Coordinate Frames

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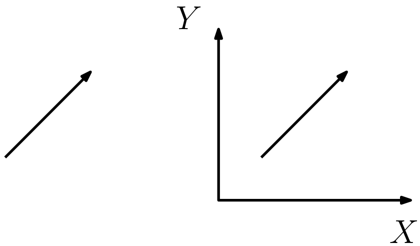
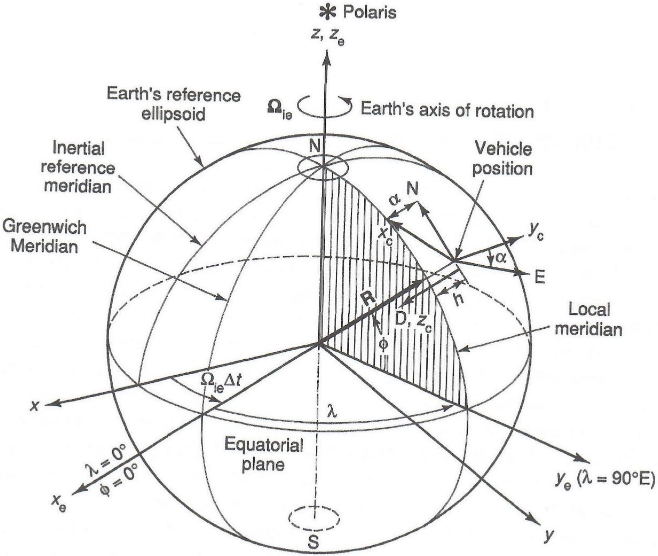
- Why do you need coordinate frames?



- To determine motion of a vehicle, it becomes necessary to relate the solution to the motion of Earth.
  - $\Rightarrow$ Define inertial reference frame w.r.t. the Earth
  - $\Rightarrow$ Obtain motion of both vehicle and Earth w.r.t. the inertial frame
- Initial orientation of reference coordinate frame, position, and velocity are required to obtain future orientation, position and velocity.

- Choice of coordinate frame
  - ⇒ Mission requirements
  - ⇒ Ease of implementations
  - ⇒ Computer storage and speed
  - ⇒ Complexity of navigation equation
- Fundamental coordinate frames
  1. True inertial
  2. Earth-centered inertial (ECI)
  3. Earth-centered Earth-fixed (ECEF)
  4. Navigation
  5. Body
  6. Wander azimuth
- All these are orthogonal and right-handed Cartesian frame.
- How these coordinate frames are different?
  - ⇒ Location of the origin
  - ⇒ Relative orientation of the axes
  - ⇒ Relative motion between the frames

- How do you define inertial frame?
  - $\Rightarrow$ The reference frame in which Newton's laws of motion are valid.
  - $\Rightarrow$ True inertial frame consists of a set of mutually perpendicular axes that neither accelerate nor rotate with respect to inertial space.
  - $\Rightarrow$ Fixed relative to the stars
- Newton's laws are also valid in Galilean frames.
- Galilean frames: Those that do not rotate with respect to one another and which are uniformly translating in space.
- True inertial frame is Galilean frames with absolute zero motion.
- Existence of true inertial frame?
  - $\Rightarrow$ Formulation of theories of relativity
  - $\Rightarrow$ Newtonian mechanics is a special case
- True inertial frame is not a practical reference frame.
- It is used only for visualization of other reference frames.



Polaris
z, z_e
Earth's reference ellipsoid
Ω_{ie}
Earth's axis of rotation
Inertial reference meridian
Greenwich Meridian
N
Vehicle position
α
N
x_c
y_c
α
E
R
h
D, z_c
ϕ
Local meridian
$\Omega_{ie}\Delta t$
x
λ
Equatorial plane
λ = 0°
ϕ = 0°
x_e
y_e (λ = 90°E)
S
y

- How do you define Earth-centered inertial (ECI) frame?
  - $\Rightarrow$ Origin at the Earth's center of mass
  - $\Rightarrow$ Nonrotating relative to the inertial space
- Is this frame accelerating?
- It accelerates with respect to inertial space since it moves with the Earth.
  - $\Rightarrow$ Due to the Earth's rotation and movement about the sun, the inertial frame appears to be rotating for an Earth-fixed observer.
- At the start of navigation mode, $x - y$ axes of this frame lie in Earth's equatorial plane with x axis typically defined toward a star and z axis is aligned with Earth's spin axis.
- For this reason, it is called Earth-centered inertial (ECI) frame.
- This frame does not rotate with the Earth.
- Are they really inertial axes?
- Theoretically, the axes that are fixed to Earth are not inertial axes.

- It is due to the various modes of motion which the Earth exhibits relative to the "fixed space".
- Most important noninertial influences
  - $\Rightarrow$ Daily rotation of the earth about its polar axis
  - $\Rightarrow$ Monthly rotation of the earth-moon system about its center of mass
  - $\Rightarrow$ Precession of the earth's polar axis about a line fixed in space
  - $\Rightarrow$ Motion of the sun with respect to the galaxy
  - $\Rightarrow$ Irregularities in the polar precession
- What about the validity of Newton's law in this frame?
- Approximately correct
- For vehicles navigating in the vicinity of the earth, computations of specific force are performed in this frame.

- How do you define Earth-centered Earth-fixed frame?
  - $\Rightarrow$ Origin at the Earth's center of mass
  - $\Rightarrow$ Rotating with Earth
  - $\Rightarrow$ Coincides with the inertial frame once every 24 hrs.
- It is also called as Earth frame or Geocenetric frame.
- Rotation of the Earth w.r.t. the ECI frame is about the same axis and in the same sense as the longitude.
- The z_e axis is directed north along the polar axis while the x_e, y_e axes are in the equatorial plane.
- The x_e axis is directed through the Greenwich Meridian (0° latitude, 0° longitude) and the y_e axis is directed through 90° East longitude.

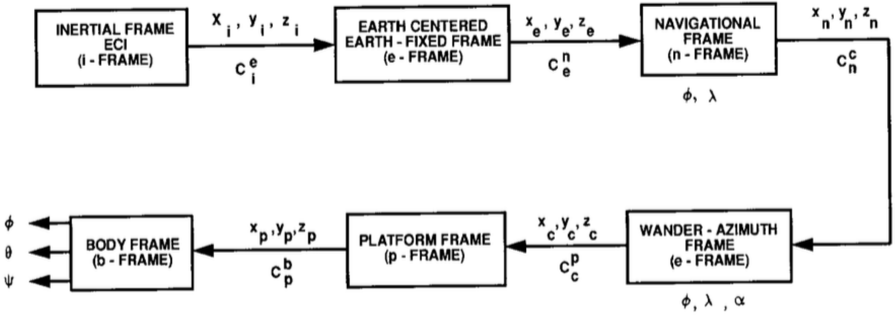
- Navigation frame (geographic frame or vehicle carried vertical frame)
  - $\Rightarrow$ Origin at the location of INS
  - $\Rightarrow$ A local-level frame with its $x - y$ axes in a plane tangent to the reference ellipsoid and z axis perpendicular to that ellipsoid.
  - $\Rightarrow$ Typically, x axis will point north, y axis east, and z axis down (or up) depending on selection of coordinate convention by the designer.
- Largest class of inertial navigation systems is the local-level type.
- Stable platform is constrained with two axes in the horizontal plane.
- Many INSs have been built using the local-level mechanization, mainly due to the error compensation simplifications of maintaining constant platform alignment to the gravity vector.
- Use of this conventional geographic set of axes leads to both hardware and computational difficulties in operation at the polar regions.

- Definition of body frame
  - $\Rightarrow$ Origin at the vehicle's center of mass
  - $\Rightarrow$ Mutually orthogonal axes along the body of vehicle
- What about the choice of axes?
- In aircraft applications, the convention is to choose the x axis pointing along the aircraft's longitudinal axis (roll axis).
- y axis out to the right wing (pitch axis), and z axis pointing down (yaw axis).
- Convenient for developing equations of motion of a vehicle
- Vehicle equations of motion are normally written in this frame.
- Typically used in the strapdown systems.

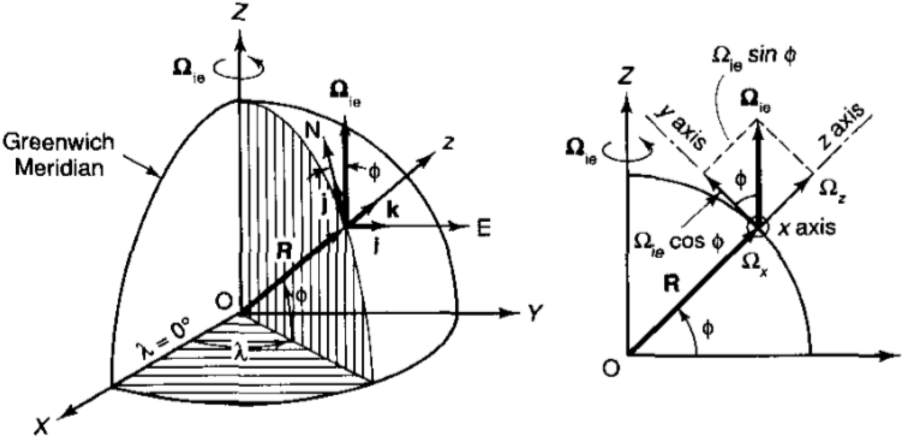
- Wander Azimuth frame: Special case of navigation frame
- Called as Computational frame
  - ⇒ Origin at the vehicle's center of mass (the system's location)
  - ⇒ Coincident with the origin of the navigation frame
- Horizontal axes of this local-level geodetic wander-azimuth frame lie in a plane tangent to the local vertical.
- Defined w.r.t. the Earth frame by three successive Eulerian angle rotations (longitude λ, latitude ϕ, and wander angle α).
- Latitude is defined to be positive in the northern hemisphere.
- Wander angle is defined to be positive west of true north and measured in geodetic horizon plane.
- What if the wander angle is made zero?
- This frame gets *aligned* with the navigational (or geographic) frame.

- What would happen if geodetic latitude, longitude, and wander angle are zero, that is, $\lambda = 0,\ \phi = 0,\ \alpha = 0$?
- Axes will be *aligned* with aligned with that of the Earth-fixed frame.
- **Geodetic coordinates**: Earth-fixed parameters defined in terms of Earth reference ellipsoid.
- **Geodetic longitude**: positive east of the Greenwich Meridian $(\lambda = 0)$, measured in reference equatorial plane.
- **Geodetic latitude**: positive north measured from the reference equatorial plane to ellipsoidal surface passing through the point of interest.
- Altitude h above reference ellipsoid measured along the normal passing through the point of interest.
- In a conventional NED mechanization, the vertical axis is precessed at a rate which keeps the two level axes pointing north and east at all times.
- However, this leads to a problem if one of the Earth's poles is traversed, in which case the required vertical precessional rate becomes infinitely large.

- Additional frames of reference
  - $\Rightarrow$ Platform
  - $\Rightarrow$ Accelerometer
  - $\Rightarrow$ Gyroscope
- Platform frame: Right-handed, orthogonal coordinate frame defined by the input axes of inertial sensors (typically gyroscope)
- Origin at the system location (INS), with its orientation in space being fixed
- This reference coordinate frame is a function of the configuration and mechanization of the particular inertial navigator under design
- If platform frame $\rightarrow$ body frame, strapdown system, whereas if platform frame $\rightarrow$ inertial frame, space-stable system.
- Accelerometer and gyroscope frames: Nonorthogonal frames defined by the input or sensitive axes of the instruments mounted on the inertial platform



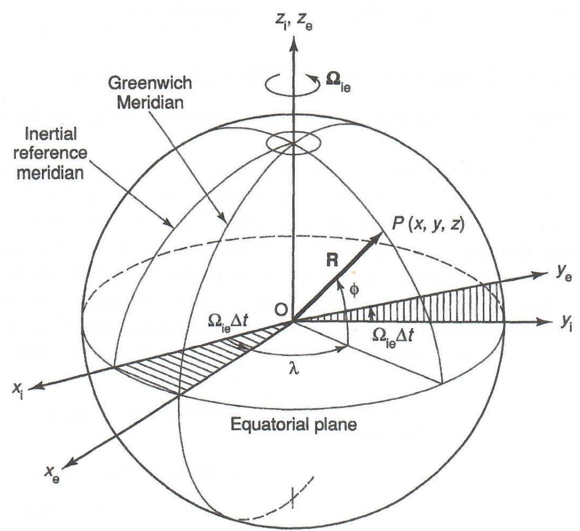
$$C_i^b = C_i^e C_e^n C_n^c C_c^p C_p^b$$



Earth rotation rate vector in NEU system

$$\boldsymbol{\Omega_{ie}} = \Omega_{ie} \cos\phi \boldsymbol{j} + \Omega_{ie} \sin\phi \boldsymbol{k}$$

Earth-Centered Inertial (ECI) to Earth-Fixed Transformation



- Both coordinate frames have their respective origin at the center of Earth.
- ECI frame has x_i axis pointing toward the true equinox of date at time t_0, z_i axis along the Earth's rotational axis, and y_i axis completes the right-handed orthogonal system.
- ECEF coordinate frame is related to the ECI frame by a single positive rotation about the z_i axis of $\Omega_{ie}\Delta t$, called as sidereal hour angle.
- Ω_{ie} is Earth's sidereal rotation rate, given by

$$\begin{aligned}\Omega_{ie} =& \frac{360}{23 + (56/60) + (4.09/3600)} = 15.04106874 \text{ deg/h} \\ =& 4.178074648 \times 10^{-3} \text{ deg/s} = 7.292115 \times 10^{-5} \text{ rad/s}\end{aligned}$$

- Δt is the time elapsed after vernal equinox.

- $\mathbf{\Omega}_{ie}$ w.r.t. ECEF frame is given by

$$\mathbf{\Omega}_{ie} = \begin{bmatrix} 0 \\ 0 \\ 7.292115 \times 10^{-35} \text{ rad/s} \end{bmatrix}$$

- Transformation matrix between ECI and ECEF frames

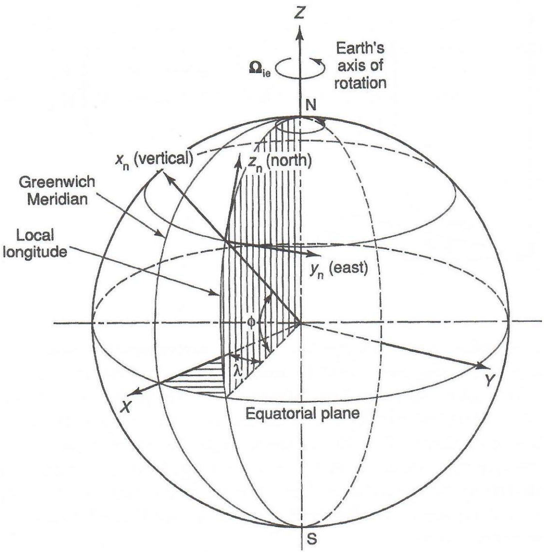
$$\begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} \cos\Omega_{ie}\Delta t & \sin\Omega_{ie}\Delta t & 0 \\ -\sin\Omega_{ie}\Delta t & \cos\Omega_{ie}\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

- Define $\Lambda = \Omega_{ie}\Delta t - 2n\pi$ where n is chosen such that $0 \leq \Lambda \leq 2\pi$.

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} \cos\Lambda & -\sin\Lambda & 0 \\ \sin\Lambda & \cos\Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix}$$

Coordinate Frames

Earth-fixed (ECEF) to Navigation Frame Transformation



- Origin at the system location.
- Navigation axes are commonly aligned with the north, east, up (or down) directions.
- For the present transformation, assume that the x axis points in the up direction, the y axis points east, and the z axis points north.
- This transformation, $\boldsymbol{C}_t$, is realized by two rotations: one through the angle λ about the z axis, and the other through the angle ϕ about the y axis.
- Rotation about z_e axis

$$\begin{bmatrix} x_e' \\ y_e' \\ z_e' \end{bmatrix} = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix}$$

- Rotation about y_e axis (look for sign of rotation angle ϕ)

$$\begin{bmatrix} x_e^{''} \\ y_e^{''} \\ z_e^{''} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x_e' \\ y_e' \\ z_e' \end{bmatrix}$$

- On combining these rotations

$$
\begin{aligned}
\begin{bmatrix} x_e^{''} \\ y_e^{''} \\ z_e^{''} \end{bmatrix} &= \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \end{bmatrix}}_{\boldsymbol{C}_e^n} \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{r}^n = \boldsymbol{C}_e^n \boldsymbol{r}^e
$$

- For transformation from ECI to navigation frame, what would be the rotation matrix?
- It is exactly same, except for λ replaced by $\Lambda = \Omega_{ie}\Delta t + (\lambda - \lambda_0)$ where λ_0 is longitude at time t_0 and $\Delta t = t - t_0$.

- Horizontal axes are displaced from east and north axes by wander angle α.
- Wander angle is taken to be positive west of true north.
- Transformation matrix

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix}$$

$$\boldsymbol{r}^c = \boldsymbol{C}_e^c \boldsymbol{r}^e$$

$$\boldsymbol{C}_e^c = \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\cos\alpha\sin\lambda - \sin\alpha\sin\phi\cos\lambda & \cos\alpha\cos\lambda - \sin\alpha\sin\phi\sin\lambda & \sin\alpha\cos\phi \\ \sin\alpha\sin\lambda - \cos\alpha\sin\phi\cos\lambda & -\sin\alpha\cos\lambda - \cos\alpha\sin\phi\sin\lambda & \cos\alpha\cos\phi \end{bmatrix}$$

- How to find latitude, longitude, and wander angles from given $\boldsymbol{C}_e^c$?

$$\alpha = \tan^{-1}\left(\frac{C_{zy}}{C_{zz}}\right), \quad \phi = \tan^{-1}\left(\frac{C_{zx}}{\sqrt{C_{xx}^2 + C_{yx}^2}}\right), \quad \lambda = \tan^{-1}\left(\frac{C_{yx}}{C_{xx}}\right)$$

- Origin at system location
- This transformation relates the north-east-up (NEU) local-vertical north-pointing frame to the ideal local-level wander azimuth frame.
- A positive, single-axis rotation about the vertical axis through the wander angle α is sufficient.

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$

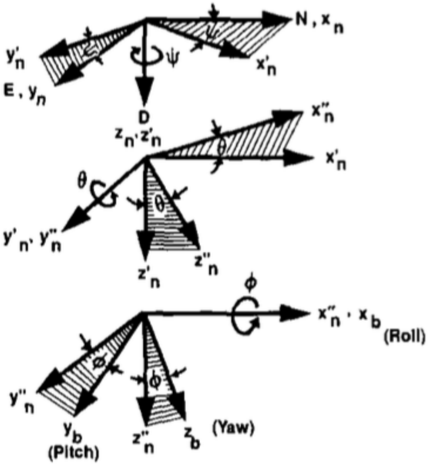
$$\boldsymbol{r}^c = \boldsymbol{C}_n^c \boldsymbol{r}^n$$

- Is this transformation matrix correct?
- Note the sign change due to negative sense of rotation with angle α.

- Origin at center of mass of vehicle
- Three successive single-axis rotations through the ordinary Euler angles of roll, pitch, and yaw
- The x_b axis points in the forward (longitudinal) direction, the y_b axis points out the right wing, and the z_b axis pointing down.
- Heading (yaw): Angle between x_n axis and the projection of the body axis on the horizontal plane. Positive when the aircraft nose is rotating from north to east (i.e., positive clockwise looking down)
- Pitch angle: Angle between the body axis and the body axis projection on the horizontal plane. Positive when the nose of the aircraft is elevated above the horizontal plane
- Roll angle: Negative of the rotation about x_b that would bring y_b into the horizontal plane. Positive when the right wing dips below the horizontal plane

Coordinate Frames

Body Frame to Navigation Frame Transformation



$$C_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$C_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \sin\phi \end{bmatrix}$$

- Overall transformation matrix

$$\boldsymbol{C}_n^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

- Direction cosine matrix

$$\boldsymbol{C}_n^b = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- Euler angles

$$\phi = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right), \quad \psi = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right), \quad \theta = \tan^{-1}\left(\frac{-C_{13}}{\sqrt{1 - C_{13}^2}}\right)$$

Reference

1. G. M. Siouris, *Aerospace Avionics Systems: A Modern Synthesis*, Academic Press, Inc. 1993.

Thank you for your attention !!!