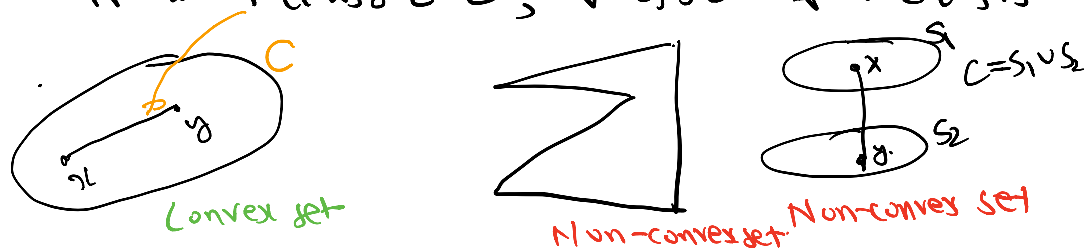
Defn B.1 [Bertsekas] A subset of C of $\mathbb{R}^n$ is called convex if $\alpha x + (1-\alpha)y \in C$, $\forall\, x, y \in C$, $\forall\, \alpha \in [0,1]$



Defn B.2 Let C be a convex subset of $\mathbb{R}^n$. A function $f: C \to \mathbb{R}$ is called convex if

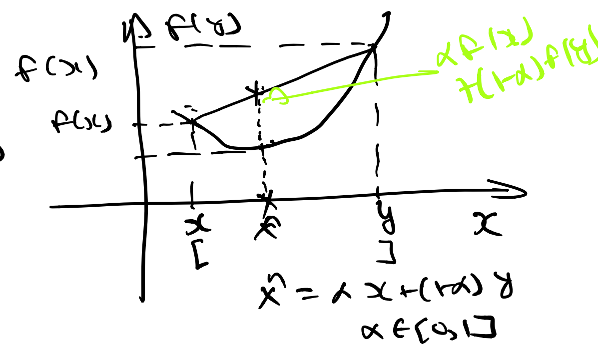
$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \forall\, x, y \in C, \ \forall\, \alpha \in [0,1].$$

(1) A function f is concave if $-f$ is convex.

(2) A function 'f' is strictly convex if above inequality is strict ($<$)

Concave function:

$$f(\alpha x + (1-\alpha) y) \geq \alpha f(x) + (1-\alpha) f(y)$$



Proposition B.1 [Bertsekas]

(a) For any collection $\{C_i \mid i \in I\}$ of convex sets, the intersection $\bigcap_{i \in I} C_i$ is convex

(b) The vector sum of two convex sets C_1 and C_2 is convex $C = C_1 \oplus C_2$ $x \in C \Rightarrow p + q$ $p \in C_1$ & $q \in C_2$

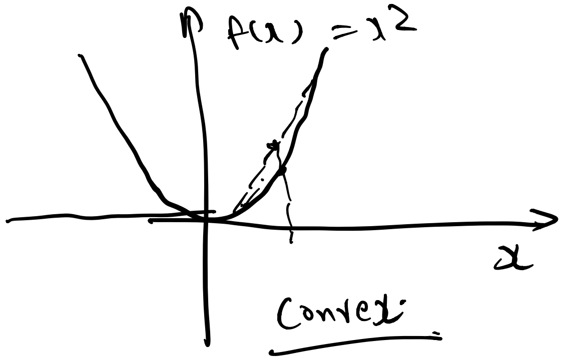
(c) The image of a convex set under a linear transformation is convex

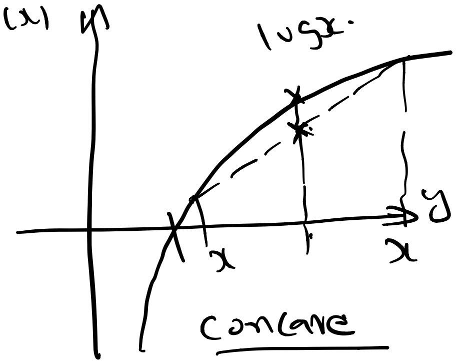
(d) If C is a convex set and $f: C \to \mathbb{R}$ is a convex function, the level set sets $\{x \in C \mid f(x) \leq \alpha\}$ and $\{x \in C \mid f(x) < \alpha\}$ are convex for all scalars α.

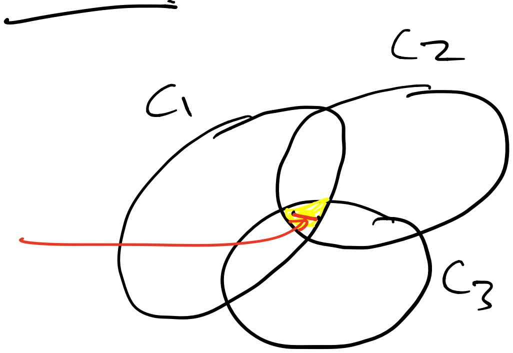
Example.

(1) $f(x) = x^2$ is a convex function

$f(x) = \log x$; $x > 0$ is a concave function





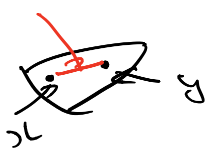


C_1 - convex

C_2 - convex

C_3 - convex

$C_1 \cap C_2 \cap C_3$ is also convex.



$$C : p + q$$

$$\underset{C_1}{p} \qquad p_{C_2}$$

$$z_1 = p_1 + q_1$$

$$z_2 = p_2 + q_2$$

$$z = \lambda z_1 + (1-\lambda) z_2 \qquad \text{T.S.T.} \in C \qquad \lambda \in [0,1]$$

$$= \lambda(p_1 + q_1) + (1-\lambda)(p_2 + q_2)$$

$$= \underbrace{\lambda p_1 + (1-\lambda) p_2}_{\text{in } C_1} + \underbrace{\lambda q_1 + (1-\lambda) q_2}_{C_2}$$

$$z = p + q \qquad p \in C_1 \; ; \; q \in C_2$$

$$z \in C = C_1 \oplus C_2$$

Q] what is a linear transformation?

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

scalars $\qquad$ R^n

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$\boxed{\text{Image of a convex set under linear transformation is convex.}}$

$\underset{m \times n}{A}$ $\qquad$ $\underset{}{A}\, \underset{\mathbb{R}^n}{x} = \underset{\mathbb{R}^m}{y}$

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

$$T(\alpha x + \beta y) = \alpha T(x) \qquad T(y)$$

$$y_1 \in Im(T) \; : \; y_1 = T(x_1)$$

$$y_2 \in Im(T) \; : \; y_2 = T(x_2)$$

T.S.T. $\alpha y_1 + \overline{1-\alpha}\, y_2$ where $\alpha \in [0,1]$

$\in Im(T)$ if set $x \in \underline{S}$ convex

$$T(\alpha x_1 + \overline{1-\alpha}\, x_2) = \alpha T(x_1) + \overline{1-\alpha}\, T(x_2)$$

$$= \alpha y_1 + \overline{1-\alpha}\, y_2$$

Range of T.

$f(x_1) \le \alpha$

$f(x_2) \le \alpha$

$\lambda x_1 + \overline{1-\lambda}\, x_2 \qquad \lambda \in [0,1]$

Q] is this point also in level set?

$$f(\lambda x_1 + \overline{1-\lambda}\, x_2)$$
$$\le \lambda f(x_1) + \overline{1-\lambda}\, f(x_2)$$
$$\le \lambda \alpha + \overline{1-\lambda}\, \alpha$$
$$= \alpha$$

$f(\lambda x_1 + \overline{1-\lambda}\, x_2) \le \alpha$

Level set L is convex.