Inequality constrained Problem (ICP)

$$\min f(x)$$
$$\text{s.t.} \quad h_1(x) = 0,\ h_2(x) = 0,\ \dots\ h_m(x) = 0;$$
$$g_1(x) \le 0,\ \dots\dots\ \dots,\ g_r(x) \le 0$$

$H = [\nabla h_1(x^*), \nabla h_2(x^*) \dots \nabla h_m(x^*)]$

$\mathrm{rank}(H) = m;$

$H = [\nabla h_1(x^*) \cdots \nabla h_m(x^*), \nabla g_1 \cdots \nabla g_t(x^*)]$

$\mathrm{rank}(H) = m+t$

$x_1^2 + x_2^2 = 1 \qquad h(x) = x_1^2 + x_2^2 - 1 = 0;$

$x_1^3 + x_2^3 \le -1 \qquad g_1(x) = x_1^3 + x_2^3 + 1 \le 0$

$x_1 + x_2 \ge 1 \qquad g_2(x) = -x_1 - x_2 + 1 \le 0$

Proposition (3.3.1) [Bertsekas]

Let x^* be a local minimum of the problem ICP where f, h_i, g_j are continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ and assume that x^* is regular. Then there exists unique Lagrange multiplier vectors

$$\lambda^* = (\lambda_1^* \dots, \lambda_m^*),\quad \mu^* = (\mu_1^*, \mu_2^* \dots \mu_r^*) \text{ s.t.}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0$$

$$\mu_j^* \ge 0 \qquad j = 1 \dots r$$

$$\mu_j^* = 0 \qquad \forall\, j \notin A(x^*) \Rightarrow$$

complementarity slackness condition. $\mu_i^* g_i(x^*) = 0$

where $A(x^*)$ is the set of active constraints at x^*.

If in addition, f, h and g are twice continuously differentiable, there holds

$$y' \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*, \mu^*)\, y \ge 0$$

for all $y \in \mathbb{R}^n$ s.t.

$$\nabla h_i(x^*)^T y = 0 \qquad \forall\, i \dots, m.$$
$$\nabla g_j(x^*)^T y = 0 \qquad \forall\, j \in A(x^*)$$

$$Z^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*, \mu^*)\, Z \text{ is}$$

symmetric positive semidefinite.

null space basis of $\underbrace{\begin{bmatrix} \nabla h_1(x)^T \\ \nabla h_m(x)^T \\ \nabla g_1(x)^T \\ \nabla g(x)^T \end{bmatrix}}_{m+t \times n}$

$$\left.\begin{array}{l} \min_x \quad f(x) \\ \lambda_1 \to h_1(x) = 0 \\ \lambda_2 \to h_2(x) = 0 \\ \quad \vdots \\ \lambda_m \to h_m(x) = 0 \\ \mu_1 \to g_1(x) \le 0 \\ \mu_2 \to g_2(x) \le 0 \\ \quad \vdots \\ \mu_r \to g_r(x) \le 0 \end{array}\right\}$$

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda_1 h_1(x) \dots + \lambda_m h_m(x) + \mu_1 g_1(x) + \dots + \mu_r g_r(x)$$

At x^*, those constraints which are inactive, $\mu_i^* = 0$ $i \notin A(x^*)$

$\min f(x)$

$h_1(x) = h_2(x) \dots = h_m(x) = 0;$

$g_1(x^*) = 0, \dots g_t(x^*) = 0;$

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = \nabla f(x^*) + \sum \lambda_i^* \nabla h_i(x^*) + \sum \mu_i^* \nabla g_i(x^*) = 0$$

$$g_i(x) \le 0$$

$$g_i(x) + s_i^2 = 0$$

Example 3.3.1 from Bertsekas

$$\min \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

subject to $x_1 + x_2 + x_3 \le -3$

$$\mathcal{L}(x, \mu) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \mu(x_1 + x_2 + x_3 + 3)$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0$$

only constraint $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ trivially regular pt.

$$\frac{\partial \mathcal{L}}{\partial x_1} \Rightarrow \quad x_1^* + \mu^* = 0;$$

$$x_2^* + \mu^* = 0;$$

$$x_3^* + \mu^* = 0$$

$$(x_1^* + x_2^* + x_3^*) + 3\mu^* = 0 \qquad 3\mu^* - 3 = 0$$

$$\mu^* = 1;$$

$$x_1^* = x_2^* = x_3^* = -1;$$