CS 747, Autumn 2020: Week 2, Lecture 1

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Autumn 2020

Multi-armed Bandits

1. Achieving sub-linear regret
2. A lower bound on regret
3. UCB, KL-UCB algorithms
4. Thompson Sampling algorithm
5. Summary and outlook

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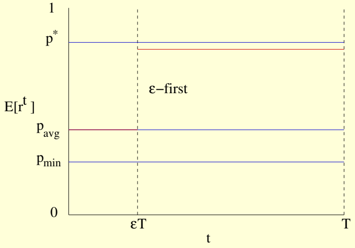

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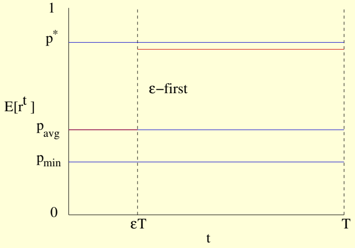

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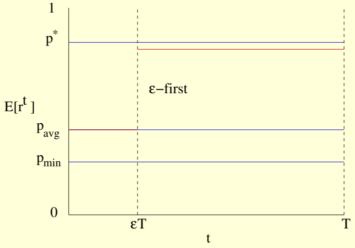

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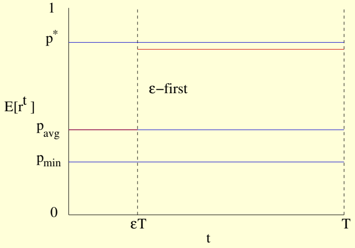

- What would happen if we ran for a horizon of $2T$ instead of T? The exploratory phase would last $2\epsilon T$ steps!
- Mathematically:

$$R_T = Tp^\star - \sum_{t=0}^{T-1} \mathbb{E}[r^t] = Tp^\star - \sum_{t=0}^{\epsilon T-1} \mathbb{E}[r^t] - \sum_{t=\epsilon T}^{T-1} \mathbb{E}[r^t]$$

$$= Tp^\star - \epsilon T p_{\text{avg}} - \sum_{t=\epsilon T}^{T-1} \mathbb{E}[r^t] \geq Tp^\star - \epsilon T p_{\text{avg}} - (T - \epsilon T)p^\star$$

$$= \epsilon(p^\star - p_{\text{avg}})T = \Omega(T).$$

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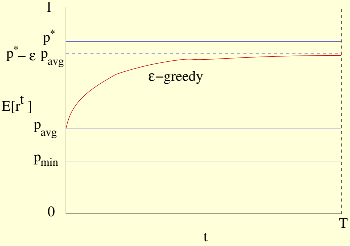

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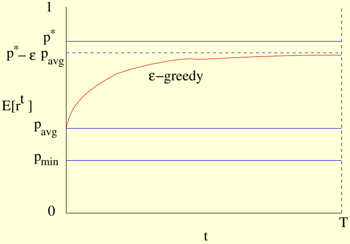

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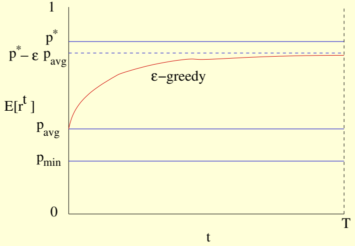

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- **Result.** An algorithm L achieves sub-linear regret on all instances $I \in \bar{\mathcal{I}}$ if and only if it satisfies C1 and C2 on all $I \in \bar{\mathcal{I}}$.

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  Question: Can we do even better than these algorithms?

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- Paraphrasing Lai and Robbins (1985; see Theorem 2).

> Let L be an algorithm such that <u>for every</u> bandit instance $I \in \bar{\mathcal{I}}$ and <u>for every</u> $\alpha > 0$, as $T \to \infty$:
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> $$R_T(L, I) = o(T^\alpha).$$

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Let L be an algorithm such that for every bandit instance $I \in \bar{\mathcal{I}}$ and for every $\alpha > 0$, as $T \to \infty$:

$$R_T(L, I) = o(T^\alpha).$$

Then, for every bandit instance $I \in \bar{\mathcal{I}}$, as $T \to \infty$:

$$\frac{R_T(L, I)}{\ln(T)} \geq \sum_{a: p_a(I) \neq p^*(I)} \frac{p^*(I) - p_a(I)}{KL(p_a(I), p^*(I))},$$

where for $x, y \in [0, 1)$, $KL(x, y) \stackrel{\text{def}}{=} x \ln \frac{x}{y} + (1 - x) \ln \frac{1-x}{1-y}$, with $0 \ln 0 \stackrel{\text{def}}{=} 0$.

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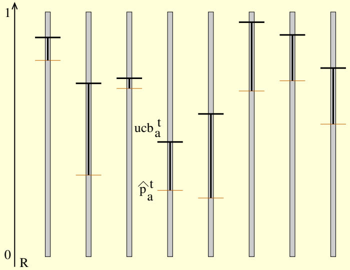
Upper Confidence Bounds

- UCB (Auer et al., 2002)
  - At time t, for every arm a, define $\text{ucb}_a^t = \hat{p}_a^t + \sqrt{\frac{2\ln(t)}{u_a^t}}$.
  - $\hat{p}_a^t$ is the empirical mean of rewards from arm a.
  - u_a^t the number of times a has been sampled at time t.



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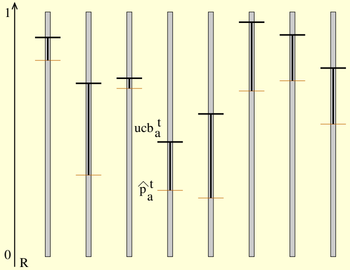

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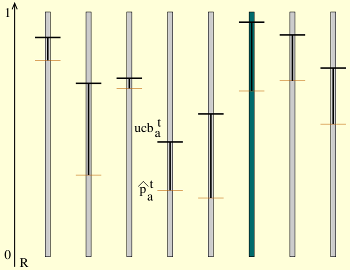

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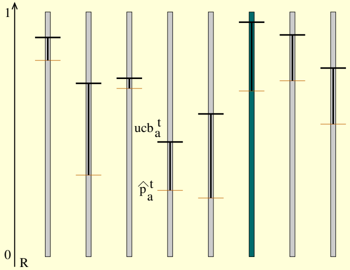

  - Sample an arm a for which ucb_a^t is maximal.
- Achieves regret of $O(\log(T))$: optimal dependence on T.

KL-UCB Algorithm (Garivier and Cappé, 2011)

- Identical to UCB algorithm on previous slide, except for a different definition of the upper confidence bound.

  $\text{ucb-kl}_a^t = \max\{q \in [\hat{p}_a^t, 1] \text{ such that } u_a^t KL(\hat{p}_a^t, q) \leq \ln(t) + c\ln(\ln(t))\}$, where $c \geq 3$.

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- Observe that $KL(\hat{p}_a^t, q)$ monotonically increases with q, and
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  Regret of KL-UCB asymptotically matches Lai and Robbins' lower bound!

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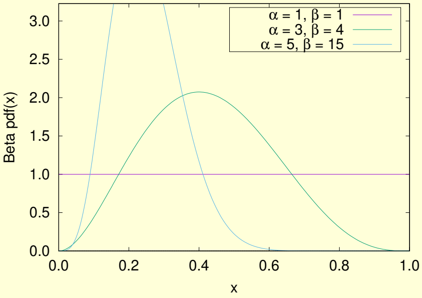
Before Moving on ... The Beta Distribution

- Beta(α, β) defined on $[0, 1]$.

  Two parameters: α and β.

$$\text{Mean} = \frac{\alpha}{\alpha + \beta}; \quad \text{Variance} = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$



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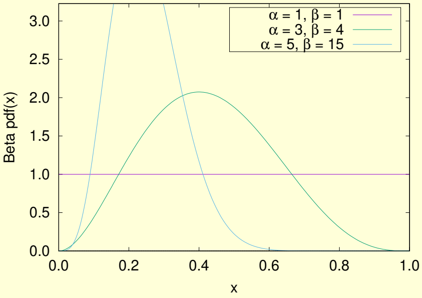

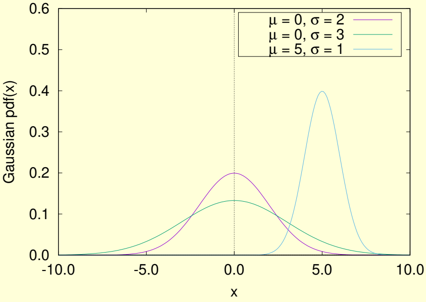

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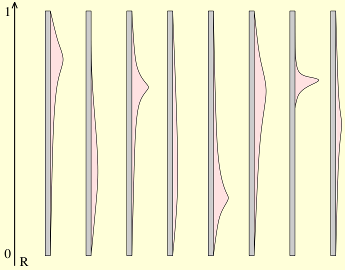
  - At time t, let arm a have s_a^t successes (ones/heads) and f_a^t failures (zeroes/tails).
  - $Beta(s_a^t + 1, f_a^t + 1)$ represents a "belief" about the true mean of arm a.
  - Mean = $\frac{s_a^t+1}{s_a^t+f_a^t+2}$; variance = $\frac{(s_a^t+1)(f_a^t+1)}{(s_a^t+f_a^t+2)^2(s_a^t+f_a^t+3)}$.



Thompson Sampling

- Thompson (Thompson, 1933)
  - At time t, let arm a have s_a^t successes (ones/heads) and f_a^t failures (zeroes/tails).
  - $Beta(s_a^t + 1, f_a^t + 1)$ represents a "belief" about the true mean of arm a.
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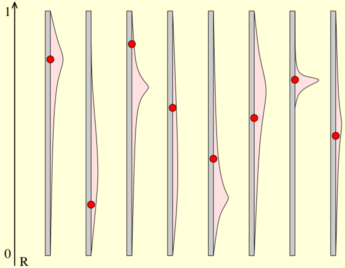

  - Computational step: For every arm a, draw a sample $x_a^t \sim Beta(s_a^t + 1, f_a^t + 1)$.
  - Sampling step: Sample an arm a for which x_a^t is maximal.

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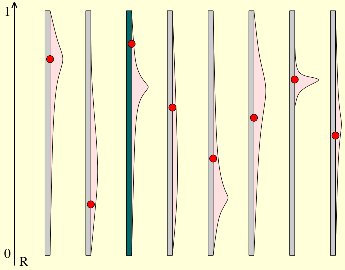

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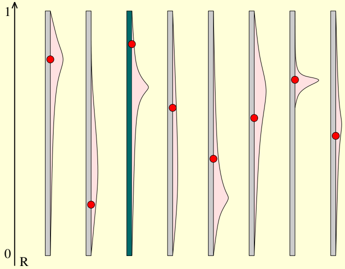

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- Achieves optimal regret (Kaufmann et al., 2012); is excellent in practice (Chapelle and Li, 2011).

Multi-armed Bandits

1. Achieving sub-linear regret
2. A lower bound on regret
3. UCB, KL-UCB algorithms
4. Thompson Sampling algorithm
5. Summary and outlook

Summary

- We desire low, sub-linear regret on all bandit instances.
- Possible if and only if algorithm satisfies GLIE conditions.
- If an algorithm gives sub-polynomial regret on all instances, it must give super-logarithmic regret on all instances (Lai and Robbins, 1985).

- UCB algorithm achieves logarithmic dependence on T.
- KL-UCB algorithm additionally improves the accompanying constant, thereby matching the lower bound (asymptotically).
- Thompson Sampling, a qualitatively different randomised algorithm, also matches regret lower bound.
- UCB, KL-UCB, Thompson Sampling all examples of optimism in the face of uncertainty principle.

- Next week: concentration inequalities, analysis of UCB, KL-UCB, Thompson Sampling, other bandit problem formulations.