MA 205 Complex Analysis: Laurent Seires and Cauchy Residue Theorem

B.K. Das
IIT Bombay

August 23, 2018

Recall

Recall that in the last class we have studied isolated singularities of a holomorphic functions. They are of three different type; namely, Removable singularity, Pole and Essential Singularity. We have also seen different methods to determine these singularities. Today, we will find a series expansion of a holomorphic function around it's isolated singular points.

Laurent Series

We would like to expand a holomorphic function around an isolated singular point, much like the power series expansion of a holomorphic function around a point. A laurent series expansion around a point P is an expression of the form

$$\sum_{-\infty}^{\infty} a_j(z-p)^j.$$

Such a laurent series converges if both the series $\sum_0^\infty a_j(z-p)^j$ and $\sum_1^\infty a_{-j}(z-p)^{-j}$ converges. A Laurent series typically converges on an annulus $\{z : r < |z| < R\}$ for some $0 \leq r < R$.

Recall how we derived the power series representation of a holomorphic function on a disc centered around z_0. We used

$$f(z) = \frac{1}{2\pi\imath} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw,$$

and manipulated $\frac{1}{w-z}$ as

$$\frac{1}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}}.$$

Now suppose z_0 is an isolated singularity for f. Consider an annulus with radii $R > r$ centered at z_0 such that f is holomorphic on $\overline{D(z_0, R)} \setminus \{z_0\}$. CIF takes the form:

$$f(z) = \frac{1}{2\pi\imath} \int_{|w-z_0|=R} \frac{f(w)}{w-z} dw - \frac{1}{2\pi\imath} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw.$$

The first integral gives rise to $\displaystyle\sum_{n=1}^{\infty} a_n (z-z_0)^n$ with

$$a_n = \frac{1}{2\pi\imath} \int_{|w-z_0|=R} \frac{f(w)}{(w-z_0)^{n+1}} dw,$$

exactly as before.

In the second integral, write

$$\frac{-1}{w-z} = \frac{1}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}}.$$

Note that $\left| \frac{w-z_0}{z-z_0} \right| < 1$ for all w with $|w-z_0| = r$.

Expand to get $\displaystyle\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ with :

$$a_{-n} = \frac{1}{2\pi\imath} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{-n+1}} dw.$$

We write both together as $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. This is the Laurent series around the isolated singularity z_0. The negative part of the series is called the **Principal part of the Laurent series**.

Singularity using Laurent series expansion

Suppose z_0 is an isolated singularity for f and $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is the Laurent series expansion of f on $r < |z - z_0| < R$. Note that the singularity at z_0 is

- removable iff principal part is zero.
- pole iff principal part is finite.
- essential iff principal part is infinite.

Residue

If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $0 < |z - z_0| < R$ for some R. The corresponding Laurent expansion is called the Laurent expansion around z_0. Consider the -1-st coefficient of this Laurent series.

$$a_{-1} = \frac{1}{2\pi\imath} \int_{\gamma} f(z)dz.$$

If you integrate a Laurent series, only a_{-1} remains; other terms vanish. What remains is usually called a residue.

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Often a_{-1} is easy to compute from $f(z)$ and if that's the case integration has become easy.

Cauchy Residue Theorem

Suppose f is given and γ is given. Suppose there are finitely many isolated singularities of f inside γ; say $z_1, z_2, \ldots, z_n$. What's $\int_\gamma f(z)dz$?

Theorem (Cauchy Residue Theorem)

$$\int_\gamma f(z)dz = 2\pi\imath \cdot \sum_{i=1}^{n} \mathrm{Res}(f, z_i).$$

Thus integral of a function on a closed curve is zero not just when the function is holomorphic throughout; isolated singularities inside are okay, provided residues are zero.

How to compute residue?

1. If z_0 is a removable singularity of f, then $\text{Res}(f; z_0) = 0$.
2. If z_0 is a simple pole of f, then $\text{Res}(f; z_0) = \lim_{z \to z_0}(z - z_0)f$.
3. If z_0 is a pole of order m, then

$$\text{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d}{dz^{m-1}}[(z - z_0)^m f(z)].$$

4. If z_0 is a simple pole of $f = \frac{f_1}{f_2}$ with f_1 and f_2 are holomorphic at z_0, then $\text{Res}(f; z_0) = \frac{f_1(z_0)}{f'(z_0)}$. (Will be proved in the tutorial).

Examples

Consider $f(z) = \frac{e^z}{z^3}$.

e^z has a Taylor series expansion $\displaystyle\sum_0^\infty \frac{z^n}{n!}$. Then the Laurent series for $f(z)$ is given by $\displaystyle\sum_0^\infty \frac{z^{n-3}}{n!}$.

Hence the residue of $f(z)$, which is the coefficient of z^{-1} is given by $1/2$.

Alternatively, we note that $f(z)$ has a pole of order 3 at $z = 0$, so we can use the general formula for the residue at a pole:

$res(f;0) = \frac{1}{2!}[\frac{d^2}{dz^2}(z^3 f(z))]_{z=0} = \frac{1}{2}[e^z]_{z=0} = \frac{1}{2}$.

Example

Lets compute the residues of $f(z) = \frac{1}{sinh(\pi z)}$ at its singularities.
$\frac{1}{sinh(\pi z)}$ has a simple pole at ni for all $n \in \mathbb{Z}$ (Note : To check this show that $\lim\limits_{z \to ni} \dfrac{z - ni}{sinh(\pi z)}$ is a non-zero number). Thus the residue at ni is given by:

$res(f; ni) = \lim\limits_{z \to ni} \dfrac{z - ni}{sinh(\pi z)}$

By L'Hospital's rule $= \lim\limits_{z \to ni} \dfrac{1}{\pi cosh(\pi z)}$

$= \frac{1}{\pi cosh(n\pi i)}$

$= \frac{1}{\pi cos(n\pi)}$

$= \frac{(-1)^n}{\pi}$

Example

$f(z) = \frac{1}{sinh^3(z)}$
We have seen that $sinh^3(z)$ has a pole of order 3 at πi with Taylor series:
$sinh^3(z) = -(z-\pi i)^3 - \frac{1}{2}(z-\pi i)^5 + ...$
Thus, $\frac{1}{sinh^3(z)} = -(z-\pi i)^{-3}(1+\frac{1}{2}(z-\pi i)^2 + ...)^{-1}$
$= -(z-\pi i)^{-3}(1-\frac{1}{2}(z-\pi i)^2 + ...)$
The coefficient of $(z-\pi i)^{-1}$ in the above expression is $1/2$ which is therefore residue of f at πi.

Example

Compute $\int_{|z|=2} \frac{(z-4)}{(z^2+2)^2} dz$.
Recall : If $f(z)$ has a pole at z_0 of order m, then the residue of f at z_0 can be computed as :
$$res(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d}{dz^{m-1}} [(z - z_0)^m (f(z))]$$

Therefore in the given example,

$res(f, \sqrt{2}i) = \frac{d}{dz}[(z - \sqrt{2}i)^2 \frac{(z-4)}{(z^2+2)^2}]_{z=\sqrt{2}i}$ and

$res(f, -\sqrt{2}i) = \frac{d}{dz}[(z + \sqrt{2}i)^2 \frac{(z-4)}{(z^2+2)^2}]_{z=-\sqrt{2}i}$
Adding the avove values, we get the final answer. I'll leave the details of the computation to you.