Univariate

$$\underset{x \in \mathbb{R}^1}{\text{minimize}}\ f(x)$$

f is everywhere twice continously differentiable and local minimum of f exists at a point x^*.

<u>Necessary Condition</u> for x^* to be optimal

A1 $\quad f'(x^*) = 0 \quad$ (stationary point)

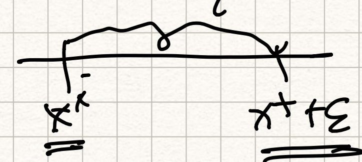
A2 $\quad f''(x^*) \geq 0$

Proof:

$$\underline{f(x^*+\varepsilon)} = \underline{f(x^*)} + \underbrace{\varepsilon f'(x^*)}_{<0} + \underline{\tfrac{1}{2}\varepsilon^2 f''(x^*+\theta\varepsilon)}$$

$f''(x^* + \theta\varepsilon)$

$0 \leq \theta \leq 1$



Proof by contradiction

Suppose. $\underline{f'(x^*) < 0}$

It implies that if I take $\varepsilon > 0$; small enough

$f'(x^*) \neq 0$

$f'(x^*) < 0 \quad f'(x^*) > 0$

So, the first derivative term will reduce the function.

$f'(x^*) < 0$

$\varepsilon > 0$

$x^* + \varepsilon$

$$\underbrace{\underbrace{\varepsilon f'(x^*)}_{<0} + \frac{\varepsilon^2}{2} f''(x^* + \theta\varepsilon)}$$

decide $\quad < 0 \quad > 0 \quad = 0$

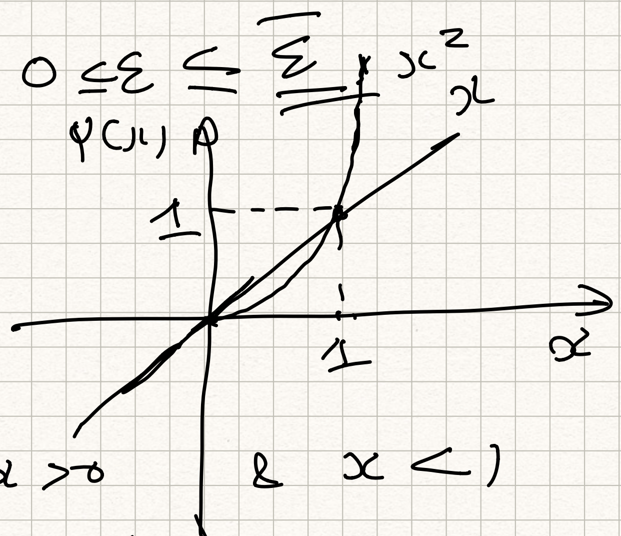
if

$\epsilon f'(x^*) < 0$

does it imply

$\epsilon f'(x^*) + \frac{\epsilon^2}{2} f''(x^* + \theta\epsilon) < 0$

Answer is yes, $0 \le \epsilon \le \bar{\epsilon}$



$\psi_1(x) = x$, $\psi_2(x) = x^2$

$\underline{x} - x^2$

then for $x < 1$

$x - x^2 > 0$ if $x > 0$ & $x < 1$

$x - x^2 < 0$ $x > 1$.

for small values of x, linear term decides the sign of $x - x^2$

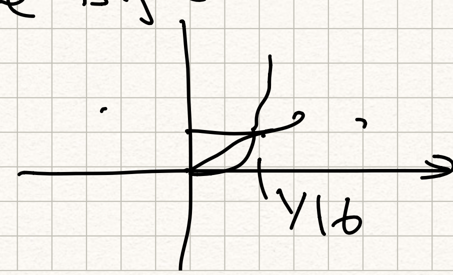
but for large x, the quadratic term decides the sign of $x - x^2$

$\underline{x} - 16\underline{x^2}$

$x = 16x^2$

$x = \frac{1}{16}$

$x < \frac{1}{16}$



$$f(x^*+\varepsilon) = f(x^*) + \varepsilon f'(x^*) \quad (2)$$
$$+ \frac{\varepsilon^2}{2} f''(x^* + \theta\varepsilon) \quad (3)$$

θ

$0 < \varepsilon < \bar{\varepsilon}$

sign of (2) + (3) will be decided by ~~two~~ the linear term

$$\underbrace{\frac{\varepsilon^2}{2} f''(x^* + \theta\varepsilon)}_{\text{cmlmo:}} < \underline{\underline{M \cdot \varepsilon^2}}$$

$\frac{1}{16}$

$$\underbrace{0 \le \theta \le 1}_{\text{closed interval}}$$

$$f(x^*+\varepsilon) < f(x^*)$$

$\therefore \quad \varepsilon f'(x^*) \underline{\underline{< 0}}$

if x^* is a local min

$f'(x^*) < 0$ X

$f'(x^*) > 0$ X

$\underline{\underline{\varepsilon < 0}}$

$\therefore \quad f'(x^*) = 0;$

$$f(x^* + \varepsilon) = f(x^*) + \frac{\varepsilon^2}{2} f''(x^* + \theta\varepsilon)$$

$0 \leq \theta \leq 1$ and unknown!

0

$+$

③

~~Suppose~~ $f''(x^*) \geq 0$

To the contrary assume that

$$f''(x^*) < 0$$

$$\Rightarrow f''(x^* + \theta\varepsilon) < 0$$

$$f(x^* + \varepsilon) < f(x^*)$$

x^* is a minimum!

Continuity argument on $f''(x^*)$

continuity argument:

If $f''(x^*) < 0$ then in a small neighborhood $f''(x)$ will continue to remain less than zero

$f''(x^*) < 0 \implies \underbrace{f''(x^* + \theta\varepsilon)}_{\uparrow} \leq 0$

$0 \leq \theta\varepsilon \leq \varepsilon$

$f(x^* + \varepsilon) = f(x^*) + \underbrace{\frac{\varepsilon^2}{2} f''(x^* + \theta\varepsilon)}_{<0}$

$f(x^* + \varepsilon) < f(x^*)$

which is a contradiction to x^* being a local min.

$f''(x^*) \geq 0$

Further, if $f''(x^*) > 0$

$f(x^* + \varepsilon) = f(x^*) + \frac{\varepsilon^2}{2} f''(x^* + \theta\varepsilon)$

$\underbrace{f(x^* + \varepsilon) > f(x^*)}_{0}$ $\quad 0 < \varepsilon < \bar{\varepsilon}$

strict local min.

sufficient condn

N C

$$\left.\begin{array}{l} f'(x^*)=0 \\ f''(x^*)\geq 0. \end{array}\right\}$$

non-negative curvature.

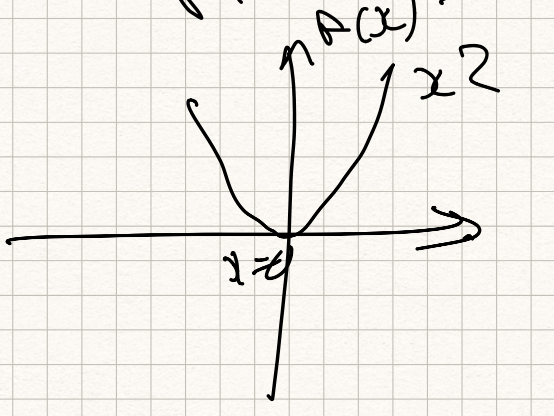
$$\left.\begin{array}{l} f'(x^*)=0 \\ f''(x^*)=0 \end{array}\right\}$$

$f(x)=x^2$

$x=0$

$f'(0)=0$

$f''(0)=2$



S C

$f'(x^*)=0$

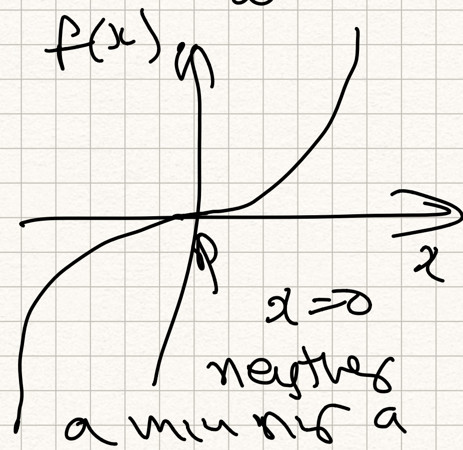
$f''(x^*)>0$

$f(x)=x^3$

$f'(0)=0$

$f''(0)=6x$
$=0$

at $=0$



$x=0$
neither a min nor a max

$f(x) = x^3$

Is $x^* = 0$ a min?

If you check necessary conds, they are satisfied.

Yet it is not a min!

$f''(x) = 0$;

More investigations in this func. are required.

$f(x) = x^3$; $x = 1$

$x = 1$ ✗ Local min.

$f'(x) = 3x^2 = 3 \neq 0$

NC ✗ local min ✗

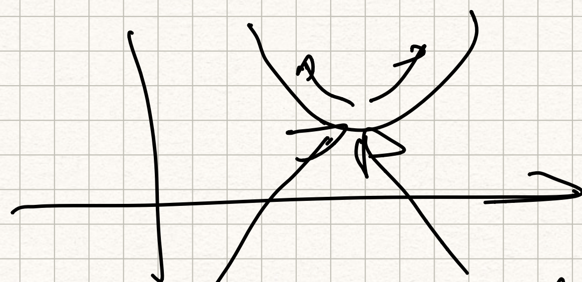
NC ✓ local min possible

Not sure

SC ✓ Sure that it is a local min.

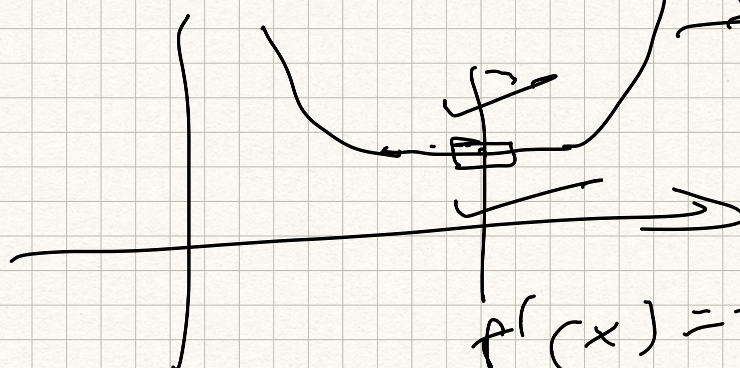
$f'(x^*) = 0$

$f''(x^*) > 0$



strict local min.

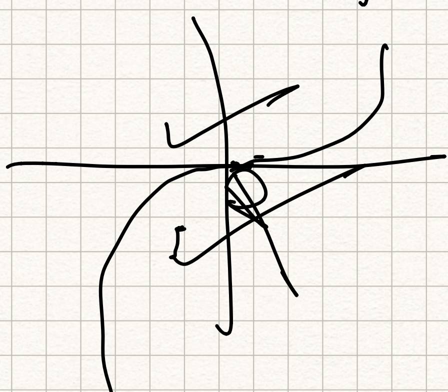
guarantee a local min



$f'(x) = 0$

$f''(x) = 0$

x is a local min.



weak

NC

SC

may SC