MA 205 Complex Analysis

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Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

Introduction

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total). Similarly, we'll be interested in $f : \mathbb{C} \mapsto \mathbb{C}$ which are differentiable in the "complex sense". If f fails to be differentiable at some points, we'll also investigate such failure.

Introduction

Thus, in a way,

> *in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.*

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further! In other words, there is MA 205 after MA 105, but no MA 305 or 405!

Furthermore complex analysis introduces techniques which are often useful in answering questions which a priori dont have anything to do with complex numbers. We will see examples of this later.
Today, I'll try to illustrate the beauty of complex analysis in two distinct ways; I'll give you two surprises!

Motivation

God made integers, all else is the work of man
- Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division. $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test, but it fails the "analysis test"! , namely Cauchy sequences of rational numbers need not converge to a rational number.

Fundamental theorem of Algebra

$\mathbb{R}$ passes both these tests and hence analysis over $\mathbb{R}$ is rich and exciting. But it fails another "algebra test" namely obtaining roots of polynomials.

Theorem

Every non-constant polynomial with complex coefficients has a complex root.

This theorem fails over all the other "number systems "we know, namely $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Today this theorem has more than hundred proofs, many of them using complex analysis. We will see at least one proof of this in this course.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root. In other words, you are imagining an $\imath$, which has this property that $\imath^2 = -1$. And then we can write $x^2 + 1 = (x - \imath)(x + \imath)$.

Numbers

We imagine $\mathbb{C}$ as
$$\mathbb{R} + \imath\mathbb{R}.$$
i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. Then we add:
$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$
If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:
$$\begin{aligned}(a + \imath b)(c + \imath d) &= ac + a\imath d + \imath bc + (\imath b) \cdot (\imath d)\\ &= (ac - bd) + \imath(ad + bc)\end{aligned}$$
which is an another complex number.

Incidentally, if $z = x + \imath y \in \mathbb{C}$, we call x to be $\mathrm{Re}(z)$ and y to be $\mathrm{Im}(z)$. So coming back to the fundamental theorem of algebra, it is interesting that just adding one root of one real polynomial, namely $X^2 + 1$ gives you all the roots of all the complex polynomials !

For $z, z_0 \in \mathbb{C}$, $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.
For $\delta > 0$

$$\mathcal{B}(z_0, \delta) := \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

is the open disc of radius δ and center at z_0 (Also known as δ-neighbourhood of z_0). Let $\Omega \subseteq \mathbb{C}$. We say that Ω is an open subset of $\mathbb{C}$ if given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that $\mathcal{B}(z_0, \delta) \subseteq \Omega$.

Connectedness

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them, i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a domain. An arbitrary open set in $\mathbb{C}$ is a disjoint union of domains. In this course we will be mostly interested in complex-valued functions defined over domains.

Limit

Let's start with recalling the notions of limit, continuity, and differentiation. Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L$$

$$\iff$$

values of $f(x)$ can be made as close to L as we like,
by taking x close enough to a, on either side of a,
but not equal to a.

$$\iff$$

$|f(x) - L|$ can be made as small as we like by taking
$|x - a|$ sufficiently small, for $x \neq a$.

Limit

$\Longleftrightarrow$

for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$, then $L \in \mathbb{C}$ is the limit of f as $z \mapsto z_0$, $z_0 \in \mathbb{C}$, if for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this!

Limit

Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ and $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then,

1. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$
2. $\lim_{z \to z_0} [cf(z)] = c \lim_{z \to z_0} f(z)$
3. $\lim_{z \to z_0} [f(z)g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$
4. $\lim_{z \to z_0} \dfrac{1}{f(z)} = \dfrac{1}{\lim_{z \to z_0} f(z)}$, *if* $\lim_{z \to z_0} f(z) \neq 0$.

Continuity

In the real case:

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is continuous on a domain if it is continuous at every point in the domain.

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Ditto in the complex case. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is said to be **differentiable**, at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

We say that f is holomorphic on an open set Ω if f is differentiable at each point of Ω. f is holomorphic (also called complex analytic) at z_0 if it is holomorphic in some neighbourhood of z_0. Similarly, holomorphicity on a non-empty open set.

Remark: Note that differentiability at a point does not imply holomorphicity at that point.

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable, always differentiable.

We'll prove this too, but later. As you know from MA 105, this is far from true in the real variable case. f' needn't even be continuous. Examples?

Can someone venture a guess as to why identical definitions lead to such extreme scenarios?

Syllabus

Definition and properties of analytic functions.
Cauchy-Riemann equations, harmonic functions.
Power series and their properties.
Elementary functions.
Cauchys theorem and its applications.
Taylor series ans Laurent expansions.
Residues and the Cauchy residue formula.
Evaluation of improper integrals.
Conformal mappings.
Inversion of Laplace transforms.

References

1. R. V. Churchill and J. W. Brown, Complex variables and applications (7th Edition), McGraw-Hill (2003).
2. J. M. Howie, Complex analysis, Springer-Verlag (2004).
3. M. J. Ablowitz and A. S. Fokas, Complex Variables-Introduction and Applications, Cambridge University Press, 1998 (Indian Edition).
4. E. Kreyszig, Advanced engineering mathematics (8th Edition), John Wiley (1999).

References

More advanced references :

1. Lars Ahlfors - Complex Analysis
2. John Conway - Functions of a Complex Variable
3. Serge Lang - Complex Analysis