Flight Mechanics/Dynamics

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- System matrix with state vector as $[v \;\; p \;\; r \;\; \phi]^T$

$$A = \begin{bmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Characteristic equation

$$\lambda^4 + 0.6358\lambda^3 + 0.9388\lambda^2 + 0.5114\lambda + 0.003682 = 0$$

- How to check stability for lateral modes?
- Stability criteria

$$E = 0.003682 > 0, \quad R = 0.04223 > 0$$

- What about stability of lateral modes? No unstable modes

- Eigenvalues

$$\lambda_1 = -0.0072973, \ \lambda_2 = -0.56248, \ \lambda_{3,4} = -0.033011 \pm 0.94655i$$

- What can you say about the lateral modes?

$$\underbrace{\lambda_1 = -0.0072973}_{\text{Spiral mode}}, \ \underbrace{\lambda_2 = -0.56248}_{\text{Roll mode}}, \ \underbrace{\lambda_{3,4} = -0.033011 \pm 0.94655i}_{\text{Dutch Roll mode}}$$

- Two modes are convergences, one very rapid, one very slow, and one lightly damped oscillation with a period similar to that of the longitudinal SP mode

Mode	Name	Period	t_{half}	N_{half}
1	Spiral		95	
2	Roll		1.23	
3	Dutch Roll	6.64	21	3.16

Eigenvectors (polar form)

	Spiral		*Rolling convergence*		*Dutch Roll*	
	Magnitude	*Phase*	*Magnitude*	*Phase*	*Magnitude*	*Phase*
$\beta = \hat{v}$	0.00119	180°	0.0198	180°	0.33	−28.1°
$\hat{p}$	1.63×10^{-4}	0°	0.0712	180°	0.12	92.0°
$\hat{r}$	9.20×10^{-4}	180°	0.0040	0°	0.037	−112.3°
ϕ	0.177	180°	1.0	0°	1.0	0°
ψ	1.0	0°	0.0562	180°	0.31	155.7°
$\frac{y_E}{u_0 t^*}$	7.772×10^3	180°	7.65	0°	1.69	−165.8°

- For spiral mode,

$$\beta : \phi : \psi = -0.001119 : -0.177 : 1$$

- Yawing at nearly zero sideslip with some rolling
- Aerodynamic variables

$$\beta : \hat{p} : \hat{r} = 1 : -0.137 : 0.773$$

- Largest of these β is negligibly small for moderate value of ϕ, ψ
- A weak mode because aerodynamic forces are very small.

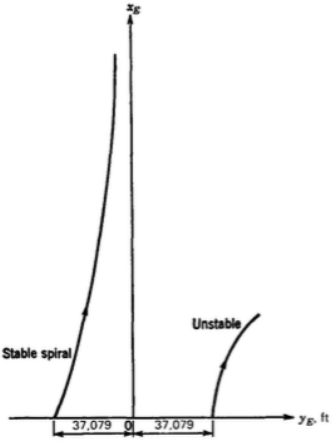
- For rolling mode,

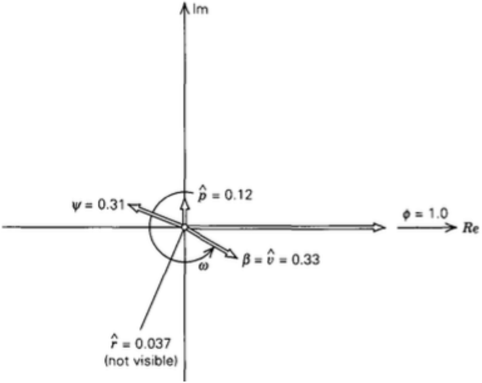
$$\beta : \phi : \psi = -0.0198 : 1 : -0.0625$$

- Pure rotation around x- axis
- Aerodynamic variables

$$\beta : \hat{p} : \hat{r} = 0.278 : 1 : -0.0561$$

- Largest rolling moment is $C_{l_p}\hat{p}$, $\hat{r}$ contribution is small.
- In DR mode, β, ϕ, ψ are of same magnitude.
- $\hat{r}$ is one order smaller while β and ψ are almost equal and opposite.
- Rectilinear motion with yawing and rolling with rolling lagging yawing by 160°


x_E
Stable spiral
Unstable
37,079
0
37,079
y_E, ft


Im
$\hat{p} = 0.12$
$\psi = 0.31$
$\phi = 1.0$
Re
$\beta = \hat{v} = 0.33$
ω
$\hat{r} = 0.037$
(not visible)

- Eigenvalue of the spiral mode is two orders of magnitude smaller than the next larger one.
- How to approximate this mode?
- **Approximation**: only two lowest-order terms in characteristic equation.

$$D\lambda_S + E = 0 \implies \lambda_S = -\frac{E}{D}$$

- System matrix for lateral modes

$$\boldsymbol{A} = \begin{bmatrix} \mathcal{Y}_v & 0 & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 1 & 1 & \tan\theta_0 & 0 \end{bmatrix}$$

- $\mathcal{L}_v = \dfrac{L_v}{I'_x} + I'_{zx} N_v$

- Values of D, E

$$E = g\left[(\mathcal{L}_v\mathcal{N}_r - \mathcal{L}_r\mathcal{N}_v)\cos\theta_0 + (\mathcal{L}_p\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_p)\sin\theta_0\right]$$
$$D = -g\left(\mathcal{L}_v\cos\theta_0 + \mathcal{N}_v\sin\theta_0\right) + \mathcal{Y}_v\left(\mathcal{L}_r\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_r\right) + \mathcal{Y}_r\left(\mathcal{L}_p\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_p\right)$$

- On neglecting smaller terms in D

$$D = -g\left(\mathcal{L}_v\cos\theta_0 + \mathcal{N}_v\sin\theta_0\right) - u_0\left(\mathcal{L}_p\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_p\right)$$

- Eigenvalue, $\lambda_S = -0.00725$, 1% different form correct value
- Condition for static stability

$$\left(\mathcal{L}_v\mathcal{N}_r - \mathcal{L}_r\mathcal{N}_v\right)\cos\theta_0 + \left(\mathcal{L}_p\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_p\right)\sin\theta_0 > 0$$

$$\left(C_{l_\beta}C_{n_r} - C_{n_\beta}C_{l_r}\right)\cos\theta_0 + \left(C_{l_p}C_{n_\beta} - C_{l_\beta}C_{n_p}\right)\sin\theta_0 > 0$$

- Stability varies with flight conditions, as some of the derivatives depend on C_{L_0}.

- **Rolling convergence**: A motion of almost a single degree of freedom, rotation about the x-axis
- Approximation: $v = r = 0$,
$$\dot{p} = \mathcal{L}_p p$$
- Eigenvalue:
$$\lambda_R = \mathcal{L}_p = \frac{L_p}{I'_x} + I'_{zx} N_p$$
- $\lambda_R = -0.434$, 23% smaller than true value -0.562.
- How can we approximate in a better way?
- Another approximation leads to a second-order system: two roots correspond to roll and spiral modes.
- Additionally, Y_p and Y_r are neglected.

- With no approximation to the rolling and yawing moment equations the system that results for horizontal flight is

$$
\begin{aligned}
0 =& -u_0 r + g\phi \\
\dot{p} =& \mathcal{L}_v v + \mathcal{L}_p p + \mathcal{L}_r r \\
\dot{r} =& \mathcal{N}_v v + \mathcal{N}_p p + \mathcal{N}_r r \\
\dot{\phi} =& p
\end{aligned}
$$

- Characteristic equation

$$C\lambda^2 + D\lambda + E = 0$$

$$C = u_0\mathcal{N}_v,\ D = u_0(\mathcal{L}_v\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_v) - g\mathcal{L}_v,\ E = g(\mathcal{L}_v\mathcal{N}_r - \mathcal{L}_r\mathcal{N}_v)$$

- For B747, $\lambda_S = -0.00734,\ \lambda_R = -0.597$, 1%, and 6% less than true values

- An approximation to lateral oscillation: A “flat” yawing/sideslipping motion with suppressed rolling
- With $p = \phi = 0$, neglecting rolling moment equation and Y_r in first equation.

$$
\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} =
\begin{bmatrix}
\dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_o\right) & g\cos\theta_0 \\
\left(\dfrac{L_v}{I'_x} + I'_{zx}N_v\right) & \left(\dfrac{L_p}{I'_x} + I'_{zx}N_p\right) & \left(\dfrac{L_r}{I'_x} + I'_{zx}N_r\right) & 0 \\
\left(I'_{zx}L_v + \dfrac{N_v}{I'_z}\right) & \left(I'_{zx}L_p + \dfrac{N_p}{I'_z}\right) & \left(I'_{zx}L_r + \dfrac{N_r}{I'_z}\right) & 0 \\
0 & 1 & \tan\theta_0 & 0
\end{bmatrix}
\begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} +
\begin{bmatrix} \dfrac{\Delta Y_c}{m} \\ \dfrac{\Delta L_c}{I'_x} + I'_{zx}N_c \\ I'_{zx}\Delta L_c + \dfrac{\Delta N_c}{I'_z} \\ 0 \end{bmatrix}
$$

- Resultant EOM

$$\dot{v} = \mathcal{Y}_v v - u_0 r, \quad \dot{r} = \mathcal{N}_v v + \mathcal{N}_r r$$

- Characteristic equation $\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r)\lambda + (\mathcal{Y}_v\mathcal{N}_r + u_0\mathcal{N}_v) = 0$
- Eigenvalues: $\lambda_{DR} = -0.1008 \pm 0.9157i$, $T = 6.86$ sec (3% error), $N_{\mathrm{half}} = 1$
- Damping is overestimated.

- What would be other approximation for damping of DR mode?
- Coefficient of second highest term in characteristic equation is sum of dampings.

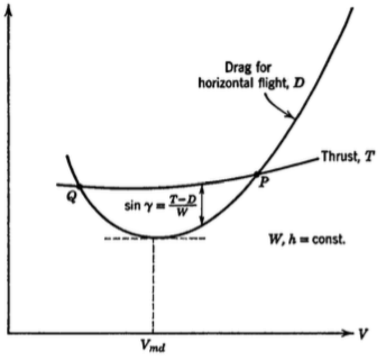
$$2n_{DR} + \lambda_R + \lambda_S = \mathcal{Y}_v + \mathcal{L}_p + \mathcal{N}_r \implies n_{DR} = \frac{1}{2}\left[\mathcal{Y}_v + \mathcal{L}_p + \mathcal{N}_r - (\lambda_R + \lambda_S)\right]$$

- As $\lambda_R + \lambda_S = -\frac{D}{C}$,

$$n_{DR} = \frac{1}{2}\left[\mathcal{Y}_v + \mathcal{N}_r + \frac{\mathcal{L}_v}{\mathcal{N}_v}\left(\mathcal{N}_p - \frac{g}{u_0}\right)\right]$$

- Damping $n_{DR} = -0.0159$, while the true value is -0.0330.
- Can the average of two approximation provide better answer?
- Average of the two approximation $n_{DR} = -0.0584$, 77% off from true value.

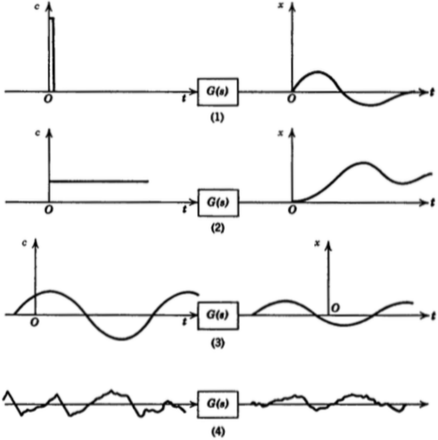
- Response of aircraft to actuation of the control inputs
- Quantities of interest in symmetric flight: Speed and flight path angle $\Longrightarrow$ Velocity vector
- Ability to apply control forces both $\parallel$ and $\perp$ to the flight path
- Thrust or drag control and lift control via elevator deflection or wing flaps
- What are the initial responses of these control inputs?
- Thrust control change forward acceleration and initial speed, while the elevator change pitch angle, angle of attack, and lift.
- Short-term and long-term effects of the control inputs (throttle and elevator angle) are quite contrary.



- Steady speed V is governed by lift coefficient, fixed by δ_e.
- A constant $\delta_e \implies$ a fixed V.
- Flight-path angle $\gamma = \theta - \alpha$ at any given speed is determined by thrust.
- Result of moving throttle at fixed δ_e is a change in γ without change in V.
- Main initial effect of moving elevator: to rotate the vehicle and influence γ, whereas ultimate effect at fixed throttle is to change both V and γ.
- Responses dominated by long-period, lightly damped phugoid oscillation, and steady state with step inputs reached only after a long time.

Linear time invariant (LTI) system:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{c}$$



- **Transfer function**: Ratio of the Laplace transform of the response to that of the input when the system is quiescent for $t < 0$.

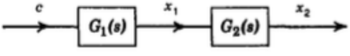
$$\boldsymbol{G}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

- Response of i^{th} state variable

$$\bar{x}_i(s) = \sum_j G_{ij}(s)\bar{c}_j(s)$$

- For single input system

$$\bar{x}(s) = G(s)\bar{c}(s)$$



- Overall transfer function

$$G(s) = G_1(s)G_2(s)$$

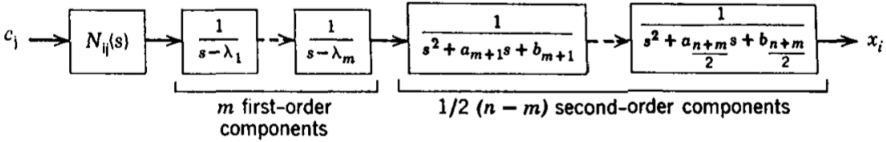
- **High-Order System**: Representation as chain of subsystems
- Elemental building blocks: first or second order systems, How?

$$(s\boldsymbol{I} - A)^{-1} = \frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}$$

- As $\boldsymbol{B}$ is constant matrix, each element of $\boldsymbol{G}(s)$: $G_{ij} = N_{ij}/f(s)$.

$$G_{ij} = \frac{N_{ij}}{(s-\lambda_1)\cdots(s-\lambda_n)} = \frac{N_{ij}}{\prod_{r=1}^{m}(s-\lambda_r)\prod_{r=m+1}^{(n+m)/2}(s^2+a_r s+b_r)}$$

- Eigenvalues may be real or occurs in complex conjugate pairs.



- **Impulse response or impulsive admittance** $h_{ij}(t)$: Response to unit impulse given to a system which is initially quiescent.

$$c_j(s) = \delta(t) \implies G_{ij}(s)\bar{\delta}(s) = G_{ij}(s)$$

- Impulse response

$$h_{ij}(s) = G_{ij}(s) \implies h_{ij}(t) = \mathcal{L}^{-1}G_{ij}(s)$$

- Consider a first order system

$$G(s) = \frac{1}{s-\lambda} = h(s) \implies h(t) = e^{\lambda t}$$

- Usually, $\lambda = -1/T$, where T is time constant, we have

$$h(t) = e^{-t/T}$$

- Consider a second order system with differential equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 = c,$$

where state-vector is given by $\boldsymbol{x} = [y \;\; \dot{y}]^T$.

- Transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Eigenvalues: $\lambda = n \pm i\omega = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$
- Impulse response

$$h(s) = \frac{1}{(s+n)^2 + \omega^2} \implies h(t) = \frac{1}{\omega}e^{nt}\sin\omega t$$

- Does this result hold for $\zeta > 1$? $h(t) = \frac{1}{\omega'}e^{nt}\sinh\omega' t, \quad \omega' = \omega_n\sqrt{\zeta^2 - 1}$

- **Step response or Indicial admittance,** $\mathcal{A}_{ij}(t)$: Response to unit step function

$$\bar{\mathcal{A}}_{ij}(s) = G_{ij}(s)I(s) = \frac{G_{ij}(s)}{s}$$

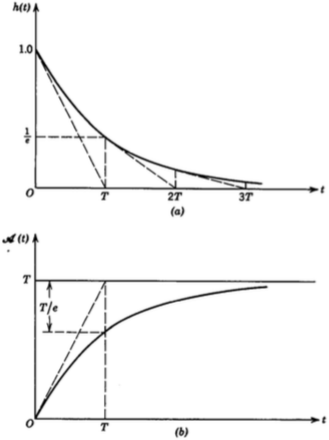
- Relation of transfer functions for impulse and step response

$$\bar{\mathcal{A}}_{ij}(s) = \frac{h_{ij}(s)}{s}$$

- Since the initial values of both h_{ij} and $\mathcal{A}_{ij}$ are zero,

$$\mathcal{A}_{ij}(t) = \int_0^t h_{ij}(\tau)d\tau \implies h_{ij}(t) = \frac{d\mathcal{A}_{ij}(t)}{dt}$$

- $\mathcal{A}_{ij}(t)$ can be found by Laplace inverse or integration of impulse response.



- First order system

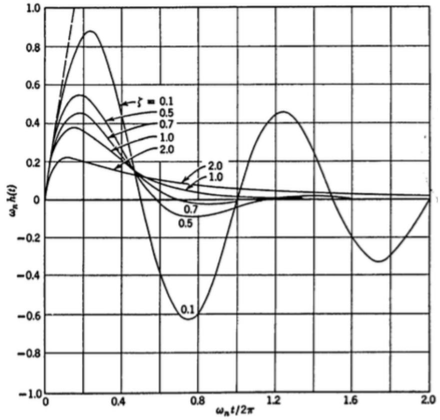
$$\mathcal{A}(t) = T(1 - e^{-t/T})$$

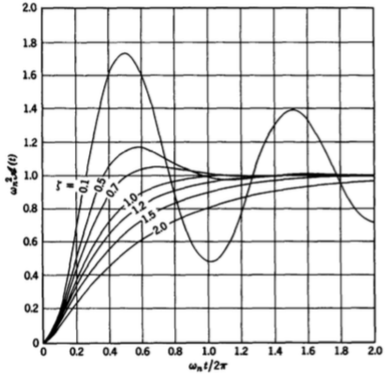
- Second order system, with $\zeta < 1$,

$$\mathcal{A}(t) = \frac{1}{\omega_n^2}\left[1 - e^{nt}\left(\cos\omega t - \frac{n}{\omega}\sin\omega t\right)\right]$$

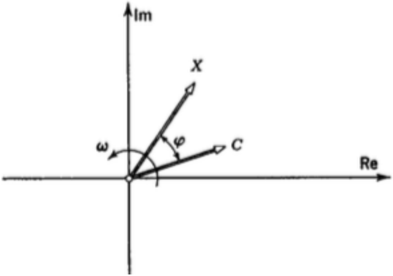
- What about $\zeta > 1$?
- What is static gain?
- Asymptotic value as $t \to \infty$

$$\lim_{t\to\infty}\mathcal{A}(t) = \lim_{s\to 0} s\mathcal{A}(s) = \lim_{s\to 0} G(s) = K$$


1.0
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1.0
$\omega_n h(t)$
ζ = 0.1
0.5
0.7
1.0
2.0
2.0
1.0
0.7
0.5
0.1
0
0.4
0.8
1.2
1.6
2.0
$\omega_n t/2\pi$


2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
ζ =
0.1
0.5
0.7
1.0
1.2
1.5
2.0
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
$\omega_n t/2\pi$

- A stable LTI system, with sinusoidal input, results in a steady-state sinusoidal response at the same frequency as that of the input.
- What about amplitude and phase of the output?
- Amplitude and phase are generally different from those of the input.



- Consider an input of the form

$$c = A_1 e^{i\omega t} \implies c(s) = \frac{A_1}{s - i\omega}$$

- Form of the response is assumed to be $x = A_2 e^{i\omega t}$.

- Response of system

$$x = A_1 \frac{G(s)}{s - i\omega} = A_1 \frac{N(s)}{(s - i\omega)f(s)}$$

- Roots of denominator: $\lambda_1 \cdots \lambda_n, i\omega$
- Response of the system

$$\begin{aligned} x(t) =& A_1 \sum_{r=1}^{n+1} \left[\frac{(s-\lambda_r)N(s)}{(s-i\omega)f(s)} \right]_{s=\lambda_r} e^{\lambda_r t} \\ =& A_1 \left[\frac{N(i\omega)}{f(i\omega)} e^{i\omega t} + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t} \right] \end{aligned}$$

- What would the response when $t \to \infty$?
- As $t \to \infty$, $e^{\lambda_r t} \to 0 \ \forall \ r$ for stable system.
- Steady state response

$$x(t) = A_1 \frac{N(i\omega)}{f(i\omega)} e^{i\omega t}, \ t \to \infty$$

- Steady state response

$$x(t) = A_1 \frac{N(i\omega)}{f(i\omega)} e^{i\omega t} = A_1 G(i\omega) e^{i\omega t} = A_2 e^{i\omega t}$$

- We have relation

$$A_2 = A_1 G(i\omega) \implies \boxed{G(i\omega) = \frac{A_2}{A_1}}$$

- Frequency response function, $G(i\omega)$,

$$G(i\omega) = KMe^{i\phi}$$

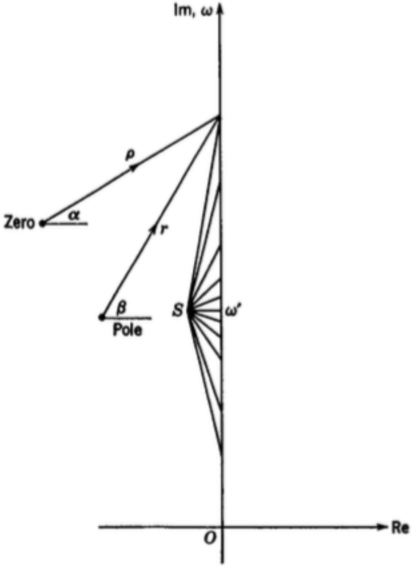
  where K and M are static and dynamic gains, and KM represents total gain.
- What about dependency of M and ϕ on frequency? Frequency-dependent

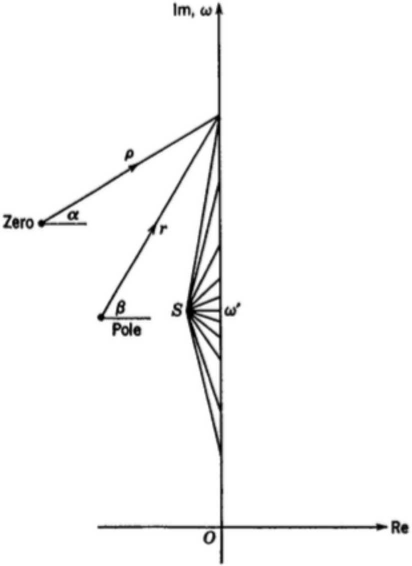
- Transfer function of system

$$G(s) = \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)}$$

  where $\lambda_i \;\forall\; i = 1\cdots n$ and $z_j \;\forall\; j = 1\cdots m$ are poles and zeros of the system, respectively.

- Assume $(s - z_k) = \rho_k e^{i\alpha_k}$ and $(s - \lambda_k) = r_k e^{i\beta_k}$
- What can we say about $|G|$ and ϕ?





- Magnitude and phase of transfer function

$$|G| = \frac{\prod_{k=1}^{m} \rho_k}{\prod_{k=1}^{n} r_k}$$

$$\phi = \sum_{1}^{m} \alpha_k - \sum_{1}^{n} \beta_k$$

- What are the effects of varying frequency when singularity is close to the axis?
- Sharp change in magnitude and about 180° change in phase
- What if there is pole or zero is in LHP?
- What if there is a zero in RHP?

- Consider a system with transfer function

$$G(s) = \frac{1}{s + 1/T}$$

- What is the value of K? $K = \lim_{s\to 0} G(s) = T$
- On substituting $s = i\omega$ in $G(s)$

$$G(i\omega) = KMe^{i\phi} = \frac{T}{1 + i\omega T} \implies Me^{i\phi} = \frac{1 - i\omega T}{1 + \omega^2 T^2}$$

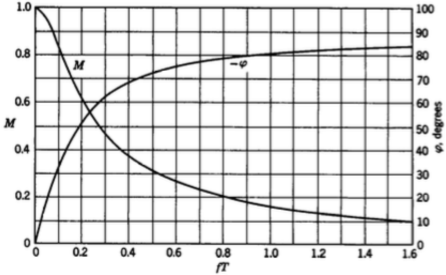
- Amplitude and phase of transfer function

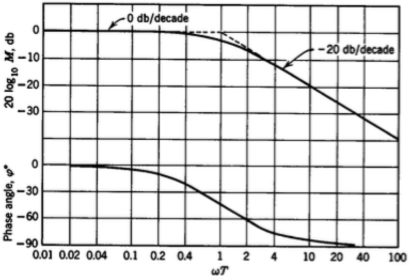
$$\boxed{M = \frac{1}{\sqrt{1 + \omega^2 T^2}}, \quad \phi = -\tan^{-1}\omega T}$$

- At $\omega = 0,\ M = 1,\ \phi = 0.$

Flight Mechanics/Dynamics

Frequency-Response Curves: First-Order System





Curve is applicable for all first order systems, with f as input frequency.

- Consider a second order system with transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- What is the value of M and ϕ?
- On substituting $s = i\omega$ in $G(s)$

$$Me^{i\phi} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2i\zeta\omega_n\omega}$$

$$\boxed{M = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + 4\zeta^2\omega^2/\omega_n^2}}, \quad \phi = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}}$$

- At $\omega = 0,\ M = 1,\ \phi = 0.$

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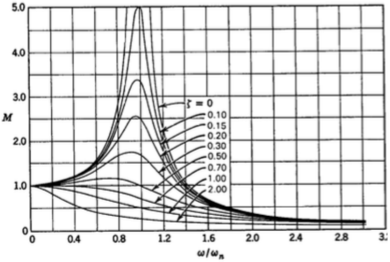
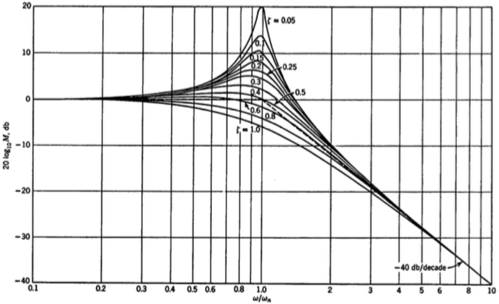
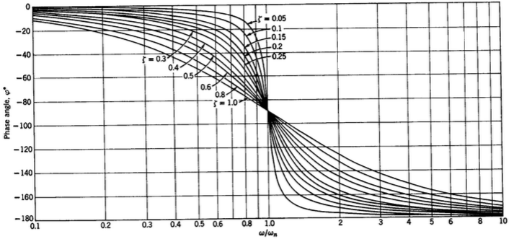
Frequency-Response Curves: Second-Order System







- What is the difference between representation of first and second order systems?
- A single pair of curves serves to define the frequency response of all first-order systems
- Two families of curves, with the damping ratio as parameter, to display the characteristics of all second-order systems.
- Damping is very important in second order system.
- Damping controls magnitude of the resonance peak which occurs near unity frequency ratio.
- At natural frequency, the phase lag is independent of ζ, as all the curves pass through $\phi = -90^\circ$. Why is it so?
- For all ζ, $M \rightarrow 1, \phi \rightarrow 0$ as $\omega/\omega_n \rightarrow 0$. What does it mean?

- How to obtain frequency response of higher order system?
- Phase changes are additive.
- Overall amplitude ratio: multiplication of individual ones of all elements.
- Assume that the system is given by

$$G(s) = G_1(s)G_2(s)\cdots G_n(s)$$

- In frequency domain,

$$\begin{aligned} G(i\omega) &= G_1(i\omega)G_2(i\omega)\cdots G_n(i\omega) \\ &= (K_1M_1K_2M_2\cdots K_nM_n)e^{\phi_1+\phi_2+\cdots+\phi_n} \\ &= KMe^{i\phi} \end{aligned}$$

- Amplitude and phase: $KM = \prod_{r=1}^{n} K_rM_r, \;\; \phi = \sum_{r=1}^{n}\phi_r$

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Thank you for your attention !!!