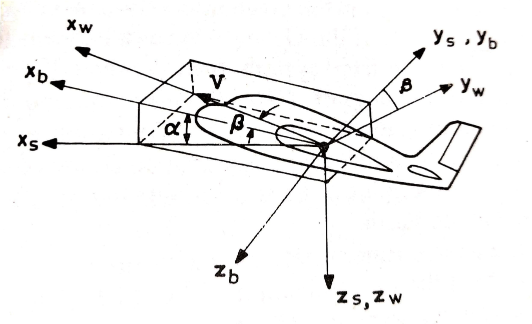
Flight Mechanics/Dynamics

Dr. Shashi Ranjan Kumar

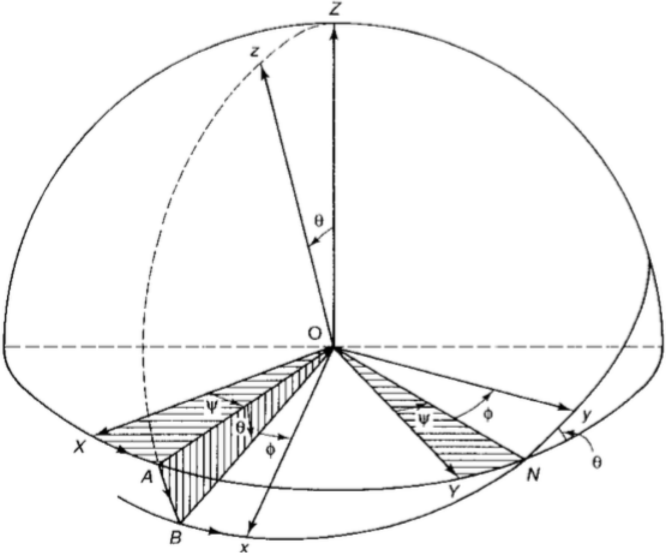
Assistant Professor
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai, 400076 India

- DCM: transforms a vector in $\mathbb{R}^3$ from one frame to other frame.
- If (X, Y, Z) and (x, y, z) are the representations of a vector in frames a and b, respectively, then

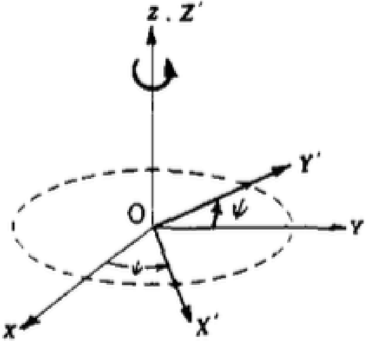
$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\boldsymbol{R}^b} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}}_{\text{Rotation Matrix}} \underbrace{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{\boldsymbol{R}^a} \Rightarrow \boldsymbol{R}^b = \boldsymbol{C}_a^b \boldsymbol{R}^a$$



- Euler angles
  - $\Rightarrow$ Method to specify the angular orientation of one coordinate frame w.r.t. another frame
  - $\Rightarrow$ A series of three ordered right-handed rotations
  - $\Rightarrow$ Correspond to the conventional roll pitch yaw angles
- Euler angles are not uniquely defined since there is an infinite set of choices.
- No standardized definitions of the Euler angles.
- For a particular choice of Euler angles, the rotation order selected and/or defined should be consistent.
- Interchange in order of rotation $\implies$ different Euler angle representation.
- Rotations are made about the Z, Y, X axes through an angle ψ, θ, ϕ angles.
- These rotations are made in the positive (anticlockwise sense) when looking down the axis of rotation toward the origin.


Z
z
θ
O
ψ
θ
φ
X
A
B
x
ψ
φ
y
θ
Y
N

- Euler angles: Three elemental rotations
- Extrinsic rotations: Rotations about the axes of the original coordinate system, which is assumed to remain motionless.
- Intrinsic rotations: Rotations about the axes of rotating coordinate system, which changes its orientation after each elemental rotation.
- Another classification
  - $\Rightarrow$ Proper Euler angles
  - $\Rightarrow$ Tait-Bryan angles
- Proper Euler angles : $(zxz, zyz, xyx, xzx, yzy, yxy)$
- Tait-Bryan angles : $(zyx, zxy, xyz, xzy, yzx, yxz)$
- What is the major difference between Proper Euler and Tait-Bryan angles?
- Tait-Bryan angles represent rotations about three distinct axes, while proper Euler angles use the same axis for both the first and third elemental rotations.



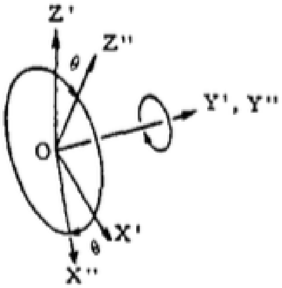
- Rotation about Z axis in anticlockwise direction by an angle ψ

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$= \boldsymbol{A} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



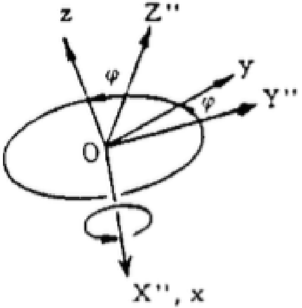
- Rotation about Y axis in anticlockwise direction by an angle θ

$$\begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

$$= \boldsymbol{B} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

where

$$\boldsymbol{B} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$



- Rotation about X axis in anticlockwise direction by an angle ϕ

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix} = \boldsymbol{D} \begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix}$$

where

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

- Consecutive rotations in the order ψ, θ, ϕ i.e., (yaw, pitch and roll) on reference frame $XYZ \implies$ new reference frame xyz.
- Rotation matrix for representing these three rotations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{D} \begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix} = \boldsymbol{DB} \begin{bmatrix} X^{'} \\ Y^{'} \\ Z^{'} \end{bmatrix} = \boldsymbol{DBA} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Equivalently,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\boldsymbol{DBA}}_{\boldsymbol{C}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Equivalent rotation matrix $\boldsymbol{C} = \boldsymbol{DBA}$ can be written as

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

- This rotation matrix is called Euler angle transformation matrix.
- Range of Euler angles:

$$-\pi \leq \psi \leq \pi, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad -\pi \leq \phi \leq \pi$$

- Is there any issue with $|\theta| > \pi/2$?

- Equivalent rotation matrices

$$\boldsymbol{C} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$
$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- How to obtain Euler angles from given DCM matrix?

$$\theta = \sin^{-1}(-C_{13})$$

$$\phi = \sin^{-1}\left(\frac{C_{23}}{\sqrt{1-C_{13}^2}}\right)$$
$$\psi = \sin^{-1}\left(\frac{C_{12}}{\sqrt{1-C_{13}^2}}\right)$$

- Are there some issues with these expressions?

- Recall about the ranges of these Euler angles
- How to determine the quadrant in which these angles lie?
- As pitch angle θ lies in $-\pi/2 \leq \theta \leq \pi/2$,

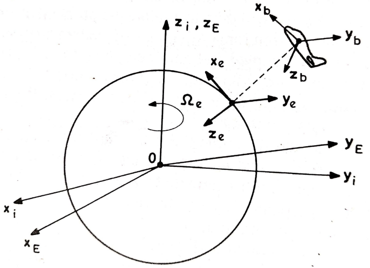
$$\theta \in \begin{cases} [0, \pi/2] & C_{13} \leq 0 \\ [-\pi/2, 0] & C_{13} \geq 0 \end{cases}$$

- What about bank angle ϕ?
- As $C_{33} = \cos\phi \cos\theta$, and $\cos\theta > 0$, sign of C_{33} is same as that of $\cos\phi$.
- Also, $C_{23} = \sin\phi \cos\theta$, and $\cos\theta > 0$, sign of C_{23} is same as that of $\sin\phi$.

$$\phi \in \begin{cases} \text{First quadrant} & C_{33} > 0 \ \& \ C_{23} > 0 \\ \text{Second quadrant} & C_{33} < 0 \ \& \ C_{23} > 0 \\ \text{Third quadrant} & C_{33} < 0 \ \& \ C_{23} < 0 \\ \text{Fourth quadrant} & C_{33} < 0 \ \& \ C_{23} < 0 \end{cases}$$

- We can also obtain the quadrants of ψ using C_{11} and C_{12} in a similar way.

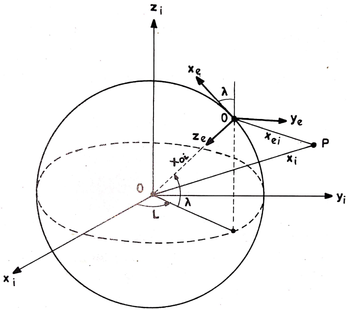
- Transformation between inertial and Earth-fixed system



- At $t = 0$, both systems coincide, and for $t > 0$, $\psi = \Omega_e t$.
- Transformation matrix

$$X_E = \begin{bmatrix} \cos\Omega_e t & \sin\Omega_e t & 0 \\ -\sin\Omega_e t & \cos\Omega_e t & 0 \\ 0 & 0 & 1 \end{bmatrix} X_i$$

- Transformation between inertial and navigation system



- What are the necessary rotations here?
- $\psi = L$, $\theta = -(\pi/2+\lambda), \phi = 0.$
- Transformation matrix

$$T_i^e = \begin{bmatrix} -\sin\lambda\cos L & -\sin\lambda\sin L & \cos\lambda \\ -\sin L & \cos L & 0 \\ -\cos\lambda\cos L & -\cos\lambda\sin L & -\sin\lambda \end{bmatrix}$$

Example

With respect to an Earth-centered inertial frame, a vehicle has the following velocity components

$$u_i(t) = u_{oi} + a_{xi}t,\ v_i(t) = a_{yi}t,\ w_i(t) = 0.$$

Assuming $u_{oi} = 100$ ft/s, $a_{xi} = 25$ ft/s^2, and $a_{yi} = 50$ ft/s^2, determine the position and velocity with reference to the Earth-fixed $(x_E y_E z_E)$ and navigation $(x_e y_e z_e)$ systems, at a time instant of 50 seconds. Assume that at $t = 0$, the vehicle is located on the equator with $L = 0$.

- At $t = 0$, $x_i = R_e = 2.0973364 \times 10^7$ ft, $y_i = z_i = 0$.
- Vehicle's coordinates

$$x_i(t) = x_{oi} + \int_0^{t_1} u_i dt = R_e + u_{0i}t_1 + \frac{1}{2}a_{xi}t_1^2, \quad y_i(t) = \frac{1}{2}a_{yi}t_1^2, \quad z_i = 0$$

- $x_i = R_e + 36250$ ft, $y_i = 62500$ ft, $z_i = 0$ ft

- Angular velocity of Earth about z-axis

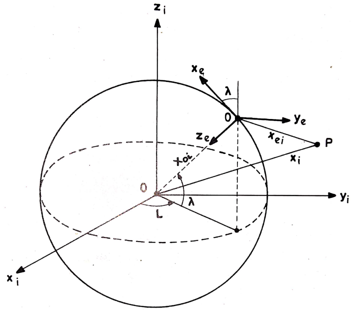
$$\Omega_e = \frac{2\pi}{24 \times 3600} = 0.7172 \times 10^{-4} \ \text{rad/s}$$

- Angle of rotation

$$\Omega_e t_1 = 0.7172 \times 10^{-4} \times 50 = 3.636 \times 10^{-3} \ \text{rad}$$

- Vehicle's coordinates in Earth-fixed axes

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} \cos\Omega_e t_1 & \sin\Omega_e t_1 & 0 \\ -\sin\Omega_e t_1 & \cos\Omega_e t_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 2.1009841 \times 10^7 \\ -13869.95 \\ 0 \end{bmatrix}$$



- Let X_i denotes the position of particle P in inertial frame of reference.
- One may write $X_i = X_{oi} + X_{ei} \implies X_{ei} = X_i - X_{oi}$.
- Components in navigation frame $X_e = T_i^e X_{ei}$

$$T_i^e = \begin{bmatrix} -\sin\lambda\cos L & -\sin\lambda\sin L & \cos\lambda \\ -\sin L & \cos L & 0 \\ -\cos\lambda\cos L & -\cos\lambda\sin L & -\sin\lambda \end{bmatrix}$$

- $L = 0°$, $\lambda = 0°$
- $X_{oi} = \begin{bmatrix} R_e \\ 0 \\ 0 \end{bmatrix}$
- Transformation matrix

$$T_i^e = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- Vehicle's coordinates in navigation frame

$$\begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_e + 36250 \\ 62500 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_e \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 62500 \\ -36250 \end{bmatrix}$$

- Transformation from inertial system to body axes system

$$\boldsymbol{C} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

- Transformation from body axes system to inertial system

$$\boldsymbol{C} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}^T$$

- Transformation from wind axes system to body axes system
- What are the rotations involved here?
- First rotation is $\psi = -\beta$, second $\theta = \alpha$, third would be any ϕ

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos(-\beta) & \sin(-\beta) & 0 \\ -\sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \cos\beta\sin\alpha\sin\phi + \sin\beta\cos\phi & -\sin\beta\sin\alpha\sin\phi + \cos\beta\cos\phi & \cos\alpha\sin\phi \\ \cos\beta\sin\alpha\cos\phi - \sin\beta\sin\phi & -\sin\beta\sin\alpha\cos\phi - \cos\beta\sin\phi & \cos\alpha\cos\phi \end{bmatrix}$$

- What are the components of velocity in the body axes frame?
- $\boldsymbol{V}_w = [V \quad 0 \quad 0]^T$
- Velocity components in body frame

$$\boldsymbol{V}_b = \boldsymbol{C}\boldsymbol{V}_w = \boldsymbol{C} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta \\ \cos\beta\sin\alpha\sin\phi + \sin\beta\cos\phi \\ \cos\beta\sin\alpha\cos\phi - \sin\beta\sin\phi \end{bmatrix} V$$

- Conventional definition of angle of attack and sideslip

$$\alpha = \tan^{-1}\left(\frac{w}{u}\right), \quad \beta = \sin^{-1}\left(\frac{v}{V}\right)$$

- This definition holds only if $\phi = 0$
- For $\phi \neq 0$, a part of angle of attack gets converted to sideslip.
- Effective α reduces and effective β is not zero, even for $\beta = 0$.
- For $\beta = 0, \phi \neq 0$,

$$\alpha_{\text{eff}} = \tan^{-1}\left(\frac{w}{u}\right) = \tan^{-1}(\tan\alpha\cos\phi)$$

$$\beta_{\text{eff}} = \sin^{-1}\left(\frac{v}{V}\right) = \sin^{-1}(\sin\alpha\sin\phi)$$

- For $\phi = 90^\circ$, $\alpha_{\text{eff}} = 0$, $\beta_{\text{eff}} = \alpha$
- All angle of attack gets converted to sideslip.

Example

An aircraft is tested in a low-speed wind tunnel at an angle of attack of 20°, sideslip of 10°, and a bank angle of 10°. An internal strain gage balance was used to measure the aerodynamic forces acting on the model, which gives components of force in the body axes system. The measurements are $F_x = 21.7$ lb, $F_y = -33$ lb, and $F_z = -91$ lb. Determine the transformation matrix from body to wind axes, and lift, drag, and side forces acting on the model.

- Transformation matrix

$$\boldsymbol{C}_w^b = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \cos\beta\sin\alpha\sin\phi + \sin\beta\cos\phi & -\sin\beta\sin\alpha\sin\phi + \cos\beta\cos\phi & \cos\alpha\sin\phi \\ \cos\beta\sin\alpha\cos\phi - \sin\beta\sin\phi & -\sin\beta\sin\alpha\cos\phi - \cos\beta\sin\phi & \cos\alpha\cos\phi \end{bmatrix}$$

- $\boldsymbol{C}_b^w = (\boldsymbol{C}_w^b)^T$

- Transformation matrix

$$\boldsymbol{C}_b^w = \begin{bmatrix} 0.9254 & 0.3188 & 0.2049 \\ -0.1631 & 0.8232 & -0.3882 \\ -0.3420 & 0.4698 & 0.8138 \end{bmatrix}$$

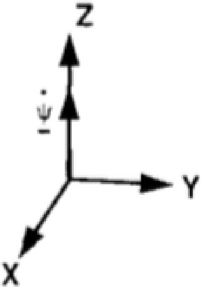
- Force components

$$\begin{bmatrix} F_{xw} \\ F_{yw} \\ F_{zw} \end{bmatrix} = \begin{bmatrix} 0.9254 & 0.3188 & 0.2049 \\ -0.1631 & 0.8232 & -0.3882 \\ -0.3420 & 0.4698 & 0.8138 \end{bmatrix} \begin{bmatrix} 21.7 \\ -33 \\ -91 \end{bmatrix} = \begin{bmatrix} -9.0809 \\ 4.6226 \\ -96.9833 \end{bmatrix}$$

- What are the forces (lift, drag, and side force) on aircraft?
- Lift $L = 96.9833$, Drag $D = 9.08909$, Side force $Y = 4.6226$.

- Similar to DCM orientation, Euler angles also vary with time when an input angular velocity vector is applied between the two reference frames.
- Angular velocity vector ω, in body-fixed coordinate system, has components p, q, and r in the x, y, and z directions, respectively.
- Consider each derivative of an Euler angle as the magnitude of the angular velocity vector in the coordinate system in which the angle is defined.
- For example, $\dot{\psi}$ is the magnitude of $\dot{\boldsymbol{\psi}}$ that lies along Z axis of the Earth-fixed coordinate system.

$$\dot{\boldsymbol{\psi}} = \begin{bmatrix} \dot{\psi}_x \\ \dot{\psi}_y \\ \dot{\psi}_z \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi \\ \dot{\psi}\cos\theta\cos\phi \end{bmatrix}$$

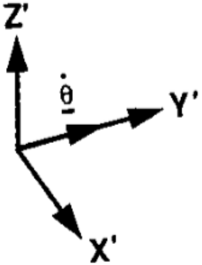


- Similarly, the components of $\dot{\boldsymbol{\theta}}$ in $X'Y'Z'$ are given by $(0, \dot{\theta}, 0)^T$.
- In body frame, it can be obtained as

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \boldsymbol{DB} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dot{\theta}\cos\phi \\ -\dot{\theta}\sin\phi \end{bmatrix}$$

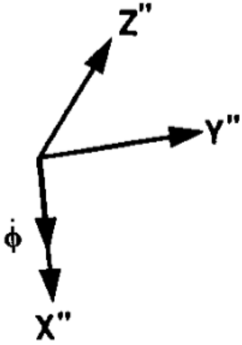


- Similarly, the components of $\dot{\boldsymbol{\phi}}$ in $X^{''}Y^{''}Z^{''}$ are given by $(\dot{\psi}, 0, 0)^T$.
- In body frame, it can be obtained as

$$\dot{\boldsymbol{\phi}} = \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \boldsymbol{D} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$



- Components of $\boldsymbol{\omega}$ in body-fixed coordinate system is given by

$$\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}}$$

- Now, we have

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\psi}_x + \dot{\theta}_x + \dot{\phi}_x \\ \dot{\psi}_y + \dot{\theta}_y + \dot{\phi}_y \\ \dot{\psi}_z + \dot{\theta}_z + \dot{\phi}_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi \\ \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \end{bmatrix}$$

- Euler angle rates

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{q\sin\phi + r\cos\phi}{\cos\theta} \\ q\cos\phi - r\sin\phi \\ p + \tan\theta(q\sin\phi + r\cos\phi) \end{bmatrix}$$

- What happen when $\theta = \pm 90^\circ$? Gimbal lock problem
- How to avoid such difficulties? Nonsingular representation, e.g., quaternions

- For $\theta = \pi/2$, $p = \dot{\phi} - \dot{\psi},\ q = \dot{\theta}\cos\phi,\ r = -\dot{\theta}\sin\phi$
- Azimuth and elevation rates

$$\dot{\psi} = \frac{q\sin\phi + r\cos\phi}{\cos\theta} = \frac{\dot{\theta}\cos\phi\sin\phi - \dot{\theta}\sin\phi\cos\phi}{\cos\theta} = \frac{0}{0}$$

$$\dot{\phi} = \ p + \frac{\sin\theta(q\sin\phi + r\cos\phi)}{\cos\theta} = p + \frac{0}{0}$$

Indeterminate forms!!!

- Using L'Hospital rule, and the fact that $\frac{d()}{d\theta} = \frac{d()}{dt}\frac{dt}{d\theta}$, we have

$$\begin{aligned}
\dot{\psi}|_{\theta=\pi/2} &= \lim_{\theta\to\pi/2} \frac{\frac{d}{d\theta}\left(q\sin\phi + r\cos\phi\right)}{\frac{d(\cos\theta)}{d\theta}} \\
&= \lim_{\theta\to\pi/2} \frac{\dot{q}\sin\phi + q\cos\phi\dot{\phi} - r\sin\phi\dot{\phi} + \dot{r}\cos\phi}{-\dot{\theta}\sin\theta} \\
&= -\frac{\dot{q}\sin\phi + \dot{r}\cos\phi + \dot{\phi}\dot{\theta}}{\dot{\theta}}
\end{aligned}$$

- Also, for $\theta = \pi/2$, $p = \dot{\phi} - \dot{\psi}$

$$\dot{\phi}|_{\theta=\pi/2} = p + \dot{\psi}|_{\theta=\pi/2} = \; p - \frac{\dot{q}\sin\phi + \dot{r}\cos\phi + \dot{\phi}\dot{\theta}}{\dot{\theta}}$$

- On solving this equation,

$$\dot{\phi}|_{\theta=\pi/2} = \frac{p}{2} - \frac{\dot{q}\sin\phi + \dot{r}\cos\phi}{2\dot{\theta}}$$

- Also, $\dot{\theta} = q\cos\phi - r\sin\phi$.
- For $\theta \approx \pi/2$, use these limiting values, else use the usual update equations.

Homework

Find out the relation between angular velocities in wind and body axes system

Reference

1. Bernard Etkin and Llyod Duff Reid, *Dynamics of Flight Stability and Control*, John Wiley and Sons, Third Edition, 1996.
2. Bandhu N. Pamadi, *Performance, Stability, and Control of Airplanes*, AIAA Education Series, 1998.
3. George M. Siouris, *Aerospace Avionics Systems: A Modern Synthesis*, Academic Press, Inc. 1993.

Thank you for your attention !!!