Indian Institute of Technology Bombay

Introduction to Navigation and Guidance

AE 410/641 Fall 2020

Solutions to Tutorial 4

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November 16, 2020

Problem 1. Consider the engagement geometry shown in Figure 1 where symbols have their usual meanings. If the target is stationary and the interceptor is guided using parallel navigation, the engagement kinematics is governed by

$$\dot{r} = -V\cos\sigma, \tag{1a}$$

$$r\dot{\theta} = -V\sin\sigma, \tag{1b}$$

$$\dot{\gamma} = N\dot{\theta}, \tag{1c}$$

$$\gamma = \theta + \sigma. \tag{1d}$$

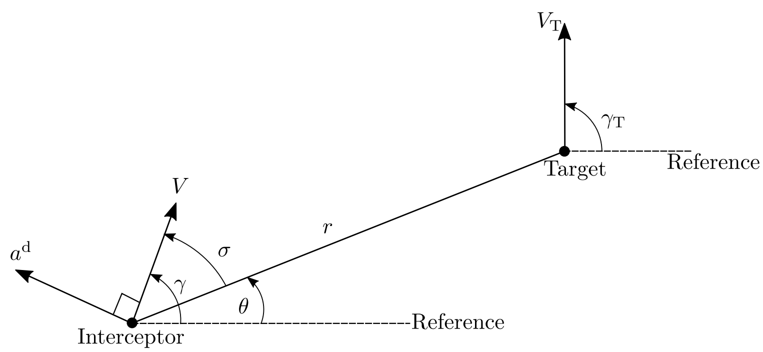


Figure 1: Interceptor-target planar engagement geometry.

(a) Prove that the angle of arrival (the angle at which the interceptor captures the target) is given by

$$\theta_f = \theta_0 - \frac{\sigma_0}{N-1}.$$

(b) Prove that the interceptor's lateral acceleration, a^d, is given by the expression

$$a^\mathrm{d} = \frac{NV^2\sin\sigma_0}{r_0}\left(\frac{r}{r_0}\right)^{N-2},$$

where $\theta_0 \triangleq 0$, r_0, and σ_0 are initial values of the corresponding variables.

(c) What happens to a^d for various values of N?

Solution. (a) Using Equations (1c) and (1d), one has

$$\dot{\gamma} = \dot{\theta} + \dot{\sigma} = N\dot{\theta} \implies \dot{\sigma} = (N-1)\dot{\theta},$$

which upon integrating yields

$$\theta_f = \theta_0 + \frac{\sigma_f - \sigma_0}{N-1}.$$

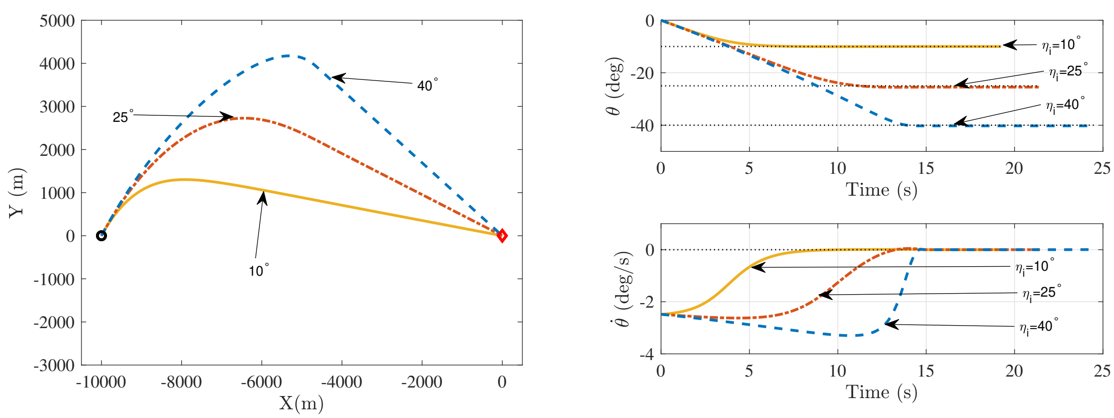
Since σ_f is zero towards the endgame when the interceptor uses PN technique (**can you prove it?**), one has the expression for angle of arrival

$$\theta_f = \theta_0 - \frac{\sigma_0}{N-1}.$$

What is the importance of having a particular angle of arrival? Can we achieve a desired angle of arrival?

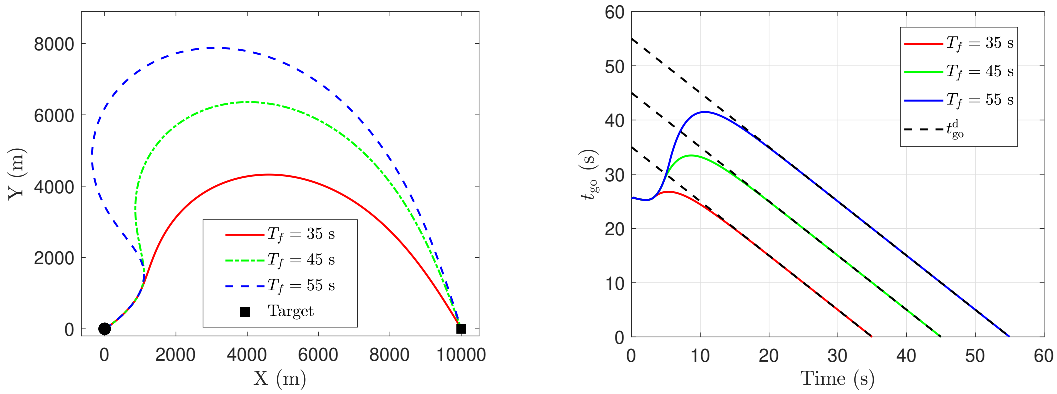
Can we have an interception at a desired time?

Can we have an interception at a desired time along with a particular angle of arrival?



(a) Trajectory. (b) LOS and its rate.

Figure 2: Interception of a stationary target for various impact angles.



(a) Trajectory. (b) Time-to-go estimate.

Figure 3: Interception of a stationary target for various impact times.

(b) Using Equation (1) and the results in part (a),

$$\dot{\sigma} = -(N-1)\frac{V}{r}\sin\sigma.$$

After some rearrangement, one may express

$$\frac{\dot{r}}{\dot{\sigma}} = \frac{-V\cos\sigma}{-(N-1)\dfrac{V}{r}\sin\sigma},$$

which upon integrating, and using the fact that $\theta_0 \triangleq 0$, results in

$$\frac{r}{r_0} = \left(\frac{\sin\sigma}{\sin\sigma_0}\right)^{\frac{1}{N-1}} \implies r = r_0\left(\frac{\sin\left[\sigma_0 + (N-1)\theta\right]}{\sin\sigma_0}\right)^{\frac{1}{N-1}}, \quad N > 1.$$

Now, $a^{\mathrm{d}} = NV\dot{\theta}$, simplifying which using the above expression, yields

$$a^{\mathrm{d}} = \frac{NV^2\sin\sigma_0}{r_0}\left(\frac{r}{r_0}\right)^{N-2}.$$

(c) From the above analyses, it is immediate that as $r \to 0$, $a^{\mathrm{d}} \to 0$ or $\to \infty$ depending on whether $N > 2$ or $N < 2$, respectively. Usually, it is a common practice to choose $N = 3$, which also turns out to be optimal (**can you prove it?**).

However, if $N = 2$, then $a^{\mathrm{d}} = \dfrac{2V^2 \sin\sigma_0}{r_0}$ is a constant value with a vehicle's turn radius of $\dfrac{r_0}{2\sin\sigma_0}$. A positive constant turn rate implies that the trajectory would be circular in counterclockwise sense.

Problem 2. Consider a modified parallel navigation law, defined as $\dot{\gamma} = \dfrac{Kr\dot{\theta}}{\cos\sigma}$, where the constant $K > 0$, against a non-maneuvering target. Prove that as the term $r^2\dot{\theta} \to 0$ as $t \to \infty$.

Hint: *Reasonably assume* $\gamma_{\mathrm{T}} = 0$ *since the target does not maneuver.*

Solution. With given information, one has

$$\begin{aligned} \dot{r} &= V_{\mathrm{T}} \cos\theta - V\cos(\gamma - \theta) \\ r\dot{\theta} &= V_{\mathrm{T}} \sin\theta - V\sin(\gamma - \theta). \end{aligned}$$

On differentiating the engagement kinematics given above, one may obtain

$$\ddot{\theta}r + \dot{\theta}\dot{r} = -\dot{\theta}\dot{r} - V\dot{\gamma}\cos(\gamma - \theta) \implies r^2\ddot{\theta} + 2\dot{\theta}\dot{r}r = -rV\dot{\gamma}\cos(\gamma - \theta).$$

The above equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = -rV\dot{\gamma}\cos(\gamma - \theta),$$

from which one obtains

$$r^2\dot{\theta} = r_0^2\dot{\theta}_0 e^{-KVt}.$$

It is worth noting that the above expression results in

$$\dot{\gamma} = \left[\frac{Kr}{\cos\sigma}\right]\dot{\theta},$$

which is a weighted PN law, referred to as *Schoen's law.* Now, it follows from Schoen's law that $r^2\dot{\theta} \to 0$ as $t \to \infty$.

What does this mean practically?

Can we have $r = 0$ but $\dot{\theta} \neq 0$ towards interception?

What does it mean to have an interception in infinite time?

Is Schoen's law implementable?

What if $\dot{\gamma} = \dfrac{K|\dot{\theta}|^{0.5}\text{sign}(\dot{\theta})}{\cos\sigma}$?

Does having $r \approx 0$ sufficient for interception?