EP 207: Introduction to Special Theory of Relativity
Tutorial 3 - Tensors and Vectors

1 Einstein summation

When a letter appears twice in a term, once as a superscript and once as a subscript, a summation is implied.

Q1: Suppose $y^r = \alpha^r_i x^i$ and $z^i = \beta^i_t y^t$. Can the transformation of x to z be represented as $z^i = \gamma^i_t x^t$? What is γ^i_t?

Q2: Let $R^2 = x^i x_i = x^1 x_1 + x^2 x_2 + ...$ Which of the following is the correct representation for R^4? $x^i x_i x^j x_j$ or $x^i x_i x^i x_i$? What is the difference between them?

2 Four-vectors

A vector is an object which follows a transformation rule keeping an invariant scalar.
For example, the lengths $|\vec{v}|$ and angles between vectors $\vec{v_1}.\vec{v_2}$ remain unchanged under rotational transformations in $\mathbb{R}^3$.
The four-vectors of special relativity are invariant under Lorentz transformation. For example,

$$\mathbb{s} = (\tau, x, y, z)$$

with $\mathbb{s}_1.\mathbb{s}_1 = \Delta\tau^2 - \Delta r^2$

Q3: a) Show that the invariants proper-time τ_o and proper-length L_o are encoded in $\mathbb{s}$. (Express τ_o and L_o in terms of $\mathbb{s}$ and other invariants).
b) One way to derive the four-vector for instantaneous velocity is to take the differential displacement $d\mathbb{s}$ and divide it by the differential proper time $d\tau_o$. Show that

$$\boldsymbol{\beta} = \frac{d\mathbb{s}}{d\tau_o} = \gamma(1, \vec{\beta})$$

(Hint: $\frac{d\tau}{d\tau_o} = \gamma$)

Q4: Derive three invariant scalars in terms of the four-vectors $\boldsymbol{\beta} = \gamma(1, \vec{\beta})$ and $\mathbb{p} = (E/c, \vec{p})$.

Q5: The propagation vector is defined as $\mathbb{k} = (\omega/c, \vec{k})$. Using the definition of $\mathbb{p}$ and the relation $E = \hbar\omega$ argue that $h = \lambda p$. [this is close to the original argument of deBroglie. For example, CLICK-THIS-LINK and find "For the purpose of generalising this result to nonuniform motion, we posit a proportionality between the momentum world vector of a particle and a propagation vector of a wave, for which the fourth component is its frequency"]

3 Tensors

Q6: Consider the coordinate transformation $x^i = x^i(x^{i'})$ such that

$$dx^i = \frac{\partial x^i}{\partial x^{i'}} dx^{i'} = a^i_{i'} dx^{i'} \qquad\qquad dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i = a^{i'}_i dx^i \tag{1}$$

Show that $a^{i'}_i a^i_{j'} = \delta^{i'}_{j'}$. Hint: Any change dx^1 can be done independent of dx^2, etc.
Did the above require assumption of orthogonal transformation?
We will call a quantity that transforms as dx^i a tensor of one contravariant rank (a contravariant vector).

Q7: Coordinates are only a labelling system and have no intrinsic geometric significance.
Suppose a particle is at position (r, θ, ϕ) (expressed in spherical coordinates) and it moves to $(r + dr, \theta + d\theta, \phi + d\phi)$. Find the displacement ds and express it in the form:

$$ds^2 = \begin{bmatrix} dr & d\theta & d\phi \end{bmatrix} \begin{bmatrix} g(r,\theta,\phi) \end{bmatrix} \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} \tag{2}$$

where $g(r, \theta, \phi)$ is a 3×3 matrix called the metric.

Q8: The invariant interval in any coordinate system can be expressed as $ds^2 = g_{ij} dx^i dx^j$.
a) Under a coordinate transformation such as (1), how would the metric g_{ij} transform? Show that it transforms as a tensor of two covariant ranks.
b) Since $\boldsymbol{A}.\boldsymbol{A} = A^i g_{ij} A^j$ for any two vectors we can define the covariant form of any vector $\boldsymbol{A}$ to be $A_i = g_{ij} A^j$. What is the covariant form for the contravariant $(dr, d\theta, d\phi)$ for spherical coordinates?
c) Define g^{ij} such that $A^i = g^{ij} A_j$. Show that $g^{ij} g_{jk} = \delta^i_k$.

Q9:
a) If A^{ij} and B_j are tensors show that $A^{ij} B_j$ is a tensor. [Contraction]
b) If $C^i = A^{ij} B_j$ is a contravariant vector and A^{ij} is an arbitrary contravariant tensor show that B_j is a covariant vector. [Quotient law]