Non-Linear Inequality Contraint Programming Tutorial

Prof. S. A. Soman

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1. Consider the half space defined by $H = \{\, x \in \Re^n \mid a^T x + \alpha \geq 0\}$ where $a \in \Re^n$ and $\alpha \in \Re$ are given. Formulate and solve the optimization problem for finding the point x in H that has the smallest Euclidean norm.[1]

   Solution:

   $$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\,\|x\|_2^2 \\ \text{subject to} \quad & a^T x + \alpha \geq 0. \end{aligned}$$

   Unconstrained global minimum is $x = 0$. If $\alpha \geq 0$, then it is feasible.

   So, the interesting case is $\alpha < 0$. Then, the inequality constraint will be active at the solution. If $a \neq 0$, then the point is regular.

   $$\mathcal{L}(x, \mu) = \frac{1}{2} x^T x - \mu(a^T x + \alpha)$$

   $$\nabla_x \mathcal{L} = x^* - a\mu^* = 0 \implies x^* = a\mu^* \tag{1}$$

   or

   $$\mu = \frac{a^T x^*}{a^T a} \tag{2}$$

[1] Exercise 12.14, Numerical Optimization, Jorge Nocedal, Stephen Wright, Second Edition.

Also, the constraint being active, we set

$$a^T x^* + \alpha = 0 \tag{3}$$

Substituting in Eqn. 2, $\mu^* = -\dfrac{\alpha}{a^T a} > 0$ (strict complimentarity condition).

Substituting μ^* in Eqn. 1, $x^* = -\dfrac{\alpha}{a^T a} a$

i) By Weirstrass theorem, the problem always has a minimum and if only one candidate exists, it is the minimum

or

ii) By coercive nature of objective function, the problem has a minimum

or

iii) We can use second order optimaity as follows:

$$\nabla_{xx} \mathcal{L}(x^*, \mu^*) = I$$

which is symmetric positive definite. Hence, second order sufficiency conditions are met.

2. Consider the problem[2]

$$\text{minimize} \quad f(x) = -2x_1 + x_2$$

subject to

$$(1 - x_1)^3 - x_2 \geq 0$$

$$x_2 + 0.25x_1^2 - 1 \geq 0$$

The optimal solution is $x^* = (0, 1)^T$, where both constraints are active.

a) Does the point meet regularity condition?

b) Are the KKT conditions satisfied?

c) Are the second order necessary conditions satisfied? Are the second order sufficiency conditions satisfied?

[2]Exercise 12.19, Numerical Optimization, Jorge Nocedal, Stephen Wright, Second Edition.

Solution: First, reverse the signs to get constraints in the standard form.

Let constraints 1 and 2 be denoted by C_1 and C_2, respectively.

a)

$$\nabla C_1 = -\begin{bmatrix} -3(1-x_1)^2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3(1-x_1)^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\nabla C_2 = -\begin{bmatrix} 0.25 * 2 * x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Since $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are linearly independent, the point is regular.

b)

$$\mathcal{L}(x, \mu_1, \mu_2) = f(x) + \mu_1 C_1(x) + \mu_2 C_2(x)$$

$$\nabla_x \mathcal{L} = \nabla f(x) + \mu_1 \nabla C_1(x) + \mu_2 \nabla C_2(x)$$

$$\nabla \mathcal{L}(x_1^*, x_2^*, \mu_1, \mu_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \mu_1^* \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \mu_2^* \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1^* \\ \mu_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \implies \mu_1^* = \frac{2}{3}, \quad \mu_2^* = 1 + \frac{2}{3} = \frac{5}{3}$$

Thus, strict complimentarity is satisfied.

c)

$$\mathcal{L}(x, \mu) = -2x_1 + x_2 - \mu_1((1-x_1)^3 - x_2) - \mu_2(x_2 + 0.25x_1^2 - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2 + 3\mu_1(1-x_1)^2 - 0.5\mu_2 x_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 + \mu_1 - \mu_2$$

$$\nabla^2 \mathcal{L} = \begin{bmatrix} -6\mu_1(1-x_1) - 0.5\mu_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{29}{6} & 0 \\ 0 & 0 \end{bmatrix}$$

Since both constraints are active, the projected Hessian is zero. Second order necessary conditions are satisfied but sufficiency conditions are not.