MA-108 Differential Equations I

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S1 - Lecture 1

Second Order Linear ODE's

We'll consider second order linear ODE.
Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.
Though there is no formula to find all the solutions of such an ODE, we will study the existence, uniqueness and number of solutions of such an ODE.

Second Order Linear ODE's

- If $r(x) \equiv 0$ in the equation above, i.e.,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

  then the ODE is said to be homogeneous.
  Otherwise it is called non-homogeneous.

- An IVP of a second order linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = r(x); \quad y(x_0) = a, \; y'(x_0) = b,$$

  where $p(x)$, $q(x)$ and $r(x)$ are assumed to be continuous on some open interval I with $x_0 \in I$.

Example

Solve the second order linear ODE

$$y'' + y = 0$$

Observe that $\sin x$ and $\cos x$ satisfy this equation.
Any scalar multiple of $\sin x$ and $\cos x$ is also a solution.
Any linear combination $c_1 \sin x + c_2 \cos x$ is a solution.

Example

Solve

$$y'' - y = 0$$

It is easy to see that e^x and e^{-x} are solutions.
Again any linear combination $c_1 e^x + c_2 e^{-x}$ is a solution.

Question. Are these all the solutions?
If not, what are the other solutions, and how to find them?
If yes, why are these the only solutions?

Solving IVP's

Let I be an open interval with $x_0 \in I$.
For an integer $n \geq 0$, let $C^n(I)$ be the set of all functions $f : I \to \mathbb{R}$ such that f is n-times differentiable on I and $f^{(n)}$ is continuous on I.
Note that $C^n(I)$ is a vector spaces over $\mathbb{R}$ with addition and scalar multiplication defined as follows. For $x \in I$ and $k \in \mathbb{R}$,

$$\begin{aligned} (f+g)(x) &:= f(x) + g(x) \\ (k \cdot f)(x) &:= kf(x) \end{aligned}$$

Let $p, q \in C(I)$ be continuous functions on I. Define

$$L : C^2(I) \to C(I)$$

by

$$L(f) = f'' + p(x)f' + q(x)f.$$

Solving IVP's

Check that L is a linear transformation; i.e.

$$L(f+g) = L(f) + L(g), \quad L(cf) = cL(f)$$

for all $c \in \mathbb{R}$ and $f, g \in C^2(I)$.
The null space of L, denoted by $N(L)$ is

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

Thus, $N(L)$ consists of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

Therefore, solutions of (*) is a vector sub-space of $C^2(I)$.

Existence and Uniqueness Theorem

Recall that in the first order case, existence and uniqueness was easy to prove in the linear case, whereas one need a nontrivial result in the non-linear case.
In the second order case, we need a non-trivial result in the linear case itself.

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = a, \; y'(x_0) = b,$$

where $p, q \in C(I)$ for an open interval I with $x_0 \in I$. Then there is a unique solution to the IVP on I.

When p, q are constant, solution exists on $\mathbb{R}$.

Example

Find the largest open interval where the ODE

$$x^2y'' + xy' - 4y = 0$$

with initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$ has a unique solution.

Write the ODE in standard form

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

Since $p(x) = \dfrac{1}{x}$ and $q(x) = \dfrac{-4}{x^2}$ are continuous on $(-\infty, 0) \cup (0, \infty)$, the IVP has has a unique solution on $(-\infty, 0)$ if $x_0 < 0$ and on $(0, \infty)$ if $x_0 > 0$.

- Verify that $y_1 = x^2$ is a solution of ODE on $(-\infty, \infty)$ and $y_2 = \dfrac{1}{x^2}$ is a solution on $(-\infty, 0) \cup (0, \infty)$.

Example

Solve IVP

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \; y'(1) = 0$$

The general solution of ODE is

$$\begin{array}{rcl} y(x) & = & c_1x^2 + c_2\dfrac{1}{x^2} \\ c_1 + c_2 = 2 & , & 2c_1 - 2c_2 = 0 \\ & \Longrightarrow & c_1 = 1, \quad c_2 = 1 \end{array}$$

Thus solution of IVP is $\quad y(x) = x^2 + \dfrac{1}{x^2}$
which is unique on the interval $(0, \infty)$.

Exercise. Solve
$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \; y'(-1) = 0$

Theorem (Dimension Theorem)

If $p(x), q(x)$ are continuous on an open interval I, then the set of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

on I is a vector space of dimension 2.
Any basis $\{y_1, y_2\}$ of solutions of (*) is called a **fundamental solutions** *of (*).*

- The theorem says that once you know that e^x and e^{-x} are solutions of $y'' - y = 0$, any other solution will be of the form $y(x) = c_1 e^x + c_2 e^{-x}$.
  Here $\{e^x, e^{-x}\}$ is a fundamental solutions.
- Similarly, any solution of $y'' + y = 0$ are of the form $y(x) = c_1 \sin x + c_2 \cos x$.
  Here $\{\sin x, \cos x\}$ is fundamental solutions.

Definition

Let us consider 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

with constant coefficients $a, b, c \in \mathbb{R}$ with $a \neq 0$.
For scalar m, e^{mx} is a solution if and only if

$$\begin{aligned} & am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \\ \implies \quad & p(m) := am^2 + bm + c = 0 \end{aligned}$$

Therefore, e^{mx} is a solutioin of ODE if and only if m is a root of the characteristic equation $p(m) = 0$.
The roots of the characteristic equation are given by

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

2nd Order Linear ODE's with constant coeff.

We consider three cases:

1. Case 1: $b^2 - 4ac > 0$.
   Then characteristic equation has two distinct real roots.
2. Case 2: $b^2 - 4ac = 0$.
   Then characteristic equation has two repeated real roots.
3. Case 3: $b^2 - 4ac < 0$.
   Then characteristic equation has two distinct complex roots which are conjugates.

Distinct real roots case

Example

Find general solution of

$$y'' + 6y' + 5y = 0$$

The roots of characteristic equation

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5) = 0$$

are -1 and -5. Thus

$$y_1 = e^{-x}, \quad y_2 = e^{-5x}$$

are fundamental solutions of ODE. Therefore, the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

Example

Solve IVP

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \; y'(0) = 1$$

The general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

$$\begin{aligned} y(0) = 3 \quad &\Longrightarrow \quad c_1 + c_2 = 3 \\ y'(0) = 1 \quad &\Longrightarrow \quad -c_1 - 5c_2 = 1 \end{aligned}$$

This gives $c_2 = -1$ and $c_1 = 4$.
Thus the solution to IVP is

$$y(x) = 4e^{-x} - e^{-5x}$$

A repeated real root case

Example

Find general solution of $y'' + 6y' + 9y = 0$
The roots of characteristic equation

$$p(m) = m^2 + 6m + 9 = (m+3)^2 = 0$$

are repeated $-3, -3$. Hence $y_1 = e^{-3x}$ is one solution.
For other solution, let $y = u(x)y_1 = u(x)e^{-3x}$. Then

$$\begin{aligned}
(u'y_1 + uy_1')' + 6(u'y_1 + uy_1') + 9(uy_1) &= 0 \\
(u''y_1 + 2u'y_1' + uy_1'') + 6(u'y_1 + uy_1') + 9(uy_1) &= 0 \\
u''y_1 + u'(2y_1' + 6y_1) + u(y_1'' + 6y_1' + 9y_1) &= 0 \\
u''y_1 + u'(2y_1' + 6y_1) &= 0
\end{aligned}$$

Example (continued ...)

$$
\begin{aligned}
\frac{u''}{u'} + \frac{2y_1'}{y_1} + 6 &= 0 \\
\ln|u'| + \ln|y_1^2| + 6x &= C \\
u' y_1^2 e^{6x} &= c \\
y_1 &= e^{-3x} \\
u' &= c \\
u &= cx \\
y_2 &= xe^{-3x}
\end{aligned}
$$

Therefore the general solution is

$$y(x) = e^{-3x}(c_1 + c_2 x)$$

Example

Solve IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \; y'(0) = 1$$

The general solution is

$$y(x) = e^{-3x}(c_1 + c_2 x)$$

$$y(0) = 3 \implies c_1 = 3$$
$$y'(0) = 1 = -3(3) + c_2 \implies c_2 = 10$$

Thus, the solution of IVP is

$$y(x) = e^{-3x}(3 + 10x)$$

two distinct complex conjugate roots case

Example

Find general solution of

$$y'' + 4y' + 13y = 0$$

Roots of the characteristic equation

$$m^2 + 4m + 13 = (m+2)^2 + 9$$

are $-2+3i$ and $-2-3i$. So

$$\begin{aligned} e^{(-2+3i)x} &= e^{-2x}(\cos 3x + i \sin 3x) \\ e^{(-2-3i)x} &= e^{-2x}(\cos 3x - i \sin 3x) \end{aligned}$$

are complex solutions of ODE.
Taking sum and difference gives real solutions

Example (continued ...)

$$y_1 = e^{-2x}\cos 3x, \quad y_2 = e^{-2x}\sin 3x$$

which are fundamental solutions.
Hence the general solution is

$$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$$

Example

Solve IVP

$$y'' + 4y' + 13y = 0, \quad y(0) = 3, \; y'(0) = 1$$

The general solution is

$$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$$

$$\begin{aligned} y(0) = 3 \quad &\Longrightarrow \quad c_1 = 3, \\ y'(0) = 1 = -2(3) + 3c_2 \quad &\Longrightarrow \quad c_2 = 7/3 \end{aligned}$$

Thus the solution of IVP is

$$y(x) = e^{-2x}\left(3\cos 3x + \frac{7}{3}\sin 3x\right)$$

Theorem

Let m_1, m_2 *be the roots of characteristic equation*

$$p(m) = am^2 + bm + c = 0 \quad \textit{of ODE}$$

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}, \;\; a \neq 0$$

Then the general solution $y(x)$ *is given as follows.*

1. *If* $m_1 \neq m_2$ *are real, then*

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

2. *If* $m_1 = m_2 \in \mathbb{R}$,

$$y(x) = e^{m_1 x}(c_1 + c_2 x)$$

3. *If* $m_1 = \lambda + i\omega$ *and* $m_2 = \lambda - i\omega$, *where* $\omega > 0$, *then*

$$y(x) = e^{\lambda x}[c_1 \cos(\omega x) + c_2 \sin(\omega x)]$$

Theorem (Dimension Theorem)

Let $p(x), q(x)$ be continuous functions on an open interval I. Then the set of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

on I is a vector space of dimension 2.

We will give a proof of this theorem using following theorem.

Theorem

Let $p(x), q(x)$ be continuous functions on an open interval I. Let $x_0 \in I$ and $y_0, y_1 \in \mathbb{R}$. Then the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = y_0, \; y'(x_0) = y_1$$

has a unique solution on I.

Proof of Dimension Theorem

If y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

then $c_1y_1 + c_2y_2$ is also a solution of ODE.
To see this,

$$\begin{aligned}
& (c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) \\
= \;& c_1[y_1'' + p(x)y_1' + q(x)y_1] + c_2[y_2'' + p(x)y_2' + q(x)y_2] \\
= \;& 0
\end{aligned}$$

Thus the solution space is a vector space. Now

1. we need to produce two linearly independent solutions, say f and g, and
2. show that any other solution is a linear combination of f and g.

Existence of f and g

Fix $x_0 \in I$. Let $y_1 = f(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 1,\ y'(x_0) = 0$$

y_1 exists on I by uniqueness theorem.
Similarly, let $y_2 = g(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0,\ y'(x_0) = 1$$

We need to show that f, g are linearly independent. Assume

$$af(x) + bg(x) \equiv 0 \implies af'(x) + bg'(x) \equiv 0$$

for some scalars a and b. Evaluate at $x = x_0$, we get

$$a = 0, \quad b = 0$$

This proves f and g are linearly independent solutions of ODE.

Any solution is a linear combination of f and g

Let $h(x)$ be an arbitrary solution of the given ODE. Define

$$\widetilde{h}(x) = h(x_0)f(x) + h'(x_0)g(x)$$

Then $\widetilde{h}(x)$ and $h(x)$ both are solutions of IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = h(x_0),\ y'(x_0) = h'(x_0)$$

on I. By uniqueness theorem,

$$\widetilde{h} \equiv h$$

Thus any solution is a linear combination of f and g.
Therefore, the solution space of ODE

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional. $\square$

Nonhomogeneous 2nd order linear ODE

Consider 2nd order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad (*)$$

with $p(x), q(x), r(x)$ continuous on an open interval I.
The solution space of its homogeneous part

$$y'' + p(x)y' + q(x)y = 0 \quad (**)$$

is a 2-dimensional vector space.

- Suppose y_1 is a solution of $(*)$ and y_2 is a solution of $(**)$. Show that $y_1 + y_2$ is a solution of $(*)$.
- Fix a solution y_1 of $(*)$. If y is any other solution of $(*)$, then $y - y_1 = y_2$ is a solution of $(**)$.

  Therefore, $y = y_1 + y_2$.

Wronskian and Linear Independence

Given two solutions f and g of

$$y'' + p(x)y' + q(x)y = 0$$

How to check whether f and g are linearly independent?

Definition

Let f and g be two differentiable functions on an open interval I. The **Wronskian** of $f(x)$ and $g(x)$ is a function on I defined by

$$W(f, g; x) := \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

- $W(e^x, e^{-x}, x) = e^x(-e^{-x}) - e^{-x}e^x = -2.$
- $W(\sin x, \cos x, x) = \sin x(-\sin x) - \cos x(\cos x) = -1.$

Theorem (Abel's Formula)

Assume $p(x)$ and $q(x)$ are continuous on some open interval I. Let $f(x)$ and $g(x)$ be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Then Wronskian of $f(x)$ and $g(x)$ is given by

$$W(f, g; x) = W(f, g; a)\, exp\left(-\int_a^x p(x)\, dx\right)$$

for any $a \in I$.

Proof.

Set $W(f,g;x) = W(x)$. Then

$$\begin{aligned}
W(x) &= (fg' - f'g)(x) \\
W'(x) &= (fg'' + f'g') - (f'g' + f''g) \\
&= (fg'' - f''g) \\
&= f(-pg' - qg) - g(-pf' - qf) \\
&= -p(fg' - f'g) \\
&= -pW \\
\implies W(x) &= C\, exp\left(-\int_a^x p(x)dx\right)
\end{aligned}$$

on I for a constant C. For $x = a$, we get $C = W(a)$. $\square$

- $W(x_0) = 0$ for some $x_0 \in I \implies W(x) \equiv 0$ on I.
- $W(x_0) \neq 0$ for some $x_0 \in I \implies W(x) \neq 0$ for all $x \in I$.

Example

Functions $y_1 = x^2$ and $y_2 = 1/x^2$ are solutions of ODE

$$
\begin{aligned}
x^2y'' + xy' - 4y &= 0 \quad \text{on} \quad (-\infty, 0) \cup (0, \infty) \\
W(y_1, y_2) &= y_1y_2' - y_1'y_2 \\
&= x^2\left(\frac{-2}{x^3}\right) - (2x)\frac{1}{x^2} = \frac{-4}{x} \\
\text{Using Formula} \quad W(x) &= W(x_0)\, exp\left[-\int_{x_0}^{x} p(x)\, dx\right] \\
&= W(x_0)\, exp\left[\int_{x_0}^{x} \frac{-1}{x}\, dx\right] \\
&= W(x_0)\, exp\left[-\left(\ln|x| - \ln|x_0|\right)\right] \\
&= W(x_0)\, exp\left(\ln\frac{x_0}{x}\right) \\
&= W(x_0)\, \frac{x_0}{x}
\end{aligned}
$$

Wronskian and Linear Independence

Proposition

Suppose $f(x)$ and $g(x)$ are linearly dependent and differentiable functions on $I = (a, b)$.
Then $W(f, g; x) = 0$ for all $x \in I$.

Proof.

As $f(x)$ and $g(x)$ are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0 \implies cf'(x) + dg'(x) = 0 \quad \text{for all } x \in I$$

$$\implies \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\begin{pmatrix} c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $W(f, g; x) = 0$ for all $x \in I$. □

Wronskian and Linear Independence

The converse is not true, i.e.
$W(f, g, x) \equiv 0$ on $I \not\Rightarrow f$ and g are linearly dependent.

Example

$f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

then, check that $W(f, g; x) = 0$ for all $x \in \mathbb{R}$.
But f and g are linearly independent.
Show that

$$af + bg \equiv 0 \implies a = 0 = b$$

Theorem

Suppose f and g are solutions on $I = (a, b)$ of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous on I.
Then f and g are linearly independent on I if and only if $W(f, g; x)$ has no zeros in I.

Equivalently, $W(f, g, x_0) = 0$ for some $x_0 \in I$ if and only if f and g are linearly dependent on I.

Proof.

$(\Rightarrow)$ Assume f, g are linearly independent on I.
We will show that $W(f, g; x)$ has no zeros in I.

If $f \equiv 0$ on I, then f, g are linearly dependent, hence $f(x') \neq 0$ for some $x' \in I$.
We get an open interval $J \subset I$, $x' \in J$ such that f does not take 0 value on J.

continued ...

On J, we have:

$$\begin{aligned}\left(\frac{g}{f}\right)'(x) &= \left(\frac{fg' - f'g}{f^2}\right)(x) \\ &= \frac{W(f,g;x)}{f^2(x)}\end{aligned}$$

If we assume that $W(x_0) = 0$ for some $x_0 \in I$, then using Abel's formula, we get

$$\begin{aligned}W(x) &= W(x_0)e^{\int p(x)\,dx} \\ &= 0 \\ \implies \left(\frac{g}{f}\right)' &\equiv 0 \quad \text{on} \quad J \\ \implies \frac{g}{f} &= k \quad \text{on} \quad J \\ \implies g &= kf \quad \text{on} \quad J\end{aligned}$$

continued ...

But we want $g = kf$ on I.
For this, consider the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \; y'(x_0) = 0 \quad (*)$$

$$y_1 \equiv 0 \;\text{ and }\; y_2 = g - kf$$

are solutions of (*). By uniqueness theorem, $y_1 = y_2$ on I.
Hence $g = kf$ on I, a contradiction.
Hence f, g linearly independent $\implies W(f, g, x) \neq 0$ on I.

$(\Leftarrow)$ We need to show that if $W(f, g; x)$ does not take 0 value on I, then f, g are linearly independent on I.

We have already proved that if f and g are linearly dependent on I, then $W(f, g, ; x) \equiv 0$ on I.
Thus, f and g are linearly independent. $\square$

Remarks:

1. The continuity of $p(x)$ and $q(x)$ is necessary in the above theorem. Consider the DE
$$x^2y'' - 4xy' + 6y = 0.$$
Note that $p = \dfrac{-4}{x}$ and $q = \dfrac{6}{x^2}$ are not continuous at 0. Here x^2 and x^3 are linearly independent solutions, but $W(x^2, x^3; 0) = 0$.

2. Let $f, g \in C^2(I)$, with I an open interval with $a \in I$. Assume f, g are linearly independent and $W(f, g; a) = 0$. Then f and g can not be a fundamental solution on I of any ODE
$$y'' + p(x)y' + q(x)y = 0$$
with $p, q \in C(I)$.

Second Order Linear ODE's

Consider second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

There is no general method to find a basis of solutions.
Given that $y_1 = f(x) \neq 0$ is a known solution of ODE.
Assume $y_2 = v(x)f(x)$ is another solution.

$$\begin{aligned}
(vf)'' + p(vf)' + q(vf) &= 0 \\
(v'f + vf')' + p(v'f + vf') + qvf &= 0 \\
(v'' + 2v'f' + vf'') + p(v'f + vf') + qvf &= 0 \\
v(f'' + pf' + qf) + v'(2f' + pf) + v''f &= 0 \\
v'(2f' + pf) + v''f &= 0 \\
\frac{v''}{v'} = -\frac{2f' + pf}{f} &= -\frac{2f'}{f} - p
\end{aligned}$$

Second Order Linear ODE's

Therefore,

$$\ln|v'| = \ln\left(\frac{1}{f^2}\right) - \int p dx;$$

$$\implies v = \int \frac{e^{-\int p dx}}{f^2} dx.$$

Let us show that f and vf are linearly independent.
We can show non-vanishing of Wronskian at a point.

$$\begin{aligned} W(f, vf) &= f(v'f + f'v) - f'vf \\ &= f^2 v' \\ &= f^2 \frac{e^{-\int p\,dx}}{f^2} \\ &= e^{-\int p\,dx} \\ &\neq 0 \end{aligned}$$

Second Order Linear ODE's

Theorem

Let p, q be continuous on some open interval I.
If $y_1 \neq 0$ is a solution of

$$y'' + p(x)y' + q(x)y = 0$$

on I, then another solution of ODE on I is given by

$$\begin{aligned} y_2(x) &= v y_1(x) \\ &= \left(\int \frac{e^{-\int p dx}}{y_1^2} dx \right) y_1(x) \end{aligned}$$

Further, y_1 and y_2 are linearly independent on I.

Example

Find all solutions of $x^2y'' + xy' - y = 0, \qquad x > 0.$
One solution $y_1 = x$ is given. The ODE in standard form is

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0.$$

Let $y_2 = vx$ be another solution. Then,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p dx}}{y_1^2} dx = \int \frac{e^{-\int (1/x) dx}}{x^2} dx \\ &= \int \frac{dx}{x^3} = -\frac{1}{2x^2}. \end{aligned}$$

Hence, $y_2 = \dfrac{-1}{2x}$. Therefore, the general solution is

$$y(x) = cx + \frac{d}{x}, \quad c, d \in \mathbb{R}.$$

Cauchy Euler Equation

Solve

$$x^2y'' + axy' + ay = 0, \quad x > 0,\ a, b \in \mathbb{R}$$

This equation can be transformed into one with constant coefficients by change of variables on the interval $(0, \infty)$.

$$\begin{aligned}
t = \ln x \quad & \Longrightarrow \quad x = e^t \\
\frac{dy}{dt} \quad & = \quad \frac{dy}{dx}\,\frac{dx}{dt} = \frac{dy}{dx}\,e^t \\
\frac{d^2y}{dt^2} \quad & = \quad \frac{d}{dt}\left[\frac{dy}{dx}e^t\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right]e^t + \frac{dy}{dx}\,e^t \\
& = \quad \frac{d}{dx}\left[\frac{dy}{dx}\right]e^{2t} + \frac{dy}{dt} \\
\frac{d^2y}{dx^2} \quad & = \quad \frac{1}{e^{2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)
\end{aligned}$$

Substituting this in the ODE

$$
\begin{aligned}
& x^2y'' + axy' + by = 0 \\
\implies \quad & e^{2t}\frac{1}{e^{2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + ae^t\frac{1}{e^t}\frac{dy}{dt} + by = 0. \\
\implies \quad & y''(t) + (a-1)y'(t) + by(t) = 0
\end{aligned}
$$

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions to this equation, then the solutions to the Cauchy-Euler equation is given by

$$y(x) = c_1y_1(\ln x) + c_2y_2(\ln x)$$

Theorem

Consider the Cauchy-Euler Equation

$$x^2y'' + axy' + by = 0, \qquad x > 0 \qquad (*)$$

Substituting $x = e^t$, ODE becomes

$$y''(t) + (a-1)y'(t) + by(t) = 0 \qquad (**)$$

Let m_1 and m_2 be the roots of the char equation of $(**)$

$$p(m) = m^2 + (a-1)m + b = 0.$$

Then the general solution of $(*)$ is given as follows.

1. *If $m_1 \neq m_2 \in \mathbb{R}$, then $y(x) = c_1x^{m_1} + c_2x^{m_2}$.*
2. *If $m_1 = m_2 \in \mathbb{R}$, $y(x) = c_1x^m + c_2x^m \ln x$.*
3. *If $m_i = \lambda \pm i\omega$ with $\omega > 0$, then*

$$y(x) = c_1x^\lambda \cos(\omega \ln x) + c_2x^\lambda \sin(\omega \ln x).$$

Example

Solve

$$x^2y'' + 7xy' + 5y = 0$$

Putting $t = \ln x$, we get

$$y''(t) + (7-1)y'(t) + 5y(t) = 0$$

It's characteristic equation is

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$$

Hence the general solution is

$$y(x) = c_1\frac{1}{x} + c_2\frac{1}{x^5}$$

Example

Solve

$$x^2y'' + 7xy' + 9y = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (7-1)m + 9 = (m+3)^2$$

Hence the general solution is

$$y(x) = c_1\frac{1}{x^3} + c_2\frac{1}{x^3}\ln x$$

Example

Solve

$$x^2y'' + 5xy' + 13y = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9$$

Hence the general solution is

$$y(x) = \frac{1}{x^2}[c_1\cos(3\ln x) + c_2\sin(3\ln x)]$$

Non-homogeneous Second Order Linear ODE's

Theorem

Let $f(x)$ be any solution of

$$y'' + p(x)y' + q(x)y = r(x) \quad (*)$$

and $y_1(x), y_2(x)$ be a fundamental solutions of homogeneous part

$$y'' + p(x)y' + q(x)y = 0$$

Then the set of solutions of the non-homogeneous ODE is

$$\{c_1y_1(x) + c_2y_2(x) + f(x) \mid c_1, c_2 \in \mathbb{R}\}.$$

Therefore, to solve $(*)$,

- get one particular solution of $(*)$ and
- general solution homogeneous part.

Method of Variation of Parameters

Assume p, q, r are continuous of an open interval I.
If we can find two linearly independent solutions y_1 and y_2 of

$$y'' + p(x)y' + q(x)y = 0 \quad (1).$$

then we can find a particular solution of

$$y'' + p(x)y' + q(x)y = r(x) \quad (2)$$

This method of finding a particular solution, using the solutions of homogeneous part, is called the method of variation of parameters.
Here, we try to find a particular solution of (2) of the form

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$
\begin{aligned}
y(x) \quad &= v_1(x)y_1(x) + v_2(x)y_2(x) \\
\Longrightarrow \; y' \quad &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2
\end{aligned}
$$

Let's further assume that v_1 and v_2 satisfy

$$
\begin{aligned}
& v_1'y_1 + v_2'y_2 = 0. \\
\Longrightarrow \quad & y' = v_1y_1' + v_2y_2' \\
\Longrightarrow \quad & y'' = v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2'.
\end{aligned}
$$

Substituting y, y', y'' in the given non-homogeneous ODE,

$$
\begin{aligned}
& (v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2') + p(v_1y_1' + v_2y_2') + q(v_1y_1 + v_2y_2) \\
& \quad = \; v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' \\
& \quad = \; r \\
& \Longrightarrow \; v_1'y_1' + v_2'y_2' = r
\end{aligned}
$$

We have

$$
\begin{aligned}
& v_1' y_1 + v_2' y_2 = 0, \quad v_1' y_1' + v_2' y_2' = r \\
\implies \quad & \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}. \\
\implies \quad & v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{W(y_1, y_2)} \\
\implies \quad & v_1 = -\int \frac{y_2 r}{W(y_1, y_2)} dx, \ v_2 = \int \frac{y_1 r}{W(y_1, y_2)} dx
\end{aligned}
$$

Hence a particular solution is

$$
y(x) = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} \, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} \, dx.
$$

Example

Solve

$$y'' + 6y' + 5y = e^x$$

Homogeneous part

$$y'' + 6y' + 5y = 0$$

has two linearly independent solutions

$$y_1 = e^{-x}, \quad y_2 = e^{-5x}$$

Wronskian of y_1 and y_2 is

$$W(e^{-x}, e^{-5x}) = e^{-x}(-5e^{-5x}) - (-e^{-x})e^{-5x} = -4e^{-6x}$$

A particular solution y_p is given by variation of parameter:

Example (continued ...)

$$\begin{aligned}
y_p &= y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx \\
&= e^{-5x} \int \frac{e^{-x} e^{x}}{-4e^{-6x}}\, dx - e^{-x} \int \frac{e^{-5x} e^{x}}{-4e^{-6x}}\, dx \\
&= -\frac{1}{4}\left[e^{-5x} \int e^{6x}\, dx - e^{-x} \int e^{2x}\, dx\right] \\
&= -\frac{1}{4}\left[\frac{1}{6}e^{x} - \frac{1}{2}e^{x}\right] \\
&= \frac{1}{12}e^{x}
\end{aligned}$$

Thus the general solution is

$$y(x) = \frac{1}{12}e^{x} + c_1 e^{-x} + c_2 e^{-5x}$$

Example

Solve

$$y'' + 6y' + 5y = e^{-x}$$

We know that

$$y_1 = e^{-x}, \quad y_2 = e^{-5x}$$

are two fundamental solutions of homogeneous part.
Its Wronskian is

$$W(e^{-x}, e^{-5x}) = -4e^{-6x}$$

A particular solution y_p is given by

$$y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx.$$

Example (continued ...)

$$\begin{aligned}
y_p &= e^{-5x}\int \frac{e^{-x}e^{-x}}{-4e^{-6x}}\,dx - e^{-x}\int \frac{e^{-5x}e^{-x}}{-4e^{-6x}}\,dx \\
&= -\frac{1}{4}\left[e^{-5x}\int e^{4x}\,dx - e^{-x}\int\,dx\right] \\
&= -\frac{1}{4}\left[e^{-x}(\frac{1}{4} - x)\right] \\
&= -\frac{1}{16}e^{-x}(1-4x)
\end{aligned}$$

Thus the general solution is given by

$$\begin{aligned}
y(x) &= -\frac{1}{16}e^{-x}(1-4x) + c_1e^{-x} + c_2e^{-5x} \\
&= \frac{1}{4}xe^{-x} + c_1e^{-x} + c_2e^{-5x}
\end{aligned}$$

Example

Find a particular solution of

$$y'' + 4y = 3\cos 2x$$

A fundamental solutions of homogeneous part is

$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

The Wronskian

$$W(y_1, y_2) = 2$$

A particular solution y_p is given

$$y_p = y_2 \int \frac{y_1 r}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r}{W(y_1, y_2)}\, dx$$

Example (continued ...)

$$
\begin{aligned}
y_p &= \sin 2x \int \frac{\cos 2x \cdot 3\cos 2x}{2} dx \\
&\quad - \cos 2x \int \frac{\sin 2x \cdot 3\cos 2x}{2} dx \\
&= \sin 2x \int \frac{3}{4}(1+\cos 4x)\,dx - \cos 2x \int \frac{3}{4}\sin 4x\,dx \\
&= \frac{3}{4}\sin 2x\left[x + \frac{1}{4}\sin 4x\right] - \frac{3}{4}\cos 2x\left[-\frac{1}{4}\cos 4x\right] \\
&= \frac{3}{4}x\sin 2x + \frac{3}{16}[\sin 2x \sin 4x + \cos 2x \cos 4x] \\
&= \frac{3}{4}x\sin 2x + \frac{3}{16}\cos 2x
\end{aligned}
$$

Example

Find a particular solution of

$$y'' + y = \csc x$$

Solutions of homogeneous part are

$$y_1 = \sin x, \quad y_2 = \cos x, \quad W(y_1, y_2) = -1$$

A particular solution y_p is given by variation of parameter:

$$\begin{aligned} y_p &= y_2 \int \frac{y_1 r}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r}{W(y_1, y_2)}\, dx \\ &= \cos x \int \frac{\sin x \csc x}{-1} dx - \sin x \int \frac{\cos x \csc x}{-1} dx \\ &= -x \cos x + \sin x \ln|\sin x| \end{aligned}$$

Example

Solve the IVP

$$(x^2-1)y''+4xy'+2y=\frac{2}{x+1},\quad y(0)=-1,\ y'(0)=-5$$

It is given that

$$y_1=\frac{1}{x-1},\quad \text{and}\quad y_2=\frac{1}{x+1}$$

are solutions of homogeneous part.

Check that the solution is given by

$$y(x)=\frac{2\ln|x+1|}{x-1}+\frac{3x+1}{x^2-1}$$

n^{th} order linear differential equations

- An n-order linear ODE is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x) \quad (*)$$

where a_i's and g are continuous on some open interval I.

- An n-order linear ODE in <u>standard form</u> is given by

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

- ODE $(*)$ is called homogeneous if $r \equiv 0$ and non-homogeneous otherwise.
- Recall that the solution space of a 2nd order linear homogenous ODE

$$y'' + p(x)y' + q(x)y = 0$$

is same as the <u>null space</u> of the linear transformation

$$(D^2 + p(x)D + q(x)Id) := L : \mathcal{C}^2(I) \to \mathcal{C}(I)$$

Definition

$C^n(I)$ is the set of all functions $f : I \to \mathbb{R}$ such that $f^{(n)}$ exists on open interval I and is continuous.
Define linear transformation

$$L : C^n(I) \to C(I)$$

$$L = D^n + p_1 D^{n-1} + \ldots + p_{n-1} D + p_n I$$

Then the solution space to the ODE

$$y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = 0$$

is given by the null space of L.
We would like to prove that Dimension of $N(L) = n$.

Theorem (Existence and Uniqueness)

Consider the initial value problem for an n-order linear ODE

$$\begin{aligned} y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y &= 0 \\ y(x_0) &= k_0 \\ y^1(x_0) &= k_1 \\ &\vdots \\ y^{n-1}(x_0) &= k_{n-1} \end{aligned}$$

where p_i's are continuous on an open interval I and $x_0 \in I$. Then the IVP has a unique solution on I.

Constant Coefficients n-th order ODE

Example

Find general solution of

$$y''' + 6y'' + 11y' - 6y = 0$$

If we write the linear operator

$$L = D^3 + 6D^2 + 11D - 6$$

then our ODE is

$$\begin{aligned} Ly &= 0, \quad \text{where} \\ L &= D^3 + 6D^2 + 11D - 6 \\ &= (D-3)(D^2+3D+2) \\ &= (D-3)(D+1)(D+2) \end{aligned}$$

Example (continued ...)

$$\begin{aligned} L &= (D-3)(D+1)(D+2) \\ &= (D+2)(D+1)(D-3) \\ &= (D-3)(D+2)(D+1) \end{aligned}$$

If y is such that

$$\begin{aligned} & (D+2)y = 0 \text{ or } (D+1)y = 0 \text{ or } (D-3)y = 0 \\ \Longrightarrow \quad & Ly = 0 \\ \Longrightarrow \quad & f(x) = e^{-x},\ g(x) = e^{-2x},\ h(x) = e^{3x} \end{aligned}$$

are all solutions to the given ODE.

By dimension theorem, if f, g, h are linearly independent, then they give a basis for all solutions.
How do we check linear independence?

Example (continues ...)

$$
\begin{aligned}
& af + bg + ch \equiv 0 \\
\implies \quad & af' + bg' + ch' \equiv 0, \quad af'' + bg'' + ch'' \equiv 0 \\
\implies \quad & \begin{bmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \forall x \in I
\end{aligned}
$$

$$
\begin{aligned}
W(f, g, h; x) &= \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} \\
W(f, g, h; 0) &= \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = -30 + 12 - 2 = -20 \neq 0 \\
\implies [a, b, c] &= [0, 0, 0]
\end{aligned}
$$

Thus, f, g, h are L.I. on any interval I containing 0.

Definition

Let $f_1, f_2, \ldots, f_n \in C^{n-1}(I)$ for some open interval I.
Define their Wronskian

$$W(f_1, \ldots, f_n; x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ldots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

Theorem (Abel's theorem)

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)I$$

be n-th order linear differential operator, where $p_1, \ldots, p_n$ are continuous on an open interval I.
Let $y_1, \ldots, y_n$ be solutions to the linear ODE

$$Ly = 0$$

Let $x_0 \in I$. Then the Wronskian $W(y_1, \cdots, y_n; x) := W(x)$ is given by

$$W(x) = W(x_0) \mathit{exp}\left(-\int_{x_0}^{x} p_1(t)\, dt\right), \quad x \in I$$

Thus, either W has no zeros on I or $W \equiv 0$ on I.

Proof of Abel's theorem

$$
\begin{aligned}
W(y_1, y_2) &= y_1 y_2' - y_1' y_2 \\
&= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
W' &= (y_1' y_2' + y_1 y_2'') - (y_1'' y_2 + y_1' y_2') \\
&= y_1 y_2'' - y_2 y_1'' \\
\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' &= \begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} \\
&= y_1 y_2'' - y_2 y_1'' \\
W(y_1, \ldots, y_n; x) &= \begin{vmatrix} y_1 & \ldots & y_n \\ y_1' & \ldots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} (x)
\end{aligned}
$$

Proof of Abel's theorem continued ...

We will use the following formula for derivative of determinant of a matrix.

$$\begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix}' = \begin{vmatrix} f'_{11} & \cdots & f'_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f'_{21} & \cdots & f'_{2n} \\ \vdots & \vdots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots \\ f'_{n1} & \cdots & f'_{nn} \end{vmatrix}$$

Proof of Abel's theorem continued ...

$$
\begin{aligned}
W(y_1,\ldots,y_n)' &= \begin{vmatrix} y_1' & \cdots & y_n' \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & \cdots & y_n \\ y_1'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \\
+ \cdots + & \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}
\end{aligned}
$$

Substitute $\quad y_i^{(n)} = -\sum_{j=1}^{n} p_j(x) y_i^{(n-j)}.$

Proof of Abel's theorem continued ...

$$
\begin{aligned}
W(y_1,\ldots,y_n)' &= \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ -\sum_{j=1}^{n} p_j y_1^{(n-j)} & \cdots & -\sum_{j=1}^{n} p_j y_n^{(n-j)} \end{vmatrix} \\
&= \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ -p_1 y_1^{(n-1)} & \cdots & -p_1 y_n^{(n-1)} \end{vmatrix} \\
&= -p_1 W(y_1,\ldots,y_n) \\
W(y_1,\ldots,y_n;x) &= W(x_0)\,\mathsf{exp}\left(-\int_{x_0}^{x} p_1(x)\,dx\right), \quad x_0 \in I
\end{aligned}
$$

Theorem

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)I$$

where $p_1, \ldots, p_n$ *are continuous on an open interval* I.
Let $y_1, \ldots, y_n$ *be solutions to the linear ODE* $Ly = 0$.
Then the following statements are equivalent.

1. *The set* $\{y_1, \ldots y_n\}$ *is a fundamental set of solutions of* $Ly = 0$ *on* I.
2. $\{y_1, \ldots, y_n\}$ *is linearly independent on* I.
3. *The Wronskian of* $\{y_1, \ldots, y_n\}$ *is nonzero at some point on* I.
4. *The Wronskian of* $\{y_1, \ldots, y_n\}$ *is nonzero for all* $x \in I$.

Note that $(1) \iff (2)$ *and* $(3) \iff (4)$.
Observe $(2) \iff (3)$.

Theorem (Dimension theorem)

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I$$

where p_i's are continuous on an open interval I.
Then the dimension of null space $N(L)$ of L is n.

Proof.

Let $\{e_1, \ldots, e_n\}$ be the standard basis vectors of $\mathbb{R}^n$.
Let $x_0 \in I$. By existence and uniqueness theorem, the IVP

$$Ly = 0; \quad (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)) = e_i$$

has a unique solution y_i on I for all $i = 1, \ldots n$.

$$W(y_1, \ldots, y_n; x_0) = 1 \neq 0$$

$\square$

continued ...

Therefore $\{y_1, \ldots, y_n\}$ are linearly independent on I.
We need to show that any solution of $Ly = 0$ is a linear combination of $y_1, \ldots, y_n$.
So let $Y(x)$ be any solution of $Ly = 0$. Write $c_i = Y^{(i-1)}(x_0)$ for $i = 1, \ldots, n$ and

$$Z(x) := c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

Then $Y(x)$ and $Z(x)$ both are solutions of IVP

$$L(y) = 0, \quad y(x_0) = c_1, \ \ldots, y^{(n-1)}(x_0) = c_n$$

Using the existence and uniqueness theorem to IVP, we get

$$Y(x) \equiv Z(x) \quad \text{on} \quad I$$

Thus, $\{y_1, \ldots, y_n\}$ is a basis for null space of L. $\square$

Constant Differential Operators

- Consider a constant coefficient differential operator

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

- For any function $f \in C^n(I)$,

$$L(f) = a_0 f^{(n)} + a_1 f^{(n-1)} + \ldots + a_{n-1} f' + a_n f$$

- If

$$\begin{aligned} & M = b_0 D^m + b_1 D^{m-1} + \ldots + b_m, \quad b_j \in \mathbb{R} \\ & D^r \cdot D^s = D^s \cdot D^r = D^{r+s}, \quad r, s \geq 0 \\ \Longrightarrow \quad & L(M(f)) = M(L(f)), \quad f \in C^{(n)}(I) \end{aligned}$$

Example

Solve $\quad y''' - 3y'' + 7y' - 5y = 0$

$$\begin{aligned} L &= D^3 - 3D^2 + 7D - 5 \\ &= (D-1)(D^2 - 2D + 5) \\ &= (D-1)((D-1)^2 + 4)) \\ &= ((D-1)^2 + 4))(D-1) \end{aligned}$$

So solutions e^x, $e^x \cos 2x$ and $e^x \sin 2x$ of ODE's

$$(D-1)y = 0, \quad [(D-1)^2 + 4]y = 0$$

give solutions to $Ly = 0$.
Verify that they are linearly independent. So general solution is

$$y(x) = e^x[c_1 + c_2 \cos 2x + c_3 \sin 2x].$$

Theorem

Consider

$$L = \sum_{i=0}^{n} a_i D^{n-i}, \quad M = \sum_{i=0}^{m} b_i D^{m-i}, \quad a_i, b_i \in \mathbb{R}$$

two constant coefficient linear differential operators.
Define the characteristic polynomials of L and M as

$$P_L(x) = \sum_{i=0}^{n} a_i x^{n-i}, \quad P_M(x) = \sum_{i=0}^{m} b_i x^{m-i}$$

1. $L = M$ *if and only if* $P_L(x) = P_M(x)$.
2. $P_{L+M}(x) = P_L(x) + P_M(x)$.
3. $P_{LM}(x) = P_L(x) \cdot P_M(x)$.
4. $P_{\lambda L}(x) = \lambda \cdot P_L(x)$, *for every* $\lambda \in \mathbb{R}$.

Proof.

(2), (3) and (4) follow from the definition of the characteristic polynomial. Let us see a proof of (1):

$$
\begin{aligned}
& P_L(x) = P_M(x) \\
\implies \quad & n = m; \quad a_i = b_i \quad \forall i \\
\implies \quad & L = M
\end{aligned}
$$

Conversely, $L = M \implies L(f) = M(f), \quad \forall f \in C^\infty(I)$

$$
\begin{aligned}
\implies \quad & L(e^{rx}) = M(e^{rx}) \\
\implies \quad & \sum_{i=0}^{n} a_{n-i} r^i e^{rx} = \sum_{i=0}^{m} b_{m-i} r^i e^{rx}. \\
\implies \quad & \sum_{i=0}^{n} a_{n-i} r^i = \sum_{i=0}^{m} b_{m-i} r^i, \qquad \forall r \in \mathbb{R} \\
\implies \quad & P_L(x) = P_M(x)
\end{aligned}
$$

Corollary

Let L, M, N be constant coefficient linear differential operators such that

$$P_L(x) = P_M(x) \cdot P_N(x) \implies L = MN.$$

In particular,

$$P_L(x) = a_0(x - r_1) \dots (x - r_n) \implies L = a_0(D - r_1) \dots (D - r_n)$$

Proof.

$$P_L(x) = P_M(x) \cdot P_N(x) = P_{MN}(x) \implies L = MN.$$

□

Example

$$D^2 - 5D + 6 = (D-3)(D-2)$$

as a linear transformation from $\mathcal{C}^2(I) \to \mathcal{C}(I)$, i.e. for any $y \in \mathcal{C}^2(I)$,

$$\begin{aligned}
(D-3)(D-2)y &= (D-3)(y' - 2y) \\
&= y'' - 5y' + 6y \\
(D-2)(D-3)y &= (D-2)(y' - 3y) \\
&= y'' - 5y' + 6y \\
\Longrightarrow \ (D-2)(D-3) &= (D-3)(D-2)
\end{aligned}$$

- The main point is that

$$D^r \circ D^s = D^s \circ D^r = D^{r+s} \implies LM = ML$$

for differential operators L and M with <u>constant coefficients</u>.

Let us see that $LM \neq ML$, if L and M are not constant coefficients.

Example

Let

$$L = D + xI, \qquad M = D + 1$$

Then

$$\begin{aligned}
LM(f) &= L(f' + f) \\
&= (f'' + f') + x(f' + f) \\
ML(f) &= M(f' + xf) \\
&= (f'' + f + xf') + (f' + xf) \\
&= LM(f) + f
\end{aligned}$$

Example

Solve $\quad y^{(3)} - 7y' + 6y = 0$

$$\begin{aligned} L &= D^3 - 7D + 6 \\ P_L(x) &= x^3 - 7x + 6 \\ &= (x-1)(x-2)(x+3) \\ \implies L &= (D-1)(D-2)(D+3) \end{aligned}$$

Since e^x, e^{2x}, e^{-3x} are in the null space of
$D - 1 = 0$, $D - 2 = 0$, $D + 3 = 0$,
they are solutions of $Ly = 0$.
Hence, $\{e^x, e^{2x}, e^{-3x}\}$ is a basis of Ker L.
Thus, the general solution is of the form

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{-3x}$$

Constant Differential Operators

Remark: The above example illustrates the general case of $P_L(x)$ having distinct real roots. If

$$P_L(x) = a_0(x - r_1)\dots(x - r_n),$$

with $r_i \in \mathbb{R}$ <u>distinct</u>, then a basis for Ker L is

$$\{e^{r_1 x},\ e^{r_2 x},\ \dots,\ e^{r_n x}\}$$

and the general solution of $Ly = 0$ is

$$y(x) = c_1 e^{r_1 x} + \dots + c_n e^{r_n x}$$

Example

Solve $\quad y^{(4)} - 16y = 0.$

$$\begin{aligned}
P_L(x) &= x^4 - 16 = (x^2-4)(x^2+4) \\
L &= (D-2I)(D+2I)(D^2+4I) \\
N(D-2I) &= \text{Span } \{e^{2x}\} \\
N(D+2I) &= \text{Span } \{e^{-2x}\} \\
N(D^2+4I) &= \text{Span } \{\cos 2x, \sin 2x\}
\end{aligned}$$

Therefore, a basis of $N(L)$ is given by

$$\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$$

The general solution is of the form

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x.$$

Theorem

Suppose

$$L = A_1 A_2 \dots A_k$$

where A_i are linear differential operators with constant coefficients and

$$N(A_i) \cap N(A_j) = 0, \quad \forall\ i \neq j$$

If $B_i =$ basis for $N(A_i)$. Then $\cup_{i=1}^{k} B_i =$ basis for $N(L)$.

Proof.

$$\begin{aligned} \dim N(L) &= \mathsf{deg} P_L(x) \\ &= \deg P_{A_1}(x) + \dots + \deg P_{A_k}(x) \\ &= \dim N(A_1) + \dots + N(A_k) \end{aligned}$$

Since $N(A_i) \cap N(A_j) = 0$ for all $i \neq j$, $\cup B_i$ is linearly independent. Therefore, $\cup B_i$ is a basis for $N(L)$.

Constant Differential Operators

Q. What if $P_L(x)$ has some repeated real roots?

In the $n = 2$ case, $m_1 = m_2$ gave us just one solution $e^{m_1 x}$. The other one we found by the method of looking for a solution of the form vf. This process gave us $xe^{m_1 x}$.

Proposition

For any real number r, the functions

$$u_1(x) = e^{rx}, u_2(x) = xe^{rx}, \dots, u_m(x) = x^{m-1}e^{rx}$$

are linearly independent and

$$u_1(x), \dots, u_m(x) \in N((D-r)^m).$$

Constant Differential Operators

Proof.

That these functions are linearly independent is obvious, since $\{1, x, x^2, \ldots, x^m\}$ is linearly independent (e^{rx} is non-zero).
We need to show that these functions are in $N((D-r)^m)$.
When $m = 1$, we need to show

$$\begin{aligned} u_1(x) &= e^{rx} \in (N(D-r)) \\ (D-r)(e^{rx}) &= re^{rx} - re^{rx} = 0 \end{aligned}$$

Suppose $m = 2$. Since u_1 is in $N(D-r)$, it's in $N((D-r)^2)$.
What about $u_2 = xe^{rx}$?

$$\begin{aligned} (D-r)^2(xe^{rx}) &= (D-r)(D-r)(xe^{rx}) \\ &= (D-r)(xre^{rx} + e^{rx} - rxe^{rx}) \\ &= (D-r)(e^{rx}) = 0. \end{aligned}$$

continued ...

Use induction to prove general case. Assume

$$u_1, u_2, \ldots, u_{m-1} \in N((D-r)^{m-1}),$$

and we need to show that

$$u_1, u_2, \ldots, u_m \in N((D-r)^m).$$

Note $\quad u_1, u_2, \ldots, u_{m-1} \in N((D-r)^{m-1}) \subseteq N((D-r)^m).$

To show that u_m is in $N((D-r)^m)$, compute

$$\begin{aligned}
& (D-r)^m(x^{m-1}e^{rx}) \\
= \ & (D-r)^{m-1}(D-r)(x^{m-1}e^{rx}) \\
= \ & (D-r)^{m-1}(x^{m-1}re^{rx} + (m-1)x^{m-2}e^{rx} - rx^{m-1}e^{rx}) \\
= \ & (D-r)^{m-1}((m-1)x^{m-2}e^{rx}) = 0.
\end{aligned}$$

$\square$

continued ...

Therefore, a basis for null space of $(D-r)^m$ is

$$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$$

Thus, if

$$P_L(x) = (x-r_1)^{e_1}(x-r_2)^{e_2}\ldots(x-r_\ell)^{e_\ell},$$

where $\sum_{i=1}^{\ell} e_i = n$, then a basis of $N(L)$ is given by

$$e^{r_1x}, \ldots, x^{e_1-1}e^{r_1x}, e^{r_2x}, \ldots, x^{e_2-1}e^{r_2x}, \ldots, e^{r_\ell x}, \ldots, x^{e_\ell-1}e^{r_\ell x}.$$

The point is that the above functions are linearly independent and since dim $N(L) = n$, these form a basis. $\square$

Ex: Check that the above functions are linearly independent.

Constant Differential Operators

Example

Find the general solution of the ODE:

$$L(y) = (D^3 - D^2 - 8D - 12)(y) = 0.$$

$$\begin{aligned} P_L(x) &= x^3 - x^2 - 8x - 12 \\ &= (x-2)^2(x+3) \\ L &= (D-2)^2(D+3) \end{aligned}$$

Thus the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x}, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Constant Differential Operators

Example

Find the general solution of the ODE:

$$L(y) = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

Now,

$$L = D^2(D-1)(D+1)^3.$$

$$N(D^2) = \{1, x\}, \quad N(D-1) = \{e^x\},$$

$$N((D+1)^3) = \{e^{-x}, xe^{-x}, x^2e^{-x}\}$$

Thus, the general solution is

$$y(x) = c_1 + c_2x + c_3e^x + c_4e^{-x} + c_5xe^{-x} + c_6x^2e^{-x}, \qquad c_i \in \mathbb{R}$$

Constant Differential Operators: complex roots

In the 2nd order case, if

$$m_i = a \pm \imath b \implies y_1 = e^{ax}\cos bx, \quad y_2 = e^{ax}\sin bx$$

were the basis for $N(L)$.

If $P_L(x)$ has a complex root $a + \imath b$, then it also has $a - \imath b$ as a root. Thus,

$$(x - (a + \imath b))(x - (a - \imath b)) = (x - a)^2 + b^2$$

is a factor of $P_L(x)$.
Null space of $(D - a)^2 + b^2$ has a basis

$$\{e^{ax}\cos bx, e^{ax}\sin bx\} \subset N(L)$$

Constant Differential Operators

If $a \pm \imath b$ is a root of $P_L(x)$ of multiplicity m, then

$$((D-a)^2 + b^2)^m$$

is a factor of $P_L(x)$.

Can we find the null space of $((D-a)^2 + b^2)^m$?

Example

Check that

$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m-1}e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m-1}e^{ax}\sin bx.$$

are a basis for null space of $((D-a)^2 + b^2)^m$.

Constant Differential Operators

Example

Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(x-1)(x^2-4x+5)^2.$$

The roots are

$$1, 2 \pm \imath, 2 \pm \imath.$$

Hence, the general solution is

$$y = c_1 e^x + e^{2x}[c_2 \cos x + c_3 \sin x + c_4 x \cos x + c_5 x \sin x],$$

where $c_i \in \mathbb{R}$.

Example

Find the fundamental set of solutions to

$$D^3(D-2I)^2(D^2+4I)^2y=0$$

The fundamental set will be given by

$$\{1, x, x^2, e^{2x}, xe^{2x}, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x\}$$

Solving the Non-homogeneous Equation

To solve non-homogeneous ODE

$$L(y) = y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

where $p_1, p_2, \ldots, p_n, r$ are continuous on an open interval I, we need to get

- a general solution of the homogeneous equation and
- one solution of the non-homogeneous equation. Why?

The variation of parameters method to find a particular solution, discussed for $n = 2$, generalises to the n-order.

But before that note that

$$\begin{aligned} y_p = ce^{x} \quad &\text{in} \quad y'' + 6y' + 5y = e^{x} \\ y_p = cxe^{-x} \quad &\text{in} \quad y'' + 6y' + 5y = e^{-x} \end{aligned}$$

Why?

Annihilator or Undetermined Coefficient Method

The Annihilator method or **method of undetermined coefficients** helps us in finding a particular solution of a non-homogeneous equation.

Example

Find a particular solution of

$$y^{(4)} - 16y = x^4 + x + 1$$

Here, $\quad L = D^4 - 16,$

and let us take $A = D^5$. Then $A(x^4 + x + 1) = 0.$

We say A annihilates or kills $x^4 + x + 1$.

Hence a solution y of $L(y) = x^4 + x + 1$ is also a solution of

$$D^5(D^4 - 16) = 0.$$

Example (continued ...)

$$AL = D^5(D^4 - 16)$$

has characteristic equation

$$x^5(x^4 - 16) = x^5(x-2)(x+2)(x^2+4).$$

Thus, a general solution of $(AL)(y) = 0$ is of the form

$$c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 e^{2x} + c_7 e^{-2x} + c_8 \cos 2x + c_9 \sin 2x.$$

Here

$$c_6 e^{2x} + c_7 e^{-2x} + c_8 \cos 2x + c_9 \sin 2x$$

is a solution of the homogeneous part

$$(D^4 - 16)y = 0$$

Example (continued ...)

We want a particular solution y_p for

$$y^{(4)} - 16y = x^4 + x + 1$$

Further, y_p will satisfy

$$ALy = 0$$

Thus, we can take

$$y_p = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$$

since all the other terms are solutions to the corresponding homogenous ODE.
To find c_i's in y_p, solve

$$y_p^{(4)} - 16y_p = x^4 + x + 1$$

Example (continued ...)

$$24c_5 - 16(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4) = x^4 + x + 1$$

Equating the coefficients, we get

$$\begin{aligned} 24c_5 - 16c_1 &= 1 \\ -16c_2 &= 1 \\ -16c_3 &= 0 \\ -16c_4 &= 0 \\ -16c_5 &= 1 \end{aligned}$$

$$\implies c_3 = c_4 = 0, \quad c_5 = c_2 = -1/16, \quad c_1 = -5/32$$

Therefore $y_p = -\dfrac{5}{32} - \dfrac{1}{16}x - \dfrac{1}{16}x^4$.

Example

Solve $\quad y^{(4)} - 4y'' = e^x + x^2$.

$$L = D^4 - 4D^2 = D^2(D-2)(D+2)$$

Let $z(x)$ and $w(x)$ be such that

$$Lz = e^x, \quad Lw = x^2 \implies L(z+w) = e^x + x^2$$

Let us first solve $Lz = e^x$.
Since $M = (D - I)$ annihilates e^x, $z(x)$ will be a solution of

$$MLz = (D-I)D^2(D-2I)(D+2I)z = 0$$

Hence z will be of the form

$$z = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} + c_5 e^x$$

Example (continued ...)

But $\{1, x, e^{2x}, e^{-2x}\}$ are all solution to $Ly = 0$ and therefore,

$$z = c_5 e^x$$

Plugging $z = c_5 e^x$ into the equation

$$y^{(4)} - 4y'' = e^x$$

we have $c_5 - 4c_5 = 1 \implies c_5 = -1/3$. Thus

$$z(x) = \frac{-1}{3} e^x$$

Let's solve $Lw = x^2$, where $L = (D^4 - 4)$.
Since $N = D^3$ annihilates x^2, w is a solution of

$$NLw = D^3 D^2 (D - 2I)(D + 2I) w = 0$$

Example (continued ...)

$$\begin{aligned} NLw &= D^5(D-2I)(D+2I)w = 0 \\ w &= c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6e^{2x} + c_7e^{-2x} \end{aligned}$$

But $c_1 + c_2x + c_6e^{2x} + c_7e^{-2x}$ are solutions to $Ly = 0$.
Therefore,

$$w = c_3x^2 + c_4x^3 + c_5x^4$$

Substituting in $Lw = (D^4 - D^2)w = x^2$, we get

$$\begin{aligned} & 24c_5 - 2c_3 + 6c_4x + 12c_5x^2 = x^2 \\ \Longrightarrow \quad & 24c_5 - 2c_3 = 0,\ c_4 = 0,\ c_5 = 1/12,\ c_3 = 1 \\ \Longrightarrow \quad & w = x^2 + \frac{1}{12}x^4 \\ \Longrightarrow \quad & y_p = z + w \\ & \quad = -\frac{1}{3}e^x + x^2 + \frac{1}{12}x^4 \end{aligned}$$

Summary: Anhilator Method

- Given a linear differential operator L with constant coefficients, we want to find a particular solution of $Ly = r(x)$.
- Find linear differential operators M which annihilates $r(x)$, i.e. $M(r(x)) = 0$.
- Find a basis for the solution space of $MLy = 0$.
- Pick those elements in the basis which are not solutions to $Ly = 0$.
- Set y_p to be a linear combination of these particular basis elements and solve $Ly_p = r(x)$ for the constants.
- A general solution to $Ly = r$ is given by $y = y_p + z$, where z is a general solution to $Ly = 0$.

Annihilator or Undetermined Coefficient Method

The Annihilator method helps us in finding a particular solution of a non-homogeneous equation with constant coefficients

$$Ly = r(x)$$

when the annihilator of $r(x)$ is a differential operator with constant coefficients.

- If $r_1(x) = a_0x^n + a_1x^{n-1} + a_n$ with $a_i \in \mathbb{R}$, $a_0 \neq 0$, then its annihilator is D^{n+1}.
- The annihilator of $r_2(x) = a_1 \cos bx + a_2 \sin bx$ is $(D^2 + b^2)$.
- The annihilator of $r_3(x) = e^{ax}(a_1 \cos bx + a_2 \sin bx)$ is $(D - a)^2 + b^2$.
- The annihilator of $r_4(x) = x^r e^{ax}(a_1 \cos bx + a_2 \sin bx)$ is $[(D - a)^2 + b^2]^{r+1}$.
- The annihilator of $r_1(x) + r_2(x)$ is $D^{n+1}(D^2 + b^2)$.

Find the form of particular solution to the following ODEs.
Let A denote the annihilator of $r(x)$.

Example

(1) $y'' + 9y' = 6, \quad L = D^2 + 9D$

$$\begin{aligned} A &= D, \\ AL &= D^2(D+9) \\ y_p &= cx \end{aligned}$$

(2) $y'' + 2y' + y = 4e^x \sin 2x, \;\; L = (D+1)^2$

$$\begin{aligned} A &= (D-1)^2 + 4, \\ AL &= (D+1)^2((D-1)^2+4) \\ y_p &= e^x[c_1 \cos 2x + c_2 \sin 2x] \end{aligned}$$

Example

(3) $y'' + y = x \sin x, \quad L = D^2 + 1$

$$\begin{aligned} A &= (D^2+1)^2, \\ AL &= (D^2+1)^3 \\ y_p &= c_1 x \cos x + c_2 x \sin x + c_3 x^2 \cos x + c_4 x^2 \sin x \end{aligned}$$

(4) $y'' + 9y = x^2 e^{3x}, \quad L = D^2 + 9$

$$\begin{aligned} A &= (D-3)^3 \\ AL &= (D^2+9)(D-3)^3 \\ y_p &= e^{3x}[c_1 + c_2 x + c_3 x^2] \end{aligned}$$

Example

(5) $y'' + 9y = xe^{3x}\cos 3x, \quad L = D^2 + 9$

$$
\begin{aligned}
A &= ((D-3)^2+9)^2 \\
AL &= (D^2+9)((D-3)^2+9)^2 \\
y_p &= e^{3x}(c_1\cos 3x + c_2\sin 3x + c_3 x\cos 3x + c_4 x\sin 3x)
\end{aligned}
$$

(6) $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x\sin x,$
$L = D^4 - D^3 - D^2 + D = D(D-1)^2(D+1),$

$$
\begin{aligned}
A &= D^3(D^2+1)^2 \\
AL &= D^4(D-1)^2(D+1)(D^2+1)^2 \\
y_p &= \text{linear combination of} \\
& \quad \{x, x^2, x^3, \cos x, \sin x, x\cos x, x\sin x\}
\end{aligned}
$$

Example

(7) $y^{(4)} - 2y'' + y = x^2e^x + e^{2x}$

$$
\begin{aligned}
L &= D^4 - 2D^2 + 1 \\
&= (D^2 - 1)^2 \\
&= (D+1)^2(D-1)^2 \\
A &= (D-1)^3(D-2), \\
AL &= (D+1)^2(D-1)^5(D-2) \\
y_p &= \text{ linear combinations of} \\
& \quad \{x^2e^x, x^3e^x, x^4e^x, e^{2x}\}
\end{aligned}
$$

Variation of Parameters

The variation of parameters method generalizes to n-th order linear ODE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

where p_i's and r are continuous on an open interval I. Let

$$\{y_1, \ldots, y_n\}$$

be a basis of solutions of homogeneous part and

$$y_p = v_1(x)y_1 + v_2(x)y_2 + \ldots + v_n(x)y_n$$

Further, assume that

$$\begin{array}{rcl} v_1'y_1 + \ldots + v_n'y_n & = & 0 \\ v_1'y_1' + \ldots + v_1'y_n' & = & 0 \\ \vdots & & \vdots \\ v_1'y_1^{(n-2)} + \ldots + v_n'y_n^{(n-2)} & = & 0 \end{array}$$

Variation of Parameters

Computing $y_p', \ldots, y_p^{(n)}$ and putting in the ODE, we get

$$v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} = r(x)$$

Thus,

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for

$$v_1', v_2', \ldots, v_n'$$

and then get

$$v_1, v_2, \ldots, v_n$$

and form

$$y_p = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n$$

Variation of Parameters

Here v_i' is given by

$$v_i' = \frac{\begin{vmatrix} y_1 & \cdot & y_{i-1} & 0 & y_{i+1} & \cdot & y_n \\ y_1' & \cdot & y_{i-1}' & 0 & y_{i+1}' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & \cdot & y_{i-1}^{(n-1)} & r(x) & y_{i+1}^{(n-1)} & \cdot & y_n^{(n-1)} \end{vmatrix}}{W(y_1, \ldots, y_n; x)}$$

Example

Solve $\quad y^{(3)} - y^{(2)} - y^{(1)} + y = r(x)$.

Here $\; L = D^3 - D^2 - D + 1 = (D-1)^2(D+1)$.

Hence, a basis of solutions of homogeneous part is

$$\{e^x, xe^x, e^{-x}\}$$

We need to calculate Wronskian $W(x)$. Use Abel's formula:

$$\begin{aligned}
W(x) &= W(0)\, exp\left(-\int_0^x p_1(t)dt\right) = W(0)\cdot e^x. \\
W(x) &= \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x + xe^x & -e^{-x} \\ e^x & 2e^x + xe^x & e^{-x} \end{vmatrix}. \\
W(0) &= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4 \implies W(x) = 4e^x
\end{aligned}$$

Example (continued ...)

$$
\begin{aligned}
W_1(x) &= \begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & e^x + xe^x & -e^{-x} \\ r(x) & 2e^x + xe^x & e^{-x} \end{vmatrix} \\
&= -r(x)(2x+1) \\
W_2(x) &= \begin{vmatrix} e^x & 0 & e^{-x} \\ e^x & 0 & -e^{-x} \\ e^x & r(x) & e^{-x} \end{vmatrix} \\
&= 2r(x) \\
W_3(x) &= \begin{vmatrix} e^x & xe^x & 0 \\ e^x & e^x + xe^x & 0 \\ e^x & 2e^x + xe^x & r(x) \end{vmatrix} \\
&= r(x)e^{2x}
\end{aligned}
$$

Example (continued ...)

Therefore, a particular solution y_p is given by

$$
\begin{aligned}
y_p &= y_1 v_1 + y_2 v_2 + y_3 v_3 \\
v_i &= \int_0^x \frac{W_i(x)}{W(x)}\, dx, \qquad i = 1, 2, 3 \\
v_1 &= \int_0^x \frac{-r(x)(2x+1)}{4e^x} dx \\
v_2 &= \int_0^x \frac{2r(x)}{4e^x} dx \\
v_3 &= \int_0^x \frac{r(x)e^{2x}}{4e^x} dx
\end{aligned}
$$

Exercise. Solve the ODE with $r(x) = 1$, $r(x) = x$, $r(x) = e^x$.