Indian Institute of Technology Bombay

Introduction to Navigation and Guidance

AE 410/641 Fall 2020

Solutions Manual to Tutorial 1

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September 3, 2020

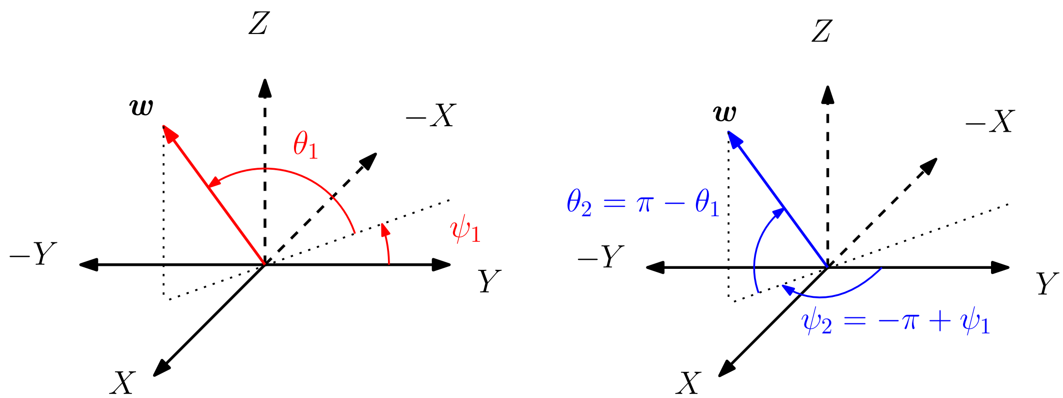
Problem 1. If the consecutive rotations are performed in the order ψ, θ and ϕ i.e., (yaw, pitch and roll) on reference frame **XYZ** then we obtain the another reference frame xyz. This rotation can be performed using a transformation matrix D. For such a transformation:

(a) What are the range for the angles ψ, θ and ϕ?

(b) Can $\pi/2 < |\theta| < \pi$? If not, Why?

(c) If $|\theta| < \pi$, what would be the resultant transformation matrix $\hat{D}$?

Solution. Before answering this question, let us discuss about uniqueness of representing orientation of a vector in 3D space using the Euler angles. As seen from Figure 1,



(a) Representation 1 (b) Representation 2

Figure 1: Dual representations.

there are at least two distinct ways of representing the orientation of a vector. In order to remove this ambiguity, we need to restrict the range of either roll, yaw or pitch angles. The standard practice is to restrict the pitch elevation between $(-\pi/2, \pi/2)$ and the roll and yaw angles be chosen from $(-\pi, \pi)$. It is not necessary for the pitch angle to be restricted. A rule of thumb is to restrict the range for the angle corresponding to the second axis of rotation. For example, the yaw angle $\in (-\pi/2, \pi/2)$, whereas roll and pitch $\in (-\pi, \pi)$ for the order of rotation as $X - Z' - Y''$.

Now that the uniqueness property is established by restricting the range of Euler angles, note that for the rotation $X - Y' - Z''$ the pitch of $\pm\pi/2$ has been excluded. In the context of Figure 1, for $\theta_1 = \theta_2 = 90$, there exist two valid yaw representation namely $\psi_1 = 0°$ and $\psi_2 = -\pi$. Therefore, for the pitch of 90 degrees there still exist ambiguity in distinctly defining the orientation of vector $\vec{w}$. The phenomenon of Gimbal lock can also be thought of in the context of losing uniqueness.

Problem 2. Prove that the sum of the squares of direction cosines is 1.

Solution. Direction cosines are the cosines of the angles made by a directed line segment with the coordinate axes. Without loss of generality, consider a directed line segment passing through origin of the coordinate system, on whose tip, there is a point **P** with coordinates (x, y, z).

The distance of $\mathbf{P}$ from the origin is $\sqrt{x^2+y^2+z^2} = r$ (say). Let the angles that the directed line segment subtends with the positive X, Y and Z axes be α, β and γ, respectively. Then, $\cos\alpha = \dfrac{x}{r}, \quad \cos\beta = \dfrac{y}{r}, \quad \cos\gamma = \dfrac{z}{r}$.

$$\implies \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1.$$

Problem 3. Consider two right handed, orthogonal, unit vectors, $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ and $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ associated with a direction cosine matrix, $\mathbf{C}$.

$$\begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix}$$

(a) Check for orthonormality of the direction cosine matrix.

(b) Prove the correctness of $|\mathbf{C}| = 1$.

Solution. (a) Taking transpose on both sides for given relation,

$$[\mathbf{m}_1 \ \mathbf{m}_2 \ \mathbf{m}_3] = [\mathbf{n}_1 \ \mathbf{n}_2 \ \mathbf{n}_3]\, \mathbf{C}^\top$$

Pre-multiplying by $[\mathbf{m}_1 \ \ \mathbf{m}_2 \ \mathbf{m}_3]^\top$ on both sides, one may obtain

$$\begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} \begin{bmatrix} \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{m}_3 \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n_2} \\ \mathbf{n_3} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \end{bmatrix} \mathbf{C}^\top$$

As the set of vectors $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ and $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ are orthogonal unit vectors, the following hold true: $\mathbf{m}_k \cdot \mathbf{m}_k = \mathbf{n}_k \cdot \mathbf{n}_k = 1, \ \forall \ k = 1,2,3$, and $\mathbf{m}_i \times \mathbf{m}_j = \mathbf{n}_i \times \mathbf{n}_j = 0, \ \ \forall \ \ i,j \in \{1,2,3\}$ and $i \neq j$. Using these relations, we get

$$\begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} \begin{bmatrix} \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{m}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vec{n}_3 \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \end{bmatrix} = \mathbf{I}.$$

Therefore, it can be inferred that $\mathbf{C}\mathbf{C}^\top = \mathbf{I}$.

(b) Using the orthogonal property of $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, e.g. $(\mathbf{m}_1 = \mathbf{m}_2 \times \mathbf{m}_3)$, one has $\mathbf{C}_{11} = \mathbf{C}_{22}\mathbf{C}_{33} - \mathbf{C}_{23}\mathbf{C}_{32}, \ \mathbf{C}_{12} = \mathbf{C}_{23}\mathbf{C}_{31} - \mathbf{C}_{21}\mathbf{C}_{33}, \ \mathbf{C}_{13} = \mathbf{C}_{21}\mathbf{C}_{32} - \mathbf{C}_{22}\mathbf{C}_{31}$. The other combinations, $\mathbf{m}_2 = \mathbf{m}_3 \times \mathbf{m}_1$ and $\mathbf{m}_3 = \mathbf{m}_1 \times \mathbf{m}_2$ can similarly be used to find other such relations, concluding that $\mathrm{adj}(\mathbf{C}) = \mathbf{C}^\top$. Using this results, we get

$$\mathbf{C}^{-1} = \frac{\mathrm{adj}(\mathbf{C})}{|\mathbf{C}|} \implies |\mathbf{C}| = \frac{\mathbf{C}^\top}{\mathbf{C}^{-1}}.$$

From the results established in part (a), one may conclude $\mathbf{C}^\top = \mathbf{C}^{-1}$. Therefore, $|\mathbf{C}| = 1$.

Problem 4. Recall that a conic section is represented by the general formula

$$\mathcal{C}: \ ax^2 + bxy + cy^2 + dx + ey + f = 0, \ a,b,c,d,e,f \in \mathbb{R}.$$

(a) Suppose that a rotation of 45° is applied to the above conic in counter-clockwise direction. What would be the representation (equation) of new conic so obtained? Denote this by $\mathcal{C}'$.

(b) Suppose $\mathcal{C}'$ is again rotated by 45° in counter-clockwise direction. What would be the representation (equation) of new conic so obtained? Denote this by $\mathcal{C}''$.

(c) Suppose a counter-clockwise rotation of 90° is directly applied to the conic $\mathcal{C}$ to obtain $\mathcal{C}^\star$. What would be the representation (equation) of new conic so obtained?

(d) Are $\mathcal{C}''$ and $\mathcal{C}^\star$ identical? Why/Why not?

Solution. A rotation by 45° in counter-clockwise direction can be obtained by using the rotation matrix

$$R_{\pi/4} = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus, the x and y coordinates are transformed as

$$x \to \frac{1}{\sqrt{2}}(x+y), \quad y \to \frac{1}{\sqrt{2}}(y-x).$$

Therefore, $\mathcal{C}'$ can be represented by

$$\begin{aligned}\mathcal{C}' : a&\left[\frac{1}{\sqrt{2}}(x+y)\right]^2 + b\left[\frac{1}{\sqrt{2}}(x+y)\right]\left[\frac{1}{\sqrt{2}}(y-x)\right] + c\left[\frac{1}{\sqrt{2}}(y-x)\right]^2 \\ &+ d\left[\frac{1}{\sqrt{2}}(x+y)\right] + e\left[\frac{1}{\sqrt{2}}(y-x)\right] + f \\ &= \left(\frac{a-b+c}{2}\right)x^2 + (a-c)\,xy + \left(\frac{a+b+c}{2}\right)y^2 + \left(\frac{d-e}{\sqrt{2}}\right)x + \left(\frac{d+e}{\sqrt{2}}\right)y + f.\end{aligned}$$

In a similar way, $\mathcal{C}''$ and $\mathcal{C}^\star$ can be obtained. Try showing that $\mathcal{C}''$ and $\mathcal{C}^\star$ are identical for rotations in plane. This happens because in 2D, rotation matrices group is commutative, thus the order of rotation does not matter. But this is not true in 3D.

Problem 5. Can we have trigonometric and hyperbolic representations of quaternions? Put differently, how can we formally *define*

$$\sin(\mathbf{q}),\ \cos(\mathbf{q}),\ \tan(\mathbf{q}),\ \cosh(\mathbf{q}),\ \sinh(\mathbf{q}),\ \tanh(\mathbf{q})$$

for a quaternion $\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = q_0 + \mathbf{v}$?

Think of this in polar form, $\mathbf{q} = re^{\theta\mathbf{u}}$, where $r = |q|$ denotes the magnitude evaluated using $|\mathbf{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$ and $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$. The angle, θ, is a convex angle ($\theta \in [0, \pi)$). Note that this form is unique for $\mathbf{q} \neq 0, \pm 1$.

Solution. A quaternion, $\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = q_0 + \mathbf{v}$, consists of a scalar component, q_0 and a vector component, $\mathbf{v} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. For ease of understanding, we can denote q_0 as the real part and $\mathbf{v}$ as the imaginary part in the definition of $\mathbf{q}$.

All quaternions have a polar representation, $\mathbf{q} = re^{\theta\mathbf{u}}$, where $r = |q|$ denotes the magnitude evaluated using $|\mathbf{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$ and square roots of -1 are the unit vectors $\mathbf{u}$, such that $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is the normalized imaginary part.

The trigonometric functions, $\sin(\mathbf{q})$, $\cos(\mathbf{q})$, $\tan(\mathbf{q})$ can be defined in many ways, but the definitions won't be equivalent because of non-symmetrization of $e^{\mathbf{iq}}$. For instance, consider the usual definition of $\sin(x)$, where x is a scalar, that is

$$\sin(x) = \frac{e^{\mathbf{i}x} - e^{-\mathbf{i}x}}{2\mathbf{i}},$$

where $e^{\mathbf{i}x} = e^{x\mathbf{i}}$, but for quaternions, $e^{\mathbf{iq}} \neq e^{\mathbf{qi}}$. Also, an ambiguity may arise as whether the factor $\dfrac{1}{2\mathbf{i}}$ should be pre-multiplied or post-multiplied. Similar arguments follow for $\cos(\mathbf{q})$ and $\tan(\mathbf{q})$. Another important thing to note that these different definitions will disagree with the Taylor's series expansion of $\sin(\cdot)$, $\cos(\cdot)$ and $\tan(\cdot)$.

An alternative would be to tackle this ambiguity by considering the definitions of $\cosh(\cdot)$ and $\sinh(\cdot)$. Note that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

give us a good way to define such functions for quaternions, without ambiguity. Thus, it follows that

$$\cosh(\mathbf{q}) = \frac{e^{\mathbf{q}} + e^{-\mathbf{q}}}{2}, \quad \sinh(\mathbf{q}) = \frac{e^{\mathbf{q}} - e^{-\mathbf{q}}}{2},$$

and their Taylor series expansions agree with their *usual* definitions when evaluated at quaternions. From these definitions, one can derive relations for $\sin(\mathbf{q})$, $\cos(\mathbf{q})$ and $\tan(\mathbf{q})$.

Now, we must evaluate $e^{\mathbf{q}}$ to get an expression for $\cosh\mathbf{q}$. We can write

$$e^{\mathbf{q}} = e^{r+\mathbf{v}} = e^r e^{\mathbf{v}},$$

which can be further simplified using Euler's formula as

$$e^{\mathbf{q}} = e^r\left[\cos(\|\mathbf{v}\|) + \cos(\|\mathbf{v}\|)\mathbf{u}\right] = e^r\left[\cos(\|\mathbf{v}\|) + \cos(\|\mathbf{v}\|)\frac{\mathbf{v}}{\|\mathbf{v}\|}\right].$$

Now, one can use the above equation to get required expressions.