Optimal Solution Strategy

Lagrange Multiplier Concept

Lagrange multiplier is that **method** which adds one **extra** unknown, in a **consistent** manner for **problems** that have **equality** constraints.

We know that **solution** will be **optimal** only at a **point** where all **derivatives** are zero **simultaneously.**

Lagrange Multiplier Concept

Therefore, it is **sufficient** if the constraint is also **exactly** satisfied only at that **single** point.

This **results** in the concept of **constraint** error that needs to be **accounted** for, while generating **derivatives.**

Lagrange Multiplier Method

This is achieved by **augmenting** the objective **function** through the **addition** of a term **corresponding** to the constraint **error**, through an additional **unknown.**

In this manner, **partial** derivatives of the augmented objective function **include** the effect of **error** due to **inexact** satisfaction of **constraint.**

Lagrange Multiplier Concept

Here, the **additional** unknown, called the **Lagrange** multiplier, acts as a **weight** for the error due to **constraint.**

It can be **clearly** see that exact **optimal** solution is obtained when all '**N+1**' equations are exactly **satisfied.**

Constrained Optimization Formulation

Objective Functions

Given below are the **basic** equations of the two **objective** functions, for a **rocket** with 'N' **stages.**

$$\ln \pi_* = \sum_{i=1}^{N} \ln \pi_i; \quad V_* = -g_0 \sum_{i=1}^{N} I_{sp_i} \ln\left[\varepsilon_i + (1-\varepsilon_i)\pi_i\right]$$

Constraint Error Definition

Further, both $\boldsymbol{\pi}_*$ and $\mathbf{V}_*$ are functions of **$\boldsymbol{\pi}_i$'s,** which are the design **variables**, so that constraint **errors** are defined as,

$$e_\pi = \ln \pi_* - \sum_{i=1}^{N} \ln \pi_i$$

$$e_V = V_* + g_0 \sum_{i=1}^{N} I_{sp_i} \ln\left[\varepsilon_i + (1-\varepsilon_i)\pi_i\right]$$

Augmented Objective Functions

The **augmented** objective **functions** are defined below.

$$\ln \pi_* = \sum_{i=1}^{N} \ln \pi_i + \lambda \left(V_* + g_0 \sum_{i=1}^{N} I_{sp_i} \ln\left[\varepsilon_i + (1-\varepsilon_i)\pi_i \right] \right) = H_\pi(\lambda, \pi_i)$$

OR

$$V_* = -g_0 \sum_{i=1}^{N} I_{sp_i} \ln\left[\varepsilon_i + (1-\varepsilon_i)\pi_i \right] + \lambda \left(\ln \pi_* - \sum_{i=1}^{N} \ln \pi_i \right) = H_V(\lambda, \pi_i)$$

Augmented Function Features

It is clear that **partial** derivatives of the above **functions** contain both **objective** & constraint related **information.**

Lastly, 'λ', a constant, is the **Lagrange** multiplier.

Augmented Function Features

We see that 'λ' couples all the **π_i's** and also **includes** the effect of **constraint** in a manner that a **consistent** solution is obtained only if the **constraint** is satisfied **exactly.**

We can then use **π_1 to π_N** to obtain **stage-wise** mass **configuration**, along with the total **lift-off mass.**

Summary

Therefore, to **summarize**, constrained optimization technique **based** on Lagrange multipliers is **adequate** for arriving at the **best** possible stage payload **solutions**.

Of course, we **note** that we need to solve for **one** additional constant in order for **us** to incorporate the **constraint**.