MA-106 Linear Algebra

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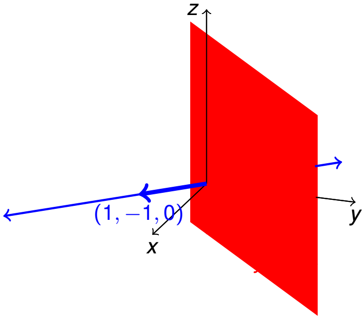
D1 - Lecture 21

Orthogonal Subspaces: Examples

If $\{w_1, \ldots, w_r\}$ is a orthogonal set then w_r is orthogonal to $\text{Span}\{w_1, \ldots, w_{r-1}\}$. Then every vector in $\text{Span}\{w_r\}$ is orthogonal to every vector in $\text{Span}\{w_1, \ldots, w_{r-1}\}$.

Example 1: $\mathbf{P} = \text{Span}\{(1,1,0),(0,0,1)\}$, $\mathbf{L} = \text{Span}(1,-1,0)$.



Observe $\begin{pmatrix} t & -t & 0 \end{pmatrix} \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} = 0$

and $\begin{pmatrix} t & -t & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} = 0.$

The subspace
$\mathbf{L} = \{(t, -t, 0) \mid t \in \mathbb{R}\}$
is orthogonal to the subspace
$\mathbf{P} = \{(a, a, b) \mid a, b \in \mathbb{R}\}$.

Observe: $\dim(\mathbf{L}) + \dim(\mathbf{P}) = \dim(\mathbb{R}^3)$.

Orthogonal Subspaces

Defn. (Orthogonal Subspaces) Let V and W be subspaces of $\mathbb{R}^n$. We say V and W are orthogonal to each other (notation: $V \perp W$) if every vector in V is orthogonal to every vector in W, i.e.,

$$\boxed{\text{for every } v \in V \text{ and } w \in W,\ v \cdot w = v^T w = 0.}$$

Note: Let $V = \text{Span of } \{v_1, \dots, v_r\}$, $W = \text{Span } \{w_1, \dots, w_s\}$.
If $v_i^T w_j = 0$ for all i, j, then $V \perp W$.

Proof. If $v = a_1 v_1 + \cdots + a_r v_r \in V$, $w = b_1 w_1 + \cdots + b_s w_s \in W$, then

$$\begin{aligned} v^T w &= (a_1 v_1^T + \cdots + a_r v_r^T)(b_1 w_1 + \cdots + b_s w_s) \\ &= a_1 b_1 v_1^T w_1 + a_1 b_2 v_1^T w_2 + \cdots + a_1 b_s v_1^T w_s \\ &\quad + a_2 b_1 v_2^T w_1 + \cdots + a_2 b_s v_2^T w_s + \cdots + a_r b_s v_r^T w_s = 0 \text{ using} \end{aligned}$$

bilinearity.

Orthogonal Subspaces: Examples

Q. Let V be the yz-plane and W be the xz-plane. Are V and W orthogonal to each other (as subspaces of $\mathbb{R}^3$)? **A.** No. The vector $e_3 = (0, 0, 1)$ lies in V and W both and $e_3^T e_3 \neq 0$.

Remark. If V and W are orthogonal subspaces of $\mathbb{R}^n$, then $V \cap W = 0$. This is a necessary condition.

Q. Is it a sufficient condition? **A.** No.

Example. If $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, and $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Then $V \cap W = 0$, but V and W are not orthogonal, since $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \neq 0$.

Orthogonal Subspaces: Examples

Ex 2: Let $V = \text{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\}$, $W = \text{Span}\left\{\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}\right\}$.

Then V is orthogonal to W.

Proof. It is enough to see that both generators of V are orthogonal to the given generator of W.

Compare: In Example 1, $\dim(L) + \dim(P) = \dim(\mathbb{R}^3)$.
In Example 2, $\dim(V) + \dim(W) = 3 < \dim(\mathbb{R}^4)$.

Q: If $W \subset \mathbb{R}^n$, can we enlarge W to W' such that $V \perp W'$ and $\dim(V) + \dim(W') = \dim(\mathbb{R}^4)$?

A: Yes! We will eventually justify why this is true.

Observe: $(0, 0, 0, 1)$ is orthogonal to both V and W.

Orthogonal Subspaces: Examples

$V = \text{Span}\left\{v_1 = (1,1,0,0)^T, v_2 = (0,1,1,0)^T\right\}$,
$W = \text{Span}\left\{(1,-1,1,0)^T\right\}$.

Let $A = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Then $C(A^T) = V$.

Note that $x \in N(A) \Leftrightarrow v_1^T x = 0 = v_2^T x$.

$\Rightarrow (1,-1,1,0)^T \in N(A) \Rightarrow W \subset N(A)$.

By *Rank-Nullity Theorem*: $\text{rank}(A) + \dim(N(A)) = 4$.

Since $\text{rank}(A) = 2$, we get $\dim(N(A)) = 2$. Therefore

if $W' = N(A)$, then $V \perp W'$ and $\dim(V) + \dim(W') = 4$.

By inspection, $W' = \text{Span}\{w_1 = (1,-1,1,0)^T, w_2 = (0,0,0,1)^T\} \perp V$.

The dimension of the two spaces sum up to $\dim(\mathbb{R}^4)$.

Random Attendance

1. 170070048 Bojja Sai Vamseedhar Reddy Absent
2. 17D070038 Divyansh Ahuja Absent
3. 17D070012 Naman Rajesh Narang
4. 170070049 Modhugu Rineeth Absent
5. 17D070015 Nikhil Arvind Bhaladhare Absent
6. 170050074 Burudi Rajesh Absent
7. 17D070030 Sarthak Jain Absent
8. 17D070042 Karan Amaliya Absent
9. 170050005 Yateesh Agrawal Absent
10. 17D070024 Prajwal Dnyaneshwar Kamble Absent

The Four Fundamental Spaces and Orthogonality

Let A be a $m \times n$ matrix.

1. The row space of A, $C(A^T)$ is orthogonal to $N(A)$.
2. $C(A)$ is orthogonal to the left nullspace of A, $N(A^T)$.

Example. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$. Then A has 1 pivot $\Rightarrow$ rank$(A) = 1 \Rightarrow$ dim $(C(A)) = 1$ and dim $(C(A^T)) = 1$.

Observe: $C(A^T)$ = Span$\{(1,2)^T\}$, $N(A)$ = Span$\{(-2,1)^T\}$,
$C(A) =$ Span$\{(1,2,3)^T\}$, $N(A^T)$ is the plane $y_1 + 2y_2 + 3y_3 = 0$.
dim$(C(A^T))$+ dim$(N(A)) = 2$,
dim$(C(A))+$ dim$(N(A^T)) = 3$.

In particular,

- $N(A) =$ set of all vectors orthogonal to $C(A^T)$.
- $N(A^T) =$ set of all vectors orthogonal to $C(A)$.

Fundamental Theorem of Orthogonality

Defn. Let W be a subspace of $\mathbb{R}^n$. Its orthogonal complement $W^\perp = \{v \in \mathbb{R}^n \mid v^T w = 0 \text{ for all } w \in W\}$.

Claim. $W^\perp$ is a subspace of $\mathbb{R}^n$.
If $v_1, v_2 \in W^\perp$ and $w \in W$, then $v_1^T w = 0 = v_2^T w$.
Hence for $c_1, c_2 \in \mathbb{R}$, $(c_1 v_1 + c_2 v_2)^T w = c_1 v_1^T w + c_2 v_2^T w = 0$.
$\Rightarrow c_1 v_1 + c_2 v_2 \in W^\perp$.

Theorem (Fundamental Theorem of Orthogonality)
Let A be an $m \times n$ matrix.
1. The row space of A = orthogonal complement of $N(A)$.
2. The column space of A = orthogonal complement of left nullspace $N(A^T)$.

i.e., $C(A^T) = N(A)^\perp$, and $C(A) = N(A^T)^\perp$.

Orthogonal Complements

Theorem (Orthogonal Complement)
Given a subspace $W \subseteq \mathbb{R}^n$, $\dim(W) + \dim(W^\perp) = n$.

Proof. Let $v_1, \ldots, v_r$ be a basis of W. Let A be a matrix with rows $v_1, \ldots, v_r$. Then $\text{rank}(A) = r$ and $W = C(A^T)$.
$W^\perp = N(A)$ is of dimension $n - r$. This proves the theorem.

Observe: Let $V = \text{Span}\left\{v_1 = (1,1,0,0)^T, v_2 = (0,1,1,0)^T\right\}$ and $W = \text{Span}\{w_1 = (1,-1,1,0)^T, w_2 = (0,0,0,1)^T\}$.

- $\{v_1, v_2\}$, $\{w_1, w_2\}$ are bases for V and W respectively.
- $V \perp W \Rightarrow V \subseteq W^\perp$ and $\dim(V) + \dim(W) = 4 = \dim(\mathbb{R}^4)$ $\Rightarrow V = W^\perp$.
- $V \cap W = 0 \Rightarrow \mathcal{B} = \{v_1, v_2, w_1, w_2\}$ is linearly independent $\Rightarrow \mathcal{B}$ is a basis of $\mathbb{R}^4 \Rightarrow V + W = \{v + w \mid v \in V, w \in W\} = \mathbb{R}^4$, and for every $x \in \mathbb{R}^4$,

$$\boxed{\exists \text{ unique } v \in V = W^\perp \text{ and } w \in W \text{ such that } x = v + w.}$$

Orthogonal Complements

To summarise: If W is a subspace of $\mathbb{R}^n$ with basis $\mathcal{B}$, and $\mathcal{B}'$ is a basis of $W^\perp$, then $\mathcal{B} \cup \mathcal{B}'$ is a basis of $\mathbb{R}^n$, $W \cap W^\perp = 0$, $\dim(W) + \dim(W^\perp) = n$, $W + W^\perp = \mathbb{R}^n$, and

$\boxed{\exists \text{ unique } w_1 \in W \text{ and } w_2 \in W^\perp \text{ such that } x = w_1 + w_2,}$

for every $x \in \mathbb{R}^n$.

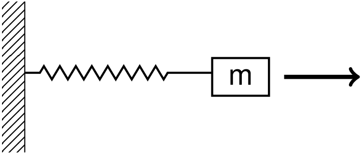
Some Consequences: Let A be $m \times n$ of rank r. Then

1. $C(A^T) \cap N(A) = 0$ and $\mathbb{R}^n = C(A^T) + N(A)$.
Similarly $C(A) \cap N(A^T) = 0$ and $\mathbb{R}^m = C(A) + N(A^T)$.

2. If $x \in \mathbb{R}^n$, there is a unique expression $x = x_r + x_n$, where $x_r \in C(A^T)$, $x_n \in N(A)$. Hence $Ax = Ax_r \in C(A)$. Thus

$\boxed{\text{The matrix } A \text{ transforms its row space into its column space.}}$

Linear Least Squares: Motivation



Hooke's Law states that displacement x of the spring is directly proportional to the load (mass) applied, i.e., $m = kx$.

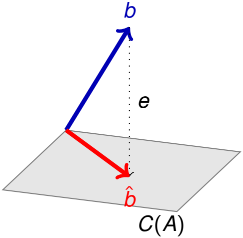
Student A performs experiments to calculate spring constant k. The data collected says for loads $4, 7, 11$ kg applied, the displacement is $3, 5, 8$ inches respectively. Hence we have:

$$\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} k = \begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} \qquad (ak = b).$$

Clearly the data is inconsistent. If we allow for various errors, how do we give an estimate of the spring constant?
Find a consistent system close to this one!

Linear Least Squares and Projections

Suppose system $Ax = b$ is inconsistent, i.e. $b \notin C(A)$.
The error $E = \|Ax - b\|$ is the distance from b to $Ax \in C(A)$.



We want the least square solution $\hat{x}$ of $Ax = b$, which minimizes E,
i.e., we want to find $\hat{b}$ closest to b such that $A\hat{x} = \hat{b}$ is a consistent system.
Therefore, $\hat{b} = \text{proj}_{C(A)}(b)$ and $A\hat{x} = \hat{b}$.

The error vector $e = b - A\hat{x}$ must be perpendicular to $C(A)$.
So $e \in C(A)^{\perp}$ = left null space of A, $N(A^T)$,
i.e.,

$$A^T(b - A\hat{x}) = 0 \text{ or } \boxed{A^T A\,\hat{x} = A^T b}$$

Therefore, to find $\hat{x}$, we need to solve $A^T A\,\hat{x} = A^T b$.
Exercise: Find the least squares solution to the Hooke's Law problem on the previous slide.