Trajectory Solution Under Gravity

Contribution of Gravity

Gravity, **imposes** a force opposite to **thrust,** reducing its **effectiveness**, and leading to lower mechanical **energy.**

Typically, **gravity** reduces the burnout velocity $\mathbf{V_b}$ for a given $(\mathbf{m_b/m_0})$ and vice versa, in **relation** to the ideal performance and **needs** to be accounted for in **design.**

Gravity Model for Sizing

In this context, it is **important** to note that this **reduction** in the terminal performance is also a **function** of the trajectory taken by the **rocket**.

However, for **initial** sizing of rocket, we **assume** the worst case **scenario**, which occurs for a **vertical** ascent, and hence, generally gives the **performance** lower bound.

Formulation for Effect of Gravity

Basic equation **governing** this effect is as given below.

$$\frac{dV}{dt} = -\frac{\dot{m}}{m} g_0 I_{sp} - \frac{\mu}{R^2}; \quad \frac{dR}{dt} = \frac{dh}{dt} = V(t); \quad R = R_E + h$$

We see that while we **need** 'R' for proceeding with the **solution**, it is available only after the **solution** is completed, leading to a **nonlinear** coupling.

In general, **such** equations are solved **using** an iterative procedure by **employing** a suitable numerical **technique**.

Solution Strategy

However, as **this** effect is secondary in **nature**, we can use **sea-level** value of gravity to generate an initial **solution** for both 'V', **'R'**, which is fairly **representative**.

Of course, **we** can now use the **above** solution to approximately **correct** the value of **gravity** and by using this **value**, we can improve the solution **accuracy**.

It is **found** that the above **process** converges quickly to the **exact** solution within a few such **cycles.**

Problem Re-formulation

Thus, **assuming** a constant sea-level **gravity**, we can **rewrite** the applicable **equations**, as follows.

$$\frac{dV}{dt} = -\frac{\dot{m}}{m} g_0 I_{sp} - \tilde{g}; \quad h(t) = \int V dt; \quad \tilde{g} = g_0$$

The **solution,** so obtained can then be **corrected** by determining the **new** value of **'g'** for the next **cycle.**

Velocity Solution

The **velocity** solution, in this technique, is as **follows**.

$$V(t) = g_0 I_{sp} \ln \frac{m_0}{m} - g_0 t; \quad V_b = g_0 I_{sp} \ln \frac{m_0}{m_b} - g_0 t_b; \quad m_p = -\int_0^{t_b} \dot{m} dt$$

$$V_b = g_0 I_{sp} \ln \frac{m_0}{m_0 - m_p} - g_0 t_b; \quad m_p, m_b \rightarrow \text{Propellant, Burnout Masses}$$

Velocity Solution Features

It is seen that **velocity** solution now needs burn **time**, which depends on **burn** rate profile.

In **launch** vehicle design, **burn** rate profile is a **design** solution / decision, either **specified**, or obtained through optimal **techniques.**

Constant Burn Rate 'V' Solution

While, **there** are many different **burn** profiles possible, the simplest is for a **constant** rate β, which is also **easy** to implement in **solid** rocket motors.

Thus, for a **constant** burn rate**,** burn **time** & burnout velocity can be **obtained** as follows.

$$m(t) = m_0 - \beta t; \quad t_b = \frac{m_p}{\beta}; \quad V_b = g_0 I_{sp} \ln \frac{m_0}{(m_0 - m_p)} - \tilde{g}\left(\frac{m_p}{\beta}\right)$$

Constant Burn Rate 'h' Solution

The **altitude** solution is now obtained as **follows**.

$$h(t) = \int V(t)dt = \int \left(g_0 I_{sp} \ln \frac{m_0}{m(t)} - \tilde{g}t \right) dt + V_0 t + C$$

$$h(t) = g_0 I_{sp} \int \ln \frac{m_0}{m_0 - \beta t} dt - \frac{1}{2}\tilde{g}t^2 + V_0 t + C; \quad \frac{m_0 - \beta t}{m_0} = x; \quad dt = -\frac{m_0}{\beta} dx$$

$$h(t) = \frac{m_0 g_0 I_{sp}}{\beta} \int \ln x \cdot dx - \frac{1}{2}\tilde{g}t^2 + V_0 t + C = \frac{m_0 g_0 I_{sp}}{\beta} [x \ln x - x] - \frac{1}{2}\tilde{g}t^2 + V_0 t + C$$

$$h(t) = \frac{m_0 g_0 I_{sp}}{\beta} \left[\left(1 - \frac{\beta}{m_0} t\right) \ln\left(1 - \frac{\beta}{m_0} t\right) - \left(1 - \frac{\beta}{m_0} t\right) \right] - \frac{1}{2}\tilde{g}t^2 + V_0 t + C, \quad t_0 = 0$$

$$h_b = \frac{m_0 g_0 I_{sp}}{\beta} [(1-\Lambda)\ln(1-\Lambda) + \Lambda] - \frac{1}{2}\tilde{g}\left(\Lambda \frac{m_0}{\beta}\right)^2 + V_0 \Lambda \frac{m_0}{\beta} + h_0; \quad \Lambda = \frac{m_p}{m_0}$$

Let us **understand** its implication through an **example**.

Constant 'g' & 'β' Example

$\mathbf{m_0}$ = 80T, m_p = 60T, I_{sp} = 240 s, $\mathbf{g_0}$ = 9.81m/s^2, R_E = 6371 km, β = 600 kg/s.

Determine V_b, h_b, for **sea-level** 'g' and compare the **values** with the ideal burnout **solution**.

$$V_{ideal} = 3264 \text{ m/s}; \quad \Lambda = \frac{60}{80} = 0.75$$

$$V_b = V_{ideal} - \tilde{g}\left(\frac{m_p}{\beta}\right) = 3264 - 9.81 \times \frac{60000}{600} = 2283 m/s$$

$$h_b = \frac{80 \times 9.81 \times 240}{0.6}\left[0.25 \ln 0.25 + 0.75\right] - \frac{1}{2} 9.81 (100)^2 = 77600 m$$

Constant 'g' & 'β' Example

Next, obtain **corrected** values of terminal parameters for the **'g'** applicable for '$\mathbf{h_b}$' and comment on the **result**.

$$g_b = \frac{g_0}{\left(1 + {h_b}/{R_E}\right)^2} = 9.575 m/s^2; \quad \tilde{g} = \frac{9.81 + 9.575}{2} = 9.693 m/s^2$$

$$V_b^{'} = 2295 m/s; \quad h_b^{'} = 78300 m$$

We see that correction of **gravity** provides marginally different **results**, which are also better, **indicating** that sea-level gravity **solutions** are quite reasonable.

Summary

Thus, to **summarize**, gravity makes the **system** of equations nonlinear, which **require** numerical solution.

However, as its **contribution** is secondary, a simplified **approach** based on sea-level **gravity** provides reasonable results for **rocket** terminal performance.