Topics in Mathematical Science V
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From quiver to quasi-hereditary algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M, +)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have*

- $m \cdot 1 = m$,
- $(m + n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r + s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $f(R)$ being in the center $Z(A) := \{z \in A \mid za = az \ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_\Bbbk A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the characteristic of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the direct product of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Definition 1.7. *A map $f : M \to N$ between right R-modules M, N is a homomorphism if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\operatorname{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$.*

Lemma 1.8. *$\operatorname{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \operatorname{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\operatorname{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.9. *$\operatorname{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the endomorphism ring of M.*

Lemma 1.10. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\operatorname{Hom}_A(M, N)$ is also an R-module (hence, $\operatorname{End}_R(M)$ is an R-algebra).*

Example 1.11. *$A \cong \operatorname{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra).*

Exercise 1.12. *Recall that R^{op} is the opposite ring of R, whose underlying set is the same as that of R with multiplication $(a \cdot^{\mathrm{op}} b) := b \cdot a$. A representation of R is a ring homomorphism*

$$\rho : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(M), \qquad r \mapsto \rho_r,$$

for some abelian group $(M, +)$. A homomorphism $f : \rho_M \to \rho_N$ of representations $\rho_M : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(M), \rho_N : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(N)$ given by an abelian group homomorphism $f : M \to N$ that intertwines R-action, i.e. $\rho_N(r) \circ f = f \circ \rho_M(r)$ for all $r \in R$.

Eplain why a representation of R is equivalent to a right R-module; and why homomorphisms correspond.

2 Indecomposable modules and Krull-Schmidt property

Recall that an R-module M is finitely generated if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x \in X} x r_x$ for some $r_x \in R$.

Notation. *We write* $\mathsf{mod}\, A$ *for the collection of all finitely generated right* A*-modules.*

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \ldots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + N_{\hat{j}} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written* <u>*uniquely*</u> *as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be unique if for any other decomposition $M = \bigoplus_{j=1}^n N_j$, we have $n = m$ and the N_j's are permutation of the M_i's.

Convention. *We write $(n_1, \ldots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose R_R is indecomposable as an R-module. Then the free module $R \oplus R \oplus \cdots \oplus R$ with R copies of R is a decomposition of R^n.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1 \le j \le n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \ge 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. By definition $(av_1 + bv_2)x = (a+b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules; a contradiction. Hence, V must be indecomposable.

A natural question is to ask when a decomposition of modules, if it exists, is unique up to permuting the direct summands.

Definition 2.5. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^m M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^n N_j$ implies that $m = n$ and there is a permutation σ such* *that* $M_i \cong N_{\sigma(i)}$ *for all* $1 \leq i \leq m$. $\mathsf{mod}\, A$ *is said to be Krull-Schmidt if every finitely generated* A*-module* M *admits a unique indecomposable decomposition.*

Theorem 2.6. *For a finite-dimensional algebra* A, $\mathsf{mod}\, A$ *is Krull-Schmidt.*

Remark 2.7. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Proposition 2.8. *There is a canonical* R*-module isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) & \xrightarrow{\cong} & \bigoplus_{i,j} \mathrm{Hom}_A(M_j, N_i) \\ f & \longmapsto & (\pi_i f \iota_j)_{i,j} \end{array}$$

where $\iota_j : N_j \to \bigoplus_j N_j$ *is the canonical inclusion for all* j *and* $\pi_i : \bigoplus_i M_i \to M_i$ *is the canonical projection for all* i.

One can think of the right-hand space above as the space of m-by-n matrix with entries in each corresponding Hom-space.

3 Extra: Krull-Schmidt theorem

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent.

Lemma 3.1. *A non-zero A-module M is indecomposable if, and only if, the endomorphism algebra* $\mathrm{End}_A(M)$ *does not contain any idempotents except* 0 *and* id_M*.*

Proof $\underline{\Leftarrow}$: Suppose $M = U \oplus V$. Then we have

$$\text{a projection map } \pi_W : M \twoheadrightarrow W,$$
$$\text{and an inclusion map } \iota_W : W \hookrightarrow M,$$

for $W \in \{U, V\}$. Both of these are clearly A-module homomorphisms. Now $e_W := \iota_W \pi_W$ is an endomorphism of M with $e_V = \mathrm{id}_M - e_U$. Since any $m \in M$ can be written as $u + v$ for $u \in U$ and $v \in V$, we have

$$e_V^2(m) = e_V^2(u + v) = e_V^2(v) = v = e_V(m);$$

and likewise for e_W, so we have idempotents different from 0 and id_M when both U and V are non-zero.

$\underline{\Rightarrow}$: Suppose that M is indecomposable, and $e \in \mathrm{End}_A(M)$ is an idempotent. Note that

$$(\mathrm{id}_M - e)^2 = \mathrm{id}_M - e \cdot \mathrm{id}_M - \mathrm{id}_M \cdot e + e^2 = \mathrm{id}_M - 2e + e = \mathrm{id}_M - e$$

is also an idempotent and $\mathrm{id}_M = e + (\mathrm{id}_M - e)$. So we have $M = e(M) + (\mathrm{id}_M - e)(M)$. We want to show that $M = e(M) \oplus (\mathrm{id}_M - e)(M)$, i.e. $e(M) \cap (\mathrm{id}_M - e)(M) = 0$. Indeed, $x \in e(M) \cap (\mathrm{id}_M - e)(M)$ means that we have $e(m) = x = (\mathrm{id}_M - e)(m')$ for some $m, m' \in M$, and so

$$x = e(m) = e^2(m) = e((\mathrm{id}_M - e)(m')) = (e(\mathrm{id}_M - e))(m') = (e - e^2)(m') = 0(m') = 0,$$

as required.

Since M is indecomposable, one of $e(M)$ or $(\mathrm{id}_M - e)(M)$ is zero. In the former case, we get $e = 0$; whereas the latter case yields $\mathrm{id}_M = e$; as required. □

The following is one of the main reasons why we like to consider finite-dimensional (or finite generated) modules over finite-dimensional $\Bbbk$-algebras.

Lemma 3.2 (Fitting's lemma (special version)). *Let M be a finite-dimensional A-module of a finite-dimensional $\Bbbk$-algebra, and $f \in \mathrm{End}_A(M)$. Then there exists $n \geq 1$ such that* $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$*.*

Remark 3.3. The general version for rings requires M to be artinian and noetherian (i.e. ascending and descending chains of submodules stabilises).

We omit the proof to save time. The point is really just take n large enough so that the chains of submodules given by $(\mathrm{Ker}(f^k))_k$ and $(\mathrm{Im}(f^k))_k$ stabilises.

Corollary 3.4. *Let M be a non-zero finite-dimensional A-module. Then M is indecomposable if, and only if, every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*

Proof By Fitting's lemma, for any $f \in \mathrm{End}_A(M)$, we have $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$ for some $n \geq 1$. So indecomposability means that one of these direct summands is is zero. If $\mathrm{Ker}(f^n) = 0$, then f^n is an isomorphism and so is f. If $\mathrm{Im}(f^n) = 0$, then $f^n = 0$ and so f is nilpotent.

Conversely, consider an idempotent endomorphism $e \in \mathrm{End}_A(M)$. The assumption says that e is either an isomorphism or nilpotent.

If e is an isomorphism, then we have $\mathrm{Im}(e) = M$, which means that for every $m \in M$, there is some $m' \in M$ with $e(m) = e^2(m') = e(m') = m$, i.e. $e = \mathrm{id}_M$.

If e is nilpotent, then $e^n = 0$ for some $n \geq 1$, but $e = e^2 = e^3 = \cdots = e^n$, and so $e = 0$.

Hence, an idempotent endomorphism of M is either 0 or id_M, which means that M is indecomposable by Lemma 3.1. □

Definition 3.5. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 3.6. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit right inverses.

Lemma 3.7. *Let A be a finite-dimensional algebra and M be a finite-dimensional A-module. Then the following hold.*

(1) *The following are equivalent.*
   - *A is local (i.e. has a unique maximal right ideal).*
   - *Non-invertible elements of A form a two-sided ideal.*
   - *For any $a \in A$, one of a or $1 - a$ is invertible.*
   - *0 and 1_A are the only idempotents of A.*
   - *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

(2) *M is indecomposable $\Leftrightarrow$ $\mathrm{End}_A(M)$ is local.*

We omit the proof to save time.

Example 3.8. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq 2} \;\middle|\; \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

Theorem 3.9 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.*

Remark 3.10. Some people refer to this result as Krull-Remak-Schmidt theorem.

For proof, interested reader can see lecture notes from last year.

4 Simple modules, Schur's lemma

Definition 4.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 4.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 4.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 4.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 4.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 4.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Proof $\Rightarrow$: $mA \subset S$ is a submodule and contains a non-zero element m, so by simplicity of S we must have $mA = S$.

$\Leftarrow$: Suppose that there is a non-zero submodule $L \subset S$. For a non-zero element $m \in L$, the assumption says that we have $mA \subset L \subset S = mA$, and so $L = S$. $\square$

Let us see how one can find a simple module.

Definition 4.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 4.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module. In particular, if A is finite-dimensional, then every simple A-module is also finite-dimensional.*

Proof Since $S = mA$, the element m defines a surjective A-module homomorphism $f : A_A \to S$ given by $a \mapsto ma$. On the other hand, we have $\mathrm{Ker}(f) = \mathrm{Ann}_A(m)$, and so $A/\mathrm{Ann}_A(m) \cong S$. $\square$

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 4.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. $\square$

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 4.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S,T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T, simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S, so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. $\square$

Example 4.12. *In Example 3.8, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x,y) \mid x, y \in \Bbbk^2\}$ given by 'row vectors'. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0,y) \mid y \in \Bbbk^2\}$.*

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A-action given by

$$w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0,y)$ for some $y \in \Bbbk$ and has to satisfy

$$(0,ya) = (0,y)a = f(wa) = f(w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = f(w) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0,y)c = (0,yc)$$

for any $a, b, c \in \Bbbk$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 4.13. *Consider a semisimple A-module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$ for some i.*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this is an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1 \le i,j \le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \xrightarrow{a_{i,j}} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. $\square$

Example 4.14. *For a tautological example, take $A = \Bbbk$ to be just a field. Then we have a 1-dimensional simple A-module $S = \Bbbk$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\Bbbk)) = \mathrm{Mat}_n(\Bbbk)$. Note that now we have an n-dimensional simple $\mathrm{Mat}_n(\Bbbk)$-module (given by the row vectors).*

5 Quiver and path algebra

Definition 5.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 5.2. *Let Q be a quiver.*

- *A trivial path on Q is a "stationary walk at i", denoted by e_i for some $i \in Q_0$.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2\cdots\alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.*

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right.*

Definition 5.3. *The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying vector space is given by $\bigoplus_{p:\text{paths of } Q} \Bbbk p$, with multiplication given by path concatenation. That is $x \in \Bbbk Q$ is a formal linear combinations of paths on Q.*

Note that $e_ie_j = \delta_{i,j}e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an idempotent of the path algebra $\Bbbk Q$. Moreover, we have an idempotent decomposition

$$1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\Bbbk Q$.

Example 5.4. *Consider the one-looped quiver, a.k.a. Jordan quiver,*

$$Q = \left(\bullet \circlearrowleft^{\alpha} \right)$$

Then $\Bbbk Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\Bbbk Q \cong \Bbbk[x]$.

An *oriented cycle* is a path of the form $v_1 \to v_2 \to \cdots v_r \to v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is *acyclic*.

Proposition 5.5. *$\Bbbk Q$ is finite-dimensional if, and only if, Q is finite acyclic.*

Proof If there is an oriented cycle c, then $c^k \in \Bbbk Q$ for all $k \geq 0$, and so $\Bbbk Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q. □

Example 5.6. *Consider the linearly oriented $\vec{\mathbb{A}}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\Bbbk Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j\alpha_{j+1}\cdots\alpha_k$.

Consider the upper triangular n-by-n matrix ring

$$\begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \;\middle|\; \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\Bbbk Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \le j < k \le n$.

From now on, we will focus in the following setting.

Assumption 5.7. (1) *Quivers are always finite.*

(2) *Modules (and representations) are finitely generated (which is equivalent to finite-dimensional when the algebra is so).*

6 Duality

For a quiver Q, the opposite quiver Q^{op} has the same set of vertices with the reverse direction of arrows, i.e. $Q_0^{\mathrm{op}} = Q_0, Q_1^{\mathrm{op}} = Q_1, s_{Q^{\mathrm{op}}} = t_Q$, and $t_{Q^{\mathrm{op}}} = s_Q$.

Exercise 6.1. *Show that there is a canonical isomorphism $(\Bbbk Q)^{\mathrm{op}} \cong \Bbbk(Q^{\mathrm{op}})$.*

Let M be a finite-dimensional A-module. Then we have a dual space

$$D(M) := M^* := \mathrm{Hom}_{\Bbbk}(M, \Bbbk),$$

which has a natural A^{op}-module structure, namely, $(a \cdot f)(m) := f(ma)$ for any $a \in A, f \in M^*, m \in M$. Moreover, for an A-module homomorphism $\theta : M \to N$, we have also an A^{op}-module homomorphism $\theta^* : N^* \to M^*$ with $\theta^*(f)(m) = f(\theta(m))$.

We note as a fact that D preserves indecomposability of (finite-dimensional) modules. This can be seen using the fact that $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$ and can be upgraded to an algebra isomorphism for the case when $N = M$; then uses characterisation of indecomposable module by local endomorphism ring.

Example 6.2. *The* left *A-module ${}_AA$ yields a right A-module structure on $D(A)$. More generally, suppose we have a left ideal Ae of A for some element $e \in A$, then $D(Ae)$ is a right ideal of A.*

Remark 6.3. There is another natural duality, which we will not used, between $\mathsf{mod}\, A$ and $\mathsf{mod}\, A^{\mathrm{op}}$ given by sending M to $\mathrm{Hom}_A(M, A)$. In general, this duality is different from the $\Bbbk$-linear dual unless A is a so-called *symmetric algebra*; interested reader can read lecture notes from last year.

7 Representations of quiver

Definition 7.1. *A $\Bbbk$-linear representation of Q is a datum $(\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$ where M_i is a $\Bbbk$-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is K-linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional if $\dim_{\Bbbk} M_i < \infty$ for all $i \in Q_0$.

Notation. *For a representation M of Q, we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

It is easy to notice that every representation of Q is equivalent to a $\Bbbk Q$-module, namely,

$$\text{representation } (\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1}) \leftrightarrow \begin{array}{c} \Bbbk Q\text{-module } \prod_{i \in Q_0} M_i \\ \text{s.t. } \sum_{p:\text{path}} \lambda_p p \text{ acts as } \sum_p \lambda_p M_p. \end{array}$$

Example 7.2 (Simple). *For $x \in Q_0$, denote by S_x (or $S(x)$) the representation given by putting a 1-dimensional space on x, zero on all other vertices, and zero on all arrows. This corresponds to a 1-dimensional $\Bbbk Q$-module and so we call it the simple at x.*

Note: at this stage, it is not clear if these are all the simple $\Bbbk Q$-modules (up to isomorphism) yet.

Example 7.3 (Projective). *For $x \in Q_0$, denote by P_x (or $P(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=x,\\ t(p)=y}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p\alpha=q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the projective at x. This corresponds to the right ideal $e_x\Bbbk Q$ of $\Bbbk Q$.

Example 7.4 (Injective). *Dual to the projective module construction, for $x \in Q_0$, denote by I_x (or $I(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=y,\\ t(p)=x}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p=\alpha q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the injective at x. This corresponds to the dual of the left ideal generated by e_x, i.e. $D(\Bbbk Q e_x)$.

Example 7.5. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to \Bbbk \xrightarrow{\text{id}} \to \cdots \xrightarrow{\text{id}} \Bbbk \to 0 \to \cdots \to 0$$

with a copy of $\Bbbk$ on vertices $i, i+1, \ldots, j$ is the uniserial $\Bbbk Q$-module corresponding to the column space (under the isomorphism of $\Bbbk Q$ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \leq k \leq j$.

Example 7.6. *Let Q be the Jordan quiver with unique arrow α. Then a representation of Q is nothing but an n-dimensional vector space equipped with a linear endomorphism, equivalently, an n-by-n matrix.*

Definition 7.7. *A homomorphism $f : M \to N$ of ($\Bbbk$-linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram*

$$\begin{tikzcd}
M_i \arrow[r, "f_i"] \arrow[d, "M_\alpha"'] & N_i \arrow[d, "N_\alpha"] \\
M_j \arrow[r, "f_j"'] & N_j
\end{tikzcd}$$

for all arrows $\alpha : i \to j$ in Q.

A homomorphism $f = (f_i)_{i\in Q_0} : M \to N$ of quiver representations is injective, resp. surjective, resp. an isomorphism, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 7.8. *Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n-by-n matrix M_α. By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α. If we assume $\Bbbk$ is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α. That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \ldots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m-by-m Jordan block with eigenvalue $\lambda \in \Bbbk$.*

Proposition 7.9. *There is an isomorphism between the category of representations of Q and* $\mathsf{mod}\,\Bbbk Q$, *where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i\in Q_0} M_i$ with $\Bbbk Q$-action given by (linear combinations of compositions of) M_α's, and isomorphism classes of Q-representations correspond to isomorphism classes of $\Bbbk Q$-modules.*

8 Idempotents

Recall that an *idempotent* of an algebra A is an element x with $x^2 = x$.

The right A-modules of the form eA and $D(Ae)$ for an idempotent $e \in A$ are of central importance in representation theory and in homological algebra.

Lemma 8.1. *The the following hold for any idempotent $e \in A$.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(eA, M) \cong Me$ *as a $\Bbbk$-vector space for all $M \in A\,\mathsf{mod}$.*

(2) *There is an isomorphism of rings $\mathrm{End}_A(eA) \cong eAe$.*

Proof For (1), check that $\mathrm{Hom}_A(eA, M) \ni f \mapsto f(e) = f(1)e \in Me$ defines a $\Bbbk$-linear map with inverse $me \mapsto (ea \mapsto mea)$. (2) follows from (1) by putting $M = eA$ with straightforward check of correspondence of multiplication on both sides. $\square$

Remark 8.2. Under the isomorphism $A \cong \mathrm{End}_A(A)$, an idempotent e of A corresponds to the 'project to direct summand $P = eA$ endomorphism', i.e. $A \twoheadrightarrow P \hookrightarrow A$. This is compatible with Yoneda lemma (think about this!) which says that there is a vector space isomorphism $fAe \cong \mathrm{Hom}_A(eA, fA)$ for any idempotents e, f.

Lemma 8.3. *For idempotents $e, f \in A$, we have $eA \cong fA$ as right A-module if and only if $f = ueu^{-1}$ for some unit $u \in A^\times$.*

Proof $\Leftarrow$: By Yoneda lemma, an isomorphism $\phi \in \mathrm{Hom}_A(fA, eA)$ corresponds to an element in $x \in eAf \subset A$; likewise an isomorphism $\psi \in \mathrm{Hom}_A((1-f)A, (1-e)A)$ corresponds to $y \in (1-e)A(1-f) \subset A$. Let $x' \in fAe$ and $y' \in (1-f)A(1-e)$ be the elements corresponding to ϕ^{-1} and ψ^{-1} respectively. Since $\phi^{-1}\phi = \mathrm{id}_{eA}$ corresponds to $e \in eAe$, we have

$$x'x = f, xx' = e, y'y = 1-f, yy' = 1-e.$$

Take $u := x + y$ and $v := x' + y'$. Then we have $vu = f + (1-f) = 1$ and $uv = e + (1-e) = 1$. Therefore, u, v are units such that $uf = x = eu$, i.e. $e = ufu^{-1}$ as required.

$\Rightarrow$: The required isomorphism $fA \to eA$ is given by $fa \mapsto eua$. $\square$

Given an idempotent $e = e^2 \in A$ in an algebra A, then eA and $(1-e)A$ are both right ideal of A. Since $e(1-e) = 0 = (1-e)e$, we have $eA \cap (1-e)A = 0$, which means that $A \cong eA \oplus (1-e)A$ as right A-module. In particular, in the setting of the above lemma, we have that $eA \cong fA$ and $(1-e)A \cong (1-f)A$ by Krull-Schmidt property.

Definition 8.4. *Two idempotents e, f are orthogonal if $ef = 0 = fe$. An idempotent e is primitive if $e \neq f + f'$ for some orthogonal (pair of) idempotents f, f'.*

It follows from the definition of primitivity that

eA and $D(Ae)$ are indecomposable A-modules for a primitive idempotent e.

Example 8.5. *The trivial paths e_x for $x \in Q_0$ is (by design) a primitive idempotent of the path algebra $\Bbbk Q$ (where Q is finite but not necessarily acyclic), and $1 = \sum_{x \in Q_0} e_x$ is an orthgonal decomposition of primitive idempotents. Hence, we have a decomposition*

$$\Bbbk Q \cong \bigoplus_{x \in Q_0} e_x \Bbbk Q = \bigoplus_{x \in Q_0} P_x \text{ and } D(\Bbbk Q) \cong \bigoplus_{x \in Q_0} D(\Bbbk Q e_x) \cong \bigoplus_{x \in Q_0} I_x.$$

9 Composition series, Jordan-Hölder Theorem

Definition 9.1. *Let A be a $\Bbbk$-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*
$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$
such that M_i/M_{i-1} is simple for all $1 \le i \le \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \le i \le \ell$ are called the composition factors of the series.

Theorem 9.2 (Jordan-Hölder Theorem). *Any two composition series have the same length and their composition factors are the same up to permutations.*

We omit the proof. The strategy is basically by induction on the length of series.

Remark 9.3. Jordan-Hölder theorem holds as long as a module, regardless of what kind of algebra, has a (finite) composition series; this condition is actually equivalent to saying that it is noetherian and artinian.

Remark 9.4. The Jordan-Hölder theorem may not hold if one relaxes the form of composition factors from simple modules to something else. There are a few active research themes, including one related to quasi-hereditary algebras, that are stemmed from this.

Lemma 9.5. *Let M be a finite-dimensional right A-module. Then M has a composition series.*

Proof Induction on $\dim_\Bbbk M$, at each step choose a maximal submodule (i.e. a submodule whose quotient is simple). □

Example 9.6. *Let $A = \Bbbk\vec{\mathbb{A}}_n$. Then the module $U_{i,j}$ has a composition series*
$$0 \subset U_{j,j} \subset U_{j-1,j} \subset \cdots \subset U_{i+1,j} \subset U_{i,j}$$
with composition factors $S_k = U_{k,j}/U_{k+1,j}$ for $i \le k \le j$. We note that this composition series is actually unique - such kind of modules are called uniserial.

Lemma 9.7. *If $M \in \mathsf{mod}\,A$ and $N \subset M$ is a submodule, then there is a composition series $(M_i)_{0\le i\le \ell}$ so that $N = M_k$ for some $0 \le k \le \ell$.*

Proof N has a composition series, say, of length k, so we take that as the first k terms of the required composition series of M. On the other hand, M/N also has a composition series, and since every submodule of M/N is of the form L/N (for a submodule U of M/N, take $L := \{m \in M \mid m+N \in U\}$; it is routine to check that this is an inverse operation as quotienting N on the submodules of M that contains N), a composition series of M/N is of the form $(L_i/N)_{0\le i\le r}$. Now take $M_{k+i} = L_i$. □

Proposition 9.8. *Suppose A is a $\Bbbk$-algebra such that A_A has a composition series. Then there are only finitely many simple A-modules up to isomorphisms, and they all appear in the form A/I for some A-submodule I of A.*

Note that while this does not require A to be finite-dimensional, it requires A_A to be of finite length (equivalently, noetherian and artinian).

Proof The final clause of the claim is just restating Lemma 4.8: any simple S is given by $A/\operatorname{Ann}_A(m)$ for any non-zero $m \in S$. Now fix such an S and $I := \operatorname{Ann}_A(m)$. Since A has a composition series, I also have one by Lemma 9.7 so that the series ends with $I \subset A$. Since this is possible for any simple S, it follows from Jordan-Hölder theorem that all simple modules other than S must appear as composition factors of I.

Since composition series is a finite chain, there must be finitely many composition factors - hence, the simple modules of A must be finite. □

10 Semisimplicity and Artin-Wedderburn theorem

In order to obtain all (isomorphism classes of) simple A-modules - or equivalently maximal right A ideal (i.e. maximal submodules of A_A) - for a finite-dimensional $\Bbbk$-algebra A, we will use the following.

Definition 10.1. *Let A be a $\Bbbk$-algebra and $M \in \mathsf{mod}\, A$.*

(1) *The (Jacobson) radical $\mathrm{rad}(A)$ (sometimes also written as $J(A)$) of A is the intersection of all maximal right ideals (i.e. maximal A-submodules) of A.*

(2) *A is semisimple if $\mathrm{rad}(A) = 0$.*

Example 10.2. *For $A = \Bbbk Q$ of a finite quiver Q and $x \in Q_0$. The projective P_x at x contains a submodule spanned by all paths starting from x with length at least 1. This is a maximal submodule of P_x since the cokernel of the natural embedding to P_x is a one-dimensional module spanned by the coset of e_x – in particular, this simple module is isomorphic to S_x. Thus, we have $\mathrm{rad}(A) = \Bbbk Q_{\geq 1}$ the submodule of A_A spanned by all paths of length at least 1.*

Example 10.3. *This example shows that we really need composition series on A_A for things to be well-behaved. Let $A = \Bbbk[x]$. Each irreducible polynomial f generates a maximal ideal $(f) \subset \Bbbk[x]$ and so $\mathrm{rad}(A) \subset \bigcap_{f:\ irred.}(f)$. Note that there are infinitely many irreducible polynomials in $\Bbbk[x]$.*

We claim that $\mathrm{rad}(A) = 0$. If, on the contrary, there is some non-zero g in this intersection of ideals, then all irreducible polynomials are factors of g; this is a contradiction as g can only has finite degree, i.e. finitely many irreducible factors.

Proposition 10.4. *Suppose A_A has a composition series. Then the following holds for the Jacobson radical $\mathrm{rad}(A)$.*

- *$\mathrm{rad}(A)$ is the intersection of finitely many maximal right ideals.*
- *$\mathrm{rad}(A)$ is the intersection of all two-sided ideals $\mathrm{Ann}_A(S) := \{a \in A \mid ma = 0 \forall m \in S\}$, in other words*
$$\mathrm{rad}(A) = \{a \in A \mid Sa = 0 \text{ for all simple } S\}.$$
- *$\mathrm{rad}(A)$ is a two-sided ideal of A.*
- *$\mathrm{rad}(A)^\ell = 0$ for ℓ at most the length of A_A.*
- *$(A/\mathrm{rad}(A))_{A/\mathrm{rad}(A)}$ is a semisimple (as a module).*
- *A_A is a semisimple (as a module) if, and only if, $\mathrm{rad}(A) = 0$ (i.e. A semisimple as an algebra).*

Proof omitted. We note that all of these claims do make use of the Jordan-Hölder theorem.

Example 10.5. (1) *Direct product of two semisimple algebras is semisimple.*

(2) *$A = \mathrm{Mat}_n(D)$ with D a division $\Bbbk$-algebra is a semisimple $\Bbbk$-algebra. We have decomposition $A_A \cong V^{\oplus n}$ into n copies of n-dimensional simple module*
$$V = \{(v_i)_{1 \leq i \leq n} \mid v_i \in D\ \forall i\}.$$

(3) *$A := \Bbbk[x]/(x^n)$ is not semisimple for any $n \geq 2$ as it has a non-trivial (unique) maximal ideal $\mathrm{rad}(A) = (x)$.*

Theorem 10.6 (Artin-Wedderburn theorem). *Let A be a finite-dimensional $\Bbbk$-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \ldots, S_r$. Let $D_i := \mathrm{End}_A(S_i)$ be the division $\Bbbk$-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of $\Bbbk$-algebras*
$$A/\mathrm{rad}(A) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r).$$

As before, if we work over algebraically closed field $\Bbbk = \overline{\Bbbk}$, then all the D_i's are just $\Bbbk$.

Proof Let $B := A/\operatorname{rad}(A)$. By definition of $\operatorname{rad}(A)$, the A-module $A/\operatorname{rad}(A)$ is semisimple, and any A-submodule M of $A/\operatorname{rad}(A)$ satisfies $M\operatorname{rad}(A) = 0$. Hence, $M = M/M\operatorname{rad}(A)$ is naturally a B-module and $\operatorname{End}_B(M) \cong \operatorname{End}_A(M)$ (even as algebras!).

By Lemma 8.1, we have $B \cong \operatorname{End}_B(B)$. Since B is semisimple, the B_B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Schur's lemma and its consequence (Lemma 4.10 and Lemma 4.13) that

$$B \cong \operatorname{End}_B(B) \cong \operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_r}(D_r),$$

where $D_i := \operatorname{End}_B(S_i)$ for all $1 \leq i \leq r$. This completes the proof. $\square$

Corollary 10.7. *For any finite-dimensional $\Bbbk$-algebra A, let $\operatorname{Sim}(A)$ be the set of isomorphism-class representatives of simple A-modules. Then there is a one-to-one correspondence*

$$\begin{array}{ccl} \operatorname{Sim}(A) & \xleftrightarrow{\;1:1\;} & \operatorname{Sim}(A/\operatorname{rad}(A)) \\ S & \longmapsto & \overline{S} := S/S\operatorname{rad}(A) \\ & & \quad (= S \textit{ as underlying vector space}) \\ \operatorname{res}T & \longleftarrow & T \end{array}$$

where $\operatorname{res}T$ is the restriction of T along $A \twoheadrightarrow A/\operatorname{rad}(A)$.

Definition 10.8. *The radical of an A-module M is $\operatorname{rad}(M) := M\operatorname{rad}(A)$. In general, take $\operatorname{rad}^0(M) := M$ and denote by $\operatorname{rad}^{k+1}(M) := \operatorname{rad}(\operatorname{rad}^k(M)) = \operatorname{rad}^k(M)\operatorname{rad}(A)$ for all $k \geq 0$.*

Successively taking the radical yields a series:

$$0 \subset \operatorname{rad}^\ell(M) \subset \cdots \subset \operatorname{rad}(M) \subset M$$

This is called the radical series. The quotient $M/\operatorname{rad}(M)$ is called the top of M, and is denoted by $\operatorname{top}(M)$.

Proposition 10.9. *The following hold for $M \in \operatorname{mod} A$.*

(1) $\operatorname{rad}(M)$ *is the intersection of all maximal submodules of M.*

(2) $\operatorname{top}(M) := M/\operatorname{rad}(M)$ *is the maximal semisimple quotient of M.*

(3) $\operatorname{rad}(M \oplus N) = \operatorname{rad}(M) \oplus \operatorname{rad}(N)$.

(4) *(Nakayama's Lemma, special case) For a submodule $N \subset M$, $(N + \operatorname{rad}(M) = M) \Rightarrow N = M$.*

Proof omitted; this follows the same kind of arguments as in the case for $\operatorname{rad}(A)$.

There is a construction dual to $\operatorname{rad}(M)$.

Definition 10.10. *The socle of an A-module M is $\operatorname{soc}(M)$, which is defined as the maximal semisimple submodule of M. More generally, take $\operatorname{soc}^0(M) = 0$ and for $k \geq 0$, let $\operatorname{soc}^{k+1}(M)$ to be the submodule of M generated by the lift of $\operatorname{soc}(M/\operatorname{soc}^k(M)) \subset M/\operatorname{soc}^k(M)$. This yields a series*

$$0 \subset \operatorname{soc}(M) \subset \operatorname{soc}^2(M) \subset \cdots \subset \operatorname{soc}^\ell(M) = M$$

called the socle series of M.

Example 10.11. *Consider a path algebra $\Bbbk Q$ of a finite acyclic (for simplicity) quiver Q, and $x \in Q_0$. The indecomposable injective $I_x = D(\Bbbk Q e_x)$ has a simple socle isomorphic to S_x. Essentially this can be seen by a dual argument in showing $\operatorname{top}(P_x) \cong S_x$.*

Lemma 10.12. *For $M \in \mathsf{mod}\, A$, the socle series and radical series has the same length, and this length is called the* Loewy length *of M.*

Note that the semisimple subquotients in (between the layers of) the socle series and the radical series of a module may not coincide.

Example 10.13. *Let Q be the quiver $1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and consider the projective P_2 which has the form*

$$\Bbbk \xleftarrow{1} \Bbbk \xrightarrow{1} \Bbbk \xrightarrow{1} \Bbbk$$

Then we have radical series

$$0 \subset S_4 = \Bbbk\beta\gamma \overset{S_1 \oplus S_3}{\subset} \mathrm{rad}(P_2) = \Bbbk\alpha + \Bbbk\beta + \Bbbk\beta\gamma \overset{S_2}{\subset} P_2$$

and socle series

$$0 \subset S_2 \oplus S_4 = \Bbbk\alpha + \Bbbk\beta\gamma \overset{S_3}{\subset} \mathrm{rad}(P_2) \subset P_2.$$

11 Bounded path algebra

For general quiver, we loses finite-dimensionality, and so many nice things we explained do not hold any more. To retain finite-dimensionality, we need to consider nice quotients of path algebras.

Definition 11.1. *An ideal $I \lhd \Bbbk Q$ is admissible if $(\Bbbk Q_1)^k \subset I \subset (\Bbbk Q_1)^2$ for some $k \geq 2$, i.e. I is generated by linear combinations of paths of finite length at least 2. The pair (Q, I) is sometimes called bounded quiver. A bounded path algebra or quiver algebra (with relations) is an algebra of the form $\Bbbk Q/I$ for some quiver Q and admissible ideal I.*

Remark 11.2. Admissiblity ensures there is no redundant arrows (which appears if there is a relation like, for example, $\alpha - \beta\gamma \in I$ for some $\alpha \neq \beta, \gamma \in Q_1$) and there is enough vertices (trivial paths may not be primitive if there is a loop x at a vertex with relation $x^2 - x \in I$).

Lemma 11.3. *A bounded path algebra is finite-dimensional.*

Proof There exists a surjective algebra homomorphism $\Bbbk Q/(\Bbbk Q_1)^k \twoheadrightarrow \Bbbk Q/I$; the former is finite-dimensional. $\square$

Example 11.4. *Let Q be the Jordan quiver with unique arrow α. Let I be the ideal of $\Bbbk Q$ generated by α^k for some $k \geq 2$. Then I is an admissible ideal and $\Bbbk Q/I \cong \Bbbk[x]/(x^k)$ is a truncated polynomial ring.*

Definition 11.5. *A representation M of a bounded quiver (Q, I) is a representation $M = (M_i, M_\alpha)_{i,\alpha}$ of Q such that $M_a = 0$ for all $a \in I$; here $M_a := \sum_p \lambda_p M_p$ for $a = \sum_p \lambda_p p$ written as a linear combinations of paths p.*

A homomorphism $f : M \to N$ of representations of (Q, I) is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' action.

As before, representations are really just synonyms of modules.

Lemma 11.6. *A representation of a bounded quiver (Q, I) is equivalent to a $\Bbbk Q/I$-module, and homomorphisms between representations are equivalent to those between $\Bbbk Q/I$-modules.*

We have seen that it is easy to write down the indecomposable decomposition of the free $\Bbbk Q$-module $\Bbbk Q_{\Bbbk Q}$, we would like such nice thing to carry over to bounded path algebras.

Theorem 11.7. *(Idempotent lifting) If I is a nilpotent ideal of A (i.e. $I^n = 0$ for some $n \geq 1$) and $\overline{e} = \overline{e}^2 \in A/I$, then there is a lift $e = e^2 \in A$ of $\overline{e}$, i.e. $\overline{e} = e + I$.*

Proof omitted.

Corollary 11.8. *Let I be an nilpotent ideal in A. Suppose that*

$$1_{A/I} = f_1 + \cdots + f_n$$

for $f_i \in A/I$ are primitive orthogonal idempotents. Then we have

$$1_A = e_1 + \cdots e_n$$

where each $e_i \in A$ is a primitive orthogonal idempotent that lifts f_i.

Notation. *As in the case of path algebra, denote by S_x or $S(x)$ the simple $\Bbbk Q/I$-module given by placing a one-dimensional vector space at vertex $x \in Q_0$ and zero everywhere else.*

Similarly, denote by P_x or $P(x)$ the indecomposable $\Bbbk Q/I$-module $e_x \Bbbk Q/I$. Likewise, by I_x or $I(x)$ the indecomposable $D((\Bbbk Q/I)e_x)$.

Proposition 11.9. *There is a decomposition of A-modules*

$$A_A = \bigoplus_{x\in Q_0} P_x, \text{ and } (DA)_A = \bigoplus_{x\in Q_0} I_x.$$

Moreover, $\{S_x \cong \operatorname{top}(P_x) \cong \operatorname{soc}(I_x) \mid x \in Q_0\}$ *form the complete set of isoclasses representatives of simple A-modules.*

Proof Each arrow $\alpha \in Q_1$ generates a maximal right ideal of A with quotient S_x for $x = s(\alpha)$. So we have $A/\operatorname{rad}(A) \cong \Bbbk Q_0 = \prod_{x\in Q_0} \Bbbk$. As primitive orthogonal decomposition of the identity element of A lifts to that of the identity element of $A/\operatorname{rad}(A)$ by Corollary 11.8, we have e_x primitive, and so P_x and I_x are indecomposable.

The simple A-modules (up to isomorphisms) correspond to those over the semisimple quotient algebra $A/\operatorname{rad}(A)$ by Corollary 10.7. Hence, there are precisely $|Q_0|$ simple modules (up to isomorphism), given by the simple top of P_x, which is also isomorphic to the simple socle of I_x. □

We give a brief justification of why quiver representations provide a good way to construct lots of algebras.

Theorem 11.10. *Suppose* $\Bbbk$ *is algebraically closed. Then every finite-dimensional* $\Bbbk$*-algebra* A *is Morita equivalent to a bounded path algebra* $\Bbbk Q/I$*. More precisely,* $\Bbbk Q/I$ *is given by* $\operatorname{End}_A(\bigoplus_e eA)$ *where* e *varies over the set of representative of equivalence classes of primitive idempotents of* A*.*

We do not explain here the precise meaning of Morita equivalent; it roughly translates to saying that understanding A-modules and homomorphisms between them is equivalently (but not necessarily 'equal to') to understanding modules and homomorphisms between a Morita equivalent bounded path algebra.

Example 11.11. *Let* $A = \operatorname{Mat}_n(\Bbbk)$ *be a matrix ring. Then the elementary matrix* $e := E_{1,1}$ *is a primitive idempotent and* $eA \cong E_{j,j}A$ *for all* $1 \leq j \leq n$*. So* A *is Morita equivalent to* $\Bbbk \cong \Bbbk Q \cong \operatorname{End}_A(eA)$ *where* Q *is a one-vertex-no-arrow quiver.*

Primitive idempotent decomposition, say, $1 = \sum_{i=1}^n e_i$, allows us to write an algebra A in matrix form $(e_iAe_j)_{1\leq i,j\leq n}$, where the 'row spaces' form the indecomposable direct summands e_iA and the dual of the 'column space' form the indecomposable direct summands $D(Ae_i)$. It could be a helpful mental exercise to think about the meaning of $eAe \cong \operatorname{End}_A(eA)$ from Yoneda lemma - this maybe a useful idea to keep in mind when one tries to understand the above theorem.

Module diagram

It is convenient to display the structure of a module via a more combinatorial form (a diagram) – if possible. [1] This is (as of today technology) a better way to display module structure – at least compare to composition series, or lattice diagram of the submodule lattice, or even, quiver representations, in some cases.

Definition 11.12. *Let $M \in \operatorname{mod} A$ be a finite-dimensional A-module for $A = \Bbbk Q/I$ a bounded path algebra. The module diagram is a (directed) graph with vertices labelled by composition factors of M (in particular, there are $\dim_{\Bbbk} Me_x$ many vertices labelled by x), and arrows labelled by those in Q_1 in such a way that $x \xrightarrow{a} y$ if for an arrow $a \in Q$ that sends (the lift of) an element in the composition factor at x to (the lift of) an element in the composition factor at y.*

Module diagram drawn in this way is not invariant under isomorphism. A connected diagram may not even implies indecomposability in general (c.f. Homework 2). Nevertheless, when the algebras or modules are well-behaved, then these diagram provide a very efficient combinatorial way to perform a lot of (linear algebra) calculation.

It is customary to draw the the module diagram flowing from top to bottom; in particular, the top (semisimple quotient) of M is placed on the top of the diagram and the socle at the bottom. We may omit a connecting line if there is no ambiguity.

Example 11.13. *The indecomposable $U_{i,j}$ of $\Bbbk\vec{\mathbb{A}}_n$ is just a column of numbers labelled from i down to j. For a concrete example, the module diagram of $U_{4,6}$ is just $\begin{smallmatrix} 4\\5\\6 \end{smallmatrix}$.*

Example 11.14. *Consider the following bounded quiver:*

$$Q: 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3, \qquad I = (\alpha_1\alpha_2, \beta_1\beta_2, \beta_1\alpha_1 - \alpha_2\beta_2).$$

Then we have

$$P_1 = \begin{smallmatrix} \Bbbk e_1 \\ \Bbbk\alpha_1 \\ \Bbbk\alpha_1\beta_1 \end{smallmatrix} = \begin{smallmatrix} 1\\2\\1 \end{smallmatrix} \quad , \quad P_2 = \begin{smallmatrix} \Bbbk e_1 \\ \Bbbk\alpha_2 \oplus \Bbbk\beta_1 \\ \Bbbk\alpha_2\beta_2 \end{smallmatrix} = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \quad , \quad P_3 = \begin{smallmatrix} \Bbbk e_3 \\ \Bbbk\beta_2 \\ \Bbbk\beta_2\alpha_2 \end{smallmatrix} = \begin{smallmatrix} 3\\2\\3 \end{smallmatrix}$$

Let us consider the two-sided ideal Ae_1A. This is spanned by all paths ('up to I') that passes through the vertex 1. As a right module, we can find its manifestation in the module diagram by picking everything below any appearance of the label 1 – in this case, it is all of P_1 and the $\begin{smallmatrix} 1\\2 \end{smallmatrix}$ part submodule of P_2. In particular, the quotient algebra $(A/Ae_1A)_{A/Ae_1A}$ has module diagram:

$$P_2^{A/Ae_1A} = e_2A/e_2Ae_1A = P_2/P_2e_1A = \begin{smallmatrix} 2\\3 \end{smallmatrix} \quad , \quad P_3^{A/Ae_1A} = P_3/P_3e_1A = P_3^A = \begin{smallmatrix} 3\\2\\3 \end{smallmatrix}$$

The bounded quiver presentation of A/Ae_1A is given by

$$Q: 2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 3, \qquad I = (\alpha\beta).$$

On the other hand, for eAe with $e = e_2 + e_3$, the module diagram is given by removing all composition factors that are not S_2, S_3, i.e.

$$e_2Ae = \begin{smallmatrix} 2\\3\\2 \end{smallmatrix} \quad , \quad e_3Ae = \begin{smallmatrix} 3\\2\\3 \end{smallmatrix}$$

and the bounded quiver presentation of eAe is given by

$$Q: 2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 3, \qquad I = (\alpha\beta\alpha, \beta\alpha\beta).$$

[1] There is no widely agreed name to these diagrams; for convenience, we just call them 'module diagram' in this notes.

12 Snippets of category theory

Some language in category will be convenient – albeit not absolutely necessary.

A *category* is a collection of *objects* along with their *morphisms* $f : X \to Y$, including all *identity morphisms* $\mathrm{id}_X : X \to X$, so that

- morphisms $f : X \to Y$, $g : Y \to Z$ can always be composed $gf : X \to Z$ to get a new morphism,
- in such a way that is associative, i.e. $h(gf) = (hg)f$ for all h, g, f,
- and has left and right unit, i.e. $f\,\mathrm{id}_X = f$ and $\mathrm{id}_Y\, f = f$.

Example 12.1. *We only really use the category* $\mathsf{mod}\,A$ *of (finitely generated) A-modules in this notes. Some results still hold for the category* $\mathsf{Mod}\,A$ *of all A-modules, but let us keep it simple.*

A *functor* $F : \mathsf{mod}\,A \to \mathsf{mod}\,B$ consists of

- an assignment of objects $M \mapsto F(M) \in \mathsf{mod}\,B$ for any $M \in \mathsf{mod}\,A$, and
- an assignment of morphisms $F(f) \in \mathrm{Hom}_B(F(X), F(Y))$ for all $f \in \mathrm{Hom}_A(X, Y)$, such that
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
- either $F(gf) = F(g)F(f)$ or $F(gf) = F(f)F(g)$.

The case when order of morphism composition does not change is called a *covariant* functor, and the other is called a *contravariant* functor. Usually, whenever we say a functor we mean a covariant one.

Functor allows us to change from the representation theory of one algebra to another. The key point is that it preserves identity and compositions.

Example 12.2. *The identity functor* $\mathrm{Id} : \mathsf{mod}\,A \to \mathsf{mod}\,A$ *is the functor given by mapping every module and homomorphism to itself.*

Example 12.3. *The ($\Bbbk$-linear) duality* $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk) : \mathsf{mod}\,A \to \mathsf{mod}\,A^{\mathrm{op}}$ *is a contravariant functor.*

To compare two functors (or compare how a pair of functors is close/far away from the identity functor), one uses *natural transformations*. More precisely, a natural transformation $\eta : F \Rightarrow G$ of functors $F, G : \mathsf{mod}\,A \to \mathsf{mod}\,B$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ such that there is the following commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

If we say that a map $\eta_X : F(X) \to G(X)$ is *natural in X*, then we mean that $\{\eta_X\}_{X \in \mathsf{mod}\,A}$ defines a natural transformation.

A *natural isomorphism* is a natural transformation η such that η_X is an isomorphism for all X; in such a case, we may simply write $F \cong G$ when η is clear from context.

13 Bimodule, tensor and Hom

13.1 Bimodule

Definition 13.1. *Let A, B be two $\Bbbk$-algebras. An A-B-bimodule is a $\Bbbk$-vector space M that has the structure of a left A-module and also the structure of a right B-module, such that $(am)b = a(mb)$ for all $a \in A, b \in B, m \in M$. In such a case, we may write $M \in A \,\mathsf{mod}\, B$ or ${}_AM_B$ to specify M is an A-B-bimodule.*

For simplicity, we assume all bimodules are $\Bbbk$-central, i.e. $\lambda m = m\lambda$ for all $\lambda \in \Bbbk$. We will omit the adjective $\Bbbk$-central from now on.

Example 13.2. *For any algebra A, both A and $D(A)$ are naturally an A-A-bimodule. Note that the right/left module structure on $D(A)$ is induced by the left/right module structure on A. (The direction of action has swapped!)*

Example 13.3. $\mathrm{Hom}_A(X, Y)$ *is naturally a* $\mathrm{End}_A(Y)$-$\mathrm{End}_A(X)$*-bimodule with action given by composition of homomorphisms.*

13.2 Tensor product

Definition 13.4. *Let V, W be finite-dimensional $\Bbbk$-vector space with bases, say, $\mathcal{B}, \mathcal{C}$ respectively. Then the tensor product $V \otimes_\Bbbk W$ (or simplifies to $V \otimes W$ if context is clear) is the finite-dimensional $\Bbbk$-vector space with bases given by*

$$\{v \otimes w \mid v \in \mathcal{B}, w \in \mathcal{C}\}.$$

In particular, note that $\dim_\Bbbk V \otimes W = (\dim_\Bbbk V) \times (\dim_\Bbbk W)$.

Proposition 13.5. *Let A, B be $\Bbbk$-algebras. Then $A \otimes_\Bbbk B$ is also a $\Bbbk$-algebra with multiplication given by extending $(a \otimes b)(a' \otimes b') \mapsto aa' \otimes bb'$ linearly. For $M \in \mathsf{mod}\, A$ and $N \in \mathsf{mod}\, B$, we have $M \otimes_\Bbbk N \in \mathsf{mod}\, A \otimes_\Bbbk B$.*

Proof Routine checking. □

Example 13.6. *Consider $A = (a_{i,j})_{1 \le i,j \le m} \in \mathrm{Mat}_m(\Bbbk)$ and $B \in \mathrm{Mat}_n(\Bbbk)$ and defines (what is sometimes called Kronecker product of matrices)*

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & \ddots & & a_{2,m}B \\ \vdots & & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,m}B \end{pmatrix}.$$

Then we have an isomorphism of algebras

$$\mathrm{Mat}_m(\Bbbk) \otimes_\Bbbk \mathrm{Mat}_n(\Bbbk) \to \mathrm{Mat}_{mn}(\Bbbk), \quad (A, B) \mapsto A \otimes B.$$

Lemma 13.7. *An idempotent $e \in A \otimes_\Bbbk B$ is primitive if and only if $e = e_l \otimes e_r$ for some primitive idempotents $e_l \in A$ and $e_r \in B$. In particular, we have $\mathrm{Sim}(A \otimes_\Bbbk B) = \{S \otimes T \mid S \in \mathrm{Sim}(A), T \in \mathrm{Sim}(B)\}$.*

Note that not all $A \otimes_\Bbbk B$-module is of the form $M \otimes N$.

Example 13.8. *Let $A = \Bbbk[x]/(x^2)$ and $A' := \Bbbk[y]/(y^2)$. Then $B := A \otimes_{\Bbbk} A' = \Bbbk[x,y]/(x^2,y^2)$. Then we have an indecomposable 2-dimensional B-module $V = \Bbbk u + \Bbbk v$ (top $S = B/\operatorname{rad}(B)$ and socle S) where both x, y acts by $u \mapsto v$. This cannot be of the form $M \otimes N$ for some $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} A'$. Indeed, as both x, y acts non-trivially, if $V = M \otimes N$ then both M, N must have dimension at least 2, and so the tensor product has dimension at least 4; but $\dim_{\Bbbk} V = 2$.*

Proposition 13.9. *An $A \otimes_{\Bbbk} B^{\mathrm{op}}$-module is the same as a ($\Bbbk$-central) B-A-bimodule. Moreover, homomorphisms of $A \otimes B^{\mathrm{op}}$-modules correspond to ($\Bbbk$-linear) homomorphisms of B-A-bimodule.*

Definition 13.10. *Let $X \in \operatorname{mod} A$ be a right A-module and $Y \in \operatorname{mod} A^{\mathrm{op}}$ be a left A-module. Then define $X \otimes_A Y$ to be the vector space $X \otimes_{\Bbbk} Y/U$ where U is the subspace consisting of $xa \otimes y - x \otimes ay$ for all $x \in X, y \in Y, a \in A$.*

In the above, if ${}_AY_B$ is, in addition, an A-B-bimodule, then $X \otimes_A Y$ has a natural right B-module structure: $(x \otimes y)b := x \otimes (yb)$. In fact, as any left A-module is also a A-$\Bbbk$-bimodule, we can $X \otimes_A Y$ being a $\Bbbk$-vector space as a special case of this observation.

Suppose we have a homomorphism $f : M \to N$ of right A-modules. Then for an A-B-bimodule ${}_AY_B$ we get a homomorphism of

$$\begin{aligned} M \otimes_A Y_B &\xrightarrow{f \otimes_A Y} N \otimes_A Y_B \\ m \otimes y &\longmapsto f(m) \otimes y \end{aligned}$$

Note that $(gf) \otimes_A Y = (g \otimes_A Y)(f \otimes_A Y)$, that is, $- \otimes_A Y$ is a (covariant) *functor*. It is also *$\Bbbk$-linear additive* in the sense that $(\lambda f + \mu g) \otimes_A Y = \lambda(f \otimes_A Y) + \mu(g \otimes_A Y)$ for all homomorphisms f, g and scalar $\lambda, \mu \in \Bbbk$.

Likewise, if X is a bimodule, then ${}_BX \otimes_A Y$ has a left module structure; mutatis mutantis.

Example 13.11. *Consider the bounded quiver*

$$Q : \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}, \qquad I = (\beta\alpha).$$

We look at the ($A \otimes A^{\mathrm{op}}$-module) structure of $Ae \otimes eA, AeA, Ae \otimes_{eAe} eA$ for $e = e_1$ in the following.

$$Ae \otimes eA = \begin{smallmatrix} & & 11 & & \\ & 21 & & 12 & \\ 11 & & 22 & & 11 \\ & 12 & & 21 & \\ & & 11 & & \end{smallmatrix} = \begin{tikzcd}[column sep=small] & & e_1 \otimes e_1 \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & & \\ & \beta \otimes e_1 \arrow[dl, dash, "\alpha^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & & e_1 \otimes \alpha \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\beta"] & \\ \alpha\beta \otimes e_1 \arrow[dr, dash, "\alpha"] & & \beta \otimes \alpha \arrow[dl, dash, "\alpha^{\mathrm{op}}"'] \arrow[dr, dash, "\beta"] & & e_1 \otimes \alpha\beta \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \\ & \alpha\beta \otimes \alpha \arrow[dr, dash, "\beta"] & & \beta \otimes \alpha\beta \arrow[dl, dash, "\alpha^{\mathrm{op}}"'] & \\ & & \alpha\beta \otimes \alpha\beta & & \end{tikzcd}$$

As a right A-module, this is $\dim_{\Bbbk} Ae = 3$ copies of $eA = P_1 = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$.

$$AeA = \begin{smallmatrix} & 11 & \\ 21 & & 12 \\ & 11 & \end{smallmatrix} = \begin{tikzcd}[column sep=small] & e_1 \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & \\ \beta \arrow[dr, dash, "\alpha^{\mathrm{op}}"'] & & \alpha \arrow[dl, dash, "\beta"] \\ & \alpha\beta & \end{tikzcd}$$

As right A-module, we have $AeA = eA \oplus \Bbbk\beta \cong P_1 \oplus S_2$.

For $Ae \otimes_{eAe} eA$, first note that $eAe = \Bbbk\{e = e_1, \alpha\beta\}$, and so $\alpha\beta \otimes e_1 = e_1 \otimes \alpha\beta$. In particular, so basis elements in $Ae \otimes eA$ vanishes, for example, $\alpha\beta \otimes \alpha = e_1 \otimes \alpha\beta\alpha = 0$ as $\beta\alpha = 0$ in A.

$$Ae \otimes_{eAe} eA = \begin{tikzcd} & 11 \arrow[dl, dash] \arrow[dr, dash] & \\ 21 \arrow[d, dash] \arrow[drr, dash] & & 12 \arrow[d, dash] \arrow[dll, dash] \\ 22 & & 11 \end{tikzcd} = \begin{tikzcd} & e \otimes e \arrow[dl, dash, "\beta^{\mathrm{op}}"'] \arrow[dr, dash, "\alpha"] & \\ \beta \otimes e \arrow[d, dash, "\alpha"] \arrow[drr, dash, "\alpha^{\mathrm{op}}"] & & e \otimes \alpha \arrow[d, dash, "\beta"'] \arrow[dll, dash, "\beta^{\mathrm{op}}"'] \\ \beta \otimes \alpha & & e \otimes \alpha\beta \end{tikzcd}$$

The right A-module structure of $Ae \otimes_{eAe} eA$ is isomorphic to $P_1 \oplus P_1/\operatorname{soc}(P_1)$.

13.3 Hom

Suppose now that we have ${}_BX_A$ a B-A-bimodule and M a right A-module. Then the space $\operatorname{Hom}_A(X, Y)$ has a natural *right* B-module structure:

$$(f : X \to Y) \cdot b := (x \mapsto f(bx))$$

Indeed, we have

$$((f \cdot b) \cdot b')(x) = (f \cdot b)(b'x) = f(bb'x) = (f \cdot (bb'))(x),$$

and other axioms are even easier to verify. $\operatorname{Hom}_A({}_BX_A, -)$ is also a $\Bbbk$-linear additive covariant functor: for $f : M \to N$ a homomorphism of A-modules, we have

$$\begin{tikzcd} \operatorname{Hom}_A(X, M) \arrow[r, "f\circ-"] & \operatorname{Hom}_A(X, N) \\ \alpha \arrow[r, mapsto] & f \circ \alpha \end{tikzcd}$$

Similarly, in the same setting, $\operatorname{Hom}_A(Y, {}_BX_A)$ also has a *left* B-module structure:

$$(b' \cdot (b \cdot f))(x) = b'((b \cdot f)(x)) = b'(bf(x)) = (b'b)f(x) = ((b'b) \cdot f)(x).$$

However, note that $\operatorname{Hom}_A(f, X) = - \circ f : \operatorname{Hom}_A(N, X) \to \operatorname{Hom}_A(M, X)$ for any $f : M \to N$, i.e. $\operatorname{Hom}_A(-, X)$ is a ($\Bbbk$-linear additive) contravariantly functor, meaning that it reverse the direction of homomorphisms.

Lemma 13.12. *Hom functor commutes with finite direct sum in both variables, i.e. there is a commutative diagram:*

$$\begin{tikzcd} \operatorname{Hom}_A(\bigoplus_{j=1}^{\ell} L_j, N) \arrow[r, "-\circ\theta"] \arrow[d, "\cong"] & \operatorname{Hom}_A(\bigoplus_{i=1}^{m} M_i, N) \arrow[d, "\cong"] \\ \bigoplus_{j=1}^{\ell} \operatorname{Hom}_A(L_j, N) \arrow[r, "(-\circ\theta\iota_j)_{i,j}"] & \bigoplus_{i=1}^{m} \operatorname{Hom}_A(M_i, N) \end{tikzcd}$$

where $\iota_j : L_j \hookrightarrow \bigoplus_k L_k$ and $\pi_i : \bigoplus_k M_k \to M_i$ are the canonical maps, and there is also a similar commutative diagram arising from $\operatorname{Hom}_A(M, \bigoplus_{i=1}^{\ell} L_i) \cong \bigoplus_{i=1}^{\ell} \operatorname{Hom}_A(M, L_i)$.

Remark 13.13. In proper functorial language, this is saying that there are natural isomorphisms

$$\operatorname{Hom}_A(\bigoplus_{j=1}^{\ell} L_j, -) \cong \bigoplus_{j=1}^{\ell} \operatorname{Hom}_A(L_j, -) \text{ and } \operatorname{Hom}_A(-, \bigoplus_{i=1}^{\ell} L_i) \cong \bigoplus_{i=1}^{\ell}(-, L_i).$$

Lemma 13.14. *For any A-module M, we have (natural) A-module isomorphisms*

$$M \otimes_A A \cong M, \quad \text{and } \operatorname{Hom}_A(A, M) \cong M$$

given by $m \otimes 1 \mapsto m$ and $f \mapsto f(1)$. Moreover, $- \otimes_A A$ and $\operatorname{Hom}_A(A, -)$ are both naturally isomorphic to the identity functor.

Proof First one follows from the construction that $ma \otimes 1 = m \otimes a$. The second one is just special case of Yoneda lemma. □

13.4 Tensor-Hom adjunction

Suppose ${}_A M_B$ is a A-B-bimodule, then we have two functors:

$$\begin{tikzcd} \mathsf{mod}\, A \arrow[r, bend left, "- \otimes_A M"] & \mathsf{mod}\, B . \arrow[l, bend left, "\operatorname{Hom}_B(M_B, -)"] \end{tikzcd}$$

These are not inverse to each other; but they form a so-called adjoint pair, which is equivalent to saying that there is the following natural isomorphisms.

Theorem 13.15 (Tensor-Hom adjunction). *Let $X \in \mathsf{mod}\, A$, $Y \in \mathsf{mod}\, B$, ${}_A M_B \in A\,\mathsf{mod}\, B$. Then there is a canonical isomorphism of $\Bbbk$-vector spaces*

$$\begin{array}{rrcl} \theta_{X,M,Y}: & \operatorname{Hom}_B(X \otimes_A M, Y) & \xrightarrow{\cong} & \operatorname{Hom}_A(X, \operatorname{Hom}_B(M, Y)) \\ & f & \longmapsto & (x \mapsto (m \mapsto f(x \otimes m))) \\ & (x \otimes m \mapsto (g(x))(m)) & \longleftarrow\!\shortmid & g \end{array}$$

that is natural in each of X, M, Y.

Proof Check that the maps written are ($\Bbbk$-linear and) mutual inverse of each other. □

In computer science, the map $\theta_{X,M,Y}$ is also called "currying".

As innocence as it looks, this isomorphism is fundamental in (homological algebra and) representation theory.

Example 13.16 (Adjoint triple (RHS)). *eA is naturally an eAe-A-bimodule. Hence, we have an adjoint pair $(- \otimes_{eAe} eA, \operatorname{Hom}_A(eA, -))$.*

On the other hand, Ae is naturally an A-eAe-bimodule, and so we have another adjoint pair $(- \otimes_A Ae, \operatorname{Hom}_{eAe}(Ae, -))$. Note that we have $\operatorname{Hom}_A(eA, -) \cong - \otimes_A Ae$ by Yoneda lemma.

Example 13.17 (Adjoint triple (LHS)). *A/I is naturally an A-A/I-bimodule for any two-sided ideal I of A, and so we have an adjoint pair $(- \otimes_A A/I, \operatorname{Hom}_{A/I}(A/I, -))$.*

A/I is also an A/I-A-bimodule, and so there is another adjoint pair $(- \otimes_{A/I} A/I, \operatorname{Hom}_A(A/I, -))$. Note that both $\operatorname{Hom}_{A/I}(A/I, -)$ and $\otimes_{A/I} A/I$ sends an A/I-module to itself (up to isomorphism) and acts identically on morphisms, i.e. $\operatorname{Hom}_{A/I}(A/I, -) \cong \operatorname{Id} \cong - \otimes_{A/I} A/I$.

14 Exactness

Definition 14.1. *Consider a sequence $M_\bullet = (M_i, d_i)_{i\in\mathbb{Z}}$ of modules and homomorphisms of modules*

$$M_\bullet : \cdots \xrightarrow{d_{i-2}} M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} \cdots .$$

We say that the sequence $M_\bullet$ is

- *a complex if $d_{i+1}d_i = 0$ for all $i \in \mathbb{Z}$. In such a case, we have $\mathrm{Im}(d_i) \subset \mathrm{Ker}(d_{i+1})$ for all $i \in \mathbb{Z}$ and the i-th cohomology of $M_\bullet$ is*

$$H^i(M_\bullet) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i-1}).$$

- *exact at M_k for some $k \in \mathbb{Z}$ if $\mathrm{Im}(d_{k-1}) = \mathrm{Ker}(d_k)$. Note that this implies $d_k \circ d_{k-1} = 0$.*
- *exact if it is so at every term.*
- *short exact (often abbreviated as s.e.s. or ses) if it is a 5-term exact sequence that starts and ends at the trivial module, i.e., of the form*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{14.1}$$

such that f is injective, g is surjective, and $\mathrm{Ker}(g) = \mathrm{Im}(f)$. In this case, M is also called an extension of N by L.

Definition 14.2. *A (covariant) functor $F : \mathsf{mod}\, A \to \mathsf{mod}\, B$ is*

- *left exact if it maps short exact sequence (such as (14.1)) to an exact sequence*

$$0 \to F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N).$$

In other words, it preserves kernel.

- *right exact if it maps short exact sequence (such as (14.1)) to an exact sequence*

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \to 0.$$

In other words, it preserves cokernel.

- *exact if it is both left exact and right exact, i.e. maps ses to ses.*

We define left/right exactness for contravariant functor analogously. In particular, left exact contravariant fucntor turns cokernel into kernel.

Lemma 14.3. *Let ${}_BX_A$ be an A-B-bimodule. Then the following hold.*

(1) $\mathrm{Hom}_A(X, -)$ *maps an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ to an exact sequence*

$$0 \to \mathrm{Hom}_A(X, L) \xrightarrow{f\circ -} \mathrm{Hom}_A(X, M) \xrightarrow{g\circ -} \mathrm{Hom}_A(X, N).$$

In particular, $\mathrm{Hom}_A(X, -)$ is left exact.

(2) $\mathrm{Hom}_A(-, X)$ *maps an exact sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ to an exact sequence*

$$0 \to \mathrm{Hom}_A(N, X) \xrightarrow{-\circ g} \mathrm{Hom}_A(M, X) \xrightarrow{-\circ f} \mathrm{Hom}_A(L, X).$$

In particular, the contravariant functor $\mathrm{Hom}_A(-, X)$ is left exact.

Proof We show (1) and leave (2) for the reader.

Exactness at $\mathrm{Hom}_A(X, L)$: we need $f \circ -$ to be injective. Indeed, if $f \circ \theta = 0$ for some $\theta : X \to L$, then $f(\theta(x))$ for all $x \in X$. This means that $\theta(x) \in \mathrm{Ker}(f) = 0$, and so $\theta = 0$.

$\mathrm{Im}(f \circ -) \subset \mathrm{Ker}(g \circ -)$: Suppose that $\theta : X \to M$ is given by $f \circ \phi$ for some $\phi : X \to L$. Then $g\phi(x) = g(f\phi(x)) = (gf)\phi(x) = 0$, which means that $\theta \in \mathrm{Ker}(g \circ -)$.

$\mathrm{Ker}(g \circ -) \subset \mathrm{Im}(f \circ -)$: Suppose that $g\theta = 0$ for some $\theta : X \to M$. Then for every $x \in X$, we have $\theta(x) \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, and so we can write $\theta(x) = f(\phi(x))$ for some $\phi(x) \in L$. Since f is injective, $\phi(x) \in L$ is uniquely determined, and so we have a well-defined function $\phi : X \to L$. We check that $\phi \in \mathrm{Hom}_A(X, L)$:

- $f(\phi(x + x')) = \theta(x + x') = \theta(x) + \theta(x') = f(\phi(x)) + f(\phi(x')) = f(\phi(x) + \phi'(x))$. Hence, f being injective implies that $\phi(x + x') = \phi(x) + \phi(x')$.
- Suppose that $\lambda \in \Bbbk$. Then $f(\phi(\lambda x)) = \theta(\lambda x) = \lambda\theta(x) = \lambda f(\phi(x)) = f(\lambda\phi(x))$. Hence, f being injective implies that $\lambda\phi(x) = \phi(\lambda x)$.

Now we have $\theta = f\phi$ as A-module homomorphism, and so $\theta \in \mathrm{Im}(f \circ -)$. □

A similar lemma for tensor product exists, and can be proved by direct verification as in the Hom functor case. Instead, we use another trick involving tensor-Hom adjunction, but first we need one more tool.

Lemma 14.4 (Yoneda embedding reflects exactness). *Consider a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathsf{mod}\, A$. If the sequence*

$$\mathrm{Hom}_A(X, L) \xrightarrow{f\circ -} \mathrm{Hom}_A(X, M) \xrightarrow{g\circ -} \mathrm{Hom}(X, N)$$

is exact for all $X \in \mathsf{mod}\, A$, then $L \xrightarrow{f} M \xrightarrow{g} N$ is also exact. Similarly, if

$$\mathrm{Hom}_A(N, X) \xrightarrow{-\circ g} \mathrm{Hom}_A(M, X) \xrightarrow{-\circ f} \mathrm{Hom}(N, X)$$

is exact for all $X \in \mathsf{mod}\, A$, then so is the original sequence.

Proof We show the first one.

$\mathrm{Im}(f) \subset \mathrm{Ker}(g)$: Take $X = L$, then we have $gf = (g \circ -)(f \circ -)(\mathrm{id}_L) = 0$.

$\mathrm{Ker}(g) \subset \mathrm{Im}(f)$: Consider $X = \mathrm{Ker}(g)$ and inclusion $\iota : \mathrm{Ker}(g) \hookrightarrow M$. Then $(g \circ -)(\iota) = g\iota = 0$, so exactness implies that $\iota = f\phi$ for some $\phi \in \mathrm{Hom}_A(\mathrm{Ker}(g), M)$. Hence, $\mathrm{Ker}(g) = \mathrm{Im}(\iota) \subset \mathrm{Im}(f)$. □

Lemma 14.5. *$- \otimes_A X$ maps an exact sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ to an exact sequence*

$$L \otimes_A X \xrightarrow{f\otimes_A X} M \otimes_A X \xrightarrow{g\otimes_A X} N \otimes_A X \to 0.$$

In particular, $- \otimes_A X$ is right exact.

Proof We apply $\mathrm{Hom}_B(-, Y)$ to the sequence (after tensoring X). By the naturality of the adjoint isomorphism, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_B(N \otimes_A X, Y) \arrow[r, "-\circ g\otimes_A X"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(M \otimes_A X, Y) \arrow[r, "-\circ f\otimes_A X"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(L \otimes_A X, Y) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_A(N, \mathrm{Hom}_B(X, Y)) \arrow[r, "-\circ g"] & \mathrm{Hom}_A(M, \mathrm{Hom}_B(X, Y)) \arrow[r, "-\circ f"] & \mathrm{Hom}_A(L, \mathrm{Hom}_B(X, Y))
\end{tikzcd}$$

The second row is exact since it is given by applying the left exact functor $\mathrm{Hom}_A(-, Z)$ for $Z = \mathrm{Hom}_B(X, Y)$. Hence, (by careful diagram chasing) the first row is also exact. Since Yoneda embedding reflects exactness, we get the claimed exactness. □