# Tutorial- 7 MA 106 (Linear Algebra)
## Most of these problems are from reference texts for this course

1. Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and $a_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and then find its projection $p$ onto the plane of $a_1$ and $a_2$.

2. Project the vector $b = (1, 2)$ onto two vectors that are not orthogonal, $a_1 = (1, 0)$ and $a_2 = (1, 1)$. Show that, unlike the orthogonal case, the sum of the two one- dimensional projections does not equal $b$.

3. Find a third column so that the matrix
$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$
is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

4. If the vectors $q_1$, $q_2$, $q_3$ are orthonormal, what combination of $q_1$ and $q_2$ is closest to $q_3$?

5. Find an orthonormal set $q_1$, $q_2$, $q_3$ for which $q_1$, $q_2$, span the column space of
$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$
Which fundamental subspace contains $q_3$ ? What is the least-squares solution of $Ax = b$ if $b = [1\ 2\ 7]^T$?

6. Apply Gram-Schmidt to $(1, -1, 0)$, $(0, 1, -1)$, and $(1, 0, -1)$, to find an orthonormal basis on the plane $x_1 + x_2 + x_3 = 0$. What is the dimension of this subspace, and how many nonzero vectors come out of Gram-Schmidt?

7. Find orthogonal vectors $A$, $B$, $C$ by Gram-Schmidt from $a = (1, -1, 0, 0)$, $b = (0, 1, -1, 0)$, $c = (0, 0, 1, -1)$. $A$, $B$, $C$ and $a$, $b$, $c$ are bases for the vectors perpendicular to $d = (1, 1, 1, 1)$.

8. Construct the projection matrix $P$ onto the space spanned by $(1, 1, 1)$ and $(0, 1, 3)$.

9. If $Q$ is orthogonal, is the same true of $Q^3$?

10. The system $Ax = b$ has a solution if and only if $b$ is orthogonal to which of the four fundamental spaces?

11. Find an orthonormal basis for the plane $x - y + z = 0$, and find the matrix $P$ that projects onto the plane. What is the nullspace of $P$.

12. Find the best straight-line fit (least squares) to the measurements: $b = 4$ at $t = -2$, $b = 3$ at $t = -1$, $b = 1$ at $t = 0$ and $b = 0$ at $t = 2$.

    Find the projection of $b = (4, 3, 1, 0)^t$ onto $\text{Span}\{(1, 1, 1, 1)^t, (-2, -1, 0, 2)^t\}$.

13. A certain experiment produces the data $(1, 7.9)$, $(2, 5.4)$and $(3, -0.9)$. Describe the model that produces a least squares fit of these points by a function of the form $y = a\cos(\frac{\pi x}{6}) + b\sin(\frac{\pi x}{6})$.

14. CT scanners examine the patient from different directions and produce a matrix giving the densities of bone and tissue at each point. Mathematically, the problem is to recover a matrix from its projections. in the 2 by 2 case, can you recover the matrix $A$ if you know the sum along each row and down each column?

15. Find an orthonormal basis for $\mathbb{R}^3$ starting with the vector $(1, 1, 1)$.

16. Let $W$ be a subspace of $\mathbb{R}^n$, $\mathcal{B}_1 = \{w_1, \ldots, w_r\}$ and $\mathcal{B}_2$ be ordered bases of $W$ and $W^\perp$ respectively, and $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\pi(v) = \text{proj}_W(v)$.

    (a) Show that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $\mathbb{R}^n$.

    (b) Show that $\pi$ is linear.

    (c) Find $N(\pi)$, and $C(\pi)$.

    (d) Find $P = [\pi]_{\mathcal{B}}^{\mathcal{B}}$.

    (e) If $\mathcal{B}_1$ is an orthogonal basis of $W$, show that for $v \in \mathbb{R}^n$, $\pi(v) = \text{proj}_{w_1}(v) + \cdots + \text{proj}_{w_r}(v)$.

    (f) Show that $(W^\perp)^\perp = W$.

    (g) Show that there is a matrix $A$ such that $W = N(A)$.

17. State true or false. If true, explain your answer and if false give a counter-example.

    (a) Any matrix with determinant 1 is a orthogonal matrix.

    (b) An orthogonal matrix cannot have eigenvalue 3.

    (c) Let $A$ be a $2 \times 2$ diagonalizable matrix. Applying Gram-Schmidt process to a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$ will give a orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

    (d) Product of orthogonal matrices is orthogonal.

    (e) Any projection matrix $P$ (that satisfies $P^2 = P$) is invertible.

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