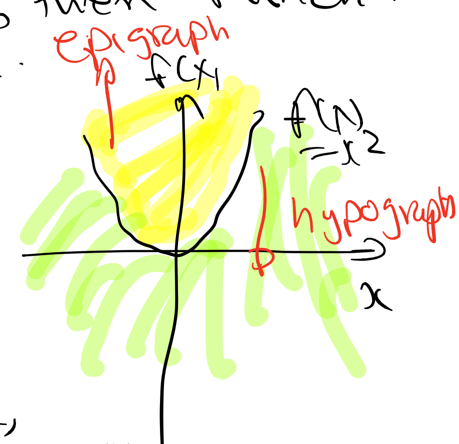
Lemma: Epigraph of a convex function over convex set 'C' is a convex set (& vice-versa)

If the epigraph of a function 'f' over a set C is convex, then the function is convex

iff result

(1) f is convex function over convex set C $\Rightarrow$ epigraph is convex set

(2) if epigraph of function 'f' over set C is convex, then function 'f' is convex.



Graph: For a univariate function it is subset of $\mathbb{R}^2$

$$\begin{pmatrix} x \\ f(x) \end{pmatrix}$$

For a function $f: \mathbb{R}^n \to \mathbb{R}$, its graph is a subset of $\mathbb{R}^{n+1}$ i.e., set of all tuples $\begin{pmatrix} \vec{x} \\ f(x) \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f(x) \end{pmatrix} \subset \mathbb{R}^{n+1}$$

Graph of (f) divides the R^{n+1} space into parts

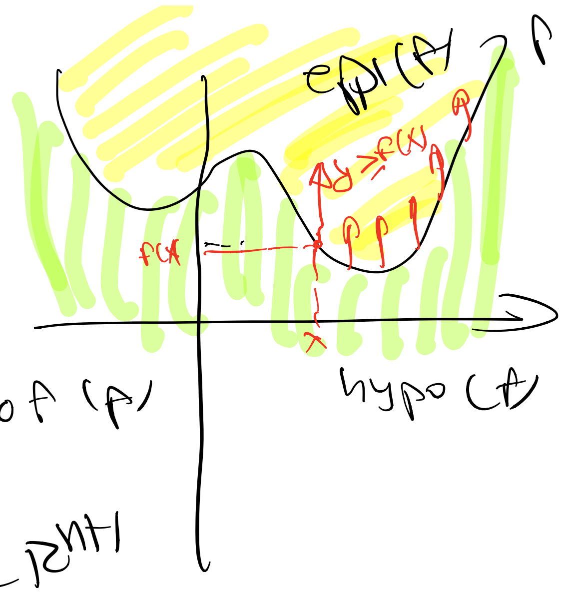
$\begin{pmatrix} x \\ y \end{pmatrix}$ s.t $y \geq f(x)$

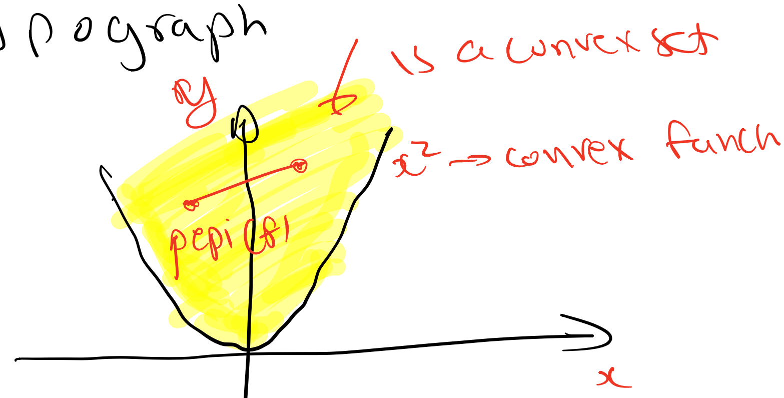
It is called epigraph of (f)

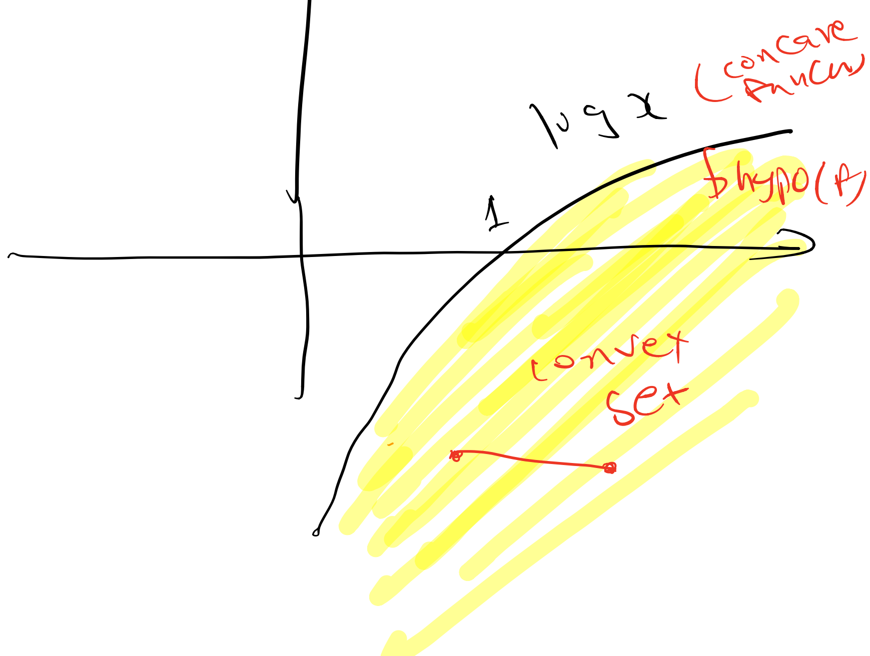
Other part

$\begin{pmatrix} x \\ y \end{pmatrix}$ $y \leq f(x)$ $\subset R^{n+1}$

is called hypograph







Proof:

part 1; Let 'f' be a convex function over a convex set (C), T.P.T.

epi (f) is a convex set

part 2: Given that epi(f) $\subset \mathbb{R}^{n+1}$ is a convex set, T.P.T 'f' is convex function.

Part 1: Given that 'f' is a convex function over convex set (C)

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in epi(f) \Rightarrow x_1 \in C$

$y_1 \geq f(x_1)$

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in epi(f)$ $x_2 \in C$

$y_2 \geq f(x_2)$.

T.S.T. $\left[\lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] \in epi(f)$

$\Rightarrow \lambda \underline{y_1} + (1-\lambda) y_2 \geq f(\lambda x_1 + \overline{1-\lambda}\, x_2)$

$\lambda y_1 + (1-\lambda) y_2 \geq \lambda f(x_1) + (1-\lambda) f(x_2)$

$\geq f(\lambda x_1 + \overline{1-\lambda}\, x_2)$

$\lambda \times y_1 \geq \lambda \times f(x_1) \qquad \lambda > 0$

$(1-\lambda) \times y_2 \geq (1-\lambda) \times f(x_2)$

Therefore epi (f) of a convex function is a convex ~~set~~.

Let f be convex.

$$\lambda \begin{bmatrix} x_1 \\ \underset{y_1}{\cancel{f(x_1)}} \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2 \\ \underset{y_2}{\cancel{f(x_2)}} \end{bmatrix} \underset{\geq}{=} \begin{bmatrix} \lambda x_1 + (1-\lambda) x_2 \\ (\lambda f(x_1) + (1-\lambda) f(x_2) \end{bmatrix}$$

$$\geq \begin{bmatrix} \lambda x_1 + (1-\lambda) x_2 \\ f(\lambda x_1 + \overline{1-\lambda}\, x_2) \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_1 \\ f(x_1) \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2 \\ f(x_2) \end{bmatrix} \geq \begin{bmatrix} \lambda x_1 + \overline{1-\lambda}\, x_2 \\ \lambda f(x_1) + (1-\lambda) f(x_2) \end{bmatrix}$$

Given that epi(f) is a convex set,
T.P.T f is a convex function

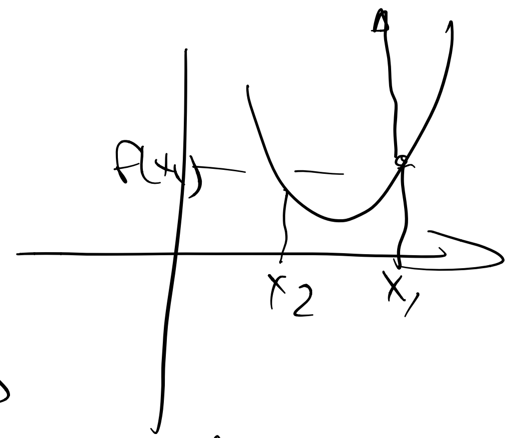
$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in epi(f)$

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in epi(f)$

$\lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in epi(f)$ $\because$ epi(f) is a convex set.

$\lambda y_1 + (1-\lambda) y_2 \geq f(\lambda x_1 + (1-\lambda) x_2)$

$y_1 = f(x_1)$
$y_2 = f(x_2)$



$\lambda f(x_1) + (1-\lambda) f(x_2)$

$\geq f(\lambda x_1 + \overline{1-\lambda} x_2)$

$\Rightarrow$ f is a convex

$\lambda x_1 + (1-\lambda) x_2 \in C$ trivially
because epi(f) is convex set