Practice Questions: CRLB and MLE
EE 223: Data Analysis and Interpretation
EE Department, IIT Bombay

1. Let $Y_1, ..., Y_n$ be random samples from the distribution:

$$f_Y(y) = \frac{2y}{\theta^2} \qquad 0 < y < \theta$$

   Can the Cramer-Rao bound be applied to the estimator $\hat{\theta} = \frac{3}{2}\bar{Y}$?

2. Consider the problem of estimating the mean μ of a Gaussian random variable from n i.i.d. samples. The variance σ^2 is known. What is the minimum possible variance of an unbiased estimator? Does the sample mean achieve this bound? Repeat for estimating μ^2.

3. Consider the problem of estimation of the phase ϕ of a signal embedded in additive Gaussian noise, $N(0, \sigma^2)$.

$$x[n] = Acos(2\pi fn + \phi) + w[n] \qquad n = 0, 1, ..., N-1$$

   Here the amplitude A and the frequency f of the sinusoid are known. Find the Cramer-Rao bound for variance of an unbiased estimator. Repeat for estimation of the frequency f while the phase ϕ is known.

4. Suppose $X_1, X_2, ..., X_n$ are i.i.d. Bernoulli samples. Find MLEs of the mean and variance.

5. Let X_i, for $i = 1, ..., n$, be n i.i.d. random variables having a uniform distribution in $[0, \theta]$. Find the MLE and UMVUE for θ. Which one among MLE and UMVUE has lower variance? What about the MSE? How does the problem change if the pdf is uniform in the open interval $(0, \theta)$?

6. Consider one sample X from the distribution:

$$f_X(x) = \begin{cases} cx(\theta - x) & \text{for} \quad 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find c such that $f_X(.)$ is a valid pdf.
   (b) Find the MLE and UMVUE for θ, θ^2 and $\sqrt{\theta}$. Compare the variance in all 3 cases. Is the UMVUE always the least variance estimator?

7. The Pareto distribution has been used as a model for a density function with a slowly decaying tail:

$$f(x|x_0, \theta) = \frac{\theta x_0^\theta}{x^{\theta+1}} \qquad x \geq x_0 \quad \theta > 1$$

   Assume that $x_0 > 0$ is given and that $X_1, X_2, ..., X_n$ are i.i.d. samples. Find the MLE for θ.

8. A 1-bit signal X taking value either 0 or 1 is transmitted via a noisy channel. The noise of the channel can be modelled as a zero-mean Gaussian random variable with variance σ^2. Design a receiver that outputs the MLE of the signal. Repeat for the case when the noise variance is σ_1^2 if the signal is 1 and is σ_0^2 if the signal is 0.

9. Suppose that $X_1, ..., X_n$ are normal with mean μ_1 ; $Y_1, ..., Y_n$ are normal with mean μ_2 ; and $W_1, ..., W_n$ are normal with mean $\mu_1 + \mu_2$. Assuming that all $3n$ random variables are independent with a common variance, find the MLEs for μ_1 and μ_2.

10. It is known that the lifetime of a bulb is an exponential random variable. Its mean is to be estimated through a burn-in test as follows. n independent samples of the bulb are turned on and the test continues till r, $r < n$, of these fail. Let x_k be the time of the k-th failure for $k = 1, ..., r$. Note that $(n - r)$ bulbs have not failed when the experiment stops at x_r. This means that if X_k is the random k-th failure time, then $X_k > x_r$ for $k = r + 1, ...n$ and it is not observed. Find the MLE for λ, the mean lifetime of a bulb. This should involve all the n failure times.