MA-106 Linear Algebra

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D1 - Lecture 14

Random Attendance

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6	170050038	Shubham Atri	
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9	170050083	Bonela Mahith	
10	170070024	Saksham Khandelwal	
11	170070028	Navneet Prabhat	
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14	17D070028	Tavish Mina	
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Summary: Linear transformations

Let $T : V \to W$ be linear.

• The nullspace of T, $N(T)$, is a subspace of V.
Its range, $C(T)$ is a subspace of W.

• T is one-one $\Leftrightarrow N(T) = 0$, and T is onto $\Leftrightarrow C(T) = W$. In particular,

$\boxed{T \text{ is an isomorphism } \Leftrightarrow N(T) = 0 \text{ and } C(T) = W.}$

• If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of V, and $v = a_1 v_1 + \cdots a_n v_n$, then $T(v) = a_1 T(v_1) + \cdots a_n T(v_n)$. Thus,

$\boxed{T \text{ is determined by its action on a basis.}}$

In particular, $T : V \to \mathbb{R}^n$ defined by $T(v_i) = e_i$, i.e., $T(v) = [v]_{\mathcal{B}}$, is an isomorphism. Thus, $\boxed{\text{if dim (V) = n, then } V \simeq \mathbb{R}^n.}$

•. Given $A \in \mathcal{M}_{m\times n}$, and $T(x) = Ax$ defines a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Conversely, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(x) = Ax$, where $A = \begin{pmatrix} T(e_1) & \cdots & T(e_n) \end{pmatrix} \in \mathcal{M}_{m\times n}$, the *standard matrix* of T.

Matrix Associated to a Linear Map: Example

$S : \mathcal{P}_2 \to \mathcal{P}_1$ given by $S(a_0 + a_1x + a_2x^2) = a_1 + 4a_2x$ is linear.

Question: Is there a matrix associated to S?
Expected size: 2×3. Why?

IDEA: Construct an associated linear map $\mathbb{R}^3 \to \mathbb{R}^2$.
Use coordinate vectors! Fix bases $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$, and $\mathcal{C} = \{1, x\}$ of $\mathcal{P}_1$ to do this.

Identify $f = a_0 + a_1x + a_2x^2 \in \mathcal{P}_2$ with $[f]_{\mathcal{B}} = (a_0, a_1, a_2)^T \in \mathbb{R}^3$,

and $S(f) \in \mathcal{P}_1$ with $[S(f)]_{\mathcal{C}} = (a_1, 4a_2)^T \in \mathbb{R}^2$.

The associated linear map $S' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by
$S'(a_0, a_1, a_2)^T = (a_1, 4a_2)^T$, i.e., $S'([f]_{\mathcal{B}}) = [S(f)]_{\mathcal{C}}$, i.e.,
S' is defined by $S'(e_1) = (0, 0)^T$, $S'(e_2) = (1, 0)^T$, $S'(e_3) = (0, 4)^T \Rightarrow$
the standard matrix of S' is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Q: How is A related to S?

Observe: $A_{*1} = [S(1)]_{\mathcal{C}}$, $A_{*2} = [S(x)]_{\mathcal{C}}$, $A_{*3} = [S(x^2)]_{\mathcal{C}}$.

Matrix Associated to a Linear Map

Example: The matrix of $S(a_0 + a_1 x + a_2 x^2) = a_1 + 4a_2 x$, w.r.t. the bases $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$ and $\mathcal{C} = \{1, x\}$ of $\mathcal{P}_1$ is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $\boxed{A_{*1} = [S(1)]_{\mathcal{C}}, A_{*2} = [S(x)]_{\mathcal{C}}, A_{*3} = [S(x^2)]_{\mathcal{C}}.}$

General Case: If $T : V \to W$ is linear, then the matrix of T w.r.t. the ordered bases $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V, and $\mathcal{C} = \{w_1, \ldots, w_m\}$ of W, denoted $[T]_{\mathcal{C}}^{\mathcal{B}}$, is

$A = \big([T(v_1)]_{\mathcal{C}} \ \cdots \ [T(v_n)]_{\mathcal{C}}\big) \in \mathcal{M}_{m \times n}$.

Example: Projection onto the line $x_1 = x_2$

$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$ has standard matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

This is the matrix of P w.r.t. the standard basis.

Q: What is $[P]_{\mathcal{B}}^{\mathcal{B}}$ where $\mathcal{B} = \{(1,1)^T, (-1,1)^T\}$?

Conclusion: The matrix of a transformation depends on the chosen basis. Some are better than others!

Determinant: Introduction

INVERTIBILITY: Very often we are interested in knowing when a matrix is invertible. Consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if A has full rank.

If a, c both are zero then clearly $\text{rank}(A) < 2 \Rightarrow A$ is not invertible.
Assume $a \neq 0$, else, interchange rows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 - c/aR_1} \begin{bmatrix} a & b \\ 0 & d - cb/a \end{bmatrix}$$

A is invertible if and only if $d - cb/a \neq 0$, i.e., $ad - bc \neq 0$.

AREA: The area of the parallelogram with sides as vectors $v = (a, b)$ and $w = (c, d)$ is equal to $ad - bc$.
Thus, a 2×2 matrix A is singular $\Leftrightarrow$ its columns are on the same line $\Leftrightarrow$ the area is zero.

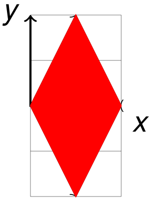
Determinant: Introduction

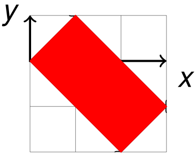
- Test for invertibility: An $n \times n$ matrix A is invertible $\Leftrightarrow \det(A) \neq 0$.
- *n*-dimensional volume: If A is $n \times n$, then $|\det(A)| =$ the volume of the box (in n-dimensional space $\mathbb{R}^n$) with edges as rows of A.

Examples: (1) The volume (area) of a line in $\mathbb{R}^2 = 0$.

(2) The determinant of the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$ is $\boxed{-4}$.

(3) Let's compute the volume of the box (parallelogram) with edges as rows of A or columns of A.





$\boxed{= 4}$

Determinants: Defining Properties

The determinant function

$$\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

can be defined (uniquely) by its three basic properties.

1. $\det(I) = 1$.
2. The sign of determinant is reversed by a row exchange.
   Thus, if $B = P_{ij}A$, i.e., B is obtained from A by exchanging two rows, then $\det(B) = -\det(A)$.
   In particular, $\det(I) = 1 \Rightarrow \det(P_{ij}) = -1$.
3. det is linear in each row separately,
   i.e. , we fix $n-1$ row vectors, say $v_2, \cdots, v_n$, then
   $\det\begin{pmatrix} - & v_2 & \cdots & v_n \end{pmatrix}^T : \mathbb{R}^n \to \mathbb{R}$ is a linear function.
   i.e., for c, d in $\mathbb{R}$, and vectors u and v, if $A_{1*} = cu + dv$, we have
   $\det\begin{pmatrix} cu+dv & A_{2*} & \cdots & A_{n*} \end{pmatrix}^T$
   $= c\det\begin{pmatrix} u & A_{2*} & \cdots & A_{n*} \end{pmatrix}^T + d\det\begin{pmatrix} v & A_{2*} & \cdots & A_{n*} \end{pmatrix}^T$.
   There are n such equations (for n choices of rows).

Determinants: Induced Properties

1. If B and C are $n \times n$, then $\det(B + C) = \det(B) + \det(C)$. False. Find examples.

   Exercise: In the 2×2 case, $\det(B + C)$
   $$= \det(B) + \det\begin{pmatrix} B_{1*} \\ C_{2*} \end{pmatrix} + \det\begin{pmatrix} C_{1*} \\ B_{2*} \end{pmatrix} + \det(C).$$

2. For a real number c, $\det(cB) = c\det(B)$ (False)

   Exercise: If B is $n \times n$, then $\boxed{\det(cB) = c^n \det(B).}$

3. If two rows of A are equal, then $\det(A) = 0$.
   Proof. Suppose i-th and j-th rows of A are equal, i.e., $A_{i*} = A_{j*}$, then $A = P_{ij}A$.

   Hence $\det(A) = \det(P_{ij}A) = -\det(A) \Rightarrow \boxed{\det(A) = 0}$.

Determinants and Row Operations

4. If B is obtained from A by $\boxed{R_i \mapsto R_i + aR_j,}$ then $\det(B) = \det(A)$.

Proof: Note that $B = E_{ij}(a)A = \begin{array}{cc} \begin{pmatrix} \vdots \\ A_{i*} + aA_{j*} \\ \vdots \\ A_{j*} \\ \vdots \end{pmatrix} & \begin{array}{l} i\text{th row} \\ \\ j\text{th row} \end{array} \end{array}$. Then

$$\det(B) = \det \begin{pmatrix} \vdots \\ A_{i*} + aA_{j*} \\ \vdots \\ A_{j*} \\ \vdots \end{pmatrix} = \det(A) + \det \begin{pmatrix} \vdots \\ aA_{j*} \\ \vdots \\ A_{j*} \\ \vdots \end{pmatrix} = \det(A).$$

In particular, $\det(E_{ij}(a)) = 1$. Why?

Determinants and Row Operations

5. If A is $n \times n$, and its row echelon form U is obtained without row exchanges, then $\det(U) = \det(A)$.

   Q: What happens if there are row exchanges? Exercise!

6. If A has a zero row, then $\det(A) = 0$.

   Proof: Let the ith row of A be zero, i.e., $A_{i*} = 0$.

   Let B be obtained from A by $\boxed{R_i = R_i + R_j}$, i.e., $B = E_{ij}(1)A$.

   Then $B_{i*} = B_{j*}$.

   Exercise: Complete the proof.

Determinants of Special Matrices

7. If $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ is diagonal, then $\det(A) = a_1 \cdots a_n$. (Use linearity).
8. If $A = (a_{ij})$ is triangular, then $\det(A) = a_{11} \dots a_{nn}$.

   Proof. If all a_{ii} are non-zero, then by elementary row operations, A reduces to the diagonal matrix $\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}$ whose determinant is $a_{11} \cdots a_{nn}$.

   If at least one diagonal entry is zero, then elimination will produce a zero row $\Rightarrow \det(A) = 0$.

Determinants and Products

9. If A and B are $n \times n$, then $\boxed{\text{det(AB) = det(A) det(B)}}$.

   Example 1: If B is obtained from A by interchanging the ith and jth row, then $\det(B) = -\det(A)$.

   In this case, $B = P_{ij}A$. Since $\det(P_{ij}) = -1$, observe that $\det(P_{ij}A) = \det(P_{ij})\det(A)$.

   Example 2: If B is obtained from A by the row operation $R_i \mapsto R_i + aR_j$, then $\det(B) = \det(A)$.

   In this case, $B = E_{ij}(a)A$. Since $\det(E_{ij}(a)) = 1$, we see that $\det(E_{ij}(a)A) = \det(E_{ij}(a))\det(A)$.

Extra Reading: $\det(AB) = \det(A)\det(B)$

$$\boxed{\det(AB) = \det(A)\det(B)}$$

We may assume that B are invertible. Else, $\text{rank}(AB) \leq \text{rank}B \neq n$ $\implies \text{rank}(AB) \neq n \implies AB$ is not invertible.

Hint: For fixed B, show that the function d defined by

$$d(A) = \det(AB)/\det(B)$$

satisfies the following properties

1. $d(I) = 1$.
2. If we interchange two rows of A, then d changes its sign.
3. d is a linear function in each row of A.

Then d is the unique determinant function det and

$$\det(AB) = \det(A)\det(B)$$