MA 205 Complex Analysis: Singularities and Laurent Seires

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Recall

We have studied zeroes of holomorphic functions. For a non-zero holomorphic function, the multiplicity of it's zeroes are finite. We have also proved that zeroes of a non-zero holomorphic function are isolated. In other words, zeroes of a non-zero holomorphic function on domain Ω can not have a limit point inside Ω but it may have a limit point on the boundary of Ω. We have also seen the following rigidity theorem known as identity theorem: If two holomorphic functions f and g on Ω are identically equal iff they are same on a sequence of points in Ω which has limit in Ω. Today, we will look at points where a function is not holomorphic and study the function on a small neighborhood of such a point.

Many times, one has a situation where Ω is an open set and f is a holomorphic function on the complement of a certain subset. The points of this subset are called **singularities** of the function. Given the rigid nature of holomorphic functions, we can get a lot of information on the nature of the singularities; essentially by looking at the function in small punctured neighborhoods of those points. Let us see this in more detail.

Definitions

Singularity of a function: The set of points in Ω where f is not defined or not holomorphic are called the singularities of Ω.
For example $1/z$ has a singularity at 0.

Singularities are of 2 types, isolated and non-isolated singularities.
A singular point is said to be isolated if the function is holomorphic in a punctured disc around that point.
For example $1/z$ is holomorphic in any punctured disc around 0.
$\frac{1}{z(z-1)}$ has 2 singular points 0 and 1, both of which are isolated singularities; the function is holomorphic in a punctured disc of radius 1 around both of them.

A singularity is non-isolated if it is not isolated ! That is, in no punctured neighborhood of the singularity is the function holomorphic.
For example $f(z) = |z|$ has all points as singularities and hence no point is an isolated singularity.

Removable and Non-Remonavle Singularities

Isolated singularities are of three types; removable singularity, pole and essential singularity.

If an isolated singularity can be removed by defining a certain value at that point, we say that the singularity is removable. For instance, the function $f(z) = \frac{sin(z)}{z}$ has a removable singularity at the origin. By redefining the function to be $f(z) = \frac{sin(z)}{z}$ for $z \neq 0$ and 1 for $z = 0$, we get a function which is holomorphic even at 0.

Note that if an isolated singularity at z_0 is removable, then $\lim_{z \to z_0} f(z)$ exists. The converse is also true and that is the Riemann's Removable Singularity Theorem.

Riemann's Removable Singularity Theorem

Theorem: An isolated singularity $z_0 \in \Omega$ of f is a removable singularity iff $\lim_{z \to z_0} f(z)$ exists.

Proof: Clearly removable singularity implies the limit exists. For the converse, suppose this limit exists. Then $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Then define g on a small open disc at z_0 by

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0. \end{cases}$$

If f is analytic in a punctured neighbourhood of z_0, then clearly g is analytic throughout that neighbourhood. Write

$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots.$$

Note that $c_0 = g(z_0) = 0$ and $c_1 = g'(z_0) = 0$. Thus,

$$g(z) = c_2(z - z_0)^2 + c_3(z - z_0)^3 + \ldots.$$

If we define $f(z_0) = c_2$, then f is holomorphic throughout. i.e., z_0 is a removable singularity.

Pole

Intuitively a pole is a point at which the function blows up from all directions. An isolated singularity z_0 is said to be a pole if $\lim_{z \to z_0} f(z)$ is ∞ (that is the function takes values outside any bounded set in any small punctured neighborhood of z_0). In this case the function $g(z) = \frac{1}{f(z)}$ is holomorphic at z_0 with $g(z_0) = 0$ (Why ?). Since $g(z)$ is not identically equal to zero, it follows that there exists a positive integer m such that $g(z) = (z - z_0)^m h(z)$ for some holomorphic function $h(z)$ defined in a neighborhood of z_0 with $h(z_0) \neq 0$. Note that such an m and therefore such a $h(z)$ is uniquely defined. Thus for all z in a punctured neighborhood of z_0, $f(z) = (z - z_0)^{-m} \frac{1}{h(z)} = (z - z_0)^{-m} f_1(z)$ for some holomorphic function $f_1(z)$. In this case, m is called the order of the pole and is a measure of how fast the function blows up at z_0. If m is one, we say that the pole is a **simple pole**.

Casorati-Weierstrass Theorem

A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

Theorem: If z_0 is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

The if part if obvious. For the only if part, suppose f has an essential singularity. Let a be any complex number. Suppose f does not attain values arbitrarily close to a, then $\lim_{z \to z_0} (z - z_0) \dfrac{1}{(f(z) - a)} = 0$. Hence by Riemann's theorem above, it has a removable singularity at z_0.

Proof cont ..

Depending on whether the singularity can be removed by assigning the value to be zero or a non-zero value, $f(z)$ will have a pole or a removable singularity at z_0. In either case we have a contradiction.

For example, the function $e^{1/z}$ has an essential singularity at 0. (Check !)