MA 205 Complex Analysis: Counting Zeros and Poles

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Multiplicity of a zero

Recall that a holomorphic function f on Ω has a zero at z_0 of multiplicity m if m is the least positive integer with $f^{(m)}(z_0)$ is non-zero. This is also equivalent to the fact that $f(z) = (z - z_0)^m h(z)$ where h is a holomorphic function on some small neighborhood of z_0 and $h(z_0) \neq 0$. Here $h(z)$ can be taken to be holomorphic on Ω (why?). Thus if f has finite number of zeros $z_1, \ldots, z_n$ inside Ω with multiplicities $m_1, \ldots, m_n$ respectively, then

$$f(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} H(z) \ (z \in \Omega)$$

for some holomorphic function H on Ω which does not vanish on Ω.

Theorem

Let f be a holomorphic function on Ω and $\bar{D}(P,r) \subset \Omega$. Suppose that f does not vanish on $\{z : |z-P| = r\}$ and that $z_1, \ldots, z_n$ are the zeros of f in $D(P,r)$ with multiplicities $m_1, \ldots, m_n$ respectively. Then

$$\frac{1}{2\pi\imath} \int_{|z-p|=r} \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{n} m_i.$$

Proof: Let

$$f(z) = \prod_{i=1}^{n} (z-z_i)^{m_i} H(z) \ (z \in \Omega)$$

for some holomorphic function H on Ω which does not vanish at $z_1, \ldots, z_{n-1}$ and z_n.

Proof Cont..

Then

$$f'(z) = \prod_{i=1}^{n}(z-z_i)^{m_i}H'(z)+\sum_{i=1}^{n} m_i(z-z_i)^{m_i-1}\prod_{1\leq j\leq n, j\neq i}(z-z_j)^{m_j}H(z)$$

and therefore,

$$\frac{f'(z)}{f(z)} = \frac{H'(z)}{H(z)} + \sum_{i=1}^{n}\frac{m_i}{z-z_i}.$$

Since $\frac{H'(z)}{H(z)}$ is holomorphic on an open set containing $\bar{D}(P,r)$,

$$\frac{1}{2\pi\imath}\int_{|z-P|=r}\frac{f'(z)}{f(z)}dz = 0 + \sum_{i=1}^{n} m_i.$$

Counting poles

Recall that if f is a meromorphic function on Ω with poles $z_1, \ldots, z_n$ of orders $m_1, \ldots, m_n$ respectively. Then

$$H(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} f(z)$$

becomes an holomorphic function on Ω.

Theorem

Let f be a meromorphic function on Ω with poles $z_1, \ldots, z_n$ of orders $m_1, \ldots, m_n$, respectively. Suppose $\bar{D}(P, r) \subset \Omega$ contains all the poles of f and f does not vanish on $\bar{D}(P, r)$. Then

$$\frac{1}{2\pi\imath} \int_{|z-P|=r} \frac{f'(z)}{f(z)} dz = -\sum_{i=1}^{n} m_i.$$

Proof: Define

$$H(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} f(z).$$

Then H is an holomorphic function on an open set containing $\bar{D}(P, r)$ and does not vanish on $\bar{D}(P, r)$. Note that for $z \in \Omega \setminus \{z_1, \dots, z_n\}$,

$$\frac{H'(z)}{H(z)} = \frac{f'(z)}{f(z)} + \sum_{i=1}^{n} \frac{m_i}{z - z_i}.$$

Then by integrating on $|z - P| = r$ we get

$$\frac{1}{2\pi\imath} \int_{|z-P|=r} \frac{f'(z)}{f(z)} dz = 0 - \sum_{i=1}^{n} m_i.$$

Combining the above results, we get a variant of the residue theorem and is known as the argument principle. It is used to count zero's and poles of a meromorphic function on a domain.

Theorem (Argument Principle)

Let f be a meromorphic function on Ω, and let γ be a closed contour contained in Ω such that γ does not pass through any of the zeros and poles of $f(z)$. Suppose, inside γ, f has zeros at $z_1, \ldots, z_n$ with multiplicities $m_1, \ldots, m_n$ respectively and has poles at $w_1, \ldots, w_k$ of orders $\ell_1, \ldots, \ell_k$ respectively. Then

$$\frac{1}{2\pi\imath}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} m_i - \sum_{j=1}^{k} \ell_j.$$

Rouche's Theorem

A nice and useful corollary of the argument principle is the following theorem:

Theorem (Rouche's Theorem)

Let γ be a simple closed contour and let $f(z)$ and $g(z)$ be two functions holomorphic on an open set containing γ and its interior. Suppose $|f(z) - g(z)| < |f(z)|$ at all points on γ. Then γ encloses the same number of zero's, counting multiplicities, of $f(z)$ and $g(z)$.

Proof: Let $h(z) = \frac{g(z)}{f(z)}$. Then h is an meromorphic function on an open set containing γ. Note that h does not have any zeros or poles on γ. Since $|h(z) - 1| < 1$ for all z on γ,

$$\int_\gamma \frac{h'}{h} dz = 0.$$

Thus number of zeros and poles of h, counting multiplicities, inside γ are same.

Example

The proof of the theorem follows easily from the fact that, after canceling common factors, the zeros of g (resp. f) are the zero's (resp. poles) of h.

Let us compute the number of zero's of
$f(z) = z^6 + 11z^4 + z^3 + 2z + 4$ inside the unit disc.
Take $g(z) = 11z^4$. Then $|g(z) - f(z)| < |g(z)|$ on the unit circle. Hence $g(z)$ has the same number of roots as $f(z)$ inside the unit circle. But the number of roots of $g(z)$ inside unit circle is 4 (counting mutiplicity) which therefore equals number of roots of $f(z)$.

Example

Lets count number of roots of $f(z) = e^z - 2z - 1$ inside the unit circle.
Let us consider $g(z) = -2z$. Then

$$|g(z) - f(z)| = |e^z - 1| = |\sum_1^\infty \frac{z^n}{n!}| \leq \sum_1^\infty \frac{|z^n|}{n!} = e - 1 < |g(z)|$$

on the unit circle. Hence by Rouche's theorem $f(z)$ and $g(z)$ have equal number of roots in the unit circle, namely 1.

Here's another quick and pretty proof of FTA using Rouche's theorem.
Let $f(z) = a_0 + a_1 z + \cdots + z^n$ be a non-constant polynomial. Take $g(z) = z^n$. Then on a sufficiently large circle around 0 of radius R, $|f(z) - g(z)| < |f(z)|$. Hence $f(z)$ and $g(z)$ have same number of zero's in the disc of radius R. Since $g(z)$ has n zero's, so does $f(z)$!

Picard's theorem

I now restate another absolutely spectacular theorem in complex analysis called Picard's theorem on the values taken by a holomorphic function.

Theorem (Big Picard's Theorem)

Let z_0 be an essential singularity of $f(z)$. Then in any punctured neighborhood of z_0, the image of $f(z)$ can miss atmost one point.

This theorem is called the Big Picard Theorem in view of what comes next.

Theorem (Little Picard theorem)

Any non-constant entire function can miss atmost one point.

The little Picard Theorem can be seen to be a corollary of the Big Picard Theorem as follows.

Picard's Theorem

Recall the following fact mentioned earlier: An entire function has a pole at infinity if and only if it is a non-constant polynomial.

Let $f(z)$ be a non-constant entire function. We wish to show it misses atmost one point. If $f(z)$ is a polynomial, then it is surjective by FTA. If $f(z)$ is not a polynomial, then it has an essential singularity at infinity (WHY ?). That is $f(\frac{1}{z})$ has an essential singularity at 0. Thus by Big Picard theorem, in any punctured neighborhood of 0, say of radius r, $f(\frac{1}{z})$ misses atmost one point. But this implies that in the complement of the circle of radius $1/r$, $f(z)$ misses atmost one point. This is what we wanted.

Exercise: If a non-constant entire function misses one point c, show that it is of the form $e^{f(z)} + c$ for some entire function $f(z)$.

Picard (1856-1941); Wiki

Picard was a top rate mathematician who did fundamental work in many disciples; analysis, function theory, differential equations, and analytic geometry to name a few. In physics he worked on elasticity, heat and electricity. Hadamard wrote about his teacher Picard:- A striking feature of Picard's scientific personality was the perfection of his teaching, one of the most marvellous, if not the most marvellous, that I have ever known.
It is a remarkable fact that between 1894 and 1937 he trained over 10000 engineers who were studying at the cole Centrale des Arts et Manufactures.