MA-106 Linear Algebra

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D1 - Lecture 2

Recall

In the last class, we saw different methods to solve 2×2 and 3×3 systems of linear equations.

1. **Cramer's Rule**: Inefficient for large systems.

2. **Geometric techniques** (row method and column method): Difficult to visualise for $n > 3$.

3. **Elimination**: Will work for any n.

Matrix notation ($A\vec{x} = \vec{b}$) for linear systems

Consider the system
$2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$
Let $\vec{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ be the unknown vector, and $\vec{b} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$.
The coefficient matrix is $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$.

If we have m equations in n variables, then A has m rows and n columns, the column vector $\vec{b}$ has size m, and the unknown vector $\vec{x}$ has size n.

Notation: From now on, we will write $\vec{x}$ as x and $\vec{b}$ as b.

Gaussian Elimination

Example: $2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$

Algorithm: Eliminate u from last 2 equations by $(2) - \frac{4}{\mathbf{2}} \times (1)$, and $(3) - \frac{-2}{\mathbf{2}} \times (1)$ to get the *equivalent system:*

$$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad 8v + 3w = 14$$

The first pivot is 2, second pivot is -8. Eliminate v from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad \mathbf{1} \cdot w = 2$$

Solve this triangular system by *back substitution*, to get the *unique solution*

$$w = 2, \quad v = 1, \quad u = 1.$$

Elimination: Matrix form

Example: $2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$

Forward elimination in the *augmented* matrix form $[A|b]$:

(NOTE: The last column is the constant vector b).

$$\left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 0 & -\mathbf{8} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 0 & -\mathbf{8} & -2 & -12 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right). \text{ Solution is: } x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Q: Is there a relation between 'pivots' and 'unique solution'?

Singular case: No solution

Example: $\mathbf{2}u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + w = 9.$

$\boxed{\text{Step 1}}$ Eliminate u (using the 1st pivot **2**) to get:

$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad 8v + 2w = 14$

$\boxed{\text{Step 2}}$: Eliminate v (using the 2nd pivot **-8**) to get:

$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad 0 \cdot w = 2.$

The last equation shows that there is no solution,
i.e., the system is *inconsistent*.

Geometric reasoning: In Step 1, notice we get two <u>distinct</u> parallel planes $8v + 2w = 12$ and $8v + 2w = 14$.
They have no point in common.

Note: The planes in the original system were not parallel, but in an equivalent system, we get two distinct parallel planes!

Singular Case: Infinitely many solutions

Example: $\mathbf{2}u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + w = 7.$

$\boxed{\text{Step 1}}$ Eliminate u (using the 1st pivot 2) to get:

$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad 8v + 2w = 12$

$\boxed{\text{Step 2}}$: Eliminate v (using the 2nd pivot -8) to get:

$\mathbf{2}u + v + w = 5, \quad -\mathbf{8}v - 2w = -12, \quad 0 \cdot w = 0.$

There are only two equations. For every value of w, values for u and v are obtained by back-substitution,
e.g., $w = 2$ gives $(1, 1, 2)$, and $w = 0$ gives $\left(\frac{7}{4}, \frac{3}{2}, 0\right)$
$\Rightarrow$ the system has infinitely many solutions.

Geometric reasoning: In Step 1, notice we get the two equations $-8v - 2w = -12$ and $8v + 2w = 12$, giving the same plane. Hence we are looking at the intersection of the two planes, $2u + v + w = 5$ and $8v + 2w = 12$, which is a line.

Singular Cases: Matrix Form

1. $2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + w = 9.$

$$\left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 1 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 0 & \mathbf{-8} & -2 & -12 \\ 0 & 0 & 0 & 2 \end{array}\right).$$

No Solution! Why?

2. $2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + w = 7.$

$$\left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 1 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 0 & \mathbf{-8} & -2 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Infinitely many solutions! Why?

Q: Is there a relation between pivots and number of solutions? THINK!

Choosing pivots: Two examples

Example 1:

$-6v + 4w = -2, \quad u + v + 2w = 5, \quad 2u + 7v - 2w = 9.$

Forward elimination in the *augmented* matrix form $[A|b]$:

$$\begin{pmatrix} \mathbf{0} & -6 & 4 & | & -2 \\ 1 & 1 & 2 & | & 5 \\ 2 & 7 & -2 & | & 9 \end{pmatrix}$$

Coefficient of u in the first equation is 0. To eliminate u, exchange the first two equations , i.e, interchange the first two rows of the matrix and get

$$\begin{pmatrix} \mathbf{1} & 1 & 2 & | & 5 \\ 0 & -6 & 4 & | & -2 \\ 2 & 7 & -2 & | & 9 \end{pmatrix}$$

Exercise: Continue using elimination method; find all solutions.

Choosing pivots: Two examples

Example 2: 3 equations in 3 unknowns (u, v, w)
$0u + 6v + 4w = -2, \quad 0u + v + 2w = 1, \quad 0u + 7v - 2w = -9.$

$$[A|b] = \begin{pmatrix} \mathbf{0} & 1 & 2 & | & 1 \\ 0 & 6 & 4 & | & -2 \\ 0 & 7 & -2 & | & -9 \end{pmatrix}$$

Coefficient of u is 0 in every equation. Start by eliminating v. Solve for v and w to get $w = 1$, and $v = -1$.

Note: $(0, -1, 1)$ is a solution of the system. So is $(1, -1, 1)$.
In general, $(*, -1, 1)$ is a solution, for any real number $*$.

Observe: Unique solution is not an option. This system has infinitely many solutions. **Q:** Does such a system always have infinitely many solutions? **A:** Depends on the constant vector b.

Exercise: Find 3 vectors b for which the above system has
(i) no solutions (ii) infinitely many solutions.

Summary: Pivots

- Can a pivot be zero? No (since we need to divide by it).
- If the first pivot (coefficient of 1st variable in 1st equation) is zero, then interchange it with next equation so that you get a non-zero first pivot. Do the same for other pivots.
- If the coefficient of the 1st variable is zero in every equation, consider the 2nd variable as 1st and repeat the previous step.
- Consider system of n equations in n variables.

  The non-singular case, i.e. the system has n pivots:
  The system has a unique solution.

  The singular case, i.e., the system has atmost $n-1$ pivots: The system has no solutions, i.e., it is inconsistent, or it will have infinitely many solutions, provided it is consistent.

What is a matrix?

I cannot tell you what the matrix is, you have to experience it for yourself – Morpheus

Unlike Morpheus, we do have a precise definition.

A *matrix* is a collection of numbers arranged into a fixed number of rows and columns.
If a matrix A has m rows and n columns, the size of A is $m \times n$.

The rows of A are denoted $A_{1*}, A_{2*}, \ldots, A_{m*}$, i.e., $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{pmatrix}$,

the columns are denoted $A_{*1}, A_{*2} \ldots, A_{*n}$, i.e.,

$A = \begin{pmatrix} A_{*1} & A_{*2} & \cdots & A_{*n} \end{pmatrix}$, and the (i, j)th entry is A_{ij} (or a_{ij}).

Operations on Matrices: Matrix Addition

Example 1. We know how to add two row or column vectors.

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}-3 & -2 & -1\end{pmatrix} = \begin{pmatrix}-2 & 0 & 2\end{pmatrix} \quad \text{(component-wise)}$$

$\boxed{\text{We can add matrices if and only if they have the same size}}$,

and the addition is component-wise.
Example 2.

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 4\end{pmatrix} + \begin{pmatrix}-1 & -4 & -2\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & -2 & 1\\ 2 & 2 & 5\end{pmatrix}$$

Thus

$$(A+B)_{i*} = A_{i*} + B_{i*} \text{ and } (A+B)_{*j} = A_{*j} + B_{*j}$$