MA 108 - Spring 2018
Tutorial Sheet 6

1. Find the Laplace transform of the following functions.

   (a) $\dfrac{\sin \omega t}{t}$, $\omega > 0$, (b) $\dfrac{e^{at} - e^{bt}}{t}$, (c) $\dfrac{\cosh t - 1}{t}$, (d) $\dfrac{\sinh^2 t}{t}$.

2. Find the Laplace transform of the following functions.

   (a) $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

   (b) $f(t) = \begin{cases} te^t, & 0 \le t < 1 \\ e^t, & t \ge 1 \end{cases}$

3. Find the inverse Laplace transform of the following functions.

   (a) $H(s) = \dfrac{e^{-\pi s}(1-2s)}{s^2+4s+5}$.

   (b) $H(s) = \dfrac{1}{s} - \dfrac{2}{s^3} + e^{-2s}\left(\dfrac{3}{s} - \dfrac{1}{s^2}\right) + e^{-3s}\left(\dfrac{4}{s} + \dfrac{3}{s^2}\right)$.

4. Solve the following IVPs using Laplace transform.

   (a) $y'' - y = \begin{cases} e^{2t}, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$, $y(0) = 3, \quad y'(0) = -1$.

   (b) $y'' + 9y = \begin{cases} \cos t, & 0 \le t < 3\pi/2 \\ \sin t, & t \ge 3\pi/2 \end{cases}$, $y(0) = 0, \quad y'(0) = 0$.

   (c) $y'' + y = \begin{cases} t, & 0 \le t < \pi \\ -t, & t \ge \pi \end{cases}$, $y(0) = 0, \quad y'(0) = 0$.

5. Solve the IVP and find a formula in terms of f for the solution that does not involve any step functions and represents y on each interval of continuity of f

   (a) $y'' + y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = m + 1, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, \ldots.$

   (b) $y'' + y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = (-1)^m, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, \ldots.$

   (c) $y'' + 2y' + 2y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = (m+1)(\sin t + 2\cos t), \quad 2m\pi \le t < 2(m+1)\pi, \quad m = 0, 1, \ldots.$

6. Express the inverse Laplace transform of following functions as an integral.

   (a) $\dfrac{1}{s^2(s^2+4)}$, (b) $\dfrac{s}{(s+2)(s^2+9)}$.

   (c) $\dfrac{1}{(s+1)^2(s^2+4s+5)}$, (d) $\dfrac{1}{s^2(s-2)^3}$

7. Find the Laplace transform of following functions.
   (a) $\int_0^t \sin(a\tau)\,\cos(b(t-\tau))\,d\tau$.
   (b) $e^t \int_0^t \sin(\omega\tau)\,\cos(\omega(t-\tau))\,d\tau$.
   (c) $\int_0^t (t-\tau)^7 e^{-\tau}\sin 2\tau\,d\tau$.
8. Find a formula for the solutions of the IVP involving $f(t)$.
   (a) $y'' + 3y' + y = f(t),\ \ y(0) = 0,\ y'(0) = 0$.
   (b) $y'' + 6y' + 9y = f(t),\ \ y(0) = 0,\ y'(0) = -2$.
   (c) $y'' - 5y' + 6y = f(t),\ \ y(0) = 1,\ y'(0) = 3$.
9. Solve the integral equation
   (a) $y(t) = t - \int_0^t (t-\tau)y(\tau)\,d\tau$.
   (b) $y(t) = 1 + 2\int_0^t \cos(t-\tau)y(\tau)\,d\tau$.
10. Use the convolution theorem to solve the integral
   (a) $\int_0^t (t-\tau)^7\tau^8\,d\tau$
   (b) $\int_0^t e^{-\tau}\sin(t-\tau)\,d\tau$