Constant Specific Thrust Solution

Constant Specific Thrust Concept

While, **constant** burn rate **design** is the **simplest** to implement, it **results** in increasing forward **acceleration,** causing large compressive **loads** on the rocket**.**

A way to **avoid** such a **situation** is to **reduce** the thrust as mass **reduces** so that net **forward** acceleration remains within acceptable **bounds.**

Specific thrust, defined as (**T/m**), is the amount of **acceleration** that propulsion **generates** and by keeping it **constant**, the above objective is **broadly** met.

Constant (T/m) Formulation

Applicable **equations** are as given below.

$$\dot{V} = \frac{T}{m} - \tilde{g}\cos\theta = n_0\tilde{g} - \tilde{g}\cos\theta$$

$$\dot{\theta} = \frac{\tilde{g}\sin\theta}{V}; \quad \frac{dV}{V} = \left(-\cot\theta + n_0 \operatorname{cosec}\theta\right)d\theta$$

$$\frac{d\theta}{dt} = \frac{\tilde{g}\sin\theta}{V} \rightarrow dt = \frac{V}{\tilde{g}\sin\theta}d\theta$$

$$n_0\tilde{g} = -\frac{\dot{m}g_0 I_{sp}}{m} \rightarrow \frac{dm}{m} = -\frac{n_0\tilde{g}}{g_0 I_{sp}}dt$$

'V' Solution

The **solution** for velocity can be **obtained** as follows.

$$\int \frac{dV}{V} = -\int \frac{\cos\theta}{\sin\theta} d\theta + n_0 \int \frac{1}{\sin\theta} d\theta = -\int \frac{d(\sin\theta)}{\sin\theta} + n_0 \int \frac{1}{2} \frac{\sec^2 \theta/2}{\tan \theta/2} d\theta$$

$$\ln V = \ln \operatorname{cosec}\theta + n_0 \ln \tan(\theta/2) + C$$

$$V = k \frac{(\tan(\theta/2))^{n_0}}{\sin\theta} = k' \left[\tan^{n_0 - 1}\left(\theta/2\right) + \tan^{n_0 + 1}\left(\theta/2\right) \right]$$

$$k' = \frac{V_0}{\left[\left(\tan\left\{\theta_0/2\right\}\right)^{(n_0 - 1)} + \left(\tan\left\{\theta_0/2\right\}\right)^{(n_0 + 1)} \right]}$$

‘t_b’ Solution

Burn time solution can be **obtained** as follows.

$$\int dt = \int \frac{V d\theta}{\tilde{g} \sin\theta} \rightarrow t = \int \frac{k'\left[\tan^{n_0-1}\left(\theta/2\right) + \tan^{n_0+1}\left(\theta/2\right)\right]}{\tilde{g}\sin\theta} d\theta$$

$$t = \frac{k'}{2\tilde{g}} \int \left[\tan^{n_0-2}\left(\theta/2\right) + \tan^{n_0}\left(\theta/2\right)\right] \times \sec^2\left(\theta/2\right) d\theta$$

$$\Delta t = \frac{k'}{\tilde{g}} \left[\frac{\left(\tan\frac{\theta}{2}\right)^{\{n_0-1\}}}{\{n_0-1\}} + \frac{\left(\tan\frac{\theta}{2}\right)^{\{n_0+1\}}}{\{n_0+1\}} \right]_{\theta_0}^{\theta_b}$$

'm' Solution

Burn profile m(t), to ensure constant $\mathbf{n_0}$, is obtained as the **direct result** of assumption of (T/m) = n_0g.

$$\int \frac{dm}{m} = -\frac{n_0 \tilde{g}}{g_0 I_{sp}} \int dt \rightarrow \ln m = -\frac{n_0 \tilde{g}}{g_0 I_{sp}} t + C$$

$$\ln\left(\frac{m_0}{m}\right) = \left(\frac{n_0 \tilde{g}}{g_0 I_{sp}}\right)\Delta t; \quad \frac{m_0}{m} = e^{\left(\frac{n_0 \tilde{g}}{g_0 I_{sp}}\right)\Delta t}$$

'h' & 'x' Formulations

Altitude and horizontal distance **solutions** are obtained from the following **equations.**

$$\frac{dh}{dt} = V\cos\theta; \quad \frac{dh}{d\theta} = \frac{V\cos\theta}{\dot{\theta}}; \quad \dot{\theta} = \frac{\tilde{g}\sin\theta}{V}$$

$$h = \int dh = \int \frac{V\cos\theta}{\dot{\theta}} d\theta + C = \frac{1}{\tilde{g}} \int \frac{V^2\cos\theta}{\sin\theta} d\theta + C$$

$$\frac{dx}{dt} = V\sin\theta; \quad \frac{dx}{d\theta} = \frac{V\sin\theta}{\dot{\theta}}$$

$$x = \int dx = \int \frac{V\sin\theta}{\dot{\theta}} d\theta + C = \frac{1}{\tilde{g}} \int V^2 d\theta + C$$

'h' & 'x' Solutions

Following are the **applicable** 'h' and 'x' **solutions.**

$$h = \frac{k'^2}{2g}\left[\frac{\left(\tan\left\{\theta/2\right\}\right)^{2(n_0-1)}}{(n_0-1)} - \frac{\left(\tan\left\{\theta/2\right\}\right)^{2(n_0+2)}}{(n_0+2)}\right]_{\theta_0}^{\theta} + h_0$$

$$x = \frac{2k'^2}{g}\left[\frac{\left(\tan\left\{\theta/2\right\}\right)^{(2n_0-1)}}{(2n_0-1)} + \frac{\left(\tan\left\{\theta/2\right\}\right)^{(2n_0+1)}}{(2n_0+1)}\right]_{\theta_0}^{\theta} + x_0$$

Implication of n_0 Value

Conceptually, for a **boosting rocket**, 'n_0' can be any **positive** real number.

However, if we intend the **velocity** to increase continuously, the net **forward** acceleration must also be **positive** at all times.

Implication of n_0 Value

This is possible if ‘$\mathbf{n_0}$’ is greater than ‘**cos θ**’ at all times.

As **maximum** value of cos θ is **1,** it logically follows that ‘$\mathbf{n_0}$’ must also be greater than **1** at all times.

n_0 Degenerate Case

n_0 = 1 represents a **singularity** in the given time **solution**. This can be **handled** in the following manner.

$$V = k'\left[\tan^{n_0-1}\left(\theta/2\right) + \tan^{n_0+1}\left(\theta/2\right)\right]$$
$$= k'\left[1 + \tan^{2}\left(\theta/2\right)\right] = k'\sec^{2}\left(\theta/2\right)$$

n_0 Degenerate Case

We can now obtain the time **solution**, as follows.

$$t = \int \frac{k' \sec^2\left(\theta/2\right)}{\tilde{g} \sin\theta} d\theta + C$$

$$t = \frac{k'}{2\bar{g}} \int \left[\frac{\sec^2\left(\theta/2\right)}{\tan\left(\theta/2\right)} + \sec^2\left(\theta/2\right) \tan\left(\theta/2\right) \right] d\theta + C$$

$$\Delta t = \frac{k'}{2\tilde{g}} \left[2\ln\left(\tan\left(\theta/2\right)\right) + \sec^2\left(\theta/2\right) \right]_{\theta_0}^{\theta_1}$$

Generation of ‘**h**’ and ‘**x**’ profiles for the **degenerate** case is left as an **exercise** for the students.

Constant (T/m) Solution Features

Constant ‘n_0’ based **solutions** show that for **larger** ‘n_0’, it takes **more** time & **propellant** to achieve same **‘θ_b’.**

However, it also **results** in larger terminal **velocity.**

Typically, ‘$\mathbf{n_0}$’ is a design solution for achieving specific terminal **parameters**, under the overall **constraint** of the vehicle **structure.** Typical values of n_0 are **~1.0-1.6.**

Summary

Therefore, to **summarize**, constant specific thrust **case** is complex from point of **view** of both solution and **implementation**.

Also, we **note** that it is **non-intuitive** from a design perspective and **requires** rigorous analysis.

However, from a **practical** view-point, it is an extremely **useful** trajectory design option.