Non-Keplerian Motion

Keplerian Solution Limitation

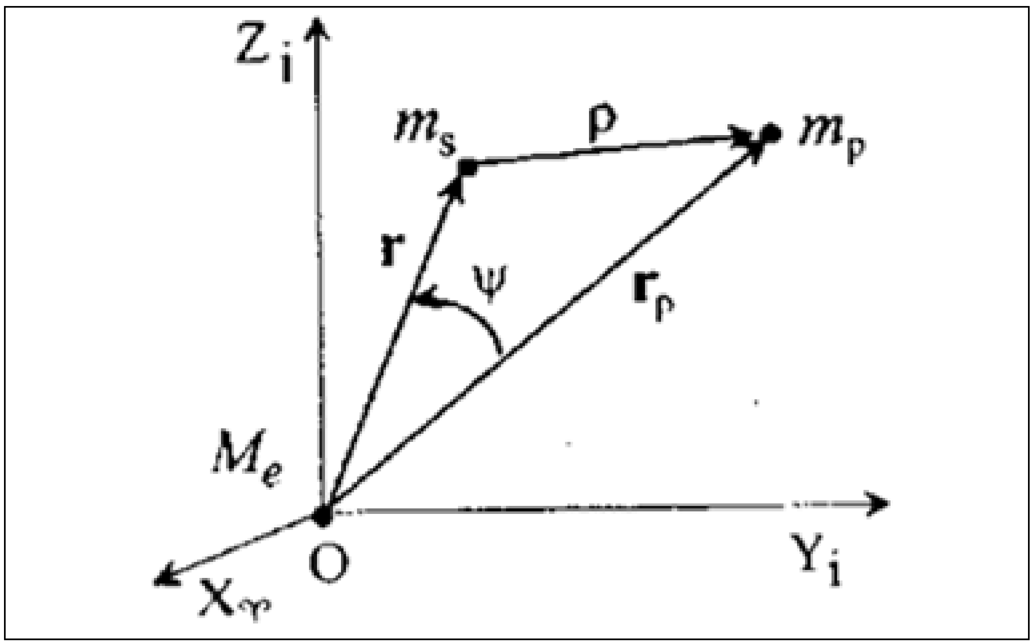
Kepler, who solved the **2-body** problem, also established the **methodology** for all such scenarios, excepting **inter-planetary** travel, when 2-body assumption **breaks** down.

In all **such** cases, correct **approach** is to employ **N-body** formulation and **solution**, even though we know that solutions for **N > 2** are not possible **directly.**

In **orbital** mechanics, solutions for a **3-body** problem are quite **useful** for inter-planetary **travel** and are obtained through certain **simplifying** assumptions.

Implication of 3rd Body in Vicinity

In general, presence of **3rd** body changes **force** system, thereby **affecting** the resulting **trajectory**. This requires consideration of a **3-body** system, as given below.



3-Body Motion Concept

Here, **earth** and satellite form a basic **2-body** system, which is **influenced** by another **planet**, or the 3^{rd} body.

We assume that **satellite** mass is much **smaller** in comparison to both **earth** and planet, while planet's **influence** is significantly **smaller** than that of **earth.**

3-Body Motion Equations

In such a case, **effect** of gravitational pull of **3rd body** can be treated like **a forcing** function, making two body problem, **non-homogeneous** in nature, as shown below.

$$\ddot{\vec{r}} + \frac{\mu_e}{r^3}\vec{r} = \mu_p\left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}_p}{r_p^3}\right) = \vec{\gamma}_p \rightarrow \text{Perturbing acceleration}$$

3-Body Formulation

Perturbation acceleration can be **obtained** in terms of a potential **function**, as follows.

$$\vec{\gamma}_p = -\frac{\partial U_p}{\partial \vec{r}}; \quad U_p = \mu_p\left(\frac{1}{\rho} - \frac{1}{r_p^3}\vec{r}\cdot\vec{r}_p\right); \quad \vec{r}\cdot\vec{r}_p = \left(r r_p\right)\cos\psi$$

$$\frac{1}{r_p^3}\vec{r}\cdot\vec{r}_p = \frac{r}{r_p^2}\cos\psi; \quad \rho^2 = r^2 + r_p^2 - r r_p \cos\psi$$

$$\rho^2 = r_p^2\left[1 + \left(\frac{r}{r_p}\right)^2 - 2\left(\frac{r}{r_p}\right)\cos\psi\right]; \quad \frac{1}{\rho} = \frac{1}{r_p}\left[1 + \left(\frac{r}{r_p}\right)^2 - 2\left(\frac{r}{r_p}\right)\cos\psi\right]^{-\frac{1}{2}}$$

3^{rd} Body Gravitation Model

3-body **equation** can be simplified for $\mathbf{r \ll r_p}$, as follows.

$$\text{For } \frac{r}{r_p} \ll 1; \quad \frac{1}{\rho} \approx \frac{1}{r_p}\left[1 + \left(\frac{r}{r_p}\right)\cos\psi - \frac{1}{2}\left(\frac{r}{r_p}\right)^2 + \frac{3}{2}\left(\frac{r}{r_p}\right)^2 \cos^2\psi\right]$$

$$U_p = \frac{\mu_p}{r_p}\left[1 - \frac{1}{2}\left(\frac{r}{r_p}\right)^2 + \frac{3}{2}\left(\frac{r}{r_p}\right)^2 \cos^2\psi\right] = \frac{\mu}{r_p}\sum_{n=0}^{\infty}\left(\frac{r}{r_p}\right)^n P_n[\cos\psi]$$

Above equation can be **solved** numerically, knowing the **initial** conditions. However, an **alternate** way of solving is through **restricted** 3-body formulation.

Restricted 3-Body Concept

Restricted 3-Body Problem

As **spacecraft** mass is much **smaller** in comparison to the **two** bodies, a simplified model is **possible.**

Further, assumption of **circular** orbit for all bodies allows us to obtain **closed** form solutions for resultant **steady-state** motion**.**

Main assumption is that **spacecraft** has no **effect** on the motion of the other two **larger** bodies.

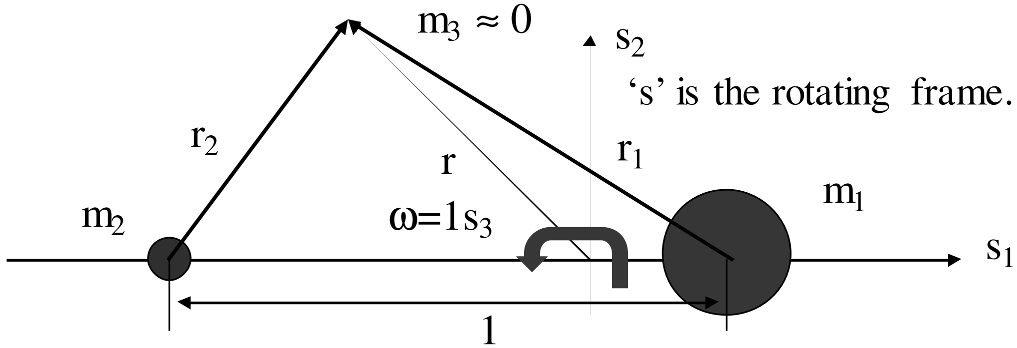
Restricted 3-Body Problem

Problem was first **formulated** and solved by **Euler**, to study motion of **moon** about earth, as perturbed by **sun**.

This **study** had a significant **impact** on the **Apollo** moon mission.

Restricted 3-Body Problem Definition

Consider the **3-body** system, as shown below.



Restricted 3-Body Problem Definition

Here, **three** masses are assumed to execute **circular** motion about the **origin**, which is the centre of mass of $\mathbf{m_1}$ **&** $\mathbf{m_2}$**.**

We also **implement** a normalizing **strategy**, discussed next.

Non-dimensionalization Strategy

We define 'λ' as **mass** ratio of the two **large** objects such that $\mathbf{m_2 = \lambda}$ **&** $\mathbf{m_1 = 1 - \lambda}$**.** ($\lambda < 1$, $m_1 + m_2 = 1$).

We define **distance** between m_1 and m_2 as **1 unit**, so that $\mathbf{m_1}$ is at a distance of '$\boldsymbol{\lambda}$' and $\mathbf{m_2}$ is at a distance of '$\mathbf{1 - \lambda}$' from the **origin** (Centre of mass)**.**

Non-dimensionalization Strategy

Angular velocity of circular **motion** is taken to be **1 unit** so that the time period is ‘**2π**’.

The above **normalization** helps up to **extend** the results to any **3-body** system, including **earth-satellite-planet.**

Summary

To **conclude**, effect of 3rd body can be **modelled** as a perturbation of the **standard** two-body problem.

Such a **restricted** 3-body formulations simplifies the **motion** modelling for inter-planetary **travel.**