Tutorial VII

1. Compute the following using residue theory:
$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\, dx.$$

2. Compute the value of $\int_0^{2\pi} \frac{d\theta}{a+1-2a\cos\theta}$, where $a < 1$, by transforming into an integral over the unit circle.

3. Let $\overline{\mathbb{D}}$ be the closed unit disc. For any $\alpha \in \mathbb{D}$, define $\varphi_\alpha : \overline{\mathbb{D}} \to \mathbb{C}$ by
$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$
(i) Show that for all $|z| = 1$, $|\varphi_\alpha(z)| = 1$.
(ii) Using (i) deduce that $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$.
(iii) Show that $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ is invertible by proving
$$\varphi_\alpha \circ \varphi_{-\alpha}(z) = z = \varphi_{-\alpha} \circ \varphi_\alpha(z) \quad (z \in \mathbb{D}).$$

4. Suppose f is an analytic function on the unit disc $\mathbb{D}$ with $|f| < M$ and $f(a) = 0$ for some $a \in \mathbb{D}$. Show that $|f(z)| \leq M|\frac{z-a}{1-\bar{a}z}|$ for all $z \in \mathbb{D}$.

5. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. If $f(a_i) = b_i$ for all $i = 1, 2$, then show that
$$\left|\frac{b_2 - b_1}{1-\bar{b_1}b_2}\right| \leq \left|\frac{a_2 - a_1}{1-\bar{a_1}a_2}\right|.$$