Example 1.1.1 (Arithmetic Geometric Mean Inequality)

[Bertsekas]

To show that

$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1+x_2+\cdots+x_n}{n} \quad \forall\, x_i > 0 \quad i=1 \ldots n$$

GM AM

[Cauchy (1821)]

Proof: $x_i = e^{y_i}$ y_i unrestricted in sign $\in \mathbb{R}$

$$GM = \left(e^{y_1} \cdot e^{y_2} \cdots e^{y_n}\right)^{1/n}$$
$$= e^{\frac{y_1+y_2+\cdots+y_n}{n}}$$

$$AM = \frac{e^{y_1}+e^{y_2}+\cdots+e^{y_n}}{n}$$

T.S.T. $\boxed{e^{\frac{\sum^n y_i}{n}} \le \sum_{i=1}^n \frac{e^{y_i}}{n}}$

$$\psi(y) = \sum_{i=1}^n \frac{e^{y_i}}{n} - e^{\sum_{i=1}^n \frac{y_i}{n}} \ge 0$$

$$\min_y \psi(y) \qquad \text{T.S.T.} \qquad \psi(y^*) = 0$$

Suppose you could fix LHS

$y_1+y_2+\cdots+y_n = s$ ← Choose any s choice is arbitrary.

LHS $e^{s/n}$

RHS $\min_{y_1, y_2, \ldots y_n} \frac{e^{y_1}+e^{y_2}+\cdots+e^{y_n}}{n} \Rightarrow e^{s/n}$

$y_1+y_2+y_3+\cdots+y_n = s$

unconstrained minimization to constrained minimization constrained min prob $\Rightarrow$ unconstrained min prob

elimination of a variable.

$$y_n = S - (y_1 + y_2 - - - + y_{n-1})$$

$$\min_{y_1, y_2, \cdots y_{n-1}} \underline{\underline{e^{y_1}}} + \underline{\underline{e^{y_2}}} - - - + \underline{\underline{e^{y_{n-1}}}} + \underline{\underline{e^{(S - \overline{y_1 + y_2 - - + y_{n-1}})}}}$$

Q1) Does it have a min.

Q2) If yes, where.

$\Rightarrow$ Yes, it has a global min.

$\hookrightarrow$ coercive in nature.

obj func. is only sum of non-negative terms

$y_i \to \infty$ either of e^{y_n}s will go ∞ & $\begin{pmatrix} x_1 \neq x_2 \\ \hline x \\ x \end{pmatrix} = 0$ $\to \infty$ to ∞ & hence func will go ∞.

$e^{(S - \overline{y_1 + y_2 - - + y_n})} \to 0$

$y_i \to -\infty$

$e^{y_i} \to 0$ $\quad e^{\overline{S - (y_1 + y_2 - - + y_{n-1})}}$ $y_i \to -\infty$

obj func. $\to \infty$

whichever way $\|y\| \to \infty$, obj func. $\to \infty$

$\Rightarrow$ obj function is coercive and has a min. (global)

$x + \frac{1}{x} \geq 2$

$x > 0$

$$\min_{y_1, y_2 \cdots y_{n-1}} \quad \underbrace{e^{y_1} + e^{y_2} \cdots + e^{y_{n-1}}}_{} + e^{s-(y_1+y_2 \cdots +y_{n-1})}$$

$$\psi(y_1, y_2 \cdots y_{n-1}) \qquad \min \psi(y_1, y_2 \cdots y_{n-1})$$

$$\nabla\psi = 0 \qquad \begin{bmatrix} \frac{\partial\psi}{\partial y_1} \\ \vdots \\ \frac{\partial\psi}{\partial y_{n-1}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\frac{\partial\psi}{\partial y_i} = e^{y_i} - e^{s-(y_1+y_2 \cdots +y_{n-1})} = 0$$

$$\left.\begin{aligned} e^{y_1} &= e^{s-(y_1+y_2 \cdots +y_{n-1})} \\ &\vdots \\ e^{y_n} &= e^{s-(y_1+y_2 \cdots +y_{n-1})} \end{aligned}\right\} \begin{matrix} n-1 \\ \text{eqn} \end{matrix}$$

$$\begin{aligned} y_1 &= s-(y_1+y_2 \cdots +y_{n-1}) \Big\} \text{ go to } \ln \\ y_2 &= s-(y_1+y_2 \cdots +y_{n-1}) \\ &\cdots \\ y_{n-1} &= s-(y_1+y_2 \cdots +y_{n-1}) \end{aligned}$$

$$\begin{aligned} 2y_1 + y_2 \cdots + y_{n-1} &= s \\ y_1 + 2y_2 \cdots + y_{n-1} &= s \\ &\vdots \\ y_1 + y_2 \cdots + 2y_{n-1} &= s \end{aligned}$$

$$\underset{n-1 \times n-1}{\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdots & & & \\ 1 & 1 & \cdots & 2 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix}$$

$$\underline{y_1^* = y_2^* = \cdots = y_{n-1}^*}$$

$$n y_1^* = S \Rightarrow y_1^* = S/n$$
$$y_2^* = S/n$$
$$y_{n-1}^* = S/n$$

$S - (y_1 + y_2 \cdots + y_{n-1})$ $\quad \therefore y_n^* = S/n$

$$y_1^* = y_2^* = \cdots = y_n^* = S/n$$

$$\frac{e^{y_1} + e^{y_2} + \cdots + e^{y_n}}{n e^{S/n}}$$

AM $\quad \dfrac{n e^{S/n}}{n} = e^{S/n}$

$$LHS = e^{S/n}$$

$$x_1^* = x_2^* = \quad = x_n^* = x$$

GM $\quad (x_1 x_2 \cdots x_n)^{1/n} \quad (x^n)^{1/n} = x$

AM $\quad \dfrac{(x_1 + x_2 \cdots + x_n)}{n} = \dfrac{nx}{n} = x$

GM = AM

and everywhere else

GM < AM

$$(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0 \quad x_1 > 0, \; x_2 > 0$$

$$x_1 + x_2 - 2\sqrt{x_1 x_2} \geq 0$$

$$\sqrt{x_1 x_2} \le \frac{x_1 + x_2}{2} \checkmark$$ 2-variable thing.

$\underline{\underline{GM \le AM}}$