Topics in Mathematical Science VI
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Group Representations and Character Theory

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Contents

Lecture 1

Throughout, 'group' means 'finite group', unless otherwise stated. K will always be a field.

Definition 1.1. *A finite-dimensional (resp. n-dimensional) K-linear representation of a group G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action of g on V.

Often, the symbol ρ is suppressed and we write $G \curvearrowright V$ instead, and say 'G acts on V'. In particular, instead of $\rho_g(v)$ for $v \in V$, we write $g(v)$ instead.

Example 1.2. *(1) The trivial representation of G is the one-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}\,.$$

(2) $G = \mathfrak{S}_n$ the symmetric group of rank n. The sign representation of $\mathfrak{S}_n$ is the one-dimensional representation

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is the parity (or sign) of the permutation σ.

Exercise 1.3. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.*

Definition 1.4. *Let KG be the K-vector space with basis G, i.e. $x \in KG \Leftrightarrow x = \sum_{g\in G} \lambda_g g$ with $\lambda_g \in K$ for all $g \in G$.*

Define a map

$$KG \times KG \to KG, \qquad (\sum_{g\in G} \lambda_g g, \sum_{h\in G} \mu_h h) \mapsto \sum_{g,h\in G} \lambda_g \mu_h (gh).$$

It is routine to check that this defines a ring structure on KG with identity given by that of G. We call this ring the group algebra of G over K.

Clearly, $G \curvearrowright KG$ naturally; this is called the regular representation.

Exercise. *Show that there is an injective ring homomorphism $K \to Z(KG) := \{x \in KG \mid xy = yx \ \forall y \in KG\}$. In other words, the group algebra KG is a K-algebra.*

Lemma 1.5. *$\rho : G \to \mathrm{GL}(V)$ is a (finite-dimensional) K-linear representation of G if, and only if, V has the structure of a (finite-dimensional) left KG-module.*

Proof $\underline{\Rightarrow:}$ For $x = \sum_g \lambda_g g \in KG, v \in V$. It is routine to check that $x \cdot v := \sum_g \lambda_g \rho_g(v)$ defines a left KG-module structure.

$\underline{\Leftarrow:}$ Define a map $\rho_g : V \to V$ by $v \mapsto gv$. Since $g^{-1}g(v) = v$, we have $\rho_{g^{-1}}\rho_g = \mathrm{id}$, and so $\rho_g \in \mathrm{GL}(V)$. It is routine to check that $g \mapsto \rho_g$ is a group homomorphism. □

Remark 1.6. One may find in older textbooks that use terminologies like 'the KG-module V is afforded by ρ' in the setting of this lemma.

Definition 1.7. *$V = (V, \rho), W = (W, \theta)$ be K-linear representations of G. A homomorphism from V to W is a K-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$, i.e. $f\rho_g = \theta_g f$ for all $g \in G$.

An isomorphism from V to W is a homomorphism that is invertible, i.e. $\exists g$ s.t. $gf = \mathrm{id}_V$ and $fg = \mathrm{id}_W$. In this case, V and W are equivalent representations, and write $V \cong W$.

Write $\mathrm{Hom}_G(V, W)$ to be the (K-vector) space of all homomorphisms from V to W.

Lemma 1.8. *$f : V \to W$ is a homomorphism of K-linear G-representations if, and only if, it is a homomorphism of left KG-modules; in other words, $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{KG}(V, W)$. Consequently, $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, $W/\mathrm{Im}(f)$ are naturally K-linear G-representations.*

Proof This first part is clear (if not, think through it).

For the second part, just recall that the kernel, image, and quotient of image of any homomorphism of modules are also modules. □

Remark. In the language of category theory, Lemma 1.5 and 1.8 together says that the category of finite-dimensional K-linear G-representations (where morphisms are homomorphisms) and the category of finitely generated left KG-modules are isomorphic (note that this is stronger than just equivalence of categories).

Exercise 1.9. *Let V be the 1-dimensional subspace spanned by $\sum_{g \in G} g \in KG$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.*

Recall that for a ring R with identity 1, either 1 has infinite order (under addition) or has prime, say p, order. The *characteristic* of R, denoted by $\mathrm{char}\, R$, is 0 in the former case, p in the latter.

Exercise. *Fix any $n \geq 2$.*

(i) Find a generator v such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g \in G} g$ of the trivial representation.)

(ii) Show that $\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ when $\mathrm{char}\, K = 2$, otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

Two classes of group representations. In the literature, by *ordinary representations* we mean K-linear representations with $\mathrm{char}\, K = 0$; by *modular representations* we mean K-linear representations with $\mathrm{char}\, K \mid |G|$.

The *Maschke's theorem* (and its consequence) justifies that ordinary representation theory is (significantly) easier to understand than modular ones - this will be our next goal. The material we will use is based on a more ring theoretic approach (from Benson's book Chapter 1) to the subject, which has the advantage of shedding some light on what happen on the modular side too. The proof of Maschke's theorem will follow the exposition of James and Liebecks.

Interlude on terminology and notation. For a field K, recall that a *K-algebra* is a ring R equipped with a ring homomorphism $K \to Z(R) := \{x \in R \mid xy = yx \ \forall y \in R\}$. This is equivalent to saying that R is a K-vector space equipped with a ring structure.

For a K-algebra A, let $A\,\mathsf{mod}$ be the category of finitely generated left A-modules. So by $M \in A\,\mathsf{mod}$ we mean that M is an A-module, and by $(f : M \to N) \in A\,\mathsf{mod}$ we mean that f is an A-module homomorphism. We will use 0 to denote either the zero homomorphism, or the zero element in a vector space, or the vector space with only the zero element; this should be clear from context.

Like numbers, we like to break down modules into simpler 'components'. The first candidate is via the notion of direct sum. Recall that an A-module M is a *direct sum*, say $M = M_1 \oplus M_2$, if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$. We will come back to this next lecture. In this lecture, we consider a more refined way to break down M into smaller modules.

Definition 1.10. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$.*

(1) *M is simple if for any submodule L of M, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 1.11. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Example 1.12. (1) *Trivial module and sign module are both simple. In general, any 1-dimensional representation of a group G will be simple for dimension reason.*

(2) *Consider the matrix ring $A := \mathrm{Mat}_n(K) := \{n \times n$ matrices with entries in $K\}$. Let V be the 'column space', i.e. $V = \{(v_j)_{1\leq j\leq n} \mid v_j \in K\}$ where $X \in \mathrm{Mat}_n(K)$ acts on $v \in V$ by $v \mapsto Xv$ (matrix multiplication from the left). Then V is an n-dimensional simple module. The regular representation A is semisimple as it is isomorphic to the direct sum of n column spaces (corresponding to the n choices of column we can cut matrix into V).*

(3) *The ring of dual numbers is $A := K[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as (x) is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by Kv where $v = 1 + (x)$. If A is semisimple, then Kv is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in K$. If $a \neq 0$, then $A(a + bx) = A$. So $a = 0$, and $Kv \cong (x)$, a contradiction.*

The following easy yet fundamental lemma describes the relation between simple modules.

Lemma 1.13 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division } K\text{-algebra}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Proof For $f \in \mathrm{Hom}_A(S,T)$, $\mathrm{Im}(f)$ is a submodule of T, and so f is either zero or a K-vector space isomorphism, and the latter case only happens when $S \cong T$. □

Remark 1.14. If K is algebraically closed, then any division K-algebra is just K itself. The complication with the divison K-algebra appearing is the reason why most literature consider only the case when K is algebraically closed. In particular, for ordinary representation one usually just consider

$K = \mathbb{C}$. In this course, this will also often be the case - perhaps the only exception is when we consider general K-algebra instead of group algebra.

Lemma 1.15. *Consider $M = S_1 \oplus \cdots S_r$ with simples $S_i \cong S_j$ for all i, j. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_r(D)$ as K-algebras, where $D := \mathrm{End}_A(S_i)$.*

Note that $\mathrm{End}_A(M)$ is a ring where multiplication is given by composition. Since A is a K-algebra, $\mathrm{End}_A(M)$ is also a K-algebra as K acts by scalar multiplications and commutes with homomorphisms, i.e. $(\lambda \cdot f)(m) := \lambda f(m) = f(\lambda m) = (f \cdot \lambda)(m)$ for all $(f : M \to M) \in A\,\mathsf{mod}$ and $m \in M$.

Proof We have canonical homomorphisms $\iota_j : S_j \hookrightarrow M$ and $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this can be identified with an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(\lambda_{i,j})_{1 \le i,j \le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \xrightarrow{\lambda_{i,j}} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. $\square$

Lecture 2

Definition 2.1. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \leq i \leq \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \leq i \leq \ell$ are called the composition factors of the series.

Composition series allows us to understand the structure of a module by simple modules. It is desirable to have a rigidity result - that composition factors do not change.

Lemma 2.2. *Let M be a finite-dimensional left A-module. Then composition series of M exists.*

Proof This is by induction on $\dim_K M$. For $\dim_K M = 0$ this is trivial. For $\dim_K M > 0$, if M is simple, then we are done. Otherwise, M proper non-zero submodule, and we pick N such a submodule so that M/N is simple. Clearly $\dim_K N < \dim_K M$ and so we can apply induction hypothesis. □

Theorem 2.3 (Jordan-Hölder Theorem). *Any two composition series have the same length and their composition factors are the same up to permutations.*

Proof Suppose we have two composition series

$$\begin{array}{ccccccc} 0 = M_0 & \subset & M_1 & \subset & \cdots & \subset & M_\ell = M, \\ 0 = N_0 & \subset & N_1 & \subset & \cdots & \subset & N_n = M. \end{array}$$

Without loss of generality, we can assume $n > \ell$. We claim that $N_\ell = M$. Indeed, we can do this by induction on ℓ. If $\ell = 0$, then clearly $M_0 = 0 = N_0$ and we are done; likewise, when $\ell = 1$, then M is simple and we have $M_1 = M = N_1$. For $\ell > 1$, we have

$$0 = M_1 \cap N_0 \subset M_1 \cap N_1 \subset \cdots \subset M_1 \cap N_n = M_1 \cap M = M_1.$$

So as M_1 simple, there is some n_0 such that $N_{n_0} \cap M_1 = M_1$ and $N_i \cap M_1 = 0$ for all $i < n_0$.

We now consider two new chains

$$\begin{array}{ccccccccccccccc} 0 & \subset & \frac{M_2}{M_1} & \subset & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \subset & \frac{M_\ell}{M_1} = \frac{M}{M_1}, \\ 0 & \subset & N_1 & \subset & \cdots & \subset & N_{n_0-1} & \subset & \frac{N_{n_0+1}}{M_1} & \subset & \frac{N_{n_0+2}}{M_1} & \subset & \cdots & \subset & \frac{N_n}{M_1} = \frac{M}{M_1}, \end{array}$$

which are both composition series of M/M_1. By induction hypothesis, we thus have $n-1 = \ell - 1$ and the composition factors of these two series coincide up to permutation. □

Remark. This (simpler) version of proof relies on M having composition series of finite length. One can expect similar more careful argument applies for modules that are both noetherian and artinian. In fact, for general K-algebra, M admits a composition series of finite length if and only if it is noetherian and artinian. In this case, Jordan-Hölder theorem also holds.

Exercise 2.4. *Let A be the algebra of upper triangular $n \times n$-matrices:*

$$A := \begin{pmatrix} K & K & \cdots & K \\ 0 & K & \cdots & K \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & K \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i,j \leq n} \,\middle|\, \begin{array}{l} a_{i,j} \in K \ \forall i,j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}$$

For $1 \leq i \leq j \leq n$, let $M_{i,j} \subset K^{\oplus n}$ be the vector space given by column vectors $v = (v_k)_{1 \leq k \leq n}$ where $v_k = 0$ for $k \notin \{i, i+1, \ldots, j\}$.

(i) *Determine which $M_{i,j}$'s are simple.*

(ii) *Describe the composition series of* $M_{i,j}$.

Jordan-Hölder theorem effectively says that the notion of length and composition factor of a module is well-defined without any reference to a chosen composition series.

Now that we no longer worries about building blocks (composition factors) of a module is non-well-defined, we can move on to understand the simplest form of algebra - where every module is semisimple.

Definition 2.5. *Let* A *be a* K*-algebra and* $M \in A\,\mathsf{mod}$.

(1) *The (Jacobson) radical of* A *is the (two-sided) ideal*

$$J(A) := \{a \in A \mid aM = 0\ \forall \textit{simple } M\}.$$

This is equivalent to saying that $J(A)$ *is the intersection of all maximal left ideals of* A*, as well as the intersection of all maximal right ideals of* A.

(2) A *is semisimple if* $J(A) = 0$*. This is equivalent to saying that left (equivalently, right) regular* A*-module* ${}_AA$ *is semisimple.*

Example 2.6. (1) *A field* K *on its own is a semisimple* K*-algebra.*

(2) *Suppose* D *is a division* K*-algebra, then* $\mathrm{Mat}_n(D) := \{n \times n$ *matrices with entries in* $D\}$ *is a semisimple* K*-algebra.*

(3) *A finite product of semisimple algebras is semisimple.*

(4) *The ring of dual numbers* $A := K[x]/(x^2)$ *is not semisimple since it has a non-trivial maximal ideal* $J(A) = (x)$*. More generally, the truncated polynomial ring* $K[x]/(x^n)$ *for any* $n \geq 2$ *is also non-semisimple.*

Theorem 2.7. (see [Benson, Lemma 1.2.4] or [Erdmann-Holm, Theorem 4.11, 4.23]) *The following are equivalent for a* K*-algebra* A.

- *A is a semisimple algebra.*
- *The regular representation* ${}_AA$ *is a semisimple module.*
- *Every* A*-module is semisimple.*

A natural question is whether all semisimple is always a product of matrix rings over division rings. To answer this question, we need some elementary (but fundamental) properties of simple modules first.

Lemma 2.8. *Let* $e \in A$ *be an idempotent, i.e.* $e = e^2 \in A$*. Then the following hold.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(Ae, M) \cong eM$ *as a* K*-vector space for all* $M \in A\,\mathsf{mod}$.

(2) *There is an isomorphism of rings* $\mathrm{End}_A(Ae)^{\mathrm{op}} \cong eAe$.

Proof (1): Check $\mathrm{Hom}_A(Ae, M) \ni f \mapsto f(e) \in Me$ defines a K-linear map with inverse $em \mapsto (ae \mapsto aem)$.

(2): Take $M = Ae$ in (1) and notice that order of multiplication in reverse that of homomorphism composition. □

Exercise. *Recall (or check any reference book) the notion of free module and the rank of it. Check that for an idempotent* $e \in A$*,* Ae *is a direct summand of* A*. In ring/module theory terms, (by definition)* Ae *is thus a projective module since it is a direct summand of a free module.*

Theorem 2.9 (Artin-Wedderburn's theorem). *Let A be a finite-dimensional K-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \ldots, S_r$. Let $D_i := \mathrm{End}_A(S_i)^{\mathrm{op}}$ be the division K-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of K-algebras*

$$A/J(A) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r).$$

Proof Let $B := A/J(A)$. By definition of $J(A)$, the A-module $A/J(A)$ is semisimple, and any A-submodule M of $A/J(A)$ satisfies $J(A)M = 0$. Hence, $M = M/J(A)M$ is naturally a B-module and $\mathrm{End}_B(M) \cong \mathrm{End}_A(M)$ (even as rings!).

By Lemma 2.8, we have $B \cong \mathrm{End}_B(B)^{\mathrm{op}}$. Since B is semisimple, the regular representation B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Lemma 1.13 and Lemma 1.15 that

$$B \cong \mathrm{End}_B(B)^{\mathrm{op}} \cong \big(\mathrm{Mat}_{n_1}(E_1) \times \cdots \times \mathrm{Mat}_{n_r}(E_r)\big)^{\mathrm{op}} \cong \mathrm{Mat}_{n_1}(E_1^{\mathrm{op}}) \times \cdots \times \mathrm{Mat}_{n_r}(E_r^{\mathrm{op}}),$$

where $E_i := \mathrm{End}_B(S_i)$ for all $1 \le i \le r$. This completes the proof. □

Theorem 2.10 (Maschke's theorem). *If $\mathrm{char}\, K \nmid |G|$, then for any KG-module V and submodule $U \subset V$, there is a KG-module W such that $V = U \oplus W$.*

Proof Let W_0 be any K-vector space complement of U in V, and $\pi : V \to U$ be the K-linear projection map. If π is a homomorphism, then W_0 is a KG-module and we are done by Lemma 1.8 – unfortunately this is not true in general. So our goal is to modify π into an idempotent homomorphism. The clever trick is to consider

$$p : V \to V, \quad v \mapsto \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(v).$$

Let us now show that $p \in \mathrm{End}_{KG}(V)$. Indeed, for any $g \in G$, we have

$$p(gv) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(gv) = \frac{1}{|G|} \sum_{h \in G} g(g^{-1}h^{-1}) \pi(hg)v = g \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h v = gp(v).$$

Now we check that $p^2 = p$. It is easy to see that, as $\mathrm{Im}(\pi) = U$, we have $\mathrm{Im}(p) \subset U$. Hence, it remains to show that $p(u) = u$ for all $u \in U$. Indeed, we have

$$p(u) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi \underbrace{h(u)}_{\in U} = \frac{1}{|G|} \sum_{h \in G} h^{-1} h(u) = \frac{1}{|G|} \sum_{h \in G} u = u.$$

This completes the proof. □

Corollary 2.11. *KG is semisimple if, and only if, $\mathrm{char}\, K \nmid |G|$.*

Proof $\underline{\Leftarrow:}$ Consequence of iteratively applying Maschke's theorem (Theorem 2.10) starting with $V = KG$.

$\underline{\Rightarrow:}$ Suppose on the contrary that KG is semisimple. Let $a := \sum_g g \in KG$. Recall that $\mathrm{triv}_G \cong V := Ka \subset KG$. So we must have $KG \cong V \oplus W$ for some left ideal W of KG.

Consider $w = \sum_h \lambda_h h \in KG$. Since W is a left ideal of KG, we have $aw \in W$. On the other hand, we also have

$$aw = (\sum_g g)(\sum_h \lambda_h h) = \sum_h \lambda_h (\sum_g gh) = \sum_h \lambda_h a,$$

which means that $aw \in V$. But $V \cap W = 0$ and so we must have $\sum_h \lambda_h = 0$, which means that

$$W \subset W' := \left\{ \sum_g \mu_g g \in KG \,\middle|\, \sum_g \mu_g = 0 \right\}.$$

The space W' can be rewritten as the kernel of the map (a.k.a. the augmentation map)

$$\epsilon : KG \to K \text{ given by } \sum_g \mu_g g \mapsto \sum_g \mu_g.$$

Thus, $\dim_K W' = |G| - 1 = \dim_K W$ which means that $W = W'$.

However, we can also see that $\epsilon(a) = 0$, and so $V \subset W$, a contradiction. □

Remark. Note that the proof of this result (both directions) relies neither on Jordan-Hölder nor Artin-Wedderburn. From ring theory perspective, it makes more sense to first talk about unicity of composition factors and structure theory for semisimple algebras, so that we know semisimple modules (and algebras) can be completely understood once we know their composition factors.