MA 205 Complex Analysis: Singularity at ∞ and Real integral

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Singularity at ∞

Isolated Singularity at Infinity: $f(z)$ is said to have an isolated singularity at ∞ if f is holomorphic outside a disc of radius R for some R. Equivalently, $f(1/z)$ has an isolated singularity at 0.
If f has an isolated singularity at ∞, we can talk about the nature of singularity at ∞.
Definition: f is said to have a zero (resp. removable singularity, pole, essential singularity) at ∞ if $f(1/z)$ has a zero (resp. removable singularity, pole, essential singularity) at 0.

Examples:

- Entire functions has isolated singularity at ∞.
- Constant function has removable singularity at ∞.
- Polynomials have pole at ∞.
- e^z has an essential singularity at ∞.
- If f is an entire function which has a zero at ∞, then f is identically zero. (Why ??)
- There are plenty of meromorphic functions which have a zero at ∞, for example $1/z$.

Theorem

An entire functions from $\mathbb{C}$ to $\mathbb{C}$ has a pole at ∞ if and only if it is a non-constant polynomial.

One of the important applications of Complex Analysis is computation of real integrals.

Let $f : [0, \infty] \to \mathbb{R}$ be a function such that $\int_0^R f(x)dx$ exists for each $R \geq 0$. One then defines the Improper integral $\int_0^\infty f(x)$ to be $\lim_{R\to\infty} \int_0^R f(x)dx$.
Similarly if $f : [-\infty, \infty] \to \mathbb{R}$ is a function such that $\int_{-a}^b f(x)dx$ exists for each $a, b \geq 0$, then the improper integral $\int_{-\infty}^\infty f(x)$ is defined as $\lim_{a,b\to\infty} \int_{-a}^b f(x)dx$. If f is integrable, then its integral can be computed as $\lim_{R\to\infty} \int_{-R}^R f(x)dx$.

Improper Integral

For instance the function $\frac{1}{1+x^2}$ is integrable on $\mathbb{R}$ while the integral $\int_{-\infty}^{\infty} sin(x)dx$ does not exist. Intuitively, for such an improper integral to exist, the function has to decay to zero sufficiently rapidly outside a "small set ". (Note that it need not quite tend to zero as $|x| \to \infty$).
Often, instead of a real variable, the function $f(z)$ with the complex variable is holomorphic outside some discrete set. This allows us to exploit Cauchy's residue formula to compute the real integral as follows.

Consider a close contour $\gamma_R \cup C_R$ where γ_R being a line segment along the real axis between $-R$ and R and C_R is the semicircle of radius R around 0. We can then evaluate $\int_{\gamma_R \cup C_R} f(z)dz$ by means of residue theorem, and show that the integral over the extra "added" part of γ_R, namely C_R asymptotically vanishes as $R \to \infty$. Thus taking the contour integral over γ_R and allowing R to tend to ∞, we get the desired answer.

Example

Compute $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$.
You might have seen the computation of this integral in MA 105 but now lets work out this computation using using MA 205 !
The idea is to compute $\int_{-r}^{r} \frac{x^2}{1+x^4} dx$ and take limit as $r \to \infty$. Fix $r > 1$. Let γ_r be $[-r, r]$ together with C_r, the upper part of the circle $|z| = r$ oriented counterclockwise. Take $f(z) = \frac{z^2}{1+z^4}$. Then f has two poles inside γ. Now,

$$\frac{1}{2\pi\imath} \int_{\gamma} f(z) dz = \mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) = \frac{-\imath}{2\sqrt{2}}.$$

This is same as

$$\frac{1}{2\pi\imath} \int_{-r}^{r} \frac{x^2}{1+x^4} dx + \frac{1}{2\pi\imath} \int_{C_r} \frac{z^2 dz}{1+z^4}.$$

Example

By changing to polar coordinates, the second integral becomes,

$$\frac{1}{2\pi}\int_0^{\pi}\frac{r^3e^{3\imath t}}{1+r^4e^{4\imath t}}dt.$$

Note that,

$$\left|r^3\int_0^{\pi}\frac{e^{3\imath t}}{1+r^4e^{4\imath t}}dt\right|\leq\frac{\pi r^3}{r^4-1}.$$

Thus, in the limit, this integral is zero. Therefore,

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx=\frac{\pi}{\sqrt{2}}.$$

Useful Estimate

If $P(z)/Q(z)$ is a rational function such that with deg $Q(z) \geq$ deg $P(z) + 2$. Then there exists a constant C such that for $|P(z)/Q(z)| \leq C/|z^2|$ for $|z|$ sufficiently large. Thus for a large real number R, $|P(z)/Q(z)| \leq \frac{C}{R^2}$ on the circle of radius R.

Example

To compute $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}$, we consider a contour γ to be the union of the segment from $-R$ to R along with the upper half semicircle C_R of radius R, oriented positively. There is just one pole inside γ which is $\imath$. Compute $\mathrm{Res}(f;\imath)$, where $f(z) = \frac{1}{(1+z^2)^n}$. This is given by $\frac{g^{(n-1)}(\imath)}{(n-1)!}$, where $g(z) = \frac{1}{(z+\imath)^n}$. Check:

$$\mathrm{Res}(f;\imath) = \frac{-\imath}{2^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}.$$

By the earlier remark, there exists a constant C , such that $\left|\frac{1}{(1+z^2)^n}\right| \leq \frac{C}{R^2}$ on C_R for large enough R. Then by ML Lemma $\int_{C_R} \frac{dz}{(1+z^2)^n}$ tends to zero as $R \to \infty$. Thus, the value of the real integral is $\frac{\pi}{4^{n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}$.

Jordan's lemma

Theorem (Jordan's Lemma)

Let f be a continuous function defined on the semicircular contour $C_R = \{Re^{\imath\theta} \mid \theta \in [0, \pi]\}$ of the form

$$f(z) = e^{\imath az} g(z),$$

where $g(z)$ is a continuous function and with $a > 0$. Then,

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} \max_{\theta \in [0,\pi]} |g(Re^{\imath\theta})|.$$

Real Integrals

Proof:

$$\int_{C_R} f(z)dz \quad = \quad \int_0^{\pi} g(Re^{\imath\theta})e^{\imath aR(\cos\theta+\imath\sin\theta)}\imath Re^{\imath\theta}d\theta.$$

Therefore,

$$\begin{aligned}
\left|\int_{C_R} f(z)dz\right| &\leq R\int_0^{\pi}\left|g(Re^{\imath\theta})e^{aR(\imath\cos\theta-\sin\theta)}\imath e^{\imath\theta}\right|d\theta \\
&= R\int_0^{\pi}\left|g(Re^{\imath\theta})\right|e^{-aR\sin\theta}d\theta \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta \quad \text{where } M_R = sup|g(Re^{i\theta})| \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{\frac{-2aR\theta}{\pi}}d\theta = \frac{\pi}{a}(1-e^{-aR})M_R \leq \frac{\pi}{a}M_R,
\end{aligned}$$

since $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta \in [0, \frac{\pi}{2}]$.

Example

Compute $\int_0^\infty \frac{\sin x}{x} dx$.
We'll consider the function

$$f(z) = \frac{e^{\imath z}}{z}.$$

Let γ be the boundary of the upper part of the annulus $A(0; r, R)$.
Then, $\int_\gamma f(z) dz = 0$, by Cauchy's theorem.

Example

But,

$$\int_\gamma f(z)dz = \int_r^R \frac{e^{\imath x}}{x}dx + \int_{\gamma_R} \frac{e^{\imath z}}{z}dz + \int_{-R}^{-r} \frac{e^{\imath x}}{x}dx + \int_{\gamma_r} \frac{e^{\imath z}}{z}dz.$$

Now,

$$\begin{aligned} \int_r^R \frac{\sin x}{x}dx &= \frac{1}{2\imath}\int_r^R \frac{e^{\imath x} - e^{-\imath x}}{x}dx \\ &= \frac{1}{2\imath}\int_r^R \frac{e^{\imath x}}{x}dx + \frac{1}{2\imath}\int_{-R}^{-r} \frac{e^{\imath x}}{x}dx. \end{aligned}$$

Thus, we only need to compute

$$\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{\imath z}}{z}dz \quad \& \quad \lim_{r\to 0}\int_{\gamma_r} \frac{e^{\imath z}}{z}dz.$$

Example

Now,

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{\imath z}}{z}dz = 0,$$

by Jordan's lemma. On the other hand, note that $\frac{e^{\imath z}-1}{z}$ has a removable singularity at 0. Thus, there is $M > 0$ such that

$$\left|\frac{e^{\imath z}-1}{z}\right| \leq M,$$

for $|z| \leq 1$. Thus,

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{\imath z}-1}{z}dz = 0,$$

by appealing to ML inequality.

Example

Therefore,

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{\imath z}}{z}dz = -\pi\imath.$$

Thus,

$$\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Example

Show that $\int_0^\infty \frac{\log x}{1+x^2} dx = 0$.
We'll work with γ as in the previous problem. We take

$$f(z) = \frac{\log z}{1+z^2},$$

where $\log z$ is a branch of the logarithm which is defined on the x-axis, so that $\int_r^R$ and $\int_{-R}^{-r}$ make sense. For instance, we can take the branch with negative y-axis as the branch cut. Then,

$$\log x = \begin{cases} \log x & \text{if } x > 0, \\ \log |x| + \imath\pi & \text{if } x < 0. \end{cases}$$

Now,

$$\int_{\gamma} \frac{\log z}{1+z^2}dz = \int_r^R \frac{\log x}{1+x^2}dx + \int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx + \int_{\gamma_r} \frac{\log z}{1+z^2}dz.$$

LHS is $2\pi\imath \cdot \mathrm{Res}(f;\imath) = 2\pi\imath \cdot \frac{\log \imath}{2\imath} = \frac{\pi^2 \imath}{2}$. Also,

$$\begin{aligned} &= \int_r^R \frac{\log x}{1+x^2}dx + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx \\ &= 2\int_r^R \frac{\log x}{1+x^2}dx + \imath\pi \int_r^R \frac{dx}{1+x^2} \\ &= 2\int_r^R \frac{\log x}{1+x^2}dx + \frac{\pi^2 \imath}{2}. \end{aligned}$$

(In the Limit)

Thus,

$$\int_r^R \frac{\log x}{1+x^2}dx = -\frac{1}{2}\left[\int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{\gamma_r} \frac{\log z}{1+z^2}dz\right].$$

Note that

$$\begin{aligned} \left|\int_{\gamma_\rho} \frac{\log z}{1+z^2}dz\right| &= \left|\rho \int_0^\pi \frac{\log \rho + \imath\theta}{1+\rho^2 e^{\imath\theta}} e^{\imath\theta} d\theta\right| \\ &\leq \frac{\rho|\log \rho|}{|1-\rho^2|}\int_0^\pi d\theta + \frac{\rho}{|1-\rho^2|}\int_0^\pi \theta d\theta \\ &= \frac{\pi\rho|\log\rho|}{|1-\rho^2|} + \frac{\rho\pi^2}{2|1-\rho^2|}. \end{aligned}$$

This is zero in the limit if $\rho \to 0+$ or $\rho \to \infty$. Thus, the given integral is zero.