Topics in Mathematical Science VII

Autumn 2024

Introduction to group representations

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Last update: October 4, 2024

Convention

Throughout the course, the symbols $K, \Bbbk, \mathbb{F}$ will always be a field. Unless otherwise stated, we assume (for simplicity) that

- all groups are finite;
- all vector spaces are finite-dimensional.

We compose maps from right to left.

We usually denote the identity element of a group G by 1 or 1_G or id_G.

1 Group action

Definition 1.1. *Let G be a group and X a set. We say that G acts on X, or X is a G-set, if there is a map $*: G \times X \to X$, with $gx := g * x := *(g, x)$ for all $g \in G$ and $x \in X$, such that*

$$1x = x, \qquad \text{and} \qquad g(hx) = (gh)x.$$

Thinking about this a little bit more, one can see that the action of G on X simply just permutes the elements of X – i.e. G is just some (sub)group of symmetries on X.

When $X = V$ is a vector-space, if we ask for G to only acts by permuting elements, then it could very well destroy the linearity – the best thing about linear algebra – and we lose all the toolkit from linear algebra. The remedy is to "linearise" the definition of action.

Definition 1.2. *For a vector space V, we say that G acts linearly on V if G acts on V and*

$$g(\lambda u + \mu v) = \lambda g(u) + \mu g(v)$$

for all $g \in G$, all $\lambda, \mu \in K$, and all $u, v \in V$.

Often in practice we just write

$$G \curvearrowright V$$

to denote the existence of linear G-action on V.

2 Linear representations

A linear g-action on V is just a linear transformation for any $g \in G$. So we can repackage the notion of linear G-action using the following.

Recall that the *general linear group* of a vector space V over K is the group of all invertible (K-)linear transformation from V to itself.

$$\mathrm{GL}(V) := \{\phi : V \to V \mid \phi \text{ invertible linear transformation}\}.$$

The group multiplication is just composition of linear transformations, and the identity element is just the identity map $\mathrm{id} : V \to V$.

More generally, one can consider $\mathrm{GL}(V)$ for some free R-module V of finite rank for some nice ring R – by nice, usually this would be at least an integral domain. We may look at some examples in the case when $R = \mathbb{Z}$ when we focus on symmetric group representations.

Now we can reformulate the notion of linear G-action as follows.

Definition 2.1. *Let G be any (not necessarily finite) group. A finite-dimensional (resp. n-dimensional) K-linear representation of G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action of g on V.

Usually, when the underlying field (or ring) is understood, we will drop the adjective 'K-linear' for representations.

Exercise 2.2. *Check that representation defines a linear G-action in the sense of Definition 1.2.*

While we assumed V is a vector space over a field K here, one can also consider more general setting of "R-linear representation" when V is an R-lattice (=free R-module of finite rank).

Example 2.3. (1) *The trivial representation of G is the 1-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}\,.$$

(2) *$G = \mathfrak{S}_n$ the symmetric group of rank n. The sign representation of $\mathfrak{S}_n$ is the 1-dimensional representation*

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is the parity (or sign) of the permutation σ.

(3) *Let X be a finite G-set (for any finite group G). Denote by KX the K-vector space with basis given by X. Then*

$$\pi_X : G \to \mathrm{GL}(KX), \qquad g \mapsto (x \mapsto gx)_{x \in X}$$

defines K-linear G-representation. Any G-representation of such a form is called a permutation representation.

Exercise 2.4. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.*

Exercise 2.5. *Consider the additive group of integer $G = (\mathbb{Z}, +)$. Let V be a fixed finite-dimensional $\mathbb{C}$-vector space. Show that every linear transformation $\phi \in \mathrm{GL}(V)$ defines a unique (not distinguished under isomorphism) $\mathbb{C}$-linear G-representation.*

Recall that for a ring R with identity 1, under addition the element 1 either has infinite or prime, say p, order. The *characteristic* of R, denoted by $\mathrm{char}\, R$, is 0 in the former case, or p in the latter.

In Example 2.3 (2), we can see that when $\mathrm{char}\, K = 2$, then sign representation is the same as trivial representation.

In general, changing characteristic drastically change the kind of representations that can appear.

- *Ordinary representation theory* studies K-linear representations over a field K with $\operatorname{char} K = 0$.
- *Modular representation theory* studies K-linear representations over a field K with $\operatorname{char} K = p > 0$ and $p | \#G$.
- *Integral representation theory* studies $\mathcal{O}$-linear representations over a (nice – such as discrete valuation ring) integral ring $\mathcal{O}$ (but sometimes including $\mathbb{Z}$) with $\operatorname{char} \mathcal{O} = 0$.

The case of K-linear representations with positive characteristic that does *not* divide the order of group is sometimes called "representations over good characteristics" but can also be regarded as a 'trivial' extension of ordinary representation theory – characteristic 0 and good characteristic cases are somewhat the same.

Most of this course will be about ordinary representation theory. We may touch on some integral and modular representation for the symmetric group later in the course.

3 Matrix representations

When V is n-dimensional K-vector space, then $\mathrm{GL}(V)$ is isomorphic to

$$\mathrm{GL}_n(K) := \{\text{invertible } n \times n\text{-matrices with entries in } K\}.$$

This isomorphism of course depends on a basis we pick for V.

Exercise 3.1. *Write down the isomorphism explicitly for a given basis of V.*

Definition 3.2. *An n-dimensional matrix representation of a group G is a group homomorphism*

$$R : G \to \mathrm{GL}_n(K), \qquad g \mapsto R_g.$$

We say that the matrix R_g represents the action of g.

It is clear that given an n-dimensional matrix representation, one obtains an n-dimensional K-linear representation (with $V = K^n$), and vice versa (by choosing a basis for V and passes through $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$).

Example 3.3. *Consider $G = C_3 = \langle x \mid x^3 = 1 \rangle$ the cyclic group of order 3. Let us try to see what matrix representations of G look like in the case when $K = \mathbb{C}$.*

Suppose that $R_x \in \mathrm{GL}_n(\mathbb{C})$ is diagonal. Since $R_x^3 = R_{x^3} = R_1 = \mathrm{id}$, the diagonal entries are in $\{\omega^k := \exp(2\pi i k/3) \mid 0 \le k < 3\}$, and we can write $R_x = \mathrm{diag}(\omega^{k_1}, \dots, \omega^{k_n})$ with any $k_i \in \{0, 1, 2\}$ for all $i = 1, \dots, n$. Note that, in this case, R_x^2 will also be a diagonal matrix $\mathrm{diag}(\omega^{2k_1}, \dots, \omega^{2k_n})$.

On the other hand, if R_x is not a diagonal matrix, since R_x is invertible and we work over $\mathbb{C}$, we can still find $P \in \mathrm{GL}_n(\mathbb{C})$ so that PR_xP^{-1} is diagonal. In other words, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\cong", "P"'] \arrow[d, "\mathrm{diag}(\omega^{ik_1}, \dots, \omega^{ik_n})"'] & \mathbb{C}^n \arrow[d, "R_x^i"] \\
\mathbb{C}^n \arrow[r, "\cong", "P"'] & \mathbb{C}^n,
\end{tikzcd}$$

i.e. the two paths from top left to bottom right resulting the same map. This amounts to say that, up to a change of basis of $\mathbb{C}^n$, the non-diagonal case is "essentially the same" as the diagonal one.

4 Homomorphism

In mathematics, the word for "essentially the same" is (usually) isomorphism; for this, we need the weaker notion of homomorphism first.

Definition 4.1. *Let $\rho : G \to \mathrm{GL}(V)$ and $\theta : G \to \mathrm{GL}(W)$ be two K-linear representations of G. A homomorphism from V to W is a K-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$, i.e. $f\rho_g = \theta_g f$ for all $g \in G$.

An isomorphism from V to W is a homomorphism that is invertible, i.e. $\exists g$ s.t. $gf = \mathrm{id}_V$ and $fg = \mathrm{id}_W$.

Write $\mathrm{Hom}_{KG}(V, W)$ *for the space of all homomorphisms from V to W.*

Remark 4.2. Older text also calls a homomorphism (sometimes, only for isomorphism) $f : V \to W$ an *intertwiner*, or that f *intertwines* ρ, θ; we will try to avoid using this and stick to homomorphism. Older text may say that V, W are equivalent if there is an isomorphism between them. We will drop this redundant language and just say V and W are isomorphic.

Example 4.3. *Let us go back to the case when $G = C_3$ and take $n = 1$. We have three representations $R^{(i)}$ with $i = 1, 2, 3$ so that $R^{(i)}_x = \omega^i$. An isomorphism on $\mathbb{C}$ is just a non-zero scalar multiplication $\lambda \cdot -$. As $\lambda R^{(i)}_x \lambda^{-1} = R^{(i)}_x = \omega^i$, we have $R^{(i)} \ncong R^{(j)}$ whenever $i \neq j$. In fact, by the same reason, we can see that*

$$\mathrm{Hom}_{\mathbb{C}G}(R^{(i)}, R^{(j)}) = \{0\}$$

for distinct i, j.

Exercise 4.4. *Verify that (a) $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space, and (b) the composition of homomorphisms is also a homomorphism of representations.*

Since $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space, we can just write $\mathrm{Hom}_{\mathbb{C}G}(R^{(i)}, R^{(j)}) = 0$ in the above example, instead of the more bulky set notation $\{0\}$.

Exercise 4.5. *Consider $G = C_3$ with generator g acting on $X = \{0, 1, 2\}$ by $gi = i + 1 \mod 3$. Recall from Example 3.3 that 3-dimensional representation of C_3 is isomorphic to a (matrix) representation $R^{(k_1,k_2,k_3)} : G \to \mathrm{GL}_3(\mathbb{C})$ with $R^{(k_1,k_2,k_3)}_g = \mathrm{diag}(\omega^{k_1}, \omega^{k_2}, \omega^{k_3})$. Find (k_1, k_2, k_3) so that $\mathbb{C}X \cong R^{(k_1,k_2,k_3)}$.*

Exercise 4.6. *Let X, Y be two G-sets. Determine the condition on a map $f : X \to Y$ so that f induces a homomorphism of permutation representations from π_X to π_Y.*