Tutorial-2, MA 106 (Linear Algebra)
Most of these problems are from reference texts for this course

1. Prove or disprove.
   (a) If a 2×2 matrix A is such that $AB = BA$ for all 2×2 matrices B, then A is a constant multiple of the identity matrix.
   (b) Let A be a matrix. There does not exist a matrix B such that $BA = 2A$.
   (c) Product of triangular matrices is triangular.
   (d) Inverse of a triangular matrix is triangular.
   (e) Inverse of a symmetric matrix is symmetric.
   (f) If u and v are solutions to $Ax = b$ then so is $(u + v)$.
   (g) Given a square matrix A, if $Ax = b$ has a solution for all b, then the solutions are all unique.
   (h) If $A^2 = A$, then $A = I$ or $A = 0$.

2. By trial and error find examples of 2 by 2 matrices such that
   (a) $A^2 = -I$, A having only real entries.
   (b) $B^2 = 0$, although $B \neq 0$.
   (c) $CD = -DC$, not allowing the case $CD = 0$.
   (d) $EF = 0$, although no entries of E or F are zero.

3. What three elementary matrices $E_{21},\ E_{31},\ E_{32}$ put $A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{pmatrix}$ into triangular form U?

   Multiply the E's to get one matrix M that does the elimination to give $MA = U$.

4. Fill in the blanks.
   (a) Let A be a 3×3 matrix, with no row exchanges are needed in elimination to get U. Suppose $a_{33} = 7$ and the third pivot is 5.
   (i) If you change a_{33} to 11, what is the third pivot?
   (ii) What should you change a_{33} to, so that there is a zero in the third pivot position?
   (b) To obtain the entry in row 3, column 4 of AB we need to multiply the ____ row of ____ with the ____ column of ____ .
   (c) If a 5×5 matrix has __ number of pivots, then it is invertible.

5. Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$. Is there a matrix B such that $AB = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$?

6. Find A such that

$$A\begin{pmatrix}0\\1\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix},\ A\begin{pmatrix}1\\0\\-3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix},\ \text{and } A\begin{pmatrix}2\\3\\8\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

How is A related to the matrix $B=\begin{pmatrix}0&1&2\\1&0&3\\4&-3&8\end{pmatrix}$?

7. Let A be $m\times n$, and b be an $m\times 1$ vector. If $Ax=0$ has a unique solution, what can you say about the number of solutions for $Ax=b$ for some b?

8. Factor A into LU and write down the upper triangular system $Ux=c$ which appears after elimination, for

$$Ax=\begin{pmatrix}2&3&3\\0&5&7\\6&9&8\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}2\\2\\5\end{pmatrix}$$

9. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A=LU$?

10. Solve as two triangular system, without multiplying LU to find A:

$$LUx=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}2&4&4\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}2\\0\\2\end{pmatrix}$$

11. For which numbers c, will A have LU decomposition ?

$$A=\begin{pmatrix}1&2&0\\3&c&1\\0&1&1\end{pmatrix}$$

12. Find the inverses of

$$A_1=\begin{pmatrix}0&0&0&1\\0&0&2&0\\0&3&0&0\\4&0&0&0\end{pmatrix},\quad A_2=\begin{pmatrix}2&3&0&0\\4&5&0&0\\0&0&2&3\\0&0&4&5\end{pmatrix},\ A=\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}.$$

13. If A, B and C are $n\times n$ matrices such that $AB=I_n$, and $CA=I_n$, then show that $B=C$.

14. (a) If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2\neq P_2P_1$ and $P_3P_4=P_4P_3$.

    (b) Find the inverses of the permutation matrices

$$P=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad\text{and}\quad P=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

(c) Explain for permutations why P^{-1} is always the same as P^T. Show that the 1's are in the right place to give $PP^T = I$.

15. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

16. Let A and B be $n \times n$. Show that $I - AB$ is invertible if $I - BA$ is invertible. Start from $B(I - AB) = (I - BA)B$.

17. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \mid I]$. Extend it to 5×5 "alternating matrix in 1, -1" and guess its inverse.

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

18. (a) There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many are invertible?

    (b) If you put 1's and 0's at random into the entries of a 10 by 10 matrix, is it more likely to be invertible or singular?

19. If A and B are $m \times n$ and $n \times m$ matrices respectively, such that $AB = I_m$ what can you say about the rank of A? Is it necessary that $BA = I_n$? Does there exists an $n \times m$ matrix C such that $CA = I_n$?

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