# Flight Mechanics/Dynamics

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- Quaternion operator

$$\begin{aligned} L_Q(\boldsymbol{v}) =& [Q]\boldsymbol{v}[Q]^\star = (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \\ =& \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\boldsymbol{v} + 2\left(\hat{q}\sin\frac{\theta}{2}.\boldsymbol{v}\right)\hat{q}\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\left(\hat{q}\sin\frac{\theta}{2} \times \boldsymbol{v}\right) \\ =& \cos\theta\boldsymbol{v} + (1-\cos\theta)\left(\hat{q}.\boldsymbol{v}\right)\hat{q} + \sin\theta\left(\hat{q} \times \boldsymbol{v}\right) \end{aligned}$$

- Quaternion operator in matrix form,

$$\begin{aligned} L_Q(\boldsymbol{v}) =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \\ =& \underbrace{\left[q_0^2 - \|\boldsymbol{q}\|^2)I_{3\times3} + 2\boldsymbol{q}\boldsymbol{q}^T + 2q_0(\boldsymbol{q}\times\right]}_{\text{Rotation Matrix}}\boldsymbol{v} \end{aligned}$$

where, matrix representing cross product is given by

$$\boldsymbol{q}\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

## Rotation of Vector using Quaternion

Consider a rotation about an axis defined by (1, 1, 1) through an angle of $2\pi/3$. Obtain the quaternion to perform this rotation. Compute the effect of rotation on the basis vector $\boldsymbol{i} = $ (1, 0, 0).

- Define unit vector $\hat{q} = \dfrac{1}{\sqrt{3}}(1, 1, 1)$.
- Quaternion

$$[Q] = \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2}$$
$$= \frac{1}{2} + \frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$$

- Actual vector $\boldsymbol{v} = \boldsymbol{i} = (1, 0, 0)$.

- By using quaternion operator on $\boldsymbol{v} = (1, 0, 0)$, we get

$$
\begin{aligned}
[w] &= \cos\theta \boldsymbol{v} + (1 - \cos\theta)\,(\hat{q}.\boldsymbol{v})\,\hat{q} + \sin\theta\,(\hat{q} \times \boldsymbol{v}) \\
&= -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(1 + \frac{1}{2}\right)\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \boldsymbol{j}
\end{aligned}
$$

- Let $[P]$ and $[Q]$ be two unit quaternions.

$$L_P(\boldsymbol{u}) = \boldsymbol{v}, \quad L_Q(\boldsymbol{v}) = \boldsymbol{w}$$

- We can rewrite

$$\begin{aligned}\boldsymbol{w} =& L_Q(\boldsymbol{v})\\ =& [Q]\boldsymbol{v}[Q]^\star\\ =& [Q][P]\boldsymbol{u}[P]^\star[Q]^\star\\ =& [QP]\boldsymbol{u}[QP]^\star\\ =& L_{QP}(\boldsymbol{u})\end{aligned}$$

- $L_{QP}$ is a unit quaternion rotation operator, with the axis and angle of the composite rotation given by the product $[QP]$.

- Consider quaternion operators $L_{P^\star}(\boldsymbol{u}) = [P]^\star \boldsymbol{u}[P]$ and $L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q]$.
- These operators define rotations of the coordinate system defined by corresponding quaternions.

$$\begin{aligned}\boldsymbol{w} =& L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q] \\ =& [Q]^\star [P]^\star \boldsymbol{u}[P][Q] = [PQ]^\star \boldsymbol{u}[PQ] \\ =& L_{(PQ)^\star}(\boldsymbol{u})\end{aligned}$$

- Quaternion product $([P][Q])^\star$ defines operator which represents a sequence of operators $L_{P^\star}$ followed by $L_{Q^\star}$.
- $L_{(PQ)^\star}$ is also a unit quaternion rotation operator, with the axis and angle of the composite rotation given by the product $[PQ]$.

## Example

Consider a rotation of the coordinate frame about the $z$-axis through an angle $\alpha$, followed by a rotation about the new $y$-axis through an angle $\beta$. By using quaternion method, find out the axis and angle of the composite rotation.

- The first rotation is about $z$-axis with an angle $\alpha$.

$$[P] = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{k}$$

- Second rotation is about $y$-axis with an angle $\alpha$.

$$[Q] = \cos\frac{\beta}{2} + \sin\frac{\beta}{2}\boldsymbol{j}$$

- As we rotate coordinate frames, the rotation operators are $L_{P^\star}$, followed by $L_{Q^\star}$, applied sequentially.
- Quaternion describing composite rotation

$$\begin{aligned}
[PQ] &= \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{k}\right)\left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\boldsymbol{j}\right) \\
&= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{j} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\boldsymbol{k} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{k}\times\boldsymbol{j} \\
&= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{i} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\boldsymbol{j} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\boldsymbol{k}
\end{aligned}$$

- Axis of composite rotation

$$\boldsymbol{v} = \begin{bmatrix} -\sin\frac{\alpha}{2}\sin\frac{\beta}{2} \\ \\ \cos\frac{\alpha}{2}\sin\frac{\beta}{2} \\ \\ \sin\frac{\alpha}{2}\cos\frac{\beta}{2} \end{bmatrix}$$

- Angle of rotation

$$\begin{aligned} \cos\frac{\theta}{2} &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} \\ \sin\frac{\theta}{2} &= \|\boldsymbol{v}\| \end{aligned}$$

- Rotational operator $L_{[PQ]^\star}$

- For unit quaternion, $\boldsymbol{p}' = [Q]^{\star}\boldsymbol{p}[Q]$.
- If $\boldsymbol{p} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ and $\boldsymbol{p}' = X'\boldsymbol{i} + Y'\boldsymbol{j} + Z'\boldsymbol{k}$ then

$$
\begin{aligned}
\boldsymbol{p}' =&(q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k})\boldsymbol{p}(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\
=&\boldsymbol{i}[X(q_0^2 + q_1^2 - q_2^2 - q_3^2) + Y(2q_3q_0 + 2q_1q_2) + Z(2q_1q_3 - 2q_0q_2)] \\
&+ \boldsymbol{j}[X(2q_1q_2 - 2q_3q_0) + Y(q_0^2 - q_1^2 + q_2^2 - q_3^2) + Z(2q_1q_0 + 2q_3q_2)] \\
&+ \boldsymbol{k}[X(2q_0q_2 + 2q_1q_3) + Y(2q_2q_3 - 2q_0q_1) + Z(q_0^2 - q_1^2 - q_2^2 + q_3^2)]
\end{aligned}
$$

- In matrix form, we have

$$
\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_3q_0 + q_1q_2) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_1q_0 + q_3q_2) \\ 2(q_0q_2 + q_1q_3) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
$$

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_0q_2 + q_1q_3) \\ 2(q_3q_0 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_1q_0 + q_3q_2) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}
$$

- Quaternion transformation matrix

$$[QT] = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_3 q_0 + q_1 q_2) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_3 q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_1 q_0 + q_3 q_2) \\ 2(q_0 q_2 + q_1 q_3) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

- Direction cosine matrix

$$[DC] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- Euler angle transformation matrix

$$[ET] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

- Compare these three matrices and get relations among these transformations.

## Rotation Rate of Quaternion

Let $[Q(t)]$ be a unit quaternion function, and $\boldsymbol{\omega(t)}$ the angular velocity. The derivative of $[Q(t)]$ is

$$[\dot{Q}(t)] = \frac{1}{2}\boldsymbol{\omega}[Q(t)]$$

- At $t + \Delta t$, the rotation is described by $[Q](t + \Delta t)$.
- This is after some extra rotation during $\Delta t$ performed on the frame that has already undergone a rotation described by $[Q(t)]$.
- This extra rotation is about the instantaneous axis $\hat{\boldsymbol{\omega}} = \dfrac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$ through the angle $\Delta\theta = \|\omega\|\Delta t$. It can be described by a quaternion.

$$\Delta[Q(t)] = \cos\frac{\Delta\theta}{2} + \hat{\boldsymbol{\omega}}\sin\frac{\Delta\theta}{2} = \cos\frac{\|\boldsymbol{\omega}\|\Delta t}{2} + \hat{\boldsymbol{\omega}}\sin\frac{\|\boldsymbol{\omega}\|\Delta t}{2}$$

- The rotation at $t + \Delta t$ is thus described by the quaternion sequence $[Q](t)$, $\Delta[Q(t)]$, implying $[Q(t+\Delta t)] = [\Delta Q(t)][Q(t)]$
- To derive $[\dot{Q}(t)]$, let us obtain the difference

$$[Q(t+\Delta t)] - [Q(t)] = \left(\cos\frac{\|\boldsymbol{\omega}\|\Delta t}{2} + \hat{\boldsymbol{\omega}}\sin\frac{\|\boldsymbol{\omega}\|\Delta t}{2}\right)[Q(t)] - [Q(t)]$$
$$= -2\sin^2\frac{\|\boldsymbol{\omega}\|\Delta t}{4}[Q(t)] + \hat{\boldsymbol{\omega}}\sin\frac{\|\boldsymbol{\omega}\|\Delta t}{2}[Q(t)]$$

- On taking the limit $\Delta t \to 0$, we have

$$[\dot{Q}(t)] = \lim_{\Delta t\to 0}\frac{[Q(t+\Delta t)] - [Q(t)]}{\Delta t} = \lim_{\Delta t\to 0}\hat{\boldsymbol{\omega}}\frac{\sin(\|\boldsymbol{\omega}\|\Delta t/2)}{\Delta t}[Q(t)]$$
$$=\frac{\hat{\boldsymbol{\omega}}\|\boldsymbol{\omega}\|}{2}[Q(t)]$$
$$=\frac{1}{2}\boldsymbol{\omega}[Q(t)]$$

- The differential equations for quaternion elements

$$\dot{q}_0 = -\frac{1}{2}\boldsymbol{q}^T\boldsymbol{\omega}$$
$$\dot{\boldsymbol{q}} = \frac{1}{2}[q_0\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{q}] = \frac{1}{2}[q_0\boldsymbol{\omega} - \boldsymbol{q}\times\boldsymbol{\omega}]$$

  where, $\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$ is the relative angular velocity vector between two coordinate frames and $\boldsymbol{q} = q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$.

- If the angular velocities are denoted in terms of the rotated frame then

$$[\dot{Q}(t)] = \frac{1}{2}[Q(t)]\boldsymbol{\omega'}, \quad \boldsymbol{\omega'} = [Q]^\star\boldsymbol{\omega}[Q]$$

- Note that $\boldsymbol{\omega} = 2[\dot{Q}(t)][Q(t)]^\star$
- Computation of angular rate with known quaternion and its rate

- The differential equations in compact form $\dfrac{d[Q]}{dt} = \dfrac{1}{2}B[Q]$

$$[B] = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -\boldsymbol{\Omega} \end{bmatrix}$$

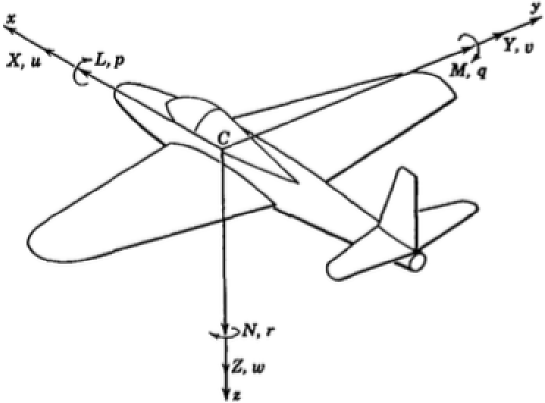
- In scalar form, above equations can be written as

$$\begin{aligned}
\dot{q}_0 &= -\frac{1}{2}[q_1\omega_x + q_2\omega_y + q_3\omega_z] \\
\dot{q}_1 &= \quad \frac{1}{2}[q_0\omega_x + q_2\omega_z - q_3\omega_y] \\
\dot{q}_2 &= \quad \frac{1}{2}[q_0\omega_y - q_1\omega_z + q_3\omega_x] \\
\dot{q}_3 &= \quad \frac{1}{2}[q_0\omega_z + q_1\omega_y - q_2\omega_x]
\end{aligned}$$

- Euler angle
  - Only 3 differential equations
  - No redundancy
  - Direct initialization from initial Euler angles
  - Nonlinear differential equations
  - Singularities
  - Gimbal lock problem
  - Transformation matrix needs to be computed
  - Order of rotation important
- Direction cosine matrix (DCM)
  - Linear differential equations
  - No singularity
  - Direct computation of DCM
  - Euler angles, required for initial calculation, are not directly available
  - Computational burden

- Quaternions
  - Only 4 linear coupled differential equations
  - No singularity thus avoids gimbal lock problem
  - Minimum redundancy to avoid singularity
  - Computationally simpler
  - If the coordinate systems do not coincides at $t = 0$ then Euler angle required for initial calculation
  - Transformation matrix needs to be computed
  - Euler angles are not directly available

- Unsteady motions of a flight vehicle
  - ⇒ Analysis, computation, or simulation
  - ⇒ Mathematical model of the vehicle and its subsystems
- Aircraft: An aggregate of elastic bodies so connected that both rigid and elastic relative motions can occur.
  - ⇒ A complicated dynamic system
- External forces: Complicated functions of its shape and its motion.
- Difficult to predict realistic analyses with a very simple mathematical model.
- Assumptions:
  - ⇒ Aircraft as a single rigid body with six degrees of freedom
  - ⇒ Free to move in the atmosphere under the actions of gravity and aerodynamic forces
  - ⇒ Flat Earth surface and stationary in the inertial space
- Nature and complexity of aerodynamic forces that distinguish flight vehicles from other dynamic systems

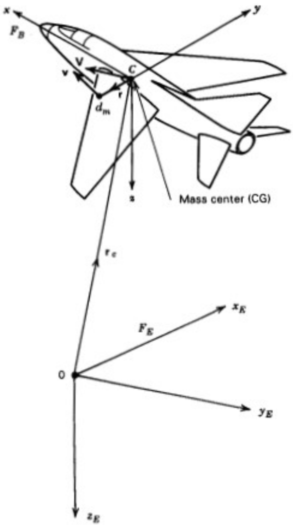



- Origin is arbitrarily located to suit the problem.
- Axis $Oz$ points vertically downward.
- Axis $Ox$ is horizontal, is chosen to point in any convenient direction.
- Axis $Oy$ is towards right wing, completing right handed rule.
- Ground speed: $V_E$

- On which speed the aerodynamic forces depend on?
- Airspeed, that is, velocity relative to surrounding air.
- If wind velocity vector is $\boldsymbol{W}$ then

$$\boldsymbol{V}^E = \boldsymbol{V} + \boldsymbol{W}$$

- Derivation of rigid-body equation: First principles
- With application of Newton's law to small element $dm$ of airplane and integration over all elements.
- Assume frame, $f_E$ $(Ox_Ey_Ez_E)$ to be inertial.



- Assuming wind velocity w.r.t $F_E$ is zero.

$$\boldsymbol{W} = 0 \implies \boldsymbol{V}^E = \boldsymbol{V}$$

where, $\boldsymbol{V}$ is the velocity of CG w.r.t. inertial frame.

- Velocity vector of CG w.r.t. the Earth, expressed in frame $F_B$,

$$\boldsymbol{V}_B = [u\ v\ w]^T$$

- Position of a "$dm$" element: $\boldsymbol{r}_c + \boldsymbol{r}$
- In frames $F_E$ and $F_B$,

$$\boldsymbol{r}_{c_E} = [x_E\ y_E\ z_E]^T,\ \boldsymbol{r}_B = [x\ y\ z]^T$$

- Position of element "$dm$" in the $F_E$ frame

$$\boldsymbol{r}_{dm} = \boldsymbol{r}_{c_E} + \boldsymbol{r}_E$$

- On differentiating $\boldsymbol{r}_{dm}$, the inertial velocity of the aircraft

$$\boldsymbol{v}_E = \dot{\boldsymbol{r}}_{c_E} + \dot{\boldsymbol{r}}_E = \boldsymbol{V}_E + \dot{\boldsymbol{r}}_E$$

- What would be the momentum of element $dm$? $d\boldsymbol{p}_E = \boldsymbol{v}_E dm$
- What would be the momentum of complete aircraft? $\boldsymbol{p}_E = \int \boldsymbol{v}_E dm$
- On substituting $\boldsymbol{v}_E$ in the momentum expression $\boldsymbol{p}_E$

$$\begin{aligned}\boldsymbol{p}_E &= \int \boldsymbol{v}_E dm = \int (\boldsymbol{V_E} + \dot{\boldsymbol{r}}_{\boldsymbol{E}}) dm \\ &= \boldsymbol{V}_E \int dm + \int \dot{\boldsymbol{r}}_E dm = m\boldsymbol{V}_E + \int \dot{\boldsymbol{r}}_E dm\end{aligned}$$

- Under rigid body assumptions with C as the CG, $\int \dot{\boldsymbol{r}}_E dm = 0$.

$$\boxed{\boldsymbol{p}_E = \int \boldsymbol{v}_E dm = m\boldsymbol{V}_E}$$

- Using Newton's second law for element $dm$,

$$d\boldsymbol{f}_E = \dot{\boldsymbol{v}}_E dm$$

where $d\boldsymbol{f}_E$ is the resultant of all forces on element $dm$.

- Similarly, under rigid body assumptions, force on the aircraft,

$$\boxed{\boldsymbol{f}_E = \int d\boldsymbol{f}_E = \int \dot{\boldsymbol{v}}_E dm = m\dot{\boldsymbol{V}}_E}$$

- What forces account for the term $d\boldsymbol{f}_E$?
- A summation of all the forces that act upon all the elements.
- What about the internal forces acting on one element upon another?
- According to Newton's third law, all internal forces act in equal and opposite pairs and their resultant is zero.
- $\boldsymbol{f}_E$: Resultant external force acting upon the airplane.
- Relation between the external force on the airplane to the motion of the CG.

- How is the moment affecting rotation of aircraft?
- Moment of momentum, also called as angular momentum
- How is the angular momentum defined for element $dm$?

$$\boxed{d\boldsymbol{h}_E = \boldsymbol{r}_E \times \boldsymbol{v}_E dm = \tilde{\boldsymbol{r}}_E \boldsymbol{v}_E dm}$$

- The cross product of two vectors $\boldsymbol{A} = [a_1 \;\; a_2 \;\; a_3]^T$ and $\boldsymbol{B} = [b_1 \;\; b_2 \;\; b_3]^T$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\tilde{\boldsymbol{A}}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Differentiating the angular momentum,

$$\boxed{\frac{d}{dt}(d\boldsymbol{h}_E) = \dot{\tilde{\boldsymbol{r}}}_E \boldsymbol{v}_E dm + \tilde{\boldsymbol{r}}_E \dot{\boldsymbol{v}}_E dm}$$

- Moment of $d\boldsymbol{f}_E$ about C

$$d\boldsymbol{G}_E = \boldsymbol{r}_E \times d\boldsymbol{f}_E = \tilde{\boldsymbol{r}}_E d\boldsymbol{f}_E = \tilde{\boldsymbol{r}}_E \dot{\boldsymbol{v}}_E dm$$

- As $\boldsymbol{v}_E = \boldsymbol{V}_E + \dot{\boldsymbol{r}}_E$, we have

$$\dot{\tilde{\boldsymbol{r}}}_E = \tilde{\boldsymbol{v}}_E - \tilde{\boldsymbol{V}}_E$$

- On substituting in the previous equation, we get

$$\boxed{d\boldsymbol{G}_{\boldsymbol{E}} = \frac{d}{dt}(d\boldsymbol{h}_{\boldsymbol{E}}) - \dot{\tilde{\boldsymbol{r}}}_E \boldsymbol{v}_E dm = \frac{d}{dt}(d\boldsymbol{h}_{\boldsymbol{E}}) - (\tilde{\boldsymbol{v}}_{\boldsymbol{E}} - \tilde{\boldsymbol{V}}_{\boldsymbol{E}})\boldsymbol{v}_E dm}$$

- As $\boldsymbol{v}_E \times \boldsymbol{v}_E = 0$, we get

$$\boxed{d\boldsymbol{G}_E = \frac{d}{dt}(d\boldsymbol{h}_E) + \tilde{\boldsymbol{V}}_E \boldsymbol{v}_E dm}$$

- On integrating previous equation, we get

$$\boxed{\int d\boldsymbol{G}_E = \frac{d}{dt}(\boldsymbol{h}_E) + \tilde{\boldsymbol{V}}_E \int \boldsymbol{v}_E dm}$$

- Note that $\boldsymbol{V}_E \times \boldsymbol{V}_E = 0$ which results in

$$\boxed{\boldsymbol{G}_E = \frac{d}{dt}(\boldsymbol{h}_E)}$$

- What about the resultant of all moments?
- Does it follow the notion same as that for forces?
- $\boldsymbol{G}_E$: Resultant of all external moments.

Two vector equations describing the motions of aircraft:

$$\boldsymbol{f}_E = m\dot{\boldsymbol{V}}_E, \quad \boldsymbol{G}_E = \dot{\boldsymbol{h}}_E, \quad \boldsymbol{h}_E = \int \tilde{\boldsymbol{r}}_E \boldsymbol{v}_E dm$$

## Remarks:

- Above equations are only valid if the moving point is the CG. This equations will *not* be valid for a moving reference point other than CG.
- Above equations are also valid if there is relative motion between parts of the plane.
- If wind vector, $\boldsymbol{W} \neq 0$, the angular momentum, $\boldsymbol{h}_E$, remains unchanged. But, the total external force is described as,

$$\boldsymbol{f}_E = m\dot{\boldsymbol{V}}^E_E, \quad \boldsymbol{G}_E = \dot{\boldsymbol{h}}_E$$

## Reference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Thank you for your attention !!!