Lecture 28: Linear Algebra A Review

How to test whether a matrix a s.p.d.?

p.d. $x^T A x > 0 \quad \forall x \neq 0 \quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$

suppose that A is not symmetric?
then what.

$$x^T A^T x > 0$$

$$x^T \underbrace{\left(\frac{A^T + A}{2}\right)}_{\text{symmetric}} x > 0 \qquad \forall x \in \mathbb{R}^n \neq 0$$

In optimization - Hessian $\left[\frac{\partial^2 f}{\partial x^2}\right]_{n \times n}$ symmetric.

Vector Space $\mathbb{R}^n$

$x \in \mathbb{R}^n,\ y \in \mathbb{R}^n \qquad x + y \to \mathbb{R}^n$ (closed)

$(\alpha x) \in \mathbb{R}^n \qquad \alpha \in \mathbb{R}$

Vector space

subspace $\to \begin{pmatrix} x_1 \\ x_2 \\ x_k \\ 0 \\ 0 \end{pmatrix}$

Yes

Range Space

Null Space

$A \in \mathbb{R}^{m \times n} \to$ is it a VS?

Ans. Yes

$\mathbb{R}^{5 \times 3}$

$A + B = C \in \mathbb{R}^{5 \times 3}$ → closed under addition

$\alpha(A) = (\alpha A) \in \mathbb{R}^{5 \times 3}$ → closed under scalar multiplication

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{y} \qquad T(x) = y$$

$A \in \mathbb{R}^{m\times n}$, $\quad x \in \mathbb{R}^{n} \quad y \in \mathbb{R}^{m}$.

$T(x) = y$

Linear transformation.

$$T(\underset{\to\,\mathbb{R}^n}{\underline{x} + \underline{y}}) = \underset{\in \mathbb{R}^m}{T(\underline{x})} + \underset{\in \mathbb{R}^m}{T(\underline{y})}$$

$$T(\alpha \underline{x}) = \alpha T(\underline{x})$$

$$T(\alpha \underline{x} + \beta \underline{y}) = \alpha T(\underline{x}) + \beta T(\underline{y})$$

$$A(\alpha x + \beta y) = \alpha(Ax) + \beta(Ay)$$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

What happens if matrix is square?

$A \in \mathbb{R}^{n\times n}$

$$T: \mathbb{R}^n \to \mathbb{R}^n.$$

Vector space 0

$$x + \underline{0} = x$$

$$x + (-x) = 0$$

$$0 \cdot x = \underline{0}$$

$A \in \mathbb{R}^{m\times n} \qquad A + [0] = A$

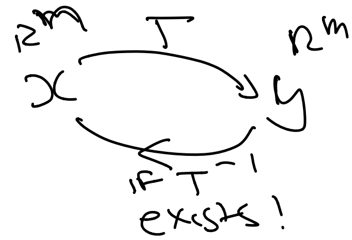
$A + (-A) = [0]$

$\to$ inverse element in V.S $\mathbb{R}^{m\times n}$.

This should not be confused with A^{-1}.

(A^{-1}) : is the transformation T invertible

Invertible transformation.



$T(x) = y$

$T^{-1}(y) = x$

$\hookrightarrow$ But this need not compulsarily exist.

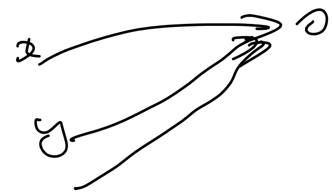
$\underset{1\times 2}{A} = [1 \quad 1]$

$Ax = 0$

$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Ay = 0$

$y = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$



one-one

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad z = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$Ax = 2$

$Ay = 2$

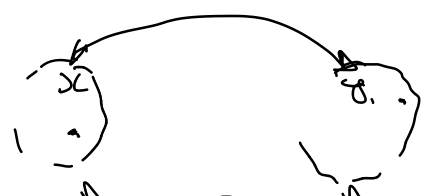
$Az = 2$

This transformation is not invertible.

Invertible transformation require

→ one-one

→ on-to



we have square matrices only $\quad \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\underset{n\times n}{A} x = y \iff x = A^{-1} y$

does there exist A^{-1}

Invertible linear transformation $\iff$

inverse of matrix $\underset{(n \times n)}{A}$

Not all $n \times n$ matrices are invertible

$A = \underset{2 \times 2}{\begin{bmatrix} 0 & 1 \\ & \end{bmatrix}}$ $\quad A \cancel{x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T(z) = 0$ $\quad z \neq 0$ then the linear transformation is not invertible

$T(x) = y$

$T(x + \alpha z) = T(x) + \alpha T(z)$

$= T(x)$.

If

$Az = 0$ $\quad z \neq 0$

then we say that A is not invertible matrix i.e., A^{-1} does not exist.

Conversely, if $\forall z \neq 0$, $\underset{n \times n}{A} z \neq 0$, then

A will be a invertible matrix.

Revisit defn. of eigen value and eigen vector.

$A x = \lambda x$ $\Rightarrow$ If the matrix is singular then

λ eigen value, x eigen vector

$Az = 0 \cdot z = 0$

Singular matrices always have zero as their eigen value.

$(A - \lambda I) x = 0$

Conversely non-singular (invertible) matrix A has all non-zero eigen values.

$$(A - \lambda I)x = 0$$
$$\det(A - \lambda I) = 0$$

If matrix A is singular $\Leftrightarrow$ $\det(A) = 0$;
Conversely if matrix A is nonsingular $\Leftrightarrow$ $\det(A) \neq 0$.

characteristic polynomial of matrix A: $\det(SI - A) = 0$

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

eigen values are the roots of characteristic polynomial.

n - roots.

n - eigen values.

And these values can repeat.

$$A = \begin{array}{ccc} e_1 & e_2 & e_3 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad (s-1)^3 = 0$$
$$s = 1, 1, 1$$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad Ae_1 = 1 \cdot e_1 \qquad \|e_1\|_2 = 1$

$Ae_2 = 1 \cdot e_2 \qquad \|e_2\|_2 = 1$

$Ae_3 = 1 \cdot e_3 \qquad \|e_3\|_2 = 1$

$$Av = \lambda v$$
$$A(\alpha v) = \lambda(\alpha v)$$

[Q] Can eigen values be complex
Yes.

but complex eigen values arrive only in pairs.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad [SI - A] = \begin{bmatrix} s & +1 \\ -1 & s \end{bmatrix}$$

$\det(sI - A) = s^2 + 1 = 0 \qquad s = \pm j$ complex.

It turns out that if
A is a real symmetric matrix
then all its eigen values will be
real.

$$\underset{n\times n}{A}\, v = \lambda v \longrightarrow \text{real} \cdot (0, +, -)$$

$$A v_1^{\in \mathbb{R}^n} = \lambda_1 v_1 \in \mathbb{R}^n \qquad \lambda_1 \neq \lambda_2$$

$$A v_2 = \lambda_2 v_2$$

$$v_1 \perp^r v_2$$

How do I prove it ?

Further, for a symmetric A real,
we can find n-eigen vectors which are
orthogonal to each other.

In other words, they form the basis
of VS $\mathbb{R}^n$.

$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2 \qquad \lambda_1 = \lambda_2 \text{ are same possible}$$
$$\vdots$$
$$A v_n = \lambda_n v_n.$$

$$v_i \perp^r v_j \qquad i \neq j$$

$$A [v_1, v_2 \cdots v_n] = [v_1, v_2, \cdots v_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 v_1\ \lambda_2 v_2 \cdots \lambda_n v_n]$$

$$A V = V \Lambda$$

$\|v_i\| = 1 \qquad v_i^T v_j$

$V \rightarrow$ orthogonal matrix

Further, $\quad V^T V = I_n \qquad V^T = V^{-1}$

$AA^{-1} = A^{-1}A = I$

$(i, i) = \|v_i\|_2^2 = 1;$

$(i, j) \rightarrow v_i^T v_j = 0 \quad i \neq j$

$$\boxed{A = V \Lambda V^T}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$V^T A V = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Q] If A is a positive definite matrix, what can you say about its eigen values?

$$Av = \lambda v$$

$$v^T A v = \lambda v^T v = \lambda > 0$$

1) So, therefore, A is a s.p.d. iff all its eigen values are +ve.

2) A is positive semidefinite matrix iff $\lambda_i \geq 0$.

2) Positive definite matrices also have a square root $A^{1/2}$.

$A = P \Lambda P^T$ $\quad \Lambda = \begin{pmatrix} \lambda_1^{1/2} & & \\ & \lambda_2^{1/2} & \\ & & \ddots \lambda_n^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1^{1/2} & & \\ & \lambda_2^{1/2} & \\ & & \lambda_n^{1/2} \end{pmatrix}$

A spd $\lambda_i > 0$

$\Rightarrow \Lambda^{1/2}$

$$\bar{A} = P \Lambda^{1/2} \times \Lambda^{1/2} P^T$$

$$= [P \Lambda^{1/2} P^T][P \Lambda^{1/2} P^T] \quad P^T P = I_n.$$

$$= A^{1/2} \times A^{1/2}$$

$\underset{\text{s.p.d}}{A} = A^{1/2} \times A^{1/2} \rightarrow$ $A^{1/2}$ has eigen values $\sqrt{\lambda_i}$, λ_i eigen value of A. and same eigen vectors

$$\therefore \quad x^T A x = x^T A^{1/2} A^{1/2} x$$

$$= x^T (A^{1/2})^T A^{1/2} x$$

$$= \| A^{1/2} x \|_2^2 \geq 0$$

To conclude:

A is s.p.d. iff all eigen values are > 0.