CS 747, Autumn 2020: Week 1, Lecture 1

Shivaram Kalyanakrishnan

Department of Computer Science and Engineering
Indian Institute of Technology Bombay

Autumn 2020

Multi-armed Bandits

1. The exploration-exploitation dilemma
2. Definitions: Bandit, Algorithm
3. ϵ-greedy algorithms
4. Evaluating algorithms: Regret

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A Game

Coin 1

$\mathbb{P}\{\text{heads}\} = p_1$

Coin 2

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Coin 3

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- p_1, p_2, and p_3 are unknown.
- You are given a total of 20 tosses.
- Maximise the total number of heads!

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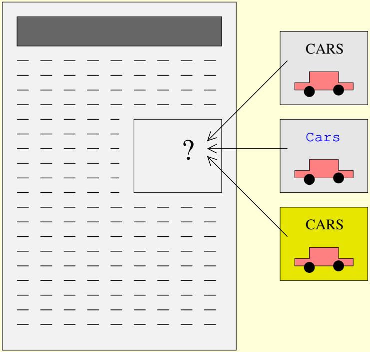
- Now we know: $p_1 = 0.6, p_2 = 0.3, p_3 = 0.8$.
- If you knew these biases beforehand, how would you have played?
- By so doing, how many heads would you have got in 20 tosses?

To Explore or to Exploit?

- On-line advertising: Template optimisation



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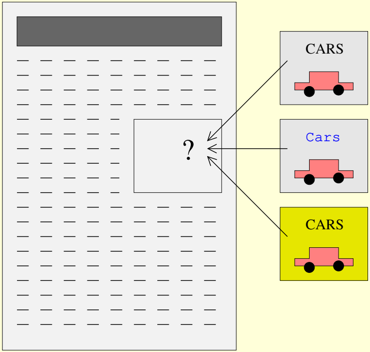

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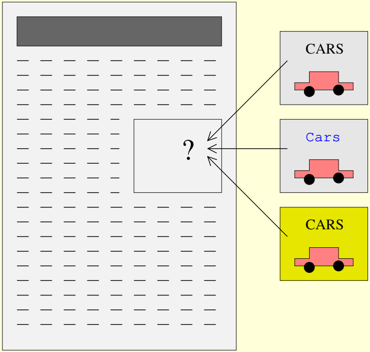

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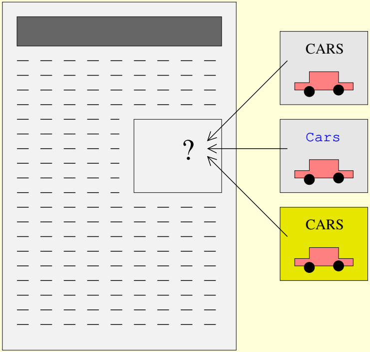

- Clinical trials
- Packet routing in communication networks
- Game playing and reinforcement learning

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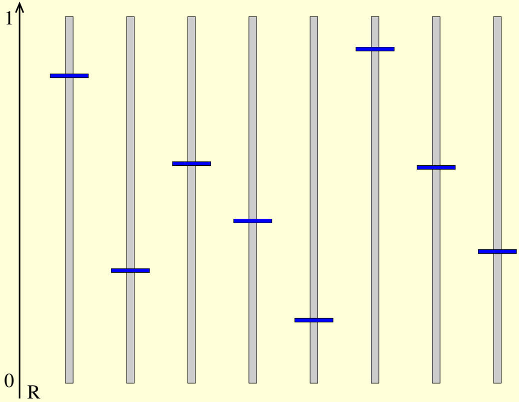
Stochastic Multi-armed Bandits



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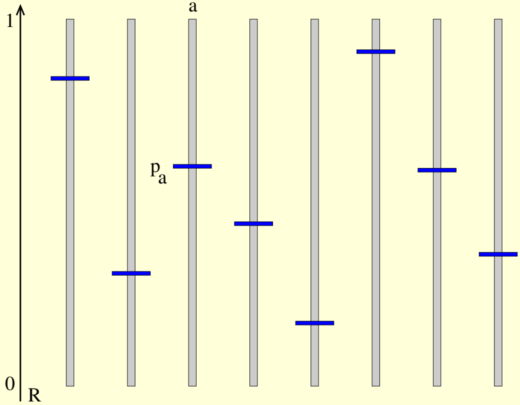

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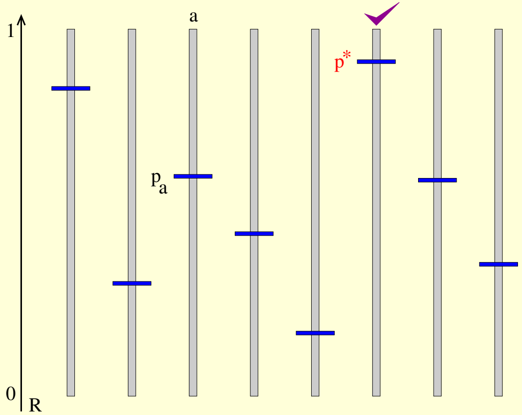

- *n* arms, each associated with a Bernoulli distribution (rewards are 0 or 1).
- Let A be the set of arms. Arm $a \in A$ has mean reward p_a.
- Highest mean is $p^\star$.

One-armed Bandits

[1]

1. https://pxhere.com/en/photo/942387.

Algorithm

- Here is what an algorithm does—

<u>For $t = 0, 1, 2, \ldots, T - 1$:</u>

- Given the history $h^t = (a^0, r^0, a^1, r^1, a^2, r^2, \ldots, a^{t-1}, r^{t-1})$,
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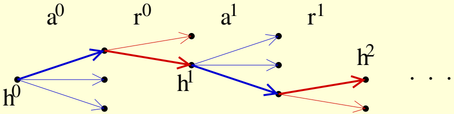
- **Note**: The algorithm picks the arm to pull; the bandit instance returns the reward.

Illustration



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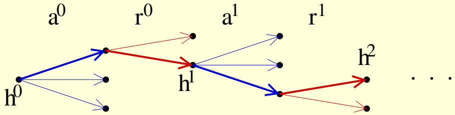

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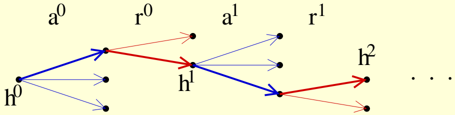

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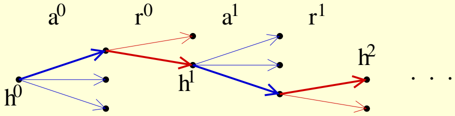

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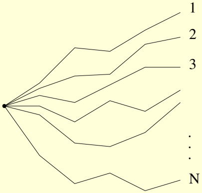
- An algorithm, bandit instance pair can generate many possible T-length histories.



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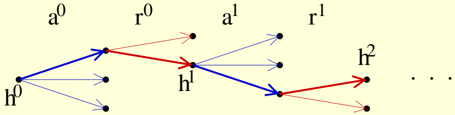

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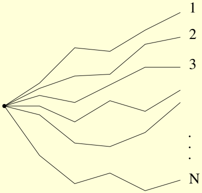

- How many histories possible if the algorithm is deterministic and rewards 0–1?

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- ϵG3
  - With probability ϵ, sample an arm uniformly at random; with probability $1 - \epsilon$, sample an arm with the highest empirical mean.

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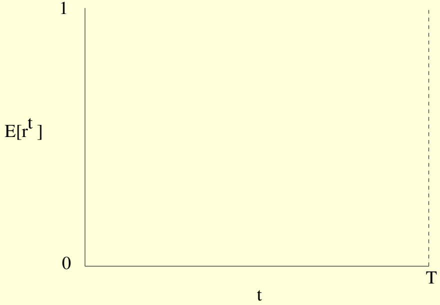
Visualising Performance

- Consider a plot of $\mathbb{E}[r^t]$ against t.



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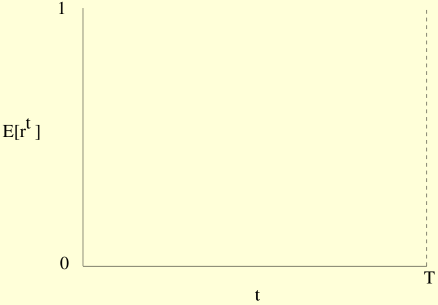

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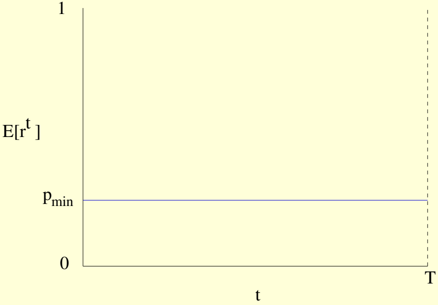

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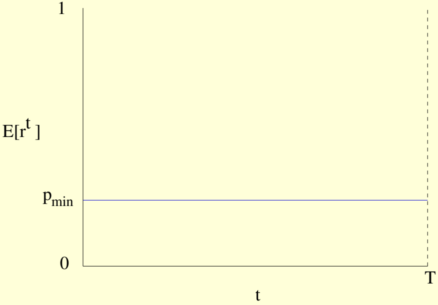

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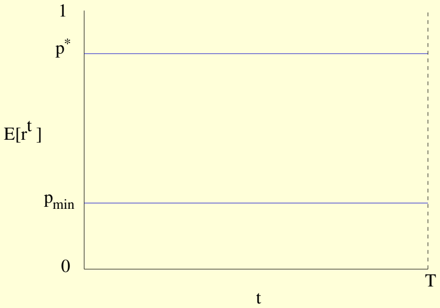

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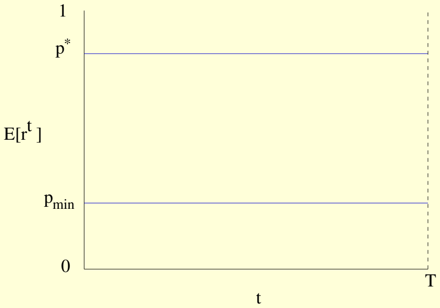

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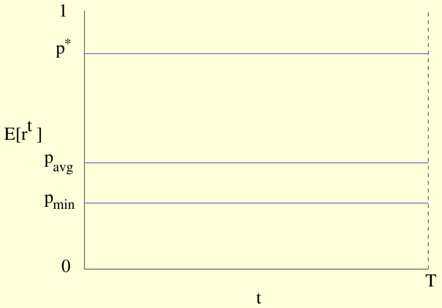

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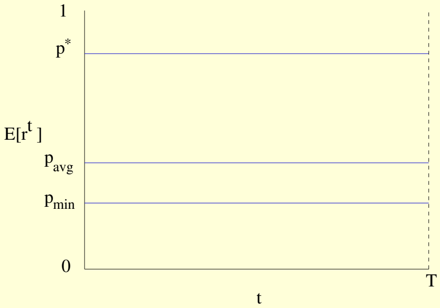

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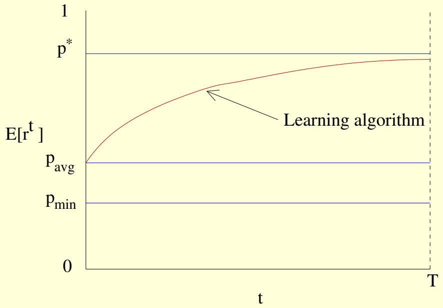

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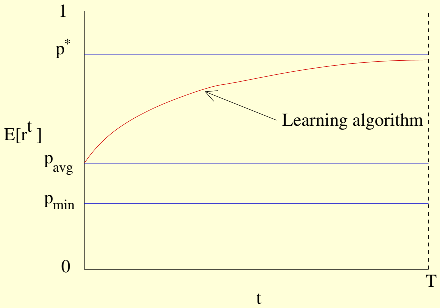

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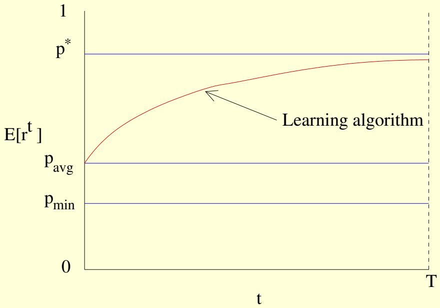

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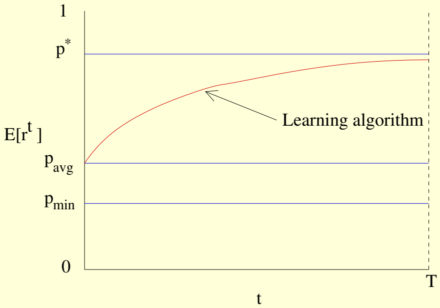

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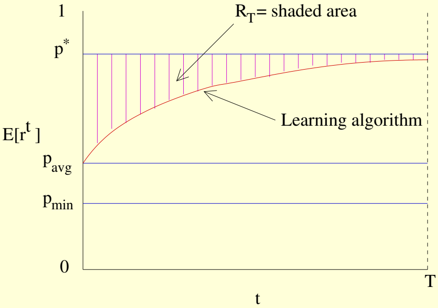

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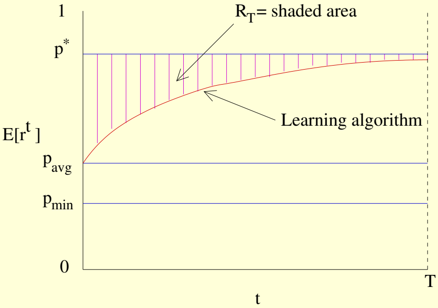

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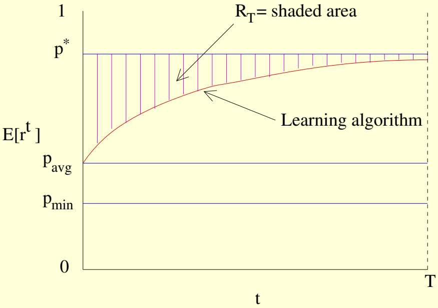

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  Does this happen for ϵG1, ϵG2, ϵG3?