Topics in Mathematical Science V
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From quiver to quasi-hereditary algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M,+)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r,s \in R$ and $m,n \in M$, we have*

- $m \cdot 1 = m$,
- $(m+n)\cdot r = m\cdot r + n\cdot r$,
- $m\cdot(r+s) = m\cdot r + m\cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $f(R)$ being in the center $Z(A) := \{z \in A \mid za = az\ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_{\Bbbk} A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the characteristic of R, denoted by char R, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the direct product of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Definition 1.7. *A map $f : M \to N$ between right R-modules M, N is a homomorphism if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\mathrm{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$.*

Lemma 1.8. *$\mathrm{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \mathrm{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\mathrm{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.9. *$\mathrm{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the endomorphism ring of M.*

Lemma 1.10. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\mathrm{Hom}_A(M, N)$ is also an R-module (hence, $\mathrm{End}_R(M)$ is an R-algebra).*

Example 1.11. *$A \cong \mathrm{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra).*

Exercise 1.12. *Recall that R^{op} is the opposite ring of R, whose underlying set is the same as that of R with multiplication $(a \cdot^{\mathrm{op}} b) := b \cdot a$. A representation of R is a ring homomorphism*

$$\rho : R^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(M), \qquad r \mapsto \rho_r,$$

for some abelian group $(M, +)$. A homomorphism $f : \rho_M \to \rho_N$ of representations $\rho_M : R^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(M), \rho_N : R^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(N)$ given by an abelian group homomorphism $f : M \to N$ that intertwines R-action, i.e. $\rho_N(r) \circ f = f \circ \rho_M(r)$ for all $r \in R$.

Eplain why a representation of R is equivalent to a right R-module; and why homomorphisms correspond.

2 Indecomposable modules and Krull-Schmidt property

Recall that an R-module M is finitely generated if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x\in X} xr_x$ for some $r_x \in R$.

Notation. *We write* $\mathsf{mod}\, A$ *for the collection of all finitely generated right* A*-modules.*

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \dots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + N_{\hat{j}} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be unique if for any other decomposition $M = \bigoplus_{j=1}^n N_j$, we have $n = m$ and the N_j's are permutation of the M_i's.

Convention. *We write $(n_1, \dots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose R_R is indecomposable as an R-module. Then the free module $R \oplus R \oplus \cdots \oplus R$ with R copies of R is a decomposition of R^n.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1\leq j\leq n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \geq 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_\Bbbk(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. By definition $(av_1 + bv_2)x = (a+b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules; a contradiction. Hence, V must be indecomposable.

A natural question is to ask when a decomposition of modules, if it exists, is unique up to permuting the direct summands.

Definition 2.5. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^m M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^n N_j$ implies that $m = n$ and there is a permutation σ such* *that $M_i \cong N_{\sigma(i)}$ for all $1 \leq i \leq m$. $\mathsf{mod}\, A$ is said to be Krull-Schmidt if every finitely generated A-module M admits a unique indecomposable decomposition.*

Theorem 2.6. *For a finite-dimensional algebra A, $\mathsf{mod}\, A$ is Krull-Schmidt.*

Remark 2.7. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Proposition 2.8. *There is a canonical R-module isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) & \xrightarrow{\cong} & \bigoplus_{i,j} \mathrm{Hom}_A(M_j, N_i) \\ f & \longmapsto & (\pi_i f \iota_j)_{i,j} \end{array}$$

where $\iota_j : N_j \to \bigoplus_j N_j$ is the canonical inclusion for all j and $\pi_i : \bigoplus_i M_i \to M_i$ is the canonical projection for all i.

One can think of the right-hand space above as the space of m-by-n matrix with entries in each corresponding Hom-space.

3 Extra: Krull-Schmidt theorem

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent.

Lemma 3.1. *A non-zero A-module M is indecomposable if, and only if, the endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*

Proof $\underline{\Leftarrow}$: Suppose $M = U \oplus V$. Then we have

$$\text{a projection map } \pi_W : M \twoheadrightarrow W,$$
$$\text{and an inclusion map } \iota_W : W \hookrightarrow M,$$

for $W \in \{U, V\}$. Both of these are clearly A-module homomorphisms. Now $e_W := \iota_W \pi_W$ is an endomorphism of M with $e_V = \mathrm{id}_M - e_U$. Since any $m \in M$ can be written as $u + v$ for $u \in U$ and $v \in V$, we have

$$e_V^2(m) = e_V^2(u+v) = e_V^2(v) = v = e_V(m);$$

and likewise for e_W, so we have idempotents different from 0 and id_M when both U and V are non-zero.

$\underline{\Rightarrow}$: Suppose that M is indecomposable, and $e \in \mathrm{End}_A(M)$ is an idempotent. Note that

$$(\mathrm{id}_M - e)^2 = \mathrm{id}_M - e \cdot \mathrm{id}_M - \mathrm{id}_M \cdot e + e^2 = \mathrm{id}_M - 2e + e = \mathrm{id}_M - e$$

is also an idempotent and $\mathrm{id}_M = e + (\mathrm{id}_M - e)$. So we have $M = e(M) + (\mathrm{id}_M - e)(M)$. We want to show that $M = e(M) \oplus (\mathrm{id}_M - e)(M)$, i.e. $e(M) \cap (\mathrm{id}_M - e)(M) = 0$. Indeed, $x \in e(M) \cap (\mathrm{id}_M - e)(M)$ means that we have $e(m) = x = (\mathrm{id}_M - e)(m')$ for some $m, m' \in M$, and so

$$x = e(m) = e^2(m) = e((\mathrm{id}_M - e)(m')) = (e(\mathrm{id}_M - e))(m') = (e - e^2)(m') = 0(m') = 0,$$

as required.

Since M is indecomposable, one of $e(M)$ or $(\mathrm{id}_M - e)(M)$ is zero. In the former case, we get $e = 0$; whereas the latter case yields $\mathrm{id}_M = e$; as required. □

The following is one of the main reasons why we like to consider finite-dimensional (or finite generated) modules over finite-dimensional $\Bbbk$-algebras.

Lemma 3.2 (Fitting's lemma (special version)). *Let M be a finite-dimensional A-module of a finite-dimensional $\Bbbk$-algebra, and $f \in \mathrm{End}_A(M)$. Then there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

Remark 3.3. The general version for rings requires M to be artinian and noetherian (i.e. ascending and descending chains of submodules stabilises).

We omit the proof to save time. The point is really just take n large enough so that the chains of submodules given by $(\mathrm{Ker}(f^k))_k$ and $(\mathrm{Im}(f^k))_k$ stabilises.

Corollary 3.4. *Let M be a non-zero finite-dimensional A-module. Then M is indecomposable if, and only if, every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*

Proof By Fitting's lemma, for any $f \in \mathrm{End}_A(M)$, we have $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$ for some $n \geq 1$. So indecomposability means that one of these direct summands is is zero. If $\mathrm{Ker}(f^n) = 0$, then f^n is an isomorphism and so is f. If $\mathrm{Im}(f^n) = 0$, then $f^n = 0$ and so f is nilpotent.

Conversely, consider an idempotent endomorphism $e \in \mathrm{End}_A(M)$. The assumption says that e is either an isomorphism or nilpotent.

If e is an isomorphism, then we have $\mathrm{Im}(e) = M$, which means that for every $m \in M$, there is some $m' \in M$ with $e(m) = e^2(m') = e(m') = m$, i.e. $e = \mathrm{id}_M$.

If e is nilpotent, then $e^n = 0$ for some $n \geq 1$, but $e = e^2 = e^3 = \cdots = e^n$, and so $e = 0$.

Hence, an idempotent endomorphism of M is either 0 or id_M, which means that M is indecomposable by Lemma 3.1. ☐

Definition 3.5. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 3.6. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit right inverses.

Lemma 3.7. *Let A be a finite-dimensional algebra and M be a finite-dimensional A-module. Then the following hold.*

(1) *The following are equivalent.*
  - *A is local (i.e. has a unique maximal right ideal).*
  - *Non-invertible elements of A form a two-sided ideal.*
  - *For any $a \in A$, one of a or $1 - a$ is invertible.*
  - *0 and 1_A are the only idempotents of A.*
  - *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

(2) *M is indecomposable $\Leftrightarrow$ $\mathrm{End}_A(M)$ is local.*

We omit the proof to save time.

Example 3.8. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq 2} \,\middle|\, \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

Theorem 3.9 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.*

Remark 3.10. Some people refer to this result as Krull-Remak-Schmidt theorem.

For proof, interested reader can see lecture notes from last year.

4 Simple modules, Schur's lemma

Definition 4.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 4.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 4.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 4.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 4.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 4.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Proof $\Rightarrow$: $mA \subset S$ is a submodule and contains a non-zero element m, so by simplicity of S we must have $mA = S$.

$\Leftarrow$: Suppose that there is a non-zero submodule $L \subset S$. For a non-zero element $m \in L$, the assumption says that we have $mA \subset L \subset S = mA$, and so $L = S$. $\square$

Let us see how one can find a simple module.

Definition 4.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 4.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module. In particular, if A is finite-dimensional, then every simple A-module is also finite-dimensional.*

Proof Since $S = mA$, the element m defines a surjective A-module homomorphism $f : A_A \to S$ given by $a \mapsto ma$. On the other hand, we have $\mathrm{Ker}(f) = \mathrm{Ann}_A(m)$, and so $A/\mathrm{Ann}_A(m) \cong S$. $\square$

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 4.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. $\square$

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 4.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S,T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T, simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S, so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. $\square$

Example 4.12. *In Example 3.8, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x,y) \mid x,y \in \Bbbk^2\}$ given by 'row vectors'. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0,y) \mid y \in \Bbbk^2\}$.*

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A-action given by

$$w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0,y)$ for some $y \in \Bbbk$ and has to satisfy

$$(0, ya) = (0,y)a = f(wa) = f(w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = f(w) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0,y)c = (0,yc)$$

for any $a,b,c \in \Bbbk$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 4.13. *Consider a semisimple A-module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$ for some i.*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this is an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \leq i,j \leq r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1 \leq i,j \leq r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \xrightarrow{a_{i,j}} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. $\square$

Example 4.14. *For a tautological example, take $A = \Bbbk$ to be just a field. Then we have a 1-dimensional simple A-module $S = \Bbbk$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\Bbbk)) = \mathrm{Mat}_n(\Bbbk)$. Note that now we have an n-dimensional simple $\mathrm{Mat}_n(\Bbbk)$-module (given by the row vectors).*

5 Quiver and path algebra

Definition 5.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 5.2. *Let Q be a quiver.*

- *A trivial path on Q is a "stationary walk at i", denoted by e_i for some $i \in Q_0$.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2\cdots\alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.*

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right.*

Definition 5.3. *The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying vector space is given by $\bigoplus_{p:\text{paths of } Q} \Bbbk p$, with multiplication given by path concatenation. That is $x \in \Bbbk Q$ is a formal linear combinations of paths on Q.*

Note that $e_i e_j = \delta_{i,j} e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an *idempotent* of the path algebra $\Bbbk Q$. Moreover, we have an idempotent decomposition

$$1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\Bbbk Q$.

Example 5.4. *Consider the one-looped quiver, a.k.a. Jordan quiver,*

$$Q = \left(\begin{tikzcd} \bullet \arrow[loop above, "\alpha"] \end{tikzcd} \right)$$

Then $\Bbbk Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\Bbbk Q \cong \Bbbk[x]$.

An *oriented cycle* is a path of the form $v_1 \to v_2 \to \cdots v_r \to v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is *acyclic*.

Proposition 5.5. *$\Bbbk Q$ is finite-dimensional if, and only if, Q is finite acyclic.*

Proof If there is an oriented cycle c, then $c^k \in \Bbbk Q$ for all $k \geq 0$, and so $\Bbbk Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q. □

Example 5.6. *Consider the linearly oriented $\vec{\mathbb{A}}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\Bbbk Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j\alpha_{j+1}\cdots\alpha_k$.

Consider the upper triangular n-by-n matrix ring

$$\begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \,\middle|\, \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\Bbbk Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \le j < k \le n$.

From now on, we will focus in the following setting.

Assumption 5.7. (1) *Quivers are always finite.*

(2) *Modules (and representations) are finitely generated (which is equivalent to finite-dimensional when the algebra is so).*

6 Duality

For a quiver Q, the opposite quiver Q^{op} has the same set of vertices with the reverse direction of arrows, i.e. $Q_0^{\mathrm{op}} = Q_0, Q_1^{\mathrm{op}} = Q_1, s_{Q^{\mathrm{op}}} = t_Q$, and $t_{Q^{\mathrm{op}}} = s_Q$.

Exercise 6.1. *Show that there is a canonical isomorphism $(\Bbbk Q)^{\mathrm{op}} \cong \Bbbk(Q^{\mathrm{op}})$.*

Let M be a finite-dimensional A-module. Then we have a dual space

$$D(M) := M^* := \mathrm{Hom}_{\Bbbk}(M, \Bbbk),$$

which has a natural A^{op}-module structure, namely, $(a \cdot f)(m) := f(ma)$ for any $a \in A, f \in M^*, m \in M$. Moreover, for an A-module homomorphism $\theta : M \to N$, we have also an A^{op}-module homomorphism $\theta^* : N^* \to M^*$ with $\theta^*(f)(m) = f(\theta(m))$.

We note as a fact that D preserves indecomposability of (finite-dimensional) modules. This can be seen using the fact that $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$ and can be upgraded to an algebra isomorphism for the case when $N = M$; then uses characterisation of indecomposable module by local endomorphism ring.

Example 6.2. *The* left *A-module ${}_AA$ yields a right A-module structure on $D(A)$. More generally, suppose we have a left ideal Ae of A for some element $e \in A$, then $D(Ae)$ is a right ideal of A.*

Remark 6.3. There is another natural duality, which we will not used, between $\mathsf{mod}\, A$ and $\mathsf{mod}\, A^{\mathrm{op}}$ given by sending M to $\mathrm{Hom}_A(M, A)$. In general, this duality is different from the $\Bbbk$-linear dual unless A is a so-called *symmetric algebra*; interested reader can read lecture notes from last year.

7 Representations of quiver

Definition 7.1. *A $\Bbbk$-linear representation of Q is a datum $(\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ where M_i is a $\Bbbk$-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is K-linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional if $\dim_{\Bbbk} M_i < \infty$ for all $i \in Q_0$.

Notation. *For a representation M of Q, we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

It is easy to notice that every representation of Q is equivalent to a $\Bbbk Q$-module, namely,

$$\text{representation } (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1}) \leftrightarrow \begin{array}{c} \Bbbk Q\text{-module } \prod_{i\in Q_0} M_i \\ \text{s.t. } \sum_{p:\text{path}} \lambda_p p \text{ acts as } \sum_p \lambda_p M_p. \end{array}$$

Example 7.2 (Simple). *For $x \in Q_0$, denote by S_x (or $S(x)$) the representation given by putting a 1-dimensional space on x, zero on all other vertices, and zero on all arrows. This corresponds to a 1-dimensional $\Bbbk Q$-module and so we call it the simple at x.*

Note: at this stage, it is not clear if these are all the simple $\Bbbk Q$-modules (up to isomorphism) yet.

Example 7.3 (Projective). *For $x \in Q_0$, denote by P_x (or $P(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=x,\\ t(p)=y}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p\alpha=q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the projective at x. This corresponds to the right ideal $e_x\Bbbk Q$ of $\Bbbk Q$.

Example 7.4 (Injective). *Dual to the projective module construction, for $x \in Q_0$, denote by I_x (or $I(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=y,\\ t(p)=x}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p=\alpha q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the injective at x. This corresponds to the dual of the left ideal generated by e_x, i.e. $D(\Bbbk Q e_x)$.

Example 7.5. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to \Bbbk \xrightarrow{\text{id}}\to \cdots \xrightarrow{\text{id}} \Bbbk \to 0 \to \cdots \to 0$$

with a copy of $\Bbbk$ on vertices $i, i+1, \ldots, j$ is the uniserial $\Bbbk Q$-module corresponding to the column space (under the isomorphism of $\Bbbk Q$ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \le k \le j$.

Example 7.6. *Let Q be the Jordan quiver with unique arrow α. Then a representation of Q is nothing but an n-dimensional vector space equipped with a linear endomorphism, equivalently, an n-by-n matrix.*

Definition 7.7. *A homomorphism $f : M \to N$ of ($\Bbbk$-linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram*

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "M_\alpha"'] & N_i \arrow[d, "N_\alpha"] \\ M_j \arrow[r, "f_j"'] & N_j \end{tikzcd}$$

for all arrows $\alpha : i \to j$ in Q.

A homomorphism $f = (f_i)_{i\in Q_0} : M \to N$ of quiver representations is injective, resp. surjective, resp. an isomorphism, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 7.8. *Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n-by-n matrix M_α. By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α. If we assume $\Bbbk$ is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α. That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \ldots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m-by-m Jordan block with eigenvalue $\lambda \in \Bbbk$.*

Proposition 7.9. *There is an isomorphism between the category of representations of Q and* $\mathsf{mod}\,\Bbbk Q$, *where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i\in Q_0} M_i$ with $\Bbbk Q$-action given by (linear combinations of compositions of) M_α's, and isomorphism classes of Q-representations correspond to isomorphism classes of $\Bbbk Q$-modules.*

8 Idempotents

Recall that an *idempotent* of an algebra A is an element x with $x^2 = x$.

The right A-modules of the form eA and $D(Ae)$ for an idempotent $e \in A$ are of central importance in representation theory and in homological algebra.

Lemma 8.1. *The the following hold for any idempotent $e \in A$.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(eA, M) \cong Me$ *as a $\Bbbk$-vector space for all $M \in A\,\mathsf{mod}$.*

(2) *There is an isomorphism of rings $\mathrm{End}_A(eA) \cong eAe$.*

Proof For (1), check that $\mathrm{Hom}_A(eA, M) \ni f \mapsto f(e) = f(1)e \in Me$ defines a $\Bbbk$-linear map with inverse $me \mapsto (ea \mapsto mea)$. (2) follows from (1) by putting $M = eA$ with straightforward check of correspondence of multiplication on both sides. $\square$

Remark 8.2. Under the isomorphism $A \cong \mathrm{End}_A(A)$, an idempotent e of A corresponds to the 'project to direct summand $P = eA$ endomorphism', i.e. $A \twoheadrightarrow P \hookrightarrow A$. This is compatible with Yoneda lemma (think about this!) which says that there is a vector space isomorphism $fAe \cong \mathrm{Hom}_A(eA, fA)$ for any idempotents e, f.

Lemma 8.3. *For idempotents $e, f \in A$, we have $eA \cong fA$ as right A-module if and only if $f = ueu^{-1}$ for some unit $u \in A^\times$.*

Proof $\Leftarrow$: By Yoneda lemma, an isomorphism $\phi \in \mathrm{Hom}_A(fA, eA)$ corresponds to an element in $x \in eAf \subset A$; likewise an isomorphism $\psi \in \mathrm{Hom}_A((1-f)A, (1-e)A)$ corresponds to $y \in (1-e)A(1-f) \subset A$. Let $x' \in fAe$ and $y' \in (1-f)A(1-e)$ be the elements corresponding to ϕ^{-1} and ψ^{-1} respectively. Since $\phi^{-1}\phi = \mathrm{id}_{eA}$ corresponds to $e \in eAe$, we have

$$x'x = f, xx' = e, y'y = 1-f, yy' = 1-e.$$

Take $u := x + y$ and $v := x' + y'$. Then we have $vu = f + (1-f) = 1$ and $uv = e + (1-e) = 1$. Therefore, u, v are units such that $uf = x = eu$, i.e. $e = ufu^{-1}$ as required.

$\Rightarrow$: The required isomorphism $fA \to eA$ is given by $fa \mapsto eua$. $\square$

Given an idempotent $e = e^2 \in A$ in an algebra A, then eA and $(1-e)A$ are both right ideal of A. Since $e(1-e) = 0 = (1-e)e$, we have $eA \cap (1-e)A = 0$, which means that $A \cong eA \oplus (1-e)A$ as right A-module. In particular, in the setting of the above lemma, we have that $eA \cong fA$ and $(1-e)A \cong (1-f)A$ by Krull-Schmidt property.

Definition 8.4. *Two idempotents e, f are orthogonal if $ef = 0 = fe$. An idempotent e is primitive if $e \neq f + f'$ for some orthogonal (pair of) idempotents f, f'.*

It follows from the definition of primitivity that

$$eA \text{ and } D(Ae) \text{ are indecomposable } A\text{-modules for a primitive idempotent } e.$$

Example 8.5. *The trivial paths e_x for $x \in Q_0$ is (by design) a primitive idempotent of the path algebra $\Bbbk Q$ (where Q is finite but not necessarily acyclic), and $1 = \sum_{x \in Q_0} e_x$ is an orthgonal decomposition of primitive idempotents. Hence, we have a decomposition*

$$\Bbbk Q \cong \bigoplus_{x \in Q_0} e_x \Bbbk Q = \bigoplus_{x \in Q_0} P_x \text{ and } D(\Bbbk Q) \cong \bigoplus_{x \in Q_0} D(\Bbbk Q e_x) \cong \bigoplus_{x \in Q_0} I_x.$$

9 Composition series, Jordan-Hölder Theorem

Definition 9.1. *Let A be a $\Bbbk$-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \le i \le \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \le i \le \ell$ are called the composition factors of the series.

Theorem 9.2 (Jordan-Hölder Theorem). *Any two composition series have the same length and their composition factors are the same up to permutations.*

We omit the proof. The strategy is basically by induction on the length of series.

Remark 9.3. Jordan-Hölder theorem holds as long as a module, regardless of what kind of algebra, has a (finite) composition series; this condition is actually equivalent to saying that it is noetherian and artinian.

Remark 9.4. The Jordan-Hölder theorem may not hold if one relaxes the form of composition factors from simple modules to something else. There are a few active research themes, including one related to quasi-hereditary algebras, that are stemmed from this.

Lemma 9.5. *Let M be a finite-dimensional right A-module. Then M has a composition series.*

Proof Induction on $\dim_\Bbbk M$, at each step choose a maximal submodule (i.e. a submodule whose quotient is simple). □

Example 9.6. *Let $A = \Bbbk\vec{\mathbb{A}}_n$. Then the module $U_{i,j}$ has a composition series*

$$0 \subset U_{j,j} \subset U_{j-1,j} \subset \cdots \subset U_{i+1,j} \subset U_{i,j}$$

with composition factors $S_k = U_{k,j}/U_{k+1,j}$ for $i \le k \le j$. We note that this composition series is actually unique - such kind of modules are called uniserial.

Lemma 9.7. *If $M \in \mathsf{mod}\,A$ and $N \subset M$ is a submodule, then there is a composition series $(M_i)_{0\le i\le \ell}$ so that $N = M_k$ for some $0 \le k \le \ell$.*

Proof N has a composition series, say, of length k, so we take that as the first k terms of the required composition series of M. On the other hand, M/N also has a composition series, and since every submodule of M/N is of the form L/N (for a submodule U of M/N, take $L := \{m \in M \mid m+N \in U\}$; it is routine to check that this is an inverse operation as quotienting N on the submodules of M that contains N), a composition series of M/N is of the form $(L_i/N)_{0\le i\le r}$. Now take $M_{k+i} = L_i$. □

Proposition 9.8. *Suppose A is a $\Bbbk$-algebra such that A_A has a composition series. Then there are only finitely many simple A-modules up to isomorphisms, and they all appear in the form A/I for some A-submodule I of A.*

Note that while this does not require A to be finite-dimensional, it requires A_A to be of finite length (equivalently, noetherian and artinian).

Proof The final clause of the claim is just restating Lemma 4.8: any simple S is given by $A/\operatorname{Ann}_A(m)$ for any non-zero $m \in S$. Now fix such an S and $I := \operatorname{Ann}_A(m)$. Since A has a composition series, I also have one by Lemma 9.7 so that the series ends with $I \subset A$. Since this is possible for any simple S, it follows from Jordan-Hölder theorem that all simple modules other than S must appear as composition factors of I.

Since composition series is a finite chain, there must be finitely many composition factors - hence, the simple modules of A must be finite. □

10 Semisimplicity and Artin-Wedderburn theorem

In order to obtain all (isomorphism classes of) simple A-modules - or equivalently maximal right A ideal (i.e. maximal submodules of A_A) - for a finite-dimensional $\Bbbk$-algebra A, we will use the following.

Definition 10.1. *Let A be a $\Bbbk$-algebra and $M \in \mathsf{mod}\, A$.*

(1) *The (Jacobson) radical $\operatorname{rad}(A)$ (sometimes also written as $J(A)$) of A is the intersection of all maximal right ideals (i.e. maximal A-submodules) of A.*

(2) *A is semisimple if $\operatorname{rad}(A) = 0$.*

Example 10.2. *For $A = \Bbbk Q$ of a finite quiver Q and $x \in Q_0$. The projective P_x at x contains a submodule spanned by all paths starting from x with length at least 1. This is a maximal submodule of P_x since the cokernel of the natural embedding to P_x is a one-dimensional module spanned by the coset of e_x – in particular, this simple module is isomorphic to S_x. Thus, we have $\operatorname{rad}(A) = \Bbbk Q_{\geq 1}$ the submodule of A_A spanned by all paths of length at least 1.*

Example 10.3. *This example shows that we really need composition series on A_A for things to be well-behaved. Let $A = \Bbbk[x]$. Each irreducible polynomial f generates a maximal ideal $(f) \subset \Bbbk[x]$ and so $\operatorname{rad}(A) \subset \bigcap_{f:\ irred.}(f)$. Note that there are infinitely many irreducible polynomials in $\Bbbk[x]$.*

We claim that $\operatorname{rad}(A) = 0$. If, on the contrary, there is some non-zero g in this intersection of ideals, then all irreducible polynomials are factors of g; this is a contradiction as g can only has finite degree, i.e. finitely many irreducible factors.

Proposition 10.4. *Suppose A_A has a composition series. Then the following holds for the Jacobson radical $\operatorname{rad}(A)$.*

- *$\operatorname{rad}(A)$ is the intersection of finitely many maximal right ideals.*
- *$\operatorname{rad}(A)$ is the intersection of all two-sided ideals $\operatorname{Ann}_A(S) := \{a \in A \mid ma = 0 \forall m \in S\}$, in other words*
$$\operatorname{rad}(A) = \{a \in A \mid Sa = 0 \text{ for all simple } S\}.$$
- *$\operatorname{rad}(A)$ is a two-sided ideal of A.*
- *$\operatorname{rad}(A)^\ell = 0$ for ℓ at most the length of A_A.*
- *$(A/\operatorname{rad}(A))_{A/\operatorname{rad}(A)}$ is a semisimple (as a module).*
- *A_A is a semisimple (as a module) if, and only if, $\operatorname{rad}(A) = 0$ (i.e. A semisimple as an algebra).*

Proof omitted. We note that all of these claims do make use of the Jordan-Hölder theorem.

Example 10.5. (1) *Direct product of two semisimple algebras is semisimple.*

(2) *$A = \operatorname{Mat}_n(D)$ with D a division $\Bbbk$-algebra is a semisimple $\Bbbk$-algebra. We have decomposition $A_A \cong V^{\oplus n}$ into n copies of n-dimensional simple module*
$$V = \{(v_i)_{1 \leq i \leq n} \mid v_i \in D\ \forall i\}.$$

(3) *$A := \Bbbk[x]/(x^n)$ is not semisimple for any $n \geq 2$ as it has a non-trivial (unique) maximal ideal $\operatorname{rad}(A) = (x)$.*

Theorem 10.6 (Artin-Wedderburn theorem). *Let A be a finite-dimensional $\Bbbk$-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \ldots, S_r$. Let $D_i := \operatorname{End}_A(S_i)$ be the division $\Bbbk$-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of $\Bbbk$-algebras*
$$A/\operatorname{rad}(A) \cong \operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_r}(D_r).$$

As before, if we work over algebraically closed field $\Bbbk = \overline{\Bbbk}$, then all the D_i's are just $\Bbbk$.

Proof Let $B := A/\operatorname{rad}(A)$. By definition of $\operatorname{rad}(A)$, the A-module $A/\operatorname{rad}(A)$ is semisimple, and any A-submodule M of $A/\operatorname{rad}(A)$ satisfies $M\operatorname{rad}(A) = 0$. Hence, $M = M/M\operatorname{rad}(A)$ is naturally a B-module and $\operatorname{End}_B(M) \cong \operatorname{End}_A(M)$ (even as algebras!).

By Lemma 8.1, we have $B \cong \operatorname{End}_B(B)$. Since B is semisimple, the B_B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Schur's lemma and its consequence (Lemma 4.10 and Lemma 4.13) that

$$B \cong \operatorname{End}_B(B) \cong \operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_r}(D_r),$$

where $D_i := \operatorname{End}_B(S_i)$ for all $1 \leq i \leq r$. This completes the proof. $\square$

Corollary 10.7. *For any finite-dimensional $\Bbbk$-algebra A, let $\operatorname{Sim}(A)$ be the set of isomorphism-class representatives of simple A-modules. Then there is a one-to-one correspondence*

$$\begin{array}{ccc} \operatorname{Sim}(A) & \xleftrightarrow{1:1} & \operatorname{Sim}(A/\operatorname{rad}(A)) \\ S & \longmapsto & \overline{S} := S/S\operatorname{rad}(A) \\ & & (= S \text{ as underlying vector space}) \\ \operatorname{res}T & \longleftarrow\!\shortmid & T \end{array}$$

where $\operatorname{res}T$ is the restriction of T along $A \twoheadrightarrow A/\operatorname{rad}(A)$.

Definition 10.8. *The radical of an A-module M is $\operatorname{rad}(M) := M\operatorname{rad}(A)$. In general, take $\operatorname{rad}^0(M) := M$ and denote by $\operatorname{rad}^{k+1}(M) := \operatorname{rad}(\operatorname{rad}^k(M)) = \operatorname{rad}^k(M)\operatorname{rad}(A)$ for all $k \geq 0$.*

Successively taking the radical yields a series:

$$0 \subset \operatorname{rad}^\ell(M) \subset \cdots \subset \operatorname{rad}(M) \subset M$$

This is called the radical series. The quotient $M/\operatorname{rad}(M)$ is called the top of M, and is denoted by $\operatorname{top}(M)$.

Proposition 10.9. *The following hold for $M \in \mathsf{mod}\, A$.*

(1) $\operatorname{rad}(M)$ *is the intersection of all maximal submodules of M.*

(2) $\operatorname{top}(M) := M/\operatorname{rad}(M)$ *is the maximal semisimple quotient of M.*

(3) $\operatorname{rad}(M \oplus N) = \operatorname{rad}(M) \oplus \operatorname{rad}(N)$.

(4) *(Nakayama's Lemma, special case) For a submodule $N \subset M$, $(N + \operatorname{rad}(M) = M) \Rightarrow N = M$.*

Proof omitted; this follows the same kind of arguments as in the case for $\operatorname{rad}(A)$.

There is a construction dual to $\operatorname{rad}(M)$.

Definition 10.10. *The socle of an A-module M is $\operatorname{soc}(M)$, which is defined as the maximal semisimple submodule of M. More generally, take $\operatorname{soc}^0(M) = 0$ and for $k \geq 0$, let $\operatorname{soc}^{k+1}(M)$ to be the submodule of M generated by the lift of $\operatorname{soc}(M/\operatorname{soc}^k(M)) \subset M/\operatorname{soc}^k(M)$. This yields a series*

$$0 \subset \operatorname{soc}(M) \subset \operatorname{soc}^2(M) \subset \cdots \subset \operatorname{soc}^\ell(M) = M$$

called the socle series of M.

Example 10.11. *Consider a path algebra $\Bbbk Q$ of a finite acyclic (for simplicity) quiver Q, and $x \in Q_0$. The indecomposable injective $I_x = D(\Bbbk Q e_x)$ has a simple socle isomorphic to S_x. Essentially this can be seen by a dual argument in showing $\operatorname{top}(P_x) \cong S_x$.*

Lemma 10.12. *For $M \in \mathsf{mod}\,A$, the socle series and radical series has the same length, and this length is called the* Loewy length *of M.*

Note that the semisimple subquotients in (between the layers of) the socle series and the radical series of a module may not coincide.

Example 10.13. *Let Q be the quiver $1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and consider the projective P_2 which has the form*

$$\Bbbk \xleftarrow{1} \Bbbk \xrightarrow{1} \Bbbk \xrightarrow{1} \Bbbk$$

Then we have radical series

$$0 \subset S_4 = \Bbbk\beta\gamma \overset{S_1 \oplus S_3}{\subset} \operatorname{rad}(P_2) = \Bbbk\alpha + \Bbbk\beta + \Bbbk\beta\gamma \overset{S_2}{\subset} P_2$$

and socle series

$$0 \subset S_2 \oplus S_4 = \Bbbk\alpha + \Bbbk\beta\gamma \overset{S_3}{\subset} \operatorname{rad}(P_2) \subset P_2.$$

11 Bounded path algebra

For general quiver, we loses finite-dimensionality, and so many nice things we explained do not hold any more. To retain finite-dimensionality, we need to consider nice quotients of path algebras.

Definition 11.1. *An ideal $I \lhd \Bbbk Q$ is admissible if $(\Bbbk Q_1)^k \subset I \subset (\Bbbk Q_1)^2$ for some $k \geq 2$, i.e. I is generated by linear combinations of paths of finite length at least 2. The pair (Q, I) is sometimes called bounded quiver. A bounded path algebra or quiver algebra (with relations) is an algebra of the form $\Bbbk Q/I$ for some quiver Q and admissible ideal I.*

Remark 11.2. Admissiblity ensures there is no redundant arrows (which appears if there is a relation like, for example, $\alpha - \beta\gamma \in I$ for some $\alpha \neq \beta, \gamma \in Q_1$) and there is enough vertices (trivial paths may not be primitive if there is a loop x at a vertex with relation $x^2 - x \in I$).

Lemma 11.3. *A bounded path algebra is finite-dimensional.*

Proof There exists a surjective algebra homomorphism $\Bbbk Q/(\Bbbk Q_1)^k \twoheadrightarrow \Bbbk Q/I$; the former is finite-dimensional. □

Example 11.4. *Let Q be the Jordan quiver with unique arrow α. Let I be the ideal of $\Bbbk Q$ generated by α^k for some $k \geq 2$. Then I is an admissible ideal and $\Bbbk Q/I \cong \Bbbk[x]/(x^k)$ is a truncated polynomial ring.*

Definition 11.5. *A representation M of a bounded quiver (Q, I) is a representation $M = (M_i, M_\alpha)_{i,\alpha}$ of Q such that $M_a = 0$ for all $a \in I$; here $M_a := \sum_p \lambda_p M_p$ for $a = \sum_p \lambda_p p$ written as a linear combinations of paths p.*

A homomorphism $f : M \to N$ of representations of (Q, I) is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' action.

As before, representations are really just synonyms of modules.

Lemma 11.6. *A representation of a bounded quiver (Q, I) is equivalent to a $\Bbbk Q/I$-module, and homomorphisms between representations are equivalent to those between $\Bbbk Q/I$-modules.*

We have seen that it is easy to write down the indecomposable decomposition of the free $\Bbbk Q$-module $\Bbbk Q_{\Bbbk Q}$, we would like such nice thing to carry over to bounded path algebras.

Theorem 11.7. *(Idempotent lifting) If I is a nilpotent ideal of A (i.e. $I^n = 0$ for some $n \geq 1$) and $\overline{e} = \overline{e}^2 \in A/I$, then there is a lift $e = e^2 \in A$ of $\overline{e}$, i.e. $\overline{e} = e + I$.*

Proof omitted.

Corollary 11.8. *Let I be an nilpotent ideal in A. Suppose that*

$$1_{A/I} = f_1 + \cdots + f_n$$

for $f_i \in A/I$ are primitive orthogonal idempotents. Then we have

$$1_A = e_1 + \cdots e_n$$

where each $e_i \in A$ is a primitive orthogonal idempotent that lifts f_i.

Notation. *As in the case of path algebra, denote by S_x or $S(x)$ the simple $\Bbbk Q/I$-module given by placing a one-dimensional vector space at vertex $x \in Q_0$ and zero everywhere else.*

Similarly, denote by P_x or $P(x)$ the indecomposable $\Bbbk Q/I$-module $e_x \Bbbk Q/I$. Likewise, by I_x or $I(x)$ the indecomposable $D((\Bbbk Q/I)e_x)$.

Proposition 11.9. *There is a decomposition of A-modules*

$$A_A = \bigoplus_{x \in Q_0} P_x, \text{ and } (DA)_A = \bigoplus_{x \in Q_0} I_x.$$

Moreover, $\{S_x \cong \operatorname{top}(P_x) \cong \operatorname{soc}(I_x) \mid x \in Q_0\}$ form the complete set of isoclasses representatives of simple A-modules.

Proof Each arrow $\alpha \in Q_1$ generates a maximal right ideal of A with quotient S_x for $x = s(\alpha)$. So we have $A/\operatorname{rad}(A) \cong \Bbbk Q_0 = \prod_{x \in Q_0} \Bbbk$. As primitive orthogonal decomposition of the identity element of A lifts to that of the identity element of $A/\operatorname{rad}(A)$ by Corollary 11.8, we have e_x primitive, and so P_x and I_x are indecomposable.

The simple A-modules (up to isomorphisms) correspond to those over the semisimple quotient algebra $A/\operatorname{rad}(A)$ by Corollary 10.7. Hence, there are precisely $|Q_0|$ simple modules (up to isomorphism), given by the simple top of P_x, which is also isomorphic to the simple socle of I_x. □

We give a brief justification of why quiver representations provide a good way to construct lots of algebras.

Theorem 11.10. *Suppose $\Bbbk$ is algebraically closed. Then every finite-dimensional $\Bbbk$-algebra A is Morita equivalent to a bounded path algebra $\Bbbk Q/I$. More precisely, $\Bbbk Q/I$ is given by $\operatorname{End}_A(\bigoplus_e eA)$ where e varies over the set of representative of equivalence classes of primitive idempotents of A.*

We do not explain here the precise meaning of Morita equivalent; it roughly translates to saying that understanding A-modules and homomorphisms between them is equivalently (but not necessarily 'equal to') to understanding modules and homomorphisms between a Morita equivalent bounded path algebra.

Example 11.11. *Let $A = \operatorname{Mat}_n(\Bbbk)$ be a matrix ring. Then the elementary matrix $e := E_{1,1}$ is a primitive idempotent and $eA \cong E_{j,j}A$ for all $1 \leq j \leq n$. So A is Morita equivalent to $\Bbbk \cong \Bbbk Q \cong \operatorname{End}_A(eA)$ where Q is a one-vertex-no-arrow quiver.*

Primitive idempotent decomposition, say, $1 = \sum_{i=1}^n e_i$, allows us to write an algebra A in matrix form $(e_i A e_j)_{1 \leq i,j \leq n}$, where the 'row spaces' form the indecomposable direct summands $e_i A$ and the dual of the 'column space' form the indecomposable direct summands $D(Ae_i)$. It could be a helpful mental exercise to think about the meaning of $eAe \cong \operatorname{End}_A(eA)$ from Yoneda lemma - this maybe a useful idea to keep in mind when one tries to understand the above theorem.

Module diagram

It is convenient to display the structure of a module via a more combinatorial form (a diagram) – if possible. [1] This is (as of today technology) a better way to display module structure – at least compare to composition series, or lattice diagram of the submodule lattice, or even, quiver representations, in some cases.

Definition 11.12. *Let $M \in \mathsf{mod}\, A$ be a finite-dimensional A-module for $A = \Bbbk Q/I$ a bounded path algebra. The module diagram is a (directed) graph with vertices labelled by composition factors of M (in particular, there are $\dim_{\Bbbk} Me_x$ many vertices labelled by x), and arrows labelled by those in Q_1 in such a way that $x \xrightarrow{a} y$ if for an arrow $a \in Q$ that sends (the lift of) an element in the composition factor at x to (the lift of) an element in the composition factor at y.*

Module diagram drawn in this way is not invariant under isomorphism. A connected diagram may not even implies indecomposability in general (c.f. Homework 2). Nevertheless, when the algebras or modules are well-behaved, then these diagram provide a very efficient combinatorial way to perform a lot of (linear algebra) calculation.

It is customary to draw the the module diagram flowing from top to bottom; in particular, the top (semisimple quotient) of M is placed on the top of the diagram and the socle at the bottom. We may omit a connecting line if there is no ambiguity.

Example 11.13. *The indecomposable $U_{i,j}$ of $\Bbbk\vec{\mathbb{A}}_n$ is just a column of numbers labelled from i down to j. For a concrete example, the module diagram of $U_{4,6}$ is just $\substack{4\\5\\6}$.*

Example 11.14. *Consider the following bounded quiver:*

$$Q : \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3, \arrow[l, bend left, "\beta_2"] \end{tikzcd} \qquad I = (\alpha_1\alpha_2, \beta_1\beta_2, \beta_1\alpha_1 - \alpha_2\beta_2).$$

Then we have

$$P_1 = \begin{smallmatrix} \Bbbk e_1 \\ \Bbbk\alpha_1 \\ \Bbbk\alpha_1\beta_1 \end{smallmatrix} = \begin{smallmatrix} 1\\2\\1 \end{smallmatrix} \quad , \quad P_2 = \begin{smallmatrix} \Bbbk e_1 \\ \Bbbk\alpha_2 \oplus \Bbbk\beta_1 \\ \Bbbk\alpha_2\beta_2 \end{smallmatrix} = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \quad , \quad P_3 = \begin{smallmatrix} \Bbbk e_3 \\ \Bbbk\beta_2 \\ \Bbbk\beta_2\alpha_2 \end{smallmatrix} = \begin{smallmatrix} 3\\2\\3 \end{smallmatrix}$$

Let us consider the two-sided ideal Ae_1A. This is spanned by all paths ('up to I') that passes through the vertex 1. As a right module, we can find its manifestation in the module diagram by picking everything below any appearance of the label 1 – in this case, it is all of P_1 and the $\substack{1\\2}$ part submodule of P_2. In particular, the quotient algebra $(A/Ae_1A)_{A/Ae_1A}$ has module diagram:

$$P_2^{A/Ae_1A} = e_2A/e_2Ae_1A = P_2/P_2e_1A = \substack{2\\3} \quad , \quad P_3^{A/Ae_1A} = P_3/P_3e_1A = P_3^A = \substack{3\\2\\3}$$

The bounded quiver presentation of A/Ae_1A is given by

$$Q : \begin{tikzcd} 2 \arrow[r, bend left, "\alpha"] & 3, \arrow[l, bend left, "\beta"] \end{tikzcd} \qquad I = (\alpha\beta).$$

On the other hand, for eAe with $e = e_2 + e_3$, the module diagram is given by removing all composition factors that are not S_2, S_3, i.e.

$$e_2Ae = \substack{2\\3\\2} \quad , \quad e_3Ae = \substack{3\\2\\3}$$

and the bounded quiver presentation of eAe is given by

$$Q : \begin{tikzcd} 2 \arrow[r, bend left, "\alpha"] & 3, \arrow[l, bend left, "\beta"] \end{tikzcd} \qquad I = (\alpha\beta\alpha, \beta\alpha\beta).$$

[1] There is no widely agreed name to these diagrams; for convenience, we just call them 'module diagram' in this notes.

12 Snippets of category theory

Some language in category will be convenient – albeit not absolutely necessary.

A *category* is a collection of *objects* along with their *morphisms* $f : X \to Y$, including all *identity morphisms* $\mathrm{id}_X : X \to X$, so that

- morphisms $f : X \to Y$, $g : Y \to Z$ can always be composed $gf : X \to Z$ to get a new morphism,
- in such a way that is associative, i.e. $h(gf) = (hg)f$ for all h, g, f,
- and has left and right unit, i.e. $f\,\mathrm{id}_X = f$ and $\mathrm{id}_Y f = f$.

Example 12.1. *We only really use the category* $\mathsf{mod}\,A$ *of (finitely generated) A-modules in this notes. Some results still hold for the category* $\mathsf{Mod}\,A$ *of all A-modules, but let us keep it simple.*

A *functor* $F : \mathsf{mod}\,A \to \mathsf{mod}\,B$ consists of

- an assignment of objects $M \mapsto F(M) \in \mathsf{mod}\,B$ for any $M \in \mathsf{mod}\,A$, and
- an assignment of morphisms $F(f) \in \mathrm{Hom}_B(F(X), F(Y))$ for all $f \in \mathrm{Hom}_A(X, Y)$, such that
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
- either $F(gf) = F(g)F(f)$ or $F(gf) = F(f)F(g)$.

The case when order of morphism composition does not change is called a *covariant* functor, and the other is called a *contravariant* functor. Usually, whenever we say a functor we mean a covariant one.

Functor allows us to change from the representation theory of one algebra to another. The key point is that it preserves identity and compositions.

Example 12.2. *The identity functor* $\mathrm{Id} : \mathsf{mod}\,A \to \mathsf{mod}\,A$ *is the functor given by mapping every module and homomorphism to itself.*

Example 12.3. *The ($\Bbbk$-linear) duality* $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk) : \mathsf{mod}\,A \to \mathsf{mod}\,A^{\mathrm{op}}$ *is a contravariant functor.*

To compare two functors (or compare how a pair of functors is close/far away from the identity functor), one uses *natural transformations*. More precisely, a natural transformation $\eta : F \Rightarrow G$ of functors $F, G : \mathsf{mod}\,A \to \mathsf{mod}\,B$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ such that there is the following commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

If we say that a map $\eta_X : F(X) \to G(X)$ is *natural in X*, then we mean that $\{\eta_X\}_{X \in \mathsf{mod}\,A}$ defines a natural transformation.

A *natural isomorphism* is a natural transformation η such that η_X is an isomorphism for all X; in such a case, we may simply write $F \cong G$ when η is clear from context.

13 Bimodule, tensor and Hom

13.1 Bimodule

Definition 13.1. *Let A, B be two $\Bbbk$-algebras. An A-B-bimodule is a $\Bbbk$-vector space M that has the structure of a left A-module and also the structure of a right B-module, such that $(am)b = a(mb)$ for all $a \in A, b \in B, m \in M$. In such a case, we may write $M \in A \operatorname{\mathsf{mod}} B$ or ${}_AM_B$ to specify M is an A-B-bimodule.*

For simplicity, we assume all bimodules are $\Bbbk$-central, i.e. $\lambda m = m\lambda$ for all $\lambda \in \Bbbk$. We will omit the adjective $\Bbbk$-central from now on.

Example 13.2. *For any algebra A, both A and $D(A)$ are naturally an A-A-bimodule. Note that the right/left module structure on $D(A)$ is induced by the left/right module structure on A. (The direction of action has swapped!)*

Example 13.3. $\operatorname{Hom}_A(X, Y)$ *is naturally a* $\operatorname{End}_A(Y)$-$\operatorname{End}_A(X)$-*bimodule with action given by composition of homomorphisms.*

13.2 Tensor product

Definition 13.4. *Let V, W be finite-dimensional $\Bbbk$-vector space with bases, say, $\mathcal{B}, \mathcal{C}$ respectively. Then the tensor product $V \otimes_\Bbbk W$ (or simplifies to $V \otimes W$ if context is clear) is the finite-dimensional $\Bbbk$-vector space with bases given by*

$$\{v \otimes w \mid v \in \mathcal{B}, w \in \mathcal{C}\}.$$

In particular, note that $\dim_\Bbbk V \otimes W = (\dim_\Bbbk V) \times (\dim_\Bbbk W)$.

Proposition 13.5. *Let A, B be $\Bbbk$-algebras. Then $A \otimes_\Bbbk B$ is also a $\Bbbk$-algebra with multiplication given by extending $(a \otimes b)(a' \otimes b') \mapsto aa' \otimes bb'$ linearly. For $M \in \operatorname{\mathsf{mod}} A$ and $N \in \operatorname{\mathsf{mod}} B$, we have $M \otimes_\Bbbk N \in \operatorname{\mathsf{mod}} A \otimes_\Bbbk B$.*

Proof Routine checking. □

Example 13.6. *Consider $A = (a_{i,j})_{1 \le i,j \le m} \in \operatorname{Mat}_m(\Bbbk)$ and $B \in \operatorname{Mat}_n(\Bbbk)$ and defines (what is sometimes called Kronecker product of matrices)*

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & \ddots & & a_{2,m}B \\ \vdots & & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,m}B \end{pmatrix}.$$

Then we have an isomorphism of algebras

$$\operatorname{Mat}_m(\Bbbk) \otimes_\Bbbk \operatorname{Mat}_n(\Bbbk) \to \operatorname{Mat}_{mn}(\Bbbk), \quad (A, B) \mapsto A \otimes B.$$

Lemma 13.7. *An idempotent $e \in A \otimes_\Bbbk B$ is primitive if and only if $e = e_l \otimes e_r$ for some primitive idempotents $e_l \in A$ and $e_r \in B$. In particular, we have $\operatorname{Sim}(A \otimes_\Bbbk B) = \{S \otimes T \mid S \in \operatorname{Sim}(A), T \in \operatorname{Sim}(B)\}$.*

Note that not all $A \otimes_\Bbbk B$-module is of the form $M \otimes N$.

Example 13.8. *Let $A = \Bbbk[x]/(x^2)$ and $A' := \Bbbk[y]/(y^2)$. Then $B := A \otimes_{\Bbbk} A' = \Bbbk[x,y]/(x^2,y^2)$. Then we have an indecomposable 2-dimensional B-module $V = \Bbbk u + \Bbbk v$ (top $S = B/\operatorname{rad}(B)$ and socle S) where both x, y acts by $u \mapsto v$. This cannot be of the form $M \otimes N$ for some $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} A'$. Indeed, as both x, y acts non-trivially, if $V = M \otimes N$ then both M, N must have dimension at least 2, and so the tensor product has dimension at least 4; but* $\dim_{\Bbbk} V = 2$.

Proposition 13.9. *An $A \otimes_{\Bbbk} B^{\mathrm{op}}$-module is the same as a ($\Bbbk$-central) B-A-bimodule. Moreover, homomorphisms of $A \otimes B^{\mathrm{op}}$-modules correspond to ($\Bbbk$-linear) homomorphisms of B-A-bimodule.*

Definition 13.10. *Let $X \in \operatorname{mod} A$ be a right A-module and $Y \in \operatorname{mod} A^{\mathrm{op}}$ be a left A-module. Then define $X \otimes_A Y$ to be the vector space $X \otimes_{\Bbbk} Y/U$ where U is the subspace consisting of $xa \otimes y - x \otimes ay$ for all $x \in X, y \in Y, a \in A$.*

In the above, if ${}_AY_B$ is, in addition, an A-B-bimodule, then $X \otimes_A Y$ has a natural right B-module structure: $(x \otimes y)b := x \otimes (yb)$. In fact, as any left A-module is also a A-$\Bbbk$-bimodule, we can $X \otimes_A Y$ being a $\Bbbk$-vector space as a special case of this observation.

Suppose we have a homomorphism $f : M \to N$ of right A-modules. Then for an A-B-bimodule ${}_AY_B$ we get a homomorphism of

$$\begin{aligned} M \otimes_A Y_B &\xrightarrow{f \otimes_A Y} N \otimes_A Y_B \\ m \otimes y &\longmapsto f(m) \otimes y \end{aligned}$$

Note that $(gf) \otimes_A Y = (g \otimes_A Y)(f \otimes_A Y)$, that is, $- \otimes_A Y$ is a (covariant) *functor*. It is also *$\Bbbk$-linear additive* in the sense that $(\lambda f + \mu g) \otimes_A Y = \lambda(f \otimes_A Y) + \mu(g \otimes_A Y)$ for all homomorphisms f, g and scalar $\lambda, \mu \in \Bbbk$.

Likewise, if X is a bimodule, then ${}_BX \otimes_A Y$ has a left module structure; mutatis mutantis.

Example 13.11. *Consider the bounded quiver*

$$Q : \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}, \qquad I = (\beta\alpha).$$

We look at the ($A \otimes A^{\mathrm{op}}$-module) structure of $Ae \otimes eA$, AeA, $Ae \otimes_{eAe} eA$ for $e = e_1$ in the following.

$$Ae \otimes eA = \begin{smallmatrix} & & 11 & & \\ & 21 & & 12 & \\ 11 & & 22 & & 11 \\ & 12 & & 21 & \\ & & 11 & & \end{smallmatrix} = \begin{tikzcd}[column sep=small] & & e_1 \otimes e_1 \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & & \\ & \beta \otimes e_1 \arrow[dl, dash, "\alpha^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & & e_1 \otimes \alpha \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\beta"] & \\ \alpha\beta \otimes e_1 \arrow[dr, dash, "\alpha"] & & \beta \otimes \alpha \arrow[dl, dash, "\alpha^{\mathrm{op}}"'] \arrow[dr, dash, "\beta"] & & e_1 \otimes \alpha\beta \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \\ & \alpha\beta \otimes \alpha \arrow[dr, dash, "\beta"] & & \beta \otimes \alpha\beta \arrow[dl, dash, "\alpha^{\mathrm{op}}"'] & \\ & & \alpha\beta \otimes \alpha\beta & & \end{tikzcd}$$

As a right A-module, this is $\dim_{\Bbbk} Ae = 3$ copies of $eA = P_1 = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$.

$$AeA = \begin{smallmatrix} & 11 & \\ 21 & & 12 \\ & 11 & \end{smallmatrix} = \begin{tikzcd}[column sep=small] & e_1 \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & \\ \beta \arrow[dr, dash, "\alpha^{\mathrm{op}}"'] & & \alpha \arrow[dl, dash, "\beta"] \\ & \alpha\beta & \end{tikzcd}$$

As right A-module, we have $AeA = eA \oplus \Bbbk\beta \cong P_1 \oplus S_2$.

For $Ae \otimes_{eAe} eA$, first note that $eAe = \Bbbk\{e = e_1, \alpha\beta\}$, and so $\alpha\beta \otimes e_1 = e_1 \otimes \alpha\beta$. In particular, so basis elements in $Ae \otimes eA$ vanishes, for example, $\alpha\beta \otimes \alpha = e_1 \otimes \alpha\beta\alpha = 0$ as $\beta\alpha = 0$ in A.

$$
Ae \otimes_{eAe} eA = \begin{tikzcd}[column sep=small]
& 11 \arrow[dl, dash] \arrow[dr, dash] & \\
21 \arrow[d, dash] \arrow[drr, dash] & & 12 \arrow[d, dash] \arrow[dll, dash] \\
22 & & 11
\end{tikzcd}
=
\begin{tikzcd}
& e \otimes e \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & \\
\beta \otimes e \arrow[d, dash, "\alpha"] \arrow[drr, dash, "\alpha^{\mathrm{op}}"] & & e \otimes \alpha \arrow[d, dash, "\beta"'] \arrow[dll, dash, "\beta^{\mathrm{op}}"'] \\
\beta \otimes \alpha & & e \otimes \alpha\beta
\end{tikzcd}
$$

The right A-module structure of $Ae \otimes_{eAe} eA$ is isomorphic to $P_1 \oplus P_1/\operatorname{soc}(P_1)$.

13.3 Hom

Suppose now that we have ${}_BX_A$ a B-A-bimodule and M a right A-module. Then the space $\operatorname{Hom}_A(X, Y)$ has a natural *right* B-module structure:

$$(f : X \to Y) \cdot b := (x \mapsto f(bx))$$

Indeed, we have

$$((f \cdot b) \cdot b')(x) = (f \cdot b)(b'x) = f(bb'x) = (f \cdot (bb'))(x),$$

and other axioms are even easier to verify. $\operatorname{Hom}_A({}_BX_A, -)$ is also a $\Bbbk$-linear additive covariant functor: for $f : M \to N$ a homomorphism of A-modules, we have

$$
\begin{tikzcd}
\operatorname{Hom}_A(X, M) \arrow[r, "f\circ-"] & \operatorname{Hom}_A(X, N) \\[-20pt]
\alpha \arrow[r, mapsto] & f \circ \alpha
\end{tikzcd}
$$

Similarly, in the same setting, $\operatorname{Hom}_A(Y, {}_BX_A)$ also has a *left* B-module structure:

$$(b' \cdot (b \cdot f))(x) = b'((b \cdot f)(x)) = b'(bf(x)) = (b'b)f(x) = ((b'b) \cdot f)(x).$$

However, note that $\operatorname{Hom}_A(f, X) = - \circ f : \operatorname{Hom}_A(N, X) \to \operatorname{Hom}_A(M, X)$ for any $f : M \to N$, i.e. $\operatorname{Hom}_A(-, X)$ is a ($\Bbbk$-linear additive) contravariantly functor, meaning that it reverse the direction of homomorphisms.

Lemma 13.12. *Hom functor commutes with finite direct sum in both variables, i.e. there is a commutative diagram:*

$$
\begin{tikzcd}
\operatorname{Hom}_A(\bigoplus_{j=1}^{\ell} L_j, N) \arrow[r, "-\circ\theta"] \arrow[d, "\cong"] & \operatorname{Hom}_A(\bigoplus_{i=1}^{m} M_i, N) \arrow[d, "\cong"] \\
\bigoplus_{j=1}^{\ell} \operatorname{Hom}_A(L_j, N) \arrow[r, "(-\circ\theta\iota_j)_{i,j}"] & \bigoplus_{i=1}^{m} \operatorname{Hom}_A(M_i, N)
\end{tikzcd}
$$

where $\iota_j : L_j \hookrightarrow \bigoplus_k L_k$ and $\pi_i : \bigoplus_k M_k \to M_i$ are the canonical maps, and there is also a similar commutative diagram arising from $\operatorname{Hom}_A(M, \bigoplus_{i=1}^{\ell} L_i) \cong \bigoplus_{i=1}^{\ell} \operatorname{Hom}_A(M, L_i)$.

Remark 13.13. In proper functorial language, this is saying that there are natural isomorphisms

$$\operatorname{Hom}_A(\bigoplus_{j=1}^{\ell} L_j, -) \cong \bigoplus_{j=1}^{\ell} \operatorname{Hom}_A(L_j, -) \text{ and } \operatorname{Hom}_A(-, \bigoplus_{i=1}^{\ell} L_i) \cong \bigoplus_{i=1}^{\ell}(-, L_i).$$

Lemma 13.14. *For any A-module M, we have (natural) A-module isomorphisms*

$$M \otimes_A A \cong M, \quad \textit{and } \operatorname{Hom}_A(A, M) \cong M$$

given by $m \otimes 1 \mapsto m$ and $f \mapsto f(1)$. Moreover, $- \otimes_A A$ and $\operatorname{Hom}_A(A, -)$ are both naturally isomorphic to the identity functor.

Proof First one follows from the construction that $ma \otimes 1 = m \otimes a$. The second one is just special case of Yoneda lemma. □

13.4 Tensor-Hom adjunction

Suppose ${}_AM_B$ is a A-B-bimodule, then we have two functors:

$$\begin{tikzcd} \mathsf{mod}\, A \arrow[r, bend left, "- \otimes_A M"] & \mathsf{mod}\, B \arrow[l, bend left, "\operatorname{Hom}_B(M_B, -)"] \end{tikzcd}.$$

These are not inverse to each other; but they form a so-called adjoint pair, which is equivalent to saying that there is the following natural isomorphisms.

Theorem 13.15 (Tensor-Hom adjunction). *Let $X \in \mathsf{mod}\, A$, $Y \in \mathsf{mod}\, B$, ${}_AM_B \in A\,\mathsf{mod}\, B$. Then there is a canonical isomorphism of $\Bbbk$-vector spaces*

$$\begin{aligned} \theta_{X,M,Y} : \quad \operatorname{Hom}_B(X \otimes_A M, Y) &\xrightarrow{\cong} \operatorname{Hom}_A(X, \operatorname{Hom}_B(M, Y)) \\ f &\longmapsto (x \mapsto (m \mapsto f(x \otimes m))) \\ (x \otimes m \mapsto (g(x))(m)) &\longleftarrow\!\shortmid\ g \end{aligned}$$

that is natural in each of X, M, Y.

Proof Check that the maps written are ($\Bbbk$-linear and) mutual inverse of each other. □

In computer science, the map $\theta_{X,M,Y}$ is also called "currying".

As innocence as it looks, this isomorphism is fundamental in (homological algebra and) representation theory.

Example 13.16 (Adjoint triple (RHS)). *eA is naturally an eAe-A-bimodule. Hence, we have an adjoint pair $(- \otimes_{eAe} eA, \operatorname{Hom}_A(eA, -))$.*

On the other hand, Ae is naturally an A-eAe-bimodule, and so we have another adjoint pair $(- \otimes_A Ae, \operatorname{Hom}_{eAe}(Ae, -))$. Note that we have $\operatorname{Hom}_A(eA, -) \cong - \otimes_A Ae$ by Yoneda lemma.

Example 13.17 (Adjoint triple (LHS)). *A/I is naturally an A-A/I-bimodule for any two-sided ideal I of A, and so we have an adjoint pair $(- \otimes_A A/I, \operatorname{Hom}_{A/I}(A/I, -))$.*

A/I is also an A/I-A-bimodule, and so there is another adjoint pair $(- \otimes_{A/I} A/I, \operatorname{Hom}_A(A/I, -))$. Note that both $\operatorname{Hom}_{A/I}(A/I, -)$ and $\otimes_{A/I} A/I$ sends an A/I-module to itself (up to isomorphism) and acts identically on morphisms, i.e. $\operatorname{Hom}_{A/I}(A/I, -) \cong \operatorname{Id} \cong - \otimes_{A/I} A/I$.

14 Exactness

Definition 14.1. *Consider a sequence*[2] $M_\bullet = (M_i, d_i)_{i\in\mathbb{Z}}$ *of modules and homomorphisms of modules*

$$M_\bullet : \cdots \xrightarrow{d_{i-2}} M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} \cdots .$$

We say that the sequence $M_\bullet$ is

- *a complex if $d_{i+1}d_i = 0$ for all $i \in \mathbb{Z}$. In such a case, we have $\mathrm{Im}(d_i) \subset \mathrm{Ker}(d_{i+1})$ for all $i \in \mathbb{Z}$ and the i-th cohomology of $M_\bullet$ is*

$$H^i(M_\bullet) := \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i-1}).$$

- *exact at M_k for some $k \in \mathbb{Z}$ if $\mathrm{Im}(d_{k-1}) = \mathrm{Ker}(d_k)$. Note that this implies $d_k \circ d_{k-1} = 0$.*
- *exact if it is so at every term.*
- *short exact (often abbreviated as s.e.s. or ses) if it is a 5-term exact sequence that starts and ends at the trivial module, i.e., of the form*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{14.1}$$

such that f is injective, g is surjective, and $\mathrm{Ker}(g) = \mathrm{Im}(f)$. In this case, M is also called an extension of N by L.

Definition 14.2. *A (covariant) functor $F : \mathsf{mod}\,A \to \mathsf{mod}\,B$ is*

- *left exact if it maps short exact sequence (such as (14.1)) to an exact sequence*

$$0 \to F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N).$$

In other words, it preserves kernel.

- *right exact if it maps short exact sequence (such as (14.1)) to an exact sequence*

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \to 0.$$

In other words, it preserves cokernel.

- *exact if it is both left exact and right exact, i.e. maps ses to ses.*

We define left/right exactness for contravariant functor analogously. In particular, left exact contravariant fucntor turns cokernel into kernel.

Lemma 14.3. *Let ${}_BX_A$ be an A-B-bimodule. Then the following hold.*

(1) $\mathrm{Hom}_A(X, -)$ *maps an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ to an exact sequence*

$$0 \to \mathrm{Hom}_A(X, L) \xrightarrow{f\circ -} \mathrm{Hom}_A(X, M) \xrightarrow{g\circ -} \mathrm{Hom}_A(X, N).$$

In particular, $\mathrm{Hom}_A(X, -)$ *is left exact.*

(2) $\mathrm{Hom}_A(-, X)$ *maps an exact sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ to an exact sequence*

$$0 \to \mathrm{Hom}_A(N, X) \xrightarrow{-\circ g} \mathrm{Hom}_A(M, X) \xrightarrow{-\circ f} \mathrm{Hom}_A(L, X).$$

In particular, the contravariant functor $\mathrm{Hom}_A(-, X)$ *is left exact.*

[2] Superscript/subscript indexing formalism only matters to topologist; I will be liberal in these notations.

Proof We show (1) and leave (2) for the reader.

Exactness at $\mathrm{Hom}_A(X, L)$: we need $f \circ -$ to be injective. Indeed, if $f \circ \theta = 0$ for some $\theta : X \to L$, then $f(\theta(x))$ for all $x \in X$. This means that $\theta(x) \in \mathrm{Ker}(f) = 0$, and so $\theta = 0$.

$\mathrm{Im}(f \circ -) \subset \mathrm{Ker}(g \circ -)$: Suppose that $\theta : X \to M$ is given by $f \circ \phi$ for some $\phi : X \to L$. Then $g\phi(x) = g(f\phi(x)) = (gf)\phi(x) = 0$, which means that $\theta \in \mathrm{Ker}(g \circ -)$.

$\mathrm{Ker}(g \circ -) \subset \mathrm{Im}(f \circ -)$: Suppose that $g\theta = 0$ for some $\theta : X \to M$. Then for every $x \in X$, we have $\theta(x) \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, and so we can write $\theta(x) = f(\phi(x))$ for some $\phi(x) \in L$. Since f is injective, $\phi(x) \in L$ is uniquely determined, and so we have a well-defined function $\phi : X \to L$. We check that $\phi \in \mathrm{Hom}_A(X, L)$:

- $f(\phi(x + x')) = \theta(x + x') = \theta(x) + \theta(x') = f(\phi(x)) + f(\phi(x')) = f(\phi(x) + \phi'(x))$. Hence, f being injective implies that $\phi(x + x') = \phi(x) + \phi(x')$.
- Suppose that $\lambda \in \Bbbk$. Then $f(\phi(\lambda x)) = \theta(\lambda x) = \lambda\theta(x) = \lambda f(\phi(x)) = f(\lambda\phi(x))$. Hence, f being injective implies that $\lambda\phi(x) = \phi(\lambda x)$.

Now we have $\theta = f\phi$ as A-module homomorphism, and so $\theta \in \mathrm{Im}(f \circ -)$. $\square$

A similar lemma for tensor product exists, and can be proved by direct verification as in the Hom functor case. Instead, we use another trick involving tensor-Hom adjunction, but first we need one more tool.

Lemma 14.4 (Yoneda embedding reflects exactness). *Consider a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathsf{mod}\, A$. If the sequence*

$$\mathrm{Hom}_A(X, L) \xrightarrow{f\circ -} \mathrm{Hom}_A(X, M) \xrightarrow{g\circ -} \mathrm{Hom}(X, N)$$

is exact for all $X \in \mathsf{mod}\, A$, then $L \xrightarrow{f} M \xrightarrow{g} N$ is also exact. Similarly, if

$$\mathrm{Hom}_A(N, X) \xrightarrow{-\circ g} \mathrm{Hom}_A(M, X) \xrightarrow{-\circ f} \mathrm{Hom}(N, X)$$

is exact for all $X \in \mathsf{mod}\, A$, then so is the original sequence.

Proof We show the first one.

$\mathrm{Im}(f) \subset \mathrm{Ker}(g)$: Take $X = L$, then we have $gf = (g \circ -)(f \circ -)(\mathrm{id}_L) = 0$.

$\mathrm{Ker}(g) \subset \mathrm{Im}(f)$: Consider $X = \mathrm{Ker}(g)$ and inclusion $\iota : \mathrm{Ker}(g) \hookrightarrow M$. Then $(g \circ -)(\iota) = g\iota = 0$, so exactness implies that $\iota = f\phi$ for some $\phi \in \mathrm{Hom}_A(\mathrm{Ker}(g), M)$. Hence, $\mathrm{Ker}(g) = \mathrm{Im}(\iota) \subset \mathrm{Im}(f)$. $\square$

Lemma 14.5. *$- \otimes_A X$ maps an exact sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ to an exact sequence*

$$L \otimes_A X \xrightarrow{f\otimes_A X} M \otimes_A X \xrightarrow{g\otimes_A X} N \otimes_A X \to 0.$$

In particular, $- \otimes_A X$ is right exact.

Proof We apply $\mathrm{Hom}_B(-, Y)$ to the sequence (after tensoring X). By the naturality of the adjoint isomorphism, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Hom}_B(N \otimes_A X, Y) \arrow[r, "-\circ g\otimes_A X"] \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(M \otimes_A X, Y) \arrow[r, "-\circ f\otimes_A X"] \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(L \otimes_A X, Y) \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowleft"] \\
0 \arrow[r] & \mathrm{Hom}_A(N, \mathrm{Hom}_B(X, Y)) \arrow[r, "-\circ g"] & \mathrm{Hom}_A(M, \mathrm{Hom}_B(X, Y)) \arrow[r, "-\circ f"] & \mathrm{Hom}_A(L, \mathrm{Hom}_B(X, Y))
\end{tikzcd}$$

The second row is exact since it is given by applying the left exact functor $\mathrm{Hom}_A(-, Z)$ for $Z = \mathrm{Hom}_B(X, Y)$. Hence, (by careful diagram chasing) the first row is also exact. Since Yoneda embedding reflects exactness, we get the claimed exactness. $\square$

15 Projective and injective

Definition 15.1. *An A-module P is projective if for any given surjective homomorphism $f : M \twoheadrightarrow M'$ and any homomorphism $p : P \to M$, we have p factors through f, i.e. $\exists q : P \to M'$ s.t. $fq = p$ there is the following commutative diagram*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists q"'] \arrow[d, "\forall p"] \\ M' \arrow[r, two heads, "f"] & M. \end{tikzcd}$$

In other words, $f \circ - = \mathrm{Hom}_A(P, f) : \mathrm{Hom}_A(P, M') \to \mathrm{Hom}_A(P, M)$ is surjective, i.e. $\mathrm{Hom}_A(P, -)$ is exact. Denote by $\mathsf{proj}\, A$ the category of finitely generated projective A-modules.

Dually, an A-module I is injective if for any given injective homomorphism $f : M' \hookrightarrow M$ and any homomorphism $i : M \to I$, i factors through f. This is equivalent to saying that $\mathrm{Hom}_A(f, I)$: $\mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(M', I)$ is surjective, i.e. $\mathrm{Hom}_A(-, I)$ is exact. Denote by $\mathsf{inj}\, A$ the category of finitely generated injective A-modules.

Example 15.2. *Take $P = A$. Then we know that $\mathrm{Hom}_A(A, Y) \cong Y$ via $\alpha \mapsto \alpha(1)$ for any $Y \in \mathsf{mod}\, A$. Hence, for any surjective $f : M' \to M$ and any $p : A \to M$, to find q we only need to show that $f(q(1)) = p(1)$, but*

$$p(1) = f({}^{\exists}x) = f({}^{\exists}q(1)).$$

That is, the free A-module A_A is projective. Note that this does not require finite-dimensionality of A. Consequently, any free A-module (of any *rank) is also projective.*

Dually, using $\mathrm{Hom}_A(X, DA) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(A, DX)$ and the same argument, we get that DA is injective. Note that this DOES *require the finite-dimensionality of A since we need to the isomorphism between the Hom-space under duality.*

Lemma 15.3. *The following are equivalent for a ses $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$.*

(1) *There is some $h : N \to N$ such that $gh = \mathrm{id}_N$.*

(2) *There is some $e : M \to L$ such that $ef = \mathrm{id}_M$.*

(3) *There is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "u"', "\cong"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "(1,0)^T"'] & L \oplus N \arrow[r, "(0,1)"'] & N \arrow[r] & 0 \end{tikzcd}$$

In the case when any of these conditions is satisfied, we say that the ses splits.

Proof See 'Splitting lemma' on Wikipedia. □

Remark 15.4. Note that (3) is strictly stronger than just having $M \cong L \oplus N$ for general modules. However, in our setting[3], having $M \cong L \oplus N$ is enough for splitness. Indeed, applying $\mathrm{Hom}_A(-, L)$ yields an exact sequence

$$0 \to \mathrm{Hom}_A(N, L) \to \mathrm{Hom}_A(L \oplus N, N) \to \mathrm{Hom}_A(N, N)$$

of left $\mathrm{End}_A(N)$-modules. Now the original ses splits is equivalent to having $hf = \mathrm{id}_N$, and so is equivalent to the last map of this induced sequence to be surjective. Since everything is finite-dimensional in our setting, and $\dim_{\Bbbk} \mathrm{Hom}_A(L \oplus N, N) = \dim_{\Bbbk} \mathrm{Hom}_A(L, N) + \dim_{\Bbbk} \mathrm{Hom}_A(N, N)$, exactness at $\mathrm{Hom}_A(L \oplus N, N)$ means that the last map must be surjective.

[3]also OK for L, N finitely generated over a Noetherian A, see `https://mathoverflow.net/questions/167701/`

The following justifies why we called eA projective before.

Lemma 15.5. *The following are equivalent of a finitely genearted A-module P.*

(1) *P is projective, i.e.* $\mathrm{Hom}_A(P, -)$ *is an exact functor.*

(2) *Any ses* $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ *splits.*

(3) *P is a direct summand of a free module of finite rank.*

Proof (1) $\Rightarrow$ (2): We have a surjective map $\mathrm{Hom}_A(P, M) \xrightarrow{f\circ -} \mathrm{Hom}_A(P, P)$, and so $\mathrm{id}_P = fq$ for some $q : P \to M$.

(2) $\Rightarrow$ (3): Since P is finitely generated, there is a surjective A-module homomorphism $\pi : A^{\oplus n} \to P$ for some n. So we have a short exact sequence

$$0 \to \operatorname{Ker} \pi \to A^{\oplus n} \xrightarrow{\pi} P \to 0.$$

Hence, it follows by (2) and Lemma 15.3 that P is a direct summand of $A^{\oplus n}$.

(3) $\Rightarrow$ (1): We have learnt that indecomposable direct summands of A_A is given by the right ideal eA of some primitive idempotent $e = e^2 \in A$. Hence, by the assumption and Krull-Schmidt property $P = \bigoplus_{i=1}^n e_i A$ with e_i primitive idempotents. Now we have a natural projection $\pi : A^{\oplus n} \to P$ given by sending the i-th identity 1_A to e_i, and a natural inclusion $\iota : P \to A^{\oplus n}$ given by $\iota|_{e_i A} = (e_i A \hookrightarrow A)$. Note that $\pi\iota = \mathrm{id}_P$.

Consider a surjective A-module homomorphism $f : M \to N$ and take any A-module homomorphism $p : P \to N$. This yields $p\pi : A^{\oplus n} \to N$, which can be lifted to some $q' : A^{\oplus n} \to M$ as $A^{\oplus n}$ is projective. Now we have

$$(fq')\iota = (p\pi)\iota = p,$$

which means that taking $q = q'\iota$ give the required lift of p. $\square$

Remark 15.6. This result do not require finiteness anywhere, nor Krull-Schmidt; but this special case yields an easier proof. For details of proof in full generality, see Proposition 3.3 and Theorem 3.5 in Rotman's book.

There is a dual result under some restriction.

Lemma 15.7. *Suppose A is finite dimensional and I is a finitely generated A-module. Then the following are equivalent.*

(1) *I is injective, i.e.* $\mathrm{Hom}_A(-, I)$ *is an exact functor.*

(2) *An ses* $0 \to I \to M \to N \to 0$ *splits.*

(3) *I is a direct summand of finite direct sum of DA.*

Definition 15.8. *For $M \in \mathsf{mod}\, A$ and simple module $S \in \mathsf{mod}\, A$, denote by $[M : S]$ the multiplicity of S as a composition factor of M.*

Lemma 15.9. *An endomorphism θ of a simple module S lifts to an endomorphism $\hat{\theta}$ on its projective cover P so that $\theta p = p\hat{\theta}$ for any non-trivial projection $p : P \to S$.*

Proof By Schur's lemma, a non-zero endomorphism θ of S has an inverse ϕ. As ϕ is an isomorphism, it is also surjective, and so the projection p lifts to $q : P \to S$ so that $\phi q = p$. Now we can lift q to $\hat{\theta}$ so that $p\hat{\theta} = q$. Hence, we have a commutative diagram

$$\begin{tikzcd} P \arrow[r, two heads, "p"] \arrow[d, "\hat{\theta}"'] \arrow[dr, "q"] & S \arrow[d, "\theta=\phi^{-1}"] \\ P \arrow[r, two heads, "p"] & S \end{tikzcd}$$

as required. ☐

Lemma 15.10. *Let P_x be the (indecomposable) projective cover of simple module S_x whose endomorphism ring is $D_x := \mathrm{End}_A(S_x)$. Then we have*

$$\dim_{D_x} \mathrm{Hom}_A(P_x, M) = [M : S_x] = \dim_{D_x} \mathrm{Hom}_A(M, I_x).$$

Proof We show by induction on the length of M.

Since P_x has a simple top S_x, we have $\mathrm{Hom}_A(P_x, S_y) \cong \mathrm{Hom}_A(S_x, S_y) \cong \delta_{x,y} D_x$. Hence, the claim holds for $M = S_y$ a simple module.

In general, we have suppose S_y is a direct summand of the top of M, then we have a short exact sequence $0 \to N \to M \to S_y \to 0$ where the length of N is strictly less than that of M. Applying $\mathrm{Hom}_A(P_x, -)$ and using exactness of the Hom-functor we have a short exact sequence

$$0 \to \mathrm{Hom}_A(P_x, N) \to \mathrm{Hom}_A(P_x, M) \to \mathrm{Hom}_A(P_x, S_y) \to 0.$$

Note that this is an exact sequence of (right) $\mathrm{End}_A(P_x)$-modules; hence, also an exact sequence of right $D_x = \mathrm{End}_A(S)$-module where $\theta \in D_x$ acts by the lift $\hat{\theta} \in \mathrm{End}_A(P_x)$ shown in Lemma 15.9. Now, we have $\dim_{D_x} \mathrm{Hom}_A(P_x, M) = \dim_{D_x} \mathrm{Hom}_A(P_x, S_y) + \dim_{D_x} \mathrm{Hom}_A(P_x, N)$, and the proof is completed by applying induction hypothesis.

The proof for the $\dim_{D_x} \mathrm{Hom}_A(M, I_x)$ side is dual. ☐

16 Resolution and Ext-group

By definition, any finitely generated module M comes a canonical surjective A-module homomorphism $A^{\oplus n} \twoheadrightarrow M$. One can expect the kernel of this map is 'too large', meaning that many direct summands of the domain appear in the kernel. For more efficient calculation, we often use the most optimal direct summand of $A^{\oplus n}$.

Definition 16.1. *A* projective cover *of an A-module M is a projective A-module P along with a surjective A-module homomorphism $p : P_M \to M$ such that the restriction $p|_Q$ for every proper submodule $Q \subset P_M$ is non-surjective.*

Dually, an injective hull *of M is an injective module I along with an injective A-module homomorphism $i : M \to I_M$ such that any proper quotient $q : I_M \twoheadrightarrow J$ yields a non-injective map qi.*

Lemma 16.2. *Projective cover and injective hull of $M \in \mathsf{mod}\, A$ exist and are unique up to isomorphism.*

Proof We show the case for projective cover; the injective hull case is dual.

Suppose $\operatorname{top}(M) = M/\operatorname{rad}(M) \cong S_1^{\oplus m_1} \oplus \cdots \oplus S_n^{\oplus m_n}$. By consequence of Artin-Wedderburn, we have $S_i = P_i/\operatorname{rad} P_i$ for each i. Take $P_M = P_1^{\oplus m_1} \oplus \cdots \oplus P_n^{\oplus m_n}$.

Since $M \twoheadrightarrow M/\operatorname{rad}(M)$, the canonical surjection $P_M \twoheadrightarrow M/\operatorname{rad}(M)$ lifts to $p : P_M \to M$. As $M \twoheadrightarrow M/\operatorname{rad}(M)$, we have $\operatorname{Im}(p)+\operatorname{rad}(M) = M$, and so it follows from Nakayama lemma (Proposition 10.9 (4)) that $\operatorname{Im}(p) = M$, meaning that p is surjective.

Let $Q \subset P_M$ be a submodule; we show that $p|_Q$ is surjective implies $Q \cong P_M$. Indeed, $p|_Q$ surjective implies that $\operatorname{top}(\operatorname{Im}(p|_Q)) = \operatorname{top}(M)$. Hence, using the definition of P_M being projective we have a commutative diagram

$$\begin{tikzcd} & P_M \arrow[dl, dashed, "\exists q"'] \arrow[d, "\overline{p}"] \\ Q \arrow[r, two heads, "g"] & \operatorname{top}(M). \end{tikzcd}$$

Since Q surjects onto $\operatorname{top}(M)$, for $\iota : Q \hookrightarrow P_M$ the canonical inclusion we get that $g\iota q = \operatorname{top}(M) = \operatorname{top}(P)$. Hence, we have $\operatorname{Im}(\iota q) + \operatorname{rad}(P) = P$. By Nakayama lemma, we have that $\operatorname{Im}(\iota q) = P$, which means that ι is also surjective; thus, ι is an isomorphism, as required. □

Remark 16.3. The claim for projective cover is still true for artinian algebras; but the claim for injective hull really needs finite-dimensionality of A.

Definition 16.4. *A* projective resolution *$P_\bullet$ of an A-module M is a sequence*

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

that is exact everywhere with P_k projective for all $k \geq 0$. It is minimal *if $P_k \twoheadrightarrow \operatorname{Ker}(d_{k-1})$ is a projective cover for all $k \geq 1$. The* n-th syzygy *of M is $\operatorname{Ker}(d_n)$ for $(P_\bullet, d_\bullet)$ the minimal projective resolution of M.*

Dually, an injective coresolution *$I_\bullet$ of M is a sequence*

$$0 \to M \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \to \cdots$$

that is exact everywhere with I_k injective for all $k \geq 0$. It is minimal *if $\operatorname{Cok}(d_{k-1}) \hookrightarrow I_k$ is an injective hull for all $k \geq 1$. The* n-th cosyzygy *of M is $\operatorname{Cok}(d_{n-1})$ for $(I_\bullet, d_\bullet)$ the minimal projective resolution of M.*

Example 16.5. **** see lecture ****

Definition 16.6. *For A-modules M, N, let $P_\bullet$ be a projective resolution of M. Define for $k \geq 0$*

$$\begin{aligned}\mathrm{Ext}_A^k(M,N) :=& H^k(\mathrm{Hom}_A(P_\bullet, N)) \\ =& H^k(\cdots \xleftarrow{-\circ d} \mathrm{Hom}_A(P_{k+1}, N) \xleftarrow{-\circ d} \mathrm{Hom}_A(P_k, N) \leftarrow \cdots) \\ =& \frac{\{f : P_k \to N \mid (fd : P_{k+1} \to N) = 0\}}{\{f : P_k \to N \mid f = gd \text{ some } g : P_{k-1} \to N\}}\end{aligned}$$

Note that $\mathrm{Ext}_A^0(M,N) = \mathrm{Hom}_A(M,N)$.

Similar to $\mathrm{Hom}_A(-,-)$, $\mathrm{Ext}_A^k(-,-)$ also commutes with finite direct sum in both variables.

Example 16.7. **** see lecture ****

There are some other ways to calculate the Ext-groups.

Proposition 16.8. *For any A-modules M, N and any $k \geq 0$, we have*

$$\mathrm{Ext}_A^k(M,N) = H^k(\mathrm{Hom}_A(M, I_\bullet))$$

where $I_\bullet$ is an injective coresolution of N.

Proposition 16.9. *For each $k \geq 1$, there are natural isomorphisms*

$$\underline{\mathrm{Hom}}_A(\Omega^k(M), N) \cong \mathrm{Ext}_A^k(M,N) \cong \overline{\mathrm{Hom}}_A(M, \Omega^{-k}N),$$

where $\underline{\mathrm{Hom}}_A(X,Y)$ (resp. $\overline{\mathrm{Hom}}_A(X,Y)$) is the quotient of $\mathrm{Hom}_A(X,Y)$ by the subspace consisting of $f : M \to N$ that factors through a projective (resp. injective) A-module, i.e. there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & Z \arrow[ur] & \end{tikzcd}$$

for some projective (resp. injective) module Z.

Proof Consider the space $Z_k := \{f : P_k \to N \mid fd_{k+1} = 0\}$ in the definition of $\mathrm{Ext}_A^k(M,N)$. Since we have a exact sequence $P_{k+1} \to P_k \xrightarrow{p} \Omega^k(M) \to 0$, applying $\mathrm{Hom}_A(-,N)$ yields an exact sequence

$$0 \to \mathrm{Hom}_A(\Omega^k(M), N) \xrightarrow{-\circ p} \mathrm{Hom}_A(P_k, N) \xrightarrow{-\circ d_{k+1}} \mathrm{Hom}_A(P_{k+1}, N).$$

By exactness, we have $Z_k = \mathrm{Ker}(- \circ d_{k+1}) \cong \mathrm{Hom}_A(\Omega^k(M), N)$ sending each $f \in Z_k$ to $\overline{f}$ so that $\overline{f}p = f$.

It remains to show that this isomorphisms restricts to one between $B_k := \mathrm{Im}(- \circ d_k)$ and $\mathcal{P} := \{f \in \mathrm{Hom}_A(\Omega^k(M), N)$ that factors through projective$\}$. Clearly, any $f \in B_k$ (by definition) factors through a projective P_{k-1} and so $B_k \subset \mathcal{P}$. For $\overline{f} : \Omega^k(M) \to N$ that factors through a projective, say, P, we want $\overline{f}p = gd_k$ some g. Consider $0 \to \Omega^k(M) \to P_k \to \Omega^{k-1}(M) \to 0$ and apply $\mathrm{Hom}_A(-,N)$ yields

$$0 \to \mathrm{Hom}_A(\Omega^{k-1}(M), N) \to \mathrm{Hom}_A(P_k, N) \xrightarrow{-\circ d_{k+1}} \mathrm{Hom}_A(P_{k+1}, N).$$

$\square$

Exercise 16.10. *In the case when M or N is simple, then we can use ordinary* Hom *instead of the underlined/overlined version.*

Proposition 16.11. *Consider indecomposable projective modules P_x, P_y with simple tops S_x, S_y respectively. Then we have an isomorphism of $\Bbbk$-vector spaces $\mathrm{Ext}_A^1(S_x, S_y) \cong \mathrm{Hom}_A(\mathrm{rad}(P_x)/\mathrm{rad}^2(P_x), S_y)$. Moreover, the $\Bbbk$-dimension of this space is the same as that of $e_x \frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)} e_y$.*

Proof By the previous exercise, we have

$$\mathrm{Ext}^1_A(S_x, S_y) \cong \mathrm{Hom}_A(\Omega(S_x), S_y) \cong \mathrm{Hom}_A(\mathrm{rad}(P_x), S_y) \cong \mathrm{Hom}_A(\mathrm{rad}(P_x)/\mathrm{rad}^2(P_x), S_y).$$

For the last part, first we have by Schur's lemma

$$\mathrm{Hom}_A(\mathrm{rad}(P_x)/\mathrm{rad}^2(P_x), S_y) \cong \mathrm{Hom}_A(S_y, \mathrm{rad}(P_x)/\mathrm{rad}^2(P_x))$$

as $\Bbbk$-vector space, which then yields

$$\begin{aligned}\mathrm{Hom}_A(S_y, \mathrm{rad}(P_x)/\mathrm{rad}^2(P_x)) &\cong \mathrm{Hom}_A(P_y, \mathrm{rad}(P_x)/\mathrm{rad}^2(P_x)) \\ &\cong \mathrm{Hom}_A(e_yA, e_x\frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)}) \cong e_x\frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)}e_y\end{aligned}$$

where the last isomorphism uses Yoneda's lemma. □

Remark 16.12. Note that when $A = \Bbbk Q/I$ a bounded path algebra, then arrows from x to y in Q correspond bijectively to basis elements of $\mathrm{Ext}^1_A(S_x, S_y)$.

16.1 Ext-group versus Extensions

The previous proposition has a better intuition using another manifestation of the Ext-groups.

Definition 16.13. *Two short exact sequences* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *and* $0 \to L \xrightarrow{f'} M' \xrightarrow{g'} N \to 0$ *are equivalent if there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "u"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & N \arrow[r] & 0 \end{tikzcd}$$

Remark 16.14. The map u is necessarily an isomorphism (as a consequence of 5-lemma (Lemma 17.7) or snake lemma).

Theorem 16.15. *The set of equivalence classes of short exact sequence with first term* L *and last term* N *form an abelian group under Baer sum, and this abelian group is isomorphic to* $\mathrm{Ext}^1_A(N, L)$, *with the zero element corresponding to the equivalence class of split short exact sequences.*

There exists similar description for $\mathrm{Ext}^n_A(N, L)$ but the notion of splitness is not as nice as in the case of ses. In any case, for us, we only need to keep in mind that $\mathrm{Ext}^1_A(N, L)$ contains information about short exact sequence of the form $0 \to L \to M \to N \to 0$; c.f. Proposition 16.11 and relation with arrows of quiver. Having said that, we should warn that equivalence classes of ses is not the same as isomorphism classes of the middle term, i.e. there exists non-equivalent ses with the same middle term.

17 Induced long exact sequence

Definition 17.1. *Suppose $C_\bullet = (C_k, d_k)_k, C'_\bullet = (C'_k, d'_k)_k$ are complexes of A-modules. A chain map is $f_\bullet : C_\bullet \to C'_\bullet$ given by A-module homomorphisms $f_k : C_k \to C'_k$ over all $k \in \mathbb{Z}$ such that $d'_k f_k = f_{k+1} d_k$.*

Theorem 17.2 (Comparison theorem). *An A-module homomorphism $f : M \to N$ extends to a chain map on their projective resolutions, as well as a chain map on their injective coresolutions.*

Proof Suppose $P_\bullet, P'_\bullet$ are projective resolutions of M and N respectively. Define the desired chain map $f_\bullet : (P_\bullet \to M \to 0) \to (P'_\bullet \to N \to 0)$ starting from $f_{-1} = f : M \to N$ inductively as follows. We take $P_{-1} = M$ and $P'_{-1} = N$.

Given $f_n : P_n \to P'_n$ defined, using the fact that P_n is projective we can lift $f_n d_{n+1}$, which yields a commutative diagram

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, "\exists f_{n+1}"'] \arrow[d, "f_n d_{n+1}"] \\ P'_{n+1} \arrow[r, two heads, "d'_{n+1}"] & \mathrm{Im}(d'_{n+1}), \end{tikzcd}$$

with the desired chain map property $d'_{n+1} f_{n+1} = f_n d_{n+1}$.

The claim for injective coresolution can be shown analogously. □

Notation. *For a complex $C_\bullet = (\cdots C_k \xrightarrow{d_k} C_{k+1} \to \cdots)$, and $z_k \in \mathrm{Ker}(d_k)$, denote by $[z_k] := z_k + \mathrm{Im}(d_{k-1})$.*

Lemma 17.3. *Suppose $C_\bullet, C'_\bullet$ are complexes of A-modules and $f_\bullet : C_\bullet \to C'_\bullet$ is a chain map. Then for each $k \in \mathbb{Z}$, we have an induced A-module homomorphism $H^k(f_\bullet) : H^k(C_\bullet) \to H^k(C'_\bullet)$ given by $[z_k] \mapsto [f_k(z_k)]$ for any $z_k \in \mathrm{Ker}(d_k : C_k \to C_{k+1})$. Moreover, H^k preserves identity map and additive, as well as intertwines with composition, i.e. H^k is a functor from the category of complexes of A-modules to the category of A-modules.*

Proof Since $d_k(z_k) = 0$, we have

$$d'_k(f_k(z_k)) = f_{k+1} d_k(z_k) = f_{k+1}(0) = 0,$$

i.e. f_k restricts to a map $\mathrm{Ker}(d_k) \to \mathrm{Ker}(d'_k)$.

Suppose now that $z_k \in \mathrm{Im}(d_{k-1})$, say, $z_k = d_{k-1}(x_{k-1})$. Then we have

$$f_k(z_k) = f_k d_{k-1}(x_{k-1}) = d'_{k-1} f_{k-1}(x_{k-1}),$$

i.e. $\mathrm{Im}(f_k|_{\mathrm{Im}(d_{k-1})}) \subset \mathrm{Im}(d'_{k-1})$. Hence, $H^k(f_\bullet) : H^k(C_\bullet) \to H^k(C'_\bullet)$ is well-defined.

We leave the rest as exercise. □

Theorem 17.4 (Induced long exact sequence). *Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence of A-modules. For any A-module M, there is the following long exact sequence:*

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, Z) \to \\ \mathrm{Ext}^1_A(M, X) \to \mathrm{Ext}^1_A(M, Y) \to \mathrm{Ext}^1_A(M, Z) \to \\ \cdots \to \mathrm{Ext}^k_A(M, X) \to \mathrm{Ext}^k_A(M, Y) \to \mathrm{Ext}^k(M, Z) \to \cdots \end{aligned}$$

Proof Our first goal is to show that there are *connecting homomorphisms* $\delta_k : \mathrm{Ext}^k_A(M, Z) \to \mathrm{Ext}^{k+1}_A(M, X)$ for all $k \geq 0$.

Setup: Let $P_\bullet$ be a projective resolution of M. Then for each $k \geq 0$, we have a short exact sequence $0 \to \mathrm{Hom}_A(P_k, X) \to \mathrm{Hom}_A(P_k, Y) \to \mathrm{Hom}_A(P_k, Z) \to 0$. Denote by $C_N^k := \mathrm{Hom}_A(P_k, N)$ for $N \in \{X, Y, Z\}$, and $i_k := \mathrm{Hom}_A(P_k, X \to Y)$ and $p_k := \mathrm{Hom}_A(P_k, Y \to Z)$. Then we have the following commutative (check!) grid with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C_X^0 \arrow[r, "i_0"] \arrow[d] & C_Y^0 \arrow[r, "p_0"] \arrow[d] & C_Z^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_X^1 \arrow[r, "i_1"] \arrow[d] & C_Y^1 \arrow[r, "p_1"] \arrow[d] & C_Z^1 \arrow[r] \arrow[d] & 0 \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_X^k \arrow[r, "i_k"] \arrow[d] & C_Y^k \arrow[r, "p_k"] \arrow[d] & C_Z^k \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

where every column $C_N^\bullet = (C_N^k, \partial_N^k)_k$ is a complex. (Be careful that the superscript does not mean taking exponent here.) Note that $\mathrm{Ext}_A^k(M, N) = H^k(C_N^\bullet)$ by definition.

Defining an assignment $\delta^k : \mathrm{Ker}(\partial_Z^k) \to \mathrm{Ker}(\partial_X^{k+1})$:

- Start with $z_k \in \mathrm{Ker}(\partial_Z^k)$. Since p_k is surjective, $z_k = p_k(y_k)$ for some $y_k \in C_k^Y$.
- By commutativity of the grid, we have $p_{k+1}\partial^k(y_k) = \partial^k p_k(y_k) = \partial^k(z_k) = 0$, i.e. $y_{k+1} := \partial^k(y_k) \in \mathrm{Ker}(p_{k+1})$.
- By the exactness of the $(k+1)$-st row, we have a unique $x_{k+1} \in C_{k+1}^X$ such that $i_{k+1}(x_{k+1}) = y_{k+1}$.
- Since $\partial^{k+1}(y_{k+1}) = \partial^{k+1}\partial_k(y_k) = 0$, it follows from the commutativity of the grid that

  $$i_{k+2}\partial^{k+1}(x_{k+1}) = \partial^{k+1} i_{k+1}(x_{k+1}) = \partial^{k+1}(y_{k+1}) = 0.$$

  Hence, we can now define an assignment $[z_k] \mapsto [x_{k+1}]$.

Showing that $\delta^k : \mathrm{Ker}(p_k) \to H^{k+1}(C_X^\bullet)$ given by $z_k \mapsto [x_{k+1}]$ is a well-defined homomorphism:

- Recall that we went through $z_k \rightsquigarrow y_k \rightsquigarrow y_{k+1} \rightsquigarrow x_{k+1}$ and that they are given by

  $$z_k = p_k(y_k) \quad \text{and} \quad y_{k+1} = \partial^k(y_k) = i_{k+1}(x_{k+1}).$$

  Hence, to show that the assignment is a well-defined map of sets, we need to show that $[x_{k+1}] = [x'_{k+1}]$ for $i_{k+1}(x'_{k+1}) = \partial^k(y'_k)$ for some other $y'_k \in C_Y^k$ with $p_k(y'_k) = z_k$.
- By assumption, we have $p_k(y_k - y'_k) = 0$. Hence, we have $y_k - y'_k \in \mathrm{Ker}(p_k) = \mathrm{Im}(i_k)$ by exactness of the k-th row at C_Y^k. Thus, we have some $x_k \in C_X^k$ so that $i_k(x_k) = y_k - y'_k$.
- By commutativity of the grid, we have

  $$i_{k+1}\partial^k(x_k) = \partial^k i_k(x_k) = \partial^k(y_k - y'_k) = \partial^k(y_k) - \partial^k(y'_k) = i_{k+1}(x_{k+1} - x'_{k+1}).$$

- As i_{k+1} is injective, we have $x_{k+1} - x'_{k+1} = \partial^k(x_k)$; hence, $[x_{k+1}] = [x'_{k+1}]$ as required.
- To show that the map is actually a homomorphism, we need to check that it preserves zero, scalar multiple, and additivity. These are all routine check and we leave these as exercise.

Showing that $\mathrm{Im}(\delta^k|_{\mathrm{Im}(\partial_Z^{k-1})}) = 0$, i.e. δ^k lifts to a homomorphism on $H^k(C_Z^\bullet)$:

- Suppose $z_k = \partial^{k-1}(z_{k-1})$ for some $z_{k-1} \in C_Z^{k-1}$.
- As p_{k-1} is surjective, we have $z_{k-1} = p_{k-1}(y_{k-1})$ some $y_{k-1} \in C_Y^{k-1}$.

- By commutativity of the grid, we have $p_k\partial^{k-1}(y_{k-1}) = \partial^{k-1}p_{k-1}(y_{k-1}) = \partial^{k-1}(z_{k-1}) = z_k$.
- This gives rise to $y_{k+1} = \partial^k(\partial^{k-1}(y_{k-1})) = 0$, and so $x_{k+1} = 0$.

The sequence of the claim is exact everywhere:

- Combining δ_k with Lemma 17.3, we obtain a sequence
$$0 \to H_X^0 \xrightarrow{H^0(i_0)} H_Y^0 \xrightarrow{H^0(p_0)} H_Z^0 \xrightarrow{\delta^0} H_X^1 \to \cdots \xrightarrow{\delta^{k-1}} H_X^k \xrightarrow{H^k(i_k)} H_Y^k \xrightarrow{H^k(p_k)} H_Z^k \xrightarrow{\delta^k} \cdots .$$
Note this is exact at H_X^0 and H_Y^0 by exactness of the Hom-functor. For convenience, we write $p_*^k := H^k(p_k), i_*^k := H^k(i_k)$.
- $H^k(p_k)H^k(i_k) = H^k(p_ki_k) = 0$ by Lemma 17.3.
- Exactness at H_Y^k: Suppose that $p_*^k([y]) = [p_k(y)] = 0$. So we have $p_k(y) = \partial^{k-1}(z)$ for some $z \in C_Z^{k-1}$. On the other hand, p_{k-1} being surjective yields some $y' \in C_Y^{k-1}$ with $p_{k-1}(y') = z$. By commutativity of the grid, we have
$$p_k(y) = \partial^{k-1}(p_{k-1}(y')) = p_k\partial^{k-1}(y').$$
Hence, we have $p_k(y - \partial^{k-1}(y')) = 0$. Exactness at C_Y^k yields $y - \partial^{k-1}(y') = i_k(x)$ for some $x \in C_X^k$. By commutativity of the gird, we have
$$i_{k+1}\partial^k(x) = \partial^k i_k(x) = \partial^k(y) - \partial^k\partial^{k-1}(y') = \partial^k(y) = 0.$$
As i_{k+1} is injective, we have $\partial^k(x) = 0$, and so $i_*^k([x]) = [i_k(x)] = [y - \partial^{k-1}(y')] = [y]$, i.e. $[y] \in \mathrm{Im}(i_*^k)$ as required.
- $\delta^k p_*^k = 0$: $\delta^k p_*^k([y]) = \delta^k([p_k(y)]) = [x]$ with $i_{k+1}(x) = \partial^k(y')$ for some y' such that $p_k(y') = p_k(y)$. Since $[x]$ is independent of choice of y', we can just take $y' = y$. As $y \in \mathrm{Ker}(\partial^k)$ by assumption, we have $i_{k+1}(x) = 0$, and so $x = 0$ by injectivity of i_{k+1}.
- Exactness at H_Z^k: Suppose that $\delta^k([z]) = 0$. Then we have some $x' \in C_X^k$ so that $i_{k+1}\partial^k(x') = \partial^k(y)$ with $p_k(y) = z$. By commutativity of the grid, we have
$$\partial^k(y) = i_{k+1}\partial^k(x') = \partial^k i_k(x')$$
and so $y - i_k(x') \in \mathrm{Ker}(\partial_Y^k)$. Hence, we have $p_*^k([y - i_k(x')]) = [p_k(y) - p_ki_k(x')] = [p_k(y)] = [z]$ as required.
- $i_*^k\delta^k = 0$: $i_*^k\delta^k([z]) = [i_k(x)]$ for $[x] = \delta^k([z])$. By definition of δ^k, we have $i_k(x) = \partial^{k-1}(y)$ with $p_{k-1}(y) = z$. Hence, we have $[i_k(x)] = [\partial^{k-1}(y)] = 0$ by the definition of H_Y^k.
- Exactness at H_X^k: Suppose that $i_*^k([x]) = [i_k(x)] = 0$. This means that there is some $y \in C_Y^{k-1}$ with $\partial^{k-1}(y) = i_k(x)$. By commutativity of the grid and $p_ki_k = 0$, we have
$$\partial^{k-1}p_{k-1}(y) = p_k\partial^{k-1}(y) = p_ki_k(x) = 0.$$
Hence, $p_{k-1}(y) \in \mathrm{Ker}(\partial_Z^{k-1})$. By definition we have $\delta^{k-1}([p_{k-1}(y)]) = [x']$ with $i_k(x') = \partial^{k-1}(y) = i_k(x)$. Hence, $[x'] = [x]$ and $[x] \in \mathrm{Im}(\delta^{k-1})$ as required.

This finishes the proof. $\square$

Remark 17.5. Careful reader should have noticed that there exists a connecting homomorphism long exact sequence by applying homology to any short exact sequence of *complexes*; there we are only consider the case when the complexes involved are all Hom-complexes-of-projective-resolutions.

17.1 Other homological lemmata

Lemma 17.6 (Horseshoe lemma). *Suppose $0 \to L \to M \to N \to 0$ is a short exact sequence. Then a projective resolution $P_\bullet$ of L and a projective resolution $Q_\bullet$ induces a projective resolution of M given by with degree $k \geq 0$ term given by $P_k \oplus Q_k$.*

In pictorial form:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & P_1^L \arrow[d] & P_1^L \oplus P_1^N \arrow[d] & P_1^N \arrow[d] & \\
 & P_0^L \arrow[d] & P_0^L \oplus P_0^N \arrow[d] & P_0^N \arrow[d] & \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Lemma 17.7 (Short 5-lemma). *Suppose there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "w"] & M \arrow[r, "g"] \arrow[d, "u"] & N \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & N' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Then the following hold.

- *If w, v are both injective, then so is u.*
- *If w, v are both surjective, then so is v.*

Proof Diagram chasing. $\square$

18 Various homological dimensions

Definition 18.1. *Let M be an A-module. The projective dimension and injective dimension of M, denoted by* $\operatorname{pdim} M$ *and* $\operatorname{idim} M$ *respectively, are the infimum of the length of the projective resolutions and of the injective coresolutions respectively, i.e.*

$$\begin{aligned}
\operatorname{pdim} M &:= \inf\{d \geq 0 \mid 0 \to P_d \to \cdots \to P_0 \to M \to 0 \textit{ is a proj. res.}\} \\
&= d \textit{ such that } 0 \to P_d \to \cdots \to P_0 \to M \to 0 \textit{ is the minimal proj. res. of } M; \\
\operatorname{idim} M &:= \inf\{d \geq 0 \mid 0 \to M \to I_0 \to \cdots \to I_d \to 0 \textit{ is an inj. cores.}\} \\
&= d \textit{ such that } 0 \to M \to I_0 \to \cdots \to I_d \to 0 \textit{ is the minimal inj. cores. of } M.
\end{aligned}$$

In case we need to clarify the ring involved, we write $\operatorname{pdim}(M_A), \operatorname{idim}(M_A)$*, etc.*

The (right) global dimension of an algebra A is

$$\operatorname{gldim} A := \sup\{\operatorname{pdim} M \mid M \in \mathsf{mod}\, A\}$$

Lemma 18.2. *For $M \in \mathsf{mod}\, A$, we have*

(1) $\operatorname{pdim} M = m \;\Leftrightarrow\; \operatorname{Ext}_A^{>m}(M, -) = 0$ *and* $\operatorname{Ext}_A^m(M, -) \neq 0$.

(2) $\operatorname{idim} M = m \;\Leftrightarrow\; \operatorname{Ext}_A^{>m}(-, M) = 0$ *and* $\operatorname{Ext}_A^m(-, M) \neq 0$.

In particular, we have

$$\begin{aligned}
\operatorname{gldim} A :=& \sup\{\operatorname{pdim} M \mid M \in \mathsf{mod}\, A\} \\
=& \sup\{m \mid \operatorname{Ext}_A^{>m}(M, N) = 0 \textit{ for } M, N \in \mathsf{mod}\, A\} \\
=& \sup\{\operatorname{idim} M \mid M \in \mathsf{mod}\, A\} \\
=& \operatorname{gldim} A^{\mathrm{op}}.
\end{aligned}$$

Proof (1) and (2) follows from the definition of Ext-groups. These yields the first three equalities of the last part; the last equality comes from duality $D : \mathsf{mod}\, A \to \mathsf{mod}\, A^{\mathrm{op}}$ which sends projectives to injectives and vice versa. □

Lemma 18.3. *For any ses $0 \to L \to M \to N \to 0$, we have*

$$\operatorname{pdim} M \leq \max\{\operatorname{pdim} L, \operatorname{pdim} M\} \textit{ and } \operatorname{idim} M \leq \max\{\operatorname{idim} L, \operatorname{idim} N\}.$$

In particular, we have

$$\begin{aligned}
\operatorname{gldim} A :=& \sup\{\operatorname{pdim} M \mid M \in \mathsf{mod}\, A\} \\
=& \sup\{\operatorname{pdim} S \mid S \textit{ simple } A\textit{-module}\} \\
=& \operatorname{gldim} A^{\mathrm{op}} \\
=& \sup\{\operatorname{idim} M \mid M \in \mathsf{mod}\, A\} \\
=& \sup\{\operatorname{idim} S \mid S \textit{ simple } A\textit{-module}\}
\end{aligned}$$

Proof Horseshoes lemma, or use long exact sequence. For the second part, second equality follows from repeatedly applying the first part, and second equality comes from the previous lemma. □

Proposition 18.4. $\operatorname{gldim} A = 0$ *if and only if A is semisimple.*

Proof Global dimension zero is equivalent to every module - in particular, simple module - being projective (i.e. direct summand of A_A). Hence, every modules are semisimple. □

Proposition 18.5. *For an acyclic quiver Q, we have* $\operatorname{gldim} \Bbbk Q \leq 1$*. Moreover, this is an equality if and only if* $Q_1 \neq \emptyset$*.*

Proof For each simple $\Bbbk Q$-module, we have a short exact sequence

$$0 \to \operatorname{rad} P_x \to P_x \to S_x \to 0,$$

so it is enough to show that $\operatorname{rad} P_x$ is projective. In fact, we show that $\operatorname{rad} P_x \cong \bigoplus_{(x \to y) \in Q_1} P_y$, and the inclusion map is given by (left-)multiplication $\alpha \cdot - : P_y \to P_x$ for each $(\alpha : x \to y) \in Q_1$.

By construction, $\operatorname{top}(\operatorname{rad} P_x) = \bigoplus_{(x \to y) \in Q_1} S_y$, and so we have $\bigoplus_{(x \to y) \in Q_1} P_y \twoheadrightarrow \operatorname{rad} P_x$. Suppose $\sum_p \lambda_p p \in \bigoplus_{(x \to y) \in Q_1} P_y$ such that

$$0 = \sum_{(\alpha : x \to y) \in Q_1} (\alpha \cdot -)(\sum_p \lambda_p p) = \sum_{(\alpha : x \to y) \in Q_1} \sum_{p\text{: path with } s(p) = y} \lambda_p (\alpha p).$$

But all the paths αp appeared are different and thus linear independent in P_x; hence, $\lambda_p \equiv 0$ - as required.

Note that if there is an arrow $\alpha \in Q_1$, then the above argument shows that $\operatorname{pdim} S_{s(\alpha)} = 1$; thus $\operatorname{gldim} \Bbbk Q = 1$. On the other hand, if there is no arrow in Q, then $\Bbbk Q = \Bbbk^{|Q_0|}$ is semisimple. $\square$

Remark 18.6. If $\Bbbk = \overline{\Bbbk}$, then $\Bbbk Q$ are the only $\Bbbk$-algebras, up to Morita equivalence, with global dimension at most one; see any book on quiver representation for proof. Otherwise, one needs to work with *valued quivers* which is enhancement of quivers with added field extension data (an example is in Homework 1 Exercise 1).

19 Hereditary algebra

Definition 19.1. *An algebra A is (right) hereditary if every submodule of a projective (right) A-module is projective.*

One can define left hereditary dually (replacing right modules by left; or equivalently, replacing projectives by injectives), but it turns out the two notions are the same - such a property is called 'left-right symmetry'.

Theorem 19.2. *The following are equivalent for an algebra A.*

(1) *A is right hereditary, i.e. $M \subset P \in \mathsf{proj}\, A \Rightarrow M \in \mathsf{proj}\, A$.*

(2) *A is left hereditary, i.e. $\mathsf{inj}\, A \ni I \twoheadrightarrow M \Rightarrow M \in \mathsf{inj}\, A$.*

(3) $\operatorname{gldim} A \leq 1$.

(4) $\operatorname{rad} P \in \mathsf{proj}\, A$ *for any* $P \in \mathsf{proj}\, A$.

(5) $I/\operatorname{soc} I \in \mathsf{inj}\, A$ *for any* $I \in \mathsf{inj}\, A$.

Proof (3) $\Rightarrow$ (1): We have a short exact sequence $0 \to M \to P \to P/M \to 0$. Take any $Y \in \mathsf{mod}\, A$ and consider the long exact sequence induced by applying $\operatorname{Hom}_A(-, Y)$ to this ses. Then, as $\operatorname{Ext}_A^{>0}(P, Y) =$, we have $\operatorname{Ext}_A^k(M, Y) \cong \operatorname{Ext}_A^{k+1}(P/M, Y)$. Since $\operatorname{pdim} P/M \leq 1$ by the global dimension assumption, the right-hand space is zero for all $k \geq 0$; hence, $\operatorname{Ext}^{>0}(M, Y) = 0$ for all $Y \in \mathsf{mod}\, A$, i.e. M is projective.

(1) $\Rightarrow$ (3): Let P be the projective cover of an A-module M. Then we have a short exact sequence $0 \to \operatorname{Ker}(p) \to P \xrightarrow{p} M \to 0$. It follows from the assumption that $\operatorname{Ker}(p)$ is projective, and thus $\operatorname{pdim} M \leq 1$.

(2) $\Leftrightarrow$ (3): (Exercise) Dual argument to the case of (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (4): Immediate.

(4) $\Rightarrow$ (1): Induction on $d = \dim_{\Bbbk} P$. If $d = 1$, then there is nothing to show. Assume $d > 1$. Write $P = P_1 \oplus P_2$ with P_1 indecomposable and let $p : P \twoheadrightarrow P_1$ be the canonical projection. Then we have a map $f = pi$ given by the composition of p with the canonical inclusion $i : M \hookrightarrow P$.

If $\operatorname{Im}(pi) = P_1$, i.e. pi is surjective, then P_1 being projective implies that pi splits; hence, $M \cong P_1 \oplus (M \cap P_2)$. Since $\dim_{\Bbbk} P_2 < d$, by the induction hypothesis we have that the submodule $M \cap P_2$ of the projective module P_2 is also projective. Hence, M is also projective.

If $\operatorname{Im}(pi) \neq P_1$, then $M \subset (\operatorname{rad} P_1) \oplus P_2$. By the assumption $\operatorname{rad} P_1$ is projective; hence, so is $(\operatorname{rad} P_1) \oplus P_2$. Since $\dim_{\Bbbk}(\operatorname{rad} P_1) \oplus P_2 < d$, by the induction hypothesis, we have M projective.

(2) $\Leftrightarrow$ (5): Dual argument to the case of (2) $\Leftrightarrow$ (5). $\square$

Corollary 19.3. *Being hereditary is left-right symmetric, i.e. A is hereditary if and only if so is A^{op}.*

Corollary 19.4. *$\Bbbk Q$ is hereditary for any finite acyclic quiver.*

Proof We have already seen that $\operatorname{gldim} \Bbbk Q \leq 1$. $\square$

Remark 19.5. A better result is that, when $\Bbbk$ is algebraically closed, then hereditary is the same as being Morita equivalent to $\Bbbk Q$. More generally, being hereditary is the same as being Morita equivalent to Dlab-Ringel's species (roughly, 'path algebra' of quiver with added field extensions datum).

20 Heredity ideal and chain

Definition 20.1. *An idempotent (two-sided) ideal $I \subset A$ is one that is generated by an idempotent, i.e. $I = AeA$ for some $e = e^2 \in A$. Such an ideal is minimal if e is primitive.*

Remark 20.2. Originally, an idempotent ideal is one which satisfies $I = I^2$. But this is equivalent to the above definition. Indeed, $I + \mathrm{rad}(A)$ is an idempotent ideal in $\overline{A} := A/\mathrm{rad}(A)$, but ideal in $\overline{A}$ is necessary of the form $\overline{A}\overline{e}\overline{A}$ for some idempotent $e \in A$ with $\overline{e} = e + \mathrm{rad}(A)$. Hence, we have $I + \mathrm{rad}(A) = AeA + \mathrm{rad}(A)$. Thus, $I^2 = I$ implies that $(I + \mathrm{rad}(A))^k = I + \mathrm{rad}^k(A)$ for all $k \geq 1$, but $\mathrm{rad}^k(A) = 0$ for large enough k, and so $I = AeA$.

Lemma 20.3. *Suppose Q is a finite acyclic quiver and $x \in Q_0$. The following hold for the idempotent $e = e_x \in \Bbbk Q$.*

(1) *eAe is a simple algebra.*

(2) *AeA is a projective right A-module.*

(3) *The quotient algebra $A/AeA \cong \Bbbk Q'$ where Q' is obtained from Q by removing x and all arrows attached to it.*

Proof (1) Since $e = e_x$ is primitive and Q is acyclic, we have eAe is given by the path algebra of a one-vertex (corresponding to x) no-loop quiver, i.e. $eAe \cong \Bbbk$.

(2) Note that AeA has a basis given by all paths that go through $x \in Q_0$, so we have a right A-module indecomposable decomposition $AeA \cong \bigoplus_p peA = \bigoplus pA$ with p varies over all paths that end at x. The claim now follows.

(3) Exercise (This is already in Homework assignment 2). □

Remark 20.4. The same result hold for any hereditary algebra A ((3) becomes "A/AeA is also hereditary").

In particular, for **any** permutation $(x_1, \dots, x_n)$ of all elements of Q_0, define $f_i = \sum_{j \geq i} e_{x_j}$ and then we have a chain of idempotent ideals

$$A = Af_1A \supset Af_2A \supset \cdots \supset Af_nA \supset 0 \tag{20.1}$$

such that $Af_iA/Af_{i+1}A$ is a projective $A/Af_{i+1}A$-module (c.f. Homework assignment 2 last question) and $f_iAf_i/f_iAf_{i+1}Af_i$ semisimple.

Definition 20.5 (Quasi-hereditary – ring theoretic version). *An idempotent ideal AeA is heredity if AeA is a projective right A-module and $eAe \cong \mathrm{End}_A(eA)$ is semisimple (or equivalently, $e\,\mathrm{rad}(A)e = \mathrm{rad}(eAe) = 0$).*

*An algebra A is quasi-hereditary if **there exists** a chain*

$$A = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq 0 \tag{20.2}$$

of (idempotent) ideals in A such that I_t/I_{t+1} is a heredity ideal of A/I_{t+1} for all t. In such a case, (20.2) is called a heredity chain. For 'simplicity', we often abbreviate quasi-hereditary algebra and quasi-hereditary to just qha and qh.

Remark 20.6. Note that we do not assume $f_i - f_{i+1}$ to be primitive (even in the case of bounded path algebra) for $I_i = Af_iA$ and $I_{i+1} = Af_{i+1}A$.

Remark 20.7. This notion is Morita-invariant – indeed, if e and e' are equivalent primitive idempotents, and $ef_i \neq f_i$ for some i, then $e' \in AeA \subset Af_iA$. Hence, for simplicity, it is safe for us to assume the algebra A is basic – especially in the case when $\Bbbk$ is algebraically closed we can then assume A to be given by a quiver algebra $\Bbbk Q/I$.

Example 20.8. *We have already seen that $\Bbbk Q$ for acyclic Q is quasi-hereditary with heredity chain* (20.1)*. Moreover, every chain of idempotent ideals of $\Bbbk Q$ is heredity. We note that the same hold for arbitrary hereditary algebra.*

Observe that if A is quasi-hereditary, then so is any algebra Morita equivalent algebra to A. Indeed, each ideal appearing in the chain must be generated by an idempotent f that is given by summing over equivalent classes of primitive idempotents. Hence, we can, for simplicity, assume $A \cong \Bbbk Q/I$ is given by a bounded path algebra.

Now we can construct a poset structure on $\Lambda := Q_0$ from the heredity chain (20.2) as follows. Define $x \lhd y$ if the smallest idempotent ideal in (20.2) that contains e_y does not contain e_x, and then adjoining reflexive relation $x \unlhd x$. This yields a poset $(\Lambda, \unlhd)$.

Conversely, given a poset structure $(\Lambda, \unlhd)$ on the set Λ of simple A-modules, we get a chain of idempotent of the form, say, (20.1) with $f_i - f_{i+1}$ (the sum of) a primitive idempotent e_i (and all those equivalent to e_i). We call this the *chain (of idempotent ideals) induced by $(\Lambda, \unlhd)$*. If, furthermore, such an induced chain is heredity, then we say that $(A, (\Lambda, \unlhd))$ is *quasi-hereditary with respect to $(\Lambda, \unlhd)$*. We may omit the set Λ and just use $\unlhd$ if context is clear.

Example 20.9. *Consider the bounded path algebra $A = \Bbbk Q/I$ given by*

$$Q: \quad \begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, "c"] & \\ 2 \arrow[dr, "b"'] & & 3 \arrow[dl, "d"] \\ & 4 & \end{tikzcd}, \quad I = \langle ab - cd \rangle, \quad A_A = \begin{smallmatrix} 1 \\ 2\,3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 .$$

Then we have a heredity chain

$$A \supset A(e_2 + e_3 + e_4)A \supset Ae_4A \supset 0$$

which yields a poset $(Q_0, \unlhd)$ whose Hasse diagram coincide with Q.

Exercise 20.10. *Let $(P, \unlhd)$ be a finite poset. Let $\vec{P}$ be the Hasse quiver of P, i.e. there is a path $x \to y$ if $x \rhd y$. Let I_P to be the admissible ideal of Q generated by $p - q$ whenever p, q are paths with the same source and target. Show that the incidence algebra $\Bbbk[P] := \Bbbk\vec{P}/I_P$ of P is quasi-hereditary with respect to P itself.*

Let us fix a few more notations for convenience.

Assumption 20.11. *Up to Morita equivalent, and by assuming $\Bbbk$ is algebraically closed, we can simplify A to a quiver algebra with the following setup.*

- *$A \cong \Bbbk Q/I$ for some bounded quiver (Q, I) with $Q_0 = \{1, 2, \ldots, n\}$.*
- *$1 = e_1 + \cdots + e_n$ is the primitive idempotent decomposition of A.*
- *For each $1 \le j \le n$, define $f_j := e_{\ge j} := e_j + e_{j+1} + \cdots + e_n$ and $e_{\le j} := e_1 + e_2 + \cdots + e_j$.*
- *For any idempotent e, define $\hat{e} := 1 - e$.*

While the definition of heredity chain does not require the difference of idempotent between each layer to be primitive even in the quiver algebra setting, it will be convenient to assume this is the case, as we can always refine the chain to such a form. Hence, we make the following assumption when A is quasi-hereditary.

- *A has a heredity chain of the form* (20.1)*, or in idempotent-free notation* (20.2)*.*
- *The associated poset structure is denoted by $(\Lambda, \unlhd)$.*

Remark 20.12. The assumption on A being a quiver algebra and $\Bbbk$ being algebraically closed are really not necessary. The argument does not change so much. The annoyance is in dealing with having multiple primitive idempotents being equivalent when working with the ring theoretic definition of qha.

Notation. *Let* $\mathrm{add}(M)$ *be the class (or subcategory) of* A*-modules given by direct summand of finite direct sum of* M*.*

Lemma 20.13. *If* $(aAeA)_A$ *is projective for some* $a, e \in A$*, then we have* $aAeA \in \mathrm{add}(eA)$*.*

Proof Proof via infinite direct sum: There is a surjective homomorphism $p : \bigoplus_{b\in aA} eA^{(b)} \twoheadrightarrow aAeA$ given by $(ea^{(b)})_{b\in aA} \mapsto \sum_{b\in eA} bea^{(b)}$ (the last summation is well-defined because the range is a direct sum – meaning that only finitely many $ea^{(b)}$ is non-zero). Hence, as $aAeA$ is projective, the short exact sequence $0 \to \mathrm{Ker}(p) \to eA^{(aA)} \xrightarrow{p} aAeA \to 0$ splits and so $aAeA$ is a direct summand of $eA^{(eA)}$. But $aAeA$ is finite-dimensional and so it must be direct summand of finitely many copies of eA.

Proof without using infinite direct sum: We want to find a split ses ending with $(eA)^{\oplus m} \to aAeA \to 0$. First consider the eAe-module aAe. As this is finitely generated, we have a surjective homomorphism $p : (eAe)^{\oplus m} \twoheadrightarrow aAe$ in $\mathsf{mod}\, eAe$. Now $p \otimes_{eAe} eA$ is surjective as tensor functor is right exact. This yields the following composition q of surjective A-module homomorphism

$$q :\ (eA)^{\oplus m} \cong (eAe)^{\oplus m} \otimes_{eAe} eA \xrightarrow{p\otimes_{eAe}eA} aAe \otimes_{eAe} eA \xrightarrow{\mu} aAeA$$

where μ is the multiplication map $abe \otimes ec \mapsto abec$. Hence, the short exact sequence $0 \to \mathrm{Ker}(q) \to eA^{\oplus m} \to aAeA \to 0$, which has a projective last term by assumption splits. $\square$

Lemma 20.14 (Left-right symmetry of heredity chain). *Let* $e = e^2$ *be idempotent of* A*.*

(1) *If* AeA *is projective as a right, or left,* A*-module, then the multiplication map* $\mu : Ae \otimes_{eAe} eA \to AeA$ *given my* $ae \otimes eb \mapsto aeb$ *is bijective.*

(2) *If* eAe *is semisimple and the multiplication map* $\mu : Ae \otimes_{eAe} eA \to AeA$ *is bijective, then both* $(AeA)_A$ *and* ${}_A(AeA)$ *are projective.*

In particular, being quais-hereditary is a left-right symmetric notion, i.e. $(A, (I_k)_{k\geq 1})$ *is quasi-hereditary with a heredity chain* $(I_k)_{k\geq 1}$ *if and only if so is* $(A^{\mathrm{op}}, (I_k)_{k\geq 1})$

Proof (1) Consider the multiplication map $\hat{\mu} : Ae \otimes_{eAe} eA \to A$ with range in A instead. Tensoring on the left by eA yields a right A-module homomorphism $\hat{\mu}_{eA} := eA \otimes_A \hat{\mu}$ of the form

$$\hat{\mu}_{eA} : eA \otimes_A Ae \otimes_{eAe} eA \to eA \otimes_A A \cong eA \quad \text{that is explicitly given by} \quad ec \otimes ae \otimes eb \mapsto ecaeb,$$

which is bijective (as rewriting $eA \otimes_A Ae \otimes_{eAe} eA \cong eAe \otimes_{eAe} eA \cong eA$ yields the identity map). This implies that $\hat{\mu}_X$ is also bijective for any $X = fA$ a direct summand of eA; hence, also for any $X \in \mathrm{add}(eA)$. Thus, it follows from Lemma 20.13 that $\hat{\mu}_{AeA} = \mu$ is also bijective. The same argument applies for the case of left modules (by tensoring on the right instead).

(2) Since eAe is semisimple, Ae is semisimple and projective as right eAe-module. Hence, $Ae \otimes_{eAe} eA \cong eA^{\oplus m}$ as right A-module with m the number of simple direct summands of Ae as right eAe-module. Hence, the assumption on μ yields an isomorphism of right A-modules $eA^{\oplus m} \cong Ae \otimes_{eA} eA \xrightarrow{\mu} (AeA)_A$. The same argument applies for the case of right modules. $\square$