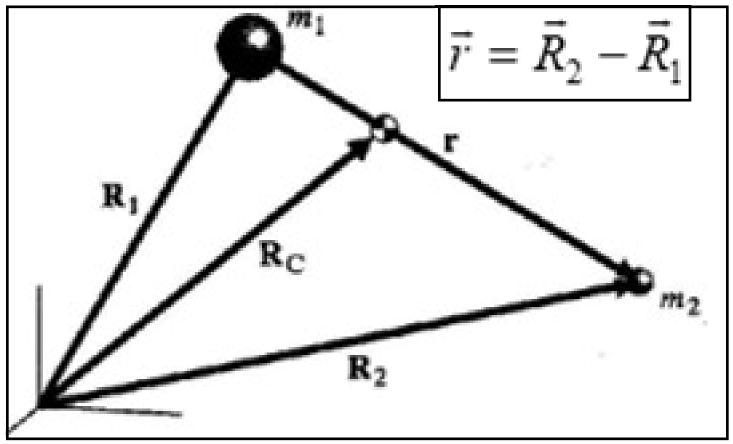
Two-body Motion Scenario

Basics of 2-Body Formulation

Consider the **two-body** system given below.



As **centre of mass** has no **acceleration**, it can be treated as the **inertial** frame origin and all position and velocity **vectors** can be defined as **relative** quantities**,** as above.

Basics of 2-Body Formulation

Basic equations for a **2-body system** are as given below.

$$m_1\ddot{\vec{R}}_1 = -\frac{Gm_1m_2\left(\vec{R}_2-\vec{R}_1\right)}{\left|\left(\vec{R}_2-\vec{R}_1\right)\right|^3}, \quad m_2\ddot{\vec{R}}_2 = +\frac{Gm_1m_2\left(\vec{R}_2-\vec{R}_1\right)}{\left|\left(\vec{R}_2-\vec{R}_1\right)\right|^3}$$

$$\vec{R}_c = \frac{m_1\vec{R}_1+m_2\vec{R}_2}{M}; \quad \vec{r} = \vec{R}_2-\vec{R}_1; \quad M = m_1+m_2$$

$$\vec{R}_1 = \vec{R}_C + \frac{m_2}{M}\vec{r}; \quad \vec{R}_2 = \vec{r}+\vec{R}_1 = \vec{R}_C + \left(1-\frac{m_2}{M}\right)\vec{r}$$

$$m_1\vec{R}_1\times\dot{\vec{R}}_1 + m_2\vec{R}_2\times\dot{\vec{R}}_2 = \vec{H}_0, \quad \frac{1}{2}m_1\dot{R}_1^2+\frac{1}{2}m_2\dot{R}_2^2-\frac{Gm_1m_2}{|\vec{r}|} = E_0$$

This system has **12 unknowns.**

2 – Body Simplification

2-body system can be solved if we **drop** 6 variables. An **acceptable** approximation is to assume that **one** body is much **larger**, leading to following **simplification.**

$$M = m_1 + m_2 \approx m_1; \quad \left({m_2}/{m_1} \right) \approx 0$$

$$\vec{R}_c = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1 + m_2} = \frac{\vec{R}_1 + \left({m_2}/{m_1} \right)\vec{R}_2}{1 + \left({m_2}/{m_1} \right)} \approx \vec{R}_1$$

Further, if **origin** is at the **centre of mass**, then we get $\mathbf{R_1} = \mathbf{0}$, and $(\mathbf{R_2} - \mathbf{R_1}) = \mathbf{r}$.

1 – Body Problem Definition

2-body simplification leads to the **case** that centre of mass of the **system** is at the **centre** of larger body.

Without loss of **generality** we can assume **larger** body to be **stationary,** so that its **centre** is a point fixed in **space.**

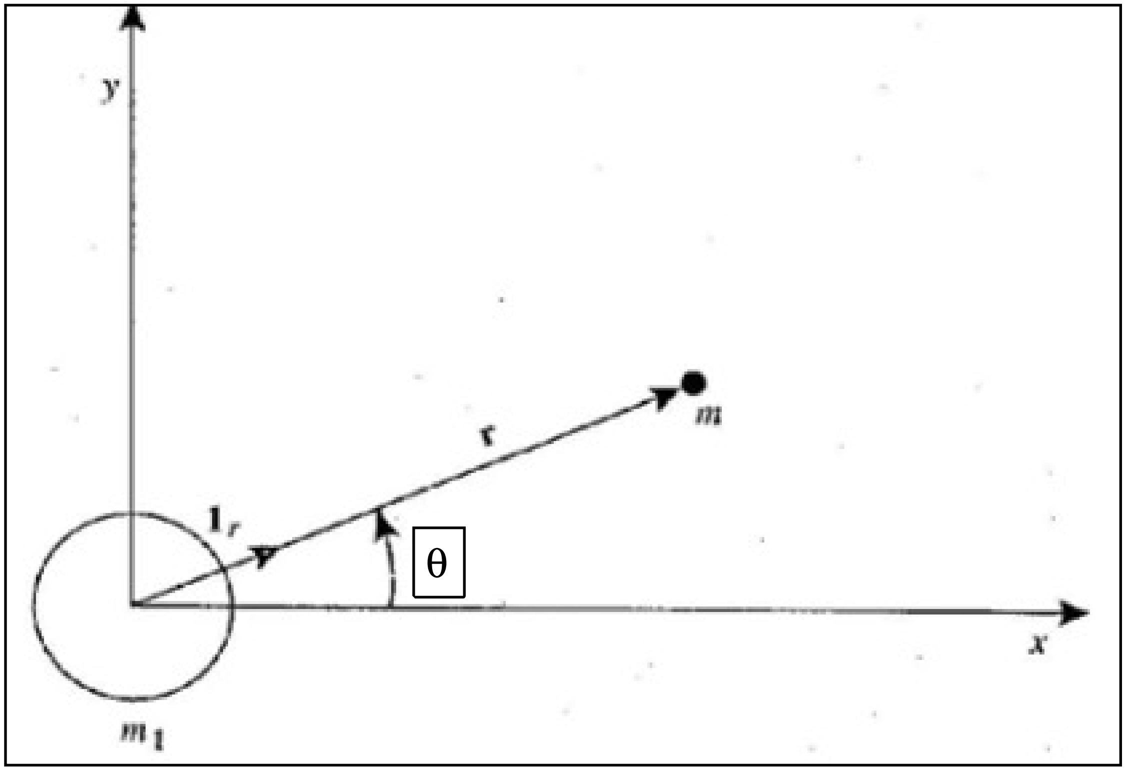
1 – Body Problem Definition

This is the case of **attractive** central force, where **smaller** body moves around a **fixed** point such that **force** is directed **along** the line joining centre and **fixed point.**

This results in what is commonly called **1-body** problem or **central** force motion.

Central Force Motion Formulation

Consider the **central** force system, as shown below.



Central Force Formulation

Assuming that $(\mathbf{m_1+m}) \approx \mathbf{m_1}$, we get the following.

$$m\ddot{\vec{r}} = -G\frac{m_1 m}{\vec{r}^3}\vec{r}; \quad \ddot{\vec{r}} = -Gm_1\frac{\vec{r}}{\vec{r}^3} \rightarrow \text{ Acceration of small body}$$

Further, we drop equation involving acceleration of large body, which is small, as follws. $m_1\ddot{\vec{r}} = G\frac{mm_1}{\vec{r}^3}\vec{r} \rightarrow \ddot{\vec{r}} = Gm\frac{\vec{r}}{\vec{r}^3} \approx 0$

Thus, we now have one vector equation in one variable, $\vec{r}$
for a spacecraft moving in Earth's gravitational field.

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0; \quad \mu = Gm_1; \quad G = 6.670\times10^{-11}Nm^2/kg^2$$

$$M_{Earth} = 5.977\times10^{24}kg; \quad \mu_E = GM_E = 3.986\times10^{14}m^3/s^2$$

Central Force Model Features

We see that we have **one** 2nd order vector differential **equation** in one variable **'r'**, so that we have **6 unknowns** and **6 initial conditions** on the smaller body.

These **6 initial conditions** are nothing but the **solution** at the end of the **ascent mission**, in terms of position and velocity **vectors** of the spacecraft.

Central Force Solution Basics

There are **many** ways in which the **central** force motion equation **can** be solved.

One way is to obtain the **closed** form time **solution** by integrating the corresponding **scalar** differential equations in (x, y, z), for specified **initial** conditions.

Central Force Solution Basics

Another way is to **numerically** integrate the **scalar** differential equations to **generate** time histories.

We can also use **vector** & scalar **products** (see earlier solutions) to **integrate** equations in terms of momentum & **energy.**

Summary

Two-body simplification is an **important** step in understanding the **nature** of motion any spacecraft.

It is to be **noted** that while, the proposed **model** is not exact, the **error** for most cases is quite **negligible**.