TOF Concept For General Conics

General Conic Problem

Δt expressions, derived **earlier,** require **knowledge** of nature of **conic** section and calculation of 'θ'.

However, in general, such **features** are not **known** and what are **actually** known are measurements of **'t' & ϕ'.**

Lambert's Theorem

In 1761, **J.H. Lambert** postulated a **theorem** that the **time** to traverse an **arc** on a **conic** section is a function only of (1) **'a',** (2) $\mathbf{r_1 + r_2}$ and (3) **chord** length of arc**.**

Lambert's **theorem** is an important aid to **characterize** the trajectories, not only of **space** objects, but also of ballistic **missiles.**

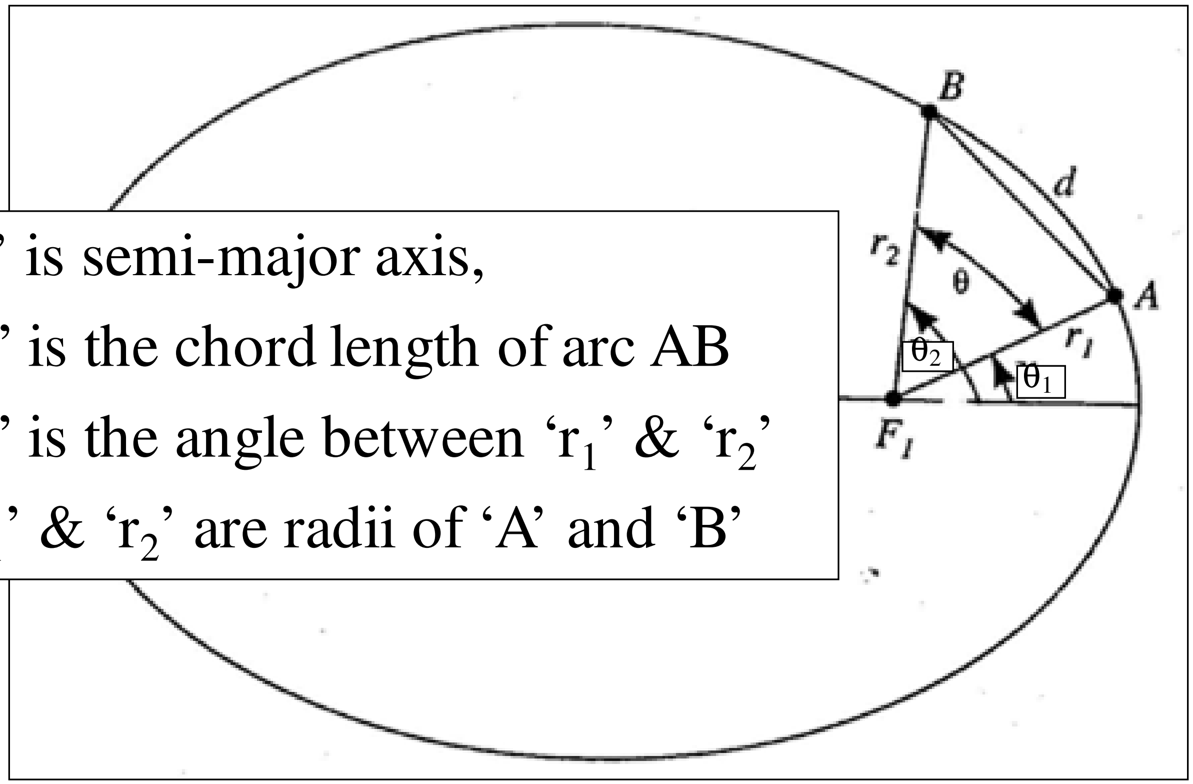
Lambert's Theorem

The above **theorem** was mathematically **derived** using a formulation given by **Lagrange.**

Basic idea of the theorem is to **observe** a small **segment** of conic **visible** from earth & **reconstruct** the trajectory.

Lambert's Δt Formulation for Ellipse

Consider **ellipse** as a representative **conic,** shown below.



'a' is semi-major axis,

'd' is the chord length of arc AB

'θ' is the angle between 'r_1' & 'r_2'

'r_1' & 'r_2' are radii of 'A' and 'B'

Lambert's Δt Formulation for Ellipse

Δt between two points **'A' & 'B'**, using mean orbital velocity and **Kepler's** equation, can be written as ,

$$\Delta t = t_B - t_A = \sqrt{\frac{a^3}{\mu}}\left[(E_B - E_A) - e(\sin E_B - \sin E_A)\right]$$

$$= \sqrt{\frac{a^3}{\mu}}\left[(E_B - E_A) - 2e\cos\frac{1}{2}(E_B + E_A)\sin\frac{1}{2}(E_B - E_A)\right]$$

Lambert's Δt Formulation for Ellipse

In the above **equation**, we need to know **eccentricity** of the conic as well as **'θ'** at 'A' and 'B'.

Lagrange gave a formulation that used the **Lambert's** condition to predict orbit **characteristics**, without 'e', 'θ'.

Lambert's Δt Formulation for Ellipse

Let us **introduce** the following **notation.**

$$\cos\frac{1}{2}(\alpha+\beta)=e\cos\frac{1}{2}\left(E_B+E_A\right); \quad 0\le\alpha+\beta\le 2\pi$$

$$\alpha-\beta=\left(E_B-E_A\right); \quad 0\le\alpha-\beta\le 2\pi$$

Lambert's Δt Formulation for Ellipse

Substituting notation into **Δt** equation, we get the **following** expression for **Δt.**

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$

We see that both **'e' & 'E'** have been replaced with **α & β**. Next step is to **relate** α & β to **a, d & (r_1 + r_2).**

Lambert's Δt Solution for Ellipse

In this **context**, we can use the **auxiliary** circle relations to arrive at the **transformation** as follows.

$$\vec{r}_1 = r_1 \cos\theta_A \hat{e}_P + r_1 \sin\theta_A \hat{e}_Q; \quad \vec{r}_2 = r_2 \cos\theta_B \hat{e}_P + r_2 \sin\theta_B \hat{e}_Q$$

$$r_1 = a(1 - e\cos E_A); \quad r_2 = a(1 - e\cos E_B); \quad \theta = \theta_B - \theta_A$$

$$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta = 4a^2\left[1 - e^2\cos^2\frac{1}{2}(E_A + E_B)\right]\sin^2\frac{1}{2}(E_B - E_A)$$

$$d = a(\cos\beta - \cos\alpha); \quad r_1 + r_2 = 2a\left[1 - \frac{1}{2}(\cos\alpha + \cos\beta)\right]$$

$$\cos\alpha = 1 - \frac{r_1 + r_2 + d}{2a}; \quad \cos\beta = 1 - \frac{r_1 + r_2 - d}{2a}$$

Lambert's Ellipse Solution

We **ca**n now estimate '**e**', as shown below.

$$\alpha - \beta = \psi = E_B - E_A \rightarrow E_B = \psi + E_A; \quad \cos E_B = \cos E_A \cos\psi - \sin E_A \sin\psi$$

$$\frac{\cos E_B}{\cos E_A} = \cos\psi - \tan E_A \sin\psi; \quad \tan E_A = \frac{1}{\sin\psi}\left(\cos\psi - \frac{\cos E_B}{\cos E_A}\right)$$

$$r = a(1 - e\cos E) \rightarrow \cos E = \frac{a-r}{ae}; \quad \frac{\cos E_B}{\cos E_A} = \frac{a-r_2}{a-r_1}; \quad e = \frac{a-r_1}{a\cos E_A}$$

Lastly, knowing '**E**, we can obtain '**θ**'.

Lambert's TOF for Parabola/Hyperbola

Lambert's Solution for Parabola

Lambert's **Δt** solution is actually valid for **all** conic sections, even though we have **derived** it only for **ellipse.**

In case of **parabola**, we know that **a** → ∞, so that angles **α & β** are small, leading to following **simplification.**

$$\cos\alpha = 1-\frac{\alpha^2}{2} = 1-\frac{r_1+r_2+d}{2a} \rightarrow \alpha^2 \cong \frac{r_1+r_2+d}{a}$$

$$\cos\beta = 1-\frac{\beta^2}{2} = 1-\frac{r_1+r_2-d}{2a} \rightarrow \beta^2 \cong \frac{r_1+r_2-d}{a}$$

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left[(\alpha-\sin\alpha)-(\beta-\sin\beta)\right] \simeq \sqrt{\frac{a^3}{\mu}}\left[\frac{\alpha^3-\beta^3}{6}\right]$$

$$\Delta t \simeq \frac{1}{6\sqrt{\mu}}\left[\left(r_1+r_2+d\right)^{3/2}-\left(r_1+r_2-d\right)^{3/2}\right]$$

Lambert's Solution for Hyperbola

Similar to **Kepler's** equation for Δt, **Lambert's** solution for a **hyperbolic** path can be obtained **simply** by employing **hyperbolic** Δt solution, as shown below.

$$\cosh\alpha = 1 + \frac{r_1 + r_2 + d}{2(-a)}; \quad \cosh\beta = 1 + \frac{r_1 + r_2 - d}{2(-a)}$$

$$\sinh\frac{\alpha}{2} = \frac{1}{2}\sqrt{\frac{r_1 + r_2 + d}{(-a)}}; \quad \sinh\frac{\beta}{2} = \frac{1}{2}\sqrt{\frac{r_1 + r_2 - d}{(-a)}}$$

$$\Delta t = \sqrt{\frac{(-a)^3}{\mu}}\left[(\sinh\alpha - \alpha) - (\sinh\beta - \beta)\right]$$

Lambert's Solution for 'a' Estimation

Preceding discussion has assumed that '**a**' is always **known** for a trajectory.

However, in a **strict** sense, 'a' is **available** only after trajectory **parameters** are known.

Lambert's Solution for 'a' Estimation

This has resulted in a **methodology**, in which '**a**' is also **estimated** from the trajectory **observations**, involving both angles and **time.**

Summary

In **conclusion**, we see that Lambert's **hypothesis** is a useful methodology for **analyzing** trajectories of general conic **sections**.