Multivariate Functions and Calculus: Review

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1 Continuous Function

- A univariate function f is *continuous* at x, if given any $\epsilon > 0$, $\exists \delta > 0$ such that $|y - x| < \delta$ implies that $|f(y) - f(x)| < \epsilon$
- Pictorially, the graph of f does not contain a 'break' at x
- A multivariate function F is *continuous* at x if, given any $\epsilon > 0$, $\exists \delta > 0$ such that if $||y - x|| < \delta$ then $|F(y) - F(x) < \epsilon$

2 Derivatives

- A univariate function f is said to be *differentiable* at $x^\star$ if following limit exists.

$$f'(x^\star) = \lim_{h \to 0} \frac{f(x^\star + h) - f(x^\star)}{h}$$

  Effectively, derivative at $x^\star$ is the rate of change of the function in the vicinity of $x^\star$, which in limit is the *slope*.

- Example of a function which is continuous but non-differentiable function $|x|$.
- Further differentiating the derivatives yields higher order derivatives
- With multivariate function F, we talk about *partial derivatives* along each coordinate

$$\left.\frac{\partial F}{\partial x_i}\right|_{x^\star} = \lim_{h \to 0} \frac{F(x_1^\star, x_2^\star, \cdots, x_i^\star + h, \cdots, x_n^\star) - F(x^\star)}{h}$$

- Generally, F is said to be *differentiable* at $x^\star$ if it all n partial derivates of F are continuous at $x^\star$
- Gradient vector of F

$$\nabla F(x) = g(x) = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{pmatrix}$$

- For a linear function F, gradient of F is a constant vector
- Similarly, we can define higher order partial derivaties

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial x_j}\right)$$

- Usually written as

$$\frac{\partial^2 F}{\partial x_i \partial x_j}, i \neq j; \qquad \frac{\partial^2 F}{\partial x_i^2}, i = j;$$

3 Hessian

- *Hessian matrix* of $F(x)$ is used to represent second order derivatives.

$$\nabla^2 F(x) = G(x) = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_1} \\ \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_2^2} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_1 \partial x_n} & \frac{\partial^2 F}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 F}{\partial x_n^2} \end{pmatrix}$$

  It can be shown that

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}$$

  Consequently, *Hessian* matrix is symmetric.

- If Hessian is constant, F is a *quadratic* function

$$F(x) = \frac{1}{2} x^T G x + c^T x + \alpha$$

- Hessian matrix of the scalar function $F(x)$ is Jacobian matrix of vector function $g(x)$

Class of fucntions with continuous derivatives of order 1 through k is denoted by $\mathcal{C}^k$. Functions with high degrees of differntiability are referred as *smooth* function.

4 Order Notation

Let $f(h)$ be a univariate function of h. Then the function $f(h)$ is said to be of order h^p ($O(h^p)$) if there exists a finite number M, $M > 0$, independent of h, such that as $|h|$ approaches zero

$$|f(h)| \leq M|h|^p.$$

For sufficiently small $|h|$, the rate at which an $O(h^p)$ term will go to zero will increase as p increases. It is often desirable to find the largest value of p for which above inequality holds.

5 Taylor's Series

Theorem 1. *If $f(x) \in \mathcal{C}^r$, then there exists a scalar $\theta (0 \leq \theta \leq 1)$, such that*

$$\begin{aligned} f(x+h) =& f(x) + hf'(x) + \frac{1}{2!} h^2 f''(x) + \cdots \\ & \frac{1}{(r-1)!} h^{r-1} f^{(r-1)}(x) + \frac{1}{r!} h^r f^{(r)}(x + \theta h) \end{aligned}$$

The above Taylor-series can also be written as

$$\begin{aligned} f(x+h) =& f(x) + hf'(x) + \frac{1}{2} h^2 f''(x) + \cdots \\ & \frac{1}{(r-1)!} h^{r-1} f^{(r-1)}(x) + O(h^r) \end{aligned}$$

assuming that $|f(r)|$ is finite in the interval $[x, x+h]$.

- For $r < p$, $\mathcal{C}^r \subset \mathcal{C}^p$. Hence, the series can be truncated at lower order.
- If function is infinitely differentiable, then we have infinite Taylor's series

Taylor theorem also holds for sufficiently smooth multivariate functions. With point x, direction p and scalar h

$$F(x+hp) = F(x) + h\nabla(x)^T p + \frac{1}{2}h^2 p^T \nabla^2 F(x) p + \cdots$$
$$+ \frac{1}{(r-1)!}h^{r-1}D^{r-1}F(x) + \frac{1}{r!}h^r D^r F(x+\theta hp)$$

where,

$$D^s F(x) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_s=1}^{n}\left\{p_{i_1}p_{i_2}\cdots p_{i_s}\frac{\partial^s F(x)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_s}}\right\}$$

Restricting to first three terms

$$F(x+hp) = F(x) + h\nabla F(x)^T p + \frac{1}{2}h^2 p^T \nabla^2 F(x) p + O(h^3)$$

6 Review Exercises

1. (a) What is the order of the funnction?

   (i) $f(x) = x + x^2$ i.e., is it $O(\epsilon^2)$ or $O(\epsilon)$?

   (ii) $f(x) = 3x^{1.5} + 5x^2, \ \ x > 0$

   where ϵ is a small number tending to zero.

   (b) What are the constants M in the above two cases.

2. Let $x \in \Re^n$; show that $\nabla \|x\|_2 = \dfrac{1}{\|x\|_2}\, x$ for $x \neq 0$.

3. Let $F(x_1\,,\,x_2) = \dfrac{x_1\,x_2}{x_1^2 + x_2^2}$ when $x \neq 0$. Show that $\nabla F(0)$ exist but, strictly, the function is not differentiable as it is not defined at $(0\,,\,0)$.

4. Let $f(x) = x^3$; we are interested in (a) linear approximation of $f(x)$ around $x = 0$ and (b) its quadratic approximation. Find them and comment.

5. (a) Let $f(x) = 2 + x - x^3$; write the Taylor Series expression of $f(x)$ around $x = 0$ i.e.,

   $$f(0+\epsilon) = f(0) + \frac{\epsilon}{1\,!}\,f'(0) + \frac{\epsilon^2}{2\,!}\,f''(0) + \frac{\epsilon^3}{3\,!}\,f'''(\theta)$$

   What would be the appropriate value of θ ?

   (b) Now consider that the Taylor Series is truncated at scond derivative i.e.,

   $$f(0+\epsilon) = f(0) + \frac{\epsilon}{1\,!}\,f'(0) + \frac{\epsilon^2}{2\,!}\,f''(\theta)$$

   What would be the value of θ you would choose? Make sure that θ is between 0 and ϵ.

References

[1] Michael T. Heath, Scientific Computing: An Introductory Survey The McGraw-Hill Companies, 2nd edition, July 17, 2002.