MA-106 Linear Algebra

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D1 - Lecture 10

Random Attendance

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Summary: Vector Spaces, Span and Independence

• Vector space: A triple $(V, +, *)$ which is closed under $+$ and $*$ with some additional properties satisfied by $+$ and $*$.

• Subspace: A non-empty subset W of V closed under linear combinations.

Let $V = \mathbb{R}^m$, $v_1, \ldots, v_n$ be in V, and $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.

• For v in V, v is in $\text{Span}\{v_1, \ldots, v_n\} \Leftrightarrow Ax = v$ is consistent

• $v_1, \ldots, v_n$ are linearly independent
$\Leftrightarrow N(A) = 0 \Leftrightarrow \text{rank}(A) = n$.

• In particular, with $n = m$, A is invertible
$\Leftrightarrow Ax = v$ is consistent for every v
$\Leftrightarrow \text{Span}\{v_1, \ldots, v_n\} = \mathbb{R}^n \Leftrightarrow \text{rank}(A) = n$
$\Leftrightarrow N(A) = 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly independent.

• Any subset of $\mathbb{R}^m$ with more than m vectors is dependent.

Minimal Spanning Set

Let $v_1 = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}^T$, $v_2 = \begin{pmatrix} 4 & 5 & 3 \end{pmatrix}^T$, $v_3 = \begin{pmatrix} 6 & 7 & 5 \end{pmatrix}^T$ and $v_4 = \begin{pmatrix} 4 & 6 & 2 \end{pmatrix}^T$. If $A = (v_1\ v_2\ v_3\ v_4)$, can $C(A) =$Span$\{v_1, v_2, v_3, v_4\}$ be spanned by less than 4 vectors?

Note: $v_3 = v_1 + v_2$ and $v_4 = -2v_1 + 2v_2 \Rightarrow C(A) = \text{Span}\{v_1, v_2\}$.

Observe:

- The span of only v_1 or only v_2 is a line. Clearly v_1 is not on the line spanned by v_2 and vice versa.
  Thus, $\{v_1, v_2\}$ is a *minimal spanning set* for $C(A)$.
- v_1 and v_2 are linearly independent and span $C(A)$.
- If v is in $C(A)$ = Span$\{v_1, v_2\}$, then v_1, v_2, v are linearly dependent. Why?
  Thus, $\{v_1, v_2\}$ is a *maximal linearly independent set* in $C(A)$.

Any such set of vectors gives a *basis* of $C(A)$.

Basis: Definition

Defn. A subset $\mathcal{B}$ of a vector space V, is said to be a *basis* of V, if it is linearly independent and Span($\mathcal{B}$) = V.

Theorem: For any subset S of a vector space V, the following are equivalent:

- S is a maximal linearly independent set in V
- S is linearly independent and Span(S) = V.
- S is a minimal spanning set of V.

Note: Every vector space V has a basis.

Examples:

- By convention, the empty set is a basis for $V = \{0\}$.
- $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.
- $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$, called the standard basis.
- A basis of $\mathbb{R}$ is just $\{1\}$.

Basis: Remarks

• Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V and v a vector in V.
Span($\mathcal{B}$) $= V \Rightarrow v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \ldots, a_n$.
Linear independence $\Rightarrow$ this expression for v is unique. Thus

Every $v \in V$ can be *uniquely* written

as a linear combination of $\{v_1, \ldots, v_n\}$.

Exercise: Prove this.

Q: Is the basis of a vector space unique? **A:** No.
e.g. $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$, so is $\left\{\begin{pmatrix}-1 & 1\end{pmatrix}^T, \begin{pmatrix}0 & 1\end{pmatrix}^T\right\}$, and so are the columns of any 2×2 invertible matrix.
Exercise: Find two different basis of $\mathbb{R}^3$.
The number of vectors in each basis of $\mathbb{R}^3$ is 3. Not a coincidence!

Dimension of a Vector Space

If $v_1, \ldots v_m$ and $w_1, \ldots, w_n$ are both basis of V, then $m = n$. This is called the *dimension* of V. Thus

$$\boxed{\text{dim}(V) = \text{number of elements in a basis of } V.}$$

Exercise: Prove that every basis of $\mathbb{R}^3$ has only three elements.

Examples:

• $\dim(\{0\}) = 0$. • $\dim(\mathbb{R}^n) = n$.

• If $\mathbf{L}$ is a line through origin in $\mathbb{R}^3$, what is its dimension as a vector space? Recall $L = \{tu \mid t \in \mathbb{R}\}$ where u is some vector in $\mathbb{R}^3$. Thus $\dim(\mathbf{L}) = 1$.

•. Dimension of a plane ($\mathbf{P}$) in $\mathbb{R}^3$ is 2. Why?

• A basis for $\mathbb{C}$ as a vector space over the scalars $\mathbb{R}$ is $\{1, i\}$.
A basis for $\mathbb{C}$ as a vector space over the scalars $\mathbb{C}$ is $\{1\}$.
i.e., $\dim(\mathbb{C})$ = 2 as a $\mathbb{R}$-vector space and 1 as a $\mathbb{C}$-vector space.
Thus, dimension depends on the choice of scalars!

Dimension and Basis

Let $\dim(V) = n$, $S = \{v_1, \dots, v_k\} \subseteq V$.

Recall: A basis is a minimal spanning set.

In particular, if $\mathrm{Span}(S) = V$, then $k \geq n$, and S contains a basis of V, i.e., there exist $\{v_{i_1}, \dots, v_{i_n}\} \subseteq S$ which is a basis of V.

Example: The columns of a 3×4 matrix A with 3 pivots span $\mathbb{R}^3$. Hence the columns contain a basis of $\mathbb{R}^3$.

Recall: A basis is a maximal linearly independent set.

In particular, if S is linear independent, then $k \leq n$, and S can be extended to a basis of V, i.e., there exist $w_1, \dots, w_{n-k}$ in V such that $\{v_1, \dots, v_k, w_1, \dots, w_{n-k}\}$ is a basis of V.

Example: The columns of a 3×2 matrix A with 2 pivots has linearly independent columns, and hence can be extended to a basis of $\mathbb{R}^3$.