Tutorial: 4

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Q: Consider a vector $P(1,0)$ lying on the x -axis of Frame $\mathcal{A}$. Rotate $\mathcal{A}$ by 15° to frame $\mathcal{B}$ and then rotate frame $\mathcal{B}$ by 30° to Frame $\mathcal{C}$. What are the new coordinates of the vector $\vec{P}$ in frame $\mathcal{C}$?

A: The rotation matrix of coordinate frame rotation is given by

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Let α and β be the angle by which we rotate frames $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, the coordinates of vector $\vec{P}$ in the resultant frame $\mathcal{C}$ is given by

$$\begin{bmatrix} x_{\mathcal{C}} \\ y_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_{\mathcal{A}} \\ y_{\mathcal{A}} \end{bmatrix}$$

Substituting the values of $\theta_{\mathcal{A}}$ and $\theta_{\mathcal{B}}$, we can simplify (29) as

$$\begin{bmatrix} x_{\mathcal{C}} \\ y_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ -\frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Q: What are the new coordinates of the vector $\vec{P}$ if the sequence of rotation is reversed?

A: Reverse the matrix operation order of part - (a)

$$\begin{bmatrix} x_{\mathcal{C}} \\ y_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ -\frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The result is same as part - (a). In general, rotational matrix in two-dimension commute. However, this is not true for three-dimensions.

Q: Consider a rotation of vector using quaternion about an axis defined by vector (1,0,1) through an angle of $2\pi/3$. Obtain the quaternion $[Q]$ to perform this rotation.

A: The axis is defined by the vector (1,0,1) and the angle is given as $\frac{2\pi}{3}$. Let us define a unit vector along the given axis as

$$\hat{\boldsymbol{q}} = \frac{1}{\sqrt{2}}(1,0,1)$$

Since a unit quaternion is described as $[Q] = \cos\frac{\theta}{2} + \hat{\boldsymbol{q}}\sin\frac{\theta}{2}$, we have the following

$$\begin{aligned}
[Q] &= \cos\frac{\theta}{2} + \hat{\boldsymbol{q}}\sin\frac{\theta}{2} \\
&= \cos\frac{\pi}{3} + \hat{\boldsymbol{q}}\sin\frac{\pi}{3} \\
&= \frac{1}{2} + \frac{1}{\sqrt{2}}(1,0,1)\frac{\sqrt{3}}{2} \\
\Rightarrow [Q] &= \frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{2}}\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + \frac{\sqrt{3}}{2\sqrt{2}}\hat{\boldsymbol{k}}
\end{aligned}$$

Q: Compute the effect of rotation on the basis vector $\boldsymbol{k} = (0, 0, 1)$.

A: The operator L_Q operates on the basis vector $\boldsymbol{k}$. Thus, $L_Q(\boldsymbol{v}) = L_Q(\boldsymbol{k})$. Therefore,

$$L_Q(\boldsymbol{v}) = [Q]\boldsymbol{v}[Q]^\star = \cos\theta \cdot \boldsymbol{v} + (1 - \cos\theta)(\hat{\boldsymbol{q}} \cdot \boldsymbol{v})\hat{\boldsymbol{q}} + \sin\theta \cdot (\hat{\boldsymbol{q}} \times \boldsymbol{v})$$
$$\text{or, } L_Q(\boldsymbol{v}) = \left(q_0^2 - \|\mathbf{q}\|^2\right)\boldsymbol{v} + 2(\mathbf{q} \cdot \boldsymbol{v})\mathbf{q} + 2q_0(\mathbf{q} \times \boldsymbol{v})$$

Evaluating dot and cross products, we have $\boldsymbol{q} \cdot \boldsymbol{v} = \boldsymbol{q} \cdot \boldsymbol{k} = \frac{\sqrt{3}}{2\sqrt{2}}$, and $\boldsymbol{q} \times \boldsymbol{v} = \boldsymbol{q} \times \boldsymbol{k} = -\frac{\sqrt{3}}{2\sqrt{2}}\boldsymbol{j}$. Further simplification yields

$$\begin{aligned} L_Q(v) &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{v} + 2(\boldsymbol{q} \cdot \boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \\ &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{k} + 2(\boldsymbol{q} \cdot \boldsymbol{k})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{k}) \\ &= \left(\frac{1}{4} - \frac{3}{4}\right)\boldsymbol{k} + \left(\frac{3}{4}\boldsymbol{i} + \frac{3}{4}\boldsymbol{k}\right) + \left(-\frac{\sqrt{3}}{2\sqrt{2}}\boldsymbol{j}\right) \end{aligned}$$

$$\therefore \quad L_Q(\boldsymbol{k}) = \frac{3}{4}\boldsymbol{i} - \frac{3}{2\sqrt{2}}\boldsymbol{j} + \frac{1}{4}\boldsymbol{k}$$

Q: Find out the conjugate $[Q]^\star$ and inverse $[Q]^{-1}$ of the quaternion $[Q]$.

A: From part(a), we have

$$[Q] = \frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{2}}\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + \frac{\sqrt{3}}{2\sqrt{2}}\hat{\boldsymbol{k}}$$

Note that the norm of $[Q]$ is 1, i.e., the quaternion is a unit quaternion. Hence, its inverse and conjugate are the same.

$$[Q]^\star = [Q]^{-1} = \frac{1}{2} - \frac{\sqrt{3}}{2\sqrt{2}}\hat{\boldsymbol{i}} - 0\hat{\boldsymbol{j}} - \frac{\sqrt{3}}{2\sqrt{2}}\hat{\boldsymbol{k}}$$

Q: Find the coordinates of above vector in new frame if we rotate the coordinate frame itself about the same axis and angle while keeping the vector constant?

A: Let the coordinates of the vector $\boldsymbol{k} = (0,0,1)$ in the new frame be given as (x,y,z). We know that for any unit quaternion, the relation

$$p' = [Q]^{\star} p [Q]$$

holds, where p and p' respectively represent the coordinates of the vector in the original frame and the rotated frame. Here $p = \boldsymbol{k} = (0,0,1)$ and $p' = (x,y,z)$. Upon simplification of (6), the same can be expressed in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\left(q_1 q_2 + q_0 q_3\right) & 2\left(q_1 q_3 - q_0 q_2\right) \\ 2\left(q_1 q_2 - q_0 q_3\right) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\left(q_0 q_1 + q_2 q_3\right) \\ 2\left(q_0 q_2 - q_1 q_3\right) & 2\left(q_2 q_3 - q_0 q_1\right) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\left(q_1 q_3 - q_0 q_2\right) \\ 2\left(q_0 q_1 + q_2 q_3\right) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{4} \end{bmatrix}$$

Q: If $\mathbf{R}$ is a general rotation matrix $R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{33} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ which represents rotation of an aircraft along three principal axes, such as about x with angle ϕ about y with angle θ, and about z with angle ψ respectively. Find the values of Euler angles (ψ, θ, ϕ) in terms of elements of $\mathbf{R}$.

A: We have the following relations for the composite rotation $\mathbf{R} = \mathbf{R}_{z_\psi} \mathbf{R}_{y_\theta} \mathbf{R}_{x_\phi}$:

$$\begin{aligned} R_{11} &= \cos\phi\cos\theta & R_{21} &= \sin\phi\cos\theta \\ R_{12} &= \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & R_{22} &= \cos\psi\sin\theta\sin\phi + \cos\phi\cos\psi \\ R_{13} &= \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & R_{23} &= \sin\phi\sin\theta\cos\psi - \cos\theta\sin\psi \end{aligned}$$

$$\begin{aligned} R_{31} &= -\sin\theta \\ R_{32} &= \cos\theta\sin\psi \\ R_{33} &= \cos\theta\cos\psi \end{aligned}$$

$$\theta = -\sin^{-1} R_{31}, \quad \psi = \tan^{-1}\frac{R_{32}}{R_{33}}, \quad \phi = \tan^{-1}\frac{R_{21}}{R_{11}}$$

Q: If

$$\mathbf{R} = \begin{bmatrix} 0.5 & -0.1464 & 0.8536 \\ 0.5 & 0.8536 & -0.1464 \\ -0.7071 & 0.5 & 0.5 \end{bmatrix}$$

Find the values of roll, pitch and yaw angles.

A: Solving the above relations, we get

$$\theta = \frac{\pi}{4}, \quad \psi = \frac{\pi}{4}, \quad \phi = \frac{\pi}{4}$$

Q: Recall that the quaternion operator with unit quaternion $[Q]$ acts on a vector $\boldsymbol{v}$ as

$$L_Q(\boldsymbol{v}) = [Q]\boldsymbol{v}[Q]^\star = \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{v} + 2(\boldsymbol{q}\cdot\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times\boldsymbol{v})$$

1. Show that length of the vector $\boldsymbol{v}$ is invariant under the operation.
2. Show that direction of the vector $\boldsymbol{v}$ remains unchanged under the operation.

A:

1. Note that $\|L_Q(\boldsymbol{v})\| = \|[Q]\boldsymbol{v}[Q]^\star\| = \|[Q]\|\|\boldsymbol{v}\|\,\|[Q]^\star\| = \|\boldsymbol{v}\|$, where $[Q]$ is the unit quaternion. Hence, length of the vector $\boldsymbol{v}$ is invariant under the operation $L_Q(\boldsymbol{v})$.
2. The vector in the direction of $\boldsymbol{v}$ is $k\boldsymbol{q}$. Thus,

$$\begin{aligned}[Q]\boldsymbol{v}[Q]^\star = [Q]k\boldsymbol{q}[Q]^\star &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)k\boldsymbol{q} + 2(\boldsymbol{q}\cdot k\boldsymbol{q})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times k\boldsymbol{q})\\ &= k\left(q_0^2 + \|\boldsymbol{q}\|^2\right)\boldsymbol{q} = k\boldsymbol{q}\end{aligned}$$

Hence, the direction of $\boldsymbol{v}$, along $\boldsymbol{q}$ is left unchanged by the operator $L_Q(\boldsymbol{v})$.

Tutorial 4
Question-4 (c)

Q: Show that the operation is a linear map over $\mathbb{R}^3$.

A: The operator L_Q is linear if it satisfies homogeneity and superposition properties. In other words, for some vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^3$, and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$

$$L_Q\left(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n\right) = \alpha_1 L_Q\left(\boldsymbol{v}_1\right) + \alpha_2 L_Q\left(\boldsymbol{v}_2\right) + \cdots + \alpha_n L_Q\left(\boldsymbol{v}_n\right)$$

must be true in order for L_Q to be a linear map. The argument made in (27) can be verified as follows.

$$\begin{aligned} L_Q\left(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n\right) &= [Q]\left(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n\right)[Q]^\star \\ &= [Q]\alpha_1 \boldsymbol{v}_1 [Q]^\star + [Q]\alpha_2 \boldsymbol{v}_2 [Q]^\star + \cdots + [Q]\alpha_n \boldsymbol{v}_n [Q] \\ &= \alpha_1 [Q]\boldsymbol{v}_1 [Q]^\star + \alpha_2 [Q]\boldsymbol{v}_2 [Q]^\star + \cdots + \alpha_n [Q]\boldsymbol{v}_n [Q] \\ &= \alpha_1 L_Q\left(\boldsymbol{v}_1\right) + \alpha_2 L_Q\left(\boldsymbol{v}_2\right) + \cdots + \alpha_n L_Q\left(\boldsymbol{v}_n\right) \end{aligned}$$

Thus, L_Q is indeed a linear operator.

Thank you for your attention !!!