Restricted 3-Body Formulation

Equations in ‘s’ Frame

We can write the **acceleration** of mass m_3 in the **rotating** frame, ‘**s**’, as follows.

$$\vec{r} = x\hat{s}_1 + y\hat{s}_2 + z\hat{s}_3; \quad \frac{{}^s d}{dt}\vec{r} = \dot{x}\hat{s}_1 + \dot{y}\hat{s}_2 + \dot{z}\hat{s}_3; \quad \frac{{}^s d^2}{dt^2}\vec{r} = \ddot{x}\hat{s}_1 + \ddot{y}\hat{s}_2 + \ddot{z}\hat{s}_3$$

Equations in ‘s’ Frame

As **gravitational** force acting on $\mathbf{m_3}$ is a function of **‘$\mathbf{r_1}$’** and **‘$\mathbf{r_2}$’,** we can write these **expressions** as follows.

$$r_1 = \left|\vec{r} - \lambda \hat{s}_1\right| = \left[(x-\lambda)^2 + y^2 + z^2\right]^{1/2}$$

$$r_2 = \left|\vec{r} + (1-\lambda)\hat{s}_1\right| = \left[(x+1-\lambda)^2 + y^2 + z^2\right]^{1/2}$$

Equations in 'i' Frame

As **Newton's** law is commonly expressed in the **inertial** frame, we need to transform the **acceleration** term in the **inertial** frame, which is **done** as shown below.

$$\frac{{}^{i}d^2\vec{r}}{dt^2} = \frac{{}^{s}d^2\vec{r}}{dt^2} + 2\vec{\omega}\times\frac{{}^{s}d\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\frac{{}^{i}d^2\vec{r}}{dt^2} = (\ddot{x} - 2\dot{y} - x)\hat{s}_1 + (\ddot{y} + 2\dot{x} - y)\hat{s}_2 + \ddot{z}\hat{s}_3$$

Equations in 'i' Frame

The **force** acting on m_3 is net **gravitational** acceleration, as given below.

$$\vec{a}_g = -\frac{(1-\lambda)\vec{r}_1}{r_1^3} - \frac{\lambda \vec{r}_2}{r_2^3}$$

Equations of Motion

We can now obtain the **governing** equation of **motion** in inertial frame for mass, $\mathbf{m_3}$, as follows.

$$\frac{{}^{i}d^2\vec{r}}{dt^2} = (\ddot{x} - 2\dot{y} - x)\hat{s}_1 + (\ddot{y} + 2\dot{x} - y)\hat{s}_2 + \ddot{z}\hat{s}_3 = \vec{a}_g$$

Scalar Equations of Motion

The **corresponding** scalar **equations** are as given below.

$$\hat{s}_1: \quad \ddot{x} - 2\dot{y} - x = -\frac{(1-\lambda)(x-\lambda)}{r_1^3} - \frac{\lambda(x+1-\lambda)}{r_2^3}$$

$$\hat{s}_2: \quad \ddot{y} - 2\dot{x} - y = -\frac{(1-\lambda)y}{r_1^3} - \frac{\lambda y}{r_2^3}$$

$$\hat{s}_3: \quad \ddot{z} = -\frac{(1-\lambda)z}{r_1^3} - \frac{\lambda z}{r_2^3}$$

Restricted 3-Body Solution

Restricted Steady-state Solutions

While, no **known** general solutions **exist**, it is still **possible** to examine the **nature** of steady-state **solution**, through the concept of **equilibrium** points. as shown below.

$$x = \frac{(1-\lambda)(x-\lambda)}{r_1^3} + \frac{\mu(x+1-\lambda)}{r_2^3}$$

$$y = \frac{(1-\lambda)y}{r_1^3} + \frac{\lambda y}{r_2^3}; \quad z\left(\frac{(1-\lambda)}{r_1^3} + \frac{\lambda}{r_2^3}\right) = 0$$

Basic Equilibrium Solutions

This gives **z = 0** as one **coordinate**, indicating that solutions are in **x-y plane**. Further, it is seen that for **y=0**, the 2nd equation is **satisfied** identically so that **x** can be **written** as**,**

$$x = \frac{(1-\lambda)(x-\lambda)}{|x-\lambda|^3} + \frac{\lambda(x+1-\lambda)}{|x+1-\lambda|^3}$$

This equation usually has **only 3** real roots (termed L_1, L_2 & L_3) for $0 \leq \lambda \leq 1$, obtained by **Euler**, lying on '**s1**' axis.

Additional Equilibrium Solutions

However, there are **additional** possible equilibrium points when $\mathbf{y \neq 0}$.

In this context, we see from the **second** equation that it will be satisfied only if $\mathbf{r_1 = r_2 = 1}$, in which case both $\mathbf{1}^{st}$ and $\mathbf{2}^{nd}$ equations are **identically** satisfied.

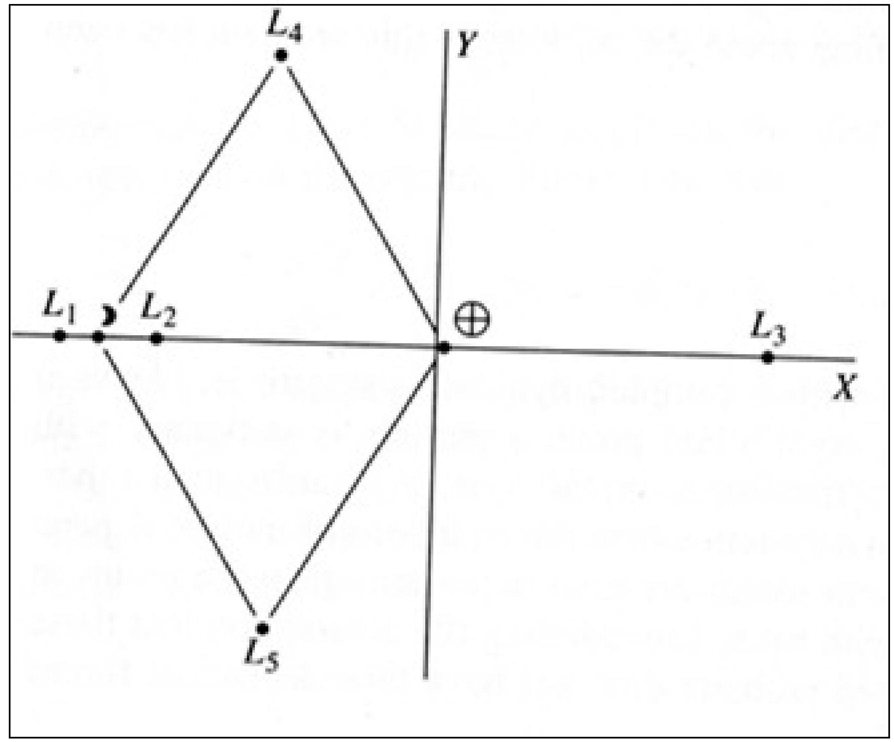
This condition represents the **equilibrium** map as an equilateral **triangle,** with two large **objects** as vertices on 's1' axis, and the **equilibrium** point as the third **vertex**.

Lagrange Points

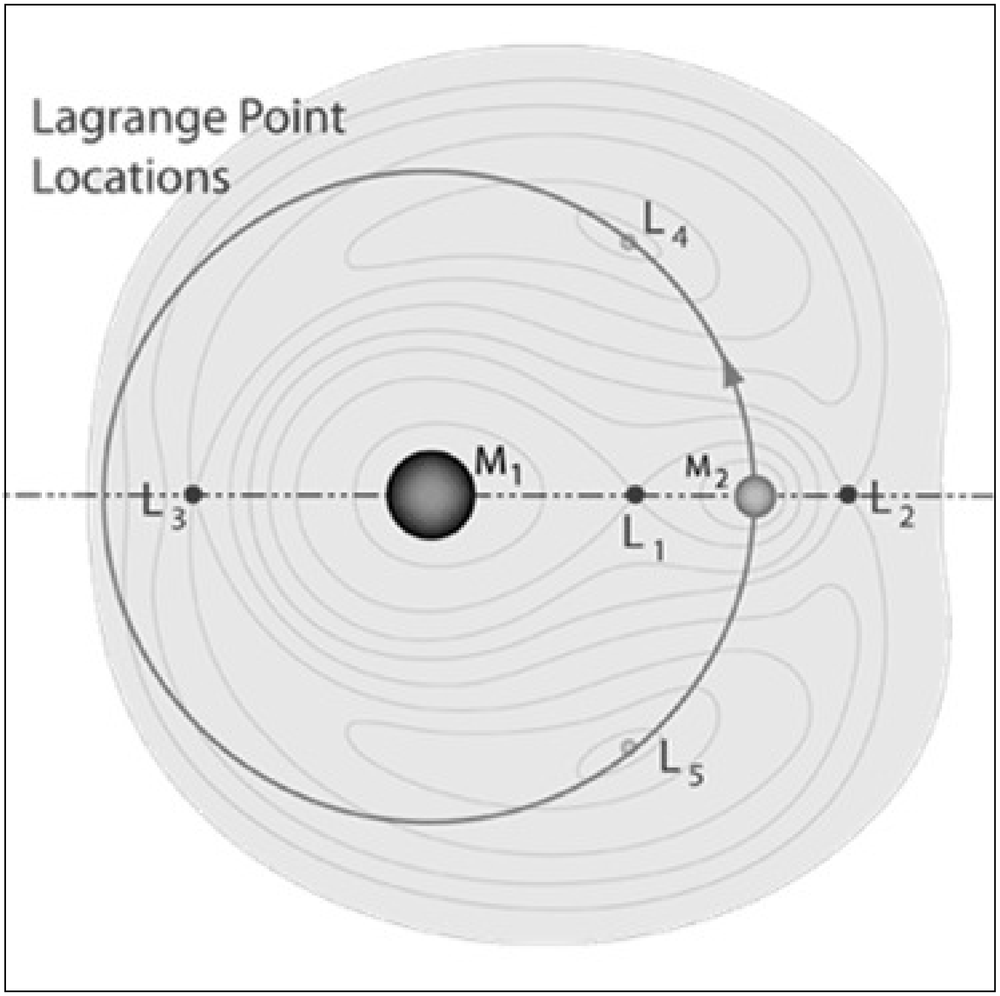
Further, as the above **condition** is also satisfied for **'-y'**, we get one more **equilibrium** point as its mirror **image**.

Both these points are termed **'L_4'** and '**L_5**' and were discovered by **Lagrange**, the **famous scientist.**

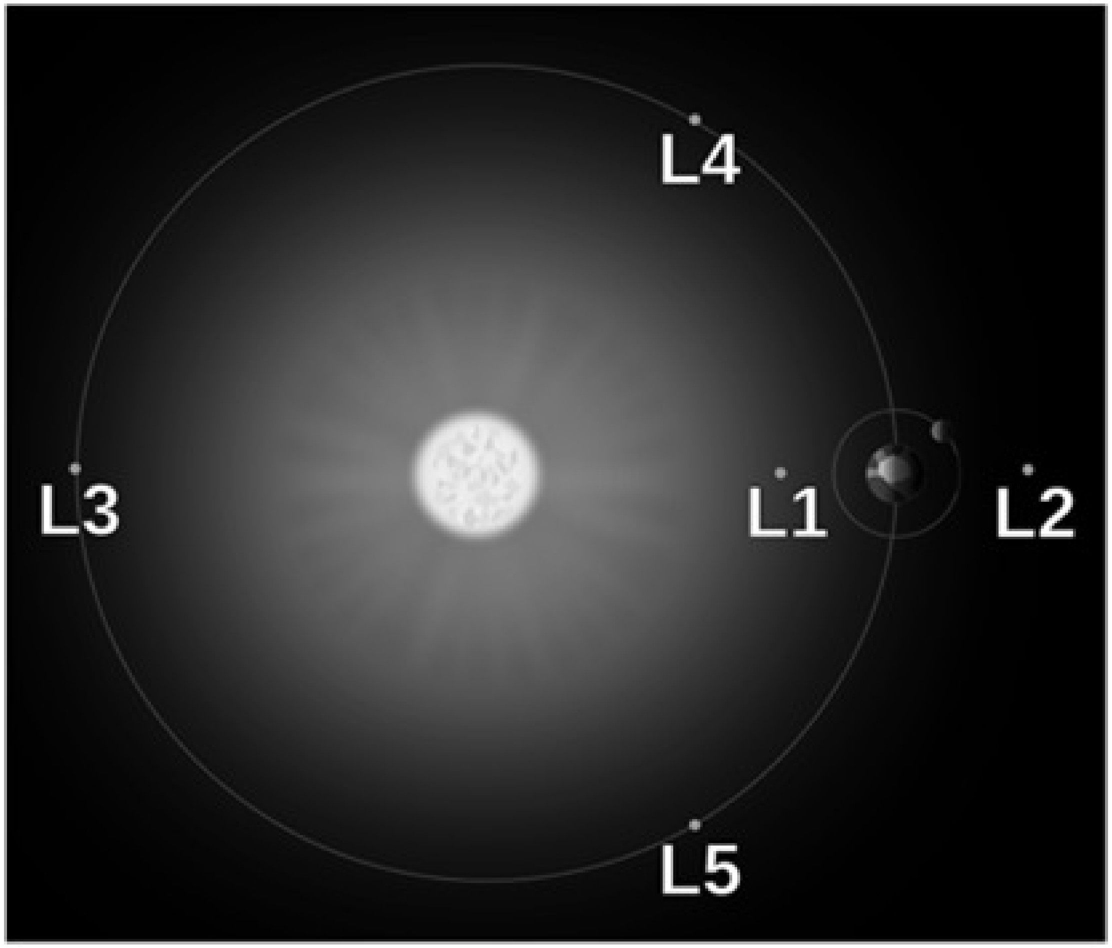
All five equilibrium points are **schematically** shown for **earth-moon-satellite** system, along side.



Earth – Moon Lagrange Points



Sun – Earth Lagrange Points



Lagrange Points Features

These 5 points are **stationary** in the **rotating** frame, but appear as **circular** orbits in the inertial **frame**.

The fact that these are **equilibrium** solutions, indicates that if an **object** is left at these **locations**, it will be **stationary** with respect to the **smaller** primary.

Lagrange Points Features

Thus, if we are **desirous** of forming **orbits** around moon, we can choose **'L_1' or 'L_2'** as possible **destinations**.

In case of **sun-earth-spacecraft**, 'L_1' or 'L_2' represent the point at which **inter-planetary** motion can begin.

Summary

In **conclusion**, restricted 3-body problem can be **solved** under equilibrium and circular **motion** conditions.

Lagrange points are important equilibrium **points** that establish the methodology for **inter-planetary** motion.