EE 622 : Optimal Control Systems

Assignment 1

1. Find the extremum of the function $f(x) = x^{\frac{1}{x}}$ and the points extrema occurs. Also compare e^π and π^e using given function.

2. Find the ratio of maximum perimeter to maximum area of a rectangle that can be inscribed in an ellipse with length of major and minor axes as 20m and 14m respectively.

3. Minimize the material required to make a cylindrical tin can of capacity 10 litres.

4. Minimize $x + y + z$ given $xyz = M$ and $x, y, z, M > 0$

5. Derive second order optimality conditions for a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$

6. Consider the optimization problem,

$$\begin{aligned} \underset{x,y}{\text{Minimize}} \quad & (x+3)^2 + (y-2)^2 \\ \text{Subject to} \quad & 3 + y - x^2 \geqslant 0 \\ & x \leqslant 0, \quad y \geqslant -1 \end{aligned} \tag{1}$$

7. Consider the optimization problem,

$$\begin{aligned} \underset{x,y}{\text{Minimize}} \quad & x^2 + y^2 \\ \text{Subject to} \quad & x^2 - (y-1)^2 = 0 \end{aligned} \tag{2}$$

   using both the method of direct substitution and Lagrange's method.

8. Consider the optimization problem,

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \quad & f(x_1, x_2) \\ \text{subject to} \quad & 3x_1 + x_2 \geqslant 1 \\ & x_1 + 2x_2 \geqslant 1 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0 \end{aligned} \tag{3}$$

   Make a sketch of the feasible set. For each of the following objective function, find the optimal set and the optimal value,

   a. $f(x_1, x_2) = x_1 + x_2$
   b. $f(x_1, x_2) = -x_1 - x_2$

c. $f(x_1, x_2) = x_2$

d. $f(x_1, x_2) = \min\{x_1, x_2\}$

e. $f(x_1, x_2) = x_1 x_2$

f. $f(x_1, x_2) = 4x_1^2 + 9x_2^2$

9. Consider the optimization problem,

$$\begin{aligned} \underset{x \in \mathbb{R}^3}{\text{minimize}} \quad & \frac{1}{2} x^T P x + q^T x + r \\ \text{subject to} \quad & -1 \leqslant x_i \leqslant 1, \quad i = 1, 2, 3. \end{aligned} \tag{4}$$

where $P = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $q = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$, and $r = 1$.

10. Consider for a set of linear equations, $Ax = b$, $b \notin \mathrm{Im}A$. Let $e = b - Ax$ be the error between b and any vector in the range space of A. The objective is to find the best possible x such that $||e||_2^2 = ||Ax - b||_2^2$ is minimized. Formulate it as an optimization problem and find the choice of x in terms of A and b for fulfilling the aforementioned objective.

11. Minimize

$$\tfrac{1}{2} x^T A x - b^T x$$

$$\text{subject to } Qx = c$$

here $x, b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times n}$.
Prove that $x^* \in \mathbb{R}^n$ is local minimum iff its a global minimum point. (No assumption of convexity)