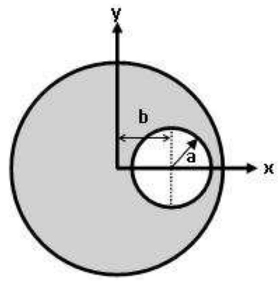
PH108 : Electricity & Magnetism : Tutorial 8

1. A long cylindrical conductor having radius R carries a current I along its length. Find the vector potential of this current distribution at all points in space $r > R$ and $r < R$.

2. An infinitely long cylindrical conductor of radius R, has a cross section with a hole as shown in the figure. A current I_0 flows along the z direction, distributed uniformly across the cross-section. The hole runs throughout its length, parallel to the z axis. The center of the hole is on the x-axis at $x = b$, and the radius of the hole is a. Determine the magnetic field at a point inside the hole.



3. The following vector identities turn out to be extremely useful in dealing with magnetic fields and potentials. Consider a volume V, whose bounding surface is S. Prove that:

   (a) If f is any scalar function then

   $$\int_V \nabla f \; d\tau \;=\; \int_S f \; d\vec{S}$$

   (b) If $\vec{A}$ is any vector function then

   $$\int_S \vec{A} \times \; d\vec{S} \;=\; -\int_V \nabla \times \vec{A} \; d\tau$$

   (c) If f, g are *any* two scalar functions and $\vec{J}$ is the steady state localised current density vector satisfying $\nabla.\vec{J} \;=\; 0$, then:

   $$\int_V \nabla . fg\vec{J} \; d\tau \;=\; \int_S fg\vec{J}.d\vec{S} \;=\; 0$$

   Then making suitable choices of f and g, prove that

   $$\int_V \vec{J} d\tau = 0 \; , \qquad \int_V \vec{r}.\vec{J} d\tau = 0 \; , \qquad \int_V (xJ_y + yJ_x) \, d\tau = 0$$

4. The magnetic vector potential produced by a magnetic dipole $\vec{m}$, placed at the origin, is given by

$$\vec{A} \;=\; \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

(a) Take $\vec{m}$ to be along $\hat{z}$ and work out $\vec{B} = \nabla \times \vec{A}$, using the expression for curl in spherical polar. You should be able to see that the expression is singular at $\vec{r} = 0$

(b) Take a sphere of radius R and evaluate $\int\limits_S \vec{A} \times d\vec{S}$. Show that this expression is independent of R.

(c) Use the previous result(s) to establish that $\int\limits_V \vec{B} d\tau$ is independent of R.

(d) How does the volume integral remain finite in the limit $R \to 0$?

(e) Use all the results to establish that the full magnetic field of $\vec{m}$ is given by

$$\vec{B} \quad = \quad \frac{\mu_0}{4\pi} \left[\frac{3\left(\vec{m}.\hat{r}\right)\hat{r} - \vec{m}}{r^3} + \frac{8\pi}{3} \vec{m} \delta^3(\vec{r}) \right]$$

The non singular part is sufficient as long as you are working at $\vec{r} \neq 0$

5. A spherical shell of radius R has uniform charge density σ on it. It is rotating with angular speed ω about an axis though its centre. Find the magnetic field at the centre.

6. A cylinder of radius a and length $2L$ is placed with its axis along $\hat{z}$ and its center at the origin. It has an uniform frozen electric polarisation $\vec{P} = P_0 \hat{z}$. It is set rotating with angular velocity ω about the direction of polarisation. Calculate the magnetic field $\vec{B}$ at a point on the $\hat{z}$ axis.