PH108 : Electricity & Magnetism : Tutorial 2

1. Check the divergence theorem for the function

$$\vec{v} = r^2 \cos\theta \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi}$$

   using the volume of one octant of a sphere of radius R.

2. Compute the unit normal vector $\hat{n}$ to the ellipsoidal surfaces defined by constant values of $\Phi(x, y, z) = V\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$. What is $\hat{n}$ when $a = b = c$?

3. Express the following derivatives in terms of linear combinations of the unit vectors of the spherical polar co-ordinate, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$:
   $\frac{\partial \hat{r}}{\partial \theta}$, $\frac{\partial \hat{r}}{\partial \phi}$, $\frac{\partial \hat{\theta}}{\partial \theta}$, $\frac{\partial \hat{\theta}}{\partial \phi}$, $\frac{\partial \hat{\phi}}{\partial \phi}$

4. Calculate the curl and divergence of the following vector functions. If the curl turns out to be zero, construct a scalar function ϕ of which the vector field is the gradient:

   (a) $F_x = x + y$; $F_y = -x + y$; $F_z = -2z$

   (b) $G_x = 2y$; $G_y = 2x + 3z$; $G_z = 3y$

   (c) $H_x = x^2 - z^2$; $H_y = 2$; $H_z = 2xz$

5. The gradient operator ∇ behaves like a vector in " some sense". For example, divergence of a curl ($\nabla . \nabla \times \vec{A} = 0$) for any $\vec{A}$, may suggest that it is just like $\vec{A}.\vec{B} \times \vec{C}$ being zero if any two vectors are equal. Prove that $\nabla \times \nabla \times \vec{F} = \nabla(\nabla . \vec{F}) - \nabla^2 \vec{F}$. To what extent does this look like the well known expansion of $\vec{A} \times \vec{B} \times \vec{C}$?

6. As a more involved example, show that the operator $\mathbf{L} = i\vec{r} \times \nabla$ where ($i = \sqrt{-1}$) satisfies $\mathbf{L} \times \mathbf{L} f = i \mathbf{L} f$ where f is an arbitrary test function. (Notice that the cross product of an operator with itself does not necessarily vanish, can you see why?)