Tutorial-3, MA 106 (Linear Algebra)
Most of these problems are from reference texts for this course

1. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = c$ (a new constant vector) to have exactly one solution? Is it possible for $Ax = c$ to be inconsistent?

2. If $Ax = b$ has two solutions x_1 and x_2, find:

   (a) two solutions to $Ax = 0$ and (b) another solution to $Ax = b$.

3. Solve the following system of equations

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 0 \\ -4x_1 - 4x_2 - 8x_3 &= 0 \\ -3x_2 + 3x_3 &= 0 \end{aligned}$$

   and

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 + 3x_3 &= 12 \end{aligned}$$

   (a) How are these two solution sets related?
   (b) Give a geometric description of the solution sets.
   (c) Are either of these solutions sets a subspace of $\mathbb{R}^3$.

4. Fill in the blanks.

   (a) Suppose column 4 of a 3×5 matrix is all 0s. Then x_4 is certainly a ____ variable. The special solution corresponding to x_4 is $x =$ ____.
   (b) If A is an invertible 8×8 matrix, then its column space is ____. Why?
   (c) If the 9×12 system $Ax = b$ is solvable for every b, then $C(A) =$ ____.
   (d) Suppose $\mathbf{P}$ is a plane in $\mathbb{R}^3$ through the origin, and $\mathbf{L}$ is a line in $\mathbb{R}^3$ through the origin. The smallest subspace containing $\mathbf{P}$ and $\mathbf{L}$ is either ____ or ____.
   (e) If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it does not.

5. If the r pivot variables come first, the reduced R must look like $R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$, where I is $r \times r$, and F is $r \times (n-r)$. What is the null space matrix containing the special solutions?

6. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Under what conditions on b does $Ax = b$ have a solution? Find two vectors in $N(A)$ and a complete solution to $Ax = b$.

7. Find q (if possible) so that the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{pmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 3 \\ q & 2 & q \end{pmatrix}.$$

8. Let $u = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$, $v = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$. Use the fact that $2u - 3v - w = 0$ to solve the system.
$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

9. Construct a matrix whose column space contains $(1,1,1)$ and whose nullspace is the line of multiples of $(1,1,1,1)$.

10. Reduce A and B to their echelon forms, find their ranks, the free and the dependent variables.
$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$
Find the special solutions to $Ax = 0$ and $Bx = 0$, and their nullspaces.

11. Reduce the matrices A and B to their echelon forms U. Find a special solution for each variable and describe all solutions in the nullspace.
$$A = \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{pmatrix}.$$
Reduce the echelon forms U to R, find the rank r and draw a box around the $r \times r$ identity matrix in R.

12. Given a 4×4 matrix A with three pivot positions,
    (a) does the equation $Ax = 0$ have a non-trivial solution?
    (b) does the equation $Ax = b$ have a least one solution for every possible b?

    Repeat the above exercise when A is a 3×2 matrix with two pivot positions.

13. Let $u = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$. Does u belong to the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

14. How many pivots should a 6×4 matrix have if its columns are linearly independent? Why?

15. Find values of h for which the following set is linearly dependent. $\left\{ \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ h \\ -9 \end{pmatrix} \right\}$

16. Mark all the correct options.
    (a) The solutions of $Ax = 0$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ form
    (i) a plane (ii) a line (iii) a point (iv) a subspace of $\mathbb{R}^2$.
    (v) a subspace of $\mathbb{R}^3$ (vi) the nullspace of A (vii) the column space of A.
    (b) A is $m \times n$ with row reduced form R. Mark all the statements that define rank of A.
    (i) The number of nonzero rows in R. (ii) $n - m$.
    (iii) $n -$ number of free columns. (iv) The number of 1's in R.
    (v) The number of dependent variables. (vi) $\min\{m, n\}$.

17. Determine by inspection whether the following sets of vectors are linearly independent.

(i) $\left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right\}$. (ii) $\left\{ \begin{pmatrix} -8 \\ 12 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \right\}$. (iii) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

18. Prove or disprove.

(a) The set of nonsingular 2×2 matrices is not a vector space

(b) The set of singular 2×2 matrices is not a vector space.

(c) Let $B = [A|b]$. The system $Ax = b$ is solvable exactly when $C(A) = C(B)$.

(d) A system of equations $Ax = 0$ where A is a square matrix has no free variables.

(e) A system of equations $Ax = 0$ where A is an invertible matrix has no free variables.

(f) An $m \times n$ matrix has no more than $\min\{m, n\}$ pivot variables.

(g) Any linear combination of vectors can always be written as Ax for appropriate choices of matrices A and column vector x.

(h) If A is a $m \times n$ matrix whose columns do not span $\mathbb{R}^m$ then the equation $Ax = b$ is consistent for every $b \in \mathbb{R}^m$.

(i) If $v_1, \ldots, v_4$ are in $\mathbb{R}^4$ and $\{v_1, v_2, v_3\}$ is linearly independent then $\{v_1, v_3, v_4\}$ is linearly independent.

(j) If S_1 and S_2 are subsets of a vector space V, then $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) \cup \text{Span}(S_2)$.

19. Let V be a vector space, $v_1, v_2, v_3 \in V$, W_1 and W_2 be subspaces of V. Prove or disprove:
(i) $W_1 \cap W_2$ is a subspace of V. (ii) $W_1 \cup W_2$ is a subspace of V.
(iii) $W_1 + W_2 = \{u + v \mid u \in W_1, v \in W_2\}$ is a subspace of V.
(iv) $V \setminus W_1 = \{u \in V \mid u \notin W_1\}$ is a subspace of V.
(v) W = set of all possible linear combinations of v_1, v_2 and v_3.
(vi) $W' = \{a_1 v_1 + a_2 v_2 + a_3 v_3 \mid a_1 \geq 0\}$.

20. Which of the following are subspaces of $\mathbb{R}^3$?
(i) The plane of vectors (b_1, b_2, b_3) with (i) $b_1 = 0$. (ii) $b_1 = 1$.
(ii) The set of vectors (b_1, b_2, b_3) with $b_2 b_3 = 0$.
(iii) All linear combinations of the vectors $(1, 1, 0)$ and $(2, 0, 1)$.
(iv) The plane of vectors (b_1, b_2, b_3) satisfying $b_3 - b_2 + 3b_1 = 0$.

21. Let M be vector of 3×3 matrices. Are the following true or false?
(i) The symmetric matrices in M (i.e., $A = A^T$) form a subspace.
(ii) The skew symmetric matrices in M (i.e., $A = -A^T$) form a subspace.
(iii) The non-symmetric matrices in M (i.e., $A \neq A^T$) form a subspace.
(iv) The set of upper triangular matrices in $\mathcal{M}$ form a subspace.
(v) The matrices that have $(1, 1, 1)$ in their nullspace form a subspace.

22. Let $V = \mathcal{C}[0, 1]$, the vector space of continuous real-valued functions on the closed interval $[0, 1]$. Which of the following are subspaces of V? Justify.
(i) $W_0 = \{f \in V \mid f(0) = 1\}$ (ii) $W_1 = \{f \in V \mid f(1) = 0\}$
(iii) W_2 = set of polynomials of degree 2.
(iv) $W_3 = \{f \mid f$ is a real valued function on $[0, 1]$ such that $\int_0^1 f(x)dx$ is finite.$\}$
(v) $\mathcal{C}^1[0, 1]$ the set of differentiable real-valued functions on $[0, 1]$
(vi) $\mathcal{C}^\infty[0, 1]$ the set of infinitely differentiable real-valued functions on $[0, 1]$.
(vii) $\mathcal{P}_2$ = set of polynomials of degree at most 2.

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