MA 205 Complex Analysis

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Zero's of holomorphic functions

Theorem

If f is a non-zero holomorphic function on a domain Ω, then each zero of f has finite multiplicity; i.e., there exists m such that $f(z) = (z - z_0)^m g(z)$ locally with $g(z_0) \neq 0$.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a domain Ω. Let z_0 be a point in Ω at which f vanishes. Then either f is identically zero or there exists a neigborhood of z_0 in which f has no other zero.

Theorem

Zeroes of a holomorphic function are isolated.

Zeros are Isolated

Theorem (Identity theorem)

If f and g are holomorphic in Ω, then $f \equiv g$ iff there exists a non-constant sequence $\{z_n\} \subseteq \{z \in \Omega \mid f(z) = g(z)\}$ such that $\lim_{n\to\infty} z_n = z_0 \in \Omega$.

Example: You cannot have two distinct holomorphic functions on a domain containing 0 which agree on all $\frac{1}{n}$.
Example: $\exp(z)$ is the only holomorphic function which agrees with e^x on the real line. Similarly, for $\sin z, \cos z$ etc.
Example: Identities like $\sin^2 z + \cos^2 z = 1$ follow without any further computation since they hold true over reals.
Example: $\exp(z+w) = \exp(z)\exp(w)$ now has another proof since this is true over reals!

Holomorphic function with pescribed zeros

One could ask if the converse is true; namely given a discrete set of points going to infinity does there exist a holomorphic function with the property that it vanishes exactly on this set ? Indeed this holds. In fact one can find a holomorphic function vanishing exactly on any discrete set with prescribed vanishing multiplicities at each of those points. This is called the **Weierstrass product theorem**.

(Karl Weierstrass (1815-1897) was a very important mathematician of the 19th century. He was responsible for giving rigorous foundations to analysis. The precise $\epsilon - \delta$ definition of limit was formulated by Weierstrass).

Failure of the property in the C^∞ case

Note that this property fails for real differentiable functions. It however works equally well for real analytic functions as well (same proof as above). The function

$$f(x) = e^{-1/x} \text{ for } x > 0$$

$$= 0 \text{ for } x < 0$$

is infinitely differentiable but vanishes along the entire negative real line.

Once we prove that holomorphic functions are analytic, the above fact will hold for holomorphic functions as well.