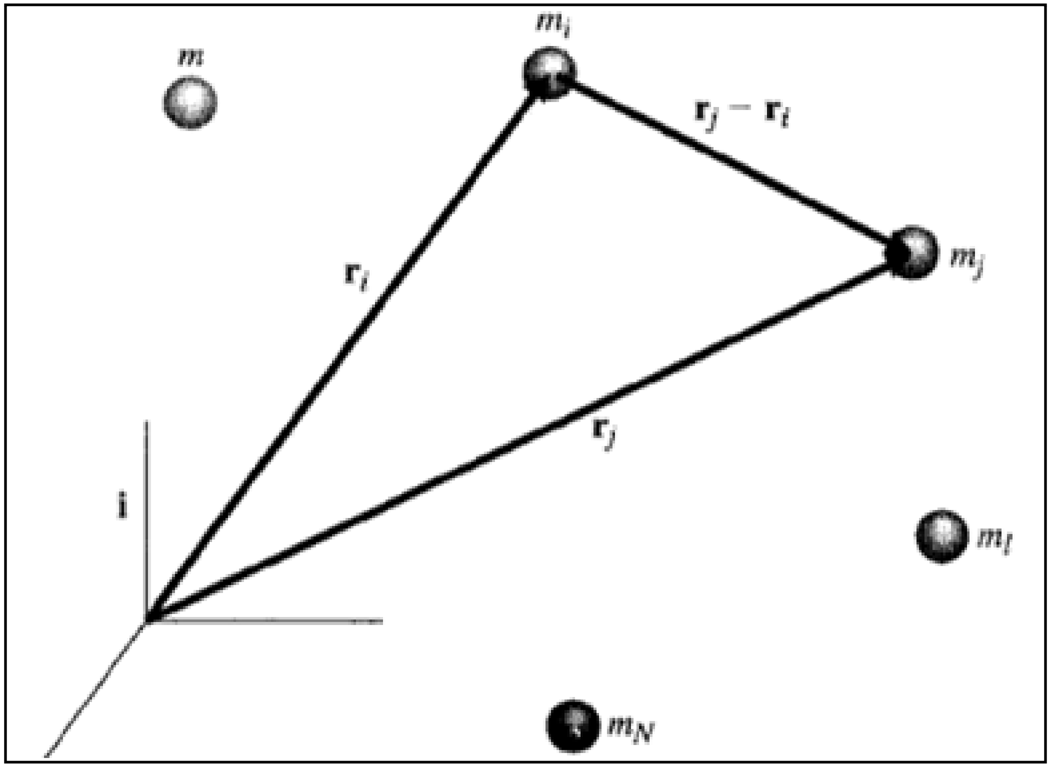
N-Body Problem Definition

Basic N – Body Gravitational System

Consider a **system** of 'N' bodies, **moving** with respect to an **arbitrarily** chosen inertial frame, under **gravity** force, as shown **alongside**.



Applicable Equations of Motion

Motion of **i^{th} particle**, can be **described** through the following **equation.**

$$m_i \ddot{\vec{r}}_i = G \sum_{j=1}^{N} \delta_{ij}^* \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij}; \quad \delta_{ij}^* = 1 - \delta_{ij}; \quad \vec{r}_{ij} = \vec{r}_j - \vec{r}_i = -\vec{r}_{ji}$$

Here, 'G' is the **universal** gravitational constant and δ_{ij} is the **kronecker delta.**

Applicable Equations of Motion

There are **(N – 1)** force terms in **RHS**, resulting in **3N** scalar **distances**, which determine **1** inertial distance.

We see that it is a **2^{nd} order** vector differential **equation,** which needs **6** initial conditions for **solution.**

Lastly, there are **N** inertial distances that **need** solution.

N-Body Problem Solution

Implicit Solution Philosophy

While, **explicit** solution of these **equations** is a numerically **intensive** exercise, we can get **useful** insight into the **solution** features through **implicit** methods.

In **implicit** technique, we solve these **equations** together as a **system** and arrive at the solution **features** of the group of **particles** under certain **simplifying** assumptions.

Implicit Solution Strategy

In this **context**, we note that terms in **RHS** in all N **equations** are the same, but with **opposite** signs, so that we can **define** a sum, as shown **below.**

$$\sum_{i=1}^{N} \mathrm{m}_i \ddot{\vec{r}}_i = \sum_{i=1}^{N} G \sum_{j=1}^{N} \delta_{ij}^{*} \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} = 0 \quad \left(\vec{r}_{ij} = -\vec{r}_{ji} \right)$$

We see that in the **absence** of any other **force**, all the gravitational **forces** add up to **zero.**

N – Body Motion Solution

This system of **equation** can be **symbolically** integrated **twice**, as shown below.

$$\int \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{N} m_i \vec{r}_i = \vec{c}_1 t + \vec{c}_2; \quad \vec{c}_1, \vec{c}_2 \rightarrow \text{Integration Constants}$$

The **result** shows that **sum** of all the 1^{st} **moments** is either **constant** or varies linearly with **time.**

N – Body Mass Solution

As 1st **moments** are directly **related** to the concept of **centre of mass**, we can rewrite the **solution** as follows.

$$\vec{r}_c = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M} = \frac{\vec{c}_1 t + \vec{c}_2}{M}; \quad M = \sum_{i=1}^{N} m_i; \quad \left(\vec{c}_1 t + \vec{c}_2\right) = M\vec{r}_c \rightarrow M\ddot{\vec{r}}_c = 0$$

The **above** solution indicates that **centre of mass** is either **stationary** or moves with **uniform** velocity, which is **valid** under the **condition** of mass **conservation.**

Solution for Momentum & Energy

As we **know** that in the presence of a **conservative** force field, **both** momentum and energy are also **conserved**, we can extend the **implicit** solution method for **these** as well.

This is **achieved** through basic **definitions** of momentum and energy **through** vector & scalar **products.**

Momentum as Vector Product

Consider the **vector** product, as shown **below.**

$$\vec{r}_i \times m_i \ddot{\vec{r}}_i = \dot{\vec{h}}_i \rightarrow \text{ Torque or Moment For i}^{th} \text{ Particle}$$

$$\dot{\vec{H}} = \sum_{i=1}^{N} \dot{\vec{h}}_i = \sum_{i=1}^{N} \vec{r}_i \times m_i \ddot{\vec{r}}_i = \sum_{i=1}^{N} m_i \left(\vec{r}_i \times \ddot{\vec{r}}_i + \dot{\vec{r}}_i \times \dot{\vec{r}}_i \right) = \sum_{i=1}^{N} m_i \frac{d}{dt}\left[\vec{r}_i \times \dot{\vec{r}}_i \right]$$

$$\dot{\vec{H}} = \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{r}_i \times \delta_{ij}^* \frac{G m_i m_j}{r_{ij}^3} \left(\vec{r}_j - \vec{r}_i \right) = \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij}^* \frac{G m_i m_j}{r_{ij}^3} \vec{r}_i \times \left(\vec{r}_j - \vec{r}_i \right) = 0$$

$$\sum_{i=1}^{N} m_i \frac{d}{dt}\left[\vec{r}_i \times \dot{\vec{r}}_i \right] = 0 \rightarrow \sum_{i=1}^{N} \left[\vec{r}_i \times \left(m_i \dot{\vec{r}}_i \right) \right] = \vec{H} = \text{ Constant}$$

This is a **statement** of angular **momentum** conservation.

Kinetic Energy as Scalar Product

Following is the **expression** for kinetic energy.

$$\sum_{i=1}^{N} \dot{\bar{r}}_i \cdot m_i \ddot{\bar{r}}_i = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N} m_i \left[\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i \right] = \frac{d}{dt}\sum_{i=1}^{N} \frac{1}{2} m_i r_i^2 = \frac{d}{dt}(T)$$

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i r_i^2 \rightarrow \text{Kinetic Energy}$$

Potential Energy as Scalar Product

Following is the **Expression** for potential energy**.**

$$\left[\dot{\vec{r}}_i \cdot \left(\vec{r}_j - \vec{r}_i\right)\right] + \left[\dot{\vec{r}}_j \cdot \left(\vec{r}_i - \vec{r}_j\right)\right] = -\left(\dot{\vec{r}}_j - \dot{\vec{r}}_i\right) \cdot \left(\vec{r}_j - \vec{r}_i\right) = -\dot{\vec{r}}_{ij} \cdot \vec{r}_{ij}$$

$$\dot{\vec{r}}_i \cdot \left(\vec{r}_j - \vec{r}_i\right) = \frac{1}{2}\left(-\dot{\vec{r}}_{ij} \cdot \vec{r}_{ij}\right); \quad \frac{d}{dt}(T) = -\frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} G\delta_{ij}^{*}\frac{m_i m_j}{r_{ij}^{3}}\left(\dot{\vec{r}}_{ij} \cdot \vec{r}_{ij}\right)\right)$$

$$\frac{d}{dt}(T) = -\frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} G\delta_{ij}^{*}\frac{m_i m_j}{r_{ij}^{2}}\dot{r}_{ij}\right) = \frac{1}{2}\frac{d}{dt}\left[\sum_{i=1}^{N}\left(\sum_{j=1}^{N} G\delta_{ij}^{*}\frac{m_i m_j}{r_{ij}}\right)\right] = -\frac{d}{dt}(-V)$$

$$V = -\sum_{i=1}^{N}\left(\sum_{j>i}^{N} G\frac{m_i m_j}{r_{ij}}\right) \rightarrow \text{Potential Energy;} \quad \dot{E} = 0 \rightarrow T - V = \text{Constant}$$

N-Body Motion Simplification

As **gravitational** pull of a body depends on its **mass** and its proximity to another **body**, presence of **large** body close by can be considered as a **dominant** effect.

E.g., **sun** is not only the **largest** body in our solar **system** but also is the **closest** to all planets and therefore, planets' **motion** are determined primarily by **sun's** gravitation.

Two-body Model Concept

Similarly, as **moon** of Earth is closest to **Earth**, its orbit is **largely** decided by the **gravitational** pull only of **Earth**.

This has resulted in **approximate**, but simplified motion models for **Earth – satellite** system.

Two-body formulation is such a **simplification.**

Summary

The **most** general description of motion of **spacecraft** is based on the **N-body** concept.

The basic **formulation** for motion is derived from the **universal** law of gravitation, under the **assumption** of spherical symmetry.