Modelling and Geometry of Optimization

1. Consider the following linear problem.

$$\text{Minimize} \quad 2x_1 + 5x_2$$
$$\text{Subject to} \quad x_1 + x_2 \geq 6$$
$$-x_1 - 2x_2 \geq -18$$
$$x_1 \geq 0 \; ; \; x_2 \geq 0$$

a) Sketch the feasible region

b) Mark the vertices

c) Find the optimal.
[Hint: It will be on one of the vertices]

2. Consider the following problem of launching a rocket to a fixed altitude 'b' in a given time T while expending a minimum amount of fuel. Let $u(t)$ be the acceleration force exerted at time t and let $y(t)$ be the rocket altitude at time t. The problem can be formulated as follows.

$$\text{Minimize} \quad \int_0^T |u(t)| \, dt$$
$$\text{Subject to} \quad \ddot{y}(t) = u(t) - g$$
$$y(T) = b$$
$$y(t) \geq 0 \quad t \in [0, T]$$

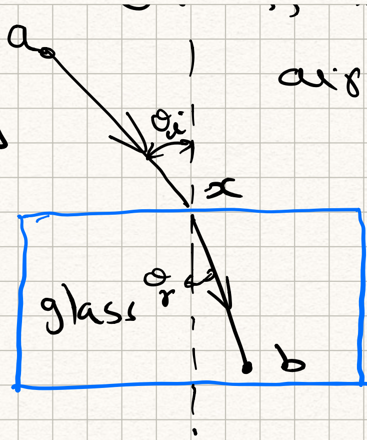
where 'g' is the gravitational force and $\ddot{y}$ is the second derivative of altitude y. Discretize the problem and reformulate it as a linear programming problem. In particular formulate the problem where $T = 10$, $b = 15$ and $g = 10$ (approximation of 9.8)

Hint: Replace the integration by proper summation and the differentiation by difference equations. Make the change of variables $|u_j| = x_j$ and note that $x_j \geq u_j$ and $x_j \geq -u_j$.

3. Consider the problem to minimize the surface area of a cylinder subject to a constraint on its volume. Hence solve it. You can fix volume to V and take cylinder as a closed one.

4. According to Fermat's Principle of Least Time, a light ray takes the path that requires the least time in going from one point to another. Many laws of geometrical optics, such as the laws of reflection and refraction, can be derived from this principle. Refraction is the "bending" of a light ray when it passes between two media in which the velocity of light differs, such as

air and glass.
The ratio of velocities is called the refractive index; for air and glass $v_a / v_g = 1.5$



(a) For given points a in the air and b in the glass, and taking the surface of glass to be the line $x_2 = 0$, formulate an optimization problem whose solution is the point (x_1, x_2) on the surface of the glass at which a light ray enters in the glass in going from a to b.

(b) For $a = (-1, 1)$ and $b = (1, -1)$, solve the optimization problem in part a to determine x.

(c) Check the consistency of your results from part b with Snell's Law of refraction, which says

$$\frac{\sin(\theta_i)}{\sin(\theta_r)} = \frac{v_a}{v_g}$$

where the angle of incidence θ_i and angle of refraction θ_r are measured with respect to the normal to the surface at x. (The dashed line in drawing).

(d) How do your results for part b and c change if instead of glass the ray strikes water, for which the refractive index is $v_a/v_w = 1.33$

5. Read up modelling of 'Location of Facilities' from Section 1.2, of Nonlinear programming, Bazaraa, Sherali and Shetty.