AE 410: Navigation and Guidance

Assignment 01

Report submitted by: Aaron John Sabu

Solution 01

A : We have the body-frame vector to be $\boldsymbol{V} = [V_m\ 0\ 0]^T$, and we have rotation about z-axis by $-\psi_m$ (negative of azimuth angle) followed by a rotation about the new x-axis by θ_m (elevation angle). The quaternion describing this rotation from the inertial frame to the body frame is given as:

$$\begin{aligned}
Q_m &= \left(\cos\frac{-\theta_m}{2} + \sin\frac{-\theta_m}{2}\boldsymbol{j}\right)\left(\cos\frac{-\psi_m}{2} + \sin\frac{-\psi_m}{2}\boldsymbol{k}\right) \\
&= \cos\frac{-\psi_m}{2}\cos\frac{-\theta_m}{2} + \sin\frac{-\psi_m}{2}\sin\frac{-\theta_m}{2}\boldsymbol{i} + \cos\frac{-\psi_m}{2}\sin\frac{-\theta_m}{2}\boldsymbol{j} + \sin\frac{-\psi_m}{2}\cos\frac{-\theta_m}{2}\boldsymbol{k} \\
&= \cos\frac{\psi_m}{2}\cos\frac{\theta_m}{2} + \sin\frac{\psi_m}{2}\sin\frac{\theta_m}{2}\boldsymbol{i} - \cos\frac{\psi_m}{2}\sin\frac{\theta_m}{2}\boldsymbol{j} - \sin\frac{\psi_m}{2}\cos\frac{\theta_m}{2}\boldsymbol{k}
\end{aligned}$$

B : We have the velocity vector in the body frame. The equivalent vector in the inertial frame is:

$$\begin{aligned}
V_{mI} &= [Q_m]^*\boldsymbol{V}[Q_m] = V_m[q_0 - \boldsymbol{q_m}]\boldsymbol{i}[q_0 + \boldsymbol{q_m}] \\
&= \left[c\frac{\psi_m}{2}c\frac{\theta_m}{2} - s\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{i} + c\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{j} + s\frac{\psi_m}{2}c\frac{\theta_m}{2}\boldsymbol{k}\right]\begin{bmatrix}0\\V_m\\0\\0\end{bmatrix}\left[c\frac{\psi_m}{2}c\frac{\theta_m}{2} + s\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{i} - c\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{j} - s\frac{\psi_m}{2}c\frac{\theta_m}{2}\boldsymbol{k}\right] \\
&= V_m\left[s\frac{\psi_m}{2}s\frac{\theta_m}{2} + c\frac{\psi_m}{2}c\frac{\theta_m}{2}\boldsymbol{i} + s\frac{\psi_m}{2}c\frac{\theta_m}{2}\boldsymbol{j} - c\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{k}\right]\left[c\frac{\psi_m}{2}c\frac{\theta_m}{2} + s\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{i} - c\frac{\psi_m}{2}s\frac{\theta_m}{2}\boldsymbol{j} - s\frac{\psi_m}{2}c\frac{\theta_m}{2}\boldsymbol{k}\right] \\
&= V_m\left[\left(s_\psi^2 s_\theta^2 + c_\psi^2 c_\theta^2 - s_\psi^2 c_\theta^2 - c_\psi^2 s_\theta^2\right)\boldsymbol{i} + \left(-s_\psi c_\psi s_\theta^2 + s_\psi c_\psi c_\theta^2 - s_\psi c_\psi s_\theta^2 + s_\psi c_\psi c_\theta^2\right)\boldsymbol{j}\right. \\
&\quad \left. + \left(-s_\psi^2 s_\theta c_\theta - c_\psi^2 s_\theta c_\theta - s_\psi^2 s_\theta c_\theta - c_\psi^2 s_\theta c_\theta\right)\boldsymbol{k}\right] \\
&= V_m\left[(c_\psi^2 - s_\psi^2)(c_\theta^2 - s_\theta^2)\boldsymbol{i} + (2s_\psi s_\psi)(c_\theta^2 - s_\theta^2) + \left(-(s_\psi^2 + c_\psi^2)(2s_\theta c_\theta\right)\boldsymbol{k}\right] \\
&= V_m\left[\cos\psi_m\cos\theta_m\boldsymbol{i} + \sin\psi_m\cos\theta_m\boldsymbol{j} - \sin\theta_m\boldsymbol{k}\right]
\end{aligned}$$

C : We can also calculate V_{mI} using Euler angle rotations as follows:

$$\begin{aligned}
V_{mI} &= \begin{bmatrix}\cos(-\psi_m) & \sin(-\psi_m) & 0\\ -\sin(-\psi_m) & \cos(-\psi_m) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(-\theta_m) & 0 & -\sin(-\theta_m)\\ 0 & 1 & 0\\ \sin(-\theta_m) & 0 & \cos(-\theta_m)\end{bmatrix}\begin{bmatrix}V_m\\0\\0\end{bmatrix} \\
&= \begin{bmatrix}\cos(-\psi_m)\cos(-\theta_m) & \sin(-\psi_m) & -\cos(-\psi_m)\sin(-\theta_m)\\ -\sin(-\psi_m)\cos(-\theta_m) & \cos(-\psi_m) & \sin(-\psi_m)\sin(-\theta_m)\\ \sin(-\theta_m) & 0 & \cos(-\theta_m)\end{bmatrix}\begin{bmatrix}V_m\\0\\0\end{bmatrix} \\
&= \begin{bmatrix}\cos\psi_m\cos\theta_m & -\sin\psi_m & \cos\psi_m\sin\theta_m\\ \sin\psi_m\cos\theta_m & \cos\psi_m & \sin\psi_m\sin\theta_m\\ -\sin\theta_m & 0 & \cos\theta_m\end{bmatrix}\begin{bmatrix}V_m\\0\\0\end{bmatrix} \\
&= V_m\cos\psi_m\cos\theta_m\boldsymbol{i} + V_m\sin\psi_m\cos\theta_m\boldsymbol{j} - V_m\sin\theta_m\boldsymbol{k}
\end{aligned}$$

Solution 02

A : We know the Euler transformation matrix for rotation in the positive directions. Hence, replacing the θ term with $-\theta$ gives us the required Euler transformation matrix:

$$\begin{aligned}
C &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos(-\theta) & 0 & -\sin(-\theta) \\ 0 & 1 & 0 \\ \sin(-\theta) & 0 & \cos(-\theta) \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(-\theta)\cos\psi & \cos(-\theta)\sin\psi & -\sin(-\theta) \\ \cos\psi\sin(-\theta)\sin\phi - \sin\psi\cos\phi & \sin\psi\sin(-\theta)\sin\phi + \cos\psi\cos\phi & \cos(-\theta)\sin\phi \\ \cos\psi\sin(-\theta)\cos\phi + \sin\psi\sin\phi & \sin\psi\sin(-\theta)\cos\phi + \cos\psi\sin\phi & \cos(-\theta)\cos\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & \sin\theta \\ -\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & -\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ -\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & -\sin\psi\sin\theta\cos\phi + \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}
\end{aligned}$$

C : We obtain the following quaternion describing composite rotation:

$$\begin{aligned}
[Q] &= \left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\boldsymbol{k}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\boldsymbol{j}\right)\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\boldsymbol{i}\right) \\
&= \left(\cos\frac{\psi}{2}\cos\frac{\theta}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\boldsymbol{i} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\boldsymbol{j} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\boldsymbol{k}\right)\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\boldsymbol{i}\right) \\
&= \left(\cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right) + \left(\cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}\right)\boldsymbol{i} \\
&\quad + \left(\sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}\right)\boldsymbol{j} + \left(\cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2}\right)\boldsymbol{k}
\end{aligned}$$

B : From the quaternion from part C, we substitute values to the quaternion transformation matrix as provided in the lecture slides:

$$[QT] = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_3q_0 + q_1q_2) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_1q_0 + q_3q_2) \\ 2(q_0q_2 + q_1q_3) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

This gives us back the Euler transformation matrix:

$$[QT] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & \sin\theta \\ -\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & -\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ -\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & -\sin\psi\sin\theta\cos\phi + \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Solution 03

A : The unit vector is $\hat{q} = \frac{1}{\sqrt{2}}(1, 0, 1)$, giving us the quaternion:

$$\begin{aligned}
[Q] &= \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2} \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{2}}(\boldsymbol{i} + \boldsymbol{k})
\end{aligned}$$

B : We have the actual vector $\boldsymbol{v} = \boldsymbol{j} = (0, 1, 0)$. By using the quaternion operator on $\boldsymbol{v}$, we get:

$$\begin{aligned}[w] = L_Q(\boldsymbol{v}) &= (q_0^2 - ||\boldsymbol{q}||^2)\boldsymbol{v} + 2(\boldsymbol{q} \cdot \boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \\ &= \left(\frac{1}{4} - \frac{3}{4}\right)\boldsymbol{j} + 2k(\boldsymbol{i} + \boldsymbol{k}) \cdot \boldsymbol{j} + 2 \cdot \frac{1}{2}\left(\frac{\sqrt{3}}{2\sqrt{2}}(\boldsymbol{i} + \boldsymbol{k}) \times \boldsymbol{j}\right) \\ &= -\frac{\sqrt{3}}{2\sqrt{2}}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{\sqrt{3}}{2\sqrt{2}}\boldsymbol{k}\end{aligned}$$

C : On rotating the coordinate frame and keeping the vector constant, we get the following new vector:

$$\begin{aligned}[w'] = L_{Q^*}(\boldsymbol{v}) &= (q_0^2 - || - \boldsymbol{q}||^2)\boldsymbol{v} + 2(-\boldsymbol{q} \cdot \boldsymbol{v})(-\boldsymbol{q}) + 2q_0(-\boldsymbol{q} \times \boldsymbol{v}) \\ &= \left(\frac{1}{4} - \frac{3}{4}\right)\boldsymbol{j} - 2k(\boldsymbol{i} + \boldsymbol{k}) \cdot \boldsymbol{j} + 2 \cdot \frac{1}{2}\left(\frac{\sqrt{3}}{2\sqrt{2}}(-\boldsymbol{i} - \boldsymbol{k}) \times \boldsymbol{j}\right) \\ &= \frac{\sqrt{3}}{2\sqrt{2}}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} - \frac{\sqrt{3}}{2\sqrt{2}}\boldsymbol{k}\end{aligned}$$

Solution 04

A : We have $R = 1000m$ and $\Omega = 100°s^{-1} = \frac{100\times\pi}{180} = \frac{5\pi}{9}$. Hence,

$$t_{\pm} = \frac{2\pi R}{c \mp R\Omega} = \frac{2 \times 1000 \times \pi}{c \pm \left(1000 \times \frac{5\pi}{9}\right)}$$

which gives $t_+ = 2.0958572235959284649072439754189 \times 10^{-5}s = 20.9585722\mu s$
and $t_- = 2.0958328204495051953294339759012 \times 10^{-5}s = 20.958328\mu s$

B : Transit time,

$$\Delta t = t_+ - t_- = 2.4403146423269577809995177 \times 10^{-10} = 0.2440314ns$$

C : Optical path difference,

$$\Delta L = c\Delta t = 0.073158792491658951282820123900098m = 7.31587924 \times 10^{-2}m$$

Solution 05

We have equilateral triangular and square ring laser gyros (RLGs) such that operating wavelength $\lambda = 0.6328\mu m$, input angular velocity, $\Omega = 1°hr^{-1} = \frac{1\times\pi}{3600s\times180} = \frac{\pi}{648000}$ and the value of the side length of the square-shaped RLG $b = 10cm$.

A : Given that the measurable beat frequencies for both RLGs are the same, we get:

$$\begin{aligned}\frac{4A_{\triangle}\Omega}{L_{\triangle}\lambda} &= \frac{4A_{\boxdot}\Omega}{L_{\boxdot}\lambda} \\ \implies \frac{A_{\triangle}}{L_{\triangle}} &= \frac{A_{\boxdot}}{L_{\boxdot}} \\ \implies \frac{\frac{\sqrt{3}a^2}{4}}{3a} &= \frac{b^2}{4b} \\ \implies a &= \sqrt{3}b\end{aligned}$$

B : Given b, we get

$$a = \sqrt{3}b = 1.732 \times 0.1m = 0.1732m = 17.32cm$$

$$S = \frac{4A_{\boxdot}}{L_{\boxdot}\lambda} = \frac{4 \times 0.1^2}{4 \times 0.1 \times 0.6328 \times 10^{-6}m} = 158027.813$$

$$\Delta\nu = S\Omega = 158027.813 \times \frac{\pi}{648000} = 0.7661405Hz$$

Solution 06

A : We can solve for C_1^2 as follows:

$$\begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix}$$

$$\begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} \begin{bmatrix} i_1 & j_1 & k_1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} \begin{bmatrix} i_1 & j_1 & k_1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies C_1^2 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} i_2 \cdot i_1 & i_2 \cdot j_1 & i_2 \cdot k_1 \\ j_2 \cdot i_1 & j_2 \cdot j_1 & j_2 \cdot k_1 \\ k_2 \cdot i_1 & k_2 \cdot j_1 & k_2 \cdot k_1 \end{bmatrix}$$

Similarly, we get C_2^1 as:

$$\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = C_2^1 \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} \begin{bmatrix} i_2 & j_2 & k_2 \end{bmatrix} = C_2^1 \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} \begin{bmatrix} i_2 & j_2 & k_2 \end{bmatrix} = C_2^1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies C_2^1 = \begin{bmatrix} i_1 \cdot i_2 & i_1 \cdot j_2 & i_1 \cdot k_2 \\ j_1 \cdot i_2 & j_1 \cdot j_2 & j_1 \cdot k_2 \\ k_1 \cdot i_2 & k_1 \cdot j_2 & k_1 \cdot k_2 \end{bmatrix}$$

We can now consider both matrices acting one after the other on frame F_1. We know that, since the first represents a transformation from frame F_1 to frame F_2 and the second represents the opposite, the resulting frame from the composite operation should be F_1:

$$\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = C_2^1 C_1^2 \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix}$$

$$\text{but } \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = \boldsymbol{I}_3 \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix}$$

$$\implies C_2^1 C_1^2 = \boldsymbol{I}_3$$

B : We know that the root mean square value of each row or column of the transformation matrix should be equal to 1:

$$c_{21}^2 + (0.8665)^2 + (-0.2496)^2 = 1 \implies c_{21} = \sqrt{1 - 0.81312} \approx \pm 0.4323$$

$$c_{32}^2 + (0.8665)^2 + (-0.4323)^2 = 1 \implies c_{21} = \sqrt{1 - 0.93771} \approx \pm 0.2496$$
$$c_{31}^2 + (0.9666)^2 + (-0.2496)^2 = 1 \implies c_{21} = \sqrt{1 - 0.99662} \approx \pm 0.0578$$

Also, due to orthogonality, each row (or column) is orthogonal to the other rows (or columns). Hence,

$$((-0.4323) \times 0.0578) + (0.8665 \times (-0.2496)) + (c_{32} \times 0.9666) = 0 \implies c_{32} = 0.2496$$
$$(0.8999 \times c_{21}) + ((-0.4323) \times (0.8665)) + (0.0578 \times (-0.2496)) = 0 \implies c_{21} = 0.4323$$
$$(0.8999 \times c_{31}) + (-0.4323 \times 0.2496) + (0.0578 \times 0.9666) = 0 \implies c_{31} = 0.0578$$

C : We know that rotation about $Z - Y' - X''$ gives the following resultant transformation matrix:

$$C_1^2 = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi + \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$
$$= \begin{bmatrix} 0.8999 & -0.4323 & 0.0578 \\ 0.4323 & 0.8665 & -0.2496 \\ 0.0578 & 0.2496 & 0.9666 \end{bmatrix}$$

From this, we get the following relations:

$$-\sin\theta = 0.0578 \implies \theta = \tan^{-1}\left(\frac{-0.0578}{\sqrt{1 - 0.0578^2}}\right) = -3.31354^\circ$$
$$\cos\theta\sin\phi = -0.2496 \text{ and } \cos\theta\cos\phi = 0.9666 \implies \phi = \tan^{-1}\left(\frac{-0.2496}{0.9666}\right) = -14.47890^\circ$$
$$\cos\theta\sin\psi = -0.4323 \text{ and } \cos\theta\cos\psi = 0.8999 \implies \phi = \tan^{-1}\left(\frac{-0.4323}{0.8999}\right) = -25.65901^\circ$$

Solution 07

We have

$$\boldsymbol{R} = \boldsymbol{R}_\beta \boldsymbol{R}_\alpha = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{bmatrix}$$

Now, the quaternions corresponding to each rotation are given as:

$$[Q1] = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{k}$$
$$[Q2] = \cos\frac{\beta}{2} + \sin\frac{\beta}{2}\boldsymbol{j}$$
$$\implies [Q] = [Q1][Q2]$$
$$= \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{k}\right)\left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\boldsymbol{j}\right)$$
$$= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{i} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{j} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\boldsymbol{k}$$

$$\text{Vector of rotation, } \hat{q} = \frac{\boldsymbol{v}}{||\boldsymbol{v}||} = \frac{1}{\sqrt{1 - \cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}}}\left(-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{i} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{j} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\boldsymbol{k}\right)$$