MA 205 Complex Analysis: Power Series

B.K. Das
IIT Bombay

July 23, 2018

Introduction

In the last lecture, you saw holomorphic functions in some detail. If $f = u + \imath v$ is holomorphic in Ω, then (i) both u and v satisfy CR equations, and (ii) $f(x, y) = (u(x, y), v(x, y))$ is real differentiable. We also saw that though neither (i) nor (ii) is sufficient to guarantee holomorphicity, both (i) and (ii) together do guarantee holomorphicity of f. Today, we'll discuss the so called <u>harmonic</u> and <u>analytic</u> functions

Harmonic Functions

A real valued function $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if it is twice continuously differentiable and satisfies $u_{xx} + u_{yy} = 0$ on U. If $f = u + \imath v$ is holomorphic on Ω, then both u and v are harmonic on Ω. Indeed, CR equations tell us that

$$u_x = v_y \ \& \ u_y = -v_x.$$

Thus,

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0.$$

Similarly for v.

Harmonic Functions

Suppose u and v are harmonic functions on Ω. We say that v is a harmonic conjugate of u if $f = u + \imath v$ is holomorphic in Ω.

Example: $v(x, y) = 2xy$ is a harmonic conjugate of $x^2 - y^2$ in any domain. Indeed, $f(z) = z^2$ is holomorphic everywhere.

Note that v is a harmonic conjugate of u does not mean that u is a harmonic conjugate of v! In fact:

Exercise: If u and v are harmonic conjugates of each other, show that they are constant functions.

Here's a general method to find a harmonic conjugate: given a harmonic u, find u_x. Equate $v_y = u_x$ and integrate wrt y. You'll get $v(x,y) = \int u_x dy + \phi(x)$. Now $v_x = \ldots + \phi'(x)$. Equate this to $-u_y$ to find $\phi(x)$. That gives you v.
This might give you an impression that you can always find a harmonic conjugate, but this is not so.

Unfortunately this method fails in general. Try and think of the reason !

But if Ω is "nice", then every harmonic u on Ω has a harmonic conjugate. Conversely, if every harmonic u on Ω has a harmonic conjugate, then Ω has to be "nice". Thus, the question in analysis: "does every harmonic function has a harmonic conjugate?" is answered by geometry: "answer depends on the shape of the domain". It's relevant at this point to recall from MA 105 that curl of grad is always zero but curl free is certainly a grad of something only when the domain is "nice" (for example $\mathbb{C}$ or a disc in $\mathbb{C}$). Remember this !

Polynomials

Now, we'll discuss the so called analytic functions. To warm up, let's first look at the simplest of all functions. What's the most trivial function? $f(z) = a_0$, i.e, constant functions. The next easiest class of functions are polynomials:
$f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_i \in \mathbb{C}$. The same polynomial $f(z)$ can be expanded along any point z_0. That is, $f(z)$ can be written as

$$b_0 + b_1(z - z_0) + \ldots + b_n(z - z_0)^n.$$

A smarter way to calculate b_i would be $b_i = \frac{f^{(i)}(z_0)}{i!}$.

A polynomial, by definition, is a "finite" polynomial; i.e., it comes with a finite degree. As the next class of functions, we consider functions defined by their power series; i.e.,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

or more generally, $\sum_{i=0}^{\infty} a_i (z - z_0)^i$. Of course one has to be careful; there are convergence issues. For example, $f(z) = 1 + z + z^2 + \dots$ makes sense for all z such that $|z| < 1$, but not when $|z| > 1$. (Why?)

It's a beautiful fact that the radius of convergence exists for any power series; i.e., there exists R such that $\sum_{i=0}^{\infty} a_i(z - z_0)^i$ converges when $|z - z_0| < R$, and diverges when $|z - z_0| > R$. In other words, the radius of convergence is the largest R such that the given power series converges inside a disc of radius R. We'll soon give a formula for R in terms of the coefficients of a given power series.

We write $a = \sum_{n=1}^{\infty} a_i$, $a_i \in \mathbb{C}$, if $\lim_{n\to\infty} s_n = a$ where $s_n = a_1 + \ldots + a_n$. The series $\sum_{n=1}^{\infty} a_i$ is said be <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_i|$ is convergent.

<u>Exercise</u>: 1. Absolute convergence $\Rightarrow$ convergence.

2. (Comparison Test) If $\sum_{n=1}^{\infty} b_i$ is absolutely convergent, and if $|a_i| \leq |b_i|$ for all large enough i, then $\sum_{n=1}^{\infty} a_i$ is absolutely convergent.

Easy Observation: If a power series $\sum_{i=0}^{\infty} a_i z^i$ converges for $z = z_0$, then it converges absolutely for any z with $|z| < |z_0|$.

Recall **Supremum**: Let $\{x_n\}$ be a sequence of real numbers. We say that a real number M is the supremum of this sequence if every term of the sequence is less than or equal to M and there exists terms of the sequence which are arbitrarily close to M. Equivalently it is the smallest real number having the property that it is greater than or equal to all the terms of the sequence. The supremum may or may not be equal to any of the terms of the sequence.

Limit Supremum

Upper Limit/ Limit Supremum: For a sequence of real numbers $x_1, x_2, \ldots$, let y_n be the supremum of the set $\{x_n, x_{n+1}, \ldots\}$. Then the sequence $y_1, y_2, \ldots$ is a monotonically decreasing sequence. The limit of $\{y_n\}$ is called the upper limit (also called limit superior, denoted lim sup) of the sequence $\{x_i\}$. It can be ∞. If limit exists, then the upper limit coincides with the usual limit.

Examples:

1. the sequence $1, 2, 3, \ldots$ has upper limit ∞.
2. the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ has upper limit 0.
3. the sequence $1, -1, 1, -1, \ldots$ has upper limit 1.

Tutors

Tutorial Batch	Tutor	Venue
T1	Deep Karman Pal Singh	LT1
T2	Archiki Prasad	LT2
T3	Shourya Pandey	LT3
T4	Shubhang Bhatnagar	LT4
T5	Reebhu Bhattacharyya	LT5

Theorem (Cauchy's Root Test)

For a series $\sum_{n=1}^{\infty} a_i$ *of complex numbers, let* $C = \limsup_{i\to\infty} \sqrt[i]{|a_i|}$*. Then the series converges absolutely if* $C < 1$ *and it diverges if* $C > 1$*.*

The test is indecisive for $C = 1$

Proof: If $C < 1$, then we can choose a k such that $\sqrt[i]{|a_i|} < k < 1$ after a stage (by the definition of the upper limit). Thus, after a stage, $|a_i| < k^i < 1$.

Now $\sum_{n=1}^{\infty} k^i$ converges absolutely, and therefore $\sum_{n=1}^{\infty} a_i$ is absolutely convergent (by comparison test). If $C > 1$, then for infinitely many i, $\sqrt[i]{|a_i|} > 1$. Hence $|a_i|$ is bigger than 1 for infinitely many i.

Thus, $\lim_{i \to \infty} a_i \neq 0$. So $\sum_{n=1}^{\infty} a_i$ diverges. (Why?)

Theorem (Ratio Test)

For a series $\sum_{n=1}^{\infty} a_i$, *let* $L = \limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. *Then, if* $L < 1$, *the series converges absolutely. The series diverges if there exists* N *such that* $\left|\frac{a_{i+1}}{a_i}\right| > 1$ *for* $i \geq N$.

Remark $L > 1$ in the above test doesn't imply that the series diverges. (Exercise !)

Proof: Let $L < 1$. Let r be such that $L < r < 1$. Then after a stage, say for $i \geq N$, $|a_{i+1}| < r|a_i|$. So $|a_{i+k}| < r^k|a_i|$. Now

$$\sum_{n=1}^{\infty}|a_i| = \sum_{0}^{N}|a_i| + \sum_{N+1}^{\infty}|a_i| = \sum_{0}^{N}|a_i| + \sum_{1}^{\infty}|a_{N+i}|$$

$$< \sum_{0}^{N}|a_i| + |a_N|\sum_{1}^{\infty}r^i = \sum_{0}^{N}|a_i| + |a_N|\frac{r}{1-r} < \infty.$$

In the other case, $|a_{i+1}| > |a_i|$ for all large enough i, so $\lim_{n\to\infty} a_i \neq 0$. Therefore the series diverges.

Theorem (Existence of Radius of Convergence)

For the power series $\sum_{i=1}^{\infty} a_i(z-z_0)^i$, let $R = \dfrac{1}{\limsup\limits_{i\to\infty} \sqrt[i]{|a_i|}}$. Then the power series converges absolutely if $|z-z_0| < R$ and diverges if $|z-z_0| > R$.

Proof: Apply root test.

Remark: (i) If $\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$ exists, then by applying the ratio test $R=\lim_{i\to\infty}\left|\frac{a_i}{a_{i+1}}\right|$.

(ii) If a series converges by the ratio test, then it converges by the root test as well. But not conversely. Thus the root test is better than the ratio test. But the ratio test is often easier to use whenever it succeeds. In fact:

$$\limsup_{i\to\infty}\sqrt[i]{|a_i|}\le\limsup_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$$

Examples:

1. $\sum_n \frac{z^n}{n!}$. Apply ratio test. $\lim_{i\to\infty}\left|\frac{a_i}{a_{i+1}}\right| = \lim_{i\to\infty} i = \infty$; i.e., the series converges everywhere.
2. $z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$ Radius of convergence is 1. Both the tests apply here.
3. $\frac{1}{2} + \frac{1}{3}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{3}\right)^2 z^3 + \ldots$. Check that the ratio test fails. Apply root test to show that the radius of convergence is $\frac{1}{\sqrt{2}}$.