MA 108 - Spring 2018
Tutorial Sheet 5

1. Determine if the following improper integrals exist.

   (a) $\int_0^\infty (t^2+1)^{-1}\, dt$.

   (b) $\int_1^\infty t^{-2}e^t\, dt$.

2. Find the Laplace transform of following functions.

   (a) $\cosh t \sin t$.

   (b) $\cosh^2 t$.

   (c) $t \sinh 2t$.

   (d) $\sin\left(t+\frac{\pi}{4}\right)$.

   (e) $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ e^{-2t}, & t \ge 1 \end{cases}$.

   (f) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$.

3. Suppose f is continuous on $[0,T]$ and $f(t+T) = f(t)$ for all $t \ge 0$. We say f is periodic with period T.

   (a) Show that the Laplace transform $L(f)$ is defined for $s > 0$.

   (b) Show that $F(s) = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt, \quad s > 0$.

4. Using 3(b), find the Laplace transform of the following periodic functions.

   (a) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \end{cases}, \quad f(t+2) = f(t),\ t \ge 0.$

   (b) $f(t) = \begin{cases} 1, & 0 \le t < 1/2 \\ -1, & 1/2 \le t < 1 \end{cases}, \quad f(t+1) = f(t),\ t \ge 0.$

   (c) $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}, \quad f(t+2\pi) = f(t),\ t \ge 0.$

   (d) $f(t) = |\sin t|$.

5. Find the inverse Laplace transform of the following functions.

   (a) $\dfrac{2s^2-s-3}{(s+1)^3}$

   (b) $\dfrac{s^2-1}{(s^2+1)^2}$.

(c) $\dfrac{s^2-4s+3}{(s^2-4s+5)^2}$,

(d) $\dfrac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$.

(e) $\dfrac{3s+2}{(s^2+4)(s^2+9)}$,

6. Solve the following IVP's using Laplace transforms.

   (a) $y''-3y'+2y=2e^{3t},\ y(0)=1,\ y'(0)=-1.$

   (b) $y''+y=t,\ y(0)=0,\ y'(0)=2,$

   (c) $y''+2y'+y=6\sin t-4\cos t,\ y(0)=-1,\ y'(0)=1.$

   (d) $y''+4y'+5y=e^{-t}(\cos t+3\sin t),\ y(0)=0,\ y'(0)=4.$

7. Suppose that $g(t)=\int_0^t f(r)\ dr$. If $G(s)$ and $F(s)$ are Laplace transforms of g and f respectively, show that $G(s)=F(s)/s$.

8. Following theorems can be used as formulas.

   (a) If $L(f(t))=F(s)$, then for positive integer k

   $$L(t^k f(t))=(-1)^k F^{(k)}(s)$$

   (b) If $f(t)$ is continuous on $[0,\infty)$ and of exponential order, then

   $$L\left(\int_0^t f(t)\,dt\right)=\frac{F(s)}{s}$$

   (c) If $f(t)$ is piece-wise continuous on $[0,\infty)$ and of exponential order, and $\lim\limits_{t\to 0+} f(t)/t$ exists, then

   $$L\left(\frac{f(t)}{t}\right)=\int_s^\infty F(r)\,dr$$

   (d) (Convolution theorem)

   $$L^{-1}(F(s)G(s))=\int_0^t f(\tau)g(t-\tau)\,d\tau$$