Convex Programming Problem.

Problem:

CP: $\min\limits_{x \in C} f(x)$

$C \subset R^n$

C → convex set.

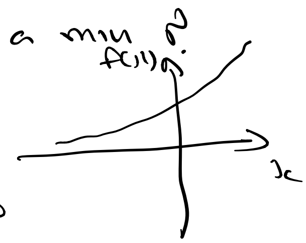
$f(x)$ → convex function

Lemma:

Let x^* be a min. Then, it is a global min.

Q] Does a CP always have a min?

$\min\limits_{x \in R} e^x$



$\lim\limits_{x \to -\infty} e^x = 0$ $\inf e^x = 0$

but it is not achieved by an finite x.

Let $x \in [-1, +1]$, $f(-1) = e^{-1}$ is the global min.

Q] When we say min, can be multiple minumums at same cost?

Proof: To the contrary, let there a local minimum at x_L s.t. $f(x_L) > f(x^*)$ $x_G = x^*$

$f(x_L) > f(x_G)$

$x := \lambda x_G + (1-\lambda) x_L$ $\lambda \in [0, 1]$

$= x_L + \lambda (x_G - x_L)$

But $f(x_G) < f(x_L)$

By convexity of f over C.

$f(\lambda x_G + (1-\lambda) x_L) \leq \lambda f(x_G) + (1-\lambda) f(x_L)$

$$f(\lambda x_G + (1-\lambda)x_L) < [\lambda + 1 - \lambda] f(x_L) = f(x_L)$$

$$f(x_L + \lambda(x_G - x_L)) < f(x_L).$$

This contradicts, the claim that $f(x_L)$ is a local minimum.

$N_\varepsilon(x_L)$ $\varepsilon > 0$

$\lambda < \varepsilon$

$x \in N_\varepsilon(x_L) \cap C$

$x_G + \lambda(x_G - x_L)$

→ Every local min is a global minimum for the convex programming problem.

→ If $\bar{x}$ & $\hat{x}$ be two minimums (global) to CP. Then $\lambda\bar{x} + (1-\lambda)\hat{x}$ is a set of minimums (global) to CP.

$$f(\lambda\bar{x} + \overline{1-\lambda}\,\hat{x}) \leq \lambda f(\bar{x}) + (1-\lambda) f(\hat{x})$$
$$= [\lambda + 1 - \lambda] f(x_G)$$
$$f(\lambda\bar{x} + \overline{1-\lambda}\,\hat{x}) = f(x_G)$$

→ further if the function f is strictly convex, then the minimum is unique.

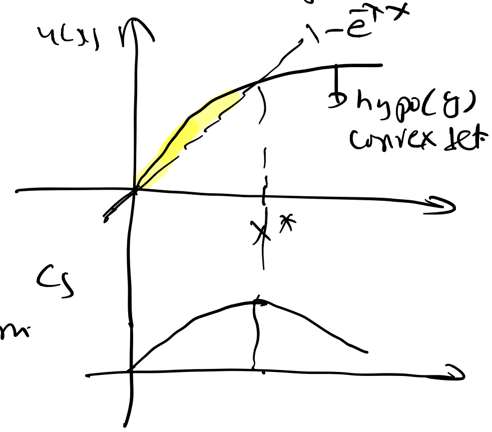
Proof: Assume to the contrary, that there are two global min x_{G_1}, x_{G_2}, $x_{G_1} \neq x_{G_2}$

$$f(\lambda x_{G_1} + (1-\lambda) x_{G_2}) < \lambda f(x_{G_1}) + \overline{1-\lambda}\, f(x_{G_2})$$
$$= f(x_{G_1}) = f(x_{G_2})$$

which is a contradiction.

Example of maximizing consumer surplus.

$$CS = u(x) - px$$



$$\max_x \; u(x) - px$$

Conversely, for a concave function, if it has a maximum then it is a global max.

$$u(x) = 1 - e^{-\lambda x}$$

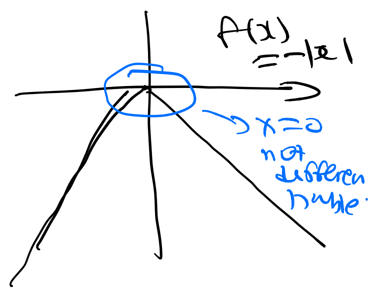
$$\max_{x \in C} f_c(x)$$

$$\min_{x \in C} \underbrace{-f_c(x)}_{\text{convex function.}}$$

Assume differentiability.

$$\max_x \; u(x) - px$$

$x \geq 0$



$$u'(x^*) - p = 0$$

$$p = u'(x^*) > 0$$

costs functions are usually convex.

$$c(P_G) = a + bP_G + cP_G^2$$
$$c > 0$$

Supplier

$$\max \; px - c(x)$$

$$c'(x^*) = p$$

discovered.

Equilibrium for market exists iff.

$$\boxed{p = c'(x^*) = u'(x^*)}$$

$$TW = CS + SP = u(x) - px + px - c(x)$$
$$SW \qquad\qquad = u(x) - c(x)$$

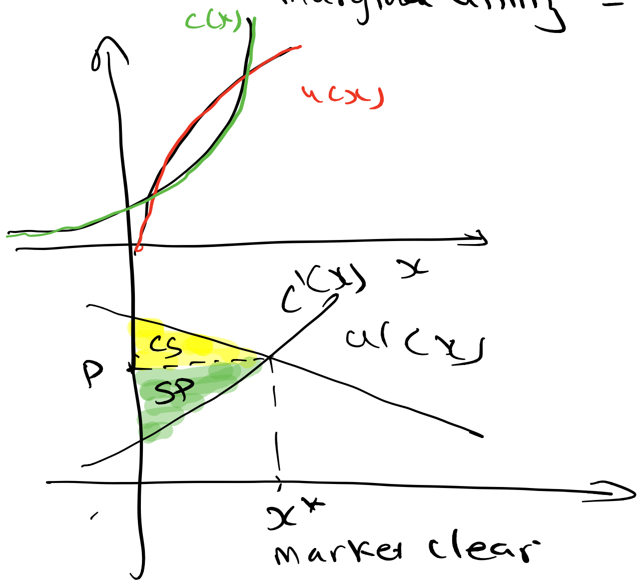
social welfare vs total welfare

$$\max_{x} \; u(x) - c(x)$$

$$u'(x^*) - c'(x^*) = 0$$

$$p = u'(x^*) = c'(x^*)$$

marginal utility = marginal cost.



$$\int (u'(x) - p)\, dx = CS$$

Linear Programming

$$\min \; c^T x \quad \leftarrow \text{convex func.}$$
$$\left.\begin{array}{l} Ax = b \\ x \geq 0 \end{array}\right\} C$$

Assume that it has soln; then the min is a global min.

$$\max \; c^T x$$
$$Ax = b$$
$$x \geq 0$$