Elementary Logic

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1 Logical Connectives

Before one starts their journey into mathematics, it becomes necessary to become acquainted with basic logic and basic proof methods. Let us first consider the logical connectives. The ones you'll encounter most often are "and", "or", "if then", "if and only if" (sometimes referred to as "iff"), and "not" (i.e. negation). Symbolically, they are $\wedge, \vee, \implies, \iff$. But how exactly do we define these symbols? Mathematical statements must always be unambiguous. In English, the word "or" could be exclusive, such as "he arrives at 9:00 am or 10:00 am" (one cannot arrive at both times), or inclusive, such as "students who complete the essay or the project will pass" (a student can complete both the essay and the project and still pass). So, how exactly do we define these logical connectives? We do so via truth tables:

P	Q	$\sim P$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Here, P and Q are propositions. Propositions are mathematical statements that are true or false. For example, $1 + 1 = 2$ is a proposition, as well as the statement $1 + 1 = 3$. Note also that statements such as $P \implies Q$ are also propositions, since they can be true or false; and that this truth table completely defines these logical connectives since a proposition can be true or false (note that this is an exclusive or and it would be incorrect to use the mathematical or $\vee$). In a statement such as $P \implies Q$, P is called the hypothesis.

For the beginner, the fact that if P is false, then $P \implies Q$ is always true might be difficult to understand. We refer to propositions where the hypothesis is false as vacuously true. One can think of the proposition $P \implies Q$ as if P is true, then Q is true. In the case where P is false, it is still the case that $P \implies Q$ since the hypothesis is not satisfied.

I would like to stress something. In more advanced mathematics, you are likely to end up in situations where the mathematics gets so abstract that your intuition simply cannot catch up. Your best bet when learning something new will be to stick to the definitions. Intuition will come later after hours of practice. Mathematics is not a sprint. In my above explanation of the symbols $\implies$ and

$\iff$, for instance, we don't arrive at those truth values because that is what we mean in English, but rather we *define* the symbols to be that way. Naturally, the symbol $\iff$ simply means $(P \implies Q) \wedge (Q \implies P)$.

Here is an example where the hypothesis being vacuous is useful in a proof:

Proposition 1: Given a set A, $\emptyset \subset A$.

Proof: By definition, $A \subset B$ if $x \in A \implies x \in B$. $x \in \emptyset \implies x \in A$.

(Since $x \notin \emptyset \ \forall x$, the proposition is vacuous) Thus $\emptyset \subset A$ $\square$

Remark: Consider the definition "given two sets A, B, $A \subset B$ if $x \in A \implies x \in B$." Note that we have only shown that if $x \in A$ implies $x \in B$ then $A \subset B$, but being pedantic, if we take this definition as is, one might ask what $A \subset B$ even tells us? For example, consider the statement "All dogs are mammals." This is true, but given an arbitrary mammal, we don't know whether the mammal is a dog. Something similar is going on here. Technically, the definition should be "given sets $A, B, A \subset B$ *iff* $x \in A \implies x \in B$. However, it is common practice in mathematics to define it with "if" rather than "iff," so don't get confused by this.

2 Proof Techniques

Direct proof The most common method of proof is directly showing that the proposition is true, such as what we did for Proposition 1. As you gain more experience, you will begin to understand which proof methods work for which kinds of problem.

Contradiction Given a proposition P, it is true or false, so if you show that it cannot be false, then it must be true. Here is how one would typically prove a proposition by contradiction:

Prove P

Proof: Assume for the sake of contradiction that we have $\sim P$, then $\sim P \implies R$, but we also have $\sim R$. We can't have $R \wedge \sim R$, thus P must be true. $\square$

Proof by contrapositive Consider the proposition $P \implies Q$, where P, Q are propositions. Note that $P \implies Q$ is logically equivalent to $\sim Q \implies \sim P$. (It is common practice to use the symbols $\cong$ or $\equiv$ to indicate that two statements are logically equivalent).

Proof: We will prove this directly with a truth table.

P	Q	$\sim P$	$\sim Q$	$P \implies Q$	$\sim Q \implies \sim P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$\square$

The contrapositive may seem complicated to work with at first, but it really isn't. It is quite intuitive, and you have probably used this identity before, though perhaps you weren't aware of our name for it. For example, consider the proposition "If n^2 is even, then n is even." How would you prove this?

Well, n is either even or odd given $n \in \mathbb{Z}$ (Can you prove this?). If we can show that if n is odd, then n^2 is odd, then it follows that if n^2 is even, n must be even (since if it is odd, n^2 would be odd, contradicting our hypothesis). This is the contrapositive. Note that proving the proposition directly is much more involved than proving its contrapositive.

Proof: It suffices to prove that if n is odd, then n^2 is odd. Suppose n is odd. By definition, we then have $n = 2k + 1$ for some $k \in \mathbb{Z}$. Hence $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2 \in \mathbb{Z}$, we have $n^2 = 2m + 1$ for some $m \in \mathbb{Z}$ (simply take $m = 2k^2 + 2k$). Thus, n^2 is odd. $\square$

Remark Note that the substitution of $n^2 = (2k+1)^2$ in our proof is true because we have that if $n = m$, then $n^2 = m^2$. This must be proven. This is because exponentiation is "well-defined." The point of this remark is to show the reader that there is almost always more math hidden under the hood. For instance, the operation of exponentiation can be defined as a function, and a function in turn can be defined as a set. For the curious, I highly recommend taking a logic/set theory course sometime in your mathematical journey.

3 Quantifiers

You may have noticed our use of "for some" in our previous proof. This is a quantifier which says there exists at least one. Other quantifiers are for all/for each/for any, there exists, and at most one. "For all" and "there exists" have their own symbols: $\forall$, $\exists$.

$\exists$ means there exists at least one, and for all is self-explanatory. Note that for $\exists$, there can be only one; and for "at most one", there can be no elements that satisfy the condition or exactly 1.

The uniqueness of a solution or variable is something that is commonly asked for in mathematics. What this means is that there is at most one element satisfying the condition. When a problem asks you to prove that there is exactly one, you're obviously proving that there exists such an element and that that element is unique. This is similar to showing an if and only if statement: You first have to show the forward implication ($\Rightarrow$) and then reverse (also known as the converse) implication ($\Leftarrow$) (though, of course, the order does not matter unless you're using one implication to prove the other).

So how would one prove a statement such as "Prove that $\forall x \in \mathbb{R},\ \ x^2 \geq 0$"? You pick an arbitrary $x \in \mathbb{R}$, and show that $x^2 \geq 0$. For example:

Proof: Given an arbitrary $x \in \mathbb{R}$, $x > 0$, $x = 0$, or $x < 0$ by order trichotomy. Thus we have three cases:

Case 1: If $x > 0$, $x^2 = x \times x > x \times 0 = 0$.

Case 2: If $x = 0$, $x^2 = 0^2 = 0 \geq 0$.

Case 3: If $x < 0$, $-x > 0 \implies x^2 = (-x)^2 = -x \times -x > -x \times 0 = 0$.

Thus $x^2 \geq 0\ \ \forall x \in \mathbb{R}$ $\square$

Since we have not assumed anything other than that $x \in \mathbb{R}$, if you take any $x \in \mathbb{R}$, you can recreate the above proof for that x. One can think of this as "applying" the proof to any x you are concerned with.