Gauss's law: Flux of Electric field through a closed surface

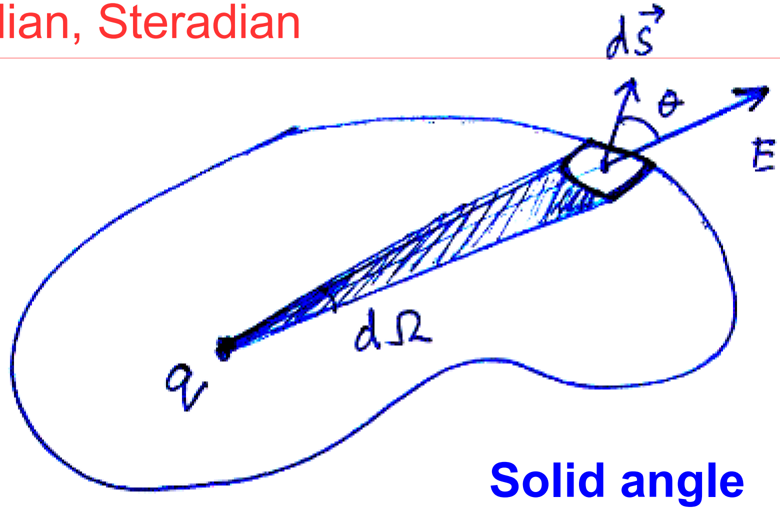
Revise the idea of planer angle and solid angle
Closed loop, Closed surface
Radian, Steradian

$$\int_{surface} \vec{E}\,.\,d\vec{S} = \frac{q}{4\pi\epsilon_0}\int_{surface}\frac{\hat{r}\,.\,d\vec{S}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\int_{surface}\frac{|d\vec{S}|\cos\theta}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\int_{surface} d\Omega$$

$$= \frac{q}{4\pi\epsilon_0}4\pi = \frac{q}{\epsilon_0}$$

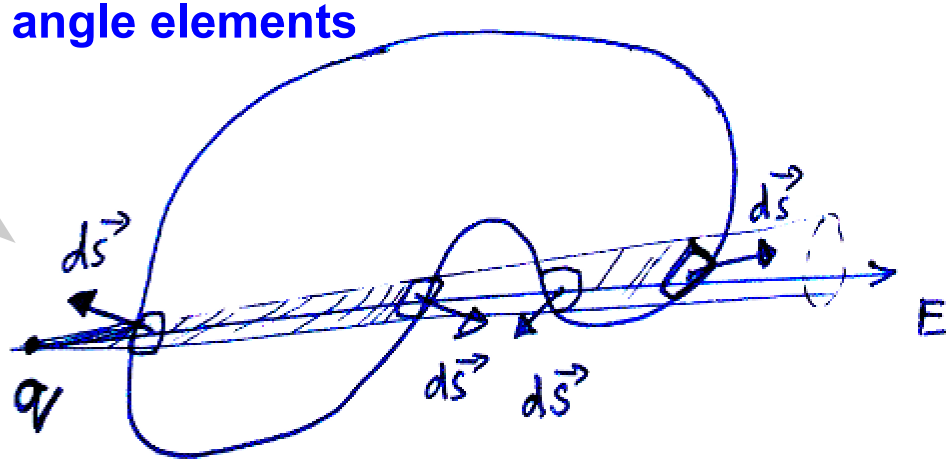


Solid angle

$$d\Omega = \frac{|ds|\cos\theta}{r^2}$$

Notice the exact cancellation of solid angle elements

If the point is located outside then the contributions exactly cancel



Use superposition principle
Add contribution from each charge

Gauss's law: Flux of Electric field through a closed surface

$$\int_{surface} \vec{E} \, . \, d\vec{S} = \int_{vol} \vec{\nabla} \, . \, \vec{E} \, d\tau = \int_{vol} \frac{\rho(\vec{r})}{\epsilon_0} d\tau$$

Q: Why do we write
This rather than leave
Coulomb force law
as it is?

$$So \quad \vec{\nabla} \, . \, \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$Also \quad \nabla^2 V = -\frac{\rho(\vec{r})}{\epsilon_0} \qquad since \quad \vec{E} = -\vec{\nabla} V$$

Only when there
is no time varying
magnetic field.
Same as saying
Curl **E** = 0

This form allows one to use the symmetry of a problem more easily
(e.g. sphere, infinite sheet, wire etc.)

Is valid even when charges are in motion.

Q: What is the problem
with moving charges
and Coulomb's law?

Fun question: If the world was 2-dimensional what would
Coulomb's law be like? (Don't take it too seriously!)

How "exact" is the inverse square force law?

The cancellation of the $1/r^2$ came from two sources:
1. The geometrical growth of the area subtended by a small solid angle (geometry)
2. The nature of the coulomb force law (experimental observation)

If the force varied as $1/r^{2.0001}$, what observational consequence would it have?

Gauss's law: divergence of $1/r^2$ and the Dirac delta function

Do the following integration over a sphere

$$\int_{vol} \vec{\nabla} \cdot \frac{\hat{r}}{r^2}\, d\tau = \int_{surface} \frac{\hat{r}}{r^2} \cdot d\vec{S} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 4\pi$$

But calculating the divergence explicitly

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = 0$$

The inconsistency comes from the singularity at **r**=0

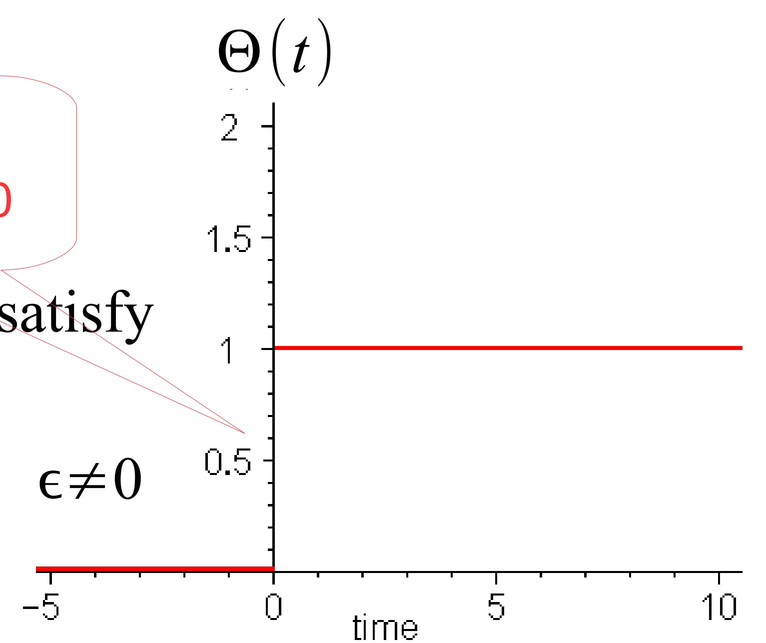
Consider a simpler example: the step function

$$\Theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Integrable singularity at **r**=0

$$\frac{d\Theta}{dt} = ?$$ looks like zero everywhere but must satisfy

$$\int_{0-|\epsilon|}^{0+|\epsilon|} \left(\frac{d\Theta}{dt}\right) dt = \Theta(|\epsilon|) - \Theta(-|\epsilon|) = 1 \quad \forall \quad \epsilon \neq 0$$



Such integrable singularities are treated by *defining the* δ *function*

Gauss's law: divergence of $1/r^2$ and the Dirac delta function

Let f(t) be a differentiable function :

$$\int_{t_1}^{t_2} \delta(t) f(t)\, dt = f(t)\Theta(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \Theta \frac{df}{dt}\, dt$$

$$\boxed{\delta(t) = \frac{d\Theta}{dt}}$$

$$\begin{array}{llll} \text{If} & t_1 < 0 \;\&\; t_2 < 0 & : & \text{RHS} = 0 - 0 \\ \text{If} & t_1 > 0 \;\&\; t_2 > 0 & : & \text{RHS} = \big(f(t_2) - f(t_1)\big) - \big(f(t_2) - f(t_1)\big) = 0 \\ \text{If} & t_1 < 0 \;\&\; t_2 > 0 & : & \text{RHS} = \,? \end{array}$$

$$\int_{t_1<0}^{t_2>0} \delta(t) f(t)\, dt = f(t)\Theta(t)\Big|_{t_1}^{t_2} - \int_{0}^{t_2} \Theta \frac{df}{dt}\, dt$$

The part less than zero contributes nothing

$$= f(t_2) - \big(f(t_2) - f(0)\big) = f(0)$$

Notice that we only needed to know how the integral of delta function behaves.
We managed to get around the singularity of the derivative.

Gauss's law: divergence of $1/r^2$ and the Dirac delta function

Such "functions" can only be defined by specifying their behaviour inside an integral. You cannot really plot such functions because they are inherently singular.

$$\int_a^b \delta(x-x_0) f(x)\, dx = \begin{cases} f(x_0), & \text{if } x_0 \text{ is within the limits} \\ 0 & \text{otherwise} \end{cases}$$

Visualise this as a huge spike at $x = x_0$ only,

Gets higher but narrower keeping the area under it, same.

Picks out the value of any $f(x)$ at the spike

Several other ways to define $\delta(x)$ *as a limit*

For our purpose, we will need to use

$$\boxed{\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\,\delta(\vec{r})}$$

Fourier & Cauchy had introduced such "functions" Before.

In physics texts It is generally associated with Dirac

Some other integral representations of δ *function*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ip(x-x_0)} = \delta(x-x_0)$$

Frequently used in Quantum mechanics

$$\lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} = \delta(x)$$

Try proving... (hint : change of variables)

$$\delta(-x) = \delta(x) \qquad \alpha \;\; is \;\; any \;\; constant$$

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

The wikipedia article is excellent! Read it.