MA 205 Complex Analysis

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Recall

In the last class, we have seen that holomorphic functions and analytic functions are same using CIF. We have derived CIF for derivatives

$$f^{(n)}(z_0) = \frac{n!}{2\pi \imath} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n \in \mathbb{N}).$$

Using Cauchy estimate we have proved a very important property of an entire function, i.e. Every bounded entire function is constant. This is known as Liouville's theorem and it gives us a proof of the FTA. We have studied many analytic properties of holomorphic functions. Today, we will see some geometric properties.

Examples

Example 1:

$$\begin{aligned}\int_{|z|=2} \frac{z^2}{z-1}dz &= 2\pi i[z^2]|_{z=1} \\ &= 2\pi i.\end{aligned}$$

Example 2:

$$\begin{aligned}\int_{|z|=2} \frac{e^z}{z^2(z-1)}dz &= \int_{|z|=\epsilon} \frac{e^z/(z-1)}{z^2}dz + \int_{|z-1|=\epsilon} \frac{e^z/z^2}{z-1}dz \\ &= 2\pi i\left[\frac{d}{dz}\left(\frac{e^z}{z-1}\right)\right]_{z=0} + 2\pi i\left[\frac{e^z}{z^2}\right]_{z=1} \\ &= -4\pi i + (2\pi i)e \\ &= 2\pi i(e-2)\end{aligned}$$

Example 3:

$$
\begin{aligned}
\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1}dz &= \int_{|z-1|=1} \frac{z^2-z-1-3z+4}{z^2-z-1}dz \\
&= \int_{|z-1|=1} [1-\frac{3z-4}{z^2-z-1}]dz \\
&= 0-\int_{|z-1|=1} \frac{3z-4}{(z-\frac{1+\sqrt{5}}{2})(z-\frac{1-\sqrt{5}}{2})} \\
&= -\int_{|z-1|=1} \frac{\left(\frac{3z-4}{z-\frac{1-\sqrt{5}}{2}}\right)}{(z-\frac{1+\sqrt{5}}{2})} \\
&= -[(3z-4)/(z-(1-\sqrt{5})/2)]_{z=(1+\sqrt{5})/2} \\
&= (\sqrt{5}-3)/2.
\end{aligned}
$$

Exercise: Let f be an entire function such that there exists a real constant C such that for all $z \in \mathbb{C}$, $|f(z)| \leq C|z|^n$, then $f(z)$ is a polynomial of degree less than or equal to n.

Logarithm Revisited

Theorem

Let Ω be a simply connected domain in $\mathbb{C}$ with $1 \in \Omega$ and $0 \notin \Omega$. Then there exists a unique holomorphic function $F(z)$ on Ω, (denoted $\log(z)$) such that:

1. $F(1) = 0$ and $F'(z) = 1/z$

2. $e^{F(z)} = z \;\; \forall z \in \Omega$

3. $F(r) = ln(r)$ when r is a positive real number close to 1. (With the usual definition of ln for real numbers)

Proof:

Since $1 \in \Omega$ and $0 \notin \Omega$, define the function $F(z) = \int_1^z \frac{1}{w} dw$.Since Ω is simply-connected, it follows by Cauchy's theorem, that this function is well defined.We have seen before that this defines a holomorphic function on Ω. Clearly $F(1) = 0$ and $F'(z) = \frac{1}{z}$ proving 1.

Proof cont ..

One checks that the function $ze^{-F(z)}$ has its derivative identically vanishing and hence is a constant. Substituting $z = 1$, this constant is seen to be 1. This proves 2. For proving 3, take a straight path joining 1 and r, for a small real number r. Then $F(r) = \int_1^r \frac{1}{t} dt = ln(r)$.

Note that such a function is unique. (Why ?)

Quiz policy and Seating arrangement

Policy: Two marks will be deducted if you don't write Tutorial Batch/Division in the answer booklet. Note that Tutorial Batch and Division are the same for this course.

Tutorial Batch/Division	**Venue**	**Block** (From the stage)
T1	LA 301	Left Block
T2	LA 301	Middle Block
T3	LA 301	Right Block
T4	LA 302	Left Block
T5	LA 302	Middle Block

Find a seat according to the above assignment and make sure that there is a gap between you and the student next to you.

Zero's of Homolorphic Functions

Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a complex analytic function defined on Ω. This means f can be expressed by a power series expanded around any point in Ω. Let z_0 be a point in Ω at which f vanishes. We will show that either f is identically zero or there exists a neigborhood of z_0 in which f has no other zero. Assume the contrary. Then f is a non-zero function for which there exists a sequence of points $\{z_n\}$ converging to z_0 such that f vanishes along this sequence. We show that $f^k(z_0) = 0$ for all $k \geq 0$. Without loss of generality, we can assume that this open set contains 0 and $z_0 = 0$. Let n be the largest natural number such that $f^i(0) = 0$ for all $0 \leq i \leq n$. Then f can be expanded in a neighborhood as

$$f(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(0) + \frac{z^{n+2}}{(n+2)!} f^{(n+2}(0) + \ldots$$

$$= z^{n+1} \left(\frac{f^{(n+1)}(0)}{(n+1)!} + \frac{z}{(n+2)!} f^{n+2}(0) + \ldots \right)$$

Now as $z \to 0$ along the sequence $\{z_n\}$, we see that the lhs is identically zero. Hence the rhs also vanishes identically along this sequence. Hence the term inside the bracket vanishes along $\{z_n\}$ and hence by continuity, vanishes at the limit namely 0, thereby showing that $f^{(n+1)}(0) = 0$. This contradicts the assumption on n. Now consider the set

$$A = \{z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

$A \neq \emptyset$ since $z_0 \in A$. We'll show that A is both open and closed, which shows that $A = \Omega$. This of course would mean $f \equiv 0$.

To show that A is closed, we need to show that A contains all its limit points. If z is a limit point of A, let $z_k \in A$ be such that $\lim z_k = z$. Since $f^{(n)}$ is continuous, it follows that $f^{(n)}(z) = 0$; i.e., $z \in A$.

To show that A is open, we need to show that every $a \in A$ has a neighborhood which is contained in A. Since Ω is open, there is a neighborhood of a which is contained in Ω. On this neighborhood, if we write $f(z) = \sum a_n(z-a)^n$, then $a_n = \frac{1}{n!}f^{(n)}(a) = 0$ for each $n \geq 0$. Thus, $f \equiv 0$ for all z in this neighborhood. Therefore, this neighborhood is in fact contained in A.

Corollary [Identity Theorem]: If f and g are holomorphic in Ω, then $f \equiv g$ iff there exists a non-constant sequence $\{z_n\} \subseteq \{z \in \Omega \mid f(z) = g(z)\}$ such that $\lim_{n\to\infty} z_n = z_0 \in \Omega$.