3] We want to find a point x in the plane whose sum of weighted distances from a given set of points $y_1, \ldots, y_m$ is minimized. Mathematically, the problem is

$$\min \sum_{i=1}^{m} w_i \|x - y_i\| \quad \text{subject to } x \in \mathbb{R}^n,$$

where $w_1, \ldots, w_m$ are positive scalars.

(a) Show that there exists a global minimum for this problem and that it can be realized by means of the mechanical model shown in Fig 1.

(b) Is the optimal solution always unique

(c) Show that an optimal solution minimizes the potential energy of the mechanical model of Fig-1 defined as $\sum_{i=1}^{m} w_i h_i$ where h_i is the height of the i^{th} weight, measured from some reference point.

Note: This problem stems from Weber's work which is generally viewed as starting point of locational theory. [5-marks]

Reference: Problem 1.1.6 from Dimitri P. Bertsekas Nonlinear Programming, Second Edition.

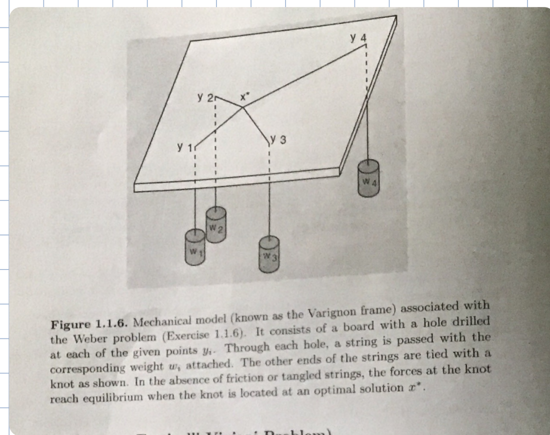


Figure 1.1.6. Mechanical model (known as the Varignon frame) associated with the Weber problem (Exercise 1.1.6). It consists of a board with a hole drilled at each of the given points y_i. Through each hole, a string is passed with the corresponding weight w_i attached. The other ends of the strings are tied with a knot as shown. In the absence of friction or tangled strings, the forces at the knot reach equilibrium when the knot is located at an optimal solution x^*.

Figure 1

SOLUTION

(a, c) Norm is a coercive function. Hence, the objective function is coercive; the weights being positive. Therefore, there is a global min to the problem.

Let $l_1, l_2, \ldots, l_m$ be the length of the strings.

Then, the potential energy of the system w.r.t. Fig-1 is

$$\rightarrow \left[m_1 g \left(l_1 - \|x - y_1\|\right) + m_2 g \left(l_2 - \|x - y_2\|\right) + \cdots + m_m g \left(l_m - \|x - y_m\|\right) \right]$$

The minus sign comes because table top is taken as reference and $(l_i - \|x - y_i\|)$ is the overhang below the table for i^{th} mass.

We use the fact that in Physics, the equilibrium position will corresponds to one which minimizes the potential energy.

Since $\sum_{i=1}^{m} m_i g l_i$ is constant it can be ignored during optimization.

∴ We solve the following unconstrained optimization problem to get equilibrium!

$$\min \; m_1 g \|x - y_1\| + \cdots + m_m g \|x - y_m\|$$

The weights $w_1 = m_1 g, \ldots, w_m = m_m g$ are all greater than zero. This mapping completes the part (a).

(b) At the optimum, the gradient vector of the objective function is zero.

$$\therefore \frac{\partial f}{\partial x_1} = \frac{w_1}{2}\frac{2(x_1^* - y_1^1)}{\|x^* - y_1\|} + \cdots + \frac{w_m}{2}\frac{2(x_1^* - y_m^1)}{\|x^* - y_m\|} = 0 \rightarrow (1)$$

$$\frac{\partial f}{\partial x_2} = \frac{w_1}{2}\frac{2(x_2^* - y_2^2)}{\|x^* - y_1\|} + \cdots + \frac{w_m}{2}\cdot\frac{2(x_2^* - y_m^2)}{\|x^* - y_m\|} = 0 \rightarrow (2)$$

where we use the notation

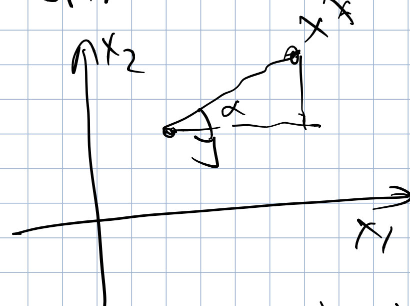
that $x^* = (x_1, x_2)$ and $y_i = (y_i^1, y_i^2)$

$$\left[\sum \frac{w_i}{\|x^* - y_i\|}\right] x_1^* = \sum \frac{w_i y_i^1}{\|x^* - y_i\|}$$

Similar eqn can be written for y-coordinate.
These are non-linear system of eqns.
These are non-linear system of eqns.

Since $\frac{x_1^* - y_1^1}{\|x^* - y_1\|}$ & $\frac{x_2^* - y_1^2}{\|x^* - y_1\|}$ can be

as $\cos\alpha$ and $\sin\alpha$. and
the weights are the tension
in the string equal to
weight mg's, it is clear
that (1) & (2) imply



that net force along x_1 & x_2 axes should be zero. So, the stationary pt condition is the equilibrium law of physics that net force acting on the knot should be zero

(b) Regarding the problem of uniqueness of soln, we know that norm is a convex function on $\mathbb{R}^n$. Positive combination of such functions is also convex. <u>If</u> a unique soln has to exist (that's my intuition - could be wrong as well!), then the objective function should be strictly convex. In general $\|\cdot\|_2$ is not strictly convex as

$$\|x+y\|_2 = \|x\|_2 + \|y\|_2 .$$

if the points are co-linear. However, in the weighted norm with origin displaced by y_i; $\xrightarrow[x]{}\xrightarrow[y]{}$

I think, it should be <u>strictly</u> convex because $y_1, y_2, \ldots y_m$ are not colinear (you will need to formalise these thoughts — or give a counter-examples).

Finally, if objective function <u>is</u> strictly convex, then optimal soln will be unique.

Prob 2.2.10 [Golub, VanLoan]

Prove or disprove

$$v \in \mathbb{R}^n \Rightarrow \|v\|_2 \|v\|_\infty \le \frac{1+\sqrt{n}}{2} \|v\|_2$$

Try a simple case of $n=1$

then, we should have

$$\|v\|_1 \|v\|_\infty \le \|v\|_2$$

But here $\|v\|_1$, $\|v\|_\infty$ and $\|v\|_2$ are all same!

Hence we should have

$$\|v\|_1 = \|v\|_2 = \|v\|_\infty \le 1$$

But then try $v = 2$ and you get a counter-example!