MA-106 Linear Algebra

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D1 - Lecture 7

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Recall: Echelon Form and Rank

Let A be an $m \times n$ matrix.

• An echelon form U (also $m \times n$) is obtained by forward elimination and has the following properties:

1. Pivots are the 1st nonzero entries in their rows.
2. Entries below pivots are zero, by elimination.
3. Each pivot lies to the right of the pivot in the row above.
4. Zero rows are at the bottom of the matrix.

• To obtain the row reduced form R of A:

1) Get the echelon form U.
2) Make the pivots 1.
3) Make the entries above the pivots 0.

U and R are used to solve $Ax = 0$ and $Ax = b$.

• Number of columns with pivots = rank(A).

Recall: Null Space of A

Given an $m \times n$ matrix A, the null space of A, denoted $N(A)$, is the set of all vectors x in $\mathbb{R}^n$ such that $Ax = 0$.

Key Point: $\boxed{N(A) = N(U) = N(R)}$

Example: If $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, then $R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$x = (t, u, v, w)^T$ is in $N(A)$ if and only if $Rx = 0$, i.e., $t = -2u - 2w$ and $v = -w$.

Thus, $x = \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2u - 2w \\ u \\ -w \\ w \end{pmatrix} = u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$,

i.e., all possible linear combinations of the special solutions.
This information is stored in a compact form in:

Null Space Matrix: Special solutions as columns.

Recall: Finding $N(A) = N(U) = N(R)$

Let A be $m \times n$. To solve $Ax = 0$, find R and solve $Rx = 0$.

1. Find free (independent) and pivot (dependent) variables:
pivot variables: columns in R with pivots ($\leftrightarrow t$ and v).
free variables: columns in R without pivots ($\leftrightarrow u$ and w).

2. No free variables, i.e., rank(A) = $n \Rightarrow N(A) = 0$.

3a. If rank(A) $< n$, obtain a special solution:
Set one free variable = 1, the other free variables = 0.
Solve $Rx = 0$ to obtain values of pivot variables.

3b. Find special solutions for each free variable.
$N(A)$ = space of linear combinations of special solutions.

Solving $Ax = b$

Caution: If $b \neq 0$, solving $Ax = b$ may not be the same as solving $Ux = b$ or $Rx = b$.

Example: $Ax = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b.$

Convert to $Ux = c$ and then $Rx = d$.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right) \to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & \mathbf{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right)$$

System is consistent $\Leftrightarrow b_3 + b_2 - 5b_1 = 0$, i.e., $b_3 = 5b_1 - b_2$

Solving $Ax = b$ or $Ux = c$ or $Rx = d$

$Ax = b$ has a solution $\Leftrightarrow b_3 = 5b_1 - b_2$.

e.g., there is no solution when $b = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}^T$.

Suppose $b = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$. Then $[A|b] \to$

$$\left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & \mathbf{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right) = \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & 1 \\ 0 & 0 & \mathbf{2} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & 1 \\ 0 & 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & 4 \\ 0 & 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$Ax = b$ is reduced to solving $Ux = c = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix}^T$,

which is further reduced to solving $Rx = d = \begin{pmatrix} 4 & -1 & 0 \end{pmatrix}^T$.

Solving $Ax = b$ or $Ux = c$ or $Rx = d$

Solving $\boxed{Ax = b}$ is reduced to solving $\boxed{Rx = d}$,
i.e., we want to solve

$$\begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

i.e., $t = 4 - 2u - 2w$ and $v = -1 - w$
Set the free variables u and $w = 0$ to get $t = 4$ and $v = -1$
A particular solution: $\mathbf{x} = \begin{pmatrix} 4 & 0 & -1 & 0 \end{pmatrix}^T$.

Ex: Check it is a solution i.e., check $Ax = b$.

Observe: In $Rx = d$, the vector d gives values for the pivot variables, when the free variables are 0.

General Solution of $Ax = b$

From $Rx = d$, we get $t = 4 - 2u - 2w$ and $v = -1 - w$, where u and w are free. Complete set of solutions to $Ax = b$:

$$\begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 - 2u - 2w \\ u \\ -1 - w \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

To solve $Ax = b$ completely, reduce to $Rx = d$. Then:

1. Find $x_{\text{NullSpace}}$, i.e., $N(A)$, by solving $Rx = 0$.
2. Set free variables = 0, solve $Rx = d$ for pivot variables.
This is a particular solution: $x_{\text{particular}}$.
3. Complete solutions: $\boxed{x_{\text{complete}} = x_{\text{particular}} + x_{\text{NullSpace}}}$

Ex: Verify geometrically for a 1×2 matrix, say $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$.

The Column Space of A

Q: Does $Ax = b$ have a solution? **A:** Not always.

Main Q2: When does $Ax = b$ have a solution?

If $Ax = b$ has a solution, then we can find numbers $x_1, \ldots, x_n$ such that

$$\begin{pmatrix} A_{*1} & \cdots & A_{*n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 A_{*1} + \cdots + x_n A_{*n} = b,$$

i.e., b can be written as a linear combination of columns of A.

The *column space* of A, denoted $C(A)$

is the set of all linear combinations of the columns of A

= $\{b \text{ in } \mathbb{R}^m \text{ such that } Ax = b \text{ is consistent}\}$.

Finding $C(A)$: Consistency of $Ax = b$

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Then $Ax = b$, where $b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$, has a solution whenever $-5b_1 + b_2 + b_3 = 0$.

• $C(A)$ is a plane in $\mathbb{R}^3$ passing through the origin with normal vector $\begin{pmatrix} -5 & 1 & 1 \end{pmatrix}^T$.

• $a = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}^T$ is not in $C(A)$ as $Ax = a$ is inconsistent.

• $b = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$ is in $C(A)$ as $Ax = b$ is consistent.

Ex: Write b as a linear combination of the columns of A.
(A different way of saying: Solve $Ax = b$).
$x = \begin{pmatrix} 4 & 0 & -1 & 0 \end{pmatrix}^T$ is a solution of $Ax = b$. Hence

$$\boxed{\begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T = 4A_{*1} + (-1)A_{*3}.}$$

Q: Can you write b as a different combination of $A_{*1}, \ldots, A_{*4}$?

Linear Combinations in $C(A)$

Let A be an $m \times n$ matrix, u and v be real numbers.

• The column space of A, $\boxed{C(A) \text{ contains vectors from } \mathbb{R}^m}$.

• If a, b are in $C(A)$, i.e., $Ax = a$ and $Ay = b$ for some x, y in $\mathbb{R}^n$, then $ua + vb = u(Ax) + v(Ay) = A(ux + vy) = Aw$, where $w = ux + vy$. Hence, if $w = \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}^T$, then $ua + vb = w_1 A_{*1} + \cdots w_n A_{*n}$, i.e., a linear combination of vectors in $C(A)$ is also in $C(A)$.

$\boxed{\text{Thus, } C(A) \text{ is \textit{closed under} linear combinations.}}$

• If b is in $C(A)$, then b can be written as a linear combination of the columns of A in as many ways as the solutions of $Ax = b$.

Summary: $N(A)$ and $C(A)$

Remark: Let A be an $m \times n$ matrix.

- The null space of A, $N(A)$ contains vectors from $\mathbb{R}^n$.
- $Ax = 0 \Leftrightarrow x$ is in $N(A)$.
- The column space of A, $C(A)$ contains vectors from $\mathbb{R}^m$.
- If B is the nullspace matrix of A, then $C(B) = N(A)$.
- $Ax = b$ is consistent $\Leftrightarrow b$ is in $C(A) \Leftrightarrow$

b can be written as a linear combination of the columns of A. This can be done in as many ways as the solutions of $Ax = b$.

- Let A be $n \times n$.

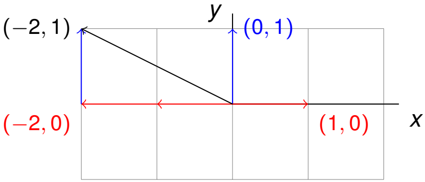
A is invertible $\Leftrightarrow N(A) = \{0\} \Leftrightarrow C(A) = \mathbb{R}^n$. Why?

- $N(A)$ and $C(A)$ are closed under linear combinations.

Vector Spaces: $\mathbb{R}^{\mathbf{n}}$

We begin with $\mathbb{R}^1, \mathbb{R}^2, \ldots, \mathbb{R}^n$, etc., where $\mathbb{R}^n$ consists of all column vectors of length n, i.e.,
$\mathbb{R}^n = \{x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$, where $x_1, \ldots, x_n$ are in $\mathbb{R}\}$.
We can add two vectors, and we can multiply vectors by scalars, (i.e., real numbers). Thus, we can take linear combinations in $\mathbb{R}^n$.

Examples:
$\mathbb{R}^1$ is the real line, $\mathbb{R}^3$ is the usual 3-dimensional space, and
$\mathbb{R}^2$ is represented by the x-y plane; the x and y co-ordinates are given by the two components of the vector.



Vector Spaces: Examples

1. $V = 0$, the space consisting of only the zero vector.
2. $V = \mathbb{R}^n$, the n-dimensional space.
3. $V = \mathbb{R}^\infty$, vectors with infinite number of components, i.e., a sequence of real numbers, e.g., $x = (1, 1, 2, 3, 5, 8, \ldots)$, with component-wise addition and scalar multiplication.
4. $V = \mathcal{M}$, the set of 2×2 matrices. What are $+$ and $*$?
   Q: Is this the 'same' as $\mathbb{R}^4$?
5. $V = \mathcal{C}[0, 1]$, the set of continuous real-valued functions on the closed interval $[0, 1]$. e.g., x^2, e^x are vectors in V.
   Q: Is $\frac{1}{x}$ a vector in V? How about $\frac{1}{x-2}$?
   Vector addition and scalar multiplication are pointwise:
   $(f + g)(x) = f(x) + g(x)$ and $(a * f)(x) = af(x)$.

Vector Spaces: Definition

Defn. A non-empty set V is a vector space if it is *closed under* vector addition (i.e., if x, y are in V, then $x + y$ must be in V) and scalar multiplication, (i.e., if x is in V, a is in $\mathbb{R}$, then $a * x$ must be in V).

Equivalently, $\boxed{x, y \text{ in } V,\ a, b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ must be in } V.}$

• A vector space is a triple $(V, +, *)$ with vector addition $+$ and scalar multiplication $*$

• The elements of V are called vectors and the scalars are chosen to be real numbers (for now).

• If the scalars are allowed to be complex numbers, then V is a *complex* vector space.

Vector Spaces: Definition continued

Let x, y and z be vectors, a and b be scalars. The vector addition and scalar multiplication are also required to satisfy:

- $x + y = y + x$ $\boxed{\text{Commutativity of addition}}$
- $(x + y) + z = x + (y + z)$ $\boxed{\text{Associativity of addition}}$
- There is a unique vector 0, such that $x + 0 = x$ $\boxed{\text{Existence of additive identity}}$
- For each x, there is a unique $-x$ such that $x + (-x) = 0$ $\boxed{\text{Existence of additive inverse}}$
- $1 * x = x$ $\boxed{\text{Unit property}}$
- $(a + b) * x = a * x + b * x, \quad a * (x + y) = a * x + a * y$
  $(ab) * x = a * (b * x)$ $\boxed{\text{Compatibility}}$

Notation: For a scalar a, and a vector x, we denote $a * x$ by ax.