Dielectric materials

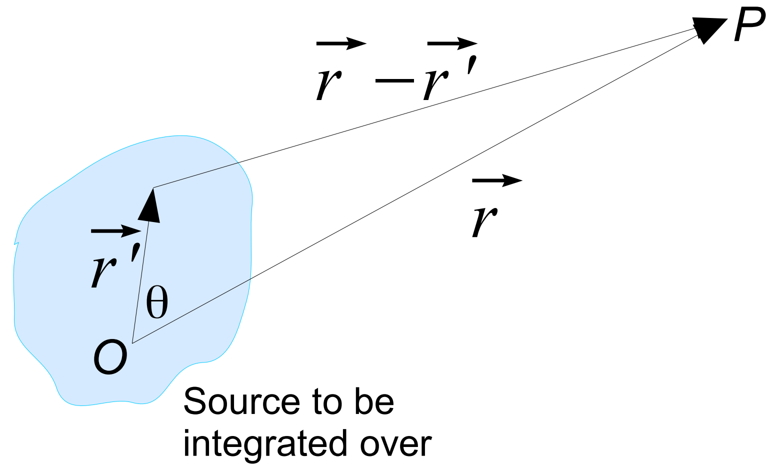
Dielectric materials:

Field of a polarised object at a large distance
Multipole expansion of scalar potential
Polar and cartesian expressions for dipole, quadrupole etc
Atomic and molecular origin of the dipole moment
Equivalent charge distribution
Force and torque on a dipole
Definition of the E D P vectors and boundary conditions
Interface of two dielectrics, sphere in an uniform field
Energy contained in Electric fields with dielectrics present

How does a charge distribution look from far away?

Quantitatively this means : With what power law does it fall off ….inverse square, cube, fourth ?

Often the charge is limited to a small area.



Answer:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

In many cases $r \gg r'$

So expand in a power series in $\frac{r'}{r}$

From the figure :

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \left[r^2 + r'^2 - 2rr'\cos\theta\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left[1 - \left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$$

Legendre polynomials again!

Binomial expansion.....

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 \ldots$$

Multipole expansion of the electrostatic potential

$$V(P) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3\vec{r}\,'\left[\rho(\vec{r}\,')\,r'^{\,l}P_l(\cos\theta)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\Bigg[\underbrace{\frac{1}{r}\int d^3\vec{r}\,'\rho(\vec{r}\,')}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int d^3\vec{r}\,' r'\cos\theta\,\rho(\vec{r}\,')}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\int d^3\vec{r}\,'(r')^2\frac{1}{2}(3\cos^2\theta-1)\rho(\vec{r}\,')}_{\text{quadrupole}} + \ldots\Bigg]$$

If the total charge is zero: Dipole term dominates.

If that is also zero Quadrupole dominates

suppose

$$\rho(\vec{r}\,') = q\,\delta(\vec{r}\,'-\vec{a}) - q\,\delta(\vec{r}\,'+\vec{a})$$

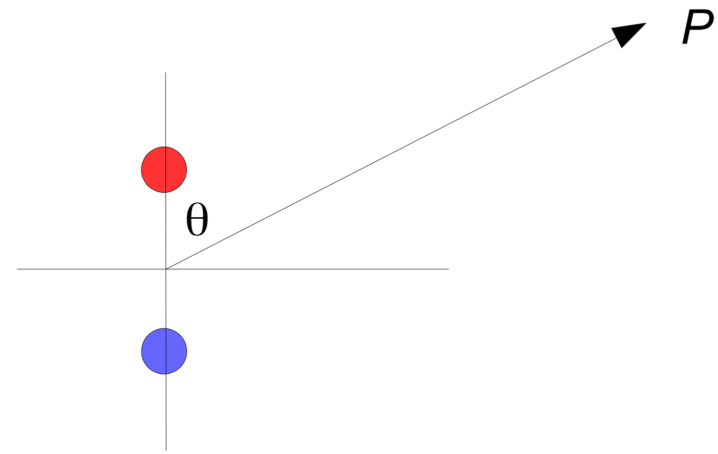
how will the dipole integral look?

can write this as

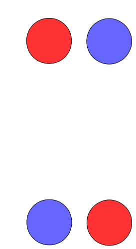
$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{\vec{r}}}{r^2}$$

with $\vec{p} = \sum q_i\vec{r}_i\,'$

and some other equivalent forms...



Stick two monopoles to get a dipole.



Stick two dipoles to get a quadrupole.

Choice of the co-ordinate system and origin in multipole expansion

We could have done the expansion in a more cartesian way...

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = [r^2 + r'^2 - 2\vec{r}.\vec{r}\,']^{-\frac{1}{2}}$$

This would have given successive terms like....

$$V_{mono} = \frac{1}{4\pi\epsilon_0}\frac{Q_{total}}{r}$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i p_i}{r^2}$$

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\frac{\sum_{ij} \hat{r}_i \hat{r}_j Q_{ij}}{r^3}$$

$$p_i = \int d^3\vec{r}\,' r_i' \rho(r')$$

$$Q_{ij} = \int d^3\vec{r}\,' \left(3r_i' r_j' - r'^2 \delta_{ij}\right)\rho(r')$$

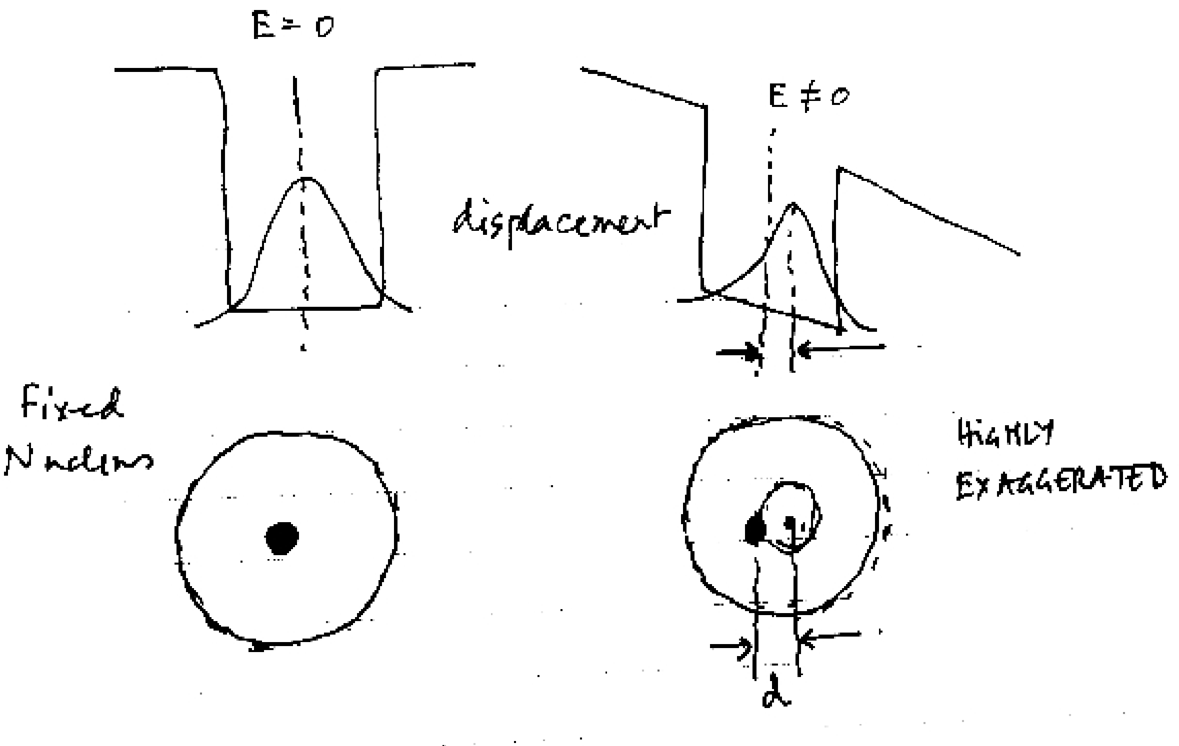
Dipole moment is a vector
Quadrupole moment is a tensor

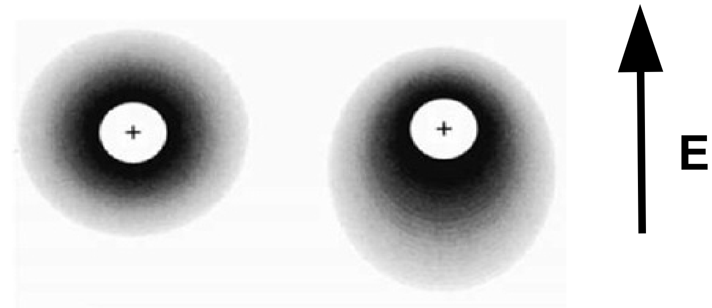
The lowest non-vanishing moment is independent of the choice of the origin.
The higher moments are NOT necessarily so.

So if the total charge (monopole) is zero then dipole term is origin-independent.
If the dipole also vanishes then quadrupole is origin independent. (Prove it!)

Dipole is more common in electronic charge distributions.
Nucleii often have quadrupole moments.
Earth's gravitational potential has a significant quadrupole component.

Response of atoms and molecules to an electric field





Q is the total charge
A is the atomic radius
p = Qd (induced moment)

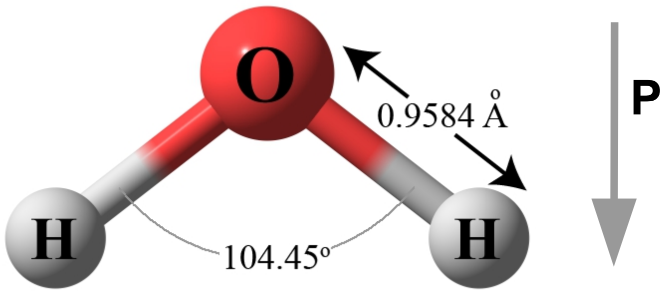
Electron cloud is an uniformly charged sphere..(assume)
Force on the nucleus due to displaced electron cloud = External force on nucleus

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qd}{a^3} \qquad hence \qquad \vec{p} = 4\pi\epsilon_0 a^3 \; \vec{E}$$

$$\sim \; 4\pi\epsilon_0 \times 10^{-30} \quad \text{in SI}$$

Atomic polarizability
Small for inert gases
Large for atoms with partially filled outer shell
Estimated values and observed value agree (order of magnitude)

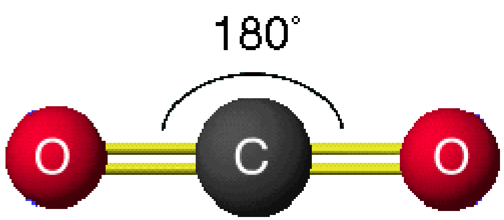
Atoms and molecules in an electric field: frozen moment of molecules



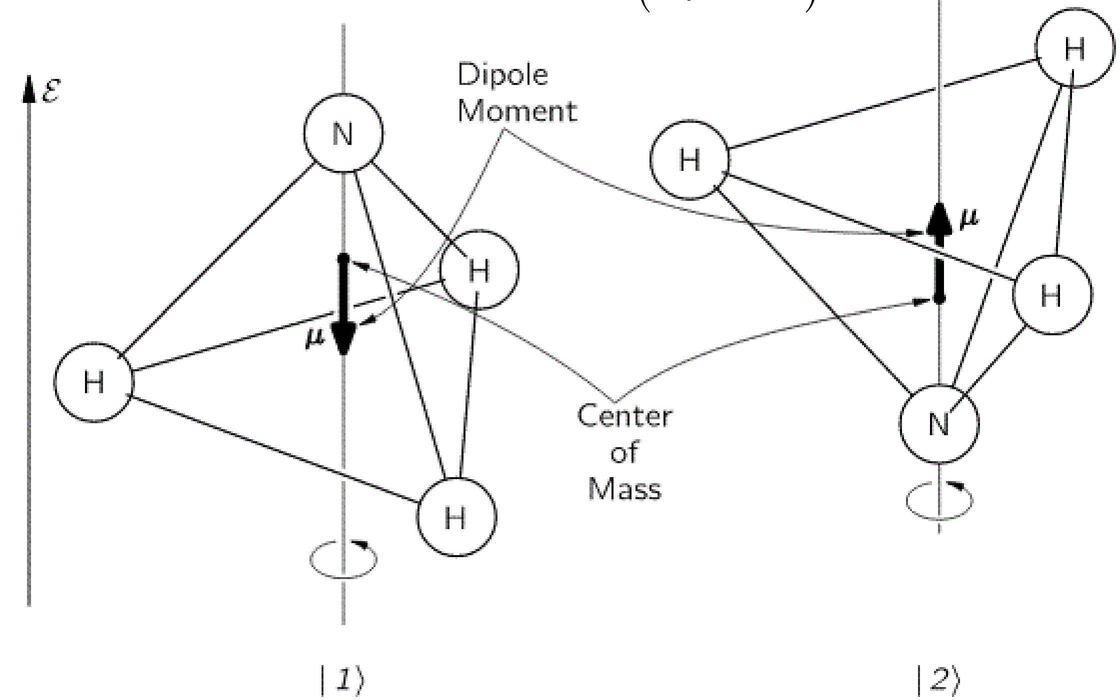
Electron distribution in the bonds can give rise to built in dipole moment

$6.2\times10^{-30}\,Coul.mt$
$(1.85\,D)$

$4.8\times10^{-30}\,Coul.mt$
$(1.42\,D)$



No dipole moment



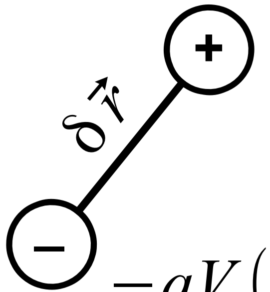
SI unit = Coul-mt.

1 Debye unit (historical but useful)
Dipole moment of 10^{-10} esu of charge separated by 1 angstrom
Useful for molecular scale since
Electron charge is 4.8 $\times10^{-10}$ esu

Induced dipole moment and electric field are not necessarily in the same direction for a molecule. Since the bonds do not shift uniformly in all directions...."easy" and "hard" directions...
P and E are related by a matrix/tensor

Force and torque on a dipole

Potential Energy
and force



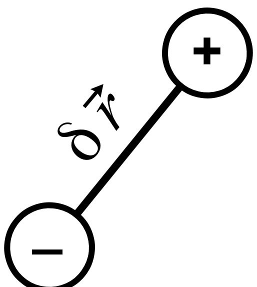
$$qV(\vec{r}+\delta\vec{r})=q(V+\delta\vec{r}\,.\,\vec{\nabla}V)$$

Add the two contributions

$$-qV(\vec{r})$$

$$\begin{aligned}\vec{p} &= q\,\delta\vec{r}\\ U_{dip} &= -\vec{p}\,.\,\vec{E}\\ \vec{F}_{dip} &= (\vec{p}\,.\,\vec{\nabla})\vec{E}\end{aligned}$$

Torque



$$(\vec{r}+\delta\vec{r})\times[q\,\vec{E}(\vec{r}+\delta\vec{r})]$$

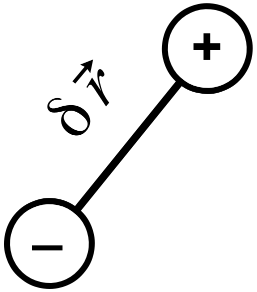
$$\vec{\tau}_{dip} = \vec{p}\times\vec{E}$$

$$\vec{r}\times[-q\,\vec{E}(\vec{r})]$$

Although the E field is different at two sites, the difference in the final expression would be second order.....

Now we can calculate the interaction force between two dipoles....easily!
If we have two dipoles...the E field of the first will act on the second and vice versa,

Correct expression for the force on a dipole



$$\vec{F} = -q\,\vec{E}(\vec{r}) + q\,\vec{E}(\vec{r}+\delta\vec{r})$$

$$F_x = -qE_x(\vec{r}) + q\,E_x(\vec{r}+\delta\vec{r})$$

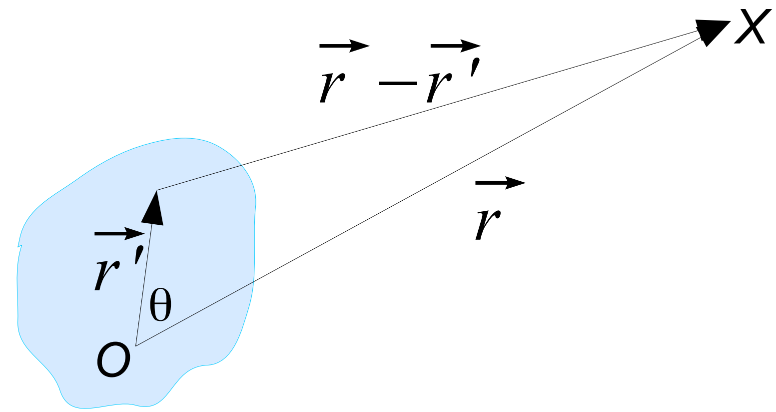
$$\left.\begin{aligned} F_x &= q\,\delta\vec{r}\,.\,\nabla E_x \\ F_y &= q\,\delta\vec{r}\,.\,\nabla E_y \\ F_z &= q\,\delta\vec{r}\,.\,\nabla E_z \end{aligned}\right\} \quad \vec{F} = (\vec{p}\,.\,\nabla)\,\vec{E}$$

Potential of an extended distribution of dipoles

$$V(X) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\vec{P}.\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\nabla_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Prove this by writing out in (x-x').....



Hence

$$V(X) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left[\nabla\frac{\vec{P}}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,'|}\nabla.\vec{P}\right]$$

dipole distribution to be integrated over

$$= \frac{1}{4\pi\epsilon_0}\left[\int\frac{d\vec{S}\,'.\vec{P}}{|\vec{r}-\vec{r}\,'|} - \int d^3\vec{r}\,'\frac{\nabla.\vec{P}}{|\vec{r}-\vec{r}\,'|}\right]$$

Here integration and differentiation are w.r.t. primed co-ordinates

Surface charge

$$\sigma = \vec{P}.\hat{n}$$

Volume charge

$$\rho = -\nabla.\vec{P}$$

Linear Dielectrics : **E P D** vectors

Linear dielectric means : Induced dipole moment (**P**) is proportional to the electric field. Hence:

$$\nabla . \vec{E} = \frac{\rho_{TOTAL}}{\epsilon_0}$$

$$\nabla . \epsilon_0 \vec{E} = \rho_{free} + \rho_{pol} \quad (since \quad \rho_{pol} = -\nabla . \vec{P})$$

$$\nabla . \left[\epsilon_0 \vec{E} + \vec{P}\right] = \rho_{free}$$

$$Use\ the\ proprotionality\ of \quad \vec{P} \quad with \quad \vec{E} : \quad \vec{P} = \epsilon_0 \chi \vec{E}$$

$$\epsilon_0 (1+\chi) \vec{E} = \epsilon \vec{E} = \vec{D}$$

Historically called electric displacement vector:
Microscopic mechanism was not known then.

susceptibility

$$\boxed{\begin{aligned} \nabla . \vec{D} &= \rho_{free} \\ \nabla . \vec{E} &= \frac{\rho_{free}}{\epsilon} \\ \nabla \times \vec{E} &= 0 \end{aligned}}$$

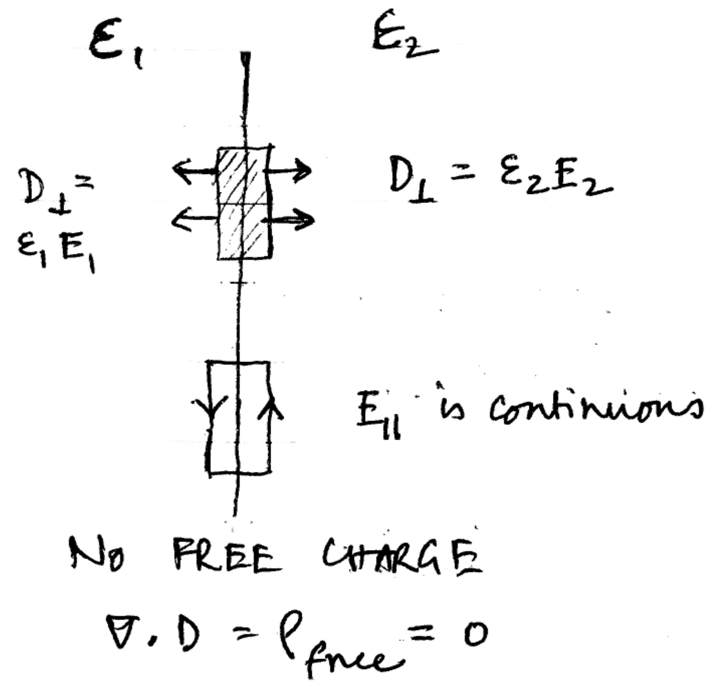
Quantities like D, ε can only be defined in an average sense.

!! One cannot talk about D or ε inside an atom!!

These only make sense if averaged over a few (~10 -100) lattice units.

Since curl **E** =0, a scalar potential is still possible.
But the "source" of this potential is reduced by a factor.
Hence the scalar potential V is also reduced by that factor.

Linear Dielectrics : **E P D** vectors : Boundary conditions and related problems



$$\rho_{free} = 0 \qquad (at \quad the \quad interface)$$
$$\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \qquad (since\, \nabla \cdot \vec{D} = 0)$$
$$E_1^\parallel = E_2^\parallel \qquad (since\, \nabla \times E = 0)$$

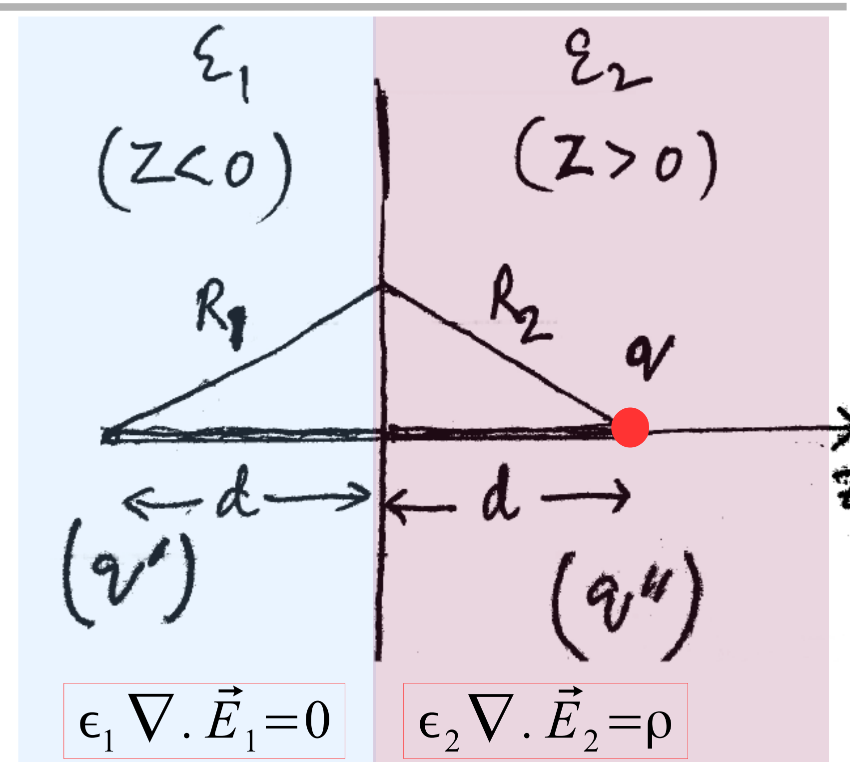
Point charge q is placed at (0,0,d) as shown near an interface of two dielectrics.
Q: What is the potential everywhere?

For z >0 : (region 2)
Image charge q' at (0,0,-d)

For z<0 : (region 1)
Charge q" at (0,0,d)

Write the potential, then the electric field. Two independent equations by matching the normal and tangential components at the boundary.

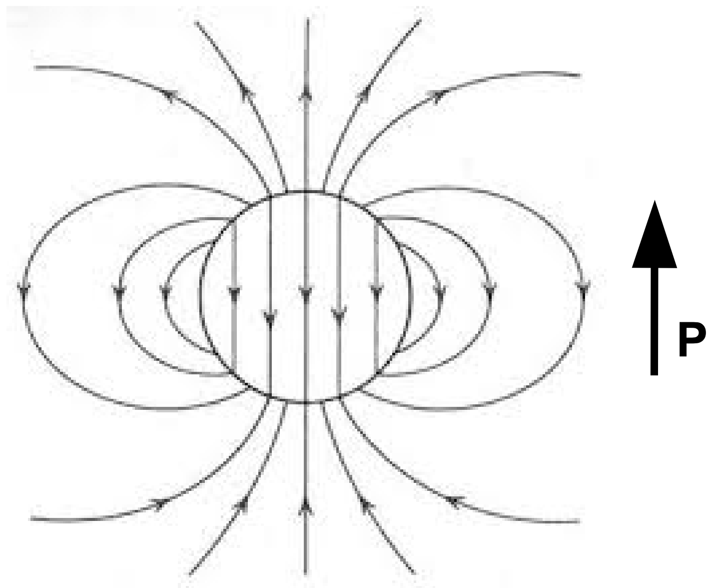
$$q' = -\left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) q$$
$$q'' = \left(\frac{2\epsilon_1}{\epsilon_1 + \epsilon_2}\right) q$$



Linear Dielectrics : A uniformly polarised sphere

Uniformly polarised sphere : (no external field)

Note the lines of force (Electric field):
Points in the opposite direction inside the sphere.



$$V(r,\theta)=\begin{cases}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & (0<r\leqslant R)\\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) & (r\geqslant R)\end{cases}$$

Use boundary conditions at r=R and
V should be well behaved at small and large r...

V(r=R) should match
E should have a discontinuity due to surface charge
Equate the coefficient of each Legendre polynomial

$$\textit{Surface charge} \;:\; \sigma_b = \vec{P}.\hat{n} = P\cos\theta$$
$$\textit{Volume charge} \;:\; \rho_b = -\nabla.\vec{P} = 0$$

$$V(r,\theta)=\begin{cases}\frac{P}{3\epsilon_0} r\cos\theta & (0<r\leqslant R)\\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & (r\geqslant R)\end{cases}$$

Looks like the field of a single (pure) dipole at r=0

The field is CONSTANT inside

Linear Dielectrics : A dielectric sphere in an uniform field

Uniform field means far from the sphere E = E_0 set externally

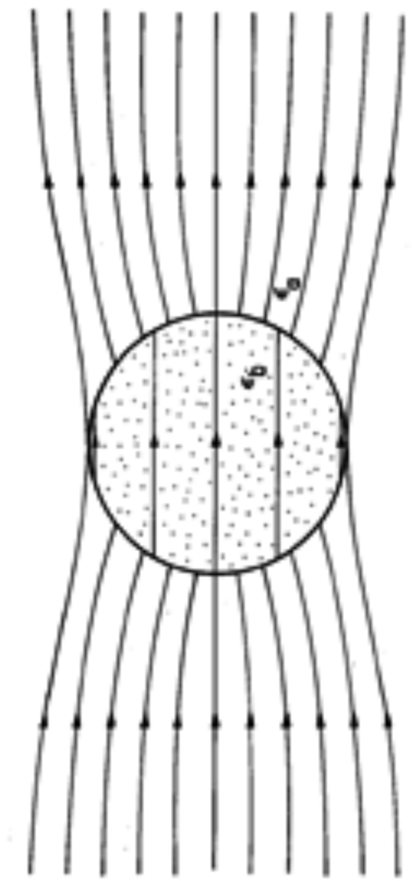
$$V(r,\theta)=\begin{cases}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & (0<r\leqslant R)\\ \sum_{l=0}^{\infty}\left[B_l r^l+\frac{C_l}{r^{l+1}}\right]P_l(\cos\theta) & (r\geqslant R)\end{cases}$$

$$-\epsilon\frac{\partial V_{\text{in}}}{\partial r}\bigg|_{r=R} = -\epsilon_0\frac{\partial V_{\text{out}}}{\partial r}\bigg|_{r=R}$$

Normal component of D is continuous

$$-\frac{1}{R}\frac{\partial V_{\text{in}}}{\partial\theta} = -\frac{1}{R}\frac{\partial V_{\text{out}}}{\partial\theta}$$

Tangential component of E is continuous



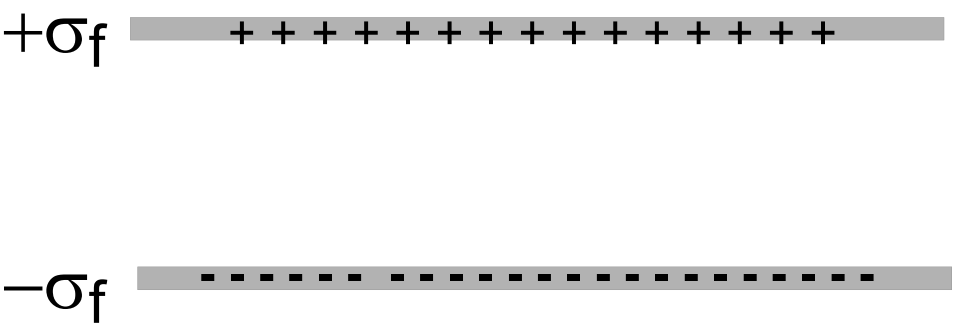
$$V_{\text{in}} = \left(\frac{3}{2+\epsilon/\epsilon_0}\right)E_0 r\cos\theta$$

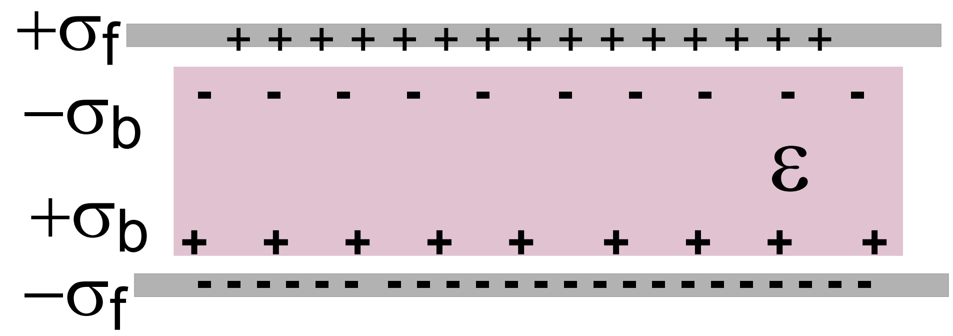
$$V_{\text{out}} = -E_0 r\cos\theta+\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)E_0\frac{R^3}{r^2}\cos\theta$$

What would $\epsilon\to\infty$ *physically mean?*
If a spherical cavity is dug out in a large slab?

Notice how the orthogonality of Legendre polynomials is crucial to solving these problems

Linear Dielectrics : A capacitor with a dielectric slab





$$\frac{V}{d} = E = \frac{\sigma_f}{\epsilon_0}$$

$$\frac{V}{d} = E = \frac{\sigma_f - \sigma_b}{\epsilon_0}$$

$$\sigma_b = \vec{P}\,.\,\hat{n} = \epsilon_0 \chi E$$

$$so \quad E = \frac{\sigma_b}{\epsilon_0 \chi}$$

$$\sigma_f - \sigma_b = \frac{\sigma_f}{1+\chi}$$

$$\frac{V}{d} = E = \frac{\sigma_f}{\epsilon_0 (1+\chi)}$$

For the same charge on the metal plate, the voltage developed is now smaller.

C=Q/V thus increases by a factor of $1 + \chi$

The energy stored in the field also increases by the same factor if the same voltage and hence the same E=V/d is established in the capacitor.

$$\boxed{W = \frac{\epsilon_0}{2}\int d^3\vec{r} \;\; \vec{E}\,.\,\vec{E} \quad \rightarrow \quad W = \frac{1}{2}\int d^3\vec{r}\,\vec{E}\,.\,\vec{D}}$$