Definitions:

Closure Point: A point 'x' is a closure point or limit point of a subset X of $\mathbb{R}^n$ if there exists a sequence $\{x_k\} \subset X$ that converges to x.

Example: Let $S = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$

Then, $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(1,0)$ and $(0,1)$ are all closure points of 'S'.

Closed set: A subset X in $\mathbb{R}^n$ is called closed if it is equal to its closure.

Open set: A subset X in $\mathbb{R}^n$ is open if its complement, $\{x \mid x \notin X\}$ is closed.

Example: (a) Let $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$

Then 'S' is a closed set.

(b) $S = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$

Then 'S' is a open set.

Interior Point: Let $X \subset \mathbb{R}^n$ and $x \in X$, we say that x is an interior point of X if there exists a neighborhood of x that is contained in 'X'.

Boundary point: A vector $x \in X$ which is not an interior point is called the boundary point of X.

The set of all boundary points of X is called the boundary of X.

Example: In the set $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ the subset $B = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ constitute the boundary.

Example: The set $S = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \cup \{(1,0)\}$ is neither open nor closed with one boundary point $(1, 0)$.

Bounded Set: A subset S in $\mathbb{R}^n$ is bounded if there exists a scalar 'c' such that $\|x\| \leq c$ for all x in X. The choice of norm is not important. For example, you could choose $\|\cdot\|_1$, $\|\cdot\|_2$ or $\|\cdot\|_\infty$ norm.

Example: All the following three sets are bounded.

$$S_1 = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$$

$$S_2 = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$$

and $$S_3 = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \cup \{(1,0)\}$$

as $\|x\|_2 \leq 1$. For x in S.

Compact Set: A set is compact if it is closed and bounded.

Example: In the above example, S_2 is the only compact set.

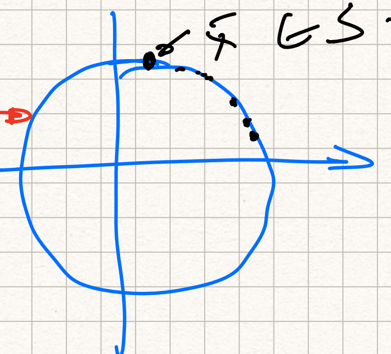
Example: closed and Bounded set, Compact set.

$$S = \{x : \|x\| = 1\}$$

$$\|x\|_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = 1$$

$x_k \in S$

$\{x_k\} \to x$



$\lim \|x_k$

$$x_1^2 + x_2^2 = 1^2$$

$$\lim_{k \to \infty} (x_1^{k\,2} + x_2^{k\,2} = 1^2) \qquad \lim_{k \to \infty}$$

$$\lim_{k \to \infty} (x_1^k)^2 + \lim_{k \to \infty} (x_2^k)^2 = \bar{x}_1^2 + \bar{x}_2^2$$

$$= 1^2$$

$S = cl(S)$

or Set is closed set

$\|x\| = 1$, it is bounded

Hence, convex.

Open Ball and a closed Ball $\varepsilon > 0$

$\{x \mid \|x - x^*\| < \varepsilon\}$ open ball or open sphere.

$\{x \mid \|x - x^*\| \le \varepsilon\}$ closed ball or a closed sphere.

Proposition (A.6: Bertsekas)

(a) The union of finitely many closed sets is closed.

(b) The intersection of closed sets is closed.

(c) The union of open sets is open.

(d) The intersection of finitely many open sets is open.

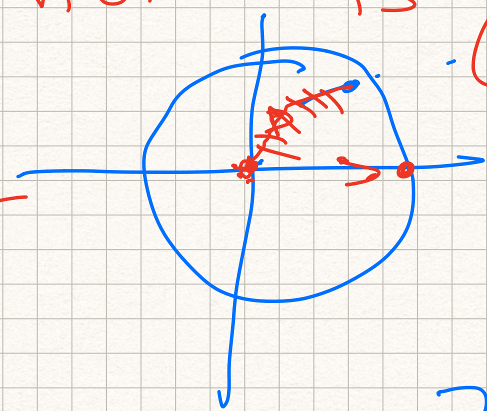
(e) A set is open if and only if all of its elements are interior points.

(f) Every subspace of $\mathbb{R}^n$ is closed.

(g) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

in not is $\begin{pmatrix}1\\0\end{pmatrix}$

SCAS



$S \subset \mathbb{R}^2$

pt.

$$S = \{(x_1, x_2) \; x_1^2 + x_2^2 \leq 1\}.$$

closure

$\{x_k\} \subset S$

$\lim_{k \to \infty} \{x_k\} = 0$

$(0,0) \longrightarrow (1,0)$

$x_1 = (0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots, \frac{9}{10},$

$0.99, 0.999, 0.9999 \ldots)$

$x_2 = (0, 0, \ldots 0, \ldots 0)$

$\lim_{k \to \infty} \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$ $\quad x^k \in S$

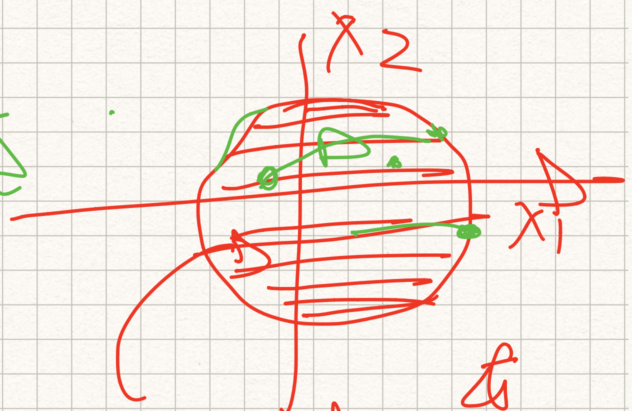
Closure → $\overline{(\ )}$

$\{x, y\}$

$S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$

$S = cl(S)$

closed



interior

but complete boundary & circle is in set

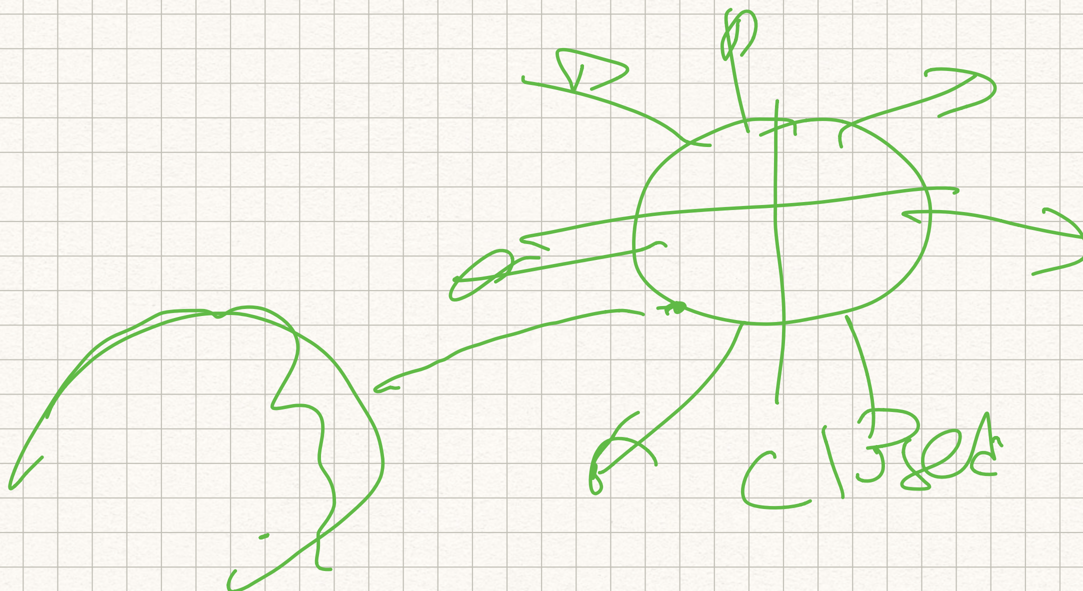
Open set

as one whose complement is a closed set.

(x_1, x_2)

$S = \{x_1^2 + x_2^2 < 1\}$

$S = \{x_1^2 + x_2^2 \geq 1\}$



Open set
has no
boundary pt
in it.

(x_1, x_2)

$$\underline{S = \{ x_1^2 + x_2^2 < 1 \}}$$

$$+ \underline{U(0,1)}$$

interior point of X
set

$x \in X$

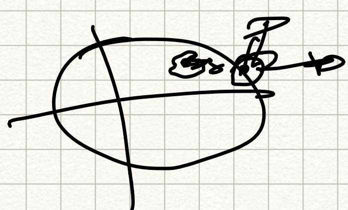
interior pt.

$N_\delta(x) \subset X$

A point $x \in X$ which is not a interior point is called ~~pt~~ a boundary of X

$S = \{ (x_1, x_2) ; x_1^2 + x_2^2 < 1 \}$

→ open



⇒ $S = \text{int}(S)$

open set

$S = \{ x_1^2 + x_2^2 \leq 1 \}$.

interior point

$\{ x_1^2 + x_2^2 < 1 \}$

Boundary pts

Bounded set.

is one in which

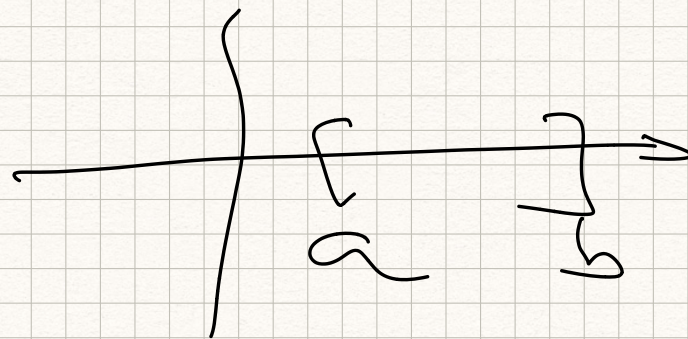
$\forall x \in S : \|x\| \leq C$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ p

the constant C

Closed $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$
is bounded.

$S = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$
open & bounded set. is also bounded

A set which is both closed and bounded is called a compact set.



$S = \{(x_1, x_2)$
s.t. $x_1^2 + x_2^2 \geq 1\}$

closed
but not
bounded

it is not compact