MA108 - Spring 2018
Tutorial 3 Final Solutions

1.

(a) Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of the ODE $(x^2-1)y'' + 4xy' + 2y = 0$ on $\mathbb{R} \setminus \{\pm 1\}$. Find the general solution.

Answer. $y(x) = \frac{A}{(x-1)} + \frac{B}{(x+1)}, \quad x \in \mathbb{R} \setminus \{\pm 1\}$.

(b) Find the solution with initial conditions $y(0) = -5, \quad y'(0) = 1$.

Answer. $y(x) = \frac{2}{(x-1)} - \frac{3}{(x+1)}$, $x \in I$.

(c) What is the interval of validity of this solution?

Answer. $(-1, 1)$.

2. Compute the Wronskian of the given set of functions.

(a) $\{e^x, e^x \sin x\}$

Answer. $e^{2x} \cos x$

(b) $\{x^{1/2}, x^{-1/3}\}$

Answer. $-\frac{5}{6}x^{-\frac{5}{6}}$

(c) $\{x \ln |x|, x^2 \ln |x|\}$

Answer. $3x^2(\ln |x|)^2 + 2x^2 \ln |x|$

3. Let y_1 and y_2 be two solutions of the given ODE. Find the Wronskian $W(y_1, y_2; x)$.

(a) $y'' + 3(x^2+1)y' - 2y = 0$ given that $W(\pi) = -1$.

Answer. $W(y_1, y_2; x) = e^{\pi^3+3\pi}e^{-x^3-3x}$.

(b) $(1-x^2)y'' - 2xy' + a(a+1)y = 0$ given that $W(0) = 1$.

Answer. $W(y_1, y_2; x) = \frac{1}{1-x^2}$.

(c) $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ given that $W(1) = 1$.

Answer. $W(y_1, y_2; x) = \frac{1}{x}$.

5. Solve the following IVP.

(a) $y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$

Answer. $y(x) = e^x(3\cos x - 5\sin x)$

(b) $y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17.$

Answer. $y(x) = e^{-7x}(2\cos x - 3\sin x)$.

(c) $6y'' - y' - y = 0, \quad y(0) = 2, \quad y'(0) = 0.$

Answer. $y(x) = 4e^{\frac{x}{2}} + 6e^{-\frac{x}{3}}$

(d) $4y'' - 4y' - 3y = 0, \quad y(0) = 13/12, \quad y'(0) = 23/24.$

Answer. $y(x) = \frac{3}{4}e^{\frac{3x}{2}} + \frac{1}{3}e^{-\frac{x}{2}}$

(e) $4y'' - 12y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = 5/2.$

Answer. $y(x) = 3e^{\frac{3x}{2}} - 2xe^{\frac{3x}{2}}$

6. Find the general solution of the following ODE.

(a) $x^2y'' + xy' - 4y = 2x^4$

Answer. $y(x) = \frac{x^6}{16} + C_1x^2 + C_2x^{-2}$

(b) $x^2y'' - 3xy' + 3y = x$

Answer. $y(x) = \frac{x^3}{2}\left(\ln|x| - \frac{1}{2}\right) + C_1x^3 + C_2x$

(c) $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$

Answer. $y(x) = (e^{2x} + e^x)(1 + \ln(1 + e^{-x})) + C_1 e^{2x} + C_2 e^x$

(d) $x^2 y'' + xy' - 4y = -6x - 4.$

Answer. $y(x) = C_1 x^2 + \frac{C_2}{x^2} + 2x + 1.$

(e) $x^2 y'' - 2xy' + 2y = x^{9/2}.$

Answer. $y(x) = \frac{4}{99} x^{\frac{13}{2}} + C_1 x^2 + C_2 x$

(f) $y'' - 2xy' + 2y = 14x^{3/2} e^x.$

Answer. $y(x) = C_1 e^x + C_2 x e^x + \frac{8}{5} x^{\frac{7}{2}} e^x.$

(g) $y'' + 4y = \sin 2x \sec^2 2x.$

Answer. $y(x) = \frac{\sin 2x}{4}(\ln(\sec 2x) - 1) + \frac{x \cos 2x}{2}$

8. Find a particular solution of $y'' + 4xy' + (4x^2 + 2)y = 4e^{-x(x+2)}$ given that $y_1 = e^{-x^2}$, $y_2 = xe^{-x^2}$ are solutions of the homogeneous part.

Answer. $y_p(x) = e^{-x(x+2)}.$

9. Find a particular solution using the variation of parameters method.

(a) $y'' - 2y' + y = 14x^{\frac{3}{2}} e^x$

Answer. $y(x) = \frac{8}{5} x^{\frac{7}{2}} e^x$

(b) $y'' - y = \frac{4e^{-x}}{1-e^{-2x}}$

Answer. $y(x) = -(e^{-x} + 2xe^x)\ln(e^{2x} - 1)$

(c) $y'' + y = \sec x \tan x$

Answer. $y(x) = (\ln(\sec x) - 1)\sin x + x \cos x$

(d) $y'' - 3y' + 2y = \sin e^{-x}$

Answer. $y(x) = (e^x + e^{-x})\cos e^{-x} - e^{-2x} \sin e^{-x}$

(e) $x^2 y'' - x(x+2)y' + (x+2)y = 2x^3, \quad x > 0,$
the fundamental set of solutions of the homogeneous part is $\{x, xe^x\}$

Answer. $y(x) = -\frac{2}{3}x^4 - 2x^3 - 4x^2 - 4x$

(f) $(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}, \quad 0 < x < 1,$
the fundamental set of solutions of the homogeneous part is $\{e^x, x\}$

Answer. $y(x) = 2x^2 e^{-x} - 2(x^2 + x + 1)$