$$\min_{x \in \mathbb{R}^n} F(x)$$ $F(x)$ is twice continuously differentiable;

The necessary conditions for x^* to be a local min are

1. $\nabla F(x^*) = 0$ [Stationary point]
2. $H(x^*) = G(x^*) = \left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right]_{x = x^*}$ is positive semidefinite [non-negative curvature]

Taylor series:

$$\underset{\substack{\text{small} \\ \varepsilon > 0}}{F(x^* + \varepsilon p)} \geq \underline{\underline{F(x^*)}} \quad 0 < \varepsilon \leq \bar{\varepsilon}$$

p in an arbitrary direction

$\underline{\underline{\text{if}}}$ x^* is a local min

If $F(x^* + \varepsilon p) < F(x^*)$ how can this local min x^*

$$F(x^* + \varepsilon p) = F(x^*) + \varepsilon \nabla F(x^*)^T p \quad O(\varepsilon^2)$$
$$+ \frac{\varepsilon^2}{2} p^T \underline{\underline{H(x^* + \theta \varepsilon p)}} p$$

(1) (1) (2) (3)

I don't this a - challenge $0 \leq \theta \leq 1$

$$\underset{n \times 1}{\nabla F(x^*)} = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix}_{x = x^*}$$

p n×1 vector

$p \in \mathbb{R}^n$

$\nabla F(x^*)^T p$ scalar

$\frac{\varepsilon^2}{2}$ (scalar) $p^T H(x^* + \theta\varepsilon p)\, p$

$x^* + \theta\varepsilon p$

$x^* \to \mathbb{R}^n$, $p \to \mathbb{R}^n$

θ scalar, ε scalar

$0 \le \theta \le 1$

$$\underbrace{\frac{\varepsilon^2}{2}\, p^T H(x^* + \varepsilon\theta p)\, p}_{\text{scalar}} = \frac{\varepsilon^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x^* + \varepsilon\theta p)\, p_i p_j$$

n^2 - terms

each of these second derivatives are continuous. functions continuous func.

$\le \sum\sum c \longleftrightarrow$

$$\frac{\varepsilon^2}{2}\, p^T H(x^* + \theta\varepsilon p)\, p \le M\varepsilon^2$$

Weierstrass

$O(\varepsilon^2)$

has soln.

$$\bar{M} = \underbrace{\max_{0 \le \theta \le 1}}_{\text{closed \& bounded set}} \underbrace{p^T H(x^* + \theta\varepsilon p)\, p}_{\text{continuous func}}$$

Proof is by contradiction

Assume first $\nabla F(x^*) \neq 0$

$$\Rightarrow \underline{\nabla F(x^*)^T p < 0}$$

descent direction

$$p = -\underbrace{\nabla F(x^*)}_{\text{steepest descent}}$$

$$-\nabla F(x^*)^T \nabla F(x^*)$$

$$-\|\nabla F(x^*)\|_2^2 < 0$$

$$\Downarrow$$

$$\cancel{\neq 0} \qquad \varepsilon > 0 \quad (2)$$

$$F(x^* + \varepsilon p) = F(x^*) + \varepsilon \underbrace{\nabla F(x^*)^T p}_{\text{descent direction}}$$

$$+ \frac{\varepsilon^2}{2} p^T H(x^* + \varepsilon\theta p)\, p \quad (3)$$

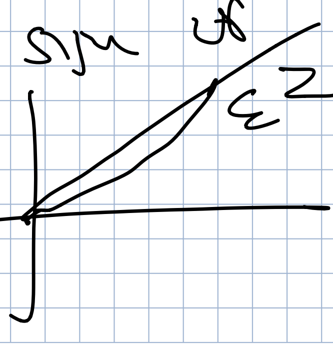
sign of (2)+(3) decides by sign of

$$\varepsilon \nabla F(x^*)^T p$$

$$0 < \varepsilon < \bar{\varepsilon}$$

$x - x^2$

$\cancel{x} - 16x^2$



$F(x^* + \epsilon p) < F(x^*)$ $0 < \epsilon < \bar{\epsilon}$

p appropriate descent direction.

which is a contradiction! Given x^* is a local min

$$\boxed{\nabla F(x^*) = 0}$$

x^* must be a stationary point.

$$F(x^* + \epsilon p) = F(x^*) + \frac{1}{2}\epsilon^2 \underbrace{p^T H(x^* + \epsilon\theta p)\, p}_{\text{quadratic term}}$$

$F(x^* + \epsilon p) < F(x^*)$
which a contradiction!

N/C

$p^T H(x^*) p \geq 0 \quad \forall p \neq 0 \in \mathbb{R}^n$

To the contrary assume that $\exists$ some p

$$p^T H(x^*) p < 0$$

$$H = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ second coordinate direction

$$p^T H p = -1 < 0$$

$$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$p^T H p > 0 \quad (=1)$$

$$H = \begin{bmatrix} 1 & 0 \\ & \end{bmatrix}$$

$$p^T H p \geq 0$$

$$H = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 \geq 0$$

$$p^T H(x) p = \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \frac{\partial^2 f}{\partial x_i \partial x_j}}_{0}$$

n^2 continuous functions

continuous.

$p^T H(x)\, p < 0$

$\Rightarrow$ in a small enough neighbourhood

$p^T H(x + \alpha \epsilon p)\, p < 0$

$\nabla F(x^*) = 0$

$\underbrace{p^T H(x^*)\, p \geq 0}_{0} \quad \forall p \neq 0$

Hessian matrix should be at least pos semidefinite

GRN.

What happens if

$H(x^*)$ is positive definite

$$p^T H(x^*) p > 0$$

$$F(x^* + \varepsilon p) = F(x^*) + \frac{\varepsilon^2}{2} p^T H(x^* + \theta \varepsilon p) p$$

For $\underline{0 < \varepsilon < \bar{\varepsilon}}$

$$\frac{\varepsilon^2}{2} p^T H(x^* + \theta \varepsilon p) p > 0$$

$$\frac{\varepsilon^2}{2} p^T H(x^*) p > 0$$

$$F(x^* + \theta \varepsilon p) > F(x^*)$$

$\Rightarrow$ strict local min.

Sufficiency conditions

$$\nabla F(x^*) = 0$$

$H(x^*)$ positive definite