Problem Set 1
Data Analysis and Interpretation (EE 223)
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# 1 Warming Up

1. Find moment generating function, expectation and variance for the following distributions:
   (a) Bernoulli($p$)
   (b) Binomial($n, p$)
   (c) Poisson($\lambda$)
   (d) Geometric($p$)
   (e) Exponential($\lambda$)
   (f) Gamma($n, \lambda$)
   (g) Uniform($a, b$)
   (h) Gaussian($\mu, \sigma^2$)
   (i) Chi-square ($\chi^2(n)$)
   (j) $f_X(x) = c_\alpha x^{-\alpha}$ for $x \geq 1$ and zero otherwise, and $\alpha \geq 2$.
2. Let $X_1, X_2, \ldots, X_n$ are independent. Define $S_n = \sum_{k=1}^{n} X_k$. Find the distribution of $S_n$ when -
   (a) $X_k$'s are i.i.d. Exponential($\lambda$).
   (b) $X_k \sim$ Gaussian($\mu_k, \sigma_k^2$)
   (c) $X_k$'s are i.i.d. Bernoulli($p$)
   (d) $X_k \sim$ Poisson($\lambda_k$)

# 2 Getting more into it

1. Let $X \sim$ Uniform$(0, 1)$, then show that $Y = -2 \log X$ is $\chi^2(2)$.
2. You are waiting on a bus stop. The inter-arrival time between the buses is Exp($\lambda$). The number of passengers in the bus is a uniform random variable taking values in $\{0, 1, \ldots, \alpha\}$. You only board a bus if it has less than $\beta$ passengers. The number of passenger across buses are independent random variables. Denote $X$ to be your waiting time until you board a bus. Find the distribution of $X$.
3. In the same setting above. Suppose you go to bus stop for observing the process. Suppose you arrive at time zero and leave at time $t$. Let $Y$ denote the number of buses that you saw. Find the distribution of $Y$.
4. Let $X_1, X_2, \ldots, X_n$ be iid random variables. Also, let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be their non-decreasing permutation. Find the distribution of $X_{(k)}$.
5. Suppose $X$ and $Y$ are random variables such that
$$f_{XY}(x, y) = \frac{e^{-x/y} e^{-y}}{y},$$
whenever $0 < x, y < \infty$. Find $\mathbb{E}[\mathbb{P}(X > 1|Y)]$.
6. Let $Y \sim$ Uniform$(0, X)$, where $X \sim$ Gamma$(2, \lambda)$. Also let $Z = X - Y$. Show that $Y$ and $Z$ are independent.

# 3 Now, That's the limit

1. Let $X_1, X_2, \ldots$ be iid random variables with density function $f(x) = e^{-x+\theta}$ for $x \geq \theta$ and zero otherwise. Show that following weak limits hold

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to 1+\theta$$
$$\min\{X_1, \ldots, X_n\} \to \theta.$$

2. Let $X_1, X_2, \ldots$ be iid random variables with distribution $F$. Show that following weak limits hold:

$$\overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k \to E[X]$$
$$\frac{1}{n+u}\sum_{k=1}^{n} X_k^2 \to E[X^2], \text{ where } u \text{ is a given constant2}$$
$$\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2 \to var(X)$$

3. You have invited 1000 friends for your wedding. Each friend gives you a gift. The expected value of a gift is 2000 Rs. and variance in the amount is 300 Rs. Find a good approximation to the probability that the total value of your gifts is less than 15,00,000 Rs. or greater than 22,00,000 Rs.

All the best!! May God be with you!!