Flight Mechanics/Dynamics

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- **Neutral point (NP)**: A CG position h, for which $C_{m_\alpha} = 0$. Also known as "vehicle aerodynamic center".
- A boundary between positive and negative pitch stiffness
- NP is given by

$$\boxed{h_n = h_{n_{wb}} - \frac{1}{C_{L_\alpha}}\left[\frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha} - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}\right]}$$

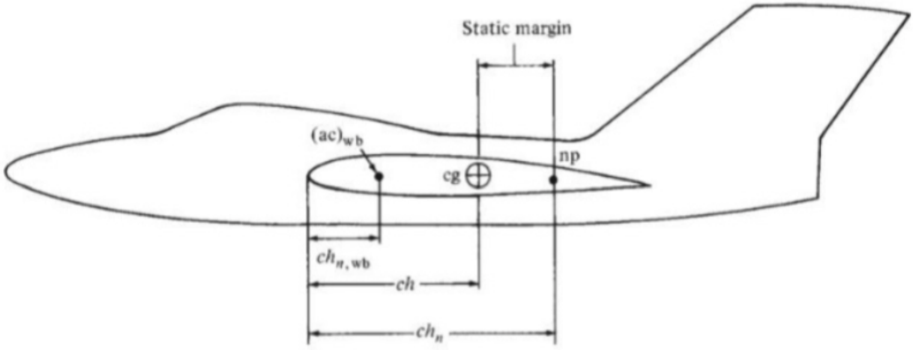
- C_{m_α} in terms of NP

$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h - h_n)}$$

- **Static margin**: Difference between CG position and NP

$$\boxed{K_n = h_n - h}$$

- $K_n > 0 \implies C_{m_\alpha} < 0$, a positive stiffness
- For positive stiffness, CG must be forward to NP



Static margin
(ac)wb
np
cg
chn,wb
ch
chn

- What if the lift and moment are linear in angle of attack?

$$C_{L_{wb}} = a_{wb}\alpha_{wb}, \quad C_{L_t} = a_t\alpha_t, \quad C_{m_p} = C_{m_{0_p}} + \frac{\partial C_{m_p}}{\partial \alpha}\alpha$$

- Angle of attack for tail

$$\alpha_t = \alpha_{wb} - i_t - \epsilon \implies C_{L_t} = a_t(\alpha_{wb} - i_t - \epsilon)$$

- Downwash angle

$$\boxed{\epsilon = \epsilon_0 + \frac{\partial \epsilon}{\partial \alpha}\alpha_{wb}}$$

- Tail lift coefficient

$$C_{L_t} = a_t\alpha_t = a_t\left[\alpha_{wb}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) - i_t - \epsilon_0\right]$$

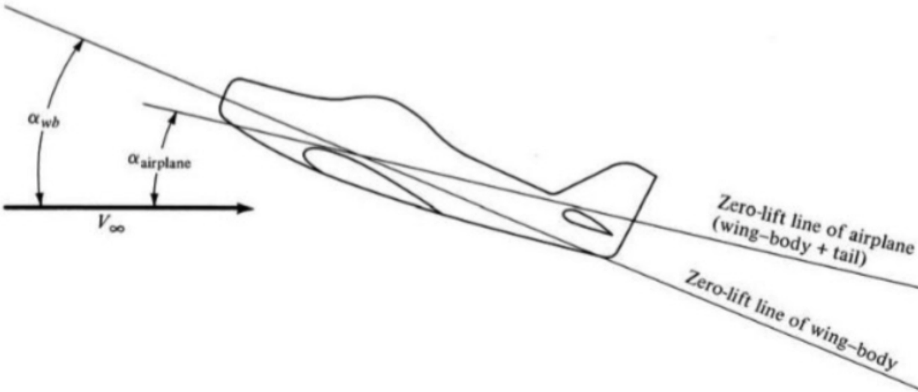
- Total lift coefficient

$$
\begin{aligned}
C_L &= C_{L_{wb}} + \frac{S_t}{S} C_{L_t} \\
&= \underbrace{a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]}_{a} \alpha_{wb} \underbrace{- a_t \frac{S_t}{S}(i_t + \epsilon_0)}_{C_{L_0}} \\
&= C_{L_0} + a\alpha_{wb}
\end{aligned}
$$

- As α_{wb} and α differ by a constant, $C_L = a\alpha$, where lift-curve slope

$$
\boxed{a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right],}
$$

and α is angle of attack of zero-lift line, of whole configuration.



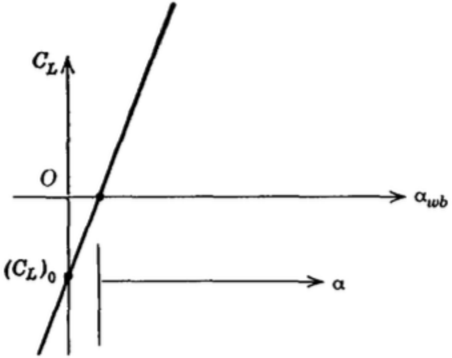
α_{wb}
$\alpha_{airplane}$
V_∞
Zero-lift line of airplane (wing–body + tail)
Zero-lift line of wing–body

- As $i_t > 0$,

$$C_{L_0} = -a_t \frac{S_t}{S}(i_t + \epsilon_0) < 0$$

- What would be the amount of difference in α and α_{wb}?

$$\alpha - \alpha_{wb} = -\frac{a_t}{a}\frac{S_t}{S}(i_t + \epsilon_0)$$



- For linear relations of C_L, C_{L_t}, C_{m_p}, we have

$$C_m = C_{m_0} + C_{m_\alpha}\alpha = \bar{C}_{m_0} + C_{m_\alpha}\alpha_{wb}$$

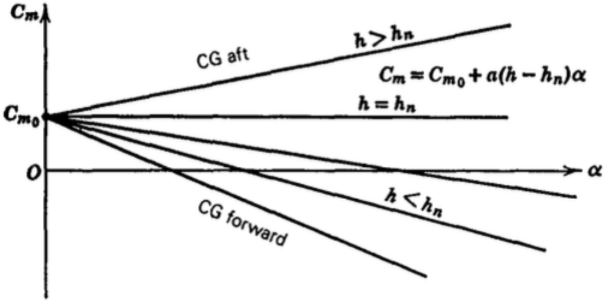
where

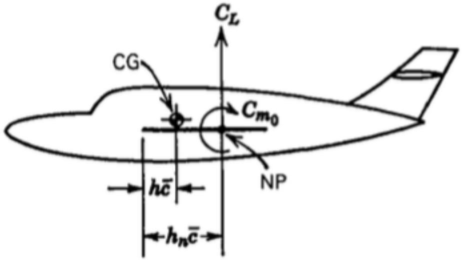
$$\begin{aligned}
C_{m_\alpha} =& a(h - h_{n_{wb}}) - a_t\bar{V}_H\left(1 - \frac{\epsilon}{\alpha}\right) + \frac{\partial C_{m_p}}{\partial \alpha} \\
=& a_{wb}(h - h_{n_{wb}}) - a_t V_H\left(1 - \frac{\epsilon}{\alpha}\right) + \frac{\partial C_{m_p}}{\partial \alpha} \\
C_{m_0} =& C_{m_{ac_{wb}}} + C_{m_{0_p}} + a_t\bar{V}_H(\epsilon_0 + i_t)\left[1 - \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right] \\
\bar{C}_{m_0} =& C_{m_{ac_{wb}}} + \bar{C}_{m_{0_p}} + a_t V_H(\epsilon_0 + i_t) \\
\bar{C}_{m_{0_p}} =& C_{m_{0_p}} + (\alpha - \alpha_{wb})\frac{\partial C_{m_p}}{\partial \alpha} \\
h_n =& h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial \alpha}
\end{aligned}$$

- What about dependence of $\bar{C}_{m_0}$ and C_{m_0} on CG location?
- As $\bar{C}_{m_0}$ is the pitching moment at zero α_{wb}, not zero total lift, its value depends on h (via V_H).
- C_{m_0} being the moment at zero total lift, represents a pure couple and is hence independent of CG position.
- Will the results remain applicable for tailless aircraft? Yes with $V_H = 0$.
- As $C_{m_\alpha} = C_{L_\alpha}(h - h_n)$, on integration we have

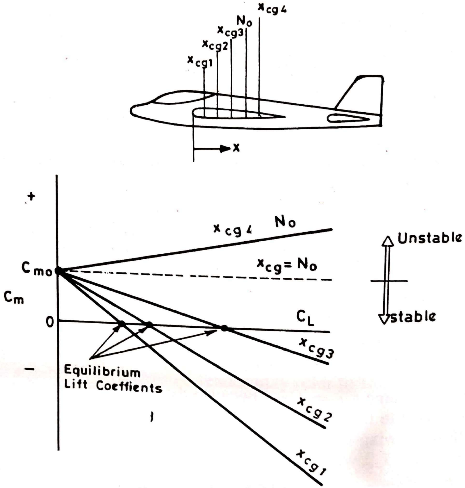
$$\begin{aligned} C_m &= C_{m_0} + C_L(h - h_n) \\ &= C_{m_0} + a\alpha(h - h_n) \end{aligned}$$

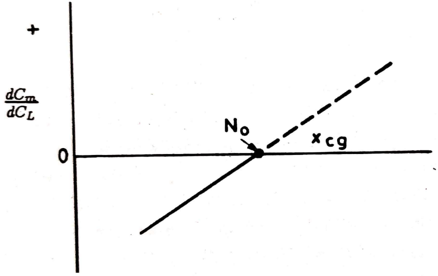
- C_{m_α} reduces to $\boxed{C_{m_\alpha} = a(h - h_n)}$

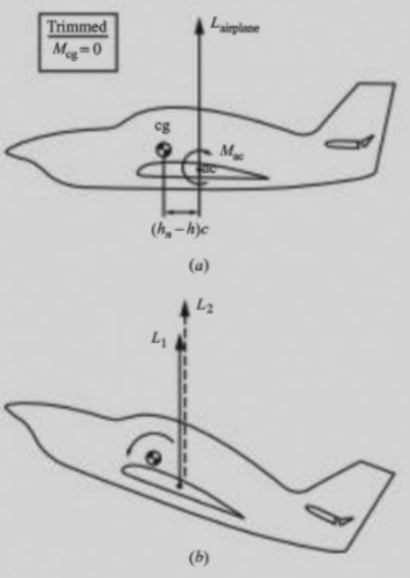

C_m
C_{m_0}
O
α
CG aft
$h > h_n$
$C_m = C_{m_0} + a(h - h_n)\alpha$
$h = h_n$
$h < h_n$
CG forward

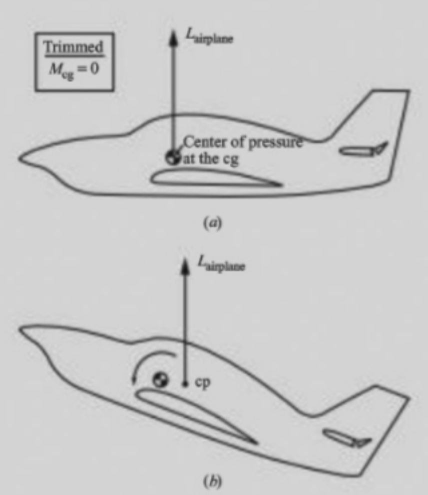

C_L
CG
C_{m_0}
NP
$h\bar{c}$
$h_n\bar{c}$

Effect of cg location on stability

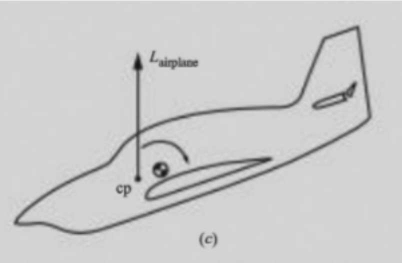





Trimmed
$M_{cg} = 0$
$L_{airplane}$
cg
M_{ac}
ac
$(h_n - h)c$
(a)
L_2
L_1
(b)

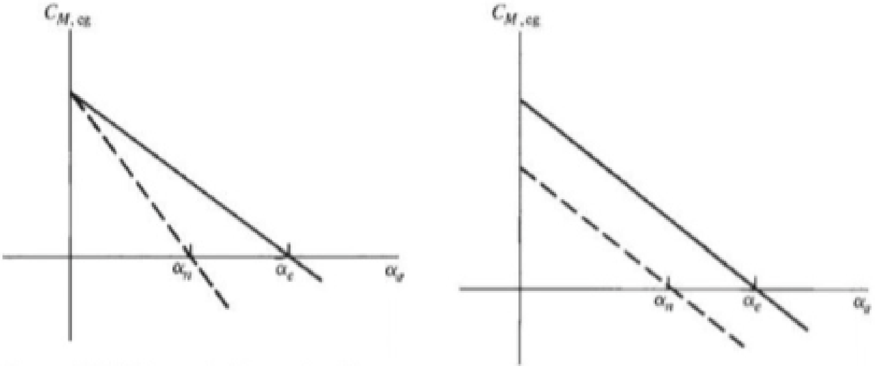


Trimmed
$M_{cg} = 0$
$L_{airplane}$
Center of pressure
at the cg
(a)
$L_{airplane}$
cp
(b)

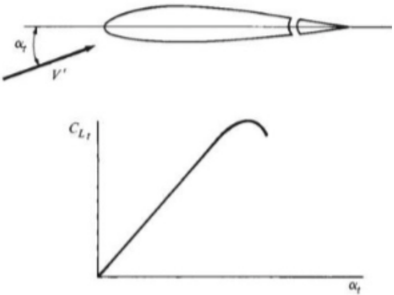


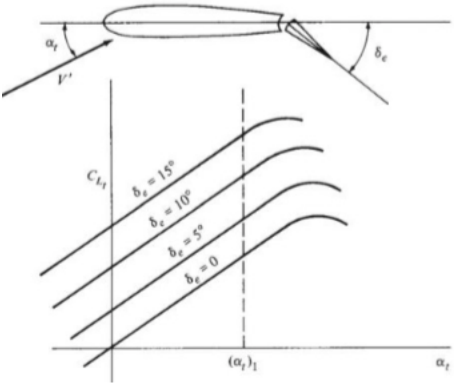
$L_{airplane}$
cp
(c)

- How to change the equilibrium state, trim angle, or aircraft speed?
- Is there a relation between trim angle and speed?
  - $\Rightarrow$ Speed is related to lift coefficient.
  - $\Rightarrow$ Lower lift coefficient implies higher speed.
- Possible ways to alter trim condition
  - $\Rightarrow$ A change in propulsive thrust
  - $\Rightarrow$ A change in configuration (aerodynamic controls)
  - $\Rightarrow$ Movement of the CG
- What is effect of moving CG forward?
- Moving CG forward reduces the trim α or C_L, and hence produces an increase in the trim speed.
- Disadvantages of changing CG location:
  - $\Rightarrow$ Practical difficulties
  - $\Rightarrow$ Reducing pitch stiffness and stability, with reduced trim speed How?



- Slope can be changed by changing CG
- Shifting of C_{m_0} can be done by changing configuration.


α_t
V'
C_{L_t}
α_t


α_t
V'
δ_e
C_{L_t}
$\delta_e = 15°$
$\delta_e = 10°$
$\delta_e = 5°$
$\delta_e = 0$
$(\alpha_t)_1$
α_t

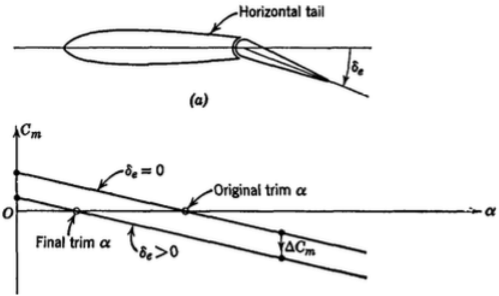
- Deflection of the elevator through an angle δ_e, produces changes in both C_m as well as C_L of the airplane.
- Lift and moment increments for both kinds of airplane (aircraft with tails or tailless one) are linear in δ_e.
- Increments in lift and moment coefficients

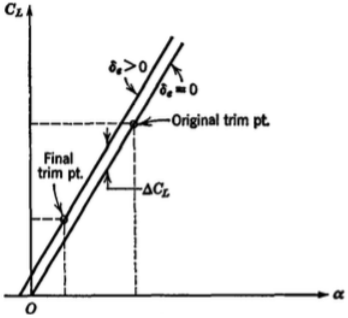
$$C_L = C_L(\alpha) + C_{L_{\delta_e}}\delta_e, \quad \Delta C_L = C_{L_{\delta_e}}\delta_e$$
$$C_m = C_m(\alpha) + C_{m_{\delta_e}}\delta_e, \quad \Delta C_m = C_{m_{\delta_e}}\delta_e$$

where $C_{L_{\delta_e}} = \dfrac{\partial C_L}{\partial \delta_e}$ and $C_{m_{\delta_e}} = \dfrac{\partial C_m}{\partial \delta_e}$

- $C_L(\alpha)$ and $C_m(\alpha)$ are lift and moment coefficients when $\delta_e = 0$.
- $\delta_e > 0$ when deflected downward $\Rightarrow C_{L_{\delta_e}} > 0, \; C_{m_{\delta_e}} < 0$
- In the case of linear lift and moment,

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta_e}}\delta_e$$
$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta_e}}\delta_e$$





C_m curve gets shifted down without change of slope, while zero-lift angle is changed.

- Lift coefficient of the tail

$$\boxed{C_{L_t} = a_t\alpha_t + a_e\delta_e}$$

- As $C_L = C_{L_{wb}} + \frac{S_t}{S}C_{L_t}$, we have

$$C_{L_{\delta_e}} = \frac{\partial C_L}{\partial \delta_e} = \frac{\partial C_{L_{wb}}}{\partial \delta_e} + \frac{S_t}{S}\frac{\partial C_{L_t}}{\partial \delta_e}$$

- Elevator lift effectiveness

$$\boxed{a_e = \frac{\partial C_{L_t}}{\partial \delta_e}} \Rightarrow \boxed{C_{L_{\delta_e}} = \frac{\partial C_{L_{wb}}}{\partial \delta_e} + a_e\frac{S_t}{S}}$$

- Total pitching moment coefficient

$$C_m = C_{m_{ac_{wb}}} + C_L(h - h_{n_{wb}}) - \bar{V}_H C_{L_t} + C_{m_p}$$

$$\Rightarrow C_{m_{\delta_e}} = \frac{\partial C_{m_{ac_{wb}}}}{\partial \delta_e} + C_{L_{\delta_e}}(h - h_{n_{wb}}) - \bar{V}_H\frac{\partial C_{L_t}}{\partial \delta_e} + \frac{\partial C_{m_p}}{\partial \delta_e}$$

Pitching moment coefficient derivative w.r.t. δ_e

$$C_{m_{\delta_e}} = \frac{\partial C_{m_{ac_{wb}}}}{\partial \delta_e} + C_{L_{\delta_e}}(h - h_{n_{wb}}) - a_e \bar{V}_H$$

Tailed aircraft

$$C_{L_{\delta_e}} = a_e \frac{S_t}{S}, \; C_{m_{\delta_e}} = C_{L_{\delta_e}}(h - h_{n_{wb}}) - a_e \bar{V}_H$$

Tailless aircraft

$$C_{L_{\delta_e}} = \frac{\partial C_L}{\partial \delta_e}, \; C_{m_{\delta_e}} = \frac{\partial C_{m_{ac}}}{\partial \delta_e} + C_{L_{\delta_e}}(h - h_n)$$

What are the important parameters to be measured?

- Trim condition: $C_m = 0$
- What would be required trim elevator deflection?

$$\boxed{\delta_{e_{\text{trim}}} = -\frac{C_m(\alpha)}{C_m\delta_e}}$$

- Lift coefficient

$$C_{L_{\text{trim}}} = C_L(\alpha) - \frac{C_{L_{\delta_e}}}{C_m\delta_e}C_m(\alpha)$$

- Can we obtain trim values of α and δ_e for linear case?

$$\begin{aligned} C_{L_\alpha}\alpha_{\text{trim}} + C_{L_{\delta_e}}\delta_{e_{\text{trim}}} =& C_{L_{\text{trim}}} \\ C_{m_\alpha}\alpha_{\text{trim}} + C_{m_{\delta_e}}\delta_{e_{\text{trim}}} =& -C_{m_0} \end{aligned}$$

- What would be trim values of angle of attack and elevator deflection?

$$\boxed{\alpha_{\text{trim}} = \frac{C_{m_0}C_{L_{\delta_e}} + C_{m_{\delta_e}}C_{L_{\text{trim}}}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}},\ \delta_{e_{\text{trim}}} = -\frac{C_{m_0}C_{L_\alpha} + C_{m_\alpha}C_{L_{\text{trim}}}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}}}$$

- We have

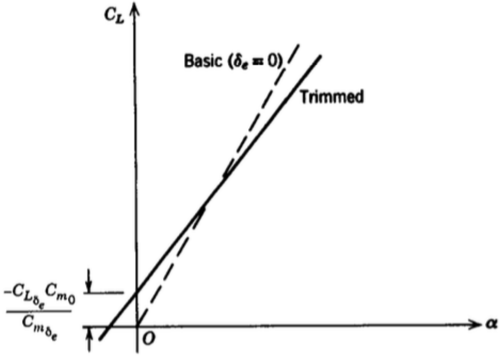
$$\frac{d\delta_{e_{\mathrm{trim}}}}{dC_{L_{\mathrm{trim}}}} = -\frac{C_{m_\alpha}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}} = -\frac{C_{L_\alpha}(h - h_n)}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}}$$

- Usually, $\Delta = C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}} < 0$
- Δ is independent of h

$$\Delta = \underbrace{C_{L_\alpha}[C_{L_{\delta_e}}(h_n - h_{n_{wb}}) - a_e\bar{V}_H]}_{\text{tailed}}, \quad \underbrace{C_{L_\alpha}\frac{\partial C_{m_{ac}}}{\partial \delta_e}}_{\text{tailless}}$$

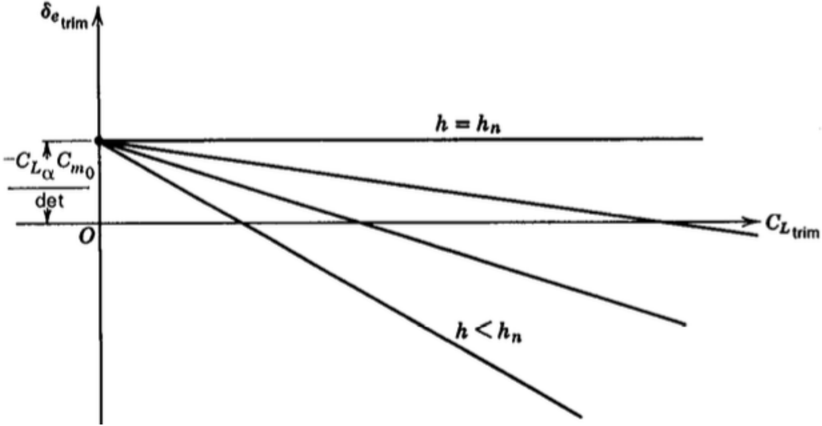
- Trimmed lift curve: $C_{L_{\mathrm{trim}}} = -\frac{C_{m_0}C_{L_{\delta_e}}}{C_{m_{\delta_e}}} + \frac{\Delta}{C_{m_{\delta_e}}}\alpha_{\mathrm{trim}}$

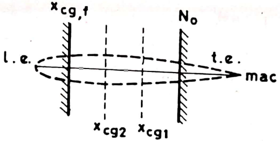
$$\left.\frac{dC_L}{d\alpha}\right|_{\text{trim}} = \frac{\Delta}{C_{m_{\delta_e}}} = C_{L_\alpha} - \frac{C_{L_{\delta_e}}}{C_{m_{\delta_e}}} C_{m_\alpha}$$

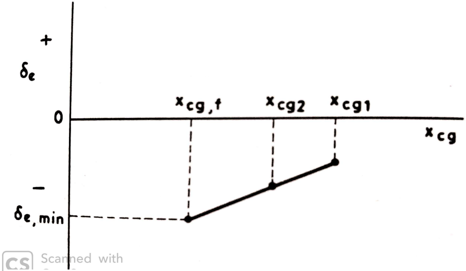


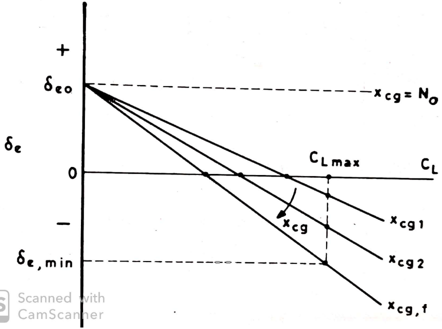
- $\left.\frac{dC_L}{d\alpha}\right|_{\text{trim}} < C_{L_\alpha}$
- Amount of reduction depends on C_{m_α} or static margin
- Reduction becomes zero when CG is at NP $h = h_n$

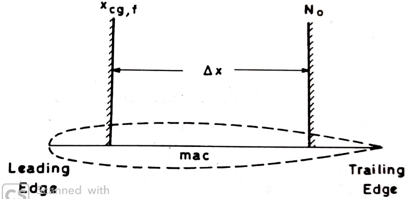
$$\boxed{\delta_{e_{\text{trim}}} = -\frac{C_{m_0}C_{L_\alpha} + C_{m_\alpha}C_{L_{\text{trim}}}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}}}$$




$x_{cg,f}$
N_o
l.e.
t.e.
mac
x_{cg2}
x_{cg1}


+
δ_e
0
$x_{cg,f}$
x_{cg2}
x_{cg1}
x_{cg}
−
$\delta_{e,min}$

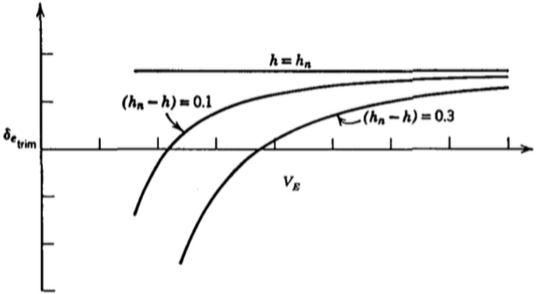

+
δ_{eo}
$x_{cg} = N_o$
δ_e
0
$C_{L\,max}$
C_L
−
x_{cg}
x_{cg1}
x_{cg2}
$\delta_{e,min}$
$x_{cg,f}$


$x_{cg,f}$
N_o
Δx
mac
Leading Edge
Trailing Edge

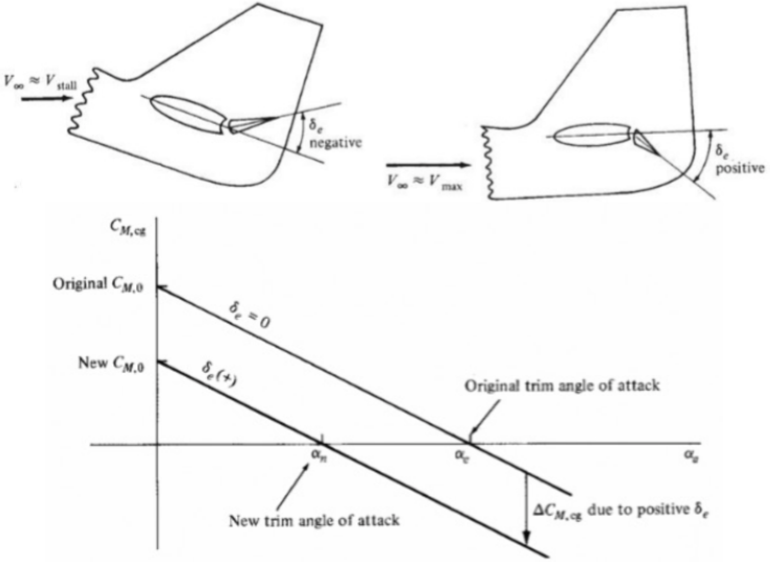
- Lift coefficient in trim condition, with equivalent airspeed V_E,

$$C_{L_{\text{trim}}} = \frac{W}{(1/2)\rho_0 V_E^2 S}$$

- $\delta_{e_{\text{trim}}}$ is a unique function of $C_{L_{\text{trim}}}$ and thus V_E.
- For any CG position, an increase in trim speed $\Rightarrow \delta_e > 0$.



- Gradient $\partial\delta_{e_{\text{trim}}}/\partial V_E$ decreases with rearward movement of the CG.
- $\partial\delta_{e_{\text{trim}}}/\partial V_E = 0$ at $h = h_n$.



$V_\infty \approx V_{stall}$
δ_e negative
$V_\infty \approx V_{max}$
δ_e positive
$C_{M,cg}$
Original $C_{M,0}$
$\delta_e = 0$
New $C_{M,0}$
$\delta_e (+)$
Original trim angle of attack
α_n
α_e
α_a
New trim angle of attack
$\Delta C_{M,cg}$ due to positive δ_e

Reference

1. John Anderson Jr., *Introduction to Flight*, McGraw-Hill Education, Sixth Edition, 2017.
2. Bernard Etkin and Llyod Duff Reid, *Dynamics of Flight Stability and Control*, John Wiley and Sons, Third Edition, 1996.

Thank you for your attention !!!