Assignment - 2 Vector norms.

Ref: Golub & Van Loan, Matrix Computations, Chap 2, Matrix Analysis, Sec. 2.2.}

1 Let $x \in \mathbb{R}^n$, Verify that $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are vector norms.

2. Show that if $x \in \mathbb{R}^n$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$

3. Show that in $\mathbb{R}^n$, sequence $x^{(k)} \to x$ iff $x_k^{(k)} \to x_k$ for $k = 1:n$.

4. Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m\times n}$. Show that if $\operatorname{rank}(A) = n$, then $\|x\|_A = \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

5. Let x and y be in $\mathbb{R}^n$ and define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(\alpha) = \|x - \alpha y\|_2$. Show that ψ is minimized when

$$\alpha = \frac{x^T y}{y^T y}.$$

6. (a) Verify that $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ is a vector norm on $\mathbb{C}^n$

(b) Show that if $x \in \mathbb{C}^n$ then

$$\|x\|_p \le c\,(\|Re(x)\|_p + \|Im(x)\|_p)$$

(c) Find a constant c_n such that

$$c_n\,(\|Re(x)\|_2 + \|Im(x)\|_2) \le \|x\|_2$$

for all $x \in \mathbb{R}^n$.

7. Prove for disprove

$$v \in \mathbb{R}^n \Rightarrow \|v\|_1 \|v\|_\infty \le \frac{1+\sqrt{n}}{2} \|v\|_2^2$$