FORMAL DIMENSION OF A P-ADIC GROUP REPRESENTATION AS A P-ADIC LIMIT

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Abstract. We define a new formal dimension of a representation of $SL_n(\mathbb{Q}_p)$ as a limit of an expression that uses its coefficient system on the Bruhat-Tits building of $SL_n(\mathbb{Q}_p)$. Because the definition is in the form of a limit, we also proof its convergence properties.

Introduction

Let V be a representation of $SL_n(\mathbb{Q}_p)$ for some $n \in \mathbb{Z}_{\geq 2}$ and some prime p. When V is non-trivial, it is not finitely dimensional [S1]. To better understand the dimension of these representations, we can define a formal dimension for each representation as a $p-$adic limit.

This paper aims at creating a new formal dimension that uses the coefficient system d of the representation on its Bruhat-Tits building $\mathcal{B}$. We define the new formal dimension $\mathrm{f.dim}(V)$ in the following way,

$$\mathrm{f.dim}(V) = \lim_{\substack{g \to Id \\ g \text{ is elliptic}}} \left(\sum_{F \in \mathcal{B}^g} (-1)^{\dim F} d(F) \right).$$

The goal of this paper is to prove that the above limit converges for any coefficient system. The result will imply that any representation will have a well-defined formal dimension. This paper will heavily use the explicit construction of $\mathcal{B}$, as described in [M1]. Readers can also read [G1] for inquiries about $p-$adic numbers, as this concept will be used extensively throughout the paper.

Before we move on, let's fix a basis $U = (u_1, ..., u_n)$ of $\mathbb{Q}_p^n$ and call it as the universal basis. From this point, unless specified otherwise, all computations will be computed with respect to U.

In section 1, we will discuss the orbits of the faces of $\mathcal{B}$. Section 2 will introduce a way of uniquely codifying each vertex of $\mathcal{B}$ with a certain type of matrix. Section 3 will use the findings in section 2 to uniquely codify each face of $\mathcal{B}$. Section 4 will introduce an equivalence sytem for some $g \in SL_n(\mathbb{Q}_p)$ such that each equivalence class is either a subset of $\mathcal{B}^g$ or have no intersection with it.

Section 5 will describe a new way to express the matrices that encodes the vertices of $\mathcal{B}$ as a product of two matrices. Section 6 will explore the requirements for a face to be in $\mathcal{B}^g$. Section 7 will be dedicated to prove a theorem in section 4 (Theorem 4.1). Lastly, section 8 will be the final steps to prove the convergence of the limit.

Acknowledgments

The author would like to thank Professor Roman Bezrukavnikov for his suggestion to study the dimension of the representation of a p-adic group. I also thank Ilya Dumanski in helping me learn the required materials (p-adic numbers, representation theory, and bruhat-tits buildings) and improve and correct the first few drafts of this paper.

1. Face Orbits

1.1. Vertex Orbits.

The orbits for vertices are quite simple, the following theorem shows how they look like.

Theorem 1.1. *There are n vertex orbits in $\mathcal{B}$, and we can enumerate them such that a vertex Λ is in the i^{th} orbit if and only if all lattices $(v_1, ..., v_n)$ in it has*

$$\nu_p(\det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}) \equiv i \mod n.$$

Proof. Consider two vertices Λ_1 and Λ_2 with lattice bases $(v_1, ..., v_n)$ and $(w_1, ..., w_n)$. Let $a = \det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ and $b = \det \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}$. If the two vertices are in an orbit, then there exists $T \in GL_n(\mathbb{Z}_p)$, $g \in SL_n(\mathbb{Q}_p)$, and $k \in \mathbb{Z}$ such that

$$p^k g \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} T = \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}. \tag{1.1}$$

Therefore,

$$p^{kn} a \det T = b.$$

Since $T \in GL_n(\mathbb{Z}_p)$, then $\nu_p(\det T) = 0$, so

$$kn + \nu_p(a) = \nu_p(b),$$
$$\nu_p(a) \equiv \nu_p(b) \mod n.$$

We claim that this is a necessary and sufficient condition for two vertices to be in the same orbit. The necessary part has been proven, now for the sufficient part, let's say that

$$\nu_p(b) - \nu_p(a) = k'n,$$

for some $k' \in \mathbb{Z}$. Therefore, (1.1) has a solution, which is

$$k = k', T = \operatorname{diag}(\frac{b}{ap^{k'n}}, 1, ..., 1), \text{ and}$$

$$g = p^{-k'} \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix} \left(\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} T \right)^{-1}.$$

From this, we can deduce that there are n vertex orbits, each corresponding to the valuation of the determinant of the lattice taken modulo n. □

For simplicity, name the i^{th} vertex orbit O_i for $i \in [n]$.

1.2. **Face Orbits in General.** We first begin with a theorem.

Theorem 1.2. *Given an $(n-1)-$face consisting of the vertices $\Lambda_1, \Lambda_2, ..., \Lambda_n$, each with lattices $L_1, L_2, ..., L_n$, respectively, such that*

$$pL_1 \subsetneq L_n \subsetneq \cdots \subsetneq L_2 \subsetneq L_1,$$

there exists a basis $(v_1, v_2, ..., v_n)$ for L_1 such that $(pv_1, ..., pv_{i-1}, v_i, ..., v_n)$ is a basis for the lattice L_1, for all $i \in [n]$.

Proof. For simplicity, let us fix a new basis $B = (b_1, ..., b_n)$ for L_1. Next, we have

$$0 \subsetneq L_n/pL_1 \subsetneq \cdots \subsetneq L_2/pL_1 \subsetneq L_1/pL_1$$

are all $\mathbb{Z}_p/p\mathbb{Z}_p-$modules, which are $\mathbb{F}_p-$vector spaces. Therefore, there exists a basis $(v_1, ..., v_n)$ of L_1/pL_1 such that $L_i/pL_1 = (v_n, ..., v_i), \forall i \in [n]$. With respect to B, $\det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ must be a unit of $\mathbb{F}_p$.

Consider the mapping $\varphi : \{1, ..., p\} \to \mathbb{Z}_p$ such that $\varphi(i) = i, \forall i \in [p]$. Let v_i' be the result of mapping each entry in v_i, with respect to the basis B, using φ. Therefore, $\det \begin{pmatrix} v_1' & \cdots & v_n' \end{pmatrix}$ must be a unit of $\mathbb{Z}_p$, and so $(v_1', ..., v_n')$ is a basis of L_1, and $(pv_1', ..., pv_{i-1}', v_i', ..., v_n')$ is a basis of L_i for $i \in [n]$.

Now, let us switch back to the universal basis. The basis for L_i would be $(pBv_1', ..., pBv_{i-1}', Bv_i', ..., Bv_n')$ for all $i \in [n]$. □

A simple consequence of this theorem is that by Theorem 1, each $(n-1)-$face has one vertex from each vertex orbit. Therefore, since any face is contained in an $(n-1)-$face, then any face cannot contain more than one vertex from any vertex orbit.

Theorem 1.3. *$SL_n(\mathbb{Q}_p)$ acts transitively on the set of $(n-1)-$faces.*

Proof. Consider two $(n-1)-$faces F_1 and F_2. Let F_1 have vertices $\Lambda_1, ..., \Lambda_n$ in orbits $O_1, ..., O_n$ with lattices $L_1, L_2, ..., L_n$ such that the basis of L_i is $(pv_1, ..., pv_{i-1}, v_i, ..., v_n)$ for all $i \in [n]$. Similarly, define $\Lambda_1', ..., \Lambda_n'$ and $(pw_1, ..., pw_{i-1}, w_i, ..., w_n)$ for F_2, such that Λ_1 and Λ_1' are in the same vertex orbit.

Let $\det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = a$ and $\det \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix} = b$. Therefore, $v_p\left(\frac{a}{b}\right) = nk$ for some $k \in \mathbb{Z}$. Let $x = \frac{a}{bp^{nk}}$. Take

$$g = p^k \begin{pmatrix} w_1 & \cdots & w_{n-1} & xw_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{-1}.$$

We can deduce that $g \in SL_n(\mathbb{Q}_p)$ and $x \in \mathbb{Z}_p^\times$. Furthermore, let P_i be a diagonal matrix where the first i entries in the diagonal is p, and the rest are 1's. Thus,

$$\begin{aligned} g\begin{pmatrix} pv_1 & \cdots & pv_{i-1} & v_i & \cdots & v_n \end{pmatrix} &= g\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} P_i \\ &= p^k \begin{pmatrix} w_1 & \cdots & w_{n-1} & \frac{a}{bp^{nk}} w_n \end{pmatrix} P_i \\ &= p^k \begin{pmatrix} pw_1 & \cdots & pw_{i-1} & w_i & \cdots & w_{n-1} & xw_n \end{pmatrix}. \end{aligned}$$

Therefore, g brings Λ_i to Λ_i' for all $i \in [n]$, and it brings F_1 to F_2. Thus, $SL_n(\mathbb{Q}_p)$ acts transitively on the $(n-1)-$faces. □

One consequence of this theorem is that for any vertex orbits $O_{i_1}, ..., O_{i_k}$, $SL_n(\mathbb{Q}_p)$ acts transitively on all faces with vertices from $O_{i_1}, ..., O_{i_k}$. Thus, there are $\binom{n}{i+1}$ $i-$face orbits.

2. Representative Matrices and Basis of A Vertex

To better understand each vertex on $\mathcal{B}$, it would be best for us to label each vertex with a unique matrix and basis. A good candidate for doing this is the localized form in [L1]. However, we need to modify its definition because the localized form does not codify each vertices uniquely. The following is a slight modification of the localized form, which we call the reduced form.

Definition 2.1. *Let ϕ be the natural embedding from $\mathbb{Z}$ to $\mathbb{Z}_p$. Define a reduced matrix as a matrix $(\phi(a_{ij}))_{1\leq i,j\leq n} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ such that there exists an associated permutation $\pi \in S_n$ that fulfills*

(i) *$a_{\pi(i)i}$ is a power of p for all $i = 1, ..., n$,*
(ii) *$\nu_p(a_{\pi(1)1}) \geq \cdots \geq \nu_p(a_{\pi(n)n}) = 0$,*
(iii) *if $a_{\pi(i)i} = a_{\pi(j)j}$ for some $i, j \in \{1, ..., n\}$, then $\pi(i) < \pi(j)$ if and only if $i < j$,*
(iv) *$a_{\pi(i)i} | a_{ji}$ for $j < \pi(i)$ and $i = 1, ..., n$,*
(v) *$pa_{\pi(i)i} | a_{ji}$ for $j > \pi(i)$ and $i = 1, ..., n$,*
(vi) *if $i > \pi^{-1}(j)$, then $0 \leq a_{ji} < a_{j\pi^{-1}(j)}$, and*
(vii) *if $\pi^{-1}(j) > i$, then $a_{ji} = 0$.*

Example 2.2. *To give an idea of how reduced form matrices looks like, the following is an example when $n = 4$ and $\pi = (1, 2)(3, 4)$,*

$$\begin{pmatrix} 0 & p^3 & 0 & 1 \\ p^4 & 0 & p^3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & p^2 & p \end{pmatrix}$$

Now that we have defined reduced matrix, we will prove that we can identify each vertex of $\mathcal{B}$ with a unique reduced matrix.

Theorem 2.1. *Any vertex $\Lambda \in \mathcal{B}$ has a unique reduced matrix $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ such that $(v_1, ..., v_n) \in \Lambda$.*

Before getting to the proof, there is a similar theorem with its own proof in [L1], but that theorem states existence, this one states both existence and uniqueness.

Proof. From condition (ii), $(v_1, ..., v_n)$ is a sublattice of $(u_1, ..., u_n)$, but not a sublattice of $p(u_1, ..., u_n)$. Therefore, we can deduce that $(v_1, ..., v_n)$ is a unique lattice in Λ, which we will name L. Let L_i be the $Z_p/(p^i)-$lattice where its elements are obtained by applying an entry-wise mod p^i to each element of L with respect to the universal basis. Furthermore, let v_{ji} be the projection of v_j to $\mathbb{Z}_p/(p^i)$. Finally, we introduce the set F, which will contain all rows k such that, at the moment, we have decided that $\pi(j) = k$ for some $j \in [n]$. Currently, F is empty. All these tools will be used to algorithmicly construct $(v_1, ..., v_n)$.

For the rightmost vector in this basis, its remainder when divided by p must be a non-zero vector because of condition (ii). Therefore, let us first observe what the basis must be when it is projected to L_1. We will construct first $v_{n1}, ..., v_{11}$ in this order. When we construct v_{j1}, we first fix its entries at the rows in F as zeros to fulfill condition (vii). Now, we have a set D_{j1} of rows of v_{j1} with entries that we have decided on, so for now it is identical to F. Next, we will decide the other entries of v_{j1} in the following way

Given that, currently k is the last row of v_{j1} that is not in D_{j1}, consider all possible values of the k^{th} entries of the vectors in L_1 that has the same entries as v_{j1} in the rows of D_{j1}. There must be at least one value, and if there is only one possible value, we decide that this value will be the k^{th} entry of v_{j1} and add row k as a new element of D_{j1}. Otherwise, there are multiple value, which means that there will be a vector $v_{j'1}$, for some $j' \leq j$, such that its k^{th} entry is 1 and its $(k + 1)^{th}$ to n^{th} entries are zero.

If this is the first time encountering this case for v_{j1}, decide that $\pi(j) = k$, the k^{th} entry is 1, and add row k as an element of F to fulfill (i) and (iii). Otherwise, the k^{th} entry has to be 0 to fulfill (vi). Of course, do not forget to add row k as an element of D_{j1}. Note that we always take the last row not in D_{j1} in each step to fulfill (iv) and (v).

Now we have constructed $(v_{11}, ..., v_{n1})$, and each element of this basis is either the zero vector or a vector with its last non-zero entry as 1. To construct $(v_1, ..., v_n)$, we will make $(v_{1i}, ..., v_{ni})$ by inducting on i.

Let's assume that we already constructed the basis $(v_{1i}, v_{2i}, ..., v_{ni})$ for L_i for some $i \in \mathbb{N}$. If none of the vectors in the basis is the zero vector, this will also be the basis for L and we are done. Otherwise, we will construct the basis $(v_{1(i+1)}, v_{2(i+1)}, ..., v_{n(i+1)})$ for L_{i+1}, which has to be congruent to $(v_{1i}, v_{2i}, ..., v_{ni})$ modulo p^i, in the following way.

Let α be the smallest index such that $v_{\alpha i}$ is not the zero vector. We will construct $v_{\alpha(i+1)}, ..., v_{n(i+1)}$ in the following way. As an example we will make $v_{j(i+1)}$, similar methods will be applied to the others. At the rows in F, make the entries of $v_{j(i+1)}$ the same as v_{ji} to fulfill (vi) and (vii). To help with the construction, make a set $D_{j(i+1)}$ that contains all rows of $v_{j(i+1)}$ which has entries that we had already decided on, so right now, it is the same as F.

Next, we inductively decide on the entries of $v_{j(i+1)}$. Given that $D_{j(i+1)}$ is not $[n]$ yet, take the last row k that is not in $D_{j(i+1)}$. Consider all possible values of the k^{th} entry of the elements of L_{i+1} that has the same entries as $v_{j(i+1)}$ at the rows in $D_{j(i+1)}$. There must be at least one possible value. If there is exactly one, we decide that this will be the k^{th} entry of $v_{j(i+1)}$. Otherwise, that there will be a vector $v_{j'(i+1)}$ where $j' < j$ such that the k^{th} entry is p^i and the $(k+1)^{th}$ to n^{th} entries are 0, so to fulfill (vi), we decide that the k^{th} entry of $v_{j(i+1)}$ is the same as that of v_{ji} . Finally, add the new element k to $D_{j(i+1)}$.

Finally, we construct $v_{(\alpha-1)(i+1)}, ..., v_{1(i+1)}$, in that order. Let's say, right now, we are constructing $v_{j(i+1)}$. The process of constructing it will be nearly identical to the process of making v_{j1}, except we use $v_{j(i+1)}$ and $v_{j'(i+1)}$ instead of v_{j1} and $v_{j'1}$, L_{i+1} instead of L_1, and instead of making the entry 1 or reasoning that a certain entry has to be 1, we use the number p^i.

Now, we have uniquely constructed $(v_{1(i+1)}, v_{2(i+1)}, ..., v_{n(i+1)})$, and this inductive process will eventually stop because $p^M\mathbb{Z}_p^n \subset L$ for some large enough M. When it stops, we get that $(v_{1(i+1)}, v_{2(i+1)}, ..., v_{n(i+1)})$ must be the unique basis $(v_1, ..., v_n)$ that we were looking for. $\square$

We will call $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ and $(v_1, ..., v_n)$ the representative matrix and basis of Λ, respectively.

3. Representatives of A Face

If a face F has vertices $\Lambda_1, ..., \Lambda_k$ from $O_{i_1}, ..., O_{i_k}$, respectively, where $i_1 < \cdots < i_k$, take lattices $L_1, ..., L_k$ such that the representative basis R_B of Λ_1 generates L_1, and

$$pL_1 \subsetneq L_k \subsetneq \cdots \subsetneq L_1.$$

Let R_M be the representative matrix of Λ_1, so $R_B = UR_M$. We can uniquely assign to this face $\mathbb{Z}_p-$lattices of rank n $L'_1, ..., L'_k$ such that $L_i = R_M L'_i$ for all $i = 1, 2, ..., k$. Each of these lattices can be represented by a reduced matrices in the same way we represent vertices of $\mathcal{B}$, and this time, when a lattice L'_i is represented by R'_i, the columns of $R_B R'_i$ generate L_i. Now, we can identify each face with reduced matrices in the following way.

Definition 3.1. *The seed of the face F is the R_M, and the vertices representatives of F as the reduced matrices $R'_1, ..., R'_k$.*

4. Equivalent Representative Matrices in $\mathcal{B}^g$

Since we are looking at the limit when $g \to Id$ with g being elliptic, then we can just focus on the case when $g - Id$ is an element of $p^n \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$ and $\mathcal{B}^g$ is finite. We call these g the good matrices. Next, we define an equivalence relationship for each good g, but before that, we need to mention one simple property of a reduced matrix, which is that the first column has only one non-zero entry, and it is a positive power of p.

Definition 4.1. *For each good g, define an equivalence relationship $\sim_g$ on reduced matrices as follows. Let m the lowest valuation among all entries of $g - Id$. For a reduced matrix $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$, if the only non-zero entry in v_1 has a valuation m' that is less than $m - n + 1$, then it is not equivalent to any other reduced matrix. Otherwise, it is equivalent to $\begin{pmatrix} v_1 & \cdots & v_{n-1} & w(i) \end{pmatrix}$ for all $i = 0, 1, ..., p^{m-n+1} - 1$, where $w(i) \equiv v_n + ip^{-m+n-1}v_1 \mod p^{m'}$ and each entry of $w(i)$ is in $\{0, 1, ..., p^{m'} - 1\}$.*

To see how this equivalence will be useful, we will state a theorem, though we will prove it later.

Theorem 4.1. *Consider a face F with vertices $\Lambda_1 \in O_{i_1}, ..., \Lambda_k \in O_{i_k}$ where $i_1 < \cdots < i_k$. If $F \in \mathcal{B}^g$, then so does every face with the same vertices representatives as F and a seed that is $\sim_g -$equivalent to the seed of F.*

We can view this as an extension of $\sim_g$ to any faces, such that two faces are equivalent when they have the same vertices representatives and equivalent seeds.

5. Split Form and Equivalence

Given a reduced matrix R, we can express it as a product of two matrices

$$R = SD,$$

where $S \in SL_n(\mathbb{Z}_p)$ and D is the diagonal matrix with the i^{th} entry $a_{\pi(i)i}$, let this be the split form.

Let s_1 be the first column of S and $a_{\pi(1)1} = p^{m'}$. If we have $g \in SL_n(\mathbb{Q}_p)$ such that $g - Id \in p^m \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$ such that $m' \geq m - n + 1$, then all equivalent reduced matrix under $\sim_g$ has a split form $S'D$, where S' has the same first $n-1$ columns as S and the last column of S' is obtained by adding integer multiples of $p^{m'-m+n-1}s_1$ to the last column of S, then taking the remainder modulo $p^{m'}$.

In short, if $R \sim_g R'$, then in the split form $R = SD$ and $R' = S'D$, $S' - S \in p^{m'-m+n-1} \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$. Furthermore, $S'^{-1} - S^{-1} \in p^{m'-m+n-1} \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$ because $S'(S'^{-1} - S^{-1})S = S - S'$.

6. Requirement for A Face to be In $\mathcal{B}^g$

It is easy to see that if a vertex Λ with representative matrix $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is in $\mathcal{B}^g$, then

$$g \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} A$$

for some $A \in SL_n(\mathbb{Z}_p)$. We can rephrase this requirement, which is sufficient and necessary, as

$$\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{-1} g \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \in SL_n(\mathbb{Z}_p).$$

Now, let us find the requirements for a face F to be in $\mathcal{B}^g$ for some good g such that $g - Id \in p^m \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$, $g = p^m X + Id$. First, let R be F's seed, and $\{R_1, ..., R_k\}$ be the set of F's vertices representatives. If $F \in \mathcal{B}^g$, then we need

$$R_i^{-1} R^{-1} g R R_i \in SL_n(\mathbb{Z}_p), \forall i = 1, ..., k.$$

The determinants of those products are obviously 1, we just need to prove that the entries are all $p-$adic integers. If the valuation of the non-zero entry of the first column of R is m', then the split form SD has $p^{m'} D^{-1} \in \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$. Furthermore, the determinant of R_i^{-1} has valuation of at most $n-1$, so $p^{n-1} R_i^{-1} \in \operatorname{Mat}_{n\times n}(\mathbb{Z}_p)$.

If $m' < m - n + 1$, then

$$\begin{aligned} R_i^{-1} R^{-1} g R R_i &= R_i^{-1} D^{-1} S^{-1} (p^m X) S D R_i + Id \\ &= p^{m-m'-n+1} (p^{n-1} R_i^{-1})(p^{m'} D^{-1}) S^{-1} X S D R_i \\ &\qquad + Id \in \operatorname{Mat}_{n\times n}(\mathbb{Z}_p). \end{aligned}$$

F is always in $\mathcal{B}^g$ in this case. If $m' \geq m-n+1$, we have that for any $Y_1, Y_2 \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$,

$$\begin{aligned} R_i^{-1}D^{-1}(S^{-1}+p^{m'-m+n-1}Y_1)p^m X(S \\ + p^{m'-m+n-1}Y_2)DR_i - R_i^{-1}R^{-1}p^m XRR_i \\ = (p^{n-1}R_i^{-1})(p^{m'}D^{-1})(Y_1XS + S^{-1}XY_2 \\ + p^{m'-m+n-1}Y_1XY_2)DR_i \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p). \end{aligned}$$

7. Proof of Theorem 4.1

Again, let m be the lowest valuation of all entries in $g - Id$ and m' be the valuation of the only non-zero entry in the first column of the seed $R = SD$ of F, a face in $\mathcal{B}^g$. The case $m' < m-n+1$ is trivial from section 6. For the case $m' \geq m-n+1$, the split form of all reduced matrices that is $\sim_g$ equivalent to R is $S'D$, where $S'-S, S'^{-1}-S^{-1} \in p^{m'-m+n-1}\,\mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ (Section 5).

Let $\{R_1, R_2, ..., R_k\}$ be the set of vertices representatives of F and $g = p^m X + Id$. Thus, from Section 6,

$$\begin{aligned} & R_i^{-1}D^{-1}S'^{-1}gS'DR_i - R_i^{-1}D^{-1}S^{-1}gSDR_i \\ &= R_i^{-1}D^{-1}S'^{-1}(p^mX)S'DR_i - R_i^{-1}D^{-1}S^{-1}(p^mX)SDR_i \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p). \end{aligned}$$

Therefore, because $F \in \mathcal{B}_g$,

$$R_i^{-1}D^{-1}S^{-1}gSDR_i \in SL_n(\mathbb{Z}_p),$$

and

$$R_i^{-1}D^{-1}S'^{-1}gS'DR_i \in SL_n(\mathbb{Z}_p),$$

and thus the face with a seed that is $\sim_g$ $-$equivalent to R and vertices representatives $\{R_1, R_2, ..., R_k\}$ are in $\mathcal{B}^g$, and we are done.

8. Final Steps

For any good g, we can see that any two $\sim_g$ $-$equivalent vertices have the same determinant for their representative matrices, so they are in the same orbit. Therefore, any equivalence class (for any kind of faces) is contained in exactly one orbit.

If $g - Id \in p^m\,\mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ for some $m > n$, then any face with a seed that has the valuation of the only non-zero entry of its first column being less than $m-n+1$ must be in $\mathcal{B}^g$ (Section 6). For the ones with this valuation being at least $m-n+1$, its face is in an equivalent class that has p^{m-n+1} elements in it, and either the entire equivalent class is in $\mathcal{B}^g$ or none of it are (Theorem 4.1).

Therefore, there are constants $a_{i_1,i_2,...,i_k}(m)$ which are equal to the remainder of the number of faces in $\mathcal{B}^g$ that consists of vertices from $O_{i_1},...,O_{i_k}$ when divided by p^{m-n+1}. Therefore, as we tend $g \to Id$, the order of the intersecion of each orbit with $\mathcal{B}^g$ converge to a $p-$adic integer, and thus the limit in section 1 must exists.

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