Flight Mechanics/Dynamics

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- What is the character of the motion following a disturbance? Does it subside or increase? If it subsides what is the final flight path?
- Stability of small disturbances from steady flight
  - ⇒ Steady flight conditions make up most of the flight time of airplanes,
  - ⇒ Disturbances in this condition must be small for a satisfactory vehicle. If not, it would be unacceptable for either commercial or military use.
- Required dynamic behavior ensured by design
  - ⇒ By making small-disturbance properties (natural modes) such that either human or automatic control can keep disturbances to an acceptably small level.
- Small-disturbance model: Valid for disturbance magnitudes that seem quite violent to human occupants.

- Small-disturbance equations

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \Delta \boldsymbol{f}_c$$

- For uncontrolled motion, with eigenvector $\boldsymbol{x}_0$, we have

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \Rightarrow \boldsymbol{x}(t) = \boldsymbol{x}_0 e^{\lambda t}$$

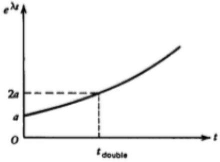
- A general solution of the system

$$\boldsymbol{x}(t) = \sum_i \boldsymbol{x}_{0_i} e^{\lambda_i t} = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \dots$$

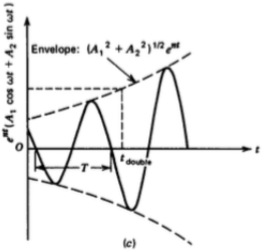
- If the eigenvalue $\lambda = n \pm i\omega$ then corresponding pair of terms

$$a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} = a_1 e^{(n+i\omega)t} + a_2 e^{(n-i\omega)t} = e^{nt}(A_1 \cos \omega t + A_2 \sin \omega t)$$

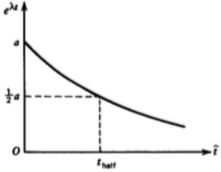
where $A_1 = a_1 + a_2, \; A_2 = i(a_1 - a_2)$ are real numbers.


$e^{\lambda t}$
$2a$
a
O
t
t_{double}

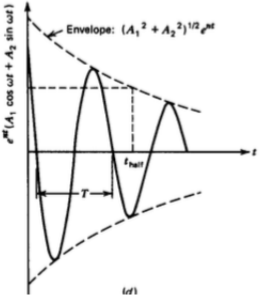
(a)


$e^{nt}(A_1 \cos \omega t + A_2 \sin \omega t)$
Envelope: $(A_1{}^2 + A_2{}^2)^{1/2} e^{nt}$
O
T
t_{double}
t

(c)


$e^{\lambda t}$
a
$\frac{1}{2}a$
O
t_{half}
$\hat{t}$

(b)


$e^{nt}(A_1 \cos \omega t + A_2 \sin \omega t)$
Envelope: $(A_1{}^2 + A_2{}^2)^{1/2} e^{nt}$
T
t_{half}
t

(d)

- Four different possible modes of solution
  - $\Rightarrow$ Static instability or divergence
  - $\Rightarrow$ Dynamic instability or divergent oscillation
  - $\Rightarrow$ Subsidence or convergence
  - $\Rightarrow$ Damped or convergent oscillation
- Quantitative characteristic for handling quality
  - $\Rightarrow$ Period $T = \dfrac{2\pi}{\omega}$
  - $\Rightarrow$ Time to double or half, t_{double} or $t_{\text{half}} = \dfrac{0.693}{|n|} = \dfrac{0.693}{|\zeta|\omega_n}$
  - $\Rightarrow$ Cycles to double or half (N_{double} or $N_{\text{half}} = 0.110\dfrac{\omega}{|n|} = 0.110\dfrac{\sqrt{1-\zeta^2}}{|\zeta|}$)
- **Time to double/half**: Times that must elapse during which any disturbance quantity will double or halve itself, respectively.
- For real root, "time to double or half" is the only parameter.
- For oscillatory modes, it is the envelope ordinate that doubles or halves.

- How to judge the stability of a system?
- Is it always necessary to find eigenvalues to check stability?
- Routh's Criteria: Number of roots of characteristic equation in RHS.
- Consider a quartic equation

$$p(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E,\ A > 0$$

- What would be the test functions to check stability?
- Test functions

$$F_0 = A,\ F_1 = B,\ F_2 = BC - AD,\ F_3 = F_2 D - B^2 E,\ F_4 = F_3 BE$$

- Necessary and sufficient conditions

$$A, B, D, E > 0$$

$$R = D(BC - AD) - B^2 E > 0$$

- R is called as Routh's discriminant.

- Vanishing of E and of R represent significant critical cases.
- If airplane is stable, and some design parameter is then varied in such a way as to lead to instability, then the following conditions hold:
  - $\Rightarrow$ If only E changes from $+$ to $-$, then one real root changes from negative to positive; that is, one divergence appears in the solution.
  - $\Rightarrow$ If only R changes from $+$ to $-$, then the real part of one complex pair of roots changes from negative to positive; that is, one divergent oscillation appears in the solution.
- Conditions $E = 0$ and $R = 0$ define boundaries between stability and instability.
- Former is the boundary between stability and static instability, and the latter is the boundary between stability and a divergent oscillation.

- Consider aircraft model (Boeing 747) with cruising in horizontal flight at approximately 40,000 ft at Mach number 0.8

$W = 636{,}636$ lb (2.83176×10^6 N)	$S = 5500$ ft^2 (511.0 m^2)
$\bar{c} = 27.31$ ft (8.324 m)	$b = 195.7$ ft (59.64 m)
$I_x = 0.183 \times 10^8$ slug ft^2 (0.247×10^8 kg m^2)	$I_y = 0.331 \times 10^8$ slug ft^2 (0.449×10^8 kg m^2)
$I_z = 0.497 \times 10^8$ slug ft^2 (0.673×10^8 kg m^2)	$I_{zx} = -.156 \times 10^7$ slug ft^2 ($-.212 \times 10^7$ kg m^2)
$u_0 = 774$ fps (235.9 m/s) $\quad \theta_0 = 0$	$\rho = 0.0005909$ slug/ft^3 (0.3045 kg/m^3)
$C_{L_0} = 0.654$	$C_{D_0} = 0.$

- Moments of inertia are for the stability axes.

- System matrix with state vector as $[\Delta u \; w \; q \; \Delta\theta]^T$

$$A = \begin{bmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Characteristic equation

$$\lambda^4 + 0.750468\lambda^3 + 0.935494\lambda^2 + 0.0094630\lambda + 0.0041959 = 0$$

- How to check stability?
- Stability criteria $\quad E = 0.0041959 > 0, \quad R = 0.004191 > 0$
- What can you say about stability? No unstable modes

- Eigenvalues

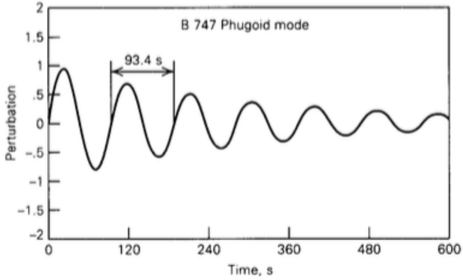
$$\lambda_{1,2} = -0.003289 \pm 0.06723i, \;\; \lambda_{3,4} = -0.3719 \pm 0.8875i$$

- Which eigenvalue corresponds to which mode and why?
- We can identify them as

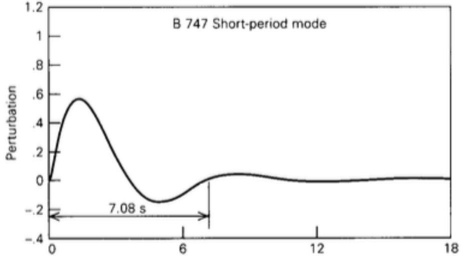
$$\underbrace{\lambda_{1,2} = -0.003289 \pm 0.06723i}_{\text{Phugoid mode}}, \;\; \underbrace{\lambda_{3,4} = -0.3719 \pm 0.8875i}_{\text{Short-period mode}}$$

- Two modes: long-period and lightly damped; short-period and heavily damped

Mode	Name	Period	t_{half}	N_{half}
1	Phugoid	93.4	211	22.5
2	Short-period	7.08	1.86	0.26


B 747 Phugoid mode
93.4 s
Perturbation
2
1.5
1
.5
0
−.5
−1
−1.5
−2
0
120
240
360
480
600
Time, s

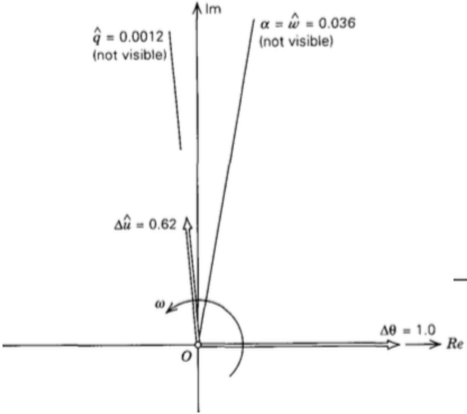
(a)

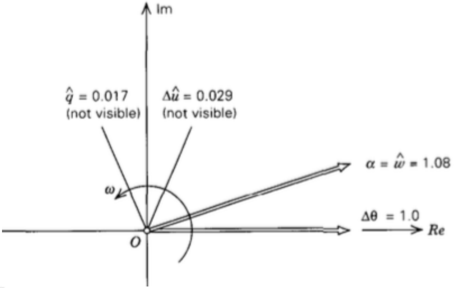

B 747 Short-period mode
7.08 s
Perturbation
1.2
1
.8
.6
.4
.2
0
−.2
−.4
0
6
12
18

Eigenvectors (polar form)

	Phugoid		*Short-Period*	
	Magnitude	*Phase*	*Magnitude*	*Phase*
$\Delta\hat{u}$	0.62	92.4°	0.029	57.4°
$\alpha = \hat{w}$	0.036	82.8°	1.08	19.2°
$\hat{q}$	0.0012	92.8°	0.017	112.7°
$\Delta\theta$	1.0	0°	1.0	0°

- Eigenvectors are normalized to see the relative magnitude.
- **Phugoid**: Small changes in q and α, but $\Delta\hat{u}$ and $\Delta\theta$ are present with significant magnitude. Speed leads $\Delta\theta$ leads by 90° in phase.
- **Short-period**: Negligible speed variation, while α oscillates with an amplitude and phase similar to θ. Two degrees of freedom, $\Delta\theta$ and α.


Im
$\hat{q}$ = 0.0012 (not visible)
$\alpha = \hat{w}$ = 0.036 (not visible)
$\Delta\hat{u}$ = 0.62
ω
$\Delta\theta$ = 1.0
Re
O


Im
$\hat{q}$ = 0.017 (not visible)
$\Delta\hat{u}$ = 0.029 (not visible)
$\alpha = \hat{w}$ = 1.08
ω
$\Delta\theta$ = 1.0
Re
O

- Differential equation for position of CG

$$\begin{aligned}\Delta\dot{x}_E =& \Delta u\cos\theta_0 - u_0\Delta\theta\sin\theta_0 + w\sin\theta_0\\ \Delta\dot{z}_E =& -\Delta u\sin\theta_0 - u_0\Delta\theta\cos\theta_0 + w\cos\theta_0\end{aligned}$$

- With $\theta_0 = 0$, we have

$$\Delta\dot{x}_E = \Delta u,\ \ \Delta\dot{z}_E = -u_0\theta + w$$

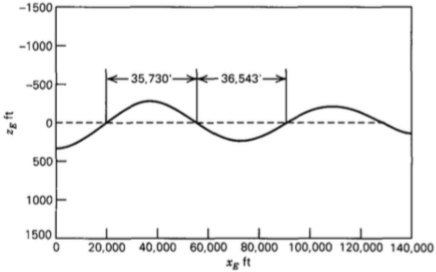
- We have now

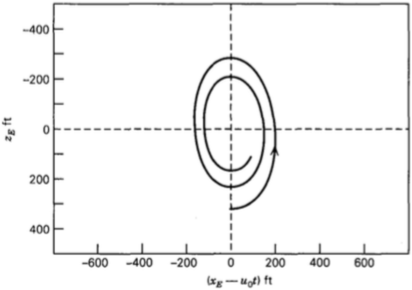
$$\Delta u = u_{1j}e^{\lambda t} + u_{1j}^{\star}e^{\lambda^{\star}t}, w = u_{2j}e^{\lambda t} + u_{2j}^{\star}e^{\lambda^{\star}t}, \theta = u_{4j}e^{\lambda t} + u_{4j}^{\star}e^{\lambda^{\star}t}$$

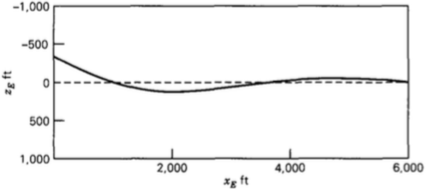
- On solving for x_E and z_E, we get

$$\begin{aligned}x_E =& u_0t + \frac{u_{1j}}{\lambda}e^{\lambda t} + \frac{u_{1j}^{\star}}{\lambda^{\star}}e^{\lambda^{\star}t} + \text{const.} = u_0t + 2e^{nt}Re\left[\frac{u_{1j}}{\lambda}e^{i\omega t}\right] + \text{const.}\\ z_E =& 2e^{nt}Re\left[\frac{u_{2j} - u_0u_{4j}}{\lambda}e^{i\omega t}\right] + \text{const.}\end{aligned}$$

Uncontrolled Motion: Flight path for Example Aircraft in Phugoid and SP Modes







- Oscillatory modes: Second order system (mass-spring-damper)
- Complete solution is sometimes difficult $\Rightarrow$ Approximate analytical solution
- Model reduction:
  - $\Rightarrow$ One small root
$$D\lambda + E = 0$$
  - $\Rightarrow$ Large complex root
$$A\lambda^2 + B\lambda + C = 0$$
- Physical insight: Some variables are negligibly small as compared to others.
- This gives us some hope to obtain approximate models with physical reasoning.
- For longitudinal, second method while for lateral, both methods are used.
- No simple analytical approximations can be relied on to give accurate results under all circumstances.

- Lanchester solution for phugoid mode: $\alpha_T = 0, \Delta\alpha = 0, T - D = 0$.
- No net aerodynamic force tangent to the flight path, and hence no work done on the vehicle except by gravity.
- Motion of constant total energy
- Total energy, with $V = u_0$ when $z_E = 0$,

$$E = \frac{1}{2}mV^2 - mgz_E \Rightarrow V^2 = u_0^2 + 2gz_E \text{ How?}$$

- Assuming $C_L = C_{L_0} = C_{W_0}$ to be constant, we have

$$L = C_{W_0}\frac{1}{2}\rho V^2 S = C_{W_0}\frac{1}{2}\rho(u_0^2 + 2gz_E)S = \underbrace{C_{W_0}\frac{1}{2}\rho u_0^2 S}_{W} + (\underbrace{C_{W_0}\rho g S}_{k})z_E$$

- Lift varies linearly with height in such a manner as always to drive the vehicle back to its reference height. How?

- Equation of motion:

$$W - L\cos\gamma = m\ddot{z}_E \Rightarrow W - L = m\ddot{z}_E, \text{ for small } \gamma$$

- On rearrangement, using $W - L = -kz_E$,

$$m\ddot{z}_E + kz_E = 0 \implies T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{C_{W_0}\rho g S}}$$

- As $C_{W_0} = \dfrac{mg}{(1/2)\rho u_0^2 S}$,

$$T = 2\pi\sqrt{\frac{m}{\frac{mg}{(1/2)\rho u_0^2 S}\rho g S}} = \frac{\sqrt{2}\pi u_0}{g} = 0.138 u_0$$

- Phugoid frequency depends only on aircraft's speed and not on any other parameter or altitude.
- For B747 aircraft, $T = 107$ seconds (Actual value is 93.2 s)

- Simplified EOM, with $q \approx 0, Z_q = Z_{\dot{w}} = 0, \theta_0 = 0$,

$$\begin{bmatrix} \Delta\dot{u} \\ \dot{w} \\ 0 \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \\ \frac{Z_u}{m} & \frac{Z_w}{m} & u_0 & 0 \\ M_u & M_w & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ w \\ q \\ \Delta\theta \end{bmatrix}$$

- To get characteristic equation

$$\begin{vmatrix} \frac{X_u}{m} - \lambda & \frac{X_w}{m} & 0 & -g \\ \frac{Z_u}{m} & \frac{Z_w}{m} - \lambda & u_0 & 0 \\ M_u & M_w & 0 & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow A\lambda^2 + B\lambda + C = 0$$

$$A = -u_0 M_w, \;\; B = gM_u + \frac{u_0}{m}(X_u M_w - M_u X_w), \;\; C = \frac{g}{m}(Z_u M_w - M_u Z_w)$$

- Characteristic equation can be written as

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\omega_n^2 = -\frac{g}{mu_0}\left(Z_u - \frac{M_u Z_w}{M-w}\right),\ \zeta = -\frac{1}{2}\frac{\frac{g}{u_0}\frac{M_u}{M_w} + \frac{1}{m}\left(X_u - \frac{M_u}{M_w}X_w\right)}{\sqrt{\frac{g}{mu_0}\left(\frac{M_u}{M_w} - Z_u\right)}}$$

- With $M_u = 0$,

$$A = -u_0 M_w,\ B = \frac{u_0}{m}X_u M_w,\ C = \frac{g}{m}Z_u M_w$$

$$\omega_n^2 = -\frac{gZ_u}{mu_0},\ \zeta = -\frac{X_u}{2}\sqrt{\frac{u_0}{-mgZ_u}} \approx \frac{1}{\sqrt{2}}\frac{C_{D_0}}{C_{L_0}}$$

- **Short-period mode**: Speed being substantially constant while airplane pitches relatively rapidly.
- Neglecting X-force equation entirely and putting $\Delta u = 0$
- $Z_{\dot{w}}$ is small compared to m and Z_q is small compared to mu_0.
- With $\theta_0 = 0$,

$$\begin{bmatrix} \dot{w} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_w}{m} & u_0 \\ \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right] & \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right] \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix}$$

- Characteristic equation

$$\lambda^2 + \left[\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right]\right]\lambda - \frac{1}{I_y}\left[u_0 M_w - \frac{M_q Z_w}{m}\right] = 0$$

- For B747, $\lambda^2 + 0.741\lambda + 0.9281 = 0 \Rightarrow \lambda = -0.371 \pm 0.889i$
- Very good for a wide range of vehicle characteristics and flight conditions.

- Static stability in longitudinal direction: $C_{m_\alpha} < 0$
- Static instability: Presence of positive root of characteristic equation
- Static stability: $E > 0$
- What is the value of E? Product of eigenvalues

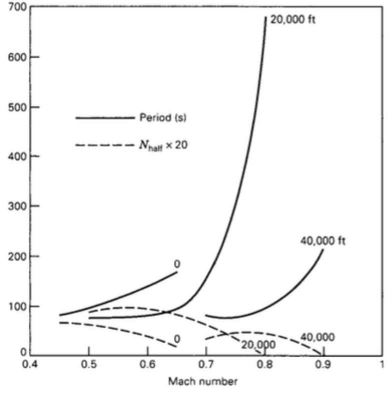
$$E = \det(\boldsymbol{A}) = \frac{g}{mI_y}(Z_u M_w - M_u Z_w)$$

- As g, m, I_y are all positive, criterion for stability

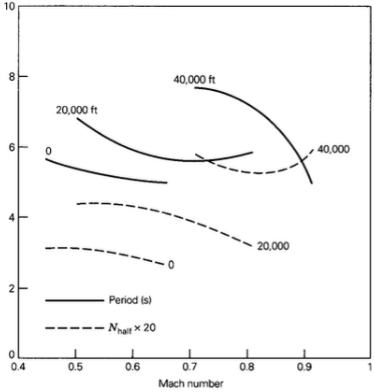
$$Z_u M_w - M_u Z_w > 0 \Rightarrow C_{m_\alpha}(C_{z_u} - 2C_{W_0}) - C_{m_u} C_{z_\alpha} > 0$$

- With no speed effect, $C_{z_u} = C_{m_u} = 0$, condition for stability

$$C_{m_\alpha} < 0$$



(a) Phugoid mode



(b) Short-period mode

Phugoid period decreases with altitude while increases with speed

SP decreases with speed while increases with altitude.

- When airplane is at the bottom of a cycle and moving fastest it is also in air of greater density, and hence additional increase in lift.
- Addition of z_E in state equation
- Equation of motion considering density gradient and $\theta_0 = 0$

$$\begin{bmatrix} \Delta\dot{u} \\ \dot{w} \\ \dot{q} \\ \Delta\dot{\theta} \\ \dot{z}_E \end{bmatrix} = \left[\begin{array}{cccc:c} & & & & \dfrac{X_z}{m} \\ & & A & & \dfrac{Z_z}{m} \\ & & & & \dfrac{1}{I_y}\left[M_z + \dfrac{M_{\dot{w}} Z_z}{m - Z_{\dot{w}}}\right] \\ & & & & 0 \\ \hdashline 0 & 1 & 0 & -u_0 & 0 \end{array}\right] \begin{bmatrix} \Delta u \\ w \\ q \\ \Delta\theta \\ z_E \end{bmatrix}$$

- Derivative $Z_z = \left.\frac{\partial Z}{\partial z_E}\right|_0$ with $Z = \frac{1}{2}\rho V^2 C_z S$.

$$\frac{\partial Z}{\partial z_E} = \frac{\partial Z}{\partial \rho}\frac{\partial \rho}{\partial z_E} = \frac{1}{2}V^2 S\left[C_z + \rho\frac{\partial C_z}{\partial \rho}\right]\frac{\partial \rho}{\partial z_E}$$

- Variation of density with height

$$\rho = \rho_0 e^{\kappa z_E} \Rightarrow \frac{\partial \rho}{\partial z_E} = \kappa \rho$$

where κ is constant over sufficient range of altitude.

- Variation of C_z with ρ is negligible $\implies \dfrac{\partial C_z}{\partial \rho} = 0.$
- Derivative Z_z now becomes

$$\frac{\partial Z}{\partial z_E} = \frac{1}{2} V^2 S \left[C_z + \rho \frac{\partial C_z}{\partial \rho} \right] \frac{\partial \rho}{\partial z_E} = \frac{1}{2} \kappa \rho V^2 S C_z$$

- Derivative $Z_z = \left. \dfrac{\partial Z}{\partial z_E} \right|_0 = \kappa Z_0$
- Similarly, $X_z = \kappa X_0, \quad M_z = \kappa M_0.$
- What would be coefficient using the reference values?
- Using reference values, we get $Z_z = -mg\kappa, X_z = M_z = 0$

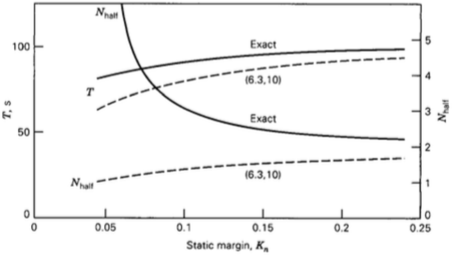
- A vertical "spring stiffness" k which governs the period.
- With variable density, lift increases when vehicle is below its reference altitude, and vice versa, resulting in a second "stiffness" k'.
- Incremental lift with variable density

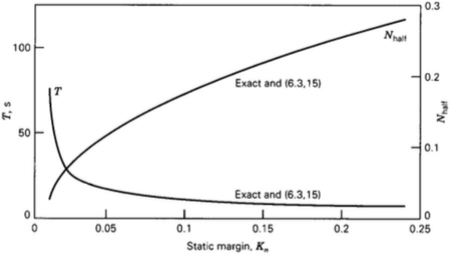
$$\Delta L = C_L \frac{1}{2} V^2 S \Delta\rho \Rightarrow k' = \frac{\partial \Delta L}{\partial z_E} = C_{L_0} \frac{1}{2} u_0^2 S \frac{\partial \rho}{\partial z_E} = \kappa W$$

- k' is approximately constant while k depends on $C_{W_0} \rho \propto V^{-2}$.
- Density gradient has its greatest relative effect at high speed.
- Correction factor for the period, which varies as $1/\sqrt{k}$,

$$F = \sqrt{\frac{k}{k + k'}} = \frac{1}{\sqrt{1 + (k'/k)}} = \frac{1}{\sqrt{1 + [\kappa u_0^2/(2g)]}}$$

- Approx. 18% reduction in phugoid period for B747, using $\kappa = 4.2 \times 10^{-5}$

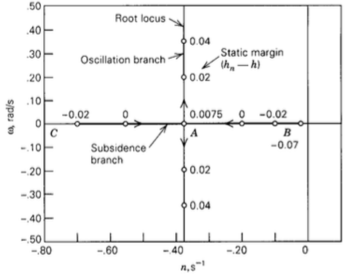


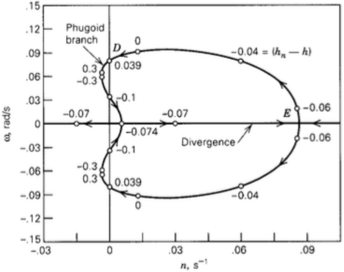


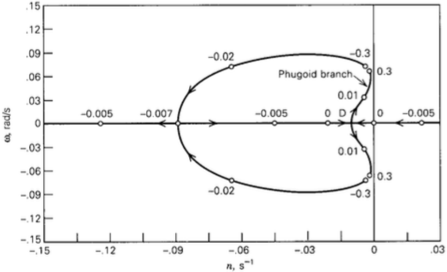
Phugoid period and damping vary rapidly at low static margin and that the approximation is useful mainly at large K_n.

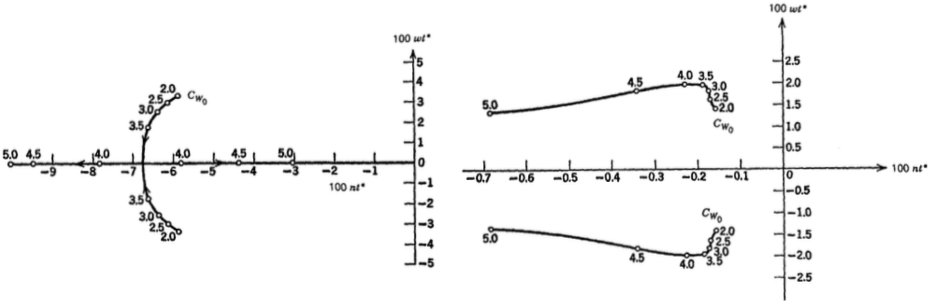
Short-period mode becoming nonoscillatory at a static margin near zero.

Uncontrolled Motion: Root Locus with Varying Static Margin

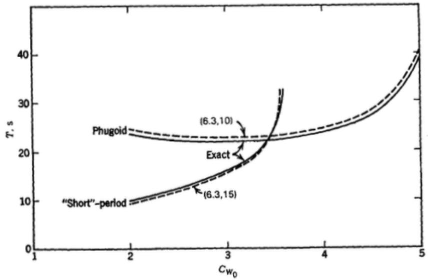


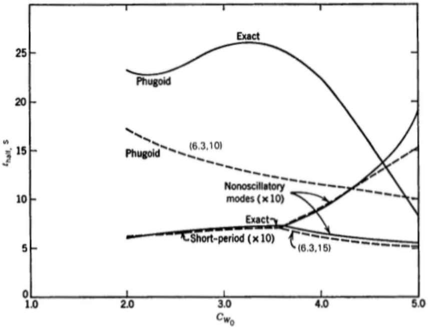






Short period mode becomes nonoscillatory at C_{W_0} more than 3.5 and damping of phugoid mode increases rapidly.





At $C_{W_0} = 3.5$ two periods are almost equal, leading to failure of long-period notion of phugoid mode.

SP mode period is well predicted by approximation, but that of phugoid is not.

References

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Thank you for your attention !!!