MA-106 Linear Algebra

H. Ananthnarayan

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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D1 - Lecture 15

Random Attendance

1	170050089	Aditya Vavre	
2	170070005	Diwan Anuj Jitendra	
3	170070008	Anugole Sai Gaurav	
4	170070014	Shriram Girish Lokhande	Absent
5	170070039	Vishesh Verma	
6	170070042	Pinnada Preetam	
7	170070048	Bojja Sai Vamseedhar Reddy	Absent
8	17D070026	Anubhav Agarwal	
9	17D070041	Tanmay Goyal	
10	170050095	Narisim Setti Charan Teja	
11	170070006	Manav Sanjay Batavia	
12	170070009	Anwesh Mohanty	
13	170070020	Ghoderao Kusumit Sudesh	
14	170070029	Chandra Shekhar	Absent
15	170070040	Akshat Singhal	
16	170070059	Ranjeet Patel	
17	17D070004	Kavishwar Mihir Vikas	
18	17D070038	Divyansh Ahuja	Absent

Determinants and Invertibility

10. A is invertible if and only if $\det(A) \neq 0$.
    By elimination, we get an upper triangular matrix U, a lower triangular matrix L with diagonal entries 1, and a product of permutation matrices P, such that $PA = LU$.

    Observation 1: If A is singular, then $\det(A) = 0$.
    This is because elimination produces a zero row in U and hence $\det(A) = \pm \det(U) = 0$.

    Observation 2: If A is invertible, then $\det(A) \neq 0$.
    This is because elimination produces n pivots, say $d_1, \ldots, d_n$, which are non-zero. Then U is upper triangular, with diagonal entries $d_1, \ldots, d_n \Rightarrow \det(A) = \pm \det(U) = \pm\, d_1 \cdots d_n \neq 0$.

Thus we have: A invertible $\Rightarrow \det(A) = \pm$(product of pivots).

Exercise: If AB is invertible, then so are A and B.

Determinants of Transposes

11.
$$\boxed{\det(A) = \det(A^T)}$$

With U, L, and P, as usual write
$$PA = LU \Rightarrow A^T P^T = U^T L^T$$
Since U and L are triangular, we get
$\det(U) = \det(U^T)$ and $\det(L) = \det(L^T)$
Since $PP^T = I$ and $\det(P) = \pm 1$, we get $\det(P) = \det(P^T)$.
Thus $\det(A) = \det(A^T)$. $\square$

Exercise: A is invertible if and only if A^T is invertible.

Formula for Determinant - 2×2 case

Write $(a, b) = (a, 0) + (0, b)$, the sum of vectors in coordinate directions. Similarly write $(c, d) = (c, 0) + (0, d)$. By linearity,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}.$$

For an $n \times n$ matrix, each row splits into n coordinate directions, so the expansion of $\det(A)$ has n^n terms.

However, when two rows are in same coordinate direction, that term will be zero, e.g.,

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = -\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = -bc \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The non-zero terms have to come in different columns. So, there will be $n!$ such terms in the $n \times n$ case.

Reading Slide - Defining Properties: 2×2 case

Known to us: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

1. $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$
2. $\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - da = -(ad - bc) = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$
3. $\det \begin{pmatrix} a + a' & b + b' \\ c & d \end{pmatrix} = (a + a')d - (b + b')c =$

   $(ad - bc) + (a'd - b'c) \quad = \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$, and

   $\det \begin{pmatrix} ta & tb \\ c & d \end{pmatrix} = t(ad - bc) = t \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
4. Similarly, check linearity property in second row.

Formula for Determinant: 3×3 case

If $A = (a_{ij})$ is 3×3 matrix, then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix}$$

$$+ a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

Formula for Determinant: 3×3 case

$$= a_{11}\, a_{22}\, a_{33}\, (1) + a_{11}\, a_{23}\, a_{32}\, (-1) + a_{12}\, a_{21}\, a_{33}\, (-1)$$
$$+ a_{12}\, a_{23}\, a_{31}\, (1) + a_{13}\, a_{21}\, a_{32}\, (1) + a_{13}\, a_{22}\, a_{31}\, (-1)$$

$$= \sum_{\text{all permutations } P} (a_{1\alpha}\, a_{2\beta}\, a_{3\gamma})\, \det(P)$$

where P runs over all permutation matrices.

If $(\alpha, \beta, \gamma) = (2, 3, 1)$, then $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = P_{13} P_{12}$.

Here $\det(P) = (-1)^2 = 1$.

Formula for Determinant: $n \times n$ case

For $n \times n$ matrix $A = (a_{ij})$,

$$\det(A) = \sum_{\text{all permutations } P} (a_{1\alpha_1} \ldots a_{n\alpha_n}) \det(P).$$

The sum is over $n!$ permutations of numbers $(1, \ldots, n)$.

Here a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ corresponds to the product of permutation matrices $P = \begin{bmatrix} e_{i_1}^T \\ \vdots \\ e_{i_n}^T \end{bmatrix}$.

Then $\det(P) = +1$ if the number of row exchanges in P needed to get I is even, and -1 if it is odd.

Reading Slide - Cofactors: 3×3 Case

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}\,a_{22}\,a_{33}\,(1) + a_{11}\,a_{23}\,a_{32}\,(-1) + a_{12}\,a_{21}\,a_{33}\,(-1) \\
&\quad + a_{12}\,a_{23}\,a_{31}\,(1) + a_{13}\,a_{21}\,a_{32}\,(1) + a_{13}\,a_{22}\,a_{31}\,(-1) \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \text{ where,}
\end{aligned}
$$

$$
C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ C_{12} = (-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.
$$

Cofactors: $n \times n$ Case

Let C_{1j} be the coefficient of a_{1j} in the expansion

$$\det(A) = \sum_{\text{all permutations } P} (a_{1\alpha_1} \ldots a_{n\alpha_n}) \det(P)$$

Then $\boxed{\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}}$ where,

$$\begin{aligned} C_{1j} &= \Sigma a_{2\beta_2} \ldots a_{n\beta_n} \det(P) \\ &= (-1)^{1+j} \det \begin{bmatrix} a_{21} & \cdot & a_{2(j-1)} & a_{2(j+1)} & \cdot & a_{2n} \\ \vdots & . & \vdots & . & \vdots & \\ a_{n1} & \cdot & a_{n(j-1)} & a_{j+1} & \cdot & a_{nn} \end{bmatrix} \\ &= (-1)^{1+j} \det(M_{1j}), \end{aligned}$$

where M_{1j} is obtained from A by deleting the 1^{st} row and j^{th} column.

Expansion of det(A) along its i-th row

If C_{ij} is the coefficient of a_{ij} in the formula of det(A), then

$\boxed{\det(A) = a_{i1}\, C_{i1} + \ldots + a_{in}\, C_{in}}$, where C_{ij} is determined as follows:

By $i-1$ row exchanges on A, get the matrix

$B = \begin{pmatrix} A_{i*} & A_{1*} & .. & A_{(i-1)*} & A_{(i+1)*} & .. & A_{n*} \end{pmatrix}^T$

Since $\det(A) = (-1)^{i-1}\det(B)$, we get

$$C_{ij}(A) = (-1)^{i-1} C_{1j}(B) = (-1)^{i-1}(-1)^{j-1}\det(M)$$

where M is obtained from B by deleting 1^{st} row and j^{th} column. Here M is obtained from B by deleting its first row, and j-th column, and hence from A by deleting i-th row and j-th column. Write M as M_{ij}.

Then $\boxed{C_{ij} = (-1)^{i+j}\det(M_{ij})}$

Expansion of $\det(A)$ along j-th column of A

Note that $\qquad C_{ij}(A^T) = C_{ji}(A).$

Hence, if we write $A^T = (b_{ij})$, then

$$\begin{aligned}\det(A) &= \det(A^T) \\ &= b_{j1}\, C_{j1}(A^T) + \ldots + b_{jn}\, C_{jn}(A^T) \\ &= a_{1j}\, C_{1j}(A) + \ldots + a_{nj}\, C_{nj}(A)\end{aligned}$$

This is the expansion of $\det(A)$ along j-th column of A.

Computing determinants: Exercise

Example: Let $F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & -1 \\ & & & 1 & 1 \end{vmatrix}$ be a $(1, 1, -1)$ tri-diagonal $n \times n$ matrix. Expanding along the first row, we get

$$F_n = F_{n-1} + (-1)^{1+2}(-1) \begin{vmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & 1 & 1 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & -1 \\ & & & & 1 & 1 \end{vmatrix} = F_{n-1} + F_{n-2},$$

by expanding along first column.

Since $F_1 =$ __, $F_2 =$ __, the sequence F_n is __, __, __, $\ldots$.

Computing determinants: Examples

Find the determinants for the following examples

- $E = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{bmatrix}.$

Find the cofactor matrix C of E. Compute EC^T

$$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$EC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note: $\det(E) = 4$ and $EC^T = 4I \Rightarrow E^{-1} = \dfrac{1}{\det(E)} C^T.$