 

The Essentials of Incompressible Flow

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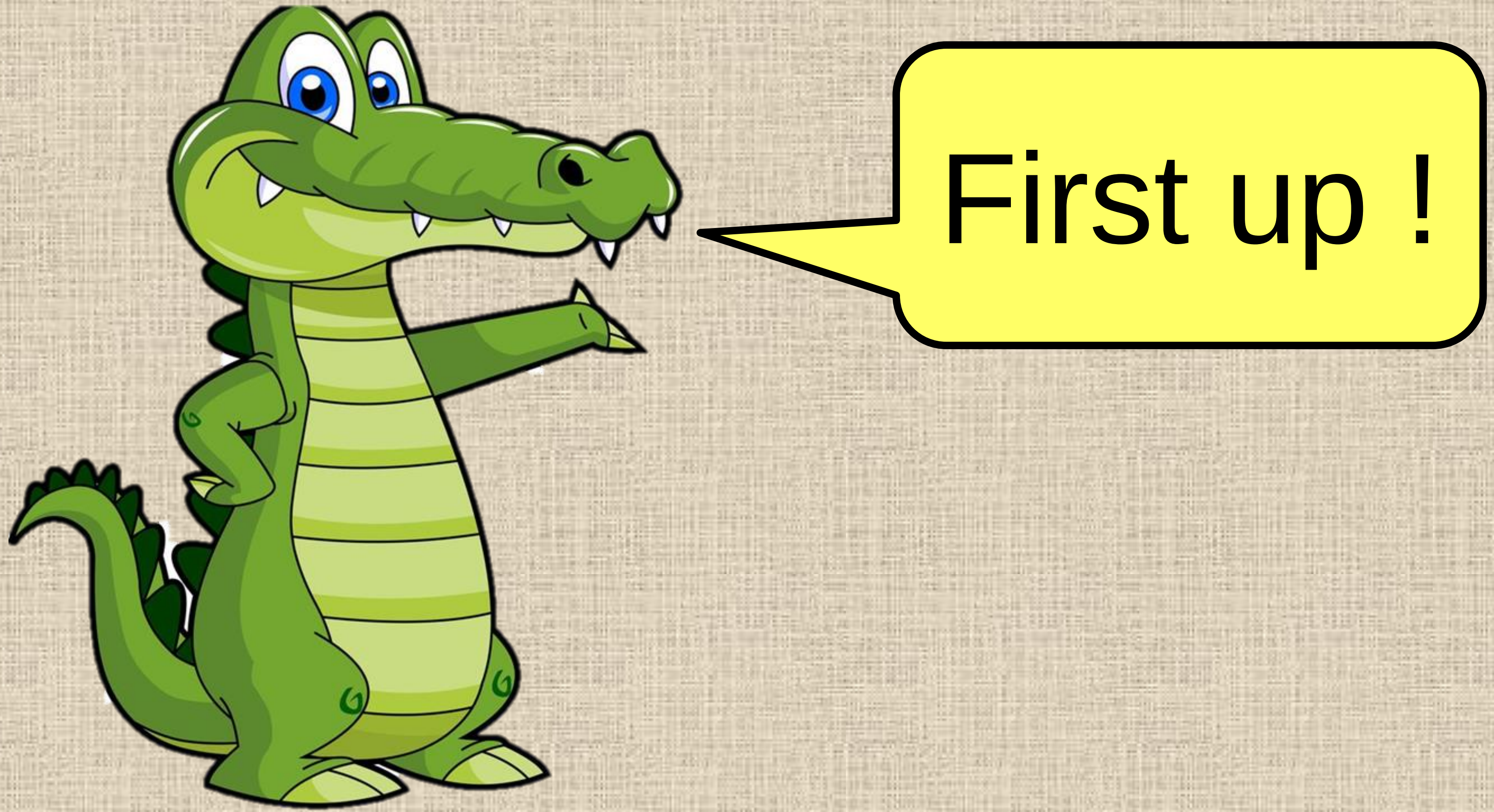


Basic Fluid Kinematics

- Euler versus Lagrange Approaches
- Material Derivative concept
- Steady and Uniform flow

In a multitude of moving cars imagine

Each Vehicle = Fluid Particle

Lets focus our attention on any vehicle

Now,
if you happen
to do
something
like this …

Lagrangian

Measurement

(literally…)

Lagrangian Approach

- Identifying and tracking a particular **Fluid Element**
- Obvious notion in the Mechanics of **Solids**
- Properties of the **element** $\rightarrow f(t)$

$$\vec{R}_{particle} \equiv position\ vector$$

$$\vec{V}_{particle}(t) = \frac{d}{dt}\vec{R}_{particle}(t)$$

$$\vec{a}_{particle}(t) = \frac{d}{dt}\vec{V}_{particle}(t)$$

Example

Helium filled balloons

Probe (measures some property)

This probe makes Lagrangian Measurements !

On the other hand ….

If you were to fix yourself at a Point and

Observe the motion at that point

Eulerian Description

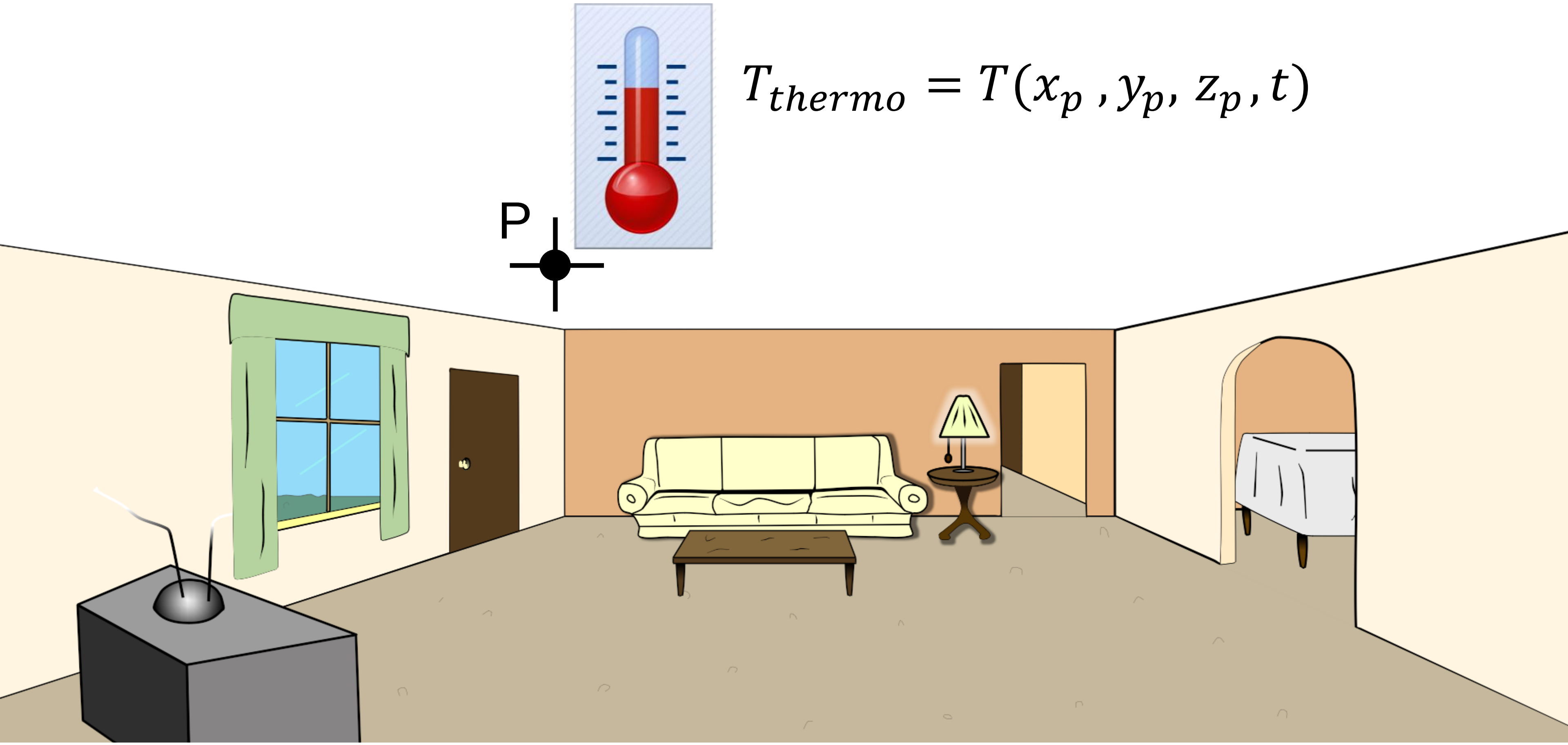
Eulerian Approach

- Observing a **spatial** point or region
- Has more mathematical implications
- Properties expressed as **field functions** of **space** and **time**

For instance, Velocity field can be written as

$$\vec{V} = \vec{V}(x, y, z, t)$$

$$T_{thermo} = T(x_p\,, y_p, z_p\,, t)$$



The Thermometer **fixed in space** makes Eulerian measurements

Physically understanding the Eulerian Approach

if it is given that

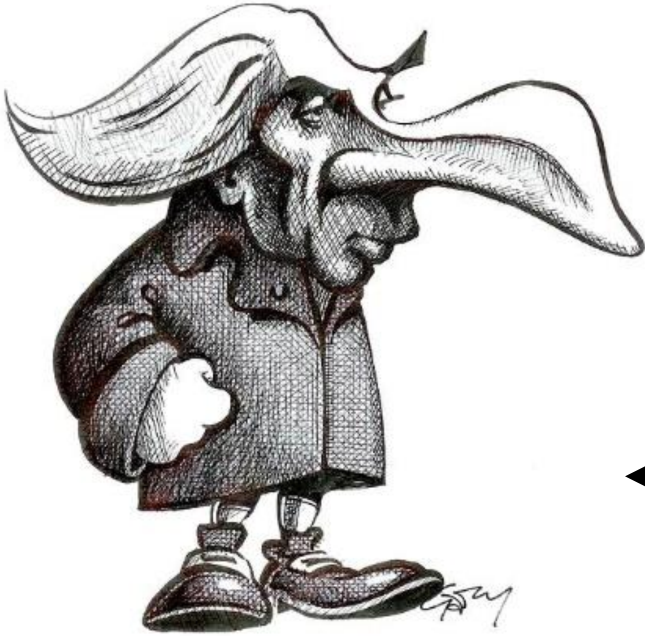
$$\vec{V}(x = 3, y = 1, z = 5, t = 3.5) = 2\,\vec{i} + 9\,\vec{j} + 7\,\vec{k}$$

What can one infer from this ?

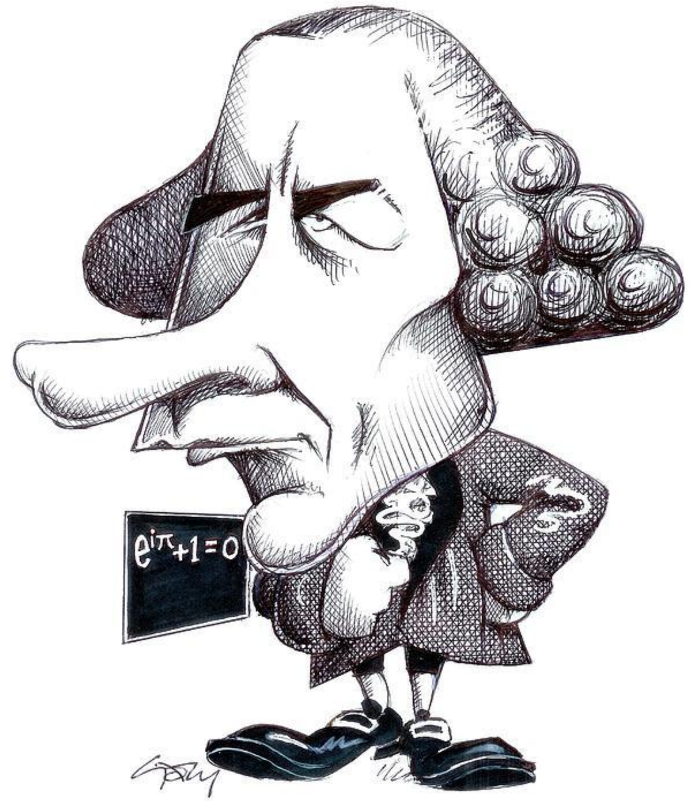
At time t=3.5

the particle present at (3,1,5)

has a velocity of $2\,\vec{i} + 9\,\vec{j} + 7\,\vec{k}$ units

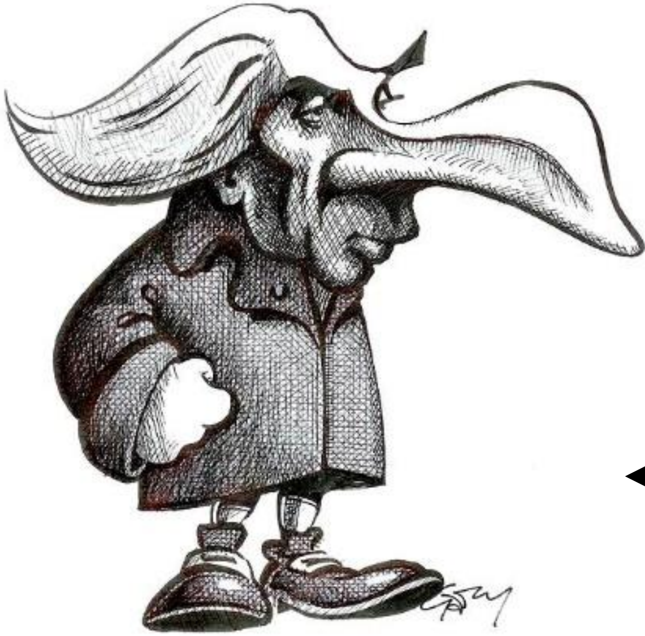


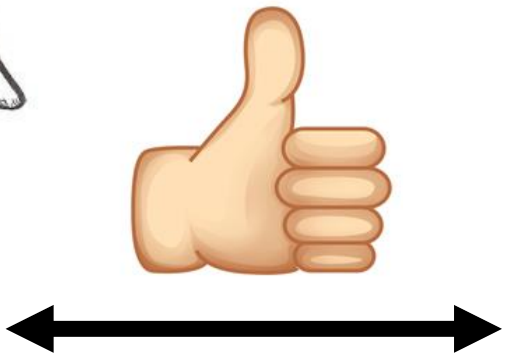
?

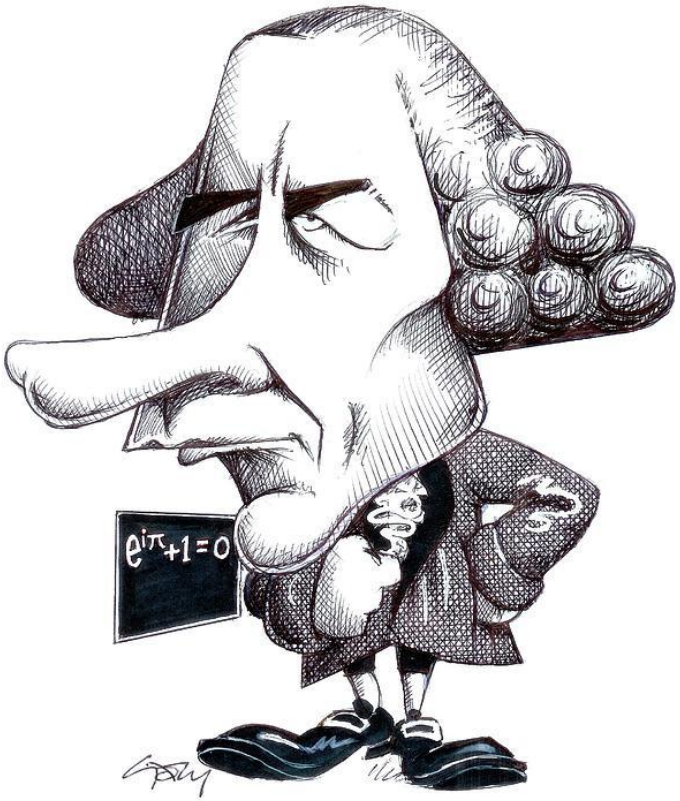


Lagrangian

Eulerian





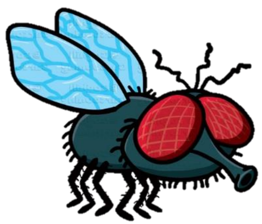


Lagrangian

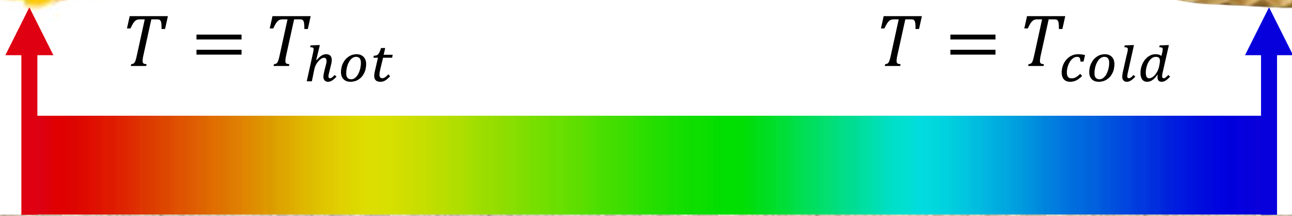
Eulerian

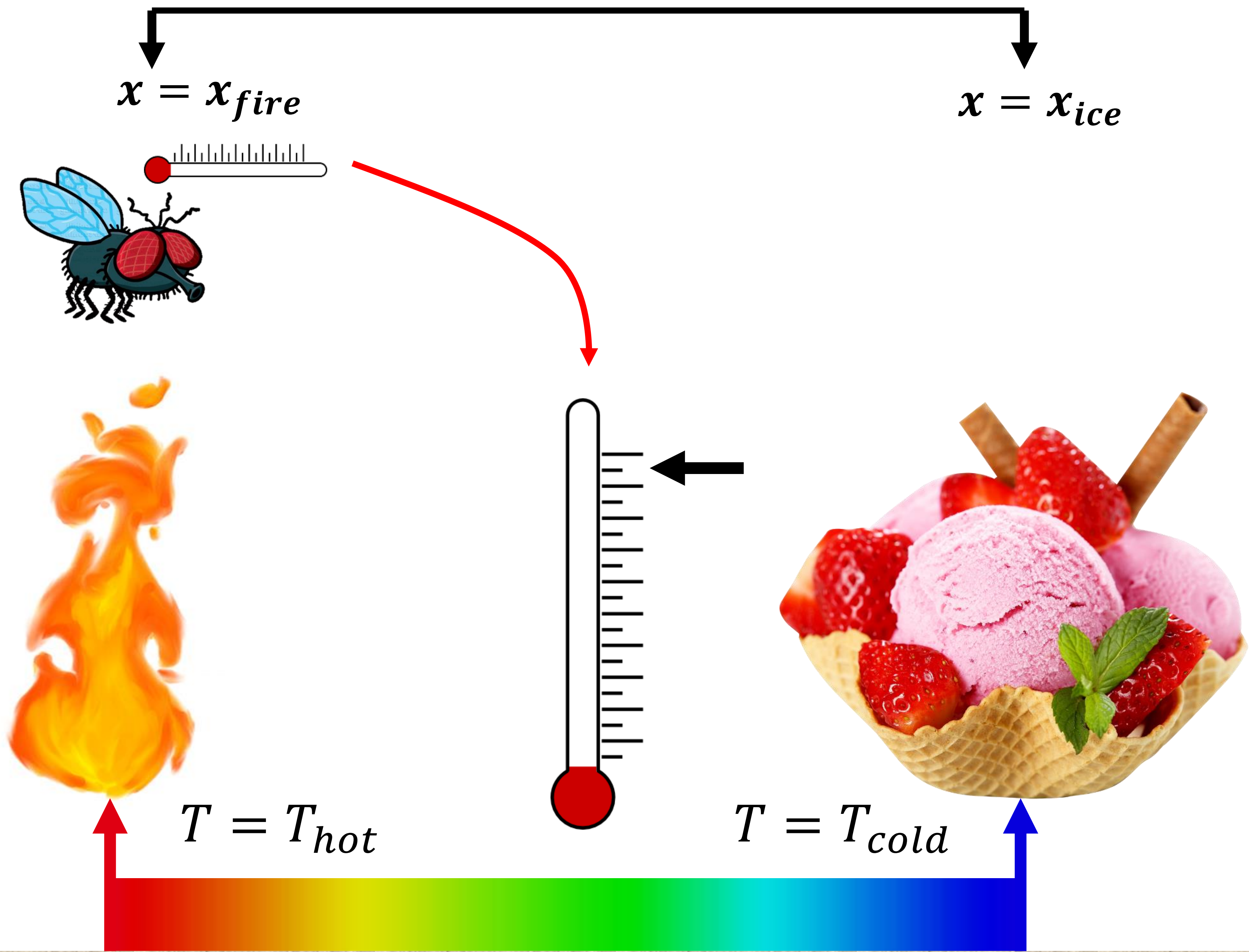
Reynold's Transport Theorem

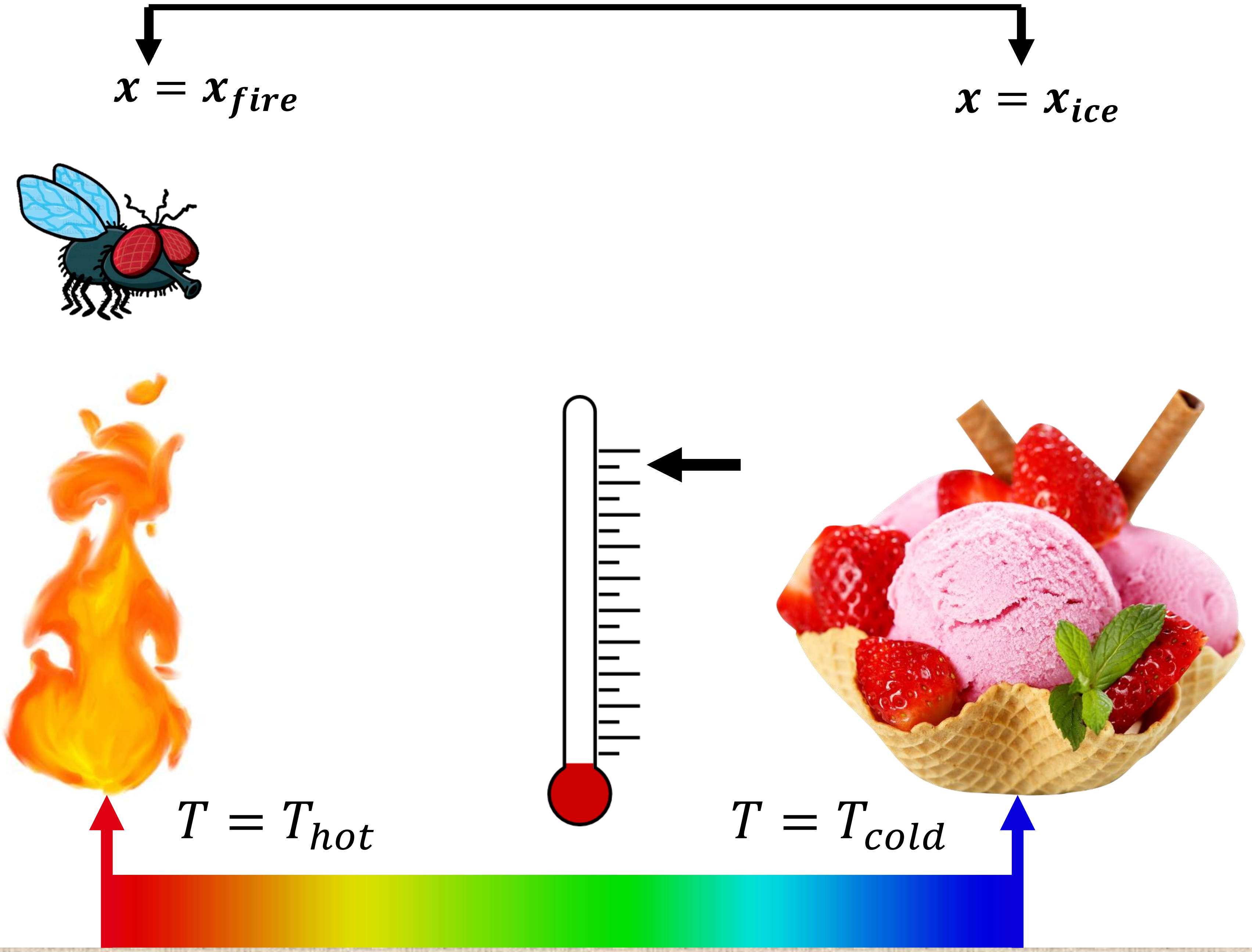
Concept of Material Derivative



$T = T_{hot}$ $\qquad$ $T = T_{cold}$

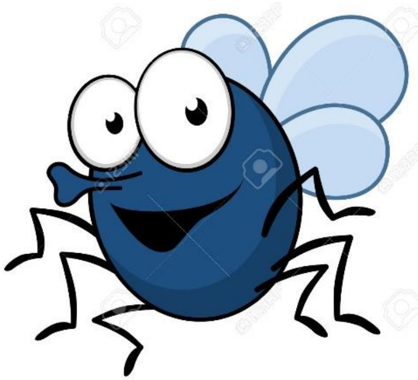



$x = x_{fire}$
$x = x_{ice}$
$T = T_{hot}$
$T = T_{cold}$


$x = x_{fire}$
$x = x_{ice}$
$T = T_{hot}$
$T = T_{cold}$

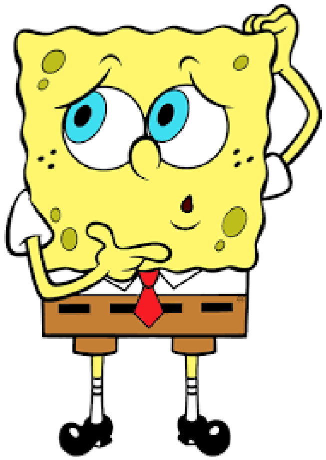
Some obvious observations

- Lagrangian approach given that we followed Mr. Fly
- Temperature changed for Mr. Fly just by virtue of his motion



- Rate of temperature change for Mr. Fly ↑ if

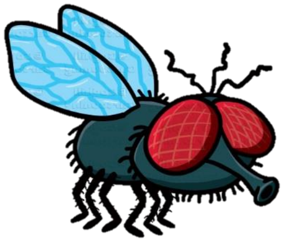
(1) ↑ Speed of Mr. Fly (2) ↑ Temperature Gradient

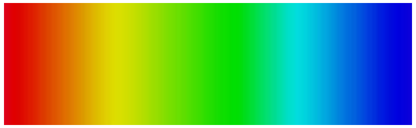


How do we put it down

Mathematically ?

$$\underbrace{\frac{dT}{dt}}_{\text{Rate of Temperature change}} = \underbrace{\frac{dT}{dx}}_{\text{Temperature Gradient}} * \underbrace{V_{fly}}_{\text{Speed of the Fly}}$$





Imagine now that the Ice cream starts melting

Temp at every point becomes Function of time

$$T = T(x, t)$$

$$T = T(x,t)$$

By Total Derivative theorem

$$\frac{DT}{Dt} = \frac{\partial T}{\partial x} * \frac{dx}{dt} + \frac{\partial T}{\partial t} * \frac{dt}{dt}$$

$$\Rightarrow \frac{DT}{Dt} = \frac{\partial T}{\partial x} * V + \frac{\partial T}{\partial t} \longrightarrow \text{Eq. 1}$$

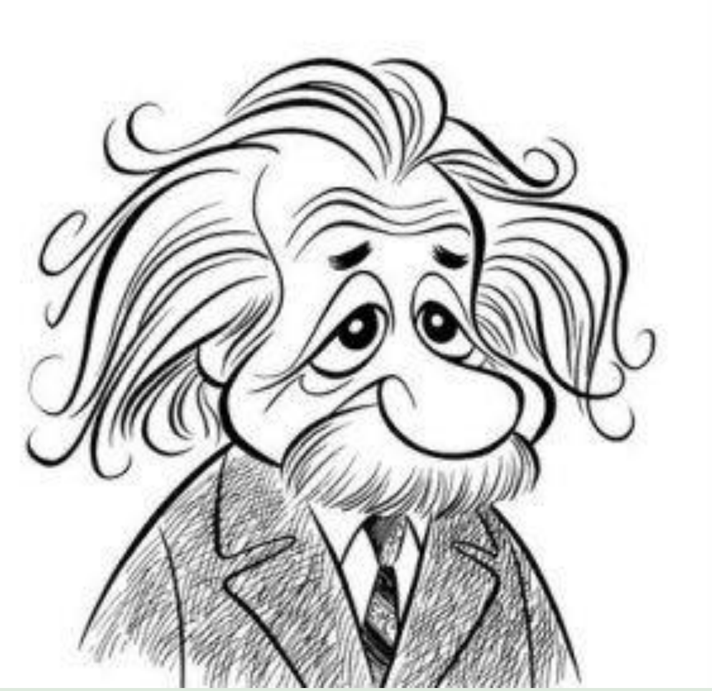
Note : Eq.1 valid only for 1d

For 3 dimensions

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V}.\nabla T$$

where

$$\nabla T \equiv \frac{\partial T}{\partial x}\,\vec{i} + \frac{\partial T}{\partial y}\,\vec{j} + \frac{\partial T}{\partial z}\,\vec{k}$$



Some Quick Facts

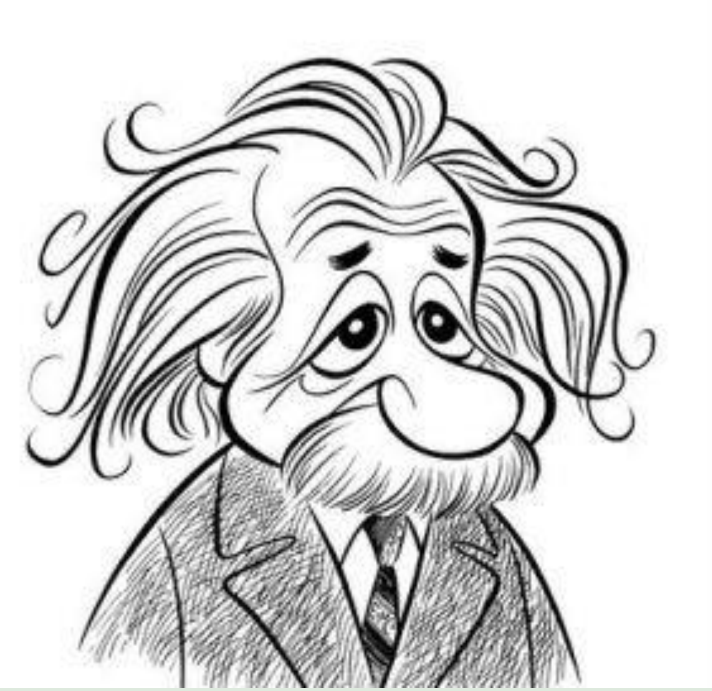
$$\underbrace{\frac{Df}{Dt}}_{1} = \underbrace{\frac{\partial f}{\partial t}}_{2} + \underbrace{\vec{V}.\nabla f}_{3}$$

1 Material / Lagrangian Derivative

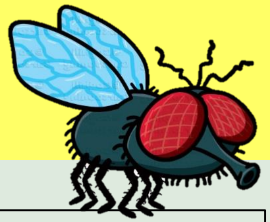
2 Local / Eulerian Derivative

3 Convective / Transport Derivative

Some Quick Facts

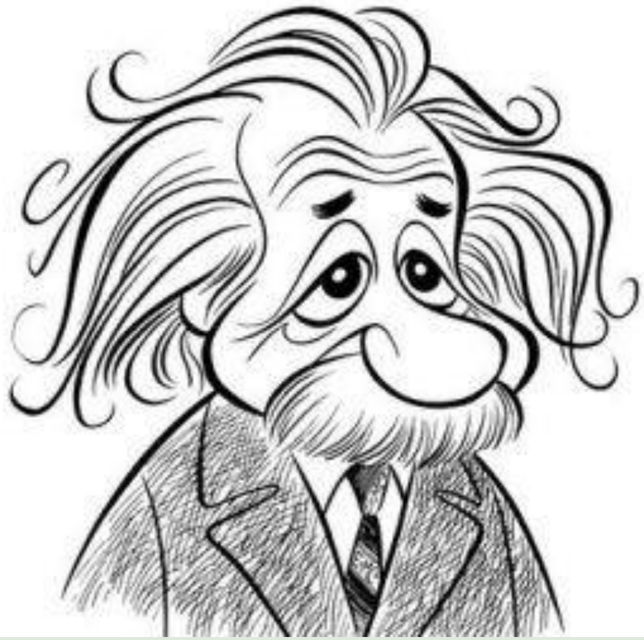


$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{V}.\nabla f$$



Material / Lagrangian Derivative

Tells us how quickly f changes
for the moving Fluid Element

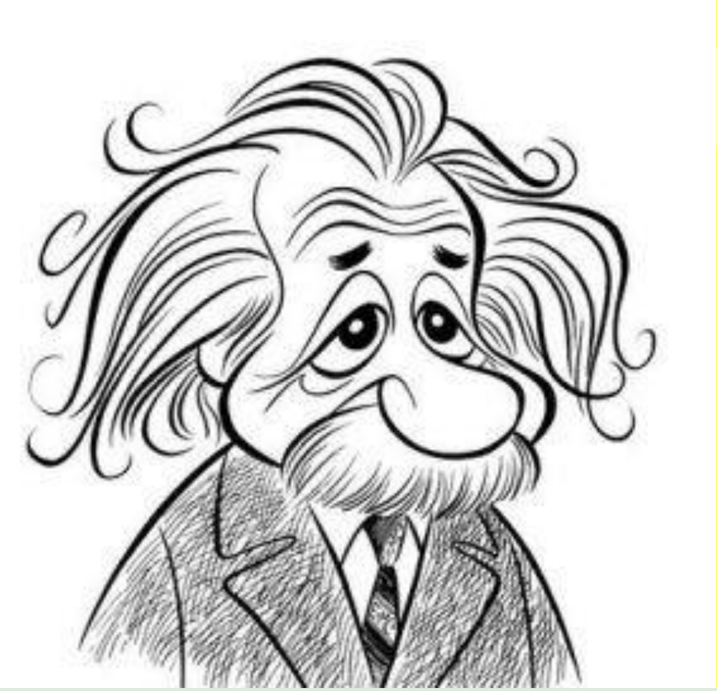


Some Quick Facts

$$\frac{Df}{Dt} = \boxed{\frac{\partial f}{\partial t}} + \vec{V}.\nabla f$$

Local / Eulerian Derivative

Tells us how quickly f changes
at a particular point in space

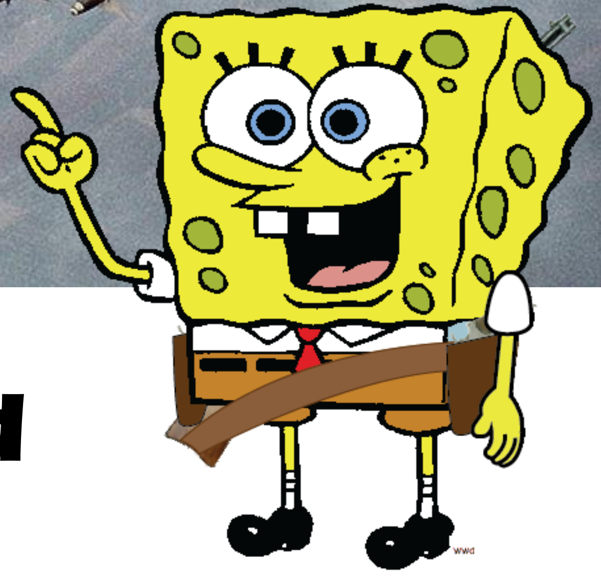


Some Quick Facts

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{V}.\nabla f$$

Convective Derivative

Tells us how quickly f changes
by virtue of the element's motion



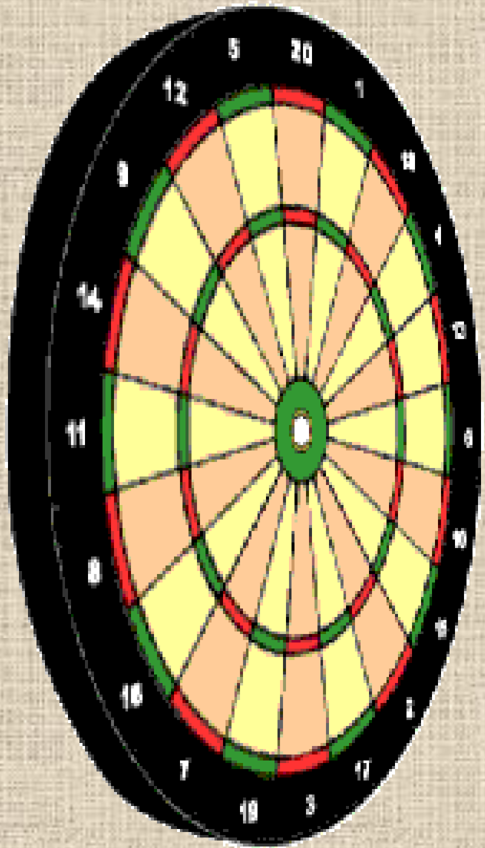
Now we are fully armed

What is Incompressible Flow ?

Density = constant ?

$$\frac{\partial \rho}{\partial t} = 0 \ ?$$

Fluid is incompressible ?



Bull’s eye !

$$\frac{D\rho}{Dt} = 0$$

The continuity equation

$$\frac{D\rho}{Dt} + \rho\left(\nabla.\vec{V}\right) = 0 \longrightarrow \text{Eq. 1}$$

Consider a flow for which $\frac{D\rho}{Dt} = 0$

Then Eq.(1) becomes

$$\rightarrow \quad \rho(\nabla.\overrightarrow{V)} = 0$$

$$\rightarrow \quad \boxed{\nabla.\vec{V} = 0}$$

Since $\rho \neq 0$

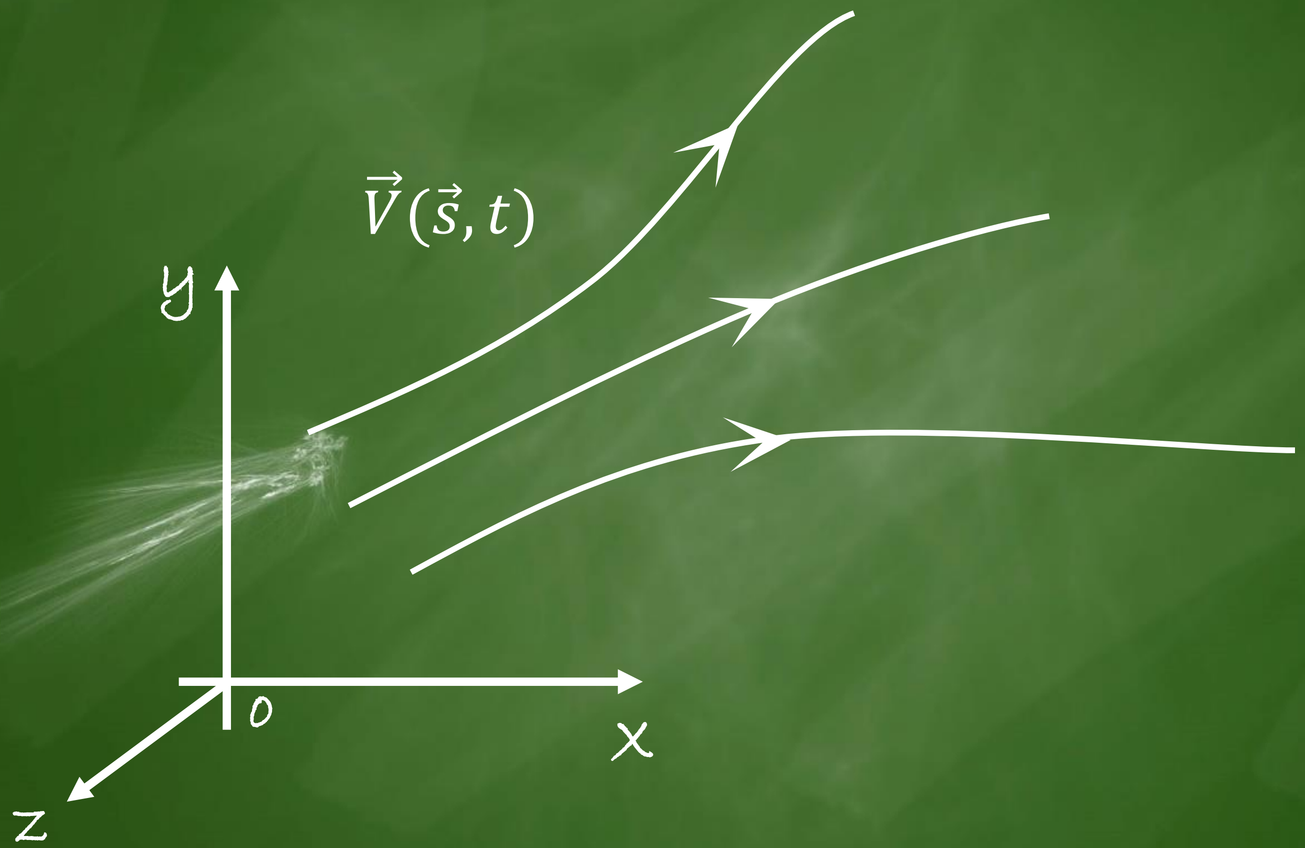
What is $\nabla.\vec{V} = 0$ telling us ???

$$\nabla.\vec{V} \equiv \text{Volumetric strain rate}$$

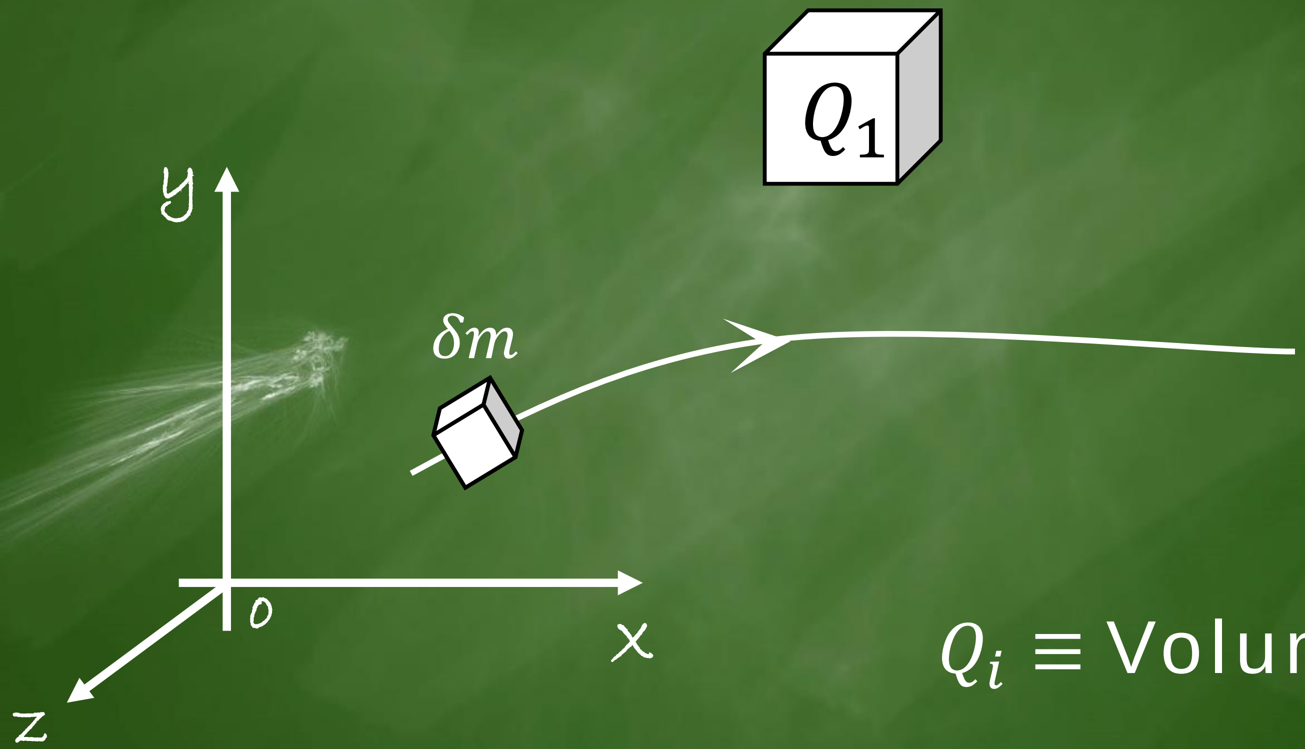
$$\nabla.\vec{V} = \frac{1}{\delta Q}\frac{D(\delta Q)}{Dt}$$

" time rate of change of the volume δQ of a moving fluid element, per unit volume."

Consider an incompressible flow field

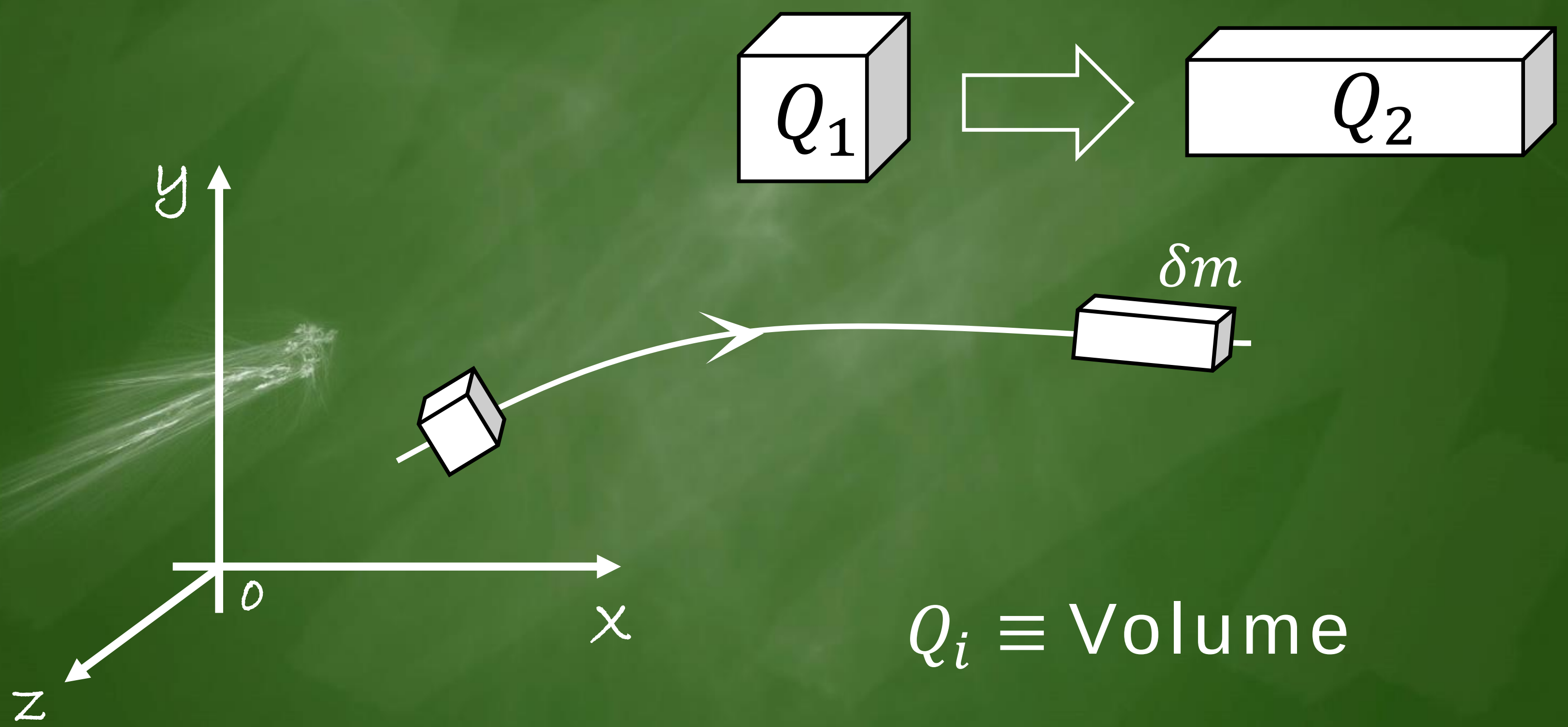


$\delta m \equiv$ infinitesimal mass element

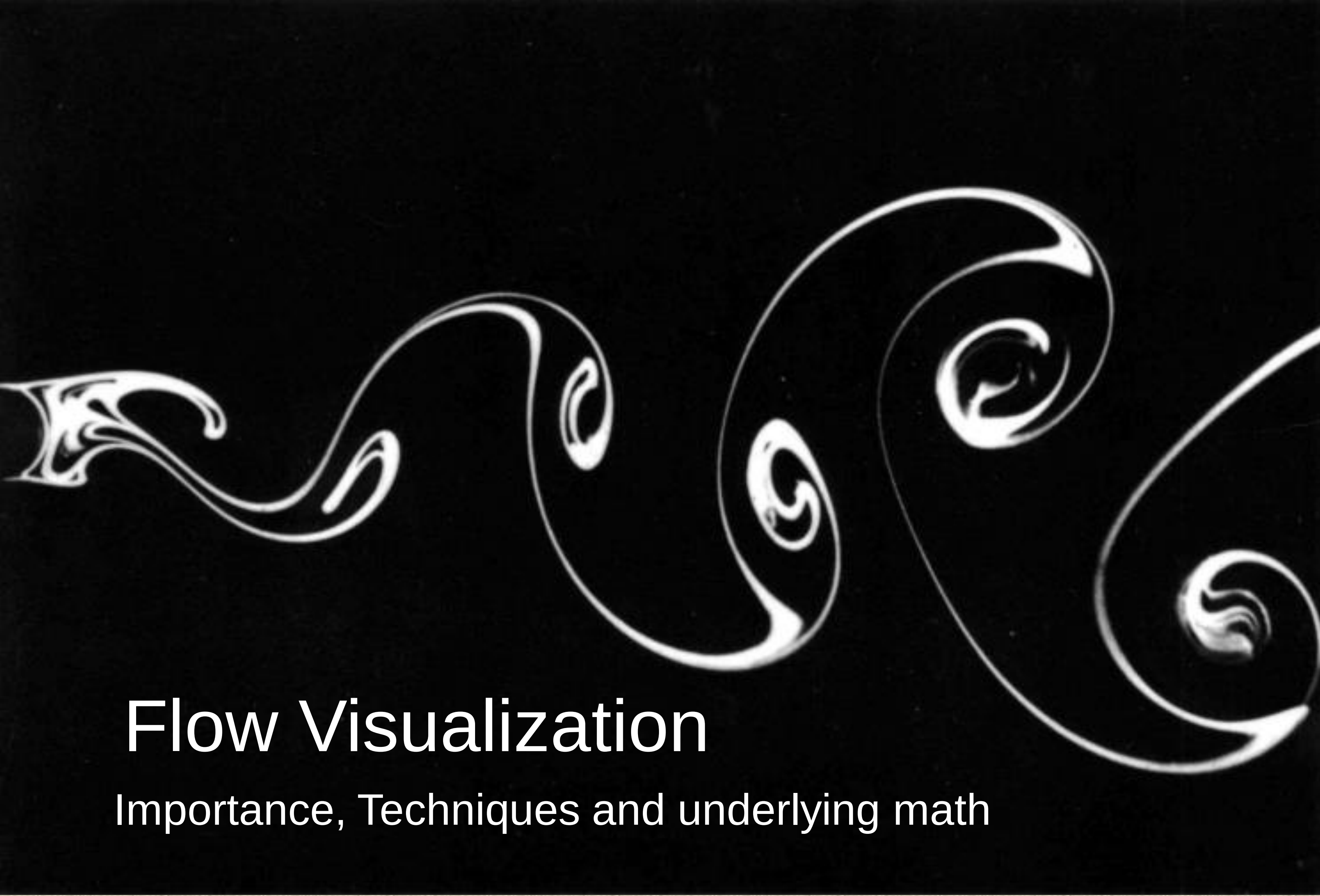


$Q_i \equiv$ Volume

$\delta m \equiv$ infinitesimal mass element



$Q_i \equiv$ Volume



Flow Visualization

Importance, Techniques and underlying math

The Inevitable tools of flow visualization

Streamlines

Streaklines

Pathlines

Timelines

Streamlines

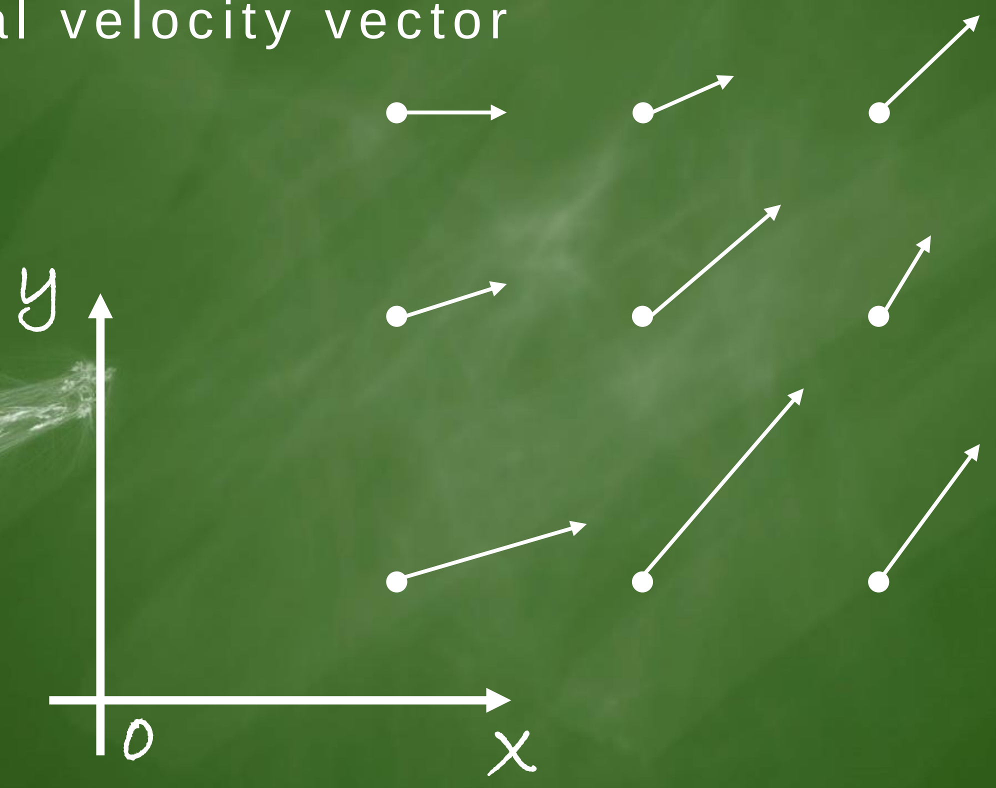
Curve that is everywhere tangent to the instantaneous local velocity vector

What does that mean ?

Consider an arbitrary flow,

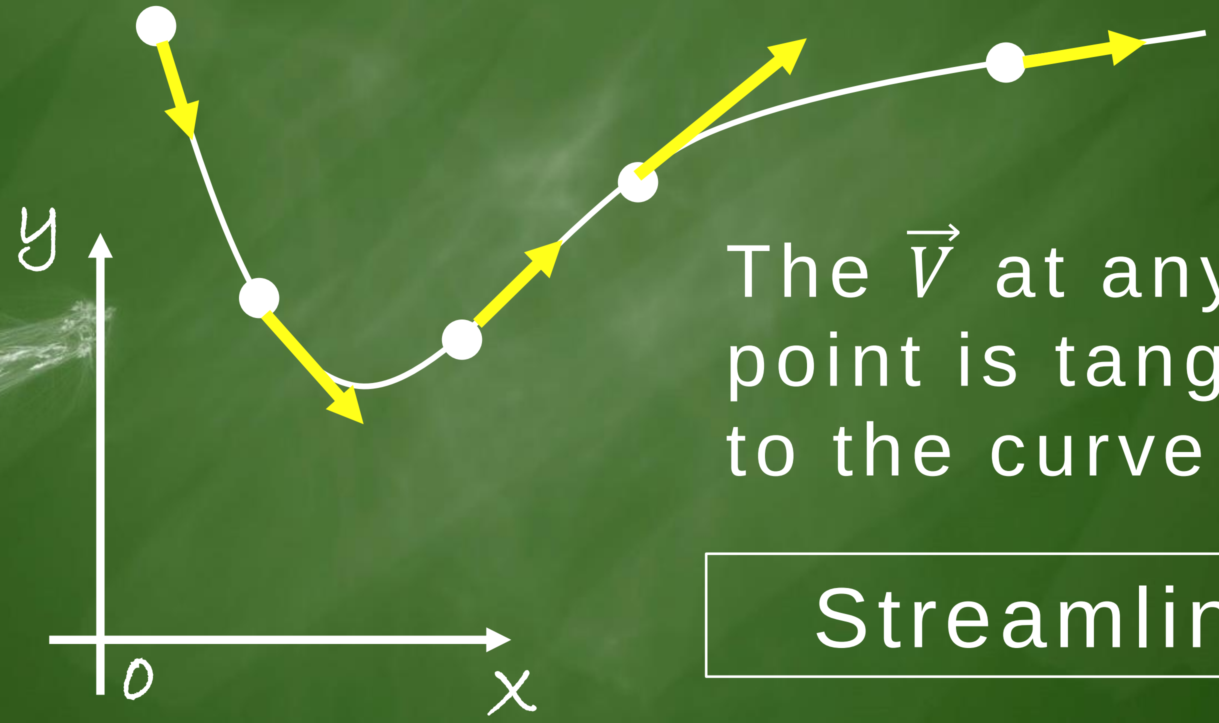
- point in space

→ local velocity vector



now,

at any instant if you draw a curve ST



The $\vec{V}$ at any point is tangent to the curve

Streamline

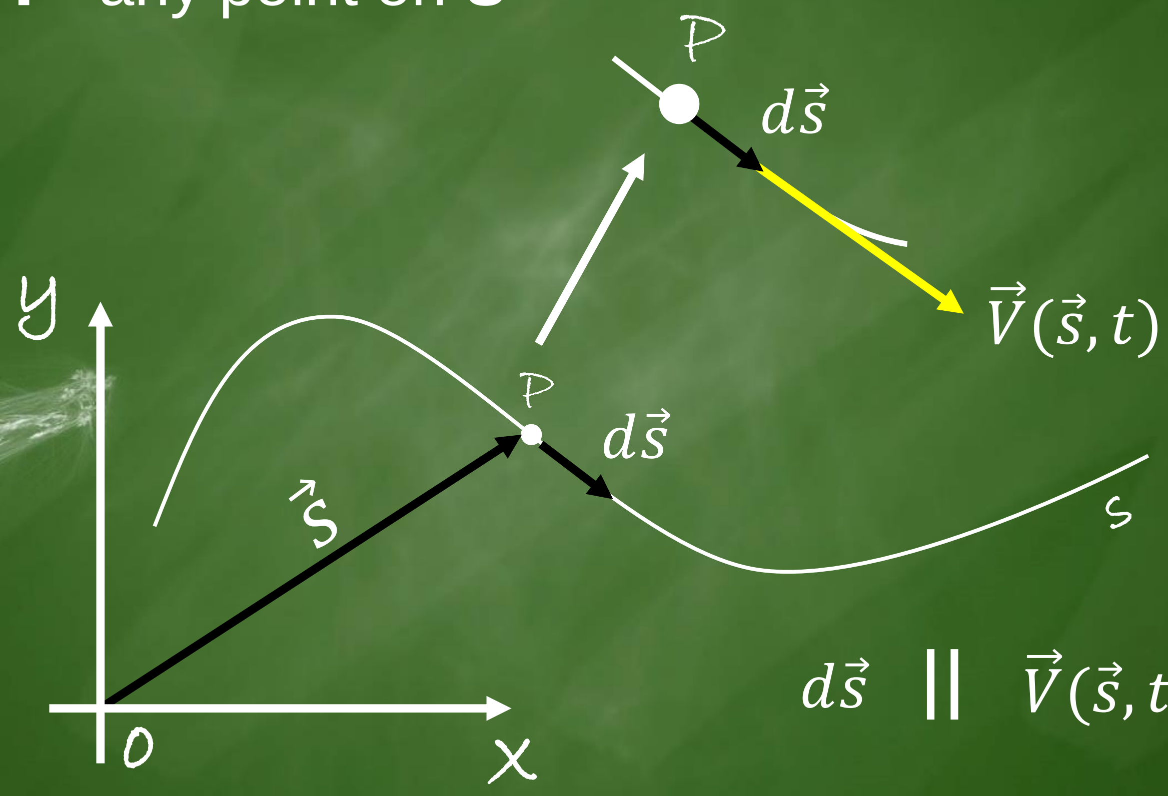
Things to keep in mind

- Defined for an instance
- never intersect
- not always the path
- mathematical notion
- Eulerian concept

Streamline equation

s $\equiv$ arbitrary streamline

Let be **P** any point on **s**

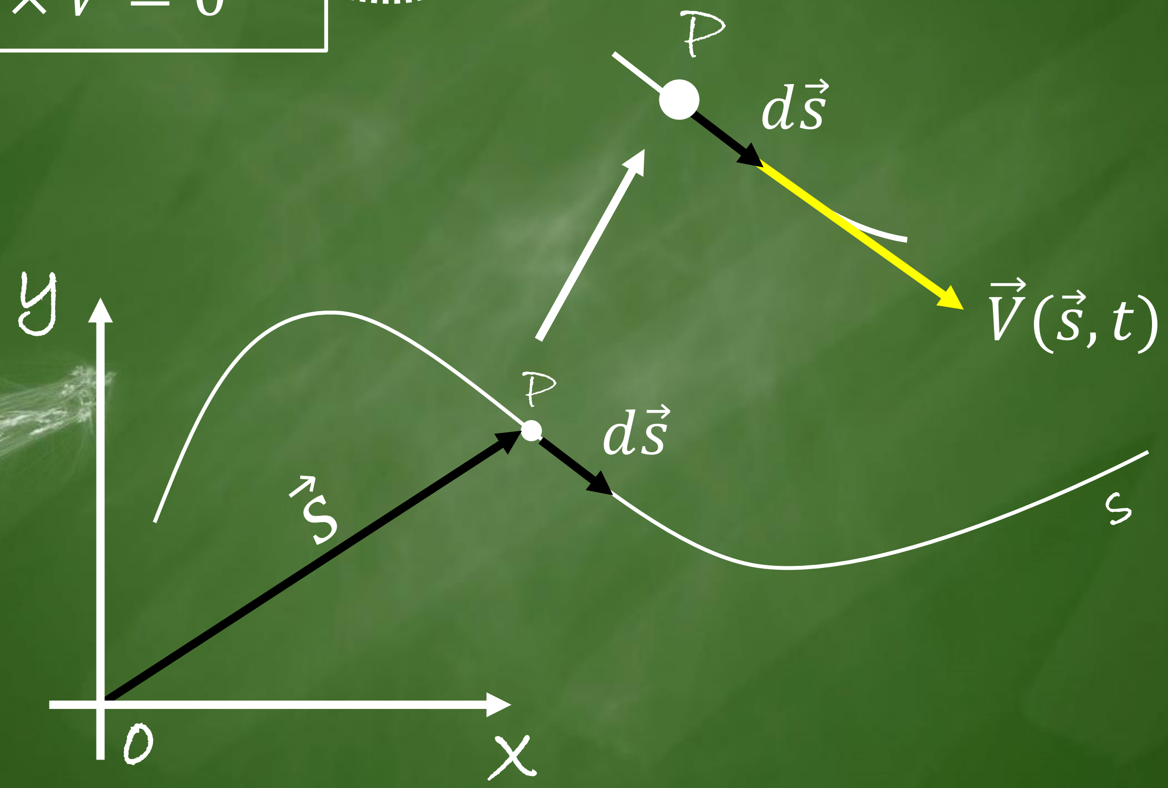


$$d\vec{s} \parallel \vec{V}(\vec{s},t)$$

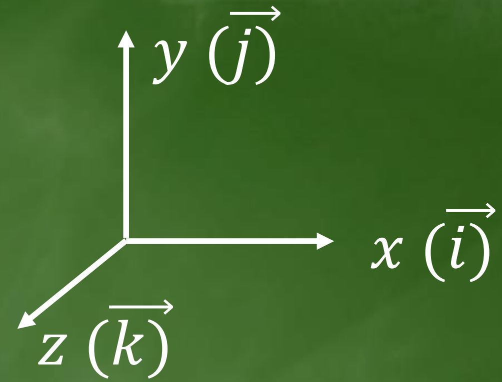
$$d\vec{s} \parallel \vec{V}(\vec{s},t)$$

$$\boxed{d\vec{s} \times \vec{V} = \vec{0}}$$

Required equation



For a Cartesian frame



$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$d\vec{s} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\Rightarrow\ d\vec{s} \times \vec{V} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ u & v & w \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ u & v & w \end{pmatrix} = \vec{0}$$

$$(wdy - vdz)\ \vec{i} + (udz - wdx)\,\vec{j} + (vdx - udy)\,\vec{k} = \vec{0}$$

simplification yields –

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}}$$

For 2d flow

$$\boxed{\frac{dy}{dx} = \frac{v}{u}}$$

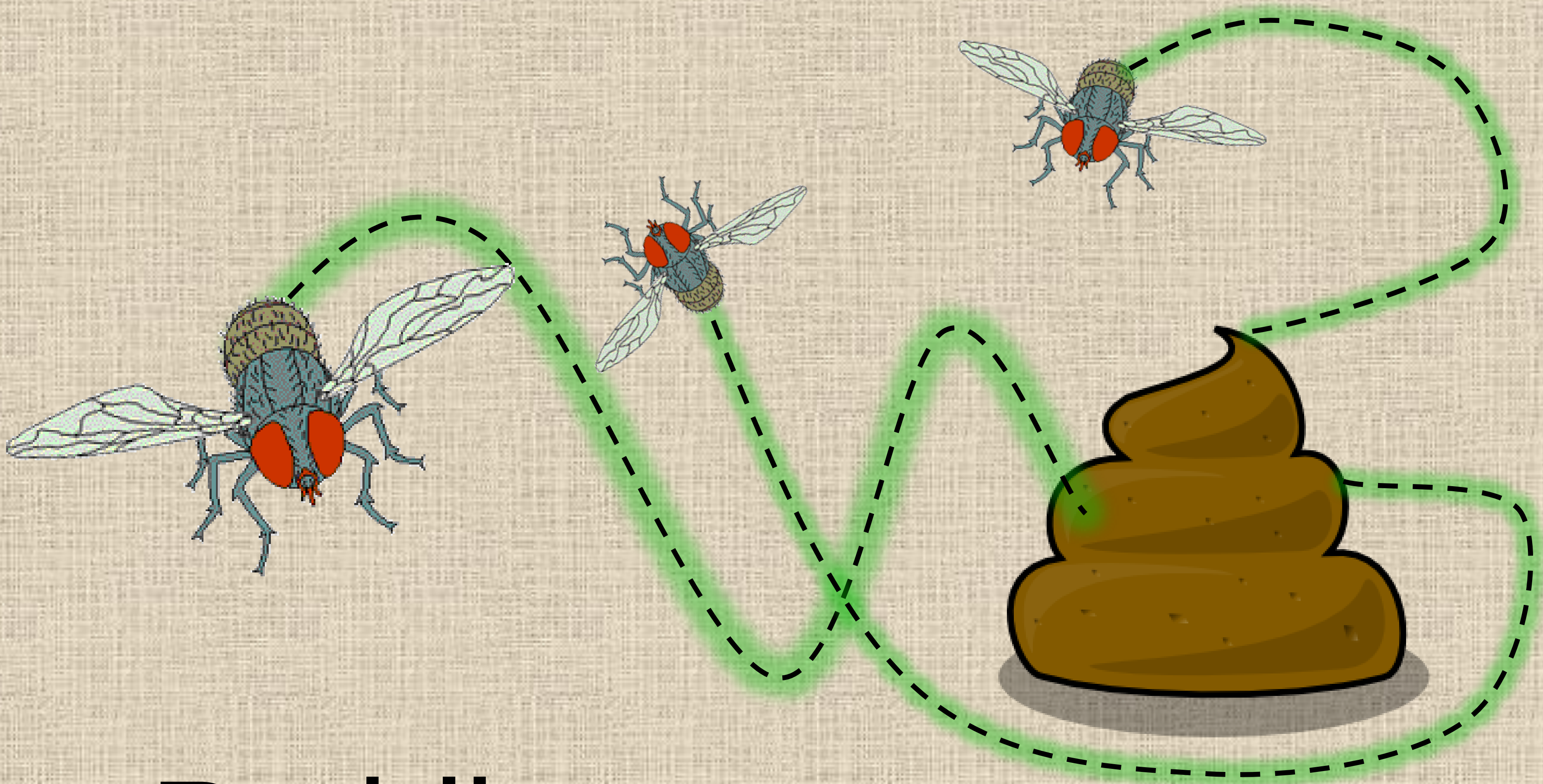
Things to keep in mind

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}}$$

⇒ System of ODEs, solved for x, y and z

⇒ time is treated as constant during integration

⇒ Different c values yield different streamlines



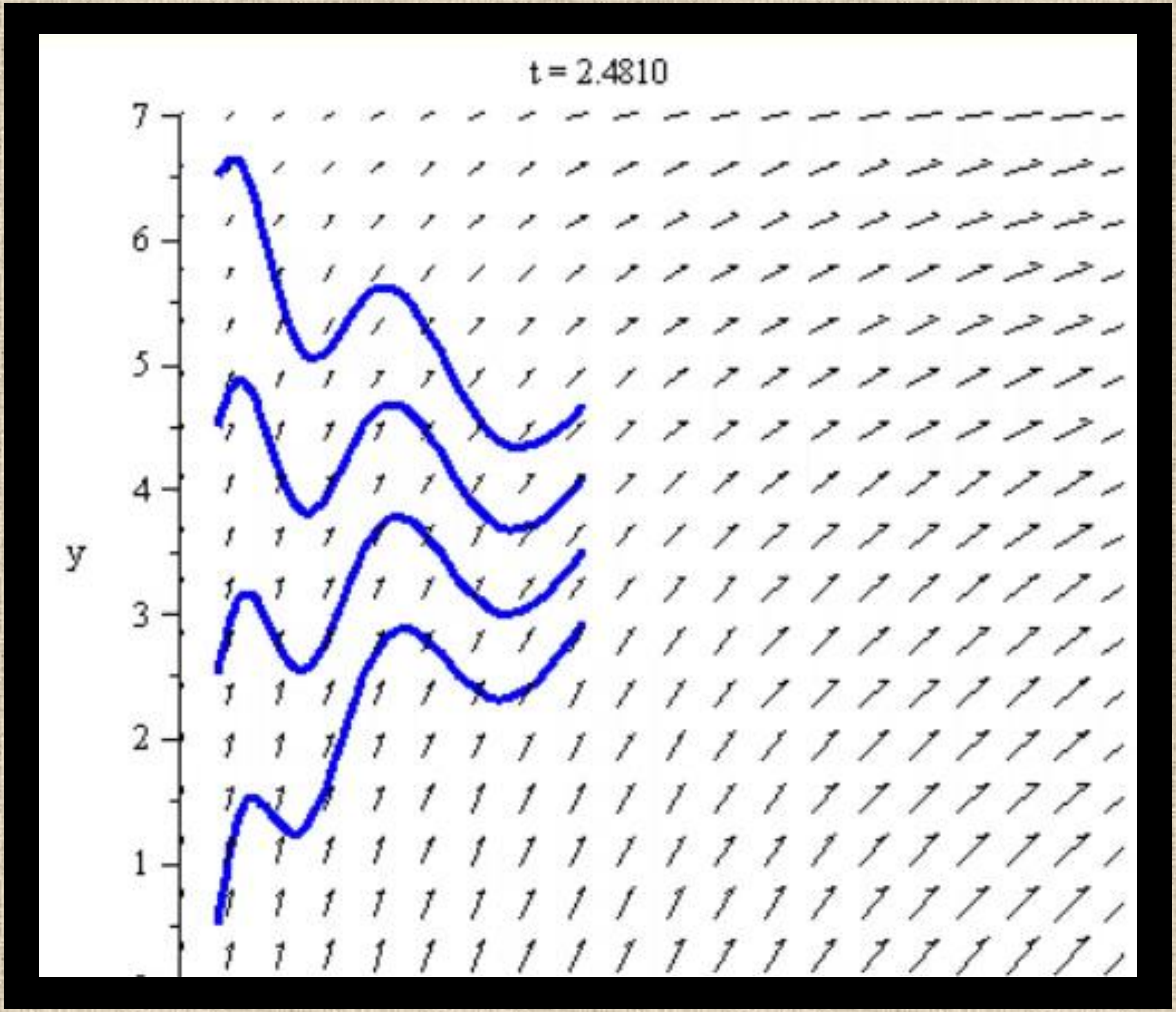
Pathlines

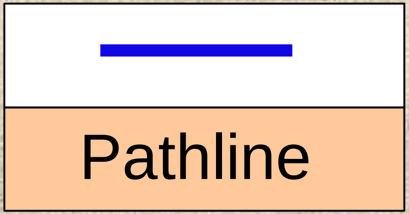
- Definitions
- Common Misconceptions
- Mathematical Expression

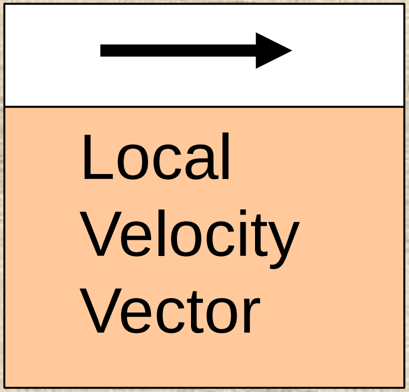
Understanding Pathlines

- Lagrangian concept
- Path traced by fluid particle in course of its motion
- Over a time interval
- Pathlines can intersect

Shows Pathlines of 4 different particles





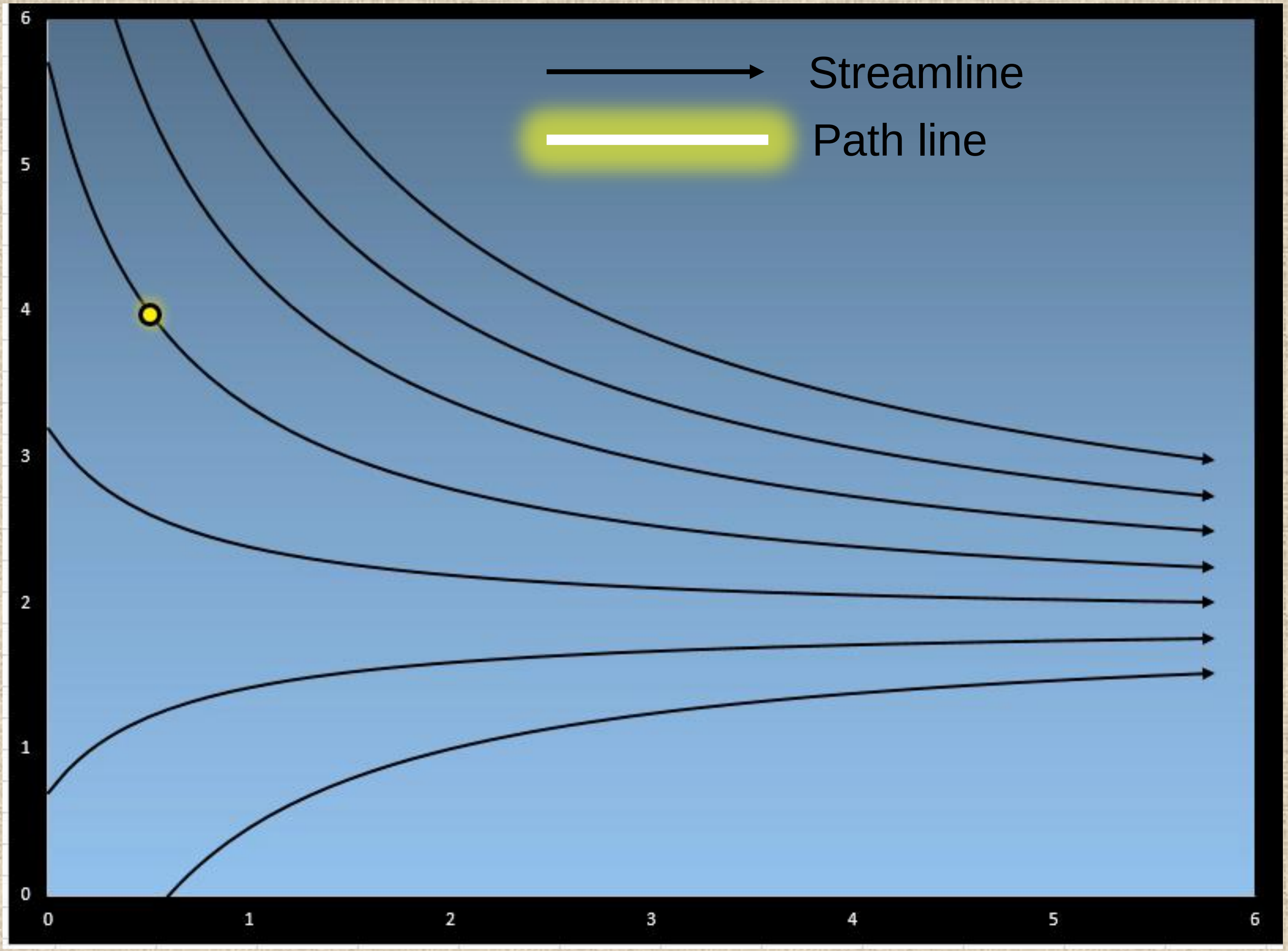


Getting the Difference right

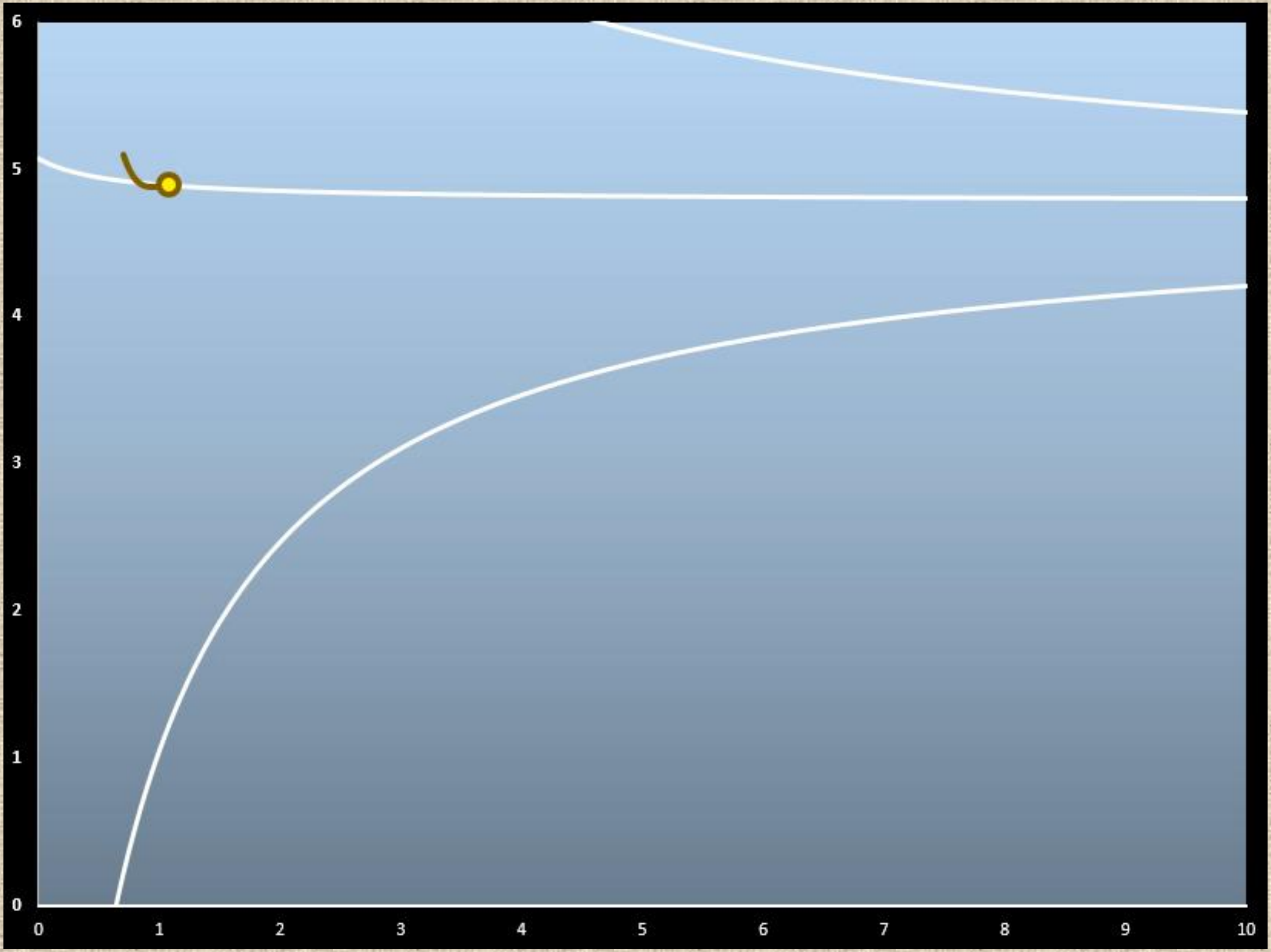
Streamlines $\neq$ Pathlines

- Only in steady flow it is that the 2 lines are same.
- In better words – particle follows the streamline in steady flow
- Streamlines – 'at an instant of time'
- Pathlines – 'over a period of time'

Steady Flow $\vec{V} = \vec{V}(\vec{s})$



Unsteady Flow $\vec{V} = \vec{V}(\vec{s}, t)$



Pathline equations

We need to solve the system

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x,y,z,t) \\ \frac{dy}{dt} &= v(x,y,z,t) \\ \frac{dz}{dt} &= w(x,y,z,t) \end{aligned}\right\}$$

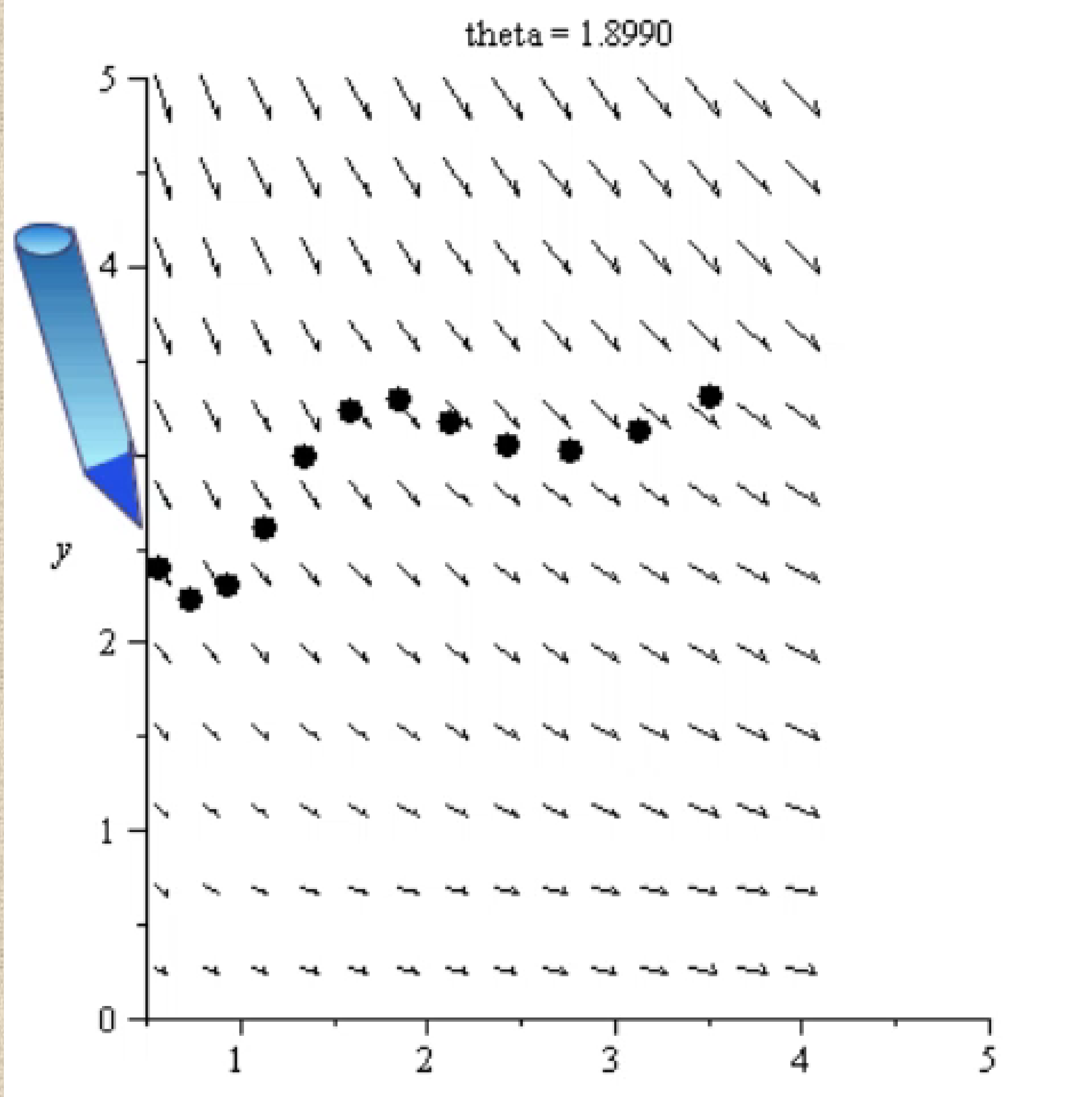
Subject to

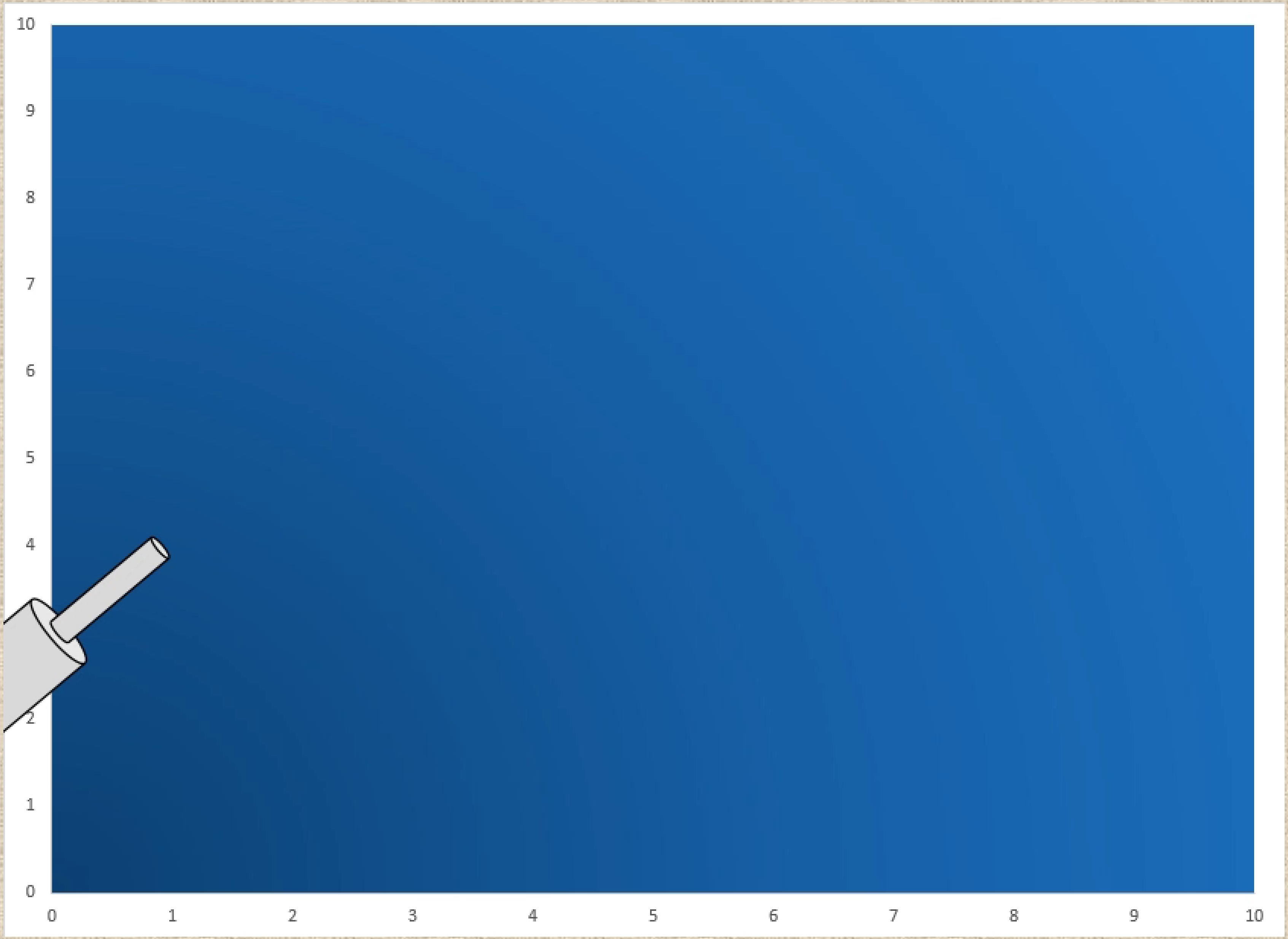
$$x(t=0) = x_0$$
$$y(t=0) = y_0$$
$$z(t=0) = z_0$$

Streaklines

What are Streak lines ?

- It is the curve **joining** all particles which have passed through a particular point in space.
- Traced over a time interval like path lines
- Smoke trails are essentially streak lines


theta = 1.8990
y
0
1
2
3
4
5


10
9
8
7
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