Proposition B.2 [Bertsekas]

(a) A linear function is convex

(b) Any vector norm is convex

(c) The weighted sum of convex functions, with positive weights is convex.

(d) If $I$ is an finite indexset, $C$ is a convex subset of $R^n$ and $f_i : C \to \mathbb{R}$ is convex for each $i \in I$, then the function $h : C \to (-\infty, +\infty]$

$$h(x) = \max_{i \in I} f_i(x)$$

is also convex

Proof

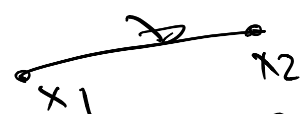
(a) $f(x) = c^T x$ where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$

cost vector; obj function in LP.

$$\boxed{\begin{aligned} LP: \min\ & c^T x. \\ & Ax = b \\ & x \geq 0 \end{aligned}}$$

$f(x_1) = c^T x_1$

$f(x_2) = c^T x_2$




$f(\lambda x_1 + \overline{1-\lambda}\, x_2) \leq \lambda f(x_1) + \overline{1-\lambda}\, f(x_2)$ $\quad \lambda \in [0,1]$

$$\begin{aligned} f(\lambda x_1 + \overline{1-\lambda}\, x_2) &= c^T(\lambda x_1 + \overline{1-\lambda}\, x_2) \\ &= \lambda c^T x_1 + \overline{1-\lambda}\, c^T x_2 \\ &= \lambda f(x_1) + (1-\lambda) f(x_2) \end{aligned}$$

$(\leq)$ $(\geq)$

$$\boxed{\begin{aligned} &\|x\| > 0 \ \forall x \neq 0, \\ &\|x+y\| \leq \|x\| + \|y\| \\ &\|\alpha x\| = |\alpha|\, \|x\| \end{aligned}}$$

$f(x) = \|x\|$ $\quad x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$

$$\begin{aligned} & f(\lambda x_1 + \overline{1-\lambda}\, x_2) \\ &= \|\lambda x_1 + \overline{1-\lambda}\, x_2\| \\ &\leq \|\lambda x_1\| + \|(1-\lambda) x_2\| \\ &= \lambda \|x_1\| + (1-\lambda)\|x_2\| = \lambda f(x_1) + (1-\lambda) f(x_2) \end{aligned}$$

(c)

$$f(x) = w_1 f_1(x) + w_2 f_2(x) + \cdots + w_p f_p(x)$$

$w_1, w_2, \cdots w_p > 0$

$f_1(x), \cdots\cdots f_p(x)$ are convex.

$$f(x_1) = w_1 f_1(x_1) + w_2 f_2(x_1) + \cdots + w_p f_p(x_1)$$

$$f(x_2) = w_1 f_1(x_2) + w_2 f_2(x_2) + \cdots + w_p f_p(x_2)$$

$$f(\lambda x_1 + \overline{1-\lambda}\, x_2) = \sum_{l=1}^{p} w_l \cdot f_l \cdot (\lambda x_1 + \overline{1-\lambda}\, x_2)$$

$$\leq \sum_{l=1}^{p} w_l \cdot \left(\lambda \cdot f_l(x_1) + \overline{1-\lambda}\, f_l(x_2)\right)$$

$$= \lambda \left(\sum_{l=1}^{p} w_l \cdot f_l(x_1)\right) + \overline{1-\lambda} \left(\sum_{l=1}^{p} w_l f_l(x_2)\right)$$

$$= \lambda f(x_1) + \overline{1-\lambda}\, f(x_2)$$

QED.

$$w_1 \times f_1(\lambda x_1 + \overline{1-\lambda}\, x_2) \leq w_1 \times \left(\lambda f_1(x_1) + \overline{1-\lambda}\, f_1(x_2)\right)$$

$$\cdots\cdots\cdots$$

$$w_p \times f_p(\lambda x_1 + \overline{1-\lambda}\, x_2) \leq w_p \times \left(\lambda f_p(x_1) + (1-\lambda) f_p(x_2)\right)$$

(d) $h(x) = \max_{i \in I} f_i(x)$

$= \max\left(f_1(x), f_2(x), \ldots f_p(x)\right)$

$h(x_1) = f_3(x_1)$

$h(x_2) = f_1(x)$

If each $f_i(x)$ is a convex function, then $h(x)$ is also convex

$x_1, x_2 \subset C \quad f_i(x) \leq h(x)$

$x\left(f_i(\lambda x_1 + \overline{1-\lambda}\, x_2)\right) \leq \lambda f_i(x_1) + \overline{1-\lambda}\, f_i(x_2)$

$\leq \left(\lambda h(x_1) + \overline{1-\lambda}\, h(x_2)\right)$

each function $\leq$ RHS

max (each func) $\leq$ RHS

$h(\lambda x_1 + \overline{1-\lambda}\, x_2) \leq \lambda h(x_1) + \overline{1-\lambda}\, h(x_2)$

Proposition B.4 [Bertsekas]

Let $C$ be a convex subset of $\mathbb{R}^n$ and let $f: \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable over $\mathbb{R}^n$.

(a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then 'f' is convex over $C$.

(b) If $\nabla^2 f(x)$ is positive definite for every $x \in C$, then $f$ is strictly convex over $C$

(c) If $C$ is open and $f$ is convex over $C$, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

(d) If $f(x) = x^T Q x$ where $Q$ is a symmetric matrix then $f$ is convex if and only if $Q$ is positive semidefinite. Furthermore, $f$ is strictly convex if and only if $Q$ is positive definite.