Modelling of Optimization Problem:

Stochastic Resource Allocation Problem

$$\max_{x} \; Pr_1 x_1 + Pr_2 x_2 \cdots + Pr_n x_n = \sum_{i=1}^{n} Pr_i x_i$$

$\rightarrow Ax \le b$

non negativity $\rightarrow x \ge 0$

$$\underset{m \times n}{A} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} [x] \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

no. of resources

ith item

$x = (x_1, x_2, \cdots, x_n)$

Decision Variables $\leftarrow$

(Activity level) s.

$Pr_i = P_i - cost_i$

profit.

price Rs per unit

Rs per unit unit quantity

$a_{ij} x_j$

Rows of $A \rightarrow$ are for resources

$a_{ij} \rightarrow$ amount of resource i consumed by unit level of jth activity

$\rightarrow a_{i1}x_1 + a_{i2}x_2 \cdots + a_{in}x_n \le b_i$

$a_i^T x \rightarrow$ net consumption of ith resource by ~~all~~ n activities. $i = 1, \cdots m$.

$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n \le b_1$

$b_i \rightarrow$ amount of resource available

$$\max_{x} \; c^T x = c_1 x_1 + c_2 x_2 \cdots + c_n x_n = \sum_{i=1}^{n} c_i x_i$$

$$\left.\begin{aligned} Ax &\le b \\ x &\ge 0 \end{aligned}\right\}$$

Linear Programming problem.

Resource allocation Problem.

random variables)

$\rightarrow$ where is the stochasticity?

random $\rightarrow Pr_i = (P_i - cost_i)$

P_i fixed? ✗

$cost_i$ random.

maximise average or mean or expected return.

$\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n$

mean or expected value for profit of item per unit quantity.

$$\boxed{\begin{aligned} &\max_x \ \bar{c}_1 x_1 + \bar{c}_2 x_2 \cdots + \bar{c}_n x_n \\ &\left.\begin{aligned} Ax &\le b \\ x &\ge 0 \end{aligned}\right\} \text{ deterministic constraints} \end{aligned}}$$

Lacuna $\longrightarrow$ risk!

$c_1, c_2, \ldots c_n$.

$\bar{c}_1 = 0.02$ $\quad x_1 + x_2 \le 100$ $\quad x_1 + x_2$ investments

$\bar{c}_2 = 1$ $\quad x_1 \ge 0, \ x_2 \ge 0$

Fixed deposited 2% $\bar{c}_1$

share of a company 100% $\bar{c}_2$

Quantify the risk!

$$z = c_1 x_1 + c_2 x_2 \cdots + c_n x_n.$$

$$- \ \bar{z} = \bar{c}_1 x_1 + \bar{c}_2 x_2 \cdots + \bar{c}_n x_n.$$

$$z - \bar{z} = (c_1 - \bar{c}_1) x_1 + (c_2 - \bar{c}_2) x_2 \cdots + (c_n - \bar{c}_n) x_n$$

like to reduce the spread of the

$$\min E \left| \sum_{i=1}^{n} (c_i - \bar{c}_i) x_i \cdots + (c_n - \bar{c}_n) x_n \right|$$

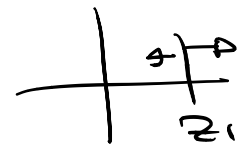
non-differentiable func.

Variance:

z — $\bar{z}$

profit — expected profit

rand. no



$$\sigma^2 = E(z-\bar{z})^2$$

$$z = c^T x = c_1x_1 + c_2x_2 - - + c_nx_n$$

$$\bar{z} = \bar{c}^T x = \bar{c}_1x_1 + \bar{c}_2x_2 - - + \bar{c}_nx_n$$

$$(z-\bar{z})^2 = \left((c_1-\bar{c}_1)x_1 + (c_2-\bar{c}_2)x_2 - - + (c_n-\bar{c}_n)x_n\right)^2$$

$$= \left(\sum_{i=1}^{n} (c_i-\bar{c}_i)x_i\right)^2$$

$$(z-\bar{z})^2 = \left(\sum_{i=1}^{n} (c_i-\bar{c}_i)x_i\right)^2$$

$$E\left[(z-\bar{z})^2\right] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} (c_i-\bar{c}_i)(c_j-\bar{c}_j)x_ix_j\right]$$

sum = 0

for $i = 1, \ldots, n$

for $j = 1, \ldots, n$

sum += $(c_i-\bar{c}_i)(c_j-\bar{c}_j)\, x_i x_j$

end

end

Example:

$$(\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3)^2$$

$$= \alpha_1^2x_1^2 + \alpha_2^2x_2^2 + 2\alpha_1\alpha_2x_1x_2 + \alpha_3^2x_3^2 + 2\alpha_1\alpha_3x_1x_3 + 2\alpha_2\alpha_3x_2x_3$$

$$3^2 = 9$$

$\sigma^2 =$

$$E(z-\bar{z})^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \, E(c_i-\bar{c}_i)(c_j-\bar{c}_j)$$

$$E\{(c_i-\bar{c}_i)(c_j-\bar{c}_j)x_i x_j\} = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

$$= x_i x_j \, E(c_i-\bar{c}_i)(c_j-\bar{c}_j)$$

$z = \sigma_{11}$

c_1

$c_n^2 = \sigma_{nn}$

covariance

$$E\,(c_i-\bar{c}_i)(c_j-\bar{c}_j)$$

$$\sigma_{ij} = E(c_i-\bar{c}_i)(c_j-\bar{c}_j) \qquad \left.\begin{array}{l}\forall\, i = 1 \dots n \\ \forall\, j = 1 \dots n\end{array}\right\} n^2 =$$

n^2 terms

1st quan c_1^2

$$V = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

c_n^2

Covariance matrix

Covariance matrix

Symmetric matrix.

$$x^T V x = (x_1 \cdots x_n) \begin{bmatrix} \sigma_{11}x_1 + \sigma_{12}x_2 \cdots + \sigma_{1n}x_n \\ \sigma_{21}x_1 + \sigma_{22}x_2 \cdots + \sigma_{2n}x_n \\ \vdots \\ \sigma_{n1}x_1 + \sigma_{n2}x_2 \cdots + \sigma_{nn}x_n \end{bmatrix}$$

col. vector.

$$= x_1(\sigma_{11}x_1 + \sigma_{12}x_2 \cdots + \sigma_{1n}x_1) \quad n\text{-term}$$

$$+ x_2(\sigma_{21}x_1 + \sigma_{22}x_2 \cdots + \sigma_{2n}x_n) \quad n\text{-term}$$

$$+ \cdots$$

$$+ x_n(\sigma_{n1}x_1 + \sigma_{n2}x_2 \cdots + \sigma_{nn}x_n) \quad n\text{ term}$$

total of n^2 terms

$$E(z-\bar{z})^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j E(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}x_i x_j$$

$$\sigma^2 = x^T \underline{V} x.$$

variance of the profit.

$x = (x_1, x_2 \cdots x_n)^T$.

covariance matrix

$$\sigma^2 \geq 0 \qquad x^T V x \geq 0 \quad \forall x \neq 0$$

positive semidefinite matrix

A sym. matrix 'A' ($n \times n$) is said to be sym. positive definite, iff $x^T A x > 0$ $\forall x \neq 0$

it is positive semidefinite $x^T A x \geq 0$ $\forall x \neq 0$.

Risk minimization formulation

$$\min_x \; x^T V x$$
$$Ax \leq b$$
$$x \geq 0$$

Investment

2% FD

100% share

P

$$\min \; x^T V x$$
$$Ax \leq b$$
$$x \geq 0$$
$$c^T x \geq z_{asp}$$

min risk but meets some aspiration level on the profits!