Coordinate Transformation Matrix

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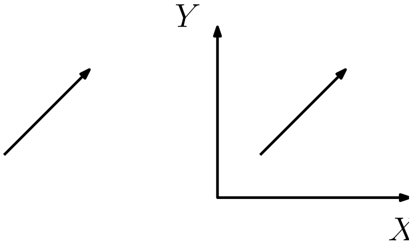
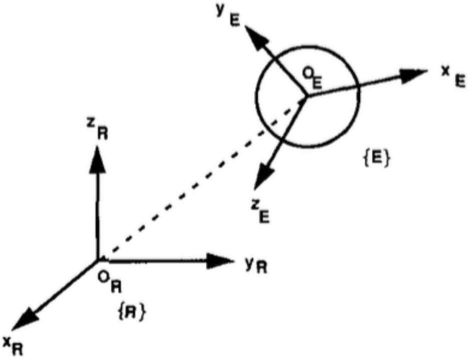
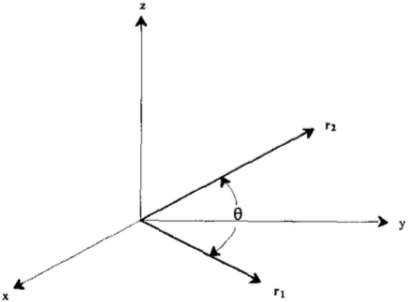
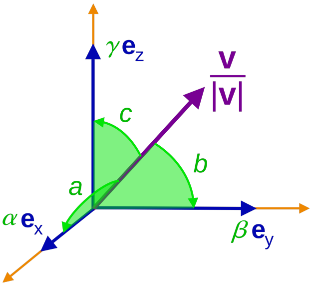




- Position of rigid body: position vector $O_R O_E$ of origin
- Orientation of rigid body: 3×3 rotation matrix
- For simplification, we assume $O_E = O_R$

- Rotation matrix approach utilizes 9 parameters, which obey the orthogonality and unit length constraints, to describe the orientation of the rigid body.
- A rigid body possesses 3 rotational DOF, 3 independent parameters are sufficient to characterize completely and unambiguously its orientation.
- Three-parameter representations are popular in engineering because they minimize the dimensionality of the rigid-body control problem
- Transformation of coordinate axes is an important necessity in resolving angular positions and rates form one coordinate system to other.
- Transformation matrix: Mapping of the components of a vector, resolved in one frame, into the same resolved into the other frame.
  - $\Rightarrow$ Direction cosine matrix (DCM)
  - $\Rightarrow$ Euler Angles
  - $\Rightarrow$ Quaternions





Angle between two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$

$$\theta = \cos^{-1}\left[\frac{\boldsymbol{r}_2^T\boldsymbol{r}_1}{\sqrt{\boldsymbol{r}_1^T\boldsymbol{r}_1}\sqrt{\boldsymbol{r}_2^T\boldsymbol{r}_2}}\right]$$

Direction cosines:

$\alpha = \cos a, \quad \beta = \cos b, \quad \gamma = \cos c$

$\cos^2 a + \cos^2 b + \cos^2 c = 1$ Proof?

Example

Find the direction cosines and direction angles of the vector $\boldsymbol{v} = -8\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.

- Assume a, b, c be the angles formed by vector w.r.t. x, y, z axes, respectively, and direction cosines are α, β, γ.
- We can write

$$\alpha = \cos a = \frac{\boldsymbol{v}^T \boldsymbol{i}}{\|\boldsymbol{v}\|} = \frac{-8}{\sqrt{77}} \implies a = 156^\circ$$

- Similarly,

$$\beta = \cos b = \frac{\boldsymbol{v}^T \boldsymbol{j}}{\|\boldsymbol{v}\|} = \frac{3}{\sqrt{77}} \implies b = 70^\circ$$

$$\gamma = \cos c = \frac{\boldsymbol{v}^T \boldsymbol{k}}{\|\boldsymbol{v}\|} = \frac{2}{\sqrt{77}} \implies c = 77^\circ$$

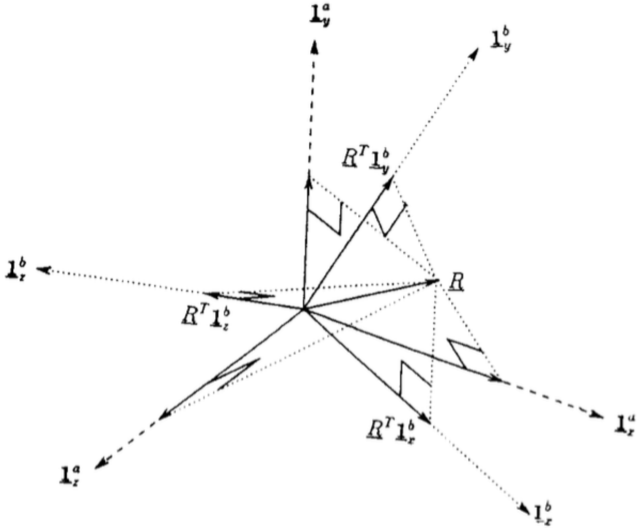
- Direction cosine matrix (DCM) transforms a vector in $\mathbb{R}^3$ from one frame to other frame.
- DCM for transformation between frames a and b

$$\boldsymbol{C}_a^b = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- Specifically, if (X, Y, Z) and (x, y, z) are the representations of a vector in frames a and b, respectively, then

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\boldsymbol{R}^b} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}}_{\text{Rotation Matrix}} \underbrace{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{\boldsymbol{R}^a} \Rightarrow \boldsymbol{R}^b = \boldsymbol{C}_a^b \boldsymbol{R}^a$$

- Matrix DCM projects the vector $\boldsymbol{R}^a$ into a reference frame b.
- For orthogonal systems, $(\boldsymbol{C}_a^b)^{-1} = (\boldsymbol{C}_a^b)^T = \boldsymbol{C}_b^a$



$\mathbf{1}_y^a$
$\mathbf{1}_y^b$
$R^T\mathbf{1}_y^b$
$\mathbf{1}_z^b$
R
$R^T\mathbf{1}_z^b$
$R^T\mathbf{1}_x^b$
$\mathbf{1}_x^a$
$\mathbf{1}_z^a$
$\mathbf{1}_x^b$

- Assume vector $\boldsymbol{R}$ coordinatized in reference frames a and b as $\boldsymbol{R}^a$ and $\boldsymbol{R}^b$, respectively.

$$\begin{aligned}\boldsymbol{R}^a =&(\boldsymbol{R}^T\mathbf{1}_x^a)\mathbf{1}_x^a + (\boldsymbol{R}^T\mathbf{1}_y^a)\mathbf{1}_y^a + (\boldsymbol{R}^T\mathbf{1}_z^a)\mathbf{1}_z^a\\ \boldsymbol{R}^b =&(\boldsymbol{R}^T\mathbf{1}_x^b)\mathbf{1}_x^b + (\boldsymbol{R}^T\mathbf{1}_y^b)\mathbf{1}_y^b + (\boldsymbol{R}^T\mathbf{1}_z^b)\mathbf{1}_z^b\end{aligned}$$

  where, $\boldsymbol{R}^T\mathbf{1}_i^a \ \forall\ i = x, y, z$ denotes scalar component of $\boldsymbol{R}$ projected along the i^{th} a-frame coordinate direction.
- Unit vectors $\mathbf{1}_i^a$ and $\mathbf{1}_j^b$ are related, for $i, j = x, y, z$, as

$$\mathbf{1}_i^b =(\mathbf{1}_i^{b^T}\mathbf{1}_x^a)\mathbf{1}_x^a + (\mathbf{1}_i^{b^T}\mathbf{1}_y^a)\mathbf{1}_y^a + (\mathbf{1}_i^{b^T}\mathbf{1}_z^a)\mathbf{1}_z^a$$

- The i^{th} component of $\boldsymbol{R}^b$ can be expressed as

$$\begin{aligned}\boldsymbol{R}^T\mathbf{1}_i^b =&\boldsymbol{R}^T[(\mathbf{1}_i^{b^T}\mathbf{1}_x^a)\mathbf{1}_x^a + (\mathbf{1}_i^{b^T}\mathbf{1}_y^a)\mathbf{1}_y^a + (\mathbf{1}_i^{b^T}\mathbf{1}_z^a)\mathbf{1}_z^a]\\ =&(\mathbf{1}_i^{b^T}\mathbf{1}_x^a)\boldsymbol{R}^T\mathbf{1}_x^a + (\mathbf{1}_i^{b^T}\mathbf{1}_y^a)\boldsymbol{R}^T\mathbf{1}_y^a + (\mathbf{1}_i^{b^T}\mathbf{1}_z^a)\boldsymbol{R}^T\mathbf{1}_z^a\end{aligned}$$

- The vector $\boldsymbol{R}^b$ can be expressed as

$$\boldsymbol{R}^b = \begin{bmatrix} \boldsymbol{R}^T \mathbf{1}_x^b \\ \boldsymbol{R}^T \mathbf{1}_y^b \\ \boldsymbol{R}^T \mathbf{1}_z^b \end{bmatrix} = \begin{bmatrix} \mathbf{1}_x^{b^T} \mathbf{1}_x^a & \mathbf{1}_x^{b^T} \mathbf{1}_y^a & \mathbf{1}_x^{b^T} \mathbf{1}_z^a \\ \mathbf{1}_y^{b^T} \mathbf{1}_x^a & \mathbf{1}_y^{b^T} \mathbf{1}_y^a & \mathbf{1}_y^{b^T} \mathbf{1}_z^a \\ \mathbf{1}_z^{b^T} \mathbf{1}_x^a & \mathbf{1}_z^{b^T} \mathbf{1}_y^a & \mathbf{1}_z^{b^T} \mathbf{1}_z^a \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T \mathbf{1}_x^a \\ \boldsymbol{R}^T \mathbf{1}_y^a \\ \boldsymbol{R}^T \mathbf{1}_z^a \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1}_x^{b^T} \mathbf{1}_x^a & \mathbf{1}_x^{b^T} \mathbf{1}_y^a & \mathbf{1}_x^{b^T} \mathbf{1}_z^a \\ \mathbf{1}_y^{b^T} \mathbf{1}_x^a & \mathbf{1}_y^{b^T} \mathbf{1}_y^a & \mathbf{1}_y^{b^T} \mathbf{1}_z^a \\ \mathbf{1}_z^{b^T} \mathbf{1}_x^a & \mathbf{1}_z^{b^T} \mathbf{1}_y^a & \mathbf{1}_z^{b^T} \mathbf{1}_z^a \end{bmatrix} \boldsymbol{R}^a$$

$$= \boldsymbol{C}_a^b \boldsymbol{R}^a = [C_{ij}] \boldsymbol{R}^a$$

- $[C_{ij}]$ represents the cosine of the angle between the unit vectors $\mathbf{1}_i^b$ and $\mathbf{1}_j^a$.

Example 1

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If $\boldsymbol{i}' = \boldsymbol{j}$, $\boldsymbol{j}' = -\boldsymbol{i}$, and $\boldsymbol{k}' = \boldsymbol{k}$ then what would be the DCM matrix?

- DCM matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If $\boldsymbol{i}' = \boldsymbol{i}$, $\boldsymbol{j}' = -\boldsymbol{k}$, and $\boldsymbol{k}' = \boldsymbol{j}$ then what would be the DCM matrix?

- DCM matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Example 3

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If the old coordinate frame is rotated with angle θ anti-clockwise w.r.t. z-axis to get new frame then what would be the DCM matrix?

- Unit vectors of new frame

$$\begin{aligned} \boldsymbol{i}' &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{j}' &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \\ \boldsymbol{k}' &= \boldsymbol{k} \end{aligned}$$

- DCM matrix

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 4

Find out the missing coefficients of DCM.

$$T = \begin{bmatrix} 0.8999 & -0.4323 & 0.0578 \\ c_{21} & 0.8665 & -0.2496 \\ c_{31} & c_{32} & 0.9666 \end{bmatrix}$$

- We can use orthogonal property of DCM.

$$0.9666c_{32} - 0.8665 \times 0.2496 - 0.4323 \times 0.0578 = 0$$

$$c_{31}c_{32} + 0.8665c_{21} - 0.8999 \times 0.4323 = 0$$

$$0.9666c_{31} - 0.2496c_{21} + 0.0578 \times 0.8999 = 0$$

- On solving, we get $c_{21} = 0.4323, \; c_{31} = 0.0578, \; c_{32} = 0.2496$
- Check for correctness

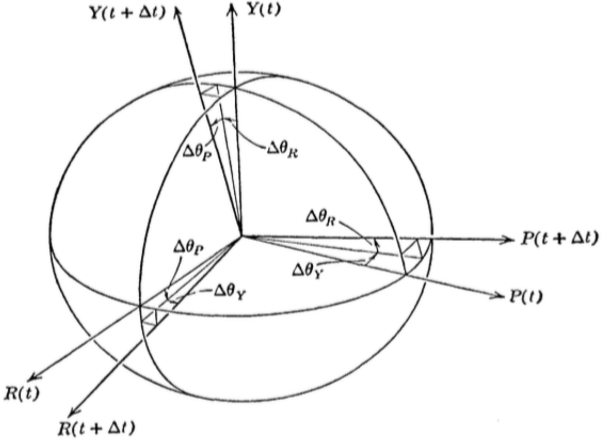
- Consider the two frames be the a and b frames.
- At time t, the frames a and b are related through the DCM $\boldsymbol{C}_b^a(t)$.
- At time $t + \Delta t$, frame b rotates to a new orientation such that the direction cosine matrix is given by $\boldsymbol{C}_b^a(t + \Delta t)$.
- Rate of change of $\boldsymbol{C}_b^a(t)$ is given by

$$\dot{\boldsymbol{C}}_b^a(t) = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{C}_b^a}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{C}_b^a(t + \Delta t) - \boldsymbol{C}_b^a(t)}{\Delta t}$$

- From geometrical considerations,

$$\boldsymbol{C}_b^a(t + \Delta t) = \boldsymbol{C}_b^a(t)(\boldsymbol{I} + \boldsymbol{\Delta\theta}^b)$$

where, $\boldsymbol{I} + \boldsymbol{\Delta\theta}^b$ is the small angle DCM relating b frame at time t to the rotated b frame at time $t + \Delta t$.



$Y(t + \Delta t)$
$Y(t)$
$\Delta\theta_P$
$\Delta\theta_R$
$\Delta\theta_R$
$P(t + \Delta t)$
$\Delta\theta_P$
$\Delta\theta_Y$
$-\Delta\theta_Y$
$P(t)$
$R(t)$
$R(t + \Delta t)$

- $\boldsymbol{\Delta\theta}^b$ is given by

$$\boldsymbol{\Delta\theta}^b = \begin{bmatrix} 0 & -\Delta\theta_Y & \Delta\theta_P \\ \Delta\theta_Y & 0 & -\Delta\theta_R \\ -\Delta\theta_P & \Delta\theta_R & 0 \end{bmatrix}, \quad \Delta\theta_k = \sin\Delta\theta_k \ \forall \ k = R, Y, P$$

- Note that because the rotation angles are small in the limit as $\Delta t \to 0$, small angle approximations are valid and the order of rotation is immaterial.
- Rate of change of $\boldsymbol{C}_b^i(t)$ is now written as

$$\dot{\boldsymbol{C}}_b^a(t) = \boldsymbol{C}_b^a(t) \lim_{\Delta t \to 0} \frac{\boldsymbol{\Delta\theta}^b}{\Delta t}$$

- In the limit $\Delta t \to 0$, $\boldsymbol{\Delta\theta^b}/\Delta t$ is the skew-symmetric form of angular velocity of the frame b relative to a frame.

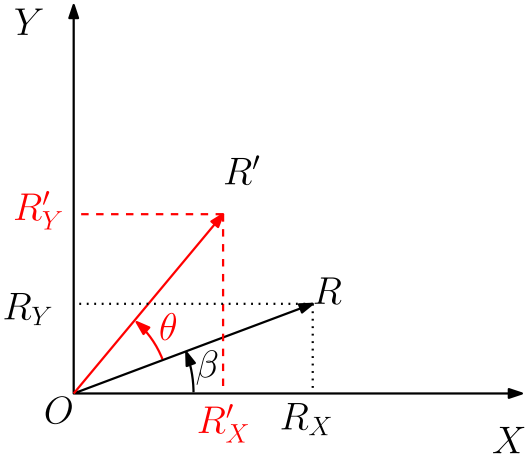
$$\dot{\boldsymbol{C}}_b^a(t) = \boldsymbol{C}_b^a(t)\boldsymbol{\Omega}_{ab}^b = \boldsymbol{C}_b^a(t) \begin{bmatrix} 0 & -\omega_Y & \omega_P \\ \omega_Y & 0 & -\omega_R \\ -\omega_P & \omega_R & 0 \end{bmatrix}$$

- DCM differential equation is a linear matrix differential equation, forced by the angular velocity vector in its skew symmetric matrix form.
- Nine scalar, linear, coupled differential equations
- This equation can be integrated with the initial conditions, which represent the initial orientation of the a-frame with respect to the b-frame.
- Differential equation

$$\dot{\boldsymbol{C}}_{i,j} = \boldsymbol{C}_{i,j+1}\omega_{j+2} - \boldsymbol{C}_{i,j+2}\omega_{j+1}, \quad i,j = 1,2,3$$

  where, second subscript is modulo 3, and $\omega_1 = \omega_R, \omega_2 = \omega_P, \omega_3 = \omega_Y$
- A first order approximation for transformation matrix, using Taylor series

$$\boldsymbol{C}_{t_k+\Delta T} = \left[\boldsymbol{I} + \boldsymbol{\Omega}_{ab}^{b}(t_k)\Delta T\right]\boldsymbol{C}_{t_k}$$



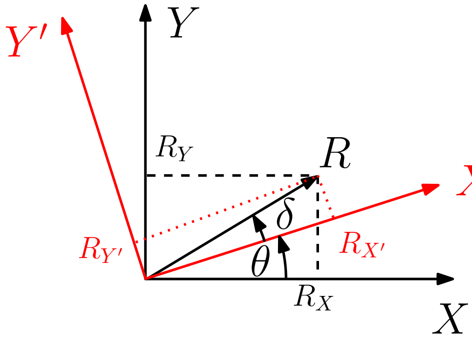
Y
R'
R'_Y
R
R_Y
θ
β
O
R'_X
R_X
X

- The position of a point R in XY coordinate frame is given by

$$\begin{bmatrix} R_X \\ R_Y \end{bmatrix} = \begin{bmatrix} R\cos\beta \\ R\sin\beta \end{bmatrix}$$

- Let us assume $\gamma = \theta + \beta$.
- Position of a point R' in XY coordinate frame is given by

$$\begin{bmatrix} R'_X \\ R'_Y \end{bmatrix} = \begin{bmatrix} R\cos\gamma \\ R\sin\gamma \end{bmatrix} = \begin{bmatrix} R\cos\theta\cos\beta - R\sin\theta\sin\beta \\ R\sin\theta\cos\beta + R\cos\theta\sin\beta \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotation matrix}} \begin{bmatrix} R_X \\ R_Y \end{bmatrix}$$



Y'
Y
R_Y
R
X'
δ
$R_{Y'}$
θ
$R_{X'}$
R_X
X

- Let us assume $\alpha = \theta + \delta$.
- The position of a point R in XY frame is given by

$$\begin{bmatrix} R_X \\ R_Y \end{bmatrix} = \begin{bmatrix} R\cos\alpha \\ R\sin\alpha \end{bmatrix}$$

- Position of a point R in $X'Y'$ frame is given by

$$\begin{bmatrix} R_{X'} \\ R_{Y'} \end{bmatrix} = \begin{bmatrix} R\cos\delta \\ R\sin\delta \end{bmatrix}$$

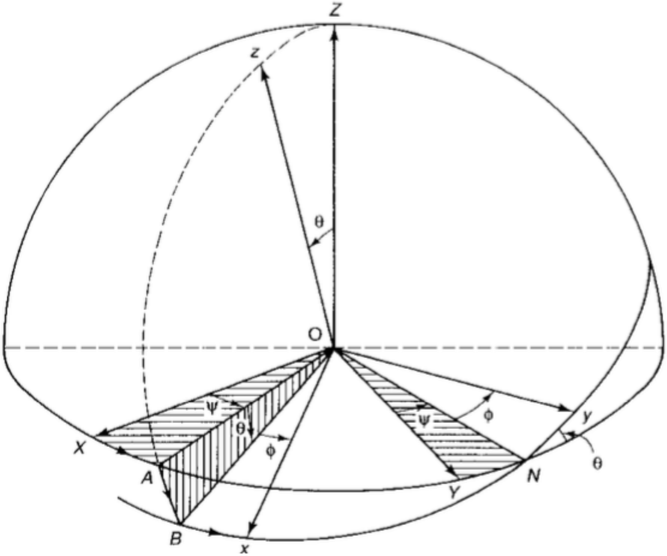
- As $\delta = \alpha - \theta$, we can also write

$$\begin{aligned} \begin{bmatrix} R_{X'} \\ R_{Y'} \end{bmatrix} &= \begin{bmatrix} R\cos(\alpha-\theta) \\ R\sin(\alpha-\theta) \end{bmatrix} = \begin{bmatrix} R\cos\alpha\cos\theta + R\sin\alpha\sin\theta \\ R\sin\alpha\cos\theta - R\cos\alpha\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} R_X\cos\theta + R_Y\sin\theta \\ R_Y\cos\theta - R_X\sin\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotation matrix}} \begin{bmatrix} R_X \\ R_Y \end{bmatrix} \end{aligned}$$

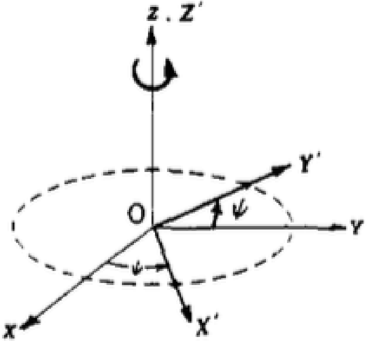
- Euler angles
  - ⇒ Method to specify the angular orientation of one coordinate frame w.r.t. another frame
  - ⇒ A series of three ordered right-handed rotations
  - ⇒ Correspond to the conventional roll pitch yaw angles
- Euler angles are not uniquely defined since there is an infinite set of choices.
- No standardized definitions of the Euler angles.
- For a particular choice of Euler angles, the rotation order selected and/or defined should be consistent.
- Interchange in order of rotation $\Rightarrow$ different Euler angle representation.
- Rotations are made about the Z, Y, X axes through an angle ψ, θ, ϕ angles.
- These rotations are made in the positive (anticlockwise sense) when looking down the axis of rotation toward the origin.

Coordinate Transformation

Euler Angle Rotations ($ZY'Z''$)



- Euler angles: Three elemental rotations
- Extrinsic rotations: Rotations about the axes xyz of the original coordinate system, which is assumed to remain motionless.
- Intrinsic rotations: Rotations about the axes of rotating coordinate system XYZ, which changes its orientation after each elemental rotation.
- Another classification
  - $\Rightarrow$ Proper Euler angles
  - $\Rightarrow$ Tait-Bryan angles
- Proper Euler angles : $(zxz, zyz, xyx, xzx, yzy, yxy)$
- Tait-Bryan angles : $(zyx, zxy, xyz, xzy, yzx, yxz)$
- What is the major difference between Proper Euler and Tait-Bryan angles?
- Tait-Bryan angles represent rotations about three distinct axes, while proper Euler angles use the same axis for both the first and third elemental rotations.



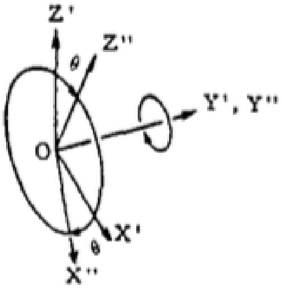
- Rotation about Z axis in anticlockwise direction by an angle ψ

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$= \boldsymbol{A} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



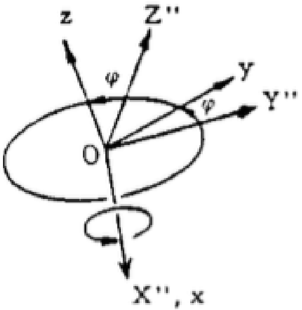
- Rotation about Y axis in anticlockwise direction by an angle θ

$$\begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

$$= \boldsymbol{B} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

where

$$\boldsymbol{B} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$



- Rotation about X axis in anticlockwise direction by an angle ϕ

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix}$$

$$= \boldsymbol{D} \begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix}$$

where

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

- If the consecutive rotations are performed in the order ψ, θ, ϕ i.e., (yaw, pitch and roll) on reference frame XYZ then we obtain the another reference frame xyz.
- Rotation matrix for representing these three rotations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{D} \begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} = \boldsymbol{DB} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \boldsymbol{DBA} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Equivalently,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\boldsymbol{DBA}}_{\boldsymbol{C}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Equivalent rotation matrix $\boldsymbol{C} = \boldsymbol{DBA}$ can be written as

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

- This rotation matrix is called Euler angle transformation matrix.
- Range of Euler angles:

$$-\pi \le \psi \le \pi, \;\; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \;\; -\pi \le \phi \le \pi$$

- Is there any issue with $|\theta| > \pi/2$?

- Equivalent rotation matrices

$$\boldsymbol{C} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$
$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- How to obtain Euler angles from given DCM matrix?

$$\theta = \sin^{-1}(-C_{13})$$

$$\phi = \sin^{-1}\left(\frac{C_{23}}{\sqrt{1 - C_{13}^2}}\right)$$
$$\psi = \sin^{-1}\left(\frac{C_{12}}{\sqrt{1 - C_{13}^2}}\right)$$

- Are there some issues with these expressions?

- Recall about the ranges of these Euler angles
- How to determine the quadrant in which these angles lie?
- As pitch angle θ lies in $-\pi/2 \leq \theta \leq \pi/2$,

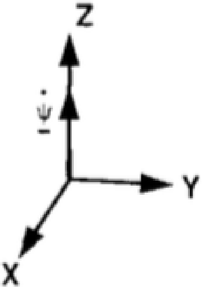
$$\theta \in \begin{cases} [0, \pi/2] & C_{13} \leq 0 \\ [-\pi/2, 0] & C_{13} \geq 0 \end{cases}$$

- What about bank angle ϕ?
- As $C_{33} = \cos\phi\cos\theta$, and $\cos\theta > 0$, sign of C_{33} is same as that of $\cos\phi$.
- Also, $C_{23} = \sin\phi\cos\theta$, and $\cos\theta > 0$, sign of C_{23} is same as that of $\sin\phi$.

$$\phi \in \begin{cases} \text{First quadrant} & C_{33} > 0 \ \& \ C_{23} > 0 \\ \text{Second quadrant} & C_{33} < 0 \ \& \ C_{23} > 0 \\ \text{Third quadrant} & C_{33} < 0 \ \& \ C_{23} < 0 \\ \text{Fourth quadrant} & C_{33} < 0 \ \& \ C_{23} < 0 \end{cases}$$

- We can also obtain the quadrants of ψ using C_{11} and C_{12} in a similar way.

- Similar to DCM orientation, Euler angles also vary with time when an input angular velocity vector is applied between the two reference frames.
- Angular velocity vector ω, in body-fixed coordinate system, has components p, q, and r in the x, y, and z directions, respectively.
- Consider each derivative of an Euler angle as the magnitude of the angular velocity vector in the coordinate system in which the angle is defined.
- For example, $\dot{\psi}$ is the magnitude of $\dot{\boldsymbol{\psi}}$ that lies along Z axis of the Earth-fixed coordinate system.



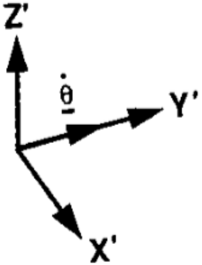
$$\dot{\boldsymbol{\psi}} = \begin{bmatrix} \dot{\psi}_x \\ \dot{\psi}_y \\ \dot{\psi}_z \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi \\ \dot{\psi}\cos\theta\cos\phi \end{bmatrix}$$

- Similarly, the components of $\dot{\boldsymbol{\theta}}$ in $X'Y'Z'$ are given by $(0, \dot{\theta}, 0)^T$.
- In body frame, it can be obtained as

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \boldsymbol{DB} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dot{\theta}\cos\phi \\ -\dot{\theta}\sin\phi \end{bmatrix}$$

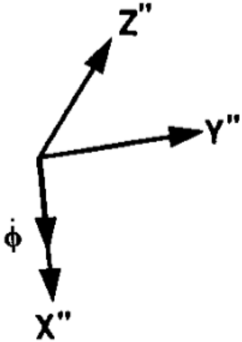


- Similarly, the components of $\dot{\phi}$ in $X^{''}Y^{''}Z^{''}$ are given by $(\dot{\psi}, 0, 0)^T$.
- In body frame, it can be obtained as

$$\dot{\boldsymbol{\phi}} = \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \boldsymbol{D} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$



- Components of $\boldsymbol{\omega}$ in body-fixed coordinate system is given by

$$\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}}$$

- Now, we have

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\psi}_x + \dot{\theta}_x + \dot{\phi}_x \\ \dot{\psi}_y + \dot{\theta}_y + \dot{\phi}_y \\ \dot{\psi}_z + \dot{\theta}_z + \dot{\phi}_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi \\ \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \end{bmatrix}$$

- Euler angle rates

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{q\sin\phi + r\cos\phi}{\cos\theta} \\ q\cos\phi - r\sin\phi \\ p + \tan\theta(q\sin\phi + r\cos\phi) \end{bmatrix}$$

- What happen when $\theta = \pm 90^\circ$? Gimbal lock problem
- How to avoid such difficulties? Nonsingular representation, e.g., quaternions

- For $\theta = \pi/2$, $p = \dot{\phi} - \dot{\psi},\ q = \dot{\theta}\cos\phi,\ r = -\dot{\theta}\sin\phi$
- Azimuth and elevation rates

$$\dot{\psi} = \frac{q\sin\phi + r\cos\phi}{\cos\theta} = \frac{\dot{\theta}\cos\phi\sin\phi - \dot{\theta}\sin\phi\cos\phi}{\cos\theta} = \frac{0}{0}$$

$$\dot{\phi} = \ p + \frac{\sin\theta(q\sin\phi + r\cos\phi)}{\cos\theta} = p + \frac{0}{0}$$

Indeterminate forms!!!

- Using L'Hospital rule, and the fact that $\frac{d()}{d\theta} = \frac{d()}{dt}\frac{dt}{d\theta}$, we have

$$\begin{aligned}
\dot{\psi}|_{\theta=\pi/2} &= \lim_{\theta\to\pi/2} \frac{\frac{d}{d\theta}\left(q\sin\phi + r\cos\phi\right)}{\frac{d(\cos\theta)}{d\theta}} \\
&= \lim_{\theta\to\pi/2} \frac{\dot{q}\sin\phi + q\cos\phi\dot{\phi} - r\sin\phi\dot{\phi} + \dot{r}\cos\phi}{-\dot{\theta}\sin\theta} \\
&= -\frac{\dot{q}\sin\phi + \dot{r}\cos\phi + \dot{\phi}\dot{\theta}}{\dot{\theta}}
\end{aligned}$$

- Also, for $\theta = \pi/2$, $p = \dot{\phi} - \dot{\psi}$

$$\begin{aligned}\dot{\phi}|_{\theta=\pi/2} &= p + \dot{\psi}|_{\theta=\pi/2} \\ &= p - \frac{\dot{q}\sin\phi + \dot{r}\cos\phi + \dot{\phi}\dot{\theta}}{\dot{\theta}}\end{aligned}$$

- On solving this equation,

$$\dot{\phi}|_{\theta=\pi/2} = \frac{p}{2} - \frac{\dot{q}\sin\phi + \dot{r}\cos\phi}{2\dot{\theta}}$$

- Also, $\dot{\theta} = q\cos\phi - r\sin\phi$.
- For $\theta \approx \pi/2$, use these limiting values, else use usual update equations.

Reference

1. George M. Siouris, *Aerospace Avionics Systems: A Modern Synthesis*, Academic Press, Inc. 1993.
2. Bandhu N. Pamadi, *Performance, Stability, and Control of Airplanes*, AIAA Education Series, 1998.

Thank you for your attention !!!