Efficient geometric multigrid methods for discontinuous Galerkin discretizations

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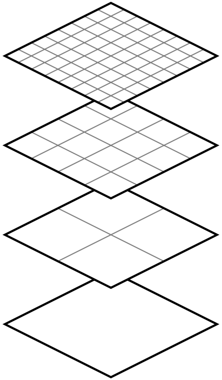
Chris Rycroft

Introduction

hp–multigrid works well for CG, but can perform poorly for DG... **why?**



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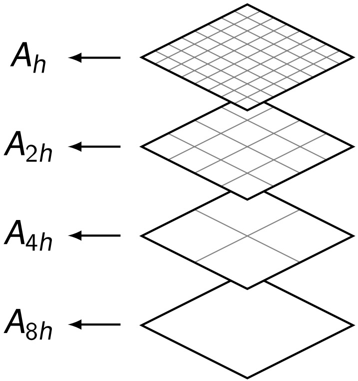

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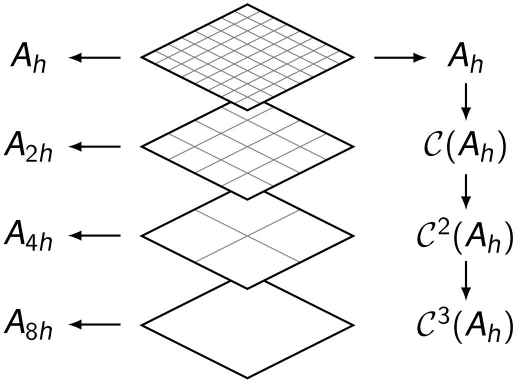
2. Automatically build coarse operators: $\mathcal{C}(A_h) \to R_h^{2h} A_h I_{2h}^h$



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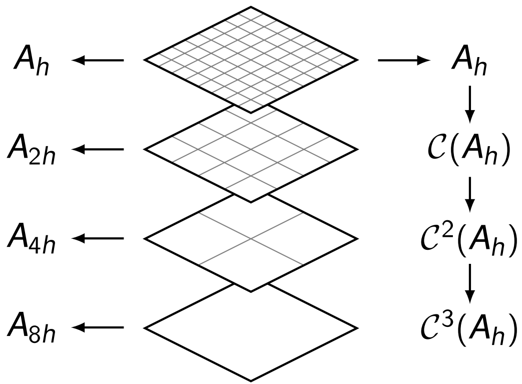

Avoids need to build coarse:
- ✓ quadrature rules
- ✓ lifting operators
- ✓ face-to-element maps
- ✓ mesh-related plumbing

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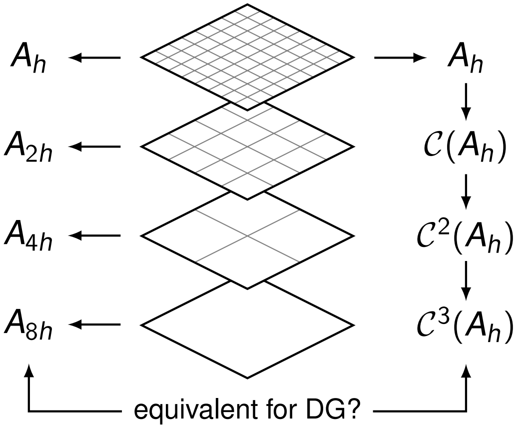

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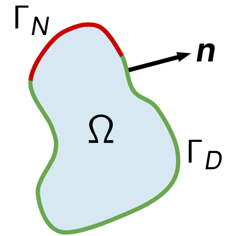
DG for elliptic problems
General approach

Consider the Poisson problem

$$\begin{aligned} -\nabla^2 u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma_D \\ \nabla u \cdot \boldsymbol{n} &= h \quad \text{on } \Gamma_N \end{aligned}$$



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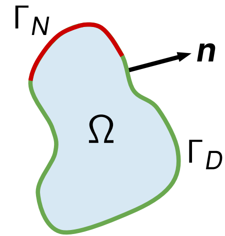

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$$

$$\downarrow$$

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\boldsymbol{q} &= \nabla u && \text{in } \Omega \\
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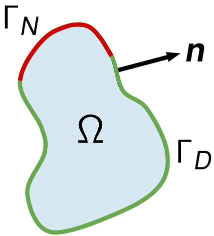

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Elements are coupled via **numerical fluxes**.

The local discontinuous Galerkin method
A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

$$\text{“}\boldsymbol{q} = \nabla u\text{”} \qquad \int_\Omega \boldsymbol{q}_h \cdot \boldsymbol{w}_h - \int_\Omega G u_h \cdot \boldsymbol{w}_h = \int_{\Gamma_D} g\, \boldsymbol{w}_h^- \cdot \boldsymbol{n}$$

$$\text{“}-\nabla \cdot \boldsymbol{q} = f\text{”} \qquad \int_\Omega \boldsymbol{q}_h \cdot G v_h + \text{penalty} = \int_\Omega f\, v_h + \int_{\Gamma_N} h\, v_h^-$$

G is the **discrete gradient operator**:

$$G = \text{broken gradient} + \text{lifting operator}$$

where the lifting operator accounts for jumps between elements.

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\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \mathrm{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
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The local discontinuous Galerkin method
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mass

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$$\underbrace{m(\boldsymbol{q}_h, \boldsymbol{w}_h)}_{\text{mass}} - \underbrace{\mathrm{grad}(u_h, \boldsymbol{w}_h)}_{\text{gradient}} = j(\boldsymbol{w}_h)$$

$$-\,\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) = k(v_h)$$

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mass gradient

$$
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m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \mathrm{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\,\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

RHS + BCs

divergence penalty

Primal formulation

“Eliminate-then-discretize”

Eliminate $\boldsymbol{q}_h$ to get bilinear form of Laplacian: $a(u_h, v_h) = \ell(v_h)$

$$
\begin{aligned}
a(u_h, v_h) &= \int_\Omega G u_h \cdot G v_h + \text{penalty} \\
\ell(v_h) &= \int_\Omega f\, v_h - \int_{\Gamma_D} g\, G v_h^- \cdot \boldsymbol{n} + \int_{\Gamma_N} h\, v_h^-
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Discretization leads to symmetric positive (semi)definite system

$$A_h u_h = \ell_h.$$

$A_h = -M_h \Delta_h$ and Δ_h is **discrete Laplacian operator**,

$$\Delta_h = D_h G_h + T_h.$$

$$
\begin{pmatrix}
M_h = \text{mass matrix} & G_h = \text{gradient matrix} \\
T_h = \text{penalty matrix} & D_h = \text{divergence matrix} = M_h^{-1} G_h^\top M_h
\end{pmatrix}
$$

Flux formulation

"Discretize-then-eliminate"

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \operatorname{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
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Flux formulation
"Discretize-then-eliminate"

$$\begin{bmatrix} M_h & -M_h G_h \\ -M_h D_h & M_h T_h \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h \\ u_h \end{bmatrix} = \begin{bmatrix} j_h \\ k_h \end{bmatrix}$$

$$\begin{aligned} M_h &= \text{mass matrix} \\ G_h &= \text{discrete gradient matrix} \\ D_h &= \text{discrete divergence matrix} = -M_h^{-1} G_h^\top M_h \\ T_h &= \text{discrete penalty matrix} \end{aligned}$$

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M_h is block diagonal, so can easily take Schur complement:

$$A_h u_h = \ell_h$$

where $A_h = -M_h \Delta_h$ and $\Delta_h = D_h G_h + T_h$.

Multigrid
Coarse operators

Primal formulation
"Eliminate-then-discretize"

$$-D_h G_h + T_h$$

$$\downarrow$$

$$\mathcal{C}(-D_h G_h + T_h)$$

Flux formulation
"Discretize-then-eliminate"

$$\begin{bmatrix} I & -G_h \\ -D_h & T_h \end{bmatrix}$$

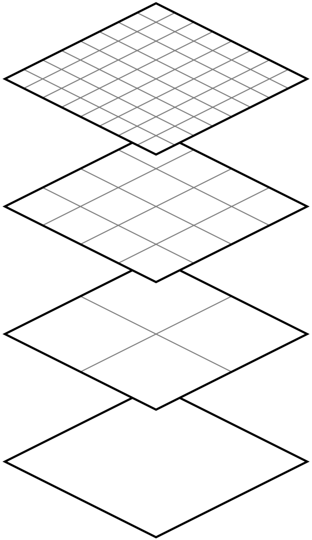
$$\downarrow$$

$$\begin{bmatrix} I & -\mathcal{C}(G_h) \\ -\mathcal{C}(D_h) & \mathcal{C}(T_h) \end{bmatrix}$$

$$\downarrow$$

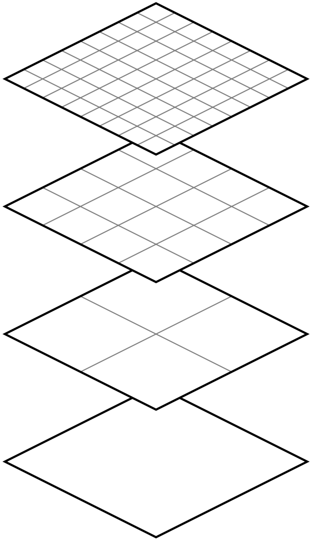
$$-\mathcal{C}(D_h)\mathcal{C}(G_h) + \mathcal{C}(T_h)$$

Multigrid
Coarse operators



Multigrid
Coarse operators

“geometric”



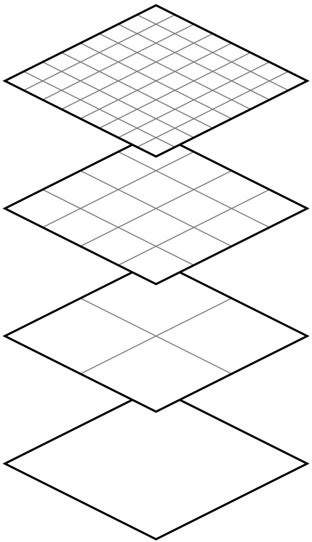
$\longrightarrow$ A_h

$\longrightarrow$ A_{2h}

$\longrightarrow$ A_{4h}

$\longrightarrow$ A_{8h}

Multigrid
Coarse operators



"geometric" "primal"

$$
\begin{tikzcd}
A_h \arrow[r] & A_h \arrow[d] \\
A_{2h} & \mathcal{C}(A_h) \arrow[d] \\
A_{4h} & \mathcal{C}^2(A_h) \arrow[d] \\
A_{8h} & \mathcal{C}^3(A_h)
\end{tikzcd}
$$

Multigrid
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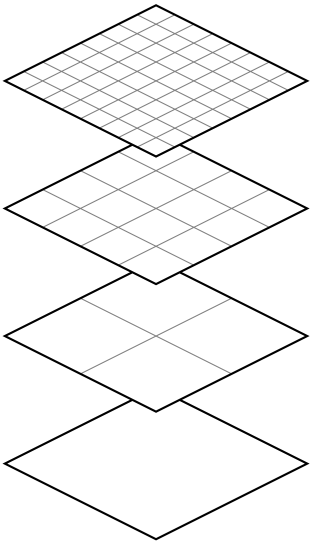

“geometric” “primal” “flux”

$$\begin{tikzcd}
A_h \arrow[r] & A_h \arrow[r] \arrow[d] & -D_h G_h + T_h \arrow[d] \\
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Multigrid
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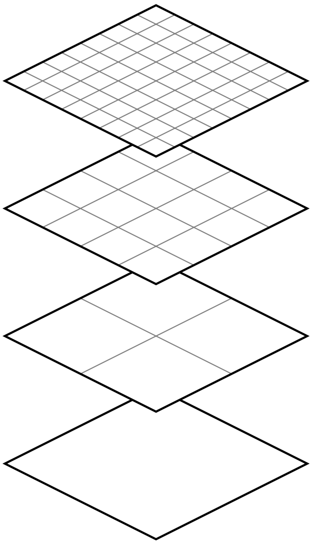

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\begin{tikzcd}
\text{“geometric”} & \text{“primal”} & \text{“flux”} \\
A_h \arrow[r] & A_h \arrow[r] \arrow[d] & -D_h G_h + T_h \arrow[d] \\
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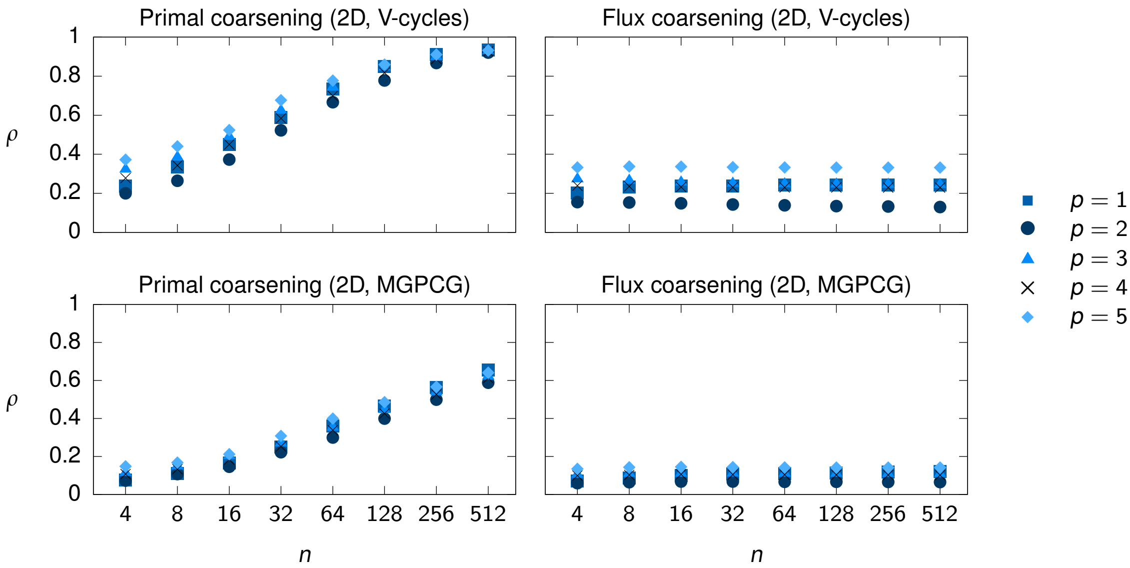
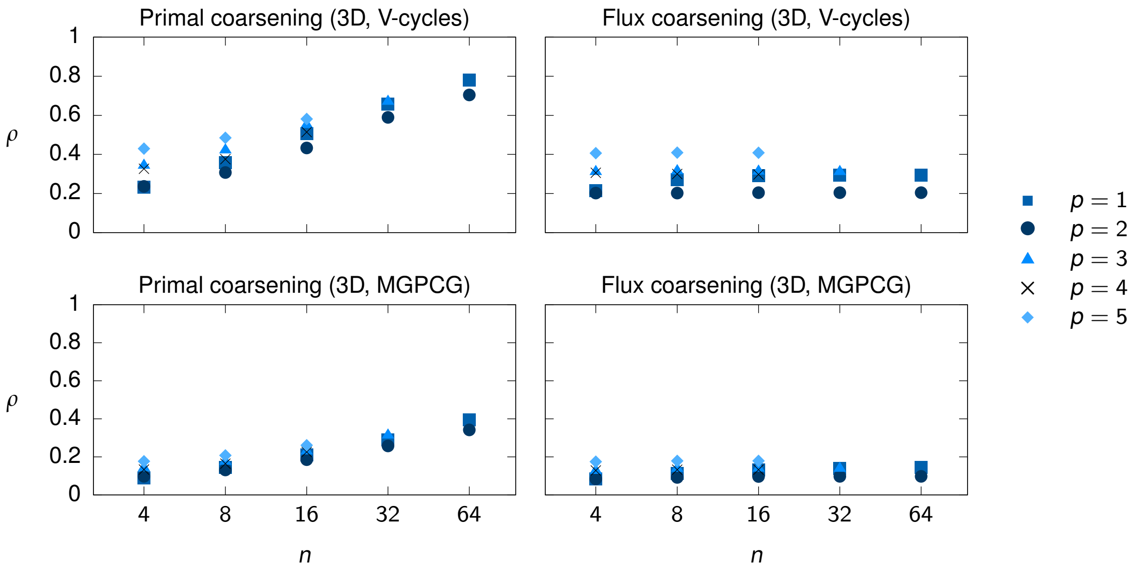
 & \text{equivalent!} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Example
Uniform Cartesian grids, 2D



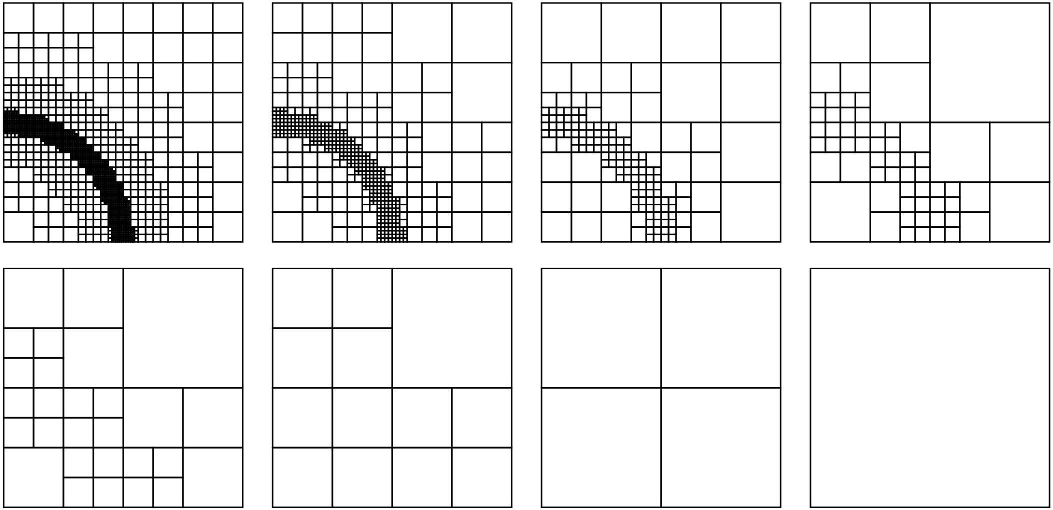
Example

Uniform Cartesian grids, 3D



Example

Adaptive grids



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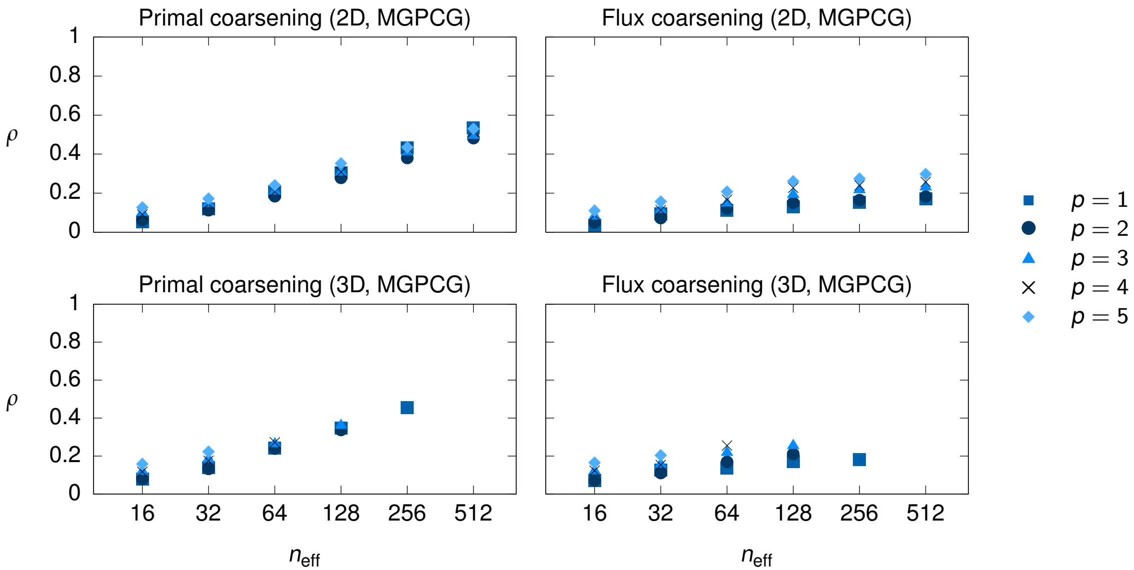
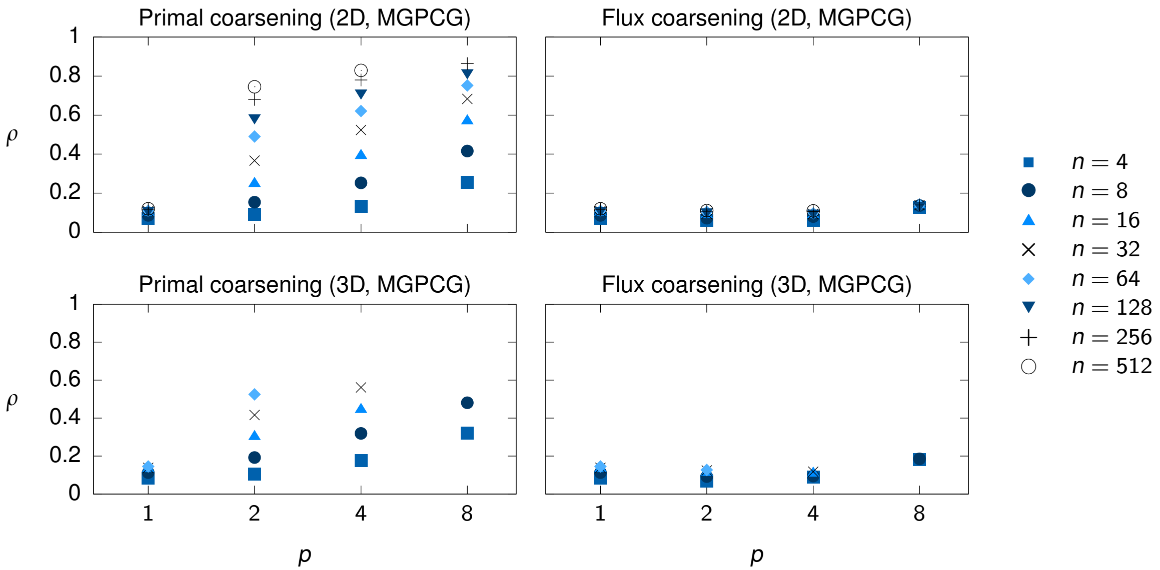
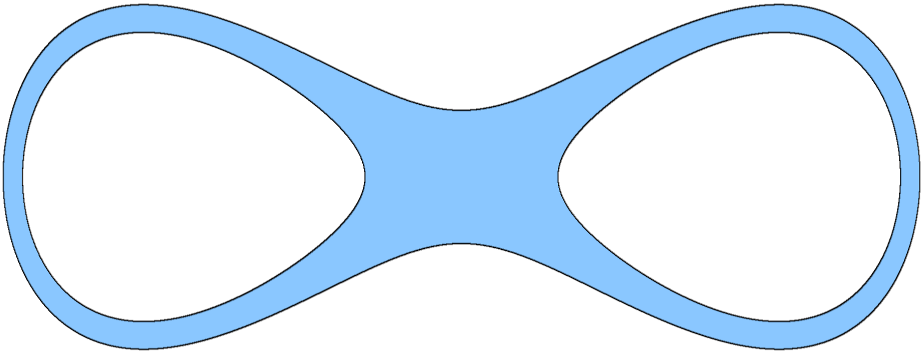


Example

p-multigrid: $p \to p/2 \to \cdots \to 1$

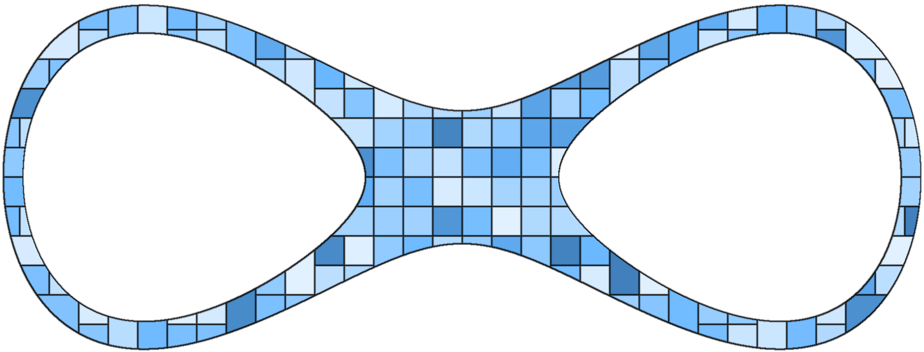


$$-\nabla^2 u = 0, \quad u|_\Gamma = 0$$



Example

Implicitly defined mesh — single phase



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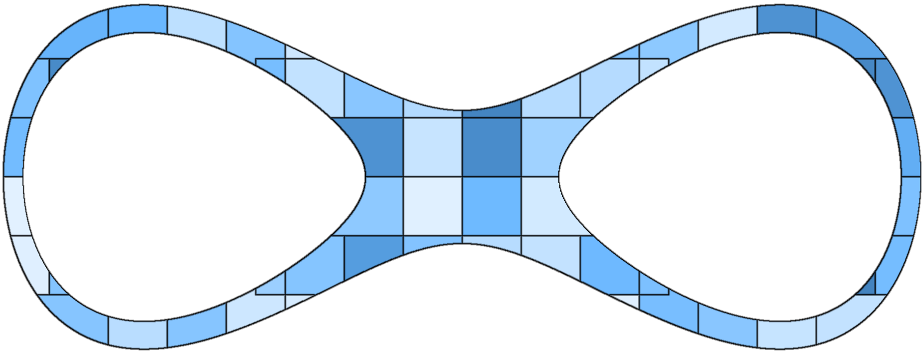

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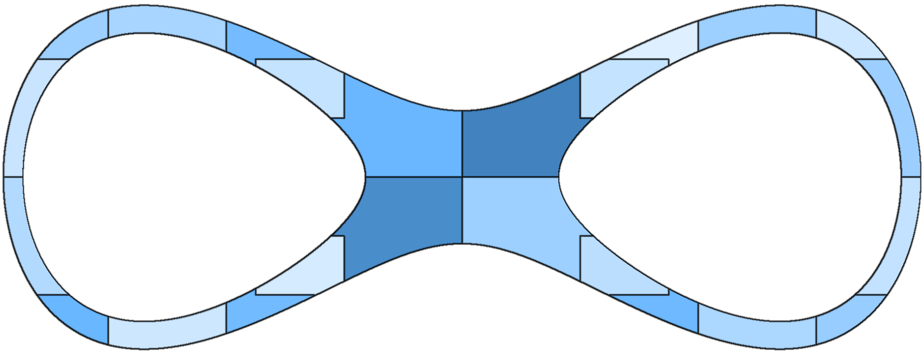

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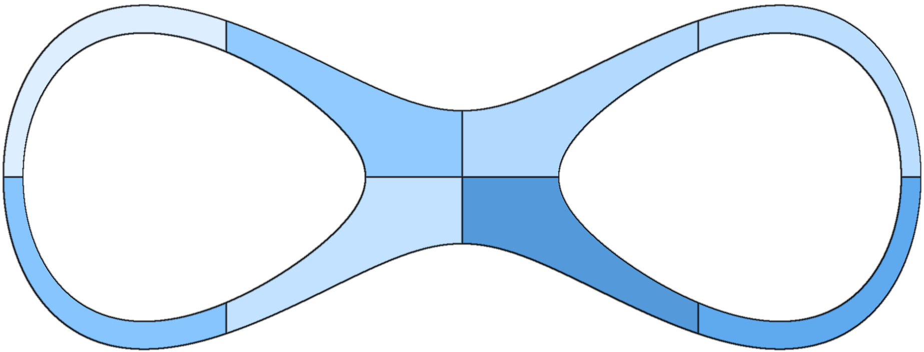

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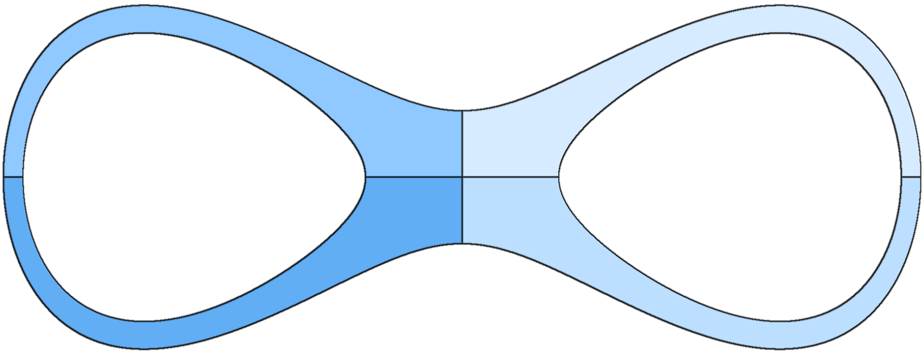

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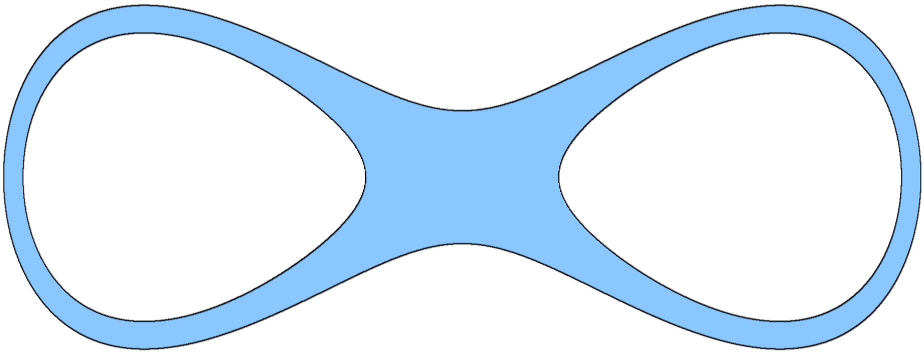
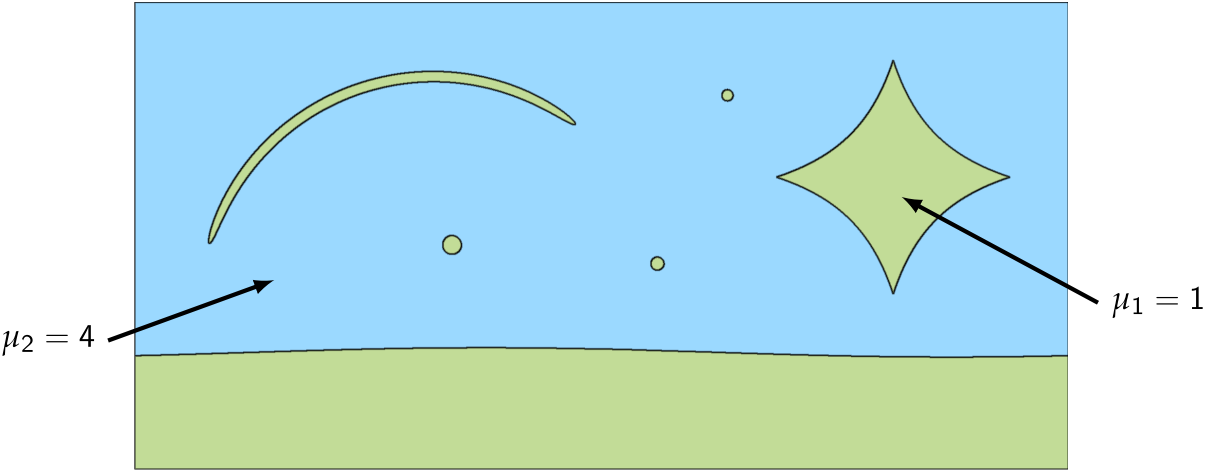


Example

Implicitly defined mesh — multiphase

$$-\nabla \cdot (\mu_i \nabla u) = 0, \quad [u] = 0, \quad u|_\Gamma = 0$$



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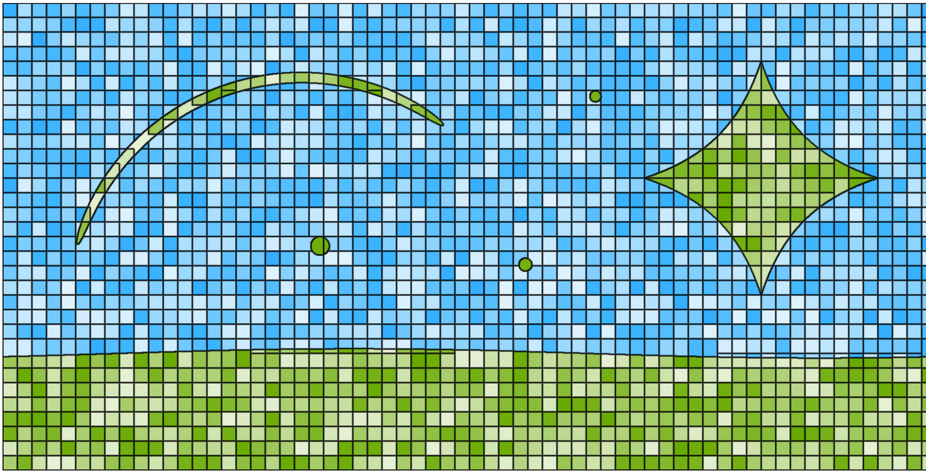

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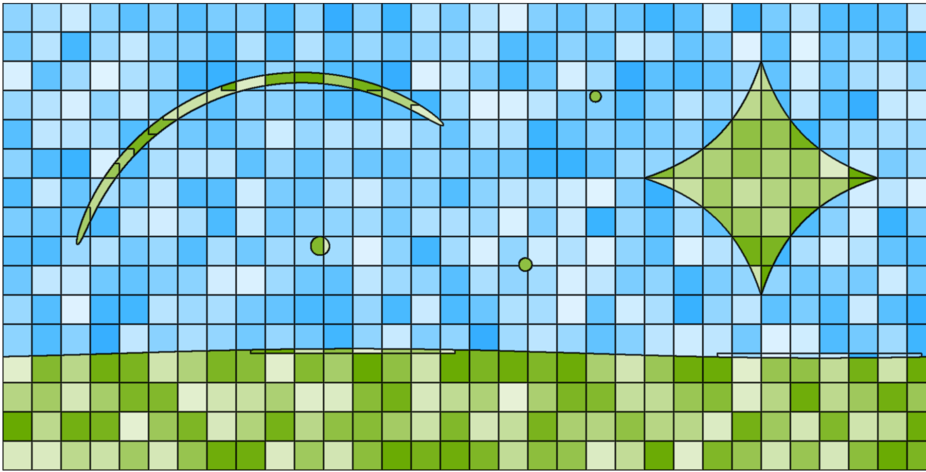

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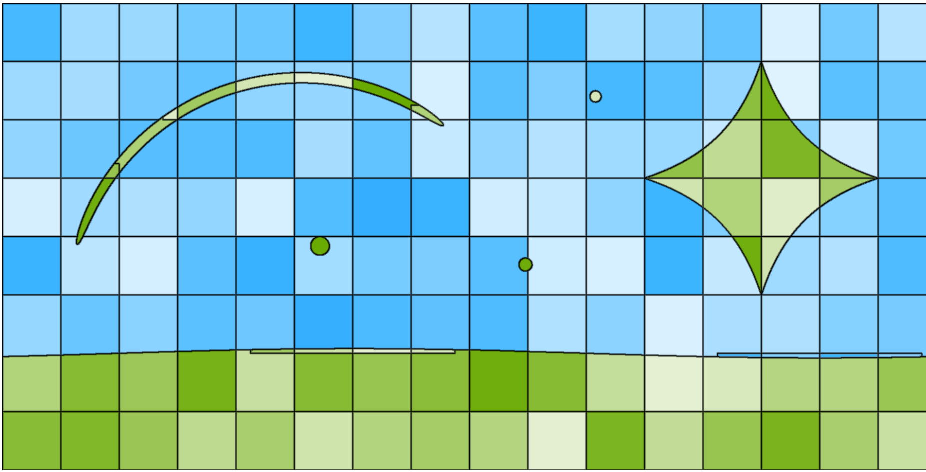

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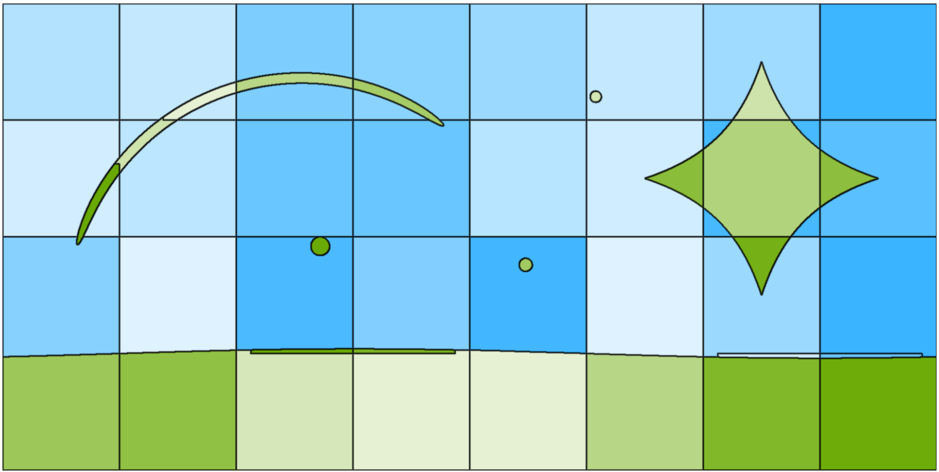

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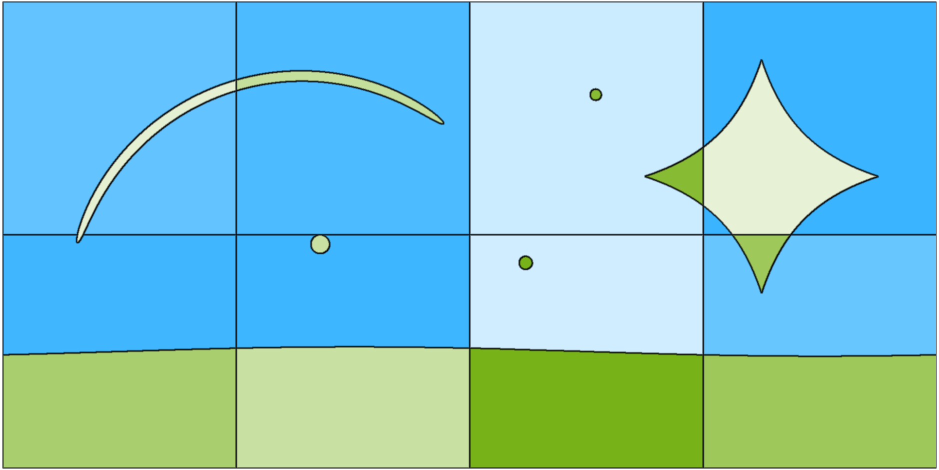

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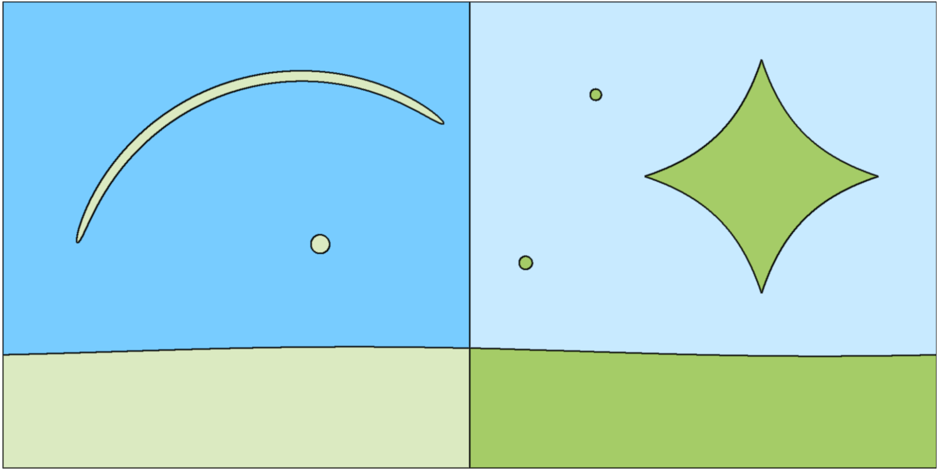

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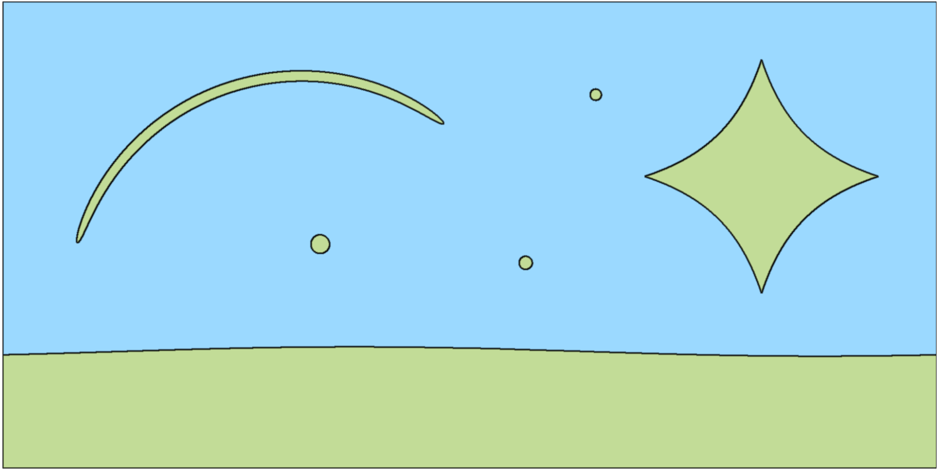

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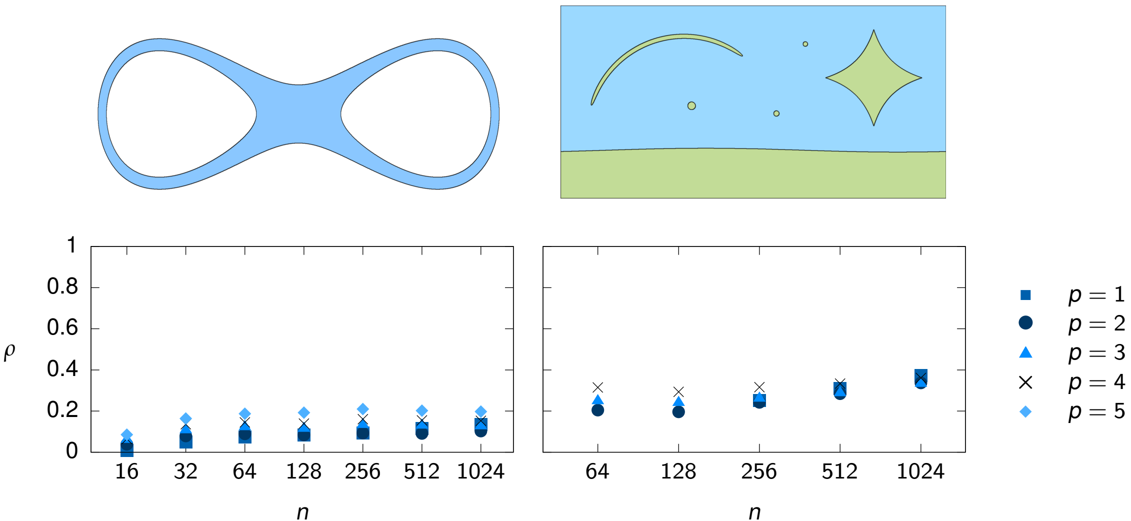

Thanks for listening!

Preprint to be posted: `danfortunato.com/multigrid-ldg`