PH108 : Electricity & Magnetism : Tutorial 6

1. A conducting sphere of radius R is grounded (held at $V = 0$) and kept in an initially uniform electric field $\vec{E} = E_0\hat{z}$. Find the induced charge as a function of θ. What is the total charge induced ?

2. A spherical shell of radius R is kept at a potential $V(\theta) = V_0 \sin^2\frac{\theta}{2}$. Where θ denotes the angle made with the z axis in standard spherical polar co-ordinates.

   (a) Show that the potential inside ($r < R$) is : $V(r,\theta) = \frac{V_0}{2}\left(1 - \frac{r}{R}\cos\theta\right)$

   (b) Show that the potential outside ($r > R$) is : $V(r,\theta) = \frac{V_0 R}{2r}\left(1 - \frac{R}{r}\cos\theta\right)$

3. If the total amount of charge (monopole) contained in a distribution is zero, show that the dipole moment is independent of the choice of the origin. Then show if the monopole and dipole moments of a charge distribution are both zero, the quadrupole moment is independent of the choice of the origin.

4. A circular disc of radius R lies in $z = 0$ plane, centred at the origin. The following charge density is frozen in it.

$$\sigma(r', \phi) \quad = \quad \sigma_0 r' \cos\phi$$

   (a) What is the monopole moment of this distribution?

   (b) Calculate the dipole contribution at a point $(0, 0, z)$, using the expression for the dipole contribution in polar form.

   (c) Then calculate all the Cartesian components of the dipole moment vector $\vec{p}$ for this charge distribution. Using this result calculate the dipole contribution at $(0, 0, z)$, using the cartesian expression for dipole contribution. Do you get the same result in both cases?

   (d) Now calculate the quadrupole contribution to the potential at $(0, 0, z)$ using the polar expression.

$$V_{quad} \quad = \quad \frac{1}{4\pi\varepsilon_0 r} \int\limits_{disc} d^2\vec{r'} \left(\frac{r'}{r}\right)^2 P_2(\cos\theta)\sigma(\vec{r'}) \tag{1}$$

   (e) Then consider the definition of the quadrupole moment matrix (or tensor) in cartesian co-ordinates. The elements are defined as

$$Q_{ij} \quad = \quad \int\limits_{disc} d^2\vec{r'} \left(3r'_i r'_j - r'^2\delta_{ij}\right)\sigma(\vec{r'}) \tag{2}$$

Where $r'_1 = x, r'_2 = y, r'_3 = z$. So some of the elements will be like:

$$
\begin{aligned}
Q_{xx} &= \int_{disc} dxdy \left(2x^2 - y^2 - z^2\right) \sigma(x,y) \\
Q_{xy} &= \int_{disc} dxdy \; 3xy \; \sigma(x,y)
\end{aligned}
$$

How many independent elements will be there in this 3×3 matrix? Calculate them.

(f) Using the result calculate the quadrupole contribution at $(0,0,z)$ again using cartesian expression.

5. Consider a charge distribution given by $-2q$ at $(0,0,0)$, $+q$ at $(0,0,a)$ and $+q$ at $(0,0,-a)$. Calculate all elements of the quadrupole moment tensor for this distribution.

6. You would have seen small specs of paper flying towards a charged object like a recently used comb in dry atmosphere or a plastic ruler rubbed with some cloth. The problem is an attempt to model this process in a slightly idealised way: Consider a metal ball of radius R is connected to a voltage source whose output is V. A very small piece of paper of thickness t and areal density σ lies on a table below the ball as shown, at a distance r from the centre of the sphere. You can treat the small piece of paper as almost a point object and ignore any screening effects due to the piece of paper itself. The presence of the table does not affect the electric field.

   (a) If the relative dielectric constant of the piece of paper is $\kappa = \dfrac{\varepsilon}{\varepsilon_0}$, calculate the voltage (V) at which the piece of paper will be lifted off the surface of the table in terms of the given quantities. Your answer should be an algebraic expression.

   (b) Estimate the minimum value of the voltage at which a small piece of typical writing paper of $t = 100\ \mu$m, $\sigma = 100$ gm.m^{-2} and $\kappa = 2$ can stick to the surface of the metal ball of $R = 1$ cm. You can take $g = 10$ m.s^{-2} and $\varepsilon_0 = 10^{-11}$ Farad.m^{-1} . Your answer should be a number expressed in volts. As usual you may consider that the electric fields and potentials go to zero at infinity.