Variants of Proportional Navigation

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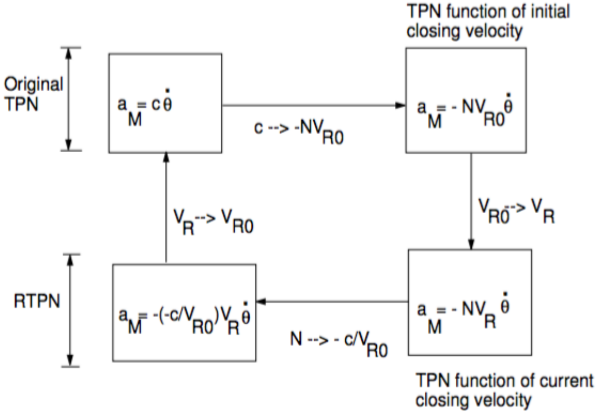
How to compare capture regions of TPN class of guidance laws?

- Guidance command for RTPN

$$a_M = N V_r \dot{\theta}$$

- Modifications to guidance command

$$a_M = -\left(-\frac{c}{V_{r_0}}\right) V_r \dot{\theta}$$

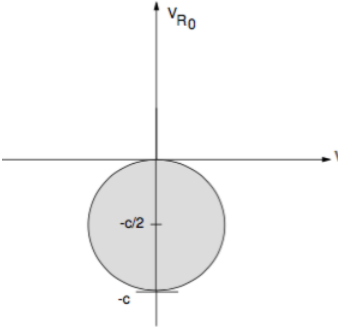


- Capture regions of RTPN guidance law

$$V_{\theta_0}^2 < (N-1)V_{r_0}^2, \quad N > 1, \; V_{r_0} < 0$$

- For $N = -\dfrac{c}{V_{r_0}}$, we have

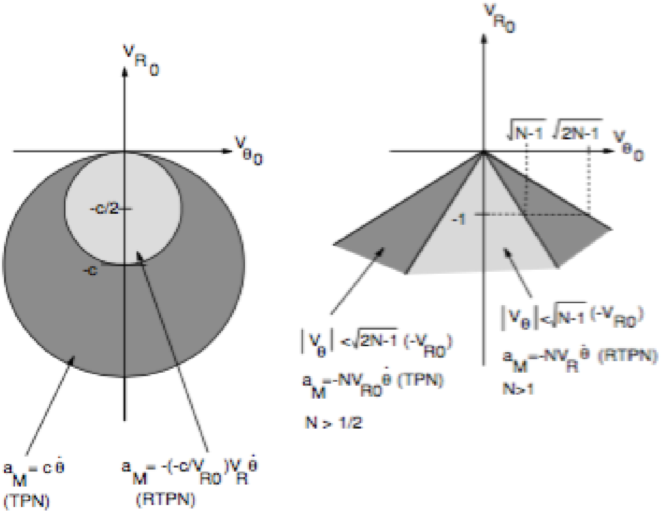
$$V_{\theta_0}^2 < -\left(\frac{c}{V_{r_0}} + 1\right) V_{r_0}^2 \Rightarrow V_{\theta_0}^2 + V_{r_0}^2 + cV_{r_0} < 0$$



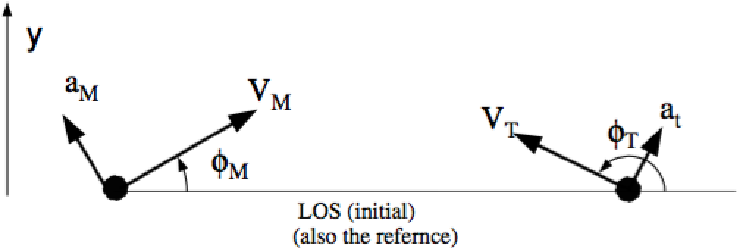
- On rearranging, we get

$$V_{\theta_0}^2 + \left(V_{r_0} + \frac{c}{2}\right)^2 < \left(\frac{c}{2}\right)^2$$

- Circle of radius $\dfrac{c}{2}$ centered at $(0, -c/2)$.


V_{R_0}
V_{θ_0}
-c/2
-c
$a_M = c\dot{\theta}$
(TPN)
$a_M = -(-c/V_{R0})V_R\dot{\theta}$
(RTPN)
V_{R_0}
$\sqrt{N-1}$ $\sqrt{2N-1}$ V_{θ_0}
-1
$|V_\theta| < \sqrt{2N-1}\,(-V_{R0})$
$a_M = -NV_{R0}\dot{\theta}$ (TPN)
$N > 1/2$
$|V_\theta| < \sqrt{N-1}\,(-V_{R0})$
$a_M = -NV_R\dot{\theta}$ (RTPN)
$N > 1$

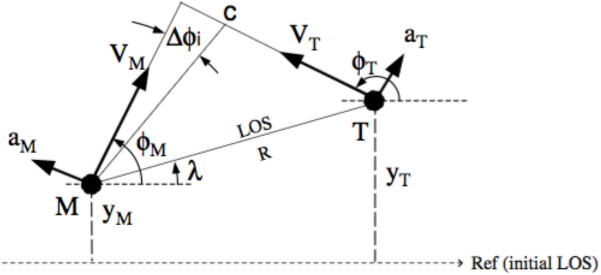
- Can we say something more about PN guidance?
- Consider a planar engagement geometry as below.



- Assume ϕ_M is small and ϕ_T is close to zero or π.
- Both a_M and a_T are perpendicular to the initial LOS.
- LOS angle is assumed to be small and its initial value is zero.

- Consider y_T and y_M are the perpendicular displacements of the missile and target, respectively, perpendicular to initial LOS.
- By defining displacement y as below and using small angle assumptions,

$$y = y_T - y_M$$
$$\ddot{y} = a_T - a_M$$



- Suppose the target does not maneuver then

$$\ddot{y} = -\, a_M = -NV_c\dot{\theta}$$

  where V_c is closing speed of missile-target engagement.
- With small angle approximation,

$$V_c \approx V_M + V_T, V_M - V_T$$

- On integration $\ddot{y}$, we get

$$\dot{y} = -\, NV_c\theta + C_1$$

- Also, $V_c = \dfrac{r}{t_f - t},\ \theta = \dfrac{y}{r},$

$$\dot{y} = -\, N\frac{r}{t_f - t}\frac{y}{r} + C_1 = -N\frac{y}{t_f - t} + C_1$$

- Suppose there is a heading error $\Delta\phi_i$, then

$$y(0) = 0, \quad \dot{y}(0) = -V_M \sin\Delta\phi_i \Rightarrow C_1 = -V_M \Delta\phi_i$$

- Substituting these values, we get

$$\frac{dy}{dt} + N\frac{y}{t_f - t} = -V_M \Delta\phi_i$$

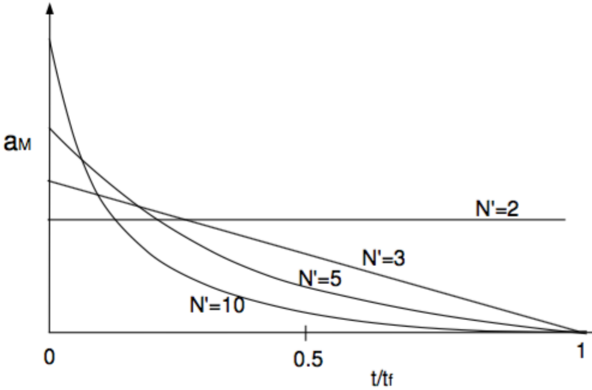
- Differential equation for a PN guided missile subject to the heading error but with a non-maneuvering target.
- On solving this equation,

$$\begin{aligned} y =& -\frac{V_M \Delta\phi_i t_f^2}{N-1}\left(1 - \frac{t}{t_f}\right)^N + K_1 t + K_2 \\ a_M =& \frac{V_M \Delta\phi_i N}{t_f}\left(1 - \frac{t}{t_f}\right)^{N-2} \end{aligned}$$

- From these equations,

$$a_M(0) = \frac{V_M \Delta\phi_i N}{t_f}$$

- $a_M \propto \Delta\phi_i$
- $a_M \propto V_M$
- $a_M \propto \dfrac{1}{t_f}$
- Similar behavior with nonlinear simulations



- Suppose there is zero heading error $\Delta\phi_i = 0$ but target maneuvers with a_T.
- Engagement equation

$$\ddot{y} = \; a_T - a_M = \; a_T - NV_c\dot{\theta}$$

with initial conditions given by

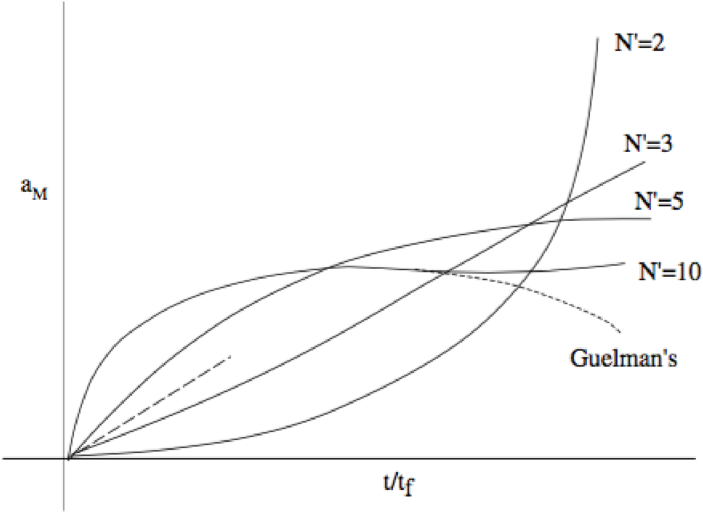
$$y(0) = 0, \quad \dot{y}(0) = 0$$

- On solving this equation,

$$a_M = \frac{N}{N-2}\left[1 - \left(1 - \frac{t}{t_f}\right)^{N-2}\right] a_T$$

- When $N \to 2$, using L'Hospital rule

$$\lim_{N\to 2} a_M = \; -2a_T \log\left(1 - \frac{t}{t_f}\right)$$



N'=2
N'=3
N'=5
N'=10
Guelman's
a_M
t/t_f

- Consider linearized engagement geometry

$$a_M = NV_c\dot{\theta} = N\frac{r}{t_{\text{go}}}\frac{d}{dt}\left(\frac{y}{r}\right)$$

where $t_{\text{go}} = t_f - t = \dfrac{r}{-\dot{r}}$ is time-to-go.

- On evaluating,

$$\begin{aligned} a_M =& N\frac{r}{t_{\text{go}}}\left(\frac{r\dot{y} - y\dot{r}}{r^2}\right) = \frac{N}{t_{\text{go}}}\left(\frac{r^2\dot{y} - yr\dot{r}}{r^2}\right) \\ =& \frac{N}{t_{\text{go}}}\left(\dot{y} + \frac{y}{t_{\text{go}}}\right) = \frac{N}{t_{\text{go}}^2}\underbrace{(y + \dot{y}t_{\text{go}})}_{\text{ZEM}} = \frac{N\text{ZEM}}{t_{\text{go}}^2} \end{aligned}$$

- **Zero-Effort-Miss (ZEM)**: Vertical separation between missile and target at t_f if missile does not use any control effort for the rest of the time and target does not maneuver.

- Suppose target maneuvers then what should be the guidance command?
- Can we use similar concept of zero-effort-miss, ZEM?
- Zero-effort-miss, with constant maneuver target, can be defined as

$$\text{ZEM} = y + \dot{y}t_{\text{go}} + \frac{1}{2}a_T t_{\text{go}}^2$$

- Guidance command for such case, called as Augmented PN (APN) guidance,

$$a_M = \frac{N\text{ZEM}}{t_{\text{go}}^2} = \frac{N}{t_{\text{go}}^2}\left[y + \dot{y}t_{\text{go}} + \frac{1}{2}a_T t_{\text{go}}^2\right] = NV_c\dot{\theta} + \frac{N}{2}a_T$$

- Missile lateral acceleration as a function of time

$$a_M = \frac{1}{2}Na_T\left[\left(1 - \frac{t}{t_f}\right)^{N-2}\right]$$

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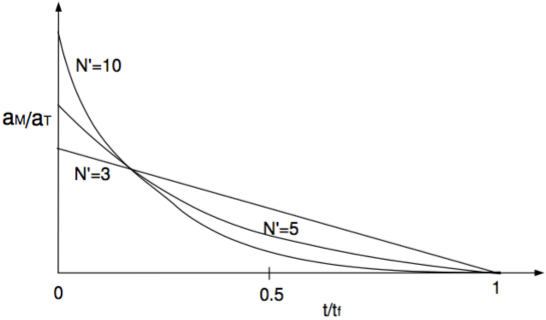
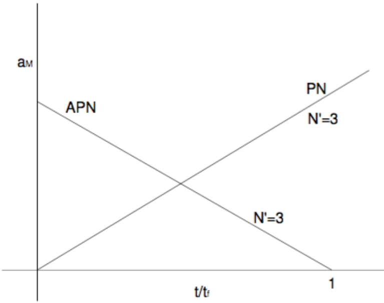
APN Guidance Law





Which one is better, PN or APN for maneuvering target?

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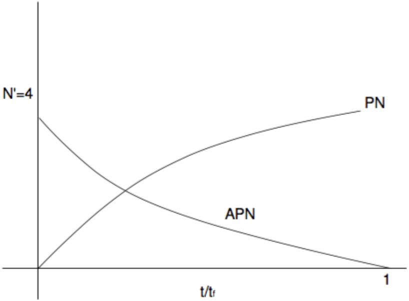
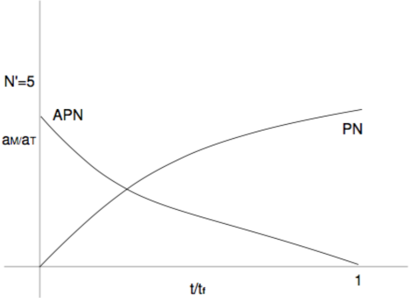
Comparison of PN and APN Guidance Laws





- APN is considered better so far as acceleration profiles are concerned.
  - $\Rightarrow$ Total acceleration needed is less
  - $\Rightarrow$ Saturation effect towards the beginning can be compensated later but not the other way.
- However, PN has its own advantage in terms of the input information requirement and implementation.

- How to quantify the acceleration demand for each guidance law?
- Area under acceleration curve
- Maneuver induced cost (MIC)

$$MIC = \int_0^{t_f} a_M dt$$

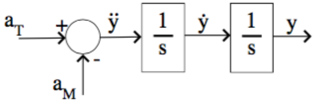
- For PN and APN,

$$MIC_{PN} = \int_0^{t_f} \frac{Na_T}{N-2}\left[\left(1-\frac{t}{t_f}\right)^{N-2}\right] dt = \frac{Na_T t_f}{N-1}$$

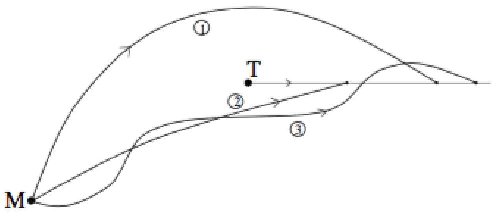
$$MIC_{APN} = \int_0^{t_f} \frac{Na_T}{2}\left[\left(1-\frac{t}{t_f}\right)^{N-2}\right] dt = \frac{1}{2}\frac{Na_T t_f}{N-1} = \frac{1}{2}MIC_{PN}$$

- This relation is true for all $N > 2$.

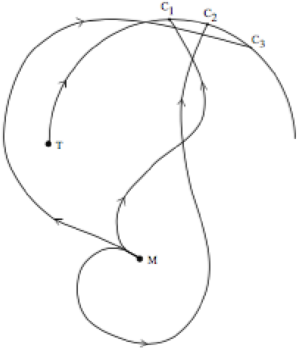
- To get a better understanding, consider the linearized model with zero lag



- We look for a guidance command as function of system states.
- How to choose a suitable guidance law?
- How to define which one is a suitable?
- Need to impose constraints and performance criteria to find unique law.
- What if we look for guidance which can ensure zero miss distance?
- There may be several such guidance laws, which satisfy this constraint.
- How to select one among them?



T
M



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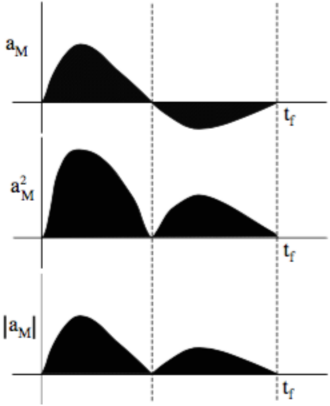
Linearized Guidance Laws

- Consider minimization of total control effort.
- **Total control effort**: Integral of square of commanded acceleration of missile.
- Why do we take square and not the actual or the absolute values?



- If we consider values of a_M^2, then

$$\int_0^{t_f} a_M^2 dt > 0$$

- Better representation of control effort. Is there any other justification for this selection?
- Analytically tractable for optimization problem, due to quadratic form.
- Maneuver induced drag

$$D = \frac{1}{2}\rho V_M^2 S C_D,\ C_D = C_{D_0} + C_{D_i},\ C_{D_i} = K C_L^2 = K \frac{m^2 a_M^2}{[(1/2)\rho V_M^2 S]^2}$$

- Guidance design as an optimal control problem.

$$\min \int_0^{t_f} a_M^2 dt, \quad y(t_f) = 0$$

subjected to state equations given by

$$\dot{y} = \dot{y}, \quad \ddot{y} = a_T - a_M, \; \dot{a}_T = 0$$

- Linear quadratic problem in optimal control theory
- System equation in state-space form

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{a}_T \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ a_T \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} a_M(t) \\ &= \boldsymbol{F}\boldsymbol{X} + \boldsymbol{G}u \end{aligned}$$

- If t_f is known then

$$\boldsymbol{X}(t_f) = \boldsymbol{\phi}(t_f - t)\boldsymbol{X}(t_f) + \int_t^{t_f} \boldsymbol{\phi}(t_f - \tau)\boldsymbol{G}(\tau)u(\tau)d\tau$$

where $\boldsymbol{\phi}(t)$ is state transition matrix given by

$$\boldsymbol{\phi}(t) = \mathcal{L}^{-1}[s\boldsymbol{I} - \boldsymbol{F}]^{-1} = \exp^{\boldsymbol{F}t}.$$

$$\begin{aligned}
\exp^{\boldsymbol{F}t} =& \boldsymbol{I} + \boldsymbol{F}t + \frac{\boldsymbol{F}^{\boldsymbol{2}}t^2}{2!} + \frac{\boldsymbol{F}^{\boldsymbol{3}}t^3}{3!} + \cdots \\
=& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
=& \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

- Using STM, we get

$$\begin{bmatrix} y(t_f) \\ \dot{y}(t_f) \\ a_T(t_f) \end{bmatrix} = \begin{bmatrix} 1 & t_f - t & \frac{(t_f-t)^2}{2} \\ 0 & 1 & t_f - t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \end{bmatrix}$$
$$+ \int_t^{t_f} \begin{bmatrix} 1 & t_f - \tau & \frac{(t_f-\tau)^2}{2} \\ 0 & 1 & t_f - \tau \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} a_M(\tau) d\tau$$

- On solving, we get

$$\begin{aligned} y(t_f) =& \underbrace{y(t) + (t_f - t)\dot{y}(t) + \frac{1}{2}(t_f - t)^2 a_T}_{f_1(t_f - t)} - \int_t^{t_f} \underbrace{[(t_f - \tau)}_{h_1(t_f - t)} a_M(\tau) d\tau \\ =& f_1(t_f - t) - \int_t^{t_f} h_1(t_f - \tau) a_M(\tau) d\tau \end{aligned}$$

- To achieve zero miss distance,

$$f_1(t_f - t) = \int_t^{t_f} h_1(t_f - \tau) a_M(\tau) d\tau$$

- There could be several a_M functions defined over $[t, t_f]$ which satisfy this expression.
- We use the minimum control effort criteria to find unique one.
- Let f and g be measurable functions, with range in $[0, \infty]$.
- According to **Schwarz' inequality**

$$\int fg dt \leq \left[\int f^2 dt\right]^{1/2} \left[\int g^2 dt\right]^{1/2}$$

and equality holds when $f = kg$ with k is a real number.

- To achieve zero miss distance,

$$f_1^2(t_f - t) \leq \int_t^{t_f} h_1^2(t_f - \tau)d\tau \int_t^{t_f} a_M^2(\tau)d\tau$$

$$\Rightarrow \int_t^{t_f} a_M^2(\tau)d\tau \geq \frac{f_1^2(t_f - t)}{\int_t^{t_f} h_1^2(t_f - \tau)d\tau}$$

- Minimimum value of LHS is the value of RHS and this occurs at equality.
- Equality hold only if

$$a_M(\tau) = kh_1(t_f - \tau)$$

- On substituting this, we get

$$k^2 \int_t^{t_f} h_1^2(t_f - \tau)d\tau = \frac{f_1^2(t_f - t)}{\int_t^{t_f} h_1^2(t_f - \tau)d\tau} \Rightarrow k = \frac{f_1(t_f - t)}{\int_t^{t_f} h_1^2(t_f - \tau)d\tau}$$

- Guidance command

$$
\begin{aligned}
a_M(\tau) = kh_1(t_f - \tau) &= \left[\frac{f_1(t_f - t)}{\int_t^{t_f} h_1^2(t_f - \tau)d\tau}\right] h_1(t_f - \tau) \\
&= \left[\frac{y(t) + (t_f - t)\dot{y}(t) + \frac{1}{2}(t_f - t)^2 a_T}{\int_t^{t_f} h_1^2(t_f - \tau)d\tau}\right](t_f - \tau) \\
&= \frac{3}{t_{\mathrm{go}}^3}[y(t) + t_{\mathrm{go}}\dot{y}(t) + \frac{1}{2}t_{\mathrm{go}}^2 a_T](t_f - \tau)
\end{aligned}
$$

- At current time $t = t$, the guidance command (APN) is given by

$$
a_M(t) = \frac{3}{t_{\mathrm{go}}^2}[y(t) + t_{\mathrm{go}}\dot{y}(t) + \frac{1}{2}t_{\mathrm{go}}^2 a_T] = \frac{3\text{ZEM}}{t_{\mathrm{go}}^2}
$$

- Equivalent to PN for $a_T = 0$.
- Optimal with $N = 3$ for linearized dynamics .
- $a_M \propto$ ZEM and inversely proportional to t_{go}^2.

Reference

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2. N. A. Shneydor, *Missile Guidance and Pursuit: Kinematics, Dynamics, and Control*, Horwood Publishing Limited. 1998.

Thank you for your attention !!!