MA108 - Spring 2018
Tutorial 2 Final Answers

1. Solve the following ODE after finding an integrating factor.

(a) $(27xy^2 + 8y^3)\,dx + (18x^2y + 12xy^2)\,dy = 0.$

Answer. $e^x\big(54y(x-1) + 24y^2 + 9x^2y^2 + 4xy^3 - 8y^2 - 27y^2(x-1)\big) = C.$

(b) $-y\,dx + (x^4 - x)\,dy = 0.$

Answer. $y(1 - x^{-3})^{-1/3} = C$, that is, $xy(x^3 - 1)^{-1/3} = C.$

(c) $y \sin y\,dx + x(\sin y - y\cos y)\,dy = 0.$

Answer. $\frac{xy}{\sin y} = C.$

(d) $y(1 + 5\ln|x|)\,dx + 4x\ln|x|\,dy = 0.$

Answer. $4x^{5/4}(\ln|x|)^{1/4} = C.$

(e) $(3x^2y^3 - y^2 + y)\,dx + (-xy + 2x)\,dy.$

Answer. $3x + \frac{1}{xy} - \frac{1}{xy^2} = C.$

(f) $y\,dx + (2x - ye^y)\,dy = 0.$

Answer. $2xy + e^y - ye^y = C.$

(g) $(a\cos xy - y\sin xy)\,dx + (b\cos xy - x\sin xy)\,dy = 0.$

Answer. $e^{ax+by}\cos xy = C.$

2. Solve the following IVP.

(a) $(4x^3y^2 - 6x^2y - 2x - 3)\,dx + (2xx^4y - 2x^3)\,dy = 0, \quad y(1) = 3.$

Answer. $x^2y^2 - 2x^3y - x^2 - 3x - 1 = 0.$

(b) $(y^3 - 1)e^x\, dx + 3y^2(e^x + 1)\, dy = 0, \quad y(0) = 0.$

Answer. $(y^3 - 1)e^x + y^3 + 1 = 0.$

(c) $(9x^2 + y - 1)\, dx - (4y - x)\, dy = 0, \quad y(1) = 0.$

Answer. $3x^3 + xy - 2y^2 - 2 = 0.$

3. Based on the existence and uniqueness theorem, (i) find all the (x_0, y_0) for which the theorem gives an interval in which the given IVP has a solution and (ii) an interval around x_0 for which it has a unique solution.

(a) $y' = \frac{e^x + y}{x^2 + y^2}.$

Answer.
(i) $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$
(ii) $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$

(b) $y' = (x^2 + y^2)y^{1/3}.$

Answer.
(i) $(x_0, y_0) \in \mathbb{R}^2.$
(ii) $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}.$

(c) $y' = \frac{1}{(x-1)\sin y}.$

Answer.
(i) $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(1, n\pi) : n \in \mathbb{N}\}.$
(ii) $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(1, n\pi) : n \in \mathbb{N}\}.$

4. Let $y' = 3x(y - 1)^{1/3}, \quad y(x_0) = y_0.$

(a) For what points (x_0, y_0) does the IVP have a solution.

Answer. $(x_0, y_0) \in \mathbb{R}^2.$

(b) For what points (x_0, y_0) does the IVP have a unique solution in an interval around x_0.

Answer. $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(x, 1) : x \in \mathbb{R}\}.$

(d) Let $(x_0, y_0) = (0, 1)$. Find four solutions for the IVP which differ from each other for values of x in every open interval that contains $x_0 = 0$.

Answer.
(i) $y(x) = 1,\ x \in \mathbb{R}.$

(ii) $y(x) = 1 + x^3$, $x \in \mathbb{R}$.
(iii) $y(x) = 1 - x^3$, $x \in \mathbb{R}$.
(iv) $y(x) = \begin{cases} 1, & x < 0 \\ 1 + x^3, & x \geq 0 \end{cases}$
(There can be other solutions as well.)

6. State on which rectangles the hypotheses of existence and uniqueness theorem for ODEs are satisfied.
(a) $y' = \frac{\ln|xy|}{1-x^2+y^2}$.

Answer. Let $S = \{(x, y) : x = 0 \text{ or } y = 0 \text{ or } 1 - x^2 + y^2 = 0\}$. Then the required rectangles are all rectangles R such that $R \cap S = \emptyset$.

(b) $y' = \frac{1+x^2}{3y-y^2}$.

Answer. Let $Y_0 = \{(x, 0) : x \in \mathbb{R}\}$ and $Y_3 = \{(x, 3) : x \in \mathbb{R}\}$. Then the required rectangles are all rectangles R such that $R \cap Y_0 = R \cap Y_3 = \emptyset$.

7. Solve the IVP and determine how the interval in which the solution exists depends on the initial value x_0.
(a) $y' + y^3 = 0, \quad y(0) = y_0$.

Answer. $y(x) = \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60}$.

(b) $y' = \frac{x^2}{y(1+x^3)}, \quad y(0) = y_0$.

Answer. If $y_0 = 0$, then $\frac{y^2}{2} = \frac{1}{3} \ln|1 + x^3|$.
If $y_0 > 0$, then $y = \sqrt{\frac{2}{3} \ln|1 + x^3| + C_1}$.
If $y_0 < 0$, then $y = -\sqrt{\frac{2}{3} \ln|1 + x^3| + C_2}$.

8. Find ϕ_1, ϕ_2, ϕ_3, the first three Picard's iterations for the following ODEs.
(a) $y' = x + y^2, \quad y(0) = 0$.

Answer. $\phi_1(x) = \int_0^x s ds = x^2/2$.

$$\phi_2(x) = \int_0^x (s + (s^2/2)^2) ds = x^2/2 + x^5/20.$$

$$\phi_3(x) = \int_0^x (s + (s^2/2 + s^5/20)^2) ds = x^2/2 + x^5/20 + x^8/160 + x^{11}/4400.$$

(b) $y' = x^2 + y^2, \quad y(0) = 0.$

(c) $y' = x^2 + y, \quad y(0) = 0.$