Central Force Motion Solution

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In **order** to solve the two-body **equations**, we first define the vector **product** concept employed for **N-body** case, as shown below.

$$\vec{H} = \vec{r} \times \dot{\vec{r}}; \quad \left(\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} \right) \times \vec{H} = 0 \rightarrow \ddot{\vec{r}} \times \vec{H} = -\frac{\mu}{r^3}\vec{r} \times \vec{H}$$

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{H}\right) = -\frac{\mu}{r^3}\left[\vec{r} \times \left(\vec{r} \times \dot{\vec{r}}\right)\right] = -\frac{\mu}{r^3}\left[\left(\vec{r} \cdot \dot{\vec{r}}\right)\vec{r} - \left(\vec{r} \cdot \vec{r}\right)\dot{\vec{r}}\right]$$

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{H}\right) = \frac{\mu}{r}\dot{\vec{r}} - \frac{\mu\dot{r}}{r^2}\vec{r} = \frac{d}{dt}\left(\frac{\mu\vec{r}}{r}\right) \rightarrow \dot{\vec{r}} \times \vec{H} - \frac{\mu\vec{r}}{r} = \mu\vec{e}$$

Central Force Motion Trajectory

Next, we take **scalar** product of the result with **vector 'r'** and **simplify** the expression, as follows.

$$\vec{r}\cdot\left(\dot{\vec{r}}\times\vec{H}-\frac{\mu\vec{r}}{r}=\mu\vec{e}\right)\rightarrow\vec{r}\cdot\left(\dot{\vec{r}}\times\vec{H}\right)-\frac{\mu}{r}=\mu\left(\vec{r}\cdot\vec{e}\right)$$

$$\vec{r}\cdot\left(\dot{\vec{r}}\times\vec{H}\right)=\left(\vec{r}\times\dot{\vec{r}}\right)\cdot\vec{H}=h^2;\quad \mu\left(\vec{r}\cdot\vec{e}\right)=\mu re\cos\theta$$

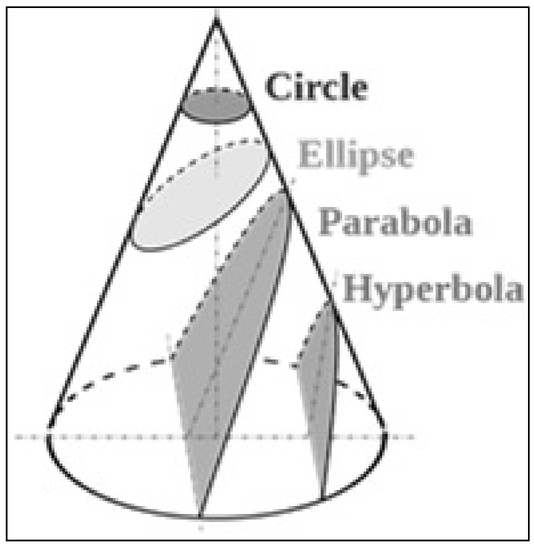
$$r=\frac{\left(h^2/\mu\right)}{1+e\cos\theta};\quad \vec{e}:\text{Constant Vector};\quad \theta:\angle\text{ between }\vec{r}\ \&\ \vec{e}$$

We now have a solution for **radius** magnitude in terms of **'θ'**, with 'e' and 'μ' as **constants.**

Central Force Motion Features

Solution obtained for ‘r’ as a **function** of ‘θ’, represents the equation of a **‘conic’** section in polar **coordinates**.

‘**Conic**’ sections are **geometries** that are created from **intersection** of a **plane** with a **cone**, as shown below.



Energy Conservation Solution

We can also obtain the **solution** for energy **conservation**, as outlined below.

$$\dot{\vec{r}}\cdot\left(\ddot{\vec{r}}+\frac{\mu}{r^{3}}\vec{r}\right)=0\rightarrow\dot{\vec{r}}\cdot\ddot{\vec{r}}+\frac{\mu}{r^{3}}\dot{\vec{r}}\cdot\vec{r}=\frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}\cdot\dot{\vec{r}}\right)+\frac{\mu}{r^{3}}\left(r\dot{r}\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}V^{2}\right)+\frac{d}{dt}\left(-\frac{\mu}{r}\right)=0\rightarrow\frac{1}{2}V^{2}-\frac{\mu}{r}=\varepsilon\rightarrow\text{ A constant}$$

Conic section solution, along with **energy** conservation solution, are sufficient to **examine** spacecraft motion.

Conic Section Features

r(t) is a **vector** drawn from **focus**, with ‘$\theta(t)$’ (Positive anti-clockwise), **measured** with respect to vector **‘e’,** which is taken as **one axis** of plane.

Ellipse is the most generic ‘**conic section**’ that is used in orbit **solutions**, as it is able to **capture** the features of the other **conic** sections e.g. circle, parabola and **hyperbola**.

Basic Orbital Solution

Elliptic Orbit Parameters

Consider **general** elliptical **geometry**, as shown below

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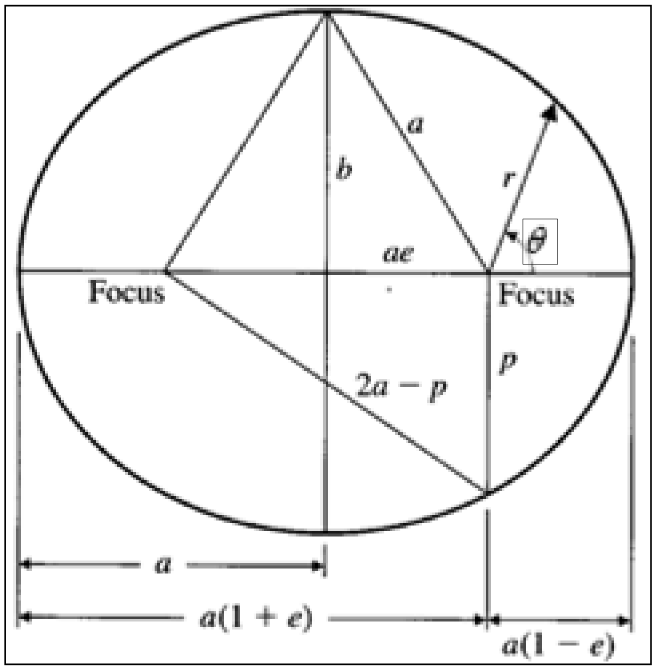

We can **relate** the ellipse **parameters** with the motion **variables**, as given below.

$$r = \frac{p\left(= h^2/\mu\right)}{1 + e\cos\theta}; \quad p = a(1 - e^2)$$

Ellipse as Generic Conic Section

Ellipse is the basic **conic** section that is applicable to **orbits**. Following are the orbit related **parameters.**

$$h = \sqrt{\mu a\left(1-e^2\right)} = r_p v_p = r_a v_a; \quad r_p = a(1-e)$$

$$r_a = a(1+e); \quad a = \frac{r_a + r_p}{2}; \quad e = \frac{r_a - r_p}{r_a + r_p}$$

$$\varepsilon = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} = \frac{1}{2}v_a^2 - \frac{\mu}{r_a} = -\frac{\mu}{2a}; \quad e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}$$

Here, **'ε' & 'h'** are related to the burnout **parameters** of the corresponding **ascent mission** performance.

Orbit Nature & Parameters

We see that **'a'**, which denotes orbit **size**, depends only on total mechanical **energy** imparted by the **ascent mission.**

However, we note that '**e**', which denotes the **shape** of the orbit, depends on both **energy** & angular momentum.

Orbit Nature & Parameters

We also **note** that all **missions** that have '**e**' between **0 and 1**, will form the **orbits.**

Therefore, we can **arrive** at the **conditions** for either $\bar{r}$ and $\bar{v}$, or for **h and ε**, for forming an **orbit**, for designing the **ascent** mission.

Bound on Orbits

It should be **noted** here that **e = 0** represents lower **limit** that degenerates **into** a circle, as shown below.

$$r_{circular} = \frac{\left(h^2/\mu\right)}{1+e\cos\theta}\Big|_{e=0} = \left(h^2/\mu\right) = \frac{r_{circular}^2 v_{circular}^2}{\mu} \rightarrow v_{circular} = \sqrt{\frac{\mu}{r_{circular}}}$$

Similarly, **e = 1** represents the upper **limit** that degenerates into a **parabola**, as shown below.

$$e=1 \rightarrow \varepsilon = 0 \rightarrow \frac{1}{2}v_{parabolic}^2 = \frac{\mu}{r_{parabolic}} \rightarrow v_{parabolic} = \sqrt{\frac{2\mu}{r_{parabolic}}}$$

Non-orbital Trajectories

In case of **e > 1**, equation of **conic** corresponds to a **hyperbola**, as follows.

$$e > 1 \rightarrow r = \infty \text{ for } 1 + e\cos\theta = 0 \rightarrow \theta < 180^0 \rightarrow \text{hyperbola}$$

In **all** such cases, the **spacecraft** escapes from earth's **gravitational** field and attains **inter-planetary** path.

Non-orbital Trajectories

However, in case of **$e < 0$**, definitions of 'r_a' and 'r_p' **inter-change**, resulting in a **reformulation** of problem.

This can be **understood** by assuming that **$(2\varepsilon h^2/\mu^2) < -1$** so that $e^2 < 0$, making **'e'** a complex number, and **leading** to situations where **$a < R_E$**.

Thus, in such **cases**, the object will **fall** back to earth, as per the various **forces** and entry initial **conditions.**

Summary

We see that two-body **equations** can be solved in the **closed** form implicitly, through **simple** vector operations.

We **further** note that ellipse is the **basic** geometry that is applicable in the **context** of space objects forming **orbits**.