# *Trade-off Ratio for Variant Design*

# *Definition of Trade-off Ratios*

Trade-off **ratios** are nothing but partial **derivatives** of the rocket performance **equations** with respect to the structural or the propellant mass **ratios.**

In general, the **aim** is to keep $V_*$ as an **invariant,** while allowing $\mathbf{m}_*$ to change.

# *Trade-off Ratio Formulation*

Thus, **we** can determine the **changes** in $m_*$ for changes in stage **masses**.

Basic procedure uses $\mathbf{V_*}$ expression to **examine** the applicable **sensitivities**.

$$\text{Let } V_* = g_0 \sum_{i=1}^{N} I_{spi} \ln \frac{m_{0i}}{m_{fi}}, \text{ which is an invariant.}$$

# ***Trade-off Ratio Formulation***

Let the **initial** and final **masses** of i$^{th}$ stage be,

$$m_{0i} = m_* + m_{pi} + m_{si} + \sum_{j=i+1}^{N} \left( m_{sj} + m_{pj} \right)$$

$$m_{fi} = m_* + m_{si} + \sum_{j=i+1}^{N} \left( m_{sj} + m_{pj} \right)$$

We can write the **ideal** burnout velocity in **terms** of the above **expressions**, as follows.

# *Trade-off Ratio Formulation*

$$V_* = g_0 \sum_{i=1}^{N} I_{spi} \ln\left( \frac{m_* + m_{pi} + m_{si} + \sum_{j=i+1}^{N} \left( m_{sj} + m_{pj} \right)}{m_* + m_{si} + \sum_{j=i+1}^{N} \left( m_{sj} + m_{pj} \right)} \right)$$

It is seen that $\mathbf{V_*}$ is a discrete **sum** of individual stage **contributions**, resulting in **sensitivities** as the **derivatives** of a piecewise continuous **function.**

# *Trade-off Ratio Solution*

We define **total** variation of $V_*$, **along** with the **condition** of invariance of $\mathbf{V_*}$, as follows.

$$dV_* = \frac{\partial V_*}{\partial m_{si}}\delta m_{si} + \frac{\partial V_*}{\partial m_*}\delta m_*; \quad dV_* = 0; \quad \frac{\delta m_*}{\delta m_{si}} = -\frac{\left(\partial V_* / \partial m_{si}\right)}{\left(\partial V_* / \partial m_*\right)}$$

$$dV_* = \frac{\partial V_*}{\partial m_{pi}}\delta m_{pi} + \frac{\partial V_*}{\partial m_*}\delta m_*; \quad dV_* = 0; \quad \frac{\delta m_*}{\delta m_{pi}} = -\frac{\left(\partial V_* / \partial m_{pi}\right)}{\left(\partial V_* / \partial m_*\right)}$$

Above **solution** establishes the possible **changes in** $\mathbf{m_*}$ due to changes in $\mathbf{m_{si}}$ **&** $\mathbf{m_{pi}}$, for a constant $V_*$.

# *Trade-off Ratio Solution*

Here, the **partial** derivatives (2 for each stage) establish the **sensitivity** of velocity to both $\mathbf{m_*}$, $\mathbf{m_{si/pi}}$.

Further, **evaluation** of these **derivatives** is to be carried out in the **context** of velocity being a discrete **function.**

This is demonstrated for a **2-stage** rocket next.

# *Trade-off Ratios for $m_{si}$*

$$V_* = g_0 I_{sp1} \ln \frac{m_{01}}{m_{f1}} + g_0 I_{sp2} \ln \frac{m_{02}}{m_{f2}}$$

$$m_{01} = m_* + m_{s1} + m_{p1} + m_{s2} + m_{p2}$$

$$m_{f1} = m_* + m_{s1} + m_{s2} + m_{p2}$$

$$m_{02} = m_* + m_{s2} + m_{p2}; \quad m_{f2} = m_* + m_{s2}$$

# *Trade-off Ratios for $m_{si}$*

$$\frac{\partial V_*}{\partial m_*} = g_0 I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right) + g_0 I_{sp2}\left(\frac{1}{m_{02}} - \frac{1}{m_{f2}}\right)$$

$$\frac{\partial V_*}{\partial m_{s1}} = g_0 I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right)$$

$$\frac{\partial V_*}{\partial m_{s2}} = g_0 I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right) + g_0 I_{sp2}\left(\frac{1}{m_{02}} - \frac{1}{m_{f2}}\right)$$

# *Trade-off Ratios for $m_{si}$*

$$\frac{\delta m_*}{\delta m_{s1}}\Big|_{dV_*=0} = -\frac{\left(\partial V_* \big/ \partial m_{s1}\right)}{\left(\partial V_* \big/ \partial m_*\right)} = -\frac{I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right)}{I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right) + I_{sp2}\left(\frac{1}{m_{02}} - \frac{1}{m_{f2}}\right)}$$

$$\frac{\delta m_*}{\delta m_{s2}}\Big|_{dV_*=0} = -\frac{\left(\partial V_* \big/ \partial m_{s2}\right)}{\left(\partial V_* \big/ \partial m_*\right)} = -1$$

# ***Trade-off Ratio for $m_{pi}$***

$$\frac{\partial V_*}{\partial m_{p1}} = g_0 I_{sp1}\left(\frac{1}{m_{01}} - 0\right)$$

$$\frac{\partial V_*}{\partial m_{p2}} = g_0 I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right) + g_0 I_{sp2}\left(\frac{1}{m_{02}} - 0\right)$$

# *Trade-off Ratio for $m_{pi}$*

$$\frac{\delta m_*}{\delta m_{p1}}\bigg|_{dV_*=0} - \frac{\left(\partial V_* / \partial m_{p1}\right)}{\left(\partial V_* / \partial m_*\right)} = -\frac{I_{sp1}\left(\dfrac{1}{m_{01}}\right)}{I_{sp1}\left(\dfrac{1}{m_{01}} - \dfrac{1}{m_{f1}}\right) + I_{sp2}\left(\dfrac{1}{m_{02}} - \dfrac{1}{m_{f2}}\right)}$$

$$\frac{\delta m_*}{\delta m_{p2}}\bigg|_{dV_*=0} = -\frac{\left(\partial V_* / \partial m_{p2}\right)}{\left(\partial V_* / \partial m_*\right)} = -\frac{I_{sp1}\left(\dfrac{1}{m_{01}} - \dfrac{1}{m_{f1}}\right) + I_{sp2}\left(\dfrac{1}{m_{02}}\right)}{I_{sp1}\left(\dfrac{1}{m_{01}} - \dfrac{1}{m_{f1}}\right) + I_{sp2}\left(\dfrac{1}{m_{02}} - \dfrac{1}{m_{f2}}\right)}$$

# *Trade-off Ratio Generalization*

$$\frac{\delta m_*}{\delta m_{si}}\Big|_{dV_*=0} = -\frac{\sum_{j=1}^{i} I_{spj}\left(\frac{1}{m_{0j}}-\frac{1}{m_{fj}}\right)}{\sum_{k=1}^{N} I_{spk}\left(\frac{1}{m_{0k}}-\frac{1}{m_{fk}}\right)}; \quad \frac{\delta m_*}{\delta m_{sN}} = -1; \quad \text{Always} < 0$$

$$\frac{\delta m_*}{\delta m_{pi}}\Big|_{dV_*=0} = -\frac{\sum_{j=1}^{i-1} I_{spj}\left(\frac{1}{m_{0j}}-\frac{1}{m_{fj}}\right)+\frac{I_{spi}}{m_{0i}}}{\sum_{k=1}^{N} I_{spk}\left(\frac{1}{m_{0k}}-\frac{1}{m_{fk}}\right)}; \text{ Always} > 0$$

# *Summary*

**We** see that trade-off **ratios** are an elegant mechanism to **understand** the launch vehicle stage **sensitivities**.

We also **note** that a small change in **configuration** based on the sensitivity, in the **vicinity** of parent configuration, has the **potential** to preserve the optimality of **solution**.