Pursuit Guidance

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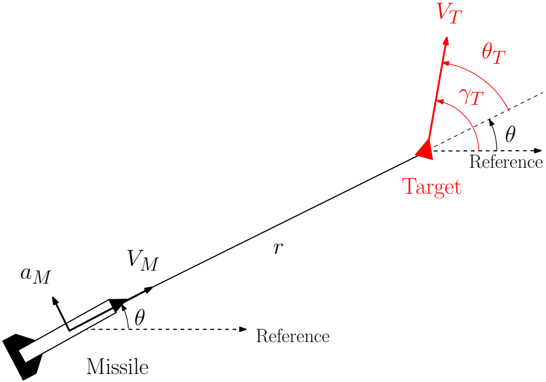
- **Philosophy**: If the missile continues to point towards the target then it is guaranteed that after a finite time the missile will intercept the target.



$\Rightarrow$ Must be true if the missile has a higher speed than that of the target.

Assumption: Non-maneuvering targets

- Engagement dynamics

$$
\begin{aligned}
\dot{r} &= V_r = V_T \cos(\gamma_T - \theta) - V_M \cos(\gamma_M - \theta) \\
r\dot{\theta} &= V_\theta = V_T \sin(\gamma_T - \theta) - V_M \sin(\gamma_M - \theta)
\end{aligned}
$$

- For a perfect pursuit guidance, missile will always point toward target.
- As $\gamma_M = \theta$, we have

$$
\begin{aligned}
\dot{r} &= V_r = V_T \cos(\gamma_T - \theta) - V_M \\
r\dot{\theta} &= V_\theta = V_T \sin(\gamma_T - \theta)
\end{aligned}
$$

- By denoting the speed ratio of target by missile as $\nu = V_M / V_T$,

$$
\frac{dr}{r} = \{\cot(\gamma_T - \theta) - \nu \csc(\gamma_T - \theta)\} d\theta \Rightarrow r = f(\theta)
$$

- Since r is a function of θ not the time, it may not provide useful insights.

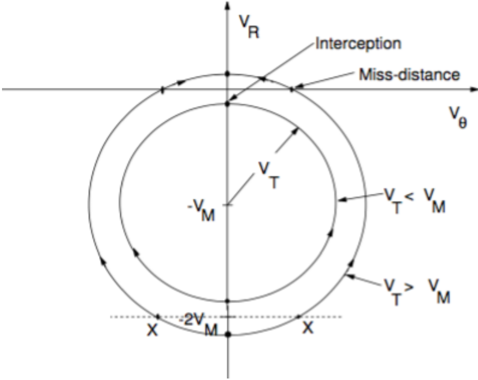
- Trajectories in (V_θ, V_r) space

$$\begin{aligned} V_r + V_M &= V_T \cos(\gamma_T - \theta) \\ V_\theta &= V_T \sin(\gamma_T - \theta) \end{aligned}$$

- Locus of (V_θ, V_r) as

$$V_\theta^2 + (V_r + V_M)^2 = V_T^2$$

- Equation of a circle with radius V_T and centered at $(0, -V_M)$.
- What about the direction of movement of the point on V_θ, V_r space?



- On differentiating V_r, V_θ

$$\dot{V}_r = -V_T \sin(\gamma_T - \theta)\left(\dot{\gamma}_T - \dot{\theta}\right) = \dot{\theta} V_\theta$$

$$\dot{V}_\theta = V_T \cos(\gamma_T - \theta)\left(\dot{\gamma}_T - \dot{\theta}\right) = -\dot{\theta}(V_r + V_M)$$

- On multiplying r on both sides,

$$r\dot{V}_r = r\dot{\theta}V_\theta = V_\theta^2, \; r\dot{V}_\theta = -r\dot{\theta}(V_r + V_M) = -V_\theta(V_r + V_M)$$

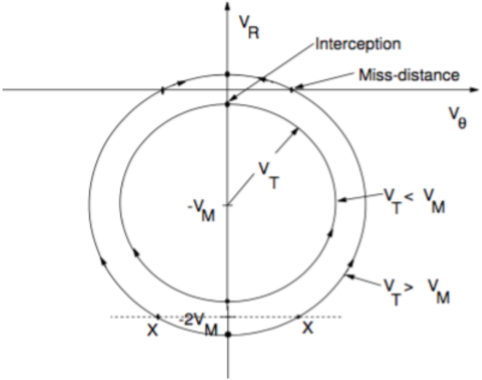
- Observations: $\dot{V}_r > 0$ as $r > 0$.

$$\dot{V}_\theta = \begin{cases} \text{Positive} & \text{if } V_\theta(V_r + V_M) < 0 \\ \text{Negative} & \text{if } V_\theta(V_r + V_M) > 0 \end{cases}$$

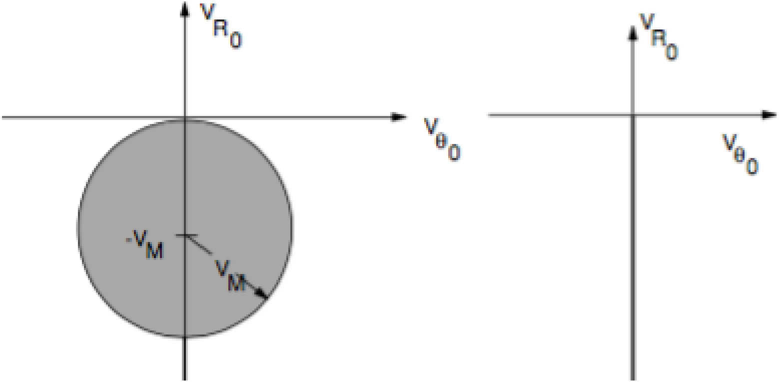
$$\dot{V}_\theta > 0 \Rightarrow V_\theta > 0 \;\&\; V_r < -V_M \text{ or } V_\theta < 0 \;\&\; V_r > -V_M$$

$$\dot{V}_\theta < 0 \Rightarrow V_\theta > 0 \;\&\; V_r > -V_M \text{ or } V_\theta < 0 \;\&\; V_r < -V_M$$

- Intersection point on V_r-axis are stationary points since at these points $V_\theta = 0 \Rightarrow \dot{V}_r = 0 = \dot{V}_\theta$.
- Points on the negative and positive V_r axes correspond to the collision and inverse collision triangles, respectively.
- Collision triangle in pure pursuit: Tail-chase mode or head-on mode.
- In the tail-chase mode, collision occurs only if $V_M > V_T$.
- In the head-on mode, collision is possible for all values of V_T and V_M.
- Collision triangle in the pure pursuit case is actually a straight line since the missile and target velocity vectors are both aligned along the LOS. How?
- What does $V_\theta = 0$ mean?
- A point on the V_r axis essentially corresponds to the situation when both the missile and target velocities are aligned with the LOS.
- What are the possible meanings of point on positive V_r axis?



- Two circles; one for each $V_T < V_M$ and $V_T > V_M$
- What is difference between the two circles?
- What about interception of target in both cases?
- If the initial point is on the negative V_r axis then the engagement is either a head-on or a tail-chase.
- If $V_T < V_M$ and the initial point is not on the negative V_R axis then the engagement always ends in a tail-chase collision.
- What if $V_M < V_T$?



Capture region: If initial point lies inside it then interception occurs.
Assumptions: V_T as free parameter and V_M points towards target initially
Expansion of capture region over unguided missile

- How to compute time of interception for a pursuit guided missile?
- Trajectories in (V_θ, V_r)-space

$$
\begin{aligned}
(V_r + V_M)^2 + V_\theta^2 =& V_T^2 \\
V_r^2 + V_M^2 + 2V_r V_M + V_\theta^2 =& V_T^2
\end{aligned}
$$

- On substituting for V_r and $V_\theta^2 = r\dot{V}_r$, we get

$$
\begin{aligned}
r\ddot{r} + \dot{r}^2 + 2V_M\dot{r} + V_M^2 =& V_T^2 \\
r\ddot{r} + \dot{r}^2 + 2V_M\dot{r} =& V_T^2 - V_M^2
\end{aligned}
$$

- We know that

$$
\frac{d\{r(V_r + 2V_M)\}}{dt} = r\ddot{r} + \dot{r}^2 + 2V_M\dot{r} = V_T^2 - V_M^2
$$

- On integration, we get

$$r(V_r + 2V_M) = (V_T^2 - V_M^2)t + b,\ b = r_0(V_{r_0} + 2V_M)$$

- Interception occurs at $t = t_f$ when $r = 0$.
- Time of interception

$$t_f = -\frac{b}{V_T^2 - V_M^2} = \frac{r_0(V_{r_0} + 2V_M)}{V_M^2 - V_T^2}$$

- In what case we would get interception of target? Can we figure out from expression of t_f?
- For $V_M > V_T$, when the initial condition lies inside the capture region, we have $t_f > 0$ and finite automatically.
- Can t_f be positive for any other condition?
- What does that implies? Is our analysis wrong????

- Fortunately, answer is no.
- To compute time of interception,

$$r(V_r + 2V_M) = (V_T^2 - V_M^2)t + b$$

- **Observation**: Equation for the final time was obtained by setting the LHS of equation to zero.
- Implicit assumption: LHS becomes zero when $r = 0$.
- LHS can also become zero when $V_r = -2V_M$.
- This condition never arises when $V_T < V_M$.
- It does occur at point X marked when $V_T > V_M$ and the initial condition lies below the line $V_r = -2V_M$.
- When $V_T > V_M$ and $V_{r0} < -2V_M$, t_f only gives the time at which the (V_θ, V_r) point crosses the point X.

- How to compute missile lateral acceleration?
- We need to look at the rate at which the missile velocity vector has to turn in order to satisfy the requirements of pure pursuit.
- If γ_M denotes the velocity direction of missile then in case of pure pursuit

$$\gamma_M = \theta$$

- This results into

$$\dot{\gamma}_M = \dot{\theta} \Rightarrow \frac{a_M}{V_M} = \dot{\theta} \Rightarrow \boxed{a_M = V_M \dot{\theta} = \frac{V_M V_T \sin(\gamma_T - \theta)}{r}}$$

- This expression can be used to implement pure pursuit guidance law, provided that initially the missile points directly towards the target.

Simulation parameters:

$V_M = 500$, $V_T = 400$, $\gamma_M = 0°$, $\gamma_T = 120°$, $r = 10$ km

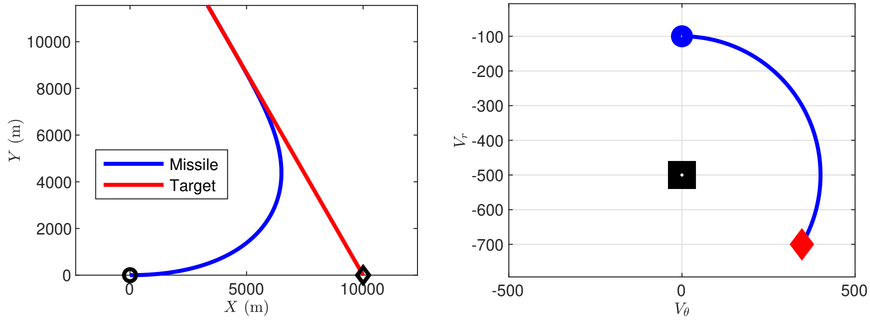


Figure: Target interception using pure pursuit guidance

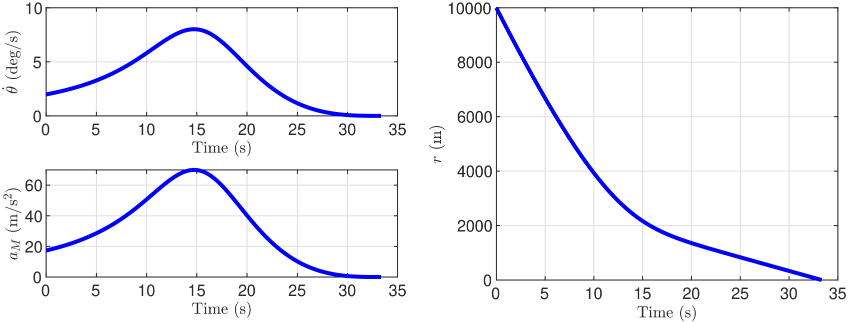


Figure: Target interception using pure pursuit guidance

- Acceleration demand and LOS rate converge to zero at interception

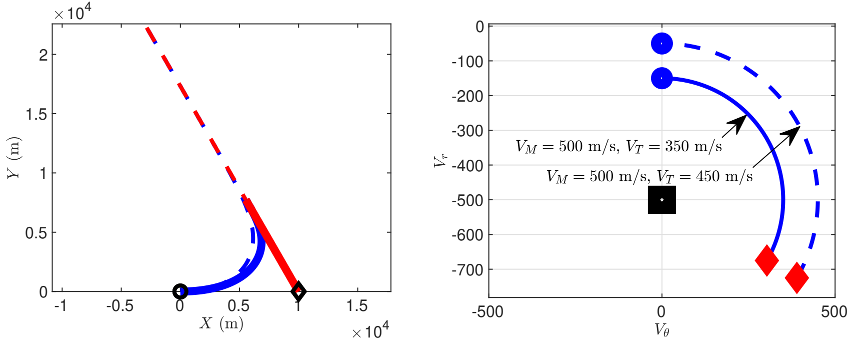


Figure: Target interception using pure pursuit guidance

- Different target speeds of 350 and 450 m/s
- Different V_r at interception, different time of interception

Pursuit Guidance

Pursuit Guidance: different target speeds

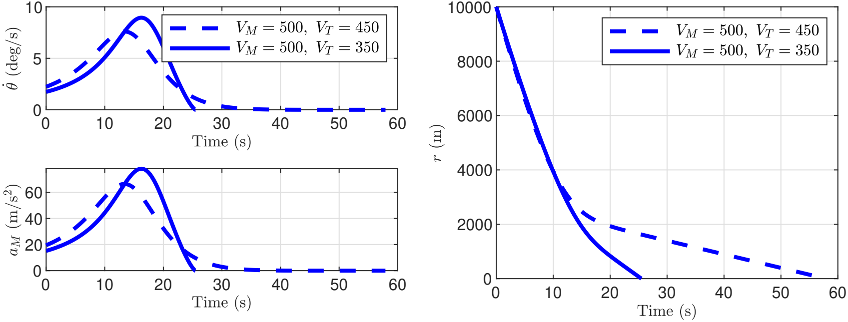


Figure: Target interception using pure pursuit guidance

- Acceleration demand and LOS rate converge to zero at interception
- Lower closing speed with higher value of target speed

- To compute missile lateral acceleration, we need to know about r and θ.
- Equation relating r and θ is given by

$$\frac{dr}{r} = \{\cot(\gamma_T - \theta) - \nu \csc(\gamma_T - \theta)\} d\theta = \frac{\cos(\gamma_T - \theta) - \nu}{\sin(\gamma_T - \theta)} d\theta$$

- On solving this equation, we get (Derive this expression by your own)

$$\boxed{r = K \frac{\left\{\tan\left(\frac{\gamma_T - \theta}{2}\right)\right\}^{\nu}}{\sin(\gamma_T - \theta)} = K \frac{\{\sin(\gamma_T - \theta)\}^{\nu - 1}}{[1 + \cos(\gamma_T - \theta)]^{\nu}}}$$

where

$$K = r_0 \frac{\sin(\gamma_T - \theta_0)}{\left\{\tan\left(\frac{\gamma_T - \theta_0}{2}\right)\right\}^{\nu}} = r_0 \frac{[1 + \cos(\gamma_T - \theta_0)]^{\nu}}{\{\sin(\gamma_T - \theta_0)\}^{\nu - 1}}$$

Pursuit Guidance

Pursuit Guidance: Lateral Acceleration History

- On substituting for r in a_M,

$$a_M = V_M\dot{\theta} = \frac{V_M V_T \sin(\gamma_T - \theta)}{r} = \frac{V_M V_T \sin^2(\gamma_T - \theta)}{K\left\{\tan\left(\frac{\gamma_T - \theta}{2}\right)\right\}^{\nu}}$$

- This does not give us the lateral acceleration history directly since we do not have any explicit expression that gives a_M as a function of time.
- Can we get a relationship that relates r and θ with time t?
- We know that

$$r(V_r + 2V_M) = (V_T^2 - V_M^2)t + b \Rightarrow t = \frac{b - r(V_r + 2V_M)}{V_T^2 - V_M^2}$$

- We can rewrite

$$t = \frac{r_0(V_{r_0} + 2V_M) - r(V_r + 2V_M)}{V_T^2 - V_M^2}$$

- Also, the time can be expressed as

$$t = \frac{r_0(V_T \cos(\gamma_T - \theta_0) + V_M) - r(V_T \cos(\gamma_T - \theta) + V_M)}{V_T^2 - V_M^2}$$

- Expression for relative range r

$$r = K \frac{\left\{\tan\left(\frac{\gamma_T - \theta}{2}\right)\right\}^{\nu}}{\sin(\gamma_T - \theta)} = K \frac{\{\sin(\gamma_T - \theta)\}^{\nu - 1}}{[1 + \cos(\gamma_T - \theta)]^{\nu}}$$

- On combining these two, we can get a_M as a function of time.

$$a_M = \frac{V_M V_T \sin^2(\gamma_T - \theta)}{K\left\{\tan\left(\frac{\gamma_T - \theta}{2}\right)\right\}^{\nu}}$$

- Not so straightforward.

- At interception, the engagement geometry is a tail-chase one.
- When initial condition lies inside capture region, interception occurs.

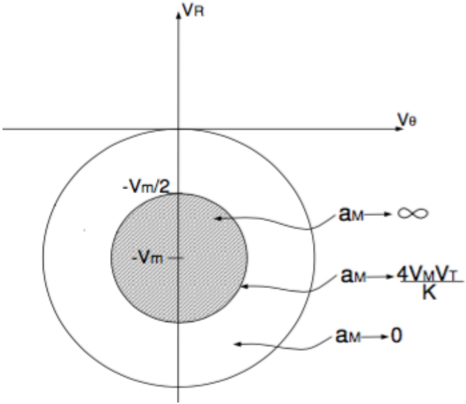
- As $t \to t_f$, $\theta \to \gamma_T$.
- Terminal value of a_M

$$\lim_{t \to t_f} a_M = \lim_{\theta \to \gamma_T} a_M$$

- For various values of ν, missile lateral acceleration

$$a_M \to \begin{cases} 0 & 1 < \nu < 2 \\ \dfrac{4V_M V_T}{K} & \nu = 2 \\ \infty & \nu > 2 \end{cases}$$

- What about high target's speed?



Pursuit Guidance

Pursuit Guidance: Miss Distance

- In the case of $V_T > V_M$, the interception does not take place and results in a miss distance.
- At the point where miss-distance occurs, $V_r = 0$.

$$V_r = V_T \cos(\gamma_{T_{\text{miss}}} - \theta_{\text{miss}}) - V_M = 0 \Rightarrow \boxed{\gamma_{T_{\text{miss}}} - \theta_{\text{miss}} = \cos^{-1}\nu}$$

- Miss distance

$$r_{\text{miss}} = K \frac{\left\{\tan\left(\dfrac{\cos^{-1}\nu}{2}\right)\right\}^{\nu}}{\sin\left(\cos^{-1}\nu\right)} = K \frac{\left\{\sin\left(\cos^{-1}\nu\right)\right\}^{\nu-1}}{(1+\nu)^{\nu}} = K \frac{\left\{1-\nu^2\right\}^{(\nu-1)/2}}{(1+\nu)^{\nu}}$$

- Time of interception with miss distance

$$2V_M r_{\text{miss}} = (V_T^2 - V_M^2) t_{\text{miss}} + b \Rightarrow t_{\text{miss}} = \frac{2V_M r_{\text{miss}} - b}{V_T^2 - V_M^2}$$

Reference

1. D. Ghose, *Lecture notes on Navigation, Guidance and Control*, Indian Institute of Science, Bangalore.