Flight Mechanics/Dynamics

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- Complex numbers $\mathbb{C} = \{a + bi | \ a, \ b \in \mathbb{R}, \ i^2 = -1\}$ form a plane.
- Their operations are related to two-dimensional geometry.
- Any complex number has a length, given by the Pythagorean formula

$$|a + bi| = \sqrt{a^2 + b^2}$$

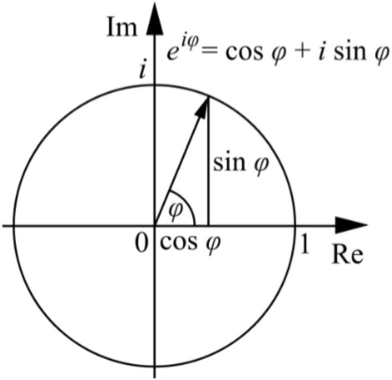
- We can add and subtract in $\mathbb{C}$.

$$a + bi + c + di = (a + c) + (b + d)i$$

- We can also multiply in $\mathbb{C}$.

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

- What does this last formula mean? difficult to interpret
- Fortunately, there is a better way to multiply complex numbers.



We can use complex arithmetic (multiplication) to perform a geometric operation (rotation).

- Geometrically, this formula says $e^{i\phi}$ lies on the unit circle in $\mathbb{C}$.
- If we multiply $e^{i\phi}$ by a positive number r, we get a complex number of length r, $re^{i\phi}$.
- If we denote $a + bi = r_1 e^{i\theta_1}$ and $c + di = r_2 e^{i\theta_2}$ then

$$(a + bi)(c + di) = r_1 r_2 e^{\theta_1 + \theta_2}$$

- To multiply two complex numbers, multiply their lengths and add their angles.
- In particular, if we multiply a given complex number z by $e^{i\phi}$ then it is rotated by ϕ degrees.

- The 19th century Irish mathematician and physicist *William Rowan Hamilton* was fascinated by the role of $\mathbb{C}$ in two-dimensional geometry.
- For years, he tried to invent an algebra of “triplets” to play the same role in three dimensions.
- On October 16th, 1843, while walking with his wife to a meeting of the Royal Society of Dublin, Hamilton discovered a 4-D division algebra called the *quaternions*.

- Although, similar concept was developed by Gauss in 1819 (but unfortunately not published).

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

- Hamilton noticed that

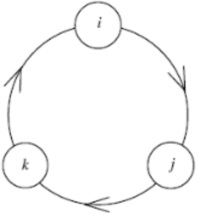
$$\boxed{\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1}$$

- The quaternions are denoted as

$$\mathbb{H} = \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} : a, b, c, d \in \mathbb{R}\}.$$

- Cyclic symmetry:

$$\begin{aligned} \boldsymbol{ij} =& \boldsymbol{k} = -\boldsymbol{ji} \\ \boldsymbol{jk} =& \boldsymbol{i} = -\boldsymbol{kj} \\ \boldsymbol{ki} =& \boldsymbol{j} = -\boldsymbol{ik} \end{aligned}$$



- Quaternions don't commute.
- $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are recognized as unit vectors.

- The quaternion product is the same as the cross product of vectors.

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{j} =& \boldsymbol{k} \\ \boldsymbol{j} \times \boldsymbol{k} =& \boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} =& \boldsymbol{j} \end{aligned}$$

- However, unlike the unit vectors $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0$, we have

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{i} =& -1 \\ \boldsymbol{j} \times \boldsymbol{j} =& -1 \\ \boldsymbol{k} \times \boldsymbol{k} =& -1 \\ \boldsymbol{i} \times \boldsymbol{j} \times \boldsymbol{k} =& -1 \end{aligned}$$

- A Hamilton quaternion can be considered as scalar part and a vector part

$$\boxed{[Q] = \langle q_0, \boldsymbol{q} \rangle = q_0 + \underbrace{q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}}_{\boldsymbol{q}}}$$

- $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ serves as basis for quaternion vector space.
- Quaternions span the space of real and imaginary numbers.
- Quaternion algebra includes scalar and vector algebra.
- Addition, subtraction and multiplication: Similar way as in vector algebra.
- **Addition**:

$$\begin{aligned} [Q] + [S] =& (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) + (s_0 + s_1\boldsymbol{i} + s_2\boldsymbol{j} + s_3\boldsymbol{k}) \\ =& (q_0 + s_0) + (q_1 + s_1)\boldsymbol{i} + (q_2 + s_2)\boldsymbol{j} + (q_3 + s_3)\boldsymbol{k} \end{aligned}$$

- **Subtraction**: Addition of negative quaternion $-[Q] = (-1)[Q]$

$$\begin{aligned}[Q]-[S] =&(q_0+q_1\boldsymbol{i}+q_2\boldsymbol{j}+q_3\boldsymbol{k})-(s_0+s_1\boldsymbol{i}+s_2\boldsymbol{j}+s_3\boldsymbol{k})\\ =&(q_0-s_0)+(q_1-s_1)\boldsymbol{i}+(q_2-s_2)\boldsymbol{j}+(q_3-s_3)\boldsymbol{k}\end{aligned}$$

- **Multiplication**:

$$\begin{aligned}[Q][S] =&(q_0+q_1\boldsymbol{i}+q_2\boldsymbol{j}+q_3\boldsymbol{k})(s_0+s_1\boldsymbol{i}+s_2\boldsymbol{j}+s_3\boldsymbol{k})\\ =&(q_0s_0-q_1s_1-q_2s_2-q_3s_3)+(q_0s_1+q_1s_0+q_2s_3-q_3s_2)\boldsymbol{i}\\ &+(q_0s_2-q_1s_3+q_2s_0+q_3s_1)\boldsymbol{j}+(q_0s_3+q_1s_2-q_2s_1+q_3s_0)\boldsymbol{k}\end{aligned}$$

- Using dot product of vectors $\boldsymbol{q}.\boldsymbol{s} = q_1s_1+q_2s_2+q_3s_3$, we have

$$\boxed{[Q][S] = \langle q_0, \boldsymbol{q}\rangle\langle s_0, \boldsymbol{s}\rangle = \langle q_0s_0 - \boldsymbol{q}.\boldsymbol{s}, q_0\boldsymbol{s}+s_0\boldsymbol{q}+\boldsymbol{q}\times\boldsymbol{s}\rangle}$$

- Product of two quaternions is still a quaternion, with scalar part $(q_0s_0 - \boldsymbol{q}\boldsymbol{s})$ and vector part $(q_0\boldsymbol{s}+s_0\boldsymbol{q}+\boldsymbol{q}\times\boldsymbol{s})$.

- The set of quaternions is closed under multiplication and addition.

Example

Consider two quaternions below and find their quaternion product.

$$\begin{aligned}[Q] =& 3 + \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} \\ [S] =& 2 - \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}\end{aligned}$$

$\Rightarrow$ $\boldsymbol{q}.\boldsymbol{s} = -2, \;\; \boldsymbol{q} \times \boldsymbol{s} = -8\boldsymbol{i} - 4\boldsymbol{j}.$

$\Rightarrow$ We know that

$$\begin{aligned}\langle q_0, \boldsymbol{q}\rangle\langle s_0, \boldsymbol{s}\rangle =& \langle q_0 s_0 - \boldsymbol{q}.\boldsymbol{s}, q_0\boldsymbol{s} + s_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s}\rangle \\ =& 6 - (-2) + 3(-\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + 2(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) + (-8\boldsymbol{i} - 4\boldsymbol{j}) \\ =& 8 - 9\boldsymbol{i} - 2\boldsymbol{j} + 11\boldsymbol{k}\end{aligned}$$

- **Scalar multiplication**:

$$\lambda[Q] = \lambda q_0 + \lambda q_1 \boldsymbol{i} + \lambda q_2 \boldsymbol{j} + \lambda q_3 \boldsymbol{k}$$

- **Conjugate**:

$$[Q]^\star = q_0 - \boldsymbol{q} = q_0 - q_1 \boldsymbol{i} - q_2 \boldsymbol{j} - q_3 \boldsymbol{k}$$

- We have the following relations

$$\begin{aligned}
([Q]^\star)^\star =& q_0 - (-\boldsymbol{q}) = [Q] \\
[Q] + [Q]^\star =& 2q_0 \\
[Q]^\star [Q] =& (q_0 + \boldsymbol{q})(q_0 - \boldsymbol{q}) \\
=& q_0 q_0 - (-\boldsymbol{q}).\boldsymbol{q} + q_0 \boldsymbol{q} + (-\boldsymbol{q}) q_0 + (-\boldsymbol{q} \times \boldsymbol{q}) \\
=& q_0^2 + \boldsymbol{q}.\boldsymbol{q} \\
=& q_0^2 + q_1^2 + q_2^2 + q_3^2 = [Q][Q]^\star
\end{aligned}$$

- **Norm of Length of quaternion**:

$$N([Q]) = [Q][Q]^\star = [Q]^\star[Q] = (q_0 + \boldsymbol{q})(q_0 - \boldsymbol{q}) = q_0^2 + \boldsymbol{q}.\boldsymbol{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

- For two quaternions $[Q]$ and $[S]$, we have

$$\boxed{([Q][S])^\star = [S]^\star[Q]^\star}$$

$\Rightarrow$ Proof:

$$\begin{aligned}([Q][S])^\star &= [(q_0 + \boldsymbol{q})(s_0 + \boldsymbol{s})]^\star \\ &= (q_0 s_0 - \boldsymbol{q}.\boldsymbol{s} + q_0\boldsymbol{s} + s_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s})^\star \\ &= (q_0 s_0 - \boldsymbol{q}.\boldsymbol{s} - q_0\boldsymbol{s} - s_0\boldsymbol{q} - \boldsymbol{q} \times \boldsymbol{s})\end{aligned}$$

$$\begin{aligned}[S]^\star[Q]^\star &= (s_0 - \boldsymbol{s})(q_0 - \boldsymbol{q}) \\ &= q_0 s_0 - (-\boldsymbol{s}).(-\boldsymbol{q}) + q_0(-\boldsymbol{s}) + s_0(-\boldsymbol{q}) + (-\boldsymbol{s}) \times (-\boldsymbol{q}) \\ &= (q_0 s_0 - \boldsymbol{q}.\boldsymbol{s} - q_0\boldsymbol{s} - s_0\boldsymbol{q} - \boldsymbol{q} \times \boldsymbol{s}) = ([Q][S])^\star\end{aligned}$$

- Norm of product of two quaternions is equal to product of their norms.

$$N([Q][S]) = N([Q])N([S])$$

$\Rightarrow$ Proof:

$$\begin{aligned} N([Q][S]) &= ([Q][S])([Q][S])^\star \\ &= [Q][S][S]^\star[Q]^\star \\ &= [Q]N([S])[Q]^\star \\ &= N([Q])N([S]) \end{aligned}$$

- Also, by using mathematical induction, one may write

$$\boxed{N([Q_1][Q_2]\dots[Q_n]) = N([Q_1])N([Q_2])\dots N([Q_n])}$$

- **Inverse of quaternion**: If $[Q] \neq 0$ then its inverse is defined by

$$[Q][Q]^{-1} = [Q]^{-1}[Q] = 1$$

- Using norm concept, $[Q]^{-1} = \dfrac{[Q]^{\star}}{N(Q)}$, $N(Q) \neq 0$. Does it make sense?

$$[Q][Q]^{-1} = [Q]^{-1}[Q] = \frac{[Q][Q]^{\star}}{N(Q)} = 1$$

- If $[Q]$ is unit quaternion then

$$[Q]^{-1} = \frac{[Q]^{\star}}{N(Q)} = [Q]^{\star}$$

- Inverse and conjugate for the unit quaternions are the same.

Identities:

- How to define zero and unit quaternions?
- A zero quaternion is quaternion with zero scalar and zero vector.
- A unit quaternion is defined as any quaternion whose norm is 1.

$$[0] = 0 + 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}, \quad [1] = 1 + 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$$

- Unlike direction cosine matrix, where six redundancies are present, the quaternion has only one.
- For unit quaternion,

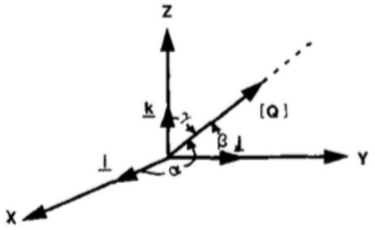
$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

- $\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ can be used for normalizing factor for each parameter.

- How to identify if two quaternions are equal?
- **Equality of quaternions**: Two quaternions are equal if both their scalars as well as their vectors are equal.

$$
\begin{aligned}
[Q] &= [S] \\
\Rightarrow q_0 &= s_0,\ q_1 = s_1,\ q_2 = s_2,\ q_3 = s_3 \\
\Rightarrow [Q]_i &= [S]_i\ \forall\ i = 0, 1, 2, 3
\end{aligned}
$$

- Can we express 3D vector as quaternion?
- Any three dimensional vector can be expressed as quaternion with zero scalar.



- Quaternions obey the associative and commutative laws of addition, and the associative and distributive laws of multiplication.
- For three quaternions, Q_1, Q_2, Q_3
  - Associative addition

$$(Q_1 + Q_2) + Q_3 = Q_1 + (Q_2 + Q_3)$$

  - Commutative addition

$$Q_1 + Q_2 = Q_2 + Q_1$$

  - Associative multiplication

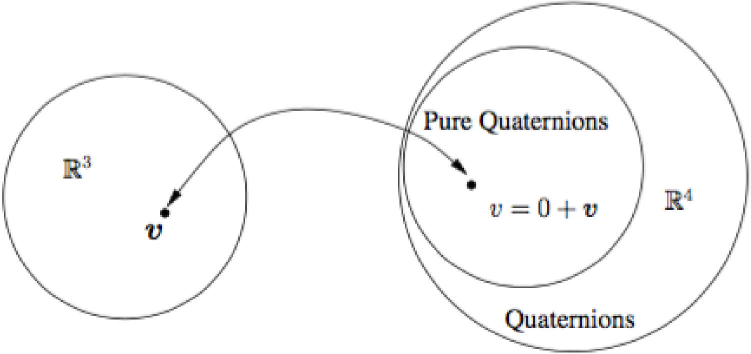
$$(Q_1 Q_2) Q_3 = Q_1 (Q_2 Q_3)$$

  - Distributive multiplication

$$Q_1 (Q_2 + Q_3) = Q_1 Q_2 + Q_1 Q_3$$

- Is the multiplication of quaternions commutative?

- Pure quaternions: Quaternion with zero real or scalar part
- Any vector in $\mathbb{R}^3$ is a pure quaternion.



- How quaternion $[Q] \in \mathbb{R}^4$ operate on a vector in $\mathbb{R}^3$?
- Define quaternion operator with unit quaternion $[Q]$ as

$$L_Q(\boldsymbol{v}) = [Q]\boldsymbol{v}[Q]^\star = (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v})$$

$\Rightarrow$ Proof:

$$\begin{aligned}
L_Q(\boldsymbol{v}) =& [Q]\boldsymbol{v}[Q]^\star = (q_0 + \boldsymbol{q})(0 + \boldsymbol{v})(q_0 - \boldsymbol{q}) \\
=& (q_0 + \boldsymbol{q})(\boldsymbol{v}.\boldsymbol{q} + \{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\}) \\
=& q_0(\boldsymbol{v}.\boldsymbol{q}) - \boldsymbol{q}.\{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} + (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0\{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} \\
& + \boldsymbol{q} \times \{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} \\
=& \boldsymbol{q}.\{\boldsymbol{v} \times \boldsymbol{q}\} + (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0\{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} + \boldsymbol{q} \times \{q_0\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{q}\} \\
=& (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0^2\boldsymbol{v} + 2q_0\{\boldsymbol{q} \times \boldsymbol{v}\} + \boldsymbol{q} \times \{\boldsymbol{q} \times \boldsymbol{v}\} \\
=& (\boldsymbol{v}.\boldsymbol{q})\boldsymbol{q} + q_0^2\boldsymbol{v} + 2q_0\{\boldsymbol{q} \times \boldsymbol{v}\} + \boldsymbol{q}(\boldsymbol{q}.\boldsymbol{v}) - \boldsymbol{v}(\boldsymbol{q}.\boldsymbol{q}) \\
=& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}.\boldsymbol{v})\boldsymbol{q} + 2q_0\{\boldsymbol{q} \times \boldsymbol{v}\}
\end{aligned}$$

Operation of Unit Quaternion on Vector

This operator L_Q does not change the length of the vector $\boldsymbol{v}$.

$$\|L_Q(\boldsymbol{v})\| = \|[Q]\boldsymbol{v}[Q]^\star\| = |[Q]|\|\boldsymbol{v}\||[Q]^\star| = \|\boldsymbol{v}\|$$

The direction of $\boldsymbol{v}$, if along $\boldsymbol{q}$ (say $\boldsymbol{v} = k\boldsymbol{q}$), is left unchanged by the operator L_Q.

$$\begin{aligned}[Q]\boldsymbol{v}[Q]^\star =& [Q](k\boldsymbol{q})[Q]^\star = (q_0^2 - |\boldsymbol{q}|^2)(k\boldsymbol{q}) + 2(\boldsymbol{q}.(k\boldsymbol{q}))\boldsymbol{q} + 2q_0(\boldsymbol{q} \times (k\boldsymbol{q})) \\ =& k(q_0^2 + \|\boldsymbol{q}\|^2)\boldsymbol{q} = k\boldsymbol{q}\end{aligned}$$

- Any vector along $\boldsymbol{q}$ is thus not changed under operator L_Q. This makes us guess that the operator L_Q acts like a rotation about $\boldsymbol{q}$.
- The operator L_Q is linear over $\mathbb{R}^3$. For any two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ and any $a_1, a_2 \in \mathbb{R}$

$$L_Q(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2) = a_1L_Q(\boldsymbol{v}_1) + a_2L_Q(\boldsymbol{v}_2).$$

Rotation of Vector using Quaternion

For any unit quaternion

$$[Q] = q_0 + \boldsymbol{q} = \cos\frac{\theta}{2} + \hat{q}\sin\frac{\theta}{2}$$

and for any vector $\boldsymbol{v} \in \mathbb{R}^3$, the action of the operator $L_Q(\boldsymbol{v}) = [Q]\boldsymbol{v}[Q]^\star$ on $\boldsymbol{v}$ is equivalent to a rotation of the vector through an angle θ, about $\hat{q}$ as the axis of rotation.

- A vector $\boldsymbol{v} \in \mathbb{R}^3$, we decompose it as $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{n}$, where $\boldsymbol{a}$ is the component along the vector $\boldsymbol{q}$ and $\boldsymbol{n}$ is the component normal to $\boldsymbol{q}$.
- Under the operator L_Q, $\boldsymbol{a}$ is invariant, while $\boldsymbol{n}$ is rotated about $\boldsymbol{q}$ through an angle θ.
- Since the operator is linear, the image $[Q]\boldsymbol{v}[Q]^\star$ is indeed interpreted as a rotation of $\boldsymbol{v}$ about $\boldsymbol{q}$ through an angle θ.

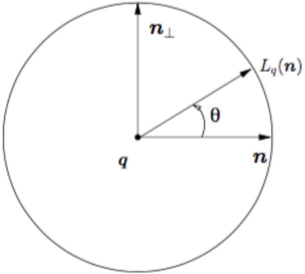
- The operator L_q on vectors $\boldsymbol{n}$

$$
\begin{aligned}
L_Q(\boldsymbol{n}) =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2(\boldsymbol{q}.\boldsymbol{n})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{n}) \\
=& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2q_0(\boldsymbol{q} \times \boldsymbol{n}) \\
=& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2q_0\|\boldsymbol{q}\|(\hat{\boldsymbol{q}} \times \boldsymbol{n})
\end{aligned}
$$

- Denote $\boldsymbol{n}_\perp = \hat{\boldsymbol{q}} \times \boldsymbol{n}$. Now,

$$
\begin{aligned}
L_Q(\boldsymbol{n}) =& (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{n} + 2q_0\|\boldsymbol{q}\|\boldsymbol{n}_\perp \\
=& \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\boldsymbol{n} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\boldsymbol{n}_\perp \\
=& \cos\theta\boldsymbol{n} + \sin\theta\boldsymbol{n}_\perp
\end{aligned}
$$

- Resulting vector is a rotation of $\boldsymbol{n}$ through an angle θ in the plane defined by $\boldsymbol{n}$ and $\boldsymbol{n}_\perp$.



- $L_{-[Q]} = [-Q]\boldsymbol{v}[-Q]^\star = [Q]\boldsymbol{v}[Q]^\star$ How?
- Negative quaternion $-[Q]$

$$-[Q] = \cos\frac{2\pi+\theta}{2} + \hat{q}\sin\frac{2\pi+\theta}{2}$$

- It represents the rotation about the same axis through the angle $2\pi+\theta$, essentially the same rotation.

Rotation of Coordinate Frame using Quaternion

For any unit quaternion $[Q] = q_0 + \boldsymbol{q} = \cos\dfrac{\theta}{2} + \hat{q}\sin\dfrac{\theta}{2}$ and for any vector $\boldsymbol{v} \in \mathbb{R}^3$ the action of the operator $L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q]^{\star\star} = [Q]^\star \boldsymbol{v}[Q]$ is a rotation of the coordinate frame about the axis $\hat{q}$ through an angle θ while $\boldsymbol{v}$ is not rotated.

- Rotation of $\boldsymbol{v}$ under the operator L_Q can also be interpreted from the perspective of an observer attached to the vector $\boldsymbol{v}$.
- What he sees happening is that the coordinate frame rotates through the angle $-\theta$ about the same axis defined by the quaternion.
- $L_{Q^\star}$ rotates the vector $\boldsymbol{v}$ with respect to the coordinate frame through an angle $-\theta$ about $\boldsymbol{q}$.
- $L_Q(\boldsymbol{v}) = [Q]v[Q]^\star$ may be interpreted as a point or vector rotation with respect to the (fixed) coordinate frame.
- $L_{Q^\star}(\boldsymbol{v}) = [Q]^\star \boldsymbol{v}[Q]$ may be interpreted as a coordinate frame rotation with respect to the (fixed) space of points.

Reference

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2. Bandhu N. Pamadi, *Performance, Stability, and Control of Airplanes*, AIAA Education Series, 1998.

Thank you for your attention !!!