MA 106 (Linear Algebra)

Tutorials- 4&5 (31st Jan and 7th Feb 2018) - Version 2

Most of these problems are from reference texts for this course

1. Are the following vectors linearly independent?
   (a) $(1,3,2)^T$, $(2,1,3)^T$, $(3,2,1)^T$.
   (b) $(1,-3,2)^T$, $(2,1,-3)^T$, $(-3,2,1)^T$.

2. Let $v_1 = (1,0,0)^T$, $v_2 = (1,1,0)^T$, $v_3 = (1,1,1)^T$ and $v_4 = (2,3,4)^T$.
   (a) v_1, v_2, v_3, v_4 are linearly dependent because ______.
   (b) Find scalars a_1, a_2, a_3, a_4, not all zero, such that $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$.
   (c) Show that v_1, v_2, v_3 are linearly independent.
   (d) Find all combinations of 3 vectors from v_1, v_2, v_3, v_4, which are linearly independent.
   (e) Compute the rank of $A = (v_1 \ \ v_2 \ \ v_3 \ \ v_4)$, and the dimensions of its four fundamental spaces.

3. Find the largest possible number of independent vectors among: $v_1 = (1,-1,0,0)^T$, $v_2 = (1,0,-1,0)^T$, $v_3 = (1,0,0,-1)^T$, $v_4 = (0,1,-1,0)^T$, $v_5 = (0,1,0,-1)^T$, $v_6 = (0,0,1,-1)^T$. How is this number related to $\text{Span}\{v_1, \ldots, v_6\}$?

4. $x = v + w$ and $y = v - w$ are combinations of v and w. Show that v and w can be written as combinations of x and y. How are $\text{Span}\{v, w\}$ and $\text{Span}\{x, y\}$ related? When is each pair of vectors a basis for its span?

5. Construct a 3×3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$, but not $(1,1,1)$. Construct a 3×3 matrix whose column space is only a line.

6. Suppose A is a 5×4 matrix with $\text{rank}(A) = 4$. Show that $Ax = v$ has no solution if and only if the 5×5 matrix $[A|v]$ is invertible. Show $Ax = v$ is solvable when $[A|v]$ is singular.

7. The matrix $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ is a vector in M, the space of all 2×2 matrices. Write the zero vector in this space, the vectors $\frac{1}{2}A$ and $-A$. What matrices are in the smallest subspace containing A?

8. Fill in the blanks.
   (a) If A is an invertible 8×8 matrix, then its column space is ____. Why?
   (b) A is a 6×4 matrix whose columns are independent. The number of pivots of A are ___. Why?
   (c) Consider $f = 1$, $g = x$, $h = x^2 \in \mathcal{C}[0,1]$. Then $\text{Span}\{f, g, h\}$ = ______. It has a basis ___ and its dimension is ___.
   (d) Let $W = \text{Span}\{\cos(x), \sin(x)\} \subset \mathcal{C}[0,1]$. A basis of W is ___, and $\dim(W)$ = ___.
   (e) A basis for the subspace of symmetric 3×3 matrices is ___, and its dimension is ___.
   Do the same for the subspaces of diagonal, skew-symmetric and lower triangular matrices respectively.

9. Are the following true or false? Briefly explain if it is true, give a counter-example if it is false.

   (a) There is a 3×3 matrix A whose column space is the same as its null space.
   (b) If the columns of a matrix are dependent, so are the rows.
   (c) The columns of a matrix are a basis for its column space.
   (d) A and A^T have the same number of pivots.
   (e) A and A^T have the same left null space.
   (f) If the vectors $v_1, \ldots, v_n$ span a subspace V, then $\dim(V) = n$.
   (g) If $v_1, \ldots, v_n$ are linearly independent in a vector space V, then $\dim(V) \geq n$.
   (h) If W is a subspace of V, then $\dim(W) \leq \dim(V)$.
   (i) The intersection of two subspaces of a vector space V cannot be empty.
   (j) If $Ax = Ay$, then $x = y$.
   (k) If a square matrix A has independent columns, then so does A^2.
   (l) If $AB = 0$, then $C(B)$ is contained in $N(A)$ (and the row space of A is contained in the left null space of B).
   (m) If the row space equals the column space, then $A = A^T$.
   (n) If $A^T = -A$, then the row space of A equals its column space.

10. A and B are 3×3 matrices. Mark all the correct options. Justify.

   (a) $C(A) = \{0\} \Rightarrow A = 0$.
   (b) $C(2A) = C(A)$.
   (c) $C(A - I) = C(A)$.
   (d) $C(A) = C(A^T)$.
   (e) $\dim(C(A)) = \dim(C(A^T))$.
   (f) $\text{rank}(AB) \leq \text{rank}(A)$. (Hint: How are $C(AB)$ and $C(A)$ related?)
   (g) $C(A + B) \subseteq C(A)$.

11. Describe the subspace of $\mathbb{R}^3$ spanned by:

   (a) $u_1 = (1, 1, -1)^T$ and $u_2 = (-1, -1, 1)^T$.
   (b) $v_1 = (0, 1, 1)^T$, $v_2 = (1, 1, 0)^T$ and $v_3 = (0, 0, 0)^T$.
   (c) The columns of a 3×5 echelon matrix with 2 pivots.
   (d) All vectors with positive components.

12. Is v in $\text{Span}\{v_1, \ldots, v_n\}$? If yes, write v as a combination of the v_i's.

   (a) $v_1 = (1, 1, 0)^T$, $v_2 = (2, 2, 1)^T$, $v_3 = (0, 0, 2)^T$; $b = (3, 4, 5)^T$.
   (b) $v_1 = (1, 2, 0)^T$, $v_2 = (2, 5, 0)^T$, $v_3 = (0, 0, 2)^T$, $v_4 = (0, 0, 0)^T$; $v = (a, b, c)^T$.

   In each case, find a basis of $\text{Span}\{v_1, \ldots, v_n\}$.

13. Let $\mathbf{P}$ be the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

   (a) Find a basis for $\mathbf{P}$.
   (b) Find a basis for the space of all the vectors perpendicular to $\mathbf{P}$.
   (c) Find a basis for the intersection of $\mathbf{P}$ with the x-y plane.

14. Find a basis for each of the following subspaces of $\mathbb{R}^4$.

    (a) All vectors whose components are equal.

    (b) All vectors whose components add to zero.

    (c) All vectors that are perpendicular to $(1,1,0,0)^T$ and $(1,0,1,1)^T$.

15. Find the echelon forms U, basis and the dimension of the four fundamental subspaces of:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}.$$

16. Without computing A, find bases for the 4 fundamental subspaces: $A = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

17. Let A be $m \times n$ with rank r. Suppose there are right hand sides b for which $Ax = b$ is not solvable.

    (a) What inequalities must be true between m, n and r?

    (b) Explain why $A^T y = 0$ has non-trivial solutions.

18. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be vectors such that:

    (i) $\text{Span}\{v_1, v_2, v_3, v_4\} = \mathbb{R}^3$ and (ii) the vectors $\{v_2, v_3, v_4\}$ are linearly independent.

    For each of the following statements, state if it is true or false. Justify.

    (a) The vectors $\{v_1, v_2, v_3, v_4\}$ are linearly independent.

    (b) The vectors $\{v_1, v_2, v_3\}$ form a spanning set for $\mathbb{R}^3$.

    (c) The vectors $\{v_2, v_3\}$ are linearly independent.

    (d) For any other vector v_5 in $\mathbb{R}^3$, the vectors $\{v_1, v_2, v_3, , v_4, v_5\}$ form a spanning set for $\mathbb{R}^3$.

    (e) The vectors $v_2 + v_3$, $v_2 + v_4$, $v_3 + v_4$ are linearly independent.

19. If A is $m \times n$, the columns of A are n vectors in $\mathbb{R}^m$. If they are linearly independent, what is $\text{rank}(A)$? If they span $\mathbb{R}^m$, what is $\text{rank}(A)$? What happens if they are a basis of $\mathbb{R}^m$?

20. Fill in the blanks: Let A be an $m \times n$ matrix, with rank r.

    (a) If A has linearly independent columns, then $r =$ ______, the nullspace is ______, and the row space is ______.

    (b) If $Ax = b$ always has at least one solution, then the solutions to $A^T y = 0$ is/are _____.
    (Hint: Find r).

    (c) If $m = n = 3$ and A is invertible, then a basis for (i) $N(A)$ is _____, (ii) $C(A)$ is _____, (iii) $N(A^T)$ is _____ and (iv) $C(A^T)$ is _____. Do the same for the 3×6 matrix $B = \begin{pmatrix} A & A \end{pmatrix}$.

    (d) If $m = 7$, $n = 9$ and $r = 5$, then the dimension of (i) $N(A)$ is _____, (ii) $C(A)$ is _____, (iii) $N(A^T)$ is _____ and (iv) $C(A^T)$ is _____.

    (e) If $m = 3$, $n = 4$ and $r = 3$, then $C(A) =$ _____ and $N(A^T) =$ _____.

    (f) If B is obtained by exchanging the first two rows of A, then the fundamental subspaces which remain unchanged are _____.

    (g) With B as above, if $(1, 2, 3, 4)$ is in the left nullspace of A, a non-zero vector in the left nullspace of B is ____.

21. Let $\mathcal{P}$ be the set of polynomials with real coefficients. Show that $\mathcal{P}$ is a real vector space under term-wise addition and scalar multiplication. Can you find a linearly independent set of size 2? 3? 50?

22. Let $\mathcal{P}_2 = \{a_0 + a_1 X + a_2 X^2 : a_0, a_1, a_2 \text{ are in } \mathbb{R}\}$ be the set of polynomials of degree two or less.

    (a) Show that $\mathcal{P}_2$ is a subspace of $\mathcal{P}$.

    (b) Show that $\text{Span}\{1, X, X^2\} = \mathcal{P}_2$.

    (c) Find a basis for $\mathcal{P}_2$ and its dimension.

23. Use the cofactor matrix C to invert matrices A and B.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

24. Compute the determinant of the following matrices.

$$A = \begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}.$$

25. Find x, y, z by Cramer's rule:

$$x + 4y - z = 1, \;\; x + y + z = 0, \;\; 2x + 3z = 0$$

26. Find the determinant when a vector x replaces j-th column of I.

27. If the right side b is the last column of A, solve the 3×3 system $Ax = b$. Explain how each determinant in Cramer's rule leads to your solution x.

28. If all the cofactors are zero, is A invertible? Why or why not?

29. Suppose $det(A) = 1$ and you know all the cofactors. How will you find A?

30. L is lower triangular and S is symmetric. Assume they are invertible.

$$L = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \quad S = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

    (a) Which three cofactors of L are zero? (L^{-1} is lower triangular)

    (b) Which three pairs of cofactors of S are equal? (S^{-1} is symmetric)

31. The parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those area by determinants and say why they must be equal.

32. (a) The corners of a triangle are $(2, 1), (3, 4), (0, 5)$. What is the area?

    (b) A new corner at $(-1, 0)$ makes it lopsided (four sides). Find the area.

33. State true or false. If true, prove the statement, and if false, explain why and give a counter-example.

    (a) Let V be a vector space. Then the union of two linearly independent subsets of V is linearly independent.

(b) Let V be a vector space. Then the intersection of two linearly independent subsets of V is linearly independent.

(c) Let $\dim(V) = n$. Any set $S \subseteq V$ with n elements spans V.

(d) Let $\dim(V) = n$ Any set $S \subseteq V$ with n elements that spans V is linearly independent.

(e) If $T : V \to V$ is a linear transformation of vector space V and $W \subseteq V$ a subspace of V. Then $T(W)$ is a subspace of V.

(f) There is no one-one linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$.

(g) There is no onto linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$.

(h) There is no one-one linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$.

(i) There is no onto linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$.

34. Let Let $\dim(V) = n$ and $\mathcal{B} = \{v_1, v_2, \cdots v_n\}$ be a basis of V.

(a) Let $T : V \to V$ be the transformation defined by $T(v_i) = v_{i+1}$ for all $i = 1, 2, \cdots, n-1$ and $T(v_n) = 0$. Find the matrix A representing T with respect to the basis $\mathcal{B}$.

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

35. Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as $Tu = Au$. Describe what T does geometrically to a point in $\mathbb{R}^2$ for A = (i) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

36. Show that the function $T : \mathcal{M}_{2\times 3} \to \mathcal{M}_{3\times 2}$ defined as $T(A) = A^t$ is a linear transformation. Show that this map is an isomorphism.

37. Show that $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined as $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + x_3, x_3 + x_4)$ is a linear transformation. Find the standard matrix of T.

38. Find the standard matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x_1, x_2) = (x_1 + 2x_2, x_2)$. Can you find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal?

39. Define $D : \mathcal{P}_3 \to \mathcal{P}_2$ as $D(f) = \frac{df}{dx}$ for all polynomials f of degree less than or equal to 3. Show that this is a linear transformation. Find $N(D)$ and $C(D)$.

40. Show that $S_1, S_2 : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ defined as $S_1(A) = A + A^T$ and $S_2(A) = A - A^T$ are linear transformations . Find $N(S_1)$, $N(S_2)$, $C(S_1)$ and $C(S_2)$.

41. Is $I : \mathcal{P} \to \mathcal{P}$ defined as $I(f) = \int f \; dx$ a linear transformation? Prove or disprove.

42. Construct a linear map T from $P_2(\mathbb{R}) \to \mathbb{R}$ such that $T(1) = 1$, $T(1-x) = 2$ and $T(x^2) = 3$. What is $N(T)$ and $C(T)$? How many such maps can you construct? Is there one with $T(x) = 0$?

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