JENSEN'S INEQUALITY: If 'f' is a convex function and X is a random variable,

$$E f(X) \geq f(EX)$$

[ref: Elements of Information Theory; Cover and Thomas IInd Edition.

Proof: Principle of Induction;

→ First, we will prove it two-mass point distribution ($n=2$), then for any n. (x_1, x_2)

$$E(X) = \underbrace{p_1 x_1 + p_2 x_2}_{\text{convex combination of } (x_1, x_2)} ; \quad p_1 + p_2 = 1 \quad p_1, p_2 \in [0,1]$$

Given that 'f' is convex over X:

$$f(p_1 x_1 + p_2 x_2) \leq \underbrace{p_1 f(x_1) + p_2 f(x_2)}_{Ef(X)}$$

$$f(EX) \leq Ef(X)$$

→ We, will assume that is true for a k-mass point distribution, and then show that it will be true for a $(k+1)$-mass point distribution.

Proposition: Inductively, $x_1, x_2, \ldots, x_n$ are n-points in a convex set, then $\lambda_1 x_1 + \lambda_2 x_2 \cdots + \lambda_n x_n$ where $\lambda_i \geq 0$ $i = 1, \ldots n$ and $\sum_{i=1}^{n} \lambda_i = 1$, is a convex combination of n-points of a convex set.

Proof: It is trivially, for $k=2$,

$$\lambda_1 x_1 + \lambda_2 x_2 \qquad x_1, x_2 \in X, \quad \lambda_1, \lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 = 1$$

→ Assume that it is true for some k $(k \geq 2)$

Then, T.P.T it is also true for $k+1$.

$$(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) + (\lambda_{k+1} x_{k+1}) \in \text{convex set } S.$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k + \lambda_{k+1} = 1.$$

$$\Rightarrow \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1 - \lambda_{k+1} \qquad \lambda_i \geq 0 \quad i = 1, \ldots, k+1$$

$$(1-\lambda_{k+1}) \left[\underbrace{\frac{\lambda_1}{1-\lambda_{k+1}}}_{\lambda_1'} x_1 + \underbrace{\frac{\lambda_2}{1-\lambda_{k+1}}}_{\lambda_2'} x_2 \cdots + \underbrace{\frac{\lambda_k}{1-\lambda_{k+1}}}_{\lambda_n'} x_k \right] + \lambda_{k+1} x_{k+1}$$

$$\lambda_1' + \lambda_2' + \cdots + \lambda_n' = \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{1-\lambda_{k+1}} = \frac{1-\lambda_{k+1}}{1-\lambda_{k+1}} = 1.$$

$\in$ convex, being convex combination of k-pts from convex set.

point in the convex set.

Proposition: Inductively, $x_i \in S$ a convex set

$$f(\lambda_1 x_1 + \lambda_2 x_2 \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) \cdots + \lambda_n f(x_n)$$

$$\lambda_i \geq 0 \quad i = 1, \ldots n.$$

$$\sum \lambda_i = 1$$

$\Rightarrow$ It is trivially true for $n = 2$;

$\Rightarrow$ Assume that it is true for some 'k', and prove it for $\underline{\underline{k+1}}$.

$\Rightarrow$

$$\lambda_1 x_1 + \lambda_2 x_2 \cdots + \lambda_n x_n + \lambda_{n+1} x_{n+1} \qquad \text{where } \sum_{i=1}^{n+1} \lambda_i = 1; \quad \lambda_i \geq 0$$

$$f\left[(1-\lambda_{n+1}) \underbrace{\frac{\lambda_1 x_1 + \lambda_2 x_2 - + \lambda_n x_n}{1-\lambda_{n+1}}}_{\in S} + \lambda_{n+1} x_{n+1}. \right]$$

$$\leq (1-\lambda_{n+1}) f(\lambda_1' x_1 + \lambda_2' x_2 - + \lambda_n' x_n) + \lambda_{n+1} f(x_{n+1})$$

$$\leq (1-\lambda_{n+1}) [\lambda_1' f(x_1) + \lambda_2' f(x_2) - + \lambda_n' f(x_n)] + \lambda_{n+1} f(x_{n+1})$$

$$= \lambda_1 f(x_1) + \lambda_2 f(x_2) \cdots + \lambda_n f(x_n) + \lambda_{n+1} f(x_{n+1})$$

$$f(\lambda_1 x_1 + \lambda_2 x_2 \cdots + \lambda_n x_n) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) \cdots + \lambda_n f(x_n)$$

Jensen's Inequality.

$\lambda_1 = p_1$, $\lambda_2 = p_2$, $\cdots \lambda_n = p_n$.

x_1, x_2 $\cdots$ x_n.

$$f(p_1 x_1 + p_2 x_2 \cdots + p_n x_n) \le p_1 f(x_1) + p_2 (x_2) + \cdots + p_n f(x_n)$$

$$f(EX) \le E f(X)$$