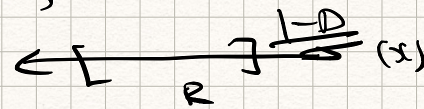
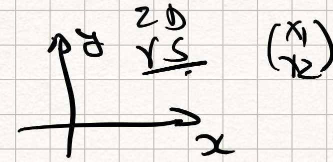
$R^n \rightarrow (x_1, x_2, \cdots x_n)^T$; $x_i \in R$, $i = 1 \text{--} n$.

vector space

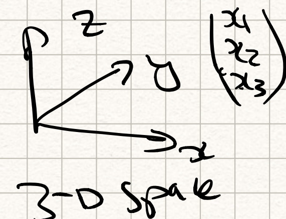


Vector norm

$\mapsto$ generalization of the distance concept.



$f : R^n \rightarrow R$. (multivariate)



is called a norm if it satisfies following conditions

1) $f(x) \geq 0 \quad \forall x \in R^n$ ($f(x) = 0$ iff $x = 0$)
   [non-negativity]

2) $f(x+y) \leq f(x) + f(y) \quad \forall x, y \in R^n$
   [Triangular inequality]

3) $f(\alpha x) = |\alpha| f(x)$
   [Scalability]

vector norm functions are shown by double bar notation. $f(x) = \|x\|$

Class of vector norms p-norms

$$\|x\|_p = \left(|x_1|^p + |x_2|^p \cdots + |x_n|^p\right)^{1/p}$$

$p \geq 1$

$p = 1, 2, 3 \ldots$

3 - vector norms are routinely used.

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| \geq 0$$

Ex $\left\| \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\|_1 = 5 + 2 + 1 = 8$

$$\|x\|_2 = \left(|x_1|^2 + |x_2|^2 \cdots + |x_n|^2\right)^{1/2}$$

$$\|x\|_2 = \left(x_1^2 + x_2^2 \cdots + x_n^2\right)^{1/2} \geq 0$$

Ex $\left\| \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\|_2 = \sqrt{5^2 + 2^2 + 1^2}$

$= \sqrt{30}$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Ex. $\left\| \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\|_\infty = 5$

HW Show that p-norm is indeed a norm.

$$\|\alpha x\|_1 = |\alpha x_1| + |\alpha x_2| + \cdots + |\alpha x_n|$$

$$= |\alpha|\,|x_1| + |\alpha|\,|x_2| \cdots + |\alpha|\,|x_n|$$

$$= |\alpha| \left(|x_1| + |x_2| \cdots + |x_n|\right)$$

$$= |\alpha|\, \|x\|_1.$$

similarly $\|\alpha x\|_2 = |\alpha| \, \|x\|_2$

$\|\alpha x\|_\infty = |\alpha| \, \|x\|_\infty$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

because it leads to different function.

→ continuity

→ And all norms are uniformly continuous on $\mathbb{R}^n$.

(1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

(2) $f(x+y) \leq f(x) + f(y)$

(3) $f(\alpha x) = |\alpha| f(x)$

→ Then $\|x\|$ is continuous function.

How do I prove it?

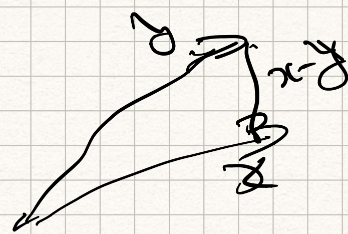
$\|x+y\| \leq \|x\| + \|y\|$

$$\begin{aligned}\|x-y\| &= \|x+(-y)\| \\ &\leq \|x\| + \|-y\| \\ &\leq \|x\| + \|y\|\end{aligned}$$

$$\|x\| - \|y\| \le \|x-y\| \le \underbrace{\|x\| + \|y\|}_{\Delta\text{-ular inequality}}$$

$$\|x\| = \|x-y+y\|$$

$$\le \|x-y\| + \underbrace{\|y\|}_{\text{to LHS}}$$



$$\boxed{\|x\| - \|y\| \le \|x-y\|}$$

T.P.T. $\|x\|$ is a continuous function.

Proof: Choose $\varepsilon > 0$, show $\exists N(x,\varepsilon) = \delta$

s.t. $\|y-x\|$ in this δ-neighborhood.

existence $\delta > 0$

$\forall y$ in this neighborhood

$$\|y - x\| < \delta$$

$$-\varepsilon < \|y\| - \|x\| < \varepsilon \qquad \varepsilon > 0$$

hold.

I am going to show that

$$\delta = \varepsilon,$$

Choose $\delta = \varepsilon$.

$$\|y\| - \|x\| \le \|y - x\| \le \varepsilon$$

$$\underline{\|y\| - \|x\| < \varepsilon}$$

J. S.T.

$$-\varepsilon < \|y\| - \|x\|$$

$$\|y\| - \|x\| \leq \|y - x\|$$

$$\|x\| - \|y\| \leq \|y - x\|$$

$$\|x - y\| = \|y - x\|$$

$$\|y\| - \|x\| \leq \|y - x\|$$

$$\|x\| - \|y\| \leq \|x - y\|$$

$$\|y\| > \|x\|$$

$$\|y\| - \|x\| = 5$$

$$\|y\| - \|x\| \leq \|y - x\|$$

$$\|x\| - \|y\| \leq \|y - x\|$$

$$5 \leq \|y - x\|$$

$$-5 \leq \|y - x\|$$

$$\|x\| - \|y\| \leq \|y - x\| < \varepsilon \quad 7$$

$$\left[\|x\| - \|y\| < \varepsilon\right] \quad x - 1$$

$$\|y\| - \|x\| > -\varepsilon.$$

$$[3 < 5] \times -1$$

$$-3 > -5$$

$$-\varepsilon < \|y\| - \|x\| < \varepsilon$$

Continuity Nbhd.

$$N.(x,\varepsilon) = \{y : \|x - y\| < \varepsilon\}$$

$$\varepsilon \to 0$$

Continuity is derived for the norm!