AE 410: Navigation and Guidance

Assignment 02

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Solution 01

A :

$$\begin{aligned} \dot{r} = V_r &= V_T\cos(\theta_T) - V_M\cos(\delta) \\ \text{and } r\dot{\theta} = V_\theta &= V_T\sin(\theta_T) - V_M\sin(\delta) \end{aligned}$$

Since γ_T is constant over time,

$$\begin{aligned} \dot{\theta_T} &= -\dot{\delta} \\ &= \frac{-V_T\sin(\theta_T) + V_M\sin(\delta)}{r} \\ \implies \frac{dr}{d\theta_T} &= \frac{r\left(V_T\cos(\theta_T) - V_M\cos(\theta_T)\right)}{-V_T\sin(\theta_T) + V_M\sin(\delta)} \\ \implies \frac{dr}{r} &= \frac{\left(V_T\cos(\theta_T) - V_M\cos(\theta_T)\right)}{-V_T\sin(\theta_T) + V_M\sin(\delta)}d\theta_T \\ &= \frac{V_T\cos(\theta_T)}{-V_T\sin(\theta_T) + V_M\sin(\delta)}d\theta_T - \frac{V_M\cos(\delta)}{-V_T\sin(\theta_T) + V_M\sin(\delta)}d\theta_T \end{aligned}$$

Let $s = -V_T\sin(\theta_T) + V_M\sin(\delta)$. This gives $ds = -V_T\cos(\theta_T)d\theta_T$ and $\theta_T =$.

$$\begin{aligned} \frac{dr}{r} &= -\frac{ds}{s} - \frac{V_M\cos(\delta)}{-V_T\sin(\theta_T) + V_M\sin(\delta)}d\theta_T \\ \int\frac{dr}{r} &= -\int\frac{ds}{s} - \int\frac{V_M\cos(\delta)}{-V_T\sin(\theta_T) + V_M\sin(\delta)}d\theta_T \\ r &= \frac{\exp\left(-\int\frac{K\cos(\delta)}{K\sin(\delta) - \sin(\theta_T)}d\theta_T\right)}{V_M\sin(\delta) - V_T\sin(\theta_T)} \end{aligned}$$

Now, we can integrate the term inside the exponential on expanding it as follows:

$$\begin{aligned} I = \frac{K\cos(\delta)}{K\sin(\delta) - \sin(\theta_T)} &= \frac{\mu\cos(\beta)}{\sin(\beta) - \sin(\theta_T)} \\ &= -\frac{\mu}{2}\frac{\cos\left(\frac{\theta_T+\beta}{2} - \frac{\theta_T-\beta}{2}\right)}{\cos\left(\frac{\theta_T+\beta}{2}\right)\sin\left(\frac{\theta_T-\beta}{2}\right)} \\ &= -\frac{\mu}{2}\left(\cot\left(\frac{\theta_T-\beta}{2}\right) + \tan\left(\frac{\theta_T+\beta}{2}\right)\right) \end{aligned}$$

On integrating, we get:

$$\begin{aligned}
-\int I d\theta_T &= \frac{\mu}{2}\left(\int \cot\left(\frac{\theta_T-\beta}{2}\right) d\theta_T + \int \tan\left(\frac{\theta_T+\beta}{2}\right) d\theta_T\right) \\
&= \frac{\mu}{2}\left(\ln\left(\sin\left(\frac{\theta_T-\beta}{2}\right)\right) - \ln\left(\cos\left(\frac{\theta_T+\beta}{2}\right)\right)\right) + C' \\
&= \frac{\mu}{2}\ln\left(\frac{\sin\left(\frac{\theta_T-\beta}{2}\right)}{\cos\left(\frac{\theta_T+\beta}{2}\right)}\right) + C'
\end{aligned}$$

From this, we can develop r as follows:

$$\begin{aligned}
r &= \frac{C''}{V_M\sin(\delta) - V_T\sin(\theta_T)}\left(\frac{\sin\left(\frac{\theta_T-\beta}{2}\right)}{\cos\left(\frac{\theta_T+\beta}{2}\right)}\right)^{\frac{\mu}{2}} \\
&= \frac{C'''}{\sin(\beta) - \sin(\theta_T)}\left(\frac{\sin\left(\frac{\theta_T-\beta}{2}\right)}{\cos\left(\frac{\theta_T+\beta}{2}\right)}\right)^{\frac{\mu}{2}} \\
&= \frac{C'''}{-2\cos\left(\frac{\theta_T+\beta}{2}\right)\sin\left(\frac{\theta_T-\beta}{2}\right)}\left(\frac{\sin\left(\frac{\theta_T-\beta}{2}\right)}{\cos\left(\frac{\theta_T+\beta}{2}\right)}\right)^{\frac{\mu}{2}} \\
&= C\frac{\sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)}{\cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}
\end{aligned}$$

given $|K\sin(\delta)| \geq 1$

B : For successful interception, $K > 1$. Now, from the basic principles of deviated pursuit, we get:

$$\begin{aligned}
a_M &= V_M\dot{\theta} \\
&= V_M\left(\frac{V_T\sin(\theta_T) - V_M\sin(\delta)}{r}\right) \\
&= \frac{V_M V_T}{C}\left(\sin(\theta_T) - K\sin(\delta)\right)\frac{\cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}{\sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)} \\
&= \frac{V_M V_T}{C}\left(\sin(\theta_T) - \sin(\beta)\right)\frac{\cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}{\sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)} \\
&= \frac{V_M V_T}{C}\left(2\cos\left(\frac{\theta_T+\beta}{2}\right)\sin\left(\frac{\theta_T-\beta}{2}\right)\right)\frac{\cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}{\sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)} \\
&= \frac{2V_M V_T}{C}\frac{\cos^{\mu+2}\left(\frac{\theta_T+\beta}{2}\right)}{\sin^{\mu-2}\left(\frac{\theta_T-\beta}{2}\right)}
\end{aligned}$$

For a_M to be bounded, $\mu - 2 \leq 0$; i.e., $\mu \leq 2$ giving:

$$\begin{aligned}
&\frac{K\cos(\delta)}{\sqrt{1-\sin^2(\beta)}} \leq 2 \\
\implies &K^2 - K^2\sin^2(\beta) \leq 4 - 4K^2\sin^2(\beta) \\
&K^2(1 + 3\sin^2(\beta)) \leq 4 \\
&K \leq \frac{2}{\sqrt{1+3\sin^2(\beta)}}
\end{aligned}$$

Hence, we get:

$$1 < K \leq \frac{2}{\sqrt{1 + 3\sin^2(\beta)}}$$

Solution 02

The miss distance is given as:

$$\begin{aligned} r_{miss} &= r_0 \sqrt{\frac{V_{\theta_0}^2}{V_{r_0}^2 + V_{\theta_0}^2}} \\ 5 &= 5000 \times \sqrt{\frac{(300 - 500\sin(\gamma_M))^2}{(500\cos(\gamma_M))^2 + (300 - 500\sin(\gamma_M))^2}} \\ \left(90000 + 250000\sin^2(\gamma_M) - 300000\sin(\gamma_M)\right) &= \frac{1}{1000000}\left(250000 + 90000 - 300000\sin(\gamma_M)\right) \end{aligned}$$

From Fig. 1, we observe that $\gamma_M = 0.6445\ rad = 36.92713°$

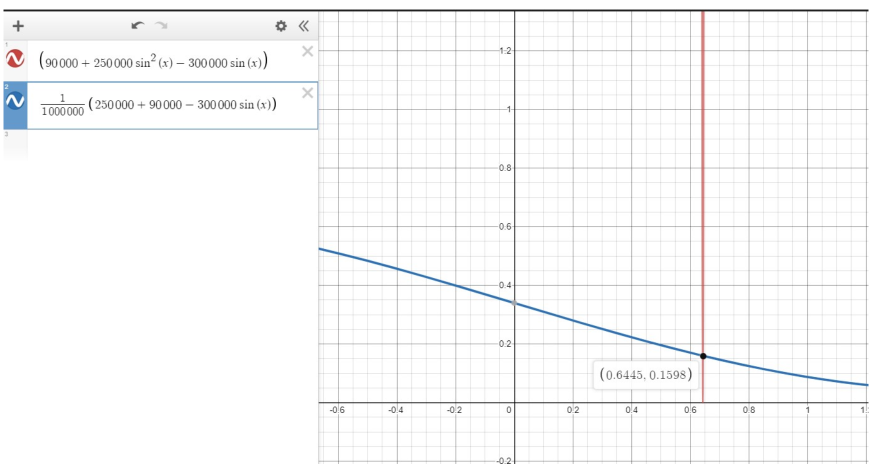


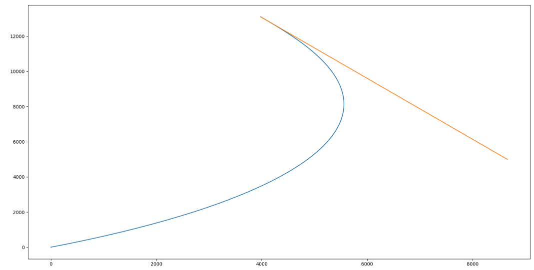
Figure 1: Solution 02

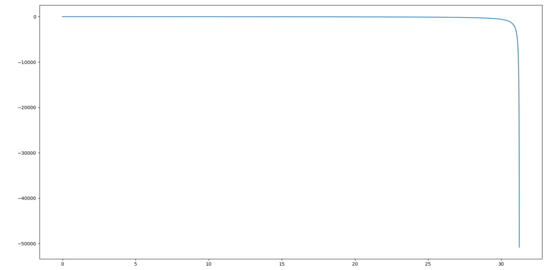
Now we need a γ_M such that $r_{miss} = 0$; i.e.,

$$\begin{aligned} V_{\theta_0} &= 0 \\ 300 - 500\sin(\gamma_M) &= 0 \\ \implies \gamma_M &= \sin^{-1}\left(\frac{300}{500}\right) \\ &= 36.86990° \end{aligned}$$

Solution 03

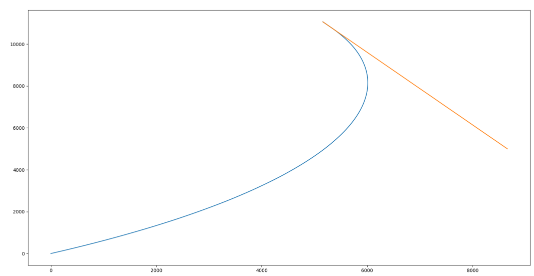
A :

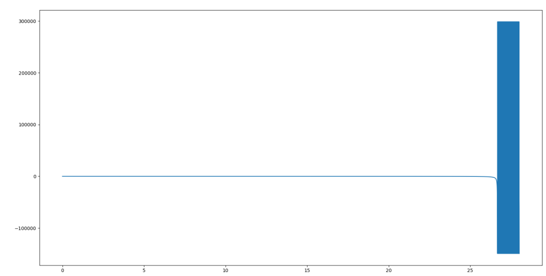




(a) Trajectory of missile and target (b) Lateral Acceleration of missile

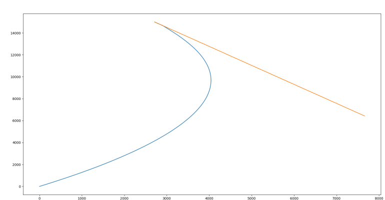
B :

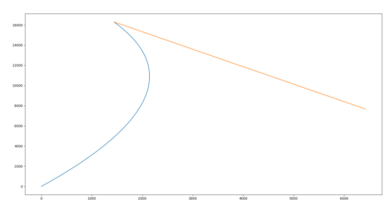


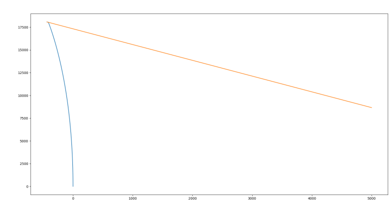


(c) Trajectory of missile and target (d) Lateral Acceleration of missile

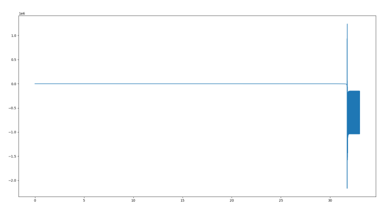
C :

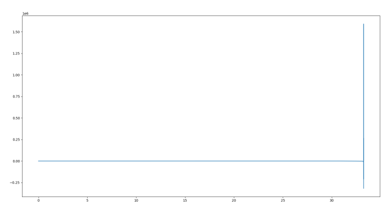


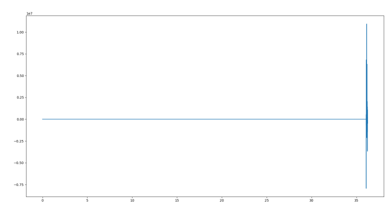




(e) Trajectory of missile and target: 10° (f) Trajectory of missile and target: 20° (g) Trajectory of missile and target: 30°







(h) Lateral Acceleration of missile: 10° (i) Lateral Acceleration of missile: 20° (j) Lateral Acceleration of missile: 30°

D :

Solution 04

A :

B :

Solution 05

A : The miss distance is given as:

$$r_{miss} = r_0 \sqrt{\frac{V_{\theta_0}^2}{V_{r_0}^2 + V_{\theta_0}^2}}$$

For interception of target, $r_{miss} = 0$ which implies that

$$V_{\theta_0} = 0$$

Also we require that the LOS reduces with time until interception. Hence,

$$V_r < 0$$

B : We are provided with a lethal radius of r_{lethal} such that $r_{miss} < r_{lethal}$. Hence,

$$\begin{aligned} r_{lethal} &> r_0 \sqrt{\frac{V_{\theta_0}^2}{V_{r_0}^2 + V_{\theta_0}^2}} \\ V_{\theta_0}^2 &< \left(\frac{r_{lethal}}{r_{miss}}\right)^2 \left(V_{r_0}^2 + V_{\theta_0}^2\right) \\ V_{\theta_0}^2 \left(1 - \left(\frac{r_{lethal}}{r_{miss}}\right)^2\right) &< V_{r_0}^2 \left(\frac{r_{lethal}}{r_{miss}}\right)^2 \\ \implies |V_{\theta_0}| &< |V_{r_0}| \sqrt{\frac{\left(\frac{r_{lethal}}{r_{miss}}\right)^2}{1 - \left(\frac{r_{lethal}}{r_{miss}}\right)^2}} \end{aligned}$$

Once again, since we require that the LOS reduces with time until interception. Hence,

$$V_r < 0$$

C : For a direct hit, $r_{lethal} = 0$. Substituting this in Part B, we get:

$$|V_{\theta_0}| < |V_{r_0}| \sqrt{\frac{\left(\frac{0}{r_{miss}}\right)^2}{1 - \left(\frac{0}{r_{miss}}\right)^2}} \qquad < |V_{r_0}| \sqrt{\frac{0}{1}} < 0$$

But, $|V_{\theta_0}| \geq 0$. Hence, $|V_{\theta_0}| = 0$; i.e., $V_{\theta_0} = 0$, giving us the condition from Part A.

Solution 06

$$H_A = \begin{bmatrix} 0.000 & 0.996 & 0.087 & 1.000 \\ 0.863 & -0.498 & 0.087 & 1.000 \\ -0.863 & -0.498 & 0.087 & 1.000 \\ 0.000 & 0.000 & 1.000 & 1.000 \end{bmatrix} \text{ and } H_B = \begin{bmatrix} 0.000 & 0.980 & 0.199 & 1.000 \\ 0.863 & -0.498 & 0.087 & 1.000 \\ -0.863 & -0.498 & 0.087 & 1.000 \\ 0.000 & 0.000 & 1.000 & 1.000 \end{bmatrix}.$$

We may assume all components of additive noise v are pairwise uncorrelated and have unit variance.

A : Covariance of the estimation error for each of the two visibility matrices is given as:

$$\begin{aligned} C_A = \mathbb{E}[\tilde{x_A}\tilde{x_A}^T] &= (H_A^T H_A)^{-1} \\ &= \begin{bmatrix} 0.7146 & 0.0426 & 0.0426 & -0.4347 \\ 0.0426 & 0.7143 & 0.0429 & -0.4347 \\ 0.0426 & 0.0429 & 0.7143 & -0.4347 \\ -0.4347 & -0.4347 & -0.4347 & 1.2087 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} C_B = \mathbb{E}[\tilde{x_B}\tilde{x_B}^T] &= (H_B^T H_B)^{-1} \\ &= \begin{bmatrix} 0.7945 & 0.0731 & 0.0731 & -0.5558 \\ 0.0731 & 0.7185 & 0.0471 & -0.4599 \\ 0.0731 & 0.0471 & 0.7185 & -0.4599 \\ -0.5558 & -0.4599 & -0.4599 & 1.3331 \end{bmatrix} \end{aligned}$$

B : Based on the covariance matrices, the dilutions of precision can be computed as:

$$\begin{aligned} \text{GDOP}_A &= \sqrt{V_{x_A} + V_{y_A} + V_{z_A} + V_{t_A}} = \sqrt{0.7146 + 0.7143 + 0.7143 + 1.2087} = 1.8308 \\ \text{PDOP}_A &= \sqrt{V_{x_A} + V_{y_A} + V_{z_A}} = \sqrt{0.7146 + 0.7143 + 0.7143} = 1.4640 \\ \text{HDOP}_A &= \sqrt{V_{x_A} + V_{y_A}} = \sqrt{0.7146 + 0.7143} = 1.1954 \\ \text{VDOP}_A &= \sqrt{V_{z_A}} = \sqrt{0.7143} = 0.8452 \\ \text{TDOP}_A &= \sqrt{V_{t_A}} = \sqrt{1.2087} = 1.0994 \\ \text{MDOP}_A &= \max\left(\sqrt{V_{x_A}}, \sqrt{V_{y_A}}\right) = \max\left(\sqrt{0.7146}, \sqrt{0.7143}\right) = 0.8453 \end{aligned}$$

$$\begin{aligned} \text{GDOP}_B &= \sqrt{V_{x_B} + V_{y_B} + V_{z_B} + V_{t_B}} = \sqrt{0.7945 + 0.7185 + 0.7185 + 1.331} = 1.8875 \\ \text{PDOP}_B &= \sqrt{V_{x_B} + V_{y_B} + V_{z_B}} = \sqrt{0.7945 + 0.7185 + 0.7185} = 1.4938 \\ \text{HDOP}_B &= \sqrt{V_{x_B} + V_{y_B}} = \sqrt{0.7945 + 0.7185} = 1.2300 \\ \text{VDOP}_B &= \sqrt{V_{z_B}} = \sqrt{0.7185} = 0.8476 \\ \text{TDOP}_B &= \sqrt{V_{t_B}} = \sqrt{1.331} = 1.1537 \\ \text{MDOP}_B &= \max\left(\sqrt{V_{x_B}}, \sqrt{V_{y_B}}\right) = \max\left(\sqrt{0.7945}, \sqrt{0.7185}\right) = 0.8913 \end{aligned}$$

C : We may compare the configurations based on the Position Dilution of Precision (PDOP). It is visible that the configuration A has a smaller PDOP, as a which it will be more accurate. Hence, configuration A gives the most accurate calculation of 3D position of the receiver.

Solution 07

Given, $V_A = 400ms^{-1}$ and $V_B = 300ms^{-1}$. Transmission frequency, $f_0 = 300MHz$. The relative velocity (V_r) of aircraft A with respect to that of aircraft B along the line of sight can be computed as:

$$\begin{aligned} V_r &= V_A \cos 0^\circ + V_B \cos 180^\circ = 400 \cos 0^\circ + 600 \cos 180^\circ \\ &= -200ms^{-2} \end{aligned}$$

A : Doppler frequency shift can be computed as:

$$\begin{aligned} f_d &= \frac{2V_r f_0}{c} = \frac{2 \times (-200) \times (300 \times 10^6)}{3 \times 10^8} \\ &= -400Hz \end{aligned}$$

B : Computations of flight directions require relative velocity to be known for different f_d i.e, $V_r = \frac{c \cdot f_d}{2 \cdot f_0}$. Hence, if $f_d = 0$, $V_r = 0$; i.e.,

$$\begin{aligned} 0 &= 400 \cos 0° + 600 \cos 180° - \alpha \\ \implies \alpha &= 180° - \arccos\left(-\frac{400}{600}\right) = 180° - \arccos\left(-\frac{2}{3}\right) \\ &= 180° - 131.8103° = 48.1897° \end{aligned}$$

C : Consider the new relative velocity to be V_{r_C}. Since no other parameter varies, we need $V_{r_C} = \frac{V_{r_A}}{2}$; i.e.,

$$\begin{aligned} 400 \cos \beta + 600 \cos 120° &= \frac{-200}{2} = -100ms^{-1} \\ \implies \beta &= \arccos \frac{-100 + 600 \cdot 0.5}{400} = \arccos \frac{1}{2} \\ &= 60° \end{aligned}$$

Solution 08

A : We know, from the principles of guidance, that

$$\begin{aligned} V_r^2 + V_\theta^2 &= c^2 \\ &= V_{r_0}^2 + V_{\theta_0}^2 \end{aligned}$$

Now, from the equations for V_r and V_θ, we get

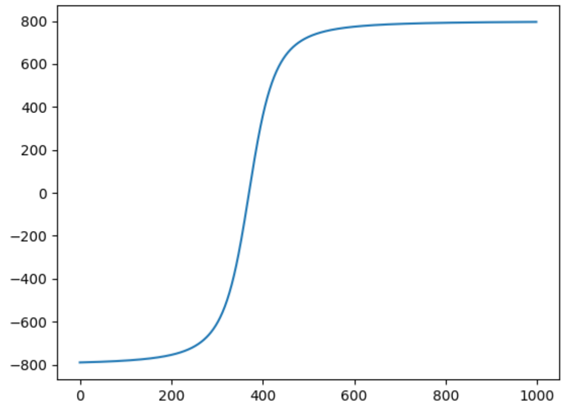
$$\begin{aligned} V_{r_0} &= 400 \cos(170° - 30°) - 500 \cos(45° - 30°) \\ &= -789.3807 \ ms^{-1} \\ \text{and} \\ V_{\theta_0} &= 400 \sin(170° - 30°) - 500 \sin(45° - 30°) \\ &= 127.7055 \ ms^{-1} \\ \implies V_{r_0}^2 + V_{\theta_0}^2 &= (-789.3807)^2 + (127.7055)^2 \\ &= 639430.5739 \\ \implies V_r^2 + V_\theta^2 &= 639430.5739 \end{aligned}$$

From the class material, we are aware that:

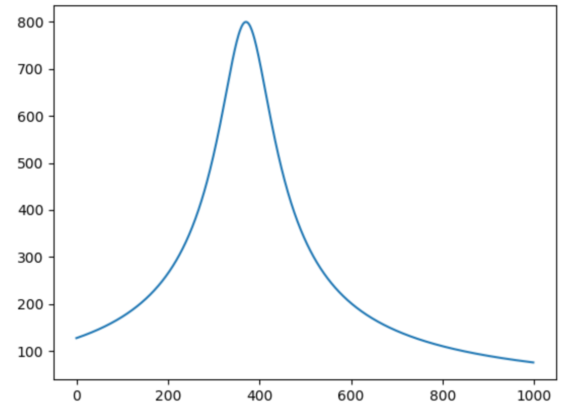
$$r\dot{V}_r = r\dot{\theta}V_\theta = V_\theta^2 > 0$$

Since $r > 0$ and $V_\theta > 0$ for the given problem as shown earlier, this is a clear indication that V_r keeps increasing, which further implies that the point (V_θ, V_r) always moves upwards on its locus.

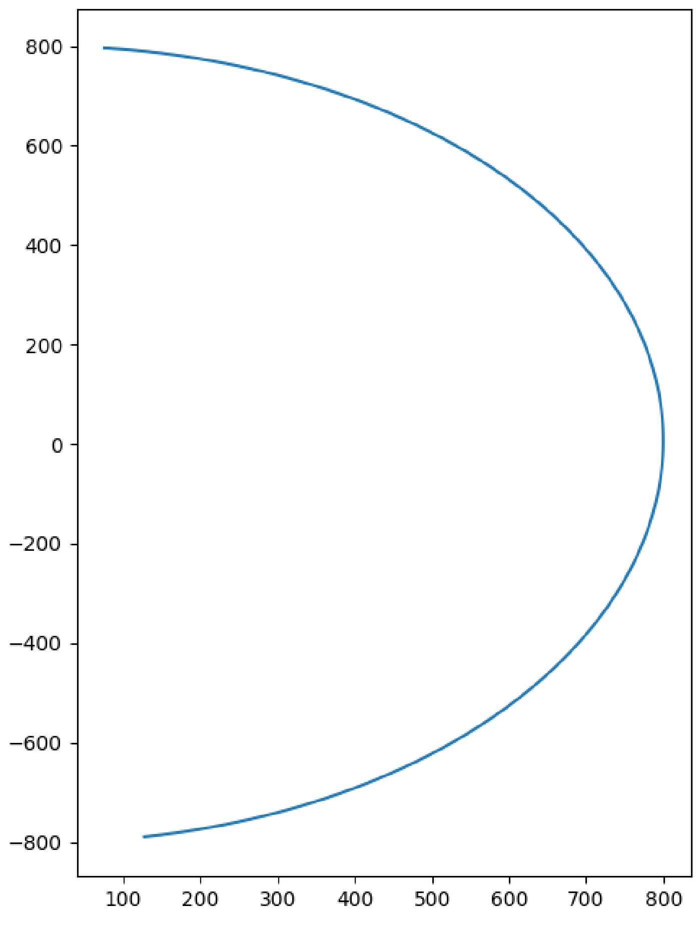
B : $V_r(t)$ converges to about 800 ms^{-1} (799.644 ms^{-1} at $t = 1000s$) and $V_\theta(t)$ converges to about 0 rad (4.976 rad at $t = 100s$ and 0.4809 rad at $t = 1000s$)



(k) Plot of $V_r(t)$



(l) Plot of $V_\theta(t)$



s

(m) Plot of the locus of (V_θ, V_r)