Introduction to Navigation & Guidance

(Course Code: AE 410/641)

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September 15, 2020

Assignment - 1

General Instructions

- It is highly recommended that students submit neatly typeset document containing answers to questions in Assignment-1.
- Your assignment needs to be submitted online in .pdf format on Moodle.
- Marks for each portion of question are given separately.
- Please show your steps clearly while solving problems.
- Do not indulge in academic dishonesty. In cases where the answers of two students are found to be copied, both of them will be awarded zero marks for that particular question. Furthermore, those students may face disciplinary actions.
- Assignment is due on **27^{th} September 2020**.
- Late assignment submission: **20% reduction** in total weightage per late day.

1. Consider a vehicle M having a velocity vector $\mathbf{V} = [V_m \ \ 0 \ \ 0]^\top$ in its body frame. The body X-axis of M is aligned at an azimuth angle, ψ_m, and an elevation angle, θ_m, with respect to the inertial frame.

   (a) Derive the quaternion rotation operator, Q_m, that is required to obtain the equivalent velocity vector, V_{mI}, in the inertial frame.

   (b) Obtain V_{mI} in inertial frame using Q_m.

   (c) Compute the value of V_{mI} in inertial frame using Euler angle rotations and compare with the results obtained in part (b).

   [5+5+5]

2. If an airplane is rotated, first along $\mathbf{Z}$ axis with an angle ψ in counter clockwise direction, then along the rotated $\mathbf{Y}$ axis with an angle θ in clockwise sense, followed by the counter clockwise rotation along newly formed $\mathbf{X}$ axis by an angle ϕ.

   (a) Compute the Euler angle transformation matrix for composite rotation.

   (b) Compute the quaternion transformation matrix for composite rotation

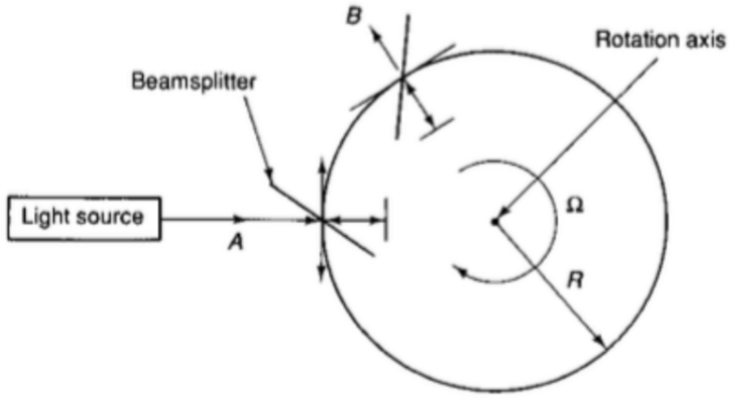

Figure 1: Circular Interferometer

(c) Obtain the values of Euler angles in terms of quaternions using the calculated transformation matrices in parts (a) and (b).

[5+5+5]

3. Consider a rotation of a vector, using quaternion, about an axis defined by the vector $(1, 0, 1)$ through an angle of $2\pi/3$.
   (a) Obtain the quaternion Q to perform this rotation.
   (b) Compute the effect of rotation on the basis vector $\mathbf{j} = (0, 1, 0)$.
   (c) Find the coordinates of above vector in the new frame if we rotate the coordinate frame itself about the same axis and angle while keeping the vector constant?

[5+5+5]

4. Consider a circular interferometer with a radius of 1 km as shown in Fig. 1. Assume that the interferometer is rotating with constant angular velocity of 100 deg/s.
   (a) What would be the closed path travel time for the beams traveling in the direction of rotation and opposite to the direction of rotation?
   (b) How much will be the difference in transit times of both beams?
   (c) Find out the optical path difference between these two beams.

[5+5+5]

5. Consider the ring laser gyro (RLG) in an equilateral triangle and a square shape with its side length given by a and b, respectively. The operating wavelength, λ, of the RLG is assumed to be $0.6328\,\mu$m. Assume that the input angular velocity, Ω, and the value of the side length of the square shaped RLG, b, are $1°$/h and 10 cm, respectively.

(a) Derive the relation between side length of an equilateral triangle and a square shaped RLGs such that the measurable beat frequency for both RLGs remain same, for common operating wavelength and input angular rate.

(b) Find the values of the side length of equilateral triangle shaped RLG, geometric scale factor, and measurable beat frequency.

[5+5]

6. Consider a frame F_1 and a frame F_2 denoted by $\{i_1, j_1, k_1\}$ and $\{i_2, j_2, k_2\}$ unit vectors, respectively. Let C_1^2 be the direction cosine matrix (DCM) for transformation from frame F_1 to frame F_2 whose coefficients are given as below:

$$C_1^2 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Prove the following:

(a) Using the unit vectors, derive C_1^2 and C_2^1 and prove $C_1^2 C_2^1 = \mathbf{I}$.

(b) Using the above results, find the missing coefficients, c_{21}, c_{31}, and c_{32} in the matrix,

$$C_1^2 = \begin{bmatrix} 0.8999 & -0.4323 & 0.0578 \\ c_{21} & 0.8665 & -0.2496 \\ c_{31} & c_{32} & 0.9666 \end{bmatrix}$$

(c) Suppose the frame, F_1, is rotated in the order $\{Z - Y' - X''\}$ where Z, Y' and X'' represent the axes after successive rotations by angles ψ, θ, ϕ, respectively. Find these Euler angles for such a transformation if the resultant composite rotation matrix is same as the one computed in part (b).

[5+5+5]

7. Consider a remote object, such as an aircraft, which is to be tracked from some point on the surface of the earth. The *Local Tangent Plane* is simply a plane tangent to the surface of the earth at this point. We define an initial coordinate frame with the **X** and **Y** axes lying in this tangent plane, pointing in directions *North* and *East* respectively. The **Z**-axis is *geocentric*, that is, it points toward the center of the earth. We then have a right-handed coordinate frame.

We define an angle α, called *Heading*, which is the angle in the tangent plane between North and the projected direction to the remote object. We also define an angle β, called *Elevation*, which is the angle between the tangent plane and the direction to the remote object being tracked, as shown in Figure 2. A *Tracking Transformation* is first a rotation about the **Z**-axis through the angle α, followed by a rotation about the new **y**-axis through the angle β. Notice that in the resulting coordinate frame, the new **x**-axis is pointing directly toward the object being tracked.

 Question 7 continues on the next page...

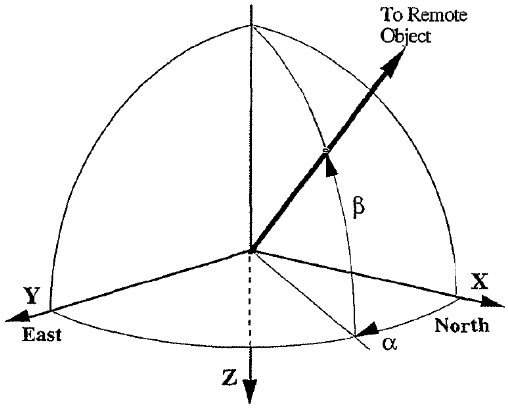

Figure 2: Tracking Transformation.

If $\mathbf{R}$ is the 3×3 matrix representing this rotation, then

$$\mathbf{R} = \mathbf{R}_\beta \mathbf{R}_\alpha = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This composite rotation (tracking transformation) may be represented as an equivalent transformation consisting of a single rotation through some angle about some axis. The axis for this single rotation is found by finding the fixed vector, $\mathbf{v} = (x_1, y_1, z_1)$, for the rotation operator. Compute $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$. [15]