MA-106 Linear Algebra

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22nd January 2018
D1 - Lecture 8

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Vector Spaces: Definition

Defn. A non-empty set V is a vector space if it is *closed under* vector addition (i.e., if x, y are in V, then $x + y$ must be in V) and scalar multiplication, (i.e., if x is in V, a is in $\mathbb{R}$, then $a * x$ must be in V).

Equivalently, $\boxed{x, y \text{ in } V,\ a, b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ must be in } V.}$

- A vector space is a triple $(V, +, *)$ with vector addition $+$ and scalar multiplication $*$, with $+$ and $*$ satisfying some additional properties (see next slide).
- The elements of V are called vectors and the scalars are chosen to be real numbers (for now).
- If the scalars are allowed to be complex numbers, then V is a *complex* vector space.

Vector Spaces: Definition continued

Let x, y and z be vectors, a and b be scalars. The vector addition and scalar multiplication are also required to satisfy:

- $x + y = y + x$ $\boxed{\text{Commutativity of addition}}$
- $(x + y) + z = x + (y + z)$ $\boxed{\text{Associativity of addition}}$
- There is a unique vector 0, such that $x + 0 = x$
  $\boxed{\text{Existence of additive identity}}$
- For each x, there is a unique $-x$ such that $x + (-x) = 0$
  $\boxed{\text{Existence of additive inverse}}$
- $1 * x = x$ $\boxed{\text{Unit property}}$
- $(a + b) * x = a * x + b * x, \quad a * (x + y) = a * x + a * y$
  $(ab) * x = a * (b * x)$ $\boxed{\text{Compatibility}}$

Notation: For a scalar a, and a vector x, we denote $a * x$ by ax.

Vector Spaces: Examples

1. $V = 0$, the space consisting of only the zero vector.
2. $V = \mathbb{R}^n$, the n-dimensional space.
3. $V = \mathbb{R}^\infty =$ sequences of real numbers, e.g., $x = (0, 1, 0, 2, 0, 3, 0, 4, \ldots)$, with component-wise addition and scalar multiplication.
4. $V = \mathcal{M}$, the set of $m \times n$ matrices, with entry-wise $+$ and $*$.
5. $V = \mathcal{P}$, the set of polynomials, e.g. $1 + 2x + 3x^2 + \cdots + 2018x^{2017}$, with term-wise $+$ and $*$.
6. $V = \mathcal{C}[0, 1]$, the set of continuous real-valued functions on the closed interval $[0, 1]$. e.g., x^2, e^x are vectors in V. Vector addition and scalar multiplication are pointwise: $(f + g)(x) = f(x) + g(x)$ and $(a * f)(x) = af(x)$.

Subspaces: Definition and Examples

If V is a vector space, and W is a non-empty subset, then W is a subspace of V if:

$$\boxed{x, y \text{ in } W, \quad a, b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ are in } W.}$$

$$\boxed{\text{i.e., linear combinations stay in the subspace.}}$$

Examples:

1. $\{0\}$: The zero subspace and $\mathbb{R}^n$ itself.
2. $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ is not a subspace of $\mathbb{R}^2$. Why?
3. The line $x - y = 1$ is not a subspace of $\mathbb{R}^2$. Why?
   Exercise: A line not passing through the origin is not a subspace of $\mathbb{R}^2$.
4. The line $x - y = 0$ is a subspace of $\mathbb{R}^2$. Why?
   Exercise: Any line passing through the origin is a subspace of $\mathbb{R}^2$.

Subspaces: Examples

5. Let A be an $m \times n$ matrix.
   The null space of A, $N(A)$, is a subspace of $\mathbb{R}^n$.
   The column space of A, $C(A)$, is a subspace of $\mathbb{R}^m$.
   Recall: They are both closed under linear combinations.
6. The set of 2×2 symmetric matrices is a subspace of $\mathcal{M}$.
   So is the set of 2×2 lower triangular matrices.
   Is the set of invertible 2×2 matrices a subspace of $\mathcal{M}$?
7. The set of convergent sequences is a subspace of $\mathbb{R}^\infty$.
   What about the set of sequences convergent to 1?
8. The set of differentiable functions is a subspace of $\mathcal{C}[0, 1]$.
   Is the same true for the set of functions integrable on $[0, 1]$?

9. - ? See the tutorial sheet for many more examples!

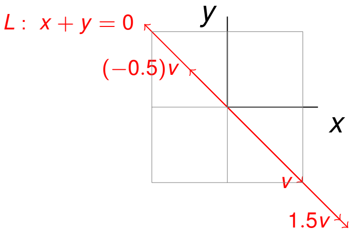
Exercise: A subspace must contain the 0 vector!

Examples: Subspaces of $\mathbb{R}^2$

What are the subspaces of $\mathbb{R}^2$?

- $V = \{\begin{pmatrix}0 & 0\end{pmatrix}^T\}$.
- $V = \mathbb{R}^2$.
- What if V is neither of the above?

Example: Suppose V contains a non-zero vector, say $v = \begin{pmatrix}-1 & 1\end{pmatrix}^T$.



V must contain the entire line $L: \ x + y = 0$, i.e., all multiples of v.

Examples: Subspaces of $\mathbb{R}^2$

Let V be a subspace of $\mathbb{R}^2$ containing $v_1 = \begin{pmatrix}-1 & 1\end{pmatrix}^T$. Then V must contain the entire line $L: \ x + y = 0$.
If $V \neq L$, it contains a vector v_2, which is not a multiple of v_1, say $v_2 = \begin{pmatrix}0 & 1\end{pmatrix}^T$.

Observe: $A = \begin{pmatrix}v_1 & v_2\end{pmatrix} = \begin{pmatrix}-1 & 0 \\ 1 & 1\end{pmatrix}$ has two pivots,
$\Leftrightarrow$ A is invertible.
$\Leftrightarrow$ for any v in $\mathbb{R}^2$, $Ax = v$ is solvable,
$\Leftrightarrow$ v is in $C(A)$,
$\Leftrightarrow$ v can be written as a linear combination of v_1 and v_2.
$\Rightarrow$ v is in V, i.e., $V = \mathbb{R}^2$

To summarise: A subspace of $\mathbb{R}^2$, which is non-zero, and not $\mathbb{R}^2$, is a line passing through the origin.