

Chapter 2

Multipole Expansion of the Electrostatic Potential

The potential of a localized charge distribution at large distance can be expanded as a series of multipole terms.¹ The terms of the series depend on the charge spatial distribution in the system and have different dependence from the distance. In this chapter we will first examine the electric dipole, the simplest system after the point charge. We will write the dipole potential and obtain the expressions of the electrostatic energy, the force and the torque acting on the dipole in an external field. Then we will derive the first terms of the multipole expansion for the potential from a charge distribution. Finally we will write the general expression for the multipole expansion together the formula for the expansion in terms of spherical harmonics.

2.1 The Potential of the Electric Dipole

The electric dipole is a rigid system of two point charges of opposite sign $+q$ and $-q$ separated by the distance δ . It is characterised by the *dipole moment* $\mathbf{p} = q\delta$ with δ oriented from the negative to the positive charge.

The potential generated by the electric dipole at point P at position \mathbf{r} from its center, is the sum of the potentials of the two point charges:

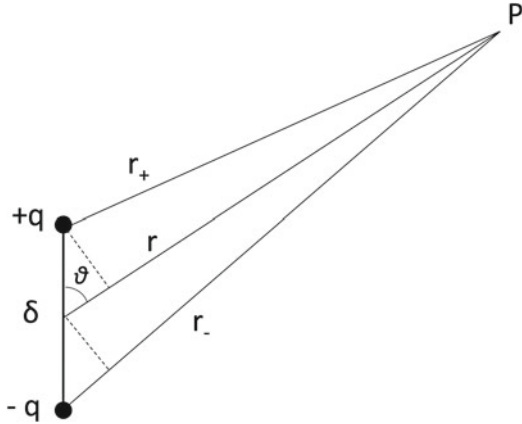
$$V(P) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_+} - \frac{q}{r_-} \right) = \frac{q}{4\pi\epsilon_0} \left(\frac{r_- - r_+}{r_+ r_-} \right)$$

where r_+ and r_- are the distances of P from $+q$ and $-q$ respectively (see Fig. 2.1).

When $r \gg \delta$ first order approximation can be used:

¹For this subject see for instance: D.J. Griffiths, *Introduction to Electrodynamics*, 4th Ed. (2013), Section 3.4, Pearson Prentice Hall; W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd Ed. (1962), Sections 1.7–8, Addison-Wesley.

Fig. 2.1 The potential of the electric dipole at a point P at distance r , is the sum of the potentials of the two opposite point charges. The distances of the point P from the two charges in the approximation $r \gg \delta$ are $r_+ = r - \frac{\delta}{2} \cos \theta$ and $r_- = r + \frac{\delta}{2} \cos \theta$



$$r_- \simeq r + \frac{\delta}{2} \cos \theta \quad r_+ \simeq r - \frac{\delta}{2} \cos \theta \quad r_- - r_+ \simeq \delta \cos \theta$$

with θ the angle between \mathbf{r} and δ , and:

$$r_+ r_- = r^2 - \frac{\delta^2}{4} \cos^2 \theta \simeq r^2$$

and the potential at a large distance from the dipole becomes:

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{q\delta \cos \theta}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}.$$

For increasing r this potential decreases as $1/r^2$ function, faster than the $1/r$ dependence of the point charge potential.

2.2 Interaction of the Dipole with an Electric Field

We can consider the interaction of an electric dipole \mathbf{p} with an external electric field \mathbf{E} that in general can be non-uniform.

For the force \mathbf{F} , the F_x component of the total force on the dipole is the sum of the x -component of the forces on the two point charges:

$$F_x = -qE_x(x, y, z) + qE_x(x + \delta_x, y + \delta_y, z + \delta_z)$$

with $\mathbf{E}(x, y, z)$ the electric field at the position of the charge $-q$ and $\mathbf{E}(x + \delta_x, y + \delta_y, z + \delta_z)$ that at the position of $+q$ respectively.

Writing the second term as:

$$E_x(x + \delta_x, y + \delta_y, z + \delta_z) = E_x(x, y, z) + \nabla E_x \cdot \boldsymbol{\delta}$$

F_x becomes:

$$F_x = q \nabla E_x \cdot \boldsymbol{\delta} = (\mathbf{p} \cdot \nabla) E_x.$$

Similar expressions can be derived for F_y and F_z and therefore we have:

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E} = \nabla(\mathbf{p} \cdot \mathbf{E}) \quad (2.1)$$

where we have used relation² $(\mathbf{p} \cdot \nabla) \mathbf{E} = \nabla(\mathbf{p} \cdot \mathbf{E}) - \mathbf{p} \times (\nabla \times \mathbf{E}) = \nabla(\mathbf{p} \cdot \mathbf{E})$ with $\nabla \times \mathbf{E} = 0$ for the electrostatic field.

The potential energy of the dipole is equal to the sum of the potential energies of the two point charges:

$$U = -qV(x, y, z) + qV(x + \delta_x, y + \delta_y, z + \delta_z) = qdV = q \nabla V \cdot \boldsymbol{\delta} = -\mathbf{p} \cdot \mathbf{E}.$$

The work $\delta W = -dU$ done by the electric field when the dipole is displaced by $d\mathbf{s}$ by a force \mathbf{F} and is rotated by $\delta\boldsymbol{\theta}$ around an axis $\hat{\boldsymbol{\theta}}$ by a torque \mathbf{M} , is:

$$\delta W = \mathbf{F} \cdot d\mathbf{s} + \mathbf{M} \cdot d\boldsymbol{\theta} = -dU = -\nabla U \cdot d\mathbf{s} - \frac{\partial U}{\partial \theta} \delta\theta$$

and from the potential energy expression:

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}) \quad \mathbf{M} = \mathbf{p} \times \mathbf{E} \quad (2.2)$$

where the expression for \mathbf{F} is that already found in (2.1).

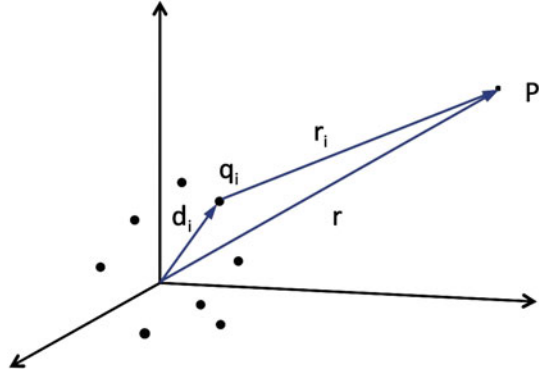
Supplemental problems are available at the end of this chapter as additional material on the interaction between two dipoles.

2.3 Multipole Expansion for the Potential of a Distribution of Point Charges

The potential $V_0(P)$ at the point $P(x, y, z)$ due to a distribution of N point charges q_i (see Fig. 2.2), is equal to the sum of the potentials at P from each charge of the system (principle of superposition of the electric potentials):

²Note that, since ∇ is a vector operator, we can get the relation used in the formula by substituting ∇ to \mathbf{B} in the vector relation $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$.

Fig. 2.2 The distribution of point charges. The position of the point P relative to the point charge q_i is $\mathbf{r}_i = \mathbf{r} - \mathbf{d}_i$



$$V_0(P) = \sum_{i=1}^{i=N} V_{0i}(P) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{i=N} \frac{q_i}{r_i} \quad (2.3)$$

where r_i is the distance between P and the i th charge.

If the distance between P and the system of the charges is much larger than the dimensions of the system, it is useful to approximate the potential at P as done for the electric dipole.

In the reference frame with origin in close proximity to the point charges system, we can write:

$$\mathbf{r} = \mathbf{r}_i + \mathbf{d}_i \quad \text{and} \quad \mathbf{r}_i = \mathbf{r} - \mathbf{d}_i$$

where \mathbf{d}_i and \mathbf{r} are the vector positions of the charge q_i and point P , respectively, and \mathbf{r}_i the vector from the i th charge to P . Then:

$$\frac{1}{r_i} = \frac{1}{|\mathbf{r} - \mathbf{d}_i|} = \frac{1}{[(\mathbf{r} - \mathbf{d}_i) \cdot (\mathbf{r} - \mathbf{d}_i)]^{\frac{1}{2}}} = \frac{1}{[r^2 - 2\mathbf{r} \cdot \mathbf{d}_i + d_i^2]^{\frac{1}{2}}} = \frac{1}{r} \cdot \frac{1}{[1 + \frac{(d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i)}{r^2}]^{\frac{1}{2}}}.$$

If we define $\alpha = \frac{d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i}{r^2}$, since $r \gg d_i$, α is very small and the fraction in the last equation can be expanded in a power series:

$$\frac{1}{[1 + \alpha]^{\frac{1}{2}}} = 1 - \frac{1}{2}\alpha + \frac{3}{8}\alpha^2 - \frac{15}{48}\alpha^3 + \dots$$

and keeping terms only to second order in α :

$$\frac{1}{r_i} = \frac{1}{r} \cdot \left(1 - \frac{1}{2} \frac{(d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i)}{r^2} + \frac{3}{8} \frac{(d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i)^2}{r^4} \right)$$

and neglecting terms with higher power than d_i/r squared we get:

$$\frac{1}{r_i} = \frac{1}{r} + \frac{(\mathbf{d}_i \cdot \hat{\mathbf{r}})}{r^2} + \frac{1}{r^3} \left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2 \right]. \quad (2.4)$$

Using this relation in Eq. 2.3, the potential in the point P becomes:

$$V_0(P) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{i=N} \frac{q_i}{r} + \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{i=N} \frac{q_i \mathbf{d}_i \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{i=N} \frac{q_i}{r^3} \left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2 \right]$$

that can be written as:

$$V_0(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \sum_{i=1}^{i=N} q_i + \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \sum_{i=1}^{i=N} q_i \mathbf{d}_i \cdot \hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \sum_{i=1}^{i=N} q_i \left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2 \right].$$

If we define the *total charge* of the system:

$$Q_{TOT} = \sum_{i=1}^{i=N} q_i \quad (2.5)$$

the *electric dipole moment*:

$$\mathbf{P} = \sum_{i=1}^{i=N} q_i \mathbf{d}_i \quad (2.6)$$

and the *electric quadrupole moment* relative to the direction $\hat{\mathbf{r}}$:

$$Q_{quadr} = \sum_{i=1}^{i=N} q_i \left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2 \right] \quad (2.7)$$

the potential of the system of point charges at the point P can be written in the form:

$$V_0(P) = \frac{1}{4\pi\epsilon_0} \frac{Q_{TOT}}{r} + \frac{1}{4\pi\epsilon_0} \frac{\mathbf{P} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{4\pi\epsilon_0} \frac{Q_{quadr}}{r^3}.$$

This relation represents the multipole expansion for the potential of the system of point charges, truncated to the second order. The first term depends on the total charge Q_{TOT} of the system and behaves as the $1/r$ potential of a point charge; the second term is related to the dipole moment \mathbf{P} of the system and behaves as the $1/r^2$ potential of an electric dipole; the third term depends on the quadrupole moment and decreases as $1/r^3$. The contribution of these terms to the total potential decreases at higher terms.

If the total charge Q_{TOT} is equal to zero, the most relevant term is the dipole term, and if also this term is null, the potential is determined by the quadrupole moment. If also this term is zero the multipole expansion should be extended to include terms of higher order.

For a continuous distribution of charge limited to a volume τ , described by the density $\rho(\mathbf{r}')$, the summation in (2.5), (2.6) and (2.7) has to be replaced by an integral:

$$Q_{TOT} = \int_{\tau} \rho(\mathbf{r}') d\tau'$$

$$\mathbf{P} = \int_{\tau} \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

$$Q_{quadr} = \int_{\tau} \rho(\mathbf{r}') \left[\frac{3}{2}(\mathbf{r}' \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}(r')^2 \right] d\tau'.$$

The measurement of the terms in the potential expansion of a charged structure gives information on the distribution of the charge.

Molecules, in which the centers of the positive and negative charge do not coincide, have a permanent electric dipole moment. The dipole moment for molecules of H_2O in its vapour state is $6.1 \times 10^{-30} \text{ C m}$, for HCl is $3.5 \times 10^{-30} \text{ C m}$ and for CO is $0.4 \times 10^{-30} \text{ C m}$. In atoms and molecules, also with a null dipole moment, the action of an external field may separate the centers of positive and negative charges and produce an *induced electric dipole*. Uncharged atoms and molecules can therefore have *dipole-dipole* or *dipole-induced dipole* interactions.

2.4 Properties of the Electric Dipole Moment

When the total charge of a system is null, the dipole moment is independent from the point (or the reference frame) chosen for the calculation of the moment, and the dipole moment becomes an intrinsic feature of the system.³ Indeed if the vector $\mathbf{a} = \overrightarrow{O'O}$ determines the position of the origin O of the first reference frame in a new frame with origin O' (see Fig. 2.3) we can write: $\mathbf{d}'_i = \mathbf{a} + \mathbf{d}_i$, and the dipole moment \mathbf{P}' is:

$$\mathbf{P}' = \sum_{i=1}^{i=N} q_i \mathbf{d}'_i = \sum_{i=1}^{i=N} q_i (\mathbf{a} + \mathbf{d}_i) = \left(\sum_{i=1}^{i=N} q_i \right) \mathbf{a} + \sum_{i=1}^{i=N} q_i \mathbf{d}_i = Q_{TOT} \mathbf{a} + \mathbf{P} = \mathbf{P}$$

because $Q_{TOT} = 0$.

³This is a general property of the first non null term in the multipole expansion.

Fig. 2.3 Position of the point charge relative to two different frames

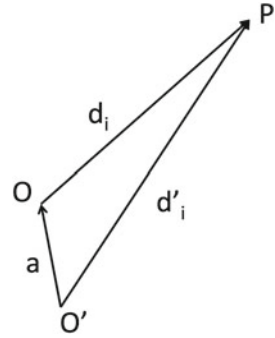
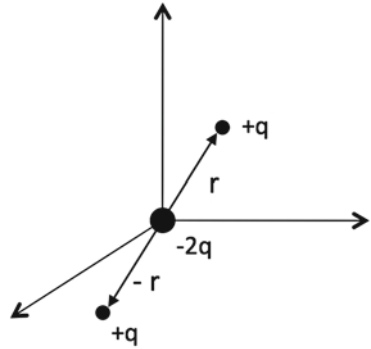


Fig. 2.4 Example of a charge distribution symmetric relative to a point



If a system of charges has a symmetry center, then the dipole moment is null. For instance for the system of three charges in Fig. 2.4:

$$\mathbf{P} = \sum_{i=1}^{i=3} q_i \mathbf{d}_i = q\mathbf{r} + q(-\mathbf{r}) + (-2q)\mathbf{0} = \mathbf{0}.$$

2.5 The Quadrupole Tensor

A rank two tensor can be associated to the quadrupole moment.

The Eq. (2.7):

$$Q_{quadr} = \frac{1}{r^2} \sum_{i=1}^{i=N} q_i \left[\frac{3}{2} (\mathbf{d}_i \cdot \mathbf{r})^2 - \frac{1}{2} d_i^2 r^2 \right]$$

can be written as:

$$Q_{quadr} = \frac{1}{r^2} \sum_{i=1}^{i=N} q_i \left[\frac{3}{2} \left(\sum_{\mu=1}^{\mu=3} x_{\mu} d_{i\mu} \right) \cdot \left(\sum_{\nu=1}^{\nu=3} x_{\nu} d_{i\nu} \right) - \frac{1}{2} d_i^2 \sum_{\mu=1}^{\mu=3} \sum_{\nu=1}^{\nu=3} x_{\mu} x_{\nu} \delta_{\mu\nu} \right]$$

where $\delta_{\mu\nu}$ is the Kronecker delta ($\delta_{\mu\nu} = 1$ if $\mu = \nu$, and $\delta_{\mu\nu} = 0$ when $\mu \neq \nu$). Assuming the sum over any index that appears twice in:

$$Q_{quadr} = \frac{1}{r^2} \sum_{i=1}^{i=N} q_i \left[\frac{3}{2} d_{i\mu} x_{\mu} d_{i\nu} x_{\nu} - \frac{1}{2} d_i^2 \delta_{\mu\nu} x_{\mu} x_{\nu} \right] = \frac{1}{r^2} \sum_{i=1}^{i=N} q_i \left[\frac{3}{2} d_{i\mu} d_{i\nu} - \frac{1}{2} d_i^2 \delta_{\mu\nu} \right] x_{\mu} x_{\nu}$$

we can finally write:

$$Q_{quadr} = \frac{1}{r^2} Q_{\mu\nu} x_{\mu} x_{\nu}$$

where we have introduced a symmetric rank two tensor, the quadrupole tensor $Q_{\mu\nu}$ given by:

$$Q_{\mu\nu} = \sum_{i=1}^{i=N} q_i \left[\frac{3}{2} d_{i\mu} d_{i\nu} - \frac{1}{2} d_i^2 \delta_{\mu\nu} \right].$$

The quadrupole term in the multipole expansion can be then expressed in the form:

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r^5} Q_{\mu\nu} x_{\mu} x_{\nu}. \quad (2.8)$$

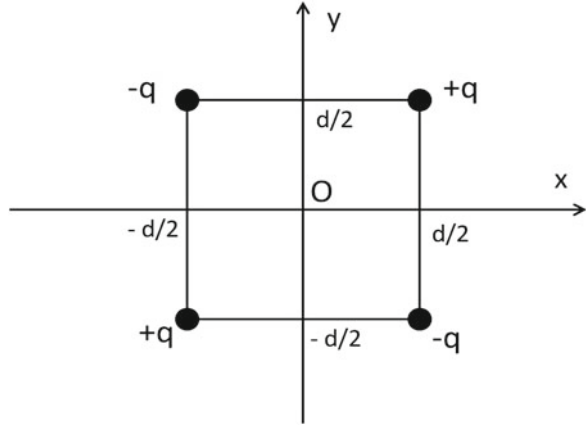
2.6 A Bidimensional Quadrupole

As an example we can calculate the potential at large distance from the quadrupole⁴ formed by four point charges as shown in Fig. 2.5.

The total charge is zero and the dipole moment is null because the point charges are placed symmetrically with respect to the origin. The first non null term in the multipole expansion is the quadrupole term.

⁴Note that the words *dipole*, *quadrupole*, etc. are used in two ways: to describe the charge distribution and secondly to designate the moment of an arbitrary charge distribution.

Fig. 2.5 The bidimensional electric quadrupole



For the tensor $Q_{\mu\nu}$ it is easy to find $Q_{xx} = Q_{yy} = Q_{zz} = Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$ and the only non null components are $Q_{xy} = Q_{yx} = \frac{3}{2}qd^2$.

Then from (2.8) the potential at a point $P(x, y, z)$ is:

$$V_0(x, y, z) = \frac{3qd^2}{4\pi\epsilon_0} \frac{xy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

that is zero in any point on the z axis.

Appendix

Higher Order Terms in the Multipole Expansion of the Potential

We have already seen the multipole expansion of the potential limited to the second order. To get the general expression⁵ with all the terms of the expansion, we write the potential at a point $P(\mathbf{r}) = P(x, y, z)$ from a continuous charge distribution, limited in space, described by the density $\rho(\mathbf{r}') = \rho(x', y', z')$. The distance of the point P from the elementary volume $d\tau'$ in the point \mathbf{r}' is:

$$|\mathbf{r} - \mathbf{r}'| = \Delta r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

⁵For this expansion see for instance: W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd Ed. (1962), Section 1.7, Addison-Wesley.

We set the origin of the frame inside the volume of the charge distribution or nearby. For a distance r large compared with the dimensions of the volume, we can expand the distance $|\mathbf{r} - \mathbf{r}'|$ as a Taylor series:

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r} + x'_\alpha \left[\frac{\partial}{\partial x'_\alpha} \left(\frac{1}{\Delta r} \right) \right]_{x'_\alpha=0} + \frac{1}{2!} x'_\alpha x'_\beta \left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \left(\frac{1}{\Delta r} \right) \right]_{x'_\alpha=x'_\beta=0} + \dots \\ &\dots + \frac{1}{n!} x'_\alpha x'_\beta x'_\gamma \dots \left[\frac{\partial^n}{\partial x'_\alpha \partial x'_\beta \partial x'_\gamma \dots} \left(\frac{1}{\Delta r} \right) \right]_{x'_\alpha=x'_\beta=x'_\gamma=\dots=0} \end{aligned}$$

where we assume the sum over $\alpha, \beta, \gamma, \dots = 1, 2, 3$, with $x_1 = x, x_2 = y, x_3 = z$.

The potential due to the charge distribution is:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad (2.9)$$

and by substituting the expression for $1/|\mathbf{r} - \mathbf{r}'|$ given before, we get:

$$\begin{aligned} V_0(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_\tau \rho(\mathbf{r}') d\tau' + \frac{1}{4\pi\epsilon_0} \left[\frac{\partial}{\partial x'_\alpha} \left(\frac{1}{\Delta r} \right) \right]_{x'_\alpha=0} \int_\tau x'_\alpha \rho(\mathbf{r}') d\tau' \\ &\quad + \frac{1}{4\pi\epsilon_0} \frac{1}{2!} \left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \left(\frac{1}{\Delta r} \right) \right]_{x'_\alpha=x'_\beta=0} \int_\tau x'_\alpha x'_\beta \rho(\mathbf{r}') d\tau' + \dots \\ &\quad + \frac{1}{4\pi\epsilon_0} \frac{1}{n!} \left[\frac{\partial^n}{\partial x'_\alpha \partial x'_\beta \partial x'_\gamma \dots} \left(\frac{1}{\Delta r} \right) \right]_{x'_\alpha=x'_\beta=x'_\gamma=\dots=0} \int_\tau x'_\alpha x'_\beta x'_\gamma \dots \rho(\mathbf{r}') d\tau'. \end{aligned}$$

In this expression we can recognise the terms corresponding to the total charge, to the dipole and to the quadrupole moments that we have already seen and then the general form of the 2^n -pole.

Expansion in Terms of Spherical Harmonics

The multipole expansion of the potential from a charge distribution limited in space, can be also expressed in series of spherical harmonics.⁶

If \mathbf{r}' gives the position of a point inside a sphere of radius R , and \mathbf{r} that of a point outside, we can write for $1/|\mathbf{r} - \mathbf{r}'|$ the expansion in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$:

⁶For an exhaustive presentation see J.D. Jackson, *Classical Electrodynamics*, cited, Chapters 3 and 4.

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{1}{2l+1} \frac{(r')^l}{r^{l+1}} Y_{lm}^* (\theta', \varphi') Y_{lm} (\theta, \varphi) .$$

If the charge distribution $\rho(r')$ is confined inside the sphere of radius R we can substitute this expansion in (2.9) and we get:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{1}{2l+1} \frac{1}{r^{l+1}} \left[\int Y_{lm}^* (\theta', \varphi') (r')^l \rho(\mathbf{r}') d\tau' \right] Y_{lm} (\theta, \varphi) \right\}$$

and introducing the *multipole moments*:

$$q_{lm} = \int Y_{lm}^* (\theta', \varphi') (r')^l \rho(\mathbf{r}') d\tau'$$

we get the expansion:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{1}{2l+1} \frac{1}{r^{l+1}} q_{lm} Y_{lm} (\theta, \varphi) \right\} .$$

Exercise With the formulas for the first spherical harmonics reported below, write the first three terms of the multipole expansion in spherical coordinates and compare with those expressed in cartesian coordinates.

$$l = 0 \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1 \quad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$l = 2 \quad Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\varphi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

with the relation:

$$Y_{l-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi) .$$

Problems

2.1 Find the dipole moment of the system of four point charges q at $(a, 0, 0)$, q at $(0, a, 0)$, $-q$ at $(-a, 0, 0)$ and $-q$ at $(0, -a, 0)$.

2.2 Write the potential for the system of three point charges: two charges $+q$ in the points $(0, 0, a)$ and $(0, 0, -a)$, and a charge $-2q$ in the origin of the frame. Find the approximate form of this potential at distance much larger than a . Compare the result with the potential from the main term in the multipole expansion.

2.3 Two segments cross each other at the origin of the frame and their ends are at the points $(\pm a, 0, 0)$ and $(0, \pm a, 0)$. They have a uniform linear charge distribution of opposite sign. Write the quadrupole term for the potential at a distance $r \gg a$.

2.4 Calculate the quadrupole term of the expansion for the potential from two concentric coplanar rings charged with q and $-q$ and with radii a and b .

2.5 Write the interaction energy of two electric dipoles \mathbf{p}_1 and \mathbf{p}_2 with their centers at distance r .

2.6 Using the result of the previous problem write the force between the electric dipoles \mathbf{p}_1 and \mathbf{p}_2 at distance r . Then consider the force when the dipoles are coplanar oriented normal to their distance and they are parallel or antiparallel. Determine also the force when the dipoles are on the same line and oriented in the same or in the opposite direction.

2.7 Two coplanar electric dipoles have their centers a fixed distance r apart. Say θ and θ' the angles the dipoles make with the line joining their centers and show that if θ is fixed, they are at equilibrium when

$$\tan \theta' = -\frac{1}{2} \tan \theta .$$

Solutions

2.1 The dipole moment of the system has components:

$$p_x = \sum_1^4 q_i x_i = 2qa \quad p_y = \sum_1^4 q_i y_i = 2qa \quad p_z = \sum_1^4 q_i z_i = 0 .$$

The dipole moment is $\mathbf{p} = (2qa, 2qa, 0)$ with module $p = 2q\sqrt{2}a$. This is the moment of an elementary dipole with opposite charges $2q$ located in the centers of the positive and the negative charges which are distant $\sqrt{2}a$.

2.2 In spherical coordinates the potential depends only on the distance r from the origin and on the angle θ . Adding the potentials from the three charges we have:

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[-\frac{2q}{r} + \frac{q}{[r^2 - 2ra \cos \theta + a^2]^{\frac{1}{2}}} + \frac{q}{[r^2 + 2ra \cos \theta + a^2]^{\frac{1}{2}}} \right]$$

and expanding in power series as in (2.4) we find:

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{qa^2}{r^3} (3 \cos^2 \theta - 1) . \quad (2.10)$$

In the multipole expansion for the system of the three charges the first non null term is the quadrupole moment. It is easy to find the components: $Q_{xz} = Q_{yz} = Q_{xy} = 0$, $Q_{xx} = Q_{yy} = -qa^2$ and $Q_{zz} = 2qa^2$ so that the potential is:

$$\begin{aligned} V(x, y, z) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^5} (Q_{xx}x^2 + Q_{yy}y^2 + Q_{zz}z^2) \\ &= \frac{1}{4\pi\epsilon_0} \frac{qa^2}{r^5} (2z^2 - x^2 - y^2) \end{aligned}$$

that is the formula (2.10) written in cartesian coordinates.

2.3 For the given charge distribution it is easy to see that $Q_{xz} = Q_{yz} = Q_{xy} = 0$ and by simple calculations we find:

$$Q_{xx} = \frac{1}{3}qa^2 \quad Q_{yy} = -\frac{1}{3}qa^2 \quad Q_{zz} = 0$$

so that the quadrupole potential is:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{qa^2}{3} \frac{(x^2 - y^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} .$$

2.4 We consider the two rings on the plane $z = 0$ with their centers in the origin. The total charge of the system is zero and, for the symmetry of the charge distribution with respect to the origin, also the dipole moment is null. The quadrupole term is the first non null term. It is evident that $Q_{xz} = Q_{yz} = 0$ and by simple integrals we can get:

$$Q_{xx} = Q_{yy} = \frac{q}{4}(a^2 - b^2) \quad Q_{zz} = \frac{q}{2}(b^2 - a^2) \quad Q_{xy} = 0$$

so that the first term of the potential expansion is:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q(a^2 - b^2)}{4} \frac{(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}.$$

2.5 The interaction energy of the two dipoles is equal to the potential energy of \mathbf{p}_2 in the field of \mathbf{p}_1 . Saying \mathbf{r} the vector from the center of \mathbf{p}_1 to that of \mathbf{p}_2 , we can write:

$$U_{21} = -\mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{r}) = \mathbf{p}_2 \cdot \nabla V_1 = \mathbf{p}_2 \cdot \nabla \left(\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_1 \cdot \mathbf{r}}{r^3} \right) = \frac{1}{4\pi\epsilon_0} \mathbf{p}_2 \cdot \left[\frac{\nabla(\mathbf{p}_1 \cdot \mathbf{r})}{r^3} + (\mathbf{p}_1 \cdot \mathbf{r}) \nabla \frac{1}{r^3} \right]$$

and since:

$$\nabla(\mathbf{p}_1 \cdot \mathbf{r}) = \mathbf{p}_1 \quad \nabla \frac{1}{r^3} = -3 \frac{\mathbf{r}}{r^5} \quad (\text{note that } \nabla \frac{1}{r^n} = -n \frac{\mathbf{r}}{r^{n+2}})$$

we get:

$$U_{21} = U_{12} = \frac{1}{4\pi\epsilon_0} \left[\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{r^3} - 3 \frac{(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})}{r^5} \right] \quad (2.11)$$

that is symmetric in \mathbf{p}_1 and \mathbf{p}_2 .

2.6 From the solution of the previous problem and from the formula (2.2) for the force on a dipole in an electric field we get:

$$\begin{aligned} \mathbf{F}_2 &= -\nabla U_{21} = \nabla(\mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{r})) \\ &= -\frac{1}{4\pi\epsilon_0} \left[(\mathbf{p}_1 \cdot \mathbf{p}_2) \nabla \frac{1}{r^3} - 3(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r}) \nabla \frac{1}{r^5} - 3 \frac{\nabla[(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})]}{r^5} \right] \end{aligned}$$

and then:

$$\mathbf{F}_2 = \frac{1}{4\pi\epsilon_0} \left[3 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{r} + \mathbf{p}_1(\mathbf{p}_2 \cdot \mathbf{r}) + (\mathbf{p}_1 \cdot \mathbf{r})\mathbf{p}_2}{r^5} - 15 \frac{(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})\mathbf{r}}{r^7} \right].$$

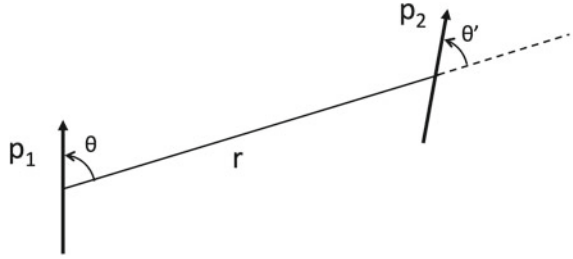
This formula is symmetric in \mathbf{p}_1 and \mathbf{p}_2 but \mathbf{r} is directed from \mathbf{p}_1 to \mathbf{p}_2 so we get \mathbf{F}_1 changing \mathbf{r} to $-\mathbf{r}$ and we have $\mathbf{F}_1 = -\mathbf{F}_2$ as expected.

For two coplanar parallel dipoles normal to the line joining their centres:

$$\text{with same direction}(\uparrow \uparrow) : \mathbf{F}_2 = \frac{3}{4\pi\epsilon_0} \frac{p_1 p_2}{r^4} \hat{\mathbf{r}} \quad \text{a repulsive force}$$

$$\text{with opposite direction}(\uparrow \downarrow) : \mathbf{F}_2 = -\frac{3}{4\pi\epsilon_0} \frac{p_1 p_2}{r^4} \hat{\mathbf{r}} \quad \text{an attractive force}$$

Fig. 2.6 Coplanar dipoles \mathbf{p}_1 and \mathbf{p}_2 with their centers at fixed distance r and orientations θ and θ' with respect to \mathbf{r}



for the dipoles on the same line

with same direction($\rightarrow \rightarrow$) : $\mathbf{F}_2 = -\frac{6}{4\pi\epsilon_0} \frac{p_1 p_2}{r^4} \hat{\mathbf{r}}$ an attractive force

with opposite direction($\rightarrow \leftarrow$) : $\mathbf{F}_2 = \frac{6}{4\pi\epsilon_0} \frac{p_1 p_2}{r^4} \hat{\mathbf{r}}$ a repulsive force.

2.7 For the two coplanar dipoles shown in Fig. 2.6 the interaction energy (2.11) becomes:

$$U_{int} = \frac{1}{4\pi\epsilon_0} \frac{p_1 p_2}{r^3} \left[\cos(\theta' - \theta) - 3 \cos \theta \cos \theta' \right].$$

At fixed θ we find the minimum of this energy solving the equation:

$$\frac{\partial U}{\partial \theta'} = \frac{1}{4\pi\epsilon_0} \frac{p_1 p_2}{r^3} \left[-\sin(\theta' - \theta) + 3 \cos \theta \sin \theta' \right] = 0.$$

The solution is:

$$\tan \theta' = -\frac{1}{2} \tan \theta$$

$$\text{with the condition } \frac{\partial^2 U}{\partial \theta'^2} = \frac{1}{4\pi\epsilon_0} \frac{p_1 p_2}{r^3} \left[2 \cos \theta \cos \theta' - \sin \theta \sin \theta' \right] > 0.$$

For $\theta = \pi/2$ the minimum is at $\theta' = -\pi/2$, for $\theta = 0$ at $\theta' = 0$ and for $\theta = \pi$ at $\theta' = -\pi$.

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