Tutorial VI

1. Locate and classify the singularities of the following :

   a) $\sin(1/z)$;

   b) $\frac{z^2+z+1}{z^3-11z+13}$;

   c) $\frac{1}{\sin(1/z)}$;

   d) $\tan(1/z)$.

2. Find the poles and their orders of the functions

   (i) $\frac{1}{(z^4+1)^2}$, (ii) $\frac{1}{z^2+z-1}$.

3. Find Laurent expansions for the function $f(z) = \frac{2(z-1)}{z^2-2z-3}$ valid on the following sets: (i) $|z| < 1$, (ii) $1 < |z| < 3$, (iii) $|z| > 3$.

4. Let Ω be a domain in $\mathbb{C}$. Suppose that $z_0 \in \Omega$ is an isolated singularity of $f(z)$ and $f(z)$ is bounded in some punctured neighborhood of z_0 (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $0 < |z - z_0| < r$). Show that $f(z)$ has a removable singularity at z_0.

5. If $f(z) = \frac{p(z)}{q(z)}$ where p, q are differentiable with $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, then show that

$$\mathrm{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

6. Calculate residue at each singular point of the functions

   (i) $\frac{1}{z^2 \sin z}$, (ii) $\frac{1}{z(1-z)^2}$, (iii) $(\frac{z+1}{z-1})^3$.