Problem 2.4.1 Optimization: Insights and Applications
Jan Brinkhuis & Vladimir Tikhomirov

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + b^T x + c$$

$Q \rightarrow$ positive definite (symmetric)

$x \in \mathbb{R}^n$ $\underline{\underline{Q}}$ $n \times n$ matrix $\underline{\underline{b}} \in \mathbb{R}^n$ $\underline{\underline{c}} \in \mathbb{R}^n$
fixed fixed

suppose $Q = -I$; [symm. negative definite matrix]

$$x^T Q x = -x^T x$$

$$\inf -x^T x \longrightarrow -\infty$$

suppose matrix is Q is indefinite.

$\bar{x}^T Q \bar{x} < 0$; $\bar{x}^T Q \bar{x} = -1$;

$\alpha \rightarrow \infty$ $(\alpha \bar{x})$

$(\alpha \bar{x})^T Q (\alpha \bar{x}) \rightarrow -\alpha^2$
$\rightarrow -\infty$;
$\inf \rightarrow -\infty$;

Q is a positive definite matrix (symmetric)

$$\min_{x} \frac{1}{2} x^T Q x + b^T x + c$$

Q] Can I say that this problem is min?

Yes $\rightarrow$ $f(x) = \frac{1}{2} x^T Q x + b^T x + c$
$\because$ Coercive function.

$\therefore$ $f(x) \rightarrow$ will have a global min;

next step, evaluate the stationary pt.

see what happens - . .

Auxilliary problem:

$$\min_{\|x\|=1} x^T Q x$$

$\|x\| = 1 \Rightarrow x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

$-1 \leq x_i \leq +1$; set bounded.

$\|x\| = 1$, a closed set;

$\{x_k\}$ s.t $\|x_k\| = 1$

$\{x_k\} \rightarrow \bar{x}$ Now if $\|\bar{x}\| = 1$

then we say that set is closed.

$\bar{x} \in S$ $\quad$ $S = \mathrm{cl}\, S$

closed set.

$\{x_k\} \rightarrow \bar{x}$ s.t $\|x_k\| = 1$

$\lim_{k \to \infty} \{\|x_k\|\} \rightarrow \{1, 1, 1, \ldots\} = 1$

Q] will limit be achieved by the function under consideration

Ans: $\|\cdot\|_2$ is continuous function

limit is indeed the value of function.

$\|\tilde{x}\| = 1$; $\|\tilde{x}\| \times$ other than 1

then $\|\cdot\|_2$ is dis continous x

set $S = \{x : \|x\| = 1\}$
closed;

$\underline{S} \Rightarrow$ closed & bounded, compact

$f(x) = x^T Q x$ is continuous

By weirstrass thm, it has min say $\gamma > 0$ [$x^T Q x > 0$ $\forall x \neq 0$]

Q is s.p.d. matrix

$$\min_{\|x\|=1} x^T Q x = \gamma > 0$$

$$x^T Q x \Rightarrow \|x\|^2 \underbrace{\left(\frac{x}{\|x\|}\right)^T}_{u} Q \underbrace{\left(\frac{x}{\|x\|}\right)}_{u}$$

$u^T Q u \geq \gamma$

$$x^T Q x \geq \gamma \|x\|^2$$

$\Rightarrow \lim_{k \to \infty} \|x_k\| \to \infty$ $\quad x^T Q x \to \infty$

$x^T Q x$ is a coercive function

$$f(x) = \underbrace{\underbrace{\frac{1}{2} x^T Q x}_{\sim} + \underbrace{b^T x}_{\sim}}_{u} + \underbrace{\underbrace{c}_{\sim}}_{v}$$

$F(x)$ $\quad\quad$ $\mathbb{R}$

$\| \quad \| \to \| \quad \|$

$$\|u - v\| \geq \|u\| - \|v\|$$

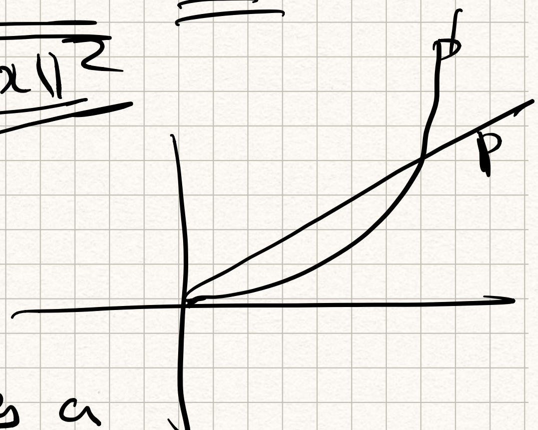
$\mathbb{R}^n$ $\quad$ $\mathbb{R}^n$ $\quad$ $\mathbb{R}^n$

$n=1$

$$|f(x)| \geq \underbrace{\tfrac{1}{2}x^T Q x}_{u} + \underbrace{b^T x}_{v} - c$$

$$\geq \underline{\tfrac{1}{2}x^T Q x} - \underline{b^T x} - c$$

$$|f(x)| \geq \underline{r\|x\|^2}$$



$$f(x) \to \infty$$
$$\|x\| \to \infty$$

This function becomes a coercive func.

It will have a global min.

$$\nabla f(x) = \nabla\left(\tfrac{1}{2}x^T Q x + b^T x + c\right)$$

$$\nabla_x c = 0 \,;$$

$$b^T x = b_1 x_1 + b_2 x_2 \cdots + b_n x_n$$

$$\frac{\partial b^T x}{\partial x_i} = b_i \,;$$

$$\nabla(b^T x) = b \,;$$

$$\nabla\left(\tfrac{1}{2}x^T Q x\right) = \underline{\underline{Qx}}$$

$$x^T Q x = q_{11}x_1^2 + q_{22}x_2^2 \cdots + q_{nn}x_n^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_i x_j \quad i \neq j$$

$$\frac{\partial (x^T Q x)}{\partial x_1} = 2q_{11}x_1 + \frac{\partial}{\partial x_1} 2(q_{12}x_1x_2 + q_{13}x_1x_3 \cdots + q_{1n}x_1x_n)$$

$$= 2q_{11}x_1 + 2q_{12}x_2 + 2q_{13}x_3 \cdots + 2q_{1n}x_n$$

$$\frac{\partial x^T Q x}{\partial x_1} = 2 q_1^T x$$

$$\nabla (x^T Q x) = 2\begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} x$$

$$= 2Qx$$

$$\nabla f(x) = \nabla\left(\tfrac{1}{2}x^T Q x + b^T x + c\right)$$

$$= \underbrace{Qx + b + 0 = 0}$$

$Qx^* = -b$ (provided it exists) will give a stationary pt.

Q] will it exist?

Q] $Q \to$ is non-singular matrix.
$\Rightarrow$ invertible matrix.

Suppose that it is not invertible!

$Qz = 0 \quad z \neq 0$

$\underbrace{z^T Q z = 0}_{0} \quad z \neq 0$

No. $z^T Q z > 0$ for ~~$z \neq 0$~~

$x^* = -Q^{-1} b$
unique soln.

$\min f(x) = \frac{1}{2}(-Q^{-1}b)^T Q (-Q^{-1}b)$
$+ b^T(-Q^{-1}b)$
$+ c$

$Q \to$ is sym.
$Q^{-1} \to$ is sym
$(Q^{-1})^T = Q^{-1}$

$= \frac{1}{2} b^T Q^{-1} b - b^T Q^{-1} b + c$

$= -\frac{1}{2} b^T Q^{-1} b + c$

$\min \frac{1}{2} x^T Q x + b^T x + c$

has a soln. at $x^* = -Q^{-1} b$.
and min is $\underline{c - \frac{1}{2} b^T Q^{-1} b}$.