Central Force Motion and Kepler's Laws

Orbit Solution

Orbital solution, in terms of **conic** section, is a geometric **static** map, in which every **point** on ellipse, denoted by **(r, θ),** also corresponds to a '**t**' which is **implied.**

However, in most **applications**, we need **positions** with respect to **time** and, therefore, we **need** to find **relations** for positions as a **function** of time.

Orbits & Kepler's Laws

In this **regard**, it is worth noting that **Kepler's** 2nd and 3rd law contain **time** as a parameter.

As **conic** section solution is **obtained** under the condition of **central** force motion, it should be **possible** to make use of **Kepler's** laws to arrive at time **solutions.**

Mathematical Form of Kepler's Laws

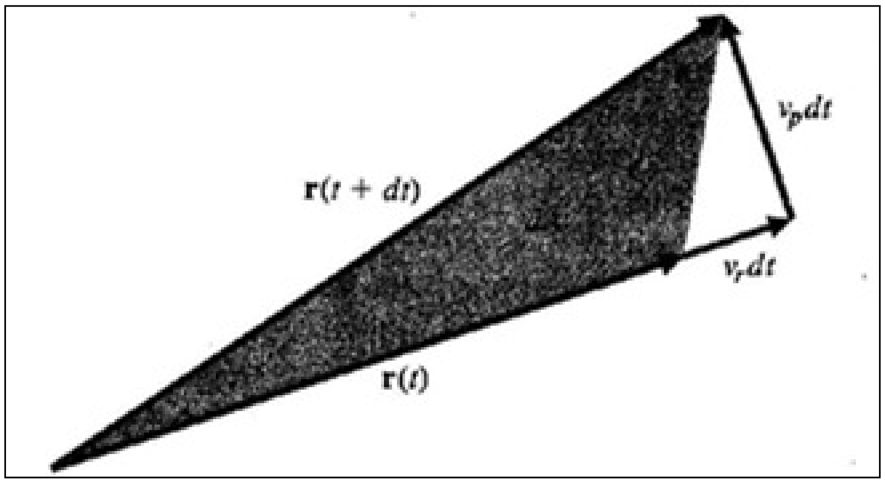
However, as **Kepler's** laws are in the form of **statements**, we need to **translate** these into mathematical **forms.**

This can be done by **invoking** the conservation of angular momentum, as **related** to the ellipse which is the **basic** orbital geometry.

Derivation of Kepler's Laws

Basic Position Model

Consider the **position** of an object at two time instants, '**t**' & '**t + dt**', along with the **velocities**, as shown below.



Kepler's 1st Law

As **H,** is a **vector** product of non-colinear vectors '**r**' and '**dr/dt**', these vectors **define** a plane, so that **r × (dr/dt)**, which is **H**, is normal to this **plane.**

Now, as **H** is constant in both **magnitude** & direction, it means that the **plane** defined by vectors '**r**' & '**dr**' is **conserved** during motion, proving the **1st Law**.

Kepler's 2nd Law

Also, we can show that **area** swept by '**r**' in **'dt'** is,

$$dA = \frac{1}{2}(r)\left(v_p dt\right); \quad \frac{dA}{dt} = \frac{1}{2} r v_p = \frac{1}{2}\left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = \frac{1}{2}\left|\bar{H}\right| = \frac{1}{2} h$$

This is **2nd law** i.e. '**equal areas**' swept in '**equal time**'.

Kepler's 3rd Law

Kepler's **3rd law** can now be **derived** by obtaining the **orbital** time period using **ellipse** relations, as follows.

$$\frac{dA}{dt} = \frac{1}{2}h = \frac{1}{2}\sqrt{\mu a(1-e^2)}; \quad T = \frac{\text{Area of ellipse}}{\text{Areal Velocity}} = \frac{\pi ab}{\left(\frac{dA}{dt}\right)}$$

$$T = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}} = \frac{2\pi}{\sqrt{\mu}}a^{(3/2)}; \quad T^2 = \frac{4\pi^2}{\mu}a^3 \rightarrow \text{Kepler's } 3^{rd} \text{ law}$$

Explicit Time Solution

Angular Motion Formulation

As **orbital** parameters, 'a' and 'e', do not involve **'t'**, we need a separate **solution** to connect **'t'** to 'r' and 'θ'.

While, **Kepler's** 3rd law does provide **time** information, it is only the **orbital** time period, which is an **average** value over one **cycle.**

t - θ Formulation

In order to fix **'t'**, we can obtain the **expression** for **(dθ/dt**) from 'h', as shown **below.**

$$h = rv_p = r\frac{rd\theta}{dt}; \quad dt = \frac{1}{h}r^2 d\theta \rightarrow t = \frac{1}{h}\int r^2 d\theta$$

$$r = \frac{\left(h^2/\mu\right)}{1 + e\cos\theta}; \quad t - t_0 = \frac{h^3}{\mu^2}\int_{\theta_A}^{\theta_B}\frac{d\theta}{(1 + e\cos\theta)^2}$$

Thus, **we** see that as '**h**' is known, we can find '**t**' for given '**θ**' or vice versa, through **above** integral.

Time Solution Strategy

In general, time **integral** can be solved in **closed** form through a **series** of trigonometric **substitutions**, though it is a bit **tedious** exercise, except for $e = 0$ or $e = 1$.

Of course, we can also **numerically** integrate the function, but would **need** to repeat the **process** for all combinations of **angles.**

't' Solution Through Transformation

Therefore, we need a **methodology** that gives the **time** solution**,** without explicit **integration** of the function.

It is **interesting** to note that Kepler was **able** to solve this problem, **without** using any **integration** at all, for **all positions** of planets.

Summary

It is seen that Kepler's laws are **directly** derivable from the basic **elliptic** solution.

Further, as the orbital **solution** is implicit in nature, the **solution** for time requires additional **formulation**.