1. A sequence $\{x_k\}$, $x_k \in \mathbb{R}$, i.e., $\{x_1, x_2, \cdots, x_n, \cdots\}$ of scalars is said to converge if there exists a scalar x such that for every $\varepsilon > 0$, $\exists$ some K (the integer K depends on ε), we have $|x_k - x| < \varepsilon$

Example 1: $\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ converges to zero.

Example 2: $\{1, 2, 1, 2, \cdots\}$ does not converge.

The scalar x is said to be the limit of $\{x_k\}$ i.e., $\lim_{k \to \infty} x_k = x$ or $x_k \to x$.

For example 1, $x_k \to 0$.

2. Limit Point: A scalar $\bar{x}$ is said to be a limit point of sequence $\{x_k\}$, if there is a subsequence $\{x_k\}_K$ within $\{x_k\}$, which limits to $\bar{x}$.

For example, in the sequence $\{1, 2, 1, 2, \cdots\}$ there are two limit points 1 and 2 but the sequence, itself does not converge.

3. When do we say $\lim_{k \to \infty} x_k = \infty$ or $\lim_{k \to \infty} = -\infty$?

We say that $\lim_{x \to \infty} x_k = \infty$ if for every scalar c, $\exists$ an integer K such that $x_k \geq c$ for all ($\forall$) $k > K$.

Obviously, K depends upon c.

Example: Sequence $\{1^2, 2^2, \ldots, n^2, \ldots\}$ goes to infinity. Similarly, does the sequence $\{1, 2, 3, \ldots, n, \ldots\}$.

Show that the sequence $\{-1, -2, \ldots, -n, \ldots\}$ goes to $-\infty$.

4. Cauchy sequence: A scalar sequence $\{x_k\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, $\exists$ some integer K (depends upon ε) such that $|x_k - x_m| < \varepsilon$ for all $k \geq K$ and $m \geq K$.

5. A sequence $\{x_k\}$ is said to be bounded above if $x_k \leq b$, for some scalar b and for all $k = 1, 2, \ldots$
Similarly a sequence can be bounded below.

6. A sequence $\{x_k\}$ is bounded if is both bounded above and below i.e., $\exists$ constants l & u s.t. $l \leq x_k \leq u$ $\forall k$.

7. A sequence $\{x_k\}$ is said to be monotonically non-increasing (oversimplifying, decreasing!) if $x_k \geq x_{k+1}$ $\forall k$. If such a sequence

converges to x i.e. $x_k \to x$ we write it $x_k \uparrow x$. Similarly, we define monotonically non-decreasing sequence ($x_k \leq x_{k+1}$).

Example sequence $\{1, \frac{1}{2}, \cdots \frac{1}{n}\}$ is monotonically non-increasing, infact decreasing with $x_k \downarrow 0$.

8. Proposition: Every bounded and monotonically non-increasing or non-decreasing scalar sequence converges

9. A sequence $\{x_k\}$ of vectors in $\mathbb{R}^n$ is said to converge to some $x \in \mathbb{R}^n$ if the ith component of x_k converges to the ith component of x. We say that $x_k \to x$ or $\lim_{k \to \infty} x_k = x$.

10. The sequence $\{x_k\}$ in $\mathbb{R}^n$ is bounded (or Cauchy) if each of its coordinate sequences is bounded (or Cauchy sequence respectively).

$\therefore$ $\{x_k\}$ is bounded iff $\|x_k\| \leq c$, for some scalar c and $\forall k$.

11. We say that a vector $x \in \mathbb{R}^n$ is a

limit point of a sequence $\{x_k\}$ in $\mathbb{R}^n$ if $\exists$ a subsequence $\{x_k\}_K$, K defining the subsequence selection rule such that $\{x_k\}_K$ converges to x.

Proposition
(a) A bounded sequence of vectors in $\mathbb{R}^n$ converges iff it has a unique limit point.
(b) A sequence in $\mathbb{R}^n$ converges iff it is a Cauchy sequence.
(c) Every bounded sequence in $\mathbb{R}^n$ has at least one limit point

Reference: Appendix A of Dimitri Bertsekas, Non-linear Programming.

Sequences.

$x_k \in \mathbb{R}$

$\{x_k\} \to \{x_1, x_2, x_3 ---, x_N, x_{N+1} ---\}$

infinite seq.

$\{1, 2, 3, 4, -- n, n+1, --- \infty\}$

$\{1, -2, 1, +5, -1, - - - - - -\}$

$\to$ I don't know the rule!

$\lim\limits_{k\to\infty} \{x_k\} = x.$ [Convergent sequence]

Given $\varepsilon > 0$, '$\underline{K}$' s.t. $k \geq \underline{\underline{K}}$ 100

you choose my. depend

$k = 100, 101, 1000,$

$$|x_k - \underline{x}| < \underline{\underline{\varepsilon}}$$

with in distance of ε

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, - - \frac{1}{n}, - - -\}$

$\lim\limits_{n\to\infty} = \underline{\underline{0}}$ [HW, prove it as definite]

$x_k \to \underline{\underline{X}}$ e.g., $x_k \to \underline{\underline{0}}$

All sequences need not converges

$\{1, 2, 1, 2, - - - \}$.

Is it convergent ? No

$\{1, 1, 1, \ldots\}$ $\lim_{k\to\infty} \{x_k\}_{k_1} = 1$

Subsequence $k \to$ only odd

$\{2, 2, 2, \ldots\}$ $\lim_{k\to\infty} \{x_k\}_{\text{even}} = 2$

subsequen

1, 2 are called limit point

<u>Cauchy Sequence</u>

$\{x_k\}$, s.t $|x_k - x_m| < \varepsilon$

and given $\varepsilon > 0$ (howsoever small) $\xrightarrow{\text{corresponding}}$ K

$k \geq K$

$m \geq K$

Bounded above

$\{x_k\}$ s.t. $|x_k| \leq C$

Bounded above

$\{1, 2, 3, 4, \ldots\}$

is not bounded above

Bounded below

$|x_k| \geq 1$

$\{-1, -2, -3, -4, \ldots\}$

Not bounded below

but it's bounded above

Bounded sequence.

$$l \leq x_k \leq u \quad \forall k$$

$\lim_{k \to \infty} \{x_k\} = \underline{\underline{\infty}}$ $\quad x_k \to \infty$

$\lim_{k \to \infty} \{x_k\} = \underline{\underline{-\infty}}$ $\quad x_k \to -\infty$

$\Rightarrow$ if for any 'M' $M > 0$,

we can 'K' s.t.

$x_k > \underline{\underline{M}} \quad \forall k > K$

$1000, \quad 100$

$10{,}000, \quad \underline{\underline{150}}$

$10^6, \quad \underline{\underline{1000}}$

Sequence of Vector $\quad \{x_k\} \quad x_k \in \mathbb{R}^n$

Herein, each coordinates creates its own seq.

$\rightarrow \quad \{x_k\} \to \underline{\underline{x}} \in \mathbb{R}^n$ provided

every seq. corresponds to coordinates

converges to x_i.

$$x_i \leftarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Bounded if $|x_i^k| \leq M$

$i = 1, \ldots n$

$\{x_k\}$ is bounded.

$$-M \leq x_i^k \leq M$$

$$\lim_{k \to \infty} \|x_k - x\|_1 = 0;$$

Monotonically decreasing seq.

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

$\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \ldots\}$

Non-increasing sequence

$\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}\}$

This Not a monotonically non-increasing seq.

Defn of continuity of univariate function, using notion of seqs.

$|f(x) - f(y)| < \varepsilon$ provides $\varepsilon > 0$

$N_\delta(x)$

$\{y : |x-y| < \delta\}$

$\|x - y\| < \delta$

sequences

$f : S \to R$

$\{x_k\}$

$x_k \in S$

Function 'f' is continuous at x

iff for every such seq

$\{x_k\} \to x$

$x_k \in S$

$\{f(x_k)\} \to f(x)$

Function is continuous.