How to test if symmetric matrix is s.p.d.?

1) $n \times n$ Matrix A (real) is s.p.d. iff all its eigen values are strictly greater than zero

2) It is positive semi definite if $\lambda_i \geq 0, i=1,\dots n$

If A s.p.d. $\Longleftrightarrow$ it will be invertible.

$$A = V \Lambda V^T$$

$V \rightarrow$ orthogonal matrix of eigen vectors

$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

If all $\lambda_i > 0$ $\Lambda^{-1} = \begin{pmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_n \end{pmatrix}$

$$A^{-1} = V \Lambda^{-1} V^T$$

s.p.d.

$A = \begin{bmatrix} 1 & \\ & -2 \end{bmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $e_2^T A e_2 = -2 < 0$

$e_1^T A e_1 = 1 > 0$

Indefinite. $A^{-1} = \begin{bmatrix} 1 & \\ & -1/2 \end{bmatrix}$

Positive semidefinite matrices are not invertible.

$A v = 0 \cdot v$

many vectors mapping to zero vector no more one-one and hence does not exist.

100×100 100 eigen values, this is enormous amount is involved

Cholesky Decomposition.

$\Rightarrow$ If it succeeds, the matrix is s.p.d.

Cholesky Decomposition

$$A = R^T R$$ where R is a upper triangular matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} \cdots & r_{2n} \\ | & | & r_{33} & \\ | & 0 & 0 & \\ | & | & | & r_{n-1,n-1} \; r_{n-1,n} \\ 0 & 0 & 0 & 0 \; r_{nn} \end{bmatrix}$$ $r_{ii} > 0$ $i = 1, \ldots n$.

$$\underline{\underline{A = LU}}$$

$$x^T A x = x^T R^T R x = \|Rx\|_2^2 > 0$$
$\forall x \neq 0$

provided that R is invertible matrix.
or all its diagonals are not equal to zero

$$\det(R) = r_{11} \times r_{22} \cdots \times r_{nn}.$$

$$Rz = 0,$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} r_{11} & \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} r_{11}^2 & r_{11} r_{12} \\ r_{11} r_{12} & r_{12}^2 + r_{22}^2 \end{bmatrix}$$

$r_{11} = \sqrt{a_{11}} \rightarrow (1)$ $a_{11} > 0$ (in general $a_{ii} > 0$)

Anyway a_{ii}'s > 0 $e_i = (0, \cdots 1, \cdots 0)^T$

$$e_i^T A e_i = a_{ii} > 0$$

$$r_{12} = \frac{a_{12}}{r_{11}} \rightarrow (2)$$

$$a_{22} = r_{12}^2 + r_{22}^2$$

$$r_{22}^2 = a_{22} - r_{12}^2 \qquad (>0)$$

$$= a_{22} - \frac{a_{12}^2}{a_{11}}$$

$$= \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{\det(A)}{a_{11}} > 0$$

$$\det(A) > 0$$

$n \times n$

extract many principal submatrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$x = e_1 \qquad e_1^T A e_1 = a_{11} = [a_{11}]$ $\quad 1 \times 1$

$x = e_2 \qquad e_2^T A e_2 = a_{22} \quad [a_{22}]$ $\quad 1 \times 1$

$x = e_3 \qquad e_3^T A e_3 = a_{33} \quad [a_{33}]$ $\quad 1 \times 1.$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$$x^T A x = (x_1 \ x_2) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3$$

$$x^T A x = [x_1 \ x_2 \ 0] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \ x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

$$x = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \qquad x^T A x = (x_2 \ x_3) \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{32} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} > 0$$

All principal submatrices of a s.p.d. matrix are also sym. positive definite.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} r_{11} & & \\ r_{12} & r_{22} & \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad l_1 \quad l_2 \quad l_3 \qquad\qquad u_1 \quad u_2 \quad u_3$$

$$= \begin{bmatrix} r_{11}^2 & r_{11} r_{12} & r_{11} r_{13} \\ r_{12} r_{11} & & \\ r_{11} r_{13} & & \end{bmatrix}$$

$$r_{11}^2 = a_{11} \qquad r_{11} = \sqrt{a_{11}}; \quad r_{12} = \frac{a_{12}}{r_{11}}; \quad r_{13} = \frac{a_{13}}{r_{11}}.$$

$$A = l_1 u_1^T + l_2 u_2^T + l_3 u_3^T$$

$$A - l_1 u_1^T = l_2 u_2^T + l_3 u_3^T.$$

$$A - l_1 u_1^T = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ | & & & \\ | & & \boxed{\boxed{(n-2) \times (n-2)}} & \\ 0 & & & \end{bmatrix}$$

$$(n-1) \times (n-1)$$

$$A - l_1 u_1^T - l_2 u_2^T$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} r_{11} & & \\ r_{12} & r_{22} & \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

$$a_{22} = r_{12}^2 + r_{22}^2$$

$$r_{22} = \sqrt{r_{12}^2 - a_{22}} \qquad \text{if} > 0$$

$$a_{23} = r_{12} r_{13} + r_{22} r_{23}$$

$$r_{23} = \frac{a_{23} - r_{12} r_{13}}{r_{22}}$$

$$a_{33} = r_{13}^2 + r_{23}^2 + r_{33}^2$$

$$r_{33} = \sqrt{a_{33} - r_{13}^2 - r_{23}^2}$$

Ex:

$$A = \begin{bmatrix} 100 & -5 & 10 \\ -5 & 100 & 10 \\ 10 & 10 & 100 \end{bmatrix}$$

check if A is positive definite

$$\begin{bmatrix} 100 & -5 & 10 \\ -5 & 100 & 10 \\ 10 & 10 & 100 \end{bmatrix} = \begin{bmatrix} r_{11} & & \\ r_{12} & r_{22} & \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

$$100 = r_{11}^2 \Rightarrow r_{11} = 10$$

$$-5 = r_{11} r_{12} \Rightarrow r_{12} = -5/10 = -1/2$$

$$10 = r_{11} r_{13} \Rightarrow r_{13} = \frac{10}{10} = 1$$

$$\begin{bmatrix} 100 & -5 & 10 \\ -5 & 100 & 10 \\ 10 & 10 & 100 \end{bmatrix} = \begin{bmatrix} 10 & & \\ -\frac{1}{2} & r_{22} & \\ 1 & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} 10 & -1/2 & 1 \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

$$100 = \left(\frac{1}{4}\right) + r_{22}^2$$

$$r_{22}^2 = 100 - \frac{1}{4} = 99.75$$

$$r_{22} = \sqrt{99.75} > 0$$

$$10 = \left(-\frac{1}{2}\right)1 + \sqrt{99.75}\, r_{23}$$

$$r_{23} = \frac{10.5}{\sqrt{99.75}} \approx 1.0513$$

$$100 = 1 + r_{23}^2 + r_{33}^2$$

$$r_{33} = \sqrt{99 - 1.0513^2} > 0$$

$$r_{11} = 10 > 0$$

$$r_{22} = \sqrt{99.75} > 0$$

$$r_{33} = \sqrt{99 - 1.0513^2} > 0$$

$$A = R^T R$$

succeeded and the matrix is s.p.d.