Coercive Function

$X \subset \mathbb{R}^n$

A function $f: X \to \mathbb{R}$ is called coercive if for every sequence $\{x_k\} \subset X$ such that $\|x_k\| \to \infty$, we have $\lim_{k\to\infty} f(x_k) = \infty$

$f(x) = x^2$ (univariate function)

x^{2n} $\quad n = 1, 2, 3 \ldots$ fixed

$x^2, x^4, x^6.$

$\{x_k\}$ scalar. $\quad \{1, 2, 3, \ldots \infty\}$

$\{-1, -2, -3, \ldots -\infty\}$

$|x| \to 1, 2, 3, \ldots, \infty$

$f(x) = x^2 \quad (1, 4, 9, 16 \ldots \infty)$

$x \to -\infty \quad x \to +\infty$

$|x| \to \infty \quad f(x) = x^2 \to \infty$

Coercive function.

$f(x) = x$? coercive function

No $\to$ $\{-1, -2, -3, -4, \ldots\}$

$f(x) \to -\infty$ ✗

$\{f(x_k)\} \to \infty$ $\quad M > 0$ large enough big M

10^5 10^6

$k \geq N$ st $f(x_k) > M$

Corollary of Weierstrass' Theorem:

If X is a nonempty subset of $\mathbb{R}^n$ and let $f: X \to \mathbb{R}$ be continuous at all points of X. Assume X is closed and 'f' is coercive

Then, the set of minima of f over X is non-empty. (and compact)

Proof: Let $\{z_k\} \subset X$ s.t. $X \subset \mathbb{R}^n$

$$\lim_{k\to\infty} f(z_k) = \inf_{z\in X} f(z) \quad \text{glb}$$

Since 'f' is coercive function. $\{z_k\}$ must be bounded.

Q] why? $\because \|z_k\| \to \infty$, $\{f(z_k)\} \to +\infty$ but $+\infty$ cannot be an infinium.

$\to$ Bounded seq. has at least one limit pt.

$\{z_k\}_{k_i} \to x^*$ must be in set X.

Because X set is closed,

$$\lim_{k\to\infty} f(z_k) \to = \underset{\text{finite}}{f(x)}$$ why because f is continuous

$x \in X$

$f(x) = \inf_{z \in X} f(z)$

$\rightarrow$ there is a global min to a coercive function.

Example 1.1.4 [Bertsekas]

$\frac{1}{x} + x \geq 2 \quad \forall \quad x > 0$

global sense!

constrained optimization problem.

-> change of variable.

-> Convert it to unconstrained optimization.

$\underline{\underline{x}} = e^y \Rightarrow \underline{\underline{y}} = \underline{\underline{\ln x}}$

$x > 0$

$e^y \quad y \to \infty \quad e^y \to \infty$

$\lim_{y \to -\infty} e^y = 0$

$e^y \quad y \to +\infty \quad e^y \to \infty$

$y = \ln e^{-1}$ +ve $\frac{1}{e}$

$y = -1$ $\frac{1}{2.73}$

$x = e^y$ y, unconstrained

$y \in \mathbb{R}. \; [-\infty, +\infty)$

model: $\underline{\underline{x > 0}}$ open, unbounded set.

$f(x) = \frac{1}{x} + x$; $f(y) = e^y + \frac{1}{e^y}$

$= e^y + e^{-y}$

$\min_{y \in \mathbb{R}} (e^y + e^{-y})$ [Equivalent unconstrained optimization problem]

→ Does it have soln.?

↳ Yes: $f(y)$ is a coercive function.

$y \to \infty$ $\quad e^{y} \to \infty$, $\quad e^{-y} \to 0$ $\Big\}$ $f(y) \to \infty$

$y \to -\infty$ $\quad e^{y} \to 0$, $\quad e^{-y} \to +\infty$ $\Big\}$ $f(y) \to \infty$

$$|y| \to \infty \qquad f(y) \to \infty$$

which means that $f(y)$ is a coercive funct and has a min (global min)

$$f(y) = e^{y} + e^{-y}$$

$$f'(y) = 0 \Rightarrow e^{y} - e^{-y} = 0$$

or $e^{y} = e^{-y}$.

$\Rightarrow y = 0$ } unique. Stationary pt.

→ only one stationary pt
+
→ guaranteed global min by Weierstrass thm.

→ Indeed, we have reached the desired global min.

$x = e^y$ $\qquad x^* = 1$

$f(x) = \frac{1}{x} + x = 2$.

$\min_{x>0} f(x) = 2$.

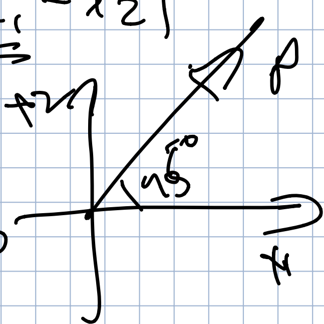
$$\boxed{\frac{1}{x} + x \geq 2} \quad \forall\, x > 0$$

Example:

$f(x_1, x_2) = |x_1 - x_2|$

Q] is it coercive fn?

Yes ✗ $\qquad$ No



$\begin{pmatrix}0\\0\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}2\\2\end{pmatrix} \begin{pmatrix}3\\3\end{pmatrix}$

$\|x\| \to \infty \qquad f(x) = 0$

$f(x_1, x_2) = |x_1 - x_2|$ is not coercive

Don't fall in to the trap

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} n \\ 0 \end{pmatrix}$

$\|x\| \to \infty \Rightarrow f(x) \to \infty$

$\Rightarrow f \to$ is coercive.

For every sequence

$$\left. \begin{array}{l} \{z_k\} \to \infty \\ \{f(z_k)\} \to \infty \end{array} \right\} \text{then is coercive.}$$