Mid-semester Examination ***AE 240/713/240M*** ***Friday February 26, 20201***
Weightage: 20% ***Time: 0915 – 1055 Hrs. (including Upload)*** ***Venue: Moodle***
Open Notes (Slides, Hand-written) ***Only Calculator permitted*** ***Model Solutions***

Q.1 A rocket is launched vertically from the surface of the Earth and expels burnt gases at a uniform rate of 0.005 times the initial mass of rocket per second. For an effective exhaust speed of 3000 m/s, and assuming a uniform sea level gravity model ($g_0 = 9.81$), find the speed, and altitude of the rocket 30s after the lift-off. Next, determine the final velocity and final altitude if the rocket increases the exhaust speed of burnt gases by 20% and continues for another 120s. What is the total energy loss due to gravity at the end of the complete mission? Lastly, what should be the propellant mass fraction to achieve the mission? (Assume the motion to be in vacuum). (6)

$$\beta_1 = 0.005 m_0; \quad m_{p1} = t_{b1} \times \beta_1 = 30 \times 0.005 m_0 = 0.15 m_0$$

$$V_{b1} = g_0 I_{sp} \ln \frac{m_0}{m_0 - m_p} - g_0 t_b = 3000 \times \ln \frac{m_0}{m_0 - 0.15 m_0} - 9.81 \times 30 = 193.2 m/s$$

$$h_{b1} = \frac{m_0 g_0 I_{sp}}{\beta} \left[(1-\Lambda)\ln(1-\Lambda) + \Lambda\right] - \frac{1}{2} g_0 \left(\Lambda \frac{m_0}{\beta}\right)^2$$

$$= \frac{m_0 \times 3000}{0.005 m_0} \left[0.85 \times \ln 0.85 + 0.15\right] - \frac{1}{2} \times 9.81 \times (30)^2 = 2700.8 m$$

$$t_{b2} = 120 s; \quad m_{p2} = t_{b2} \times \beta_1 = 120 \times 0.005 m_0 = 0.6 m_0$$

$$V_{b2} = V_{b1} + g_0 I_{sp} \ln \frac{0.85 m_0}{0.85 m_0 - m_p} - g_0 t_b = 193.2 + 3600 \times \ln \frac{0.85 m_0}{0.85 m_0 - 0.6 m_0} - 9.81 \times 120 = 3421.6 m/s$$

$$h_{b2} = h_{b1} + V_{b1} t_{b2} + \frac{0.85 m_0 g_0 I_{sp}}{\beta} \left[(1-\Lambda)\ln(1-\Lambda) + \Lambda\right] - \frac{1}{2} g_0 \left(\Lambda \frac{m_0}{\beta}\right)^2$$

$$= 2700.8 + 23184 + \frac{0.85 m_0 \times 3600}{0.005 m_0} \left[0.4 \times \ln 0.4 + 0.6\right] - \frac{1}{2} \times 9.81 \times (120)^2 = 168776.8 m$$

$$V_{b1-ideal} = g_0 I_{sp} \ln \frac{m_0}{m_0 - m_{p1}} = 3000 \times \ln \frac{m_0}{m_0 - 0.15 m_0} = 487.5 m/s$$

$$V_{b2-ideal} = V_{b1-ideal} + g_0 I_{sp} \ln \frac{0.85 m_0}{0.85 m_0 - m_{p2}} = 487.5 + 3600 \times \ln \frac{0.85 m_0}{0.85 m_0 - 0.6 m_0} = 4893.1 m/s$$

$$E_{ideal} = \frac{1}{2} V_{b2-ideal}^2 = 11.9712 \times 10^6; \quad E_{Actual} = \frac{1}{2} V_{b2-actual}^2 + g_0 h_{b2} = 7.5094 \times 10^6; \quad \Delta E = 31.3\%$$

$$\Lambda_{Total} = \Lambda_{p1} + \Lambda_{p2} = 0.15 + 0.6 = 0.75$$

Q.2 A rocket is to be designed to inject a spacecraft of 2.5 Tons mass in a geo-stationary circular equatorial orbit at 36000 km altitude above earth's surface. (R_E = 6371 km). Further, it is known that spacecraft will need 3070 m/s of speed parallel to local horizon at the injection point to achieve the stated mission. Assuming that all the losses are restricted to 15% of the ideal mechanical energy, determine the lower bound on m_0 if I_{sp} is 450s. Lastly, if 50 Tons of inert mass is unavoidable, what would be m_0? ($g_0 = 9.81$ m/s^2). (3)

$$\frac{1}{2}V^2 + g_0 h = \frac{1}{2}\times 3070^2 + 9.81\times 42.371\times 10^6 = 4.2037\times 10^8$$

$$\frac{1}{2}V_0^2 = \frac{4.2037\times 10^8}{0.85} = 4.9455\times 10^8 \rightarrow V_0 = 31450 m/s$$

$$m_b = m_0 e^{-\frac{3205.9}{450}} = 0.0008 m_0 \rightarrow m_0 > 3104.3 Tons$$

$$m_b = 2.5 + 50 = 52.5 \rightarrow m_0 > 65625T$$

Q.3 A sounding rocket is launched at an angle of 30° from the local horizon with an initial velocity of 250m/s, in a constant specific thrust gravity turn manoeuvre with $n_0 = 1.5$, and travels till it becomes parallel to local horizon of the launch point. Determine the time taken and the velocity achieved at this point. Also, how much fraction of propellant (m_p/m_0) having $I_{sp} = 300$s will be needed to complete the manoeuvre? ($g_0 = 9.81$ m/s^2). (5)

$$V_0 = 250 m/s; \quad \theta_0 = 60^o; \quad \theta_b = 90^o; \quad I_{sp} = 300 s; \quad n = 1.5$$

$$k' = \frac{V_0}{\left[\left(\tan\left\{\theta_0/2\right\}\right)^{(n_0-1)} + \left(\tan\left\{\theta_0/2\right\}\right)^{(n_0+1)}\right]} = \frac{250}{[0.76 + 0.253]} = 246.7$$

$$V_{\theta=90} = k'\left[\tan^{n_0-1}\left(\theta/2\right) + \tan^{n_0+1}\left(\theta/2\right)\right] = 246.7\times[1+1] = 493.4$$

$$\Delta t = \frac{k'}{\tilde{g}}\left[\frac{\left(\tan\frac{\theta}{2}\right)^{\{n_0-1\}}}{\{n_0-1\}} + \frac{\left(\tan\frac{\theta}{2}\right)^{\{n_0+1\}}}{\{n_0+1\}}\right]_{\theta_0}^{\theta_b} = \frac{246.7}{9.81}\left[2 + \frac{1}{2.5} - \frac{0.76}{0.5} - \frac{0.253}{2.5}\right] = 19.6 s$$

$$\frac{m_0}{m} = e^{\left(\frac{n_0 \tilde{g}}{g_0 I_{sp}}\right)\Delta t} = e^{\left(\frac{1.5}{300}\right)\times 19.6} = 1.1029 \rightarrow \Lambda = 0.093$$

Q.4 Consider a 3-stage rocket of the following overall configuration. $m_0 = 300$T, $m_p = 270$T, $g_0 = 9.81$m/s^2. Assume that individual stage masses are in the ratios 10:4:1 starting from the first stage and the corresponding I_{sp} of stages 1, 2 and 3 are 250s, 350s and 450s respectively. Further, assume that mission payload is 5T. Determine the ideal burnout velocity for a 3-stage operation. Further, determine the propellant mass, inert mass and the total mass of each of three stage and determine the corresponding structural and payload ratios of the three stages. (6)

$$m_0 = 300; \quad m_p = 270; \quad m_* = 5; \quad m_{inert} = 25$$

$$m_{p1} = \frac{10}{15}\times 270 = 180; \quad m_{p2} = \frac{4}{15}\times 270 = 72; \quad m_{p3} = \frac{1}{15}\times 270 = 18$$

$$m_{inert1} = \frac{10}{15}\times 25 = 16.67; \quad m_{inert2} = \frac{4}{15}\times 25 = 6.67; \quad m_{inert3} = \frac{1}{15}\times 25 = 1.66$$

$$m_{stage-1} = 196.67; \quad m_{stage-2} = 78.67; \quad m_{stage-3} = 19.66$$

$$V_{b-ideal} = 9.81\times\left[250\times\ln\frac{300}{120} + 350\times\ln\frac{103.33}{31.33} + 450\times\ln\frac{24.66}{6.66}\right] = 12126.7 m/s$$

$$\varepsilon_1 = \frac{16.67}{196.67} = 0.0848; \quad \varepsilon_2 = \frac{6.67}{78.67} = 0.0848; \quad \varepsilon_3 = \frac{1.66}{19.66} = 0.0844$$

$$\pi_1 = \frac{103.33}{300} = 0.344; \quad \pi_2 = \frac{24.66}{103.33} = 0.239; \quad \pi_3 = \frac{5}{24.66} = 0.203$$

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