Ref: Vector Norms sec, 2.2
Gene Golub & Van Loan Matrix Computations

Recap:

1) $\cos\theta = \dfrac{x^T y}{\|x\|_2 \|y\|_2}$ $x \in R^n$, $y \in R^n$, $x \neq 0$, $y \neq 0$

$\|x\|_2 = (x_1^2 + x_2^2 - + x_n)^{1/2}$ $\angle\theta$ x & y vectors

2) orthogonal vectors

$\theta = 90^\circ \Rightarrow x^T y = 0$

Topics for today

1] Prove triangular inequality for $\|\cdot\|_2$

2] Equivalence of norms,

$$c_1 \|x\|_\alpha \le \|x\|_\beta \le c_2 \|x\|_\alpha$$

$c_1$ & $c_2$ are positive constants

$$\boxed{\begin{aligned} \|x\|_2 &\le \|x\|_1 \le \sqrt{n}\,\|x\|_2 \\ \|x\|_\infty &\le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty \\ \|x\|_\infty &\le \|x\|_1 \le n\,\|x\|_\infty \end{aligned}}$$

$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$ $n = 100$ $R^{100}$

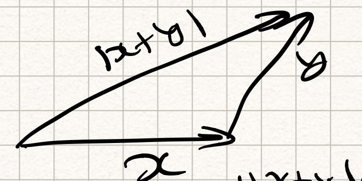
$$\|x\|_2 \le \|x\|_1 \le 10\,\|x\|_2$$

$$\|x\|_2 = 1$$

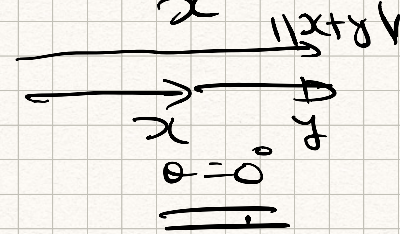
$$\|x\|_1 = 10 \times 1$$

Δ-ular inequality for $\|x\|_2$

$$\|x+y\|_2 \le \|x\|_2 + \|y\|_2$$




$$\|x+y\|_2 = \|x\|_2 + \|y\|_2$$




$$\|x+y\|_2^2 = (x+y)^T(x+y)$$

$$= x^Tx + y^Ty + 2x^Ty$$

$$\max_\theta \|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 \cos\theta$$

$$\cos\theta \in [-1, +1]$$

max RHS

$$\cos\theta = 1$$

$$\theta = 0°$$

$$\max \|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$

$$= (\|x\|_2 + \|y\|_2)^2$$

$$\max \|x+y\|_2 = \|x\|_2 + \|y\|_2$$

in other words

$$\|x+y\|_2 \le \|x\|_2 + \|y\|_2$$

QED

#HW Prove $\Delta^{ular}$ inequality for $\|\cdot\|_1$ & $\|\cdot\|_\infty$.

Proving equivalence relationship

$$\|x\|_2 \le \underbrace{\|x\|_1 \le \sqrt{n}\,\|x\|_2}_{\mathbb{P}} \qquad x \in \mathbb{R}^n.$$

$$P_{max}\begin{cases} \max\ x_1+x_2+x_3 \cdots + x_n \\ \|x\|_2 = x_1^2+x_2^2 \cdots + x_n^2 = 1 \\ x_1 \ge 0,\ x_2 \ge 0,\ \cdots,\ x_n \ge 0 \end{cases}$$

$$P_{min}. \quad \min\ x_1+x_2+x_3 \cdots + x_n$$
$$x_1^2+x_2^2 \cdots + x_n^2 = 1$$
$$x_1 \ge 0;\ x_2 \ge 0,\ \cdots x_n \ge 0$$

$$\max\ x_1+x_2+x_3 \cdots + x_n$$
$$x_1^2+x_2^2 \cdots + x_n^2 = 1$$

~~redundant~~ relaxed $\rightarrow$ $\underline{x_1 \ge 0,\ x_2 \ge 0,\ \cdots x_n \ge 0}$

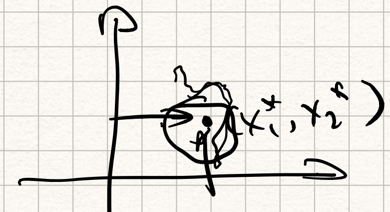
$$x_1 > 0,\ x_2 > 0,\ \cdots x_n > 0.$$

$$\max\ x_1+x_2+x_3 \cdots + x_n$$
$$x_1^2+x_2^2+\cdots+x_n^2 = 1$$

$\underline{\max f(x)}$

$\nabla_x f(x) = 0$; [Fermat's theorem]

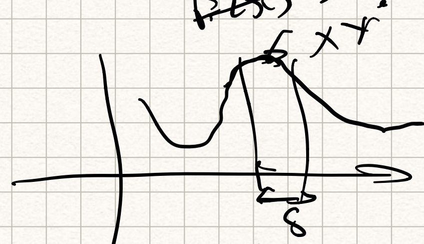
unconstrained problem.




$x \in N_\delta(x^*)$

$f(x^*) \geq f(x)$

~~$f(x) \geq f(x^*)$~~




hunch.

$$\max \; x_1 + x_2 \cdots + x_n$$
$$x_1^2 + x_2^2 \cdots + x_n^2 = 1$$

of one ~~two~~ equality

$$x_n = \left(1 - \overline{x_1^2 + x_2^2 - + x_{n-1}^2}\right)^{1/2}$$

$$\max_{x_1, x_2, \ldots, x_{n-1}} \; \underbrace{x_1 + x_2 + \cdots + x_{n-1} + \left(1 - \overline{x_1^2 + x_2^2 \cdots + x_{n-1}^2}\right)^{1/2}}_{f(x_1, x_2 \ldots, x_{n-1})}$$

Stationary pt.

$$\frac{\partial f}{\partial x_1} = 0, \; \frac{\partial f}{\partial x_2} = 0 \; - \; \cdots, \; \frac{\partial f}{\partial x_{n-1}} = 0;$$

$x^{1/2}$ $\quad \frac{1}{2\sqrt{x}}$

$$\frac{\partial f}{\partial x_i} = 1 + \frac{1}{2}\frac{(-1)\cdot 2x_i}{\left(1-x_1^2+x_2^2 - \cdots + x_{n-1}^2\right)^{1/2}} = 0$$

$$1 - \frac{x_i}{\left(1-x_1^2+x_2^2-\cdots+x_{n-1}^2\right)^{1/2}} = 0$$

$$1 - x_1^2+x_2^2 - \cdots + x_{n-1}^2 = x_i^2 \quad i=1\cdots n-1$$

$$2x_1^2 + x_2^2 \cdots + x_{n-1}^2 = 1$$

$$x_1^2 + 2x_2^2 \cdots + x_{n-1}^2 = 1$$

$$\cdots$$

$$x_1^2 + x_2^2 \cdots + 2x_{n-1}^2 = 1$$

$$\begin{bmatrix} 2, & 1, & \cdots & ,1 \\ 1, & 2, & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_{n-1}^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$Ax = b$$

$$x_1^2 = x_2^2 = \cdots = x_{n-1}^2 = x_n^2$$

$$\therefore x_1^2 + x_2^2 \cdots + x_n^2 = 1$$

$$x_i^2 = \frac{1}{n}$$

$$x_1 = x_2 = \cdots = x_n = \frac{1}{\sqrt{n}}$$

$$\|x\|_1 = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

$Df(x) = 0$

$$\|x\|_2 \le \|x\|_1$$

$$\|x\|_1 = \left[|x_1| + |x_2| \cdots + |x_n|\right]^2$$

$$\|x\|_1^2 = |x_1|^2 + |x_2|^2 \cdots + |x_n|^2 + \underbrace{2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} |x_i||x_j|}_{p}$$

$\ge 0$ $\qquad$ $\ge 0$ $\qquad$ $n^2 - n$

$$\|x\|_1^2 = \|x\|_2^2 + (p)$$

$\ge 0$

$$\|x\|_1^2 \ge \|x\|_2^2$$

$$\|x\|_1 \ge \|x\|_2$$

$$\underbrace{\|x\|_2 \le \|x\|_1}_{} \le \sqrt{n}\,\|x\|_2$$

$$\left\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|_2 = 1 \qquad \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|_1 = 1$$