PH108 : Electricity & Magnetism : Tutorial 4

1. Measurements show that an electric field of approximately 150 Newtons/Coulomb pointing towards the center of the earth exists all around us. Given that the radius of the earth is 6400 km, estimate the number of extra electrons per cubic meter of the earth's volume? You can assume that the charge is distributed uniformly throughout the volume of the earth.

2. Consider a conducting sphere A which is initially uncharged. Another conducting sphere B is given a charge $+Q$, brought into contact with A and then moved far away. The charge on B is then increased to its original value $+Q$ and again brought into contact with A. Show that if this process is repeated many times, the charge on A will tend to the limit $\frac{Qq}{Q-q}$, where q is the charge acquired by A after its first contact with B.

3. A thin conducting spherical shell of radius R, has a charge $+Q$. The exact thickness of the shell is of no consequence in the problem. Another *point charge* $+Q$ is located at a distance b from the center of the sphere ($b > R$, of course).

   Let $F(b)$ denote the force between the two objects and F_0 denote the quantity $\frac{Q^2}{4\pi\epsilon_0 b^2}$.

   Calculate $\frac{F(b)}{F_0}$, when

   (a) $b = 10R$. Is the force attractive or repulsive?

   (b) $b = 1.1R$. Is the force attractive or repulsive?

   (c) Why is it that the force between two $+Q$ charges turns out to be attractive sometimes?

4. Consider a function of a complex variable z, $F(z) = (z-1)^2$. Write down the real and imaginary parts of this function in the following manner

$$F(x, y) = u(x, y) + iv(x, y)$$

   (a) Show by direct calculation that $u(x, y)$ and $v(x, y)$ both satisfy the $2D$ Laplace equation.

   (b) What is the value $u(0, 0)$ and $v(0, 0)$?

   (c) Now show that

$$u(r=0) = \frac{1}{2\pi}\oint u(r,\theta)d\theta$$
$$v(r=0) = \frac{1}{2\pi}\oint v(r,\theta)d\theta$$

   irrespective of the radius r of the circle, over which the integral is performed.

   (d) Can you show (you will have to use the Cauchy relations for complex functions) that $u(x, y)$ and $v(x, y)$ will always satisfy the Laplace's equation if they are real and imaginary parts of any differentiable $F(z)$?

   (e) Now suppose you have a more complicated function $F(z) = k \cosh z = u + iv$, where k is a constant. What are the shapes of $u = \text{const}$ and $v = \text{const}$ curves?

5. Try to construct a general proof of what you found in the previous problem, by following the steps given below:

   - Consider a function $\phi(r, \theta, z)$ satisfying $\nabla^2 \phi \;=\; 0$ that has no z dependence, so it is effectively a function $\phi(r, \theta)$
   - Now write down $\vec{F} = \vec{\nabla}\phi$ and $\nabla^2\phi$ in r, θ, z system.
   - Then consider the flux of $\vec{F}$ over a cylinder stretching from $z = +h$ to $z = -h$, centered at the origin and of radius R. Is there any flux through the flat (top and bottom) surfaces?
   - Apply the divergence theorem and determine the value of the volume integral. Equate this to the surface integral. Show that the value of the function at the origin multiplied by 2π is same as the value of the integral over the perimeter of the circle.
   - This should prove the mean value theorem in 2D. There could of course be alternative proofs as well.