Flight Mechanics/Dynamics

(Course Code: AE 305/305M/717)

Department of Aerospace Engineering
Indian Institute of Technology Bombay

Instructor: Shashi Ranjan Kumar
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Time: **180 Minutes**

End-Semester Examination

Total Points: 100

Instructions

- All questions are mandatory.
- In case a question is missing some data/information, assume the same suitably and clearly mention it in your answer sheet.
- You are only allowed to open lecture slides of the course, any other form of help/reference is not permitted.
- In cases where the answers of two students are found to be copied, both of them will be awarded zero marks for that particular question.
- Answer sheets need to be submitted in a single "Roll_Number.pdf" format on Moodle.
- You will get 15 minutes duration for submission of your answer sheet on Moodle after the exam time.

1. The coefficients of the characteristic polynomial corresponding to lateral–directional stability of an aircraft are

$$A = 1,\ B = 9.42,\ C = 9.48 + N_v,\ D = 10.29 + 8.4N_v,\ E = 2.24 - 0.39N_v\,.$$

Find the range of values of N_v for which the aircraft will be laterally dynamically stable.

[10]

Solution: For dynamic stability of an aircraft, the coefficients of the characteristic equation are required to satisfy the Routh's criteria of stability. As per the Routh's criteria, we get $A, B, D, E > 0$ and $R > 0$ where R is the Routh's discriminant. It can be noted from the given data that $A > 0$ and $B > 0$. Enforcing the conditions on D and E, we get

$$N_v > -\frac{10.29}{8.4},\ \text{and}\ N_v < \frac{2.24}{0.39} \implies N_v \in (-1.225, 5.7435)\,.$$

Furthermore, enforcing the necessary and sufficiency conditions on

$$R = D(BC - AD) - B^2E$$

by substituting for the given data, we get

$$\begin{aligned}(10.29 + 8.4N_v)(9.42 \times (9.48 + N_v) - 10.29 - 8.4N_v) - 9.42^2(2.24 - 0.39N_v) &> 0\\ (10.29 + 8.4N_v)(79.01 + 1.02N_v) - 198.77 + 34.61N_v &> 0\\ 813.01 + 674.18N_v + 8.568N_v^2 - 198.77 + 34.61N_v &> 0\\ 8.568N_v^2 + 708.79N_v + 614.24 &> 0\\ N_v^2 + 82.72N_v + 71.69 &> 0\end{aligned}$$

Analyzing the roots of the above polynomial, one can conclude that $R > 0 \ \forall \ N_v \in (-\infty, -81.85) \bigcup (-0.8759, \infty)$. Therefore, the feasible set of N_v for which the aircraft has lateral dynamical stability is

$$N_v \in (-1.225, 5.7435) \bigcap (-0.8759, \infty) \implies N_v \in (-0.8759, 5.7435)\,.$$

2. Consider an aircraft equipped with accelerometers and gyroscopes to measure accelerations and body rates.

   (a) If the aircraft is undergoing a steady rotation with angular velocity components in body axes system $p = 10$ deg/s, $q = 2$ deg/s, and $r = 5$ deg/s, then determine the corresponding Euler angle rates at the time instant where the Euler angles are $\psi = -30$ deg, $\theta = 10$ deg, and $\phi = 15$ deg.

   (b) If the aircraft is flying at an angle of attack of 10°, sideslip of 5°, and a bank angle of 10° and the onboard accelerometers record $a_{xb} = 10$ ft/s^2, $a_{yb} = 5$ ft/s^2, and a_{zb} = -5 ft/s^2, then determine the acceleration components in the wind axes system.

[7.5+7.5]

Solution:

(a) The given data is as follows:

$$p = 10°/s,\ q = 2°/s,\ r = 5°/s, \quad \psi = -30°,\ \theta = \ 10°,\ \phi = \ 15°\,.$$

The Euler angle rates can be computed using

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{q\sin\phi + r\cos\phi}{\cos\theta} \\ q\cos\phi - r\sin\phi \\ p + \tan\theta(q\sin\phi + r\cos\phi) \end{bmatrix}.$$

On substitution of the given date, we get

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 5.429 \\ 0.637 \\ 10.942 \end{bmatrix} {}^\circ/s\,.$$

(b) The rotation matrix from the wind axes to body axes is given by

$$C_w^b = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \cos\beta\sin\alpha\sin\phi + \sin\beta\cos\phi & -\sin\beta\sin\alpha\sin\phi + \cos\beta\cos\phi & \cos\alpha\sin\phi \\ \cos\beta\sin\alpha\cos\phi - \sin\beta\sin\phi & -\sin\beta\sin\alpha\cos\phi - \cos\beta\sin\phi & \cos\alpha\cos\phi \end{bmatrix}.$$

The side-slip angle β, angle of attack α, and bank angle ϕ are given as 5°, 10°, and 10°, respectively. Substituting the same in the above transformation matrix, we get

$$C_w^b = \begin{bmatrix} 0.9811 & -0.0858 & -0.1736 \\ 0.1159 & 0.9784 & 0.1710 \\ 0.1552 & -0.1879 & 0.9698 \end{bmatrix}.$$

The acceleration in the wind axes can be written as $a_w = C_b^w a_b$, where a_w and a_b are the acceleration in the wind and body axes. Therefore,

$$a_w = C_b^w a_b = \left[C_w^b\right]^{-1} a_b = \begin{bmatrix} 9.6138 & 4.9733 & -5.7307 \end{bmatrix}^T \text{ ft/s}^2\,.$$

3. Answer the following:

(a) Determine the missing elements of the following direction cosine matrix:

$$C = \begin{bmatrix} 0.1587 & c_{12} & 0.4858 \\ 0.8595 & -0.1218 & c_{23} \\ c_{31} & 0.4963 & 0.7195 \end{bmatrix}$$

(b) Consider a rotation of a vector, using quaternion, about an axis defined by the vector $(1, 0, 0)$ through an angle of $2\pi/3$.

(i) Obtain the quaternion Q to perform this rotation.

(ii) Compute the effect of rotation on the basis vector $\mathbf{k} = (0, 0, 1)$.

[5+10]

Solution:

(a) Using the properties that the norm of any column or row of a DCM matrix is unity. Moreover, these column vectors of a DCM matrix are linearly independent. Therefore,

$$\begin{aligned} c_{12} &= \sqrt{1 - 0.1587^2 - 0.4858^2} \approx \pm 0.8595 \\ c_{23} &= \sqrt{1 - 0.8595^2 - 0.1218^2} \approx \pm 0.4963 \\ c_{31} &= \sqrt{1 - 0.4963^2 - 0.7195^2} \approx \pm 0.4858 \end{aligned}$$

However, utilizing the linear independence property, we can eliminate one of the roots. Hence, after some analysis, we get

$$c_{12} \approx -0.8595, \; c_{23} \approx -0.4963, \; c_{31} \approx 0.4858\,.$$

(b) Given axis of rotation : $\hat{q} = (1, \; 0, \; 0)$ and angle of rotation: $\theta = \dfrac{2\pi}{3}$. Therefore,

(i) The resultant Quaternion Q is given by

$$\begin{aligned} Q &= \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\hat{i} \end{aligned}$$

(ii) Effect of rotation on $\mathbf{k} = (0, \; 0, \; 1)$ is given by $\mathbf{k}' = [Q] \; \mathbf{k} \; [Q^\star]$, which can be simplified to

$$\begin{aligned} \mathbf{k}' &= \cos\theta\boldsymbol{v} + (1 - \cos\theta)\,(\hat{q}.\boldsymbol{v})\hat{q} + \sin\theta(\hat{q} \times \boldsymbol{v}) \\ &= -\frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \left(1 + \frac{1}{2}\right).(0) + \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= \begin{pmatrix}0\\0\\-1/2\end{pmatrix} - \frac{\sqrt{3}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} \\ &= -\frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{k} \end{aligned}$$

4. Consider the speed controller, as shown in Fig. 1, with the system output, $\boldsymbol{y} = [u \quad \gamma]^T$ and the control vector $\boldsymbol{c} = [\delta_e \quad \delta_p]^T$. Note that only the elevator input is in feedback

 Question 4 continues on the next page. . .

loop. The desired and actual speeds are denoted by u_c and u, respectively. Transfer function matrix $\boldsymbol{G}(s)$ which relates the control vector, $\boldsymbol{c}$, to the output, $\boldsymbol{y}$, is represented as

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{u\delta_e} & G_{u\delta_p} \\ G_{\gamma\delta_e} & G_{\gamma\delta_p} \end{bmatrix}.$$

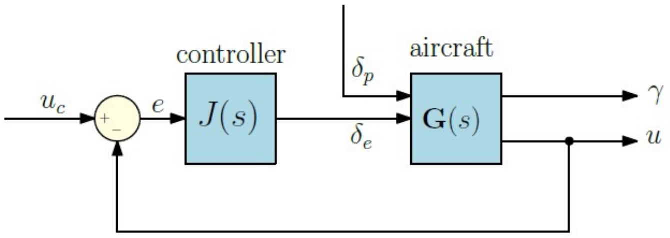

Figure 1: Speed controller

Explicitly derive the *closed-loop transfer functions*, denoted by $G^{\star}_{u\delta_p}$ and $G^{\star}_{\gamma\delta_p}$, corresponding to the throttle input and the outputs, i.e., $\delta_p \to u$ and $\delta_p \to \gamma$.

[10+10]

Solution: Using the transfer function between the control vector and output vector, we can write

$$u(s) = G_{u\delta_e}\delta_e + G_{u\delta_p}\delta_p\,, \tag{1}$$
$$\gamma(s) = G_{\gamma\delta_e}\delta_e + G_{\gamma\delta_p}\delta_p\,. \tag{2}$$

From Fig. 1 and the above result, we can write

$$\delta_e = J(s)e = J(s)\left[u_c(s) - u(s)\right]\,. \tag{3}$$

Substituting (3) in (1), we get

$$u(s) = G_{u\delta_e}J(s)\left[u_c(s) - u(s)\right] + G_{u\delta_p}\delta_p\,, \tag{4}$$
$$\left[G_{u\delta_e}J(s) + 1\right]u(s) = G_{u\delta_e}J(s)u_c(s) + G_{u\delta_p}\delta_p\,, \tag{5}$$
$$\Longrightarrow \quad u(s) = \left[\frac{G_{u\delta_e}J(s)}{G_{u\delta_e}J(s)+1}\right]u_c(s) + \left[\frac{G_{u\delta_p}}{G_{u\delta_e}J(s)+1}\right]\delta_p\,. \tag{6}$$

Therefore, the transfer function between the u and δ_p is given by

$$G^{\star}_{u\delta_p} = \frac{G_{u\delta_p}}{G_{u\delta_e}J(s)+1}\,.$$

From (2) and (3), we can write

$$\gamma(s) = G_{\gamma\delta_e} J(s)\left[u_c(s) - u(s)\right] + G_{\gamma\delta_p}\delta_p\,.$$

Substituting for (6), we get

$$\begin{aligned}
\gamma(s) &= G_{\gamma\delta_e} J(s)\left[\left(\frac{1}{G_{u\delta_e}J(s)+1}\right)u_c(s) - G^{\star}_{u\delta_p}\delta_p\right] + G_{\gamma\delta_p}\delta_p\,,\\
&= \left[\frac{G_{\gamma\delta_e}J(s)}{G_{u\delta_e}J(s)+1}\right]u_c(s) - G_{\gamma\delta_e}J(s)G^{\star}_{u\delta_p}\delta_p + G_{\gamma\delta_p}\delta_p\,,\\
&= \left[\frac{G_{\gamma\delta_e}J(s)}{G_{u\delta_e}J(s)+1}\right]u_c(s) + \left[G_{\gamma\delta_p} - G_{\gamma\delta_e}J(s)G^{\star}_{u\delta_p}\right]\delta_p\,.
\end{aligned}$$

Therefore, the transfer function between the γ and δ_p is given by

$$G^{\star}_{\gamma\delta_p} = G_{\gamma\delta_p} - G_{\gamma\delta_e}J(s)G^{\star}_{u\delta_p}\,.$$

5. Show that the small-disturbance equations, with $\theta_0 = 0$ and neglecting all Y force derivatives, yields the following approximation for the lateral displacement:

$$\Delta y_E(t) = g\int_0^t\int_0^t \phi(\tau)d\tau dt\,.$$

[10]

Solution: After ignoring all the Y derivative terms in the lateral small-disturbance equation and substituting for $\theta_0 = 0$, we get

$$\begin{aligned}
\dot{v} &= -u_0 r + g\phi\,,\\
\dot{\psi} &= r\,,\\
\Delta\dot{y}_E &= u_0\psi + v\,.
\end{aligned}$$

Differentiating $\Delta\dot{y}_E$ with respect to time, we get

$$\begin{aligned}
\Delta\ddot{y}_E &= u_0\dot{\psi} + \dot{v}\\
&= u_0 r - u_0 r + g\phi\\
&= g\phi\,.
\end{aligned}$$

Therefore, we can write the displacement in the y direction as

$$\Delta y_E = g\int_0^t\int_0^t \phi(\tau)d\tau dt\,.$$

6. Consider a stable system whose transfer function is

$$\boldsymbol{G}(s) = \frac{Y(s)}{U(s)} = \frac{N(s)}{f(s)} = \frac{1}{(s^2+3s+2)(s^2+7s+12)}\,.$$

The roots of the characteristic equation are denoted as $\lambda_1, \lambda_2, \lambda_3$ and λ_4. Find these roots and derive the expression for output $y(t)$ of the system in time domain, using the partial fraction expansion, for an input given by $u(t) = e^{i2t}$. Also, show that the *steady state* output becomes a scaled version of the input along with some phase change depending on the transfer function. [10]

Solution: The roots of the characteristic equation can be found as $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, and $\lambda_4 = -4$. The Laplace transformation of the input is given by

$$u(s) = \frac{1}{s-2i}\,.$$

The output can be written as

$$Y(s) = G(s)u(s) = \frac{1}{(s+1)(s+2)(s+3)(s+4)(s-2i)}\,.$$

Therefore, the control input adds a root $\lambda_5 = 2i$. Using partial fraction expansion, the output in time domain can be obtained as

$$\begin{aligned} y(t) &= \sum_{r=1}^{5}\left[\frac{(s-\lambda_r)}{(s+1)(s+2)(s+3)(s+4)(s-2i)}\right]_{s=\lambda_r} e^{\lambda_r t} \\ &= \frac{1}{(2i+1)(2i+2)(2i+3)(2i+4)}e^{i2t} + \frac{1}{6(-1-2i)}e^{-t} \\ &\quad - \frac{1}{2(-2-2i)}e^{-2t} + \frac{1}{2(-3-2i)}e^{-3t} - \frac{1}{6(-4-2i)}e^{-4t},. \end{aligned}$$

As $t \to \infty$, the terms e^{-t}, e^{-2t}, e^{-3t} and e^{-4t} tend to zero. Therefore the steady state response of the system is given by

$$\begin{aligned} y(t) &= \frac{1}{(2i+1)(2i+2)(2i+3)(2i+4)}e^{i2t}\,, \\ &\approx 0.0098 \times e^{-i(\phi_1+\phi_2+\phi_3+\phi_4)}e^{i2t}\,, \\ &\approx 0.0098 \times e^{-i(2.944)} \;\times\; u(t)\,, \end{aligned}$$

where ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are the phase angles corresponding to the polar representation of $2i+1,\ 2i+2,\ 2i+3,$ and $2i+4$, respectively. Therefore, it can be seen that the steady state response is a scaled version of the input with the output magnitude 0.0098 times the input and phase lagging by $\phi_1+\phi_2+\phi_3+\phi_4 \approx 168.69$ degrees.

7. Consider the following approximate system of equations, corresponding to the short period mode of an aircraft, given by

$$\dot{w} = \frac{Z_w}{m} w + u_0 q\,,$$
$$\dot{q} = \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right] w + \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right] q\,,$$

where u_0 is the nominal speed and m is the mass of the aircraft.

(a) Find the natural frequency and damping ratio corresponding to the short period mode as a function of $m, u_0, I_y, M_w, M_q, M_{\dot{w}}$ and Z_w.

(b) If an aircraft weighing 2×10^6 N is moving at a nominal speed of 230 m/s has the following structural and aerodynamic parameters: $I_y = 0.5 \times 10^8$ kg m^2, $M_w = -1.563 \times 10^4$ Nm, $M_q = -1.521 \times 10^7$ Nm, $M_{\dot{w}} = -1.702 \times 10^4$ Nm and $Z_w = -9.030 \times 10^4$ N, then use the relationships derived in the previous part to compute the natural frequency and damping ratio corresponding to short-period mode of the aircraft. Assume the gravitational acceleration to be 9.81 m/s^2.

[20]

Solution:

(a) The short period dynamics given in the question can be written in the state space form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, where the matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix} Z_w/m & u_0 \\ \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right] & \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right] \end{bmatrix}.$$

The corresponding characteristic equation can be obtained by simplifying

$$\begin{vmatrix} \lambda - Z_w/m & -u_0 \\ -\frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right] & \lambda - \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right] \end{vmatrix} = 0$$

$$\implies \lambda^2 - \left(\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right]\right)\lambda + \frac{Z_w}{m I_y}\left[M_q + M_{\dot{w}} u_0\right] - \frac{u_0}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right] = 0$$

$$\implies \lambda^2 - \left(\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right]\right)\lambda + \frac{1}{I_y}\left[\frac{M_q Z_w}{m} - u_0 M_w\right] = 0$$

On comparison with $\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0$, we get the natural frequency as

$$\omega = \sqrt{\frac{1}{I_y}\left[\frac{M_q Z_w}{m} - u_0 M_w\right]}\,.$$

Question 7 continues on the next page. . .

Moreover, the damping coefficient is given by

$$\zeta = -\frac{1}{2\omega}\left(\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right]\right) = -\frac{1}{2\omega}\left(\frac{Z_w I_y + M_q m + M_{\dot{w}}u_0 m}{m I_y}\right)$$
$$= -\frac{1}{2\sqrt{m I_y}}\left(\frac{Z_w I_y + M_q m + M_{\dot{w}}u_0 m}{\sqrt{M_q Z_w - u_0 M_w m}}\right)$$

(b) The mass of the aircraft is given by $m = (2/9.81) \times 10^6$. The natural frequency and damping coefficient of the short period are

$$\omega = \sqrt{\frac{1}{I_y}\left[\frac{M_q Z_w}{m} - u_0 M_w\right]} = 0.4546\,,$$
$$\zeta = -\frac{1}{2\sqrt{m I_y}}\left(\frac{Z_w I_y + M_q m + M_{\dot{w}}u_0 m}{\sqrt{M_q Z_w - u_0 M_w m}}\right) = 0.9079\,.$$