Conformal Mappings and Riemann Mapping Theorem

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Motivation

Today we look at conformal mappings. Roughly speaking a conformal map between two subsets U and V of $\mathbb{R}^n$ is a differentiable mapping that preserves magnitude and orientation of angles between directed curves.

A more general class of mappings which only preserve magnitude of angles between directed curves but not neccessarily their orientation are called isogonal mappings. We will focus our attention on studying conformal mappings between open subsets of $\mathbb{C}$.

Let Ω be a domain in $\mathbb{C}$ and let $f(z)$ be a holomorphic function on Ω. C be a smooth parametrized curve in Ω represented by the equation $z(t); a \leq t \leq b$. Consider the image of C under f , say γ. It is parametrized by $w = f[z(t)]$. Suppose C passes through $z_0 = z(t_0);\ a < t_0 < b$ at which $f(z)$ is analytic and $f'(z_0) \neq 0$. Then by chain rule,

$$w'(t_0) = f'[z(t_0)]z'(t_0)$$

and hence arg $w'(t_0) =$ arg $f'[z(t_0)] +$ arg $z'(t_0)$

Preservation of Angles

Thus if the directed tangent to C at z_0 makes an angle θ with the x-axis, then γ makes an angle $\theta + \arg f'(z_0)$ with the x-axis.Consequently if C_1 and C_2 are two smooth, parametrized (hence also directed) curves passing through z_0 and intersecting at an angle ϕ at z_0 (meaning their tangents at z_0 make an angle ϕ), then their images also make an angle ϕ at $f(z_0)$.

Because of this angle preserving property, a holomorphic function $w = f(z)$ is said to be conformal at z_0 if $f'(z_0)$ is non-zero.

The mapping $w = e^z$ is conformal at all points since the derivative is everywhere non-vanishing. Consider two lines $C_1 = \{x = c_1\}$ and $C_2 = \{y = c_2\}$ in the domain; the first one directed upwards and the second one directed to the right. These lines intersect at the point (c_1, c_2) at right angles. Under this transformation, these lines get mapped to a positively oriented circle around origin and a ray from the origin respectively.Thus the images also intersect at right angles. Also note that the orientation is respected under the mapping; the angle between C_1 and C_2 and well as their images is -90°.

Counterexample

The mapping $z \to \bar{z}$ which is reflection about real line is isogonal but not conformal.

The mapping $z \to z^2$ is not conformal at 0 and does not preserve angles; the images of the real and imaginary axis are the real axis and the real axis with opposite orientation resp. Thus the angle between curves through 0 gets doubled. This is true for any two smooth curves passing through 0. This is a special case of the following more general fact:

If z_0 is a point at which first $m-1$ derivatives vanish, then then angle between two smooth curves passing through z_0 gets multiplied by m.

Inverse Function Theorem (Special Case)

Let us understand holomorphic mappings in a neighborhood of a conformal point. Recall the basic fact from calculus: If $f \in C^1(\mathbb{R})$ and $f'(x_0) \neq 0$, then in a neighborhood of x_0, $f(x)$ is either strictly increasing or strictly decreasing. In particular it is injective in a neighborhood of x_0. Converse is not true : Even if the derivative vanishes at a point, the function could be injective in a neighborhood (Example ?).It is natural to ask for the analogues statement for functions of a complex variable. A special case of the inverse function theorem provides the answer:

Let Ω be a domain in $\mathbb{C}$ and let $f(z)$ be a holomorphic function on Ω such that for some $z_0 \in \Omega$, $f'(z_0) \neq 0$. Then in a neighborhood of z_0, $f(z)$ is injective.

In this setting even the converse is true: If z_0 is a point such that $f'(z_0) = 0$ then in no neighborhood of z_0 is $f(z)$ injective.
For example note that while $x \to x^3$ is injective in a neighborhood of 0, $z \to z^3$ is not injective in any neighborhood.

Policy

- Two marks will be deducted if you don't write Tutorial Batch/Division in the answer booklet. Note that Tutorial Batch and Division are the same for this course.
- Use only results which are discussed in the class or given in the tutorial sheet.
- Please write short answers. Short and correct answers would be given high value.
- Answers with missing justification/argument/step will loose marks.

A very important subclass of conformal mappings are what are called Biholomorphisms.

Definition: If U and V are open subsets of $\mathbb{C}$ (not necessarily domains), a biholomorphism from U to V is a mapping $f : U \to V$ which is bijective and holomorphic.
If such a mapping exists U and V are said to be biholomorphic. Note in particular by the earlier remark that such an $f(z)$ is conformal at all points in U. An easy exercise show that the inverse mapping is automatically holomorphic.
Thus a biholomorphism is a bijective map, holomorphic both ways. Clearly composite of biholomorphisms is a biholomorphism and the inverse of a biholomorphism is a biholomorphism. In view of the special case of the inverse function theorem stated earlier, a holomorphic map which is conformal at a point z_0 is a biholomorphism in a neighborhood of z_0 (what's called a local biholomorphism).

Motivation for this notion

The motivation for this definition is simple: If two open subsets are biholomorphic, then (loosely speaking) studying complex analysis on one of them is equivalent to studying it on the other. For example if $f : U \to V$ is a biholomorphism, then $g : V \to \mathbb{C}$ is holomorphic on V if and only if $g \circ f$ is a holomorphic function on U.

Examples

1. The identity map $f : U \to U$ is clearly a biholomorphism. More generally, multiplication by a non-zero scalar defines a biholomorphism from $\mathbb{C}$ to $\mathbb{C}$. Similarly open discs of any two radii are biholomorphic.

2. The mapping from the open unit disc to the upper half plane $\mathbb{D} \to \mathbb{H}$ given by $z \to i\frac{1-z}{1+z}$ is a biholomorphism as one can check.

If $U \subseteq \mathbb{C}$ is open, then a biholomorphism from U to U is called an automorphism.
3. A basic fact is that the only automorphisms of $\mathbb{C}$ are of the form $az + b$ with $a \neq 0$. This is because biholomorphisms can be easily seen to have poles at infinity and hence polynomial. But the only injective polynomial functions are linear polynomials with non-zero linear coefficient !

Examples

4. As a consequence of Schwartz lemma, one can show that the automorphisms of the unit disc are

$$z \to \lambda \frac{z-a}{1-\bar{a}z}$$

where $|\lambda| = 1$ and $|a| < 1$.

I now state the deep, fundamental and spectacular theorem of Riemann:

Riemann Mapping Theorem: Any open, simply-connected subset of $\mathbb{C}$ other than $\mathbb{C}$ is biholomorphic to the open unit disc.

Note that by Lioville's theorem, the plane and the disc are not biholomorphic.
Given the rigid nature of holomorphic functions, the theorem is hugely surprising and beautiful. This theorem was conjecture by Riemann in 1851 in his thesis. He gave an incomplete proof based on Dirichlet principle stated roughly as : Minimizer of a certain energy functional is a solution to Poisson's equation.
Weierstrass found an error in the proof. The first complete proof was due to Constantin Carathodory in 1922 and simplified by Paul Koebe 2 years later. Here is a link to the history of the Riemann mapping theorem.
https://www.math.stonybrook.edu/ bishop/classes/math401.F09/GrayRMT.pdf