Introduction to the Serre Spectral Sequence

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Let X be some space which is filtered by $\{X_p\}$, so

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For example lets take X to be a CW-complex with X_p the p-skeleton of X.

For the various pairs (X_p, X_{p-1}) we have long exact sequences in homology.

$$\ldots \to H_{n+1}(X_p) \xrightarrow{j} H_{n+1}(X_p, X_{p-1}) \xrightarrow{k} H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \to \ldots$$

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We can fit the sequences for different pairs together in a 'staircase diagram'.

$$
\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H_{n+1}(X_p) & \xrightarrow{j} & H_{n+1}(X_p, X_{p-1}) & \xrightarrow{k} & H_n(X_{p-1}) & \to & H_n(X_{p-1}, X_{p-2}) & \to \\
 & \downarrow & & \downarrow & & \downarrow i & & \downarrow & \\
\to & H_{n+1}(X_{p+1}) & \to & H_{n+1}(X_{p+1}, X_p) & \to & H_n(X_p) & \to & H_n(X_p, X_{p-1}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H_{n+1}(X_{p+2}) & \to & H_{n+1}(X_{p+2}, X_{p+1}) & \to & H_n(X_{p+1}) & \to & H_n(X_{p+1}, X_p) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H_{n+1}(X_{p+3}) & \to & H_{n+1}(X_{p+3}, X_{p+2}) & \to & H_n(X_{p+2}) & \to & H_n(X_{p+2}, X_{p+1}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}
$$

From this diagram we can form an exact couple.

Let $A_1 = \bigoplus_{n,p} H_n(X_p)$ and $E_1 = \bigoplus_{n,p} H_n(X_p, X_{p-1})$.

$$\begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j"] \\ & E_1 \arrow[ul, "k"] & \end{tikzcd}$$

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$$\begin{array}{ccc} A_1 & \xrightarrow{i} & A_1 \\ {}_k\nwarrow & & \swarrow_j \\ & E_1 & \end{array}$$

This is clearly exact at the corners since the maps are just those from the exact sequences.

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In particular $d_1 : H_n(X_p, X_{p-1}) \to H_{n-1}(X_{p-1}, X_{p-2})$.
So d_1 drops a degree in homology so we have an E^1 page with entries,

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}),$$

and,

$$d_1 : E^1_{p,q} \to E^1_{p-1,q}.$$

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- $E^2 = \ker(d_1)/\mathrm{im}(d_1)$,
- $A^2 = i(A^1)$,
- $i_2 := i|_{A^1}$,
- $j_2(ia) = [ja]$,
- $k_2([e]) = k(e)$ and
- $d_2 = k_2 \circ j_2$.

Repeating this gives us subsequent pages E^m with

$$\begin{aligned} E^m_{p,q} &= \ker(d_{m-1})/\mathrm{im}(d_{m-1}), \\ d_m : E^m_{p,q} &\to E^m_{p-m,q+m-1}. \end{aligned}$$

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For cohomology we essentially get the same results but with the arrows turned around and the maps d increase the total degree by 1. We also gain some further structure,

$$E_r^{p,q} \otimes E_r^{s,t} \to E_r^{p+s,q+t}.$$

Convergence Theorem

If only finitely many terms in each E column are non zero in the staircase diagram and;

- $A^1_{n,-\infty} = 0$ for all n then $E^\infty_{n,p} \cong F^p_n / F^{p-1}_{n-1}$ where $\ldots \subseteq F^{p-1}_n \subseteq F^p_n \subseteq \ldots$ is the filtration of $A^1_{n,\infty}$ with $F^p_n = \mathrm{im}(A^1_{n,p} \to A^1_{n,\infty})$.
- $A^1_{n,\infty} = 0$ for all n then $E^\infty_{n,p} \cong F^{n-1}_p / F^{n-1}_{p-1}$ where $\ldots \subseteq F^{n-1}_{p-1} \subseteq F^{n-1}_p \subseteq \ldots$ is the filtration of $A^1_{n-1,-\infty}$ with $F^{n-1}_p = \ker(A^1_{n-1,-\infty} \to A^1_{n-1,p})$.

Serre's Theorem

If $F \to X \to B$ is a fibration with B path-connected and $\pi_1(B)$ acting trivially on $H_*(F; G)$ then there is a spectral sequence with d_r and E^m as before and $E^\infty = H_*(X; G)$. Furthermore the E^2 page has entries,

$$E^2_{p,q} = H_p(B; H_q(F, G))$$

Example

Show that

$$H^m(K(\mathbb{Z},2);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } m \text{ is even} \\ 0 & \text{else} \end{cases}$$

Use the path-space fibration

$$S^1 = K(\mathbb{Z}, 1) \to * \to K(\mathbb{Z}, 2).$$

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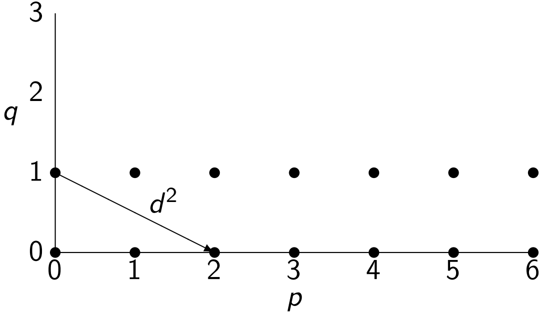
We know that the cohomology of S^1 is 0 everywhere except in degrees 0 and 1 where we have a $\mathbb{Z}$.

So the E_2 page looks something like this:



- $E_2^{0,0} = H^0(K(\mathbb{Z}, 2); H^0(S^1; \mathbb{Z})) = \mathbb{Z}$
- $E_2^{0,1} = H^0(K(\mathbb{Z}, 2); H^1(S^1; \mathbb{Z})) = \mathbb{Z}$

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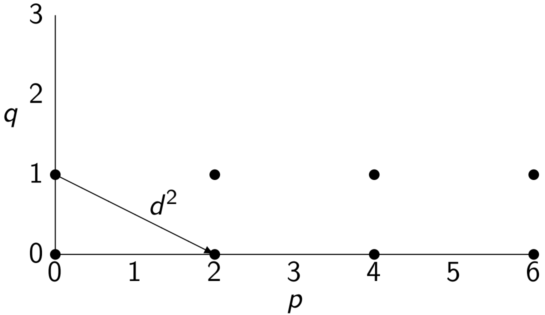
- A little thought shows that d^2 is the only non-zero differential.
- Since $*$ is contractible only the $\mathbb{Z}$ at $(0,0)$ survives to E_∞
- d^2 must be an isomorphism everywhere except when entering/leaving $(0,0)$.



3
2
q
1
d^2
0
0
1
2
3
4
5
6
p

So

- $E_2^{2p+1,0} = H^{2p+1}(K(\mathbb{Z},2);\mathbb{Z}) = 0$ for all p
- $H^{2p}(K(\mathbb{Z},2);\mathbb{Z}) = E_2^{2p,1} = E_2^{2(p+1),0} = H^{2(p+1)}(K(\mathbb{Z},2);\mathbb{Z})$ for all p

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Since $H^0(K(\mathbb{Z},2);\mathbb{Z}) = \mathbb{Z}$ we get the result. Using the multiplication of the E_2 page we could show that $H^*(K(\mathbb{Z},2);\mathbb{Z})$ is in fact the polynomial algebra $\mathbb{Z}[\alpha]$ where the degree of α is 2.