MA 205 Complex Analysis: Logarithm and Integration

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Introduction

So last lecture, almost all the time we spent discussing the exponential function. We defined it by a power series which converges everywhere. We characterized the exponential function in two distinct ways. It's the only function which is invariant under differentiation modulo the normalization that it takes the value 1 at the point 0. It's essentially the only function with the property of converting addition in $\mathbb{C}$ to multiplication in $\mathbb{C}^{\times}$. We checked that, for a real variable, the exponential function matches with real exponential function e^x. e^x is monotonic increasing, hence one-to-one, hence invertible. This inverse is the logarithm.

Introduction

To define complex logarothm, we tried to construct an analytic function $g(z) = log(z)$ such that $\exp(g(z)) = z$. Solution of the above equation is a multi-valued function

$$g(z) = log|z| + i(\theta + 2\pi n) \quad (z \neq 0, n \in \mathbb{Z}),$$

where θ is the argument of z. So we need to choose values so that g become single-valued and continuous (hence analytic) function.

Definition

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f(z)$ be a continuous function on Ω such that $\exp(f(z)) = z \ \forall z \in \Omega$. Then f is called a branch of the logarithm.

Logarithm cont...

Lemma

Let $\Omega \subseteq \mathbb{C}$ be a domain and let f be a branch of the logarithm. Then any other branch of the logarith differs from f by an integral multiple of $2\pi i$.

Proof: Let $g(z)$ be a branch of the logarithm. Then $f(z) - g(z)$ is an integral multiple of $2\pi i$ for all $z \in \Omega$. Since Ω is connected, and $f(z) - g(z)$ is continuous while integral multiples of $2\pi i$ is a discrete set, it follows that $f(z) - g(z)$ is an integral multiple of $2\pi i$.

We will usually work with a fixed branch of the logarithm called the **Principal Branch**. This is defined as follows:

Let $\Omega \subset \mathbb{C}$ be the open subset defined by $\mathbb{C}$ minus the negative real line. For any $z \in \Omega$, $z = |z|e^{i\theta} : -\pi < \theta < \pi$, define $f(z) = \log r + i\theta = \log|z| + iArg(z)$.

One checks that $f(z)$ is a branch of $\log(z)$ on Ω. In fact, f is analytic and $f'(z) = \frac{1}{z}$.

Now that we've differentiated enough, it's time to integrate. Recall integration from MA 105. You first integrated real valued functions on an interval. Remember Riemann sum, Riemann integration, area under a curve, etc? Integrals had nice properties: well behaved under addition and scalar multiplication. In the language of MA 106, integral is a linear functional from the vector space of integrable functions to $\mathbb{R}$. Integral also respected monotonicity. But in spite of all the nice properties, one struggled hard to integrate. To effortlessly integrate, we needed the fundamental theorem.

So that was a quick summary of real integration from MA 105. Now to complex integration in MA 205. First we integrated $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$. Now let's start with a continuous function $f : [a, b] \to \mathbb{C}$. Let $f(t) = u(t) + iv(t)$. We define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

where both these integrals are defined to be the usual Riemann Integrals.

Some basic properties:

$$1. Re \int_a^b f(t)dt = \int_a^b Ref(t)dt = \int_a^b u(t)dt$$

$$2. Im \int_a^b f(t)dt = \int_a^b Imf(t)dt = \int_a^b v(t)dt$$

$$3. \int_a^b (c_1 f_1(t) + c_2 f_2(t))dt = c_1 \int_a^b f_1(t) + c_2 \int_a^b f_2(t)dt$$

$$4. \quad |\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$$

Proof: f is complex valued and so $\int_a^b f(t)dt \in \mathbb{C}$, say w_0. Let $c = \frac{|w_0|}{w_0}$. Thus, $|c| = 1$ and $\mathrm{Re}(cf(t)) \leq |cf(t)| = |f(t)|$. Thus,

$$\begin{aligned} \left|\int_a^b f(t)dt\right| &= c\int_a^b f(t)dt \\ &= \int_a^b \mathrm{Re}(cf(t))dt \\ &\leq \int_a^b |f(t)|dt. \end{aligned}$$

Do we have a complex version of the fundamental theorem?

Theorem (Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{C}$ be a continuous function. Then,

$$x \mapsto \int_a^x f(t)dt$$

is an anti-derivative (or a primitive) of f. If F is any anti-derivative of f, then for any $a \leq r < s \leq b$,

$$\int_r^s f(t)dt = F(s) - F(r).$$

What about the proof? It's easy. Just apply the fundamental theorem from MA 105 to real and imaginary parts of f.

Countour Integration

We now discuss the complex analogue of line integrals form calculus.

We say a curve $\gamma(t) = x(t) + iy(t)$ is C^1 if both $x(t)$ and $y(t)$ are C^1 functions of t. A **contour** is a curve consisting of a finite number of C^1 curves joined end to end. A curve is said to be **simple** if the parametrization map is one to one except possibly at the end-points. (Intuitively it means that the curve does not cross itself). It is said to be **closed** if the initial and end-point are the same. i.e, $\gamma(a) = \gamma(b)$.

Jordan Curve Theorem

Any simple closed curve in $\mathbb{R}^2$ seperates the plane in to two connected components. The curve is the common boundary of both of them. Exactly one of the components is bounded.

The theorem was first discovered by Camille Jordan in 1887 although his proof was not rigorous. The first rigorous proof was due to Oswald Veblenin 1905. Although intuitively very believable the proof of this theorem is non-trivial. We shall not prove this here.

Let $f : \Omega \to \mathbb{C}$ be a complex function defined on a domain Ω and let $\gamma(t) = x(t) + iy(t), t \in [a, b]$ be a contour. The integral of f along γ is defined as

$$\int_\gamma f(z)dz \stackrel{def}{=} \int_a^b f(\gamma(t))\gamma'(t)dt$$
$$= \int_a^b [(u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt$$
$$+i \int_a^b [(u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)]dt,$$

where $f(z) = u(x, y) + iv(x, y)$.

Properties

The usual properties of real line integrals get carried over to the complex analogues:

1. This integral is independent of parametrization.
2. $\int_{-C} f(z)dz = -\int_C f(z)dz$ where $-C$ is the opposite curve, i,e curve with the opposite parametrization.
3. $\int_{C_1 \cup C_2 \cup \ldots C_n} f(z)dz = \int_{C_1} f(z)dz + \ldots + \int_{C_n} f(z)dz$
4. $|\int_C f(z)dz| \leq \int_C |f(z)|dz$

Basic Example

Consider $f(z) = \int_C \frac{1}{z-z_0} dz$ where C is any circle around z_0.

We can parametrize C as $z(t) = z_0 + re^{it}$ with $0 \leq t \leq 2\pi$.
Then $\int_C f(z)dz = \int_C \frac{1}{z-z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i$

Note that the integral is independent of the circle chosen around z_0.

Path independence

We will show that a function f defined on a domain Ω has a primitive iff $\int f(z)dz$ is path independent. Suppose f has a primitive; i.e., there is F such that $F' = f$. Then,

$$\begin{aligned}\int_C f(z)dz = \int_C F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \left[\frac{d}{dt}F(\gamma(t))\right] dt \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

Thus, the integral depends only on the end points.

Proof of Path Independence

On the other hand, suppose the integral depends only on the end points of the path and not the path itself. This means that the integral is independent of the path on which you integrate. We need to find an F, show that it is differentiable, and $F'(z) = f(z)$ for all $z \in \Omega$. How do we go about getting such an F? Intuitively, something whose derivative is the given function should be an integral of that function! To get a function of z, we'll integrate "up to" z. Fix z_0. Let z be an arbitrary point in Ω. Choose <u>any</u> path joining z_0 to z; this exists since Ω is path connected.

Proof of Path Independence

Thus, our candidate for the primitive is

$$F(z) = \int_{\gamma(z_0,z)} f(z)dz.$$

This function is well defined because of the hypothesis of independence of integral on the path. We have a good candidate for the primitive. We only have to check that it is indeed a primitive. To this end, consider a small neighborhood of z which is completely contained in Ω. Let $h \in \mathbb{C}$ be such that the straight line $z + th$, $(t \in [0, 1])$, joining z and $z + h$ lie in Ω.

Proof of Path Independence

Then,

$$\begin{aligned}
& \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \\
= \;& \lim_{h\to 0} \frac{1}{h}\left[\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw\right] \\
= \;& \lim_{h\to 0} \frac{1}{h}\int_{\gamma(z,z+h)} f(w)dw \\
= \;& \lim_{h\to 0} \frac{1}{h}\int_0^1 f(z+ht)hdt \\
= \;& f(z).
\end{aligned}$$

This finishes the proof.