Well-definedness

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1 Functions

Functions are objects pertaining to two sets X, Y such that for each $x \in X$, there exists a unique $y \in Y$ according to some rule. Terence Tao defines functions in his Analysis I using predicates $P(x, y)$ pertaining to the given domain X and codomain Y. Given a predicate $P(x, y)$ and the two sets X, Y, he then defines a function $f : X \to Y$ by $f(x) = y \iff P(x, y)$. What this means for $P(x, y)$ is that it must be well-defined. But what does being well-defined mean? Something being well-defined can mean several things depending on the context. For example, for functions, we must have $x = x' \implies f(x) = f(x')$ and if $y = f(x)$ and $y' = f(x)$ then $y = y'$. That is, equal inputs mean equal outputs. Thus, if you are going to define functions by a rule, call it a predicate $P(x, y)$, we must have $x = x' \implies P(x, y) \iff P(x', y)$ and $P(x, y) \wedge P(x, y') \implies y = y'$. This justifies whenever we substitute one value for another. For example, if we define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$, then indeed $f(1) = f(0+1) = f(2-1) = f(3-1) = ...$

To be able to consider functions, it then becomes necessary to either construct it, or to take the object as one of our axioms. In Terence Tao's Analysis I, the reader is led through an exercise of constructing functions as ordered triples (X, Y, G) where $G \subset X \times Y$ such that given $x \in X$, $|\{y \in Y : (x, y) \in G\}| = 1$ is the graph of f. Here we use $|A|$ to denote the cardinality of a set A. The exercise lets us conclude that given sets X, Y and a predicate $P(x, y)$ we can construct the ordered triple (X, Y, G) which we define to be the function $f : X \to Y$ where $f(x) \in \{y \in Y : (x, y) \in G\}$.

One can construct the graph using the axiom of specification (Axiom 4 in the appendix) and then the construction of the ordered triple is trivial. However, there is a very sensitive part in this construction. Consider the real numbers, or even the integers. It is the case that equality on these sets is defined by an equivalence relation, so when you define the predicate $P(x, y)$, it must be well-defined with respect to that equivalence relation. That is, you're checking that $x \sim x' \implies P(x, y) \iff P(x', y)$ and $P(x, y) \wedge P(x, y') \implies y \sim y'$. But then, given $x \in X$, $\{y \in Y : (x, y) \in G\}$ is an equivalence class (that is a set of elements of X such that every one of them are equivalent with respect to the equivalence relation). So then we must rework our concept of

$|\{y \in Y : (x, y) \in G\}| = 1$ by considering that the cardinality is 1 with respect to the equivalence relation.

So, it seems that functions must also come with an equivalence relation if we want to consider its well-definedness. Note that it need not be well-defined. There is nothing stopping us from constructing a different kind of function than what we are used to. We only want it to be well-defined because it suits our needs.

Now let us consider addition. Addition on naturals can be constructed by a recursion function:

Definition of Addition: Given $m \in \mathbb{N}$, we recursively construct the function $f_m : \mathbb{N} \to \mathbb{N}$ by $f_m(0) = m$ and $f_m(S(n)) = S(f_m(n)) \;\; \forall n \in \mathbb{N}$. We define define $n + m$ by $n + m := f_m(n)$.

Remark: $S(n)$ denotes the primitive successor function as per the Peano axioms. Assume the recursion theorem. We then have that f_m is well-defined, so $f_m(n)$ outputs one and only one output and $n_1 = n_2 \implies f_m(n_1) = f_m(n_2)$. Note that this means that $n_1 + m = n_2 + m$. Isn't that cool?

We then have $0 + 1 = 1$ and $1 + 1 = (S(0)) + 1 = f_1(S(0)) = S(f(0)) = S(1) := 2$. Note that the only way we were able to write $1 + 1 = (S(0)) + 1$ is because $+$ is well-defined (that is $n_1 + m = n_2 + m$ where $n_1 = n_2$, which was because f_m is well-defined). Note that it is well-defined only with respect to the equivalence relation on $\mathbb{N}$!

So now we've encountered another meaning of well-definedness, but this time with respect to operators such as $+$. But, with our current paradigm, $+$ is a function, so it really is just well-definedness of a function. In lecture 0, I made the point of asking you to think of proofs regarding arbitrary variables as stating that given any variable, we can follow that exact same procedure to prove it for that variable, so then we have a trivial extension to proving the proposition for that particular variable. So, you might think of a proof as a predicate $P(x)$ and applying it to any x that fits the assumptions you made concerning your proof. In the same way, you may regard our construction of addition as whenever you need to add two numbers $n, m \in \mathbb{N}$, because of how our construction only assumed that our two numbers $k_1, k_2 \in \mathbb{N}$, we can simply take $n = k_1$ and $m = k_2$ and then we have $f_{k_2}(k_1) = k_1 + k_2$.

But there is a point to be made! How is it that we are able to say that by simply setting $k_2 = m$ and $k_1 = n$ we then have $f_m(n) = f_{k_2}(k_1)$? Well, this trivially follows by simply constructing f_m and f_{k_2} and showing that $f_m = f_{k_2}$ (note that equality between function is an equivalence relation distinct from that between $\mathbb{N}$) and showing that $f_m(n) = f_{k_2}(n')$ whenever $n = n'$. But the point I want to make is about the symbolic representation. The visual difference between m and k_2, and for that the mathematician must set down rules for semantics. In fact, if I recall correctly, this is due to $\alpha-$invariance. The point of this tangent was simply to let the reader see how much hidden machinery there is even if you think you're being rigorous or pedantic. I highly recommend someone who is fascinated by this to take a logic theory course.

A self-studier should realize that when you're reading a non-logic theory text, an author will take such things for granted. The reason for this, I believe, is simply because these things aren't the focus of say a topology textbook and they usually handle themselves.

2 Appendix

The ZF axioms (Incomplete list):

Axiom 1 Two sets A and B are equal iff $x \in A \implies x \in B$ and $x \in B \implies x \in A$.

Axiom 2 There exists a set called the empty set $\emptyset$ such that $\forall x \ \ x \notin \emptyset$

Axiom 3 Given an object x there exists a set A such that $y \in A \iff y = x$

Axiom 4 Given a set A and an unary predicate P(x) pertaining to $x \in A$ there exists a set S such that $x \in S$ iff $x \in A$ and $P(x)$

Axiom 5 Given sets A, B and a predicate $P(x, y)$ pertaining to $x \in A$ and to $y \in B$ there exists a set S such that $y \in S \iff P(x, y)$ for some $x \in A$ and some $y \in B$

Axiom 6 Given sets A, B there exists a set S such that $x \in S \iff x \in A \ \vee \ x \in B$

Axiom 7 Given a set A, there exists a set 2^A such that $A' \in 2^A \iff A' \subset A$.

Remark The set defined by Axiom 6 can be represented as $A \cup B$, the set defined in axiom 5 as $\{y : P(x, y) \ s.t. \ x \in A\}$ or $\{y : f(x) \ s.t. \ x \in A\}$, and the set defined in axiom 4 as $\{x \in A : P(x)\}$. Note the use of iff so that the sets contain exactly the elements we want.