MA-106 Linear Algebra

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12th February 2018
D1 - Lecture 17

Random Attendance

1. 170050016 Himanshu Vinayrao Bhoyar
2. 170050018 Shubhamkar Bajrang Ayare Absent
3. 170050060 Tushar Agarwal
4. 170050061 Sameer Prajapati
5. 170050074 Burudi Rajesh Absent
6. 170050094 Poorvi R Hebbar
7. 170050100 Ramya Narayanasamy
8. 170070035 Farhan Ali
9. 17D070019 Siddharth Chandak
10. 17D070021 Sakshee Anil Pimpale
11. 17D070030 Sarthak Jain Absent
12. 17D070039 Yagya Mundra
13. 17D070042 Karan Amaliya Absent
14. 17D070053 Aryan Lall
15. 17D070056 Tirupati Saketh Chandra
16. 17D070064 Manas Vashistha

Summary: Determinants

Let A and B be $n \times n$, and c a scalar.

- $\det(A+B) \neq \det(A) + \det(B)$, and $\det(cA) = c^n \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(A) = \det(A^T)$.
- If A is orthogonal, i.e., $AA^T = I$, then $\det(A) = \pm 1$.
- If $A = [a_{ij}]$ is triangular, then $\det(A) = a_{11} \cdots a_{nn}$
- A is invertible $\Leftrightarrow \det(A) \neq 0$. If this happens, then $\det(A^{-1}) = 1/\det(A)$
- If A and B are similar, i.e., $B = P^{-1}AP$ for an invertible matrix P, then $\det(B) = \det(A)$
- If A is invertible, and $d_1, \ldots, d_n$ are the pivots of A, then $\det(A) = \pm(d_1 \cdots d_n)$

Summary: Eigenvalues and Characteristic Polynomial

Let A be $n \times n$.

1. The *characteristic polynomial* of A is $\det(A - \lambda I)$ (of degree n) and its roots are the *eigenvalues* of A.
2. For each eigenvalue λ, the associated *eigenspace* is $N(A - \lambda I)$. To find it, solve $(A - \lambda I)v = 0$. Any non-zero vector in $N(A - \lambda I)$ is an *eigenvector* associated to λ.
3. If A is a diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then its eigenvalues are $\lambda_1, \cdots, \lambda_n$ with associated eigenvectors $e_1, \cdots, e_n$ respectively.
4. Write $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$ and expand.

$$\begin{aligned} \text{Trace of } A &= a_{11} + \cdots + a_{nn} \quad \text{(sum of diagonal entries)} \\ &= \lambda_1 + \ldots + \lambda_n \quad \text{(sum of eigenvalues)} \\ \det(A) &= \lambda_1 \cdots \lambda_n \quad \text{(product of eigenvalues)} \end{aligned}$$

Examples

Example: Projection onto the line $x = y$: $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

$v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ projects onto itself $\Rightarrow \lambda_1 = 1$ with eigenvector v_1.
$v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T \mapsto 0 \Rightarrow \lambda_2 = 0$ with eigenvector v_2.
Further, $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$.

Q: Do a collection of eigenvectors always form a basis of $\mathbb{R}^n$?

A: No! **Example:** For $c \in \mathbb{R}$, let $A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$.

Characteristic Polynomial: $\det(A - \lambda I) = (c - \lambda)^2$.

Eigenvalues: $\lambda = c$.

Eigenvectors: $(A - I)v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or a multiple.

Eigenspace of A is 1 dimensional $\Rightarrow \mathbb{R}^2$ has no basis of eigenvectors of A.

Q: What is the advantage of a basis of $\mathbb{R}^n$ consisting of eigenvectors?

Similarity and Eigenvalues

Defn. Two $n \times n$ matrices A and B are *similar*, if $P^{-1}AP = B$ for an invertible matrix P.

Observe: If $B = P^{-1}AP$, then $B^n = P^{-1}A^nP$ for each n.

Theorem: If A and B are similar, then they have the same characteristic polynomial.

In particular, they have the same eigenvalues, $\det(A) = \det(B)$ and Trace(A) = Trace(B).

Proof. Given: $B = P^{-1}AP$. Want to prove: $\det(A - \lambda I) = \det(B - \lambda I)$.
Indeed, $\det(B - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}P)$
$$= \det(P^{-1}(A - \lambda I)P) = \det(A - \lambda I). \qquad \square$$

Observe: $A - \lambda I$ and $B - \lambda I$ are similar.

Diagonizability

Definition: An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix Λ, i.e., there is an invertible matrix P and a diagonal matrix Λ such that $P^{-1}AP = \Lambda$.
Note: Finding roots of characteristic polynomials is difficult in general.
For $n \geq 5$, no formula exists for roots. (Abel, Galois)
For $n = 3, 4$, formulae for root exist, but not easy to use.

Importance of Diagonalizability:
Let the $n \times n$ matrix A be diagonalizable, i.e., $P^{-1}AP = \Lambda$, where P is invertible and Λ is diagonal. If this happens,

- The eigenvalues of A are the diagonal entries of Λ,
- $\det(A)$ is the product of the diagonal entries of Λ, and
- $\text{Trace}(A) =$ sum of the diagonal entries of Λ.
- **Other Information:** e.g., what is $\text{Trace}(A^n)$?

Diagonalization: Example

Example: $A = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{pmatrix}$ is triangular.

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda).$$
Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Note: If A is triangular, its eigenvalues are sitting on the diagonal

Eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -7 \\ -4 \\ 1 \end{pmatrix}$. (**Exercise**)

Further, $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
Hence $P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is invertible , and
$AP = \begin{pmatrix} Av_1 & Av_2 & Av_3 \end{pmatrix} = \begin{pmatrix} v_1 & 2v_2 & 3v_3 \end{pmatrix} = P\Lambda$, where
$\Lambda = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$. Thus $P^{-1}AP = \Lambda$, i.e., A is diagonalizable.

Diagonalization of a Matrix

Question: What is the advantage of a basis of $\mathbb{R}^n$ consisting of eigenvectors?

Let A be an $n \times n$ matrix with n eigenvectors $v_1, \ldots, v_n$, associated to eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, then the matrix $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is invertible.

Moreover, $AP = A\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} Av_1 & \cdots & Av_n \end{pmatrix}$

$$= \begin{pmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{pmatrix} = P\Lambda, \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Therefore $P^{-1}AP = \Lambda$, i.e., A is similar to a diagonal matrix.

Thus: $\boxed{\text{Eigenvectors diagonalize a matrix}}$

Caution: $\Lambda P \neq P\Lambda$ in general.

Q: When is A diagonalizable?