EE 659: A First Course in Optimization
End Semester Examination

27 November, 2020

1. Determine if the function $f(x, y) = x^4 - 2xy + y^4$ is coercive on $\Re^2$. [5 marks]

2. Determine whether the function $f(x) = e^{-x}$ is convex, strictly convex or non-convex on $\Re$. [5 marks]

3. For function $f(x) = x^3$, what do the first- and second-order optimality conditions say about whether 0 is a minimum on $\Re$. [5 marks]

4. Determine the critical points of the function $f(x) = x^3 + 6x^2 - 15x + 2$ and characterize as minimum, maximum, or inflection point. Also determine whether the function has a global minimum or maximum on $\Re$. [5 marks]

5. Determine the critical points of the Lagrangian function $f(x, y) = 2x + y$ subject to $g(x, y) = x^2 + y^2 - 1 = 0$ and determine whether each is a constrained minimum, a constrained maximum, or neither. [5 marks]

6. Let $\mathbf{Q}$ be a $n \times n$ real orthogonal matrix. [5 marks]

   (a) Is it necessarily positive definite?

   (b) What is $\|\mathbf{Q}\|_2$?

7. Prove or disprove:
$$\|\mathbf{A}\|_2 = \|\mathbf{Q}\,\mathbf{A}\|_2$$
   where $\mathbf{A}$ is $m \times n$ real matrix and $\mathbf{Q}$ is a $m \times m$ real orthogonal matrix.

8. Do the following problems have solutions? Explain. [6 marks]

   (a) $\min_{x_1,\, x_2} x_1 + x_2$, subject to $x_1^2 + x_2^2 = 2; \quad 0 \le x_1 \le 1; \quad 0 \le x_2 \le 1.$

   (b) $\min_{x_1,\, x_2} x_1 + x_2$, subject to $x_1^2 + x_2^2 \le 1; \quad x_1 + x_2 = 3.$

   (c) $\min_{x_1,\, x_2} x_1\, x_2$, subject to $x_1 + x_2 = 2.$

9. Let $f : \Re^2 \to \Re$ be a real-valued function of two variables. What is the geometrical interpretation of the gradient vector [5 marks]

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f(\mathbf{x})}{\partial x_1} \\ \\ \dfrac{\partial f(\mathbf{x})}{\partial x_2} \end{bmatrix} ?$$

Specifically, explain the meaning of the direction and magnitude of $\nabla f(\mathbf{x})$.

10. [5 marks]

    (a) If $f : \Re^n \to \Re$, what do we call the Jacobian matrix of the gradient $\nabla f(\mathbf{x})$?

    (b) What special property does this matrix have, assuming f is twice continuously differentiable?

    (c) What additional special property does this matrix have near a local minimum of f?

11. Let $f : \Re^2 \to \Re$ be a nonlinear function. Since $\|f(\mathbf{x})\| = 0$ if, and only if $f(\mathbf{x}) = \mathbf{0}$, does this mean that searching for a minimum of $\|f(\mathbf{x})\|$ is equivalent to solving the nonlinear system $f(\mathbf{x}) = \mathbf{0}$? Why? [5 marks]

12. What is a good way to test a symmetric matrix to determine whether it is positive definite? [5 marks]

13. A spherical segment is the intersection of a sphere with one of the halfspaces corresponding to a 2-dimensional plane that meets the sphere. The area of a spherical segment is the area of the portion of its surface that is also part of the surface of the sphere. Note that if r is the radius of the sphere and h is the height of spherical segment, the volume enclosed by the segment is $\pi h^2 (r - h/3)$ and its spherical area is $2\pi r h$. Show that among all spherical segments with the same spherical area, the one that encloses the largest volume is hemisphere.
    . [5 marks]

14. Consider the two-dimensional problem with two inequality constraints.
    . [7 marks]

$$\min f(x_1, x_2)$$

$$g_1(x_1, x_2) \leq 0$$
$$g_2(x_1, x_2) \leq 0$$

The functions are defined as follow.

$$f(x_1, x_2) = x_1 + x_2,$$

$$g_1(x_1, x_2) = \begin{cases} -x_1^2 + x_2 & \text{if } x_1 \geq 0, \\ -x_1^4 + x_2 & \text{if } x_1 \leq 0, \end{cases}$$

$$g_2(x_1, x_2) = \begin{cases} x_1^4 - x_2 & \text{if } x_1 \geq 0, \\ x_1^2 - x_2 & \text{if } x_1 \leq 0. \end{cases}$$

(a) Is (0, 0) a local optimal?

(b) Can we apply Lagrangian multiplier theorem/KKT conditions at (0, 0)?

15. Consider the function $f : \Re^3 \to \Re$ given by $f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$, where, [7 marks]

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & \theta \end{bmatrix}$$

What is the Hessian of f? For what values of θ is f strictly convex?