Energy and Momentum of particle + EM field system

Conservative field $\rightarrow$ KE + PE (scalar potential) conserved.
EM fields are in general not conservative, so what is conserved?

So may be : KE of particles + "something" will be conserved ?

$$\delta W_M = \int_{all\,vol} \rho(\vec{E}+\vec{v}\times\vec{B}).\vec{v}\,\delta t\, d\tau$$

Work done on the charges= Force x displacement (integrated over all vol)

$$\frac{dW_M}{dt} = \int \vec{E}.\vec{j}\, d\tau$$

Use $\nabla\times\vec{B} = \mu_0\vec{j}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$ to replace $\vec{j}$

$$\nabla.(\vec{E}\times\vec{B}) = \vec{B}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{B}$$

$$= \frac{1}{\mu_0}\int(\vec{E}.\nabla\times\vec{B})d\tau - \frac{\partial}{\partial t}\int\frac{\epsilon_0 E^2}{2}d\tau$$

$$= -\frac{1}{\mu_0}\int\nabla.(\vec{E}\times\vec{B})d\tau + \frac{1}{\mu_0}\int\vec{B}.(\nabla\times\vec{E})d\tau$$

$$- \frac{\partial}{\partial t}\int\frac{\epsilon_0 E^2}{2}d\tau$$

compare with $\nabla.\vec{j}+\frac{\partial\rho}{\partial t} = 0$: OR : $\frac{dQ_{\text{in}}}{dt} = -\int_{surf}\vec{j}.d\vec{a}$

Energy and Momentum of particle + EM field system

$$\frac{dW_M}{dt} = -\frac{1}{\mu_0}\int \nabla.(\vec{E}\times\vec{B})\,d\tau + \frac{1}{\mu_0}\int \vec{B}.(\nabla\times\vec{E})\,d\tau - \frac{\partial}{\partial t}\int \frac{\epsilon_0 E^2}{2}\,d\tau$$

$$= -\frac{1}{\mu_0}\int \nabla.(\vec{E}\times\vec{B})\,d\tau - \frac{\partial}{\partial t}\int\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)d\tau$$

$$Hence \quad \frac{d}{dt}\left[W_M + \int_{vol}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)d\tau\right] = -\frac{1}{\mu_0}\int_{surf} \nabla.(\vec{E}\times\vec{B})\,d\tau$$

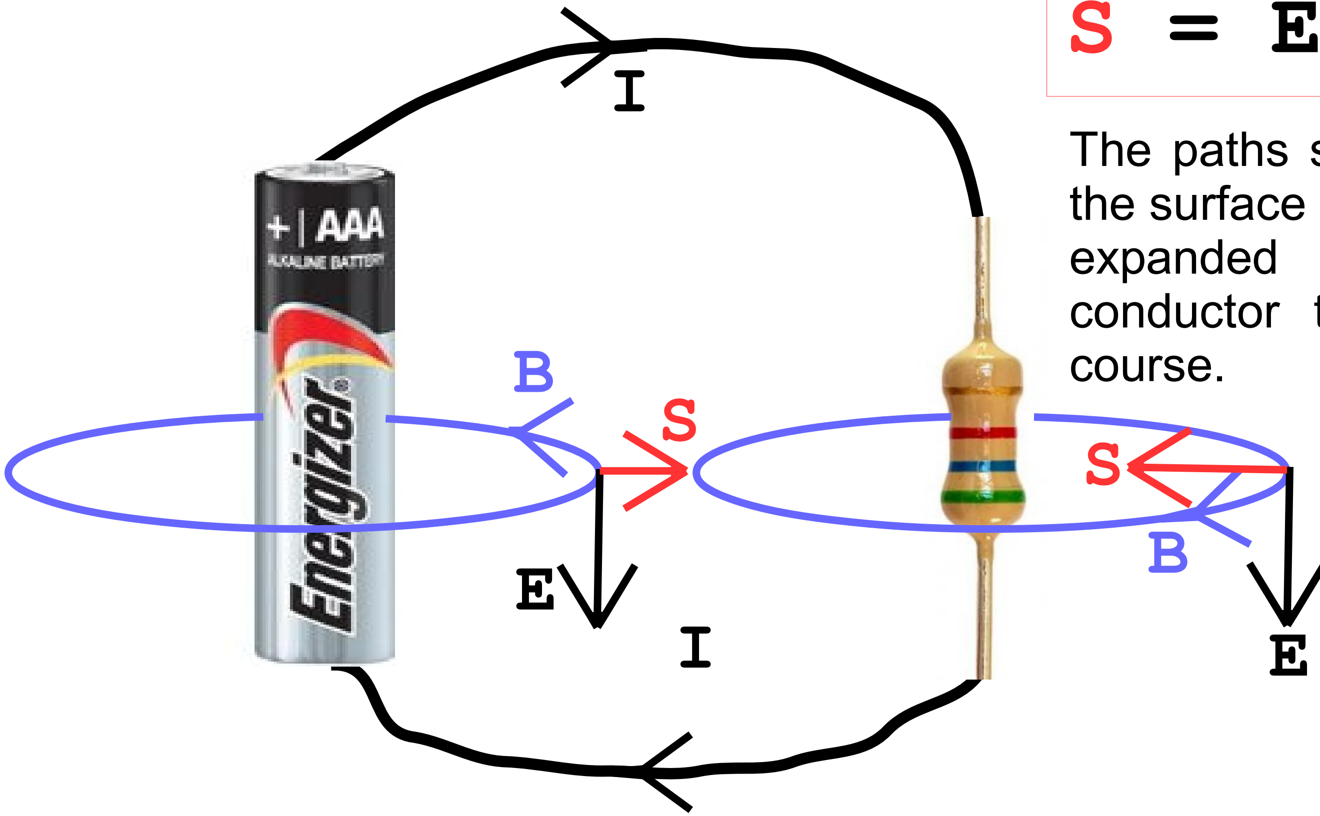
$$\boxed{\frac{d}{dt}\left[W_M + \int_{vol}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)d\tau\right] = -\int_{surf} \frac{1}{\mu_0}(\vec{E}\times\vec{B}).d\vec{a}}$$

Energy of particles + field

Poynting vector : The energy flux

Poynting vector : some examples

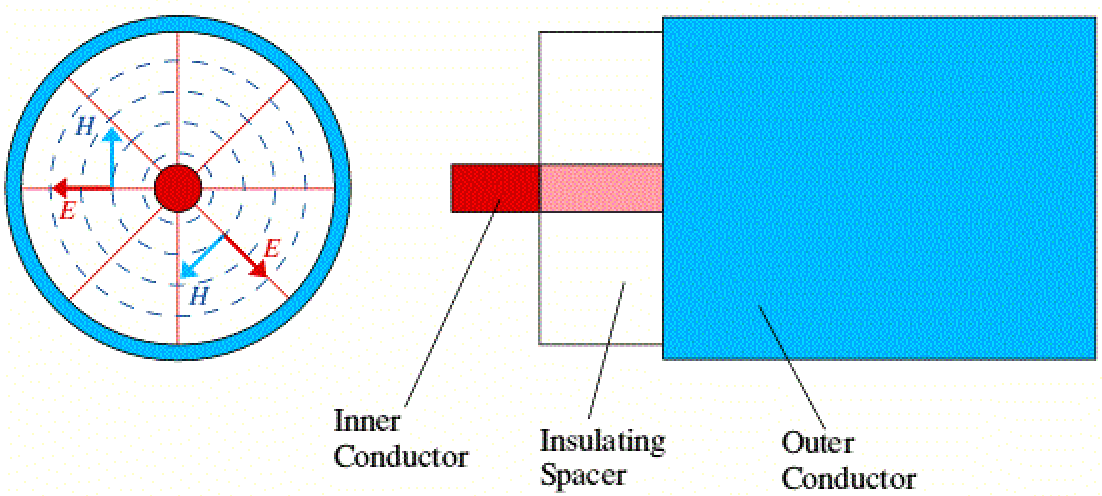
Energy flows OUT from the battery (or capacitor) INTO the resistance



Poynting vector also written as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

The paths shown should be taken just at the surface of the battery/resistor. It is expanded out for clarity. Outside the conductor there is no electric field, of course.



In a coaxial cable Poynting vector is along the direction of the current.

For an EM wave?

Energy and Momentum of particle + EM field system

We find that the EM field contains energy and we can identify the energy flux/flow/current term as well.

Natural question: Can we do the same for momentum of the particles? This is more invloved, becuase momentum is a vector and forming the continuity equation for a vector would require a "tensor".

Apart from that the reasoning is very similar...

Replace charge and current by field terms using Maxwell's equations

$$\frac{d}{dt}\sum_{all}\vec{p}_i = \vec{F} = \int_{all\ vol}\rho(\vec{E}+\vec{v}\times\vec{B})\,d\tau$$

$$= \int\left[\left(\epsilon_0\nabla.\vec{E}\right)\vec{E} + \left(\frac{\nabla\times\vec{B}}{\mu_0}-\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]d\tau$$

$$\text{Since}\ :\ (\nabla\times\vec{B})\times\vec{B} = (\vec{B}.\nabla)\vec{B}-\nabla\frac{B^2}{2}$$

$$\text{And}\ :\ \left(\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\nabla\times\vec{E})$$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B})-\left[(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]$$

we have used Faraday's law

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

and a similar expansion again

Energy and Momentum of particle + EM field system

RHS becomes :

$$\epsilon_0\left[(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right] + \frac{1}{\mu_0}\left[(\nabla.\vec{B})\vec{B}+(\vec{B}.\nabla)\vec{B}-\nabla\frac{B^2}{2}\right]-\frac{1}{c^2}\frac{\partial}{\partial t}\frac{(\vec{E}\times\vec{B})}{\mu_0}$$

The integrand is now symmetric in E and B although the initial expression was not. The extra term we have added is div B which is always zero.

S = E x **B** emerges again

$$\frac{d}{dt}\left[\sum_{particles}\vec{p}_i+\frac{1}{c^2}\int\vec{S}\,d\tau\right] = \int\left[\epsilon_0\left\{(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right\} + \frac{1}{\mu_0}\left\{(\nabla.\vec{B})\vec{B}+(\vec{B}.\nabla)\vec{B}-\nabla\frac{B^2}{2}\right\}\right]d\tau$$

Question : Is RHS the divergence of something? Then the form of the continuity equation will emerge again.

But the RHS is already a vector, so it can only be the divergence of tensor (if at all)

Energy and Momentum of particle + EM field system

$$\left[(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]_i$$

$$=\frac{\partial E_j}{\partial x_j}E_i+E_j\frac{\partial E_i}{\partial x_j}-\frac{1}{2}\frac{\partial E^2}{\partial x_i}$$

$$=\frac{\partial}{\partial x_j}\left(E_iE_j-\delta_{ij}\frac{E^2}{2}\right)$$

Repeated index j is summed over, there is no summation over i

Hence the entire RHS integrand is a divergence of the following quantity

$$T_{ij} = \epsilon_0\left(E_iE_j-\delta_{ij}\frac{E^2}{2}\right)+\frac{1}{\mu_0}\left(B_iB_j-\delta_{ij}\frac{B^2}{2}\right)$$

Formally called the Electromagnetic (Maxwell) stress tensor

$$\frac{d}{dt}\left[\sum_{particles}\vec{p}_i+\frac{1}{c^2}\int\vec{S}\,d\tau\right] = -\int_{vol}\nabla.(-\underline{\underline{T}})\,d\tau = -\int_{surf}(-\underline{\underline{T}}).d\vec{a}$$

compare with $\frac{d}{dt}Q_{\text{inside}} = -\int_{vol}\nabla.\vec{j}\,d\tau = -\int_{surf}\vec{j}.d\vec{a}$

Q: Why would you call it a stress tensor?

Energy and Momentum of particle + EM field system

$$\left[(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]_i$$

$$=\frac{\partial E_j}{\partial x_j}E_i+E_j\frac{\partial E_i}{\partial x_j}-\frac{1}{2}\frac{\partial E^2}{\partial x_i}$$

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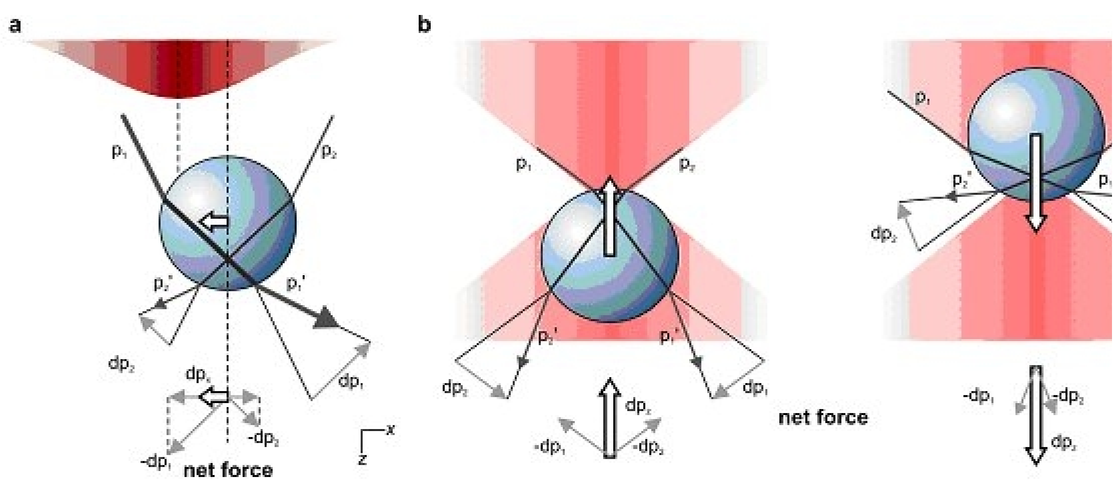
Energy and Momentum of particle + EM field system

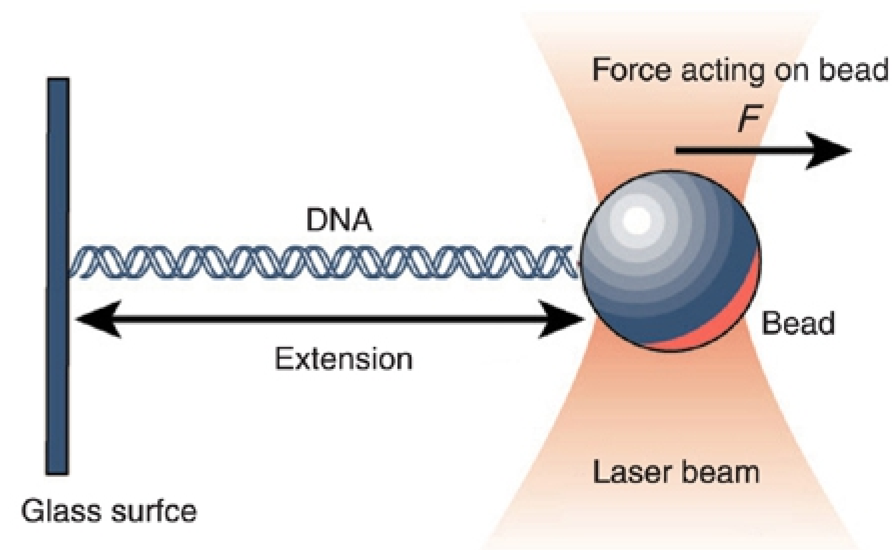
$$\frac{d}{dt}\sum_{particles}\vec{p}_i = \left[-\frac{1}{c^2}\frac{d}{dt}\int \vec{S}\,d\tau + \int_{surf} \underline{T}\,.\,d\vec{a}\right]$$

If we take the volume to include all the particles (or a solid object) then the RHS tells us the total force on that volume.

If there is no t dependence then the integral of T gives the force. In a "mechanical" or "fluid" situation, this is exactly what the stress tensor would have given us.

This formulation can also be used to analyse cases where a focussed beam of light is used to hold/pull a particle...."optical tweezer"..





Physics world: Optical tweezers: where physics meets biology : Nov 13, 2008

Electromagnetic Waves in free space: What is a plane EM wave ?

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)}$$
$$\vec{B} = \vec{B}_0 e^{i(\vec{k}.\vec{r}-\omega t)}$$

We start by assuming that Eo and Bo have no spatial dependence, they do NOT depend on x,y,z. All spatial dependence comes from the exponential.

Of course Eo and Bo can have x,y,z components, but they are all constants. These must satisfy Maxwell's equations.

$$\nabla . \vec{E} = 0$$
$$\nabla . \vec{B} = 0$$

$$\nabla . \vec{E}_0 e^{i(k.r-\omega t)}$$
$$= \frac{\partial}{\partial x} E_{0x} e^{i(k.r-\omega t)} + \frac{\partial}{\partial y} E_{0y} e^{i(k.r-\omega t)} \cdots$$
$$= i\left(E_{0x} k_x + E_{0y} k_y + E_{0z} k_z\right) e^{i(k.r-\omega t)}$$
$$= i \vec{k} . \vec{E}_0 e^{i(k.r-\omega t)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Similarly Prove that

$$\nabla \times E_0 e^{i(k.r-\omega t)}$$
$$= i(k \times E_0) e^{i(k.r-\omega t)}$$

And $\frac{\partial}{\partial t} \vec{E}_0 e^{i(k.r-\omega t)} = -i\omega \vec{E}_0 e^{i(k.r-\omega t)}$

$$\textit{Hence} \quad \vec{k} . \vec{E}_0 = 0$$
$$\& \quad \vec{k} . \vec{B}_0 = 0$$

We would not have got this "transverse" condition without the assumption of Eo and Bo being constant....in waveguides the condition does NOT hold.

Electromagnetic Waves in free space : What is a plane EM wave?

The third equation gives:

$$\vec{k}\times\vec{E}_0 = \omega\vec{B}_0$$

Hence

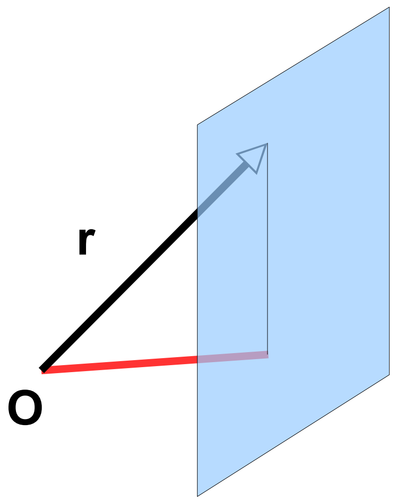
$$\begin{aligned}
\vec{E}_0\times(\vec{k}\times\vec{E}_0) &= \omega\vec{E}_0\times\vec{B}_0\\
\vec{k}(\vec{E}_0.\vec{E}_0)-\vec{E}_0(\vec{k}.\vec{E}_0) &= \omega\vec{E}_0\times\vec{B}_0\\
\vec{k} &= \omega\frac{\vec{E}_0\times\vec{B}_0}{E_0^2}\\
|B_0| &= \frac{|E_0|}{c}
\end{aligned}$$

The wave propagates in the direction of E x B.

The relative magnitudes :
For reasonably strong E = 1000V/m
B ~ 3 microTesla very weak .

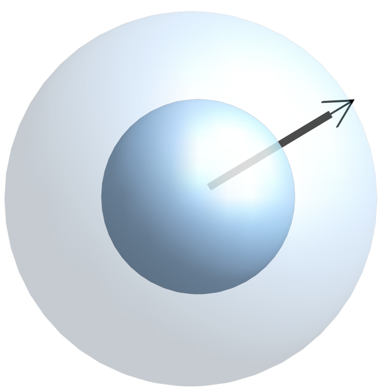
That's why we mostly talk about coupling with the electric field of light.

Electromagnetic Waves in free space : Wavefronts and their shapes



Wavefront of plane waves

Plane normal to **k**
k.r = length of the red line x magnitude of k as long as the tip of **r** lies in the plane.
Surfaces of constant **k.r** at a certain time t are called wavefronts.
For plane waves the wavefronts are planes.
For sphereical waves these would be spherical surfaces.



Simple spherical wavefront described by

$$V(r,t) = \frac{A}{r} e^{i(kr - \omega t)}$$

It is NOT $\vec{k}.\vec{r} - \omega t$

Any wave coming from a source (like light from a point) is in reality spherical. But at large distances it is approximated by a plane wave very well.

This is similar to neglecting the earth's curvature over a small region....

Electromagnetic Waves in free space :Energy, momentum density and Intensity

$$\vec{E}(r,t) = \vec{E}_0 \cos(\vec{k}.\vec{r} - \omega t)$$

$$Hence \ \langle E^2 \rangle = \frac{1}{T}\int_0^T E_0^{\,2} \cos^2(\vec{k}.\vec{r} - \omega t)\, dt$$

$$= \frac{E_0^{\,2}}{2}$$

$$Energy \quad U = \left(\frac{\epsilon_0 \langle E^2 \rangle}{2} + \frac{\langle B^2 \rangle}{2\mu_0} \right) = \frac{\epsilon_0 E_0^{\,2}}{2}$$

Using the earlier results

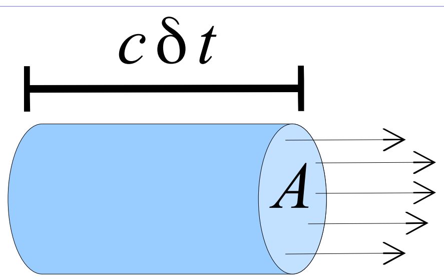
$$|B| = \frac{|E|}{c}$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

Although the B field is much weaker, E and B components make equal contributions to the field energy.

$$Momentum \ \vec{p} = \frac{\vec{S}}{c^2} = \frac{1}{\mu_0 c^2} \langle \vec{E} \times \vec{B} \rangle$$

$$|p| = \frac{U}{c}$$



Intensity : Energy passing through per unit area per unit time.

$$Intentsity \quad I = \frac{A(c\delta t)U}{A\delta t} = Uc$$

All the energy in the volume will pass through the cross section in time dt

Maxwell's equation in "linear" matter : what happens to the wave equation?

We consider an insulator first, so there are no free charges in the material

$$\vec{D} = \epsilon \vec{E} \qquad \nabla . \vec{D} = 0$$

$$\vec{B} = \mu \vec{H} \qquad \nabla . \vec{B} = 0$$

But now both magnetisation and electric polarisation can simultaneously change. So the "bound" current will result from change in **M** as well as **P**.

$$\sigma_b = \vec{P} . \hat{n} \quad : \quad \text{Then consider } \vec{P} \rightarrow \vec{P} + \delta \vec{P}$$

This change causes some amount of charge to flow in/out

$$\delta Q = \delta(\vec{P} . \hat{n}) \delta a$$

$$\vec{J}_p . \delta \vec{a} = \frac{\delta Q}{\delta t} = \frac{\partial \vec{P}}{\partial t} . \delta \vec{a}$$

Total bound current flow

$$\vec{J}_b = \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Show that this interpretation is consistent with the continuity equation

$$\nabla \times \vec{B} = \mu_0 \vec{J}_{total} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times [\mu_0 (\vec{H} + \vec{M})] = \mu_0 \left[\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right] + \mu_0 \frac{\partial}{\partial t} [\vec{D} - \vec{P}]$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's equation in "linear" matter : what happens to the wave equation?

$$
\begin{aligned}
\nabla . \vec{D} &= 0 \\
\nabla . \vec{B} &= 0 \\
\nabla \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{B} &= \mu_0 (\vec{H} + \vec{M}) \\
\epsilon_0 \vec{E} &= (\vec{D} - \vec{P})
\end{aligned}
$$

$$
\begin{aligned}
\vec{D} &= \epsilon \vec{E} \\
\vec{B} &= \mu \vec{H}
\end{aligned}
$$

With $\vec{J}_f = 0$ *we will get* $\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t}$

The wave will propagate with speed $v^2 = \frac{1}{\mu \epsilon}$

Refractive index of the medium $n = \frac{c}{v} = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$

Maxwell's equation in "linear" matter : The boundary conditions

Consider a boundary between two media 1 and 2

Since div D = 0, the normal component of D must be continuous.
div B = 0, always (so normal component of B is continuous)

Since curl H has no singularities … the tangential component of H is continuous
curl E has no singularities … the tangnetial component of E is continuous

$$D_1^{\perp} = D_2^{\perp} \qquad Hence \quad \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$

$$B_1^{\perp} = B_2^{\perp}$$

$$H_1^{\parallel} = H_2^{\parallel} \qquad Hence \quad \frac{B_1^{\parallel}}{\mu_1} = \frac{B_2^{\parallel}}{\mu_2}$$

$$E_1^{\parallel} = E_2^{\parallel}$$

These boundary conditions govern the reflection and transmission of electromagnetic waves at an interface and hence the laws of reflection and refraction (optics)

Electromagnetic waves at an interface : reflection and transmission

Normal incidence

x=0

The incident wave propagating to the right

$$\vec{E}_I = E_{0I}\, e^{i(k_1 x - \omega t)}\,\hat{y}$$

$$\vec{B}_I = \frac{1}{v_1} E_{0I}\, e^{i(k_1 x - \omega t)}\,\hat{z}$$

The reflected wave propagating to the left

$$\vec{E}_R = E_{0R}\, e^{i(-k_1 x - \omega t)}\,\hat{y}$$

$$\vec{B}_R = -\frac{1}{v_1} E_{0R}\, e^{i(-k_1 x - \omega t)}\,\hat{z}$$

The transmitted wave propagating to the right

$$\vec{E}_T = E_{0T}\, e^{i(k_2 x - \omega t)}\,\hat{y}$$

$$\vec{B}_T = \frac{1}{v_2} E_{0T}\, e^{i(k_2 x - \omega t)}\,\hat{z}$$

1 2

Tangential E
Tangential H
are continuous

$$E_{0I} + E_{0R} = E_{0T}$$

$$\frac{1}{\mu_1}\left(\frac{E_{0I}}{v_1} - \frac{E_{0R}}{v_1}\right) = \frac{1}{\mu_2}\frac{E_{0T}}{v_2}$$

$$\mathit{define} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Need to solve for the ratios only....

$$\frac{E_{0R}}{E_{0I}} = \frac{1-\beta}{1+\beta} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right|$$

$$\frac{E_{0T}}{E_{0I}} = \frac{2}{1+\beta} = \left(\frac{2n_1}{n_1 + n_2}\right)$$

An useful result with three "phasor" s:

$$A\,e^{iax} + B\,e^{ibx} = C\,e^{icx} \quad \forall \quad x$$

$$Then \quad a = b = c$$

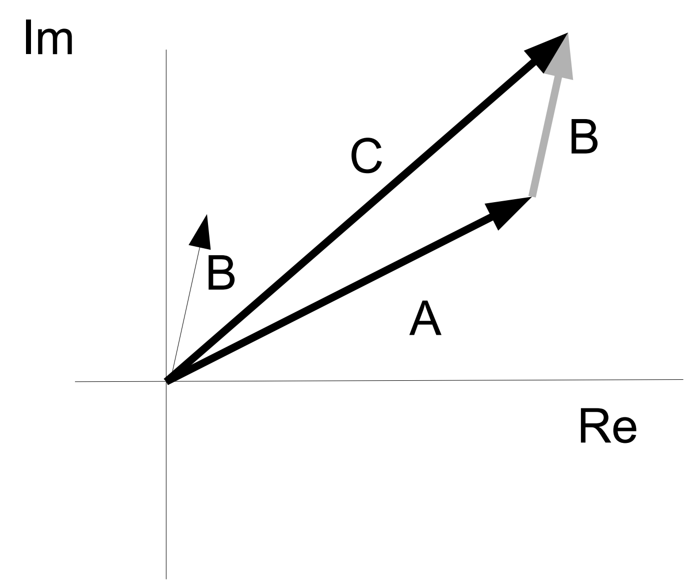
$$set\, x = 0 \quad : \quad this\; gives \quad A + B = C$$

This condition determines the length of the phasors, which must be satisfied at all times

$$Now \quad draw \quad the \quad three \quad phasors \quad when \quad x \neq 0$$

$$Two\; sides\; of\;\; a\; traingle\; are\; together$$
$$greater\; than\; the\; third\;\; side$$

$$The\; equality\; can\; only\; hold\; if$$
$$A\,,B\,,C\; are\; along\; the\; same\; ray..$$
$$The\; phase\; angle\; also\; must\; be\; same$$
$$implies \quad a = b = c$$

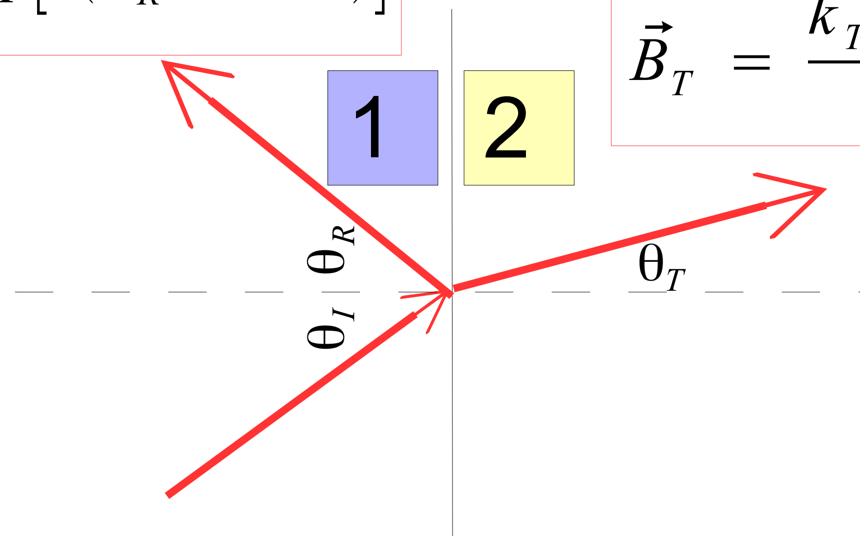


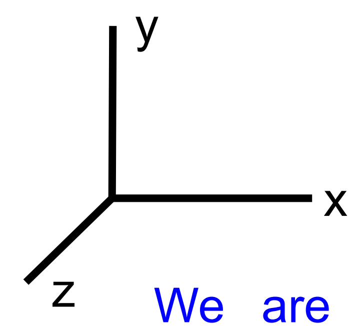
Electromagnetic waves at an interface : reflection and refraction

$$\vec{E}_R = \vec{E}_{0R} \exp\left[i\left(\vec{k}_R . \vec{r} - \omega t\right)\right]$$
$$\vec{B}_R = \frac{\hat{k}_R \times \vec{E}_{0R}}{v_1} \exp\left[i\left(\vec{k}_R . \vec{r} - \omega t\right)\right]$$

Oblique incidence at an interface (general case)

$$\vec{E}_T = \vec{E}_{0T} \exp\left[i\left(\vec{k}_T . \vec{r} - \omega t\right)\right]$$
$$\vec{B}_T = \frac{\hat{k}_T \times \vec{E}_{0T}}{v_2} \exp\left[i\left(\vec{k}_T . \vec{r} - \omega t\right)\right]$$





We are looking at the x=0 plane sideways

$$\vec{E}_I = \vec{E}_{0I} \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right]$$
$$\vec{B}_I = \frac{\hat{k}_I \times \vec{E}_{0I}}{v_1} \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right]$$

Notice how unit vectors have been used to fix the relative directions

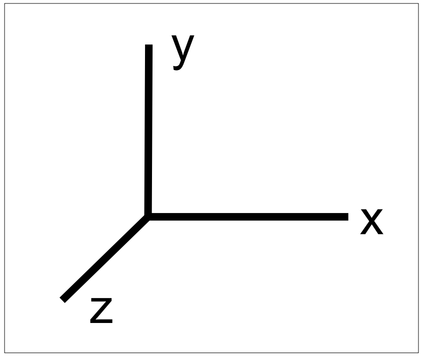
$$\omega = |\vec{k}| v \quad : \quad Hence \quad k_I v_1 = k_R v_1 = k_T v_2$$

Use the result derived just before

$$\vec{k}_I . \vec{r} = \vec{k}_R . \vec{r} = \vec{k}_T . \vec{r} \quad must \quad hold \quad \forall r \quad \text{on} \, the \; x = 0 \; plane$$

Electromagnetic waves at an interface : reflection and refraction

$$k_I = k_R = \frac{v_2}{v_1} k_T \quad \text{in magnitude}$$



$$\left.\begin{aligned}(k_I)_y y + (k_I)_z z &= (k_R)_y y + (k_R)_z z \\ (k_I)_y y + (k_I)_z z &= (k_T)_y y + (k_T)_z z\end{aligned}\right\} \quad holds \ \forall \ y, z$$

This means all the coefficients (y,z components) must be equal

Form the triple product of k_I, k_R, k_T : this must vanish since two row/columns are identical.

The three vectors are co-planer [Law of reflection and refraction]
Let this be the x-y plane.

Since $|\boldsymbol{k_I}| = |\boldsymbol{k_R}|$ and y components are equal, the other (x) component is exactly reversed.
No other possibility can satisfy all these conditions.

Equality of the y-components require

$$\theta_I = \theta_R$$

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \qquad \frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$