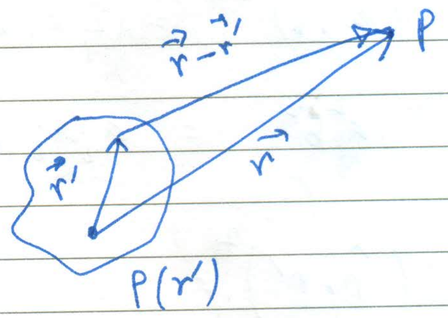
The "source co-ordinate" is $\vec{r}'$

The potential at point P



$$V(P) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{|\vec{r}-\vec{r}'|} d^3r' \quad \text{- - - - - - - -} \quad (1)$$

We want to:

1. "Unmix" $\vec{r}$ and $\vec{r}'$ to the extent possible.

2. For $\frac{r'}{r} \ll 1$ want to separate out the leading dependence on $\frac{1}{r}$

But our "small parameter" is not exactly $\frac{r'}{r}$

Notation: In Cartesian co-ordinate the x, y, z components will be denoted by r_i, r_j etc.

So $r_1 = x$ $\quad r_1' = x_1'$ and so on.

$r_2 = y$

$r_3 = z$ $\quad i, j, k$ runs ~~from~~ over $1, 2, 3$

We need to expand

$$\frac{1}{|\vec{r}-\vec{r}'|} = \left[r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}' \right]^{-1/2} \qquad (2)$$

$$= \frac{1}{r}\left[1 - \left(2\,\hat{r}\cdot\frac{\vec{r}'}{r} - \frac{r'^2}{r^2} \right) \right]^{-1/2} . \qquad (3)$$

Since $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$ (4)

we take $x = 2\frac{\hat{r}\cdot\vec{r}'}{r} - \frac{r'^2}{r^2}$ as our "small parameter"

$$\frac{1}{2}x = \frac{\hat{r}\cdot\vec{r}'}{r} - \frac{1}{2}\frac{r'^2}{r^2} \quad (5)$$

$$\frac{3}{8}x^2 = \frac{3}{8}\left(\frac{4}{r^2}(\hat{r}\cdot\vec{r}')(\hat{r}\cdot\vec{r}') + \frac{r'^4}{r^4} - 4\frac{r'^2}{r^2}\cdot\frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

$$= \frac{1}{2}\left[\frac{3}{r^2}(\hat{r}\cdot\vec{r}')(\hat{r}\cdot\vec{r}') + \frac{3}{4}\frac{r'^4}{r^4} - 3\frac{r'^2}{r^3}\hat{r}\cdot\vec{r}'\right] \quad (6)$$

The only term which is of the order $\frac{r'}{r}$ is the first term in (5)

There are two terms of the order of $\frac{r'^2}{r^2}$, one coming from (5) & one from (6).

The two other terms in (6) are of cubic and fourth order in $\frac{r'}{r}$. So we ignore them.

No term of first & second order can come from $\frac{5}{16}x^3$ & higher terms as in (4).

So the first two terms in the expansion of (1) would be

$$V(P) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(r')\,d^3r' + \frac{1}{r^2}\hat{r}\cdot\underbrace{\int \vec{r}'\rho(r')\,d^3r'}_{\vec{p}\text{ the dipole moment.}} \cdots\right] \quad (7)$$

The next term will come from

$$\frac{1}{2r^2}\left[3(\hat{r}\cdot\vec{r}\,')(\hat{r}\cdot\vec{r}\,') - r'^2\right] \quad \cdots\cdots (8)$$

Now we write $\hat{r}\cdot r' = \hat{r}_i r'_i$. (summation over i)

So (8) can be written as

$$\frac{1}{2r^2}\left[3.(\hat{r}_i r'_i)(\hat{r}_j r'_j) - r'^2(\hat{r}^2)\right] \quad --- (9)$$

We can put the extra $\hat{r}^2$ because $\hat{r}^2 = 1$ since it is an unit vector $\hat{r} = \frac{\vec{r}}{r}$.

So (9) can be written as .

$$\frac{1}{2r^2}\left[3\,\hat{r}_i\hat{r}_j\,r'_i r'_j - r'^2\,\hat{r}_j\hat{r}_j\right]$$

$$= \frac{1}{2r^2}\left[\hat{r}_i\hat{r}_j\,3r'_i r'_j - r'^2\,\hat{r}_i\hat{r}_j\,\delta_{ij}\right].$$

$$= \frac{1}{2r^2}\left[\hat{r}_i\hat{r}_j\left(3r'_i r'_j - r'^2\delta_{ij}\right)\right].$$ summed over i & j

The contribution to $V(P)$ would be

$$\frac{1}{4\pi\epsilon_0}\cdot\frac{1}{r^3}\,\hat{r}_i\hat{r}_j\underbrace{\int\frac{1}{2}\left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(r')\,d^3r'}_{Q_{ij}}$$

The contribution of the ~~dipol,~~ ~~monopole~~ terms:

$\frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{r}$ Monopole.

$\frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \vec{p}$ Dipole.

$\frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^3} \sum_{ij} \hat{r}_i \hat{r}_j Q_{ij}$ Quadrupole.

Where $\vec{p} = \int \vec{r}' \rho(r') d^3r'$

$$Q_{ij} = \frac{1}{2} \int \left(3 r_i' r_j' - r'^2 \delta_{ij}\right) \rho(r') d^3r'$$

So Q_{ij} is like a matrix.

Now you should be able to prove.

1. If the monopole contribution to $V(P)$ is zero i.e. $Q_{total} = 0$, then $\vec{p}$ is independent of where the origin is chosen. Otherwise it depends on the position of the origin.

2. If the monopole & ~~so~~ dipole terms are both zero, then shifting the origin leaves all the Q_{ij} terms unchanged.
Prove this by replacing $\vec{r}$ by $\vec{r} - \vec{r}_0$, hence $r_1 = x$ by $(x - x_0)$ and so on and computing each element of Q_{ij}.