Flight Mechanics/Dynamics

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- Longitudinal system dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\Delta\boldsymbol{c}$$

- Control component vector

$$\boldsymbol{B}\Delta\boldsymbol{c} = \begin{bmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{\Delta M_{\dot{w}}}{I_y} + \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ 0 \end{bmatrix}$$

- Longitudinal control and control derivatives

$$\boldsymbol{c} = \begin{bmatrix} \delta_e \\ \delta_p \end{bmatrix}, \quad \begin{bmatrix} \Delta X_c \\ \Delta Z_c \\ \Delta M_c \end{bmatrix} = \begin{bmatrix} X_{\delta_e} & X_{\delta_p} \\ Z_{\delta_e} & Z_{\delta_p} \\ M_{\delta_e} & M_{\delta_p} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_p \end{bmatrix}$$

- Is this representation of system always sufficient? May not be
- Constant derivatives to describe the force output in response to throttle input does not allow for any time lag.
- Constant derivatives implies that the thrust is instantaneously proportional to the throttle position.
- Reasonable for propeller airplanes, but not a good model for jets in situations when the short-term response is important.
- In the Laplace domain, one can use control transfer functions instead of control derivatives, X_{δ_p} by $G_{x_{\delta_p}}(s)$.
- What to do in time domain to achieve similar result?
- Addition of an extra differential equation and an additional variable.

- System input matrix, for the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\delta$,

$$\boldsymbol{B} = \begin{bmatrix} \frac{X_{\delta_e}}{m} & \frac{X_{\delta_p}}{m} \\ \frac{Z_{\delta_e}}{m - Z_{\dot{w}}} & \frac{Z_{\delta_p}}{m - Z_{\dot{w}}} \\ \frac{M_{\delta_e}}{I_y} + \frac{M_{\dot{w}} Z_{\delta_e}}{I_y(m - Z_{\dot{w}})} & \frac{M_{\delta_p}}{I_y} + \frac{M_{\dot{w}} Z_{\delta_p}}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 \end{bmatrix}$$

- For example aircraft B747

$$\boldsymbol{A} = \begin{bmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} -0.000187 & 9.66 \\ -17.85 & 0 \\ -1.158 & 0 \\ 0 & 0 \end{bmatrix}$$

- How to obtain the transfer functions relating elevator input to the states?
- With such cases, matrix $\boldsymbol{B}$ reduces to consist of only first column.
- Four transfer functions:

$$G_{u\delta_e} = \frac{\Delta \bar{u}}{\bar{\delta}_e}, \;\; G_{w\delta_e} = \frac{\bar{w}}{\bar{\delta}_e}, \;\; G_{q\delta_e} = \frac{\bar{q}}{\bar{\delta}_e}, \;\; G_{\Delta\theta\delta_e} = \frac{\Delta\bar{\theta}}{\bar{\delta}_e}$$

- Denominators of all transfer functions are same, given by

$$D(s) = s^4 + 0.750468s^3 + 0.93549s^2 + 0.0094630s + 0.0041959$$

- Numerators

$$\begin{aligned}
N_{u\delta_e} &= -0.000188s^3 - 0.02491s^2 + 24.68s + 11.16 \\
N_{w\delta_e} &= -17.85s^3 - 904.s^2 - 6.208s - 3.445 \\
N_{q\delta_e} &= -1.158s^3 - 0.354s^2 - 0.003873s \\
N_{\theta\delta_e} &= -1.158s^2 - 0.3545s - 0.003873
\end{aligned}$$

- Are there any other transfer functions of interest?
- Flight path angle and load factor. How to obtain them?
- With $\theta_0 = 0$, $\Delta\theta = \theta$ and $\Delta\gamma = \Delta\theta - \Delta\alpha$

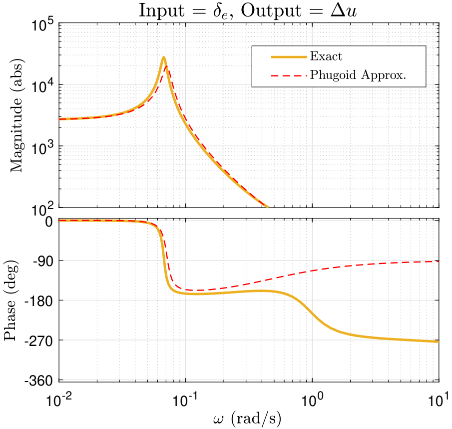
$$G_{\gamma_{\delta_e}} = G_{\theta_{\delta_e}} - G_{\alpha_{\delta_e}}$$

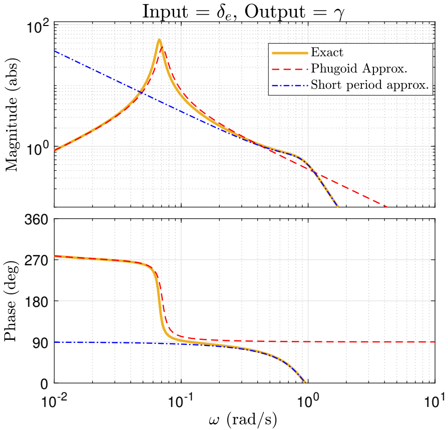
- Load factor, $n_z = -\frac{Z}{W}$, which is unity in a level flight.
- Incremental value of load factor during response to elevator input

$$\Delta n_z = -\frac{\Delta Z}{W} = -\frac{Z_u \Delta u + Z_w w + Z_q q + Z_{\dot{w}} \dot{w} + Z_{\delta_e} \Delta\delta_e}{W}$$

- Transfer function for the load factor

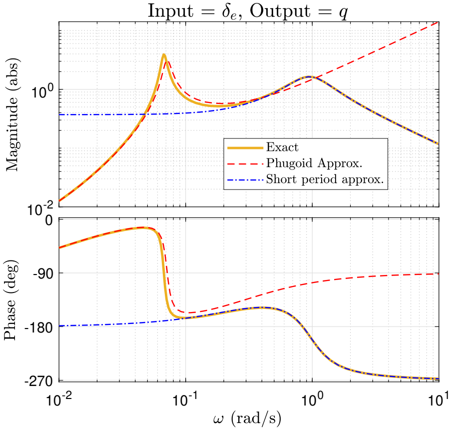
$$G_{n_z \delta_e} = -\frac{Z_u G_{u\delta_e} + Z_w G_{w\delta_e} + Z_q G_{q\delta_e} + Z_{\dot{w}} G_{\dot{w}\delta_e} + Z_{\delta_e}}{W}$$

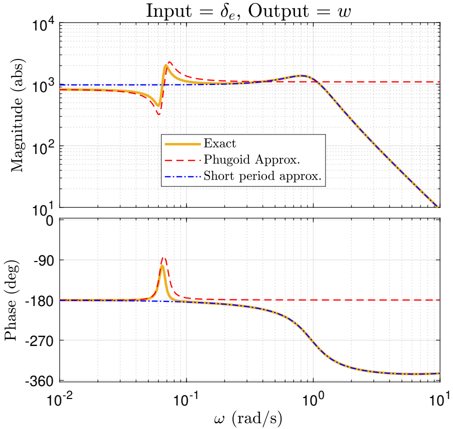




Large peak at low frequency due to light damping of phugoid mode and diminishing response at high frequency

Phase also changes sharply at low frequency and second drop at high frequency

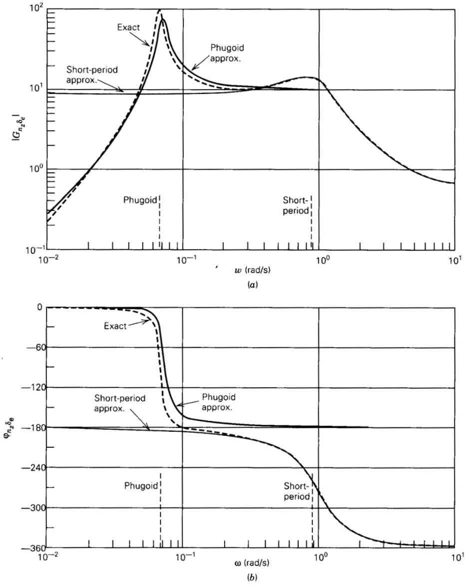




Complicated behavior near phugoid frequency due to pole-zero location close to each other

Diminishing response for high frequency

Frequency-Response Curves: Load Factor



- Large resonant peak at phugoid frequency in amplitude of load factor Δn_z.
- What is disadvantage due to this property?
- Large elevator deflection may cause structural failure.

- Differential equations

$$\begin{bmatrix} \Delta\dot{u} \\ \dot{w} \\ 0 \\ \Delta\dot{\theta} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \Delta u \\ w \\ q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} \frac{X_{\delta_e}}{m} \\ \frac{Z_{\delta_e}}{m} \\ M_{\delta_e} \\ 0 \end{bmatrix} \Delta\delta_e$$

- Transfer functions

$$G_{u\delta_e} = \frac{a_1 s + a_0}{f(s)}$$

$$G_{w\delta_e} = \frac{b_2 s^2 + b_1 s + b_0}{f(s)}$$

$$G_{\theta\delta_e} = \frac{c_2 s^2 + c_1 s + c_0}{f(s)}$$

where $f(s) = As^2 + Bs^2 + C$ is the characteristic polynomial.

- Differential equations

$$(s\boldsymbol{I} - \boldsymbol{A})\begin{bmatrix} \bar{w} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_{\delta_e}}{m} \\ \frac{M_{\delta_e}}{I_y} + \frac{M_{\dot{w}}}{I_y}\frac{Z_{\delta_e}}{m} \end{bmatrix}\delta_e$$

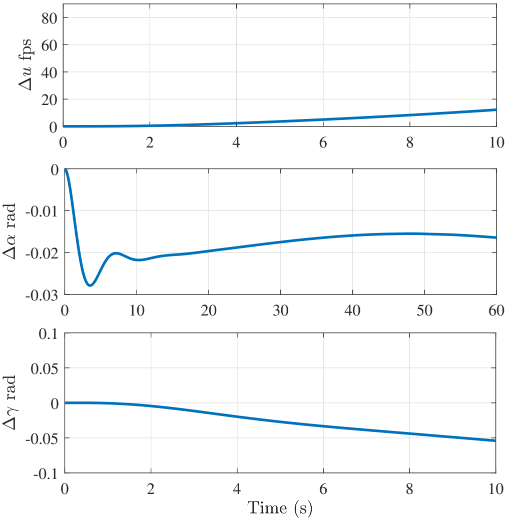
- Transfer functions

$$G_{w\delta_e} = \frac{a_1 s + a_0}{f(s)}, \quad G_{\theta\delta_e} = \frac{b_1 s + b_0}{sf(s)} = \frac{G_{q\delta_e}}{s}$$

where $f(s) = s^2 + c_1 s^2 + c_0$ is the characteristic polynomial.

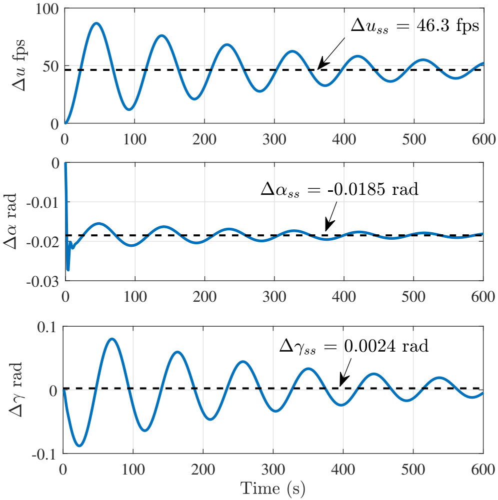
- Phugoid approximation is exact at very low frequencies and the short-period approximation is exact in the high-frequency limit.
- For frequencies between those of phugoid and short-period modes, one approximation or the other can give reasonable results.

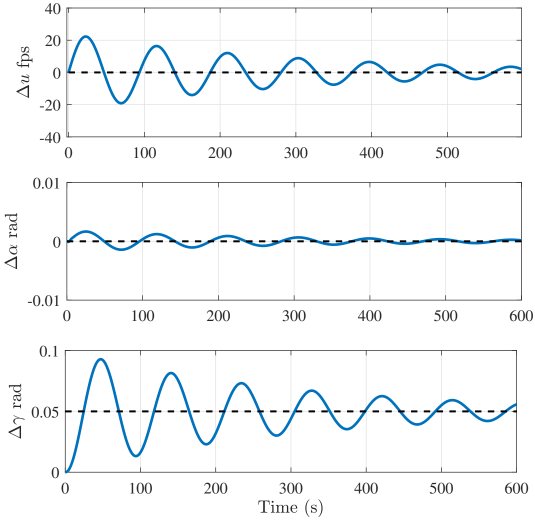
- What about the step responses of the aircraft?
- Angle of attack responds quickly to elevator motion during first 10 sec.
- Its variation is dominated by the rapid, well-damped short-period mode.
- Speed and flight path angle, respond much more slowly.



Long Term Response to Elevator Input of 1 Degree

- Dynamic response persists for a very long time due to the phugoid mode
- **Steady state**: Slightly higher speed and a slightly smaller angle of attack than the original flight condition
- Expected from a down movement of the elevator.
- Flight path angle is seen to be almost unchanged.
- Not a successful way to change steady-state flight condition





- V and α do not change but γ settles at a new steady state.
- What would happen if the thrust line does not pass through CG?
- α changes fast and decay to new value, $\Delta\alpha_{ss} \neq 0$, $\Delta u_{ss} \neq 0$.

- How to obtain response of aircraft to rudder or aileron inputs?
- Procedure is similar to that in longitudinal direction.
- Lateral controls and their derivatives

$$\begin{bmatrix} \Delta Y_c \\ \Delta L_c \\ \Delta N_c \end{bmatrix} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

- Laplace transform of system

$$(s\boldsymbol{I} - \boldsymbol{A})[\bar{v} \quad \bar{p} \quad \bar{r} \quad \bar{\phi}]^T = \boldsymbol{B}[\delta_a \quad \delta_r]^T$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{Y_{\delta_a}}{m} & \frac{Y_{\delta_r}}{m} \\ \frac{L_{\delta_a}}{I'_x} + I'_{zx}N_{\delta_a} & \frac{L_{\delta_r}}{I'_x} + I'_{zx}N_{\delta_r} \\ \frac{N_{\delta_a}}{I'_z} + I'_{zx}L_{\delta_a} & \frac{N_{\delta_r}}{I'_x} + I'_{zx}L_{\delta_r} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5.642 \\ -0.1431 & 0.1144 \\ 0.003741 & -0.4859 \\ 0 & 0 \end{bmatrix}$$

- Denominators for all transfer functions are same and is given by

$$D(s) = s^4 + 0.6358s^3 + 0.9388s^2 + 0.5114s + 0.003682$$

- Numerators are given by

$$\begin{aligned}
N_{v\delta_a} &= 2.896s^2 + 6.542s + 0.6220 \\
N_{v\delta_r} &= 5.642s^3 + 379.4s^2 + 167.9s - 5.934 \\
N_{p\delta_a} &= 0.1431s^3 + 0.02630s^2 + 0.1102s \\
N_{p\delta_r} &= 0.1144s^3 - 0.1997s^2 - 1.368s \\
N_{r\delta_a} &= -0.003741s^3 - 0.002708s^2 - 0.0001394s + 0.004539 \\
N_{r\delta_r} &= -0.4859s^3 - 0.2327s^2 - 0.009018s - 0.05647 \\
N_{\phi\delta_a} &= 0.1431s^2 + 0.02730s + 0.1102 \\
N_{\phi\delta_r} &= 0.1144s^2 - 0.1997s - 1.368
\end{aligned}$$

- Approximate transfer function might be useful in designing controller.
- Two second order system as approximation to fourth order complete system
- **Sprial/Roll approximation**:

$$\begin{bmatrix} 0 & 0 & u_0 & -g \\ -\mathcal{L}_v & (s-\mathcal{L}_p) & -\mathcal{L}_r & 0 \\ -\mathcal{N}_v & -\mathcal{N}_p & (s-\mathcal{N}_r) & 0 \\ 0 & -1 & 0 & s \end{bmatrix} \begin{bmatrix} \bar{v} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_{\delta_a} & \mathcal{Y}_{\delta_r} \\ \mathcal{L}_{\delta_a} & \mathcal{L}_{\delta_r} \\ \mathcal{N}_{\delta_a} & \mathcal{N}_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

- Denominators for all transfer functions are same, and is $Cs^2 + Ds + E$.
- Numerators, with $\delta = \delta_a, \delta_r$, are given by

$$\begin{aligned} N_{v\delta} =& a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ N_{\phi\delta} =& b_1 s + b_0 \\ N_{r\delta} =& d_2 s^2 + d_1 s + d_0 \\ N_{p\delta} =& s N_{\phi\delta} \end{aligned}$$

- **Dutch Roll approximation**:

$$\begin{aligned}\dot{v} =& \mathcal{Y}_v v - u_0 r + \Delta \mathcal{Y}_c \\ \dot{r} =& \mathcal{N}_v v - \mathcal{N}_r r + \Delta \mathcal{N}_c\end{aligned}$$

- We can rewrite system of equations as

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 & \mathcal{Y}_{\delta_r} \\ \mathcal{N}_{\delta_a} & \mathcal{N}_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

where

$$\mathcal{Y}_{\delta_r} = \frac{Y_{\delta_r}}{m}, \quad \mathcal{N}_{\delta_a} = I'_{zx} L_{\delta_a} + \frac{N_{\delta_a}}{I'_z}, \quad \mathcal{N}_{\delta_r} = I'_{zx} L_{\delta_r} + \frac{N_{\delta_r}}{I'_z}$$

- Denominators for all transfer functions are same and is

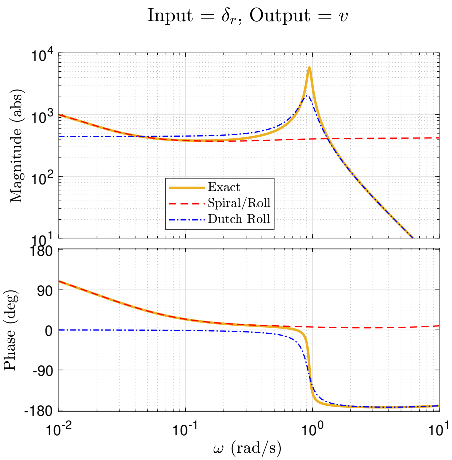
$$f(s) = s^2 - (\mathcal{Y}_v + \mathcal{N}_r)s + (\mathcal{Y}_v\mathcal{N}_r + u_0\mathcal{N}_v)$$

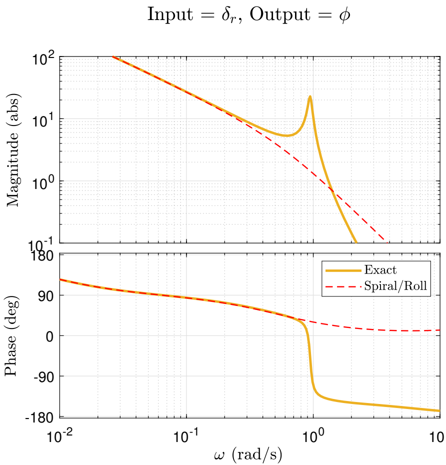
- Numerators

$$\begin{aligned}
N_{v\delta_a} &= -\, u_0\mathcal{N}_{\delta_a} \\
N_{r\delta_a} &= \mathcal{N}_{\delta_a}s - \mathcal{Y}_v\mathcal{N}_{\delta_a} \\
N_{v\delta_r} &= \mathcal{Y}_{\delta_r}s - (\mathcal{Y}_{\delta_r}\mathcal{N}_r + u_0\mathcal{N}_{\delta_r}) \\
N_{r\delta_r} &= \mathcal{N}_{\delta_r} - (\mathcal{Y}_v\mathcal{N}_{\delta_r} - \mathcal{Y}_{\delta_r}\mathcal{N}_v)
\end{aligned}$$

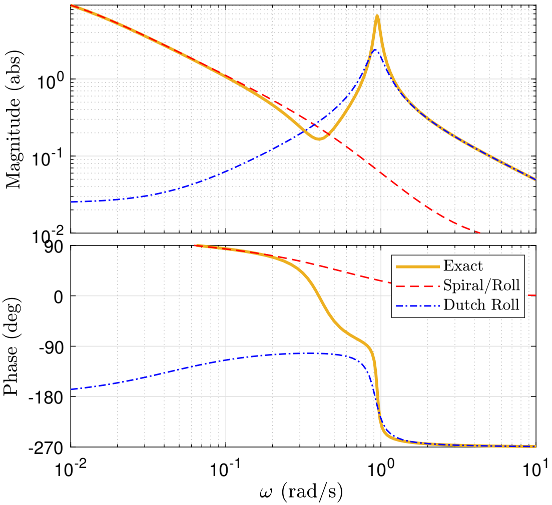
- DR mode is exact in the limit of high frequency while Spiral/Roll is exact as $\omega \rightarrow 0$.
- What similarity do you observe in approximation w.r.t. longitudinal direction?
- Analogous to longitudinal case, Spiral/Roll $\rightarrow$ Phugoid and DR $\rightarrow$ SP mode.

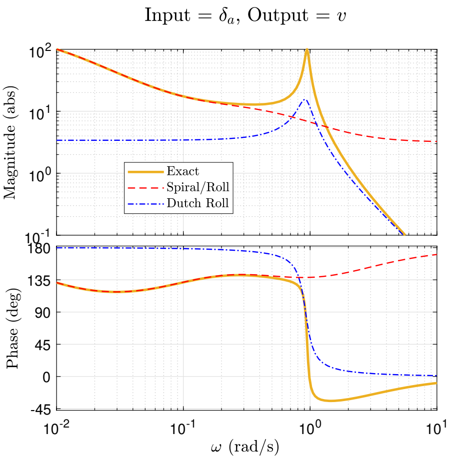
Frequency Response to Rudder Input: Sideslip and Roll

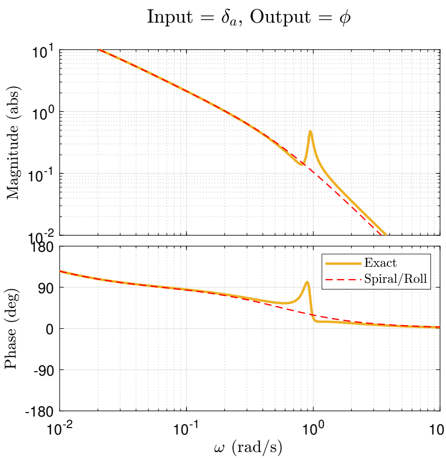




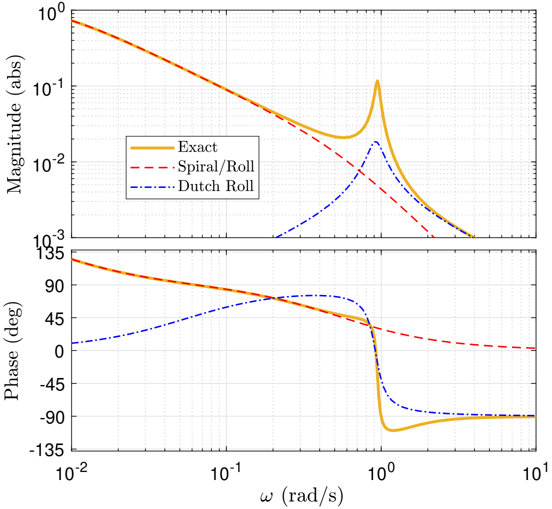
Input $= \delta_r$, Output $= r$




Input = δ_a, Output = v
Magnitude (abs)
Phase (deg)
Exact
Spiral/Roll
Dutch Roll
ω (rad/s)


Input = δ_a, Output = ϕ
Magnitude (abs)
Phase (deg)
Exact
Spiral/Roll
ω (rad/s)

Input $= \delta_a$, Output $= r$



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Thank you for your attention !!!