Flight Mechanics/Dynamics

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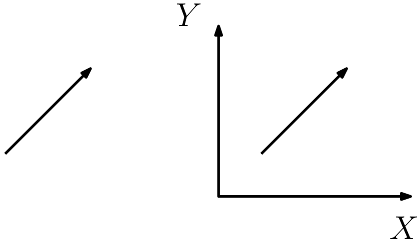
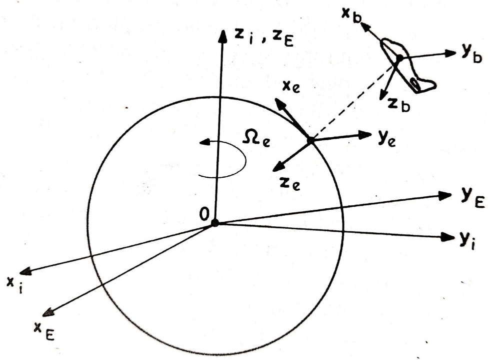
- Why do you need coordinate frames?



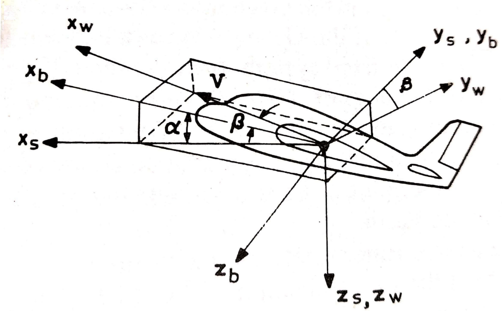
- To determine motion of a vehicle, it becomes necessary to relate the solution to the motion of Earth.
  - $\Rightarrow$ Define inertial reference frame w.r.t. the Earth
  - $\Rightarrow$ Obtain motion of both vehicle and Earth w.r.t. the inertial frame
- Initial orientation of reference coordinate frame, position, and velocity are required to obtain future orientation, position and velocity.

- How do you define inertial frame?
  - ⇒ Reference frame in which Newton's laws of motion are valid.
  - ⇒ A set of mutually perpendicular axes that neither accelerate nor rotate with respect to inertial space.
  - ⇒ Fixed relative to the stars
- Newton's laws are also valid in Galilean frames.
- Galilean frames: Those which do not rotate w.r.t. one another, and are uniformly translating in space.
- True inertial frame is Galilean frames with absolute zero motion.
- True inertial frame is not a practical reference frame.
- It is used only for visualization of other reference frames.

- Inertial axes system
- Earth-Fixed axes system: Fixed at center of Earth and rotates with it.
- Navigation system: Located at surface of Earth and origin is directly beneath the vehicle.



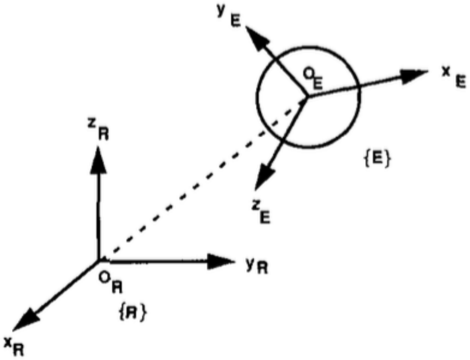
- Body axes system: Any set of axes fixed to vehicle, moving with it.
- Origin: CG location of vehicle
- Ox_bz_b coincides with plane of symmetry
- Ox_b axis along longitudinal centerline or zero-lift line
- Oy_b axis $\perp$ towards right side, Oz_b axis downward to complete RH system



- Special cases of body axes system: Stability and Wind axes systems

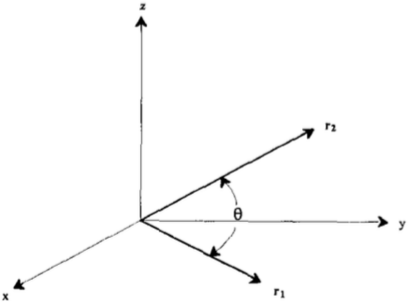
- Stability axes system: $Ox_sy_sz_s$
  - $\Rightarrow$ Ox_s lies in plane of symmetry (PoS)
  - $\Rightarrow$ For $\beta = 0$, Ox_s points in opposite direction of relative wind
  - $\Rightarrow$ For $\beta \neq 0$, Ox_s chosen to coincide the projection of relative wind in PoS
  - $\Rightarrow$ $Oy_s \perp$ PoS towards right side and Oz_s downward to complete RH system
  - $\Rightarrow$ How to locate stability axes w.r.t. body axes? Angle of attack
  - $\Rightarrow$ What would be the direction of D and L for $\beta = 0$? Opposite to Ox_s and Oz_s

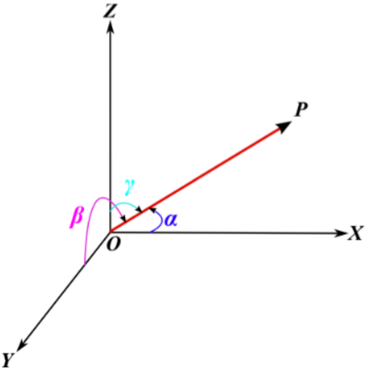
- Wind axes system: $Ox_wy_wz_w$
  - $\Rightarrow$ Ox_w points in opposite direction of relative wind
  - $\Rightarrow$ Oz_w lies in PoS
  - $\Rightarrow$ $Oy_s \perp Ox_wz_w$ towards right side
  - $\Rightarrow$ For $\beta \neq 0$, Ox_wz_w will not coincide with PoS
  - $\Rightarrow$ How to locate wind axes w.r.t. body axes?Angle of attack and sideslip angle
  - $\Rightarrow$ What would be the direction of drag and lift? Opposite to Ox_w and Oz_w



- Position of rigid body: position vector $O_R O_E$ of origin
- Orientation of rigid body: 3×3 rotation matrix
- For simplification, we assume $O_E = O_R$

- Rotation matrix approach utilizes 9 parameters, which obey the orthogonality and unit length constraints, to describe the orientation of the rigid body.
- A rigid body possesses 3 rotational DOF, 3 independent parameters are sufficient to characterize completely and unambiguously its orientation.
- Three-parameter representations are popular in engineering because they minimize the dimensionality of the rigid-body control problem
- Transformation of coordinate axes is an important necessity in resolving angular positions and rates from one coordinate system to other.
- Transformation matrix: Mapping of the components of a vector, resolved in one frame, into the same resolved into the other frame.
  - $\Rightarrow$ Direction cosine matrix (DCM)
  - $\Rightarrow$ Euler Angles
  - $\Rightarrow$ Quaternions





Angle between two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$

$$\theta = \cos^{-1}\left[\frac{\boldsymbol{r}_2^T\boldsymbol{r}_1}{\sqrt{\boldsymbol{r}_1^T\boldsymbol{r}_1}\sqrt{\boldsymbol{r}_2^T\boldsymbol{r}_2}}\right]$$

Direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ Proof?

Example

Find the direction cosines and direction angles of the vector $\boldsymbol{v} = -8\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.

- Assume α, β, γ be the angles formed by vector w.r.t. x, y, z axes, respectively.
- We can write

$$\cos\alpha = \frac{\boldsymbol{v}^T\boldsymbol{i}}{\|\boldsymbol{v}\|} = \frac{-8}{\sqrt{77}} \implies \alpha = 156^\circ$$

- Similarly,

$$\cos\beta = \frac{\boldsymbol{v}^T\boldsymbol{j}}{\|\boldsymbol{v}\|} = \frac{3}{\sqrt{77}} \implies \beta = 70^\circ$$

$$\cos\gamma = \frac{\boldsymbol{v}^T\boldsymbol{k}}{\|\boldsymbol{v}\|} = \frac{2}{\sqrt{77}} \implies \gamma = 77^\circ$$

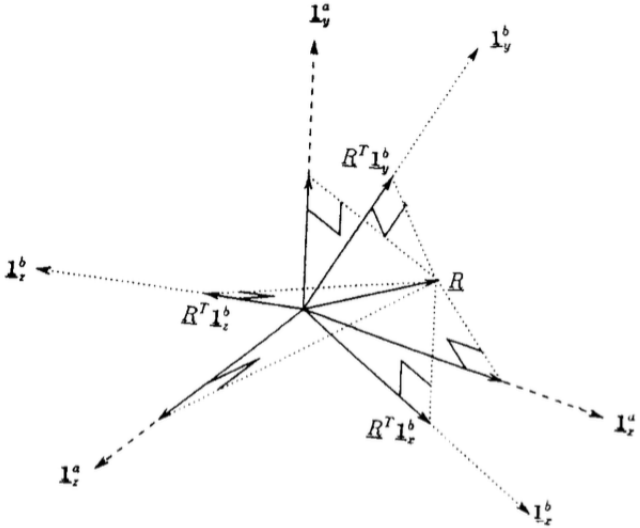
- Direction cosine matrix (DCM) transforms a vector in $\mathbb{R}^3$ from one frame to other frame.
- DCM for transformation between frames a and b

$$\boldsymbol{C}_a^b = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- Specifically, if (X, Y, Z) and (x, y, z) are the representations of a vector in frames a and b, respectively, then

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\boldsymbol{R}^b} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}}_{\text{Rotation Matrix}} \underbrace{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{\boldsymbol{R}^a} \Rightarrow \boldsymbol{R}^b = \boldsymbol{C}_a^b \boldsymbol{R}^a$$

- Matrix DCM projects the vector $\boldsymbol{R}^a$ into a reference frame b.
- For orthogonal systems, $(\boldsymbol{C}_a^b)^{-1} = (\boldsymbol{C}_a^b)^T = \boldsymbol{C}_b^a$


$\mathbf{1}_y^a$
$\mathbf{1}_y^b$
$R^T\mathbf{1}_y^b$
$\mathbf{1}_z^b$
R
$R^T\mathbf{1}_z^b$
$R^T\mathbf{1}_x^b$
$\mathbf{1}_x^a$
$\mathbf{1}_z^a$
$\mathbf{1}_x^b$

- Assume vector $\boldsymbol{R}$ coordinatized in reference frames a and b as $\boldsymbol{R}^a$ and $\boldsymbol{R}^b$, respectively.

$$\begin{aligned}\boldsymbol{R}^a =& (\boldsymbol{R}^T \mathbf{1}_x^a)\mathbf{1}_x^a + (\boldsymbol{R}^T \mathbf{1}_y^a)\mathbf{1}_y^a + (\boldsymbol{R}^T \mathbf{1}_z^a)\mathbf{1}_z^a \\ \boldsymbol{R}^b =& (\boldsymbol{R}^T \mathbf{1}_x^b)\mathbf{1}_x^b + (\boldsymbol{R}^T \mathbf{1}_y^b)\mathbf{1}_y^b + (\boldsymbol{R}^T \mathbf{1}_z^b)\mathbf{1}_z^b\end{aligned}$$

where, $\boldsymbol{R}^T \mathbf{1}_i^a \; \forall \; i = x, y, z$ denotes scalar component of $\boldsymbol{R}$ projected along the i^{th} a-frame coordinate direction.

- Unit vectors $\mathbf{1}_i^a$ and $\mathbf{1}_j^b$ are related, for $i, j = x, y, z$, as

$$\mathbf{1}_i^b = (\mathbf{1}_i^{b^T} \mathbf{1}_x^a)\mathbf{1}_x^a + (\mathbf{1}_i^{b^T} \mathbf{1}_y^a)\mathbf{1}_y^a + (\mathbf{1}_i^{b^T} \mathbf{1}_z^a)\mathbf{1}_z^a$$

- The i^{th} component of $\boldsymbol{R}^b$ can be expressed as

$$\begin{aligned}\boldsymbol{R}^T \mathbf{1}_i^b =& \boldsymbol{R}^T [(\mathbf{1}_i^{b^T} \mathbf{1}_x^a)\mathbf{1}_x^a + (\mathbf{1}_i^{b^T} \mathbf{1}_y^a)\mathbf{1}_y^a + (\mathbf{1}_i^{b^T} \mathbf{1}_z^a)\mathbf{1}_z^a] \\ =& (\mathbf{1}_i^{b^T} \mathbf{1}_x^a)\boldsymbol{R}^T \mathbf{1}_x^a + (\mathbf{1}_i^{b^T} \mathbf{1}_y^a)\boldsymbol{R}^T \mathbf{1}_y^a + (\mathbf{1}_i^{b^T} \mathbf{1}_z^a)\boldsymbol{R}^T \mathbf{1}_z^a\end{aligned}$$

- The vector $\boldsymbol{R}^b$ can be expressed as

$$\boldsymbol{R}^b = \begin{bmatrix} \boldsymbol{R}^T\mathbf{1}_x^b \\ \boldsymbol{R}^T\mathbf{1}_y^b \\ \boldsymbol{R}^T\mathbf{1}_z^b \end{bmatrix} = \begin{bmatrix} \mathbf{1}_x^{b^T}\mathbf{1}_x^a & \mathbf{1}_x^{b^T}\mathbf{1}_y^a & \mathbf{1}_x^{b^T}\mathbf{1}_z^a \\ \mathbf{1}_y^{b^T}\mathbf{1}_x^a & \mathbf{1}_y^{b^T}\mathbf{1}_y^a & \mathbf{1}_y^{b^T}\mathbf{1}_z^a \\ \mathbf{1}_z^{b^T}\mathbf{1}_x^a & \mathbf{1}_z^{b^T}\mathbf{1}_y^a & \mathbf{1}_z^{b^T}\mathbf{1}_z^a \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T\mathbf{1}_x^a \\ \boldsymbol{R}^T\mathbf{1}_y^a \\ \boldsymbol{R}^T\mathbf{1}_z^a \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1}_x^{b^T}\mathbf{1}_x^a & \mathbf{1}_x^{b^T}\mathbf{1}_y^a & \mathbf{1}_x^{b^T}\mathbf{1}_z^a \\ \mathbf{1}_y^{b^T}\mathbf{1}_x^a & \mathbf{1}_y^{b^T}\mathbf{1}_y^a & \mathbf{1}_y^{b^T}\mathbf{1}_z^a \\ \mathbf{1}_z^{b^T}\mathbf{1}_x^a & \mathbf{1}_z^{b^T}\mathbf{1}_y^a & \mathbf{1}_z^{b^T}\mathbf{1}_z^a \end{bmatrix} \boldsymbol{R}^a$$

$$= \boldsymbol{C}_a^b \boldsymbol{R}^a = [C_{ij}]\boldsymbol{R}^a$$

- $[C_{ij}]$ represents the cosine of the angle between the unit vectors $\mathbf{1}_i^b$ and $\mathbf{1}_j^a$.

Example 1

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If $\boldsymbol{i}' = \boldsymbol{j}$, $\boldsymbol{j}' = -\boldsymbol{i}$, and $\boldsymbol{k}' = \boldsymbol{k}$ then what would be the DCM matrix?

- DCM matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If $\boldsymbol{i}' = \boldsymbol{i}$, $\boldsymbol{j}' = -\boldsymbol{k}$, and $\boldsymbol{k}' = \boldsymbol{j}$ then what would be the DCM matrix?

- DCM matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Example 3

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If the old coordinate frame is rotated with angle θ anti-clockwise w.r.t. z-axis to get new frame then what would be the DCM matrix?

- Unit vectors of new frame

$$\begin{aligned}
\boldsymbol{i}' &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\
\boldsymbol{j}' &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \\
\boldsymbol{k}' &= \boldsymbol{k}
\end{aligned}$$

- DCM matrix

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 4

Find out the missing coefficients of DCM.

$$T = \begin{bmatrix} 0.8999 & -0.4323 & 0.0578 \\ c_{21} & 0.8665 & -0.2496 \\ c_{31} & c_{32} & 0.9666 \end{bmatrix}$$

- We can use orthogonal property of DCM.

$$0.9666c_{32} - 0.8665 \times 0.2496 - 0.4323 \times 0.0578 = 0$$

$$c_{31}c_{32} + 0.8665c_{21} - 0.8999 \times 0.4323 = 0$$

$$0.9666c_{31} - 0.2496c_{21} + 0.0578 \times 0.8999 = 0$$

- On solving, we get $c_{21} = 0.4323,\ c_{31} = 0.0578,\ c_{32} = 0.2496$
- Check for correctness

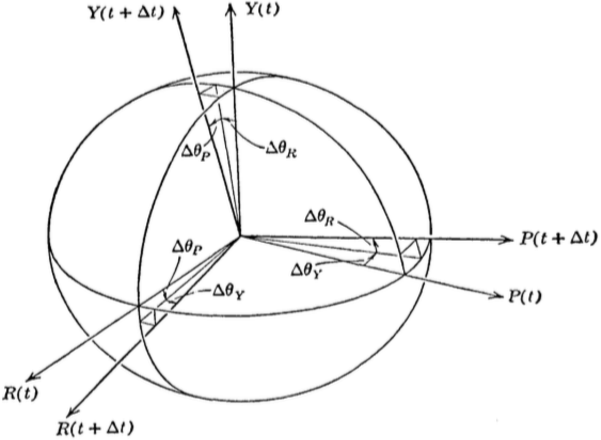
- Consider the two frames be the a and b frames.
- At time t, the frames a and b are related through the DCM $\boldsymbol{C}_b^a(t)$.
- At time $t + \Delta t$, frame b rotates to a new orientation such that the direction cosine matrix is given by $\boldsymbol{C}_b^a(t + \Delta t)$.
- Rate of change of $\boldsymbol{C}_b^a(t)$ is given by

$$\dot{\boldsymbol{C}}_b^a(t) = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{C}_b^a}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{C}_b^a(t + \Delta t) - \boldsymbol{C}_b^a(t)}{\Delta t}$$

- From geometrical considerations,

$$\boldsymbol{C}_b^a(t + \Delta t) = \boldsymbol{C}_b^a(t)(\boldsymbol{I} + \boldsymbol{\Delta\theta}^b)$$

where, $\boldsymbol{I} + \boldsymbol{\Delta\theta}^b$ is the small angle DCM relating b frame at time t to the rotated b frame at time $t + \Delta t$.



$Y(t+\Delta t)$
$Y(t)$
$\Delta\theta_P$
$\Delta\theta_R$
$\Delta\theta_R$
$P(t+\Delta t)$
$\Delta\theta_P$
$\Delta\theta_Y$
$-\Delta\theta_Y$
$P(t)$
$R(t)$
$R(t+\Delta t)$

- $\boldsymbol{\Delta\theta}^b$ is given by

$$\boldsymbol{\Delta\theta}^b = \begin{bmatrix} 0 & -\Delta\theta_Y & \Delta\theta_P \\ \Delta\theta_Y & 0 & -\Delta\theta_R \\ -\Delta\theta_P & \Delta\theta_R & 0 \end{bmatrix}, \quad \Delta\theta_k = \sin\Delta\theta_k \ \forall \ k = R, Y, P$$

- Note that because the rotation angles are small in the limit as $\Delta t \to 0$, small angle approximations are valid and the order of rotation is immaterial.
- Rate of change of $\boldsymbol{C}_b^i(t)$ is now written as

$$\dot{\boldsymbol{C}}_b^a(t) = \boldsymbol{C}_b^a(t) \lim_{\Delta t \to 0} \frac{\boldsymbol{\Delta\theta}^b}{\Delta t}$$

- In the limit $\Delta t \to 0$, $\boldsymbol{\Delta\theta^b}/\Delta t$ is the skew-symmetric form of angular velocity of the frame b relative to a frame.

$$\dot{\boldsymbol{C}}_b^a(t) = \boldsymbol{C}_b^a(t)\boldsymbol{\Omega}_{ab}^b = \boldsymbol{C}_b^a(t) \begin{bmatrix} 0 & -\omega_Y & \omega_P \\ \omega_Y & 0 & -\omega_R \\ -\omega_P & \omega_R & 0 \end{bmatrix}$$

- DCM differential equation is a linear matrix differential equation, forced by the angular velocity vector in its skew symmetric matrix form.
- Nine scalar, linear, coupled differential equations
- This equation can be integrated with the initial conditions, which represent the initial orientation of the a-frame with respect to the b-frame.
- Differential equation

$$\dot{\boldsymbol{C}}_{i,j} = \boldsymbol{C}_{i,j+1}\omega_{j+2} - \boldsymbol{C}_{i,j+2}\omega_{j+1}, \quad i,j = 1,2,3$$

  where, second subscript is modulo 3, and $\omega_1 = \omega_R, \omega_2 = \omega_P, \omega_3 = \omega_Y$
- A first order approximation for transformation matrix, using Taylor series

$$\boldsymbol{C}_{t_k+\Delta T} = \left[\boldsymbol{I} + \boldsymbol{\Omega}^b_{ab}(t_k)\Delta T\right]\boldsymbol{C}_{t_k}$$

- Consider transformation of unit vectors from body to inertial axes system using DCM

$$\begin{bmatrix} \hat{i}_i \\ \hat{j}_i \\ \hat{k}_i \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \hat{i}_b \\ \hat{j}_b \\ \hat{k}_b \end{bmatrix} = C_b^i \begin{bmatrix} \hat{i}_b \\ \hat{j}_b \\ \hat{k}_b \end{bmatrix}$$

- Consider first equation

$$\hat{i}_i = C_{11}\hat{i}_b + C_{12}\hat{j}_b + C_{13}\hat{k}_b$$

- On differentiation, we get

$$\dot{\hat{i}}_i = \dot{C}_{11}\hat{i}_b + \dot{C}_{12}\hat{j}_b + \dot{C}_{13}\hat{k}_b + C_{11}\dot{\hat{i}}_b + C_{12}\dot{\hat{j}}_b + C_{13}\dot{\hat{k}}_b$$

- We know that

$$\dot{\hat{i}}_b = \omega_b \times \hat{i}_b, \quad \dot{\hat{j}}_b = \omega_b \times \hat{j}_b, \quad \dot{\hat{k}}_b = \omega_b \times \hat{k}_b$$

- Now, one may write

$$\begin{aligned}\dot{\hat{i}}_i =& \dot{C}_{11}\hat{i}_b + \dot{C}_{12}\hat{j}_b + \dot{C}_{13}\hat{k}_b + C_{11}\omega_b \times \hat{i}_b + C_{12}\omega_b \times \hat{j}_b + C_{13}\omega_b \times \hat{k}_b \\ =& \dot{C}_{11}\hat{i}_b + \dot{C}_{12}\hat{j}_b + \dot{C}_{13}\hat{k}_b + \omega_b \times (C_{11}\hat{i}_b + C_{12}\hat{j}_b + C_{13}\hat{k}_b) \\ =& \dot{C}_{11}\hat{i}_b + \dot{C}_{12}\hat{j}_b + \dot{C}_{13}\hat{k}_b + \omega_b \times \hat{i}_i\end{aligned}$$

- As $\hat{i}_i$ is a vector of unit magnitude and fixed direction in inertial frame, $\dot{\hat{i}}_i = 0$, which implies

$$\dot{C}_{11}\hat{i}_b + \dot{C}_{12}\hat{j}_b + \dot{C}_{13}\hat{k}_b + \omega_b \times \hat{i}_i = 0$$

- Similarly

$$\dot{C}_{21}\hat{i}_b + \dot{C}_{22}\hat{j}_b + \dot{C}_{23}\hat{k}_b + \omega_b \times \hat{j}_i = 0$$
$$\dot{C}_{31}\hat{i}_b + \dot{C}_{32}\hat{j}_b + \dot{C}_{33}\hat{k}_b + \omega_b \times \hat{k}_i = 0$$

- Vector product

$$
\begin{aligned}
\omega_b \times \hat{i}_i = & \begin{vmatrix} \hat{i}_b & \hat{j}_b & \hat{k}_b \\ p & q & r \\ C_{11} & C_{12} & C_{13} \end{vmatrix} \\
= & \hat{i}_b(C_{13}q - C_{12}r) + \hat{j}_b(C_{11}r - C_{13}p) + \hat{k}_b(C_{12}p - C_{11}q)
\end{aligned}
$$

- Thus, $\dot{C}_{11}\hat{i}_b + \dot{C}_{12}\hat{j}_b + \dot{C}_{13}\hat{k}_b + \omega_b \times \hat{i}_i = 0$ leads to

$$\hat{i}_b(\dot{C}_{11} + C_{13}q - C_{12}r) + \hat{j}_b(\dot{C}_{12} + C_{11}r - C_{13}p) + \hat{k}_b(\dot{C}_{13} + C_{12}p - C_{11}q) = 0$$

- Now we have

$$\dot{C}_{11} = C_{12}r - C_{13}q, \ \dot{C}_{12} = C_{13}p - C_{11}r, \ \ \dot{C}_{13} = C_{11}q - C_{12}p$$

- Similarly

$$\dot{C}_{21} = C_{22}r - C_{23}q,\ \dot{C}_{22} = C_{23}p - C_{21}r,\ \ \dot{C}_{23} = C_{21}q - C_{22}p$$

$$\dot{C}_{31} = C_{32}r - C_{33}q,\ \dot{C}_{32} = C_{33}p - C_{31}r,\ \ \dot{C}_{33} = C_{31}q - C_{32}p$$

- In compact form,

$$\begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b$$

where

$$\Omega_{ib}^b = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

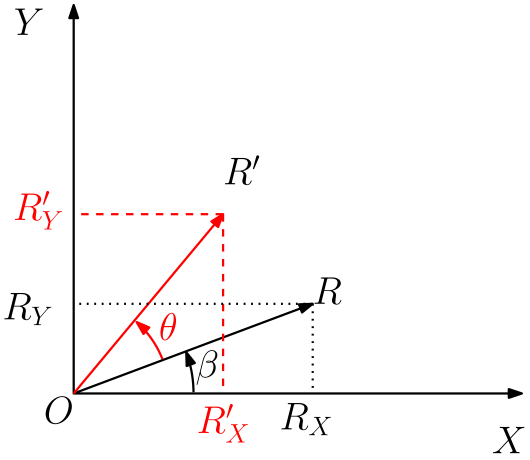
- On taking transpose and using the fact that $(\Omega_{ib}^b)' = -\Omega_{ib}^b$, we get

$$\dot{C}_i^b = \quad -\Omega_{ib}^b C_i^b$$

- This equation is of the form of $\dot{X} = AX, \quad A = -\Omega_{ib}^b$.
- Characteristic equation: $\Delta(\lambda I - A) = 0$, where Δ denotes the determinant of matrix.

$$\lambda(\lambda^2 + p^2 + q^2 + r^2) = 0$$

- Eigenvalues: $\lambda = 0, \pm j\sqrt{p^2 + q^2 + r^2} = \pm j|\omega_b|$
- Neutrally stable system
- Care needs to be taken during updating these coefficients to avoid accumulation of rounding off error, and prevent the system from blow out.

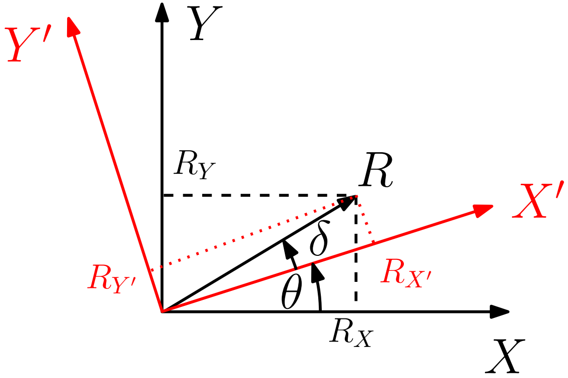

Y
R'
R'_Y
R
R_Y
θ
β
O
R'_X
R_X
X

- The position of a point R in XY coordinate frame is given by

$$\begin{bmatrix} R_X \\ R_Y \end{bmatrix} = \begin{bmatrix} R\cos\beta \\ R\sin\beta \end{bmatrix}$$

- Let us assume $\gamma = \theta + \beta$.
- Position of a point R' in XY coordinate frame is given by

$$\begin{bmatrix} R'_X \\ R'_Y \end{bmatrix} = \begin{bmatrix} R\cos\gamma \\ R\sin\gamma \end{bmatrix} = \begin{bmatrix} R\cos\theta\cos\beta - R\sin\theta\sin\beta \\ R\sin\theta\cos\beta + R\cos\theta\sin\beta \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotation matrix}} \begin{bmatrix} R_X \\ R_Y \end{bmatrix}$$


Y'
Y
R_Y
R
X'
δ
$R_{Y'}$
$R_{X'}$
θ
R_X
X

- Let us assume $\alpha = \theta + \delta$.
- The position of a point R in XY frame is given by

$$\begin{bmatrix} R_X \\ R_Y \end{bmatrix} = \begin{bmatrix} R\cos\alpha \\ R\sin\alpha \end{bmatrix}$$

- Position of a point R in $X'Y'$ frame is given by

$$\begin{bmatrix} R_{X'} \\ R_{Y'} \end{bmatrix} = \begin{bmatrix} R\cos\delta \\ R\sin\delta \end{bmatrix}$$

- As $\delta = \alpha - \theta$, we can also write

$$\begin{aligned} \begin{bmatrix} R_{X'} \\ R_{Y'} \end{bmatrix} &= \begin{bmatrix} R\cos(\alpha-\theta) \\ R\sin(\alpha-\theta) \end{bmatrix} = \begin{bmatrix} R\cos\alpha\cos\theta + R\sin\alpha\sin\theta \\ R\sin\alpha\cos\theta - R\cos\alpha\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} R_X\cos\theta + R_Y\sin\theta \\ R_Y\cos\theta - R_X\sin\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotation matrix}} \begin{bmatrix} R_X \\ R_Y \end{bmatrix} \end{aligned}$$

Reference

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Thank you for your attention !!!