MA 205 Complex Analysis: CR Equations

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Introduction

In the last class, we introduced complex numbers and studied complex valued functions defined on a domain in $\mathbb{C}$. We stated the fact that every polynomial with complex coefficients has a complex roots. This is called the fundamental theorem of algebra. We introduced complex-differentiability of a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, where Ω is an open subset of $\mathbb{C}$. We also stated the fact that if f is once differentiable in Ω, then it is infinitely many times differentiable in Ω.

Today, first we'll derive the so called Cauchy-Riemann equations. There are two Cauchy-Riemann equations, and these are partial differential equations; i.e., equations containing partial derivatives. If f is complex differentiable at a point $z_0 = a + \imath b$, then these two equations will be satisfied at the point (a, b).

Cauchy-Riemann Equations

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$. Thus,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists. In the last class, we have stressed the point that the existence of this complex limit means a lot; the limit exists as z approaches z_0 along any path. To derive the CR equations, we'll in particular look at the existence of this limit as $z \to z_0$ along the x-direction and the y-direction.

Cauchy-Riemann Equations

Let $z = x + \imath y$ and $f(z) = u(x, y) + \imath v(x, y)$. Now, as $z \to z_0$ in the x-direction:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

$$= \lim_{h \to 0} \left[\frac{u(a + h, b) - u(a, b)}{h} + \imath \frac{v(a + h, b) - v(a, b)}{h} \right]$$

$$= \lim_{h \to 0} \frac{u(a + h, b) - u(a, b)}{h} + \imath \lim_{h \to 0} \frac{v(a + h, b) - v(a, b)}{h}$$

$$= u_x(a, b) + \imath v_x(a, b).$$

In writing the limit of a sum as the sum of the limits, we have used the fact that the individual limits exist. Why is this true in our situation?

Cauchy-Riemann Equations

Similarly, in the y-direction, we get

$$f'(z_0) = \lim_{k \to 0} \frac{f(z_0 + \imath k) - f(z_0)}{\imath k} = v_y(a, b) - \imath u_y(a, b).$$

Thus, differentiability of $f = u + \imath v$ at $z_0 = a + \imath b$ implies that u_x, u_y, v_x, v_y exist at (a, b) and they satisfy

$$u_x = v_y \;\&\; u_y = -v_x$$

at (a, b). These are the CR equations. If CR equations are not satisfied at a point, then f is not differentiable at that point.

Example: Consider $f(z) = |z|^2$. Here, $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. Thus CR equations are satisfied only at the point $(0, 0)$. We conclude that f is not differentiable at any point other than $(0, 0)$. Can we conclude that f is differentiable at $(0, 0)$? Well, we need to check; CR equations give only one direction. In other words, real and imaginary parts of f satisfying CR equations at a point is necessary but not sufficient for f to be differentiable at that point. In this example:

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{|z|^2}{z} = 0.$$

Example:

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Here,

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

$$v(x, y) = \begin{cases} \frac{-3x^2y + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Check that CR equations are satisfied at $(0, 0)$. You'll get $u_x = v_y = 1$ and $u_y = -v_x = 0$ at $(0, 0)$.

But, f is not differentiable at 0.

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{\bar{z}^2}{z^2}=\lim_{(x,y)\to(0,0)}\frac{(x-\imath y)^2}{(x+\imath y)^2}.$$

If $(x,y)\to(0,0)$ via either of the axes, this limit is 1. If $(x,y)\to(0,0)$ via $y=x$, this limit is -1. So limit does not exist.

If $z = x + \imath y$, then,

$$x = \frac{z + \bar{z}}{2}, \; y = \frac{z - \bar{z}}{2\imath}.$$

Suppose for a moment that z and $\bar{z}$ are independent variables! Formally applying chain rule:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2\imath} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \imath \frac{\partial f}{\partial y}\right).$$

Similarly,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \imath \frac{\partial f}{\partial y}\right).$$

Cauchy-Riemann Equations

Motivated by this, we introduce the symbols:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right);\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right).$$

Note that CR equations now can be written as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Cauchy-Riemann Equations

We can of course view $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ as a function of two real variables;

$$f(x, y) = (u(x, y), v(x, y)).$$

For such functions, in MA 105, you have seen the notion of the total derivative. Recall: f is differentiable at (a, b) if there exists a 2×2 matrix $Df(a, b)$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(a+h, b+k) - f(a,b) - Df(a,b) \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h,k)\|} = 0.$$

Cauchy-Riemann Equations

Of course, if total derivative exists, then all the partial derivatives exist, and

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Existence of partial derivatives does not imply the existence of total derivative, but existence of partial derivatives which are continuous throughout the domain does imply the existence of total derivative.

Exercise

<u>Exercise</u>: Show that if f is complex differentiable, then f is real differentiable; i.e., f has a total derivative as a function of two real variables. Show that the converse is not true.
(At the moment solve this exercise assuming the continuity of the first partial derivatives of u and v. We shall see later that this assumption can be removed (it is automatic)).

Cauchy-Riemann Equations

Thus, complex differentiability implies:

- real differentiability
- real and imaginary parts satisfy CR.

What if we assume both these? Can we then say f is complex differentiable? <u>And the answer is Yes</u>.

Proof: Since $f = u + \imath v$ is real differentiable,

$$\lim_{(x,y)\to(a,b)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} - \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \right\|}{\|(x-a,\ y-b)\|} = 0.$$

Note that the numerator is nothing but

$$|f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)|,$$

where $\alpha = u_x + \imath v_x$, $\beta = u_y + \imath v_y$.

Define

$$\eta(z) = \frac{f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)}{z - z_0}.$$

Observe that

$$\lim_{z \to z_0} \eta(z) = 0.$$

Thus,

$$f(z) - f(z_0) = \alpha(x-a) + \beta(y-b) + \eta(z)(z - z_0),$$

with $\eta(z) \to 0$ as $z \to z_0$.

Cauchy-Riemann Equations

Then

$$f(z) - f(z_0) = \frac{\alpha - \imath\beta}{2}(z - z_0) + \frac{\alpha + \imath\beta}{2}\overline{z - z_0} + \eta(z)(z - z_0).$$

Thus,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\frac{\overline{z - z_0}}{z - z_0} + \eta(z).$$

Question is whether the lhs limit exists as $z \to z_0$. This exists if and only if the rhs limit exists. Note that

$$\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}$$

does not exist (why?) and $\lim\limits_{z \to z_0} \eta(z)$ exists.

Thus the rhs limit exists if and only if

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

i.e., CR equations are satisfied at z_0. Also, if this is the case,

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0),$$

since $\lim_{z \to z_0} \eta(z) = 0$.

Corollary: Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be such that the partial derivatives exists in a neighborhood of z_0 and continuous at z_0. If they satisfy the CR equations at z_0, f is differentiable at z_0. (Proof?)

Cauchy-Riemann Equations

The assumptions in the statement of the corollary can be weakened. In fact, the following is true:

Theorem

Let f be continuous on Ω. Suppose the partial derivatives exist and satisfy the Cauchy-Riemann equations <u>at every point</u> in Ω. Then f is holomorphic in Ω.

We shall not prove this theorem.

Exercise

Exercise: Show that $f(z) = e^x(\cos y + \imath \sin y)$ is holomorphic throughout $\mathbb{C}$.
Note that $f'(z) = f(z)$. This is the complex exponential function.
Exercise: Show that the CR equations take the form

$$u_r = \frac{1}{r} v_\theta \ \& \ v_r = -\frac{1}{r} u_\theta$$

in polar coordinates.