A fast spectrally-accurate Poisson solver on rectangular domains

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SIAM CSE, February 28th 2017

Introduction
A long-standing question

Consider Poisson's equation on $[-1,1]^2$ with homogeneous Dirichlet conditions,

$$u_{xx} + u_{yy} = f, \quad (x,y) \in [-1,1]^2, \qquad u(\pm 1, \cdot) = u(\cdot, \pm 1) = 0.$$

The classic fast Poisson solver using finite differences:

$$\underbrace{KX + XK = F,}_{\text{solve with FFT, } \mathcal{O}(n^2 \log n)} \qquad K = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

[Buzbee et al, 1970]

- Based on **structured eigenvectors**
- Complexity increases with order of accuracy

$$\boxed{\text{Can we make a spectrally-accurate Poisson solver with } \mathcal{O}(n^2 \log n) \text{ complexity?}}$$

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A clever choice of basis
The ultraspherical polynomials

Dirichlet on $[-1,1]$ $\longleftrightarrow$ Pick a basis that vanishes at ± 1

The classical orthogonal polynomials, f_k, satisfy

$$A(x)f_k''(x) + B(x)f_k'(x) = q_k f_k(x), \qquad x \in [-1,1].$$

The second derivative of $(1-x^2)C_k^{(\lambda)}(x)$ is given by

$$\frac{\partial^2}{\partial x^2}\left[(1-x^2)C_k^{(\lambda)}(x)\right] = (1-x^2)C_k^{(\lambda)''}(x) - 4xC_k^{(\lambda)'}(x) - 2C_k^{(\lambda)}(x).$$

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$\boxed{C_k^{(3/2)}(x) \text{ is an eigenfunction of the differential operator } u \mapsto \frac{\partial^2}{\partial x^2}(1-x^2)u}$

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Therefore, represent the solution in the basis

$$u(x,y) \approx \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} X_{jk}(1-x^2)(1-y^2)C_j^{(3/2)}(x)C_k^{(3/2)}(y), \qquad (x,y)\in[-1,1]^2.$$

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Can we "diagonalize" Poisson?

$$\nabla^2 u = f$$

We know the action of ∇^2 on this basis:

$$\nabla^2 \left[(1-x^2)(1-y^2)C_j^{(3/2)}(x)C_k^{(3/2)}(y) \right] = -(j(j+3)+2)\,(1-y^2)C_j^{(3/2)}(x)C_k^{(3/2)}(y) \\ -(k(k+3)+2)\,(1-x^2)C_j^{(3/2)}(x)C_k^{(3/2)}(y).$$

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diagonal

$$MXD^T + DXM^T = F$$

pentadiagonal

[NIST DLMF, 18.9.7-8]

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$$TX + XT^T = D^{-1}FD^{-1}, \qquad T = D^{-1}M$$

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The Alternating Direction Implicit (ADI) method

(for solving matrix equations) [Wachspress, 1987]

$$TX + XT^T = F$$

- Based on **structured eigenvalues**
- Optimal parameters known [Lu & Wachspress, 1991]

still works for spectral

set $\quad X_0 = 0$

pick shift parameters p_j

for $j = 0, \ldots, J$

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Thomas algorithm $\mathcal{O}(n^2)$

If eigenvalues of T lie in $[a, b]$, then for $0 < \epsilon < 1$, $\dfrac{\|X - X_J\|_2}{\|X\|_2} \leqslant \epsilon$ when $J > \dfrac{1}{\pi^2} \log \dfrac{4b}{a} \log \dfrac{4}{\epsilon}$

[Lu & Wachspress, 1991]

Theorem

Every eigenvalue of a complex $n \times n$ matrix A lies within at least one disc centered at a_{ii} of radius $\sum_{j \neq i} |a_{ij}|$.

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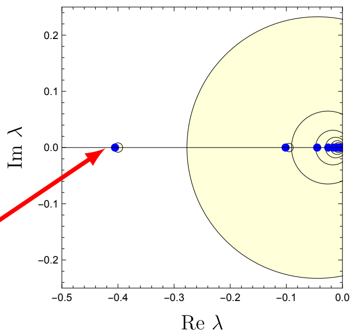
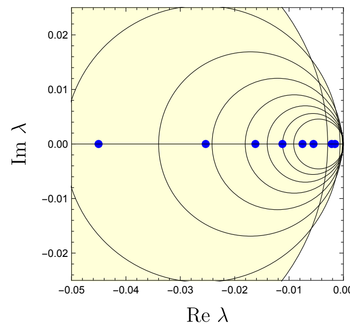


$\mathcal{O}(1)$



never crosses origin

$\mathcal{O}(n^{-4})$

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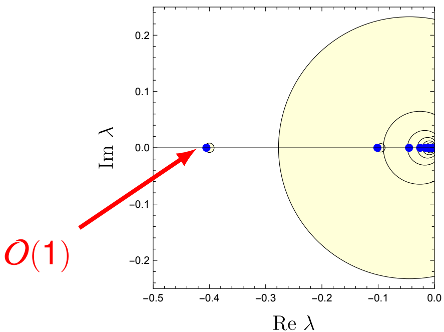
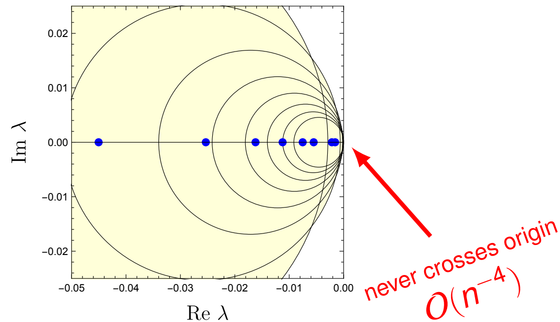
Every eigenvalue of a complex $n \times n$ matrix A lies within at least one disc centered at a_{ii} of radius $\sum_{j \neq i} |a_{ij}|$.





$$J \sim \mathcal{O}\left(\log n \log \frac{1}{\epsilon}\right)$$

A fast spectrally-accurate Poisson solver

For a given error tolerance $0 < \epsilon < 1$: **Cost**

1. Compute $C^{(3/2)}$ coefficients of f — $\mathcal{O}(n^2(\log n)^2 \log 1/\epsilon)$ [Hale & Townsend, 2014]
2. Solve matrix equation using ADI — $\mathcal{O}(n^2 \log n \log 1/\epsilon)$
   - $\mathcal{O}(n^2)$ per iteration
   - $\mathcal{O}(\log n \log 1/\epsilon)$ iterations
3. Convert solution to Chebyshev — $\mathcal{O}(n^2(\log n)^2 \log 1/\epsilon)$ [Hale & Townsend, 2014]

$$\mathcal{O}(n^2(\log n)^2 \log 1/\epsilon)$$

A similar method in 1979

...but a different conclusion!

“The accurate solution of poisson’s equation by expansion in chebyshev polynomials”

[Haidvogel & Zang, 1979]

Dale Haidvogel

$$D_2 X + X D_2^T = F$$

Chebyshev differentiation

inverse is tridiagonal

- Concluded **ADI is too slow to be practical!**

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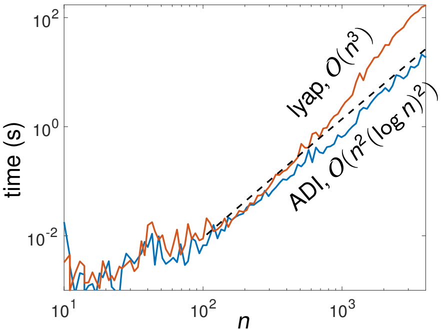

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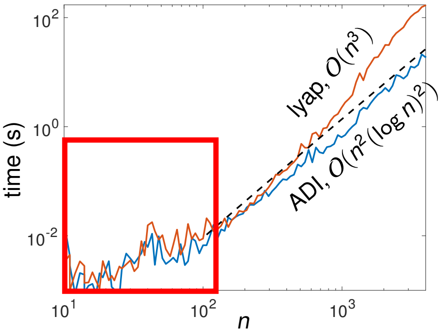

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Extras

Our fast solver can also...

- ✓ exploit low rank right-hand sides using factored ADI
- ✓ handle arbitrary Dirichlet BCs
- ✓ handle more complex BCs (e.g. Neumann)
- ✓ apply to other strongly elliptic PDEs with nice spectra

Extras

Alex Townsend

Heather Wilber

Our fast solver can also...

low-rank RHS $\Rightarrow$ low-rank solution

- ✓ exploit low rank right-hand sides using factored ADI
- ✓ handle arbitrary Dirichlet BCs
- ✓ handle more complex BCs (e.g. Neumann)
- ✓ apply to other strongly elliptic PDEs with nice spectra

Thank you

Thanks for listening!

Thanks also to Chris Rycroft, Sheehan Olver, Heather Wilber, & Grady Wright.