Constant Pitch Formulation and Rate Solution

'V' and 'θ' Solutions

In this case, rocket is **commanded** to track a specified pitch **rate** i.e. ($d\theta/dt$), so that velocity **solution** is obtained **directly**, as shown below.

$$\dot{\theta} = q_0 \rightarrow \theta(t) = q_0 t + \theta_0; \quad \dot{\theta} = q_0 = \frac{\tilde{g}\sin\theta}{V(t)} \rightarrow V(t) = \frac{\tilde{g}\sin\theta}{q_0}$$

We see that '**V**' is a sinusoidal function of 'θ' and is also **inversely** proportional to '$\mathbf{q_0}$', indicating a higher **'V'** for both higher 'θ' and lower '$\mathbf{q_0}$'.

'V' & 'θ' Solution Constraints

As q_0 is **constant**, we can write the following **relation** also at the **start** of the gravity **turn.**

$$q_0 = \frac{\tilde{g}\sin\theta_0}{V_0} \neq 0 \text{ or } \infty \rightarrow \theta_0 = 0 \text{ and/or } V_0 = 0 \text{ Not admissible.}$$

This means that **gravity** turn can only be **started** from a non-zero 'θ', after it acquires a finite '**V**'.

'V' & 'θ' Solution Constraints

This **requirement** is usually met by giving a **'pitch kick'** to the vehicle at **appropriate** time.

Also, out of **'q_0',** 'V_0' & 'θ_0', only **two** can be specified.

'm' Solution

The **tangential** equilibrium can then be **rewritten** as,

$$\dot{V} = -\frac{\dot{m} g_0 I_{sp}}{m(t)} - \tilde{g}\cos\theta; \quad \frac{dV}{dt} = \dot{V} = \frac{d}{dt}\left(\frac{\tilde{g}\sin\theta}{q_0}\right) = \tilde{g}\cos\theta$$

$$\frac{dm}{m} = -\frac{2\tilde{g}\cos\theta dt}{g_0 I_{sp}} = -\frac{2\tilde{g}\cos\theta d\theta}{q_0 g_0 I_{sp}}$$

'm' Solution

Integrating both the sides, we get,

$$\boxed{\ln m = -\frac{2\tilde{g}\sin\theta}{q_0 g_0 I_{sp}} + C \rightarrow \ln\frac{m_0}{m} = \frac{2\tilde{g}}{q_0 g_0 I_{sp}}(\sin\theta - \sin\theta_0)}$$

It is seen that, **similar** to velocity, **mass** is also a **sinusoidal** function of 'θ' as well as **inversely** proportional to $\mathbf{q_0}$.

'm' Solution Features

This **indicates** that for small **'q_0',** large 'm_0' is **needed** to achieve **terminal** conditions.

Also, a **large** 'q_0' will **reduce** terminal **velocity.**

Therefore, the **design** of ascent mission under **constant** q_0 needs its **careful** selection.

't' & 'θ' Solutions as Function of 'm'

We can **solve** for the flight **time**, as follows.

$$t_b = \frac{(\theta_b - \theta_0)}{q_0}$$

We can also **determine** the final burnout inclination**, 'θ_b',** from mass **solution**, as follows.

$$\theta_b = \sin^{-1}\left\{\left(\frac{g_0 q_0 I_{sp}}{2\tilde{g}}\ln\frac{m_0}{m_b}\right) + \sin\theta_0\right\}$$

'h' Solution as Function of 'θ'

The **altitude** profile can be obtained by **resolving** the velocity **V** in vertical direction as **follows**.

$$\frac{dh}{dt} = V\cos\theta \rightarrow \frac{dh}{d\theta} = \frac{V\cos\theta}{q_0} = \frac{\tilde{g}\sin\theta\cos\theta}{q_0^2}$$

$$\frac{dh}{d\theta} = \frac{\tilde{g}\sin 2\theta}{2q_0^2} \rightarrow h(\theta) = \frac{\tilde{g}}{4q_0^2}(\cos 2\theta_0 - \cos 2\theta) + h_0$$

We find that **altitude** is a cosine function of '**2θ**', indicating that it **reaches** its peak value for θ **= 90°**.

'x' Solution as Function of 'θ'

Final parameter of interest is the **horizontal** distance, **'x'** along **Earth's** surface, which also impacts **'θ'** solution. The requisite **relation** can be obtained as **follows.**

$$\frac{dx}{dt} = V\sin\theta \rightarrow \int dx = \int V\sin\theta dt = \int \frac{\tilde{g}\sin^2\theta}{q_0^2}d\theta$$

$$x = \frac{1}{2q_0^2}\int \tilde{g}(1-\cos 2\theta)d\theta + C = \frac{\tilde{g}}{2q_0^2}\left[\theta - \left(1 - \frac{\sin 2\theta}{2}\right)\right] + C$$

$$x(\theta) = \frac{\tilde{g}}{2q_0^2}\left[(\theta-\theta_0) - \frac{(\sin 2\theta - \sin 2\theta_0)}{2}\right] + x(\theta_0)$$

Summary

Therefore, to **summarize**, it is possible to obtain **closed** form solutions for **trajectory** under the assumption of **constant** pitch rate.

We also **note** that the solution, so **obtained**, fixes the trajectory **time** once the starting and **terminal** angles are specified.