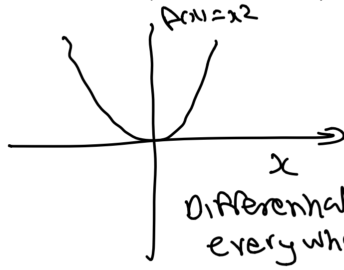
Proposition B.3 [Bertsekas] Let C be a convex subset of $\mathbb{R}^n$ and let $f:\mathbb{R}^n \to \mathbb{R}$ be a differentiable over $\mathbb{R}^n$.

(a) f is convex over C if and only if

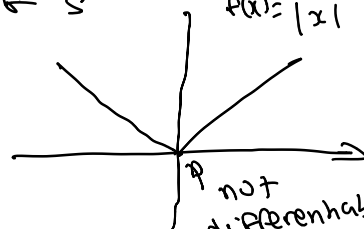
$$f(z) \geq f(x) + (z-x)^T \nabla f(x) \quad \forall\, x, z \in C$$

(b) f is strictly convex over C if and only if the above inequality is strict whenever $x \neq z$.

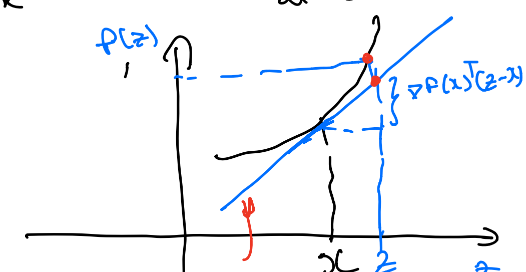
Theorem 3.13 [Bazaraa et. al.] Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f: S \to \mathbb{R}$ be convex. Then f is continuous on the interior of S



Differentiable everywhere



not differentiable at $x=0$



gradient vector supports the convex from below. Hence, it can be used for outer approximations convex function.

Proof: Assume that the inequality

$$f(z) \geq f(x) + \nabla f(x)^T (z-x)$$

is true.

T.P.T: function is convex

$x, y \quad x \neq y, \in C$

$z = \alpha x + (1-\alpha) y$

$\alpha \times \left[f(x) \geq f(z) + \nabla f^T(z)(x-z) \right]$

$(1-\alpha) \times \left[f(y) \geq f(z) + \nabla f^T(z)(y-z) \right] \qquad \alpha \in [0,1]$

Add.

$$\alpha f(x) + (1-\alpha) f(y) \geq [\alpha + 1 - \alpha] f(z) + \nabla f^T(z) \left[\alpha x - \alpha z + (1-\alpha) y - (1-\alpha) z \right]$$

$$\alpha x + (1-\alpha) y - (\alpha + 1 - \alpha) z = z - z = 0$$

$$f(\alpha x + \overline{1-\alpha}\, y) \leq \alpha f(x) + (1-\alpha) f(y)$$

which means that the function is convex

Conversely, assume that 'f' is convex, let x and z be any vectors in C with $x \neq z$ and for $\alpha \in (0,1)$, consider the function

$$g(\alpha) = \frac{f(x+\alpha(z-x)) - f(x)}{\alpha}, \quad \alpha \in [0,1]$$

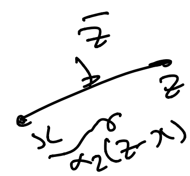
We will show that $g(\alpha)$ is monotonically increasing with α,

This will imply that

$$(z-x)^T \nabla f(x) = \lim_{\alpha \downarrow 0} g(\alpha) \leq g(1) = f(z) - f(x)$$

Indeed, consider any α_1, α_2, $0 < \alpha_1 < \alpha_2 < 1$ and let

$$\bar{\alpha} = \frac{\alpha_1}{\alpha_2}, \quad \bar{z} = x + \alpha_2 (z-x)$$



We have, by convexity of 'f',

$$f(x + \bar{\alpha}(\bar{z}-x)) \leq \bar{\alpha} f(\bar{z}) + (1-\bar{\alpha}) f(x)$$

$$\frac{f(x+\bar{\alpha}(\bar{z}-x)) - f(x)}{\bar{\alpha}} \leq f(\bar{z}) - f(x)$$

$$\bar{\alpha} = \frac{\alpha_1}{\alpha_2}$$

$$\frac{[f(x+\bar{\alpha}(\bar{z}-x)) - f(x)]}{\alpha_1} \leq \frac{f(\bar{z}) - f(x)}{\alpha_2}$$

$$\bar{\alpha}\,\bar{z} = \frac{\alpha_1}{\cancel{\alpha_2}} (\cancel{x} + \cancel{\alpha_2}(z-x) - \cancel{x})$$

$$= x + \alpha_1 (z-x)$$

$$\frac{f(x + \alpha_1(z - x))}{\alpha_1} \leq \frac{f(x + \alpha_2(z - x)) - f(x)}{\alpha_2}$$

or $\quad g(\alpha_1) \leq g(\alpha_2)$