You may assume all algebras are finite-dimensional over a field $\Bbbk$. You may attempt the exercises with the additional assumption of $\Bbbk$ being algebraically closed.

Throughout, unadorned tensor product over assumed to be taken over a field, i.e. $\otimes = \otimes_{\Bbbk}$.

Ex 1. Let e be an idempotent of an algebra A.

1. Show that $\mathrm{Hom}_A(eA, \mathrm{Hom}_{eAe}(Ae, M)) \cong M$ as eAe-module.
2. Show that indecomposable projective (right) eAe-modules are of the form fAe for an idempotent $f \in A$ with $fe \neq 0$.
3. Show that indecomposable injective (right) eAe-modules are of the form $D(fAe)$ for an idempotent $f \in A$ with $fe \neq 0$.
4. Show that $-\otimes_{eAe} eA$ sends projective A-modules to projective A-modules, and $\mathrm{Hom}_{eAe}(Ae, -)$ sends injective A-modules to injective A-modules.
   *** Ideal solution is to prove this directly using part 1 and 2. If you only present this as a consequence of property of adjointness, no mark will be awarded.*
5. Suppose that $M \in \operatorname{\mathsf{mod}} eAe$ has a projective resolution $\cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$. Show that there is a projective resolution of $M \otimes_{eAe} eA \in \operatorname{\mathsf{mod}} A$ where the first two term are given by direct sums of direct summands of eA.
6. Suppose that $M \in \operatorname{\mathsf{mod}} A$ has a projective resolution $\cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$ such that, for both $i \in \{0, 1\}$, the projective module P_i is given by direct sums of direct summands of eA. Show that $Me \otimes_{eAe} eA \cong M$.
   Hint: Use part 2 and find an appropriate commutative diagram.

Ex 2.

1. Show that $\mathrm{Hom}_A(M, N) \cong D(M \otimes_A DN)$ as vector spaces.
2. Let $P_\bullet = (P_i, d_i : P_i \to P_{i-1})_{i \geq 0}$ be a projective resolution of an A-module M, and define
$$\mathrm{Tor}_1^A(M, N) := H_1(P_\bullet \otimes_A N) = \frac{\mathrm{Ker}(d_1 \otimes_A N)}{\mathrm{Im}(d_2 \otimes_A N)}$$
   the first homology group of the complex $P_\bullet \otimes_A N$. Show that $\mathrm{Ext}_A^1(M, N) \cong D\,\mathrm{Tor}_1^A(M, DN)$ as $\Bbbk$-vector spaces.
3. Show that $D\,\mathrm{Hom}_A(M, A) \cong M \otimes_A DA$ as right A-modules.
4. Let ${}_AX_B$ be an A-B-bimodule. If M is a C-A-bimodule and N is a C-B-bimodule. Show that $\mathrm{Hom}_{C^{\mathrm{op}} \otimes B}(M \otimes_A X, N) \cong \mathrm{Hom}_{C^{\mathrm{op}} \otimes A}(M, \mathrm{Hom}_B(X, N))$ as vector spaces.
5. Let $B := A^{\mathrm{op}} \otimes A$. Show that $\mathrm{Hom}_B(A, B) \cong \mathrm{Hom}_A(DA, A)$ as A-A-bimodules.
   Hint: $B \cong (DDA) \otimes A \cong \mathrm{Hom}_{\Bbbk}(DA, A)$ as B-modules.

Ex 3. Consider the quiver algebra $A = \Bbbk Q/I$ given by

$$Q:\ \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] & 4 \arrow[l, bend left, "\beta_3"] \end{tikzcd}, \quad I = (\beta_3\alpha_3, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid i = 1, 2)$$

For $i \in \{1, 2, 3, 4\}$, let $\Delta(i) := P_i/\alpha_i A$ (with $\alpha_4 := 0$ as a convention).

(i) Write down the minimal projective resolution of $\Delta(1)$.

(ii) Show that $\mathrm{Ext}^k_A(\Delta(i), \Delta(j)) = 0$ whenever $i > j$ for any $k \geq 0$.

(iii) Show that $\mathrm{Ext}^k_A(\Delta(i), \Delta(j)) = 0$ whenever $k > 3$ for any i, j.

(iv) Compute $\dim_\Bbbk \mathrm{Ext}^k_A(\Delta(i), \Delta(j))$ for all possible i, j, k. Show your working.

(v) Consider the chain of ideals

$$A = Af_1A \supset Af_2A \supset Af_3A \supset Af_4A \supset Af_5A = 0$$

where $f_i = \sum_{j=i}^4 e_j$ for $i < 4$ and $f_5 = 0$. Let $A_i := A/Af_{i+1}A$. Compute the A_i-A_i-bimodule structure of $\overline{I}_i := Af_iA/Af_{i+1}A$ and show that

(i) $\overline{I}_i$ is projective as a right A_i-module, and

(ii) $\overline{I}_i \,\mathrm{rad}(A_i)\overline{I}_i = 0$.

Ex 4. Let e be an idempotent of an algebra A.

1. Show that indecomposable projective $\mathrm{End}_A(eA)$-modules are of the form $\mathrm{Hom}_A(eA, fA)$ for an primitive idempotent $f \in A$ with $fe \neq 0$.
2. Show that if $(AeA)_A$ is projective, then Ae is a projective (right) eAe-module.
   Hint (i): Assumption implies that $AeA \cong (eA)^{\oplus m}$ (since $eA^{\oplus A} \twoheadrightarrow AeA$ splits).
   Hint (ii): $Ae = AeAe$ and use part 1.
3. Show that if $(Ae)_{eAe}$ is projective, then $\mathrm{gldim} eAe \leq \mathrm{gldim} A$ for any simple A-module S.
   Hint: $- \otimes_{eAe} Ae$ takes simple module to simple module or zero.

Let I be a two-sided ideal of A such that I_A is projective, and take $B := A/I$.

3. Show that $\mathrm{pdim}(B_A) \leq 1$.
4. Show that $\mathrm{pdim}(M_A) \leq 1 + \mathrm{pdim}(M_B)$.
   Hint (i): Prove by induction.
   Hint (ii): Construct a short exact sequence in $\mathsf{mod}\, A$ involving M and a projective B-module.

Deadline: 29th December, 2022

Submission: E-mail to (replace at by @) aaron.chan at math.nagoya-u.ac.jp