Proposition 3.2.1 Consider minimize $f(x)$ $f: \mathbb{R}^n \to \mathbb{R}$ and constraints $h_1(x)=0, \dots h_m(x)=0$. All function f & h_i's are twice continously differentiable. Let $x^* \in \mathbb{R}^n$ satisfy the following. (1, 2)

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T h(x)$$
$$= f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \qquad \nabla_\lambda \mathcal{L}(x^*, \lambda^*) = 0 \quad [h(x^*) = [0]] \rightarrow (1)$$

$$\underset{n \times n}{y^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) y} > 0 \quad \forall\, y \neq 0 \text{ with } \underset{m \times n}{\nabla h(x^*)^T} y = 0 \rightarrow (2)$$

$$\nabla h(x^*)^T = \underset{m \times n}{\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}}$$

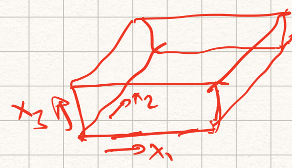
$\nabla h(x^*)^T y = 0 \implies y \in$ Null Space of $\nabla h(x^*)^T$ matrix

$Z^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) Z$ is p.d.

$$\nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) = \underset{n \times n}{\begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n^2} \end{bmatrix}_{x^*, \lambda^*}}$$

Then x^* is a strict local min of f subject to $h(x)=0$.

Example 3.2.1 [Bertsekas]



$$\max_{x_1, x_2, x_3} \; x_1 x_2 + x_2 x_3 + x_3 x_1$$

subject $\quad x_1 + x_2 + x_3 = 3$

$$\min \; -(x_1 x_2 + x_2 x_3 + x_3 x_1)$$

s.t. $\quad x_1 + x_2 + x_3 = 3$

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = -x_1 x_2 - x_2 x_3 - x_3 x_1 + \lambda(x_1 + x_2 + x_3 - 3)$$

$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0;$

$$\frac{\partial f}{\partial x_1} = -x_2^* - x_3^* + \lambda^* = 0; \qquad (1)$$

$$\frac{\partial f}{\partial x_2} = -x_1^* - x_3^* + \lambda^* = 0; \qquad (2)$$

$$\frac{\partial f}{\partial x_3} = -x_2^* - x_1^* + \lambda^* = 0; \qquad (3)$$

$$x_1^* + x_2^* + x_3^* = 3$$

$$-2(x_1^* + x_2^* + x_3^*) + 3\lambda^* = 0 \qquad (1)+(2)+(3)$$

$$3\lambda^* = 6 \Rightarrow \lambda^* = 2;$$

$$x_2^* + x_3^* = 2 \Rightarrow x_1^* = x_2^* = x_3^* = 1;$$

$$\nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} + \lambda^* \frac{\partial^2 h}{\partial x_i \partial x_j} \right]_{x = x^*}$$

$$h(x) = x_1 + x_2 + x_3 - 3$$

$$\frac{\partial^2 h}{\partial x_i \partial x_j} = 0$$

$$\nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]$$

$$\nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$y^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) y$$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} -(y_2+y_3) \\ -(y_1+y_3) \\ -(y_1+y_2) \end{bmatrix} = (y_1 \; y_2 \; y_3) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + y_2^2 + y_3^2 > 0$$

$\forall y \neq 0$

$\nabla h(x^*)^T y = 0$

$$h(x) = x_1 + x_2 + x_3 - 1$$

$$\nabla h^T(x^*) y = [1 \quad 1 \quad 1] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\Rightarrow y_1 + y_2 + y_3 = 0 \quad \& \; y \neq 0$$

Hence maximum occurs at $x_1^* = x_2^* = x_3^* = 1;$