AE 227: Mechanics of Solids

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1 General motivation

Mechanics of solids is a part of continuum mechanics that deals with the changes of place and shape of things we see every day. One aspect of mechanics represents the mathematical construction of surrounding objects that composed of earth, water, air, and fire. It also describes the observation in the laboratories where scientists perform experiments through an 'approximate' theory of nature. Mechanics not only describes but also predicts phenomena.

Mechanics does not study natural things directly. Instead, it considers bodies, which are mathematical concepts of some common features of many natural things, e.g., mass assigned to each body. A body at any instant of time occupies a set of places or geometry (shape). The change of shape/geometry of a body from one instant to another is called the motion of the body, and the description of motion is called kinematics. The motion of a body is conceived as resulting by the action of forces. Thus mechanics provides a mathematical model for certain aspects of nature. The mathematical equations that describe the physical laws are called the field equations.

The assignment of mass m to a body might be discrete, however, in the mechanics of continuum the masses are an absolutely continuous function with volume. A small volume ΔV enclosed within a surface ΔS possesses a mass density defined by the following limit:

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \geq 0,$$

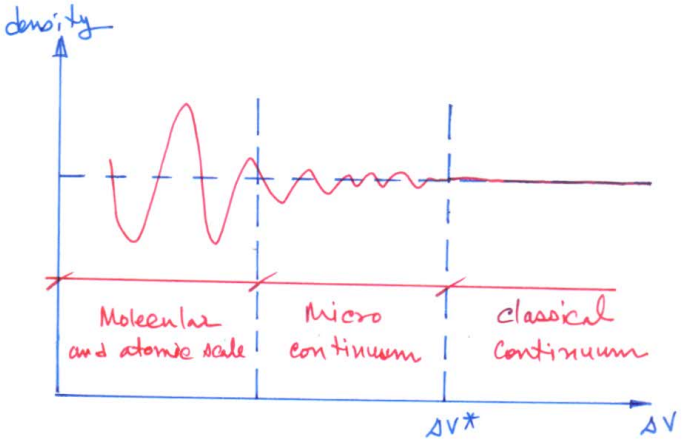


Figure 1: Mass density versus volume.

where Δm is the total mass contained in ΔV. In order to judge the physical validity of the above mathematical definition, consider the following thought experiment. Suppose you have a large number of pieces of different volume of a (homogeneous) material. The mass density for each piece is then calculated from the ration $\Delta m/\Delta V$. If the resulting numbers ρ are plotted versus ΔV, one finds a nearly constant ratio when ΔV is greater than a certain critical volume ΔV^* which roughly corresponds to a critical length $L \approx (\Delta V^*)^{1/3}$, which is of course, not precisely defined. The size of ΔV^* depends on the constitution of the material. As ΔV approaches zero, these dependence becomes extreme (Fig. 1). The classical continuum theory may therefore not be a good approximation of a physical theory in the range $\Delta V << \Delta V^*$. Examples of continuous description, in excellent agreement with the experimental data in the range $\Delta V >> \Delta V^*$, are those corresponding to the classical nonlinear and linear theory of solids and fluids.

More involved physical theories based on continuum hypothesis have also been successful. Among many you can find: theories of magnetic solids and fluids, piezoelectricity, photo-mechanical etc. where deformations and electromagnetic fields play prominent roles. The concept of balance equations comes from free-body

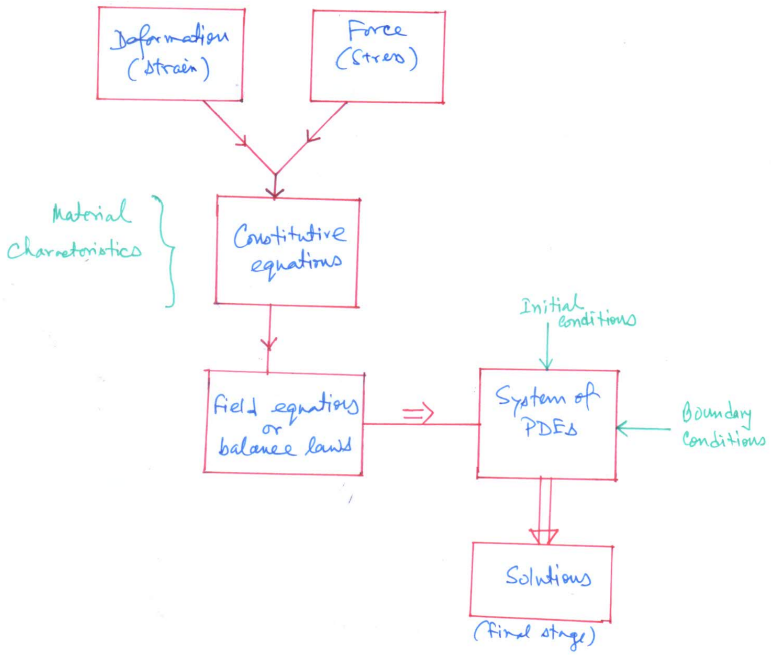


Figure 2: Steps of mathematical modeling.

principle. The influence of the external world (by means of volume and surface forces) on the motion of the body is expressed in terms of balance relations for

mass, linear momentum, and angular momentum.

The particular behavior patterns of any given material body are described by statements that have to be formulated individually for each and every material, whereas the balance relations are true for each and every material. Representations of material behaviors are known as constitutive equations, which relate between the processes of deformation and internal forces/stresses. They do not come under the laws of nature but consists of a mathematical model to reproduce typical behavior patterns of real materials.

The mathematical modeling of material properties from the point of view of general principles and systematic methods is one of the objectives of the Mechanics of Solids. Such theories cover elasticity, viscoelasticity, plasticity and viscoplasticity. Every mathematical model evolves from a combination of universal balance relations with individually valid constitutive equations. The aim of each analysis is to calculate the fields, which are expressed through systems of partial differential equations with appropriate initial conditions and boundary conditions. The steps are shown in Fig. 2.

2 Deformation of bodies

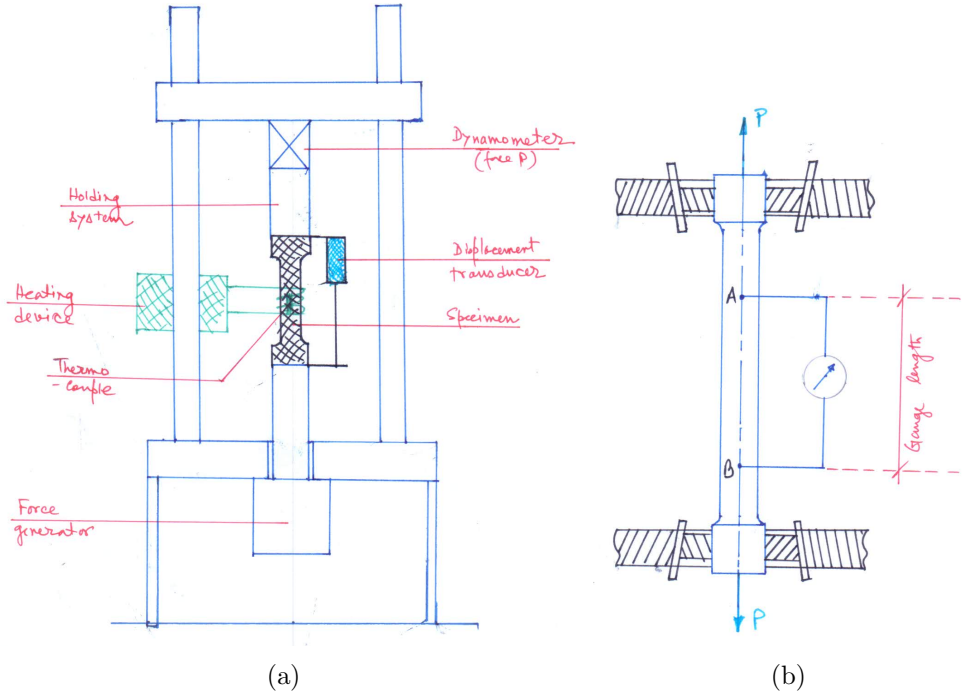
2.1 Testing machine

The phenomenological method requires the experiments to be made on a volume element of matter. The most common test is the simple tension test (Fig. 3a) in which the useful part of the specimen (also known as gage length, Fig. 3b) is subjected to a uniform uniaxial stress field. The force (or load) could be applied by a continuous screw-nut system driven by electric motor or hydraulic system. The specimen consists of a cylindrical shaft, end grips, and between the two, 'shoulders', to avoid stress concentration.

During an experiment, the change in gage length is noted as a function of the applied force. With the same load and a longer gage length, a larger deformation is observed, than when the gage length is small. Therefore it is more fundamental to refer to the observed deformation per unit of the gage length. If L_0 is the initial gage length and L is the observed length under a given load, the gage elongation $\Delta L = L - L_0$. The elongation ε per unit of the initial gage length is then given as

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0} \tag{1}$$

The above expression defines the extensional strain or normal strain. It is a dimensionless quantity and often it is given as a percentage (%). The quantity ε is generally very small (0.1%) in most of the engineering applications. In this



(a) (b)

Figure 3: (a) A typical experiment setup and (b) a typical specimen for uniaxial testing.

course, we will confine ourself in this small strain regime. However, in some applications, e.g. metal forming,[1] the strain may be large. For such purposes, one defines natural or true strain $\bar{\varepsilon}$. The strain increment $d\varepsilon$ for such a strain measure is defined as dL/L, where L is the instantaneous length of the specimen, and dL is the incremental change in length L. We write

$$\bar{\varepsilon} = \int_{L_0}^{L} d\varepsilon = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0} = \ln(1+\varepsilon)$$
$$= \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \cdots (-1 < \varepsilon < 1). \qquad (2)$$

Note that for small strain, where the higher order terms do not contribute, we simply have $\bar{\varepsilon} = \varepsilon$.

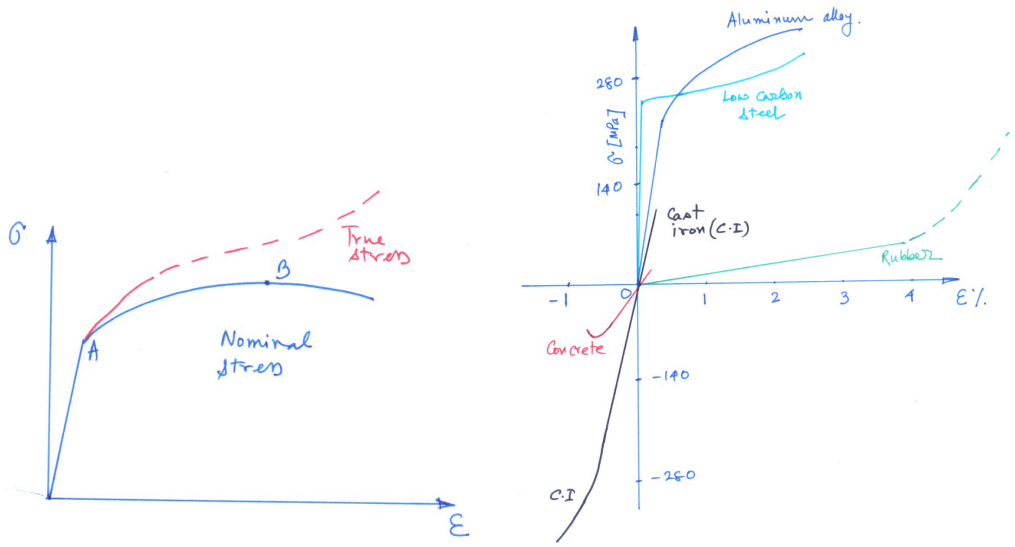
2.2 Force (stress)-deformation (strain) relationship

In solid mechanics, the mechanical behavior of real materials under load is of primary interest. Experiments provide basic information on this behavior. In the

[1]It is a metalworking process where the workpiece is reshaped without adding or removing material through large deformation.

experiments, macroscopic (overall) responses of a specimen to the applied loads are observed to determine the empirical force-deformation relationships.

Like strain, 'stress' is a more significant parameter than force since the effect on a material of an applied force P depends primarily on the cross-sectional area of the member. Let us quickly define stress as force per unit area. The stress measure corresponding to the actual undeformed cross-sectional area is known as nominal stress, where true stress consider the current cross-sectional area (Fig. 4a). We'll study the concept of stress with more detail in the next section.



(a) (b)

Figure 4: (a) True stress vs nominal stress and (b) different material responses.

In the experimental study, it is customary to plot diagrams of the relationship between stress and strain in a particular test. Such plots are, in general, independent of the size of the specimen and its gage length. It is customary to use the ordinate scale for the stress and abscissa for the strain. Experimentally determined stress-strain diagrams differ widely for different materials (Fig. 4b). It is the material microstructures that are responsible for such diverse responses. We'll now look briefly some microstructural aspect of mechanical responses.

2.3 Material microstructures

2.3.1 Single crystal

The crystalline state is characterized by the regularity of the atomic arrangement. An elementary pattern or unit cell (Fig. 5) repeats itself periodically in all the

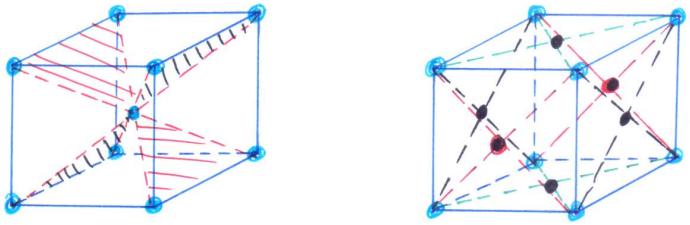


Figure 5: BCC and FCC unit cell.

three directions to form a space lattice, an infinite array of points (Fig. 6). A crystalline solid can be a single crystal, where the entire solid consists of only one crystal.

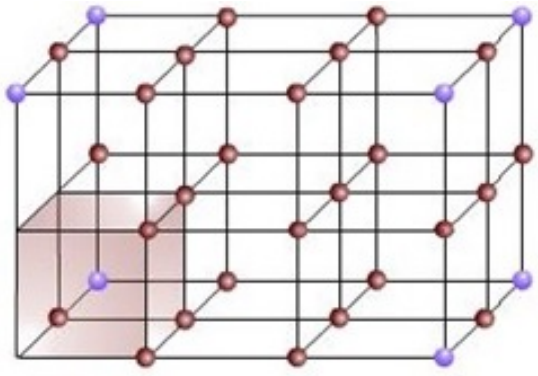


Figure 6: Representation of space lattice with unit cell.

2.3.2 Polycrystals

Metals are generally produced in a liquid state, and their structure is formed as they solidify when cooled. As the temperature of the liquid decreases, the interatomic distances become smaller. The critical distance, at which bonding occurs, is reached at several, randomly distributed sites, and constitute the first germs or nuclei of crystal growth. The lattices are formed in the same crystalline system but in random directions. A polycrystal is made up of several single-crystals oriented randomly (Fig. 7).

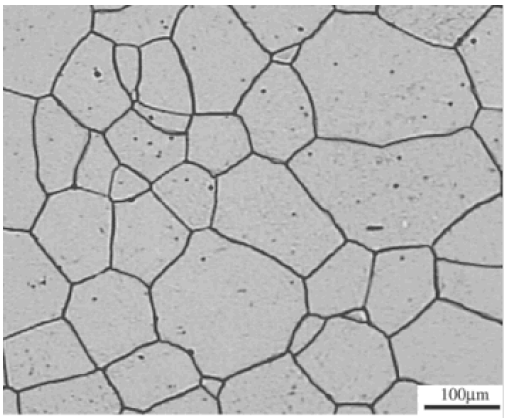


Figure 7: Randomly oriented single crystals in a polycrystal.

2.4 Crystal defects

2.4.1 Point defects (0-d imperfection)

As the name suggests, they are imperfect point-like regions in the crystal. Point defects consist of atoms inserted or substituted or missing (one or two atomic diameters, Fig. 8) in the crystal. They result in a local distortion of the lattice.

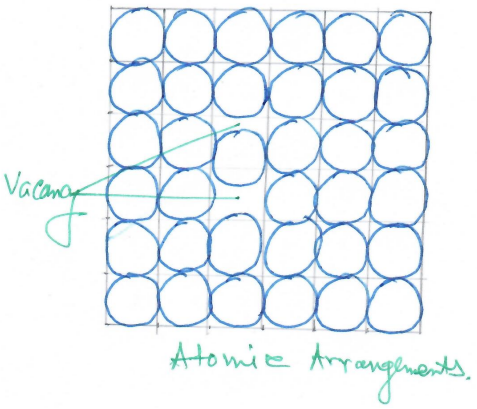


Figure 8: Point defects in an elemental crystal due to vacancy.

2.4.2 Line defects or dislocations (1-d imperfection)

Line defects, a one-dimensional imperfections in geometrical sense, are called dislocations. Dislocations are created during the growth of the crystals. These

are the defects which are responsible for plasticity of metals. Let us consider a schematic representation of such defects in the case of simple cubic crystals.

Edge dislocation

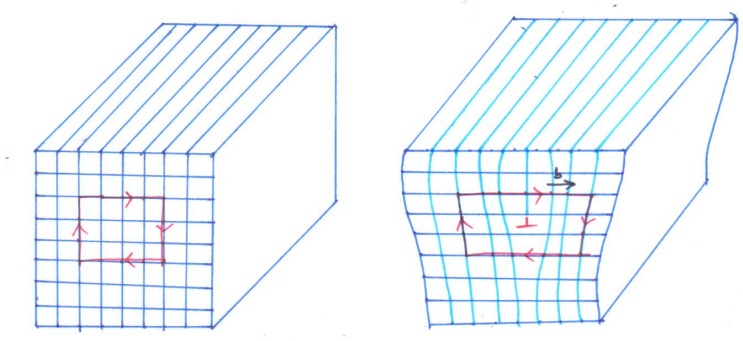


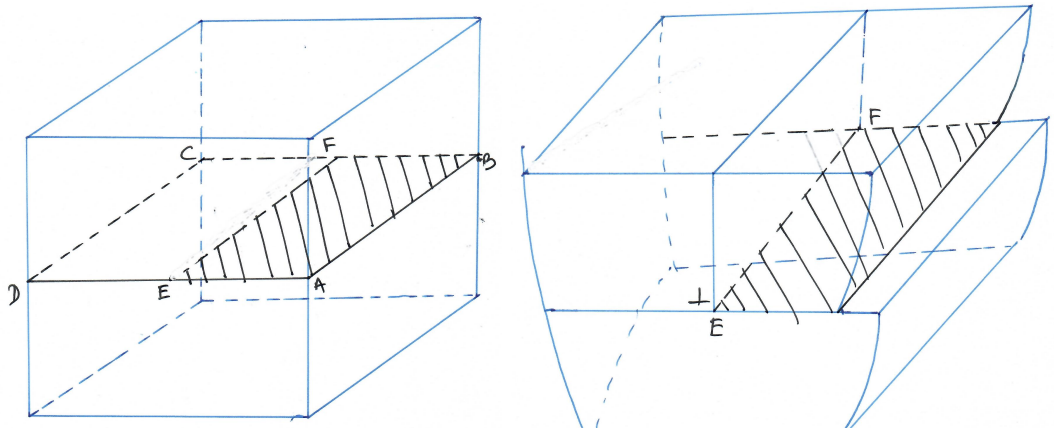
Figure 9: An incomplete plane in a crystal results in an edge dislocation. **b** is the measure of lattice distortion resulting from dislocation and is known as burger vector.

We can think about a perfect crystal to be made up of vertical planes parallel to one another and to the side faces. If one of these vertical planes does not reach from the top to the bottom of the crystal but ends part away within the crystal, as in Fig. 9, a dislocation is created. Thus there is a distortion in the region immediately surrounding the edge of the incomplete plane. Edge dislocations are symbolically represented by $\perp$ or $\top$ depending on wheather the incomplete plane starts from the top or from the bottom of the crystal.

An alternative way of looking at an edge dislocation is illustrated in Fig. 10. The top part of the the crystal above the hatched area $ABFE$ is displaced to the left with respect to the bottom part. No such displacement takes place over the unhatched area $EFCD$. The displacement has thus resulted in an incomplete vertical plane over the edge EF, which thus becomes an edge dislocation. The $ABCD$ is called a slip plane. The region $ABEF$ over which the displacement is done is called the slipped part of the slip plane.

Screw dislocation

A screw dislocation is created by a local rotation of the upper part of the crystal (Fig. 11b).



(a) (b)

Figure 10: Displacement of the top part with respect to the bottom part introduces an edge dislocation EF.

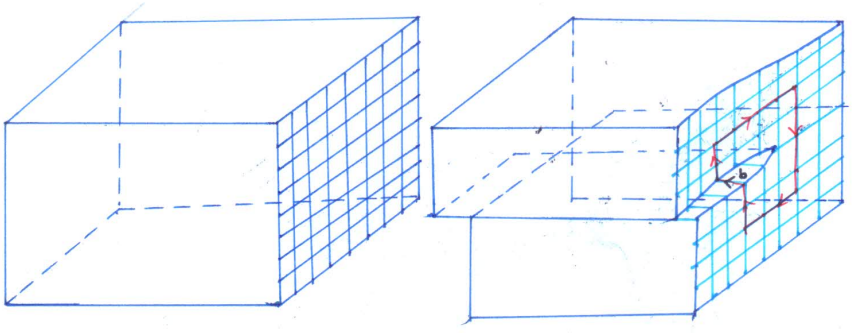


Figure 11: Screw dislocation.

2.4.3 Surface defects (2-d imperfection)

Surface defects are the surfaces of separation between crystals or parts of a crystal with different orientations. Their thickness is of the order of 4-5 atomic 'diameters'. For example, grain boundaries (or crystal boundaries) in polycrystals, interfaces between two phases, twin boundaries, etc.

2.4.4 Volume defects (3-d imperfection)

For example cracks, precipitates, large voids in materials, etc.

3 Physical mechanisms of deformation

3.0.1 Elastic deformation

Elastic deformations occur at the atomic level. The observed macroscopic effect is the result of the variations in the interatomic spacing necessary to balance the external loads, and also the reversible movement of the dislocations. These geometrical adjustments are essentially reversible. In a purely elastic deformation, the initial configuration of atoms is restored upon the removal of the load.

3.0.2 Permanent/Inelastic deformation

Inelastic deformations (plastic and viscoplastic) occur at the crystal level. In addition to the elastic deformations, inelastic deformation corresponds to a relative displacement of atoms which remains when the load is removed.

Consider deformation by dislocations. The presence of dislocations considerably reduces the stability of the crystal lattice. Their mobility is the essential cause of permanent deformations, homogeneous at the macroscopic scale. This slip displacement mechanism requires the breaking of bonds only in the vicinity of the dislocation line. The rate of dislocation displacement can be very low or very high depending on the applied stress, but it can not be higher than the speed of sound in the material under consideration.

3.1 Simple tension test

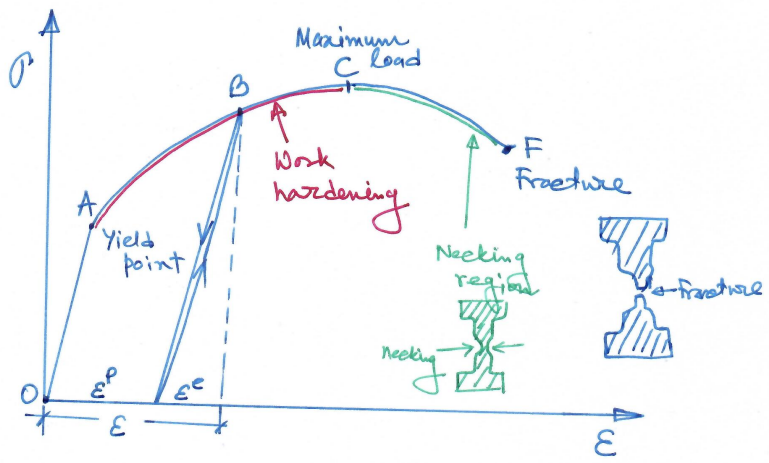


Figure 12: Typical stress-strain curve in simple tension test for a ductile material.

Consider a metallic rod is pulled in a testing machine at room temperature and the load applied on the test specimen is recorded against strain. Numerous experiments show typical load-elongation relationships, as indicated in Fig. 12. The initial region OA appears as a straight line. This is the linear elastic region. Point A is the elastic limit or the yield point. The elastic limit is characterized by the state of stress or strain which causes the first irreversible movements of dislocations. The plastic deformation starts after A.

Upon unloading at any stage (say at B) in the deformation, the strain is reduced along an elastic unloading line, and reloading retraces the unloading curve with relatively minor deviation and then produces further plastic deformation when approximately the previous maximum stress is exceeded. In such a zone, the dislocations started interact with each other. This interaction generally increases the stress requirement to cause plastic deformation. The increased resistance to slip deformation known as work hardening.

If one continues loading, the stress-strain slope gradually decreases and levels off at C. This point corresponds to the ultimate tensile strength, where the deformation becomes highly localized and the state of stress changes from uniaxial to biaxial at the surface, and triaxial in the interior of the specimen. There is also pronounced lateral contraction, known as necking. Thus the area of cross-section decreases and the stress response starts dropping until the specimen fractures at F.

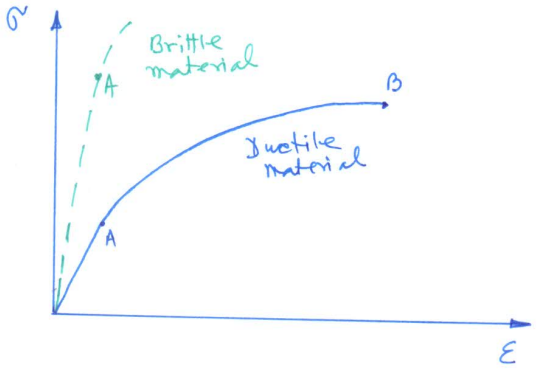


Figure 13: Brittle and ductile material response.

A brittle material breaks without significant plastic deformation (ex: cast iron, concrete, and some glass products). Brittle materials fracture by cleavage or separation and it breaks suddenly. Breaking is often accompanied by a snapping sound. Ductile materials show a prominent plastic zone. They can be bent, twist or role significantly before reaching the fracture point F.

APPENDIX

A Index notation

Consider the following sets of numbers:

$$a_{11}, \quad a_{22}, \quad a_{12}, \quad a_{21}$$

or

$$a^{11}, \quad a^{22}, \quad a^{12}, \quad a^{21}$$

or

$$a^1_1, \quad a^2_2, \quad a^1_2, \quad a^2_2$$

In the index notation, they are represented by: a_{ij}, a^{ij} and a^i_j respectively, where each of the indices (i, j) takes values 1 and 2.

A.1 Summation convention

Let n quantities be denoted by $x_1, x_2, x_3, ..., x_n$. Consider the expression:

$$\sum_i^n a_i x_i = a_1 x_1 + a_2 x_2 + ... + a_n x_n$$

This expression will be written $a_i x_i$ by introducing the summation convention of EINSTEIN. When an index appears twice in a term, a summation is implied. The range of the summation is known from the context. Note carefully that:

1. The convention does not apply to numerical indices. For instant $a_2 x_2$ stands for single term.
2. The repeated index may be replaced by any other index. For instance $a_i x_i = a_t x_t$. For this reason, the repeated index is called a dummy index.
3. If an index is not dummy, it is called free index. Thus in the expression $a_{ik} x_i$, k is the free index.

Some examples:

1. If $n = 3$, $a_k x_k = a_1 x_1 + a_2 x_2 + a_3 x_3$.
2. If ϕ is a function of $x_1, x_2, x_3, ..., x_n$, then $d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + ... + \frac{\partial \phi}{\partial x_n} dx_n = \frac{\partial \phi}{\partial x_i} dx_i$

3. Let $y_i = \alpha_{it} x_t$ and $z_i = \beta_{it} y_t$. We can rewrite $y_t = \alpha_{tk} x_k$ and now we can substitute back as $z_i = \beta_{it}(\alpha_{tk} x_k) = [\beta_{it}\alpha_{tk}]x_k = \gamma_{ik} x_k$. Here $\gamma_{ik} = \beta_{it}\alpha_{tk}$, where t is the dummy index and i, k are the free index.

4. $A^k_{mn} B^l_{kn} = (A^1_{mn} B^l_{1n}) + (A^2_{mn} B^l_{2n}) + ... + (A^N_{mn} B^l_{Nn}) = (A^1_{m1} B^l_{11} + A^1_{m2} B^l_{12} + ..) + (A^2_{m1} B^l_{21} + A^2_{m2} B^l_{22} + ...) + ...$

5. Consider the summation $(a_{1j}x_j)^2 + (a_{2j}x_j)^2 + ... + (a_{nj}x_j)^2$. This expression is written as $(a_{is}x_s)(a_{it}x_t)$ or $a_{is}a_{it}x_s x_t$.

A.2 Kronecker delta

This is defied as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{A-1}$$

Note that,

$$\begin{aligned} \delta_{1j}x_j &= \delta_{11}x_1 + \delta_{12}x_2 + ... + \delta_{1n}x_n = x_1 \\ \delta_{2j}x_j &= \delta_{21}x_1 + \delta_{22}x_2 + ... + \delta_{2n}x_n = x_2 \\ &\vdots \\ \delta_{ij}x_j &= \delta_{i1}x_1 + \delta_{i2}x_2 + ... + \delta_{in}x_n = x_i \end{aligned} \tag{A-2}$$

Thus if we multiply x_k by δ_{ik}, we simply replace the index k of x_k by i. Moreover,

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \tag{A-3}$$

A.3 Dot product

If the set $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ contains a set of orthonormal unit vectors, then

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij} \tag{A-4}$$

For a Cartesian system, you can think as

$$\hat{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{\boldsymbol{e}}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $\boldsymbol{a} = a_i\, \hat{\boldsymbol{e}}_i$ and $\boldsymbol{b} = b_j\, \hat{\boldsymbol{e}}_j$

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_i \hat{\boldsymbol{e}}_i) \cdot (b_j \hat{\boldsymbol{e}}_j) = a_i b_j (\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j) = a_i b_j \delta_{ij} = a_i b_i \tag{A-5}$$

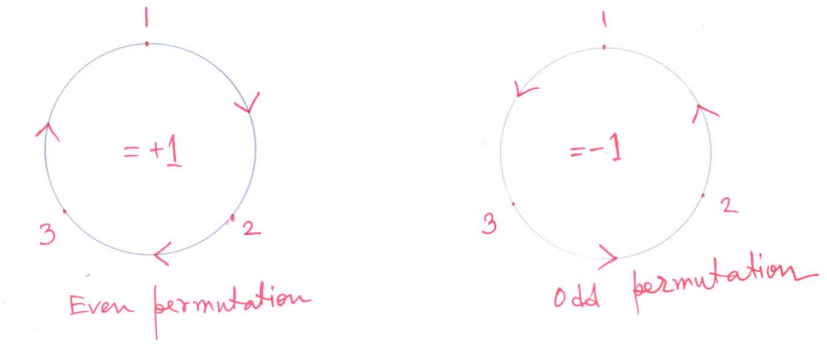


Figure 14: Even and odd permutations.

A.4 Permutation symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ form an even permutation} \\ -1 & \text{if } i,j,k \text{ form an odd permutation} \\ 0 & \text{if two or more indices are equal} \end{cases} \tag{A-6}$$

A.5 Cross product

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\hat{\boldsymbol{e}}_1 + (a_3b_1 - a_1b_3)\hat{\boldsymbol{e}}_2 + (a_1b_2 - a_2b_1)\hat{\boldsymbol{e}}_3 \\ &= \epsilon_{ijk} a_j b_k \, \hat{\boldsymbol{e}}_i \end{aligned} \tag{A-7}$$

Basically we represents $\boldsymbol{a} = a_j \, \hat{\boldsymbol{e}}_j$ and $\boldsymbol{b} = b_k \, \hat{\boldsymbol{e}}_k$. So we can also write the component form as:

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k \tag{A-8}$$

If we replace $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\boldsymbol{b} \times \boldsymbol{a} = \epsilon_{ikj} b_k a_j \, \hat{\boldsymbol{e}}_i = -\epsilon_{ijk} a_j b_k \, \hat{\boldsymbol{e}}_i = -\boldsymbol{a} \times \boldsymbol{b}$$

Let $\boldsymbol{c} = a_r \, \hat{\boldsymbol{e}}_r$, then

$$\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \epsilon_{ijk} c_r a_j b_k \, (\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_r) = \epsilon_{ijk} c_r a_j b_k \delta_{ir} = \epsilon_{ijk} c_i a_j b_k$$

A.6 Dyadic product

Let $\boldsymbol{a} = a_i\,\hat{\boldsymbol{e}}_i$ and $\boldsymbol{b} = b_j\,\hat{\boldsymbol{e}}_j$, then the dyadic product is defined as

$$\boldsymbol{a} \otimes \boldsymbol{b} = a_i b_j \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$$

In the matrix notation

$$(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

This operation constructs a second-order tensor from two vectors. In general we have $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$.

A.7 Matrices

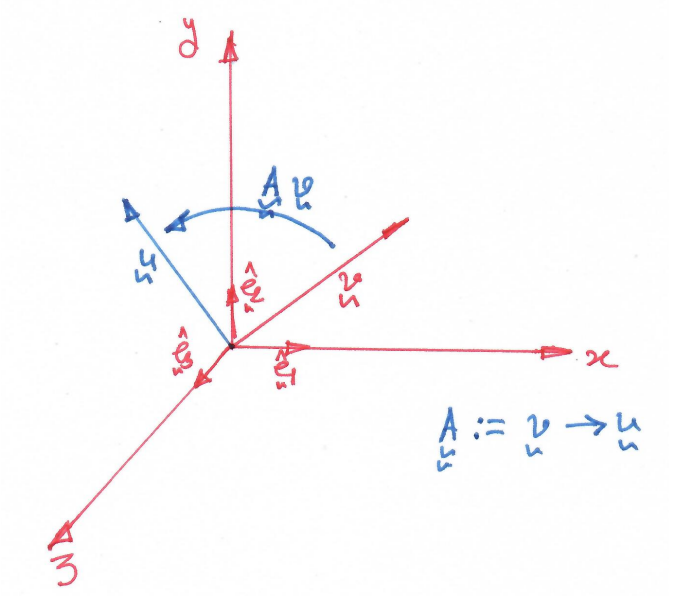


Figure 15: Second order tensor $\boldsymbol{A}$ as a mapping of a vector $\boldsymbol{v}$ to a vector $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}$.

We can think of a seconder tensor as a mapping of a vector to a vector (see Fig. 15) and write $\boldsymbol{A} : \boldsymbol{v} \to \boldsymbol{u}$, i.e., $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{u}$. Consider a particular situation where $\boldsymbol{v} = \hat{\boldsymbol{e}}_1$. Then the transformed vector $\boldsymbol{A}\hat{\boldsymbol{e}}_1$ could be again written in terms of the unit vectors. Say,

$$\boldsymbol{A}\hat{\boldsymbol{e}}_1 = \alpha\,\hat{\boldsymbol{e}}_1 + \beta\,\hat{\boldsymbol{e}}_2 + \gamma\,\hat{\boldsymbol{e}}_3.$$

Here α, β, γ are some real numbers, i.e., components of $\boldsymbol{A}\hat{\boldsymbol{e}}_1$. We can rename them as A_{11}, A_{21} and A_{31} and can write the above equation as

$$\boldsymbol{A}\hat{\boldsymbol{e}}_1 = A_{11}\hat{\boldsymbol{e}}_1 + A_{21}\hat{\boldsymbol{e}}_2 + A_{31}\hat{\boldsymbol{e}}_3.$$

Similarly, we can write

$$\begin{aligned} \boldsymbol{A}\hat{\boldsymbol{e}}_2 &= A_{12}\hat{\boldsymbol{e}}_1 + A_{22}\hat{\boldsymbol{e}}_2 + A_{32}\hat{\boldsymbol{e}}_3, \\ \boldsymbol{A}\hat{\boldsymbol{e}}_3 &= A_{13}\hat{\boldsymbol{e}}_1 + A_{23}\hat{\boldsymbol{e}}_2 + A_{33}\hat{\boldsymbol{e}}_3. \end{aligned}$$

We write $\boldsymbol{A}\hat{\boldsymbol{e}}_j = A_{ij}\hat{\boldsymbol{e}}_i$. Components of a second order tensor $\boldsymbol{A}$ could be represented by a matrix A_{ij} with respect to a coordinate system (in our case, it is cartesian). Let for tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, the components are A_{ij} and B_{kl}. Then

$$\begin{aligned} \boldsymbol{A} &= A_{ij} \\ \boldsymbol{A}^T &= A_{ji} \\ tr(\boldsymbol{A}) &= A_{ii} = A_{11} + A_{22} + A_{33} \\ (\boldsymbol{AB})_{ik} &= A_{ij}B_{jk} \\ (\boldsymbol{AB}^T)_{ik} &= A_{ij}B_{kj} \\ (\boldsymbol{A}^T\boldsymbol{B})_{ik} &= A_{ji}B_{jk} \end{aligned}$$

Note that:

$$tr(\boldsymbol{a} \otimes \boldsymbol{b}) = a_1b_1 + a_2b_2 + a_3b_3 = \boldsymbol{a} \cdot \boldsymbol{b}$$

Symmetric matrix: $\boldsymbol{A} = \boldsymbol{A}^T$ or $A_{ij} = A_{ji}$
Skew symmetric matrix: $\boldsymbol{A} = -\boldsymbol{A}^T$ or $A_{ij} = -A_{ji}$

Any matrix $\boldsymbol{B}$ could be decomposed as a symmetric and skew-symmetric part as:

$$\begin{aligned} \text{Sym}\,\boldsymbol{B} &= \frac{1}{2}\Big(\boldsymbol{B} + \boldsymbol{B}^T\Big), \Rightarrow [\text{Sym}\,B]_{ij} = \frac{1}{2}\Big(B_{ij} + B_{ji}\Big) \\ \text{Skw}\,\boldsymbol{B} &= \frac{1}{2}\Big(\boldsymbol{B} - \boldsymbol{B}^T\Big), \Rightarrow [\text{Skw}\,B]_{ij} = \frac{1}{2}\Big(B_{ij} - B_{ji}\Big) \end{aligned}$$

A.7.1 skw-dual relationship

For a skew symmetric matrix

$$[\boldsymbol{W}] = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix},$$

its dual vector or axial vector is given by,

$$\text{dual } [\boldsymbol{W}] = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}.$$

Note that for an arbitrary vector $\boldsymbol{b}$,

$$\boldsymbol{W}\boldsymbol{x} = \text{dual } [\boldsymbol{W}] \times \boldsymbol{x}, \rightarrow W_{ij}x_j = \epsilon_{ijk}\text{dual } [\boldsymbol{W}]_j \, x_k$$

A.8 Differentiation notation

Let

$$\begin{aligned} a &= a(x_1, x_2, x_3) \\ a_i &= a_i(x_1, x_2, x_3) \\ a_{ij} &= a_{ij}(x_1, x_2, x_3) \end{aligned}$$

Then

$$\begin{aligned} a_{,\,i} &= \frac{\partial a}{\partial x_i} = \partial_i a \\ a_{i,\,j} &= \frac{\partial a_i}{\partial x_j} = \partial_j a_i \\ a_{ij,\,k} &= \frac{\partial a_{ij}}{\partial x_k} = \partial_k a_{ij} \end{aligned} \tag{A-9}$$

A.9 Coordinate transformations

Let a vector $\boldsymbol{v}$ is expressed with respect to Oxy system as:

$$\boldsymbol{v} = v_1\,\hat{\boldsymbol{e}}_1 + v_2\,\hat{\boldsymbol{e}}_2 \tag{A-10}$$

where, v_1, v_2 are the components and $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ are the unit vectors . Let the components with respect to the $Ox'y'$ system are v_1', v_2', where the unit vectors are $\hat{\boldsymbol{e}}_1'$ and $\hat{\boldsymbol{e}}_2'$. Since the vector does not change (only the components change), we can write

$$\begin{aligned} \boldsymbol{v} &= v_1\,\hat{\boldsymbol{e}}_1 + v_2\,\hat{\boldsymbol{e}}_2 \\ \boldsymbol{v} &= v_1'\,\hat{\boldsymbol{e}}_1' + v_2'\,\hat{\boldsymbol{e}}_2' \end{aligned}$$

Now,

$$\begin{aligned} \hat{\boldsymbol{e}}_1 &= \cos\theta\,\hat{\boldsymbol{e}}_1' - \sin\theta\,\hat{\boldsymbol{e}}_2' \\ \hat{\boldsymbol{e}}_2 &= \sin\theta\,\hat{\boldsymbol{e}}_1' + \cos\theta\,\hat{\boldsymbol{e}}_2' \end{aligned}$$

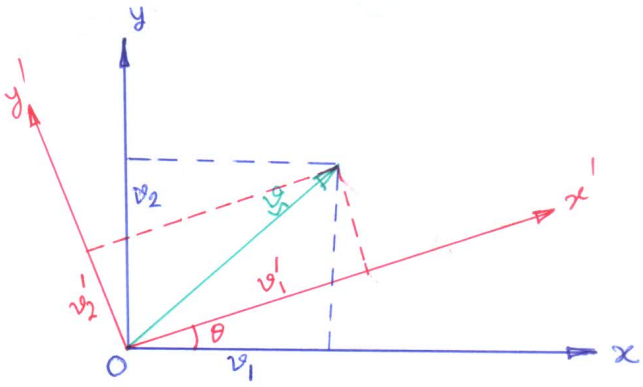


Figure 16: Coordinate transformation.

Substituting back we get,

$$\begin{aligned} \boldsymbol{v} &= v_1\left(\cos\theta\,\hat{\boldsymbol{e}}_1' - \sin\theta\,\hat{\boldsymbol{e}}_2'\right) + v_2\left(\sin\theta\,\hat{\boldsymbol{e}}_1' + \cos\theta\,\hat{\boldsymbol{e}}_2'\right) \\ &= \left(v_1\cos\theta + v_2\sin\theta\right)\hat{\boldsymbol{e}}_1' + \left(-v_1\sin\theta + v_2\cos\theta\right)\hat{\boldsymbol{e}}_2' \\ \boldsymbol{v} &= v_1'\,\hat{\boldsymbol{e}}_1' + v_2'\,\hat{\boldsymbol{e}}_2' \end{aligned}$$

This implies:

$$\begin{aligned} v_1' &= v_1\cos\theta + v_2\sin\theta \\ v_2' &= -v_1\sin\theta + v_2\cos\theta \end{aligned}$$

In the matrix notation:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

and in the index notation

$$v_i' = Q_{ij}v_j \tag{A-11}$$

where, the elements of the transformation matrix $Q_{ij} = \cos\angle(x_i', x_j)$ denotes the cosine of the angle between the x_i'-axis and x_j-axis. Similarly,

$$v_i = Q_{ji}v_j' \tag{A-12}$$

Lets start from

$$\begin{aligned} v_i &= Q_{mi}v_m' \\ &= Q_{mi}(Q_{mj}v_j) \\ &= (Q_{mi}Q_{mj})v_j \\ \implies \delta_{ij}v_j &= (Q_{mi}Q_{mj})v_j \\ \implies (\delta_{ij} - Q_{mi}Q_{mj})v_j &= 0 \end{aligned}$$

The above equation is true for all vectors v_j. Then

$$Q_{mi}Q_{mj} = \delta_{ij} \quad \text{or in direct notation} \quad \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I} \tag{A-13}$$

Here $\boldsymbol{I}$ is the identity tensor. In a similar way, starting from $v_i' = Q_{ij}v_j$, we can show that

$$Q_{im}Q_{jm} = \delta_{ij} \qquad \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I} \tag{A-14}$$

Let us now consider a second order tensor in the $x-y$ frame $\boldsymbol{A} = A_{ij}$ and when it operates on a vector $\boldsymbol{v}$ gives a vector $\boldsymbol{u}$. That is

$$u_i = A_{ij}v_j \qquad \boldsymbol{u} = \boldsymbol{A}\boldsymbol{v} \tag{A-15}$$

In the $x'-y'$ frame the above relation becomes

$$\begin{aligned} u_i' &= A_{ij}'v_j' \\ (Q_{im}u_m) &= A_{ij}'(Q_{jk}v_k) \\ Q_{im}(A_{mk}v_k) &= A_{ij}'(Q_{jk}v_k) \\ (Q_{im}A_{mk})v_k &= (A_{ij}'Q_{jk})v_k \end{aligned} \tag{A-16}$$

This implies

$$\begin{aligned} Q_{im}A_{mk} &= A_{ij}'Q_{jk} \\ Q_{im}A_{mk}Q_{pk} &= A_{ij}'(Q_{jk}Q_{pk}) \\ Q_{im}A_{mk}Q_{pk} &= A_{ij}'\delta_{jp} \\ Q_{im}A_{mk}Q_{pk} &= A_{ip}' \\ Q_{im}A_{mk}Q_{jk} &= A_{ij}' \end{aligned}$$

or

$$A_{ij}' = Q_{ip}Q_{jq}A_{pq}. \tag{A-17}$$

In matrix (tensor) notation:

$$\boldsymbol{A}' = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T. \tag{A-18}$$