MA 205 Complex Analysis

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Recall

We have seen two important theorems in Complex Analysis, namely; Cauchy's theorem and Cauchy integral formula. Today we will consider some examples to demonstrate how these theorems can be use to evaluate line integrals and will see some beautiful consequences of these theorems.

Example:

(i) $\int_{|z|=2} \frac{e^z dz}{(z+1)(z-3)^2} = \int_{|z|=2} \frac{f(z)dz}{z+1}$, where $f(z) = \frac{e^z}{(z-3)^2}$. So by CIF, answer is $2\pi\imath f(-1) = \frac{\pi\imath}{8e}$.

(ii) $\int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz = \frac{1}{2} \int_{|z|=3} \left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1} \right] dz = 0$

OR

$$\begin{aligned} \int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz &= \int_{|z-1|=\varepsilon} \frac{\frac{\cos \pi z}{z+1}}{z-1} dz + \int_{|z+1|=\varepsilon} \frac{\frac{\cos \pi z}{z-1}}{z+1} dz \\ &= 0. \end{aligned}$$

Example

Here is an example where the Cauchy Integral formula can be used to compute a seemingly hard real integral.
Let k be a real constant. Show that $\int_0^{2\pi} e^{k\cos\theta}\sin(k\sin\theta)d\theta = 0$ and $\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)d\theta = 2\pi$.
Applying CIF to $\int_{|z|=1} \frac{e^{kz}}{z}dz = (2\pi i)e^{k.0} = 2\pi i$.

Hence

$$\begin{aligned} 2\pi i &= \int_0^{2\pi} \frac{e^{k(\cos\theta + i\sin\theta)}}{e^{i\theta}} ie^{i\theta}d\theta \\ &= i\int_0^{2\pi} e^{k\cos\theta}[\cos(k\sin\theta) + i\sin(k\sin\theta)]d\theta \end{aligned}$$

Equating the real and imaginary parts gives us the answer.

Summing Up

We saw two very important theorems, namely Cauchy's theorem and the Cauchy integral formula. The first said that the integral along a closed curve of a function is zero if the function is holomorphic on and within the curve. The second said:

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz$$

if f is holomorphic on and within the simple closed curve γ. We derived Cauchy's theorem by appealing to Green's theorem after assuming Goursat's theorem. We derived CIF from Cauchy's theorem by making use of the computation $\int_\gamma \frac{dz}{z-z_0} = 2\pi\imath$. In fact, Cauchy's theorem is equivalent to CIF. (Try and prove this little fact).

Holomorphic $\implies$ Analytic

Let f be holomorphic on Ω and $z_0 \in \Omega$. Let $R > 0$ be such that f is holomorphic in $|z - z_0| < R$. Let γ be a circle of radius r with $r < R$ centered at z_0. CIF gives us:

$$f(z) = \frac{1}{2\pi\imath} \int_{\gamma} \frac{f(w)}{w - z} dw,$$

for any z such that $|z - z_0| < r$. Note that we have chosen $r < R$ so that f is holomorphic on and within γ. The aim is to prove that f is analytic at z_0; i.e., to show that f can be expanded as a power series around z_0 in a neighborhood of z_0. The idea is to get a power series in $(z - z_0)$ from the rhs of CIF. The term in CIF which looks amenable to some manipulation is

$$\frac{1}{w - z}.$$

Also always keep in mind that the only series that we know well is the geometric series! Let's look at it closely.

Now,

$$
\begin{aligned}
\frac{1}{w-z} &= \frac{1}{w-z_0}\cdot\frac{w-z_0}{w-z} \\
&= \frac{1}{w-z_0}\cdot\frac{1}{1-\left[\frac{z-z_0}{w-z_0}\right]} \\
&= \frac{1}{w-z_0}\cdot\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots\right]
\end{aligned}
$$

since $\left|\frac{z-z_0}{w-z_0}\right| < 1$ for every $w \in \gamma$. We plug this in CIF.

Holomorphic $\implies$ Analytic

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z_0}\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\dots\right]dw \\
&= \left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)}dw\right]+\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^2}dw\right](z-z_0) \\
&\quad + \left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^3}dw\right](z-z_0)^2+\dots \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n,
\end{aligned}$$

for $|z-z_0|<r$ where

$$a_n = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw.$$

Remark: Integral of sum needn't be sum of integrals in general, but in the previous slide it can be justified. The key word is "uniform convergence". We'll skip the details.

Holomorphic $\implies$ Analytic

Thus, we have proved that if f is holomorphic in the disc $|z - z_0| < R$, then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi \imath} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{n+1}} dw,$$

for any $r < R$. Since the power series converges to $f(z)$ for $|z - z_0| < r$, the radius of convergence is at least r. We also know that

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

whenever $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. In particular, a_n does not depend on r. Any $r < R$ gives same a_n. Thus the radius of convergence is at least R.

Examples:
(i) $f(z) = \frac{e^z}{\sin z + \cos z}$ expanded as a power series centered at 0 has radius of convergence $= \frac{\pi}{4}$.
(ii)

$$f(z) = \begin{cases} \frac{z}{e^z - 1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

expanded as a power series centered at 0 has radius of convergence $= 2\pi$.
Thus, we have proved:

$$\text{holomorphic} \implies \text{analytic.}$$

Since analytic functions are infinitely differentiable (term by term differentiation), we have proved :

A holomorphic function is infinitely differentiable - a long-awaited claim !

Cauchy's Estimate

We have also concluded that if $f : \Omega \to \mathbb{C}$ is holomorphic, and if $\{z \mid |z - z_0| \leq r\} \subset \Omega$, then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi\imath} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where γ is $|z - z_0| = r$. Now suppose f is holomorphic in $|z - z_0| < R$ and suppose f is bounded by $M > 0$ there. Can apply ML inequality in the above formula to get

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Since this is true for any $r < R$, we get, $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$. This is called Cauchy's estimate.

Liouville's Theorem

A function defined all over $\mathbb{C}$ is called entire if it is holomorphic everywhere in $\mathbb{C}$. Examples? Polynomials, $\exp(z)$, $\sin z$, $\cos z$, etc. Clearly, sums and products of entire functions are entire. The fact that the function is defined and holomorphic everywhere puts strong restrictions on the function. For instance, we have the so called Liouville's theorem, which says:

a bounded entire function is a constant.

A non-constant entire function has to be unbounded. As we have seen $\exp(z)$ takes all values in $\mathbb{C}$ except 0, sin and cos are surjective, in particular these are all unbounded. Polynomials are also clearly unbounded (Why?).

<u>Proof of Liouville's theorem</u>: Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We need to show that f is a constant. We'll show this by showing that $f' \equiv 0$. It's enough to show $|f'(z)|$ can be made arbitrarily small. By Cauchy's estimate.

$$|f'(z)| \leq \frac{M}{R},$$

if f is holomorphic in a disc with center z and radius R. But R can be as large as we want, since f is entire. So, $f'(z) = 0$ for all $z \in \mathbb{C}$ and hence f is a constant.

Fundamental Theorem of Algebra

Recall that the fundamental theorem of algebra asserts that every non-constant polynomial with complex coefficients has a complex root.

Proof: We first show that $|f(z)|$ tends to ∞ as $|z|$ tends to ∞. If $f(z) = a_0 + a_1 z + \cdots a_n z^n$, then

$$\begin{aligned} |f(z)| &= |z|^n (|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + a_n|) \\ &\geq |z^n| \left(\left| |a_n| - |\frac{a_0}{z^n} + \ldots + \frac{a_{n-1}}{z}| \right| \right) \end{aligned}$$

As $|z|$ tends to ∞, $|\frac{a_0}{z^n} + \ldots + \frac{a_{n-1}}{z}|$ tends to zero and hence the above quantity clearly tends to infinity.
Suppose $f(z)$ does not have any zero, then $\frac{1}{f(z)}$ defines a holomorphic function which, by the above computation, is bounded and hence by Lioville's theorem is constant. A contradiction.