MA 205 Complex Analysis: Real integral and Maximum modulus principle

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Recall

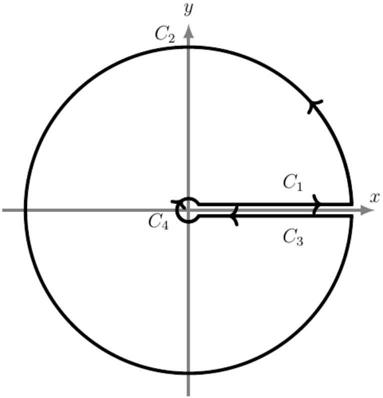
Last time we began by understanding the notion of an isolated singularity at ∞. Much the same way as isolated singularity at a point in $\mathbb{C}$, we can classify isolated singularities at infinity into removable, pole and essential singularity.
We then looked at various examples of computing residues and contour integrals. Let us begin by looking at some more today.

Examples

Show that $\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.
We'll integrate

$$f(z) = \frac{z^{-c}}{1+z},$$

where z^{-c} is the branch corresponding to branch cut being the positive real axis. Consider the contour $\gamma = C_1 \cup C_2 \cup C_3 \cup C_4$:



Real Integrals

By residue theorem,

$$\int_\gamma \frac{z^{-c}}{1+z}dz = 2\pi\imath e^{-\imath\pi c}.$$

Note that

$$\int_r^R \frac{t^{-c}}{1+t}dt = \lim_{\delta\to 0}\int_{C_1} \frac{z^{-c}}{1+z}dz.$$

Similarly,

$$\lim_{\delta\to 0}\int_{C_3} \frac{z^{-c}}{1+z}dz = -e^{-2\pi\imath c}\int_r^R \frac{t^{-c}}{1+t}dt.$$

Let γ_ρ be the ρ radius of circle. Then,

$$\left| \int_{\gamma_\rho} \frac{z^{-c}}{1+z} dz \right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho.$$

This is zero in the limit as $\rho \to 0$ or $\rho \to \infty$. Thus we get:

$$2\pi\imath e^{-\imath\pi c} = (1 - e^{-2\imath\pi c}) \int_0^\infty \frac{t^{-c}}{1+t} dt.$$

Thus,

$$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi\imath e^{-\imath\pi c}}{1 - e^{-2\imath\pi c}} = \frac{\pi}{\sin \pi c}.$$

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

By residue theorem, $\int_\gamma \frac{e^{z/2}dz}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.
Now $|cosh(L + iy)| = |e^{L+iy} + e^{-L-iy}|/2 \geq \frac{1}{2}(|e^{L+iy}| - |e^{-L-iy}|) = (e^L - e^{-L})/2 \geq e^L/4$ (for L sufficiently large)
From this it follows from the ML-inequality that as L tends to ∞, the integral along the right vertical side tends to zero. Similarly one checks that the integral along the left vertical side also tend to zero.

Example cont ..

Now since $cosh(x + i\pi) = -coshx$, the integrals along the horizontal sides are related by

$$\int_{L}^{-L} \frac{e^{(x+i\pi)/2}dx}{cosh(x+i\pi)} = e^{i\pi/2} \int_{-L}^{L} \frac{e^{x/2}dx}{coshx}$$

Taking L tending to ∞ , we see that
$I = \frac{2\pi e^{i\pi/4}}{(1+e^{i\pi/2})} = \frac{\pi}{cos(\pi/4)} = \pi\sqrt{2}$.

Choice of contour for integration

It might be a bit of a mystery as to which contour one should consider for a given contour integral. Here is a general recipe. Suppose the improper integral is of the form $\int_{-\infty}^{\infty} f(x)dx$. The general idea of course is to find a contour which contains the real line as part of the contour in the "limit". The choice should be made so that by residue theory one knows the integral over the full contour and such that the integal over the extra added part goes to zero in the limit.

I. For instance suppose there exists a constant C such that $|f(z)| \leq \frac{C}{|z^r|}$ for sufficiently large $|z|$ and for some $r > 1$ (here $f(z)$ is an extension of $f(x)$ to a function of the complex variable). Note that this happens for instance in the case when $f(x) = P(x)/Q(x)$ where $\deg Q(x) \geq \deg\, P(x) + 2$. Then close up the interval with a semicircle into the upper half plane and integrate along the contour and take limit as the radius of semicircle goes to infinity. Use ML inequality to show that the integral along the semicircle goes to zero as radius goes to ∞.

Choice of contour for integration

In case the integral is from 0 to ∞ , try and relate it to some integral from $-\infty$ to ∞. For instance the function may have a natural continuation to the negative reals. In case this is not possible, often because $f(z)$ has a singularity at origin; usually a pole, then try using a half annular region $A(0; r, R)$ like we have done in earlier examples. This will avoid the pole and then show that in the limit as R tends to ∞ and r tends to zero, the integrals along the larger and smaller circle tend to zero. Hence in the limit we will get integral over the real line. Then try and relate the integral over the positive real to that over the negative reals. See the example of $\int_0^\infty \frac{log(x)}{1+x^2}$ as an illustration of this.

Choice of contour for integration

II. If the integrand is of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} sin(x)dx$, where $P(x)$ and $Q(x)$ are polynomials with $degQ(x)$ atleast one more than that of $P(x)$, close up the interval by the semicircular region in the upper half plane and use Jordan's lemma to show that the integral over the semicircle goes to zero using Jordan's lemma. (We have seen this when we integrated $sin(x)/x$).

III. If the integrand is of the type $\int_0^{2\pi} P(cos(t), sin(t))dt$, set $z = e^{it}$ and use $cos(t) = \frac{z+z^{-1}}{2}$ and $sin(t) = \frac{z-z^{-1}}{2i}$. dt becomes $\frac{dz}{iz}$ and then the integral assumes the form $\int_{|z|=1} P(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{dz}{iz}$ which can then be computed by using residue theorem.

IV. If the integrand has infinitely many poles going to infinity, you are usually better off using a rectangular contour which emcompasses only finitely many poles.

Choice of contour for integration

As before one tries to show that in the limit, the integral over the extra added vertical sides goes to zero in the limit and the intergals over the two horizontal sides are related ; usually proportional to each other. Thus taking limit as the length of the rectangular sides goes to infinity, one gets the desired answer.

V. In case the function involves a branch cut, choose a contour which avoids (goes around) the branch cut like in the earlier example.

Maximum Modulus Theorem

An important theorem in Complex Analysis states that a non-constant holomorphic function on an open connected domain never attains its maximum modulus at any point in the domain. This is called the maximum modulus theorem. Once again, this is vastly different from what happens to real differentiable functions; in fact even for real analytic functions. Real analytic functions can achieve maximum anywhere inside the interval. We'll use CIF and the identity theorem to prove MMT.

Proof: Suppose there is $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. Then we'll prove that f is a constant. Let γ be a small circle around z_0 with radius r such that the closed disc with boundary γ is contained in Ω. CIF gives,

$$f(z_0) = \frac{1}{2\pi\imath}\int_\gamma \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{\imath\theta})d\theta.$$

Hence,

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{\imath\theta})|d\theta \leq |f(z_0)|,$$

since $|f(z_0)|$ is assumed to be the maximum value.

Thus,

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{\imath\theta})| \right] dt = 0.$$

Note that the integrand is non-negative. Therefore it has to be zero; i.e., $|f(z_0)| = |f(z_0 + re^{\imath\theta})|$ for all θ. Since this is true for each small r, we see that $|f(z)|$ is a constant on a small disc around z_0. This means that $f(z)$ is a constant, say c, on this small disc. (Why?) This implies that $f \equiv c$ on Ω by the identity theorem, since a disc has limit points!

Schwartz lemma

A nice consequence of the Maximum modulus principle is the following lemma of Schwartz.

<u>**Schwarz Lemma :**</u> Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ on $\mathbb{D}$.
Then, $|f(z)| \leq |z| \; \forall z \in \mathbb{D}$ and $|f'(0)| \leq 1$.
Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$.
Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that
$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$
As $r \to 1$ we get $|g(z)| \leq 1$.

Moreover, suppose $|f(z)| = |z|$ for some non-zero z in $\mathbb{D}$, or $|f'(0)| = 1$.Then, $|g(z)| = 1$ at some point of $\mathbb{D}$. Hence by Maximum Modulus Principle, $g(z)$ is a constant , say a with $|a| = 1$. Therefore, $f(z) = az$, as desired.

Open Mapping Theorem

The maximum modulus theorem is a special case of a even more powerful theorem called the Open Mapping Theorem.

Theorem: Any non-constant holomorphic function defined on a domain $\Omega \subseteq \mathbb{C}$ is open; i.e, maps open subsets of $\mathbb{C}$ contained in Ω to open subsets of $\mathbb{C}$.

The theorem has an interesting proof which unfortunately we will skip due to lack of time.