MA-106 Linear Algebra

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D1 - Lecture 3

Recall: Matrices

A *matrix* is a collection of numbers arranged into a fixed number of rows and columns.
The (i, j)th entry is A_{ij} (or a_{ij}),
the ith row is denoted A_{i*}, and jth column is A_{*j}.

Row form: $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{pmatrix}$,

Column form: $A = \begin{pmatrix} A_{*1} & A_{*2} & \cdots & A_{*n} \end{pmatrix}$,

We can add matrices only if they have the same size, and the addition is component-wise.

In particular, $(A + B)_{i*} = A_{i*} + B_{i*}$ and $(A + B)_{*j} = A_{*j} + B_{*j}$

Linear Systems: Multiplying a Matrix and a Vector

One row at a time (dot product): The system

$2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9$

can be rewritten using dot product as follows:

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 5, \quad \begin{pmatrix} 4 & -6 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = -2 \quad \text{and}$$

$$\begin{pmatrix} -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 9.$$

Write the system in the $Ax = b$ form:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u + v + w \\ 4u - 6v \\ -2u + 7v + 2w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

Note: No. of columns of A = length of the vector x.

Multiplication of a Matrix and a Vector

Dot Product (row method): Ax is obtained by taking dot product of each row of A with x.

If $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ A_{3*} \end{pmatrix}$, then $\boxed{Ax = \begin{pmatrix} A_{1*} \cdot x \\ A_{2*} \cdot x \\ A_{3*} \cdot x \end{pmatrix}}$

Linear Combinations (column method):

The column form of the system

$2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9$ is:

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Thus Ax is a linear combination of columns of A, with the coordinates of x as weights, i.e., $\boxed{Ax = uA_{*1} + vA_{*2} + wA_{*3}}$.

An Example

Let $A = \begin{pmatrix} 1 & 3 & -3 & -1 \\ 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, and $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$A_{1*} = \begin{pmatrix} 1 & 3 & -3 & -1 \end{pmatrix}$, $A_{2*} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix}$ $A_{3*} = ?$.

Then $A_{1*} \cdot x = ?$, $A_{2*} \cdot x = 0$, $A_{3*} \cdot x = 0$, hence $Ax = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$.

Q: What is Ae_1? **A:** The first column A_{*1} of A.

Exercise:

What should e_2, e_3, e_4 be so that $Ae_j = A_{*j}$, the jth column of A?

Observe: No. of rows of Ax = No. of rows of A,
and No. of columns of Ax = No. of columns of x.

Question: What can you say about the solutions of $Ax = 0$?

Operations on Matrices: Matrix Multiplication

$\boxed{\text{Two matrices } A \text{ and } B \text{ can be multiplied if and only if}}$

$\boxed{\text{no. of columns of } A = \text{no. of rows of } B}$.

If A is $m \times \underline{n}$ and B is $\underline{n} \times r$, then AB is $m \times r$.

Key Idea: We know how to multiply a matrix and a vector.

Column wise: Write B column-wise, i.e., let
$B = \begin{pmatrix} B_{*1} & B_{*2} & \cdots & B_{*r} \end{pmatrix}$. Then

$$AB = \begin{pmatrix} AB_{*1} & AB_{*2} & \cdots & AB_{*r} \end{pmatrix}$$

Note: Each B_{*j} is a column vector of length n. Hence, AB_{*j} is a column vector of length m. So, the size of AB is $m \times r$.

Operations on Matrices: Matrix Multiplication

Row wise: Write A row-wise, i.e., let $A_{1*}, \ldots, A_{m*}$ be the rows of A. Then

$$AB = \begin{pmatrix} A_{1*} \\ \vdots \\ A_{m*} \end{pmatrix} B = \begin{pmatrix} A_{1*}B \\ \vdots \\ A_{m*}B \end{pmatrix}$$

Note: Each A_{i*} is a row vector of size $1 \times n$. Hence, $A_{i*}B$ is a row vector of size $1 \times r$. So, the size of AB is $m \times r$.

<u>WORKING RULE</u>:
The entry in the ith row and jth column of AB is the dot product of the ith row of A with the jth column of B, i.e., $(AB)_{ij} = A_{i*} \cdot B_{*j}$.

Properties of Matrix Multiplication

If A is $m \times n$ and B is $n \times r$.
a) $(AB)_{ij} = A_{i*} \cdot B_{*j} =$ (ith row of A) $\cdot$ (jth column of B)
b) jth column of AB $= A \cdot$ (jth column of B) , i.e., $(AB)_{*j} = AB_{*j}$.
c) ith row of AB $=$(ith row of A) $\cdot B$, i.e., $(AB)_{i*} = A_{i*}B$.

Properties of Matrix Multiplication:

- (associativity) $(AB)C = A(BC)$
- (distributivity) $A(B + C) = AB + AC$

  $(B + C)D = BD + CD$
- (non-commutativity) $AB \neq BA$, in general.
  Find examples.

Matrix Multiplication: Examples

Examples:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ (Identity)}$$

- $AB = ??$
- size of BA is __ $\times$ __
- $BA = \begin{pmatrix} 4 & 10 & 7 \\ 4 & 18 & 10 \end{pmatrix}$,
- and $IA = A = AI$.

Matrix Multiplication: Examples

Examples:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}, \; E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{(Permutation)} \;\; P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_2 & e_1 & e_3 \end{pmatrix}$$

Then $AP = \begin{pmatrix} Ae_2 & Ae_1 & Ae_3 \end{pmatrix} = \begin{pmatrix} A_{*2} & A_{*1} & A_{*3} \end{pmatrix}$

Exercise: Find EA and PA.

Question: How can you obtain EA and PA directly from A?

Transpose A^T of a Matrix A

Defn. The i-th row of A is the i-th column of A^T and vice-versa. Hence if $A_{ij} = a$, then $(A^T)_{ji} = a$.

Example: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 0 \\ 2 & -2 \\ 3 & 1 \end{pmatrix}$.

- If A is $m \times n$, then A^T is $n \times m$.
- If A is upper triangular, then A^T is lower triangular.
- $(A^T)^T = A$, $\boxed{(A+B)^T = A^T + B^T}$.
- $\boxed{(AB)^T = B^T A^T}$.

Proof. Exercise.

Symmetric Matrix

Defn. If $A^T = A$, then A is called a *symmetric* matrix.

Note: A symmetric matrix is always $n \times n$.

Examples: $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are symmetric.

• If A, B are symmetric, then AB may NOT be symmetric.

In the above case, $AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

• If A and B are symmetric, then so is $A + B$.

• If A is $n \times n$, $A + A^T$ is symmetric.

• For any $m \times n$ matrix B, BB^T and $B^T B$ are symmetric.

Exercise: If $A^T = -A$, we say that A is *skew-symmetric*.
Verify if similar observations are true for skew-symmetric matrices.