Supremum:

- Let $X \subset \mathbb{R}$, $X \neq \{\phi\}$. The supremum of set X, i.e., $\sup X$ is defined as the smallest scalar y such that $y \geq x$ for all $x \in X$.
  - If no such scalar exists, we say $\sup X = \infty$.
- $\sup X$ is also called a least upper bound or lub.

Example:

(1) Let $X = (1, 2)$
$\sup X = 2$;

(2) Let $X = [1, 2]$
$\sup X = 2$;

Q] What is difference between Ex (1) & (2).

Ans: In (1) $\sup X \notin X$ as X is open set

In (2) $\sup X \in X$

(3) Let $X = [1, \infty)$
Then $\sup X = \infty$.

Q] Supremum relates to global maximum!

Infimum:

- Let $X \subset \mathbb{R}$, and $X = \phi$;
  Then infimum of X is the largest scalar y s.t. $y \leq x \;\; \forall x \in X$.
- In other words, it is greatest lower bound (glb).
- If no such scalar exists, it is $-\infty$.

Examples:

- In both $X = (1, 2)$ and $X = [1, 2]$
  $\inf X = 1$.
- In the first case $\inf X \notin X$
  In the second case $\inf X \in X$.
- For $X = (-\infty, 1]$,
  $\inf X = -\infty$.

Q] What do we do with empty set?

For $X = \{\phi\}$; by convention we write $\inf X = +\infty$
and $\sup X = -\infty$.

This is the otherway round, indicating that we have a nullset and NOT an unbounded set.

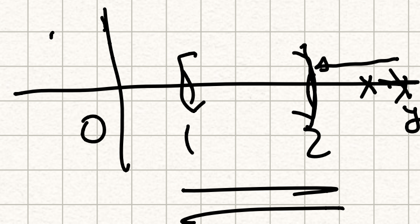
Supremum, $X \subset R$, non empty.

smallest scalar y s.t $y \geq x \ \forall x \in X$.

In other words, it least upper bound on set X.

$X = [1, 2]$

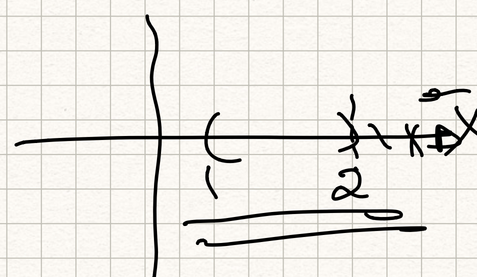
$\sup X = 2$;



$X = (1, 2)$

least upper bound

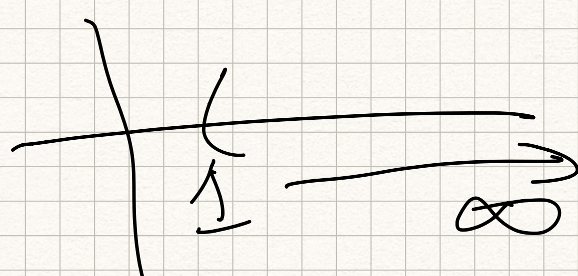
$\sup X = 2$;



$2 - \varepsilon$

$2 - \varepsilon + \frac{\varepsilon}{2}$

$2 - \frac{\varepsilon}{2} \in X$.



$X = (1, \infty)$

$cl\ X = [1, \infty]$

lub ∞;

$\sup X = \infty$;

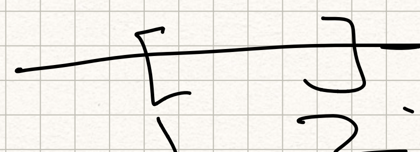
Infimum;

Let $X \subset \mathbb{R}$, let $X \neq \{\phi\}$,

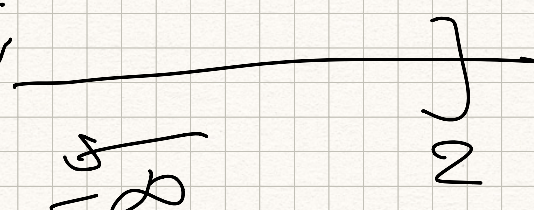
infimum of X, also known as greatest lower bound (glb) is that scalar y s.t it is largest s.t. $y \le x \ \forall x \in X$

$\inf X = \underline{\underline{1}}$;

$\inf X = \underline{1}$



$\inf X = -\infty$;



unbounded below

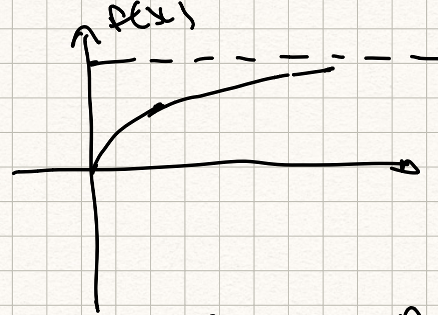
What are the infimum and supremum of an empty set?

By convention:

$$\left.\begin{aligned} \inf\{\phi\} &= \infty \\ \sup\{\phi\} &= -\infty \end{aligned}\right\} \rightarrow \text{avoid any contradiction with unbounded sets.}$$

What is the role of infimum & supremum in optimization

$$\max f(x) = 1 - e^{-x}$$
$$\text{s.t } x \geq 0 \; \} \; S$$



$$\lim_{x \to \infty} f(x) = 1$$

$$\left\{ \sup_{x \geq 0} f(x) \right\}$$

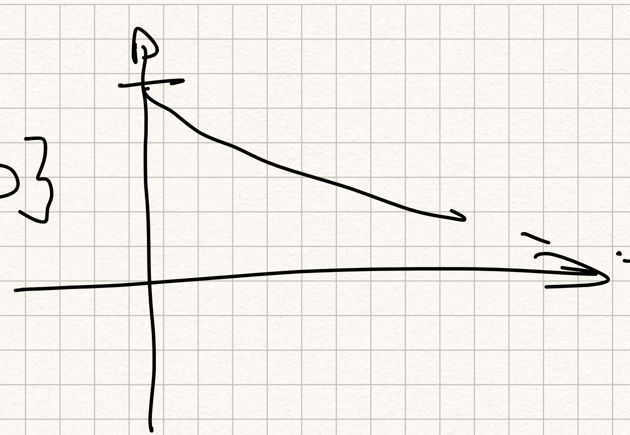
but is it max? Technically No!

→ why because no $x^* \in S$ achieves the value of 1 which is supremum.

→ while supremum will also exist, a max need not.

$f(x) = e^{-x}$

$S = \{x : x \geq 0\}$



$\min_{x \in S} f(x)$

$\inf_{x \geq 0} f(x) = 0;$

$\min f(x) \neq 0,$

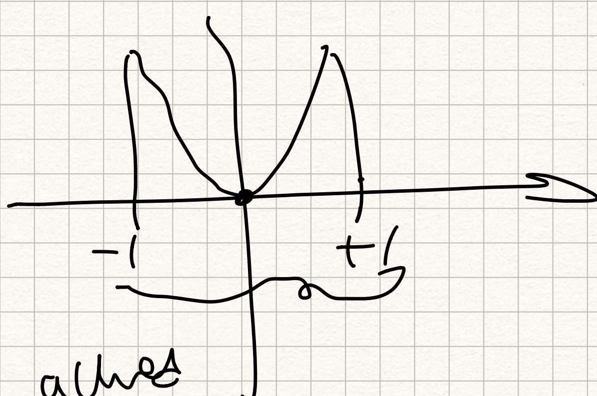
because no x achieves it.

inf always exists,

but min need not.

$f(x) = x^2$

$-1 \leq x \leq +1$



$\inf = 0;$

and it is achieved

at $x = 0 \in S$

$0 = \arg\min_{x \in [-1,+1]} x^2$

If X is a non-empty of $\mathbb{R}^n$ and f is a real values function whose domain contains X, we say that vector $x^* \in X$ is a minimum of f over X if

$$f(x^*) = \inf_{x \in X} f(x).$$

we call x^* a minimizing point.

$$x^* \in \arg\min_{x \in X} f(x).$$

If it is unique

$$x^* = \arg\min f(x)$$

Can the infimum be translated to min

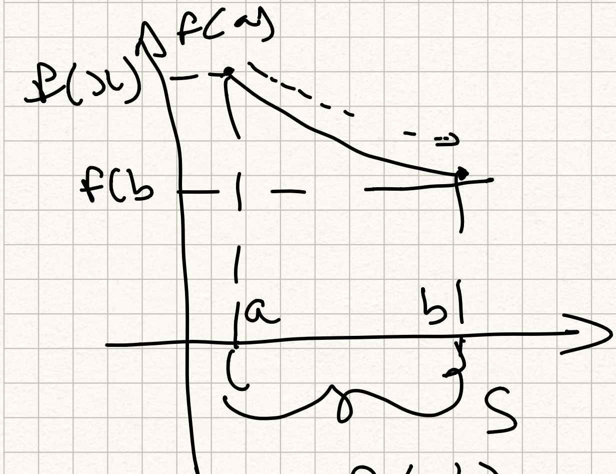
$$\inf_{x\in S} f(x) \Longrightarrow \min_{x\in S} f(x) \; ?$$

Or in other words.

is $x^* \in S$ which achieves this infimum?

$x^* \in \arg\min f(x)$

$x^* = \arg\min f(x)$



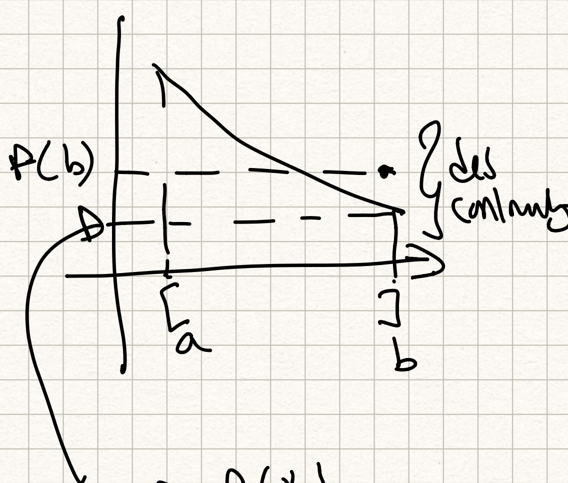
inf f(b)

but it cannot be realized.

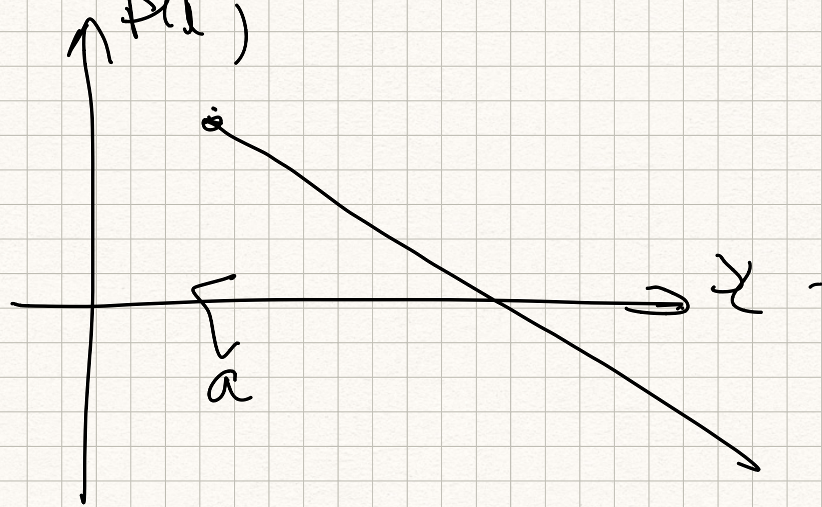
$b \notin S$

min $\inf_{x\in S} f(x)$

open.



$\inf_{x\in[a,b]} f(x)$



inf $-\infty$

is there any x which reaches it?

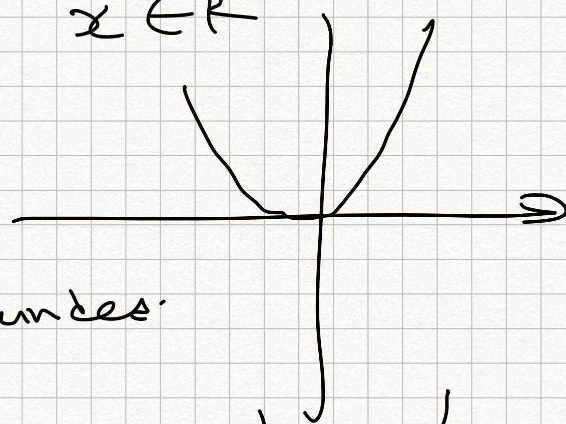
2.3.1 Theorem (Bazaraa, Sherali & Shetty)

Let S be a nonempty, compact set (closed & bounded) and let $f: S \to \mathbb{R}$. Then, the problem $\min\{f(x) : x \in S\}$ attains its minimum, that is, there exists a minimizing soln to this problem.

[sufficiency condition on existence of min]

[global min.]

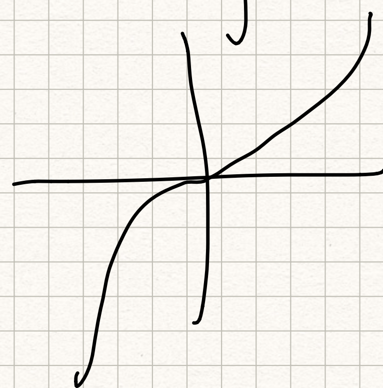
$f(x) = x^2 \quad x \in R$



$f(0), = 0$, global min.

$R \to$ not bounded.

$f(x) = x^3$



$f(x) \to -\infty$

$x \to -\infty$