Coordinate Transformation: Quaternions

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- Complex numbers $\mathbb{C} = \{a + bi|\ a,\ b \in \mathbb{R},\ i^2 = -1\}$ form a plane.
- Their operations are related to two-dimensional geometry.
- Any complex number has a length, given by the Pythagorean formula

$$|a + bi| = \sqrt{a^2 + b^2}$$

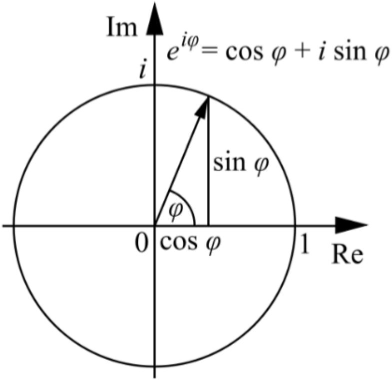
- We can add and subtract in $\mathbb{C}$.

$$a + bi + c + di = (a + c) + (b + d)i$$

- We can also multiply in $\mathbb{C}$.

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

- What does this last formula mean? difficult to interpret
- Fortunately, there is a better way to multiply complex numbers.



We can use complex arithmetic (multiplication) to perform a geometric operation (rotation).

- Geometrically, this formula says $e^{i\phi}$ lies on the unit circle in $\mathbb{C}$.
- If we multiply $e^{i\phi}$ by a positive number r, we get a complex number of length r, $re^{i\phi}$.
- If we denote $a + bi = r_1 e^{i\theta_1}$ and $c + di = r_2 e^{i\theta_2}$ then

$$(a + bi)(c + di) = r_1 r_2 e^{\theta_1 + \theta_2}$$

- To multiply two complex numbers, multiply their lengths and add their angles.
- In particular, if we multiply a given complex number z by $e^{i\phi}$ then it is rotated by ϕ degrees.

- The 19th century Irish mathematician and physicist *William Rowan Hamilton* was fascinated by the role of $\mathbb{C}$ in two-dimensional geometry.
- For years, he tried to invent an algebra of "triplets" to play the same role in three dimensions.
- On October 16th, 1843, while walking with his wife to a meeting of the Royal Society of Dublin, Hamilton discovered a 4-D division algebra called the *quaternions*.

- Although, similar concept was developed by Gauss in 1819 (but unfortunately not published).


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

- Hamilton noticed that

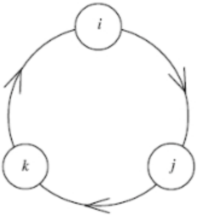
$$\boxed{\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1}$$

- The quaternions are denoted as

$$\mathbb{H} = \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} : a, b, c, d \in \mathbb{R}\}.$$

- Cyclic symmetry:

$$\begin{aligned} \boldsymbol{ij} &= \boldsymbol{k} = -\boldsymbol{ji} \\ \boldsymbol{jk} &= \boldsymbol{i} = -\boldsymbol{kj} \\ \boldsymbol{ki} &= \boldsymbol{j} = -\boldsymbol{ik} \end{aligned}$$



- Quaternions don't commute.
- $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are recognized as unit vectors.

- The quaternion product is the same as the cross product of vectors.

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{j} &= \boldsymbol{k} \\ \boldsymbol{j} \times \boldsymbol{k} &= \boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} &= \boldsymbol{j} \end{aligned}$$

- However, unlike the unit vectors $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0$, we have

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{i} &= -1 \\ \boldsymbol{j} \times \boldsymbol{j} &= -1 \\ \boldsymbol{k} \times \boldsymbol{k} &= -1 \\ \boldsymbol{i} \times \boldsymbol{j} \times \boldsymbol{k} &= -1 \end{aligned}$$

- A Hamilton quaternion can be considered as scalar part and a vector part

$$\boxed{[Q] = \langle q_0, \boldsymbol{q} \rangle = q_0 + \underbrace{q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}}_{\boldsymbol{q}}}$$

- $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ serves as basis for quaternion vector space.
- Quaternions span the space of real and imaginary numbers.
- Quaternion algebra includes scalar and vector algebra.
- Addition, subtraction and multiplication: Similar way as in vector algebra.
- **Addition**:

$$\begin{aligned}[Q] + [S] =&(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) + (s_0 + s_1\boldsymbol{i} + s_2\boldsymbol{j} + s_3\boldsymbol{k})\\ =&(q_0 + s_0) + (q_1 + s_1)\boldsymbol{i} + (q_2 + s_2)\boldsymbol{j} + (q_3 + s_3)\boldsymbol{k}\end{aligned}$$

- **Subtraction**: Addition of negative quaternion $-[Q] = (-1)[Q]$

$$\begin{aligned}[Q] - [S] =& (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) - (s_0 + s_1\boldsymbol{i} + s_2\boldsymbol{j} + s_3\boldsymbol{k}) \\ =& (q_0 - s_0) + (q_1 - s_1)\boldsymbol{i} + (q_2 - s_2)\boldsymbol{j} + (q_3 - s_3)\boldsymbol{k}\end{aligned}$$

- **Multiplication**:

$$\begin{aligned}[Q][S] =& (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(s_0 + s_1\boldsymbol{i} + s_2\boldsymbol{j} + s_3\boldsymbol{k}) \\ =& (q_0s_0 - q_1s_1 - q_2s_2 - q_3s_3) + (q_0s_1 + q_1s_0 + q_2s_3 - q_3s_2)\boldsymbol{i} \\ & + (q_0s_2 - q_1s_3 + q_2s_0 + q_3s_1)\boldsymbol{j} + (q_0s_3 + q_1s_2 - q_2s_1 + q_3s_0)\boldsymbol{k}\end{aligned}$$

- Using dot product of vectors $\boldsymbol{q}.\boldsymbol{s} = q_1s_1 + q_2s_2 + q_3s_3$, we have

$$\boxed{[Q][S] = \langle q_0, \boldsymbol{q}\rangle\langle s_0, \boldsymbol{s}\rangle = \langle q_0s_0 - \boldsymbol{q}.\boldsymbol{s}, q_0\boldsymbol{s} + s_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s}\rangle}$$

- Product of two quaternions is still a quaternion, with scalar part $(q_0s_0 - \boldsymbol{q}\boldsymbol{s})$ and vector part $(q_0\boldsymbol{s} + s_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s})$.

- The set of quaternions is closed under multiplication and addition.

Example

Consider two quaternions below and find their quaternion product.

$$
\begin{aligned}
[Q] =& 3 + \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} \\
[S] =& 2 - \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}
\end{aligned}
$$

$\Rightarrow$ $\boldsymbol{q}.\boldsymbol{s} = -2, \;\; \boldsymbol{q} \times \boldsymbol{s} = -8\boldsymbol{i} - 4\boldsymbol{j}.$

$\Rightarrow$ We know that

$$
\begin{aligned}
\langle q_0, \boldsymbol{q}\rangle\langle s_0, \boldsymbol{s}\rangle =& \langle q_0 s_0 - \boldsymbol{q}.\boldsymbol{s}, q_0\boldsymbol{s} + s_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s}\rangle \\
=& 6 - (-2) + 3(-\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + 2(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) + (-8\boldsymbol{i} - 4\boldsymbol{j}) \\
=& 8 - 9\boldsymbol{i} - 2\boldsymbol{j} + 11\boldsymbol{k}
\end{aligned}
$$

- **Scalar multiplication**:

$$\lambda[Q] = \lambda q_0 + \lambda q_1 \boldsymbol{i} + \lambda q_2 \boldsymbol{j} + \lambda q_3 \boldsymbol{k}$$

- **Conjugate**:

$$[Q]^\star = q_0 - \boldsymbol{q} = q_0 - q_1 \boldsymbol{i} - q_2 \boldsymbol{j} - q_3 \boldsymbol{k}$$

- We have the following relations

$$\begin{aligned}
([Q]^\star)^\star =& q_0 - (-\boldsymbol{q}) = [Q] \\
[Q] + [Q]^\star =& 2q_0 \\
[Q]^\star [Q] =& (q_0 + \boldsymbol{q})(q_0 - \boldsymbol{q}) \\
=& q_0 q_0 - (-\boldsymbol{q}).\boldsymbol{q} + q_0 \boldsymbol{q} + (-\boldsymbol{q}) q_0 + (-\boldsymbol{q} \times \boldsymbol{q}) \\
=& q_0^2 + \boldsymbol{q}.\boldsymbol{q} \\
=& q_0^2 + q_1^2 + q_2^2 + q_3^2 = [Q][Q]^\star
\end{aligned}$$

- **Norm of Length of quaternion**:

$$N([Q]) = [Q][Q]^\star = [Q]^\star[Q] = (q_0 + \boldsymbol{q})(q_0 - \boldsymbol{q})$$
$$= q_0^2 + \boldsymbol{q}.\boldsymbol{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

- For two quaternions $[Q]$ and $[S]$, we have

$$\boxed{([Q][S])^\star = [S]^\star[Q]^\star}$$

$\Rightarrow$ Proof:

$$\begin{aligned}([Q][S])^\star &= [(q_0 + \boldsymbol{q})(s_0 + \boldsymbol{s})]^\star \\ &= (q_0 s_0 - \boldsymbol{q}.\boldsymbol{s} + q_0\boldsymbol{s} + s_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s})^\star \\ &= (q_0 s_0 - \boldsymbol{q}.\boldsymbol{s} - q_0\boldsymbol{s} - s_0\boldsymbol{q} - \boldsymbol{q} \times \boldsymbol{s})\end{aligned}$$

$$\begin{aligned}[S]^\star[Q]^\star &= (s_0 - \boldsymbol{s})(q_0 - \boldsymbol{q}) \\ &= q_0 s_0 - (-\boldsymbol{s}).(-\boldsymbol{q}) + q_0(-\boldsymbol{s}) + s_0(-\boldsymbol{q}) + (-\boldsymbol{s}) \times (-\boldsymbol{q}) \\ &= (q_0 s_0 - \boldsymbol{q}.\boldsymbol{s} - q_0\boldsymbol{s} - s_0\boldsymbol{q} - \boldsymbol{q} \times \boldsymbol{s}) = ([Q][S])^\star\end{aligned}$$

- Norm of product of two quaternions is equal to product of their norms.

$$N([Q][S]) = N([Q])N([S])$$

$\Rightarrow$ Proof:

$$\begin{aligned} N([Q][S]) &= ([Q][S])([Q][S])^\star \\ &= [Q][S][S]^\star[Q]^\star \\ &= [Q]N([S])[Q]^\star \\ &= N([Q])N([S]) \end{aligned}$$

- Also, by using mathematical induction, one may write

$$\boxed{N([Q_1][Q_2]\ldots[Q_n]) = N([Q_1])N([Q_2])\ldots N([Q_n])}$$

- **Inverse of quaternion**: If $[Q] \neq 0$ then its inverse is defined by

$$[Q][Q]^{-1} = [Q]^{-1}[Q] = 1$$

- Using norm concept, $[Q]^{-1} = \dfrac{[Q]^\star}{N(Q)}$, $N(Q) \neq 0$. Does it make sense?

$$[Q][Q]^{-1} = [Q]^{-1}[Q] = \frac{[Q][Q]^\star}{N(Q)} = 1$$

- If $[Q]$ is unit quaternion then

$$[Q]^{-1} = \frac{[Q]^\star}{N(Q)} = [Q]^\star$$

- Inverse and conjugate for the unit quaternions are the same.

Identities:

- How to define zero and unit quaternions?
- A zero quaternion is quaternion with zero scalar and zero vector.
- A unit quaternion is defined as any quaternion whose norm is 1.

$$[0] = 0 + 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}, \quad [1] = 1 + 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$$

- Unlike direction cosine matrix, where six redundancies are present, the quaternion has only one.
- For unit quaternion,

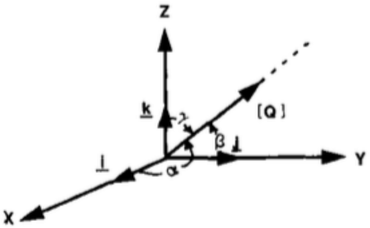
$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

- $\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ can be used for normalizing factor for each parameter.

- How to identify if two quaternions are equal?
- **Equality of quaternions**: Two quaternions are equal if both their scalars as well as their vectors are equal.

$$
\begin{aligned}
[Q] =& [S] \\
\Rightarrow q_0 =& s_0,\ q_1 = s_1,\ q_2 = s_2,\ q_3 = s_3 \\
\Rightarrow [Q]_i =& [S]_i \ \forall\ i = 0, 1, 2, 3
\end{aligned}
$$

- Can we express 3D vector as quaternion?
- Any three dimensional vector can be expressed as quaternion with zero scalar.



- Quaternions obey the associative and commutative laws of addition, and the associative and distributive laws of multiplication.
- For three quaternions, Q_1, Q_2, Q_3
  - Associative addition

$$(Q_1 + Q_2) + Q_3 = Q_1 + (Q_2 + Q_3)$$

  - Commutative addition

$$Q_1 + Q_2 = Q_2 + Q_1$$

  - Associative multiplication

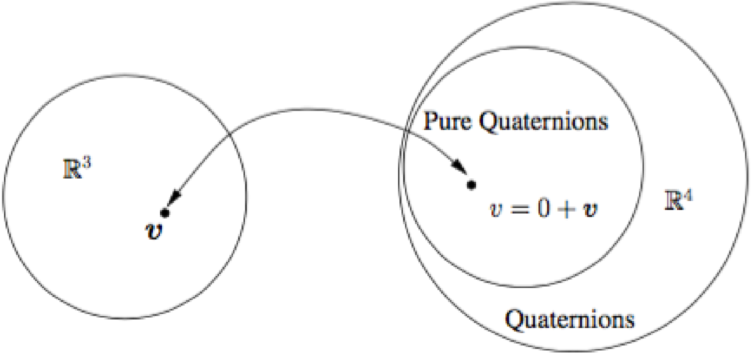
$$(Q_1 Q_2) Q_3 = Q_1 (Q_2 Q_3)$$

  - Distributive multiplication

$$Q_1 (Q_2 + Q_3) = Q_1 Q_2 + Q_1 Q_3$$

- Is the multiplication of quaternions commutative?

- Pure quaternions: Quaternion with zero real or scalar part
- Any vector in $\mathbb{R}^3$ is a pure quaternion.



- How quaternion $[Q] \in \mathbb{R}^4$ operate on a vector in $\mathbb{R}^3$?
- Define quaternion operator with unit quaternion $[Q]$ as

$$L_Q(\boldsymbol{v}) = [Q]\boldsymbol{v}[Q]^\star = (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v})$$

$\Rightarrow$ Proof:

$$\begin{aligned}
L_Q(\boldsymbol{v}) =& [Q]\boldsymbol{v}[Q]^\star = (q_0 + \boldsymbol{q})(0 + \boldsymbol{v})(q_0 - \boldsymbol{q}) \\
=& (q_0 + \boldsymbol{q})(\boldsymbol{v}.\boldsymbol{q} + \{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\}) \\
=& q_0(\boldsymbol{v}.\boldsymbol{q}) - \boldsymbol{q}.\{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} + (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0\{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} \\
& + \boldsymbol{q} \times \{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} \\
=& \boldsymbol{q}.\{\boldsymbol{v} \times \boldsymbol{q}\} + (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0\{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} + \boldsymbol{q} \times \{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} \\
=& (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0^2\boldsymbol{v} + 2q_0\{\boldsymbol{q} \times \boldsymbol{v}\} + \boldsymbol{q} \times \{\boldsymbol{q} \times \boldsymbol{v}\} \\
=& (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0^2\boldsymbol{v} + 2q_0\{\boldsymbol{q} \times \boldsymbol{v}\} + \boldsymbol{q}(\boldsymbol{q}.\boldsymbol{v}) - \boldsymbol{v}(\boldsymbol{q}.\boldsymbol{q}) \\
=& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0\{\boldsymbol{q} \times \boldsymbol{v}\}
\end{aligned}$$

Operation of Unit Quaternion on Vector

This operator L_Q does not change the length of the vector $\boldsymbol{v}$.

$$\|L_Q(\boldsymbol{v})\| = \|[Q]\boldsymbol{v}[Q]^\star\| = |[Q]|\|\boldsymbol{v}\||[Q]^\star| = \|\boldsymbol{v}\|$$

The direction of $\boldsymbol{v}$, if along $\boldsymbol{q}$ (say $\boldsymbol{v} = k\boldsymbol{q}$), is left unchanged by the operator L_Q.

$$\begin{aligned}[Q]\boldsymbol{v}[Q]^\star =& [Q](k\boldsymbol{q})[Q]^\star = (q_0^2 - |\boldsymbol{q}|^2)(k\boldsymbol{q}) + 2(\boldsymbol{q}.(k\boldsymbol{q}))\boldsymbol{q} + 2q_0(\boldsymbol{q} \times (k\boldsymbol{q})) \\ =& k(q_0^2 + \|\boldsymbol{q}\|^2)\boldsymbol{q} = k\boldsymbol{q}\end{aligned}$$

- Any vector along $\boldsymbol{q}$ is thus not changed under operator L_Q. This makes us guess that the operator L_Q acts like a rotation about $\boldsymbol{q}$.
- The operator L_Q is linear over $\mathbb{R}^3$. For any two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ and any $a_1, a_2 \in \mathbb{R}$

$$L_Q(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2) = a_1 L_Q(\boldsymbol{v}_1) + a_2 L_Q(\boldsymbol{v}_2).$$

Rotation of Vector using Quaternion

For any unit quaternion

$$[Q] = q_0 + \boldsymbol{q} = \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2}$$

and for any vector $\boldsymbol{v} \in \mathbb{R}^3$, the action of the operator $L_Q(\boldsymbol{v}) = [Q]\boldsymbol{v}[Q]^\star$ on $\boldsymbol{v}$ is equivalent to a rotation of the vector through an angle θ, about $\hat{q}$ as the axis of rotation.

- A vector $\boldsymbol{v} \in \mathbb{R}^3$, we decompose it as $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{n}$, where $\boldsymbol{a}$ is the component along the vector $\boldsymbol{q}$ and $\boldsymbol{n}$ is the component normal to $\boldsymbol{q}$.
- Under the operator L_Q, $\boldsymbol{a}$ is invariant, while $\boldsymbol{n}$ is rotated about $\boldsymbol{q}$ through an angle θ.
- Since the operator is linear, the image $[Q]\boldsymbol{v}[Q]^\star$ is indeed interpreted as a rotation of $\boldsymbol{v}$ about $\boldsymbol{q}$ through an angle θ.

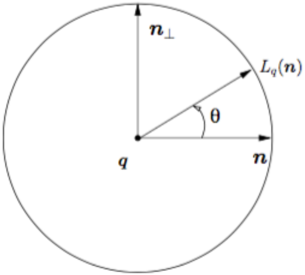
- The operator L_q on vectors $\boldsymbol{n}$

$$\begin{aligned} L_Q(\boldsymbol{n}) =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2(\boldsymbol{q}.\boldsymbol{n})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{n}) \\ =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2q_0(\boldsymbol{q} \times \boldsymbol{n}) \\ =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2q_0\|\boldsymbol{q}\|(\hat{\boldsymbol{q}} \times \boldsymbol{n}) \end{aligned}$$

- Denote $\boldsymbol{n}_\perp = \hat{\boldsymbol{q}} \times \boldsymbol{n}$. Now,

$$\begin{aligned} L_Q(\boldsymbol{n}) =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2q_0\|\boldsymbol{q}\|\boldsymbol{n}_\perp \\ =& \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\boldsymbol{n} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\boldsymbol{n}_\perp \\ =& \cos\theta\boldsymbol{n} + \sin\theta\boldsymbol{n}_\perp \end{aligned}$$

- Resulting vector is a rotation of $\boldsymbol{n}$ through an angle θ in the plane defined by $\boldsymbol{n}$ and $\boldsymbol{n}_\perp$.



- $L_{-[Q]} = [-Q]\boldsymbol{v}[-Q]^\star = [Q]\boldsymbol{v}[Q]^\star$ How?
- Negative quaternion $-[Q]$

$$-[Q] = \cos\frac{2\pi+\theta}{2} + \hat{q}\sin\frac{2\pi+\theta}{2}$$

- It represents the rotation about the same axis through the angle $2\pi+\theta$, essentially the same rotation.

Rotation of Coordinate Frame using Quaternion

For any unit quaternion $[Q] = q_0 + \boldsymbol{q} = \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2}$ and for any vector $\boldsymbol{v} \in \mathbb{R}^3$ the action of the operator $L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q]^{\star\star} = [Q]^\star \boldsymbol{v}[Q]$ is a rotation of the coordinate frame about the axis $\hat{q}$ through an angle θ while $\boldsymbol{v}$ is not rotated.

- Rotation of $\boldsymbol{v}$ under the operator L_Q can also be interpreted from the perspective of an observer attached to the vector $\boldsymbol{v}$.
- What he sees happening is that the coordinate frame rotates through the angle $-\theta$ about the same axis defined by the quaternion.
- $L_{Q^\star}$ rotates the vector $\boldsymbol{v}$ with respect to the coordinate frame through an angle $-\theta$ about $\boldsymbol{q}$.
- $L_Q(\boldsymbol{v}) = [Q]v[Q]^\star$ may be interpreted as a point or vector rotation with respect to the (fixed) coordinate frame.
- $L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q]$ may be interpreted as a coordinate frame rotation with respect to the (fixed) space of points.

- Quaternion operator

$$
\begin{aligned}
L_Q(\boldsymbol{v}) =& [Q]\boldsymbol{v}[Q]^\star = (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \\
&= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\boldsymbol{v} + 2\left(\hat{q}\sin\frac{\theta}{2}.\boldsymbol{v}\right)\hat{q}\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\left(\hat{q}\sin\frac{\theta}{2} \times \boldsymbol{v}\right) \\
&= \cos\theta\boldsymbol{v} + (1-\cos\theta)\left(\hat{q}.\boldsymbol{v}\right)\hat{q} + \sin\theta\left(\hat{q} \times \boldsymbol{v}\right)
\end{aligned}
$$

- Quaternion operator in matrix form,

$$
\begin{aligned}
L_Q(\boldsymbol{v}) =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \\
&= \underbrace{\left[q_0^2 - \|\boldsymbol{q}\|^2)I_{3\times3} + 2\boldsymbol{q}\boldsymbol{q}^T + 2q_0(\boldsymbol{q}\times\right]}_{\text{Rotation Matrix}}\boldsymbol{v}
\end{aligned}
$$

where, matrix representing cross product is given by

$$
\boldsymbol{q}\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}
$$

Rotation of Vector using Quaternion

Consider a rotation about an axis defined by (1, 1, 1) through an angle of $2\pi/3$. Obtain the quaternion to perform this rotation. Compute the effect of rotation on the basis vector $\boldsymbol{i}$ = (1, 0, 0).

- Define unit vector $\hat{q} = \frac{1}{\sqrt{3}}(1, 1, 1)$.
- Quaternion

$$\begin{aligned}[Q] &= \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2} \\ &= \frac{1}{2} + \frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}\end{aligned}$$

- Actual vector $\boldsymbol{v} = \boldsymbol{i} = (1, 0, 0)$.

- By using quaternion operator on $\boldsymbol{v} = (1, 0, 0)$, we get

$$
\begin{aligned}
[w] &= \cos\theta \boldsymbol{v} + (1 - \cos\theta)\,(\hat{q}.\boldsymbol{v})\,\hat{q} + \sin\theta\,(\hat{q} \times \boldsymbol{v}) \\
&= -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(1 + \frac{1}{2}\right)\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \boldsymbol{j}
\end{aligned}
$$

- Let $[P]$ and $[Q]$ be two unit quaternions.

$$L_P(\boldsymbol{u}) = \boldsymbol{v}, \quad L_Q(\boldsymbol{v}) = \boldsymbol{w}$$

- We can rewrite

$$\begin{aligned}\boldsymbol{w} =& L_Q(\boldsymbol{v})\\ =& [Q]\boldsymbol{v}[Q]^{\star}\\ =& [Q][P]\boldsymbol{u}[P]^{\star}[Q]^{\star}\\ =& [QP]\boldsymbol{u}[QP]^{\star}\\ =& L_{QP}(\boldsymbol{u})\end{aligned}$$

- L_{QP} is a unit quaternion rotation operator, with the axis and angle of the composite rotation given by the product $[QP]$.

- Consider quaternion operators $L_{P^\star}(\boldsymbol{u}) = [P]^\star \boldsymbol{u}[P]$ and $L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q]$.
- These operators define rotations of the coordinate system defined by corresponding quaternions.

$$\begin{aligned}\boldsymbol{w} =& L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q] \\ =& [Q]^\star [P]^\star \boldsymbol{u}[P][Q] = [PQ]^\star \boldsymbol{u}[PQ] \\ =& L_{(PQ)^\star}(\boldsymbol{u})\end{aligned}$$

- Quaternion product $([P][Q])^\star$ defines operator which represents a sequence of operators $L_{P^\star}$ followed by $L_{Q^\star}$.
- $L_{(PQ)^\star}$ is also a unit quaternion rotation operator, with the axis and angle of the composite rotation given by the product $[PQ]$.

Example

Consider a rotation of the coordinate frame about the z-axis through an angle α, followed by a rotation about the new y-axis through an angle β. By using quaternion method, find out the axis and angle of the composite rotation.

- The first rotation is about z-axis with an angle α.

$$[P] = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{k}$$

- Second rotation is about y-axis with an angle α.

$$[Q] = \cos\frac{\beta}{2} + \sin\frac{\beta}{2}\boldsymbol{j}$$

- As we rotate coordinate frames, the rotation operators are $L_{P^\star}$, followed by $L_{Q^\star}$, applied sequentially.
- Quaternion describing composite rotation

$$\begin{aligned}
[PQ] &= \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{k}\right)\left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\boldsymbol{j}\right) \\
&= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{j} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\boldsymbol{k} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{k}\times\boldsymbol{j} \\
&= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{i} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{j} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\boldsymbol{k}
\end{aligned}$$

- Axis of composite rotation

$$\boldsymbol{v} = \begin{bmatrix} -\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2} \\ \\ \cos\dfrac{\alpha}{2}\sin\dfrac{\beta}{2} \\ \\ \sin\dfrac{\alpha}{2}\cos\dfrac{\beta}{2} \end{bmatrix}$$

- Angle of rotation

$$\begin{aligned} \cos\frac{\theta}{2} &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} \\ \sin\frac{\theta}{2} &= \|\boldsymbol{v}\| \end{aligned}$$

- Rotational operator $L_{[PQ]^\star}$

- For unit quaternion, $\boldsymbol{p}' = [Q]^\star \boldsymbol{p}[Q]$.
- If $\boldsymbol{p} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ and $\boldsymbol{p}' = X'\boldsymbol{i} + Y'\boldsymbol{j} + Z'\boldsymbol{k}$ then

$$
\begin{aligned}
\boldsymbol{p}' =& (q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k})\boldsymbol{p}(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\
=& \boldsymbol{i}[X(q_0^2 + q_1^2 - q_2^2 - q_3^2) + Y(2q_3q_0 + 2q_1q_2) + Z(2q_1q_3 - 2q_0q_2)] \\
& + \boldsymbol{j}[X(2q_1q_2 - 2q_3q_0) + Y(q_0^2 - q_1^2 + q_2^2 - q_3^2) + Z(2q_1q_0 + 2q_3q_2)] \\
& + \boldsymbol{k}[X(2q_0q_2 + 2q_1q_3) + Y(2q_2q_3 - 2q_0q_1) + Z(q_0^2 - q_1^2 - q_2^2 + q_3^2)]
\end{aligned}
$$

- In matrix form, we have

$$
\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_3q_0 + q_1q_2) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_1q_0 + q_3q_2) \\ 2(q_0q_2 + q_1q_3) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
$$

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_0q_2 + q_1q_3) \\ 2(q_3q_0 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_1q_0 + q_3q_2) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}
$$

- Quaternion transformation matrix

$$[QT] = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_3 q_0 + q_1 q_2) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_3 q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_1 q_0 + q_3 q_2) \\ 2(q_0 q_2 + q_1 q_3) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

- Direction cosine matrix

$$[DC] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- Euler angle transformation matrix

$$[ET] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

- Compare these three matrices and get relations among these transformations.

Rotation Rate of Quaternion

Let $[Q(t)]$ be a unit quaternion function, and $\boldsymbol{\omega(t)}$ the angular velocity. The derivative of $[Q(t)]$ is

$$[\dot{Q}(t)] = \frac{1}{2}\boldsymbol{\omega}[Q(t)]$$

- At $t + \Delta t$, the rotation is described by $[Q](t + \Delta t)$.
- This is after some extra rotation during Δt performed on the frame that has already undergone a rotation described by $[Q(t)]$.
- This extra rotation is about the instantaneous axis $\hat{\boldsymbol{\omega}} = \dfrac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$ through the angle $\Delta\theta = \|\omega\|\Delta t$. It can be described by a quaternion.

$$\Delta[Q(t)] = \cos\frac{\Delta\theta}{2} + \hat{\boldsymbol{\omega}}\sin\frac{\Delta\theta}{2} = \cos\frac{\|\boldsymbol{\omega}\|\Delta t}{2} + \hat{\boldsymbol{\omega}}\sin\frac{\|\boldsymbol{\omega}\|\Delta t}{2}$$

- The rotation at $t + \Delta t$ is thus described by the quaternion sequence $[Q](t)$, $\Delta[Q(t)]$, implying $[Q(t+\Delta t)] = [\Delta Q(t)][Q(t)]$
- To derive $[\dot{Q}(t)]$, let us obtain the difference

$$[Q(t+\Delta t)] - [Q(t)] = \left(\cos\frac{\|\boldsymbol{\omega}\|\Delta t}{2} + \hat{\boldsymbol{\omega}}\sin\frac{\|\boldsymbol{\omega}\|\Delta t}{2}\right)[Q(t)] - [Q(t)]$$
$$= -2\sin^2\frac{\|\boldsymbol{\omega}\|\Delta t}{4}[Q(t)] + \hat{\boldsymbol{\omega}}\sin\frac{\|\boldsymbol{\omega}\|\Delta t}{2}[Q(t)]$$

- On taking the limit $\Delta t \to 0$, we have

$$[\dot{Q}(t)] = \lim_{\Delta t\to 0}\frac{[Q(t+\Delta t)] - [Q(t)]}{\Delta t} = \lim_{\Delta t\to 0}\hat{\boldsymbol{\omega}}\frac{\sin(\|\boldsymbol{\omega}\|\Delta t/2)}{\Delta t}[Q(t)]$$
$$= \frac{\hat{\boldsymbol{\omega}}\|\boldsymbol{\omega}\|}{2}[Q(t)]$$
$$= \frac{1}{2}\boldsymbol{\omega}[Q(t)]$$

- The differential equations for quaternion elements

$$\dot{q}_0 = -\frac{1}{2}\boldsymbol{q}^T\boldsymbol{\omega}$$
$$\dot{\boldsymbol{q}} = \frac{1}{2}[q_0\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{q}] = \frac{1}{2}[q_0\boldsymbol{\omega} - \boldsymbol{q} \times \boldsymbol{\omega}]$$

  where, $\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$ is the relative angular velocity vector between two coordinate frames and $\boldsymbol{q} = q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$.
- If the angular velocities are denoted in terms of the rotated frame then

$$[\dot{Q}(t)] = \frac{1}{2}[Q(t)]\boldsymbol{\omega'}, \quad \boldsymbol{\omega'} = [Q]^\star\boldsymbol{\omega}[Q]$$

- Note that $\boldsymbol{\omega} = 2[\dot{Q}(t)][Q(t)]^\star$
- Computation of angular rate with known quaternion and its rate

- The differential equations in compact form $\dfrac{d[Q]}{dt} = \dfrac{1}{2}B[Q]$

$$[B] = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -\boldsymbol{\Omega} \end{bmatrix}$$

- In scalar form, above equations can be written as

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2}[q_1\omega_x + q_2\omega_y + q_3\omega_z] \\ \dot{q}_1 &= \frac{1}{2}[q_0\omega_x + q_2\omega_z - q_3\omega_y] \\ \dot{q}_2 &= \frac{1}{2}[q_0\omega_y - q_1\omega_z + q_3\omega_x] \\ \dot{q}_3 &= \frac{1}{2}[q_0\omega_z + q_1\omega_y - q_2\omega_x] \end{aligned}$$

- Euler angle
  - Only 3 differential equations
  - No redundancy
  - Direct initialization from initial Euler angles
  - Nonlinear differential equations
  - Singularities
  - Gimbal lock problem
  - Transformation matrix needs to be computed
  - Order of rotation important
- Direction cosine matrix (DCM)
  - Linear differential equations
  - No singularity
  - Direct computation of DCM
  - Euler angles, required for initial calculation, are not directly available
  - Computational burden

- Quaternions
  - Only 4 linear coupled differential equations
  - No singularity thus avoids gimbal lock problem
  - Minimum redundancy to avoid singularity
  - Computationally simpler
  - If the coordinate systems do not coincides at $t = 0$ then Euler angle required for initial calculation
  - Transformation matrix needs to be computed
  - Euler angles are not directly available

Reference

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Thank you for your attention !!!