Introduction to Navigation & Guidance

(Course Code: AE 410/641)

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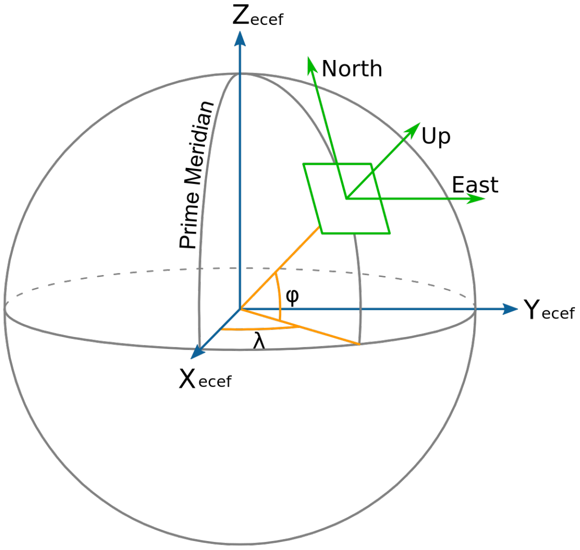
Tutorial-2 Solutions

1. Consider two coordinate systems, one a fixed rectangular system with unit vectors $I,\ J,\ K$ and the other rotating on a local tangent plane of a sphere. The latter has unit vectors $i,\ j,\ k$. The sphere is rotating at a constant rate of Ω. In the inertial coordinate system the rotation vector is $\Omega\ K$. In the rotating coordinate system, the unit vector i is to the east, the unit vector j is to the north and the unit vector k is vertically normal to the surface. Based on this information,



(a) Transform the unit vectors $i,\ j,\ k$ of rotating frame in fixed frame.

(b) Show that :

$$\frac{di}{dt} = \Omega \times i$$
$$\frac{dj}{dt} = \Omega \times j$$
$$\frac{dk}{dt} = \Omega \times k$$

Solution

(a) From above figure, it can be seen that the ENU coordinates can be transformed to the ECEF by two rotations:

- An anticlockwise rotation by $(\pi/2 - \phi)$ about east axis in order to align up axis unit vector k with the Z axis of ECEF.
- A clockwise rotation by $(\pi/2 + \lambda)$ about Z axis in order to align east axis unit vector i with the X axis of ECEF.

The composite transformation matrix is obtained as,

$$\begin{aligned} R &= \begin{bmatrix} \cos(\pi/2+\lambda) & -\sin(\pi/2+\lambda) & 0 \\ \sin(\pi/2+\lambda) & \cos(\pi/2+\lambda) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/2-\phi) & \sin(\pi/2-\phi) \\ 0 & -\sin(\pi/2-\phi) & \cos(\pi/2-\phi) \end{bmatrix} \\ &= \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \cos\phi \end{bmatrix} \end{aligned}$$

The unit vectors in local East, North and Up directions as expressed in ECEF cartesian coordinates are given by the columns of the transformation matrix. Hence,

$$i = -\sin\lambda \ I + \cos\lambda \ J \tag{1}$$

$$j = -\sin\phi\cos\lambda \ I - \sin\phi\sin\lambda \ J + \cos\phi \ K \tag{2}$$

$$k = \cos\phi\cos\lambda \ I + \cos\phi\sin\lambda \ J + \cos\phi \ K \tag{3}$$

(b) Since ϕ is constant,

$$\frac{di}{dt} = (-\cos\lambda \ I - \sin\lambda J)\frac{d\lambda}{dt} = -\Omega\left(\cos\lambda \ I + \sin\lambda J\right) \tag{4}$$

The cross product $\Omega \times i$ can be obtained using

$$\Omega \times i = \begin{bmatrix} I & J & K \\ 0 & 0 & \Omega \\ -\sin\lambda & \cos\lambda & 0 \end{bmatrix} = -\Omega\left(\cos\lambda \ I + \sin\lambda J\right) \tag{5}$$

From equations (4) and (5) it can be seen clearly that

$$\frac{di}{dt} = \Omega \times i$$

Similarly we can prove,

$$\frac{dj}{dt} = \Omega \times j$$
$$\frac{dk}{dt} = \Omega \times k$$

2. Consider a square shaped RLG with input angular velocity equal to 1°/hr. The operating wavelength is assumed to be 0.6328μm. If the measurable beat frequency is obtained to be 0.35 Hz,

(a) Calculate the scale factor, side length and optical path length of given RLG.

(b) Calculate the radius for the circular RLG equivalent to given square shaped RLG with same operating wavelength.

Solution

(a) From the ideal RLG equation, the measurable beat frequency is given by,

$$\Delta\nu = S\Omega \tag{6}$$

$$S = \frac{\Delta\nu}{\Omega} \tag{7}$$

Substituting the values,

$$S = \frac{0.35}{4.8481 \times 10^{-6}} \tag{8}$$

$$S = 7.2193 \times 10^{4} \tag{9}$$

For a square shaped RLG of side length b, total optical length is given by

$$L = 4\ b$$

. Also, the scale factor is obtained by

$$S = \frac{4A}{L\lambda} \tag{10}$$

$$= \frac{4b^2}{4b\lambda} = \frac{b}{\lambda} \tag{11}$$

$$7.2193 \times 10^{4} = \frac{b}{0.6328 \times 10^{-6}} \tag{12}$$

$$b = 0.0457 \tag{13}$$

Thus, the side length of square shaped RLG is $b = 4.57$cm. Also, the optical length is obtained by,

$$L = 4 \times 4.57 = 18.28\text{cm}$$

(b) For an equivalent circular RLG, the scale factor needs to be equal to the given square shaped RLG.
Area of circular RLG = πR^2
Optical path length of circular RLG = $2\pi R$
The scale factor of circular RLG is given by

$$S = \frac{4\pi R^2}{2\pi R\lambda} = \frac{2R}{\lambda}$$

Hence,

$$R = \frac{S\lambda}{2} = \frac{7.2193 \times 10^{4} \times 0.6328 \times 10^{-6}}{2} = 0.0228$$

. Hence, required radius for equivalent circular RLG is $R = 2.28$cm.

(c) Optical path length needs to satisfy

$$L_{\pm} = N\lambda_{\pm} \qquad \text{or}$$
$$\lambda_{\pm} = \frac{L_{\pm}}{N}$$

where N is a large integer.
Thus, length change of ΔL will cause wavelength change of

$$\Delta\lambda = \frac{\Delta L}{N}$$

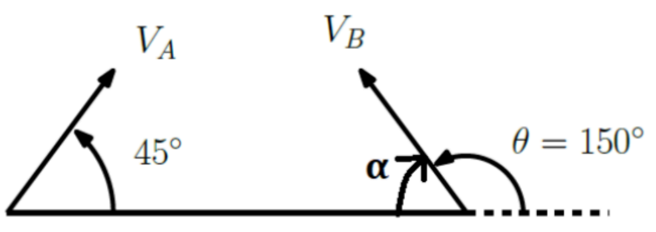
Also, as

$$\nu\lambda = c = 3 \times 10^8$$

is constant,

$$\frac{\Delta L}{L} = \frac{\Delta\lambda}{\lambda} = \frac{\Delta\nu}{\nu}$$

3. Consider the positions of two aircraft, A and B, as shown in the figure below. Aircraft A has a speed of 400 m/s and carries a radar transmitting signal at frequency of 300 MHz. Aircraft A is tracking aircraft B which has a speed of 300 m/s. The direction of aircraft are the same as shown in figure below.



(a) How much is the doppler frequency shift observed by the radar in aircraft A?

(b) What should be the flight directions θ of aircraft B for the doppler frequency shift to be equal to 2000 Hz and 0 Hz?

(c) Find the value of maximum doppler shift for the above given speeds of aircraft if it is allowed to change directions of both the aircraft ?

Solution : Given, $V_A = 400$ m/s and $V_B = 300$m/s. Transmission frequency, $f_0 = 300$ MHz. We compute the relative velocity (V_r) of aircraft A with respect to that of aircraft B along the line of sight as follows :

$$\begin{aligned} V_r &= V_A \ \cos 45^\circ + V_B \ \cos 30^\circ \\ &= 400 \ \cos 45^\circ + 300 \ \cos 30^\circ \\ V_r &= 542.65 m/s. \end{aligned}$$

1. Doppler frequency shift is now computed as

$$\begin{aligned} f_d &= \frac{2V_r f_0}{c} \\ &= \frac{2 \times 542.65 \times 300 \times 10^6}{3 \times 10^8} \\ &= 1085.30Hz \end{aligned}$$

2. Computations of flight directions require relative velocity to be known for different f_d i.e, $V_r = \dfrac{c\ f_d}{2f_0}$
   - For $f_d = 2000Hz$, $V_r = 1000$m/s. Thus,

$$\begin{aligned} 1000 &= 400\ \cos 45^\circ + 300\ \cos\theta \\ \cos\theta &= 2.39 \end{aligned}$$

     This scenario is not possible.
   - For $f_d = 0Hz$, $V_r = 0$m/s,

$$\begin{aligned} 0 &= 400\ \cos 45^\circ + 300\ \cos\theta \\ \theta &= 160.53^\circ \end{aligned}$$

3. In order to evaluate $f_{d_{max}}$,

$$\begin{aligned} f_{d_{max}} &= \frac{2V_{r_{max}} f_0}{c} \\ &= \frac{2 \times (400\ \cos 0^\circ + 300\ \cos 0^\circ) \times 300 \times 10^6}{3 \times 10^8} \\ &= 1400Hz. \end{aligned}$$

4. A position vector in navigation frame is obtained as $r_n = 0.6124\hat{i}\ -\ 0.7071\hat{j}\ -\ 0.3536\hat{k}$. If the corresponding position vector in earth frame represented by unit vectors (I, J, K) is obtained as $r_e = \hat{I}$, find out the location of INS with respect to earth frame. Assume that the navigation frame follows ENU convention.

   Solution The transformation matrix from the ECEF frame to navigation frame is given by,

$$r_n = C_e^n\ r_e \qquad \text{where,} \tag{14}$$

$$C_e^n = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\cos\lambda & \cos\phi \end{bmatrix} \tag{16}$$

Based on given information,

$$\begin{bmatrix} 0.6124 \\ -0.7071 \\ -0.3536 \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\cos\lambda & \cos\phi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

Above equation reduces to following three equations,

$$\cos\phi\cos\lambda = 0.6124 \tag{18}$$

$$-\sin\lambda = -0.7071 \tag{19}$$

$$-\sin\phi\cos\lambda = -0.3536 \tag{20}$$

Solving above equations we get $\lambda = 45°$ and $\phi = 30°$

5. A vehicle is elevating in earth atmosphere, consisting of the INS devices to measure different parameters. A vertical accelerometer is mounted on the vehicle to obtain the acceleration values.

   Derive the relation between altitude error (Δh) and time (t), where h is altitude from the surface of the earth. Use the radius of earth as $Z_0 = 6378$ km and the gravitational acceleration at earth's surface as $g_0 = 9.8$ m/s^2. Assume the error in acceleration measurement to be $10^{-5}g_0$, and initial conditions $\Delta h(0) = \Delta\dot{h}(0) = 0$. Also,

   (a) Comment on stability of the vehicle.

   (b) Plot the relation derived between Δh and t, to draw the relevant inference.

Solution :

(a) Assuming a vertical accelerometer having an input axis along Z-axis.
Dynamics of Z can be written as,

$$\ddot{Z} = A - g$$

We know from Newton's law from gravitation,

$$g = g_0\left(\frac{Z_0}{Z}\right)^2$$

Where, Z_0 is the arbitrary initial point and g_0 is the acceleration due to gravity at Z_0.
Let,

$$Z = Z_0 + h$$

hence,

$$g = g_0\left(1 + \frac{h}{Z_0}\right)^{-2} = g_0\left(1 - \frac{2h}{Z_0}\right) = g_0 - \frac{2g_0h}{Z_0} \quad \text{(using binomial property)}$$

Dynamics of h can be written as,

$$\ddot{h} = A - g_0 + \frac{2g_0 h}{Z_0} \implies \boxed{\ddot{h} - \frac{2g_0 h}{Z_0} = A - g_0}$$

After implementing Laplace transformation on both the sides and finding the poles of the system, one of the poles lies on the right hand side of the complex s-plane which gives rise to the instability of the system.

(b) Now, If there is an error ΔA in measurement of A, dynamics of error in h would be,

$$\Delta\ddot{h} - \frac{2g_0}{Z_0}\Delta h = \Delta A$$

General solution of the above equation can be written as,

$$\Delta h = A\cosh\left(\sqrt{\frac{2g_0}{Z_0}}t\right) + B\sinh\left(\sqrt{\frac{2g_0}{Z_0}}t\right) - \frac{\Delta A}{\frac{2g_0}{Z_0}}$$

Also, for given initial conditions in the form of,

$$\Delta h(0) = \Delta\dot{h}(0) = 0$$

Particular Solution becomes,

$$\boxed{\Delta h = \frac{Z_0 \Delta A}{2g_0}\left[\cosh\left(\sqrt{\frac{2g_0}{Z_0}}t\right) - 1\right]}$$

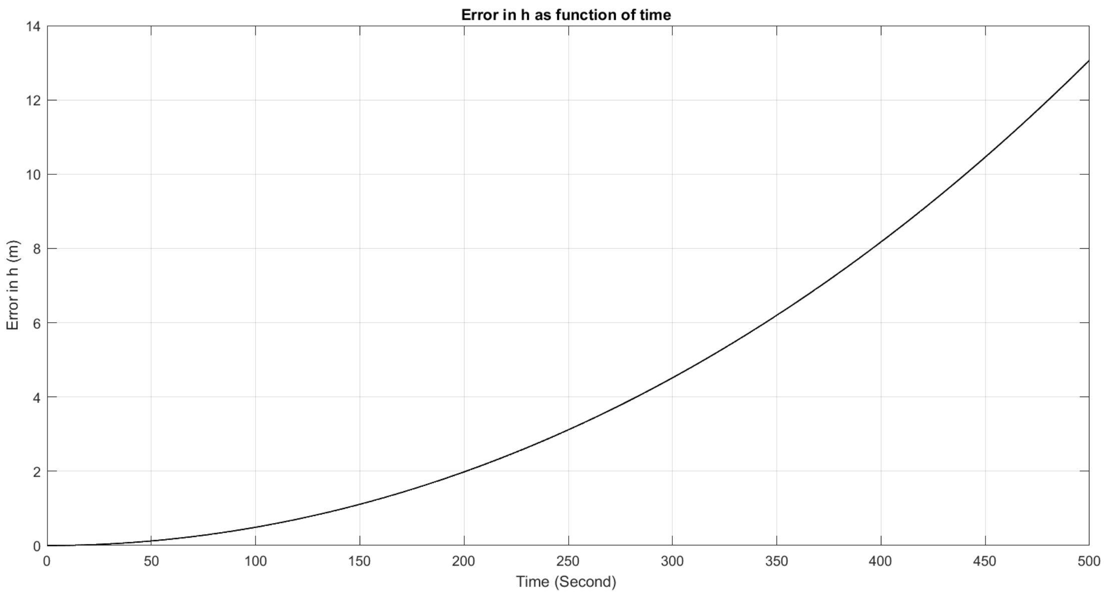
Putting the given values in the equation,

$$\boxed{\Delta h = \frac{6378000 \times 10^{-5}}{2}\left[\cosh\left(\sqrt{\frac{2 \times 9.8}{6378000}}t\right) - 1\right]}$$

$$\boxed{\Delta h = 31.89\left[\cosh\left(1.7530 \times 10^{-3}t\right) - 1\right]}$$

Δh vs time is plotted for a duration of 500s. As can be seen in the figure, even for the small measurement error in the acceleration, error in the measurement of height is huge. As the time increases, error in the altitude increases and the plot diverges. An important conclusion to draw from the plot is, as the error is getting accumulated with time, the measurement would be inaccurate & hence the parameters can only be measured for a short time span.

Question 5 continues on the next page...



Error in h as function of time
Error in h (m)
Time (Second)
0
2
4
6
8
10
12
14
0
50
100
150
200
250
300
350
400
450
500