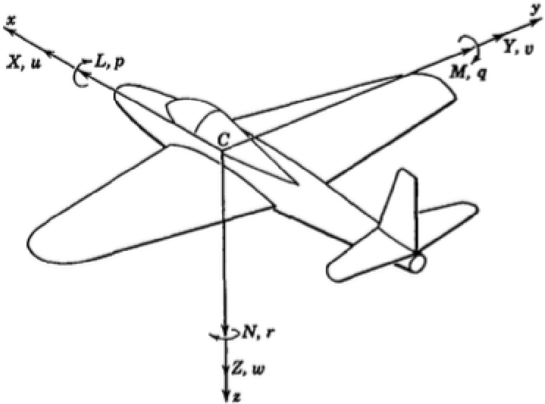
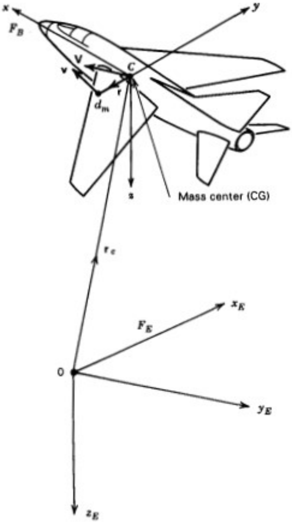
Flight Mechanics/Dynamics

Dr. Shashi Ranjan Kumar

Assistant Professor
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai, 400076 India

- Unsteady motions of a flight vehicle
  - ⇒ Analysis, computation, or simulation
  - ⇒ Mathematical model of the vehicle and its subsystems
- Aircraft: An aggregate of elastic bodies so connected that both rigid and elastic relative motions can occur.
  - ⇒ A complicated dynamic system
- External forces: Complicated functions of its shape and its motion.
- Difficult to predict realistic analyses with a very simple mathematical model.
- Assumptions:
  - ⇒ Aircraft as a single rigid body with six degrees of freedom
  - ⇒ Free to move in the atmosphere under the actions of gravity and aerodynamic forces
  - ⇒ Flat Earth surface and stationary in the inertial space
- Nature and complexity of aerodynamic forces that distinguish flight vehicles from other dynamic systems


x
X, u
L, p
y
Y, v
M, q
C
N, r
Z, w
z


x
F_B
y
V
C
d_m
r
z
Mass center (CG)
r_c
x_E
F_E
0
y_E
z_E

Two vector equations describing the motions of aircraft:

$$\boldsymbol{f}_E = m\dot{\boldsymbol{V}}_E, \quad \boldsymbol{G}_E = \dot{\boldsymbol{h}}_E, \quad \boldsymbol{h}_E = \int \tilde{\boldsymbol{r}}_E \boldsymbol{v}_E dm$$

Remarks:

- Above equations are only valid if the moving point is the CG. This equations will *not* be valid for a moving reference point other than CG.
- Above equations are also valid if there is relative motion between parts of the plane.
- If wind vector, $\boldsymbol{W} \neq 0$, the angular momentum, $\boldsymbol{h}_E$, remains unchanged. But, the total external force is described as,

$$\boldsymbol{f}_E = m\dot{\boldsymbol{V}}_E^E, \quad \boldsymbol{G}_E = \dot{\boldsymbol{h}}_E$$

- How to compute angular momentum?

$$\boldsymbol{h} = \int d\boldsymbol{h} = \int \boldsymbol{r} \times \boldsymbol{v} dm$$

- In body frame F_B, angular momentum

$$\boldsymbol{h}_B = \int \boldsymbol{r}_B \times \boldsymbol{v}_B dm = \int \tilde{\boldsymbol{r}}_B \boldsymbol{v}_B dm$$

- In body frame F_B, the angular velocity of airplane w.r.t. inertial space

$$\boldsymbol{\omega}_B = [p \quad q \quad r]^T$$

- The velocity of a point on the rigid rotating body

$$\boldsymbol{v}_B = \boldsymbol{V}_B + \tilde{\boldsymbol{\omega}}_B \boldsymbol{r}_B$$

$$\tilde{r}_B = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}, \quad \tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

- Angular momentum

$$\boldsymbol{h}_B = \int \tilde{\boldsymbol{r}}_{\boldsymbol{B}} \boldsymbol{v}_{\boldsymbol{B}} dm = \int \tilde{\boldsymbol{r}}_{\boldsymbol{B}} (\boldsymbol{V}_{\boldsymbol{B}} + \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}} \boldsymbol{r}_{\boldsymbol{B}}) dm$$

$$= \left(\int \tilde{\boldsymbol{r}}_{\boldsymbol{B}} dm \right) \boldsymbol{V}_B + \int \tilde{\boldsymbol{r}}_{\boldsymbol{B}} \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}} \boldsymbol{r}_{\boldsymbol{B}} dm$$

- Angular momentum can be obtained as

$$\boldsymbol{h}_B = \int \tilde{\boldsymbol{r}}_{\boldsymbol{B}} \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}} \boldsymbol{r}_{\boldsymbol{B}} dm = \boldsymbol{I}_{\boldsymbol{B}} \boldsymbol{\omega}_{\boldsymbol{B}}$$

$$\boldsymbol{I}_B = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}$$

$$I_x = \int (y^2 + z^2) dm, \quad I_y = \int (x^2 + z^2) dm, \quad I_z = \int (y^2 + x^2) dm$$

$$I_{xy} = I_{yx} = \int xy dm, \quad I_{yz} = I_{zy} = \int yz dm, \quad I_{xz} = I_{zx} = \int xz dm$$

- To calculate net torque on aircraft, time derivative of $\boldsymbol{h}_B$, which involves time derivative of the inertia matrix, is required.
- If the equations of motion are expressed in the frame F_B, then inertia matrix is constant.
- A desirable feature of the body frame.
- When xz is plane of symmetry,

$$I_{xy} = I_{yz} = 0, \ I_{zx} \neq 0$$

- Can we choose axes in a way to make $I_{zx} = 0$?
- If axes are chosen such that $I_{zx} = 0$, they are called principal axes.

- Expressing the equations of motion in the body frame using the transformation matrix $\boldsymbol{L}_B^E$, we get

$$\boldsymbol{f}_E = m\dot{\boldsymbol{V}}_E^E$$
$$\boldsymbol{L}_B^E \boldsymbol{f}_B = m\frac{d}{dt}(\boldsymbol{L}_B^E \boldsymbol{V}_B^E) = m(\dot{\boldsymbol{L}}_B^E \boldsymbol{V}_B^E + \boldsymbol{L}_B^E \dot{\boldsymbol{V}}_B^E)$$

- How to compute $\dot{\boldsymbol{L}}_B^E$?

$$\boxed{\dot{\boldsymbol{L}}_B^E = \boldsymbol{L}_B^E \tilde{\boldsymbol{\omega}}_B}$$ How?

$$\boldsymbol{L}_B^E \boldsymbol{f}_B = m(\boldsymbol{L}_B^E \tilde{\boldsymbol{\omega}}_B \boldsymbol{V}_B^E + \boldsymbol{L}_B^E \dot{\boldsymbol{V}}_B^E)$$

- Pre-multiplying by $\boldsymbol{L}_E^B$, we get

$$\boxed{\boldsymbol{f}_B = m(\dot{\boldsymbol{V}}_B^E + \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}} \boldsymbol{V}_B^E)}$$

- Consider two frames a and b where frame b is rotating with angular velocity ω w.r.t. frame a.

$$\boldsymbol{v}_b = \boldsymbol{L}_a^b \boldsymbol{v}_a, \quad \boldsymbol{v}_a = \boldsymbol{L}_b^a \boldsymbol{v}_b$$

- On differentiating, we get

$$\dot{\boldsymbol{v}}_b = \boldsymbol{L}_a^b \dot{\boldsymbol{v}}_a + \dot{\boldsymbol{L}}_a^b \boldsymbol{v}_a, \quad \dot{\boldsymbol{v}}_a = \boldsymbol{L}_b^a \dot{\boldsymbol{v}}_b + \dot{\boldsymbol{L}}_b^a \boldsymbol{v}_b$$

- $\dot{\boldsymbol{L}}_a^b$ must be independent of $\boldsymbol{v}$, and thus same for constant $\boldsymbol{v}_b$.
- For $\boldsymbol{v}_b =$ const. we have $\dot{\boldsymbol{v}}_a = \dot{\boldsymbol{L}}_b^a \boldsymbol{v}_b$
- Also, $\dot{\boldsymbol{v}}_a = \boldsymbol{\omega}_a \times \boldsymbol{v}_a = \tilde{\boldsymbol{\omega}}_a \boldsymbol{v}_a$.
- We have $\dot{\boldsymbol{v}}_a = \dot{\boldsymbol{L}}_b^a \boldsymbol{v}_b = \tilde{\boldsymbol{\omega}}_a \boldsymbol{v}_a = \tilde{\boldsymbol{\omega}}_a \boldsymbol{L}_b^a \boldsymbol{v}_b$
- As this is true for all $\boldsymbol{v}_b$, the following relation must be true.

$$\dot{\boldsymbol{L}}_b^a \boldsymbol{v}_b = \tilde{\boldsymbol{\omega}}_a \boldsymbol{L}_b^a \boldsymbol{v}_b \ \forall \ \boldsymbol{v}_b \implies \boxed{\dot{\boldsymbol{L}}_b^a = \tilde{\boldsymbol{\omega}}_a \boldsymbol{L}_b^a}$$

- Similarly, $\boxed{\dot{\boldsymbol{L}}_a^b = -\tilde{\boldsymbol{\omega}}_b \boldsymbol{L}_a^b = \boldsymbol{L}_a^b \tilde{\boldsymbol{\omega}}_b} \quad \boxed{\dot{\boldsymbol{L}}_b^a = \boldsymbol{L}_b^a \tilde{\boldsymbol{\omega}}_b}$

- Similarly, the moment equation can be expressed as

$$\boxed{\boldsymbol{G}_B = (\dot{\boldsymbol{h}}_B + \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}} \boldsymbol{h}_B)}$$ Try to derive yourself.

- In the inertial frame, the force acting on the aircraft can be divided in gravitational, $m\boldsymbol{g}$, and aerodynamic forces, $\boldsymbol{A}$.
- Let the above forces expressed in the frame F_B be,

$$\boldsymbol{A}_B = [X \quad Y \quad Z]^T$$
$$m\boldsymbol{g}_B = m\boldsymbol{L}_E^B \boldsymbol{g}_E = m\boldsymbol{L}_E^B [0 \quad 0 \quad g]^T$$

- Also $\dot{\boldsymbol{I}}_B = 0$,

$$\boldsymbol{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}, \quad \boldsymbol{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\boxed{\boldsymbol{f}_B = m\boldsymbol{g}_B + \boldsymbol{A}_B = m(\dot{\boldsymbol{V}}_B^E + \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}}\boldsymbol{V}_B^E)}$$

$$\boxed{\boldsymbol{G}_B = \dot{\boldsymbol{h}}_B + \tilde{\boldsymbol{\omega}}_{\boldsymbol{B}}\boldsymbol{h}_B}$$

Equations of Motion of Aircraft

$$\begin{aligned} X - mg\sin\theta =& m\left(\dot{u}^E + qw^E - rv^E\right) \\ Y + mg\cos\theta\sin\phi =& m\left(\dot{v}^E + ru^E - pw^E\right) \\ Z + mg\cos\theta\cos\phi =& m\left(\dot{w}^E + pv^E - qu^E\right) \end{aligned}$$

$$\begin{aligned} L =& I_x\dot{p} - I_{yz}\left(q^2 - r^2\right) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) - \left(I_y - I_z\right)qr \\ M =& I_y\dot{q} - I_{zx}\left(r^2 - p^2\right) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) - \left(I_z - I_x\right)rp \\ N =& I_z\dot{r} - I_{xy}\left(p^2 - q^2\right) - I_{yz}(\dot{q} + rp) - I_{zx}(\dot{p} - qr) - \left(I_x - I_y\right)pq \end{aligned}$$

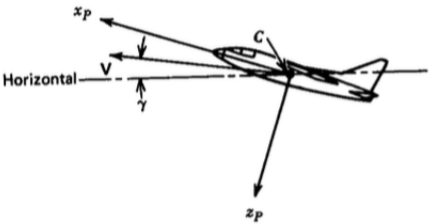
- EOM are valid for any orthogonal axes fixed in airplane, with origin at CG.
- These axes are called as body axes.
- We use symmetry property to define axes for simplification.
- If x, y, z are chosen along forward, downward, and to the right directions, respectively, then

$$I_{xy} = 0 = I_{yz}$$

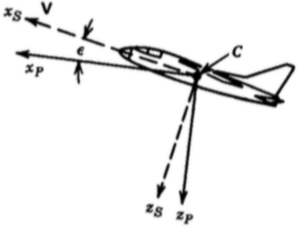
- How to choose directions of x and z axes?
  - $\Rightarrow$ Principal axes
  - $\Rightarrow$ Stability/wind axes
  - $\Rightarrow$ Body axes
- **Principal axes**: Axes coincide with principal axes of vehicle (all product of inertia are zero)

$$h_x = I_x p, \quad h_y = I_y q, \quad h_z = I_z r$$

- **Stability/wind axes**: x- axis aligned with V in reference condition of steady symmetric flight.
- For symmetric flight condition both stability and wind axes coincide.
- Velocity components of aircraft: $v = 0$, $w = 0$
- Simplification in EOM and expressions for aerodynamic forces
- I_x, I_z, I_{xz} vary from problem to problem.



Principal axes



Stability axes

$\epsilon > 0$

Relation between moment of inertias for stability and principal axes

$$
\begin{aligned}
I_x =& I_{x_p}\cos^2\epsilon + I_{z_p}\sin^2\epsilon \\
I_z =& I_{x_p}\sin^2\epsilon + I_{z_p}\cos^2\epsilon \\
I_{zx} =& \frac{1}{2}(I_{z_p} - I_{x_p})\sin 2\epsilon
\end{aligned}
$$

where ϵ is the angle between x_p and x_s.

- How to obtain this relation between moment of inertia?

$$\boldsymbol{I}_a = \boldsymbol{T}\boldsymbol{I}_b\boldsymbol{T}^{-1} = \boldsymbol{T}\boldsymbol{I}_b\boldsymbol{T}^T$$

- Body Axes: If axes (fixed to body) are neither principal nor stability/wind axes.

- EOM was derived with rigid body assumption for aircraft.
- Spinning portions of airplane relative to the body axes: propellers
- Each rotor has angular momentum relative to body axes.
- Assume resultant relative angular momentum of all rotors in F_B

$$\boldsymbol{h}' = (h'_x, h'_y, h'_z)$$

- Total angular momentum of an airplane with spinning rotors

$$\boldsymbol{h}_B = \boldsymbol{I}_B \boldsymbol{\omega}_B + \boldsymbol{h}'_B$$

- Extra terms in L, M, N equations

$$qh'_z - rh'_y, \;\; rh'_x - ph'_z, \; ph'_y - qh'_x$$

- What if the rotor axes are parallel to Cx with angular momentum $\boldsymbol{h}' = \boldsymbol{i} I\Omega$?

$$0, I\Omega r, -I\Omega q$$

Complete Equations of Motion: I

$$
\begin{aligned}
X - mg\sin\theta &= m\left(\dot{u}^E + qw^E - rv^E\right)\\
Y + mg\cos\theta\sin\phi &= m\left(\dot{v}^E + ru^E - pw^E\right)\\
Z + mg\cos\theta\cos\phi &= m\left(\dot{w}^E + pv^E - qu^E\right)
\end{aligned}
$$

$$
\begin{aligned}
L &= I_x\dot{p} - I_{zx}\dot{r} + \left(I_z - I_y\right)qr - I_{zx}pq + qh'_z - rh'_y\\
M &= I_y\dot{q} - I_{zx}\left(r^2 - p^2\right) - \left(I_z - I_x\right)rp + rh'_x - ph'_z\\
N &= I_z\dot{r} - I_{zx}(\dot{p} - qr) - \left(I_x - I_y\right)pq + ph'_y - qh'_x
\end{aligned}
$$

$$
\begin{aligned}
\dot{\phi} &= p + \left(q\sin\phi + r\cos\phi\right)\tan\theta\\
\dot{\theta} &= q\cos\phi - r\sin\phi\\
\dot{\psi} &= \left(q\sin\phi + r\cos\phi\right)\sec\theta
\end{aligned}
$$

Complete Equations of Motion: II

$$\begin{aligned}
\dot{x}_E = \; & u^E \cos\theta\cos\psi + v^E(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\
& + w^E(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\
\dot{y}_E = \; & u^E \cos\theta\sin\psi + v^E(\sin\phi\sin\theta\sin\psi - \cos\phi\cos\psi) \\
& + w^E(\cos\phi\sin\theta\sin\psi + \sin\phi\cos\psi) \\
\dot{z}_E = \; & -u^E\sin\theta + v^E\sin\phi\cos\theta + w^E\cos\phi\cos\theta
\end{aligned}$$

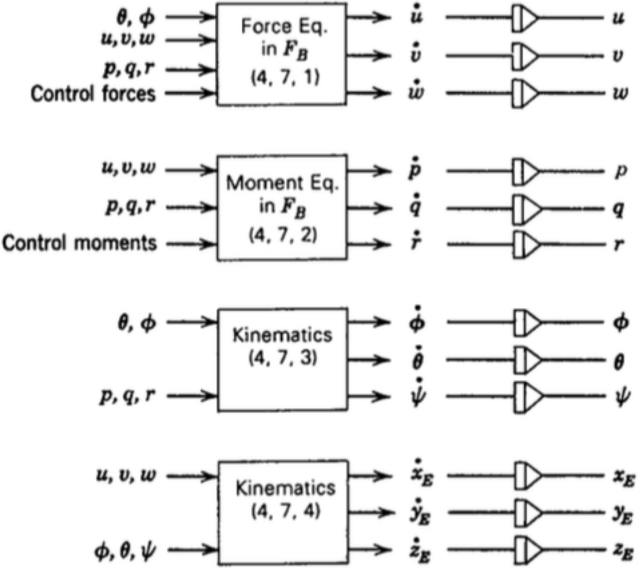
$$u^E = u + W_x,\; v^E = v + W_y,\; w^E = w + W_z$$

$$\begin{aligned}
p &= \dot{\phi} - \dot{\psi}\sin\theta \\
q &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\
r &= \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi
\end{aligned}$$

- 15 coupled nonlinear ordinary differential equations
- 3 Algebraic equations
- Dependence of aerodynamic forces
  - $\Rightarrow$ Relative motion of aircraft w.r.t. the air (given by $\boldsymbol{V}$ and $\boldsymbol{\omega}$)
  - $\Rightarrow$ Control variables which fixes control (movable) surface deflections
  - $\Rightarrow$ Throttle of propulsion controls
  - $\Rightarrow$ Forces and moments are functions of linear and angular velocities, and the control vector

$$u, v, w, p, q, r, \quad \boldsymbol{c} = [\delta_a, \delta_e, \delta_r, \delta_p]^T$$

- True implicit variables:
  - $\Rightarrow$ CG position (x_E, y_E, z_E)
  - $\Rightarrow$ Attitude (ψ, θ, ϕ)
  - $\Rightarrow$ Velocity (u^E, v^E, w^E)
  - $\Rightarrow$ Angular velocity (p, q, r)



θ, ϕ
u, v, w
p, q, r
Control forces
Force Eq. in F_B (4, 7, 1)
$\dot{u}$
$\dot{v}$
$\dot{w}$
u
v
w
u, v, w
p, q, r
Control moments
Moment Eq. in F_B (4, 7, 2)
$\dot{p}$
$\dot{q}$
$\dot{r}$
p
q
r
θ, ϕ
p, q, r
Kinematics (4, 7, 3)
$\dot{\phi}$
$\dot{\theta}$
$\dot{\psi}$
ϕ
θ
ψ
u, v, w
ϕ, θ, ψ
Kinematics (4, 7, 4)
$\dot{x}_E$
$\dot{y}_E$
$\dot{z}_E$
x_E
y_E
z_E

- Stability and control analysis: Linearization
- Reference flight condition: symmetric and with no angular velocity

$$v_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0$$

- Due to stability axes choice, $w_0 = 0$
- u_0 and θ_0 are reference flight speed and angle of climb, respectively.
- We have the relations, for small angle assumption,

$$\begin{aligned}
\sin(\theta_0 + \Delta\theta) &= \sin\theta_0 \cos\Delta\theta + \cos\theta_0 \sin\Delta\theta \\
&= \sin\theta_0 + \Delta\theta \cos\theta_0 \\
\cos(\theta_0 + \Delta\theta) &= \cos\theta_0 \cos\Delta\theta - \sin\theta_0 \sin\Delta\theta \\
&= \cos\theta_0 - \Delta\theta \sin\theta_0
\end{aligned}$$

- Δ is omitted if reference value is zero.

Linearization of Equations of Motion

$$
\begin{aligned}
X_0 + \Delta X - mg(\sin\theta_0 + \Delta\theta\cos\theta_0) &= m\Delta\dot{u} \\
Y_0 + \Delta Y + mg\phi\cos\theta_0 &= m\left(\dot{v} + u_0 r\right) \\
Z_0 + \Delta Z + mg(\cos\theta_0 - \Delta\theta\sin\theta_0) &= m\left(\dot{w} - u_0 q\right) \\
L_0 + \Delta L &= I_x\dot{p} - I_{zx}\dot{r} \\
M_0 + \Delta M &= I_y\dot{q} \\
N_0 + \Delta N &= -I_{zx}(\dot{p} - qr) + I_z\dot{r}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\phi} = p + r\tan\theta_0,\ \dot{\theta} &= q,\ \dot{\psi} = r\sec\theta_0 \\
\dot{x}_E &= (u_0 + \Delta u)\cos\theta_0 - u_0\Delta\theta\sin\theta + w\sin\theta_0 \\
\dot{y}_E &= u_0\psi\cos\theta_0 + v \\
\dot{z}_E &= -(u_0 + \Delta u)\sin\theta_0 - u_0\Delta\theta\cos\theta_0 + w\cos\theta_0
\end{aligned}
$$

- If all disturbance terms are zero then we obtain equations for reference flight conditions.
- In reference steady-state

$$\boxed{X_0 - mg\sin\theta_0 = 0,\ Y_0 = 0,\ Z_0 + mg\cos\theta_0 = 0}$$

- Moments are given by

$$\boxed{L_0 = M_0 = N_0 = 0}$$

- Also, kinematic equation is given by

$$\boxed{\dot{x}_E = \ u_0\cos\theta_0, \quad \dot{y}_E = \ 0, \quad \dot{z}_E = \ -u_0\sin\theta_0}$$

Linearization of Equations of Motion

$$
\begin{aligned}
\Delta\dot{u} &= \frac{\Delta X}{m} - g\Delta\theta\cos\theta_0 \\
\dot{v} &= \frac{\Delta Y}{m} + g\phi\cos\theta_0 - u_0 r \\
\dot{w} &= \frac{\Delta Z}{m} - \Delta\theta\sin\theta_0 + u_0 q \\
\dot{p} &= \frac{I_z\Delta L + I_{zx}\Delta N}{I_x I_z - I_{zx}^2} \\
\dot{q} &= \frac{\Delta M}{I_y} \\
\dot{r} &= \frac{I_{zx}\Delta L + I_x\Delta N}{I_x I_z - I_{zx}^2}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\dot{\theta} &= q \\
\dot{\phi} &= p + r\tan\theta_0 \\
\dot{\psi} &= r\sec\theta_0 \\
\Delta\dot{x}_E &= \Delta u\cos\theta_0 - u_0\Delta\theta\sin\theta_0 + w\sin\theta_0 \\
\Delta\dot{y}_E &= u_0\psi\cos\theta_0 + v \\
\Delta\dot{z}_E &= -\Delta u\sin\theta_0 - u_0\Delta\theta\cos\theta_0 + w\cos\theta_0
\end{aligned}
$$

- Problem of determining and describing aerodynamic forces and moments that act on a given body in arbitrary motion
- Aerodynamic forces and moments are functions of the state variables
- Consider lift force dependence on angle of attack
- Wing $\Rightarrow$ vortex wake $\Rightarrow$ induced velocity field at the wing.
- Due to hysteresis in flow separation processes, lift is dependent not only on the instantaneous value of α, but its entire past history.
- This functional relation is expressed by

$$L(t) = L(\alpha(\tau)) \ \forall \ -\infty \leq \tau \leq t$$

- Using Taylor's series

$$\alpha(\tau) = \alpha(t) + (\tau - t)\dot{\alpha}(t) + \frac{1}{2}(\tau - t)^2\ddot{\alpha}(t) + \dots$$

- At time instant $t = t_0$, lift forces depend on angle of attack and its derivatives.

$$L(t_0) = L(\alpha, \dot{\alpha}, \ddot{\alpha}, \cdots)$$

- Using Taylor's series around $\alpha(t_0), \dot{\alpha}(t_0), \ddot{\alpha}(t_0), \ldots$ yields

$$\Delta L(t_0) = L_\alpha \Delta\alpha + \frac{1}{2} L_{\alpha\alpha} (\Delta\alpha)^2 + \cdots + L_{\dot{\alpha}} \Delta\dot{\alpha} + \frac{1}{2} L_{\dot{\alpha}\dot{\alpha}} (\Delta\dot{\alpha})^2 + \cdots$$

where stability or aerodynamic derivative

$$L_\alpha = \left. \frac{\partial L}{\partial \alpha} \right|_{\alpha_0(t_0)}$$

- Lift force

$$\Delta L(t_0) = L_\alpha \Delta\alpha + L_{\dot{\alpha}} \Delta\dot{\alpha} + L_{\ddot{\alpha}} \Delta\ddot{\alpha} + \cdots$$

- For most of the quantities, the first term suffices to capture the behavior.

- For a truly symmetric configuration, $Y = L = N = 0$ in any condition of symmetric flight
- Also, β, p, r, ϕ, ψ, and y_E are all identically zero.
- The derivatives of asymmetric or lateral forces and moments, Y, L, N w.r.t. the symmetric or longitudinal motion variables u, w, q are zero.
- Neglect all the derivatives of symmetric forces and moments w.r.t. asymmetric motion variables.
- Neglect all derivatives w.r.t. rates of change of motion variables, except for $Z_{\dot{w}}$ and $M_{\dot{w}}$.
- $X_q \approx 0$ and $\rho =$ constant.
- None of these assumptions are necessary for solution of airplane dynamics problem.

- Linear forces and moments

$$
\begin{aligned}
\Delta X =& X_u \Delta u + X_w \Delta w + \Delta X_c \\
\Delta Y =& Y_v v + Y_p p + Y_r r + \Delta Y_c \\
\Delta Z =& Z_u \Delta u + Z_w w + Z_{\dot{w}} \dot{w} + Z_q q + \Delta Z_c \\
\Delta L =& L_v v + L_p p + L_r r + \Delta L_c \\
\Delta M =& M_u \Delta u + M_w w + M_{\dot{w}} \dot{w} + M_q q + \Delta M_c \\
\Delta N =& N_v v + N_p p + N_r r + \Delta N_c
\end{aligned}
$$

- Is there any issue if we substitute these equations in linearized version of equations?
- Solve for $\dot{w}$ and $\dot{q}$ and then we get linearized EOM.

Longitudinal Equations, Eq. (4.9,18):

$$\begin{bmatrix}\Delta\dot{u}\\ \dot{w}\\ \dot{q}\\ \Delta\dot{\theta}\end{bmatrix}=\begin{bmatrix}\dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_o\\ \dfrac{Z_u}{m-Z_{\dot{w}}} & \dfrac{Z_w}{m-Z_{\dot{w}}} & \dfrac{Z_q+mu_o}{m-Z_{\dot{w}}} & \dfrac{-mg\sin\theta_o}{m-Z_{\dot{w}}}\\ \dfrac{1}{I_y}\left[M_u+\dfrac{M_{\dot{w}}Z_u}{(m-Z_{\dot{w}})}\right] & \dfrac{1}{I_y}\left[M_w+\dfrac{M_{\dot{w}}Z_w}{(m-Z_{\dot{w}})}\right] & \dfrac{1}{I_y}\left[M_q+\dfrac{M_{\dot{w}}(Z_q+mu_o)}{(m-Z_{\dot{w}})}\right] & -\dfrac{M_{\dot{w}}\,mg\sin\theta_o}{I_y(m-Z_{\dot{w}})}\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ w\\ q\\ \Delta\theta\end{bmatrix}+\begin{bmatrix}\dfrac{\Delta X_c}{m}\\ \dfrac{\Delta Z_c}{m-Z_{\dot{w}}}\\ \dfrac{\Delta M_c}{I_y}+\dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m-Z_{\dot{w}})}\\ 0\end{bmatrix}$$

$$\Delta\dot{x}_E=\Delta u\cos\theta_o+w\sin\theta_o-u_o\,\Delta\theta\sin\theta_o$$

$$\Delta\dot{z}_E=-\Delta u\sin\theta_o+w\cos\theta_o-u_o\,\Delta\theta\cos\theta_o$$

Lateral Equations, Eq. (4.9,19):

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_o\right) & g\cos\theta_0 \\ \left(\dfrac{L_v}{I'_x} + I'_{zx}N_v\right) & \left(\dfrac{L_p}{I'_x} + I'_{zx}N_p\right) & \left(\dfrac{L_r}{I'_x} + I'_{zx}N_r\right) & 0 \\ \left(I'_{zx}L_v + \dfrac{N_v}{I'_z}\right) & \left(I'_{zx}L_p + \dfrac{N_p}{I'_z}\right) & \left(I'_{zx}L_r + \dfrac{N_r}{I'_z}\right) & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} \dfrac{\Delta Y_c}{m} \\ \dfrac{\Delta L_c}{I'_x} + I'_{zx}N_c \\ I'_{zx}\Delta L_c + \dfrac{\Delta N_c}{I'_z} \\ 0 \end{bmatrix}$$

$$\dot{\psi} = r\sec\theta_o$$

$$\Delta\dot{y}_E = u_o\psi\cos\theta_o + v$$

$$I'_x = (I_xI_z - I_{zx}^2)/I_z$$

$$I'_z = (I_xI_z - I_{zx}^2)/I_x$$

$$I'_{zx} = I_{zx}/(I_xI_z - I_{xz}^2)$$

- Suppose that $\beta, v, p, r, \Delta Y_c, \Delta L_c$, and ΔN_c are identically zero.
- Latter equations are all identically satisfied.
- Former equations form a complete set for the six homogeneous variables $\Delta u, w, q, \Delta\theta, \Delta x_E, \Delta z_E$.
- **Longitudinal or symmetric mode**: Motion in which only these variables differ from zero.
- **Lateral mode**: Motion where only ϕ, ψ, v, p, r, y_E differ from zero and zero longitudinal variables.

Reference

1. Bernard Etkin and Llyod Duff Reid, *Dynamics of Flight Stability and Control*, John Wiley and Sons, Third Edition, 1996.