Recap:

Norm: $\|x\|$ in $x \in \mathbb{R}^n$ $\quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$f(x) \to \mathbb{R}^+ \qquad \mathbb{R}^+$ 0 or +ve

1) $\|x\| \geq 0 \quad \|x\| = 0$ iff $x = 0$

2) $\|x+y\| \leq \|x\| + \|y\|$

3) $\|\alpha x\| = |\alpha| \, \|x\|$

Examples:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$$\|x\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_p = \{|x_1|^p + |x_2|^p + \cdots + |x_n|^p\}^{1/p}$$

$$\lim_{p \to \infty} \|x\|_p = x_\infty \quad \# \text{HW}$$

Notion of Angle in $\mathbb{R}^n$.

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Cauchy-Scwartz inequality

$$\frac{|x^T y|}{\|x\|_2 \, \|y\|_2} \leq 1 \quad \leftarrow \text{HOW?}$$

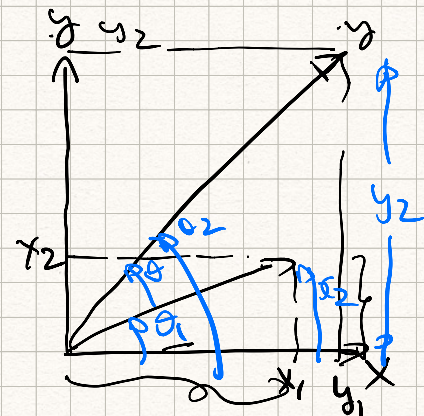
$\|x\|_2 \neq 0, \ \|y\|_2 \neq 0$

$$|x^T y| \le \|x\|_2 \|y\|_2$$

$\cos\theta$

$$-1 \le \frac{x^T y}{\|x\|_2 \|y\|_2} \le +1$$

$\cos\theta$, $\sin\theta$ $\in [-1, +1]$



$$\theta = \theta_2 - \theta_1$$

$$\cos\theta = \cos(\theta_2 - \theta_1)$$

$$= \cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1$$

$$\cos\theta_2 = \frac{y_1}{\sqrt{y_1^2 + y_2^2}} \qquad \sin\theta_2 = \frac{y_2}{\sqrt{y_1^2 + y_2^2}}$$

$$\cos\theta_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \qquad \sin\theta_1 = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$\cos\theta = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}} = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

$$\cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2} \qquad x \neq 0, y \neq 0$$

HW: Molecular Biology

$$\min \sum\sum\sum_{ijk} A \left| \theta_{ijk} - \bar{\theta}_{ijk} \right|^2$$

$$\cos^{-1} \frac{x^T y}{\|x\| \|y\|_2}$$

Complete & write the angle expression!

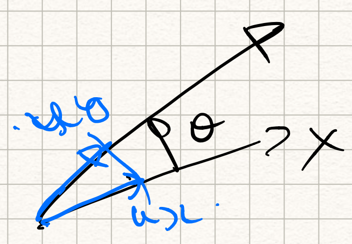
Orthogonal vectors:

$\vec{x} \perp^r \vec{y}$ i.e. $\theta = \pm 90^\circ$

$\cos\theta = 0$

$$\frac{x^T y}{\|x\|_2 \|y\|_2} = 0 \Rightarrow x^T y = 0$$

$$\cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2} = \left(\frac{x}{\|x\|_2}\right)^T \left(\frac{y}{\|y\|_2}\right)$$



$$\boxed{\cos\theta = u_x^T u_y}$$

$$-1 \le \frac{x^T y}{\|x\|_2 \|y\|_2} \le +1$$

$\Rightarrow$

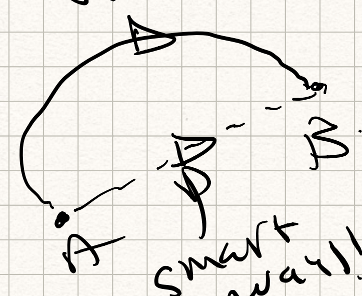
$$\max_{x,y} \frac{x^T y}{\|x\|_2 \|y\|_2} \qquad \min_{x,y} \frac{x^T y}{\|x\|_2 \|y\|_2}$$

Show that global max is $+1$

Show that global min is -1

x^*

$\nabla f(x^*) = 0$; at least!



$$\underset{1\times n}{(ax+by)^T}\underset{n\times 1}{(ax+by)} \ge 0$$

1×1

a, b R^n

constants or scalars.

$$\|ax+by\|_2^2 \ge 0$$

Choose $a = \frac{1}{\|x\|_2}$ $b = \frac{1}{\|y\|_2}$

$$a^2 x^T x + b^2 y^T y + 2ab\, x^T y \ge 0$$

$$\frac{x^T x}{\|x\|_2^2} + \frac{y^T y}{\|y\|_2^2} + 2\frac{x^T y}{\|x\|_2 \|y\|_2} \ge 0$$

$\|x\|_2^2$

$$1 + 1 + 2\frac{x^T y}{\|x\|_2 \|y\|_2} \ge 0$$

$$\frac{x^T y}{\|x\|_2 \|y\|_2} \geq -1$$

$$(ax + by)^T (ax + by) \geq 0$$

$$a = \frac{1}{\|x\|_2} \qquad b = -\frac{1}{\|y\|_2}$$

$$\underbrace{\frac{x^T x}{\|x\|_2}}_{1} + \underbrace{\frac{y^T y}{\|y\|_2^2}}_{1} - 2 \frac{1}{\|x\|_2} \cdot \frac{1}{\|y\|_2} x^T y \geq 0$$

$$2 - 2 \frac{x^T y}{\|x\|_2 \|y\|_2} \geq 0$$

$$\frac{x^T y}{\|x\|_2 \|y\|_2} \leq 1$$

$$-1 \leq \cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2} \leq +1$$

$$x \perp^r y \Rightarrow x^T y = 0$$

orthogonal vectors

$\|x\|_2 = 1$; $\|y\|_2 = 1$ $\qquad x \perp^r y$

orthonormal vectors.

In R^n $\quad Q = [q_1 \ q_2 \ \cdots \ q_n]$

$n \times n$

orthognoal matrix

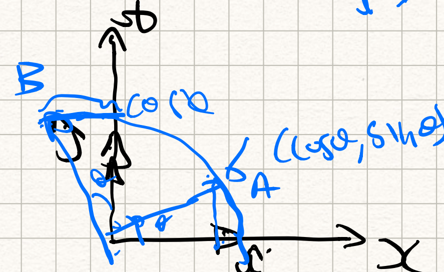
$$q_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} \quad q_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad q_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$q_i^T q_j = 0 \qquad i \neq j$$

$Q = I_n$ is trivial orthogonal matrix.

Non trivial orthogonal matrix.

$A \ (\cos\theta, \sin\theta)$



$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$q_1 \perp q_2$

infinitely many choices

$0 \leq \theta \leq 2\pi$

$q_1^T q_2 = 0$

$\|q_1\|_2 = 1$

$\|q_2\|_2 = 1$

$$Q = [q_1 \ \ q_2]$$

$$Q^T Q = I_2 \Rightarrow \underline{\underline{Q^{-1} = Q^T}}$$

Given's matrix

$$G_{i,j,\theta} = \begin{array}{c} \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & \cdots & s & 0 & 0 \\ \vdots & \vdots & 0 & 1 & \vdots & \vdots & \vdots \\ \vdots & \vdots & s & \cdots & c & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$c = \cos\theta$