MA-108 Differential Equations I

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Laplace Transforms

- This method illustrates a general problem solving technique in mathematics; transform a difficult problem into an easier one, solve the later problem and use its solution to get a solution of the original problem.
- Laplace tranform converts an IVP for a constant coefficient ODE, into an algebraic equation whose solution is used to solve the IVP.
- We have already seen some methods to solve IVP. Laplace transform is especially useful when we are dealing with discontinuous functions $r(x)$.
- For example, when $r(x)$ is piece-wise continuous function, by earlier method, we need to solve IVP on each piece where $r(x)$ is continuous. Laplace transform gives solution in one step.

Laplace Transforms

Let's first define an improper integral.
If g is integrable over the interval $[a, T]$ for every $T > a$, then an improper integral of g over $[a, \infty)$ is defined as

$$\int_a^{\infty} g(t)\, dt := \lim_{T \to \infty} \int_a^T g(t)\, dt$$

We say that the improper integral converges to the limit value, if the limit exists and is finite;
Otherwise we say that the improper integral diverges or does not exist.

Example

1. Let $f(t) = e^{ct}$, $t \geq 0$ and $c \neq 0$ constant. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{T\to\infty} \int_0^T e^{ct}\,dt = \lim_{T\to\infty} \frac{1}{c}(e^{cT} - 1)$$

    - the integral converges to $-1/c$ if $c < 0$;
    - the integral diverges if $c > 0$.
    - If $c = 0$, then $f(t) = 1$ and the integral again diverges.

2. Let $f(t) = 1/t$ for $t \geq 1$. Then

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{T\to\infty} \int_1^T \frac{dt}{t} = \lim_{T\to\infty} \ln T$$

the improper integral diverges.

Laplace Transforms

Definition

Let $f(t)$ be defined for $t \geq 0$ and let s be a real number. The **Laplace transform** of f, denoted by $F(s)$, is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for those values of s for which the improper integral converges.

Note that s is a parameter and t is a variable of integration.

The Laplace transform can be thought of as an operator L that transforms a function $f(t)$ into the function $F(s)$.
We will write it as

$$F = L(f) \quad \text{or} \quad F(s) = L(f(t)) \quad \text{or} \quad f(t) \leftrightarrow F(s).$$

Let us discuss existence of Laplace transform.

Definition

A function f is of exponential order (s_0), if there exist constants M and t_0 such that

$$|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0.$$

1. If $f(t)$ is a bounded function, then f is of exponential order. In particular, $\sin at, \cos at$ are of exponential order for any $a \in \mathbb{R}$.
2. $f(t) = t^2$ is of exponential order.
3. $f(t) = e^{t^2}$ is <u>not</u> of exponential order.

$$\lim_{t \to \infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t \to \infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty$$

So $e^{t^2} > Me^{s_0 t}$ for large t, for any fixed s_0, M.

Definition

A function $f : [a, b] \to \mathbb{R}$ is called **piecewise continuous**, if there exist finitely many points $a = t_0 < t_1 < \ldots < t_n = b$ in $[a, b]$ such that

- f is continuous on each open sub-interval (t_{i-1}, t_i).
- $f(t_{i-1}+)$ and $f(t_i-)$ exists and are finite for all i.

• A function $f : [0, \infty) \to \mathbb{R}$ is called **piecewise continuous**, if it is piecewise continuous on $[0, T]$ for every $T > 0$.

• For $f : [a, b] \to \mathbb{R}$ to be piecewise continuous, we may not define f at finitely many points.

Example

Define $f : [-1, 2] \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x^2 - 2, & 0 < x < 1 \\ 0, & 1 < x \leq 2 \end{cases}$$

Then f is continuous on $(-1, 0) \cup (0, 1) \cup (1, 2)$ and

$$\begin{aligned} f(-1+) = 0 \quad &, \quad f(0-) = 1, \\ f(0+) = -2 \quad &, \quad f(1-) = -1, \\ f(1+) = 0 \quad &, \quad f(2-) = 0 \end{aligned}$$

Therefore, f is piecewise continuous on $[-1, 2]$.

Example

If

$$f(x) = \begin{cases} \dfrac{1}{x-1}, & 0 \le x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

Then f is not piecewise continuous on $[0, 2]$, since f is not continuous at 1, and $f(1+)$ does not exist (is not finite).

Example

If $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = \dfrac{1}{x}$, then f is continuous on $(0, 1)$, but not piecewise continuous on $[0, 1]$, since $f(0+)$ does not exists.

Existence of Laplace transform

- Assume f is piecewise continuous on closed interval $[a, b]$. Then there exists

$$t_0 = a < t_1 < \ldots < t_n = b$$

such that f is continuous on (t_{i-1}, t_i) and $f(t_{i-1}+)$ and $f(t_i-)$ exists and are finite for all i. Therefore,

$$\int_a^b f(t)dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(t)\, dt$$

exists and is finite.

- If f is piecewise continuous on $[0, \infty)$, then so is $e^{-st} f(t)$. Hence

$$\int_0^T e^{-st} f(t)dt$$

exists for every $T > 0$.

Theorem

If f is piecewise continuous on $[0,\infty)$ and of exponential order s_0, then the Laplace transform $F(s) = L(f)$ exists for $s > s_0$.

Proof.

Assume $|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0$.
We need to show that the integral

$$\int_0^\infty e^{-st} f(t) dt = \int_0^{t_0} e^{-st} f(t) dt + \int_{t_0}^\infty e^{-st} f(t) dt$$

converges. The first integral exists and is finite, since $e^{-st} f(t)$ is piecewise continuous. For $t > t_0$,

$$|e^{-st} f(t)| \leq e^{-st} M e^{s_0 t} = M e^{-(s-s_0)t}$$

Thus the second integral converges, since it is dominated by a convergent integral for $s > s_0$. Therefore $L(f)$ exists. $\square$

Example

- Show that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite, then f is of exponential order s_0.
- If $\alpha \in \mathbb{R}$ and $s_0 > 0$, then $\lim_{t\to\infty} e^{-s_0 t} t^\alpha = 0$
  Hence t^α is of exponential order s_0 for any $s_0 > 0$.
- **Question.** Does this mean $L(t^\alpha)$ exists for any $\alpha \in \mathbb{R}$.
  No. We need piecewise continuity for $t \geq 0$.
- If $\alpha \geq 0$, then t^α is piecewise continuous on $[0, \infty]$, hence $L(t^\alpha)$ exists for $\alpha \geq 0$.

Example

Find the Lapalace transform $F(s)$ of $f(t) = 1$.

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \frac{1}{s}(1 - e^{-sT})$$

$F(s) \to \dfrac{1}{s}$ for $s > 0$ and diverges for $s < 0$.
For $s = 0$ also $F(s)$ diverges. We write this as

$$L(1) = \frac{1}{s}, \quad s > 0 \quad \text{or} \quad 1 \leftrightarrow \frac{1}{s}, \quad s > 0$$

Convention. Instead of writing $\lim_{T\to\infty}$ everytime, we will write directly as

$$\int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}|_0^\infty = \begin{cases} \dfrac{1}{s} & , \quad s > 0 \\ \infty & , \quad s < 0 \end{cases}$$

Example

Find Laplace transform of $f(t) = t$.

For $s \leq 0$, $F(s)$ diverges. For $s > 0$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} t \, dt \\ &= -\frac{1}{s} t e^{-st} |_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt \\ &= \frac{1}{s^2}, \quad s > 0 \end{aligned}$$