Variants of Proportional Navigation

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- Parallel navigation: Constant bearing
- LOS is kept parallel to initial LOS, that is, angular rate of LOS $\boldsymbol{\omega} = \mathbf{0}$.
- In three dimensional engagement,

$$\boldsymbol{r} = \boldsymbol{r}_T - \boldsymbol{r}_M, \quad \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_T - \dot{\boldsymbol{r}}_M, \quad \ddot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_T - \ddot{\boldsymbol{r}}_M$$

- We can also write

$$\dot{\boldsymbol{r}} = \dot{r}\mathbf{1}_r + r\dot{\mathbf{1}}_r = \dot{r}\mathbf{1}_r + \boldsymbol{\omega} \times \boldsymbol{r}$$

- On cross-multiplying with $\boldsymbol{r}$, we have

$$\boldsymbol{r} \times \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{r} \times \mathbf{1}_r + \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = r^2\boldsymbol{\omega}$$

- For parallel navigation, $\boldsymbol{r} \times \dot{\boldsymbol{r}} = \mathbf{0} \Rightarrow \boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ must be colinear.
- Also, $\boldsymbol{r} \cdot \dot{\boldsymbol{r}} < 0$ for positive closing speed.

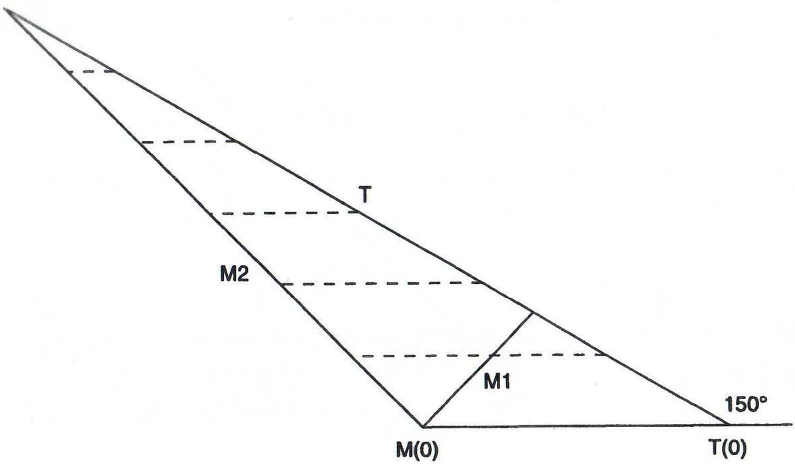
- For planar engagement, the relative kinematics

$$\dot{r} = V_r = V_T \cos(\gamma_T - \theta) - V_M \cos(\gamma_M - \theta)$$
$$r\dot{\theta} = V_\theta = V_T \sin(\gamma_T - \theta) - V_M \sin(\gamma_M - \theta)$$

- **Parallel navigation**: LOS rate $\dot{\theta} = 0$ and $\dot{r} < 0$.
- To satisfy these conditions,

$$V_M \sin(\gamma_M - \theta) = V_T \sin(\gamma_T - \theta)$$
$$V_M \cos(\gamma_M - \theta) > V_T \cos(\gamma_T - \theta)$$

- Can we always get a solution?
- What about $V_M/V_T \leq 1$ and $|\gamma_T - \theta| \leq \pi/2$?
- What about $V_M/V_T < 1$ and $|\gamma_T - \theta| > \pi/2$?
- What about $V_M/V_T = 1/\sqrt{2}$ and $(\gamma_T - \theta) = 150^\circ$?



$$(\gamma_M - \theta) = 45^\circ, 135^\circ$$

- Parallel navigation is optimal for nonmaneuvering targets.
- Interception is achieved in minimum time, with $a_M = 0$.
- Problems with manevering target and time varying speed targets
- Missile acceleration differ from zero most of the time.
- For nonplanar engagements,

$$\boldsymbol{r} \times \dot{\boldsymbol{r}} = \boldsymbol{r} \times (\boldsymbol{V}_T - \boldsymbol{V}_M) = \mathbf{0} \Rightarrow \mathbf{1}_r \times \boldsymbol{V}_M = \mathbf{1}_r \times \boldsymbol{V}_T$$

- On cross-multiplying with $\mathbf{1}_r$,

$$\mathbf{1}_r \times (\mathbf{1}_r \times \boldsymbol{V}_M) = \mathbf{1}_r \times (\mathbf{1}_r \times \boldsymbol{V}_T) \Rightarrow \boldsymbol{V}_{M_\perp} = \boldsymbol{V}_{T_\perp}$$

- Component of closing velocity perpendicular to LOS $V_{c_\perp} = \mathbf{0}$.
- Component of missile velocity along LOS

$$\boldsymbol{r} \cdot \dot{\boldsymbol{r}} < \mathbf{0} \Rightarrow \mathbf{1}_r \cdot \boldsymbol{V}_M > \mathbf{1}_r \cdot \boldsymbol{V}_T$$

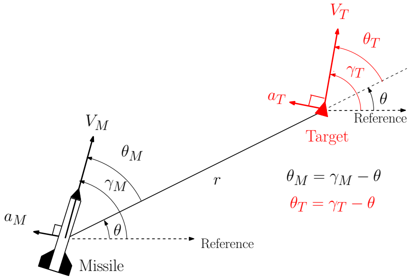
- Engagement dynamics

$$\dot{r} = V_r = V_T \cos(\gamma_T - \theta) - V_M \cos(\gamma_M - \theta)$$
$$r\dot{\theta} = V_\theta = V_T \sin(\gamma_T - \theta) - V_M \sin(\gamma_M - \theta)$$

- **PN Guidance**: A law that generates a guidance command to ensure the rate of rotation of missile velocity vector $\propto$ LOS rate.

$$\dot{\gamma}_M = N\dot{\theta}$$

where N is the navigation constant.



- Types of PN guidance
  - $\Rightarrow$ Pure Proportional Navigation (PPN)
  - $\Rightarrow$ True Proportional Navigation (TPN)
  - $\Rightarrow$ Generalized True Proportional Navigation (GTPN)
  - $\Rightarrow$ Ideal Proportional Navigation (IPN)
- **PPN Guidance**: Most natural type of PN guidance

$$\dot{\gamma}_M = N\dot{\theta} \Rightarrow a_M = NV_M\dot{\theta}$$

- $\dot{\gamma}_M = \dfrac{a_M}{V_M}$ is valid only when lateral acceleration $a_M \perp V_M$.
- Acceleration a_M applied perpendicular to the velocity vector of the missile.
- On ignoring angle-of-attack of the missile, direction of lateral acceleration is also the natural direction of the lift force.
- **Issue**: Angle-of-attack of a missile is never zero and for many highly maneuverable missiles it turns out to be quite high.

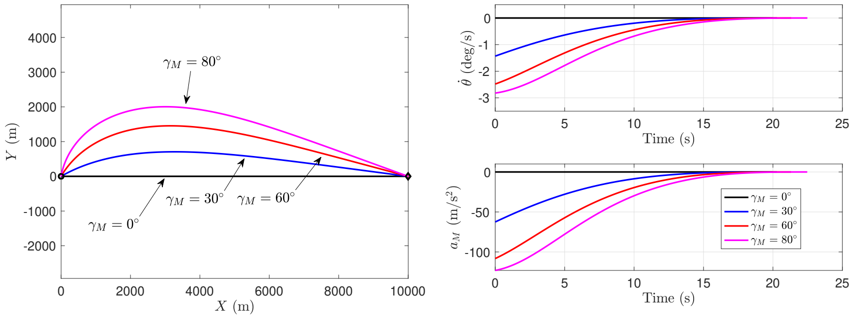


Figure: Target interception using PPN guidance

- $V_M = 500$ m/s, $\theta(0) = 0^\circ$
- Initial launch angles of $0^\circ, 30^\circ, 60^\circ, 80^\circ$
- $a_M = NV_M\dot{\theta}$, $N = 5$
- Acceleration demand and LOS rate converge to zero at interception

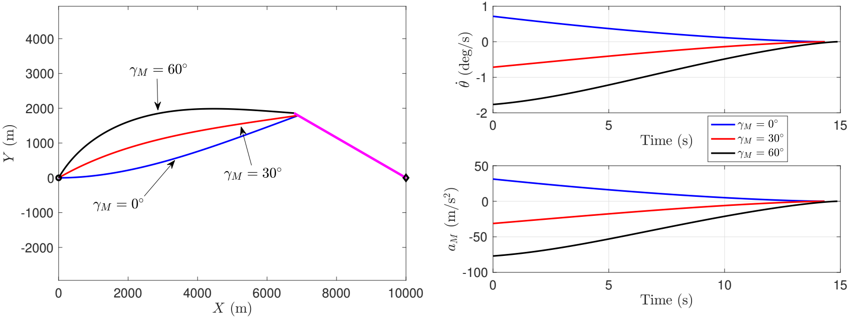
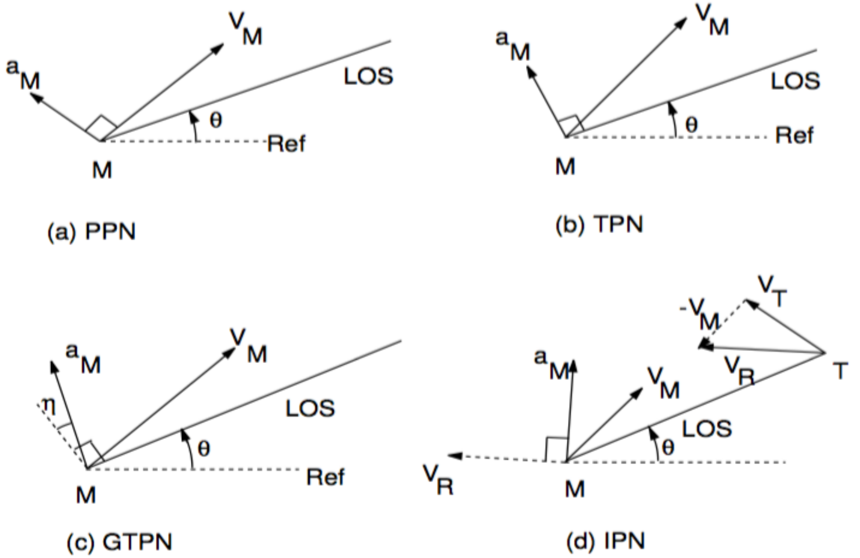


Figure: Target interception using PPN guidance

- $V_M = 500$ m/s, $\theta(0) = 0^\circ$,$\gamma_T = 150^\circ$
- Initial launch angles of $0^\circ, 30^\circ, 60^\circ$
- $a_M = NV_M\dot{\theta}$, $N = 5$
- Acceleration demand and LOS rate converge to zero at interception

Variants of Proportional Navigation

Variants of PN Guidance



- Velocity of interest is actually closing velocity and not V_M itself, because it is the closing velocity which ultimately drives the LOS separation to zero.
- Moreover, it is also the LOS rate which we are trying to drive to zero.
- **TPN Guidance**:
  - ⇒ Guidance command based on closing velocity
  - ⇒ Guidance command $\perp$ LOS
- V_M is not directly available unless the missile carries an inertial navigation unit, but V_c is easily available from the doppler data of the seeker.
- Guidance command of TPN

$$a_M = N' V_c \dot{\theta} = -N' V_r \dot{\theta}$$

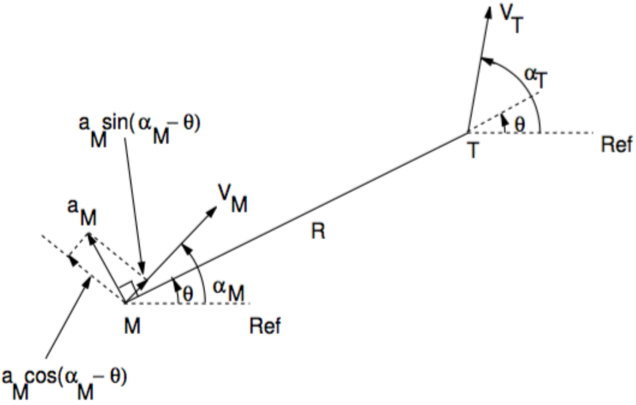
  where N' is effective navigation constant.
- Note the difference between PPN and TPN.

- **Issues in implementation of TPN**: Direction of TPN lateral acceleration is not in lift's natural direction.
- Use of thrusters either in the forward or aft direction accordingly.
- Not a very practical solution due to requirement of extra thrusters.
- Useful for exo-atmospheric interception as aerodynamic forces are non-existent at very high altitudes.
- **GTPN**: lateral acceleration direction was defined as being deviated by some angle from the normal to LOS.
- Idea was to increase the capturability performance of the guidance law further and (hopefully!) make it comparable to the PPN law.
- Considering closing velocity term as a time-varying one and not constant deteriorates the performance even further.

- **Ideal Proportional Navigation (IPN)**: Commanded lateral acceleration was applied perpendicular to relative velocity between missile and target.
- Arguments in favour of this are similar to the arguments in favour of TPN.
- **Capturability of IPN**: comparable to that of PPN and much better than that of TPN or its many generalizations.
- IPN is difficult to implement like TPN or any of its generalizations.

- **True Proportional Navigation**: Missile lateral acceleration commanded by it is proportional to the LOS angular rate and is applied normal to the LOS

$$a_M = c\dot{\theta}, \quad c > 0$$



Variants of Proportional Navigation

TPN Guidance

- Target is assumed to be non-maneuvering and also having a constant speed.
- Equation of motion

$$
\begin{aligned}
\dot{r} =& V_r = V_T \cos(\gamma_T - \theta) - V_M \cos(\gamma_M - \theta) \\
r\dot{\theta} =& V_\theta = V_T \sin(\gamma_T - \theta) - V_M \sin(\gamma_M - \theta) \\
\dot{V}_M =& a_M \sin(\gamma_M - \theta) \\
\dot{\gamma}_M =& \frac{a_M \cos(\gamma_M - \theta)}{V_M}
\end{aligned}
$$

- Missile velocity is a time-varying quantity as a_M is not normal to V_M.
- Closed-form expression for the guidance command is pre-determined.
- In case of LOS and pursuit guidance, the guidance was specified by certain specific requirements (what were those?) and $a_M \perp V_M$.

- On differentiating V_r

$$\begin{aligned}\dot{V}_r =& -V_T\sin(\gamma_T-\theta)(-\dot{\theta}) + V_M\sin(\gamma_M-\theta)(\dot{\gamma}_M-\dot{\theta}) - \dot{V}_M\cos(\gamma_M-\theta)\\ =&\dot{\theta}V_\theta + a_M\sin(\gamma_M-\theta)\cos(\gamma_M-\theta) - a_M\sin(\gamma_M-\theta)\cos(\gamma_M-\theta)\\ =&\dot{\theta}V_\theta\end{aligned}$$

- On differentiating V_θ

$$\begin{aligned}\dot{V}_\theta =&V_T\cos(\gamma_T-\theta)(-\dot{\theta}) - V_M\cos(\gamma_M-\theta)(\dot{\gamma}_M-\dot{\theta}) - \dot{V}_M\sin(\gamma_M-\theta)\\ =& -\dot{\theta}V_r - a_M\cos^2(\gamma_M-\theta) - a_M\sin^2(\gamma_M-\theta)\\ =& -\dot{\theta}V_r - a_M\\ =& -\dot{\theta}(V_r+c)\end{aligned}$$

- Using these equations, we can write

$$\dot{V}_\theta V_\theta + \dot{V}_r V_r + c\dot{V}_r = 0$$

- On integration of previous equation

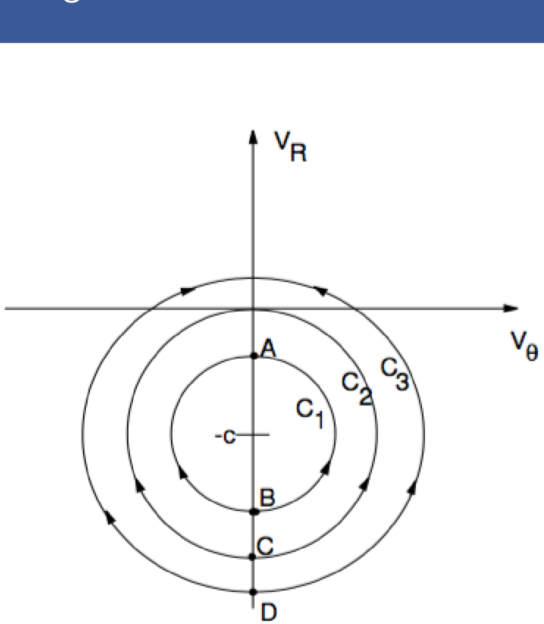
$$V_\theta^2 + V_r^2 + 2cV_r = k$$

where

$$k = V_{\theta_0}^2 + V_{r_0}^2 + 2cV_{r_0}$$

- On rearrangement,

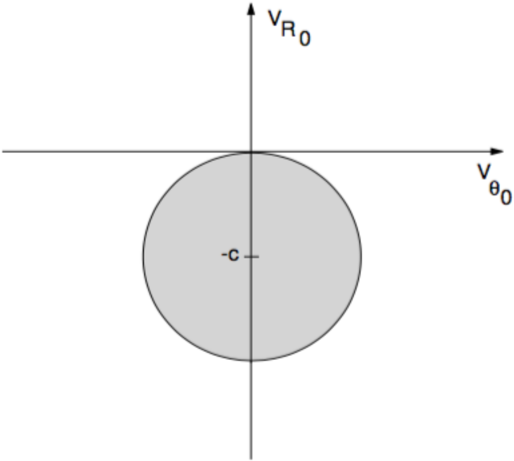
$$\begin{aligned} V_\theta^2 + (V_r + c)^2 &= k + c^2 \\ &= V_{\theta_0}^2 + (V_{r_0} + c)^2 \end{aligned}$$



- Any point on negative V_r axis leads to interception.
- Circle $\mathcal{C}_1$: Always results in capture
- Circle $\mathcal{C}_3$: No capture and results into miss distance
- Circle $\mathcal{C}_2$: Capture if

$$V_{\theta_0}^2 + V_{r_0}^2 + 2cV_{r_0} < 0$$

- Capture equation
- Parameter c plays an important role in the determination of the capturability of TPN.



- Assume that it is a function of the initial closing velocity $V_{c_0} = -V_{r_0}$.

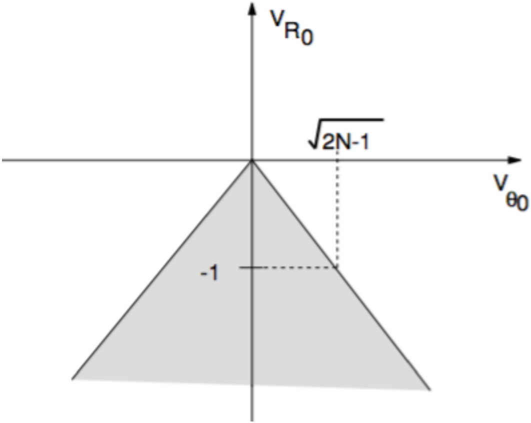
$$a_M = c\dot{\theta},\ c = -NV_{r_0}$$

- Capture equation

$$V_{\theta_0}^2 < (2N-1)V_{r_0}^2$$

- Alternatively,

$$|V_{\theta_0}| < \sqrt{2N-1}|V_{r_0}|$$



Condition for existence of capture circle? $N > 1/2$

- How to obtain time of interception in TPN case?
- We know that

$$V_\theta^2 + V_r^2 + 2cV_r = k, \quad \dot{V}_r = \dot{\theta} V_\theta$$

- On using above equations

$$r\dot{V}_r + V_r^2 + 2cV_r = k \Rightarrow \frac{d(rV_r)}{dt} + 2c\dot{r} = k$$

- On integration, we get

$$rV_r + 2cr = kt + b, \; b = r_0(V_{r_0} + 2c)$$

- To get time of interception, at $t = t_f, \; r = 0$.

$$t_f = -\frac{b}{k} = -\frac{r_0(V_{r_0} + 2c)}{V_{\theta_0}^2 + V_{r_0}^2 + 2cV_{r_0}}$$

- **TPN law**: Guidance command as a function of the initial closing velocity.
- Closing velocity during the missile-target engagement actually varies with time.
- TPN guidance command should be of the form

$$a_M = NV_c\dot{\theta} = -NV_r\dot{\theta}, \quad N > 0$$

  where N is navigation constant.
- Current closing velocity is used for guidance command computation.
- This guidance is called as Realistic True Proportional Navigation (RTPN) Guidance.

- On differentiating V_r

$$
\begin{aligned}
\dot{V}_r =& -V_T\sin(\gamma_T-\theta)(-\dot{\theta}) + V_M\sin(\gamma_M-\theta)(\dot{\gamma}_M-\dot{\theta}) - \dot{V}_M\cos(\gamma_M-\theta) \\
=& \dot{\theta}V_\theta + a_M\sin(\gamma_M-\theta)\cos(\gamma_M-\theta) - a_M\sin(\gamma_M-\theta)\cos(\gamma_M-\theta) \\
=& \dot{\theta}V_\theta
\end{aligned}
$$

- On differentiating V_θ

$$
\begin{aligned}
\dot{V}_\theta =& V_T\cos(\gamma_T-\theta)(-\dot{\theta}) - V_M\cos(\gamma_M-\theta)(\dot{\gamma}_M-\dot{\theta}) - \dot{V}_M\sin(\gamma_M-\theta) \\
=& -\dot{\theta}V_r - a_M\cos^2(\gamma_M-\theta) - a_M\sin^2(\gamma_M-\theta) \\
=& -\dot{\theta}V_r - a_M \\
=& -\dot{\theta}V_r + NV_r\dot{\theta} \\
=& -(1-N)\dot{\theta}V_r
\end{aligned}
$$

- We have the derivative of V_r and V_θ as

$$\begin{aligned}\dot{V}_r =& \dot{\theta} V_\theta \\ \dot{V}_\theta =& -\dot{\theta} V_r (1-N)\end{aligned}$$

- Using previous equations, we get

$$\dot{\theta} = \frac{\dot{V}_r}{V_\theta} = -\frac{\dot{V}_\theta}{(1-N)V_r} \Rightarrow V_\theta \dot{V}_\theta + (1-N) V_r \dot{V}_r = 0$$

- On integration, we get

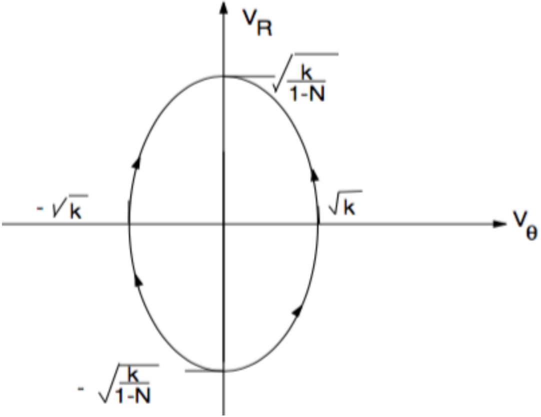
$$V_\theta^2 + (1-N)V_r^2 = k = V_{\theta_0}^2 + (1-N)V_{r_0}^2$$

- What does this equation represents in (V_θ, V_r) space?

- **Case 1**: $0 < N < 1$

  $$k = V_{\theta_0}^2 + (1-N)V_{r_0}^2 > 0$$

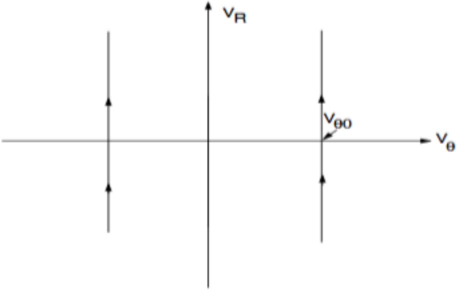
- For $V_r = 0$, $V_\theta^2 = k$.
- For $V_\theta = 0$, $V_r^2 = \dfrac{k}{1-N}$.
- Equation of an ellipse.
- Direction of movement: $r\dot{V}_r = V_\theta^2 > 0$.
- Condition of capturability for case $0 < N < 1$?



- **Case 2**: $N = 1$

$$V_\theta^2 = k = V_{\theta_0}^2$$

- What does it represent in (V_θ, V_r) space?
- Condition of capturability for case $N = 1$?



- For $N = 1$, capture is not possible except for negative V_r axis.
- **Case 3**: $N > 1$

$$V_\theta^2 - (N-1)V_r^2 = k = V_{\theta_0}^2 - (N-1)V_{r_0}^2$$

- What does it represent in (V_θ, V_r) space?
- Condition of capturability for case $N > 1$?

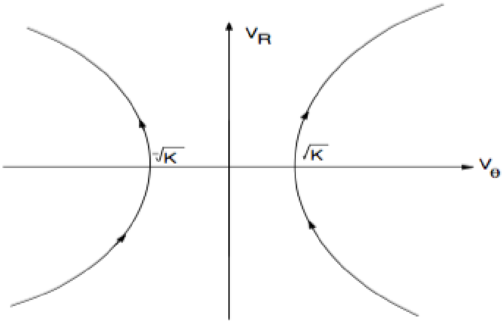
- **Case 3a**: $N > 1$

$$V_\theta^2 - (N-1)V_r^2 = k$$

where

$$k = V_{\theta_0}^2 - (N-1)V_{r_0}^2.$$

- For $k > 0$,
  $V_r = 0 \Rightarrow V_\theta^2 = k$.
- There are no value of V_r for which $V_\theta = 0$.
- What about interception of target?
- No interception



- **Case 3b**: $N > 1$

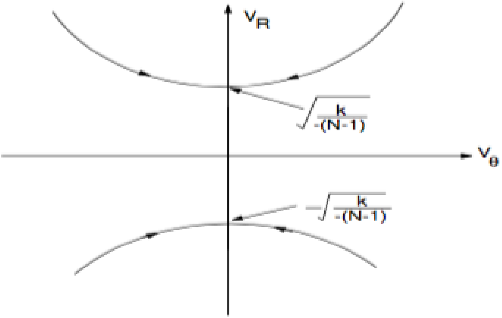
$$V_\theta^2 - (N-1)V_r^2 = k$$

where

$$k = V_{\theta_0}^2 - (N-1)V_{r_0}^2$$

- For $k < 0$,

  $V_\theta = 0 \Rightarrow V_r = \pm\sqrt{\dfrac{k}{N-1}}$.
- There are no value of V_θ for which $V_r = 0$.
- What about interception of target?
- Interception occurs for $V_{r_0} < 0$.



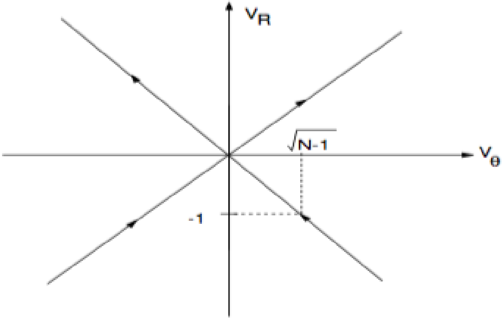
- **Case 3c**: $N > 1$

$$V_\theta^2 - (N-1)V_r^2 = k$$

where

$$k = V_{\theta_0}^2 - (N-1)V_{r_0}^2$$

- $k = 0$, $V_\theta^2 = (N-1)V_r^2$
- What about interception of target?
- Interception does not occur for positive V_r region.
- Nothing can be said about negative V_r region.



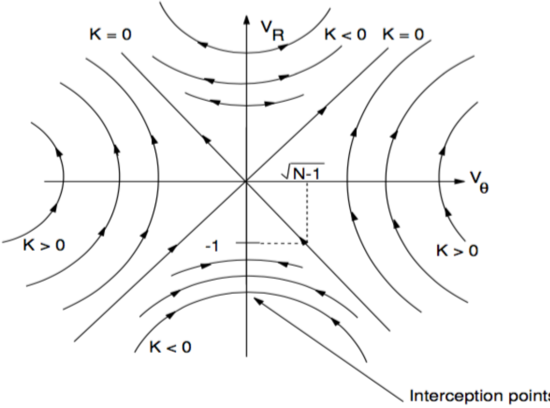
- Capture region can be obtained as

$$V_{\theta_0}^2 + (1-N)V_{r_0}^2 < 0,\ V_{r_0} < 0$$

- Alternatively,

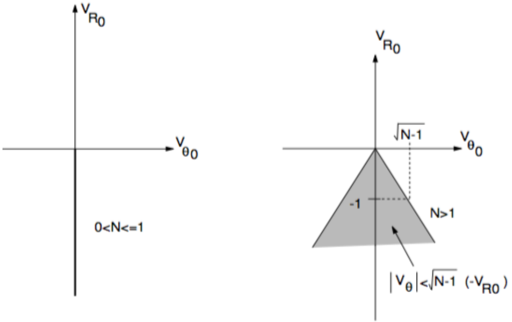
$$|V_{\theta_0}| < \sqrt{N-1}|V_{r_0}|,\ V_{r_0} < 0$$

- For $N < 1$, it shrinks to negative V_r axis.
- Capture region is smaller than that of original TPN.



Variants of Proportional Navigation

Realistic TPN Guidance: Capturability Regions



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