Mean Angular Motion Concept

Kelper's Implicit Strategy

Kepler, while **working** with elliptic **orbits**, found that **circle** was a degenerate form of **ellipse** and, hence proposed a mapping **strategy.**

Through this **strategy**, Kepler was able to **introduce** the concept of **mean** angular motion for an **elliptic** orbit and obtain **Δt** between any **two points** on ellipse.

Mean Angular Velocity Concept

Mean angular velocity ('n') is defined as the **average** of angular velocity function, **($d\theta/dt$),** taken over **one** complete **cycle.**

Thus, '**n**' is nothing but '**$2\pi/T$**', where 'T' is the **orbital** time period and is **average** (or mean) angular **velocity**.

Mean Angular Velocity Solution

We can obtain **‘n’** through **application** of Kepler’s **3rd law**, as follows.

$$\text{Angular Velocity (rad/s): } n = \frac{2\pi}{T} = \frac{2\pi}{\left(2\pi\sqrt{a^3} \big/ \sqrt{\mu}\right)} = \sqrt{\frac{\mu}{a^3}}$$

Mean Time Concept

We **define** mean time '$\Delta \bar{t}$' in which a **mean** angle '$\Delta \bar{\theta}$' is travelled, in terms of '**n**', as follows.

$$\boxed{\Delta \bar{t} = \frac{\Delta \bar{\theta}}{n} \rightarrow \Delta \bar{\theta} = n \cdot \Delta \bar{t}}$$

Here, $\Delta \bar{\theta}$ is the average value of **θ(t)**, travelled **during** the time interval, $\Delta \bar{t}$.

Auxiliary Circle Concept

As **average** time ‘$\Delta \overline{t}$’ is an **exact** solution for a **circle** on which the **exact** angle ‘$\Delta \overline{\theta}$’ is **travelled**, we invoke this **analogy** to arrive at the **actual** time solution.

Kepler **evolved** a methodology to **connect** mean motion to actual **motion**, through concept of an **auxiliary** circle.

Auxiliary Circle Concept

Auxiliary Circle Concept

Kepler defined the **auxiliary** circle as a **circle** whose radius was same as '**a**' of the ellipse and whose **rotated** form (about **major** axis) was the applicable **ellipse**.

Though Kepler **worked** only with the **elliptic** geometries, his strategy was **later** found to be **applicable** even to hyperbolic **trajectories.**

Auxiliary Circle Concept

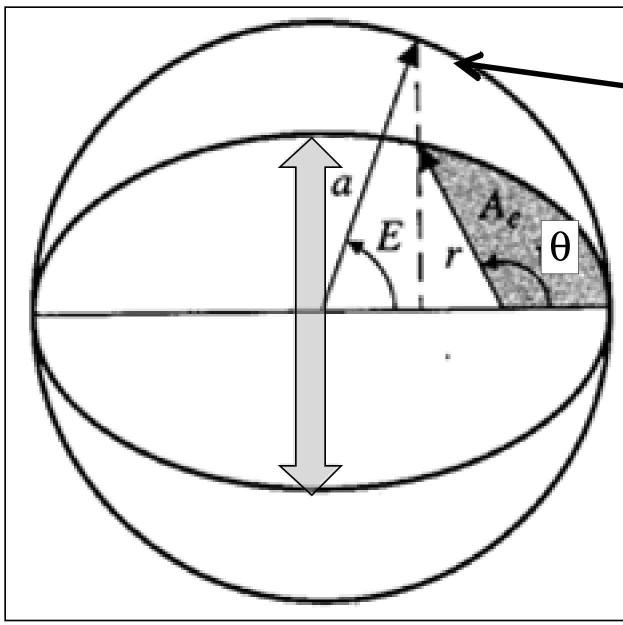
Auxiliary circle is the **circle** which results in the required **ellipse** under a rigid body **rotation** about major **axis.**

Therefore, we can **reverse** the above rigid body **rotation** to arrive at the **circle** from a given **ellipse.**

The **circle-ellipse** mapping is geometrically shown **next.**

Auxiliary Circle Concept

Consider ellipse, along with projected **auxiliary circle.**

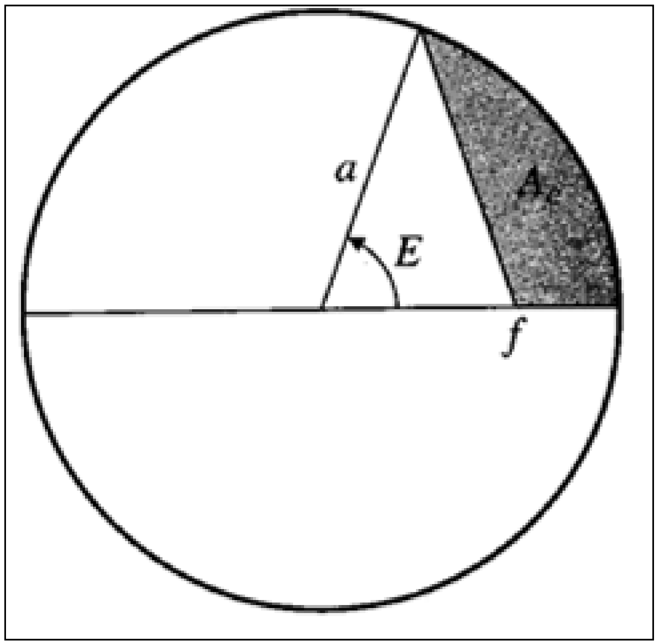


Auxiliary Circle

Here, **E** is the angle made by ‘**a**’ whose end-point is the **projected** point of ‘**r**’ on the **circle.**

Auxiliary Circle – Ellipse Mapping

Under the rotation, **area** segment **'A_e'** of ellipse becomes **area** segment **'A_c',** in auxiliary **circle**, as shown below**.**



$$A_c = \text{Sector Area - Triangle Area}$$

$$= \frac{1}{2}a(aE) - \frac{1}{2}(ae)(a\sin E)$$

$$A_c = \frac{1}{2}a^2(E - e\sin E)$$

E: Angle travelled by spacecraft on auxiliary circle in time 'Δt'

Area Swept by Radius Vector

It should be **noted** that areas $\mathbf{A_e}$, and $\mathbf{A_c}$, are areas **swept** by the **actual** and projected radius **vectors.**

Further, we know that **areal** velocity is **constant** (and also same) for **both** ellipse and circle**,** while **angular** velocity is **also** constant in case of **circle.**

Actual Time Solution

Therefore, we can relate **'Δt'** to **' $\overline{\Delta\theta}$ (or M)',** as follows.

$$A_e = \frac{b}{a} A_c = \frac{ab}{2}(E - e\sin E); \quad \Delta t = \frac{A_e}{\pi ab} T = \frac{A_e}{\pi ab}\left(2\pi \sqrt{a^3} \Big/ \sqrt{\mu} \right)$$

$$\Delta t = \frac{(E - e\sin E)}{n} = \frac{\Delta\overline{\theta}}{n} = \frac{M}{n}; \quad \frac{b}{a} \rightarrow \text{Mapping Ratio}$$

Kepler's Equation

Mean Angle Features

The expression **(E – e sin E)** is the amount of mean **angle (M)** traversed by the radius **vector 'r'**, in time interval **from** t_A to t_B**,** in respect of **any** two points 'A' and 'B'.

The equation relating **'M'** to **'E',** is called the **Kepler's equation**, which he used in order to **provide** accurate **solutions** for time in elliptic **orbits.**

Kepler's Equation Features

Kepler's equation is one of the earliest **transcendental** equations, which cannot be **solved** in closed form and only **numerical** solutions are **possible.**

Kepler's equation provides the **solution** to angle **'E',** also called '**eccentric**' angle, for a given **mean** angle, M, orbit eccentricity, **e**, or vice versa.

Kepler's Equation Features

In this **context**, we see that for **e = 0**, which represents a **circular** orbit, 'M' is same as 'E', resulting the **exact** solution.

However, in general, **M** and **E** are different for **e > 0**, with **M < E.**

Solution of Kepler's Equation

Mean angle **'M',** in a sense, is the **average** of 'θ' function, over time **interval** 'Δt'.

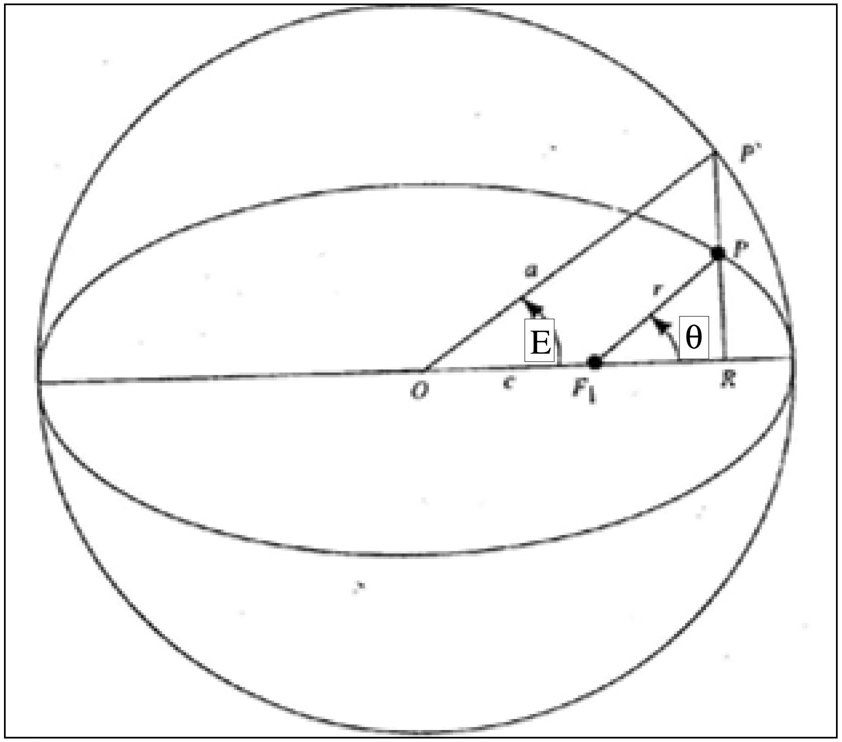
However, we **need** to know the **actual** or 'true' **angle,** '$\Delta\theta$', travelled during **'Δt'.**

This can be done by **mapping** 'E' with 'θ', through a geometric **relation**, as shown next.

Time Solutions for Orbits

$E - \theta$ Geometric Mapping

Consider **ellipse** and applicable **circle** as shown below.



E - θ Relation

We can obtain **'θ'** from '**E**', as follows.

$$\cos E = \frac{OR}{OP'} = \frac{OF_1 + FR}{a}; \quad \cos E = \frac{c + r\cos\theta}{a}$$

$$\cos E = \frac{ae + r\cos\theta}{a}; \quad r = \frac{p}{1 + e\cos\theta} = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

$$\cos E = \frac{ae(1 + e\cos\theta) + a\left(1 - e^2\right)\cos\theta}{a(1 + e\cos\theta)}$$

$$CosE = \frac{e + \cancel{e^2\cos\theta} + \cos\theta - \cancel{e^2\cos\theta}}{1 + e\cos\theta} = \frac{e + \cos\theta}{1 + e\cos\theta}$$

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E}; \quad \tan\frac{E}{2} = \left(\sqrt{\frac{1 - e}{1 + e}}\right)\tan\frac{\theta}{2}; \quad r = a(1 - e\cos E)$$

Explicit Time Expression

Combining 'M' and 'E', 'Δt' between two points '**A**' and '**B**', can be expressed as a **function of 'θ',** as follows.

$$\Delta t = \frac{M_B - M_A}{n} = \sqrt{\frac{a^3}{\mu}}\left[E_B - e\sin E_B - E_A + e\sin E_A\right]$$

$$\Delta t = \left(\sqrt{\frac{a^3}{\mu}}\right)\left[\begin{array}{l} 2\left(\tan^{-1}\left\{\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_B}{2}\right\} - \tan^{-1}\left\{\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_A}{2}\right\}\right) \\ -e\left(\frac{\left(\sqrt{1-e^2}\right)\sin\theta_B}{1+e\cos\theta_B} - \frac{\left(\sqrt{1-e^2}\right)\sin\theta_A}{1+e\cos\theta_A}\right) \end{array}\right]$$

Exact Δt Based on Integral

In this regard, it is **worth** recalling the following **integral** for Δt**,** which is applicable to any **conic section.**

$$t_B - t_A = \frac{h^3}{\mu^2} \int_{\theta_A}^{\theta_B} \frac{d\theta}{(1 + e\cos\theta)^2}$$

As the **previously** derived time solution is an **explicit** function of 'θ' and also **exact**, we can interpret that the **Δt** solution is also the **solution** of the above **integral.**

Solution for Time of Flight

Solution Process for Δt

Typical problems involving Δt evaluation are **posed** in terms of **achieving** a specified angular travel**, $\Delta\theta$.**

We use the **above** specification to evaluate '**ΔE**' using E - θ mapping, and arrive at 'Δ**M**' using the relation **$M = E - e \sin E$,** at both locations, A and B.

Once we get 'Δ**M**', we use the relation **$\Delta t = (\Delta M/n)$** to arrive at the **Δt.**

Broad Solution Steps for Δt

<u>'Δt' Calculations for given 'Δθ':</u>

Determine 'E_A' & 'E_B' from relation: $\boxed{\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}}$

Evaluate 'M_A, M_B' from relation: $M = E - e \sin E$

Obtain 'Δt' from relation: $\Delta t = \Delta M / n$

Solution for Angular Travel

Solution Process for $\Delta\theta$

Conversely, when we need to **determine '$\Delta\theta$'** for a specified **Δt**, we use **'n'** to first calculate **'ΔM'** as '$n \times \Delta t$'.

Next, we convert **ΔM** into **ΔE**, using transcendental relation **$M = E - e \sin E$,** at the two **locations** A and B, through an iterative **numerical** procedure.

Solution Process for $\Delta\theta$

Once '$\mathbf{\Delta E}$' is obtained, we can use **E - θ** mapping to solve for '$\Delta\theta$'.

It is to be **noted** that this **process** does not have any closed form **expression**, contrary to the **Δt** solution.

Broad Solution Steps for $\Delta\theta$

Determine 'ΔM' from relation: $n \cdot \Delta t = \Delta M$

Solve for 'E_A, E_B' from relation: $M = E - e \sin E$

Obtain '$\Delta\theta$' from relation: $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan\frac{E}{2}$ at A, B.

Summary

We note that concept of **mean** angular velocity and **auxiliary** circle provide a simple and **elegant** methodology for determining the **time** solution.

We **also** see that applicable relation is **reversible** so that we can obtain **angular** travel for a given time **interval**.