AE 240 /AE 240M /AE 713 Practice Session's Solutions Multi-Staging Design

1. Design a two-stage rocket to achieve an ideal burnout speed of 7924 m/s if the I_{sp} of both stages is 250s, while $\epsilon_1 = 0.18$ $\epsilon_2 = 0.15$ and determine the stage-wise payload fractions,π_1, π_2 as well as mission payload fraction, m*/m0. Also, determine stage-wise mass for structure and propellant, along with the lift-off mass if the mission payload is 1 T. Compare the solutions obtained from the Lagrange and the approximate design methods.
   Sol.
   $I_{sp} = 250\ s; \varepsilon_1 = 0.18;\ \varepsilon_2 = 0.15; V_* = 7924\ m/s;$
   Using Approximate Design Method,
   $\ln\left(\varepsilon_1 + (1-\varepsilon_1)\pi_1\right) = -\left\{\frac{V_*}{I_{sp}g_0} + \ln\left(\varepsilon_2 + (1-\varepsilon_2)\pi_2\right)\right\}$
   $\ln(0.18 + 0.82\pi_1) = -\left\{\frac{7924}{250\times 9.81} + \ln(0.15+0.85\pi_2)\right\}$
   $\ln\left((0.18+0.82\pi_1)(0.15+0.85\pi_2)\right) = -3.231$
   $(0.18+0.82\pi_1)(0.15+0.85\pi_2) = e^{-3.231}$
   $\pi_1 = \frac{0.0482}{0.15+0.85\pi_2} - 0.2195$
   $\ln(\pi_*) = \ln(\pi_1 \times \pi_2); \frac{\partial \pi_*}{\partial \pi_2} = 0$
   $\pi_* = \left\{\frac{0.0482}{0.15+0.85\pi_2} - 0.2195\right\} \times \pi_2$
   $\frac{\partial \pi_*}{\pi_2} = \left\{\frac{0.0482}{0.15+0.85\pi_2} - 0.2195\right\} + \pi_2 \times \left\{-\frac{0.04097}{(0.15+0.85\pi_2)^2}\right\} = 0$
   $0.1586\pi_2^2 + 0.056\pi_2 - 0.002291 = 0$
   $\pi_2 = 0.03703; \pi_1 = 0.0461; \pi_* = 0.001707$
   $\pi_* = \frac{m_*}{m_0} \rightarrow\ m_0 = 585.7\ Tons$
   $0.03703 = \frac{m_*}{m_{s2}+m_{p2}+m_*} \rightarrow\ m_{s2} + m_{p2} = 26\ Tons$
   $\varepsilon_2 = \frac{m_{s2}}{m_{s2}+m_{p2}} \rightarrow\ m_{s2} = 3.9\ Tons\ ; m_{p2} = 22.1\ Tons$
   $0.0461 = \frac{27}{27+m_{s1}+m_{p1}} \rightarrow\ m_{s1} + m_{p1} = 558.7\ Tons$
   $\varepsilon_1 = \frac{m_{s1}}{m_{s1}+m_{p1}} \rightarrow\ m_{s1} = 100.56\ Tons\ ; m_{p1} = 458.14$

2. A rocket, with a uniform structural ratio of 0.1 and under the assumption of no gravity and aerodynamic drag, is required to achieve an escape speed of 12,500 m/s at the end of burnout. Determine the payload ratios for 3- stage and 4-stage configurations that will make it possible. Assume all stages to be equal with I_{sp} of 185s. Can this be done using only 2 stages? Give reasons.
   Sol.
   Given $\epsilon = 0.1,\ V^* = 12500\ m/s,\ I_{sp} = 185s$
   Using:
   $$\beta = \frac{V^*}{N g_0 I_{sp}}$$

$$\pi = \frac{e^{-\beta} - \epsilon}{1 - \epsilon}$$

3 stage rocket

$\beta = 2.295$, $\pi = 0.000748$

4 stage rocket

$\beta = 1.72125$, $\pi = 0.08747$

2 stage rocket

Not possible, $\pi < 0$.

3. A multistage rocket needs to be designed to give a velocity of 15, 250 m/s to a spacecraft of 1814 kg mass. Assuming the structural ratio of 0.2 and specific impulse of 346s for all the stages, determine the minimum number of stages required for an optimum rocket to achieve the mission. Also, what would be the corresponding lift-off mass?
Sol.

$$\beta = \frac{V^*}{N g_0 I_{sp}} = \frac{4.4928}{N}$$

We require $\pi > 0$ for to design a rocket.

$$\begin{aligned}\pi &= \frac{e^{-\beta} - \epsilon}{1 - \epsilon} > 0 \\ &= \frac{e^{-\beta} - 0.2}{0.8} > 0 \\ \Rightarrow & N > 2.79\end{aligned}$$

This gives us $N_{min} = 3$ For N = 3: $\pi = 0.029582, \pi^* = 0.0000258894, m_0 = 70,067$ tonnes.

4. Consider the Von Braun Rocket as a potential booster for space shuttle, as given below.

Vehicle	Payload	Stage-1	Stage-2	Stage-3
VON	$m_* = 35.4$	$m_{01} = 6349$	$m_{02} = 898$	$m_{03} = 130$
BRAUN		$m_{f1} = 1587$	$m_{f2} = 199$	$m_{f3} = 60$
		$m_{s1} = 698$	$m_{s2} = 70$	$m_{s3} = 22$
		$I_{sp1} = 230$	$I_{sp2} = 286$	$I_{sp3} = 286$
		$t_b = 84s$	$t_b = 124s$	$t_b = 84s$

If the I_{sp} of 2^{nd} and 3^{rd} stages are increased to 295s, at the cost of 7% increase in structural mass of both these stages, would the payload increase or decrease? Also, the change would be by how much? Assume propellant mass to remain constant in all stages. (Note: All mass values are in Tons).

Sol.

First we calculate m_{pi} as $m_{0i} - m_{fi}$, and the calculate π_i, ϵ_i

Vehicle	Payload	Stage-1	Stage-2	Stage-3
VON	$m_* = 35.4$	$m_{01} = 6349$	$m_{02} = 898$	$m_{03} = 130$
BRAUN		$m_{f1} = 1587$	$m_{f2} = 199$	$m_{f3} = 60$
		$m_{s1} = 698$	$m_{s2} = 70$	$m_{s3} = 22$
		$I_{sp1} = 230$	$I_{sp2} = 286$	$I_{sp3} = 286$
		$t_b = 84s$	$t_b = 124s$	$t_b = 84s$
		$m_{p1} = 4762$	$m_{p2} = 699$	$m_{p3} = 70$
		$\pi_1 = 0.141439$	$\pi_2 = 0.14476$	$\pi_3 = 0.2723$
		$\epsilon_1 = 0.127838$	$\epsilon_2 = 0.09678$	$\epsilon_3 = 0.2391304$

We now find V^* as:

$$V^* = -g_0 \sum_{i=1}^{N} ln\left(\pi_i + \epsilon_i\left(1 - \pi_i\right)\right)$$

We get $V^* = 10101.2094788 m/s$

We now make the modifications to our rocket.

Vehicle	Payload	Stage-1	Stage-2	Stage-3
VON	$m_* = ?$	$m_{01} = m_{02} + 5460$	$m_{02} = m_{03} + 773.9$	$m_{03} = m_* + 93.54$
BRAUN		$m_{s1} = 698$	$m_{s2} = 74.9$	$m_{s3} = 23.54$
		$I_{sp1} = 230$	$I_{sp2} = 295$	$I_{sp3} = 295$
		$t_b = 84s$	$t_b = 124s$	$t_b = 84s$
		$m_{p1} = 4762$	$m_{p2} = 699$	$m_{p3} = 70$

We find the new $\pi_i' s$ in terms of m_* and substitute it into the V^* equation, and use the constraint that terminal velocity remains the same to get:

$$m_* = 38.713 tonnes.$$

Thus the payload increases by 3.313 tonnes.