MA-108 Differential Equations I

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Jump discontinuity

Definition

Let $f : [a, b] \to \mathbb{R}$ is not continuous at $x \in (a, b)$ and $f(x+)$ and $f(x-)$ exists.

- f has jump discontinuity at x if $f(x+) \neq f(x-)$.
  In this case, f can not be made continuous by re-defining the value $f(x)$.
- f has removable discontinuity at x if $f(x+) = f(x-)$.
  In this case, f can be made continuous by re-defining the value $f(x)$.

IVP with piecewise continuous forcing functions

Let $f(t)$ be a piecewise continuous function on $[0, \infty)$.
Let us consider the following IVP, here $a, b, k_0, k_1 \in \mathbb{R}$,

$$y'' + ay' + by = f(t), \quad y(0) = k_0,\ y'(0) = k_1$$

Above IVP has no solution $y \in C^2(I)$ on an open interval I that contains a jump discontinuity of $f(t)$.

Since the ODE is defined on $[0, \infty)$, $y'(0) := y'(0+)$.

Let us define, what we mean by a solution of IVP when $f(t)$ has a jump discontinuity.

Theorem

Let $f(t)$ *be a piecewise continuous function on* $[0,\infty)$ *with jump discontinuities at* $t_1, t_2, \ldots, t_n$.
Consider the IVP, here $a, b, k_0, k_1 \in \mathbb{R}$,

$$y'' + ay' + by = f(t), \quad y(0) = k_0, \; y'(0) := y'(0+) = k_1$$

Then there exists a unique function y *defined on* $[0,\infty)$, *called the solution of IVP, such that*

1. $y(0) = k_0$ *and* $y'(0) = k_1$.
2. y *and* y' *are continuous on* $[0,\infty)$.
3. *If* I *is an open sub-interval of* $[0,\infty)$ *that does not contain any of the points* $t_1, \ldots, t_n$, *then* $y \in C^2(I)$ *and* y *is a solution of ODE on* I.
4. y'' *has left and right limits at* $t_1, \ldots, t_n$.

IVP with piecewise continuous forcing functions

Consider the IVP of the form

$$y''+ay'+by = r(t) = \begin{cases} f_0(t), & 0 \le t < t_1 \\ f_1(t), & t \ge t_1, \end{cases}, \quad y(0) = k_0,\ y'(0) = k_1$$

where $r(t)$ has a single jump discontinuity at t_1.

We can solve the IVP as follows.

1. Find the solution y_0 of the IVP
$$y'' + ay' + by = f_0(t), \quad y(0) = k_0,\ y'(0) = k_1$$
2. Compute $c_0 = y_0(t_1), \quad c_1 = y_0'(t_1)$.
3. Find the solution y_1 of the IVP
$$y'' + ay' + by = f_1(t), \quad y(t_1) = c_0,\ y'(t_1) = c_1$$
4. The solution of original IVP is
$$y(t) = \begin{cases} y_0(t), & 0 \le t < t_1 \\ y_1(t), & t \ge t_1 \end{cases} \quad \in C^1[0, \infty)$$

Example

Find the solution of the IVP

$$y'' + 3y' + 2y = \begin{cases} e^t, & 0 \le t < 2 \\ e^{-t}, & t \ge 2 \end{cases}$$

$$y(0) = 1 \quad , \quad y'(0) = -1$$

Let y_1 be the unique solution of the IVP

$$y'' + 3y' + 2y = e^t, \;\; y(0) = 1, \; y'(0) = -1$$

$$y_1(t) = \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + \frac{1}{6}e^t$$

$$y_1(2) = \frac{1}{2e^2} + \frac{1}{3e^4} + \frac{e}{6} = c_1, \quad y_1'(2) = -\frac{1}{2e^2} - \frac{2}{3e^4} + \frac{e}{6} = c_2$$

Example (continued ...)

Let $y_2(t)$ be the solution of the IVP

$$y'' + 3y' + 2y = e^{-t}, \quad y(2) = c_1, \quad y'(2) = c_2$$

A particular solution of ODE is $y_p = cte^{-t}, \quad c = 1$.

$$y_2(t) = d_1 e^{-t} + d_2 e^{-5t} + te^{-t}$$

$$d_2 = -\frac{(c_1 + c_2)e^{10} - e^8}{4}, \quad d_1 = \frac{1}{4}e^2(5c_1 + c_2) - \frac{9}{4}$$

The solution $y(t)$ of original IVP on $[0, \infty)$ is

$$y(t) = \begin{cases} y_1(t), & 0 \le t < 2 \\ y_2(t), & t \ge 2 \end{cases}$$

Example

Solve the IVP

$$y'' + y = \begin{cases} 1, & 0 \le t < \pi/2 \\ -1, & \pi/2 \le t < \infty \end{cases}$$
$$y(0) = 2 \quad , \quad y'(0) = -1$$

Let $y_1(t)$ be the solution of

$$y'' + y = 1, \; y(0) = 2, \;\; y'(0) = -1$$

Then

$$y_1(t) = 1 + \cos t - \sin t$$

Compute

$$y_1(\pi/2) = 0, \;\; y_1'(\pi/2) = -1$$

Example (continued ...)

Let $y_2(t)$ be solution of

$$y'' + y = -1, \quad y(\pi/2) = 0, \; y'(\pi/2) = -1$$

Then

$$y_2(t) = \; -1 + \cos t + \sin t$$

The solution of original IVP is

$$y(t) = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \frac{\pi}{2} \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases}$$

Example

Let us solve the same problem using Laplace transform.

$$y'' + y = \begin{cases} 1, & 0 \leq t < \pi/2 \\ -1, & \pi/2 \leq t < \infty \end{cases}$$

$$y(0) = 2 \quad , \quad y'(0) = -1$$

$$f(t) = 1 + (-1-1)u\left(t - \frac{\pi}{2}\right) = 1 - 2u\left(t - \frac{\pi}{2}\right)$$

Assume $y = \phi(t)$ is the solution of IVP. Then

$$L(\phi'') + L(\phi) = L(f(t))$$

$$(s^2+1)L(\phi) + 1 - 2s = \frac{1}{s} - 2e^{-\pi s/2}\frac{1}{s}$$

Example (continued ...)

$$L(\phi) = (1 - 2e^{-\pi s/2})\frac{1}{s(s^2+1)} + \frac{2s-1}{s^2+1}$$

$$L(\phi) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - 2e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{2s-1}{s^2+1}$$

$$\phi(t) = 1 - \cos t - 2u\left(t - \frac{\pi}{2}\right)\left[1 - \cos\left(t - \frac{\pi}{2}\right)\right] + 2\cos t - \sin t$$

$$= 1 + \cos t - \sin t - 2u\left(t - \frac{\pi}{2}\right)(1 - \sin t)$$

$$= \begin{cases} 1 + \cos t - \sin t, & 0 \leq t < \frac{\pi}{2} \\ -1 + \cos t + \sin t, & t \geq \frac{\pi}{2} \end{cases}$$

Check that ϕ and ϕ' are continuous and ϕ'' has left and right limit at $\pi/2$.

Example

Solve the following IVP using Laplace transform.

$$y'' + y = f(t), \quad y(0) = 0,\ y''(0) = 0$$

$$f(t) = \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4} \\ \cos 2t, & \dfrac{\pi}{4} \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

$$= u\left(t - \frac{\pi}{4}\right)\cos 2t - u(t-\pi)\cos 2t$$

$$L(f) = L\left(u\left(t - \frac{\pi}{4}\right)\cos 2t\right) - L(u(t-\pi)\cos 2t)$$

$$= e^{-\pi s/4} L\left(\cos 2\left(t + \frac{\pi}{4}\right)\right) - e^{-\pi s} L\left(\cos 2(t+\pi)\right)$$

Example (continued ...)

$$L(f) = e^{-\pi s/4} L(-\sin 2t) - e^{-\pi s} L(\cos 2t)$$

$$= -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}$$

Let us come back to our IVP

$$y'' + y = f(t), \quad y(0) = 0,\; y'(0) = 0$$

$$L(y)(s^2+1) = -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}$$

$$L(y) = \frac{1}{s^2+1}\left[-\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}\right]$$

$$= e^{-\pi s/4} H_1(s) + e^{-\pi s} H_2(s)$$

Example (continued ...)

$$H_1(s) = \frac{-2}{(s^2+1)(s^2+4)} = \frac{-2}{3(s^2+1)} + \frac{2}{3(s^2+4)}$$

$$L^{-1}(H_1(s)) = h_1(t) = \frac{-2}{3}\sin t + \frac{1}{3}\sin 2t$$

$$H_2(s) = \frac{-s}{(s^2+1)(s^2+4)} = \frac{-s}{3(s^2+1)} + \frac{s}{3(s^2+4)}$$

$$L^{-1}(H_2(s)) = h_2(t) = \frac{-1}{3}\cos t + \frac{1}{3}\cos 2t$$

$$L(y(t)) = e^{-\pi s/4} H_1(s) + e^{-\pi s} H_2(s)$$

$$y(t) = u\left(t - \frac{\pi}{4}\right) h_1\left(t - \frac{\pi}{4}\right) + u(t-\pi)h_2(t-\pi)$$

Example (continued ...)

$$y(t) = u\left(t - \frac{\pi}{4}\right) h_1 \left(t - \frac{\pi}{4}\right) + u(t-\pi)h_2(t-\pi)$$

$$= u\left(t - \frac{\pi}{4}\right) \left[\frac{-2}{3} \sin\left(t - \frac{\pi}{4}\right) + \frac{1}{3} \sin 2\left(t - \frac{\pi}{4}\right)\right]$$

$$+ u(t-\pi)\left[\frac{-1}{3}\cos(t-\pi) + \frac{1}{3}\cos 2(t-\pi)\right]$$

$$= u(t-\pi/4)\left[\frac{-\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t\right]$$

$$+\frac{1}{3}u(t-\pi)(\cos t + \cos 2t)$$

Example (continued ...)

$$y(t) = u(t - \pi/4)\left[\frac{-\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t\right]$$

$$+\frac{1}{3}u(t-\pi)(\cos t + \cos 2t)$$

$$= \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4} \\ \dfrac{-\sqrt{2}}{3}(\sin t - \cos t) - \dfrac{1}{3}\cos 2t, & \dfrac{\pi}{4} \le t < \pi \\ \dfrac{-\sqrt{2}}{3}\sin t + \dfrac{1+\sqrt{2}}{3}\cos t, & t \ge \pi \end{cases}$$

Check that y, y' are continuous and y'' has left and right limits at $\pi/4$ and π.

Convolution

Consider IVP

$$ay'' + by' + cy = f(t),\ y(0) = 0,\ y'(0) = 0$$

Taking Laplace transform gives

$$(as^2 + bs + c)Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{as^2 + bs + c}$$

For known $f(t)$, we were finding $y(t) = L^{-1}(Y(s))$ by partial fraction method.

Question. What if $f(t)$ is unknown function?
Can we get a formula for the solution $y(t)$

$$y(t) = L^{-1}(F(s)G(s))$$

in terms of $f(t)$?

Convolution : $L^{-1}(FG)$

Example

Consider IVP

$$y' - ay = f(t), \quad y(0) = 0, \qquad a \in \mathbb{R}$$

$$e^{-at}(y' - ay) = f(t)e^{-at}$$

$$(e^{-at}y)' = f(t)e^{-at}$$

$$e^{-at}y(t) = \int_0^t e^{-a\tau} f(\tau)\, d\tau$$

$$y(t) = e^{at}\int_0^t e^{-a\tau} f(\tau)\, d\tau = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau$$

Example (continued ...)

Let us use Laplace transform to solve same IVP.

$$y' - ay = f(t), \quad y(0) = 0$$

$$(s-a)Y(s) = F(s) \implies$$

$$Y(s) = F(s)\frac{1}{s-a} = F(s)G(s), \qquad g(t) = e^{at}$$

$$y(t) = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau$$

$$y(t) := L^{-1}(F(s)G(s)) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

Definition (Convolution)

The convolution $f * g$ of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

We saw that when $g(t) = e^{at}$, then

$$L^{-1}(F(s)G(s)) = f * g, \qquad F(s)G(s) = L(f * g)$$

This is true in general.

Show the followings.

1. $f * g = g * f$.
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$
5. $f * 1 \neq f$,
   e.g. $\sin t * 1 = 1 - \cos t$.