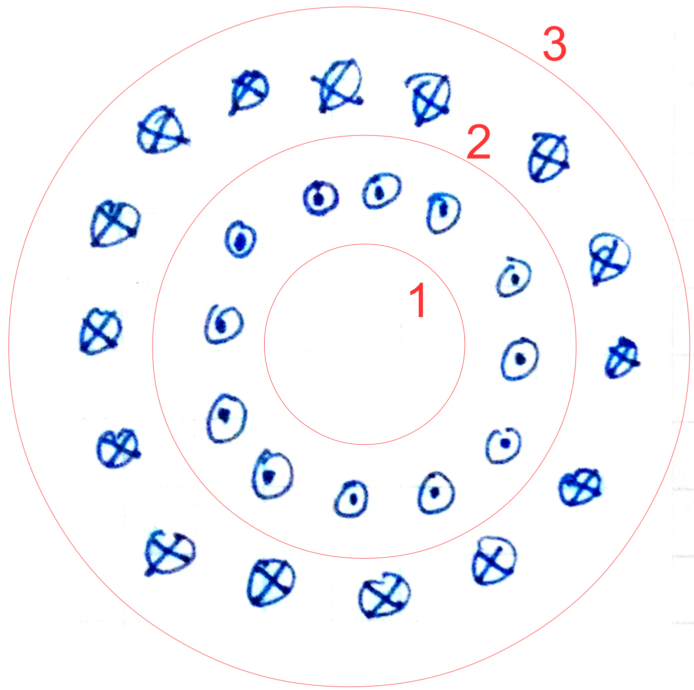


$$B\,L = \mu_0 N I$$

$$B = \mu_0 \frac{N}{L} I$$

Using the integral form in symmetrical cases: long wire, coil, full toroid





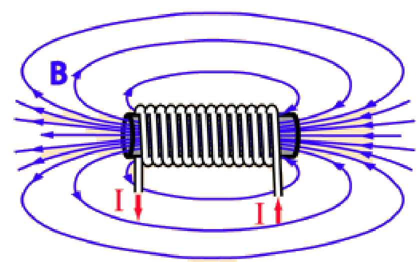
Path 1, 3 $\oint \vec{B}\cdot d\vec{l} = 0$

Path 2 $\oint \vec{B}\cdot d\vec{l} = \mu_0 N I$.

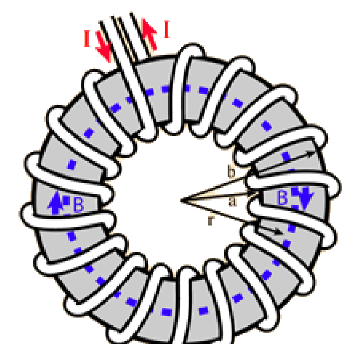
$$B_r = \frac{\mu_0 N I}{2\pi r}$$

Geometries where one can apply Ampere's Law quickly.

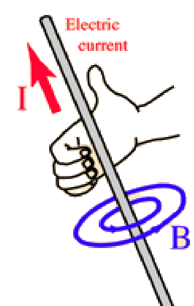
Similar to using Gauss's law in some symmetrical situations.



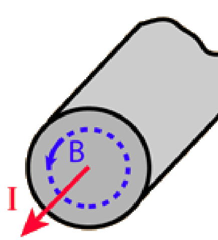
Magnetic field inside a long solenoic.



Magnetic field inside a toroidal coil.



Magnetic field from a long straight wire.



Magnetic field inside a conductor.

The magnetic vector potential : the formal solution

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$
$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$$
$$\nabla(\nabla . \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

The choice $\nabla . \vec{A} = 0$
is called a gauge choice

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Since $\nabla . \vec{B} = 0$ we can write $\vec{B} = \nabla \times \vec{A}$
Does it make things simpler?

There can be other possible choices.
For each gauge A and V will be different
But they will give the same **E** and **B**.

is like three Poisson's equation $\left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$ put together

Call $\vec{A}$ the vector potential.
$\vec{A}$ has no simple interpretation like potential energy

The magnetic vector potential : the formal solution

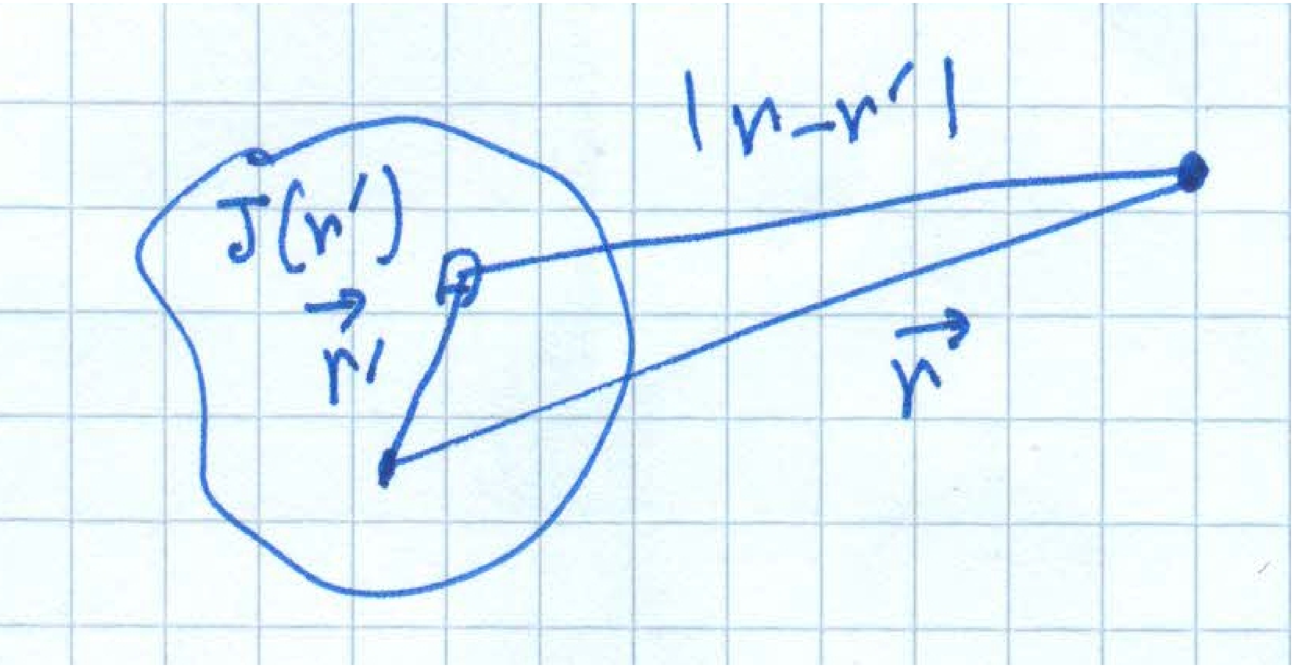
$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,'}$$

$$\vec{B} = \nabla \times \vec{A}(\vec{r})$$

$$= \nabla \times \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,'$$

The curl is to be taken w.r.t. $\vec{r}$

The integration is w.r.t. $\vec{r}\,'$



what is $\nabla \times \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$?

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{J_k(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= \epsilon_{ijk} J_k(\vec{r}\,') \frac{\partial}{\partial x_j} \frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= \epsilon_{ikj} J_k(\vec{r}\,') \frac{x_j - x'_j}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \left[\vec{J}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right]_i$$

We can interchange the order of integration and differentiation.

We recover Biot-Savart Law, which is an important consistency check!

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} d^3\vec{r}\,'}$$

The magnetic vector potential : the choice of div A and its consequences

Our choice of A cannot affect the final result for B.

But is does effect the solution for BOTH the scalar and the vector potential.

Notice that A,V suffer from "instantaneous change at a distance" problem.

We do not need to care as long as it is a static/steady state solution.

But what if charge and current densities (hence A and V) are both varying arbitrarily?

The choice of A and V : how much freedom is there?

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\vec{A}$$

$$\nabla\times\left(\vec{E}+\frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\vec{E}+\frac{\partial \vec{A}}{\partial t} = \nabla(\mathit{some\ scalar})$$

$$\vec{E} = -\nabla V-\frac{\partial \vec{A}}{\partial t}$$

$$\left.\begin{aligned}\vec{E} &= -\nabla V-\frac{\partial \vec{A}}{\partial t}\\ \vec{B} &= \nabla\times\vec{A}\end{aligned}\right\}$$

It is possible to set V=0 and still have an electric field via time varying **A**

V and **A** has to change in such a way that E and B remain same.

V, **A** and V', **A'** will have to be related

$$\vec{A}' = \vec{A}+\nabla\lambda$$

$$V' = ?$$

Suppose $\nabla V'+\frac{\partial \vec{A}'}{\partial t} = \nabla V+\frac{\partial \vec{A}}{\partial t}$

$$\nabla(V-V') = \frac{\partial}{\partial t}(\vec{A}-\vec{A}') = -\frac{\partial}{\partial t}\nabla\lambda$$

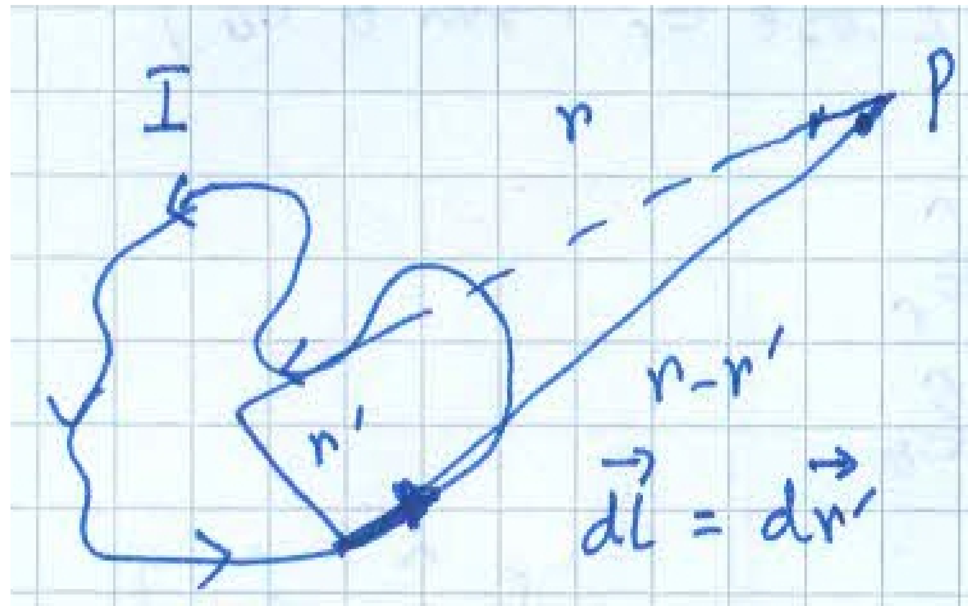
$\Rightarrow$

$$\boxed{\begin{aligned}\vec{A}' &= \vec{A}+\nabla\lambda\\ V' &= V-\frac{\partial\lambda}{\partial t}\end{aligned}}$$

λ is a scalar fn of x, y, z, t

Multipole expansion of the magnetic vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}}{|\vec{r}-\vec{r}\,'|}$$



$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} (r')^l P_l(\cos\theta)$$

$$(r \gg r')$$

$$l=0 \qquad \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\vec{l}$$

Always zero: no magnetic monopoles

$$l=1 \qquad \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta\, d\vec{l}$$

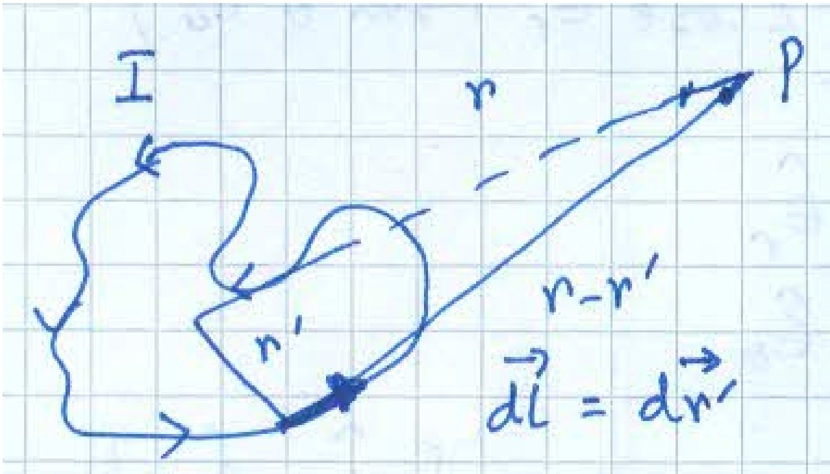
Origin independent: magnetic dipole

$$l=2 \qquad \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint r'^2 \frac{1}{2}\left(3\cos^2\theta - 1\right) d\vec{l}$$

quadrupole

Multipole expansion of the magnetic vector potential

$$\oint r'\cos\theta\, d\vec{r}' = \frac{1}{r}\oint \left(\vec{r}'.\vec{r}\right) d\vec{r}'$$



Write the integral componentwise as :

$$C_i = \oint \left(x_j x'_j\right) dx'_i = x_j \oint x'_j dx'_i$$

$$= x_j\left[\oint x'_j dx'_i - \frac{1}{2}\oint d\left(x'_j x'_i\right)\right]$$

$$= x_j \frac{1}{2}\oint \left(x'_j dx'_i - x'_i dx'_j\right)$$

Exact differential hence zero over a closed path

This looks like the k^{th} component of cross product $\left[-\frac{1}{2}\oint \vec{r}'\times d\vec{r}'\right]$

$$C_i = \epsilon_{ijk}\, x_j \left[-\frac{1}{2}\oint \vec{r}'\times d\vec{r}'\right]_k$$

vector area of the loop, area x current = dipole moment

$$\boxed{\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\left[\frac{1}{2} I \oint \vec{r}'\times d\vec{r}'\right]\times\frac{\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\nabla\times\frac{\vec{m}}{r}}$$

Force on a small current loop or a magnetic dipole

$$\vec{F} = I \oint d\,\vec{l} \times \vec{B} \quad \Rightarrow \quad F_i = \epsilon_{ijk}\, I \oint d\,x_j\, B_k$$

If B is constant then there is no force on the loop.
Field gradient is essential to create force on a dipole.

Take r to be small but finite.
Then lim r → 0

$$\begin{aligned}
B_k(\vec{r}) &= B_k(0) + \vec{r}\,.\,\nabla B_k \\
F_i &= \epsilon_{ijk}\, I \oint dx_j \left[B_k(0) + x_p \frac{\partial B_k(0)}{\partial x_p} \right] \\
&= \epsilon_{ijk}\, I \oint d\,x_j\, x_p \left[\left. \frac{\partial B_k}{\partial x_p} \right|_0 \right]
\end{aligned}$$

Constant, since the derivative is evaluated at r=0

We need a general expression for $\oint d\,x_j\, x_p$

Recall : $\oint d\left(x_j x_p\right) = 0$ always

Force on a small current loop or a magnetic dipole

$$\oint d\left(x_j x_p\right) = 0 \Rightarrow \frac{1}{2}\oint\left(x_j dx_p + x_p dx_j\right) = 0$$

$$\therefore I\oint dx_j \; x_p = \frac{I}{2}\oint\left(x_p dx_j - x_j dx_p\right)$$

RHS has the form of a cross product like $\vec{r}\times d\vec{r}$

Dipole moment of a current loop $\vec{m} = \frac{I}{2}\oint \vec{r}\times d\vec{r}$

$$F_i = \epsilon_{ijk}\left[\frac{I}{2}\oint\left(x_p dx_j - x_j dx_p\right)\right]\left[\frac{\partial B_k}{\partial x_p}\right]_0$$

$$= \epsilon_{ijk}\epsilon_{qpj} m_q \left[\frac{\partial B_k}{\partial x_p}\right]_0 = -\left(\delta_{iq}\delta_{kp} - \delta_{ip}\delta_{kq}\right) m_q \left[\frac{\partial B_k}{\partial x_p}\right]_0$$

Force on a small current loop or a magnetic dipole

$$F_i = -\left(\delta_{iq}\delta_{kp} - \delta_{ip}\delta_{kq}\right) m_q \left[\frac{\partial B_k}{\partial x_p}\right]_0$$

$$= -m_i \boxed{\frac{\partial B_k}{\partial x_k}} + m_k \frac{\partial B_k}{\partial x_i}$$

div B =0

If $\vec{m}$ is constant : $\vec{F} = \nabla\left(\vec{m}.\vec{B}\right)$

otherwise use : $F_i = m_k \dfrac{\partial B_k}{\partial x_i}$

If $\vec{m} = \alpha \vec{B}$: $\vec{F} = \alpha \nabla \left(\dfrac{B^2}{2}\right)$

Torque on a small current loop or a magnetic dipole

$$\vec{\tau} = \oint \vec{r} \times \left(I\, d\,\vec{l} \times \vec{B} \right)$$

B(r=0) term will not vanish after integration.
So no need to consider the first order expansion.

$$\vec{\tau} = I\left[\oint d\,\vec{r}\left(\vec{r}\,.\,\vec{B}(0)\right) - \vec{B}(0)\oint\left(\vec{r}\,.\,d\,\vec{r}\right)\right]$$

Second term is a perfect differential. So goes to zero

$$\tau_i = I\,B_j \oint x_j\, dx_i$$

$$= IB_j \frac{1}{2}\oint\left(x_j\, dx_i - x_i\, dx_j\right)$$

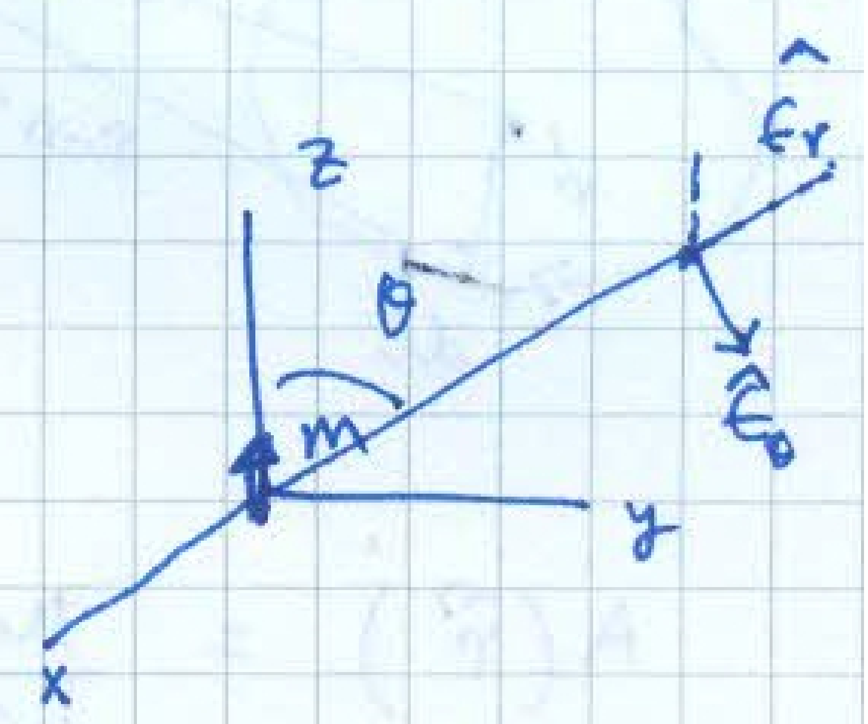
$$\vec{\tau} = \vec{m} \times \vec{B}$$

$$\boxed{\vec{m} = \frac{I}{2}\oint \vec{r} \times d\,\vec{r}}$$

The magnetic vector potential and field of a perfect dipole

$$\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\epsilon}_\phi$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{\epsilon}_r + \sin\theta\,\hat{\epsilon}_\theta\right)$$



$$Since \begin{cases} m\cos\theta = \vec{m}.\hat{\epsilon}_r \\ m\sin\theta = -\vec{m}.\hat{\epsilon}_\theta \end{cases}$$

$\vec{m}$ lies in the plane defined by $[\epsilon_r, \epsilon_\theta]$

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}.\hat{r})\hat{r}-\vec{m}}{r^3}\right] \qquad (r \neq 0)$$

If the r=0 point is to be correctly handled then the delta fn is needed.

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}.\hat{r})\hat{r}-\vec{m}}{r^3}\right] + \mu_0\frac{2}{3}\vec{m}\,\delta(\vec{r})}$$

Correct solution of the interaction of electron spin and nuclear spin (hyperfine) requires this.

Calculate $\int \vec{A}\times d\vec{S}$ over a sphere with a dipole $(0,0,m)$ at $\vec{r}=0$

Then calculate $\int \vec{B}\,d\tau$ over the volume of the sphere

Do you see why the $\delta(\vec{r})$ is necessary

Multipole expansion : an useful identity for volume currents

For a localised current distribution (J) with zero divergence at all points, and any two scalar functions f,g

$$\int_{vol} \nabla . (\vec{J} fg) d\tau = \int_{surf} \vec{J} fg . d\vec{S} = 0$$

Since J is localised it vanishes on the surface everywhere.

$$\int_{vol} \left[fg (\nabla . \vec{J}) + \vec{J} . \nabla fg \right] d\tau = 0$$

$$\int_{vol} \left[f \vec{J} . \nabla g + g \vec{J} . \nabla f \right] d\tau = 0$$

Take f=1 g=x,y,z in turn & prove $\int_{vol} \vec{J} d\tau = 0$

Take f= g=x,y,z in turn & prove $\int_{vol} \vec{r} . \vec{J} d\tau = 0$

Take f= x, g=y and other permutaions $\int_{vol} (xJ_y + yJ_x) d\tau = 0$

Multipole expansion of the magnetic vector potential with volume current

$$A_i(\vec r) = \frac{\mu_0}{4\pi}\int \frac{J_i(\vec r\,')}{|\vec r - \vec r\,'|} d^3\vec r\,'$$

$$= \frac{\mu_0}{4\pi}\left[\frac{1}{r}\underbrace{\int J_i(\vec r\,') d^3\vec r\,'}_{=0} + \frac{1}{r^3}\vec r . \int \vec r\,' J_i(\vec r\,') d^3\vec r\,' + \cdots\right]$$

$$= \frac{\mu_0}{4\pi r^3}\vec r . \int \vec r\,' J_i(\vec r\,') d^3\vec r\,' + \cdots$$

Need to use the identities derived just before to obtain the result in the next step

$$= \frac{\mu_0}{4\pi r^3}\left[-\frac{1}{2}\vec r \times \int \vec r\,' \times \vec J\, d^3\vec r\,'\right]_i + \cdots$$

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m \times \vec r}{r^3} + \cdots$$

$$\int d^3\vec r\,'(xx' + yy' + zz')J_x(\vec r\,')$$

$$= x\int d^3\vec r\,' x' J_x(\vec r\,') + y\int d^3\vec r\,' y' J_x(\vec r\,') + z\int d^3\vec r\,' z' J_x(\vec r\,')$$

$$= 0 + y\int d^3\vec r\,'\left[y' J_x(\vec r\,') - \frac{1}{2}\left(y' J_x(\vec r\,') + x J_y(\vec r\,')\right)\right] + z\int d^3\vec r\,'\left[z' J_x(\vec r\,') - \frac{1}{2}\left(z' J_x(\vec r\,') + x' J_z(\vec r\,')\right)\right]$$

$$= y\int d^3\vec r\,'\left[-\frac{1}{2}\left(x' J_y(\vec r\,') - y' J_x(\vec r\,')\right)\right] + z\int d^3\vec r\,'\left[-\frac{1}{2}\left(x' J_z(\vec r\,') - z' J_x(\vec r\,')\right)\right]$$

$$= -\frac{1}{2}\left[y\int d^3\vec r\,'[\vec r\,' \times \vec J(\vec r\,')]_z - z\int d^3\vec r\,'[\vec r\,' \times \vec J(\vec r\,')]_y\right] = -\frac{1}{2}\left[\vec r\,' \times \int d^3\vec r\,' \vec r\,' \times \vec J(\vec r\,')\right]_x \text{............ result follows}$$

Forces and torques on current loops and dipoles with volume current

Force on a current distribution will also vanish if all the current loops are closed and the fields are constant

$$\vec{F} = q\left(\vec{E}+\vec{v}\times\vec{B}\right) \qquad \left(\textit{single particle}\right)$$
$$\delta\vec{F} = (n\,\delta\tau)\,q\left(\vec{E}+\vec{v}\times\vec{B}\right) \qquad \left(\textit{many particles}\right)$$
$$\delta\vec{F} = \vec{J}\times\vec{B}\,\delta\tau = I\,\delta\vec{l}\times\vec{B} \qquad \left(\textit{current line, distrib}\right)$$

A current carrying wire is electrically neutral because it always has equal number of electrons and positive ions in lattice. An electric field does not create a net force on it.

Magnetic field does, becuase the electrons are moving and the fixed ions in the lattice are not – so the lattice sees no Lorentz force.

Useful facts to remember....

Also $\nabla\cdot\vec{J} = 0,\quad$ since $\dfrac{\partial\rho}{\partial t}=0$

$$\oiiint \vec{J}\,d\tau = 0$$

(for a localised charge distribution)... *why?*

Consider the expression

$$\int_{vol}\nabla\cdot(x\vec{J})\,d\tau = \int\left[x\nabla\cdot\vec{J}+\vec{J}\cdot\nabla x\right]d\tau$$

$$\textit{But}\quad \vec{J}\cdot\nabla x = J_x$$

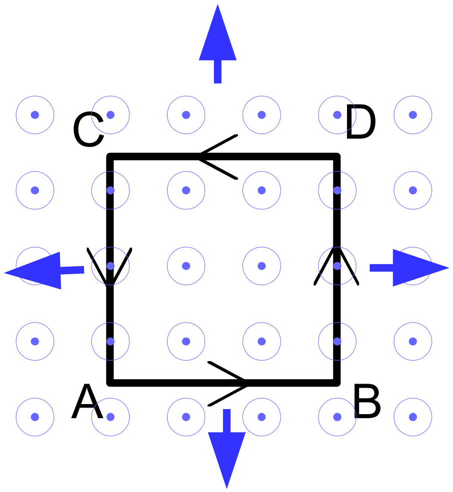
$$\textit{Also}\quad \vec{J} = 0$$

on a large bounding surface

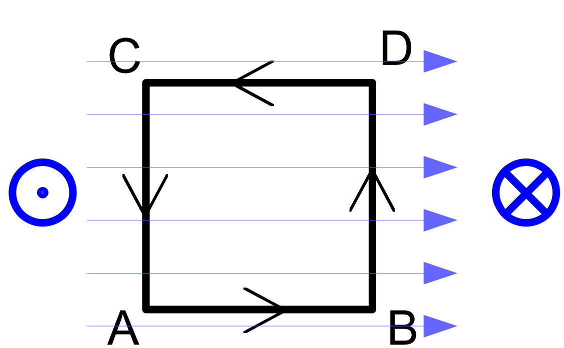
So the result follows

Forces and torques on current loops and dipoles

What is the force and torque on the square loop?



F=0
Torque = 0

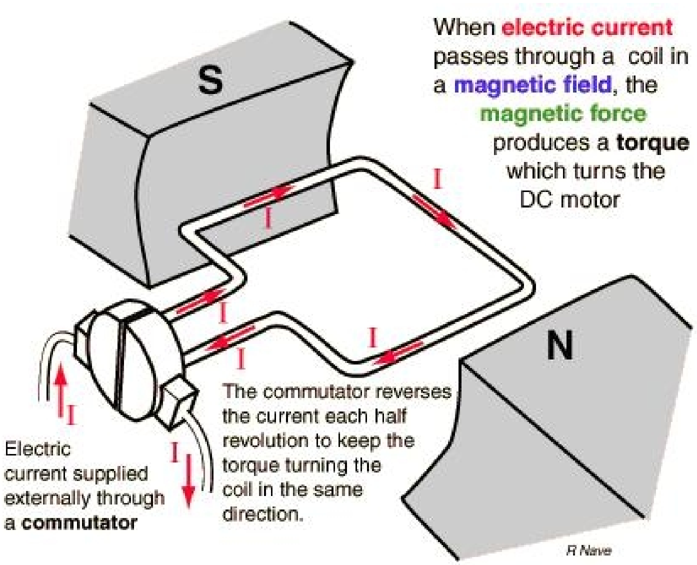


$$\vec{F} = \oint I\,\delta\vec{l}\times\vec{B}$$

Total F=0, still, since opposite sides have exactly opposite dl vector

Torque = BI x area (in magnitude)

The basis for electric motor winding, pointer type current measuring meters etc.



Only inhomogeneous magnetic field can create a force on a current loop/ dipole.

Torque is possible with uniform fields.

Forces and torques on current loops and dipoles

Consider an arbitrary current distribution in a spatially varying field **B**.
Question: What is the force and torque on it?
Assume that the current density is confined to a small volume.

$$\vec{F} = \int \left(\vec{J} \times \vec{B}\right) d^3\vec{r}\,' \quad \text{and} \quad B_k(\vec{r}) = B_k(0) + \vec{r}\,.\nabla B_k$$

$$F_i = \epsilon_{ijk}\Big[B_k(0)\underbrace{\int J_j(\vec{r}\,')\, d^3\vec{r}\,'}_{=0} + \underbrace{\int J_j(\vec{r}\,')\vec{r}\,'.\nabla' B_k(0)\, d^3\vec{r}\,'}_{\text{Field inhomogeneity giving rise to force}} + \ldots\Big]$$

We need to simplify: $\epsilon_{ijk} \nabla B_k(0).\int \vec{r}\,' J_j(\vec{r}\,') d^3\vec{r}\,'$

This will give : $\left[\underbrace{\left(\frac{1}{2}\int (\vec{r}\,' \times \vec{J}) d^3\vec{r}\,'\right)}_{\text{Dipole moment of the current distribution}} \times \nabla B_k(0)\right]_i$

The proof is similar to the one given for the dipole moment calculation before.

Forces and torques on current loops and dipoles

The torque will be given by

$$\begin{aligned}
\vec{\tau} &= \int \left[\vec{r}\,' \times \left(\vec{J} \times \vec{B}\right)\right] d^3 \vec{r}\,' \\
&= \int \left[\vec{J}\,(\vec{r}\,'.\vec{B}) - \vec{B}(\vec{r}\,'.\vec{J})\right] d^3 \vec{r}\,' \\
&\approx \vec{B}(0).\int \vec{r}\,' \vec{J}\, d^3 \vec{r}\,' + \vec{B}(0) \underbrace{\int (\vec{r}\,'.\vec{J})\, d^3 \vec{r}\,'}_{=0} \\
&= \left[\frac{1}{2} \int (\vec{r}\,' \times \vec{J})\, d^3 \vec{r}\,'\right] \times \vec{B}(0) \\
&= \vec{m} \times \vec{B}
\end{aligned}$$

Since the zeroth order term does not vanish, we take the value of B at a fixed point in the distribution and treat it as a constant

Electric dipole

$$\begin{aligned}
\vec{F} &= (\vec{p}.\nabla)\vec{E} \\
\vec{\tau} &= \vec{p} \times \vec{E}
\end{aligned}$$

Magnetic dipole

$$\begin{aligned}
\vec{F} &= \nabla(\vec{m}.\vec{B}) \\
\vec{\tau} &= \vec{m} \times \vec{B}
\end{aligned}$$

Only if m is constant

The end result is very similar, though the internal mechanism is quite different.

How does matter acquire magnetic polarisation (dipole moment)

The mechansims by which matter acquires a magnetic "dipole moment" per unit volume is more complex than the way electric polarisation is acquired.

A classical description of this is not really possible.

The magentic moment acquired may be

1. In the direction of the magnetic field but very weak. (paramagnetism)
2. OPPOSITE to the direction of the applied field and also very weak (diamagnetism)

   Para & dia magnetic effects disappear when the applied field is removed.

3. In the direction of the applied field but very strong and remains even after the initial field is removed .(ferromagnetism)
   This is characterised by hysteresis effects/loops/

These effects involve the dynamics of orbital electrons of an atom/ free electrons in a metal in a magnetic field, which requires a quantum mechanical description.

We will not focus on "how" the polarisation is acquired.

Magnetic polarisation and its description

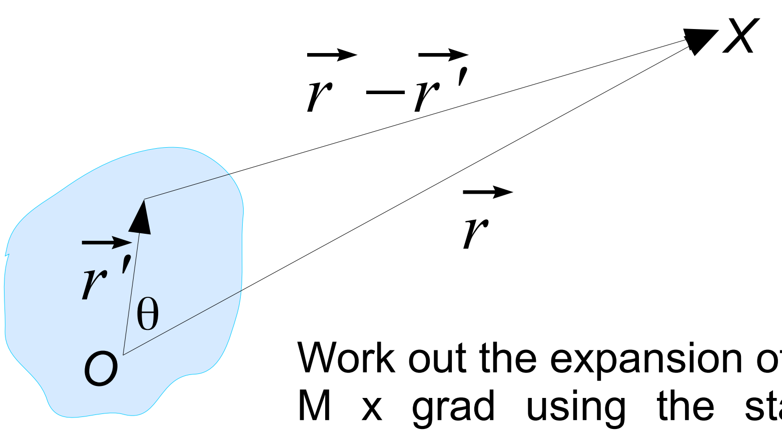
$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol} \frac{\vec{M}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} d^3\vec{r}\,'$$

Recall that

$$\nabla_{r'} \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Hence

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol} \vec{M}(\vec{r}\,')\times\nabla_{r'} \frac{1}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,'$$



Work out the expansion of M x grad using the standard rules. Convert one volume integral of a curl to a surface integral.....

M is the magnetic moment per unit volume.

The unit of **M** can be defined in two ways:

[m] = current x area (so Ampere .m²)
[M] = Am^{-1}

Also [m.B] = energy, hence
[m] = Joule/Tesla
[M] = J/Tesla/m^{-3}

But historically an unit emu has been used. emu =1erg/gauss

From which we can define emu/cc or emu/gm

1 erg/gauss = 10^{-3} Ampere .m²

Atomic magnetic moments are of the order of a "Bohr magneton"

Magnetic polarisation and its description

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol} M(\vec{r}\,')\times\nabla_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|}d^3\vec{r}\,'$$

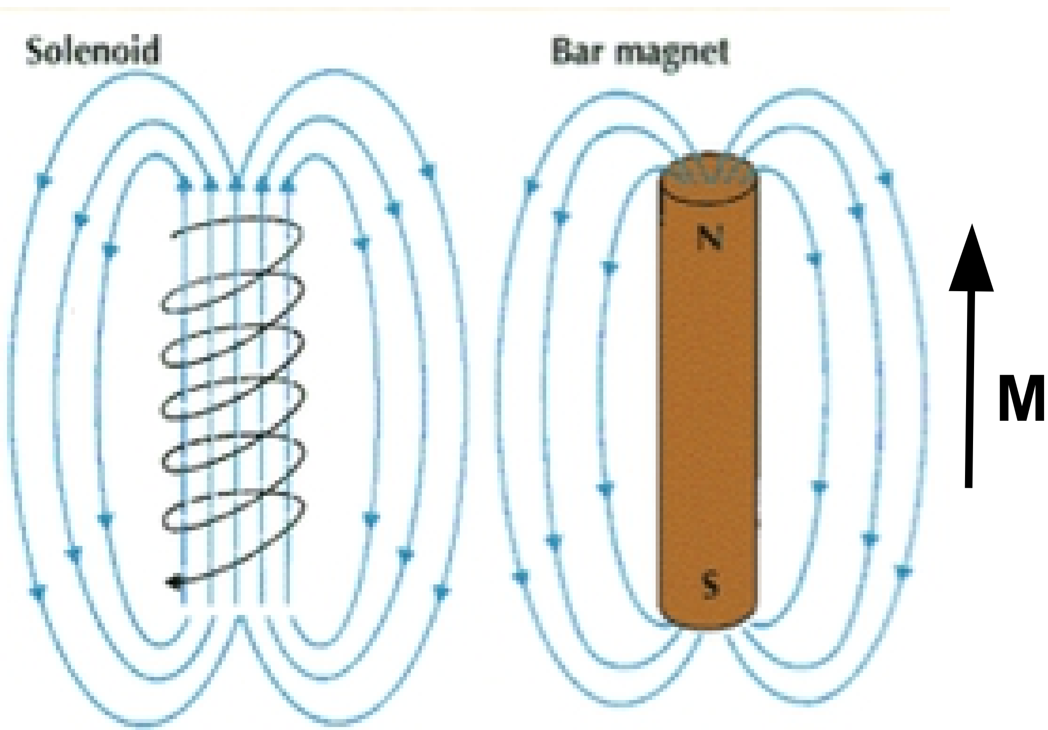
$$= \frac{\mu_0}{4\pi}\left[\int_{vol}\frac{1}{|\vec{r}-\vec{r}\,'|}(\nabla\times\vec{M})\,d\tau + \int_{surf}\frac{1}{|\vec{r}-\vec{r}\,'|}\vec{M}\times d\vec{S}\right]$$

$$\vec{J}_b = \nabla\times\vec{M} \qquad \& \qquad \sigma_b = \vec{M}\times\hat{n}$$

need to use the relation

$$\int_{vol}\nabla\times\vec{A}\,d\tau = -\int_{surf}\vec{A}\times d\vec{S}$$

Contribution from a volume current and a surface current density.



$$\vec{M} = constant$$

$$\nabla\times\vec{M} = 0$$

$$\vec{M}\times\hat{n} = M\,\hat{\epsilon}_\phi$$

mimics the solenoid current

equivalent ampere turns per mt

Magnetic polarisation and its description: **B H M** vectors

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 \underbrace{\left(\vec{J}_f + \vec{J}_b\right)}$$

"Free" current put in by wires, solenoids etc.

"Bound" current due to induced or frozen magnetic dipoles

$$\vec{J}_b = \nabla \times \vec{M} \quad \textit{hence}$$

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f$$

$$\textit{call} \quad \frac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}$$

$$\boxed{\begin{aligned} \nabla \times \vec{H} &= \vec{J}_f \\ \nabla . \vec{H} &= ? \end{aligned}}$$

!! NOT NECESSARILY ZERO!!

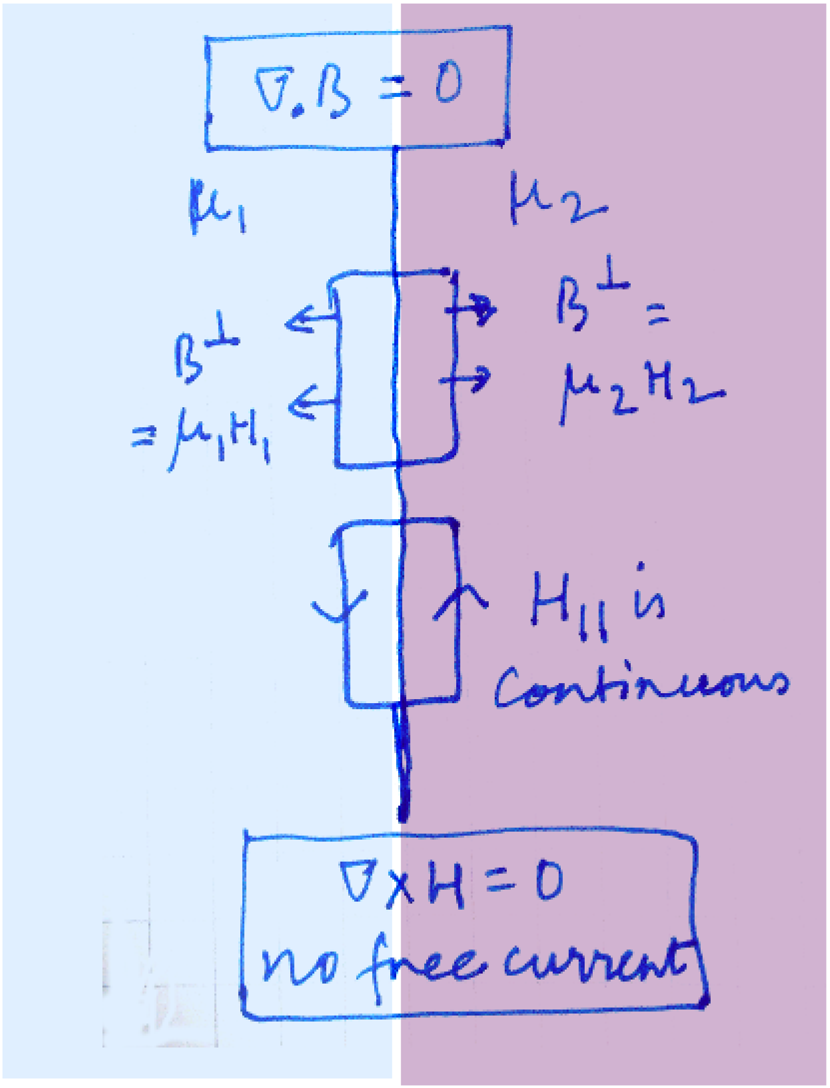
Historically a proportionality between **M** and **H** was emphasized as a material property. This leads to:

$$\begin{aligned} \vec{B} &= \mu_0 \left(\vec{H} + \vec{M} \right) \\ \vec{M} &= \chi \vec{H} \\ \vec{B} &= \mu_0 (1 + \chi) \vec{H} \\ \vec{B} &= \mu \vec{H} \end{aligned}$$

χ is called susceptibility

μ is called permeability

Magnetic polarisation and its description: **B H M** vectors : Boundary conditions



$$\vec{J}_{free} = 0 \qquad (at \quad the \quad interface)$$

$$\mu_1 H_1^{\perp} = \mu_2 H_2^{\perp} \qquad (since\, \nabla . \vec{B} = 0)$$

$$H_1^{\parallel} = H_2^{\parallel} \qquad (since\, \nabla \times \vec{H} = 0)$$

The unit of H is same as the unit of M.
In SI ampere-turn per meter is generally used. It is dimensionally different from Tesla.

In cgs unit of B and H have same dimensionality.
Gauss is used for B
Oersted is used for H

Confusion is very common between B and H

Divergence of H is not necessarily zero : An example

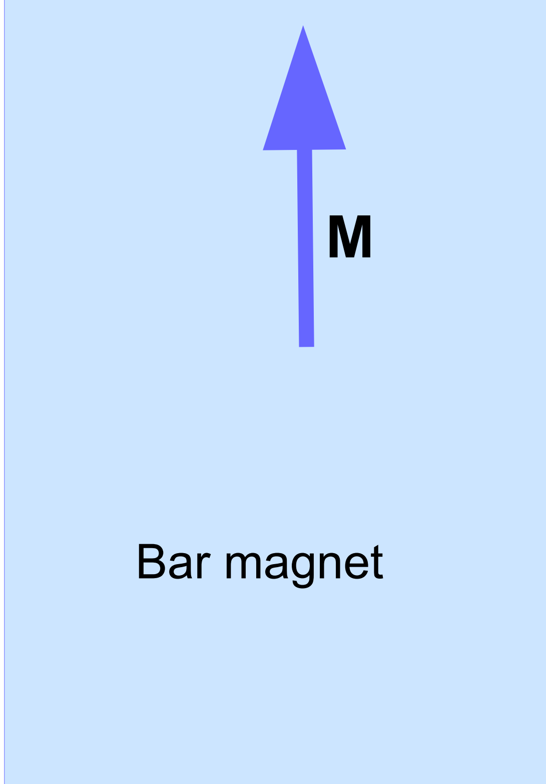
Consider a bar magnet with magnetisation M

$$Air: \vec{H} = \frac{\vec{B}}{\mu_0} \quad (M=0)$$

$$B_{\text{air}}^{\perp} = B_{\text{bar}}^{\perp}$$
$$H_{\text{air}}^{\perp} \neq H_{\text{bar}}^{\perp}$$

$$\nabla \times \vec{H} = 0$$
$$\nabla . \vec{H} = -\nabla . \vec{M} = \rho_m$$
is sometimes useful to describe an assembly of magnets via a potential ϕ such that $\vec{H} = -\nabla \phi$

$$\text{Bar}: \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$



So H can have "sources" and "sinks" like the electric field.

In cases where curl **H** = 0, it is possible to construct a magnetic scalar potential, whose gradient would give **H**.

In older texts "magnetic pole density" etc are used. These are the sources and sinks of H, like electric charge is the source and sink of E.

This leads to some confusion about H being the "real" field, which is wrong!!

Field of an uniformly magnetised sphere

$$\vec{J}_b = \nabla \times M = 0$$

$$\vec{\sigma}_b = \vec{M} \times \hat{n} = M \sin\theta\, \epsilon_\phi$$

Integrate directly to find $\vec{A}$ and then $\vec{B}$

$$\vec{A} = \frac{\mu_0 M R^3}{3} \frac{\sin\theta}{r^2} \hat{\epsilon}_\phi \qquad (r > R)$$

$$\vec{A} = \frac{\mu_0 M}{3} r \sin\theta\, \hat{\epsilon}_\phi \qquad (r < R)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{4\pi R^3 M}{3}\right)\left(\frac{2\cos\theta\,\hat{\epsilon}_r + \sin\theta\,\hat{\epsilon}_\theta}{r^3}\right)$$

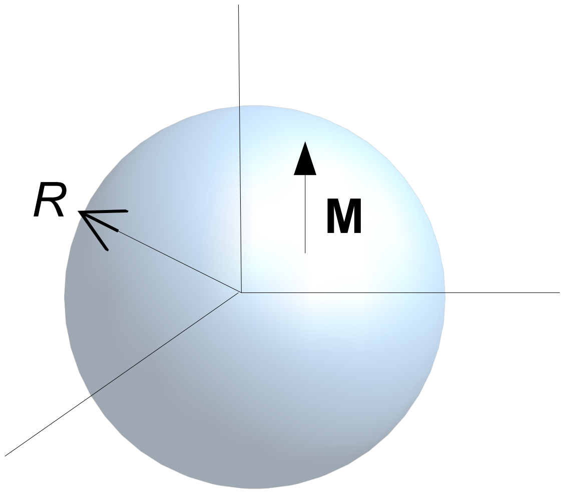
$$\vec{B} = \frac{2\mu_0}{3}\vec{M}$$

Inside the sphere

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = -\frac{\vec{M}}{3}$$

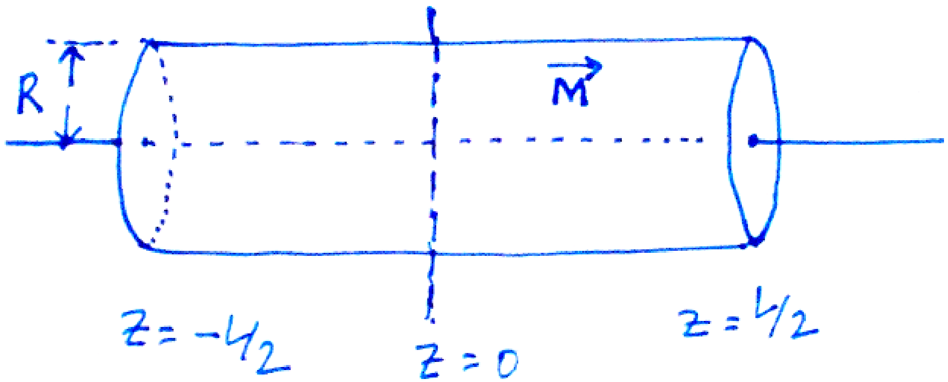
directed opposite to $\vec{B}$ and $\vec{M}$

A somewhat counter-intuitive result!



Field inside is constant
Field outside is that of an equivalent dipole placed at origin.

Field of a cylindrical bar magnet



What happens to $\vec{H}$ inside the bar as $L \to \infty$

Replace the Magnetisation by an equivalent current: M=NI and calculate the axial field due to all the current loops, using the result for a single loop

$$B(0,0,z) = \frac{\mu_0 M}{2}\left[\frac{z+L/2}{\sqrt{R^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{R^2+(z-L/2)^2}}\right]$$

Field just outside : $B \approx \dfrac{\mu_0 M}{2}$ $\quad (z=\pm L/2 \;:\; L \gg R)$

Field very far away : $B \approx \dfrac{\mu_0}{2\pi z^3}.(M\pi R^2 L)$ $\quad (z \gg L)$

Inverse cube fall off of a dipole field

Calculate H and show that it points opposite to M inside the bar.
Typical strong permanent magnets have remnance $\mu_0 M \sim 1\, Tesla$

Typical values of susceptibilities

TABLE 6.1 MAGNETIC SUSCEPTIBILITIES

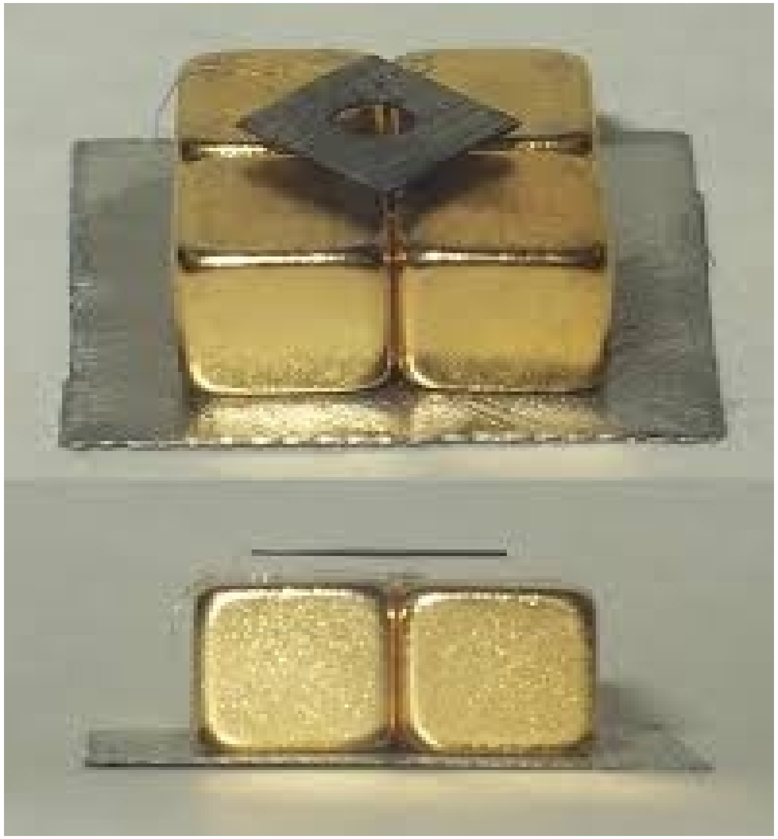
Material	Magnetic Susceptibility
Diamagnetic:	
Bismuth	-16.5×10^{-5}
Gold	-3.0×10^{-5}
Silver	-2.4×10^{-5}
Copper	-0.96×10^{-5}
Water	-0.90×10^{-5}
Carbon Dioxide	-1.2×10^{-8}
Hydrogen	-0.22×10^{-8}
Paramagnetic:	
Oxygen	190×10^{-8}
Sodium	0.85×10^{-5}
Aluminum	2.1×10^{-5}
Tungsten	7.8×10^{-5}
Gadolinium	$48{,}000 \times 10^{-5}$

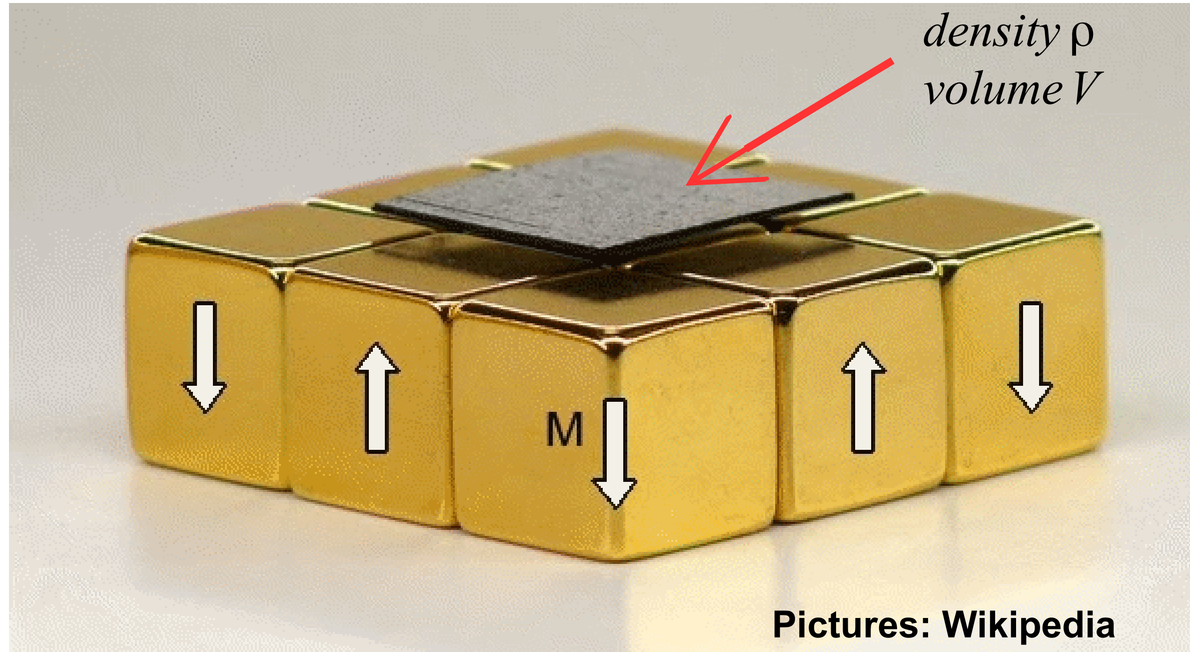
Source: Handbook of Chemistry and Physics, 67th ed. (Cleveland: CRC Press, Inc., 1986–87.) All figures are for atmospheric pressure and room temperature.

Graphite's suscpetibility can be -6e-4 to -1e-5 depending on orientation in SI units

Stability of levitation: Why does it require diamagnets ?

Since magnetic field repels "diamagnets", it can be made to float in a region of strongly varying (large gradient) magnetic field.





Pieces of graphite floating on a strong magnets. The magnets are typically 5-10mm cubes and would have a remnance of 1-2 Tesla.

The height at which these float are typically 1-2 mm

$$\vec{F} = \frac{1}{2}\nabla(\vec{m}.\vec{B}) = \rho V g, \quad where \quad \vec{m} \approx V\frac{\chi}{\mu_0}\vec{B}$$

Stability requires that $\nabla.(\nabla \chi \vec{B}.\vec{B}) < 0$

$But \quad \nabla^2 \vec{B}.\vec{B} = \nabla^2(B_x^2 + B_y^2 + B_z^2)$

$$Show\ that\ : \sum_{ij} \frac{\partial^2}{\partial x_i^2} B_j B_j = 2\sum_j |\nabla B_j|^2$$

$Hence \quad \nabla^2 \vec{B}.\vec{B} > 0 \quad always$

stability requires $\nabla.\vec{F} < 0$ possible only if $\chi < 0$