$A = \underline{\underline{R^T}}\ \underline{\underline{R}}$ [possible if A is positive semidefinite and rank$(A) \geq n-1$]
s.p.d. $\quad L \quad U$

rank $(A) = n$ & positive semidefiniteness $\Rightarrow$ p.d.

$Az = 0 \quad z \neq 0$

$Ax = 0 \iff x = 0;$

l.c. of vectors

$(\vec{a}_1 x_1 + \vec{a}_2 x_2 + \cdots + \vec{a}_m x_m) \rightarrow$ linear combinat of $\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n$ $\quad a_i \in \mathbb{R}^n$ scalars $x = \begin{pmatrix} x_1 \\ x_2 \\ x_m \end{pmatrix}$ $\underline{\underline{\text{vector}}}$

$A = [\vec{a}_1, a_2, - \vec{a}_m]$

l.c. $\quad Ax$

l.i. of vectors or linear independence of vectors

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \qquad \underline{\underline{\vec{a}_3}} = \underline{\underline{2\vec{a}_1}} - \underline{\underline{\vec{a}_2}}$ linearly dependent

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \sum_{i=1}^{m} a_i x_i = 0$$

all $x_i \neq 0;$

$Ax = 0$ has a nontrivial soln $x \neq 0;$

$Ax = 0$ trivial soln $x = 0;$

A set of vectors $a_1, a_2, - a_m$, $a_i \in \mathbb{R}^n$ are linearly independent if

$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0$ iff $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

otherwise, it is linearly dependent.

$\mathbb{R}^n$; dimension of $\mathbb{R}^n$ is n!

vector space $\mathbb{R}^n$ can be derived from n-l.i. vectors, which constitutes its Basis

Basis is that minimal set of l.i. vectors that generates a given vector space.

$\mathbb{R}^n$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ $\cdots e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$e_1 x_1 + e_2 x_2 \cdots + e_n x_n = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

$x_1 = \alpha_1$
$x_2 = \alpha_2$
$\vdots$
$x_n = \alpha_n$

That a set of vectors generate a VS is stated as the vectors $\{e_1, e_2 \cdot e_n\}$ span the vector space $(\mathbb{R}^n)$.

If this spanning set of vectors is a minimal set, then it is called a Basis.

And obviously, in a basis the vectors should be l.i.

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{spans } \mathbb{R}^2}$$

is A minimal ?

Remove any one vector

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{span } \mathbb{R}^2} \text{ basis} \quad \text{or} \quad \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{spans } \mathbb{R}^2} \text{ basis}$$

Further, if I remove a vector from here $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\underbrace{\times \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{span } \mathbb{R}^2}$$

→ Basis is not unique.

→ No. of vectors in a basis are identical.

No. of vectors in a basis is called dimension of VS.

$$\dim(\mathbb{R}^n) = n;$$

$$A = [a_1, a_2, \cdots a_m]$$

What is the maximal set of l.i vectors in A?

Rank is defined as the no. of linearly independent vectors in A.

col. rank; row rank — the no of maximal set of l.i rows of A;

col rank = row rank

= rank; of a matrix

$\underset{5\times5}{A}$ has rank = 5; ⇒ all col.s are l.i and all rows are l.i

$\underset{5\times5}{A}$ has rank = 3; the maximal set of l.i cols. or rows has cardinality = 3.

$\underset{5\times3}{A}$ → it can atmost have 3 l.i colums

$\text{rank}(A) \le 3;$

$\underset{m\times n}{A}$ $\quad \text{rank}(A) \le \min(m, n)$

You can associate now with a matrix A;

→ A range space. (vector space)

→ Null space. (vector space)

$\underset{m\times n}{A}$ Range Space $\subset \mathbb{R}^m$

Null Space $\subset \mathbb{R}^n$.

If the matrix square, then both are subspaces of $\mathbb{R}^n$.

→ What is a range space?

set of vectors $y = Ax$, $x \in \mathbb{R}^n$ constitute the range space of A.

$Ax = b$; and if it has soln, $b \in$ Range Space(A)

$y_1 \in$ Range Space

$y_2 \in$ Range Space.

$y_1 = Ax_1$; $y_2 = Ax_2$.

$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 A x_1 + \alpha_2 A x_2$

$= A(\alpha_1 x_1 + \alpha_2 x_2)$ belongs to that range set

set is closed under addition.

$Ax = y$ $\quad \alpha y = A(\alpha x)$

closed under scalar multiplication

Hence, is a VS by itself, subspace of $\mathbb{R}^m$!

$Ax = b$ has soln. iff $b \in$ Range Space(A).

To the unknowns $b \in$ $\quad Ax = b$

Consistency condition for l.s. of eqns to have soln.

$\operatorname{rank}(A) = \operatorname{rank}(A, b)$

$\operatorname{rank}(A) = r$

$$\begin{array}{c} r+1 \\ \begin{bmatrix} A & b \end{bmatrix} \begin{bmatrix} x \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

$\underset{m \times n}{A}$

rank $(A) < n$.

$\underset{3 \times 5}{A}$ rank$(A) \leq 3$

rectangular matrix

$A z = 0$ with $z \neq 0$.

$A z_1 = 0$

$A z_2 = 0$

$A(\alpha_1 z_1 + \alpha_2 z_2) = 0$

All these null space taking vectors constitute a VS. which is called as null space of A. $\subset \mathbb{R}^n$.

dim(Null space) $< n$.

$A = [1 \quad 1 \quad 1]$

$Az = 0 \qquad z_1 + z_2 + z_3 = 0$

$$\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{2}$$

dim range space (1)

$Ax = x_1 + x_2 + x_3 \in \underline{\mathbb{R}} \qquad \mathbb{R}^1$

dim nullspace(A) + dim Range space (A) = max(m,n)

$\qquad 2 \qquad\qquad\qquad 1 \qquad\qquad\qquad 3$

If a A is a square matrix

$\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = n$.

Rank-Nullity Thm. of l.a.

If dim of null space is 1 or more, then if $Ax = b$ has a soln, then it has infinitely many solns.

$Ax_1 = b$ $\qquad z \in$ Null space

$A(x_1 + \alpha z)$ is also a soln. for all α.

If dim null space is zero & $Ax = b$ has a soln, then it must be unique.

To the contrary, $Ax_1 = b$ $\qquad x_1 \neq x_2$

$Ax_2 = b$

$A(\underbrace{x_1 - x_2}_{z \neq 0}) = 0$ $\qquad Az = 0 \quad z \neq 0$

$\Rightarrow \dim \mathcal{N}(A) \geq 1$, which is a contradiction

$\underset{n \times n}{A} x = b$ $\qquad \text{rank}(A) = n$,

$\dim \mathcal{N}(A) = 0$;

$\dim \mathcal{R}(A) = n$;

$\mathbb{R}^n$ is in the range space.

non-singular.

$Ax = b$ and it is unique.

$x = A^{-1} b$ (one-one & onto)

matrix is invertible.

singular matrix $\Leftrightarrow$ atleast one zero eigen value

non singular matrices $\Leftrightarrow$ all non-zero eigen values