MA 205 Complex Analysis: Cauchy Integral Theorems

B.K. Das
IIT Bombay

August 2, 2018

Recall

Last time we defined Principle Branch of Logarithm and also remarked that a Branch of logarithm can be defined by removing a ray originating from the origin. We have defined integral of complex valued functions defined on an interval. Also studied line integral of a complex valued function.

Today, we will see one of the most important theorems in Complex Analysis known as Cauchy's theorem and its beautiful consequences.

Recall that for a function $f : \Omega \to \mathbb{C}$ defined on a domain Ω and a parametrized C^1 curve $\gamma(t) = x(t) + iy(t), t \in [a, b]$. We define

$$\int_\gamma f(z)dz \overset{def}{=} \int_a^b f(\gamma(t))\gamma'(t)dt$$

$$= \int_a^b [(u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt$$

$$+i \int_a^b [(u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)]dt,$$

where $f(z) = u(x, y) + iv(x, y)$.

Properties

The usual properties of real line integrals get carried over to the complex analogues:

1. This integral is independent of parametrization.
2. $\int_{-C} f(z)dz = -\int_C f(z)dz$ where $-C$ is the opposite curve, i,e curve with the opposite parametrization.
3. $\int_{C_1 \cup C_2 \cup \ldots \cup C_n} f(z)dz = \int_{C_1} f(z)dz + \ldots + \int_{C_n} f(z)dz$
4. $|\int_C f(z)dz| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt$

Path independence

We will show that a function f defined on a domain Ω has a primitive iff $\int f(z)dz$ is path independent. Suppose f has a primitive; i.e., there is F such that $F' = f$. Then,

$$\begin{aligned}\int_C f(z)dz = \int_C F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \left[\frac{d}{dt}F(\gamma(t))\right] dt \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

Thus, the integral depends only on the end points.

Proof of Path Independence

On the other hand, suppose the integral depends only on the end points of the path and not the path itself. This means that the integral is independent of the path on which you integrate. We need to find an F, show that it is differentiable, and $F'(z) = f(z)$ for all $z \in \Omega$. How do we go about getting such an F? Intuitively, something whose derivative is the given function should be an integral of that function! To get a function of z, we'll integrate "up to" z. Fix z_0. Let z be an arbitrary point in Ω. Choose <u>any</u> path joining z_0 to z; this exists since Ω is path connected.

Proof of Path Independence

Thus, our candidate for the primitive is

$$F(z) = \int_{\gamma(z_0,z)} f(z)dz.$$

This function is well defined because of the hypothesis of independence of integral on the path. We have a good candidate for the primitive. We only have to check that it is indeed a primitive. To this end, consider a small neighborhood of z which is completely contained in Ω. Let $h \in \mathbb{C}$ be such that the straight line $z + th$, $(t \in [0, 1])$, joining z and $z + h$ lie in Ω.

Proof of Path Independence

Then,

$$\begin{aligned} & \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \\ = & \lim_{h\to 0} \frac{1}{h}\left[\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw\right] \\ = & \lim_{h\to 0} \frac{1}{h}\int_{\gamma(z,z+h)} f(w)dw \\ = & \lim_{h\to 0} \frac{1}{h}\int_0^1 f(z+ht)hdt \\ = & f(z). \end{aligned}$$

This finishes the proof.

Cauchy's theorem

We now come to the most important, central theorem in this subject on which most of complex analysis depends, namely Cauchy's theorem.

Theorem

Let C be a simple closed contour and let f be a holomorphic function defined on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.

Remark Note that by Jordan curve theorem, interior of C makes sense.

Proof: Let $f(z) = u(x, y) + iv(x, y)$. The proof uses Green's theorem.

Theorem

If P and Q are two real valued functions with continuous first partial derivatives on an open set containing C and its interior, then

$$\int_C (Pdx + Qdy) = \int\int_\Omega (Q_x - P_y)dxdy$$

Note that a priori we do not have the hypothesis to guarantee continuity of the first partial derivatives of u and v since we do not know whether $f'(z)$ is continuous. However it is a fact that if $f(z)$ is holomorphic, then $f'(z)$ is continuous.

This is called Goursat's theorem. We will assume this theorem without proof.
Thus by Green's theorem,

$$
\begin{aligned}
& \int_C f(z)dz \\
= \; & \int_C (udx - vdy) + i \int_C (vdx + udy) \\
= \; & \int \int_\Omega (-v_x - u_y)dxdy + i \int \int_\Omega (u_x - v_y)dxdy \\
= \; & 0 \quad \text{(By CR equations).}
\end{aligned}
$$

Cauchy's theorem- for simply connected domain

Definition

An open subset $\Omega \subseteq \mathbb{C}$ is said to be **simply connected** if every simple closed curve in Ω has all its interior points belonging to Ω.

Examples: $\mathbb{C}$, any open disc, $\mathbb{C}$ minus negative reals etc. Open annulus i.e, area between two concentric circle is **NOT** simply connected.

Theorem

(Cauchy's theorem for simply connected domain) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then* $\int_C f(z)dz = 0$

Basic Example

Consider $f(z) = \int_C \frac{1}{z-z_0} dz$ where C is any positively oriented circle around z_0.

We can parametrize C as $z(t) = z_0 + re^{it}$ with $0 \leq t \leq 2\pi$. Then $\int_C f(z)dz = \int_C \frac{1}{z-z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i$

Note that the integral is independent of the circle chosen around z_0. This is an instance of the following more general situation:

Theorem

(More General form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. If γ and γ' are two closed contours in Ω which can be "continuously deformed" into each other and f is a holomorphic function on Ω, then $\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$.*

Remark: A similar computation shows that $\int_C \frac{1}{(z-z_0)^m} = 0$ for all $m \neq 1$. This follows from the fact that $\frac{1}{(z-z_0)^m}$ admits a primitive in $\mathbb{C} - \{z_0\}$ when $m \neq 1$. Note that for $m = 1$, $\frac{1}{(z-z_0)^m}$ does not admit a primitive; $log(z - z_0)$ does not define a holomorphic function on any open set in $\mathbb{C}$ containing C.

Cauchy Integral Formula

We'll now use Cauchy's theorem to prove another very important, closely related theorem in this subject, namely the Cauchy Integral Formula.

Theorem (Cauchy Integral Formula)

Let f be holomorphic everywhere within and on a simple closed curve γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

We use the following fact in the proof, often called ML inequality : If γ is a contour of length L and $|f(z)| \leq M$ on γ, then $|\int_\gamma f(z)dz| \leq ML$.