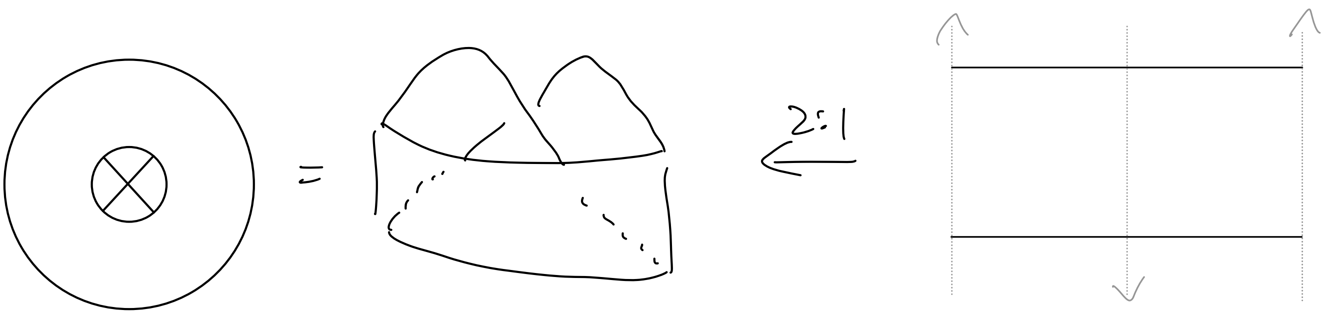
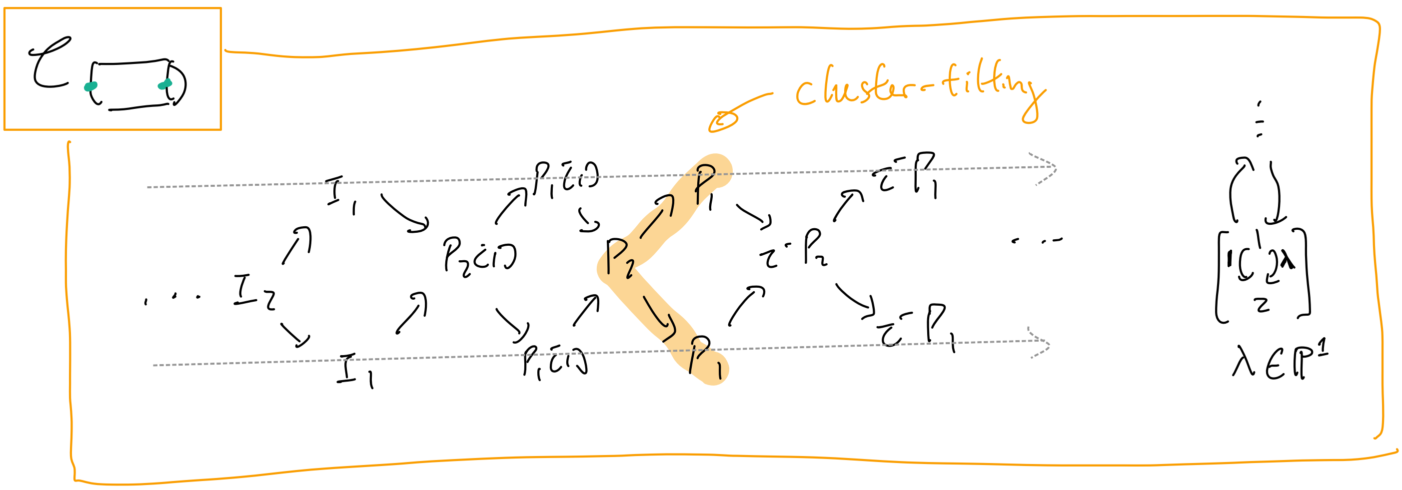
Categorification of (quasi-)triangulation of unpunctured non-orientable marked surfaces

jt. work with V. Bazier-Matte, K. Wright





- Surface combinatorics

★ Green texts = extra info

- Detour: Relation to clusters
- Cluster categories of orientable marked surface
- Triangulations vs. CT

(- Symmetric representations)

(- The case of quasi-triangulations)

§ Surface combinatorics

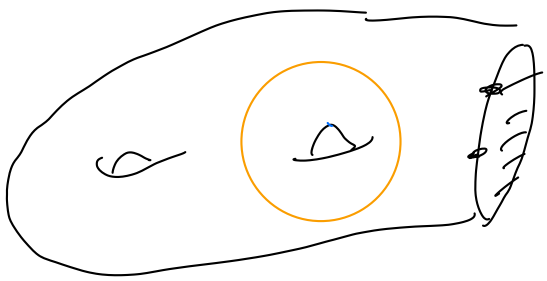
$\partial S \neq \emptyset$

S: Surface = compact 2-dim$^{l}/\mathbb{R}$ w/ non-empty boundary

$\mathbb{M}$: finite set of marked points in ∂S ($\therefore$ unpunctured)

s.t. each boundary component contains $\geqslant 1$ marked pt.

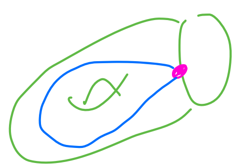
e.g.



$|\mathbb{M}| = 2$.

γ: Curve on $(S, \mathbb{M})$ $:\Longleftrightarrow$ either $\begin{cases} \text{closed} & \gamma \simeq S^1 \\ \text{s.t.} & \gamma \cap \mathbb{M} = \emptyset \\ & \text{non-contractible} \end{cases}$

or $\begin{cases} \text{non-closed} & \gamma : [0,1] \to S \\ & \gamma(0),\ \gamma(1) \in \mathbb{M} \\ & \gamma(0,1) \subset S \setminus \partial S \end{cases}$



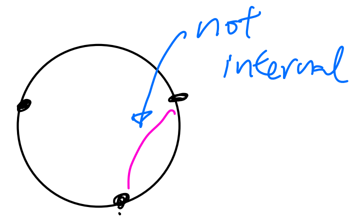
$\in$

up to isotopies that fix ∂S pointwise

$$\boxed{\mathbb{C}_{nc}(S, \mathbb{M}) := \left\{ \begin{matrix} \text{non-closed} \\ \text{curves} \end{matrix} \right\} \qquad \mathbb{C}_{cc}(S, \mathbb{M}) := \left\{ \begin{matrix} \text{closed} \\ \text{curve} \end{matrix} \right\}}$$

Arc :$\Longleftrightarrow$ non-closed, no self intersection except possibly at its endpoints

Internal arc :$\Longleftrightarrow$ arc $\neq$ boundary interval



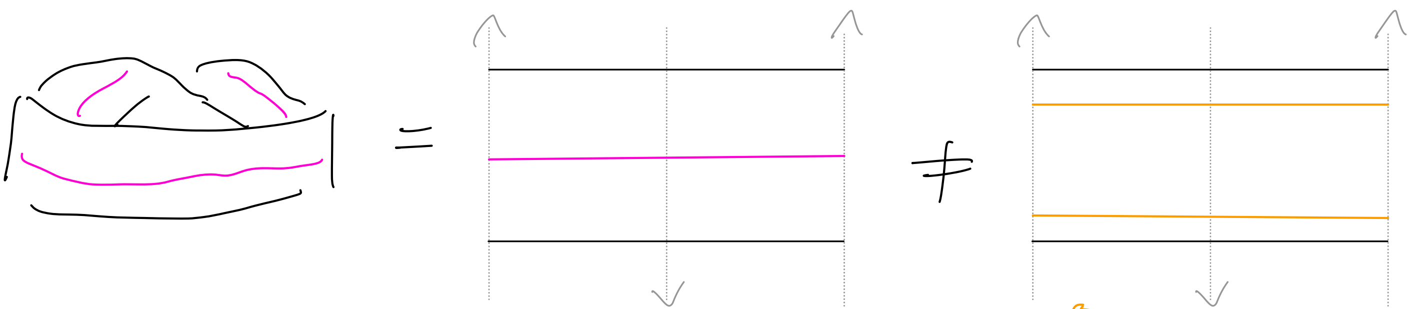
$$\boxed{A(S,M) := \{\text{internal arcs}\}}$$

quasi-arc :$\Longleftrightarrow$ either internal arc,

or 1-sided closed curve

(the curve is non-orientable)

$$\boxed{A^{\otimes}(S,M) := \{\text{quasi-arcs}\}}$$



2-sided closed curve
||
orientable

(quasi-)triangulation :$\Longleftrightarrow$ maximal non-crossing collection of (quasi-)arcs

(pairwise non-intersecting except possibly at their endpt's)

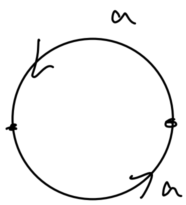
Working with non-orientable surfaces (NoS)

orientable of genus g ; $k' = k - 2g \geq 1$

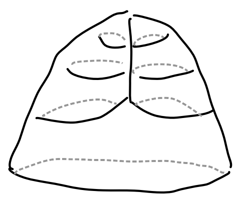
<u>Fact</u> S : Compact closed NoS

$$\iff S \simeq N_k := (\mathbb{RP}^2)^{\# k} \simeq S' \# (\mathbb{RP}^2)^{\# k'}$$

Euler genus of S

- $\mathbb{RP}^2 =$ 

Youtube
"The cross-cap"

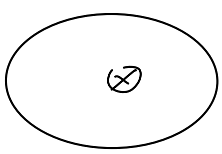
- <u>crosscap</u> $:= \mathbb{RP}^2 \setminus \text{disc}$  $=$ Möbius strip.

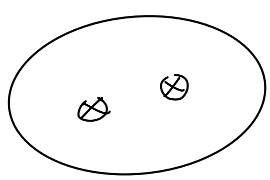
<u>Rmk</u>: Some literature call $\mathbb{RP}^2$ the crosscap instead.

In practice, crosscap is drawn as $\otimes$ (to represent the antipodal map)

(or $\circledast$, $\otimes$, etc...)

$N_k := (\mathbb{RP}^2)^{\# k} \setminus \text{disc}$

$N_1 =$  $=$ Möbius strip ,

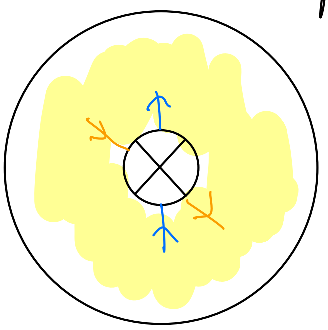
$N_2 =$  $=$ Klein bottle $\setminus$ disc

$(\tilde{\mathbb{S}}, \tilde{\mathbb{M}}) :=$ Orientable double cover of a marked nos. $(\mathbb{S}, \mathbb{M})$

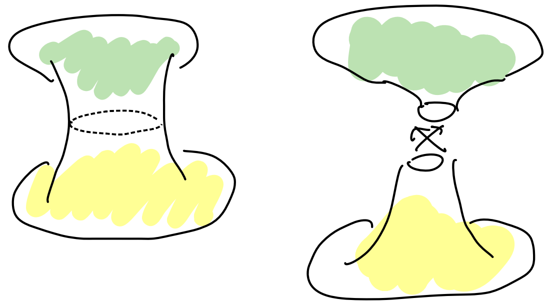
M_n := Möbius strip with n marked pt's. on ∂M_n.

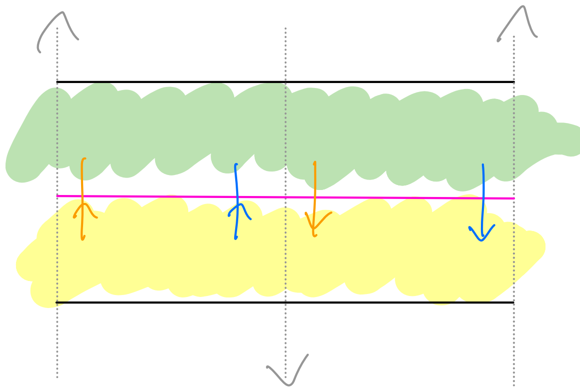
<u>e.g.</u>

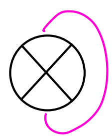
$N_1 \xleftarrow{2:1} \tilde{N}_1$



non-crossing

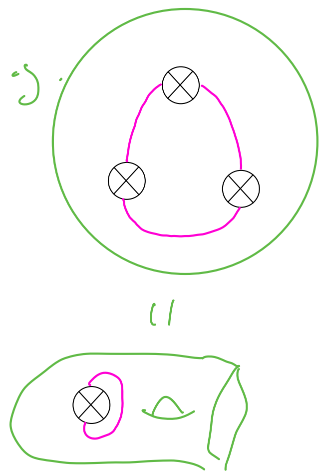


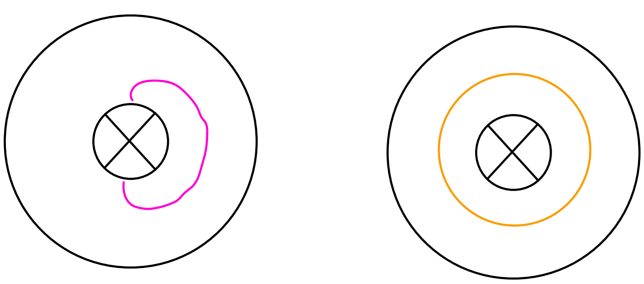


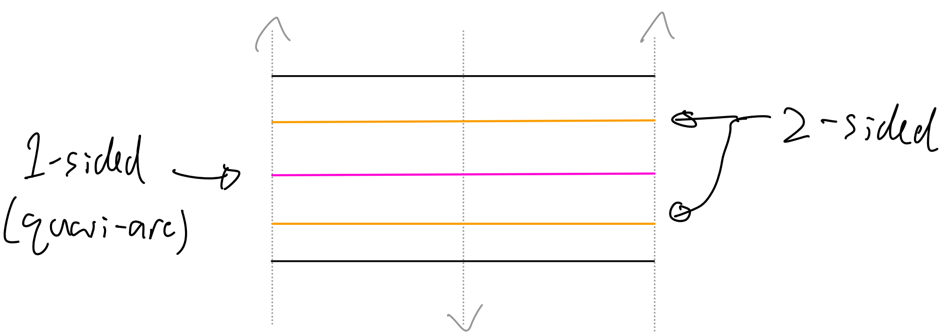
• Quasi-arcs are of the form: 

up to changes of pictorial representation

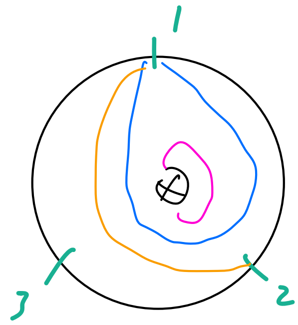
e.g.



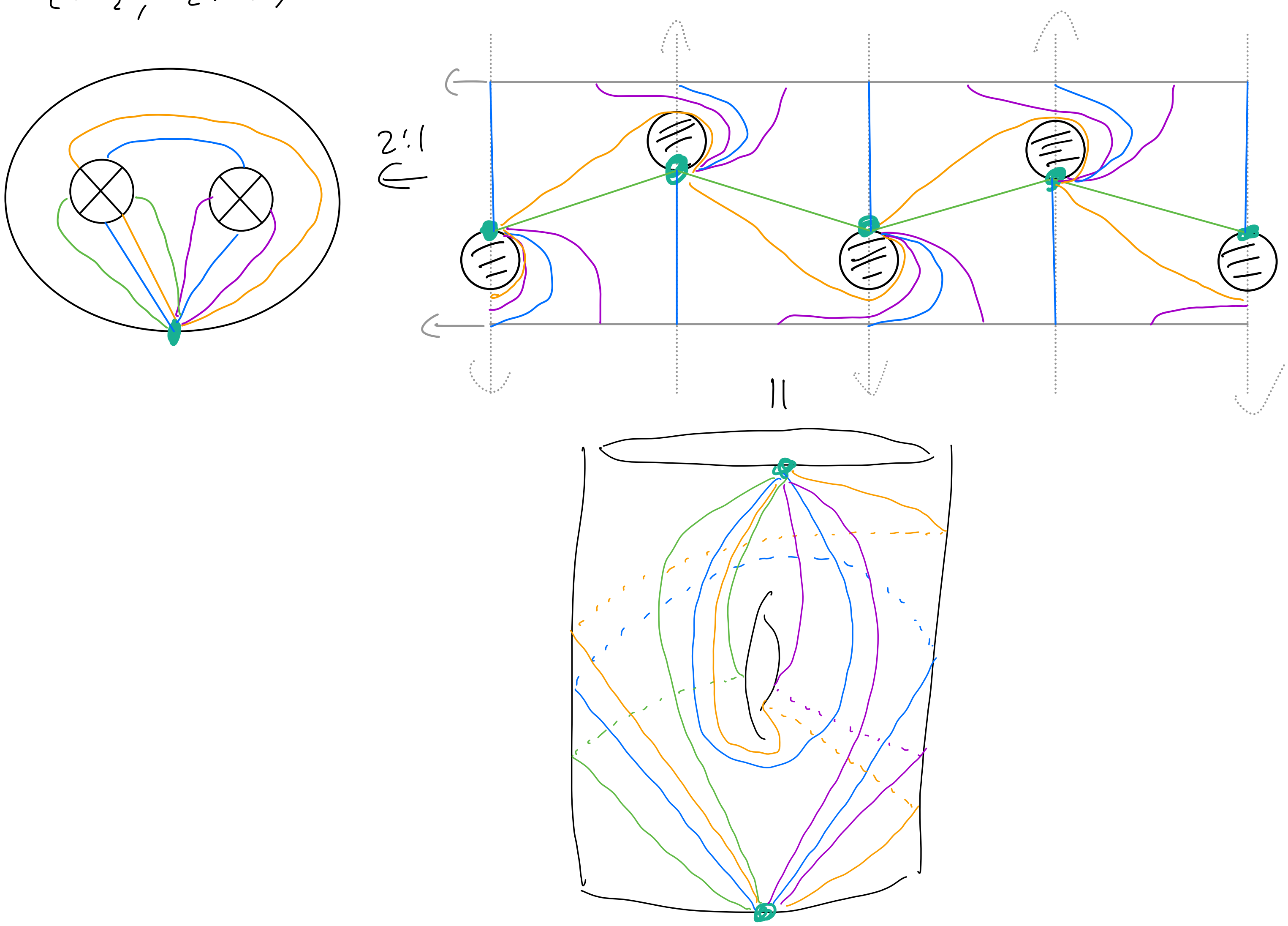




M_3 :

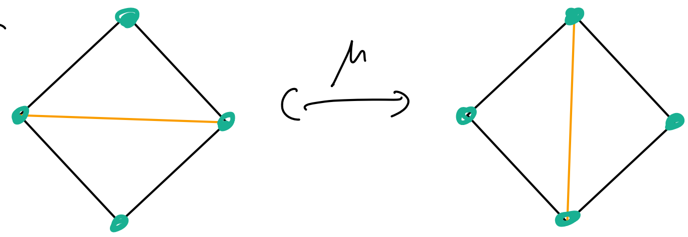


$(N_2, \{*\})$:

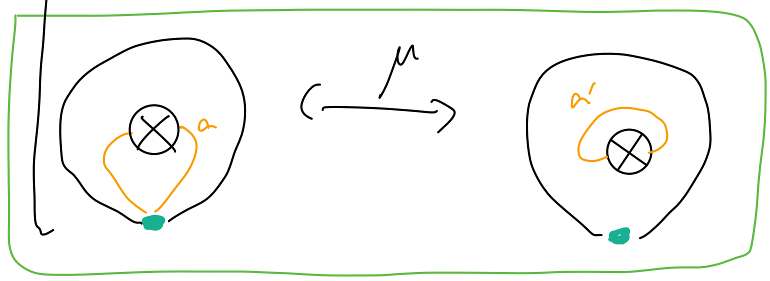


§1.2) Mutation

Flip of quasi-triang^n :



Flip graph of M_1 =

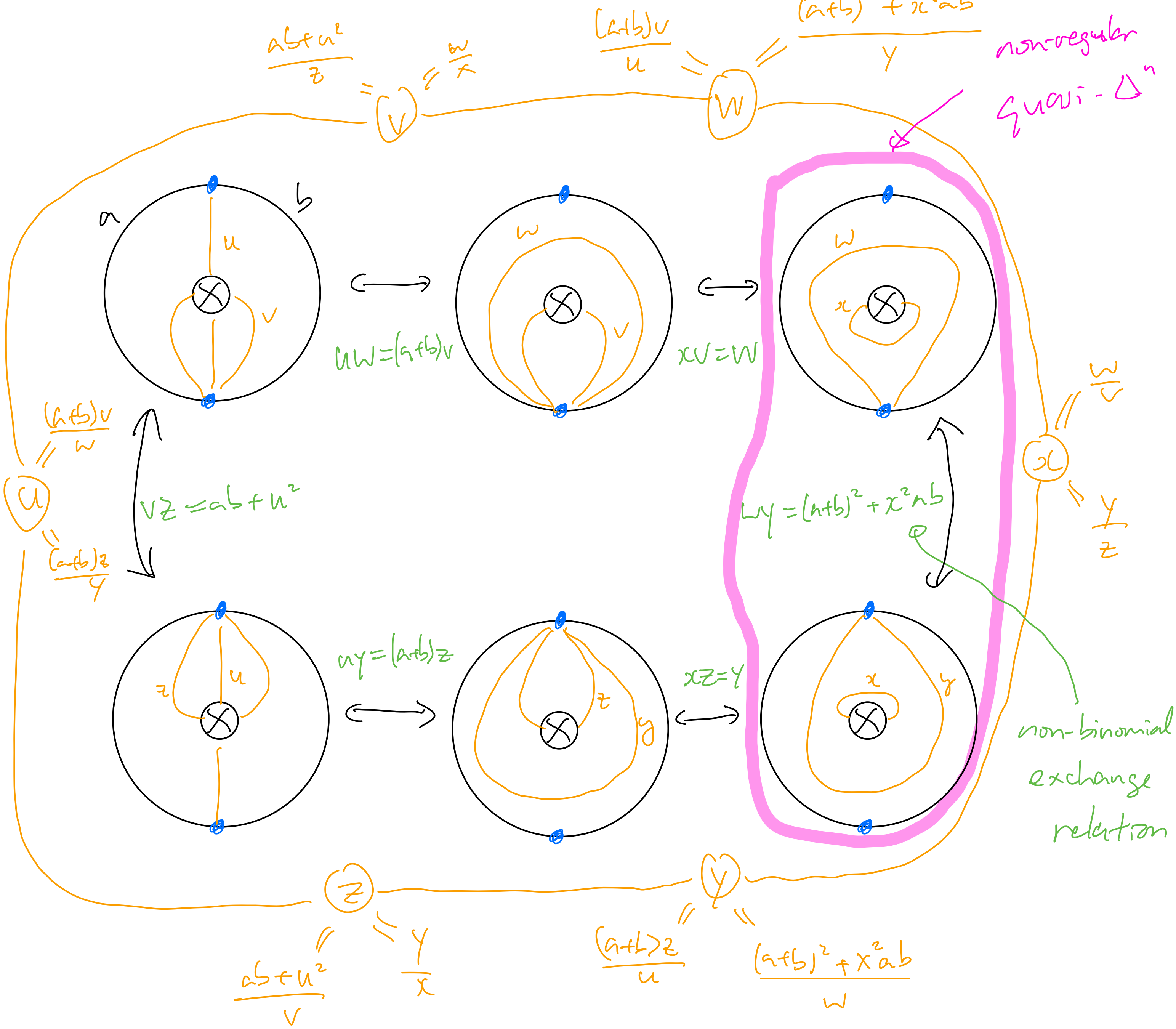


Flip graph := $\begin{cases} \text{vx's} = \text{quasi-triang}^n\text{'s} \\ \text{edge} \leftrightarrow \text{flip} \end{cases}$

Quasi-arc complex := simplicial complex "dual" to flip graph

i.e. $\begin{cases} \text{vx's} = \text{quasi-arcs} \\ \text{faces} = \text{NC sets of arcs} \\ \text{facets} = \text{max. NC sets} = \text{quasi-triang}^n\text{'s} \end{cases}$

e.g. Quasi-arc cpx + flip graph of M_2



§§2 Detour: Relation to cluster algebras

Fomin-Shapiro-Thurston, Fock-Goncharov, Gekhtman-Shapiro-Vainshtein :

$\mathcal{S}$: orientable

"=" moduli of decorated twisted SL_2-local sys

(c.f. Ishibashi's talk for the cluster Poisson side)

$\Rightarrow$ $\exists$ cluster algebra $\mathcal{A}_{(\mathcal{S}, \mathbb{M})}$

Teichmüller theory

where
- cluster var $\leftrightarrow$ (λ-length of) arc
- cluster $\leftrightarrow$ triangulation

  μ $\circlearrowleft$ $\qquad$ flip $\circlearrowleft$
- exch. relation = Ptolemy relation

non-orientable version

Dupont-Palesi

Quasi-cluster algebra $\mathcal{A}_{(\mathcal{S}, \mathbb{M})}$ $\leftarrow$
- λ-length of q. arc
- q. "triang" $\circlearrowleft$ flip
- Ptolemy relation & "quadratic relation"

(= FST's cluster algebra $\mathcal{A}_{(\mathcal{S}, \mathbb{M})}$ when $\mathcal{S}$ orientable)

See also: [Wilson]

for
- $\cong$ LP algebras,
- positivity, etc.

§3 Categorification

§3.1 Motivation

$(\mathbb{S},\mathbb{M})$: orientable

$$\rightsquigarrow \mathcal{A}(\mathbb{S},\mathbb{M}) \xrightarrow{\text{additive categorify (naive)}} \mathcal{C}(\mathbb{S},\mathbb{M}) \quad \text{cluster category}$$

[CCS, BMRRT, Amiot, Keller+...]

(2-CY) triangulated cat.

$$\begin{matrix} \mathcal{A}(\mathbb{S},\mathbb{M}) \\ \cup \\ \{\text{cluster variables}\} \hookrightarrow \mathcal{A}(\mathbb{S},\mathbb{M}) \longleftrightarrow \{\text{indec. rigid obj's}\}/\cong \end{matrix}$$

$\exists$ cluster character

[Caldero-Chamboton, Palu, ...]

rigid: $\mathrm{Ext}^1(M,M)=0$

$\cong \mathrm{Hom}_{\mathcal{C}}(M,M[1])$

inducing:

$$\{\text{clusters}\} \hookrightarrow \{\text{triang}^n\} \longleftrightarrow \{\boxed{\text{basic}}\ \text{cluster-tilting obj. (CTO)}\}/\cong$$

basic: indec direct summands are pairwise non-isom.

"$\Leftrightarrow$" max rigid obj.

require 2-CY in order to use this def^n

Q: Can be done also for quasi-cluster alg?

Today: $\longleftrightarrow$ part for NoS's.

§3.2 Orientable surface to cluster cat. / f.d. algebra

⚠ From now on, $\Bbbk = \mathbb{C}$ (or $\Bbbk = \bar{\Bbbk}$ with char $\Bbbk \neq 2$)

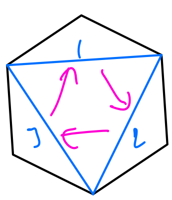
Key points:

- orientable $(\tilde{S},\tilde{M}) \to$ category $\mathcal{C}_{(\tilde{S},\tilde{M})}$
- $\tilde{T} \to$ QP (Q,W) and a f.d. algebra $\mathrm{Jac}(Q,W)$
- curves (+...) $\to$ indec obj's
- crossings $\to$ extensions (∴ No crossing $\iff$ rigidity)

$Q_0 = \{$internal arcs of $\tilde{T}\}$

$Q_1 =$ CCW oriented angles of triangles of $\tilde{T}$

$$W = \sum_{\substack{\text{internal}\\ \text{triangles}}} \triangle$$

e.g. $(\tilde{S},\tilde{M})$  $\tilde{T}$

$$Q = \begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd}, \quad W = abc$$

- For convenience, call such a QP of surface type.
- QP $(Q,W) \to$ Jacobian algebra $\mathrm{Jac}(Q,W) := \Bbbk Q / (\partial_a W)$

e.g. above: $\mathrm{Jac} = \begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd} \Big/ (ab, bc, ca)$

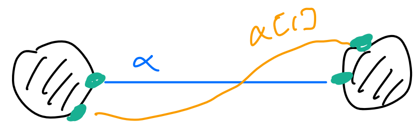
• $\exists$ Hom-finite KS 2-CY triang. cat. $\mathcal{C} = \mathcal{C}_{(\tilde{\mathbb{S}},\tilde{\mathbb{M}})}$

s.t. $\forall$ triangulation $\tilde{T}$ of $(\tilde{\mathbb{S}},\tilde{\mathbb{M}})$, $\exists$ CTO $\tilde{T}$

with $\mathrm{End}_{\mathcal{C}}(\tilde{T}) \cong \mathrm{Jac}(Q,W)$ & is a f.d. gentle algebra

See [Brustle-Zhang] for details

c.f. Ikeda's talk



shift of $\mathcal{C}_{(\tilde{\mathbb{S}},\tilde{\mathbb{M}})}$ (= CW rotation of α)

• indec $\mathcal{C}_{(\tilde{\mathbb{S}},\tilde{\mathbb{M}})} \overset{1:1}{\longleftrightarrow} \left\{ \alpha[1] \;\middle|\; \begin{array}{l} \alpha : \text{arc} \\ \text{in } \tilde{T} \end{array} \right\} \sqcup \underbrace{\text{indec } \mathrm{Jac}(Q,W)\text{-mod}}_{}$

$= \{\text{strings}\} \sqcup \boxed{\{\text{band}\}}$

c.f. Ikeda's talk

$\hookrightarrow \mathcal{C}_{nc}(\tilde{\mathbb{S}},\tilde{\mathbb{M}}) \sqcup \left(\mathcal{C}_{cc}(\tilde{\mathbb{S}},\tilde{\mathbb{M}}) \times k^{\times}\right)$

$\cup|$

$\mathcal{A}(\tilde{\mathbb{S}},\tilde{\mathbb{M}})$

comes from local system on S^1

$k[T,T^{-1}]$

• non-crossing between arcs $\Longleftrightarrow$ rigidity

i.e. $\mathrm{cross}(\alpha,\beta) = \emptyset \iff \mathrm{Ext}^1_{\mathcal{C}}(X,Y) = 0$

$\left(\underset{\mathcal{C} \text{ is 2-CY}}{\Longleftrightarrow} \mathrm{Ext}^1_{\mathcal{C}}(Y,X) = 0 \right)$

§3.3 QP with involution

Def. • Q: quiver

An involution $\sigma : Q \to Q$

$:\Longleftrightarrow$ $\sigma = (\sigma_0 : Q_0 \to Q_0,\ \sigma_1 : Q_1 \to Q_1)$

s.t. $\begin{cases} \sigma^2 = 1 \\ \sigma(v \xrightarrow{\alpha} w) = \left(\sigma(v) \xleftarrow{\sigma(\alpha)} \sigma(w)\right) \end{cases}$

In [Derksen-Weyman] language: (Q,σ) is a symmetric quiver

N.B. Specifying involution σ $\Leftrightarrow$ Specifying $kQ \overset{\text{alg}}{\xrightarrow{\sim}} kQ^{op}$

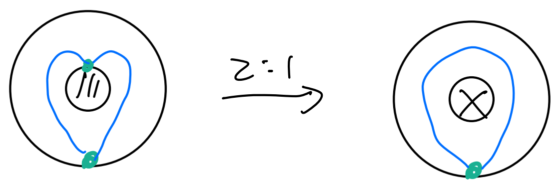
• An involution on a QP (Q,W).

$:\Longleftrightarrow$ an involution σ on Q s.t. $\sigma W = W$

• A ring with involution is a pair (R,σ) of ring R + a ring-antiautomorphism σ of order 2

e.g. • $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet}$ $\quad \sigma : \begin{matrix} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \alpha \end{matrix}$

• $\overset{1}{\bullet} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \overset{2}{\bullet}$ $\quad \sigma : \begin{matrix} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \beta \end{matrix}$



• $Q = 1 \rightleftarrows \cdots \rightleftarrows n$, $I =$ (mesh relation)

$\Rightarrow$ $\sigma : k \longleftrightarrow n-k+1$ $\quad$ involution that "defines" ("but $\neq$") the Nakayama auto on $\Pi(A_n)$.

$(\rightarrow) \longleftrightarrow (\rightarrow)$

$(\leftarrow) \longleftrightarrow (\leftarrow)$

§3.4) Main result

$(\mathbb{S}, \mathbb{M}; T)$: NoS & Triangulation T

$\uparrow 2:1$

$(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}}; \widetilde{T})$: double cover & lift $\widetilde{T}$ of triang^n

$\rightsquigarrow (Q, W)$

For simplicity : $(\widetilde{\mathbb{S}, \mathbb{M}})$

$\Rightarrow \exists\, \sigma \in \mathrm{Aut}_-(\widetilde{\mathbb{S}, \mathbb{M}})$: $(\mathbb{S}, \mathbb{M}) = (\widetilde{\mathbb{S}, \mathbb{M}})/\sigma$

Obs : This induces an involution $\sigma : (Q, W) \to (Q, W)$

Moreover, σ is fixed-point free (FPF)

$\sigma(v) \neq v \quad \forall v \in Q_0$

$\sigma(\alpha) \neq \alpha \quad \forall v \in Q_1$

Prop

$$\left\{ \begin{array}{c} \text{triangulations} \\ \text{of NoS.} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} (Q, W; \sigma) \text{ s.t. } (Q, W) \text{ is} \\ \text{of surface type } \& \ \sigma \text{ is a FPF involution} \end{array} \right\}$$

$$(\mathbb{S}, \mathbb{M}; T) \longmapsto (Q, W; \sigma)$$

Rmk Argument works for dissections vs locally gentle quivers.

Categorical meaning of the $\sigma \in \mathrm{Aut}_-(\widetilde{(\mathbb{S},\mathbb{M})})$:

<u>Prop</u> [BM-C-W]

Notation as before; $\Lambda := \mathrm{Jac}(Q,W)$

$\nabla^2 \cong \mathrm{Id}$

$\Rightarrow$ $\exists$ (exact contravar.) <u>duality</u> functor ∇

$$\begin{tikzcd}
\mathcal{C}_{\widetilde{(\mathbb{S},\mathbb{M})}} \arrow[r, "\nabla", "\sim"'] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}_{\widetilde{(\mathbb{S},\mathbb{M})}} \arrow[d, "\wr"] \\
\mathrm{mod}\,\Lambda \arrow[r, "\nabla", "\sim"'] & \mathrm{mod}\,\Lambda
\end{tikzcd}$$

s.t. ① $\forall \gamma \in \mathbb{C}_{nc}(\widetilde{\mathbb{S},\mathbb{M}})$

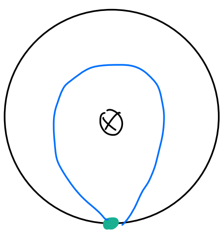
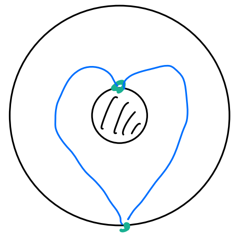
$\Rightarrow$ $\{\gamma, \nabla(\gamma)\}$ = lift of some $\bar{\gamma} \in \mathbb{C}_{nc}(\mathbb{S},\mathbb{M})$

② $\forall \gamma \in \mathbb{C}_{cc}(\widetilde{\mathbb{S},\mathbb{M}})$
$\lambda \in \mathbb{k}^\times$

$\Rightarrow$ $\nabla((\gamma,\lambda)) \cong (\gamma', \lambda^{-1})$

with $\{\gamma, \gamma'\}$ = lift of some $\bar{\gamma} \in \mathbb{C}_{cc}(\mathbb{S},\mathbb{M})$

e.g.

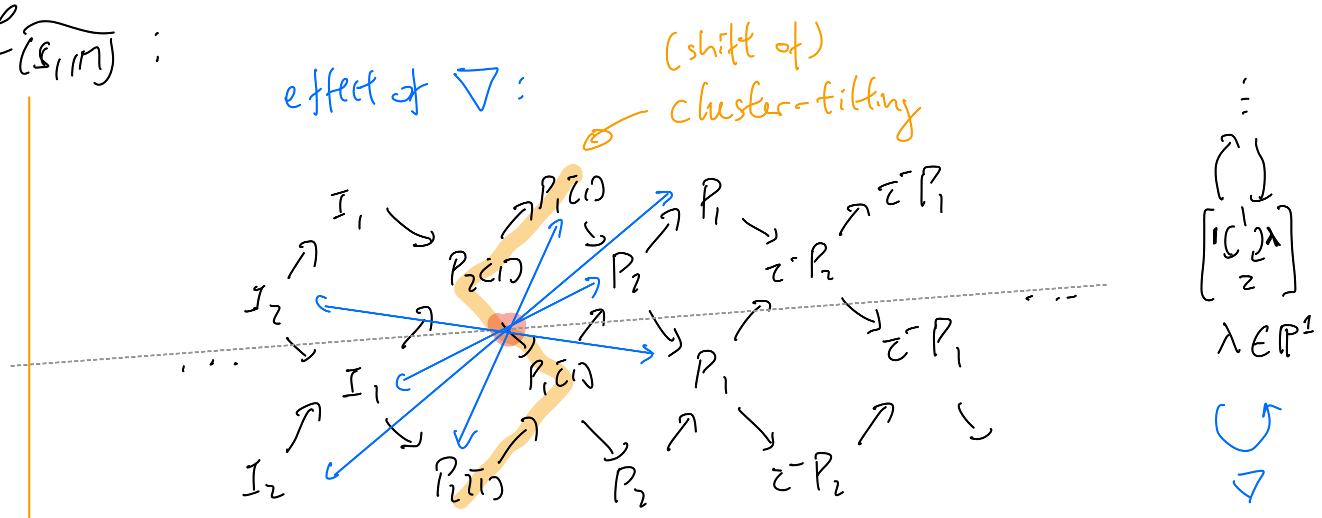
$(\mathbb{S}, \mathbb{M}; T)$:  $\xleftarrow{2:1}$ $(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}}, \widetilde{T})$: 

$\underbrace{\mathbb{S}, \mathbb{M}}_{= M_1}$

$\Lambda = k\left(1 \overset{a}{\underset{b}{\rightrightarrows}} 2\right)$

$\sigma : 1 \leftrightarrow 2, \quad a \leftrightarrow b$

$\mathcal{C}_{(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}})}$:



Thm [BM-C-W]

$(\mathbb{S}, \mathbb{M})$: non-orientable, unpunctured

$\mathcal{C} := \mathcal{C}_{(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}})}$ cluster cat. of the double cover $(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}})$ of $(\mathbb{S}, \mathbb{M})$

$\Rightarrow \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } (\mathbb{S}, \mathbb{M}) \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{basic CTO's of } \mathcal{C} \\ \text{that are } \underline{\text{self-dual}} \end{array} \right\} =: \mathrm{CT}^{\nabla}(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}})$

i.e. $\nabla M \cong M$

Moreover, this bijection is compatible with mutations.

Rmk Mutation RHS is given by composing two mutations of "dual" direct summands : $\mu_{\nabla(\alpha)}(\mu_\alpha(\widetilde{T}))$.

<u>Rem</u> Above Prop & Thm

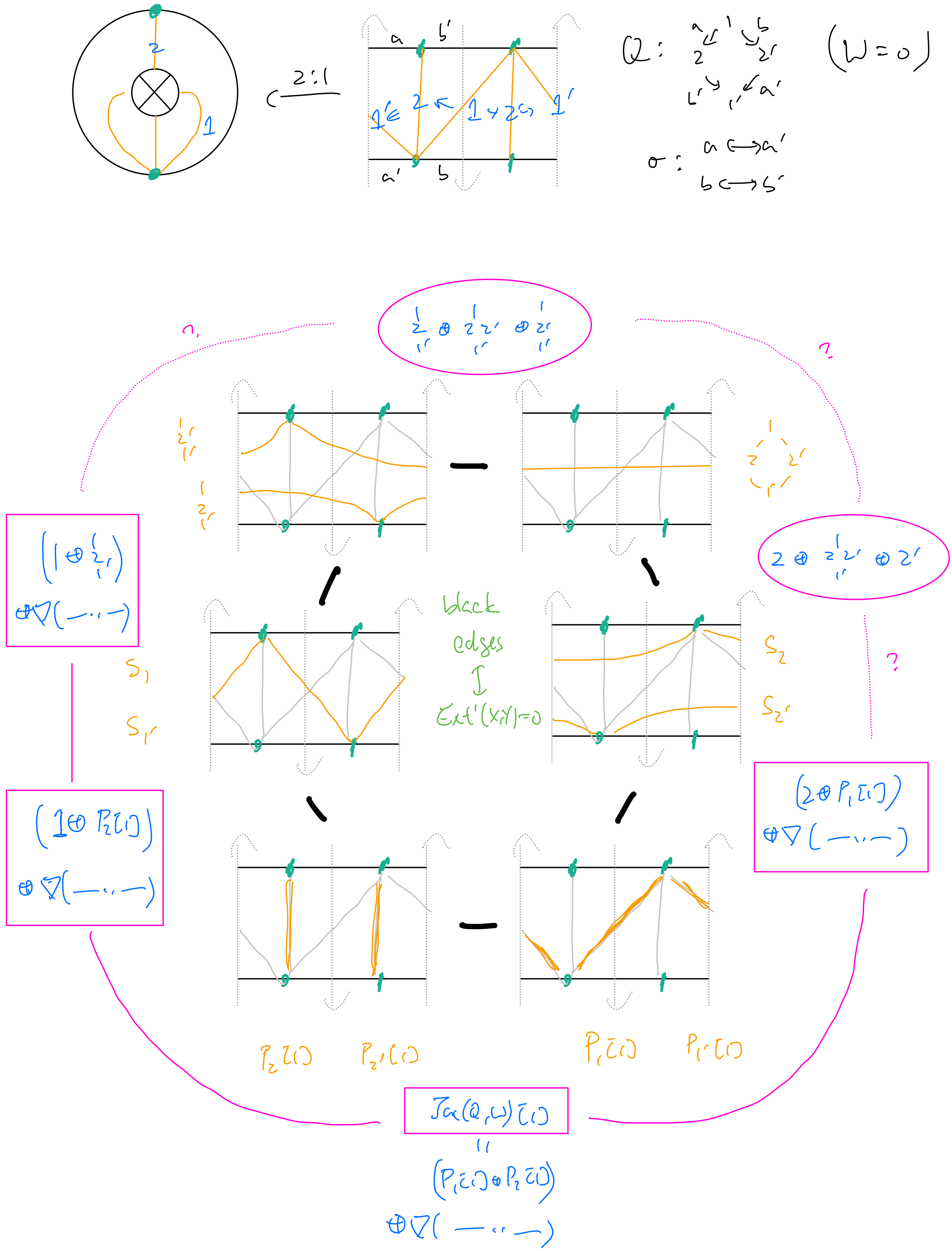
<u>should</u> work for any (A, σ) : gentle alg + FPF involution

by replacing $\mathcal{C}$ with the extriangulated cat. $K^{[-1,0]}(\mathrm{proj}\, A)$,

& CTO by ($\mathbb{E}$-) tilting

& triang^n by (pair of dual) dissections

(likewise for quasi-triang^n's & their categorical analogue)



2:1
Q:
(W=0)
σ: a ⟷ a'
b ⟷ b'
black edges
Ext¹(X,Y)=0
S1
S1'
S2
S2'
P2[1]
P2'[1]
P1[1]
P1'[1]
Jac(Q,W)[1]
(P1[1] ⊕ P2[1])
⊕ ∇(— ,, —)
(1 ⊕ P2[1])
(2 ⊕ P1[1])
?

§§4 Symmetric rep^n's

(⚠ char $k \neq 2$ is important here)

So far:

- arc on $(\Sigma, \mathbb{M})$ $\longleftrightarrow$ pair (γ, γ') of obj's w/ $\nabla\gamma = \gamma'$

$\underbrace{\text{arc on } (\Sigma,\mathbb{M})}_{\text{"index"}}$ $\longleftrightarrow$ pair $\underbrace{(\gamma, \gamma') \text{ of obj's w/ } \nabla\gamma = \gamma'}_{\text{not index}}$

WANT: "∇-orbit" as index objects (+ special treatment to ∇-fixed pt.)

To fix these, use ~~symmetric representations~~ (= ε-rep^n $\varepsilon \in \{\pm 1\}$)

over (A, σ): ring with involution.

[Boos-Cerulli Irelli]

orthogonal rep^n ($\varepsilon = +1$)

symplectic rep^n ($\varepsilon = -1$)

Recall: ε-bilinear form :$\Leftrightarrow$ $\begin{cases} \text{symmetric bil. if } \varepsilon = +1, \\ \text{antisymm. bil. if } \varepsilon = -1. \end{cases}$

"Def" An ε-representation over (A, σ)

:$\Leftrightarrow$ ordinary representation + non-degen. ε-bilinear form with "compatible" A-action

$\Rightarrow$ ① $\begin{matrix} M_i \\ \downarrow M_\alpha \\ M_j \end{matrix}$ and $-\begin{pmatrix} M_{\sigma(i)} \\ \uparrow M_{\sigma(\alpha)} \\ M_{\sigma(j)} \end{pmatrix}$ are adjoint w.r.t. ε-form $\forall (i \xrightarrow{\alpha} j) \in Q_1$

② $\underline{\dim} M$ symm w.r.t. σ

③ M is self-dual as ordinary rep^n

☆ Indecomposability makes sense for ε-rep^n's.

Prop [Derksen-Weyman, Boos-Cerelli Irelli] (char $k \neq 2$ is needed)

f.d. only

M: indec. ε-rep^n / (A, σ): ring with involution

$\Rightarrow$ $\exists\, \overline{M}$: indec A-module s.t. exactly one of the following holds

a) $\nabla\overline{M} \not\cong \overline{M}$, $\quad M = \overline{M} \oplus \nabla\overline{M}$ as A-module $\rightarrow$ M split

b1) $\nabla\overline{M} \cong \overline{M}$, $\quad M = \overline{M} \oplus \nabla\overline{M}$ —— " —— $\rightarrow$ M ramified

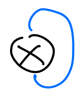
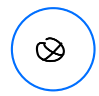
b2) $\nabla\overline{M} \cong \overline{M}$, $\quad M = \overline{M}$ —— " —— $\rightarrow$ M type I

"Slogan" (for people familiar with skew-group ring setting):

ε-rep^n $\approx$ anti-version of rep^n over skew-gp ring $A * \mathbb{Z}/2$

Notation : $\lambda \in \Bbbk^\times$, $[\lambda]$ elt. of $\Bbbk^\times / \lambda \sim \lambda^{-1}$

Def^n : γ : closed curve in $(\mathbb{S}, M)$

is · special $:\Longleftrightarrow$ $\gamma =$  or 

1-sided ... 2-sided

(Assuming $\exists$ min. # of $\otimes$ in the pictorial representation of $\mathbb{S}$)

$\Downarrow$

· primitive $:\Longleftrightarrow$ $\gamma \neq \omega^n$ some $n > 1$,

ω : closed curve

Thm [BM-C-W]

$(\mathbb{S}, M)$: non-orientable, unpunc. ; T : triang.^{n} of $(\mathbb{S}, M)$.

A : Jacobian alg of surface type assoc. to $\widehat{T}$ = double cover of T

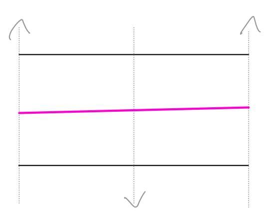
M : indec ε-rep^n / A

$\Rightarrow$ M can be distinguished by exactly one of the following :

- γ : non-closed curves on $(\mathbb{S}, M)$
- $(\gamma^n, [\lambda])$ where γ : primitive non-special cc, $\lambda \in \Bbbk^\times$
  or γ : special cc, $\lambda \in \Bbbk^\times \setminus \{\pm 1\}$

  split type

- $(\gamma^n, -\varepsilon)$ where γ : special 2-sided cc $\leftarrow$ ramified type
- (γ^n, ε) where γ : special 1-sided cc $\leftarrow$ type I

 vs.