MA108 - Spring 2018
Tutorial 4 Final Solutions

1. Given one solution y_1 of homogeneous part, find a particular solution of the ODE by putting $y = vy_1$ into the ODE and solving for v.

(a) $y'' + 4xy' + (4x^2 + 2)y = 8e^{-x(x+2)}, \quad y_1 = e^{-x^2}$.

Answer. General solution is $y(x) = C_1 e^{-x^2} + C_2 x e^{-x^2} + 2e^{-x(x+2)}$.
Particular solution is $y_p(x) = 2e^{-x(x+2)}$.

(b) $4x^2y'' - 4x(x + 1)y' + (2x + 3)y = 4x^{5/2}e^{2x}. \quad y_1 = x^{1/2}$.

Answer. General solution is $y(x) = C_1 e^x x^{1/2} + C_2 x^{1/2} + \frac{1}{2}x^{1/2}e^{2x}$.
Particular solution is $y_p(x) = \frac{1}{2}x^{1/2}e^{2x}$.

(c) $xy'' - y' + 4x^3y = 0, \quad u_1 = \sin x^2$.

Answer. General solution is $y(x) = C_1 \cos x^2 + C_2 \sin x^2$.
Particular solution is $y_p(x) = 0$.

2. Find the general solution of the ODE and then solve the IVP. Here y_1 is a solution of the homogeneous part.

(a) $x^2y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7,\ y'(-1) = 8,\ y_1 = x^2$.

Answer. $y(x) = 6x^2 - 24x^2 \ln |x| + x^4$.

(b) $(3x - 1)y'' - (3x + 2)y' - (6x - 8)y = 0, \quad y(0) = 2,\ y'(0) = 3,\ y_1 = e^{2x}$.

Answer. $y(x) = 2e^{2x} - xe^{-x}$.

3. Find a general solution to the following ODE and IVP where mentioned.

(a) $y''' - y = 0$.

Answer. $y(x) = C_1 e^x + C_2 e^{-\frac{x}{2}} \cos \frac{\sqrt{3}x}{2} + C_3 e^{-\frac{x}{2}} \sin \frac{\sqrt{3}x}{2}$.

(b) $y^{(4)} + 64y = 0$.

Answer. $y(x) = e^{2x}\big(C_1 \cos 2x + C_2 \sin 2x\big) + e^{-2x}\big(C_3 \cos 2x + C_4 \sin 2x\big)$.

(c) $y^{(5)} + y^{(4)} + y'' + y'' + y' + y = 0$.

Answer. $y(x) = C_1 e^{-x} + e^{x/2}\big(C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x\big) + e^{-x/2}\big(C_4 \cos \frac{\sqrt{3}}{2}x + C_5 \sin \frac{\sqrt{3}}{2}x\big)$.

(d) $y^{(4)} + 2y'' + y = 0$.

Answer. $(C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x$.

(e) $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 0,\ y'(0) = -2,\ y''(0) = 0$.

Answer. $y(x) = -\sin 2x$.

(f) $y'' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1,\ y'(0) = -1,\ y''(0) = -4$.

Answer. $y(x) = e^{2x}\big(1 - 3x + 2x^2\big)$.

(g) $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5,\ y'(0) = -2,\ y''(0) = 6,\ y'''(0) = 8$.

Answer. $y(x) = e^{2x} + e^{-2x} + e^{-x}\big(3\cos x + \sin x\big)$.

4. Find the fundamental set of solutions for the following equations.

(a) $(D^2 + 9)^3 D^2 y = 0$

Answer. $\{1, x, \cos 3x, x\cos 3x, x^2 \cos 3x, \sin 3x, x \sin 3x, x^2 \sin 3x\}$.

(b) $D^3(D-2)^2(D^2+4)^2 y = 0$.

Answer. $\{1, x, x^2, e^{2x}, xe^{2x}, \cos 2x, \sin 2x, x\cos 2x, x \sin 2x\}$.

(c) $\big((D-1)^4 - 16\big)y = 0$.

Answer. $\{e^{3x}, e^{-x}, e^x \cos 2x, e^x \sin 2x\}$.

5. Using the annihilator method, find the general solution (i.e. $y = y_p + C_1 y_1 + C_2 y_2$, where y_p is a particular solution of ODE and y_1, y_2 is a basis of solutions of homogeneous part.)

(a) $y'' - 2y' - 3y = e^x(-8 + 3x)$.

Answer. $y(x) = C_1 e^{-x} + C_2 e^{3x} + 2e^x \big(1 - \frac{3}{8}x\big)$.

(b) $y'' + 5y' + 6y = \cos x + \sin x$.

Answer. $y(x) = C_1 e^{-3x} + C_2 e^{-2x} + \frac{1}{5}\sin x$.

(c) $y''' - y'' - y' + y = 2e^{-x} + 3$.

Answer. $y(x) = e^x(C_1 + C_2 x) + C_3 e^{-x} + \frac{1}{2}xe^{-x} + 3$.

6. Find the form of the particular solution (without explicitly finding the particular solution) of ODEs.

(a) $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$.

Answer. $y_p(x) = e^{-x}(a_0 + a_1 x + a_2 x^2) + e^{3x}(b_0 + b_1 x + b_2 x^2)$.

(b) $y'' + 6y' + 13y = e^{-2x}\big(e^{-2x}(4 + 20x)\cos 2x + (26 - 32x)\sin 3x\big)$.

Answer.
$y(x) = e^{-3x}\big(C_1 \cos 2x + C_2 \sin 2x\big) + e^{-2x}\big((A + Bx)\cos 3x + (C + Dx)\sin 3x\big)$.

(c) $y'' + 2y' + y = 8x^2 \cos x - 4x \sin x$.

Answer. $(a_0 + a_1 x + a_2 x^2)\cos x + (b_0 + b_1 x + b_2 x^2)\sin x$.

(d) $y^{(4)} - 4y'' = 3x + \cos x$.

Answer. $a_1 x^2 + a_2 x^3 + a_3 \cos x + a_4 \sin x$.

(e) $y''' - y'' - y' + y = e^x(7 + 6x)$.

Answer. $y_p(x) = a_1 x^2 e^x + a_2 x^3 e^x$.

(f) $4y^{(4)} - 11y'' - 9y' - 2y = -e^x(1 - 6x)$.

Answer. $y_p(x) = a_1 e^x + a_2 x e^x$.

(g) $y''' + 3y'' + 4y' + 12y = 8\cos 2x - 16\sin 2x$.

Answer. $y_p(x) = a_1 x \cos 2x + a_2 x \sin 2x$.

(h) $y^{(4)} + 3y''' + 2y'' - 2y' - 4y = -e^{-x}(\cos x - \sin x)$.

Answer. $y_p(x) = a_1 e^{-x} x \cos x + a_2 e^{-x} x \sin x$.

7. Consider the ODE $Ly = f$. Find the annihilator A for f. Then write down a basis for the solutions of the equation $ALy = 0$.

(a) $y''' - 2y'' + y' = x^3 + 2e^x$.

Answer. $A = D^4(D-1)$.
Basis is $\{1, x, x^2, x^3, x^4, e^x, xe^x, x^2e^x\}$.

(b) $y^{(4)} - y''' + y'' + y' = x^2 + 4 + x\sin x$.

Answer. $A = D^3(D^2+1)^2$.
Basis is $\{1, x, x^2, x^3, e^x, xe^x, e^{-x}, \cos x, x\cos x, \sin x, x\sin x\}$.

(c) $y^{(4)} + 4y'' = \sin 2x + xe^x + 4$.

Answer. $A = (D^2+4)(D-1)^2 D$.
Basis is $\{1, x, x^2, e^x, xe^x, \cos 2x, x\cos 2x, \sin 2x, x\sin 2x\}$.

(d) $y''' - 2y'' + y' - 2y = -e^x\big((9 - 5x + 4x^2)\cos 2x - (6 - 5x - 3x^2)\sin 2x\big)$.

Answer. $A = \big((D-1)^2 + 4\big)^3$.
Basis is $\{e^{2x}, \cos x, \sin x, e^x\cos 2x, xe^x\cos 2x, x^2e^x\cos 2x, e^x\sin 2x, xe^x\sin 2x, x^2e^x\sin 2x\}$.

(e) $y^{(4)} - 7y''' + 18y' - 20y' + 8y = e^{2x}(3 - 8x - 5x^2)$.

Answer. $A = (D-2)^3$.
Basis is $\{e^{2x}, xe^{2x}, x^2e^{2x}, x^3e^{2x}, x^4e^{2x}, x^5e^{2x}, e^x\}$.

(f) $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x) - e^{-2x}$.

Answer. $A = (D+1)^2(D+2)$.
Basis is $\{e^{-x}, xe^{-x}, x^2e^{-x}, x^3e^{-x}, x^4e^{-x}, e^{-2x}, xe^{-2x}\}$.