Partial differential equations

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1 Power series

For a real number x_0 and a sequence (a_n) of real numbers, consider the expression

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0+a_1(x-x_0)+a_2(x-x_0)^2+\dots.$$

This is called a power series in the real variable x.

The number a_n is called the n-th coefficient of the series and x_0 is called its center.

For instance,

$$\sum_{n=0}^{\infty} \frac{1}{n+1}(x-1)^n = 1+\frac{1}{2}(x-1)+\frac{1}{3}(x-1)^3+\dots$$

is a power series in x centered at 1 and with n-th coefficient equal to $\frac{1}{n+1}$.

A power series centered at 0 has the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots.$$

For instance,

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^n = 1 + \frac{1}{2} x + \frac{1}{3} x^3 + \ldots$$

is a power series in x centered at 0 and with n-th coefficient equal to $\frac{1}{n+1}$.

What can we do with a power series?

We can substitute a value for the variable x and get a series of real numbers.

We say that a power series converges at a point x_1 if substituting x_1 for x yields a convergent series, that is,

$$\sum_{n=0}^{\infty} a_n (x_1 - x_0)^n$$

converges.

A power series always converges at its center x_0 to the constant term a_0 since

$$\sum_{n=0}^{\infty} a_n (x_0 - x_0)^n = a_0.$$

(By convention $0^0 = 1$.)

We would like to know the set of values of x where a power series converges.

How far can we move away from the center?

Lemma. *Suppose a power series centered at* x_0 *converges for some real number* $x_1 \neq x_0$*. Let* $|x_1 - x_0| = r$*.*

Then the power series is convergent for all x *such that* $|x - x_0| < r$*.*

That is, the power series is convergent in the open interval of radius r *centered at* x_0*.*

The radius of convergence of the power series is the *largest* number R such that the power series converges in the open interval

$$\{x \mid |x - x_0| < R\}.$$

The latter is called the interval of convergence of the power series.

A power series can converge at all points in which case we take R to be ∞.

The radius of convergence can also be 0.

We assume from now on that $R > 0$.

A power series determines a function in its interval of convergence.

Denoting this function by f, we may write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad |x - x_0| < R.$$

Fact. *The function f is infinitely differentiable in the interval of convergence.*

The successive derivatives of f can be computed by differentiating the power series on the right termwise.

Hence

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2,$$

and in general,

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Corollary. *Two power series both centered at x_0 take the same values in some open interval around x_0*

iff

all the corresponding coefficients of the two power series are equal.

2 Real analytic functions

Let $\mathbb{R}$ denote the set of real numbers.

A subset U of $\mathbb{R}$ is said to be open if for each $x_0 \in U$, there is a $r > 0$ such that the open interval $|x - x_0| < r$ is contained in U.

For example, open intervals are open sets.

The interval $(0, \infty)$ is also an open set.

Union of open sets is again an open set.

$$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$$

The interval $[0, 1]$ or $(0, 1]$ is *not* an open set.

Let $f : U \to \mathbb{R}$ be a real-valued function on an open set U.

We say f is real analytic at a point $x_0 \in U$

if there exists a power series centered at x_0 which converges to f in some open interval around x_0,

that is,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

holds in some open interval around x_0.

We say f is real analytic on U if it is real analytic at all points of U.

In general, we can always consider the set of all points in the domain where f is real analytic. This is called the domain of analyticity of f.

Corollary. *Suppose f is real analytic on U.*

Then f is infinitely differentiable on U.

Its power series representation around x_0 is necessarily the Taylor series of f around x_0,

that is, the coefficients a_n are given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Just like continuity or differentiability, real analyticity is a local property.

That is, to know whether f is real analytic at a point, we only need to know the values of f in an open interval around that point.

Example. Polynomials such as $x^3 - 2x + 1$ are real analytic on all of $\mathbb{R}$.

A polynomial is a truncated power series

(so there is no issue of convergence).

By writing $x = x_0 + (x - x_0)$ and expanding, we can rewrite any polynomial using powers of $x - x_0$.

This will be a truncated power series centered at x_0.

For instance,

$$1 + 2x + x^2 = 4 + 4(x - 1) + (x - 1)^2.$$

Fact. *A power series is real analytic in its interval of convergence.*

Thus, if a function is real analytic at a point x_0,

then it is real analytic in some open interval around x_0.

Thus, the domain of analyticity of a function is an open set.

Fact. *If f and g are real analytic functions on U, then so are*

$$cf, \quad f+g, \quad fg \quad \textit{and} \quad f/g.$$

Here cf is the multiple of f by the scalar c.

Also we wrote f/g above, it is implicit that $g \neq 0$ on U.

Example. The sine, cosine and exponential functions are real analytic on all of $\mathbb{R}$.

Their Taylor series around 0 are

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

Using the Taylor remainder formula, one can show that these identities are valid for all x.

For example,

$$\sin(\pi) = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} + \dots.$$

Example. Consider the function

$$f(x) = \frac{1}{1+x^2}.$$

It is real analytic on all of $\mathbb{R}$.

The Taylor series around 0 is

$$\frac{1}{1+x^2} = (1 - x^2 + x^4 - x^6 + \dots).$$

This series has a radius of convergence 1 and the identity only holds for $|x| < 1$.

At $x = 1$, note that $f(1) = 1/2$ while the series oscillates between 1 and 0.

Thus from here we can only conclude that f is real analytic in $(-1, 1)$.

If we want to show that f is real analytic at 1,

then we need to need to find another power series centered at 1 which converges to f in an interval around 1.

This is possible.

Even though a function is real analytic, one may not be able to represent it by just one power series.

Example. The function $f(x) = x^{1/3}$ is defined for all x.

It is not differentiable at 0 and hence not real analytic at 0.

However it is real analytic at all other points.

For instance,

$$x^{1/3} = (1+(x-1))^{1/3} = 1+\frac{1}{3}(x-1)+\frac{1}{3}(\frac{1}{3}-1)\frac{(x-1)^2}{2!}+\ldots$$

is valid for $|x-1| < 1$, showing analyticity in the interval $(0, 2)$.

(This is the binomial theorem which we will prove later.)

Analyticity at other nonzero points can be established similarly.

Thus, the domain of analyticity of $f(x) = x^{1/3}$ is $\mathbb{R} \setminus \{0\}$.

Example. The function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is infinitely differentiable.

But it is not real analytic at 0.

This is because $f^{(n)}(0) = 0$ for all n and the Taylor series around 0 is identically 0. So the Taylor series around 0 does not converge to f in any open interval around 0.

The domain of analyticity of f is $\mathbb{R} \setminus \{0\}$.

For a function f, never say “convergence of f”, instead say “convergence of the Taylor series of f”.

3 Solving a linear ODE by the power series method

Example. Consider the first order linear ODE

$$y' - y = 0, \quad y(0) = 1.$$

We know that the solution is $y = e^x$.

Let us use power series to arrive at this result.

We assume that the solution has the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The initial condition $y(0) = 1$ implies $a_0 = 1$.

Substitute $y = \sum_n a_n x^n$ in the ODE $y' = y$ and compare the coefficient of x^n on both sides.

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \ldots$$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots.$$

This yields the equations

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2, \ldots.$$

In general,

$$(n+1)a_{n+1} = a_n \text{ for } n \geq 0.$$

Such a set of equations is called a recursion.

Since $a_0 = 1$, we see that

$$a_1 = 1, \quad a_2 = 1/2, \quad a_3 = 1/6, \ldots.$$

In general,

$$a_n = 1/n!.$$

Thus

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

which we know is e^x.

Example. Consider the first order linear ODE

$$y' - 2xy = 0, \quad y(0) = 1.$$

We proceed as in the previous example.

The initial condition $y(0) = 1$ implies $a_0 = 1$.

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \ldots$$
$$2xy = 2a_0x + 2a_1x^2 + 2a_2x^3 + \cdots + 2a_{n-1}x^n + \ldots.$$

This time we get the recursion:

$$(n+2)a_{n+2} = 2a_n \quad \text{for } n \geq 0,$$

and $a_1 = 0$.

So all odd coefficients are zero.

For the even coefficients, put $n = 2k$.

This yields

$$(2k+2)a_{2k+2} = 2a_{2k} \quad \text{for } k \geq 0.$$

This is the same as

$$(k+1)a_{2k+2} = a_{2k} \quad \text{for } k \geq 0.$$

This yields

$$a_{2k} = 1/k!.$$

Thus

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$$

which we know is e^{x^2}.

Example. Consider the second order linear ODE

$$y'' + y = 0.$$

(The initial conditions are left unspecified.)

Proceeding as before, we get

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \text{for } n \geq 0,$$

and a_0 and a_1 are arbitrary.

Solving we obtain

$$y(x) = a_0(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots) + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots).$$

Thus,

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

Consider the following initial value problem.

$$p(x)y'' + q(x)y' + r(x)y = g(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Here x_0, y_0 and y_1 are fixed real numbers.

We assume that the functions p, q, r and g are real analytic in an interval containing the point x_0.

Let $r > 0$ be less than the minimum of the radii of convergence of the functions p, q, r and g expanded in power series around x_0.

We also assume that $p(x) \neq 0$ for all x in the interval $(x_0 - r, x_0 + r)$.

Theorem. *Under the above conditions, there is a unique solution to the initial value problem in the interval* $(x_0 - r, x_0 + r)$*, and moreover it can be represented by a power series*

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

whose radius of convergence is at least r*.*

Moreover, there is an algorithm to compute the power series representation of y.

It works as follows.

- Plug a general power series into the ODE.
- Take derivatives of the power series formally.
- Equate the coefficients of $(x - x_0)^n$ for each n.
- Obtain a recursive definition of the coefficients a_n.
- The a_n's are uniquely determined and we obtain the desired power series solution.

In most of our examples, $x_0 = 0$ and the functions p, q, r and g will be polynomials of small degree.

This result generalizes to any n-th order linear ODE with the first $n-1$ derivatives at x_0 specified.

For the first order linear ODE, the initial condition is simply the value at x_0. Suppose the first order ODE is

$$p(x)y' + q(x)y = 0.$$

Then, by separation of variables, the general solution is

$$ce^{-\int \frac{q(x)}{p(x)}dx},$$

provided $p(x) \neq 0$ else the integral may not be well-defined.

This is an indicator why such a condition is required in the hypothesis of the theorem.

Example. Consider the function

$$f(x) = (1+x)^p$$

where $|x| < 1$ and p is any real number.

Note that it satisfies the linear ODE.

$$(1+x)y' = py, \quad y(0) = 1.$$

Let us solve this using the power series method around $x = 0$.

Since $1 + x$ is zero at $x = -1$, we are guaranteed a solution only for $|x| < 1$.

The initial condition $y(0) = 1$ implies $a_0 = 1$.

To calculate the recursion, express each term as a power series:

$$\begin{aligned}
y' &= a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \ldots \\
xy' &= \qquad\quad a_1x + 2a_2x^2 + \cdots + na_nx^n + \ldots \\
py &= pa_0 + pa_1x + pa_2x^2 + \cdots + pa_nx^n + \ldots .
\end{aligned}$$

Comparing coefficients yields the recursion:

$$a_{n+1} = \frac{p-n}{n+1}\, a_n \quad \text{for } n \geq 0.$$

Since $a_0 = 1$, we see that

$$a_1 = p, \quad a_2 = \frac{p(p-1)}{2}, \quad a_3 = \frac{p(p-1)(p-2)}{6}, \ldots.$$

This shows that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \ldots.$$

This is the binomial theorem and we just proved it.

The most well-known case is when p is a positive integer (in which case the power series terminates to a polynomial of degree p).

Given a function f, it is useful to know whether it satisfies an ODE.

Example. Consider the second order linear ODE

$$y'' + y' - 2y = 0.$$

(The initial conditions are left unspecified.)

Proceeding as before, we get the recursion:

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - 2a_n = 0 \quad \text{for } n \geq 0,$$

and a_0 and a_1 are arbitrary.

The recursion involves three terms.

How do we solve it?

The key idea is to make the substitution

$$b_n = n!a_n,$$

so we obtain

$$b_{n+2} + b_{n+1} - 2b_n = 0 \quad \text{for } n \geq 0,$$

We now guess that $b_n = \lambda^n$ is a solution.

This yields the quadratic

$$\lambda^2 + \lambda - 2 = 0$$

(which can be written directly by looking at the constant-coefficient ODE).

Its roots are 1 and -2.

Thus the general solution is

$$b_n = \alpha + \beta\,(-2)^n,$$

where α and β are arbitrary.

The general solution to the original recursion is

$$a_n = \alpha\,\frac{1}{n!} + \beta\,\frac{(-2)^n}{n!}.$$

So the general solution to the ODE is

$$y(x) = \alpha e^x + \beta e^{-2x}.$$

4 Inner product spaces

4.1 Vector spaces

Recall the notion of a vector space V over $\mathbb{R}$.
Elements of V are called vectors, and

elements of $\mathbb{R}$ are called scalars.

There are two operations in a vector space, namely,

- addition

$$v + w, \quad v, w \in V,$$

  and

- scalar multiplication

$$cv, \quad c \in \mathbb{R}, v \in V.$$

Any vector space V has a dimension, which may *not* be finite.

4.2 Inner product spaces

Let V be a vector space over $\mathbb{R}$ (not necessarily finite-dimensional).

A bilinear form on V is a map

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R},$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for $a \in \mathbb{R}$ and $u, v \in V$.

An inner product on V is a bilinear form on V which is

- symmetric: $\langle v, w \rangle = \langle w, v \rangle$.
- positive definite: $\langle v, v \rangle \geq 0$ for all v, and $\langle v, v \rangle = 0$ iff $v = 0$.

A vector space with an inner product is called an inner product space.

4.3 Orthogonality

In an inner product space, we have the notion of orthogonality.

We say vectors u and v are orthogonal if $\langle u, v \rangle = 0$.

More generally, a set of vectors forms an orthogonal system if they are mutually orthogonal.

An orthogonal basis is an orthogonal system which is also a basis.

Example. Consider the vector space $\mathbb{R}^n$.

Vectors are given by n tuples $(a_1, \ldots, a_n)$.

Addition and scalar multiplication are done coordinatewise. That is,

$$(a_1, \ldots, a_n)+(b_1, \ldots, b_n) := (a_1+b_1, \ldots, a_n+b_n)$$

$$c(a_1, \ldots, a_n) := (ca_1, \ldots, ca_n).$$

The rule

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle := \sum_{i=1}^{n} a_i b_i$$

defines an inner product on $\mathbb{R}^n$.

Let e_i be the vector which is 1 in the i-th position, and 0 in all other positions.

Then $\{e_1, \ldots, e_n\}$ is an orthogonal basis.

For example,

$$\{(1,0,0), \quad (0,1,0), \quad (0,0,1)\}$$

is an orthogonal basis of $\mathbb{R}^3$.

The previous example can be formulated more abstractly as follows.

Example. Let V be a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$.

For $u = \sum_{i=1}^n a_i e_i$ and $v = \sum_{i=1}^n b_i e_i$, let

$$\langle u, v \rangle := \sum_{i=1}^n a_i b_i.$$

This defines an inner product on V.

Further, $\{e_1, \ldots, e_n\}$ is an orthogonal basis.

Lemma. *Suppose V is a finite-dimensional inner product space, and $e_1, \ldots, e_n$ is an orthogonal basis. Then for any $v \in V$,*

$$v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i.$$

Proof. To see this, write $v = \sum_{i=1}^{n} a_i e_i$.

We want to find the coefficients a_i.

For this we take inner product of v with e_i one by one:

$$\begin{aligned} \langle v, e_j \rangle &= \langle \sum_{i=1}^{n} a_i e_i, e_j \rangle \\ &= \sum_{i=1}^{n} a_i \langle e_i, e_j \rangle \\ &= a_j \langle e_j, e_j \rangle. \end{aligned}$$

Thus,

$$a_j = \frac{\langle v, e_j \rangle}{\langle e_j, e_j \rangle}$$

as required. $\square$

Lemma. *In a finite-dimensional inner product space, there always exists an orthogonal basis.*

You can start with any basis and modify it to an orthogonal basis by Gram-Schmidt orthogonalization.

This result is not necessarily true in infinite-dimensional inner product spaces.

In this generality, we can only talk of a maximal orthogonal set.

4.4 Length of a vector and Pythagoras theorem

In an inner product space, we define for any $v \in V$,

$$\|v\| := \langle v, v\rangle^{1/2}.$$

This is called the norm or length of the vector v.

It verifies the following properties.

$$\|0\| = 0 \text{ and } \|v\| > 0 \text{ if } v \neq 0,$$

$$\|v + w\| \leq \|v\| + \|w\|,$$

$$\|av\| = |a|\|v\|,$$

for all $v, w \in V$ and $a \in \mathbb{R}$.

Theorem. *For orthogonal vectors v and w in any inner product space,*

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2.$$

This is the modern avatar of Pythagoras theorem.

Proof. The proof is as follows.

$$\begin{aligned}
\|v+w\|^2 &= \langle v+w, v+w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \langle v, v\rangle + \langle w, w\rangle \\
&= \|v\|^2 + \|w\|^2.
\end{aligned}$$

$\square$

More generally, for any orthogonal system $\{v_1, \ldots, v_n\}$,

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2.$$

5 Legendre equation

Consider the following second order linear ODE.

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0.$$

Here p denotes a fixed real number.

This is known as the Legendre equation.

This ODE is defined for all real numbers x.

The Legendre equation can also be written in the form

$$((1-x^2)y')' + p(p+1)y = 0.$$

We assume $p \geq -1/2$.

5.1 General solution

The coefficients

$$(1 - x^2), \quad -2x \quad \text{and} \quad p(p+1)$$

are polynomials (and in particular real analytic).

However $1 - x^2 = 0$ for $x = \pm 1$.

The points $x = \pm 1$ are the singular points of the Legendre ODE.

Our theorem guarantees a power series solution of the Legendre ODE around $x = 0$ in the interval $(-1, 1)$.

We apply the power series method to the Legendre ODE

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0.$$

The recursion obtained is

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0.$$

That is,

$$(n+2)(n+1)a_{n+2} - [n(n-1) + 2n - p(p+1)]a_n = 0.$$

That is,

$$(n+2)(n+1)a_{n+2} - [n(n+1) - p(p+1)]a_n = 0.$$

That is,

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)}a_n.$$

This is valid for $n \geq 0$, with a_0 and a_1 arbitrary.

Thus, the general solution to the Legendre equation in the interval $(-1, 1)$ is given by

$$y(x)=a_0(1-\frac{p(p+1)}{2!}x^2+\frac{(p(p-2)(p+1)(p+3)}{4!}x^4+\dots)$$
$$+a_1(x-\frac{(p-1)(p+2)}{3!}x^3+\frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5+\dots).$$

It is called the Legendre function.

The first series is an even function while the second series is an odd function.

5.2 Legendre polynomials

Now suppose the parameter p in the Legendre equation is a nonnegative integer.

Then one of the two series in the general solution terminates, and is a polynomial.

Thus, for each nonnegative integer m, we obtain a polynomial $P_m(x)$ (up to multiplication by a constant).

It is traditional to normalize the constants so that $P_m(1) = 1$.

These are called the Legendre polynomials.

The m-th Legendre polynomial $P_m(x)$ solves the Legendre equation

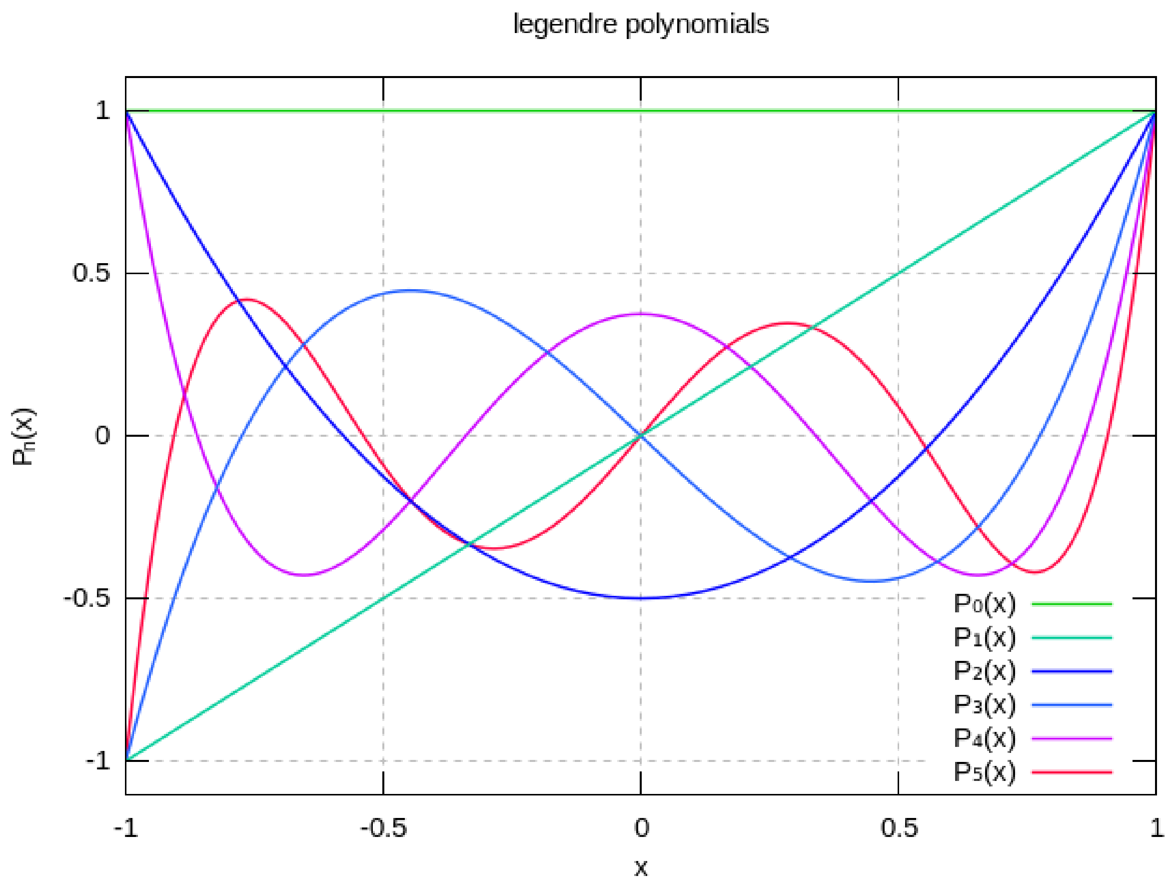
$$(1 - x^2)y'' - 2xy' + m(m+1)y = 0.$$

This solution is valid for all x, not just for $x \in (-1, 1)$.

The first few values are as follows.

m	$P_m(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$

Their graphs in the interval $(-1, 1)$ are given below.



5.3 Second solution to the Legendre equation when p is an integer

Now let us consider the second independent solution. It is a honest power series (not a polynomial).

For $p = 0$, it is

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \frac{1}{2} \log \left(\frac{1+x}{1-x}\right).$$

For $p = 1$, it is

$$1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \cdots = 1 - \frac{1}{2} x \log \left(\frac{1+x}{1-x}\right).$$

These nonpolynomial solutions for any nonnegative integer p always have a log factor of the above kind and hence are unbounded near both $+1$ and -1.

(Since they are either even or odd, the behavior at $x = 1$ is reflected at $x = -1$.)

They are called the Legendre functions of the second kind.

5.4 The vector space of polynomials

The set of all polynomials in the variable x is a vector space.

(We can add polynomials and multiply a polynomial by a scalar.)

The set

$$\{1, x, x^2, \dots\}$$

is a basis of the vector space of polynomials.

Note: This vector space is *not* finite-dimensional.

Every polynomial is a 'vector' in this vector space.

The vector space of polynomials carries an inner product defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx.$$

Note that we are integrating only between -1 and 1. This ensures that the integral is always finite.

The norm of a polynomial is defined by

$$\|f\| := \left(\int_{-1}^{1} f(x)f(x)dx \right)^{1/2}.$$

5.5 Technique of derivative transfer

Note this simple consequence of integration by parts:

Fact. *For differentiable functions f and g, if*

$$(fg)(b) = (fg)(a),$$

then

$$\int_a^b fg'dx = -\int_a^b f'gdx.$$

(This process transfers the derivative from g to f.)

5.6 Orthogonality of the Legendre polynomials

Since $P_m(x)$ is a polynomial of degree m, it follows that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of the vector space of polynomials.

Proposition. *We have*

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \textit{if } m \neq n, \\ \frac{2}{2n+1} & \textit{if } m = n. \end{cases}$$

The first alternative says that Legendre polynomials form an orthogonal basis for the vector space of polynomials.

The second alternative can be rewritten as

$$\|P_n(x)\|^2 = \frac{2}{2n+1}.$$

Proof. We will only prove orthogonality.

We use the technique of derivative-transfer.

Suppose $m \neq n$.

Since $P_m(x)$ solves the Legendre equation for $p = m$, we have

$$((1-x^2)P_m')' + m(m+1)P_m = 0.$$

Multiply by P_n and integrate to get

$$\int_{-1}^{1} ((1-x^2)P_m')'P_n + m(m+1)\int_{-1}^{1} P_m P_n = 0.$$

By derivative transfer, we get

$$-\int_{-1}^{1} (1-x^2)P_m'P_n' + m(m+1)\int_{-1}^{1} P_m P_n = 0.$$

Interchanging the roles of m and n,

$$-\int_{-1}^{1}(1-x^2)P'_m P'_n + n(n+1)\int_{-1}^{1} P_m P_n = 0.$$

Subtracting the two identities, we obtain

$$[m(m+1)-n(n+1)]\int_{-1}^{1} P_m P_n = 0.$$

Since $m \neq n$, the scalar in front can be canceled and we get

$$\int_{-1}^{1} P_m P_n = 0.$$

Thus, P_m and P_n are orthogonal. $\square$

5.7 Rodrigues formula

Consider the sequence of polynomials

$$q_n(x) := (\frac{d}{dx})^n (x^2-1)^n = D^n(x^2-1)^n, \quad n \geq 0.$$

Observe that $q_n(x)$ has degree n.

For instance,

$$q_1(x) = \frac{d}{dx}(x^2 - 1) = 2x.$$

The first few polynomials are $1, 2x, 4(3x^2 - 1), \ldots$.

[Note the similarity with the Legendre polynomials.]

Proposition. *We have*

$$P_n(x) = \frac{1}{2^n n!}\Big(\frac{d}{dx}\Big)^n (x^2 - 1)^n.$$

Equivalently,

$$P_n(x) = \frac{1}{2^n n!}\, q_n(x).$$

This is known as Rodrigues formula.

Proof. We sketch a proof of Rodrigues formula.

We first claim that

$$\int_{-1}^{1} q_m(x) q_n(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1}(2^n n!)^2 & \text{if } m = n. \end{cases}$$

This can be established using the technique of derivative-transfer.

We prove orthogonality. That is,

$$\int_{-1}^{1} q_m(x) q_n(x) dx = 0 \quad \text{if } m \neq n.$$

Assume without loss of generality that $m < n$.

To understand the method, let us take $m = 2$ and $n = 5$.

$$\begin{aligned}
&\int_{-1}^{1} D^2(x^2-1)^2 D^5(x^2-1)^5 dx \\
&= -\int_{-1}^{1} D^3(x^2-1)^2 D^4(x^2-1)^5 dx \\
&= +\int_{-1}^{1} D^4(x^2-1)^2 D^3(x^2-1)^5 dx \\
&= -\int_{-1}^{1} D^5(x^2-1)^2 D^2(x^2-1)^5 dx \\
&= 0.
\end{aligned}$$

Thus, q_2 and q_5 are orthogonal.

Transfer $m + 1$ of the n derivatives in $q_n(x)$ to $q_m(x)$.

$$\int_{-1}^{1} D^m(x^2 - 1)^m D^n(x^2 - 1)^n dx$$

$$= -\int_{-1}^{1} D^{m+1}(x^2 - 1)^m D^{n-1}(x^2 - 1)^n dx$$

$$= \cdots = (-1)^m \int_{-1}^{1} D^{2m}(x^2-1)^m D^{n-m}(x^2-1)^n dx$$

$$= (-1)^{m+1} \int_{-1}^{1} D^{2m+1}(x^2-1)^m D^{n-m-1}(x^2-1)^n dx = 0,$$

since $D^{2m+1}(x^2 - 1)^m \equiv 0$.

Thus, q_m and q_n are orthogonal for $m < n$.

We deduce that $P_n(x)$ and $q_n(x)$ are scalar multiples of each other.

By convention $P_n(1) = 1$ while $q_n(1) = 2^n n!$. Why?

So Rodrigues formula is proved. $\square$

5.8 Square-integrable functions

A function $f(x)$ on $[-1, 1]$ is square-integrable if

$$\int_{-1}^{1} f(x)f(x)dx < \infty.$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

The set of all square-integrable functions on $[-1, 1]$ is a vector space.

The inner product on polynomials extends to square-integrable functions. That is, for square-integrable functions f and g, we define their inner product by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx.$$

5.9 Fourier-Legendre series

The Legendre polynomials no longer form a basis for the vector space of square-integrable functions.

But they form a maximal orthogonal set in this larger space.

This means that there is no nonzero square-integrable function which is orthogonal to all Legendre polynomials.

This is a nontrivial fact.

This allows us to expand any square-integrable function $f(x)$ on $[-1, 1]$ in a series of Legendre polynomials

$$\sum_{n \geq 0} c_n P_n(x),$$

where

$$\begin{aligned} c_n &= \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} \\ &= \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx. \end{aligned}$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

Example. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ -1 & \text{if } -1 < x < 0. \end{cases}$$

The Legendre series of $f(x)$ is

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \ldots.$$

By the Legendre expansion theorem (stated below), this series converges to $f(x)$ for $x \neq 0$ and to 0 for $x = 0$.

Let us go back to the general case.

The Fourier-Legendre series of $f(x)$ converges in norm to $f(x)$, that is,

$$\|f(x) - \sum_{n=0}^{m} c_n P_n(x)\| \to 0 \quad \text{as} \quad m \to \infty.$$

Pointwise convergence is more delicate.

There are two issues here:

Does the series converge at x?

If yes, then does it converge to $f(x)$?

A useful result in this direction is the Legendre expansion theorem:

Theorem. *If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1, 1]$, then the Legendre series converges to*

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \textit{for } -1 < x < 1,$$

converges to $f(-1_+)$ at $x = -1$, and

converges to $f(1_-)$ at $x = 1$.

In particular, the series converges to $f(x)$ at every point of continuity.

6 Ordinary and singular points

Consider the second-order linear ODE

$$p(x)y'' + q(x)y' + r(x)y = 0,$$

where p, q, and r are real analytic functions with no common zeroes.

A point x_0 is called an ordinary point if $p(x_0) \neq 0$, and a singular point if $p(x_0) = 0$.

A singular point x_0 is called regular if after dividing throughout by $p(x)$ the ODE can be written in the form

$$y'' + \frac{b(x)}{x - x_0}\, y' + \frac{c(x)}{(x - x_0)^2}\, y = 0,$$

where $b(x)$ and $c(x)$ are real analytic around x_0.

A singular point which is not regular is called irregular.

For instance, in the equation $x^3 y'' + x y' + y = 0$, $x = 0$ is an irregular singular point.

7 Cauchy-Euler equation

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0,$$

where b_0 and c_0 are constants.

$x = 0$ is a regular singular point since we can rewrite the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0.$$

The rest are all ordinary points.

Assume $x > 0$.

Note that $y = x^r$ solves the equation if

$$r(r-1) + b_0r + c_0 = 0,$$

that is,

$$r^2 + (b_0 - 1)r + c_0 = 0.$$

Let r_1 and r_2 denote the roots of this quadratic equation.

- If the roots are real and unequal, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

  are two independent solutions.

- If the roots are real and equal, then

$$x^{r_1} \quad \text{and} \quad (\log x)x^{r_1}$$

  are two independent solutions.

- If the roots are complex (written as $a \pm ib$), then

$$x^a \cos(b \log x) \quad \text{and} \quad x^a \sin(b \log x)$$

  are two independent solutions.

This gives us a general idea of the behavior of the solution of an ODE near a regular singular point.

8 Fuchs-Frobenius theory

Consider the ODE

$$x^2 y'' + x b(x)\, y' + c(x)\, y = 0,$$

where

$$b(x) = \sum_{n \geq 0} b_n x^n \quad \text{and} \quad c(x) = \sum_{n \geq 0} c_n x^n$$

are real analytic in a neighborhood of the origin.

Restrict to $x > 0$.

Assume a solution of the form

$$y(x) = x^r \sum_{n \geq 0} a_n x^n, \quad a_0 \neq 0,$$

with r fixed.

Substitute in the ODE and equate the coefficient of x^r to obtain

$$r(r-1) + b_0 r + c_0 = 0,$$

that is,

$$r^2 + (b_0 - 1)r + c_0 = 0.$$

Let us denote the quadratic by $I(r)$.

It is called the indicial equation of the ODE.

Let r_1 and r_2 be the roots of $I(r) = 0$ with $r_1 \geq r_2$.

Equating the coefficient of x^{r+1}, we obtain

$$[(r+1)r + (r+1)b_0 + c_0]a_1 + a_0(rb_1 + c_1) = 0.$$

More generally, equating the coefficient of x^{r+n}, we obtain the recursion

$$I(r+n)a_n + \sum_{j=0}^{n-1} a_j((r+j)b_{n-j} + c_{n-j}) = 0, \quad n \geq 1.$$

This is a recursion for the a_n's.

One can solve it provided $I(r+n) \neq 0$ for any n.

Theorem. *The ODE has as a solution for $x > 0$*

$$y_1(x) = x^{r_1}(1 + \sum_{n \geq 1} a_n x^n)$$

where a_n's solve the recursion for $r = r_1$ and $a_0 = 1$.

The power series converges in the interval in which both $b(x)$ and $c(x)$ converge.

The term fractional power series solution is used for such a solution.

8.1 Roots not differing by an integer

Recall that r_1 and r_2 are the roots of $I(r) = 0$ with $r_1 \geq r_2$.

Theorem. *If $r_1 - r_2$ is not an integer, then a second independent solution for $x > 0$ is given by*

$$y_2(x) = x^{r_2}(1 + \sum_{n \geq 1} A_n x^n)$$

where A_n's solve the recursion for $r = r_2$, $a_0 = 1$.

The power series in these solutions converge in the interval in which both $b(x)$ and $c(x)$ converge.

This solution is also a fractional power series.

Example. Consider the ODE

$$2x^2y'' - xy' + (1+x)y = 0.$$

Observe that $x = 0$ is a regular singular point since the ODE can be written as

$$y'' - \frac{1}{2x}y' + \frac{(1+x)}{2x^2}y = 0.$$

(The functions $b(x) = -1/2$ and $c(x) = (1+x)/2$ are in fact polynomials, so they converge everywhere.)

Let us apply the Frobenius method.

The indicial equation is

$$r^2 - \frac{3r}{2} + \frac{1}{2} = 0.$$

The roots are $r_1 = 1$ and $r_2 = 1/2$.

Their difference is not an integer.

So we will get two fractional power series solutions converging everywhere.

Let us write them down explicitly.

The general recursion is

$$(2(r+n)(r+n-1)-(r+n)+1)a_n+a_{n-1}=0, \quad n\geq 1.$$

That is,

$$a_n = \frac{-1}{(r+n-1)(2r+2n-1)}a_{n-1}, \quad n\geq 1$$

For the root $r=1$, the recursion simplifies to

$$a_n = \frac{-1}{(2n+1)n}a_{n-1}, \quad n\geq 1$$

leading to the solution for $x>0$

$$y_1(x)=x\left(1+\sum_{n\geq 1} \frac{(-1)^n x^n}{(2n+1)n(2n-1)(n-1)\ldots(5\cdot 2)(3\cdot 1)}\right).$$

Similarly, for the root $r=1/2$, the recursion simplifies to

$$a_n = \frac{-1}{2n(n-1/2)}a_{n-1}, \quad n\geq 1$$

leading to the second solution for $x>0$

$$y_2(x)=x^{1/2}\left(1+\sum_{n\geq 1} \frac{(-1)^n x^n}{2n(n-1/2)(2n-2)(n-3/2)\ldots(4\cdot 3/2)(2\cdot 1/2)}\right).$$

8.2 Repeated roots

Theorem. *If the indicial equation has repeated roots, then there is a second solution of the form*

$$y_2(x) = y_1(x) \log x + x^{r_1} \sum_{n \geq 1} A_n x^n$$

with $y_1(x)$ *as before.*

The power series converges in the interval in which both $b(x)$ *and* $c(x)$ *converge.*

Let us see where this solution comes from.

In doing so, we will also derive a formula for A_n.

Treating r as a variable, one can uniquely solve

$$I(r+n)a_n+\sum_{j=0}^{n-1} a_j((r+j)b_{n-j}+c_{n-j}) = 0, \quad n \geq 1$$

starting with $a_0 = 1$.

Since the a_n depend on r, let us write $a_n(r)$.

Now consider

$$\varphi(r,x) := x^r\Big(1+\sum_{n\geq 1} a_n(r)x^n\Big).$$

Note that the first solution is $\varphi(r_1, x)$.

For the second solution, take partial derivative of $\varphi(r,x)$ with respect to r, and then put $r = r_1$.

$$\begin{aligned}
y_2(x) &= \left.\frac{\partial \varphi(r,x)}{\partial r}\right|_{r=r_1} \\
&= \left.\frac{\partial}{\partial r}\left(x^r \sum_{n\geq 0} a_n(r)x^n\right)\right|_{r=r_1} \\
&= x^{r_1}\log x \sum_{n\geq 0} a_n(r_1)x^n + x^{r_1}\sum_{n\geq 1} a_n'(r_1)x^n \\
&= y_1(x)\log x + x^{r_1}\sum_{n\geq 1} a_n'(r_1)x^n.
\end{aligned}$$

In particular,

$$A_n = a_n'(r_1).$$

Example. Consider the ODE

$$x^2y'' + 3xy' + (1-2x)y = 0.$$

This has a regular singularity at 0 with $b(x) = 3$ and $c(x) = 1 - 2x$.

The indicial equation is $r^2 + 2r + 1 = 0$ which has a repeated root at -1.

One may check that the recursion is

$$(n+r+1)^2 a_n = 2a_{n-1}, \quad n \geq 1.$$

Hence

$$a_n(r) = \frac{2^n}{[(r+2)(r+3)\dots(r+n+1)]^2}\, a_0.$$

Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x}\sum_{n\geq 0} \frac{2^n}{(n!)^2}\, x^n.$$

The power series converges everywhere.

For the second solution:

$$a_n'(r) = -2a_n(r)\left(\frac{1}{r+2} + \frac{1}{r+3} + \cdots + \frac{1}{r+n+1}\right), \quad n \geq 1.$$

Evaluating at $r = -1$,

$$a_n'(-1) = -\frac{2^{n+1} H_n}{(n!)^2},$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(These are the partial sums of the harmonic series.)

So the second solution is

$$y_2(x) = y_1(x) \log x - \frac{1}{x} \sum_{n \geq 1} \frac{2^{n+1} H_n}{(n!)^2} x^n.$$

The power series converges everywhere.

8.3 Roots differing by an integer

Theorem. *If $r_1 - r_2$ is a positive integer, say N, then there is a second solution of the form*

$$y_2(x) = K y_1(x) \log x + x^{r_2}(1 + \sum_{n \geq 1} A_n x^n),$$

with

$$A_n = \frac{d}{dr}((r - r_2)a_n(r))\Big|_{r=r_2}, \qquad n \geq 1.$$

and

$$K = \lim_{r \to r_2} (r - r_2)a_N(r).$$

The power series converges in the interval in which both $b(x)$ and $c(x)$ converge.

It is possible that $K = 0$.

Example. Consider the ODE

$$xy'' - (4+x)y' + 2y = 0.$$

The indicial equation is $r(r-5) = 0$, with the roots differing by a positive integer.

The recursion is

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}, \quad n \geq 1.$$

Hence

$$a_n(r) = \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)}a_0.$$

Setting $r = 5$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \sum_{n \geq 0} \frac{60}{n!(n+5)(n+4)(n+3)} x^{n+5}.$$

For the second solution, the ‘critical’ function is

$$a_5(r) = \frac{(r+2)(r+1)r(r-1)(r-2)}{(r+5)\dots(r+1)r(r-1)\dots(r-4)}.$$

Note that there is a factor of r in the numerator also.

So this function does not have a singularity at $r = 0$, and $K = 0$, and $a_5(0) = 1/720$.

$$y_2(x) = (1 + \tfrac{1}{2}x + \tfrac{1}{12}x^2) + a_5(0)\Big(x^5 + \textstyle\sum_{n\geq 6} \frac{60}{(n-5)!n(n-1)(n-2)}\, x^n\Big).$$

Here the first solution $y_1(x)$ is clearly visible.

Thus,

$$1 + \frac{1}{2}x + \frac{1}{12}x^2$$

also solves the ODE.

8.4 Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer with **no** log term
- roots differing by a positive integer with log term

The larger root always yields a fractional power series solution.

In the first and third cases, the smaller root also yields a fractional power series solution.

In the second and fourth cases, the second solution involves a log term.

9 Gamma function

Define for all $p > 0$,

$$\Gamma(p) := \int_0^\infty t^{p-1} e^{-t} dt.$$

There is a problem at $p = 0$ since $1/t$ is not integrable in an interval containing 0.

The same problem persists for $p < 0$.

For large values of p, there is no problem because e^{-t} is rapidly decreasing.

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

For any integer $n \geq 1$,

$$\begin{aligned}
\Gamma(n+1) &= \lim_{x \to \infty} \int_0^x t^n e^{-t} dt \\
&= n \left(\lim_{x \to \infty} \int_0^x t^{n-1} e^{-t} dt \right) \\
&= n\Gamma(n).
\end{aligned}$$

This yields

$$\Gamma(n) = (n-1)!.$$

Thus the gamma function extends the factorial function to all positive real numbers.

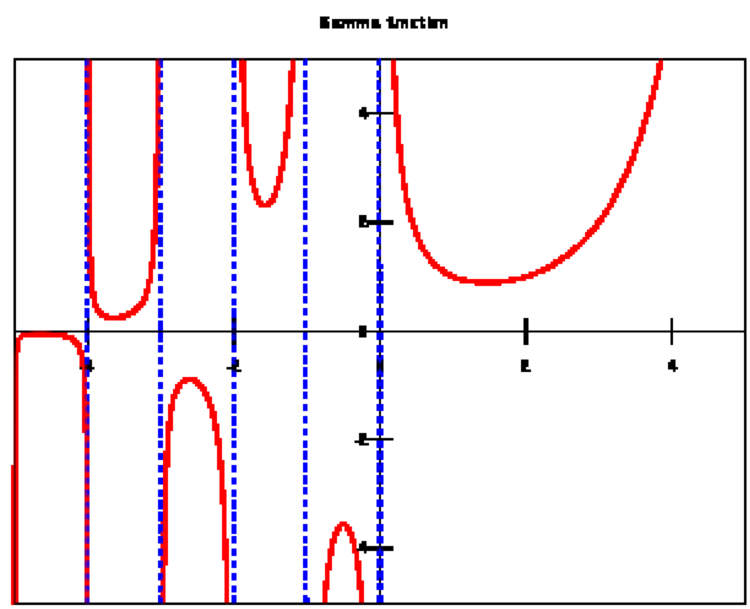
The above calculation is valid for any real $p > 0$, so

$$\Gamma(p+1) = p\,\Gamma(p).$$

We use this identity to extend the gamma function to all real numbers except 0 and the negative integers:

First extend it to the interval $(-1, 0)$, then to $(-2, -1)$, and so on.

The graph is shown below.



Though the gamma function is now defined for all real numbers (except the nonpositive integers), remember that the integral representation is valid only for $p > 0$.

It is useful to rewrite

$$\frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)}.$$

This holds for all p if we impose the natural condition that the reciprocal of Γ evaluated at a nonpositive integer is 0.

A well-known value of the gamma function at a non-integer point is

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = 2 \int_0^\infty e^{-s^2} ds = \sqrt{\pi}.$$

(We used the substitution $t = s^2$.) By translating,

$$\begin{aligned}
\Gamma(-3/2) &= \tfrac{4}{3}\sqrt{\pi} && \approx 2.363 \\
\Gamma(-1/2) &= -2\sqrt{\pi} && \approx -3.545 \\
\Gamma(3/2) &= \tfrac{1}{2}\sqrt{\pi} && \approx 0.886 \\
\Gamma(5/2) &= \tfrac{3}{4}\sqrt{\pi} && \approx 1.329 \\
\Gamma(7/2) &= \tfrac{15}{8}\sqrt{\pi} && \approx 3.323.
\end{aligned}$$

10 Bessel functions

Consider the second-order linear ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

This is known as the Bessel equation.

Here p denotes a fixed real number.

We may assume $p \geq 0$.

There is a regular singularity at $x = 0$.

All other points are ordinary.

10.1 Bessel functions of the first kind

We apply the Frobenius method to find the solutions.

In previous notation, $b(x) = 1$ and $c(x) = x^2 - p^2$.

The indicial equation is

$$r^2 - p^2 = 0.$$

The roots are $r_1 = p$ and $r_2 = -p$.

The recursion is

$$(r + n + p)(r + n - p)a_n + a_{n-2} = 0, \quad n \geq 2,$$

and $a_1 = 0$.

So all odd terms are 0.

Let us solve this recursion with r as a variable.

The even terms are

$$a_2(r) = -\frac{1}{(r+2)^2 - p^2}\, a_0,$$

$$a_4(r) = \frac{1}{((r+2)^2 - p^2)((r+4)^2 - p^2)}\, a_0,$$

and in general

$$a_{2n}(r) = \frac{(-1)^n}{((r+2)^2 - p^2)((r+4)^2 - p^2)\ldots((r+2n)^2 - p^2)}\, a_0.$$

The fractional power series solution for the larger root $r_1 = p$ obtained by setting $a_0 = 1$ and $r = p$ is

$$y_1(x) = x^p \sum_{n\geq 0} \frac{(-1)^n}{((p+2)^2-p^2)((p+4)^2-p^2)\dots((p+2n)^2-p^2)} x^{2n}$$
$$= x^p \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n}.$$

The power series converges everywhere.

Multiplying by $\frac{1}{2^p \Gamma(1+p)}$, we define

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n}, \quad x > 0.$$

This is called the Bessel function of the first kind of order p.

It is a solution of the Bessel equation.

The Bessel function of order 0 is

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots$$
$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!3!}\left(\frac{x}{2}\right)^6 + \dots$$

This is similar to $\cos x$.

The Bessel function of order 1 is

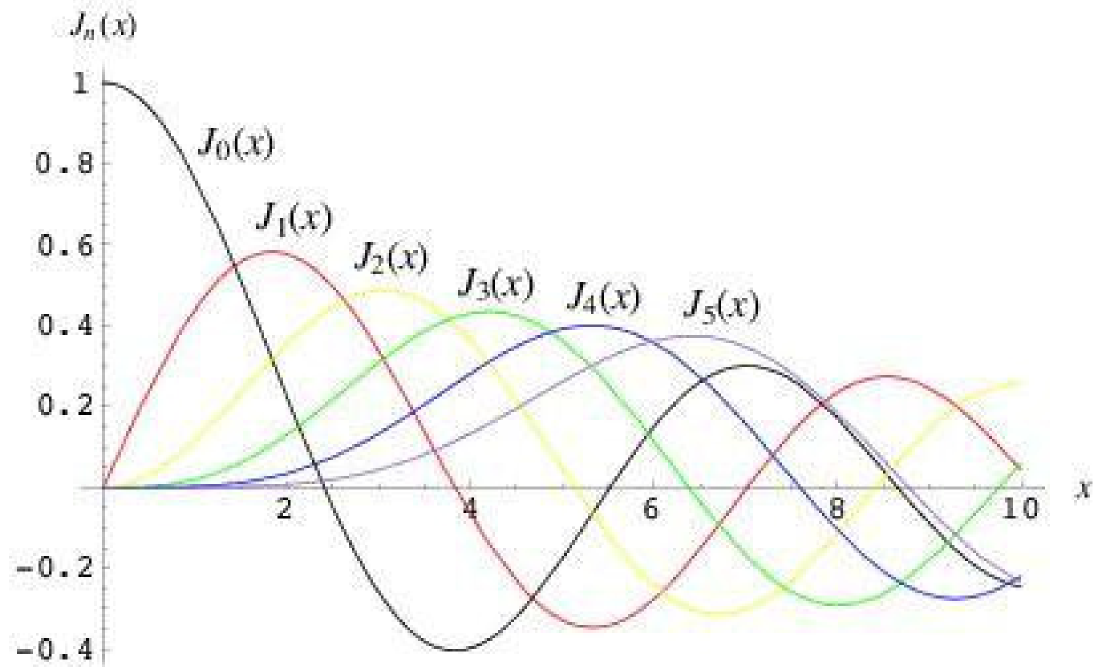
$$J_1(x) = \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots.$$

This is similar to $\sin x$.

Both $J_0(x)$ and $J_1(x)$ have a damped oscillatory behavior having an infinite number of zeroes, and these zeroes occur alternately.

Further, they satisfy derivative identities similar to $\cos x$ and $\sin x$.

$$J_0'(x) = -J_1(x) \quad \text{and} \quad [xJ_1(x)]' = xJ_0(x).$$



These functions are real analytic representable by a single power series centered at 0.

10.2 Second independent solution

Observe that $r_1 - r_2 = 2p$.

The analysis to get a second independent solution of the Bessel equation splits into four cases.

- $2p$ is not an integer:

  We get a second fractional power series solution.

- $p = 0$:

  We get a log term.

- p is a positive half-integer $1/2, 3/2, 5/2, \dots$:

  We get a second fractional power series solution.

- p is a positive integer:

  We get a log term.

We discuss the first three cases.

Case : Suppose $2p$ is not an integer.

Solving the recursion uniquely for $a_0 = 1$ and $r = -p$, we obtain

$$y_2(x) = x^{-p} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n! (1-p) \ldots (n-p)} x^{2n}.$$

Normalizing by $\frac{1}{2^{-p}\Gamma(1-p)}$, define

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n \geq 0} \frac{(-1)^n}{n!\, \Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n}, \; x > 0.$$

This is a second solution of the Bessel equation linearly independent of $J_p(x)$.

It is clearly unbounded near $x = 0$, behaving like x^{-p} as x approaches 0.

Case : Suppose p is a positive half-integer.

The same solution as in the previous case works in the half-integer case because $N = 2p$ is an odd integer, and the recursion only involves the even coefficients a_0, a_2, a_4, etc.

Remark. $J_{-p}(x)$ make sense for all p. However, when p is an integer, $J_{-p}(x) = (-1)^p J_p(x)$.

<u>Case</u> : Suppose $p = 0$.

In this case,

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2 \dots (r+2n)^2},$$

and $a_n(r) = 0$ for n odd. The first solution is a power series with coefficients $a_n(0)$:

$$y_1(x) = J_0(x) = \sum_{n \geq 0} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}, \quad x > 0.$$

Differentiating the $a_{2n}(r)$ with respect to r, we get

$$a'_{2n}(r) = -2a_{2n}(r)\Big(\frac{1}{r+2} + \frac{1}{r+4} + \dots + \frac{1}{r+2n}\Big).$$

Now setting $r = 0$, we obtain

$$a'_{2n}(0) = \frac{(-1)^{n-1} H_n}{2^{2n}(n!)^2}, \qquad H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$

Thus, the second solution is

$$y_2(x) = J_0(x) \log x - \sum_{n \geq 1} \frac{(-1)^n H_n}{2^{2n}(n!)^2} x^{2n}, \quad x > 0.$$

10.3 Summary of $p = 0$ and $p = 1/2$

For $p = 0$, we found two independent solutions.

The first function $J_0(x)$ is a real analytic function for all $\mathbb{R}$, while the second function has a logarithmic singularity at 0.

For $p = 1/2$, two independent solutions are $J_{1/2}(x)$ and $J_{-1/2}(x)$.

These can be expressed in terms of the trigonometric functions:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Both exhibit singular behavior at 0.

The first function is bounded near 0 but does not have a bounded derivative near 0, while the second function is unbounded near 0.

The substitution $u(x) = \sqrt{x}\, y(x)$ transforms the Bessel equation into

$$u'' + \big(1 + \frac{1 - 4p^2}{4x^2}\big)u = 0.$$

For $p = 1/2$, this specializes to

$$u'' + u = 0,$$

whose solutions are $\sin x$ and $\cos x$.

10.4 Bessel identities

Proposition. *For any real number p,*

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$$

The first two identities can be directly established by manipulating the respective power series.

These can then be used to prove the next two identities.

We prove the first identity.

Recall

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n}.$$

$$\begin{aligned}
(x^p J_p(x))' &= \left(2^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+2p}\right)' \\
&= 2^p \sum_{n\geq 0} \frac{(-1)^n (2n+2p)}{n!\,\Gamma(n+p+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{2n+2p-1} \\
&= 2^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p)} \left(\frac{x}{2}\right)^{2n+2p-1} \\
&= x^p \left(\frac{x}{2}\right)^{p-1} \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p)} \left(\frac{x}{2}\right)^{2n} \\
&= x^p J_{p-1}(x).
\end{aligned}$$

Using the first two identities:

$$
\begin{aligned}
J_{p-1}(x) + J_{p+1}(x) &= x^{-p}[x^p J_p(x)]' - x^p[x^{-p} J_p(x)]' \\
&= J_p'(x) + \frac{p}{x} J_p(x) - \left[J_p'(x) - \frac{p}{x} J_p(x)\right] \\
&= \frac{2p}{x} J_p(x).
\end{aligned}
$$

The identity

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

can be thought of as a recursion in p.

In other words, $J_{p+n}(x)$ can be computed algorithmically from $J_p(x)$ and $J_{p+1}(x)$ for all integer n.

Recall that we had nice formulas for $J_{\frac{1}{2}}(x)$ and $J_{\frac{-1}{2}}(x)$.

Using them, we get formulas for $J_p(x)$ for half-integer values of p.

For instance:

$$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \Big(\frac{\sin x}{x} - \cos x \Big)$$

$$J_{\frac{-3}{2}}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \Big(\frac{\cos x}{x} + \sin x \Big)$$

$$J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \Big(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x \Big)$$

10.5 Zeroes of Bessel function

Fix $p \geq 0$.

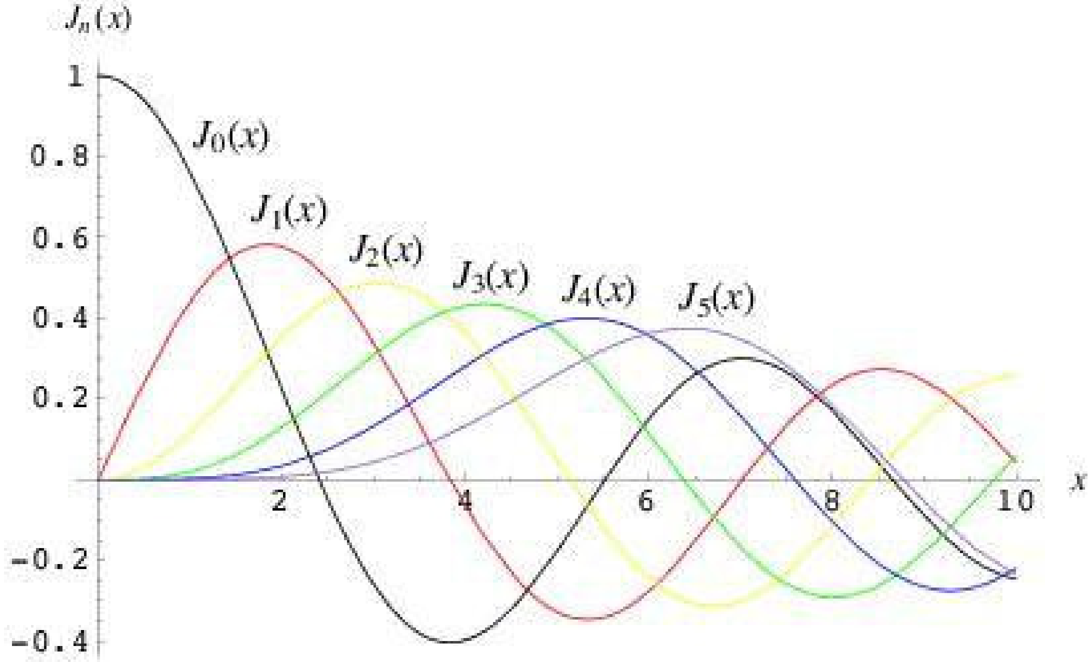
Let $Z^{(p)}$ denote the set of zeroes of $J_p(x)$.

Fact. *The set of zeroes is a sequence increasing to infinity.*

Let x_1 and x_2 be successive positive zeroes of $J_p(x)$.

- *If $0 \leq p < 1/2$, then $x_2 - x_1$ is less than π and approaches π as $x_1 \to \infty$.*
- *If $p = 1/2$, then $x_2 - x_1 = \pi$.*
- *If $p > 1/2$, then $x_2 - x_1$ is greater than π and approaches π as $x_1 \to \infty$.*

This fact can be proved using the Sturm Comparison theorem.



The first few zeroes are tabulated below.

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	J
1	2.4048	3.8317	5.1356	6.3802	7.5883	8
2	5.5201	7.0156	8.4172	9.7610	11.0647	12
3	8.6537	10.1735	11.6198	13.0152	14.3725	15
4	11.7915	13.3237	14.7960	16.2235	17.6160	18
5	14.9309	16.4706	17.9598	19.4094	20.8269	22

10.6 Scaled Bessel equation

Consider the second-order linear ODE

$$x^2y'' + xy' + (a^2x^2 - p^2)y = 0.$$

This is known as the scaled Bessel equation.

The usual Bessel equation has been scaled by the scalar a.

The function $J_p(ax)$ solves this equation.

It is called the scaled Bessel function.

10.7 Orthogonality

Define an inner product on square-integrable functions on $[0, 1]$ by

$$\langle f, g \rangle := \int_0^1 x f(x) g(x) dx.$$

This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by x, and the interval of integration is from 0 to 1.

The multiplying factor x is called a weight function.

Fix $p \geq 0$.

The set of scaled Bessel functions

$$\{J_p(zx) \mid z \in Z^{(p)}\}$$

indexed by the zero set $Z^{(p)}$ form an orthogonal family:

Proposition. *If k and ℓ are any two positive zeroes of the Bessel function $J_p(x)$, then*

$$\int_0^1 x J_p(kx) J_p(\ell x) dx = \begin{cases} \frac{1}{2}[J_p'(k)]^2 & \textit{if } k = \ell, \\ 0 & \textit{if } k \neq \ell. \end{cases}$$

In this result, p is fixed.

Proof. We prove orthogonality.

For any positive constants a and b, the functions $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfy the scaled Bessel equations

$$u'' + \frac{1}{x}u' + \big(a^2 - \frac{p^2}{x^2}\big)u = 0 \quad \text{and} \quad v'' + \frac{1}{x}v' + \big(b^2 - \frac{p^2}{x^2}\big)v = 0$$

respectively.

Multiplying the first equation by v, the second by u and subtracting,

$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv.$$

Multiplying by x, we obtain

$$[x(u'v - v'u)]' = (b^2 - a^2)xuv.$$

Note that $x(u'v - v'u)$ is bounded on the interval $(0, 1)$:

The only problem is near 0 where u' and v' may blow up if $0 < p < 1$, however multiplying by x tempers this, and the value goes to 0 as x goes to 0.

Integrating from 0 to 1, we get

$$(b^2 - a^2) \int_0^1 xuv\, dx = u'(1)v(1) - v'(1)u(1).$$

Suppose $a = k$ and $b = \ell$ are distinct zeroes of $J_p(x)$.

Then $u(1) = v(1) = 0$, so the rhs is zero.

Further, $b^2 - a^2 \neq 0$, so the integral in the lhs is zero.

This proves orthogonality. $\square$

10.8 Fourier-Bessel series

Fix $p \geq 0$.

Any square-integrable function $f(x)$ on $[0, 1]$ can be expanded in a series of scaled Bessel functions $J_p(zx)$ as

$$\sum_{z \in Z^{(p)}} c_z J_p(zx),$$

where

$$c_z = \frac{2}{[J_p'(z)]^2} \int_0^1 x\, f(x)\, J_p(zx)\, dx.$$

This is called the Fourier-Bessel series of $f(x)$ for the parameter p.

Example. Let us compute the Fourier-Bessel series (for $p = 0$) of the function $f(x) = 1$ in the interval $0 \leq x \leq 1$.

$$\int_0^1 x\, J_0(zx)\, dx = \frac{1}{z} x\, J_1(zx)\Big|_0^1 = \frac{J_1(z)}{z},$$

so

$$c_z = \frac{2}{zJ_1(z)}.$$

Thus, the Fourier-Bessel series is

$$\sum_{z \in Z^{(0)}} \frac{2}{zJ_1(z)} J_0(zx).$$

By theorem below, this converges to 1 for all $0 < x < 1$.

The Fourier-Bessel series of $f(x)$ converges to $f(x)$ in norm.

For pointwise convergence, we have the Bessel expansion theorem:

Theorem. *If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[0, 1]$, then the Bessel series converges to*

$$\frac{1}{2}(f(x_-) + f(x_+))$$

for $0 < x < 1$.

11 Fourier series

11.1 Orthogonality of the trigonometric family

Consider the space of square-integrable functions on $[-\pi, \pi]$.

Define an inner product by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x) dx.$$

The norm of a function is then given by

$$\|f\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) f(x) dx \right)^{1/2}.$$

Proposition. *The set* $\{1, \cos nx, \sin nx\}_{n\geq 1}$ *is a maximal orthogonal set with respect to this inner product.*

Explicitly,

$$\langle 1, 1\rangle = 1.$$

$$\langle \cos mx, \cos nx\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1/2 & \text{if } m = n. \end{cases}$$

$$\langle \sin mx, \sin nx\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1/2 & \text{if } m = n. \end{cases}$$

$$\langle \sin mx, \cos nx\rangle = \langle 1, \cos nx\rangle = \langle 1, \sin mx\rangle = 0.$$

This can be proved by a direct calculation.

11.2 Fourier series

Any square-integrable function $f(x)$ on $[-\pi, \pi]$ can be expanded in a series of the trigonometric functions

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where the a_n and b_n are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \quad n \geq 1,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \quad n \geq 1.$$

This is called the Fourier series of $f(x)$.

The a_n and b_n are called the Fourier coefficients.

The above formulas are sometimes called the Euler formulas.

The Fourier series of $f(x)$ converges to $f(x)$ in norm.

11.3 Pythagoras theorem or Parseval's identity

Suppose V is a finite-dimensional inner product space and $\{v_1, \ldots, v_k\}$ is an orthogonal basis.

If $v = \sum_{i=1}^{k} a_i v_i$, then $\|v\|^2 = \sum_{i=1}^{k} a_i^2 \|v_i\|^2$.

This is the Pythagoras theorem.

There is an infinite-dimensional analogue which says that the square of the norm of f is the sum of the squares of the norms of its components with respect to any maximal orthogonal set.

Thus, we have

$$\|f\|^2 = a_0^2 + \frac{1}{2} \sum_{n \geq 1} (a_n^2 + b_n^2).$$

This is known as the Parseval identity.

11.4 Pointwise convergence

A function $f:\mathbb{R}\to\mathbb{R}$ is periodic (of period 2π) if

$$f(x+2\pi)=f(x) \qquad \text{for all } x.$$

Theorem. *Let $f(x)$ be a periodic function of period 2π which is integrable on $[-\pi,\pi]$. Then at a point x, if the left and right derivative exist, then the Fourier series of f converges to*

$$\frac{1}{2}[f(x^+)+f(x^-)].$$

Example. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x < 0. \end{cases}$$

The value at 0, π and $-\pi$ is left unspecified. Its periodic extension is the square-wave.

Since f is an odd function, a_0 and all the a_n are zero. The b_n for $n \geq 1$ can be calculated as follows.

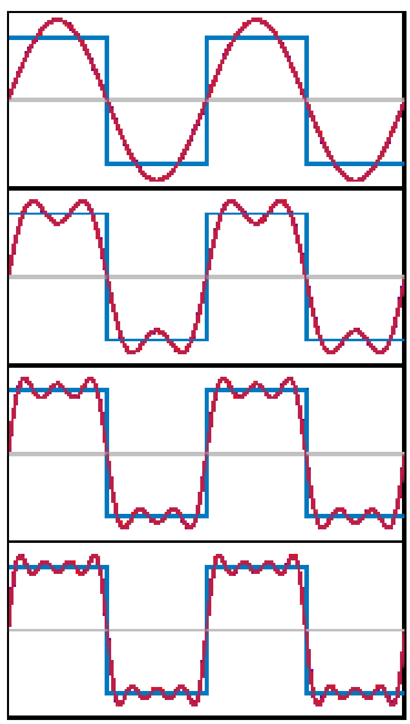
$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx \\ &= \frac{2}{n\pi}(1 - \cos n\pi) \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus the Fourier series of $f(x)$ is

$$\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right).$$

This series converges to $f(x)$ at all points except integer multiples of π where it converges to 0.

The partial sums of the Fourier series wiggle around the square wave.



In particular, evaluating at $x = \pi/2$,

$$f(\frac{\pi}{2}) = 1 = \frac{4}{\pi}\big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\big).$$

Rewriting,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Example. Consider the function

$$f(x) = x^2, \quad -\pi \le x \le \pi.$$

Since f is an even function, the b_n are zero.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{4}{n^2}\cos n\pi = \begin{cases} \frac{4}{n^2} & \text{if } n \text{ is even,} \\ -\frac{4}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} - 4\Big(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \ldots\Big).$$

This series converges to $f(x)$ at all points.

Evaluating at $x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{4} + \frac{1}{9} + \dots\right).$$

This yields the identity

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

11.5 Fourier sine and cosine series

If f is an odd (even) function, then its Fourier series has only sine (cosine) terms.

This allows us to do something interesting.

Suppose f is defined on the interval $(0, \pi)$.

Then we can extend it as an odd function on $(-\pi, \pi)$ and expand it in a Fourier sine series,

or extend it as an even function on $(-\pi, \pi)$ and expand it in a Fourier cosine series.

For instance, consider the function

$$f(x) = x, \quad 0 < x < \pi.$$

The Fourier sine series of $f(x)$ is

$$2\Big(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\Big)$$

while the Fourier cosine series of $f(x)$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\Big(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots\Big)$$

The two series are equal on $0 < x < \pi$

(but different on $-\pi < x < 0$).

The Fourier cosine series above is the same as the Fourier series of $g(x) = |x|$. Note that $g'(x)$ equals the square wave function $f(x)$ discussed before and the Fourier series of the square wave is precisely the term-by-term derivative of the Fourier series of $g(x)$.

This is a general fact which can be seen by applying derivative transfer on the Euler formulas.

For instance,

$$\begin{aligned}\int_{-\pi}^{\pi} f'(x) \sin nx dx &= -\int_{-\pi}^{\pi} f(x)(\sin nx)' dx \\ &= -n \int_{-\pi}^{\pi} f(x) \cos nx dx\end{aligned}$$

shows that the b_n of $f'(x)$ are related to the a_n of $f(x)$.

11.6 Fourier series for arbitrary periodic functions

One can also consider Fourier series for functions of any period not necessarily 2π.

Suppose the period is 2ℓ.

Then the Fourier series is of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}.$$

The Fourier coefficients are given by

$$a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) dx, \quad a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx,$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx, \quad n \geq 1.$$

By scaling the independent variable, one can transform the given periodic function to a 2π-periodic function, and then apply the standard theory.

12 One-dimensional heat equation

The one-dimensional heat equation is given by

$$u_t = k u_{xx}, \quad 0 < x < \ell,\ t > 0,$$

where k is a fixed positive constant.

There are two variables, x is the space variable and t is the time variable.

This PDE describes the temperature evolution of a thin rod of length ℓ.

More precisely, $u(x, t)$ is the temperature at point x at time t.

The temperature at $t = 0$ is specified.

This is the initial condition.

We write it as

$$u(x, 0) = u_0(x).$$

In addition to the initial condition, there are conditions specified at the two endpoints of the rod.

These are the boundary conditions.

We consider four different kinds of boundary conditions one by one.

In each case, we employ the method of separation of variables:

Suppose $u(x,t) = X(x)T(t)$.

Substituting this in the PDE, we get

$$T'(t)X(x) = kX''(x)T(t).$$

Observe that we can now separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \lambda \text{ (say)}.$$

The equality is between a function of x and a function of t, so both must be constant. We have denoted this constant by λ.

12.1 Dirichlet boundary conditions

$$u(0,t) = u(\ell,t) = 0.$$

In other words, the endpoints of the rod are maintained at temperature 0 at all times t.

(The rod is isolated from the surroundings except at the endpoints from where heat will be lost to the surroundings.)

We need to consider three cases:

1. $\lambda > 0$:

   Write $\lambda = \mu^2$ with $\mu > 0$.

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \mu^2.$$

   Then

$$X(x) = Ae^{\mu x} + Be^{-\mu x} \quad \text{and} \quad T(t) = Ce^{\mu^2 kt}.$$

   Hence

$$u(x,t) = e^{\mu^2 kt}(Ae^{\mu x} + Be^{-\mu x}),$$

   where the constant C has been absorbed in A and B.

   The boundary conditions imply that $A = 0 = B$.

   So there is no nontrivial solution of this form.

2. $\lambda = 0$:

   In this case,

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = 0.$$

   Thus, we have $X(x) = Ax + B$ and $T(t) = C$.

   Hence

$$u(x, t) = Ax + B.$$

   The boundary conditions give $A = 0 = B$.

   Thus this case also does not yield a nontrivial solution.

3. $\lambda < 0$:

Write $\lambda = -\mu^2$ with $\mu > 0$.

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\mu^2.$$

Then

$$X(x) = A\cos\mu x + B\sin\mu x \quad \text{and} \quad T(t) = Ce^{-\mu^2 kt}.$$

Hence

$$u(x,t) = e^{-\mu^2 kt}[A\cos\mu x + B\sin\mu x].$$

The boundary condition at $x = 0$ yields $A = 0$, and at $x = \ell$ yields

$$B\sin\mu\ell = 0.$$

Thus $B = 0$ unless $\mu = n\pi/\ell,\ n = 1, 2, 3, \dots$. Hence

$$u_n(x,t) = e^{-n^2(\pi/\ell)^2 kt}\sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \dots$$

are the nontrivial solutions.

The general solution is obtained by taking an infinite linear combination of these solutions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2(\pi/\ell)^2 kt} \sin \frac{n\pi x}{\ell}.$$

The coefficients b_n remain to be found.

For this we finally make use of the initial condition:

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}, \quad 0 < x < \ell.$$

It is natural to let the rhs be the Fourier sine series of $u_0(x)$ over the interval $(0, \ell)$.

Conclusion: The solution for Dirichlet boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2(\pi/\ell)^2 kt} \sin \frac{n\pi x}{\ell},$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} u_0(x) \sin \frac{n\pi x}{\ell} dx.$$

As t increases, the temperature of the rod rapidly approaches 0 everywhere.

12.2 Neumann boundary conditions

$$u_x(0,t) = 0 = u_x(\ell,t).$$

In other words, there is no heat loss at the endpoints. Thus the rod is completely isolated from the surroundings.

As in the Dirichlet case, we need to consider three cases:

1. $\lambda > 0$:

   Write $\lambda = \mu^2$ with $\mu > 0$.

   Then

   $$X(x) = Ae^{\mu x} + Be^{-\mu x} \quad \text{and} \quad T(t) = Ce^{\mu^2 kt}.$$

   The boundary conditions imply that $A = 0 = B$.

   So there is no nontrivial solution of this form.

2. $\lambda = 0$:

In this case we have $X(x) = Ax + B$ and $T(t) = C$.

Hence

$$u(x, t) = Ax + B.$$

The boundary conditions give $A = 0$.

Hence this case contributes the solution $u(x, t) = \text{constant}$.

3. $\lambda < 0$:

Write $\lambda = -\mu^2$ with $\mu > 0$.

It follows that

$$u(x,t) = e^{-\mu^2 kt}[A\cos\mu x + B\sin\mu x].$$

The boundary conditions now imply that $B = 0$.

Also $A = 0$ unless $\mu = n\pi/\ell,\ n = 1, 2, 3, \ldots$.

Thus

$$u_n(x,t) = e^{-n^2(\pi/\ell)^2 kt}\cos\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots$$

are the nontrivial solutions.

The general solution will now be of the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2(\pi/\ell)^2 kt} \cos \frac{n\pi x}{\ell}.$$

The coefficients a_n remain to be determined.

For this we finally make use of the initial condition:

$$u(x,0) = u_0(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}, \quad 0 < x < \ell.$$

Now we let the rhs must be the Fourier cosine series of $u_0(x)$ over the interval $(0,\ell)$.

Conclusion: The solution for Neumann boundary conditions is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2(\pi/\ell)^2 kt} \cos\frac{n\pi x}{\ell},$$

where

$$a_0 = \frac{1}{\ell}\int_0^{\ell} u_0(x)dx, \quad a_n = \frac{2}{\ell}\int_0^{\ell} u_0(x)\cos\frac{n\pi x}{\ell}dx.$$

All terms except for the first one tend rapidly to zero as $t \to \infty$.

So one is left with a_0, which is the mean or average value of u_0.

Physically, this means that an isolated rod will eventually assume a constant temperature, which is the mean of the initial temperature distribution.

12.3 Mixed boundary conditions

$$u(0,t) = 0 = u_x(\ell,t).$$

Thus, the left endpoint is maintained at temperature 0 (so there will be heat loss from that end), while there is no heat loss at the right endpoint.

The solution is

$$u(x,t) = \sum_{n\geq 0} b_n e^{-(n+1/2)^2(\pi/\ell)^2 kt} \sin \frac{(n+1/2)\pi x}{\ell},$$

where

$$b_n = \frac{2}{\ell}\int_0^\ell u_0(x) \sin \frac{(n+1/2)\pi x}{\ell} dx.$$

For the mixed boundary conditions

$$u_x(0,t) = 0 = u(\ell,t),$$

the solution is similar with a cosine series instead.

12.4 Periodic boundary conditions

$$u(0,t) = u(\ell,t), \quad u_x(0,t) = u_x(\ell,t).$$

The solution is

$$u(x,t) = a_0 + \sum_{n\geq 1} e^{-4n^2(\pi/\ell)^2 kt}\Big[a_n \cos\tfrac{2n\pi x}{\ell} + b_n \sin\tfrac{2n\pi x}{\ell}\Big],$$

where

$$a_0 = \frac{1}{\ell}\int_0^\ell u_0(x)dx, \quad a_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{2n\pi x}{\ell}dx$$

and

$$b_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin\frac{2n\pi x}{\ell}dx.$$

12.5 Dirichlet boundary conditions with heat source

We now solve

$$u_t - k u_{xx} = f(x,t), \quad 0 < x < \ell,\ t > 0,$$

with $u(x,0) = u_0$ specified, and with Dirichlet boundary conditions.

For simplicity, we assume $\ell = 1$.

Expand everything in a Fourier sine series over $(0,1)$:

$$u(x,t) = \sum_{n\geq 1} Y_n(t)\sin n\pi x,$$

$$f(x,t) = \sum_{n\geq 1} B_n(t)\sin n\pi x,$$

$$u_0(x) = \sum_{n\geq 1} b_n \sin n\pi x,$$

where the $B_n(t)$ and the b_n are known while the $Y_n(t)$ are to be determined.

Substituting, we obtain

$$\sum_{n\geq 1}[\dot{Y}_n(t)+kn^2\pi^2 Y_n(t)]\sin n\pi x = \sum_{n\geq 1} B_n(t)\sin n\pi x.$$

This implies that $Y_n(t)$ solves the ODE

$$\dot{Y}_n(t) + kn^2\pi^2 Y_n(t) = B_n(t), \quad Y_n(0) = b_n.$$

Example. Suppose

$$f(x,t) = \sin \pi x \sin \pi t \quad \text{and} \quad u_0(x) = 0.$$

Thus, the initial temperature of the rod is 0 everywhere but there is a heat source.

Then $b_n = 0$ for all $n \geq 1$, and

$$B_n(t) = \begin{cases} 0 & \text{for } n \neq 1, \\ \sin \pi t & \text{for } n = 1. \end{cases}$$

Therefore, for $n \neq 1$,

$$\dot{Y}_n(t) + kn^2\pi^2 Y_n(t) = 0, \quad Y_n(0) = 0$$

which implies $Y_n \equiv 0$.

For $n = 1$,

$$\dot{Y}_1(t) + k\pi^2 Y_1(t) = \sin \pi t, \quad Y_1(0) = 0.$$

Solving, we get

$$u(x,t) = \frac{1}{\pi(k^2\pi^2+1)} \left[e^{-\pi^2 kt} - \cos \pi t + k\pi \sin \pi t \right] \sin \pi x.$$

13 One-dimensional wave equation

The one-dimensional wave equation is given by

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \ell,\, t > 0,$$

where c is a fixed constant.

This describes the vibrations of a string of length ℓ.

More precisely, $u(x, t)$ represents the deflection from the mean position of the string at position x at time t.

The initial position and velocity is specified.

These are the initial conditions.

We write them as

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x).$$

In addition to the initial conditions, there are conditions specified at the two endpoints of the string.

These are the boundary conditions.

Adopting the method of separation of variables, let
$u(x,t) = X(x)T(t)$.

Substituting this in the PDE, we get

$$T''(t)X(x) = c^2 X''(x)T(t).$$

Observe that we can now separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \lambda \text{ (say)}.$$

13.1 Dirichlet boundary conditions

$$u(0,t) = u(\ell,t) = 0.$$

The boundary conditions can be rewritten as

$$X(0) = X(\ell) = 0.$$

As in the case of the heat equation we consider three cases.

1. $\lambda > 0$: Write $\lambda = \mu^2$ with $\mu > 0$. Then

$$X(x) = Ae^{\mu x} + Be^{-\mu x}.$$

   The boundary conditions imply that $A = 0 = B$. So there is no nontrivial solution of this form.

2. $\lambda = 0$: In this case we have $X(x) = Ax + B$. Again the boundary conditions give $A = 0 = B$. Thus this case also does not yield a nontrivial solution.

3. $\lambda < 0$: Write $\lambda = -\mu^2$ with $\mu > 0$.

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = -\mu^2.$$

Then

$$X(x) = A\cos\mu x + B\sin\mu x$$

and

$$T(t) = C\cos c\mu t + D\sin c\mu t.$$

The boundary conditions now imply that $A = 0$.

Also $B = 0$ unless $\mu = n\pi/\ell,\ n = 1, 2, 3, \ldots$.

Thus

$$u_n(x,t) = [C\cos\frac{cn\pi t}{\ell} + D\sin\frac{cn\pi t}{\ell}]\sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots$$

are the nontrivial solutions.

These are called the pure harmonics or simply the harmonics or the modes of vibration.

$\frac{cn\pi}{\ell}$ is called the frequency of the n-th harmonic.

The general solution (ignoring initial conditions) is

$$u(x,t) = \sum_{n \geq 1} [C_n \cos \frac{cn\pi t}{\ell} + D_n \sin \frac{cn\pi t}{\ell}] \sin \frac{n\pi x}{\ell}.$$

Now from the initial conditions,

$$C_n = \frac{2}{\ell} \int_0^\ell u_0(x) \sin \frac{n\pi x}{\ell} dx$$

and

$$D_n = \frac{2}{cn\pi} \int_0^\ell u_1(x) \sin \frac{n\pi x}{\ell} dx.$$

Thus the Fourier sine series of $u_0(x)$ and $u_1(x)$ enter into the solution.

Conclusion: The solution is

$$u(x,t) = \sum_{n\geq 1} [C_n \cos \frac{cn\pi t}{\ell} + D_n \sin \frac{cn\pi t}{\ell}] \sin \frac{n\pi x}{\ell},$$

where

$$C_n = \frac{2}{\ell} \int_0^\ell u_0(x) \sin \frac{n\pi x}{\ell} dx$$

and

$$D_n = \frac{2}{cn\pi} \int_0^\ell u_1(x) \sin \frac{n\pi x}{\ell} dx.$$

13.2 Neumann boundary conditions

$$u_x(0,t) = 0 = u_x(\ell, t).$$

They describe a a vibrating string free to slide vertically at both ends,

OR shaft vibrating torsionally in which the ends are held in place by frictionless bearings so that rotation at the ends is permitted but all other motion is prevented,

OR longitudinal waves in an air column open at both ends.

Conclusion: The solution is

$$u(x,t) = C_0 + D_0 t + \sum_{n\geq 1} [C_n \cos \frac{cn\pi t}{\ell} + D_n \sin \frac{cn\pi t}{\ell}] \cos \frac{n\pi x}{\ell},$$

where

$$C_0 = \frac{1}{\ell} \int_0^{\ell} u_0(x) dx, \quad C_n = \frac{2}{\ell} \int_0^{\ell} u_0(x) \cos \frac{n\pi x}{\ell} dx, \quad n \geq 1$$

and

$$D_0 = \frac{1}{\ell} \int_0^{\ell} u_1(x) dx, \quad D_n = \frac{2}{cn\pi} \int_0^{\ell} u_1(x) \cos \frac{n\pi x}{\ell} dx, \quad n \geq$$

Note the presence of the linear term $D_0 t$.

It says that the whole wave or the vibrating interval drifts in the direction of u-axis at a uniform rate.

Imposing the condition $\int_0^\ell u_1(x)dx = 0$ can prevent this drift.

(Note that the integral represents the "net" velocity. So things are at rest in an average sense when this integral is zero.)

At the other extreme, if $u_0(x) = 0$ and $u_1(x)$ is a constant, then $u(x,t) = D_0 t$, in the shaft example, the shaft will rotate with uniform angular velocity D_0, there will be no vibrational motion.

13.3 Mixed boundary conditions

$$u(0,t) = 0 = u_x(\ell, t).$$

The solution is

$$u(x,t)=\sum_{n\geq 0}[C_n \cos c(n+1/2)\pi t + D_n \sin c(n+1/2)\pi t] \sin(n+1/2)\pi x,$$

where

$$C_n = \frac{2}{\ell}\int_0^\ell u_0(x) \sin(n+1/2)\frac{\pi}{\ell} x dx$$

and

$$D_n = \frac{2}{c(n+1/2)\pi}\int_0^\ell u_1(x) \sin(n+1/2)\frac{\pi}{\ell} x dx.$$

13.4 Periodic boundary conditions

$$u(0,t) = u(\ell,t) \quad \text{and} \quad u_x(0,t) = u_x(\ell,t).$$

The solution is

$$u(x,t) = C_0 + D_0 t + \sum_{n \ge 1} [C_n \cos \tfrac{2cn\pi t}{\ell} + D_n \sin \tfrac{2cn\pi t}{\ell}][A_n \cos \tfrac{2n\pi x}{\ell} + B_n \sin \tfrac{2n\pi x}{\ell}]$$

where

$$C_0 = \frac{1}{\ell} \int_0^\ell u_0(x) dx,$$

$$C_n A_n = \tfrac{2}{\ell} \int_0^\ell u_0(x) \cos \tfrac{2n\pi x}{\ell} dx, \quad \text{and} \quad C_n B_n = \tfrac{2}{\ell} \int_0^\ell u_0(x) \sin \tfrac{2n\pi x}{\ell} dx.$$

And also

$$D_0 = \frac{1}{\ell} \int_0^\ell u_1(x) dx,$$

and

$$D_n A_n = \tfrac{1}{cn\pi} \int_0^\ell u_1(x) \cos \tfrac{2n\pi x}{\ell} dx, \quad \text{and} \quad D_n B_n = \tfrac{1}{cn\pi} \int_0^\ell u_1(x) \sin \tfrac{2n\pi x}{\ell} dx.$$

Note that A_n, B_n, C_n and D_n are not uniquely defined, but the above products are.

14 Nonhomogeneous case

Consider the nonhomogeneous wave equation

$$u_{tt} - u_{xx} = f(x,t), \quad 0 < x < 1,\, t > 0$$

with Dirichlet boundary conditions.

Expand everything in a Fourier sine series on $(0,1)$:

$f(x,t) = \sum_{n \geq 1} B_n(t) \sin n\pi x$ and $u(x,t) = \sum_{n \geq 1} Y_n(t) \sin n\pi x.$

Then the functions $Y_n(t)$ must satisfy

$$\ddot{Y}_n(t) + n^2\pi^2 Y_n(t) = B_n(t), \quad n = 1, 2, 3, \ldots.$$

Also let

$u_0(x) = \sum_{n \geq 1} b_n \sin n\pi x$ and $u_1(x) = \sum_{n \geq 1} b_{1n} \sin n\pi x.$

These lead to the initial conditions

$$Y_n(0) = b_n \quad \text{and} \quad \dot{Y}_n(0) = b_{1n}.$$

They determine the $Y_n(t)$ uniquely.

Example. Suppose

$$f(x,t) = \sin \pi x \sin \pi t \quad \text{and} \quad u_0(x) = u_1(x) = 0.$$

This problem has homogeneous Dirichlet boundary conditions and zero initial conditions. Thus $b_n = 0 = b_{1n}$ for all $n \geq 1$, and

$$B_n(t) = \begin{cases} 0 & \text{for } n \neq 1, \\ \sin \pi t & \text{for } n = 1. \end{cases}$$

Therefore we have $Y_n(t) = 0$ for $n \geq 2$ while $Y_1(t)$ is the solution to the IVP

$$\ddot{Y}_1(t) + \pi^2 Y_1(t) = \sin \pi t, \quad Y_1(0) = 0 = \dot{Y}_1(0).$$

Solving we find $Y_1(t) = \frac{\sin \pi t - \pi t \cos \pi t}{2\pi^2}$ and hence

$$u(x,t) = \frac{(\sin \pi t - \pi t \cos \pi t)}{2\pi^2} \sin \pi x.$$

15 Vibrations of a circular membrane

The two-dimensional wave equation is given by

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

There are three variables, x and y are the space variables and t is the time variable.

Recall the polar coordinates (r, θ) in $\mathbb{R}^2$:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The wave equation written in polar coordinates is

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

We consider the wave equation for a circular membrane of radius R.

The initial conditions are

$$u(r, \theta, 0) = f(r, \theta) \quad \text{and} \quad u_t(r, \theta, 0) = g(r, \theta).$$

We assume Dirichlet boundary conditions

$$u(R, \theta, t) = 0.$$

Physically $u(r, \theta, t)$ represents the displacement of the point (r, θ) at time t in the z-direction. These are transverse vibrations.

15.1 Radially symmetric solutions

We first find solutions which are radially symmetric, that is, independent of θ.

Thus $u = u(r,t)$ with initial and boundary conditions

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{and} \quad u(R,t) = 0.$$

The wave equation simplifies to

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r).$$

Substituting $u(r,t) = X(r)Z(t)$, we get

$$Z''(t)X(r) = c^2(X''(r) + r^{-1}X'(r))Z(t).$$

Separating variables, we obtain

$$\frac{Z''(t)}{c^2Z(t)} = \frac{X''(r) + r^{-1}X'(r)}{X(r)} = \lambda.$$

What can λ be?

It has to be negative. (We will address this point later.) Put $\lambda = -\mu^2$.

The equation in the r variable is

$$r^2 X''(r) + rX'(r) + \mu^2 r^2 X(r) = 0.$$

This is the scaled Bessel equation of order 0.

This implies that $X(r)$ is a multiple of the scaled Bessel function of the first kind $J_0(\mu r)$.

(The second solution of the Bessel equation is unbounded at $r = 0$. Since we require $u(0, t)$ to be finite, this solution has to be discarded.)

The Dirichlet boundary condition now implies
$J_0(\mu R) = 0$.

So μ must be $1/R$ times one of the countably many positive zeroes of J_0.

These are the fundamental modes of vibration of the membrane.

For any such μ (there are countably many of them), $Z(t)$ is given by

$$Z(t) = A \cos c\mu t + B \sin c\mu t.$$

The elementary solutions or pure harmonics are

$$u_n(r,t) = (A_n \cos c\mu_n t + B_n \sin c\mu_n t) J_0(\mu_n r),$$

where $\mu_n R$ is the n-th positive zero of J_0.

The center of the membrane vibrates with the maximum amplitude.

For $n = 1$, the membrane moves fully up and then fully down. All points except those on the boundary vibrate.

For $n = 2$, there is a circle of radius $\frac{\mu_1}{\mu_2} R$ which does not move at all. This is called a nodal line. When the portion inside the nodal line moves up, the portion outside moves down, and vice versa.

What happens for $n = 3$?

There are two nodal lines.

Visualize this vibration.

Conclusion: The solution is given by

$$u(r,t) = \sum_{n\geq 1} (A_n \cos c\mu_n t + B_n \sin c\mu_n t) J_0(\mu_n r).$$

The constants can be calculated by expanding the initial position $f(r)$ and initial velocity $g(r)$ into Fourier-Bessel series. Thus,

$$A_n = \frac{2}{R^2 J_1^2(\mu_n R)} \int_0^R r f(r) J_0(\mu_n r) dr$$

and

$$B_n = \frac{2}{c\mu_n R^2 J_1^2(\mu_n R)} \int_0^R r g(r) J_0(\mu_n r) dr.$$

15.2 Reason for not allowing λ to be zero or positive

- Suppose $\lambda = 0$.

  Then we need to solve

  $$r^2 X''(r) + rX'(r) = 0.$$

  This is a Cauchy-Euler equation.

  The general solution is $A \log r + B$.

  It must be finite at $r = 0$ which gives $A = 0$, while it is zero at $r = R$ which gives $B = 0$.

  Thus, $A = B = 0$.

- Suppose $\lambda = \mu^2 > 0$.

  Then we get the Bessel equation scaled by the imaginary number $i\mu$.

  Since u must be bounded as $r \to 0$, discarding the unbounded solution, we are left with

$$J_0(i\mu r) = \sum_{k \geq 0} \frac{(\mu r)^{2k}}{4^k (k!)^2}.$$

  But since this is a series of positive terms only, it cannot satisfy $X(R) = 0$.

15.3 General solution

We solve for $u(r, \theta, t) = X(r)Y(\theta)Z(t)$.

Substituting in $u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta})$,

we get

$$Z''(t)X(r)Y(\theta)$$
$$= c^2(X''(r)Y(\theta)+r^{-1}X'(r)Y(\theta)+r^{-2}X(r)Y''(\theta))Z(t).$$

The method of separation of variables now proceeds in two steps:

We have a total of 3 variables and we need to separate one variable at a time.

Separating θ,

$$\frac{Y''(\theta)}{Y(\theta)} = -\frac{r^2X''(r) + rX'(r)}{X(r)} + \frac{r^2Z''(t)}{c^2Z(t)}.$$

What can this constant be?

Periodic boundary conditions in θ force the constant to be $-n^2$ for $n = 0, 1, 2, 3, \ldots$. We have

$$Y(\theta) = A \cos n\theta + B \sin n\theta.$$

($n = 0$ recovers the radially symmetric case.)

Putting the constant equal to $-n^2$ and separating r and t,

$$\frac{Z''(t)}{c^2 Z(t)} = \frac{X''(r) + r^{-1} X'(r)}{X(r)} - \frac{n^2}{r^2}.$$

For reasons similar to the radially symmetric case, this constant must be negative. So we write it as $-\mu^2$.

The equation in the r variable

$$r^2 X''(r) + rX'(r) + (\mu^2 r^2 - n^2)X(r) = 0.$$

This is the scaled Bessel equation of order n.

The elementary solutions are

$$(A\cos n\theta + B\sin n\theta)(C\cos c\mu t + D\sin c\mu t)J_n(\mu r),$$

where μR is a positive zero of J_n.

First visualize the amplitude $\cos\theta J_1(\mu r)$, where μR is the first positive zero of J_1.

Now visualize the corresponding vibration.

There is a nodal like along a diameter, when the portion on one side of the diameter is up, the other portion is down, and vice versa.

16 Coordinate systems

Polar coordinates (r, θ) in $\mathbb{R}^2$:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Spherical polar coordinates (r, θ, φ) in $\mathbb{R}^3$:

$$x = r\cos\theta\sin\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\varphi.$$

In all three cases, $0 \leq \theta \leq 2\pi$, while in the last case $0 \leq \varphi \leq \pi$.

17 Laplacian operator

The Laplacian on the line is $\Delta_{\mathbb{R}}(u) = u''$.

The Laplacian on the plane is

$$\begin{aligned}\Delta_{\mathbb{R}^2}(u) &= u_{xx} + u_{yy} \\ &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.\end{aligned}$$

These are standard coordinates and polar coordinates respectively.

The Laplacian on three-dimensional space is

$$\begin{aligned}\Delta_{\mathbb{R}^3}(u) &= u_{xx} + u_{yy} + u_{zz}, \\ &= (u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) + u_{zz}, \\ &= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\varphi\varphi} + \cot\varphi u_{\varphi} + \frac{1}{\sin^2\varphi}u_{\theta\theta}).\end{aligned}$$

These are standard coordinates, cylindrical coordinates and spherical polar coordinates respectively.

The Laplacian on the sphere of radius R is

$$\Delta(u) = \frac{1}{R^2}(u_{\varphi\varphi} + \cot\varphi u_{\varphi} + \frac{1}{\sin^2\varphi} u_{\theta\theta}).$$

When $R = 1$, we write $\Delta_{\mathbb{S}^2}(u)$ instead of $\Delta(u)$.

17.1 General heat and wave equation

The Laplacian can be used to define the heat equation in general:

$$u_t = k\Delta(u),$$

where Δ is the Laplacian on the domain under consideration.

Similarly, the general wave equation can be written as

$$u_{tt} = c^2\Delta(u).$$

17.2 Eigenfunctions for Laplacian on unit sphere

Consider the eigenvalue problem on the unit sphere:

$$\Delta_{\mathbb{S}^2}(u) = \mu\, u.$$

Any solution u is called an eigenfunction and the corresponding μ is called its eigenvalue.

Explicitly, we want to solve

$$\frac{\partial^2 u}{\partial \varphi^2} + \cot\varphi \frac{\partial u}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} = \mu\, u.$$

Put $u(\theta, \varphi) = X(\theta)Y(\varphi)$.

Substituting in the PDE,

$$X(\theta)Y''(\varphi) + \cot\varphi X(\theta)Y'(\varphi) + \frac{1}{\sin^2\varphi}X''(\theta)Y(\varphi)$$
$$= \mu\, X(\theta)Y(\varphi).$$

Dividing by $X(\theta)Y(\varphi)$ and separating variables,

$$\sin^2\varphi\left[\frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi\frac{Y'(\varphi)}{Y(\varphi)} - \mu\right] = -\frac{X''(\theta)}{X(\theta)}.$$

The constant is forced to be m^2, for some nonnegative integer m because of periodic boundary conditions in θ.

The equation in φ becomes

$$Y''(\varphi) + \cot\varphi Y'(\varphi) - (\mu + \frac{m^2}{\sin^2\varphi})Y(\varphi) = 0.$$

The change of variable $x = \cos\varphi$ yields

$$(1-x^2)y'' - 2xy' - (\mu + \frac{m^2}{1-x^2})y = 0.$$

(Here x is the coordinate along the z-axis.)

This is the associated Legendre equation.

Putting $m = 0$ recovers the Legendre equation.

The associated Legendre equation has a bounded solution iff $\mu = -n(n+1)$ for some nonnegative integer n.

The bounded solution is given by

$$y(x) = (1-x^2)^{m/2} P_n^{(m)}(x).$$

Going back to the φ coordinate,

$$Y(\varphi) = \sin^m \varphi \, P_n^{(m)}(\cos \varphi).$$

Thus

$$\cos m\theta \sin^m \varphi \, P_n^{(m)}(\cos \varphi), \quad m = 0, 1, 2, \ldots n$$
$$\sin m\theta \sin^m \varphi \, P_n^{(m)}(\cos \varphi), \quad m = 1, 2, \ldots n$$

are eigenfunctions for the Laplacian operator on the unit sphere with eigenvalues $-n(n+1)$.

In particular,

$$\cos \varphi, \quad \sin \theta \sin \varphi, \quad \cos \theta \sin \varphi$$

are eigenfunctions with eigenvalue -2.

(These are nothing but z, y and x in spherical-polar coordinates.)

17.3 Vibrations of a spherical membrane

The wave equation on the unit sphere is

$$u_{tt} = \Delta_{\mathbb{S}^2} u.$$

Putting $u(\theta, \varphi, t) = X(\theta)Y(\varphi)Z(t)$, we obtain

$$\frac{Z''(t)}{Z(t)} = \frac{\Delta_{\mathbb{S}^2}(X(\theta)Y(\varphi))}{X(\theta)Y(\varphi)} = \mu.$$

What can μ be?

Since μ is an eigenvalue of the Laplacian on the sphere, it has the form

$$\mu = -n(n+1).$$

The elementary solutions or the pure harmonics are

$$(A\cos m\theta + B\sin m\theta)\sin^m\varphi P_n^{(m)}(\cos\varphi)(C\cos(\sqrt{n(n+1)}t) + D\sin(\sqrt{n(n+1)}t)),$$

for nonnegative integers $0 \le m \le n$ and $n \ne 0$.

The case $m = 0$ corresponds to the harmonics which do not depend on θ. These are called latitudinally symmetric solutions. We elaborate on these below.

The maximum amplitudes of vibrations are at the north and south poles. For $n = 1$, the equator is a nodal line, the maximum amplitudes are at the poles. When the north pole starts moving in towards the center along the z-axis, the south pole starts moving down the z-axis. The opposite happens in the other part of the cycle.

What happens for $n = 2$?

General case: There are n values of φ for which $P_n(\cos\varphi) = 0$, the corresponding latitudes are the nodal lines.

What happens for $n = 0$?

$$Z(t) = A + Bt$$

17.4 Laplace equation in three space

Recall

$$\Delta_{\mathbb{R}^3}(u) = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\varphi\varphi} + \cot\varphi u_\varphi + \frac{1}{\sin^2\varphi}u_{\theta\theta}).$$

Consider the Laplace equation in three space:

$$\Delta_{\mathbb{R}^3}(u) = 0.$$

Putting $u(r,\theta,\varphi) = Z(r)X(\theta)Y(\varphi)$, we obtain

$$\frac{r^2 Z''(r)}{Z(r)} + \frac{2rZ'(r)}{Z(r)} = -\frac{\Delta_{\mathbb{S}^2}(X(\theta)Y(\varphi))}{X(\theta)Y(\varphi)}.$$

From the eigenvalue problem for the Laplacian, we know that this constant must be $n(n+1)$ for some nonnegative integer n.

The equation in the r variable is

$$r^2 Z''(r) + 2rZ'(r) - n(n+1)Z(r) = 0.$$

This is a Cauchy-Euler equation with solutions r^n and $\frac{1}{r^{n+1}}$.

The elementary solutions are

$$(Cr^n + D\frac{1}{r^{n+1}})(A\cos m\theta + B\sin m\theta)\sin^m \varphi P_n^{(m)}(\cos\varphi),$$

where $0 \leq m \leq n$ are nonnegative integers.

17.5 Other domains

In the circular membrane case, we did not consider the second solution to the Bessel equation since it is unbounded at $r = 0$. However, this solution would have to be considered if we are looking at an annulus.

Similar remark applies while working with a spherical cap (such as the upper hemisphere) as opposed to the full sphere.