MA-106 Linear Algebra

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D1 - Lecture 18

Random Attendance: 13th Feb

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Diagonalization: Example

Example: $A = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{pmatrix}$ is triangular.

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -7 \\ -4 \\ 1 \end{pmatrix}$.

Further, $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. Hence $P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is invertible , and $AP = \begin{pmatrix} Av_1 & Av_2 & Av_3 \end{pmatrix} = \begin{pmatrix} v_1 & 2v_2 & 3v_3 \end{pmatrix} = P\Lambda$,

where $\Lambda = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$. Thus $P^{-1}AP = \Lambda$, i.e., A is diagonalizable.

Diagonalization of a Matrix

Question: What is the advantage of a basis of $\mathbb{R}^n$ consisting of eigenvectors?

Let A be an $n \times n$ matrix with n eigenvectors $v_1, \ldots, v_n$, associated to eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, then the matrix $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is invertible.

Moreover, $AP = A\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} Av_1 & \cdots & Av_n \end{pmatrix}$

$$= \begin{pmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{pmatrix} = P\Lambda, \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Therefore $P^{-1}AP = \Lambda$, i.e., A is similar to a diagonal matrix.

Thus: $\boxed{\text{Eigenvectors diagonalize a matrix}}$

Caution: $\Lambda P \neq P\Lambda$ in general.

When is A Diagonalizable?

Thus, we have proved: If an $n \times n$ matrix A has n linearly independent eigenvectors $v_1, \ldots, v_n$, then A is diagonalizable.

Moreover, if $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, then $P^{-1}AP = \Lambda$, where the diagonalizing matrix is $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$, and the diagonal matrix is $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, i.e., $\boxed{P^{-1}AP = \Lambda}$, where

$$\boxed{\text{The diagonal entries of } \Lambda \text{ are eigenvalues of } A \text{ and}}$$

$$\boxed{\text{The columns of } P \text{ are corresponding eigenvectors of } A.}$$

Exercise: With the above notation, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(v) = Av$, and $\mathcal{B} = \{v_1, \ldots, v_n\}$, then $[T]_{\mathcal{B}}^{\mathcal{B}} =$ ___.

Note: P need not be unique, e.g., replace v_1 by $2v_1$, etc.

When is A Diagonalizable?

Q: When does A have n linearly independent eigenvectors?

- If $v_1, \ldots, v_r$ are eigenvectors of A associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then $v_1, \ldots, v_r$ are linearly independent.

Proof. Suppose $v_1, \ldots, v_r$ are linearly dependent. Choose a linear relation involving minimum number of v_i's, say

(1) $\quad a_1 v_1 + \cdots + a_t v_t = 0. \quad (1 < t \leq r,\ t$ is minimal, $a_i \neq 0)$

Apply A to get $\quad a_1 \lambda_1 v_1 + \cdots + a_t \lambda_t v_t = 0 \qquad (2)$

$\lambda_1(1) - (2)$ gives $\quad a_2(\lambda_1 - \lambda_2)v_2 + \cdots + a_t(\lambda_1 - \lambda_t)v_t = 0,$

which contradicts the minimality of t. $\square$

- If A has n distinct eigenvalues, then A is diagonalizable.

Proof. If $v_1, \ldots, v_n$ are eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\{v_1, \ldots, v_n\}$ is linearly independent.

Then $P = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$ is invertible, and $P^{-1}AP = \Lambda$ as earlier. Hence A is diagonalizable. $\square$

When is A Diagonalizable?

Theorem A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors. In particular, $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Proof. ($\Leftarrow$): Done! To prove ($\Rightarrow$), assume $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is an invertible matrix such that $P^{-1}AP = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Then

$AP = P\Lambda$, i.e. $\begin{pmatrix} Av_1 & \dots & Av_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \dots & \lambda_n v_n \end{pmatrix}$. Therefore

$v_1, \dots, v_n$ are eigenvectors of A. They are linearly independent since P is invertible. $\square$

Question: Is every matrix is diagonalizable? **A:** No.

Examples: $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ no eigenvalues (over $\mathbb{R}$)!

$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ not enough eigenvectors!

Summary: Diagonalizability

Let A be $n \times n$.

• A is diagonalizable $\Leftrightarrow \mathbb{R}^n$ has a basis, say $\mathcal{B} = \{v_1, \ldots, v_n\}$, of eigenvectors of A, associated to eigenvalues $\lambda_1, \ldots, \lambda_n$. In this case, $P^{-1}AP = \Lambda$, where $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ and Λ is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$.

• If A is diagonalizable, and T is the linear operator defined by $Tx = Ax$, then $[T]_{\mathcal{B}}^{\mathcal{B}} = \Lambda$. Thus diagonalization of A is the same as finding a basis w.r.t. which the matrix of T (defined by $Tx = Ax$) is diagonal.

• Eigenvectors associated to distinct eigenvalues are linearly independent. In particular, if A has n distinct eigenvalues, A is diagonalizable.

Question: If λ and μ are eigenvalues of A and B respectively, then are $\lambda\mu$ and $\lambda + \mu$ eigenvalues of AB and $A + B$ respectively?

Reading Slide - Eigenvalues of AB and $A+B$

- If λ is an eigenvalue of A, μ is an eigenvalue of B, is $\lambda\mu$ an eigenvalue of AB?

  False Proof. $ABx = A(\mu x) = \mu(Ax) = \lambda\mu x$.

  This is false since A and B may not have same eigenvector x.

- **Ex:** $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

  The eigenvalues of A and B are $0, 0$ and that of AB are $1, 0$.

- Eigenvalues of $A+B$ are NOT $\lambda + \mu$.

  In above example, $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $1, -1$.

- If A and B have same eigenvectors associated to λ and μ, then $\lambda\mu$ and $\lambda + \mu$ are eigenvalues of AB and $A+B$ respectively.

Q: When do A and B have the same eigenvectors?

Extra Reading: Simultaneous Diagonalizability

- Assume $\boxed{A \text{ and } B \text{ are diagonalizable}}$. Then A and B have same eigenvector matrix S if and only if $AB = BA$.
- **Proof.** ($\Rightarrow$) Assume $S^{-1}AS = \Lambda_1$ and $S^{-1}BS = \Lambda_2$, where Λ_1 and Λ_2 are diagonal matrices.

  Then $AB = (S\Lambda_1 S^{-1})(S\Lambda_2 S^{-1}) = S(\Lambda_1\Lambda_2)S^{-1}$
  and $BA = S(\Lambda_2\Lambda_1)S^{-1}$.

  Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$, we get $AB = BA$.
- (Part of $\Leftarrow$) Assume $AB = BA$.
  If $Ax = \lambda x$, then $ABx = B(Ax) = B(\lambda x) = \lambda Bx$.
  If $Bx = 0$, then x is an eigenvector of B, associated to $\mu = 0$.
  If $Bx \neq 0$, then x and Bx both are eigenvectors of A, associated to λ.

  Special case: Assume all the eigenspaces of A are one dimensional. Then $Bx = \mu x$ for some scalar $\mu \Rightarrow x$ is an eigenvector of B. We will not prove the general case. $\square$

Eigenvalues of A^k

- If $Av = \lambda v$, then $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$.

  Similarly $A^k v = \lambda^k v$ for any $k \geq 0$.

  Thus if v is an eigenvector of A with associated eigenvalue λ, then v is also an eigenvector of A^k with associated eigenvalue λ^k for $k \geq 0$. If A is invertible, then $\lambda \neq 0$. Hence, the same also holds for $k < 0$ since $A^{-1} v = \lambda^{-1} v$.

- If A is diagonalizable , then $P^{-1}AP = \Lambda$ is diagonal where columns of P are eigenvectors of A.

  Since $(P^{-1}A^k P) = \Lambda^k$, which is diagonal, we see that A^k is diagonalizable, and the eigenvectors of A^k are same as eigenvectors of A. Similarly, the same also holds for $k < 0$ if A is invertible.

Reading Slide - Application: Fibonacci Numbers

Let $F_0 = 0$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$ define the Fibonacci sequence. What is the kth term?

If $u_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$, then $\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}$, i.e.,

$u_k = Au_{k-1}$ for $n \geq 1$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow u_k = A^k u_0$ for $k \geq 1$.

Characteristic polynomial of A: $\lambda^2 - \lambda - 1$

Eigenvalues: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$.

There are 2 distinct eigenvalues $\Rightarrow$ the associated eigenvectors x_1 and x_2 are linearly independent $\Rightarrow \{x_1, x_2\}$ is a basis for $\mathbb{R}^2$.

Write $u_0 = c_1 x_1 + c_2 x_2$. Then $u_k = A^k u_0 = A^k(c_1 x_1 + c_2 x_2)$

$$= c_1 A^k x_1 + c_2 A^k x_2 = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^k x_1 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^k x_2.$$

Q: Find x_1, x_2, c_1 and c_2 and get the exact formula for F_k.

An Application: Predator-Prey Model

Let the owl and rat populations at time k be O_k and R_k respectively. Owls prey on the rats, so if there are no rats, the population of owls will go down by 50%. If there are no owls to prey on the rats, then the rat population will increase by 10%.

In particular, the rat and owl populations dependence is as follows:

$$\begin{array}{rcl} O_{k+1} & = & 0.5O_k + 0.4R_k \\ R_{k+1} & = & -pO_k + 1.1R_k \end{array}$$

The term $-p$ calculates the rats preyed by the owls.

Thus, if $P_k = \begin{pmatrix} O_k \\ R_k \end{pmatrix}$ and $A = \begin{pmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{pmatrix}$, then $P_{k+1} = AP_k$ for all k. In particular, $P_k = A^k P_0$.

Exercise: If we start with a certain initial population of owls and rats, how many will be there in, say, 50 years, i.e., given P_0, what is P_{50}? What is the steady state, i.e., what is $\lim_{k\to\infty} P_k$?

An Application: Steady State

Suppose we have a system where the current state u_k depends on the previous one u_{k-1} linearly, i.e., $u_k = Au_{k-1}$. Then observe that $u_k = A^k u_0$. The steady state of the system is $u_\infty = \lim_{k\to\infty}(u_k)$. How do we find this?

- If u_0 is an eigenvector of A associated to λ, then $u_k = \lambda^k u_0$.
- Let $v_1, \ldots, v_r$ be eigenvectors of A associated respectively to $\lambda_1, \ldots, \lambda_r$. If $u_0 \in \text{Span}\{v_1, \ldots, v_r\}$, i.e., $u_0 = c_1 v_1 + \cdots + c_r v_r$ for scalars $c_1, \ldots, c_r$, then $u_k = A^k u_0 = c_1 A^k v_1 + \cdots + c_r A^k v_r = c_1 \lambda_1^k v_1 + \cdots + c_r \lambda_r^k v_r$. In particular, if A is diagonalizable, then there is a basis of $\mathbb{R}^n$ of eigenvectors of A. Hence, this is applicable to every $u_0 \in \mathbb{R}^n$.

Let A be diagonalizable, and u_k represent population.

- Under what conditions will there be a population explosion?
- What conditions will force the population to become extinct?
- When does it stabilise (to a non-zero value)?

Hint: Depends on $|\lambda_i|$.

Complex Eigenvalues

Ex: Rotation by 90^o in $\mathbb{R}^2$ is given by $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It has no eigenvectors since rotation by 90^o changes the direction. It has no real eigenvectors.

Q has eigenvalues, but they are not real. $\det(Q - \lambda I) = \lambda^2 + 1 \Rightarrow$
$\lambda_1 = i$ and $\lambda_2 = -i$, where $i^2 = -1$.
Let us compute the eigenvectors.

$$(Q - iI)x_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$(Q + iI)x_2 = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The eigenvalues, though imaginary, are distinct, hence eigenvectors are linearly independent.

If $P = \begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, then $P^{-1}QP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.