LEP $\min_{x} f(x)$ $x \in \mathbb{R}^n$

$\underset{m \times n}{A} x = \underset{m \times 1}{b}$

Convex Programming Problem $\min_{x \in S} f(x)$ → convex function

S → Convex set

Assume that these constraints are consistent.

$A\bar{x} = b$; set of feasible solns is non-empty

$m < n$; $Ax = b$ underdetermined set of eqns.

$\text{rank}(A) < n$;

$\min \; x_1^2 + x_2^2 + x_3^2$

$x_1 + x_2 + x_3 = 3$ $A = [1, 1, 1]$

$\text{rank}(A) = 1$;

$x_1 = 3 - x_2 - x_3$

choose arbitrarily.

Further, I will assume that $\text{rank}(A) = m$; the no. of rows.

$x_1 + x_2 + x_3 + x_4 = 5$ → ①

$2x_1 - x_2 + x_3 - x_4 = -6$ → ②

$3x_1 + 2x_3 = -1$ → ③

$\underset{3 \times 4}{A}$ $\text{rank}(A) = 2$

$\min f(x)$ $x \in \mathbb{R}^n$

$\underset{m \times n}{A} x = \underset{m \times 1}{b}$ $\text{rank}(A) = m$

Possible alternatives

$\min \; x_1^2 + x_2^2 + x_3^2$

$x_1 = 3 - x_2 - x_3$

Elimination of Variables

$$\min_{x_2, x_3} \quad x_2^2 + x_3^2 + (3 - x_2 - x_3)^2$$

Penalty Function Approach

$$\min \quad (x_1^2 + x_2^2 + x_3^2) + \frac{\rho}{2}(x_1 + x_2 + x_3 - 3)^2$$

$\rho >$ large +ve no.

Null Space Basis:

$$A Z = [0]$$

$\underset{n \times n-m}{[Z]}$ basis of null space of A

$Z = [z_1, z_2, \cdots z_{n-m}]$

$\text{rank}(Z) = n-m$.

$\left[\underset{n \times m}{A^T} \quad \underset{n \times n-m}{Z}\right]$ constitute a basis of $\mathbb{R}^n$

$p = Zy \quad \in$ Null space of A

$A p = \underset{m \times 1}{\underset{\text{vector}}{[0]}}$

Let $\bar{x}$ be a feasible soln. of LEP

$$A\bar{x} = b$$

Let $\hat{x}$ be another feasible soln. of LEP

$$A\hat{x} = b.$$

$$A(\hat{x} - \bar{x}) = 0$$

$\Rightarrow \hat{x} - \bar{x} \in$ Null Space of A;

$$\hat{x} - \bar{x} = Z\bar{y}$$

In other words, if I have some feasible soln. to LEP, then set of all solns of LEP $\quad x = \bar{x} + Zy$

given $A\bar{x} = b$

Necessary condition for soln to LEP problem. Let x^* be a local min of LEP problem. Then, we must have.

(1) $Ax^* = b$; [feasibility]

(2) $Z^T \nabla f(x^*) = 0$ or $\nabla f(x^*) + A^T \lambda^* = 0$

(3) $Z^T H(x^*) Z$ [projected Hessian] is positive semidefinite.

Sufficiency conditions for a min of LEP

(1) $Ax^* = b$

(2) $Z^T \nabla f(x^*) = 0 \iff \nabla f(x^*) + A^T \lambda^* = 0$

(3) $Z^T H(x^*) Z$ is positive definite.

Proof:

$$f(x^* + \varepsilon Zy) = f(x^*) + \varepsilon \nabla f(x^*)^T Z y + \frac{\varepsilon^2}{2} y^T Z^T H(x^* + \theta \varepsilon p) Z y$$

$0 \le \theta \le 1$

$\underline{\varepsilon p} \quad p = \underline{Zy}$

$$\frac{1}{2} p^T H(x^* + \theta \varepsilon p) p$$

$0 \le \theta \le$

$\underline{p = Zy}$

$f(x^* + \varepsilon p)$

$\nabla f(x^*)^T Z$ or $Z^T \nabla f(x^*) = [0]$

projected gradient on the Null space should be zero

or

gradient vector is orthogonal to null space.

why?

To the contrary

$$Z^T \nabla f(x^*) \neq 0$$

$$\underset{1 \times n-m}{y^T} \quad \underset{n-m \times n}{Z^T} \underset{n \times 1}{\nabla f(x^*)} < 0$$

$$y = - Z^T \nabla f(x^*)$$

$$Z^T \nabla f(x^*) = 0$$

$p = Zy$

$$f(x^* + \varepsilon Z y) = f(x^*) + \frac{\varepsilon^2}{2} y^T Z^T H(x^* + \varepsilon \theta p) Z y$$

To the contrary assume that $Z^T H(x^*) Z$ is not semidefinite

$$y^T \underset{n-m \times n}{Z^T} \underset{n \times n}{H(x^*)} \underset{n \times n-m}{Z} y < 0$$

$p = Zy$

$$n-m \times n-m$$

$f \rightarrow$ twice continuously differentiable function.

$p = Zy$

$p^T H(x) p < 0$ is a continuous

$$1 \times 1 \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} p_i p_j$$

therefore, I reach contradiction

$Z^T H(x^*) Z$ at least positive semidefinite.

$[Z \quad A^T]$

Basis of $\mathbb{R}^n$.

$$\nabla f(x^*) = A^T \mu^* + Z \alpha^*$$

$$Z^T \nabla f(x^*) = Z^T A^T \mu^* + (Z^T Z) \alpha^*$$

$$0 = [0] \mu^* + (Z^T Z) \alpha^*$$

$AZ = [0]$ $\qquad$ $(Z^T Z) \alpha^* = 0$

$Z^T Z$ is $n-m \times n-m$ positive definite matrix, hence is it invertible.

$\alpha^* = 0$;

$$\nabla f(x^*) = A^T \mu^*$$

$$\nabla f(x^*) + A^T(-\mu^*) = [0]$$

$$\boxed{\underline{\nabla f(x^*)} + A^T \lambda^* = 0}$$