Partial Differential Equations

Contents

CHAPTER 1

Tutorial Problems

1.1. Power series and series solutions

Problems.

(1) Find the radius of convergence of the following power series:

SOLUTION. Method used: $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{R}$

(a) $\sum x^n$ SOLUTION. $R = 1$

(b) $\sum \frac{x^m}{m!}$ SOLUTION. $R = \infty$

(c) $\sum m!x^m$ SOLUTION. $R = 0$

(d) $\sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1)x^m$ SOLUTION. $R = 1$

(e) $\sum \frac{(2n)!}{2^{2n}(n!)^2}x^n$ SOLUTION. $R = 1$

(f) $\sum_1^{\infty} \frac{x^m}{m(m+1)\cdots(m+k+1)}$ SOLUTION. $R = 1$

(g) $\sum_1^{\infty} \frac{n^n}{n!}x^n$ SOLUTION. $R = e^{-1}$

(h) $\sum_1^{\infty} \frac{(2n)!}{n^n}x^n$ SOLUTION. $R = 0$

(i) $\sum_1^{\infty} \frac{(3n)!}{2^n(n!)^3}x^n$ SOLUTION. $R = 2/27$

(2) Determine the radius of convergence of

$$\sum n!x^{n^2} \quad \text{and} \quad \sum x^{n!}.$$

SOLUTION. (i) Let $a_n = n!x^{n^2}$. Then $|\frac{a_{n+1}}{a_n}| = (n+1)|x|^{2n+1}$. Hence

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \begin{cases} 0 & \text{if } |x| < 1, \\ \infty & \text{if } |x| \geq 1. \end{cases}$$

Therefore, convergence if $|x| < 1$ and divergence otherwise. Hence $R = 1$.
(ii) Let $b_n = x^{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} |x|^{n!n} = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| = 1 \\ \infty & \text{if } |x| > 1. \end{cases}$$

(3) Show that if $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence R, then $\sum_{n=1}^{\infty} a_n x^{2n}$ has radius of convergence $\sqrt{R}$ and $\sum_{n=1}^{\infty} a_n^2 x^n$ has radius of convergence R^2.

SOLUTION. (i) Let $x^2 = z$. Then $\sum a_n x^{2n} = \sum a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$. Equivalently, $\sum a_n x^{2n}$ converges for $|x| < \sqrt{R}$ and diverges for $|x| > \sqrt{R}$. Hence the radius of convergence is $\sqrt{R}$.
(ii) We know that $\limsup |a_n|^{1/n} = 1/R$. So $\limsup |a_n^2|^{1/n} = 1/R^2$. Hence the radius of convergence is R^2.

(4) Apply the power series method around $x = 0$ to solve the following differential equations.
(a) $(1 - x^2)y' = y$

SOLUTION. Let $y = \sum a_n x^n$. Substitution yields $a_0 = a_1$ and

$$(n + 1)a_{n+1} = (n - 1)a_{n-1} + a_n, \; n \geq 1.$$

By induction on k, one can show that

$$a_{2k} = a_{2k+1} \quad \text{and} \quad 2k a_{2k} = (2k - 1)a_{2k-2}.$$

Now

$$a_{2k} = \frac{2k - 1}{2k} a_{2k-2} = \cdots = \frac{(2k)!}{(2^k k!)^2} a_0.$$

Combining with $a_{2k+1} = a_{2k}$,

$$y = a_0 \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2}(x^{2k} + x^{2k+1}) = a_0(x + 1) \sum \frac{(2k)! x^{2k}}{(2^k k!)^2}.$$

This can be written in closed form as follows.

$$y = a_0 \sqrt{\frac{1 + x}{1 - x}} = a_0(1 + x)(1 - x^2)^{-1/2}.$$

(b) $y' = xy, \; y(0) = 1$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$(n + 1)a_{n+1} = a_{n-1}, \quad n \geq 0, \; a_{-1} = 0.$$

The initial condition $y(0) = 1$ implies $a_0 = 1$. Since $a_{-1} = 0$, we have $a_{od} = 0$, and for the even coefficients

$$a_{2n} = \frac{a_0}{2^n n!} = \frac{1}{2^n n!}.$$

Therefore,

$$y = \sum \frac{x^{2n}}{2^n n!} = e^{x^2/2}.$$

(c) $(1-x^2)y' = 2xy$

SOLUTION. Let $y = \sum a_n x^n$. Then $a_1 = 0$ and the recursion is $a_{n+1} = a_{n-1}$. Hence $a_{od} = 0$ and $a_{2n} = a_0$ and

$$y = a_0 \sum x^{2n} = \frac{a_0}{1-x^2}.$$

(d) $y' - 2xy = 1,\ y(0) = 0$. Use the solution to deduce the Taylor series for $e^{x^2} \int_0^x e^{-t^2}\, dt$.

SOLUTION. Let $y = \sum a_n x^n$. The initial condition $y(0) = 0$ implies $a_0 = 0$. Further $a_1 - 2a_0 = 1$ which implies $a_1 = 1$. The general recursion is

$$(n+1)a_{n+1} = 2a_{n-1}, \quad n \geq 1.$$

Hence $a_{2n} = 0$ and

$$a_{2n+1} = \frac{2a_{2n-1}}{2n+1} = \cdots = \frac{2^n a_1}{(2n+1)(2n-1)\ldots 3} = \frac{2^{2n} n!}{(2n+1)!}$$

since $a_1 = 1$. Hence

$$y = \sum \frac{2^{2n} n! x^{2n+1}}{(2n+1)!}.$$

Using integrating factor e^{-x^2}, the differential equation can be written in an exact form to yield the solution

$$y = e^{x^2} \int_0^x e^{-t^2} dt.$$

By uniqueness of solutions, we conclude that the above power series is the Taylor series of this function.

(5) Find the power series solutions for the following differential equations around $x = 1$, that is in powers of $(x-1)$.

(a) $y'' + y = 0$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$n(n-1)a_n + a_{n-2} = 0, \quad n \geq 2$$

with a_0, a_1 arbitrary. This gives

$$y = a_0 \sum \frac{(-1)^k (x-1)^{2k}}{(2k)!} + a_1 \sum \frac{(-1)^k (x-1)^{2k+1}}{(2k+1)!}$$
$$= a_0 \cos(x-1) + a_1 \sin(x-1).$$

(b) $y'' - y = 0$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$n(n-1)a_n - a_{n-2} = 0, \quad n \geq 2$$

with a_0, a_1 arbitrary. This gives

$$y = a_0 \sum \frac{(x-1)^{2k}}{(2k)!} + a_1 \sum \frac{(x-1)^{2k+1}}{(2k+1)!} = a_0 \cosh(x-1) + a_1 \sinh(x-1).$$

(6) Find the power series solutions for the following differential equations around $x = 0$.

(a) Tchebychev equation:

$$(1-x^2)y'' - xy' + p^2 y = 0.$$

When do we have polynomial solutions?

SOLUTION. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{n^2 - p^2}{(n+2)(n+1)} a_n, \quad n \geq 0.$$

This implies a_0 and a_1 are arbitrary. Explicitly,

$$a_2 = -\frac{p^2}{2!} a_0, \ a_4 = +\frac{p^2(p^2-2^2)}{4!} a_0, \ a_6 = -\frac{p^2(p^2-2^2)(p^2-4^2)}{6!} a_0, \ldots$$

and

$$a_3 = -\frac{p^2-1^2}{3!} a_1, \ a_5 = \frac{(p^2-1^2)(p^2-3^2)}{5!} a_1, \ a_7 = -\frac{(p^2-1^2)(p^2-3^2)}{7!} a_1, \ldots$$

Write $y = a_0 y_0 + a_1 y_1$, with

$$y_0(x) = 1 - \frac{p^2}{2!} x^2 + \frac{p^2(p^2-2^2)}{4!} x^4 - \ldots$$

and

$$y_1(x) = x - \frac{p^2-1^2}{3!} x^3 + \frac{(p^2-1^2)(p^2-3^2)}{5!} x^5 - \ldots.$$

We have polynomial solutions if and only if p is an integer. (Suppose p is an integer. Then either the series y_0 or the series y_1 terminates, according as p is even or odd. Accordingly, on setting either $a_1 = 0$ or $a_0 = 0$, we get a polynomial solution of degree p.)

(b) Airy equation:

$$y'' - xy = 0.$$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}, \quad n \geq 0.$$

This implies a_0 and a_1 are arbitrary. Further, since $a_{-1} = 0$, $a_2 = a_5 = \cdots = a_{3n-1} = \cdots = 0$. The remaining coefficients are

$$a_{3n} = \frac{1.4.7\ldots(3n-2)}{(3n)!} a_0 \quad \text{and} \quad a_{3n+1} = \frac{2.5.8\ldots(3n-1)}{(3n+1)!} a_1.$$

Hence

$$y(x) = a_0 \left[1 + \frac{1}{3!} x^3 + \frac{1.4}{6!} x^6 + \frac{1.4.7}{9!} x^9 + \ldots\right] + a_1 \left[x + \frac{2}{4!} x^4 + \frac{2.5}{7!} x^7 + \frac{2.5.8}{10!} x^{10} + \ldots\right].$$

Note:

$$a_n = \frac{(n-2)(n-5)\dots(0 \textit{ or } 1 \textit{ or } 2)a_{-1 \textit{ or } 0 \textit{ or } 1}}{n!}$$

according as $n \equiv (-1 \textit{ or } 0 \textit{ or } 1) \bmod 3$.

(c) Hermite equation :

$$y'' - x^2 y = 0.$$

Solution. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{a_{n-2}}{(n+1)(n+2)}, \quad n \geq 0.$$

This implies a_0 and a_1 are arbitrary. Further, since $a_{-1} = 0 = a_{-2}$, $a_n = 0$ for $n \equiv 2, 3 \bmod 4$. The remaining coefficients are

$$a_{4n} = \frac{[1.5.9\dots(4n-3)][2.6.10\dots(4n-2)]}{(4n)!}a_0$$

and

$$a_{4n+1} = \frac{[2.6.10\dots(4n-2)][3.7.11\dots(4n-1)]}{(4n+1)!}a_1.$$

(7) Show that the function $(\sin^{-1} x)^2$ satisfies the initial value problem (IVP):

$$(1-x^2)y'' - xy' = 2, \quad y(0) = 0,\, y'(0) = 0.$$

Hence find the Taylor series for $(\sin^{-1} x)^2$ around 0. What is its radius of convergence ?

Solution. Direct substitution gives the first part. To find the Taylor series, let us apply the power series method. Accordingly let $y = \sum a_n x^n$ be a solution of the IVP. Then $a_0 = a_1 = 0$ due to the initial conditions, and $a_2 = 1$ and

$$a_{n+2} = \frac{n^2 a_n}{(n+1)(n+2)}, \quad n \geq 1.$$

This implies $a_{od} = 0$ and

$$a_{2n} = \frac{2^2.4^2\dots(2n-2)^2}{(2n)!}a_2 = \frac{2^{2n-1}((n-1)!)^2}{(2n)!}$$

on substituting $a_2 = 2$. For the radius of convergence, let $a_{2n} = b_n$ and $x^2 = z$. The radius of convergence of $\sum b_n z^n$ is

$$\lim \frac{b_n}{b_{n+1}} = \lim \frac{(2n+2)(2n+1)}{4n^2} = 1.$$

Hence radius of convergence is unity for both the series since $|z| < 1$ is equivalent to $|x| < 1$.

(8) Show that the even and odd parts of the binomial series of $(1-x)^{-m}$ are two linearly independent power series solutions of

$$(1-x^2)y'' - 2(m+1)xy' - m(m+1)y = 0$$

around $x = 0$. Hence deduce that $\{(1-x)^{-m}, (1+x)^{-m}\}$ is another linearly independent set of solutions.

SOLUTION. Let $y = \sum_{n \geq 0} a_n x^n$ be a power series solution. Substitution in the equation gives the recursion

$$a_{n+2} = \frac{(m+n+1)(m+n)}{(n+2)(n+1)} a_n, \quad n \geq 0,$$

with a_0, a_1 arbitrary. For $n \geq 2$,

$$\begin{aligned} a_n &= \frac{(m+n-1)(m+n-2)}{n(n-1)} a_{n-2} \\ &= \frac{(m+n-1)(m+n-2)(m+n-3)(m+n-4)}{n(n-1)(n-2)(n-3)} a_{n-4} = \dots \\ &= \begin{cases} \dfrac{(m+n-1)(m+n-2)\dots(m+1)m}{n!} a_0 & \text{if } n \text{ is even,} \\ \dfrac{(m+n-1)(m+n-2)\dots(m+2)(m+1)}{n!} a_1 & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} \dbinom{m+n-1}{n} a_0 & \text{if } n \text{ is even} \\ \dbinom{m+n-1}{n} \dfrac{a_1}{m} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Replace a_1/m by a new constant a_1 to conclude that

$$a_n = \begin{cases} \dbinom{m+n-1}{n} a_0 & \text{if } n \text{ is even,} \\ \dbinom{m+n-1}{n} a_1 & \text{if } n \text{ is odd.} \end{cases}$$

The general solution therefore, is

$$y(x) = a_0 \sum_{n \text{ even}} \binom{m+n-1}{n} x^n + a_1 \sum_{n \text{ odd}} \binom{m+n-1}{n} x^n.$$

The n-th coefficient of $(1-x)^{-m}$ equals

$$(-1)^n \frac{-m(-m-1)\dots(-m-n+1)}{n!} = \frac{m(m+1)\dots(m+n-1)}{n!} = \binom{m+n-1}{n}.$$

This proves the first part. Setting $a_0 = 1 = a_1$, we get $(1-x)^{-m}$ as a solution, while on letting $a_0 = 1 = -a_1$, we get $(1+x)^{-m}$ as another independent solution. This proves the last statement.

1.2. Legendre equation and Legendre polynomials

Problems.

(1) Express x^2, x^3, and x^4 as a linear combination of the Legendre polynomials. (This is possible since the Legendre polynomials form a basis for the vector space of polynomials.)

SOLUTION. We first express x^2 and x^4 using the Legendre polynomials of even degree. Since $P_0 = 1$ and $P_2 = \frac{3}{2}x^2 - \frac{1}{2}$,

$$x^2 = \frac{2}{3}P_2 + \frac{1}{3}P_0.$$

Substituting this,

$$P_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} = \frac{35}{8}x^4 - \frac{15}{4}(\frac{2}{3}P_2 + \frac{1}{3}P_0) + \frac{3}{8}P_0 = \frac{35}{8}x^4 - \frac{5}{2}P_2 - \frac{7}{8}P_0.$$

Therefore

$$x^4 = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0.$$

Similarly, x^3 can be expressed in terms of the Legendre polynomials of odd degree. Since $P_1 = x$ and $P_3 = \frac{1}{2}(5x^3 - 3x)$,

$$x^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1.$$

(2) Show that

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \sum_{m=0}^{[n/2]}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}\, x^{n-2m}$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$.

Both expressions equal $P_n(x)$, the n-th Legendre polynomial. The expression in the lhs is known as the Rodrigues formula.

SOLUTION. Start with the lhs. The binomial expansion gives

$$(x^2-1)^n = \sum_{m=0}^{n}\binom{n}{m}(-1)^m x^{2n-2m}.$$

Differentiating n times,

$$\begin{aligned}\frac{d^n}{dx^n}(x^2-1)^n &= \sum_{m=0}^{[n/2]}\binom{n}{m}(-1)^m(2n-2m)(2n-2m-1)\ldots(n-2m+1)\, x^{n-2m}\\ &= \sum_{m=0}^{[n/2]}\frac{n!}{m!(n-m)!}(-1)^m\frac{(2n-2m)!}{(n-2m)!}\, x^{n-2m}.\end{aligned}$$

Dividing both sides by $2^n n!$ yields the required identity.

(3) Show that if $f(x)$ is a polynomial with double roots at a and b then $f''(x)$ vanishes at least twice in (a,b). (This is also true if $f(x)$ is a smooth function.)

Generalize this and show (using Rodrigues' formula) that $P_n(x)$ has n distinct roots in $(-1,1)$.

SOLUTION. Let $f(a) = f'(a) = 0 = f(b) = f'(b)$. By Rolle's theorem, there is a $c \in (a,b)$ such that $f'(c) = 0$. Applying Rolle's theorem to $f'|_{[a,c]}$ and $f'|_{[c,b]}$, we get $c_1 \in (a,c)$ and $c_2 \in (c,b)$ where f'' vanishes. More generally: If $f(x)$ is a smooth function with roots of multiplicity n at both a and b, then $f^{(n)}$ vanishes at least n times in (a,b). (The hypothesis says $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0 = f(b) = f'(b) = \cdots = f^{(n-1)}(b)$.) We prove this result by induction. Assuming the result for $n-1$, there are $n-1$ points $a < t_1 < \cdots < t_{n-1} < b$ where $f^{(n-1)}(t_i) = 0$. Applying Rolle's theorem to $f^{(n-1)}|_{[t_{i-1},t_i]}$, we get n distinct zeroes of $f^{(n)}$ in the intervals (t_{i-1}, t_i). (Here $t_0 = a$ and $t_n = b$ is implicit.) This completes the induction step.
Now consider

$$f(x) = \frac{(x^2-1)^n}{2^n n!}.$$

This polynomial has roots of multiplicity n at $x = \pm 1$, Therefore, by the above result $P_n(x) = f^{(n)}(x)$ has at least n distinct zeroes in $(-1,1)$. Being a polynomial of degree n, these can be the only zeroes and each of them must be simple.

(4) Take the Rodrigues formula as the definition for $P_n(x)$, and show the following relations.

(a) $P_n(-x) = (-1)^n P_n(x)$

SOLUTION. Note that $P_n(x)$ is an even or an odd function according as n is even or odd. Hence $P_n(-x) = (-1)^n P_n(x)$.

(b) $P_n'(-x) = (-1)^{n+1} P_n'(x)$

SOLUTION. Note that $P_n'(x)$ is an even or an odd function according as n is odd or even. Hence $P_n'(-x) = (-1)^{n+1} P_n'(x)$.

(c) $P_n(1) = 1$ and $P_n(-1) = (-1)^n$

SOLUTION.

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n = \frac{1}{2^n n!} \sum_{r=0}^{n} \binom{n}{r} D^r (x-1)^n D^{n-r} (x+1)^n.$$

Now,

$$D^r (x-1)^n \big|_{x=1} = \begin{cases} 0 & \text{if } r < n, \\ n! & \text{if } r = n. \end{cases}$$

Hence evaluating at $x = 1$,

$$P_n(1) = \frac{1}{2^n n!} n! (1+1)^n = 1.$$

Similarly, or by part (a), $P_n(-1) = (-1)^n$.

(d) $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$

SOLUTION. $P_{2n+1}(0) = 0$ since it is an odd function, while

$$P_{2n}(0) = \frac{1}{4^n (2n)!} D^{2n} (x^2-1)^{2n} \Big|_{x=0} = \frac{1}{4^n (2n)!} \times \text{ the constant term in } D^{2n}(x^2-1)^{2n}.$$

The constant term in $D^{2n}(x^2-1)^{2n}$ is

$$(2n)! \times \text{ the coefficient of } x^{2n} \text{ in } (x^2-1)^{2n} = (2n)!\binom{2n}{n}(-1)^n.$$

Hence

$$P_{2n}(0) = \frac{1}{4^n(2n)!}(2n)!\frac{(2n)!}{n!n!}(-1)^n = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}.$$

(e) $P_n'(1) = \frac{1}{2}n(n+1)$ and $P_n'(-1) = (-1)^{n-1}\frac{1}{2}n(n+1)$

SOLUTION.

$$\begin{aligned}
P_n'(1) &= \frac{1}{2^n n!}D^{n+1}(x^2-1)^2\big|_{x=1} \\
&= \frac{1}{2^n n!}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}D^r(x-1)^n \cdot D^{n+1-r}(x+1)^n\right]_{x=1} \\
&= \frac{1}{2^n n!}\binom{n+1}{n}n! \cdot D(x+1)^n\big|_{x=1} \\
&= \frac{n+1}{2^n}\cdot n(1+1)^n = \frac{n(n+1)}{2}
\end{aligned}$$

The main point to note is that only the n-th term in the summation survives when we substitute $x = 1$.

Similarly, or by part (b),

$$P_n'(-1) = (-1)^{n+1}P_n'(1) = (-1)^{n+1}\frac{n(n+1)}{2}.$$

(f) $P_{2n}'(0) = 0$ and $P_{2n+1}'(0) = (-1)^n\dfrac{(2n+1)!}{2^{2n}(n!)^2}$.

SOLUTION. Since P_{2n}' is an odd function, $P_{2n}'(0) = 0$.

$$\begin{aligned}
P_{2n+1}'(0) &= \frac{1}{2^{2n+1}(2n+1)!}D^{2n+2}(x^2-1)^{2n+1}\big|_{x=0} \\
&= \frac{1}{2^{2n+1}(2n+1)!}\binom{2n+1}{n+1}(2n+2)!(-1)^n \\
&= (-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}
\end{aligned}$$

(5) Show that

$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx = \begin{cases}\dfrac{2n(n+1)}{2n+1} & \text{if } m = n,\\ 0 & \text{otherwise.}\end{cases}$$

SOLUTION. Recall the self-adjoint form of the Legendre equation

$$[(1-x^2)P_n']' + n(n+1)P_n = 0.$$

Multiplying by P_m and integrating over $[-1,1]$,

$$\int_{-1}^{1} P_m[(1-x^2)P_n']' + n(n+1)P_mP_n = 0.$$

Integrating the first term by parts,

$$-\int_{-1}^{1} P_m'[(1-x^2)P_n']dx + n(n+1)\int_{-1}^{1} P_m P_n dx = 0.$$

Now use

$$\int_{-1}^{1} P_m P_n dx = \begin{cases} \frac{2}{2n+1} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

(6) Show the following relations when $n - m$ is even and nonnegative.

(a) $\int_{-1}^{1} P_m' P_n' dx = m(m+1)$

SOLUTION.

$$\int_{-1}^{1} P_m' P_n' dx = P_m' P_n \Big|_{-1}^{1} - \int_{-1}^{1} P_m'' P_n dx$$
$$= 2P_m'(1) - \int_{-1}^{1} P_m'' P_n dx$$

since $n - m$ being even makes $P_m' P_n$ odd and $P_n(1) = 1$

$$= m(m+1) - \int_{-1}^{1} P_m'' P_n dx.$$

To evaluate the integral, we repeatedly integrate by parts

$$\int_{-1}^{1} P_m'' D^n (x^2-1)^n dx$$

to get

$$(-1)^n \int_{-1}^{1} (D^{n+2} P_m)(x^2-1)^n dx.$$

Now $n - m \geq 0$ implies $n + 2 > m$ so that $D^{n+2} P_m \equiv 0$. This makes the integral vanish.

(b) $\int_{-1}^{1} x^m P_n'(x) dx = 0$. What is the value of the integral if $n - m$ is odd (instead of even)?

SOLUTION. (i) Since $n - m$ is even, $x^m P_n'$ is an odd function. So the integral is zero. (ii) If $n - m$ is odd, then $n > m$ and $x^m P_n'$ is an even function. Now

$$\int_{-1}^{1} x^m P_n'(x) dx = 2\int_{0}^{1} x^m P_n' dx = 2\Big[x^m P_n \Big|_{0}^{1} - m\int_{0}^{1} x^{m-1} P_n dx\Big]$$
$$= 2 - 2m\int_{0}^{1} x^{m-1} P_n dx = 2.$$

In the last step, we used that x^{m-1} belongs to the span of $P_0,\dots,P_{m-1}$, and hence is orthogonal to P_n. Alternatively,

$$2^n n! \int_{-1}^{1} x^{m-1} P_n dx = \int_{-1}^{1} x^{m-1} D^n (x^2-1)^n dx$$
$$= (-1)^{m-1}(m-1)! \int_{-1}^{1} D^{n-m+1}(x^2-1)^n dx$$
$$= (-1)^{m-1}(m-1)! D^{n-m}(x^2-1)^n \Big|_{-1}^{1} = 0.$$

(7) If $x^n = \sum_{r=0}^{n} a_r P_r(x)$, then show that $a_n = \dfrac{2^n (n!)^2}{(2n)!}$.

SOLUTION.

$$\begin{aligned} a_n &= \frac{2n+1}{2}\frac{1}{2^n n!}\int_{-1}^{1} x^n D^n (x^2-1)^n dx \\ &= (-1)^n \frac{2n+1}{2}\cdot\frac{1}{2^n n!}\cdot n! \int_{-1}^{1} (x^2-1)^n dx \\ &= \frac{2n+1}{2^{n+1}}\int_{-1}^{1}(1-x)^n(1+x)^n dx \\ &= \frac{2n+1}{2^{n+1}}\frac{n(n-1)\dots 1}{(n+1)(n+2)\dots(2n)}\int_{-1}^{1}(1+x)^{2n}dx \\ &= \frac{2n+1}{2^{n+1}}\frac{n(n-1)\dots 1}{(n+1)(n+2)\dots(2n)}\frac{2^{2n+1}}{2n+1} \\ &= \frac{2^n(n!)^2}{(2n)!} \end{aligned}$$

(8) Expand the following functions $f(x)$ in a series of Legendre polynomials:

$$f(x) \approx \sum_{n\geq 0} c_n P_n \quad \text{with} \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx.$$

The Rodrigues formula is useful to evaluate these integrals. The Legendre expansion theorem (stated in the lecture notes) applies in each case.

(a)

$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

SOLUTION. Since $f(x) =: \operatorname{sgn}(x)$ is an odd function, $c_{even} = 0$. The odd coefficients are computed below.

$$\begin{aligned} c_{2k+1} &= \frac{4k+3}{2}\int_{-1}^{1}\operatorname{sgn}(x)P_{2k+1}(x)dx = (4k+3)\int_{0}^{1}P_{2k+1}(x)dx \\ &= (4k+3)\frac{D^{2k}(x^2-1)^{2k+1}}{2^{2k+1}(2k+1)!}\Big|_{0}^{1} \\ &= \frac{2k+1}{2^{2k+1}(2k+1)!}[-D^{2k}(x^2-1)^{2k+1}\big|_{x=0}] \end{aligned}$$

$$= \frac{2k+1}{2^{2k+1}(2k+1)!}[-(2k)!\binom{2k+1}{k}(-1)^{k+1}]$$
$$= \frac{(-1)^k(4k+3)}{2^{2k+1}(k+1)}\binom{2k}{k}.$$

(b)
$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

SOLUTION. Note that
$$f(x) = \frac{1}{2}(\operatorname{sgn}(x) + 1) = \frac{1}{2}(\operatorname{sgn}(x) + P_0).$$

Hence, using part (a),
$$f(x) = \frac{1}{2}P_0 + \frac{1}{2}\sum_{k\geq 0} c_{2k+1}P_{2k+1}, \quad \text{where} \quad c_{2k+1} = \frac{(-1)^k(4k+3)}{2^{2k+1}(k+1)}\binom{2k}{k}.$$

(c)
$$f(x) = \begin{cases} -x & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1. \end{cases}$$

SOLUTION. Since $f(x) = |x|$ is an even function, $c_{od} = 0$. Now
$$c_{2k} = (4k+1)\int_0^1 xP_{2k}(x)dx.$$

Clearly, $c_0 = 1/2$. For $k \geq 1$,
$$c_{2k} = -\frac{4k+1}{2^{2k}(2k)!}[-D^{2k-2}(x^2-1)^{2k}]\big|_{x=0} = (-1)^{k-1}\frac{4k+1}{4^k k(k+1)}\binom{2k-2}{k-1}.$$

Explicitly, for $k = 1$,
$$c_2 = 5\int_0^1 \frac{1}{2}x(3x^2-1)dx = 5/8.$$

(d)
$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$$

SOLUTION. Using
$$f(x) = x/2 + |x|/2 = \frac{1}{2}P_1 + \frac{1}{2}|x|,$$

we obtain from part (c),
$$f(x) = \frac{1}{2}P_0 + \frac{1}{2}P_1 + \frac{1}{2}\sum_{k\geq 1}\frac{(-1)^{k-1}(4k+1)}{4^k k(k+1)}\binom{2k-2}{k-1}P_{2k}.$$

(9) Consider the *associated Legendre equation*

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0 \tag{1}$$

which occurs in quantum physics. Substituting

$$y(x) = (1-x^2)^{m/2} v(x),$$

show that v satisfies

$$(1-x^2)v'' - 2(m+1)xv' + [n(n+1) - m(m+1)]v = 0 \tag{2_m}$$

Show that $v = D^m P_n$ satisfies equation (2_m). Thus

$$y(x) = (1-x^2)^{m/2} D^m P_n(x)$$

is the bounded solution of (1) and is called an *associated Legendre function*.

SOLUTION. Given $y(x) = (1-x^2)^{m/2} v(x)$. Then

$$y'(x) = -mx(1-x^2)^{m/2-1} v(x) + (1-x^2)^{m/2} v'(x),$$

$$\begin{aligned} y''(x) = -m(1-x^2)^{m/2-1} v(x) &+ m(m-2)x^2(1-x^2)^{m/2-2} v(x) \\ &- mx(1-x^2)^{m/2-1} v'(x) - mx(1-x^2)^{m/2-1} v'(x) \\ &\qquad + (1-x^2)^{m/2} v''(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &(1-x^2)y''(x) - 2xy'(x) \\ &\qquad = (1-x^2)^{m/2+1} v''(x) - 2(m+1)x(1-x^2)^{m/2} v'(x) \\ &\qquad\qquad + (1-x^2)^{m/2-1}\Big[m(m-2)x^2 - m(1-x^2) + 2mx^2\Big] v(x). \end{aligned}$$

The lhs is

$$\left[\frac{m^2}{1-x^2} - n(n+1)\right](1-x^2)^{m/2} v(x).$$

Now simplify to obtain (2_m). Let n be a fixed natural number. That (2_m) is satisfied by $D^m P_n$ is obviously true for $m = 0$. Assume for m and check for $m+1$ by substituting $D^m P_n$ in (2_m) and differentiating once to check the validity of (2_{m+1}).

Remark: By applying the solution to the last problem of Section 1.1, one can show that

$$\left(\frac{1-x}{1+x}\right)^{\pm m/2}$$

form a basis of the solution of equation (1) in the special case when $n = 0$. Clearly, the only bounded solution $D^m P_0$ is identically zero, if $m > 0$.

1.3. Frobenius method for regular singular equations

Problems.

(1) Attempt a power series solution around $x = 0$ for

$$x^2y'' - (1+x)y = 0.$$

Explain why the procedure does not give any nontrivial solutions.

SOLUTION. Write $y = \sum_{n\geq 0} a_n x^n$. Hence

$$n(n-1)a_n = a_n + a_{n-1},$$

or

$$a_n = \frac{1}{n^2 - n - 1} a_{n-1}.$$

This holds for all $n \geq 0$ with the convention $a_{-1} = 0$. This implies

$$a_0 = 0, a_1 = 0, \ldots, a_n = 0, \ldots.$$

Reason: The differential equation can be written as $y'' - \dfrac{1+x}{x^2} y = 0$ and the coefficient $-\dfrac{1+x}{x^2}$ does not have a power series around $x = 0$. In fact 0 is a regular singular point.

(2) Attempt a Frobenius series solution for the differential equation

$$x^2y'' + (3x-1)y' + y = 0.$$

Why does the method fail?

SOLUTION. Write

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

This implies $ra_0 = 0$ and hence $r = 0$ since $a_0 \neq 0$. Further with $r = 0$, we get

$$a_{n+1} = (n+1)a_n.$$

The radius of convergence of the resulting power series is 0. The method fails because $x = 0$ is a not a regular singular point.

(3) Locate and classify the singular points for the following differential equations. (All letters other than x and y such as p, λ, etc are constants.)

(a) Bessel equation:

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

SOLUTION. $x = 0$ is the only singular point and it is regular singular. We can write

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

and both 1 and $(x^2 - p^2)$ are real analytic everywhere, in fact polynomials.

(b) Laguerre equation:

$$xy'' + (1-x)y' + \lambda y = 0.$$

SOLUTION. $x = 0$ is the only singular point and it is regular singular.

(c) Jacobi equation:

$$x(1-x)y'' + (\gamma - (\alpha+1)x)y' + n(n+\alpha)y = 0.$$

SOLUTION. $x = 0$ and $x = 1$ are the only singular points and both are regular singular.

(d) Hypergeometric equation:

$$x(1-x)y'' + [c - (a+b+1)x)]y' - aby = 0.$$

SOLUTION. $x = 0$ and $x = 1$ are the only singular points and both are regular singular.

(e) Associated Legendre equation:

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$$

SOLUTION. $x = \pm 1$ are the singular points and both are regular singular.

(f)

$$xy'' + (\cot x)y' + xy = 0 \quad \text{on the interval} \quad (-\pi, \pi).$$

SOLUTION. $x = 0$ is the only singular point and it is not regular singular. We can write

$$y'' + \frac{\cot x}{x}y' + \frac{x^2}{x^2}y = 0.$$

Though the second coefficient x^2 is a polynomial, the first coefficient $\cot x$ cannot be expanded as a power series about $x = 0$.

(4) In Problem (3) above find the indicial equations corresponding to all the regular singular points.

SOLUTION. The basic method is as follows: If x_0 is a regular singular point of a second order linear ODE, first write it in the form

$$y'' + \frac{b(x)}{(x-x_0)}y' + \frac{c(x)}{(x-x_0)^2}y = 0.$$

Now the indicial equation for the purpose of expanding in fractional powers of $(x - x_0)$ is

$$r(r-1) + b(x_0)r + c(x_0) = 0.$$

(a) $x_0 = 0$ is the only singular point which is regular. $b(x) = 1,\ c(x) = x^2 - p^2$. The indicial equation is $r^2 - p^2 = 0$.

(b) $x_0 = 0$ is the only singular point which is regular. $b(x) = 1-x,\ c(x) = \lambda x$. The indicial equation is $r^2 = 0$.

(c) $x_0 = 0$ and $x_0 = 1$ are both regular singular points. For $x_0 = 0$,

$$b(x) = \frac{\gamma - (\alpha+1)x}{1-x} \quad \text{and} \quad c(x) = n(n+\alpha)x^2.$$

The indicial equation is $r(r-1) + \gamma r = 0$.
For $x_0 = 1$,

$$b(x) = \frac{\gamma - (\alpha+1)x}{-x} \quad \text{and} \quad c(x) = n(n+\alpha)(x-1)^2.$$

The indicial equation is $r(r-1) + (\alpha+1-\gamma)r = 0$.

(d) Again $x_0 = 0$ and $x_0 = 1$ are both regular singular points. For $x_0 = 0$,

$$b(x) = \frac{c - (a+b+1)x}{1-x} \quad \text{and} \quad c(x) = -abx^2.$$

The indicial equation is $r(r-1) + cr = 0$.
For $x_0 = 1$,

$$b(x) = \frac{c - (a+b+1)x}{-x} \quad \text{and} \quad c(x) = -ab(x-1)^2.$$

The indicial equation is $r(r-1) + (a+b+1-c)r = 0$.

(e) $x_0 = \pm 1$ are regular singular. For $x_0 = 1$,

$$b(x) = \frac{2x}{x+1} \quad \text{and} \quad c(x) = \frac{[n(n+1)(1-x^2) - m^2}{(1+x)^2}.$$

The indicial equation is $r(r-1) + r - m^2/4 = 0$, that is, $r^2 = m^2/4$.
By symmetry, the same is true for $x_0 = -1$.

(f) $x_0 = 0$ is the only singular point. It is not regular, so no indicial equation.

(5) Find two linearly independent solutions of the following differential equations.

(a) $x(x-1)y'' + (4x-2)y' + 2y = 0$.

SOLUTION. Observe that

$$x(x-1)y'' + (4x-2)y' + 2y = D^2[x(x-1)y].$$

Hence

$$x(x-1)y = Ax + B$$

is the general solution with A and B arbitrary constants, and

$$y = \frac{1}{x-1} \quad \text{and} \quad y = \frac{1}{x(x-1)}$$

are two linearly independent solutions. The first one has a singularity at 1, while the second has a singularity at both 1 and -1.
(One may also attempt a Frobenius series solution.)

(b) $(1-x^2)y'' - 2xy' + 2y = 0$.

SOLUTION. This is the Legendre equation for $p = 1$ which was solved in class.
Aliter: Guess that $y_1(x) = x$ is a solution. Employing the method of variation of parameters, let the second solution be $y_2(x) = xu(x)$. Substituting in the ODE and simplifying, we get

$$x(1-x^2)u'' = (4x^2-2)u'.$$

So

$$\frac{u''}{u'} = \frac{1}{1-x} - \frac{2}{x} - \frac{1}{1+x}.$$

So

$$u' = \frac{1}{x^2(1+x)(1-x)} = \frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}.$$

So

$$u(x) = -\frac{1}{x} + \frac{1}{2}\log\frac{1+x}{1-x}.$$

So

$$y_2(x) = -1 + \frac{1}{2}x\log\frac{1+x}{1-x}.$$

(c) $x^2y'' + x^3y' + (x^2-2)y = 0$.

SOLUTION. Applying the Frobenius method, gives the indicial equation $r(r-1) - 2 = 0$, which implies $r_2 = -1$ and $r_1 = 2$. The recursion is

$$[(n+r)(n+r-1)-2]a_n = -(n+r-1)a_{n-2}.$$

For $r = -1$,

$$a_n = -\frac{(n-2)}{n(n-3)}a_{n-2} \quad \text{for } n \geq 0,\ n \neq 0, 3$$

and with $a_{-1} = a_{-2} = 0$. Now $n = 1, 2$ yield $a_1 = 0 = a_2$. Thus we see that along with a_2, $a_4 = a_6 = \cdots = a_{2k} = \cdots = 0$. Further,

$$a_{2k+1} = (-1)^{k-1}\frac{3a_3}{(2k+1)2^{k-1}(k-1)!} \quad \text{for } k \geq 1.$$

Thus

$$y_1(x) = a_0/x + a_3x^2 \times \text{ an even power series}$$

is the form of the solution with a_0, a_3 arbitrary. Thus we already get the general solution. If we try $r = 2$ now we will get

$$x^2\sum_{k\geq 0} A_{2k}x^{2k}, \quad A_{2k} = (-1)^k\frac{3A_0}{(2k+3)2^k k!}$$

which is aready present in y_1 if we take $a_0 = 0$, $a_3 = A_0$.
Aliter: Guess that $y_1(x) = 1/x$ is a solution. By the method of variation of parameters, let the second solution be $y_2(x) = u(x)/x$.

On substitution, we find that u satisfies $xu'' + (x^2 - 2)u' = 0$ whence $u' = x^2 e^{-x^2/2}$. Therefore,

$$y_2(x) = \frac{1}{x} \int x^2 e^{-x^2/2} dx.$$

On expanding we see that it matches the power series part of the solution obtained above.

(d) $xy'' + 2y' + xy = 0$.

SOLUTION. $0 = xy'' + 2y' + xy = (xy)'' + (xy)$. Hence two linearly independent solutions are

$$\frac{\cos x}{x} \quad \text{and} \quad \frac{\sin x}{x}.$$

(6) While solving $x^2y'' + 2x(x-2)y' + 2(2-3x)y = 0$ by the Frobenius method around the point $x = 0$, which one of the following four cases will we encounter?

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer with **no** log term
- roots differing by a positive integer with log term

SOLUTION. The indicial equation is $(r-1)(r-4) = 0$, so roots differ by a positive integer. The recursion is $(r+n-1)(r+n-4)a_n = -2(n+r-4)a_{n-1}$. Since the problematic factor $(r+n-4)$ from the lhs (when $r = 1$ and $n = 3$) also appears in the rhs, it can be canceled, and we will not see any log term. Thus, we will be in the third alternative.

1.4. Bessel equation and Bessel functions

Problems.

(1) Using the indicated substitutions, reduce the following differential equations to the Bessel equation and find the general solution in term of the Bessel functions.

(a) $x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \quad (\lambda x = z)$

Solution. Let $\lambda x = z$ and $u(z) = y(x)$. Then

$$\frac{du}{dz} = \frac{1}{\lambda}\cdot\frac{dy}{dx} \quad \text{and} \quad \frac{d^2u}{dz^2} = \frac{1}{\lambda^2}\frac{d^2y}{dx^2}.$$

Hence

$$z\frac{du}{dz} = \frac{x}{d}ydx \quad \text{and} \quad z^2\frac{d^2u}{dz^2} = x^2\frac{d^y}{dz^2}.$$

The given equation transforms to

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - p^2)u = 0.$$

The general solution is $u(z) = c_1J_p(z) + c_2Y_p(z)$ which gives

$$y(x) = c_1J_p(\lambda x) + c_2Y_p(\lambda x).$$

(b) $xy'' - 5y' + xy = 0, \quad (y = x^3u).$

Solution. $y = x^3u(x)$. Therefore,

$$y' = 3x^2u + x^3u' \quad \text{and} \quad y'' = x^3u'' + 6x^2u' + 6xu.$$

So

$$\begin{aligned} 0 = xy'' - 5y' + xy &= (x^4u'' + 6x^3u' + 6x^2u) - 5(3x^2u + x^3u') + x^4u \\ &= x^4u'' + x^3u' + (x^4 - 9x^2)u. \end{aligned}$$

This implies $x^2u'' + xu' + (x^2 - 3^2)u = 0$. The general solution therefore, is $u(x) = c_1J_3(x) + c_2Y_3(x)$, or equivalently,

$$y(x) = x^3[c_1J_3(x) + c_2Y_3(x)].$$

(2) Show that

(a) $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$

Solution. From the expression for J_p,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\sum_{m\geq 0}\frac{(ix/2)^{2m}}{m!(m+\frac{1}{2})!}.$$

Also,

$$m!(m+\frac{1}{2})! = m!(m+\frac{1}{2})(m-\frac{1}{2})\dots\frac{1}{2}\Gamma(1/2) = \frac{(2m+1)!}{2^{2m+1}}\sqrt{\pi}.$$

This implies,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\sum_{m\geq 0}\frac{(ix/2)^{2m}2^{2m+1}}{(2m+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\sin x.$$

(b) $J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x$

SOLUTION. Similarly, using

$$m!(m - \frac{1}{2})! = \frac{(2m-1)!\sqrt{\pi}}{2^{2m}},$$

one can see that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

(c) $J_{3/2} = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

SOLUTION. From 2(a),

$$[x^{-1/2} J_{1/2}(x)]' = -x^{-1/2} J_{3/2}(x).$$

Therefore

$$\sqrt{\frac{2}{\pi}} \left[\frac{\sin x}{x} \right]' = -x^{-1/2} J_{3/2}(x).$$

This implies

$$J_{3/2}(x) = -\sqrt{\frac{2x}{\pi}} \left(\frac{\cos x}{x} - \frac{\sin x}{x^2} \right) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right).$$

(d) $J_{-3/2} = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$

SOLUTION. Similarly from 2(a) again,

$$[x^{-1/2} J_{-1/2}(x)]' = x^{-1/2} J_{-3/2}(x).$$

Therefore

$$J_{-3/2}(x) = \sqrt{\frac{2x}{\pi}} \left[\frac{\cos x}{x} \right]' = \sqrt{\frac{2x}{\pi}} \left[-\frac{\sin x}{x} - \frac{\cos x}{x^2} \right] = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right).$$

(3) For an integer n show that $J_n(x)$ is an even (resp. odd) function if n is even (resp. odd).

SOLUTION. It is clear for nonnegative integers directly from

$$J_n(x) = \left(\frac{x}{2} \right)^n \sum_{m \geq 0} \frac{(ix/2)^{2m}}{m!(m+n)!}.$$

We only have to observe that for $n < 0$, the terms corresponding to $0 \leq m \leq |n| - 1$ will vanish due to the presence of the factorials of negative integers in the denominators.

(4) Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$ and one zero of $J_{n-1}(x)$.

SOLUTION. Let $J_n(a) = J_n(b) = 0$, where $0 < a < b$ are consecutive zeroes of J_n. Then $x^{\pm n} J_n(x) = 0$ for $x = a,\ b$. By Rolles' theorem there exist $c_{\pm} \in (a, b)$ such that $[x^{\pm n} J_n(x)]'(c_{\pm}) = 0$. This implies $\pm x^{\pm n} J_{n \mp 1}(c_{\pm}) = 0$. (We take corresponding signs only.) In other words,

$J_{n-1}(c_+) = 0$ and $J_{n+1}(c_-) = 0$. If possible let there be $c < d$ in (a, b) such that $J_{n+1}(c) = 0 = J_{n+1}(d)$. Then there is another $k \in (c, d)$ where $J_{[(n+1)-1]}(k) = 0$. This contradicts that a and b are consecutive zeroes of J_n. Therefore, J_{n+1} vanishes exactly once in (a, b). Similarly, J_{n-1} vanishes exactly once in (a, b).
Remark: n need not be an integer in this problem.

(5) Show the following.
(a) $J_3 + 3J_0' + 4J_0''' = 0$.

SOLUTION. The relation $J_0' = \dfrac{1}{2}(J_{-1} - J_1) = -J_1$ implies that

$$J_3 + 3J_0' + 4J_0''' = J_3 - 3J_1 - 4J_1'' = J_3 - 3J_1 - 2(J_0 - J_2)'$$
$$= J_3 - 3J_1 + 2J_1 + (J_1 - J_3) = 0.$$

(b) $J_2 - J_0 = aJ_c''$ find a and c.
SOLUTION. $J_2 - J_0 = -2J_1' = +2J_0''$. Thus $a = 2$ and $c = 0$.

(c) $\int J_{p+1}dx = \int J_{p-1}dx - 2J_p$.
SOLUTION. $2J_p' = J_{p-1} - J_{p+1}$. This implies $2J_p = \int J_{p-1} - \int J_{p+1}$ (indefinite integrals) and the result follows.

(6) If y_1 and y_2 are any two solutions of the Bessel equation of order p, then show that $y_1y_2' - y_1'y_2 = c/x$ for a suitable constant c.

SOLUTION. Let $W(y_1, y_2) = y_1y_2' - y_1'y_2$. This is called the Wronskian. Then

$$W' = y_1y_2'' - y_1''y_2$$
$$= -y_1(y_2'/x + (x^2 - p^2)y_2^2/x^2) + (y_1'/x + (x^2 - p^2)y_1^2/x^2)y_2 = -\frac{W}{x}.$$

Integrating, $\log W = -\log x + \log c$ or $W = c/x$.

(7) Show that

$$\int x^\mu J_p(x)dx = x^\mu J_{p+1}(x) - (\mu - p - 1)\int x^{\mu-1}J_{p+1}(x)dx.$$

SOLUTION.

$$\int x^\mu J_p(x)dx = \int x^{\mu-p-1}(x^{p+1}J_p(x))dx = \int x^{\mu-p-1}(x^{p+1}J_{p+1}(x))'dx$$
$$= x^{\mu-p-1}x^{p+1}J_{p+1} - \int(\mu - p - 1)x^{\mu-p-2}x^{p+1}J_{p+1}dx$$
$$= x^\mu J_{p+1} - (\mu - p - 1)\int x^{\mu-1}J_{p+1}dx.$$

In one of the steps, we used integration by parts.

(8) Expand the indicated function in Fourier-Bessel series over the given interval and in terms of the Bessel function of given order. (The Bessel expansion theorem applies in each case.)

(a) $f(x) = 1$ over $[0, 3]$, $p = 0$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(0)}} c_z J_0(zx/3), \quad 0 \le x \le 3,$$

where

$$\begin{aligned} c_z &= \frac{2}{9J_1(z)^2} \int_0^3 f(x) J_0(zx/3) x dx \\ &= \frac{2}{z^2 J_1(z)^2} \int_0^z t J_0(t) dt \quad \text{(on setting } x = 3t/z,\ f(x) = 1) \\ &= \frac{2}{z^2 J_1(z)^2} z J_1(z) = \frac{2}{z J_1(z)} \quad \text{(since } xJ_0(x) = [xJ_1(x)]'). \end{aligned}$$

Sample values from the tables

$z \in Z^{(0)}$	2.405	5.52	8.65	11.79	14.93
$J_1(z)$	0.52	-0.34	0.27	-0.23	0.21
c_z	1.60	-1.07	0.86	-0.74	0.64

Explicitly, substituting the first few values, the Bessel series is

$$1 \approx 1.60 J_0(0.80x) - 1.07 J_0(1.84x) + 0.86 J_0(2.88x) - \dots$$

for $0 \le x \le 3$.

(b) $f(x) = x$ over $[0, 1]$, $p = 1$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(1)}} c_z J_1(zx), \quad 0 \le x \le 1,$$

where

$$c_z = \frac{2}{J_0(z)^2} \int_0^1 f(x) J_1(zx) x dx$$

On setting $x = t/z = f(x)$

$$= \frac{2}{z^3 J_0(z)^2} \int_0^z t^2 J_1(t) dt$$

Integrating by parts using $J_1 = -J_0'$

$$\begin{aligned} &= \frac{2}{z^3 J_0(z)^2} [-z^2 J_0(z) + \int_0^z 2t J_0(t) dt] \\ &= \frac{2}{z^3 J_0(z)^2} [-z^2 J_0(z) + 2z J_1(z)] \quad \text{(since } xJ_0(x) = [xJ_1(x)]') \\ &= \frac{-2}{z J_0(z)}, \quad z \in Z^{(1)}. \end{aligned}$$

Sample values from the tables

$z \in Z^{(1)}$	3.83	7.02	10.17	13.32
$J_0(z)$	-0.40	0.30	-0.25	0.22
c_z	1.31	-0.95	0.79	-0.68

Explicitly, substituting the first few values, the Bessel series is

$$x \approx 1.31J_1(3.83x) - 0.95J_1(7.02x) + 0.79J_1(10.17x) - 0.68J_1(13.32x) + \dots$$

for $0 \le x \le 1$.

(c) $f(x) = x^3$ over $[0, 3]$, $p = 1$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(1)}} c_z J_1(zx/3), \quad 0 \le x \le 3,$$

where

$$c_z = \frac{2}{9J_0(z)^2} \int_0^3 f(x) J_1(zx/3) x dx$$

On setting $x = 3t/z,\ f(x) = x^3 = 27t^3/z^3$

$$= \frac{18}{z^4 J_0(z)^2} \int_0^z t^4 J_1(t) dt$$

Integrating by parts using $x^2 J_1 = [x^2 J_2]'$

$$= \frac{18}{z^4 J_0(z)^2} [-z^2(z^2 J_2(z)) - \int_0^z 2t.t^2 J_2(t) dt]$$
$$= \frac{18}{z^4 J_0(z)^2} [-z^4 J_0(z) - 2z^3 J_3(z)]$$

since $x^3 J_2(x) = [x^3 J_3(x)]'$ and $J_2(z) = -J_0(z);\ z \in Z^{(1)}$

$$= \frac{-18}{J_0(z)} - \frac{36 J_3(z)}{z J_0(z)^2}, \quad z \in Z^{(1)}.$$

Further,

$$J_1(z) + J_3(z) = \frac{4}{z} J_2(z) = -\frac{4}{z} J_0(z).$$

Hence $J_3(z) = -4J_0(z)/z$ and

$$c_z = \frac{18}{J_0(z)} \left[\frac{8}{z^2} - 1 \right];\ z \in Z^{(1)}.$$

Sample values from the tables

$z \in Z^{(1)}$	3.83	7.02	10.17	13.32
$J_0(z)$	-0.40	0.30	-0.25	0.22
c_z	??	??	??	??

(d) $f(x) = x^2$ over $[0, 2]$, $p = 2$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(2)}} c_z J_1(zx/2), \quad 0 \le x \le 2,$$

where

$$c_z = \frac{2}{4J_1(z)^2}\int_0^2 f(x)J_2(zx/2)xdx$$

On setting $x = 2t/z,\ f(x) = x^2 = 4t^2/z^2$

$$\begin{aligned}
&= \frac{8}{z^4 J_1(z)^2}\int_0^z t^3 J_2(t)dt \\
&= \frac{8}{z^4 J_1(z)^2} z^3 J_3(z) \quad (\text{Since } x^3 J_2 = [x^3 J_3]') \\
&= \frac{8J_3(z)}{zJ_1(z)^2} = \frac{-8}{zJ_1(z)}, \qquad z \in Z^{(2)}.
\end{aligned}$$

The last equality is due to

$$J_3(z) + J_1(z) = \frac{4}{z}J_2(z) = 0.$$

Sample values from the tables

$z \in Z^{(2)}$	5.1356	8.4172	11.6198	14.7960
$J_1(z)$	-0.3397	0.2713	-0.2324	0.2065
c_z	4.5857	-3.5033	2.9625	-2.6183

Explicitly, substituting the first few values, the Bessel series is

$$x^2 \approx 4.5857J_1(2.5678x) - 3.5033J_1(4.2086x) + 2.9625J_1(5.8099x) - 2.6183J_1(7.3980x) + \ldots$$

for $0 \le x \le 2$.

(e) $f(x) = \sqrt{x}$ over $[0, \pi]$, $p = \frac{1}{2}$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(1/2)}} c_z J_{1/2}(zx/\pi), \quad 0 \le x \le \pi,$$

where

$$c_z = \frac{2}{\pi^2 J_{-1/2}(z)^2}\int_0^\pi f(x)J_{1/2}(zx/\pi)xdx.$$

Now

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad Z^{(1/2)} = \{\pi, 2\pi, 3\pi, \ldots\}.$$

Hence writing c_n for $z = n\pi$, we have

$$c_n = \frac{2}{\pi^2 J_{-1/2}(n\pi)^2}\int_0^\pi \sqrt{x}\sqrt{\frac{2}{n\pi x}}\sin nx.xdx = \frac{2n\pi^2}{\pi^2.2}\int_0^\pi \sqrt{\frac{2}{n\pi}}x\sin nxdx.$$

In the last equality we have evaluated $[J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}\cos^2 x$ at $x = n\pi$.

Thus

$$c_n = \sqrt{\frac{2n}{\pi}}\int_0^\pi x\sin xdx = \sqrt{\frac{2n}{\pi}}\left(\left|x\frac{-\cos x}{n}\right|_0^\pi + \int_0^\pi \frac{\cos nx}{n}dx\right) = \frac{2}{n\pi}(-1)^{n+1}.$$

Hence

$$\sqrt{x} = \sqrt{2\pi} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} J_{1/2}(nx).$$

Remark: Putting $J_{1/2}(nx) = \sqrt{\frac{2}{n\pi x}} \sin nx$ and simplifying, we get

$$x = 2 \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin nx,$$

the Fourier sine series of x over $[0, \pi]$.

(9) Show Schlömilch's formula

$$\exp\left(\frac{tx}{2} - \frac{x}{2t}\right) = \sum_{-\infty}^{\infty} J_n(x) t^n.$$

Use this formula to show that

$$J_0^2 + 2 \sum_{n=1}^{\infty} J_n^2 = 1.$$

Deduce that $|J_0| \leq 1$ and $|J_n| \leq \frac{1}{\sqrt{2}}$.

SOLUTION.

$$e^{tx/2 - x/2t} = e^{tx/2} e^{-x/2t} = \Big[\sum_{k\geq 0} \frac{(tx)^k}{2^k k!}\Big] \Big[\sum_{j\geq 0} \frac{(-1)^j x^j}{2^j t^j j!}\Big].$$

For $n \in \mathbb{Z}$, the coefficient of t^n in the above is

$$\sum_{k-j=n} \frac{(-1)^j x^{j+k}}{2^{j+k} j! k!} = \sum_{j\geq 0} \frac{(-1)^j x^{2j+n}}{2^{2j+n} (j+n)! j!} = \left(\frac{x}{2}\right)^n \sum_{j\geq 0} \frac{(ix/2)^j}{j!(j+n)!} = J_n(x).$$

This proves Schlömilch's formula. Now replace t by $-t$ and take product to get

$$1 = \left[\sum_{-\infty}^{\infty} J_n(x) t^n\right] \left[\sum_{-\infty}^{\infty} (-1)^m J_m(x) t^m\right] = \left[\sum_{-\infty}^{\infty} J_n(x) t^n\right] \left[\sum_{-\infty}^{\infty} J_{-m}(x) t^m\right].$$

This shows that $J_0^2 + 2 \sum_{n=1}^{\infty} J_n^2 = 1$, along with a sequence of identities:

$$\sum_{j\in\mathbb{Z}} J_{m+j} J_m = 0 \text{ for } m \in \mathbb{Z} \setminus \{0\}.$$

(Just look at the coefficients of various powers of t.) The bounds on $|J_n|$ are now obvious.

(10) Show that

$$\begin{aligned}
\int J_0(x)dx &= J_1(x) + \int \frac{J_1(x)dx}{x} \\
&= J_1(x) + \frac{J_2(x)}{x} + 1.3\int \frac{J_2(x)dx}{x^2} \\
&= J_1(x) + \frac{J_2(x)}{x} + \frac{1.3J_3(x)}{x^2} + 1.3.5\int \frac{J_3(x)dx}{x^3} \\
&. \\
&. \\
&. \\
&= J_1(x) + \frac{J_2(x)}{x} + \frac{1.3J_3(x)}{x^2} + \cdots + \frac{1.3.5\ldots(2n-3)J_n(x)}{x^{n-1}} \\
&\qquad + 1.3.5\ldots(2n-1)\int \frac{J_n(x)dx}{x^n}
\end{aligned}$$

SOLUTION. We use induction on n.

$$\begin{aligned}
\int \frac{J_n(x)dx}{x^n} &= \int \frac{x^{n+1}J_n(x)dx}{x^{2n+1}} = \int \frac{[x^{n+1}J_{n+1}(x)]'dx}{x^{2n+1}} \\
&= x^{-2n-1}[x^{n+1}J_{n+1}(x)] - \int (-2n-1)x^{-2n-2}[x^{n+1}J_{n+1}(x)]dx \\
&= \frac{J_{n+1}(x)}{x^n} + (2n+1)\int \frac{J_{n+1}dx}{x^{n+1}}.
\end{aligned}$$

Substituting in the n-th step, assumed to be valid by induction hypothesis, we get the validity of the $(n+1)$-th step.