MA-108 Differential Equations I

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D1-D3 - 1st week lectures

Class Information

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- Reference Text : Elementary Differential Equations by William Trench available at ramanujan.math.trinity.edu/wtrench/texts/index.shtml
- Two short quiz of 5 marks each on 21st March and 18th April in the tutorial classes during 3:00-3:10 PM.
- Main quiz of 30 marks on 4th April from 8:15-9:15 AM.
- End Semester exam of 60 marks.
- Minimum passing marks is 30.
- Be Honest. Cheating in exams will give you atleast an FR grade in the course.

Definition

Let $y = y(x)$ be an unknown function of x.
An Ordinary differential equation (ODE) is an equation involving atleast one derivative of y.
The order of an ODE is the highest order of derivative of y occuring in the ODE.

Example

(1) $y' = x^2y^2 + x$ is a 1st order ODE.
(2) $y'' + 2xy' + y = \sin x$ is a 2nd order (linear) ODE.

Definition

An ODE of order n is called linear if it can be written as

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x),$$

If $a < b$ are real numbers, then

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

is an open interval.
$\mathbb{R} = (-\infty, \infty)$ is also an open interval.
$\mathbb{R} - \{0\}$ is not an open interval. It is union of two open intervals.

$$\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$$

Definition

An (explicit) solution of an ODE is a function $y = f(x)$ which satisfies the ODE on some open interval.

First simple example of an ODE

Consider the linear (homogeneous) ODE $y' + ay = 0$, $a \in \mathbb{R}$.
Note that $y \equiv 0$ is the (trivial) solution.
Let $y = y(x)$ be a non-trivial solution, i.e. $y(x) \neq 0$.
Since y is a continuous function, there exists an open interval, say I in $\mathbb{R}$ such that y does not take 0 value on I.
Let us solve the ODE on I.

$$
\begin{aligned}
y' + ay = 0 &\implies \frac{y'}{y} = -a \\
&\implies \frac{d}{dx} \ln |y| = -a \\
&\implies \ln |y| = -ax + c \\
&\implies |y| = e^c e^{-ax} \\
&\implies y(x) = Ce^{-ax},
\end{aligned}
$$

is a solution of $y' + ay = 0$ on $I = (-\infty, \infty)$, where $C = e^c$ when $y(x) > 0$ and $C = -e^c$ when $y(x) < 0$ on I.

1st order linear homogeneous ODE

Consider the ODE with $a(x)$ continuous on an open interval I,

$$y' + a(x)y = 0 \qquad (1)$$

Let $y = y(x)$ be a non-trivial solution, i.e. $y(x) \neq 0$.
Since y is a continuous function, there exists an open interval, say $J \subset I$ such that $y(x)$ does not take 0 value on J.

$$\begin{aligned} y' + a(x)y = 0 &\implies y'/y = -a(x) \\ &\implies \ln|y| = -\int a(x)\,dx + c \\ &\implies |y| = e^c e^{-\int a(x)\,dx} \\ &\implies y(x) = Ce^{-\int a(x)\,dx} \end{aligned}$$

$C = e^c$ when $y(x) > 0$ and $C = -e^c$ when $y(x) < 0$ on J.
Thus, $y(x) = Ce^{-\int a(x)\,dx}$ is a solution of (1) on $J = I$.

Theorem

Let $p(x)$ be a continuous function on an open interval (a, b). Then the general solution of

$$y' + p(x)y = 0 \qquad (1)$$

on the interval (a, b) is $\boxed{y(x) = Ce^{-P(x)}}$*,*
where $P(x)$ is any anti-derivative of $p(x)$ on (a, b), i.e.

$$P'(x) = p(x), \quad x \in (a, b)$$

- *General solution means $y(x) = Ce^{-P(x)}$ is a solution of (1) for all choices of $C \in \mathbb{R}$.*
- *Further, any solution of (1) can be obtained from the general solution for some choice of C.*
- *This may not be true for non-linear ODEs.*

Second simple example of an ODE

Consider the linear (non-homogeneous) ODE

$$y' + ay = f(x) \qquad (1)$$

where $f(x)$ is continuous on some open interval I.
The solution of $y' + ay = 0$ is $y_1(x) = e^{-ax}$ on $\mathbb{R}$.
Let us try to look for a solution of (1) of the type
$y = u(x)e^{-ax}$.

Substituting into the differential equation (1), we get on I

$$\begin{aligned} & u'e^{-ax} - aue^{-ax} + aue^{-ax} = f(x) \\ \Longrightarrow\ & u' = f(x)e^{ax} \\ \Longrightarrow\ & u(x) = \int f(x)e^{ax}\,dx + C \end{aligned}$$

Thus

$$y(x) = e^{-ax}\Big(\int f(x)e^{ax}\,dx + C\Big)$$

is a solution of (1) on the (open) interval I.

1st order Linear non-homogeneous ODE

Let $p(x)$ and $f(x)$ be continuous on (a, b). Let us solve

$$y' + p(x)y = f(x) \qquad (1)$$

$y' + p(x)y = 0$ is the Complementary equation of (1).
Let $y_1(x) = e^{-\int p(x)\,dx}$ be a solution of C.E.
Substitute $y(x) = u(x)y_1$ into ODE, we get

$$\begin{aligned} & u'y_1 + uy_1' + p(x)uy_1 = f(x) \\ \Longrightarrow\ & u'y_1 = f(x) \\ \Longrightarrow\ & u(x) = \int f(x)e^{\int p(x)dx} + C \end{aligned}$$

$$\Longrightarrow \ y(x) = e^{-\int p(x)dx}\left(\int f(x)e^{\int p(x)dx} + C\right)$$

is the general solution of (1) on (a, b).

Theorem (Existence Theorem)

Let $p(x)$ and $f(x)$ be continuous functions on an open interval (a, b). Then the general solution of

$$y' + p(x)y = f(x) \qquad (1)$$

on the interval (a, b) is

$$y(x) = e^{-\int p(x)} \left(\int f(x) e^{\int p(x)dx}\, dx + C \right) \qquad (2)$$

- *General solution means $y(x)$ in (2) is a solution of (1) for all choices of $C \in \mathbb{R}$.*
- *Further, any solution of (1) can be obtained from the general solution for some choice of C.*
- *This may not be true for non-linear ODEs.*

Example

Solve $\quad y' + 2y = x^3 e^{-2x}$. $\qquad$ (1)

C.E. $\quad y' + 2y = 0$ has a solution $y_1(x) = e^{-2x}$.

The solution of (1) is $y = uy_1$

$$\begin{aligned} & u' y_1 = x^3 e^{-2x} \\ \Longrightarrow\ & u' = x^3 \\ \Longrightarrow\ & u(x) = x^4/4 + C \end{aligned}$$

Therefore,

$$y(x) = e^{-2x}(x^4/4 + C)$$

is a solution of ODE on $\mathbb{R}$.

Example

(1) Solve $y' - 2xy = 1$.
C.E. $y' - 2xy = 0$ has a solution $y_1(x) = e^{\int 2x\,dx} = e^{x^2}$.
The solution of ODE is $y = uy_1$, where

$$\begin{aligned} u'y_1 &= 1 \\ \Longrightarrow\ u(x) &= \int e^{-x^2} dx + C \\ \Longrightarrow\ y(x) &= e^{x^2}\left(\int e^{-x^2} dx + C\right) \end{aligned}$$

(2) Solve the IVP $\boxed{y' - 2xy = 1,\ y(0) = y_0}$.
Write the solution of ODE as

$$\begin{aligned} y(x) &= e^{x^2}\left(\int_0^x e^{-x^2} dx + C\right) \\ y(0) = y_0 &\Longrightarrow C = y_0 \end{aligned}$$

Definition

An Initial value problem (IVP) for 1st order ODE is

$$y' = F(x, y), \quad y(x_0) = y_0.$$

A function $y = y(x)$ defined on some open interval (a, b) containing x_0 is a solution of the IVP if y satisfies the ODE on (a, b) and $y(x_0) = y_0$.

Theorem (Existence and Uniqueness Theorem for IVP)

Let $p(x)$ and $f(x)$ be continuous functions on an interval (a, b). Let $x_0 \in (a, b)$ and $y_0 \in \mathbb{R}$. Then the IVP

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a unique solution on (a, b).

Definition

Let $y(x)$ be an explicit solution of IVP

$$y' = F(x,y), \quad y(x_0) = y_0$$

on some open interval containing x_0.
The interval of validity of $y(x)$ is the largest open interval containing x_0 where $y(x)$ is a solution of IVP.

Example

The function

$$y = (x^2/3) + (1/x)$$

satisfies

$$xy' + y = x^2$$

on $(-\infty, 0) \cup (0, \infty)$.

- For IVP
$$xy' + y = x^2, \quad y(1) = 4/3$$
the interval of validity of $y(x)$ is $(0, \infty)$.
- For IVP
$$xy' + y = x^2, \quad y(-1) = -2/3$$
the interval of validity of $y(x)$ is $(-\infty, 0)$.

Definition

- An explicit solution of an ODE is a function $y = y(x)$ which satisfies the ODE on some open interval (a, b).
- A solution curve of an ODE is the graph of an explicit solution of the ODE.
- An implicit solution of an ODE is an equation $g(x, y) = 0$ that gives an explicit solution of the ODE on some open interval.
- A curve C is an integral curve of an ODE if the following holds: If the graph of a function $y = f(x)$ is a portion of the curve C, then $y = f(x)$ is a solution of the ODE.
- An integral curve C of an ODE is the curve defined by an implicit solution of the ODE.

Note that a solution curve is also an integral curve, but an integral curve may not be a solution curve, since an integral curve C may not be the graph of a single function.

Example

Circle C defined by $x^2 + y^2 = 1$ is an integral curve of

$$y' = -x/y$$

Only functions whose graph is a segment of C are

$$y_1 = \sqrt{1 - x^2}, \quad y_2 = -\sqrt{1 - x^2}$$

on $(-1, 1)$.
So graphs of y_1 and y_2 are solution curves.
But C is not a solution curve as C is not the graph of a function.

Separation of variable method : 1st order ODE

Assume that the ODE can be written in the form

$$h(y)y' = g(x)$$

Let $H(y)$ and $G(x)$ be antiderivatives of $h(y)$ and $g(x)$ respectively. Then

$$\frac{d}{dy}H(y) = H'(y) = h(y), \quad G'(x) = g(x)$$

Then our ODE is

$$\frac{d}{dx}H(y) = H'(y)y' = \frac{d}{dx}G(x)$$

Integrating, we get

$$H(y) = G(x) + C$$

This is an implicit solution of ODE.

Separable ODE's

Example

Solve $\qquad y' = 2xy^2$.

Assume $y \neq 0$. Rewrite ODE as

$$\frac{1}{y^2} y' = 2x$$

Integrating, we get

$$\begin{aligned} \frac{-1}{y} &= x^2 + C \\ \implies y &= \frac{-1}{x^2 + C} \end{aligned}$$

The solution $y \equiv 0$ cannot be obtained for any choice of C.

Example

Solve IVP

$$y' = 2xy^2, \quad y(0) = y_0$$

and find the interval of validity.
The solution is

$$y = \frac{-1}{x^2 + C}$$

- If $y_0 = 0$, the solution is $y \equiv 0$ and the interval of validity is $\mathbb{R}$.
- If $y_0 \neq 0$, then $C = -\dfrac{1}{y_0}$. Hence $y = \dfrac{-y_0}{y_0 x^2 - 1}$.
- If $y_0 < 0$, the solution is defined for all x. Hence the interval of validity is $\mathbb{R}$.
- If $y_0 > 0$, the solution is valid when $x \in \mathbb{R} - \{\pm 1/\sqrt{y_0}\}$. Hence the interval of validity is $\left(\dfrac{-1}{\sqrt{y_0}}, \dfrac{1}{\sqrt{y_0}}\right)$.

Example

Solve IVP

$$y' = \frac{y\cos x}{1+2y^2}; \quad y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\begin{array}{rcl} \frac{1+2y^2}{y}y' & = & \cos x \\ \ln|y| + y^2 & = & \sin x + c \\ y(0) = 1 & \Longrightarrow & c = 1 \\ \ln|y| + y^2 & = & \sin x + 1 \end{array}$$

is an implicit solution of IVP.
Note: $y \equiv 0$ is a solution to the ODE, but it is not a solution to the given IVP.

Linear vs Non-Linear ODE

Theorem (Existence and Uniqueness of solution : $y' = f(x,y)$)

Let $D = (a,b) \times (c,d)$ be an open rectangle containing the point (x_0, y_0) and consider the IVP

$$y' = f(x,y), \quad y(x_0) = y_0$$

- *(Existence) Assume $f(x,y)$ is continuous on D. Then IVP has at least one solution on some interval $(a_1, b_1) \subset (a,b)$ containing x_0.*
- *(Uniqueness) If both $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution on some interval $(a', b') \subset (a,b)$ containing x_0.*

• Let us prove that if $p(x), f(x)$ are continuous on (a,b), and $x_0 \in (a,b)$ then IVP

$$y' + p(x)y = f(x), \quad y(x_0) = y_0 \qquad (*)$$

has a unique solution on (a,b).
We know that IVP has one solution

$$y(x) = e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx} f(x)\,dx + C_0\right)$$

on (a,b) for some C_0.
Let $Y(x)$ be another solution of IVP on (a,b). Let us write the ODE as

$$y' = F(x,y) := f(x) - p(x)y$$

Then $F(x,y)$ and $\dfrac{\partial F}{\partial y} = -p(x)$ are continuous on the open rectangle $(a,b) \times (-\infty, \infty)$.

By previous Existence and Uniqueness theorem $(*)$ has a unique solution on some interval $x_0 \in (a', b') \subset (a, b)$. Therefore,

$$y(x) \equiv Y(x) \quad x \in (a', b')$$

We need to show that $a = a'$ and $b = b'$.
Let $a < a'$. This means uniqueness holds only in the interval (a', b'). Let

$$\lim_{x \to a'+} y(x) = \lim_{x \to a'+} Y(x) = c$$

The IVP

$$y' + p(x)y = f(x), \quad y(a') = c$$

has a unique solution $y(x)$ on some interval $(a' - \epsilon, a'')$. This means uniqueness of $y(x)$ holds on $(a' - \epsilon, b')$. This contradicts that $a < a'$.
Similarly, prove that $b = b'$.

- Consider

$$y' = F(x, y), \quad y(x_0) = y_0$$

with $F(x, y)$ and $\dfrac{\partial F}{\partial y}$ continuous on $\mathbb{R}^2$.

Then it does not give that the solution is defined on $\mathbb{R}$.

For an example, the IVP

$$y' = 2xy^2, \;\; y(0) = 1$$

the solution

$$y(x) = \frac{-1}{x^2 - 1}$$

is defined on $(-1, 1)$ only.

But on whatever interval the solution is defined, it will be unique.

Linear vs Non-Linear ODE

- For the solution of a non-linear ODE, the interval where the solution exists, depends on the choice of our initial condition.
- The general solution of a non-linear ODE involving an arbitrary constant, may not give all solutions.
- For example, for non-linear ODE $y' = 2xy^2$, our solution $y = -1/(x^2 + C)$ does not give the solution $y \equiv 0$ for any value of C.
- In an implicit solution of a non-linear ODE, not every value of C will give an actual solution.

Example

The circle $x^2 + y^2 = C$ is an implicit solution of $yy' = -x$. For $C = -1$, it does not give any solution to ODE, since the curve $x^2 + y^2 = -1$ is empty.

Example

Consider the IVP

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0 \quad (*)$$

$$\begin{aligned} f(x,y) &= \frac{x^2 - y^2}{1 + x^2 + y^2}, \\ \frac{\partial f}{\partial y} &= \frac{-2y}{1 + x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(1 + x^2 + y^2)^2} \\ &= \frac{-2y(1 + 2x^2)}{(1 + x^2 + y^2)^2} \end{aligned}$$

Since $f(x,y)$ and $\partial f / \partial y$ are continuous for all $(x,y) \in \mathbb{R}^2$,

by existence and uniqueness theorem, for any $(x_0, y_0) \in \mathbb{R}^2$, IVP has a unique solution on some open interval containing x_0.

Example

Consider the IVP $\quad y' = f(x, y), \quad y(x_0) = y_0 \quad (*)$

$$\begin{aligned} f(x,y) &= \frac{x^2 - y^2}{x^2 + y^2} \\ \frac{\partial f}{\partial y} &= \frac{-2y}{x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(x^2 + y^2)^2} \\ &= \frac{-4x^2 y}{(x^2 + y^2)^2} \end{aligned}$$

f and $\partial f / \partial y$ are continuous for all $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$.

Assume $(x_0, y_0) \neq (0, 0)$.

- There is an open rectangle R containing (x_0, y_0) but not containing $(0, 0)$.
- $f(x, y)$ and $\partial f / \partial y$ are continuous on R.
- By existence and uniqueness theorem, $(*)$ has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{x+y}{x-y}, \quad y(x_0) = y_0 \qquad (*)$$

If

$$f(x,y) = \frac{x+y}{x-y}, \quad \text{then} \quad \frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}$$

Here $f(x,y)$ and $\partial f/\partial y$ are continuous everywhere except on the line $y = x$.

Assume $x_0 \neq y_0$.

- There is an open rectangle R containing (x_0, y_0) that does not intersect with the line $y = x$.
- $f(x,y)$ and $\partial f/\partial y$ are continuous on R.
- By existence and uniqueness theorem, $(*)$ has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{10}{3}\,xy^{2/5}, \quad y(x_0) = y_0 \quad (*)$$

$$f(x,y) = \frac{10}{3}\,xy^{2/5} \ \text{ and } \ \frac{\partial f}{\partial y} = \frac{4}{3}\,xy^{-3/5}$$

- Since $f(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$, IVP $(*)$ has atleast one solution for all $(x_0, y_0) \in \mathbb{R}^2$.
- If $y \neq 0$, then $f(x,y)$ and $\partial f/\partial y$ both are continuous for all $(x,y) \in \mathbb{R}^2$.
- If $y_0 \neq 0$, there is an open rectangle R containing (x_0, y_0) s.t. f and $\partial f/\partial y$ are continuous on R.
  Hence IVP $(*)$ has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{10}{3}\, x y^{2/5}, \quad y(0) = 0 \qquad (*)$$

Since $\dfrac{\partial f}{\partial y} = \dfrac{4}{3}\, x y^{-3/5}$ is not continuous if $y = 0$,
(*) may have more than one solution on every open interval containing $x_0 = 0$.
$y \equiv 0$ is one solution of IVP $(*)$.

Let y be a non-zero solution of ODE.

$$\begin{aligned} \frac{y'}{y^{2/5}} &= (10/3)\, x \\ (5/3)\, y^{3/5} &= (5/3)\,(x^2 + C) \\ y(x) &= (x^2 + C)^{5/3} \end{aligned}$$

Example (continued ...)

Note that $y(x) = (x^2 + C)^{5/3}$ is defined for all (x, y) and

$$y' = \frac{5}{3}\,(x^2 + C)^{2/3}(2x) = \frac{10}{3}\,xy^{2/5}, \qquad \forall\, x \in (-\infty, \infty)$$

Thus $y(x)$ is a solution on $\mathbb{R}$ for all C.

$$y(0) = 0 \implies C = 0$$

Thus, the IVP

$$y' = \frac{10}{3}y^{2/5}, \quad y(0) = 0 \qquad\qquad (*)$$

has two solutions, $y_1 \equiv 0$ and $y_2(x) = x^{10/3}$.

We can construct two more solutions of IVP (*). How?

Example

Consider the IVP

$$y' = \frac{10}{3}\, xy^{2/5}, \quad y(0) = -1 \qquad (*)$$

$$f(x, y) = \frac{10}{3}\, xy^{2/5}, \quad \frac{\partial f}{\partial y} = \frac{4}{3} xy^{-3/5}$$

are continuous in an open rectangle containing $(0, -1)$. Hence the IVP has a unique solution on some open interval containing $x_0 = 0$.

Question. Find the unique solution and its interval of validity.

Let $y \neq 0$ be the solution of $y' = (10/3)\, xy^{2/5}$. Then

$$\begin{aligned} y(x) &= (x^2 + C)^{5/3} \\ y(0) &= -1 \implies C = -1 \\ \implies y(x) &= (x^2 - 1)^{5/3} \end{aligned}$$

Example (continued ...)

- $y(x) = (x^2 - 1)^{5/3}$ is a solution on $(-\infty, \infty)$ of IVP

$$y' = (10/3)\, xy^{2/5}, \quad y(0) = -1$$

  Hence interval of validity of this solution is $\mathbb{R}$.

- We have seen that if $y_0 \neq 0$, then the IVP

$$y' = (10/3)\, xy^{2/5}, \quad y(x_0) = y_0$$

  has a unique solution on some open interval around x_0.

- $y(x) = (x^2 - 1)^{5/3}$ is non-zero on $(-1, 1)$. Therefore, $y(x)$ is the unique solution on $(-1, 1)$.

  To see this, If $w(x)$ is another solution on $(-1, 1)$. Then $w(x) \equiv y(x)$ on some interval (ϵ', ϵ) containing 0.
  We need to show that $\epsilon = 1$ and $\epsilon' = -1$.

Example (continued ...)

- If $\epsilon \neq 1$, then $w(\epsilon) = y(\epsilon) = c \neq 0$ as w and y are continuous. Hence there exists a unique solution of ODE with IV $y(\epsilon) = c \neq 0$. Hence $w \equiv y$ on an open interval around ϵ. Thus $\epsilon = 1$. Similarly, $\epsilon' = -1$.
- $(-1, 1)$ is the largest interval on which the ODE with IV $y(0) = -1$ has a **unique** solution.
  To see this, we can define another solution

$$y_1(x) = \begin{cases} (x^2 - 1)^{5/3} & , \quad -1 \leq x \leq 1 \\ 0 & , \quad |x| > 1 \end{cases}$$

Exercise. Find largest interval where the IVP

$$y' = \frac{10}{3} xy^{2/5}, \quad y(0) = 1$$

has a unique solution.

Transforming Non-Linear into Separable ODE

A non-linear differential equation

$$y' + p(x)y = f(x)y^r$$

where $r \in \mathbb{R} - \{0, 1\}$ is said to be a **Bernoulli Equation**.
For $r = 0, 1$, it is linear.
If $y_1 = e^{-\int p(x)\,dx}$ is a non-zero solution of $y' + p(x)y = 0$, then putting $y = u(x)y_1$ in ODE, we get

$$\begin{aligned} & u'y_1 + uy_1' + puy_1 = fu^r y_1^r \\ \implies & u'y_1 = fu^r y_1^r \\ \implies & \frac{u'}{u^r} = f(x)(y_1(x))^{r-1} \\ \implies & \frac{u^{-r+1}}{-r+1} = \int f(x)(y_1(x))^{r-1}\,dx + C \end{aligned}$$

Example (Bernoulli Equation)

Consider

$$y' + y = xy^2$$

Set $y = u(x)e^{-x}$, where $y_1 = e^{-x}$ is solution of homogeneous part.

$$\begin{aligned}
& u'e^{-x} - ue^{-x} + ue^{-x} = u^2e^{-2x}x \\
\implies & u'e^{-x} = u^2e^{-2x}x \\
\implies & \frac{u'}{u^2} = xe^{-x} \\
\implies & \frac{-1}{u} = -(1+x)e^{-x} + C \\
\implies & u = \frac{1}{(1+x)e^{-x} - C} \\
\implies & y = \frac{e^{-x}}{(1+x)e^{-x} - C} = \frac{1}{1 + x - Ce^{x}}
\end{aligned}$$

Example

Consider Bernoulli equation

$$xy' - 2y = \frac{x^2}{y^6} \implies y' - \frac{2}{x}y = \frac{x}{y^6}$$

The solution to homogeneous part is $y_1 = x^2$.
Set $y = u(x)y_1$,

$$\begin{aligned}
u'y_1 &= x(uy_1)^{-6} \\
u^6u' &= x(x^2)^5 = x^{11} \\
\frac{1}{7}u^7 &= \frac{1}{12}x^{12} + C \\
(1/7)y^7 &= \left[(1/12)x^{12} + C\right] y_1^7 \\
y^7 &= \left[(7/12)x^{12} + 7C\right] x^{14}
\end{aligned}$$

is an implicit solution.
We do not have an explicit solution.

Homogeneous Non-Linear Equations

Definition

An ODE

$$y' = f(x, y)$$

is said to be **homogeneous** if it can be written as

$$y' = q(y/x)$$

Substitute $y = v(x)x$ in homogeneous ODE, we get

$$v'x + v = q(v)$$

This is a separable ODE.

Example

Solve

$$xy' = y + x$$

Rewrite it as

$$y' = \frac{y}{x} + 1$$

This is homogeneous ODE.

Substitute $y = vx$. We get

$$\begin{aligned} & v'x + v = v + 1 \\ \Longrightarrow\ & v'x = 1 \\ \Longrightarrow\ & v' = 1/x \\ \Longrightarrow\ & v(x) = \ln|x| + C \\ \Longrightarrow\ & y = x(\ln|x| + C) \end{aligned}$$