Prob ICP: $\min f(x)$

Proposition 3.3.2 [Bertsekas]

$h_1(x)=0;\ h_2(x)=0;\ \ldots;\ h_m(x)=0$

$g_1(x)\le 0;\ g_2(x)\le 0;\ \ldots, g_r(x)\le 0.$

Assume f, h and g functions are twice continuously differentiable.

Let $x^*\in\mathbb{R}^n$, $\lambda^*\in\mathbb{R}^m$ and $\mu^*\in\mathbb{R}^r$

satisfy.

$$\nabla_x \mathcal{L}(x^*,\lambda^*,\mu^*)=0;\quad h(x^*)=0\quad g(x^*)\le 0.$$

$$\mu_j^*\ge 0\qquad j=1\ldots r$$

$$\mu_j^*=0,\ \forall j\notin A(x^*)$$

$$y^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*,\mu^*)\,y>0$$

for all $y\ne 0$ s.t.

$$\nabla h_i(x^*)^T y=0;\ \forall\ i=1\ldots m.$$

$$\nabla g_j(x^*)^T y=0\ \ \forall j\in A(x^*)$$

Assume that $\mu_j^*>0\ \ \forall j\in A(x^*)$

Then x^* is a strict local minimum of problem ICP.

Example 3.3.2 [Bertsekas]

$$\min\ \tfrac{1}{2}(x_1^2-x_2^2)$$

subject to $x_2\le 0$

Observation 1: $x_2\to(-\infty)$, $x_1=0$; $f(x)\to-\infty$

so, there is no global min.

$$\mathcal{L}(x_1, x_2, \mu) = \frac{1}{2}(x_1^2 - x_2^2) + \mu x_2$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow \qquad x_1^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow \qquad -x_2^* + \mu^* = 0$$

$\therefore \quad x_1^* = x_2^* = 0 \quad ; \quad \mu^* = 0 \,;$

satisfies all the above conditions except $\mu^* > 0$.

$$\nabla^2_{xx}\mathcal{L}(x^*, \mu^*) = \nabla^2_{xx} f(x^*) + \mu^* \nabla^2_{xx} g(x^*)$$
$$= \nabla^2_{xx} f(x^*)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$f(x^*) = \frac{1}{2}(x_1^2 - x_2^2)$

$$g(x^*) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\nabla g^T(x^*) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\nabla g^T(x^*) y = 0 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$
$$\Rightarrow \quad y_2 = 0$$

$$y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$$

$$y^T \nabla^2_{xx}\mathcal{L}(x^*, \mu^*) y = (y_1 \;\; 0) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix}$$
$$= y_1^2 > 0$$

$y \neq 0 \Rightarrow y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \quad y \neq 0 \Rightarrow y_1 \neq 0.$

$f(0, 0) = 0 \qquad \frac{1}{2}(x_1^2 - x_2^2) \qquad \varepsilon > 0$

$f(0, -\varepsilon) = \frac{1}{2}(-\varepsilon^2) < 0$ Not a local min

Prob: 3.3.2 [Kantorvitch Inequality]

Given a vector 'y',

$$\max_{x} \; y^T x$$

$$\text{s.t.} \quad x^T Q x \leq 1 \qquad Q \Rightarrow \text{s.p.d.}$$

Show:

(1) optimal value is $\sqrt{y^T Q^{-1} y}$

(2) $(x^T y)^2 \leq (x^T Q x)(y^T Q^{-1} y)$

$$\mathcal{L}(x, \mu) = -y^T x + \mu (x^T Q x - 1)$$

$x = \alpha y \quad \alpha > 0 \qquad y^T x = \alpha y^T y > 0$

$\alpha \to \infty \qquad y^T x \to \infty$

$\alpha^2 \, y^T Q y \to \infty \Rightarrow x^T Q x \leq 1$ will violated in our journey towards ∞.

Hence, at the optimal

$$x^{*T} Q x^* = 1,$$

$-y_1 x_1 - y_2 x_2 - \cdots - y_n x_n$

$$\mathcal{L}(x, \mu) = -y^T x + \mu (x^T Q x - 1)$$

$$\nabla_x \mathcal{L}(x^*, \mu^*) = -y + \mu \nabla_x (x^T Q x)$$

$$x^T Q x = q_{11}^2 x_1^2 + q_{22} x_2^2 \overset{q_{kk} x_k^2}{+} \cdots + q_{nn} x_n^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} q_{ij} x_i x_j$$

$$\frac{\partial x^T Q x}{\partial x_k} = 2 q_{kk} x_k + \sum_{\substack{j=1 \\ j \neq k}}^{n} q_{kj} x_j + \sum_{\substack{i=1 \\ i \neq k}}^{n} q_{ik} x_i$$

$$q_{kj} = q_{jk}$$

$$= 2 q_{kk} x_k + 2 \sum_{i=1, i \neq k}^{n} q_{ik} x_i$$

$$\frac{\partial x^T Q x}{\partial x_k} = 2 \left[q_k^T \right] \left[x \right]$$

$q_k^T \Rightarrow$ k^{th} row of Q matrix.

$$\nabla_x (x^T Q x) = 2 Q x$$

$$\nabla_x \mathcal{L}(x^*, \mu^*) = -y + \mu \nabla_x (x^T Q x)$$

$$= -y + 2\mu^* Q x^* = 0;$$

$$2\mu^* Q x^* = y$$

$$x^* = \frac{1}{2\mu^*} Q^{-1} y$$

$$x^{*T} Q x^* = 1$$

$$\frac{1}{4\mu_*^2} y^T Q^{-1} Q Q^{-1} y = 1$$

$$4\mu^{*2} = y^T Q^{-1} y.$$

$$\mu^{*} = \frac{\sqrt{y^T Q^{-1} y}}{2} > 0$$

$$x^* = \frac{1}{\sqrt{y^T Q^{-1} y}} Q^{-1} y.$$

$$y^T x^* = \frac{y^T Q^{-1} y}{\sqrt{y^T Q^{-1} y}} = \sqrt{y^T Q^{-1} y}.$$

$$z^T \nabla_{xx}^2 \mathcal{L}(x^*, \mu^*) z$$

null space of

$\nabla_x g(x^*) = \underline{\underline{Q x^*}} = a$

$a^T z = 0$

$$f(x) = -y^T x$$

$$\nabla_{xx}^2 f(x) = [0]$$

$$\mathcal{L}(x, \mu) = f(x) + \mu(x^T Q x - 1)$$

$$\nabla^2_{xx} \mathcal{L}(x^*, \mu^*) = 2\mu^* Q$$

$$z^T (2\mu^* Q) z > 0 \quad \forall z \neq 0$$

$$x^{*T} y = \sqrt{y^T Q^{-1} y}$$

By weirstrass; theorm, $\sqrt{y^T Q^{-1} y}$ is a global maximum.

Kantorovitch Inequality:

$$(x^T y)^2 \leq (x^T Q x)(y^T Q^{-1} y)$$

$$y^T x = y^T \frac{x}{\sqrt{x^T Q x}} \sqrt{x^T Q x}$$

$$\bar{x} = \frac{x}{(x^T Q x)^{1/2}} \qquad Q \bar{x} = \frac{Q x}{(x^T Q x)^{1/2}}$$

$$\bar{x}^T Q \bar{x} = \frac{x^T Q x}{x^T Q x} = 1,$$

$$y^T x = \sqrt{(x^T Q x)} \left(\frac{y^T x}{\sqrt{x^T Q x}} \right) \leq \sqrt{(x^T Q x)} \sqrt{y^T Q^{-1} y}$$

$$(y^T x)^2 \leq (x^T Q x)(y^T Q^{-1} y)$$