# EE 659: A First Course in Optimization
# Practice Problems

November 11, 2020

1. Determine whether each of the following functions is coercive on $\Re^2$.

   (a) $f(x, y) = x + y + 2$.

   (b) $f(x, y) = x^2 + y^2 + 2$.

   (c) $f(x, y) = x^2 - 2xy + y^2$.

2. Determine whether each of the following function is convex, strictly convex or nonconvex on $\Re$.

   (a) $f(x) = x^2$.

   (b) $f(x) = x^3$.

   (c) $f(x) = |x|$.

3. For each of the following functions, what do the first- and second-order optimality conditions say about whether 0 is a minimum on $\Re$.

   (a) $f(x) = x^2$.

   (b) $f(x) = x^4$.

   (c) $f(x) = -x^4$.

4. Determine the critical points of each of the following functions and characterize each as minimum, maximum, or infection point. Also determine whether each function has a global minimum or maximum on $\Re$.

   (a) $f(x) = 2x^3 - 25x^2 - 12x + 15$.

   (b) $f(x) = 3x^3 + 7x^2 - 15x - 3$.

   (c) $f(x) = x^2\, e^x$.

5. Determine the critical points of each of the following functions and characterize each as a minimum, maximum, or saddle point. Also determine whether each function has a global minimum or maximum on $\Re^2$.

   (a) $f(x, y) = x^2 - 4xy + y^2$.

(b) $f(x, y) = x^4 - 4xy + y^4$.

(c) $f(x, y) = (x - y)^4 + x^2 - y^2 - 2x + 2y + 1$.

6. Use the first- and second-order optimality conditions to show that $x^* = [2.5\ -1, 5\ -1]^T$ is a constrained local minimum for the function

$$f(x) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$$

subject to

$$g(x) = x_1 - x_2 + 2x_3 - 2 = 0$$

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7. Consider the function $f : \Re^2 \to \Re$ defined by

$$f(x) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2.$$

At what point does $f$ attain a minimum?

8. Prove that if a continuous function $f : S \subseteq \Re^n \to \Re$ has a nonempty sublevel set that is closed and bounded, then $f$ has a global minimum on $S$.

9. (a) Prove that any local minimum of a convex functions $f$ on a convex set $S \subseteq \Re^n$ is a global minimum of $f$ on $S$. (Hint: If local minimum $\mathbf{x}$ is not a global minimum, then let $\mathbf{y}$ be a point in $S$ such that $f(\mathbf{y}) < f(\mathbf{x})$ and consider the line segment between $\mathbf{x}$ and $\mathbf{y}$ to obtain a contradiction.)

   (b) Prove that any local minimum of a strictly convex function $f$ on a convex set $S \subseteq \Re^n$ is the unique global minimum of $f$ on $S$. (Hint: Assume there are two minima $\mathbf{x},\ \mathbf{y} \in S$ and consider the line segment between $\mathbf{x}$ and $\mathbf{y}$ to obtain a contradiction.)

10. A function $f : \Re^n \to \Re$ is said to be quasiconvex on a convex set $S \subseteq \Re^n$ if for any $\mathbf{x}, \mathbf{y} \in S$,

$$f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \leq \max\{f(\mathbf{x}),\ f(\mathbf{y})\}$$

for all $\alpha \in (0, 1)$, and $f$ is *strictly quasiconvex* if strict inequality holds when $\mathbf{x} \neq \mathbf{y}$. If $f : \Re \to \Re$ has a minimum on an interval $[a, b]$, show that $f$ is unimodal on $[a, b]$ if, and only if, $f$ is strictly quasiconvex on $[a, b]$.

11. Consider the linear programming problem

$$\min_{\mathbf{x}} f(\mathbf{x}) = -3x_1 - 2x_2$$

subject to

$$5x_1 + x_2 \leq 6, \quad 3x_1 + 4x_2 \leq 6, \quad 4x_1 + 3x_2 \leq 6 \quad x_1 \geq 0, \quad x_2 \geq 0.$$

   (a) How many vertices does the feasible region have?

(b) Since the solution must occur at a vertex, solve the problem by evaluating the objective function at each vertex and choosing the one that gives the lowest value.

(c) Obtain the graphical solution to the problem by drawing the feasible region and contours of the objective function.

12. How can the linear programming problem given in question 11 be stated in the standard form. (Hint: Standard form is minimize $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.)

13. What is the Cholesky factorization of the following matrix?

    (a)
$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

    (b)
$$\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

    (c)
$$\begin{bmatrix} 10 & 20 & 30 \\ 20 & 45 & 80 \\ 30 & 80 & 171 \end{bmatrix}$$

14. Let $\mathbf{A}$ be a symmetric positive definite matrix. Show that the function
$$\|\mathbf{x}\|_A = (\mathbf{x}^T\mathbf{A}\mathbf{x})^{1/2}$$
satisfies the three properties of a vector norm. This vector norm is said to be induced by the matrix $\mathbf{A}$.

15. Suppose that $\mathbf{A}$ is a positive definite matrix.

    (a) Show that $\mathbf{A}$ must be nonsingular.

    (b) Show that $\mathbf{A}^{-1}$ must be positive definite.

16. Show that if $\mathbf{A}$ is $n \times n$ is real symmetric positive definite matrix and $\mathbf{X}$ is a real $n \times k$ matrix with rank $k$, then $\mathbf{B} = \mathbf{X}^T\mathbf{A}\mathbf{X}$ is also positive definite.