Optimal Staging Solution

Optimal Staging Solution Steps

The **procedure** for solving **optimal** rocket sizing problem is given below.

1. All the ‘**N**’ partial derivative **equations** are solved for ‘$\boldsymbol{\pi_i}$’ in terms of **Lagrange** parameter **‘λ’.**

2. Next, all **solutions** for ‘π_i’ are substituted into the **constraint** equation and value of ‘λ’ is obtained.

3. Once **‘λ’** is obtained, it is used to **obtain** all the **‘$\boldsymbol{\pi_i}$’.**

Optimal Velocity Solution

Given below is **solution** for maximizing $\mathbf{V}_*$ with m_* **constraint.**

$$H_V(\lambda,\pi_i) = -g_0 \sum_{i=1}^{n} I_{sp_i} \ln\left[\varepsilon_i + \left(1-\varepsilon_i\right)\pi_i\right] + \lambda\left(\ln \pi_* - \sum_{i=1}^{n} \ln \pi_i\right)$$

$$\frac{\partial H_V}{\partial \pi_i} = \frac{g_0 I_{spi}(1-\varepsilon_i)}{\varepsilon_i + (1-\varepsilon_i)\pi_i} + \frac{\lambda}{\pi_i} = 0; \quad \pi_i = \frac{-\lambda\varepsilon_i}{(1-\varepsilon_i)\left(\lambda + g_0 I_{spi}\right)}$$

$$\pi_{*-\mathrm{con}} = \prod_{i=1}^{n} \frac{-\lambda\varepsilon_i}{(1-\varepsilon_i)\left(\lambda + g_0 I_{spi}\right)}; \quad V_{*-\mathrm{optim}} = -g_0 \sum_{i=1}^{n} I_{sp_i} \ln\left[\varepsilon_i + \left(1-\varepsilon_i\right)\pi_i\right]$$

Here, known $\boldsymbol{\pi}_*$ **fixes** the value of ‘λ’.

Optimal Payload Ratio Solution

Given below is **solution** for maximizing $\mathbf{m}_*$ with V_* **constraint.**

$$H_\pi(\lambda,\pi_i) = \sum_{i=1}^{n} \ln \pi_i + \lambda\left(V_* + g_0 \sum_{i=1}^{n} I_{sp_i} \ln\left[\varepsilon_i + \left(1-\varepsilon_i\right)\pi_i\right]\right)$$

$$\frac{\partial H_\pi}{\partial \pi_i} = \frac{1}{\pi_i} + \frac{\lambda g_0 I_{spi}(1-\varepsilon_i)}{\varepsilon_i + (1-\varepsilon_i)\pi_i} = 0; \quad \pi_i = \frac{-\varepsilon_i}{(1-\varepsilon_i)\left(1+\lambda g_0 I_{spi}\right)}$$

$$V_{*-\text{con}} = -g_0 \sum_{i=1}^{n} I_{sp_i} \ln\left[\frac{\varepsilon_i \lambda g_0 I_{spi}}{\left(1+\lambda g_0 I_{spi}\right)}\right]; \quad \pi_{*-\text{optim}} = \prod_{i=1}^{n} \frac{-\varepsilon_i}{(1-\varepsilon_i)\left(1+\lambda g_0 I_{spi}\right)}$$

Here, known $\mathbf{V}_*$ **fixes** the value of ‘λ’.

Equal Stages - V_* Constraint Solution

$\varepsilon_i = \varepsilon$ and $I_{spi} = I_{sp}$.

$$\boxed{\beta = \frac{V_*}{N g_0 I_{sp}}; \quad \pi = \frac{e^{-\beta} - \varepsilon}{1-\varepsilon}; \quad \pi_* = \left(\frac{e^{-\beta} - \varepsilon}{1-\varepsilon} \right)^N = \pi^N}$$

Equal Stages - π_* Constraint Solution

$\varepsilon_i = \varepsilon$ and $I_{spi} = I_{sp}$.

$$\boxed{\pi = \sqrt[N]{\pi_*}; \quad V_* = -g_0 I_{sp} \, \mathrm{N} \ln\left\{\varepsilon + \pi(1-\varepsilon)\right\}}$$

Limitation of Lagrange Procedure

Lagrange multiplier based method requires the **solution** of 'λ', before we can **get** the solution for $\boldsymbol{\pi_i}$.

In addition, we find that **equation** for 'λ' is an **'Nth'** order algebraic equation, so that **solution** effort is higher for more **number** of stages.

Alternate Solution Methodology

Lastly, when **both** 'ε_i' and 'π_i' are **distinct**, the solution of the algebraic equation **requires** additional effort.

Therefore, it would be **useful** if we can set up a **simpler process,** which does not **compromise** significantly on the **accuracy.**

Summary

Therefore, to **summarize**, Lagrange's multiplier based **technique** is capable of providing optimal **multi-stage** solutions that are also in the **closed** form.

However, **we** also note that we **need** to solve a slightly more **complicated** N^{th} order algebraic **equation** for the Lagrange multiplier.