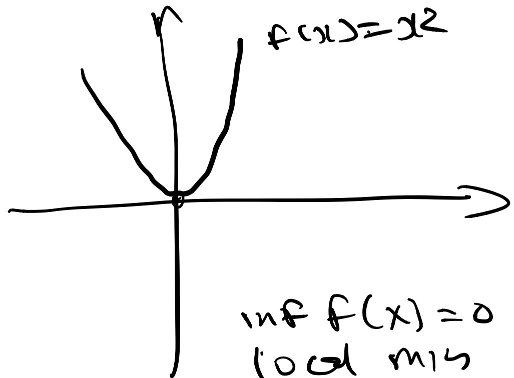
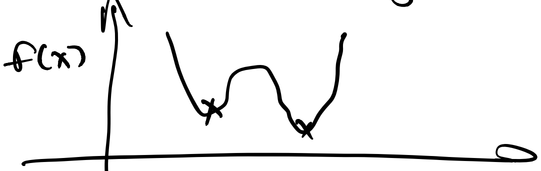
Lecture -21: Convex Sets and Functions

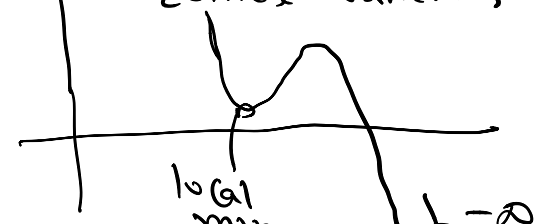
- Convex optimization (local min and global min)



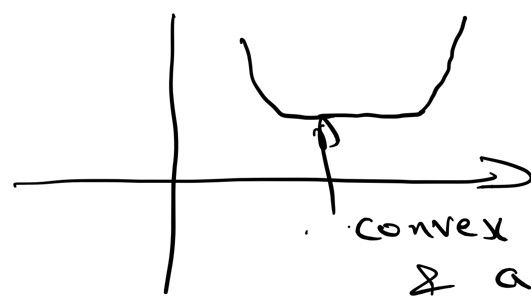
$\inf f(x) = 0$
local min
= global min!



this is not convex function;



$\inf f(x) = -\infty$;
this is not a convex function



convex function
& all minima are at same cost.

→ social welfare maximization problem (convex programming)

convex sets & functions.

First define convex sets:

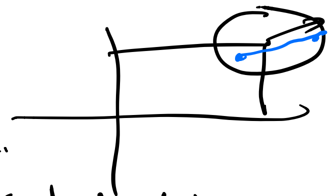
Then convex functions:

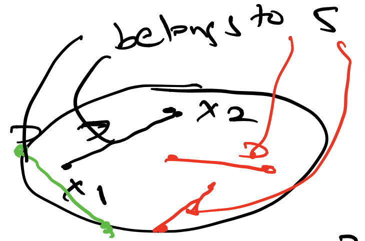
Finally optimizing functions on convex sets which is convex programming.

Defn: A set S in $\mathbb{R}^n$ is said to be convex, if the line segment joining two points of the set, belongs to the set S.

$(x_1-a)^2+(x_2-b)^2 \leq r^2$

↳ also a convex set. (Closed set)





$S \subset \mathbb{R}^2$

Convex set.

$(x_1-a)^2+(x_2-b)^2 < r^2$

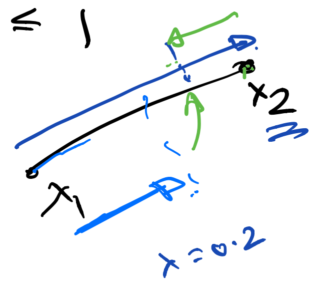
Q] is this convex. Yes?

convex set → closed, → open ✓

convex set: let $x_1 \in S$, $x_2 \in S$, arbitrary points

$L=\{x : x=\lambda x_1+(1-\lambda)x_2\}$ $0 \leq \lambda \leq 1$

should $\subset S$.



$\lambda x_1+(1-\lambda)x_2$

$\underline{x_2} - \lambda(x_2-x_1)$ ↳ $\in [0,1]$

$\lambda x_2+(1-\lambda)x_1$ ✓ $\lambda \in [0,1]$

↳ alternate description.

points on the line segment are not in S

line segment created from points

every linesegment

in set S should be in set S

Affine combinations.

Examples of convex sets:

1. $S = \{(x_1, x_2, x_3) : 3x_1 + 2x_2 - x_3 = 4\}$

Plane

$(a_1, a_2, a_3) \in S$, $(b_1, b_2, b_3) \in S$.

$\lambda a + (1-\lambda) b \qquad \lambda \in [0, 1]$

Check that it belong to S.

$$\lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + (1-\lambda) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$x_1 = \lambda a_1 + (1-\lambda) b_1$

$x_2 = \lambda a_2 + (1-\lambda) b_2$

$x_3 = \lambda a_3 + (1-\lambda) b_3$.

Check $3x_1 + 2x_2 - x_3 = 4$? If yes then S - convex

$3(\lambda a_1 + \overline{1-\lambda}\, b_1) + 2(\lambda a_2 + \overline{1-\lambda}\, b_2)$
$+ 3(\lambda a_3 + \overline{1-\lambda}\, b_3)$

$\lambda(3a_1 + 2a_2 + 3a_3) + (1-\lambda)(3 b_1 + 2b_2$
$+ 3b_3)$

$= \lambda 4 + (1-\lambda) 4 = 4$.

$\lambda \vec{a} + \overline{1-\lambda}\, \vec{b} \qquad \lambda \in [0, 1]$

$\in$ plane given! and hence

plane → convex set.

$\underline{\underline{\mathbb{R}^n}}$ Hyperplane:

$$S = \{x : p^t x = \alpha\}$$

$$p = \begin{pmatrix}3\\2\\3\end{pmatrix}$$

$$\alpha = 4;$$

$p_1x_1 + p_2x_2 \cdots + p_nx_n = \alpha \leftarrow$ constant.

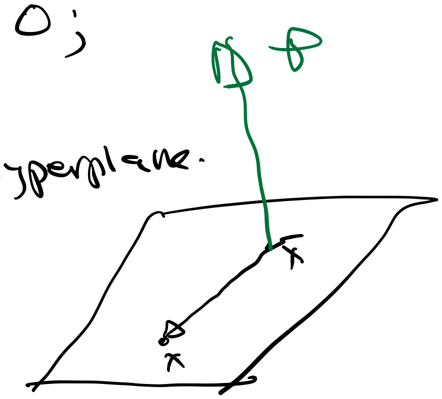
$$p^t \bar{x} = \alpha$$

$$p^t x = \alpha = p^t \bar{x}$$

$$p^t (x - \bar{x}) = 0;$$

p

gradient or normal to the hyperplane.



$$f(x) = p^t x$$

$$\nabla f(x) = p$$

Check: $S = \{x : p^t x = \alpha\}$ is a convex set.

$p^t \hat{x} = \alpha$ $\quad \hat{x} \in S$

$p^t \bar{x} = \alpha$ $\quad \bar{x} \in S$

$x = \lambda \hat{x} + (1-\lambda)\bar{x}$ $\quad \lambda \in [0, 1]$

$p^t x = p^t[\lambda \hat{x} + \overline{1-\lambda}\;\bar{x}]$ $\quad p^t\hat{x} = \alpha$, $p^t \bar{x} = \alpha$

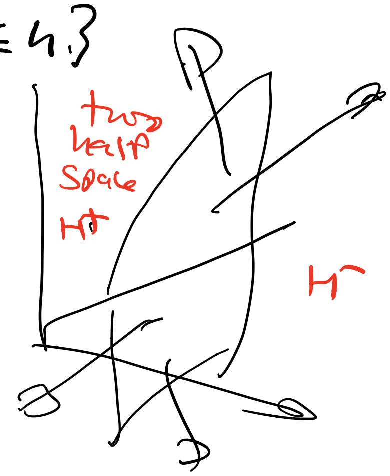
$= \lambda\alpha + \overline{1-\lambda}\;\alpha = \alpha$

$p^t x = \alpha$ [on the hyperplane]

hyperplane is a convex set.

$S = \{(x_1, x_2, x_3) : 3x_1 + 2x_2 - x_3 \le 4\}$

half space.



$H^- S = \{x : p^t x \le \alpha\}$

$H^+ S = \{x : p^t x \ge \alpha\}$

$$H^+ \cup H^- = R^n$$

Q] Is the half space convex?

Yes

$p^t \hat{x} \le \alpha$; $\quad \hat{x} \in H^-$

$p^t \bar{x} \le \alpha$; $\quad \bar{x} \in H^-$

$x = \lambda \hat{x} + (1-\lambda)\bar{x} \qquad \lambda \in (0,1]$

$$
\begin{aligned}
p^t x &= \lambda p^t \hat{x} + (1-\lambda) p^t \bar{x} \\
&\le \lambda \alpha + (1-\lambda)\alpha \\
&= \alpha
\end{aligned}
$$

$p^t x \le \alpha$

Half space is also convex.

$3 \le 5$

$0.2 \times 3 \le 0.2 \times 5$

$-0.2 \times 3 \ge -0.2 \times 5$

3. $S = \{ (x_1, x_2, x_3) : x_1 + 2x_2 - x_3 \le 5$
$3x_1 - 2x_2 + x_3 \le 6 \}$

Intersection of two half spaces.

This is also convex.

$\hat{x} \in S \qquad \bar{x} \in S$

$p_1^t \hat{x} \le \alpha_1 \qquad p_1^t \bar{x} \le \alpha_1$

$p_2^t \hat{x} \le \alpha_2 \qquad p_2^t \bar{x} \le \alpha_2.$

$x : \lambda \hat{x} + (1-\lambda) \bar{x}$

$p_1^t x \in H_1^-$ (first half space

$p_2^t x \in H_2^-$ (second half spaces

$\therefore x \in S \}.$

and hence intersection of half spaces is a convex set.

$S = \{ x : \underset{m\times n}{A} \underset{n\times 1}{x} \le \underset{m\times 1}{b} \}.$ also convex

$$Ax \le b = \left\{ \begin{array}{l} a_1^T x \le b_1 \ \cap \\ a_2^T x \le b_2 \ \cap \\ \vdots \\ a_m^T x \le b_m \end{array} \right]$$

a_1^T + first row of A;

Intersection of m-half spaces
polyhedral set is also convex.

4. $S = (x_1, x_2 : x_2 \geq |x_1| \}$
convex cone

5- $x_1^2 + x_2^2 \leq 9$
closed disc