Tutorial V

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1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that f is constant in the following cases:

   (i) There exists a $c > 0$ such that $|f(z)| > c > 0$ for all $z \in \mathbb{C}$.

   (ii) Re $f(z) \geq 0$ for all $z \in \mathbb{C}$.

   (ii) $\overline{f(\mathbb{C})} \neq \mathbb{C}$, i.e. $\mathbb{C} \setminus \overline{f(\mathbb{C})}$ is a non-empty open set.

2. Does there exists a holomorphic function f on the open unit disc such that

$$f\left(\frac{1}{n}\right) = \begin{cases} 1/n & \text{if } n \text{ is even;} \\ -1/n & \text{if } n \text{ is odd} \end{cases} ?$$

3. Let f be an entire function such that $f(\frac{1}{n^2}) = 1/n$ for all $n \in \mathbb{N}$. What can be said about f?

4. The following identity is called Taylor series with remainder :

$$f(z) = f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+\cdots+\frac{z^N}{N!}f^N(0)+\frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N f^{N+1}(tz)\, dt.$$

   Use this to prove the following inequalities :

   a) $\left| e^z - \sum_0^N \frac{z^n}{n!} \right| \leq \frac{|z|^{N+1}}{(N+1)!}$, $\mathrm{Re}(z) \leq 0$;

   b) $\left| \cos z - \sum_0^N (-1)^n \frac{z^{2n}}{2n!} \right| \leq \frac{|z|^{2N+2} \cosh R}{(2N+2)!}$, $\mathrm{Im}(z) \leq R$.

5. By computing $\int_{|z|=1} (z + 1/z)^{2n} \frac{dz}{z}$, show that $\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \frac{2\pi}{4^n} \frac{2n!}{n!^2}$.