MA 205 Complex Analysis: Exponential Function

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July 26, 2018

So what did we do in the last class? We looked at power series, saw the existence of radius of convergence for any power series, remember that this was given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

and let me repeat once again, it's limsup and not the limit in the above and thus it always exists. If

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists, then this too gives R.

Power series can be added, subtracted, and multiplied in the obvious way. It can also be differentiated term by term, in its domain of convergence. Indeed, if $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, then,

$$
\begin{aligned}
&\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \\
&= \lim_{z\to z_0} \sum_n a_n \left(\frac{(z-a)^n - (z_0-a)^n}{z-z_0} \right) \\
&= \sum_n a_n \left(\lim_{z\to z_0} \frac{(z-a)^n - (z_0-a)^n}{z-z_0} \right) \\
&= \sum_n n a_n (z_0-a)^{n-1}.
\end{aligned}
$$

Apply root test to check that the radius of convergence of $\sum_n n a_n (z-a)^{n-1}$ is same as the given power series.

Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is said to be **analytic** if it is locally given by a convergent power series; i.e., every $z_0 \in \Omega$ has a neighbourhood contained in Ω such that there exists a power series centered at z_0 which converges to $f(z)$ for all z in that neighbourhood. Analytic functions are infinitely differentiable; you only have to differentiate the power series term by term. Also, if $f(z) = \sum_{n=1}^{\infty} a_i (z - z_0)^i$, then $a_i = \frac{f^{(i)(z_0)}}{i!}$. Thus, an analytic function is given by its Taylor series. We'll later prove:

$$\textit{holomorphic} \implies \textit{analytic.}$$

This would prove our statement from Lecture 1 that once differentiable is always differentiable!

Just as in the complex case, power series and analytic functions can be defined in the real case too. But unlike in the complex case, differentiable does not mean real analytic. In fact, even infinitely differentiable does not mean real analytic. For example, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable but not real analytic. In this example, $f^{(i)}(0) = 0$ for all i, and thus the Taylor series of f is the zero function.

Now, we'll use our knowledge of power series to construct a few basic functions. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.$$

We've seen that its radius of convergence is ∞; i.e., this function is well-defined for any $z \in \mathbb{C}$. This function will keep our company throughout this course. So let's befriend it a bit more ! By term by term differentiation, one observes that

$$f'(z) = f(z),$$

and $f(bz)' = bf(bz)$. We'll denote this function by $\exp(z)$, also denoted e^z.

Now consider the function

$$h(z) = \exp(z) \cdot \exp(-z).$$

This is defined throughout $\mathbb{C}$. What's $h'(z)$?

$$h'(z) = \exp(z) \cdot (-\exp(-z)) + \exp(-z) \cdot \exp(z) = -h + h = 0.$$

Therefore $h(z) \equiv c$ and since $h(0) = 1$, it is identically equal to 1.
Thus, we have proved two things:
(i) $\exp(z)$ is never vanishing.
(ii) $\exp(-z) = \frac{1}{\exp(z)}$.

Note that the derivative of $f(z) = a\exp(bz)$ is $f'(z) = bf(z)$. Interestingly, the converse is also true. Thus,

$$f(z) = a\exp(bz) \textit{ for } a, b \in \mathbb{C} \iff f'(z) = bf(z).$$

Proof: Assume $f'(z) = bf(z)$ for $b \in \mathbb{C}$. Now consider

$$h(z) = f(z)\exp(-bz).$$

Then, $h'(z) = -bh + bh = 0$, for all z in the domain. So, $h(z) \equiv a$ for some $a \in \mathbb{C}$. Therefore,

$$f(z) = \frac{a}{\exp(-bz)} = a\exp(bz),$$

by what we already know.

Exponential Function

Corollary: $f' = f$ and $f(0) = 1$ characterizes the exponential function. The function

$$f(z) = e^x(\cos y + \imath \sin y),$$

is holomorphic throughout $\mathbb{C}$ and $f' = f$. Clearly, $f(0) = 1$ as well. Thus,

$$\exp(z) = e^x(\cos y + \imath \sin y).$$

Remark: e^x here is e to the power of x, and e is the number that you know from MA 105 (base of natural logarithm). We'll try and reconstruct everything about logarithm and exponential function from scratch.

By now we know that exp is defined throughout $\mathbb{C}$ and that 0 is not in the range of $\exp(z)$. , exp is a map from $\mathbb{C} \to \mathbb{C}^\times$. Now exp has this wonderful property that it takes the "correct" operation in $\mathbb{C}$ to the "correct" operation in $\mathbb{C}^\times$.

$$\exp(w + z) = \exp(w) \cdot \exp(z).$$

Thus in the language of group theory exp is a homomorphism from $\mathbb{C}$ to $\mathbb{C}^\times$.

Proof: Fix $w \in \mathbb{C}$. Then the function $f(z) = \exp(w + z)$ is holomorphic in $\mathbb{C}$ and $f'(z) = f(z)$. So, $f(z) = a \exp(z)$ for some constant a. By evaluating f at 0, see that $a = \exp(w)$. Thus, $f(z) = \exp(w) \cdot \exp(z)$.

This property of converting addition into multiplication also characterizes $\exp(z)$.

Let $0 \in \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is such that f is differentiable at 0 and $f(0) \neq 0$. Suppose $f(w+z) = f(w)f(z)$ whenever $w, z, w+z \in \Omega$. Then, $f(z) = \exp(bz)$, where $b = f'(0)$.

Trigonometric Functions

Recall the Taylor expansions:

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots = \frac{\exp(\imath y) - \exp(-\imath y)}{2\imath}$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots = \frac{\exp(\imath y) + \exp(-\imath y)}{2}.$$

Motivated by this, we define complex trigonometric functions:

$$\sin z = \frac{\exp(\imath z) - \exp(-\imath z)}{2\imath} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\cos z = \frac{\exp(\imath z) + \exp(-\imath z)}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots.$$

Hyperbolic Functions

Define hyperbolic sine and hyperbolic cosine by:

$$cos(hz) = \frac{\exp(z) + \exp(-z)}{2}.$$

Its power series is given by

$$coz(hz) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

Simillarly,

$$sin(hz) = \frac{\exp(z) - \exp(-z)}{2}.$$

Its power series is given by

$$sin(hz) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Exercise:

(i) Define other trigonometric functions.

(ii) Show that $\exp : \mathbb{C} \to \mathbb{C}^\times$ is onto. Is it one-to-one ? (Show that $\exp(z) = \exp(z + 2\pi i)$)

(iii) Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. In particular, note the difference with real sine and cosine which were bounded by 1.

(iv) Show that $\sin^2 z + \cos^2 z = 1$,

$\sin(z + w) = \sin z \cos w + \cos z \sin w$,

$\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Logarithms cont...

We have seen that exp is not a 1-1 function. Hence its inverse is not defined everywhere. Nevertheless we would like to construct an analytic inverse function, called logarithm on certain subsets, i.e, a analytic function $g(z) = log(z)$ such that $\exp(g(z)) = z$. As remarked before, log will be undefined at 0. Let z be any complex number. Then $z = |z|(cos(\theta) + isin(\theta)) = |z| \exp(i\theta) = \exp(log|z| + i\theta)$. Here $|z|$ is the magnitude and θ is the argument. Note that θ is defined only upto an integer multiple of $2\pi i$. Then the solutions of $\exp(g(z)) = z$ are given by $g(z) = log|z| + i(\theta + 2\pi n)$.

Definition

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f(z)$ be a continuous function on Ω such that $\exp(f(z)) = z \ \forall z \in \Omega$. Then f is called a branch of the logarithm.

Logarithm cont...

Lemma

Let $\Omega \subseteq \mathbb{C}$ be a domain and let f be a branch of the logarithm. Then any other branch of the logarith differs from f by a constant multiple multiple of $2\pi i$.

Proof.

Let $g(z)$ be a branch of the logarithm. Then $f(z) - g(z)$ is a constant multiple of $2\pi i$ for all $z \in \Omega$. Since Ω is connected, and $f(z) - g(z)$ is continuous while integral multiples of $2\pi i$ is a discrete set, it follows that $f(z) - g(z)$ is a constant multiple of $2\pi i$

$\square$

We will usually work with a fixed branch of the logarithm called the **Principal Branch**. This is defined as follows:

Let $\Omega \subset \mathbb{C}$ be the open subset defined by $\mathbb{C}$ minus the negative real line. For any $z \in \Omega$, $z = |z|e^{i\theta} : -\pi < \theta < \pi = re^{i\theta}$, define $f(z) = logr + i\theta = log|z| + iArg(z)$.

One checks that $f(z)$ is a branch of $log(z)$ on Ω. In fact, f is analytic and $f'(z) = \frac{1}{z}$.