MA 205 Complex Analysis: Cauchy Integral Formula and its Beautiful Consequences

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August 6, 2018

Recall

We saw several versions of Cauchy's theorem in the last class.

Theorem (Cauchy's theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let C be a simple closed contour and let f be holomorphic on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.

Theorem

(Cauchy's theorem for simply connected domain) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then $\int_C f(z)dz = 0$*

Theorem

(More General form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. If γ and γ' are two closed contours in Ω which can be "continuously deformed" into each other and f is a holomorphic function on Ω, then $\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$.*

Examples

Let C_1 be the line segment joining -1 and i and let C_2 be the arc of the unit circle with initial point -1 and end point i.

$$C_1 : z_1(t) = -1 + (1+i)t = (-1+t) + it : 0 \leq t \leq 1$$

$-C_2 : z_2(t) = e^{it} : \pi/2 \leq t \leq \pi.$

Then

$$\begin{aligned} \int_{C_1} |z|^2 dz &= \int_0^1 ((-1+t)^2 + t^2)(1+i)dt \\ &= (1+i)\int_0^1 (2t^2 - 2t + 1)dt \\ &= \frac{2(1+i)}{3}. \end{aligned}$$

$$\int_{-C_2} |z|^2 dz = \int_{\pi/2}^{\pi} ie^{it} dt = -1 - i.$$

Hence the results do not agree and this is consistent with the fact that $|z|^2$ is not holomorphic.

Cauchy integral formula

Theorem (Cauchy Integral Formula)

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let f be holomorphic everywhere within and on a simple closed contour γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

Recall that $\int_C \frac{1}{z-z_0} dz = 2\pi\imath$ for any positivily oriented curve C with z_0 is interior to C.
If $|f| < M$ on C and $\gamma : [a,b] \to \mathbb{C}$ is a parametrization of C, then

$$|\int_C f(z)dz| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq M \int_a^b |\gamma'(t)|dt = M\ell(C).$$

Proof

Proof: We need to show that

$$\int_\gamma \frac{f(z)}{z-z_0}dz = \int_\gamma \frac{f(z_0)}{z-z_0}dz,$$

since the latter integral is $2\pi\imath \cdot f(z_0)$. Thus, we need to show that

$$\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz = 0.$$

Proof cont...

Since f is continuous at z_0, given $\epsilon > 0$, there is $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Choose $r < \delta$ and consider the circle $C_r : |z - z_0| = r$. By Cauchy's theorem applied to γ and C_r, we get

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

Now,

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon.$$

Thus, $\left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz \right|$ can be made arbitrarily small; i.e., it is zero.