Topics in Mathematical Science VII
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Introduction to group representations

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Convention

Throughout the course, the symbols $K, \Bbbk, \mathbb{F}$ will always be a field. Unless otherwise stated, we assume (for simplicity) that

- all groups are finite;
- all vector spaces are finite-dimensional.

We compose maps from right to left.

We usually denote the identity element of a group G by 1 or 1_G or id_G.

1 Group action

Definition 1.1. *Let G be a group and X a set. We say that G acts on X, or X is a G-set, if there is a map $*: G \times X \to X$, with $gx := g * x := *(g,x)$ for all $g \in G$ and $x \in X$, such that*

$$1x = x, \qquad \text{and} \qquad g(hx) = (gh)x.$$

Thinking about this a little bit more, one can see that the action of G on X simply just permutes the elements of X – i.e. G is just some (sub)group of symmetries on X.

When $X = V$ is a vector-space, if we ask for G to only acts by permuting elements, then it could very well destroy the linearity – the best thing about linear algebra – and we lose all the toolkit from linear algebra. The remedy is to "linearise" the definition of action.

Definition 1.2. *For a vector space V, we say that G acts linearly on V if G acts on V and*

$$g(\lambda u + \mu v) = \lambda g(u) + \mu g(v)$$

for all $g \in G$, all $\lambda, \mu \in K$, and all $u, v \in V$.

Often in practice we just write

$$G \curvearrowright V$$

to denote the existence of linear G-action on V.

2 Linear representations

A linear g-action on V is just a linear transformation for any $g \in G$. So we can repackage the notion of linear G-action using the following.

Recall that the *general linear group* of a vector space V over K is the group of all invertible (K-)linear transformation from V to itself.

$$\mathrm{GL}(V) := \{\phi : V \to V \mid \phi \text{ invertible linear transformation}\}.$$

The group multiplication is just composition of linear transformations, and the identity element is just the identity map $\mathrm{id} : V \to V$.

More generally, one can consider $\mathrm{GL}(V)$ for some free R-module V of finite rank for some nice ring R – by nice, usually this would be at least an integral domain. We may look at some examples in the case when $R = \mathbb{Z}$ when we focus on symmetric group representations.

Now we can reformulate the notion of linear G-action as follows.

Definition 2.1. *Let G be any (not necessarily finite) group. A finite-dimensional (resp. n-dimensional) K-linear* representation *of G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action *of g on V.*

Usually, when the underlying field (or ring) is understood, we will drop the adjective 'K-linear' for representations.

Exercise 2.2. *Check that representation defines a linear G-action in the sense of Definition 1.2.*

While we assumed V is a vector space over a field K here, one can also consider more general setting of "R-linear representation" when V is an R-lattice (=free R-module of finite rank).

Example 2.3. (1) *The* trivial representation *of G is the* 1*-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}.$$

(2) *$G = \mathfrak{S}_n$ the symmetric group of rank n. The* sign representation *of $\mathfrak{S}_n$ is the* 1*-dimensional representation*

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is the parity (or sign) of the permutation σ.

(3) *Let X be a finite G-set (for any finite group G). Denote by KX the K-vector space with basis given by X. Then*

$$\pi_X : G \to \mathrm{GL}(KX), \qquad g \mapsto (x \mapsto gx)_{x \in X}$$

defines K-linear G-representation. Any G-representation of such a form is called a permutation representation*.*

Exercise 2.4. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.*

Exercise 2.5. *Consider the additive group of integers $G = (\mathbb{Z}, +)$. Let V be a fixed finite-dimensional $\mathbb{C}$-vector space. Show that every linear transformation $\phi \in \mathrm{GL}(V)$ defines a unique (not distinguished under isomorphism) $\mathbb{C}$-linear G-representation.*

Recall that for a ring R with identity 1, under addition the element 1 either has infinite or prime, say p, order. The *characteristic* of R, denoted by $\mathrm{char}\, R$, is 0 in the former case, or p in the latter.

In Example 2.3 (2), we can see that when $\mathrm{char}\, K = 2$, then sign representation is the same as trivial representation.

In general, changing characteristic drastically change the kind of representations that can appear.

- *Ordinary representation theory* studies K-linear representations over a field K with $\operatorname{char} K = 0$.
- *Modular representation theory* studies K-linear representations over a field K with $\operatorname{char} K = p > 0$ and $p | \#G$.
- *Integral representation theory* studies $\mathcal{O}$-linear representations over a (nice – such as discrete valuation ring) integral ring $\mathcal{O}$ (but sometimes including $\mathbb{Z}$) with $\operatorname{char} \mathcal{O} = 0$.

The case of K-linear representations with positive characteristic that does *not* divide the order of group is sometimes called "representations over good characteristics" but can also be regarded as a 'trivial' extension of ordinary representation theory – characteristic 0 and good characteristic cases are somewhat the same.

Most of this course will be about ordinary representation theory. We may touch on some integral and modular representation for the symmetric group later in the course.

3 Matrix representations

When V is n-dimensional K-vector space, then $\mathrm{GL}(V)$ is isomorphic to

$$\mathrm{GL}_n(K) := \{\text{invertible } n \times n\text{-matrices with entries in } K\}.$$

This isomorphism of course depends on a basis we pick for V.

Definition 3.1. *An n-dimensional matrix representation of a group G is a group homomorphism*

$$R : G \to \mathrm{GL}_n(K), \qquad g \mapsto R_g.$$

We say that the matrix R_g represents the action of g.

It is clear that given an n-dimensional matrix representation, one obtains an n-dimensional K-linear representation (with $V = K^n$), and vice versa (by choosing a basis for V and passes through $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$).

Example 3.2. *Consider $G = C_3 = \langle x \mid x^3 = 1 \rangle$ the cyclic group of order 3. Let us try to see what matrix representations of G look like in the case when $K = \mathbb{C}$.*

Suppose that $R_x \in \mathrm{GL}_n(\mathbb{C})$ is diagonal. Since $R_x^3 = R_{x^3} = R_1 = \mathrm{id}$, the diagonal entries are in $\{\omega^k := \exp(2\pi i k/3) \mid 0 \le k < 3\}$, and we can write $R_x = \mathrm{diag}(\omega^{k_1}, \dots, \omega^{k_n})$ with any $k_i \in \{0,1,2\}$ for all $i = 1, \dots, n$. Note that, in this case, R_x^2 will also be a diagonal matrix $\mathrm{diag}(\omega^{2k_1}, \dots, \omega^{2k_n})$.

On the other hand, if R_x is not a diagonal matrix, since R_x is invertible and we work over $\mathbb{C}$, we can still find $P \in \mathrm{GL}_n(\mathbb{C})$ so that $P R_x P^{-1}$ is diagonal. In other words, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\cong", "P"'] \arrow[d, "\mathrm{diag}(\omega^{ik_1},\dots,\omega^{ik_n})"'] & \mathbb{C}^n \arrow[d, "R_x^i"] \\
\mathbb{C}^n \arrow[r, "\cong", "P"'] & \mathbb{C}^n,
\end{tikzcd}$$

i.e. the two paths from top left to bottom right resulting the same map. This amounts to say that, up to a change of basis of $\mathbb{C}^n$, the non-diagonal case is "essentially the same" as the diagonal one.

4 Homomorphism

In mathematics, the word for "essentially the same" is (usually) isomorphism; for this, we need the weaker notion of homomorphism first.

Definition 4.1. *Let $\rho : G \to \mathrm{GL}(V)$ and $\theta : G \to \mathrm{GL}(W)$ be two K-linear representations of G. A homomorphism from V to W is a K-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$, i.e. $f\rho_g = \theta_g f$ for all $g \in G$.

An isomorphism from V to W is a homomorphism that is invertible, i.e. $\exists g$ s.t. $gf = \mathrm{id}_V$ and $fg = \mathrm{id}_W$.

Write $\mathrm{Hom}_{KG}(V, W)$ for the space of all homomorphisms from V to W.

Remark 4.2. Older text also calls a homomorphism (sometimes, only for isomorphism) $f : V \to W$ an *intertwiner*, or that f *intertwines* ρ, θ; we will try to avoid using this and stick to homomorphism. Older text may say that V, W are equivalent if there is an isomorphism between them. We will drop this redundant language and just say V and W are isomorphic.

Example 4.3. *Let us go back to the case when $G = C_3$ and take $n = 1$. We have three representations $R^{(i)}$ with $i = 1, 2, 3$ so that $R^{(i)}_x = \omega^i$. An isomorphism on $\mathbb{C}$ is just a non-zero scalar multiplication $\lambda \cdot -$. As $\lambda R^{(i)}_x \lambda^{-1} = R^{(i)}_x = \omega^i$, we have $R^{(i)} \not\cong R^{(j)}$ whenever $i \neq j$. In fact, by the same reason, we can see that*

$$\mathrm{Hom}_{\mathbb{C}G}(R^{(i)}, R^{(j)}) = \{0\}$$

for distinct i, j.

Exercise 4.4. *Verify that (a) $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space, and (b) the composition of homomorphisms is also a homomorphism of representations.*

Since $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space, we can just write $\mathrm{Hom}_{\mathbb{C}G}(R^{(i)}, R^{(j)}) = 0$ in the above example, instead of the more bulky set notation $\{0\}$.

Exercise 4.5. *Consider $G = C_3$ with generator g acting on $X = \{0, 1, 2\}$ by $gi = i + 1 \mod 3$. Recall from Example 3.2 that 3-dimensional representation of C_3 is isomorphic to a (matrix) representation $R^{(k_1,k_2,k_3)} : G \to \mathrm{GL}_3(\mathbb{C})$ with $R^{(k_1,k_2,k_3)}_g = \mathrm{diag}(\omega^{k_1}, \omega^{k_2}, \omega^{k_3})$. Find (k_1, k_2, k_3) so that $\mathbb{C}X \cong R^{(k_1,k_2,k_3)}$.*

Exercise 4.6. *Let X, Y be two G-sets. Determine the condition on a map $f : X \to Y$ so that f induces a homomorphism of permutation representations from π_X to π_Y.*

5 Group algebra

Definition 5.1. *Let KG be the K-vector space with basis G, i.e. $x \in KG \Leftrightarrow x = \sum_{g \in G} \lambda_g g$ with $\lambda_g \in K$ for all $g \in G$.*

Define a map

$$KG \times KG \to KG, \qquad (\sum_{g \in G} \lambda_g g, \sum_{h \in G} \mu_h h) \mapsto \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

It is routine to check that this defines a ring structure on KG with identity given by that of G. We call this ring the group algebra of G over K.

Exercise. (1) *Show that there is an injective ring homomorphism $K \to Z(KG) := \{x \in KG \mid xy = yx \ \forall y \in KG\}$. In other words, the group algebra KG is a K-algebra.*

(2) *Let R be a commutative ring and A be another (possibility non-commutative) ring. Show that if there is an injective ring homomorphism $R \to Z(A)$, then any A-module is also an R-module.*

Lemma 5.2. *$\rho : G \to \mathrm{GL}(V)$ is a (finite-dimensional) K-linear representation of G if, and only if, V has the structure of a (finite-dimensional) left KG-module.*

Proof $\underline{\Rightarrow:}$ For $x = \sum_g \lambda_g g \in KG, v \in V$. It is routine to check that $x \cdot v := \sum_g \lambda_g \rho_g(v)$ defines a left KG-module structure.

$\underline{\Leftarrow:}$ From the previous exercise, we checked that there is an injective ring homomorphism $K \hookrightarrow Z(KG)$. Hence, we have

$$(\lambda g)(v) = g(\lambda v)$$

for all $g \in G, \lambda \in K, v \in V$. By the axiom of module, V is an abelian group, and so there $0 \in V$ and also well-defined addition operation. Taking $g = 1$ in the above equation, we get that $\lambda v \in V$ for all $\lambda \in K$. Hence, V is a K-vector space.

Now for $g \in G$, define a map $\rho_g : V \to V$ given by $v \mapsto gv$. We then have

$$g(\lambda u + \mu v) = (\lambda g)(u) + (\mu g)(v) = \lambda \rho_g(u) + \mu \rho_g(v),$$

and so ρ_g is a linear transformation. Since $g^{-1}(g(v)) = (g^{-1}g)v = 1_G \cdot v = v$, we have $\rho_{g^{-1}}\rho_g = \mathrm{id}$, and so $\rho_g \in \mathrm{GL}(V)$.

Finally, the axiom of module says that $(gh)(v) = g(hv)$, which means that $\rho_{gh} = \rho_g \rho_h$. Thus, $g \mapsto \rho_g$ is a group homomorphism. □

Remark 5.3. One may find in older textbooks that use terminologies like 'the KG-module V is afforded by ρ' in the setting of this lemma. We will just used *ρ is the representation associated/corresponding to V*, or vice versa, to keep the language simple.

Example 5.4. *KG is clearly a KG-module where the (left) action is given by (left) multiplication. Thus, we have a G-representation $\rho : G \to \mathrm{GL}(KG)$ with $\rho_g(\sum_{h\in G} \lambda_h h) := \sum_{h\in G} \lambda_h gh$. This representation is usually called regular representation of G.*

Exercise 5.5. *Let V be the 1-dimensional subspace of KG spanned by $\sum_{g\in G} g$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.*

Lemma 5.6. *$f : V \to W$ is a homomorphism of K-linear G-representations if, and only if, it is a homomorphism of left KG-modules. Consequently, $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, $W/\mathrm{Im}(f)$ are naturally K-linear G-representations.*

Proof First part: Exercise.

For the second part, just recall that the kernel, image, and quotient of image of any homomorphism of modules are also modules. □

Remark. In the language of category theory, Lemma 5.2 and 5.6 together says that the category of finite-dimensional K-linear G-representations (where morphisms are homomorphisms) and the category of finitely generated left KG-modules are isomorphic (note that this is stronger than just equivalence of categories).

Exercise 5.7. *Verify the first part of Lemma 5.6.*

Exercise 5.8. *Fix any $n \geq 2$.*

(i) Find a generator v such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g\in G} g$ of the trivial representation.)

(ii) Show that $\mathrm{Hom}_{K\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ when $\mathrm{char}\, K \neq 2$; otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

6 Subrepresentation, indecomposable, irreducible

Definition 6.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a K-linear G-representation. A subpace W of V is G-invariant if $\rho_g(W) \subset W$. In this case we call the homomorphism $\theta : G \to \mathrm{GL}(W)$ given by $\theta_g := \rho_g|_W$ a subrepresentation of ρ. It is non-trivial, or proper, if W is non-zero and $W \neq V$.*

We say that ρ is irreducible (or that V is simple) if it admits no proper subrepresentation.

We will use both the terminologies irreducible and simple for representations and modules since they are 'the same' notion.

Exercise 6.2. *Let $f : V \to W$ be a homomorphism of representations from $\rho : G \to \mathrm{GL}(V)$ to $\phi : G \to \mathrm{GL}(W)$. Show the following directly without using the language of KG-modules.*

- $\mathrm{Ker}(f)$ *is a G-invariant subspace of V.*
- $\mathrm{Im}(f)$ *is a G-invariant subspace of W.*

Example 6.3. (1) *Any 1-dimensional representation is irreducible.*

(2) triv_G *is a 1-dimensional irreducible subrepresentation of the regular representation; see Exercise 5.5.*

(3) *Consider $G = D_6 = \langle a, b \mid b^2 = 1 = a^3, abab = 1\rangle$ and $K = \mathbb{C}$. Consider a 2-dimensional representation $\rho : G \to \mathrm{GL}(V)$ so that under the basis $\{u, v\}$ we have its matrix representation form given by*

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If there is a non-trivial subrepresentation, then it will be 1-dimensional spanned by $w := \lambda u + \mu v$ for some scalar $\lambda, \mu \in K$. Being G-invariant means that $aw, bw \in Kw$. Writing the action out:

$$\begin{cases} bw = b(\lambda u + \mu v) = \mu u + \lambda v, \\ aw = a(\lambda u + \mu v) = \omega\lambda u + \omega^{-1}\nu v \end{cases}$$

Looking at b-action we have some $c \in K$ so that $c\lambda = \mu$ and $c\mu = \lambda$, which yields $\lambda = \pm\mu$.

Looking at a-action we have $aw = \omega^{-1}w$ which means that $\mu\omega^{-2} = \mu$ and so $\mu = 0$. (If we take $aw = \omega w$ then we get $\lambda = 0$.) Hence, combining with $\lambda = \pm\mu$, we have $\lambda = 0$. Thus, $w = 0$. This shows that there is no non-trivial G-invariant subspace and so R is irreducible.

If $\rho : G \to \mathrm{GL}(V)$ is a G-representation has a subrepresentation with corresponding module W. Then natural inclusion map $W \hookrightarrow V$ naturally defines an injective homomorphism of KG-module. Hence, we know already from module theory that there is a KG-module structure on the quotient space V/W.

Definition 6.4. *If ϕ is a subrepresentation of $\rho = \rho_V$, with corresponding KG-modules $W \subset V$ respectively, then the quotient representation is the induced homomorphism $\rho_{V/W} : G \to \mathrm{GL}(V/W)$, i.e. $\rho_{V/W}(g)(v + W) := \rho_g(v) + W$.*

Exercise 6.5. *Check that quotient representation is indeed a representation of G directly (without using module theory).*

Lemma 6.6 (First isomorphism theorem). *Let $f : V \to W$ be a homomorphism of representations $V = (V, \rho), W = (W, \phi)$. Then the quotient representation $V/\mathrm{Ker}(f)$ is isomorphic to the subrepresentation $\mathrm{Im}(f)$ of W.*

Proof Just use first isomorphism theorem for KG-modules. $\square$

Looking back at Example 6.3, one can see that looking at matrix really helps to determine subrepresentations. Formulating this more precisely we have the following simple observation.

Lemma 6.7. *Suppose W is a G-invariant subspace of V for a G-representation $\rho : G \to \mathrm{GL}(V)$. If $\{w_1, \dots, w_m\}$ is a basis of W, then we can extends it to a basis $\mathcal{B} = \{v_1, \dots, v_k, w_1, \dots, w_m\}$ of V so that, for every $g \in G$, the matrix form R_g of ρ_g with respect to $\mathcal{B}$ is a lower block-triangular matrix*

$$R_g = \begin{pmatrix} * & 0 \\ * & R_g|_W \end{pmatrix}. \tag{6.1}$$

For ordinary vector space, having a subspace U, we can immediately get $V = U \oplus V/U$, i.e. there is a complement W of U in V such that $W \cong V/U$. However, this is not true for G-representations (and KG-modules, and also modules over a ring in general) in general.

Definition 6.8. *A representation $\rho : G \to \mathrm{GL}(V)$ is decomposable if there are non-trivial G-invariant subspaces (=subrepresentations) $U, W \subset V$ such that $V = U \oplus W$ (i.e. $V = U + W$ and $U \cap W = 0$ as vector spaces). In this case, we can write $\rho = \rho|_U \oplus \rho|_W$ and call U, W the direct summands of V. If no such pair of G-invariant subspace exists, then we say that ρ is indecomposable.*

We can formulate this in terms of matrices like Lemma 6.7.

Lemma 6.9. *$\rho = \rho|_U \oplus \rho|_W$ if and only if there is a basis $\mathcal{B}_V := \{u_1, \dots, u_m, w_1, \dots, w_k\}$ so that $\mathcal{B}_U := \{u_1\}_{1 \le i \le m}$ is a basis of U and $\mathcal{B}_W := \{w_i\}_{1 \le i \le k}$ is a basis of W, and the lower block-triangular matrix R_g in (6.1) has the lower-left corner being 0 for all g:*

$$R_g^V = \begin{pmatrix} R_g^U & 0 \\ 0 & R_g^W \end{pmatrix}.$$

Here R_g^X is the matrix form of $\rho|_X$ with respect to the basis $\mathcal{B}_X$ for $X \in \{V, U, W\}$.

The more compact way to say the right-hand side of this lemma is that 'we can *simultaneously block-diagonalize* ρ_g for all g'.

Of course, direct sum is not just an operation on subspaces. If we have two representations $\rho : G \to \mathrm{GL}(V), \phi : G \to \mathrm{GL}(W)$, then we have a new representation $\rho \oplus \phi : G \to \mathrm{GL}(V \oplus W)$ given by

$$(\rho \oplus \phi)_g(v + w) := \rho_g(v) + \phi_g(w)$$

for any $v \in V$ and $w \in W$.

Exercise 6.10. *If X, Y are two finite G-sets, then we have a new G-set $Z := X \sqcup Y$ given by the disjoint union. The associated permutation representation π_Z is then the direct sum $\pi_X \oplus \pi_Y$.*

Exercise 6.11. *Suppose that X is a finite G-set with G-orbit decomposition $X = O_1 \sqcup \cdots \sqcup O_m$. Then we have $\pi_X = \pi_{O_1} \oplus \cdots \oplus \pi_{O_m}$.*

Some natural questions once we have the notion of indecomposable and irreducible.

Question. (1) *Can we classify all irreducibles?*

(2) *Can we classify all indecomposables?*

(3) *How to build indecomposable representations from irreducibles?*

(4) *When does being indecomposable imply irreducible?*

(5) *Is there any criteria to guarantee a representation can be decomposed into a direct sum of irreducibles?*

(6) *Is decomposition of representation into direct sum of indecomposable direct summand unique? That is, for a representation V with decompositions $U_1 \oplus \cdots U_m$ and $W_1 \oplus \cdots W_n$ with U_i, W_j's all indecomposable, do we have $m = n$ and $\sigma \in \mathfrak{S}_n$ such that $U_i \cong W_{\sigma(i)}$?*

(7) *If we 'divide' a representation into subquotients of irreducibles, is the resulting multi-set of irredcuible contribution 'unique'?*

Our plan is to answer Questions (4) first – this is given by the Maschke's theorem. And use it, and other tools, to give answers to other questions in the case of ordinary representation theory. We will not give any account for the case of modular representation theory, but just minor remarks here: Question (1) has an answer similar to that of the ordinary case. Question (2) is almost always impossible (for interested audience, search on 'tame-wild dichotomy of representation-type'). Question (3) can only be studied by looking at the homological algebra of KG, which is beyond the scope of this text. Question (4) and (5) does not have any good answer in general. Question (6) and (7) actually have affirmative answer as they are consequence of classical result in ring and module theory (namely, Krull-Schmidt theorem and Jordan-Hölder theorem); these are also beyond the scope of this text.

Before we move on, let us have a look when the Question (4) fails.

Example 6.12. *Take $G = C_2 = \langle g \mid g^2 = 1 \rangle$.*

First consider the case when $\operatorname{char} K \neq 2$ *(e.g. $K = \mathbb{C}$). Recall that the trivial representation* $\mathrm{triv}_G \cong K(1+g)$ *is a subrepresentation of the regular representation KG. On the other hand, $C_2 = \mathfrak{S}_2$ has a* 1*-dimensional representation* $\mathrm{sgn} \cong K(1-g)$*. Clearly $\{1+g, 1-g\}$ is a basis of KG. This yields a direct sum decomposition*

$$KG = K(1+g) + K(1-g) = K(1+g) \oplus K(1-g) \cong \mathrm{triv} \oplus \mathrm{sgn}\,.$$

Consider $G = C_2$ with $\operatorname{char} K = 2$ *(e.g. $K = \mathbb{F}_2$). Consider regular representation $C_2 \curvearrowright KC_2$. With respect to the canonical basis $\{1, g\}$, the matrix of g-action is given by $R_g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Suppose we can change the basis via $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to diagonalise R_g. Then R_g becomes*

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} bd-ac & a^2-b^2 \\ d^2-c^2 & ac-bd \end{pmatrix}.$$

Hence, we have $b = \pm a$ and $d = \pm c$. Since we are working over characteristic 2*, we just get $b = a$ and $d = c$. But in this case the above matrix becomes* 0*. Hence, R_g cannot be diagonalised and so it is <u>not</u> a direct sum of two* 1*-dimensional subrepresentations. In particular, it is a* 2*-dimensional indecomposable. As mentioned,* triv *is always a subrepresentation and so we have a* 1*-dimensional subrepresentation* triv *of KG. One can check that the quotient representation is isomorphic to* triv *as well, i.e. in pictorial form, we can write:*

$$KG = \begin{matrix} \mathrm{triv} \\ \mathrm{triv} \end{matrix}\,.$$

Exercise 6.13. *Complete the argument in the example above by showing that $KG/\,\mathrm{triv} \cong \mathrm{triv}$ when* $\operatorname{char} K = 2$.

Exercise 6.14. *Let $A = K[x]/(x^2)$ for any field K. Check that the left A-module ${}_AA$ is indecomposable, i.e. $A \ncong X \oplus Y$ for some non-trivial submodules X, Y of A.*

7 Maschke's theorem

We introduce the following notion to help talking about the Question (5) above.

Definition 7.1. *A representation is completely reducible, or semisimple if it is a direct sum of irreducible representations.*

The main aim of this section is to explain the following foundational result of group representation theory, which is the answer to Question (5).

Theorem 7.2. (Maschke) *Suppose that G is finite and* $\operatorname{char} K$ *is coprime to the order of G. For any KG-module V, every submodule U of V admits a G-invariant complement, i.e. $V = U \oplus V/U$ as KG-module.*

Proof Let W_0 be any K-vector space complement of U in V, and $\pi : V \to V$ be the K-linear projection map that projects onto U (i.e. write $v \in V$ as $u + w$ for $u \in U, w \in W_0$, then $\pi(v) = u$). If π is a homomorphism of KG-modules, then W_0 is a KG-module and we are done by Lemma 5.6 – unfortunately this is not true in general. So our goal is to modify π into an idempotent homomorphism. The clever trick is to consider

$$p : V \to V, \quad v \mapsto \frac{1}{|G|} \sum_{h\in G} h^{-1}\pi h(v).$$

Let us now show that p is a KG-module homomorphism. Indeed, for any $g \in G$, we have

$$p(gv) = \frac{1}{|G|} \sum_{h\in G} h^{-1}\pi h(gv) = \frac{1}{|G|} \sum_{h\in G} g(g^{-1}h^{-1})\pi(hg)v = g\frac{1}{|G|} \sum_{h\in G} h^{-1}\pi hv = gp(v).$$

The averaging by $|G|$ bit seems very unnecessary so far, but we will see soon that this averaging operation makes p a projection onto U. Indeed, first, $\operatorname{Im}(\pi) = U$ implies that $\operatorname{Im}(p) \subset U$, and so it remains to show that $p(u) = u$ for all $u \in U$. Indeed, we have

$$p(u) = \frac{1}{|G|} \sum_{h\in G} h^{-1}\pi \underbrace{h(u)}_{\in U} = \frac{1}{|G|} \sum_{h\in G} h^{-1}h(u) = \frac{1}{|G|} \sum_{h\in G} u = u.$$

Now that we have $p : V \to V$ a KG-module projection onto U, we get that $\operatorname{Ker}(p)$ is a KG-submodule of V. Hence, we have by first isomorphism theorem that $V/\operatorname{Ker}(p) \cong \operatorname{Im}(p) = U \subset V$ and so $V = \operatorname{Ker}(p) \oplus U$. □

Corollary 7.3. *Every K-linear representation of G semisimple if, and only if,* $\operatorname{char} K \nmid |G|$.

Proof $\Leftarrow$: Consequence of iteratively applying Maschke's theorem (Theorem 7.2).

$\Rightarrow$: It is enough to show that KG is not semisimple. Suppose on the contrary that KG is semisimple. Let $a := \sum_g g \in KG$ and $V := Ka \subset KG$. Recall that $\mathrm{triv}_G \cong V$. So KG being semisimple means that we must have $KG \cong V \oplus W$ for some left ideal W of KG.

Consider $w = \sum_h \lambda_h h \in W$. Since W is a left ideal of KG, we have $aw \in W$. On the other hand, we also have

$$aw = (\sum_g g)(\sum_h \lambda_h h) = \sum_h \lambda_h(\sum_g gh) = \sum_h \lambda_h a,$$

which means that $aw \in V$. But $V \cap W = 0$ and so we must have $\sum_h \lambda_h = 0$, which means that

$$W \subset W' := \left\{ \sum_g \mu_g g \in KG \,\middle|\, \sum_g \mu_g = 0 \right\}.$$

The space W' can be rewritten as the kernel of the map (a.k.a. the augmentation map) given by

$$\epsilon : KG \to K, \qquad \sum_g \mu_g g \mapsto \sum_g \mu_g.$$

Thus, $\dim_K W' = |G| - 1 = \dim_K W$ which means that $W = W'$. However, we can also see that $\epsilon(a) = 0$, and so $V \subset W$, a contradiction. $\square$

Exercise 7.4. *Let G be the subgroup of $\mathrm{GL}_n(\mathbb{C})$ given by*

$$\left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

Let V be the 2-dimensional $\mathbb{C}$-vector space. Then we have a natural $\mathbb{C}$-linear representation $\rho : G \to \mathrm{GL}(V)$ given by $g \mapsto gv$ (usual applying matrix on vector). Show that V is indecomposable but not irreducible. In particular, Maschke's theorem fails for infinite group even for $K = \mathbb{C}$.

8 Schur's lemma

Definition 8.1. *A division ring, or a skew field, is a ring whose non-zero elements are invertible.*

Remark 8.2. A field is a division ring where multiplication is commutative.

The following easy yet fundamental lemma describes the relation between simple modules.

Lemma 8.3 (Schur's lemma). *Suppose S, T are simple KG-modules, then*

$$\mathrm{Hom}_{KG}(S, T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

If, moreover, K is algebraically closed, then

$$\dim_K \mathrm{Hom}_{KG}(S, T) = \begin{cases} 1, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Proof We prove the first part by showing that any homomorphism $f : S \to T$ is either zero or an isomorphism. Indeed, for $f \in \mathrm{Hom}_{KG}(S, T)$, we have submodules $\mathrm{Ker}(f) \subset S$ and $\mathrm{Im}(f) \subset T$. Since S is simple, either $\mathrm{Ker}(f) = 0$ or $\mathrm{Ker}(f) = S$. Similarly, since T is simple, either $\mathrm{Im}(f) = T$ or $\mathrm{Im}(f) = 0$. Thus we have

	$\mathrm{Ker}(f) = 0$	$\mathrm{Ker}(f) = S$
$\mathrm{Im}(f) = T$	f isom.	impossible
$\mathrm{Im}(f) = 0$	impossible	$f = 0$.

Assume now that K is algebraically closed, and that $S = T$. We claim that any non-zero homomorphism $f : S \to S$ is given by a scalar multiple $\lambda\,\mathrm{id}_S$ of the identity map. Indeed, K being algebraically closed implies that f has an eigenvalue λ, and so $f - \lambda\,\mathrm{id}_S$ is a non-invertible linear endomorphism on S. It follows from the first part that $f - \lambda\,\mathrm{id}_S = 0$, and so $f = \lambda\,\mathrm{id}_S$.

For the case $S \cong T$, we can fix any pair of isomorphisms $f, g : S \to T$, and so $g^{-1}f : S \to S$ is an endomorphism. By the previous paragraph, we have $g^{-1}f = \lambda\,\mathrm{id}_S$ and so $f = \lambda g$. Thus any homomorphism in $\mathrm{Hom}_{KG}(S, T)$ is a scalar multiple of any other non-zero homomorphism. $\square$

We will now address Question (6). We start with a preliminary lemma.

Lemma 8.4. *For any finite-dimensional KG-modules U, V, W, we have*

(1) $\mathrm{Hom}_{KG}(U \oplus V, W) \cong \mathrm{Hom}_{KG}(U, W) \oplus \mathrm{Hom}_{KG}(V, W)$.

(2) $\mathrm{Hom}_{KG}(U, V \oplus W) \cong \mathrm{Hom}_{KG}(U, V) \oplus \mathrm{Hom}_{KG}(U, W)$.

Proof Exercise (consider the natural projection map $\pi_X : X \oplus Y \to X$). $\square$

Notation. *For a semisimple KG-module M and a simple KG-module S, denote by $[M : S]$ the multiplicity of S as a direct summand, up to isomorphism, of M, i.e. the maximal number m such that $M \cong S^{\oplus m} \oplus M'$.*

Proposition 8.5 (Krull-Schmidt property). *Suppose that K is algebraically closed and $\operatorname{char} K \nmid |G|$. For a finite-dimensional KG-module M and simple KG-module S, we have*

$$[M : S] = \dim_K \mathrm{Hom}_{KG}(M, S) = \dim_K \mathrm{Hom}_{KG}(S, M).$$

In particular, if $M \cong S_1 \oplus \cdots S_s$ and $M \cong T_1 \oplus \cdots \oplus T_t$ are two decomposition of M into direct sum of simple KG-modules, then we have $s = t$ and a permutation $\sigma \in \mathfrak{S}_t$ so that $S_i \cong T_{\sigma(i)}$ for all $1 \leq i \leq t$.

This is only a (very) special case for the Krull-Schmidt theorem, which says that the Krull-Schmidt property (=unique decomposition into direct sum of indecomposables) holds for any finite-dimensional K-algebras (without assumption on the field K); we provide a group representation theoretic proof of this instead.

Proof By Maschke's theorem, we can write $M = S_1 \oplus \cdots \oplus S_s$ for simple modules $S_1, \ldots, S_s$. Hence, we have

$$\dim_K \mathrm{Hom}_{KG}(M, S) = \sum_{i=1}^{s} \dim_K \mathrm{Hom}_{KG}(S_i, S) = \#\{i \in [1, s] \mid S_i \cong S\} = [M : S],$$

where the first equality comes from repeatedly applying Lemma 8.4, and the second comes from Schur's lemma. The proof for $\dim_K \mathrm{Hom}_{KG}(S, M)$ is similar. One can then show the final statement using the formula and induction on s. $\square$

9 Representations of finite abelian groups

One application of Schur's lemma is that it allows us to say a very useful fact about irreducible representations of a finite abelian group.

Recall that the center of a group G is the subgroup

$$Z(G) := \{z \in G \mid zg = gz \ \ \forall g \in G\}.$$

Likewise, the center of the group algebra KG is the (commutative) subring

$$Z(KG) := \{z \in KG \mid zx = xz \ \ \forall x \in KG\}.$$

Note that it is enough to check $zg = gz$ for all $g \in G$ when calculating $Z(KG)$. Also, we have natural inclusion (of sets) $Z(G) \hookrightarrow Z(KG)$.

Exercise 9.1. *If $H \trianglelefteq G$ is a normal subgroup of G, then $\sum_{h \in H} h \in Z(KG)$.*

Lemma 9.2. *Let $\rho : G \to \mathrm{GL}(V)$ be a G-representation. If V is simple and K is algebraically closed, then for each $z \in Z(KG)$, there is a canonical $\lambda_{V,z} \in K^\times$ such that the assignment $z \mapsto \lambda_{V,z}$ restricts to a group homomorphism $\xi_V : Z(G) \to K^\times$.*

Proof It is routine to check that the map

$$f_z : V \to V, \qquad v \mapsto zv(:= \rho_z(v))$$

is K-linear. Since $zg = gz$ for all $g \in G$, we have $f_z\rho_g = \rho_g f_z$ for all $g \in G$. Thus, f_z satisfies the condition of being a KG-homomorphism (note that this is possible without V being simple nor K being algebraically closed).

Suppose now that V is simple and K is algebraically closed. It then follows from Schur's lemma (Lemma 8.3) that $f_z = \lambda_{V,z}\,\mathrm{id}_V$ for some $\lambda_{V,z} \in K^\times$. It is routine (Exercise) to check that ξ_V is a group homomorphism. (More generally, $Z(KG) \to K^\times$ is a semigroup homomorphism.) $\square$

Proposition 9.3. *For K algebraically closed, every irreducible K-linear representation of a finite abelian group is 1-dimensional.*

Proof Let G be a finite abelian group and V a simple KG-module. As in Lemma 9.2, for each $z \in G = Z(G)$, we have $f_z = \lambda_{V,z}\,\mathrm{id}_V \in \mathrm{End}_{KG}(V) := \mathrm{Hom}_{KG}(V,V)$. Hence, for any non-zero $v \in V$, Kv is a non-zero G-invariant subspace of V, and so irreducibility of V implies that $Kv = V$. $\square$

Remark 9.4. One can prove this without so much representation theory. Just use the fact that commuting diagonalizable matrices can be simultaneously diagonalized.

Exercise 9.5. *Proposition 9.3 can fail without the algebraically closed assumption. Consider $G = C_3 = \langle g \mid g^3 = 1\rangle$ and K be a field with $\mathrm{char}\,K = 0$. Define a matrix G-representation $R : G \to \mathrm{GL}_2(K)$ given by*

$$R_g := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

(1) *Show that when $K = \mathbb{R}$, R is an irreducible $\mathbb{R}$-linear C_3-representation.*

(2) *For $K = \mathbb{C}$, find $i, j \in \{1,2,3\}$ so that $R \cong R^{(i)} \oplus R^{(j)}$ (for the $R^{(a)}$'s given in Example 4.3).*

Recall that every finite abelian group G is isomorphic to the direct product $C_{n_1} \times \cdots C_{n_r}$ of cyclic groups. Also, over an algebraically closed field K, the n-th root of 1 forms the cyclic group C_n of order n:

$$C_n \cong \{x \in K^\times \mid x^n = 1_K\} =: \mu_n.$$

Proposition 9.6. *Over an algebraically closed field K with $\mathrm{char}\,K \nmid |G|$, A finite abelian group $G \cong C_{n_1} \times \cdots \times C_{n_r}$ has exactly $|G|$ irreducible K-linear representations, each of which is labelled by a tuple $(\lambda_1, \ldots, \lambda_r) \in \prod_{i=1}^r \mu_{n_i}$.*

Proof A finite abelian group G is of the form $C_{n_1} \times \cdots \times C_{n_r}$. Let g_i be the generator of the factor C_{n_i}. Take an irreducible representation $\rho : G \to \mathrm{GL}(V)$. It follows from Lemma 9.2 Proposition 9.3 that $\dim_K V = 1$ with each g_i acts by multiplying a scalar $\lambda_{V,i} \in K$. Since $g_i^{n_i} = 1$, we have $\lambda_{V,i}^{n_i} = 1 \in \mu_{n_i}$. Thus, $V \mapsto (\lambda_{V,1}, \ldots, \lambda_{V,r})$ defines a map α from the set of (representative of) isomorphism classes of irreducible representations

$$\alpha : \{\text{irreducible representation } V\}/\cong \to \mu_{n_1} \times \mu_{n_2} \times \cdots \times \mu_{n_r}.$$

α injective: Suppose that $(\lambda_{V,i})i = (\lambda_{V',i})_i$, then g_i acts the same way for all i, and so $V \cong V'$.

α surjective: Given $(\lambda_i)_i \in \mu_{n_1} \times \cdots \times \mu_{n_r}$. Define a map $\rho : G \to \mathrm{GL}_1(K) = K^\times$ as follows. Take $\rho(g_i) := \lambda_i$ for all $i = 1, \ldots, r$. In general, any $g \in G$ is of the form $g = g_1^{a_1} \cdots g_r^{a_r}$, and we define $\rho(g) := \lambda_1^{a_1} \cdots \lambda_r^{a_r}$. It is clear that ρ is a group homomorphism. $\square$

Remark 9.7. α fails to be injective when $p := \mathrm{char}\,K$ divides $|G|$ as $\#\{x \in K \mid x^n = 1\} < n$ when $p|n$ (note: $x^p - 1 = (x-1)^p$ over such a field). Nevertheless, a similar argument can still applies (note: Proposition 9.3 still holds) – for example, there is *only* one irreducible representation over a p-group (i.e. a group where every element has order p^k for some k), namely, the trivial representation.

Example 9.8. *For $G = C_3$ and $K = \mathbb{C}$, it follows from Proposition 9.6 that the three (pairwise non-isomorphic) irreducible 1-dimensional $R^{(i)}$ from Example 4.3 are all the irreducible (hence, indecomposable, by Maschke) representations up to isomorphism.*

Example 9.9. *Recall that the Klein 4-group V_4 is the abelian group of order 4 given by $\langle a, b \mid a^2 = 1 = b^2, ab = ba\rangle \cong C_2 \times C_2$. Thus we have 4 (isomorphism classes of) irreducible representations $\rho_{(0,0)}, \rho_{(1,0)}, \rho_{(0,1)}, \rho_{(1,1)}$ where*

$$\rho_{(i,j)} : \begin{cases} a \mapsto (-1)^i, \\ b \mapsto (-1)^j, \\ ab \mapsto (-1)^{i+j} \end{cases}$$

for all $i, j \in \{1, 2\}$.

Proposition 9.10. *Let G be a finite group and K be an algebraically closed field with $\operatorname{char} K \nmid |G|$. If every irreducible K-linear G-representation is 1-dimensional, then G is abelian.*

Proof By Maschke's theorem (Theorem 7.2), we have $KG = S_1 \oplus \cdots \oplus S_n$ for simple KG-modules $S_1, \ldots, S_n$. By assumption, we have $\dim_K S_i = 1$ and so we can write $S_i = Kv_i$ with $\mathcal{B} := \{v_i\}_{1 \le i \le n}$ forming a K-basis of KG. Thus, with respect to this basis, the matrix of every $g \in G$ of the regular representation is a diagonal matrix and pairwise commute. Note that the regular representation $\rho : G \to \mathrm{GL}(KG)$ has $\mathrm{Ker}(\rho) = 1$ (the matrix of ρ_g with respect to the canonical basis G is a non-trivial permutation matrix for all element $g \neq 1_G$ of G), and so $\mathrm{Im}(\rho) \cong G$ has pairwise commuting elements, i.e. G is abelian. $\square$

Finally, we show one small application of representation theory on group theory – how existence of certain type of representations guarantee a finite abelian group is cyclic.

Proposition 9.11. *For a finite group G and K algebraically closed, if there is an irreducible representation $\rho : G \to \mathrm{GL}(V)$ with $\mathrm{Ker}(\rho) = 1_G$ (i.e. ρ is faithful), then the center $Z(G)$ of G is cyclic.*

Proof Consider the group homomorphism $\xi_V : Z(G) \to K^\times$ of Lemma 9.2. $\xi_V(z) = 1$ implies that z acts trivially on V. Since $\mathrm{Ker}(\rho) = 1_G$, we have $\xi_V(z) = 1$ implies that $z = 1_G$. Hence, ξ_V is injective, which means that $\mathrm{Im}(\xi_V) \cong Z(G)$. Since $K^\times$ is abelian and $Z(G)$ is finite, $\mathrm{Im}(\xi_V) \cong Z(G)$ is isomorphic to product of cyclic groups, say, $C_{p_1^{n_1}} \times \cdots \times C_{p_r^{n_r}}$ with p_i primes.

> **Claim:** p_i's are pairwise distinct.
>
> **Proof of Claim:** Consider $m := \mathrm{lcm}(p_1^{n_1}, \ldots, p_r^{n_r})$, which has $m \le p_1^{n_1} \cdots p_r^{n_r}$ always.
>
> For any generator g_i of the factor $C_{p_i^{n_i}}$ (any $1 \le i \le r$), we have $(g_i)^{p_i^{n_i}} = 1$, and so $(g_i)^m = 1$. However, $\{x \in K^\times \mid x^m = 1\}$ is a group (under multiplication) of order at most m, and so we have $m = n = p_1^{n_1} \cdots p_r^{n_r}$. ■

It follows from the claim and the Chinese Remainder theorem that $\mathrm{Im}(\xi_V) \cong C_n$ for $n := p_1^{n_1} \cdots p_r^{n_r}$, and now we are done. $\square$

10 Irreducible and regular representations

Over a field in good characteristic, we have completely answered Question (1) now for finite abelian groups in the previous section; we will give some partial progress towards Question (1) for other finite groups now. (If you are ring theorists, then this section is just a corollary of the Artin-Wedderburn theorem combined with Maschke's theorem.)

Careful audience may notice from the previous two propositions that "everything" is encoded within the regular representation $V = KG$.

Lemma 10.1 (Yoneda). *Let M be any KG-module. Then we have a K-vector space isomorphism $\mathrm{Hom}_{KG}(KG, M) \cong M$.*

Proof Take any $m \in M$, define a map $f_m : KG \to M$ that maps $x \mapsto xm$. It is routine to check that this is a homomorphism of KG-modules. Now we have a K-linear map

$$\alpha : M \to \mathrm{Hom}_{KG}(KG, M), \qquad m \mapsto (f_m : x \mapsto xm).$$

α is injective: $f_m = 0$ means that $m = f(1_{KG}) = 0$.
α is surjective: For any $f \in \mathrm{Hom}_{KG}(KG, M)$, f is determined by the image of 1_{KG} under f, since f is a K-linear map, G is a basis of KG, and $g(f(1)) = f(g1) = f(g)$ holds for all $g \in G$. Hence, $f = f_m$ where $m = f(1_{KG})$, and so α is surjective. □

Remark 10.2. (1) Actually, $\mathrm{Hom}_{KG}(KG, M)$ can be equipped with a KG-module structure as KG is a KG-bimodule (see later section) and the isomorphism is actually a KG-module isomorphism.

(2) For category theorist: we view KG as a category $\mathcal{C}$ with single object $*$ and morphisms $\mathcal{C}(*, *) := KG$. A KG-module M is the same as a functor $F : \mathcal{C} \to \mathrm{Vec}_K$ valued in the category of K-vector spaces via $F(*) := M$. Homomorphisms between KG-modules are just natural transformations of such functors.

Proposition 10.3. *Up to isomorphism, every irreducible G-representation is a quotient representation of the regular representation.*

Proof Let V be a simple KG-module and $v \in V$ a non-zero element. Consider the KG-module homomorphism $f_v : KG \to V$ that maps $f_v(x) := xv$ as in Lemma 10.1. Then $\mathrm{Im}(f_v) \subset V$ is a quotient of the KG-module KG, and also a KG-submodule of V. As V is simple, and $f_v =\neq 0$, we have $\mathrm{Im}(f_v) \cong V$. □

Remark 10.4. The same result actually holds without the assumption on characteristic and also holds if we replace 'quotient' by 'sub'. Under good characteristic, we can deduce the 'sub' version of the lemma as $\mathrm{Im}(f_v)$ is a direct summand of V. For the case when $\mathrm{char}\, K$ divides $|G|$, we can either use the fact that KG is a so-called 'symmetric algebra' (meaning that $(KG)^* \cong KG$, see later section on 'Dual representation'), which allows us to dualise a surjective homomorphism $KG \twoheadrightarrow V$ to an injective one $V^* \hookrightarrow (KG)^* \cong KG$. Then use the fact that dual representation preserves irreducibility and the fact that dualisation is an involutive operation on the set of (isomorphism classes of) irreducible representations.

Corollary 10.5. *For a finite group G, there are only finitely many irreducible representations up to isomorphism when* $\mathrm{char}\, K \nmid |G|$.

Proof This is because KG is a finite-dimensional KG-module, so we can only have finitely many quotients of KG. The claim now follows from Proposition 10.3. □

Corollary 10.6. *Suppose K is algebraically closed with* $\mathrm{char}\, K \nmid |G|$. *Let $\{S_1, \ldots, S_r\}$ be the complete set of isomorphism classes of simple KG-modules. Then we have KG-module isomorphism*

$$KG \cong S_1^{d_1} \oplus \cdots \oplus S_r^{d_r}$$

with $d_i = \dim_K S_i$ for all $1 \le i \le r$.

Proof By Maschke's theorem and Proposition 10.3, we have a (unique, by Krull-Schmidt property Proposition 8.5) decomposition

$$KG \cong S_1^{[KG:S_1]} \oplus \cdots \oplus S_r^{[KG:S_r]}$$

of KG into a direct sum of simple modules $S_1, \dots, S_r$ with $[KG : S_i] \geq 1$. In fact, Proposition 8.5 already tells us that $d_i = [KG : S_i] = \dim_K \operatorname{Hom}_{KG}(KG, S_i)$. By Lemma 10.1, we have $[KG : S_i] = \dim_K S_i$, and the assertion follows. □

Our next goal is to relate the number r with group-theoretic information of G. On the way, we will also show a ring-theoretic description of KG – in ring theoretic terms, what we want to do is to find Artin-Wedderburn decomposition of KG.

Definition 10.7. *Let C be a conjugacy class in G. The class sum is the element $\overline{C} := \sum_{g \in C} g \in KG$.*

Recall that the center $Z(A) := \{a \in A \mid ab = ba \forall b \in A\}$ of a ring A is commutative.

Proposition 10.8. *Suppose $C_1, \dots, C_r$ are all conjugacy classes of G. Then $\{\overline{C}_1, \dots, \overline{C}_r\}$ is a K-basis of $Z(KG)$.*

Note that Proposition 10.8 requires no characteristic assumption.

Proof (1) $\overline{C}_i \in Z(KG)$ for all i: By definition, $g\overline{C}_i g^{-1} = \overline{C}_i$ for any $g \in G$, so we have $g\overline{C}_i = \overline{C}_i g$ which implies, by linearity, that $\overline{C}_i \in Z(KG)$.

(2) $\{\overline{C}_i\}_i$ is linear independent: Simply because each $g \in G$ lies in precisely one conjugacy class.

(3) Spanning: Suppose that $v = \sum_g \lambda_g g \in Z(KG)$. Then for all $h \in G$ we have

$$v = hvh^{-1} = \sum_g \lambda_g hgh^{-1} = \sum_{k \in G} \lambda_{h^{-1}kh} k.$$

Hence, as G is the basis of KG, comparing coefficients yields $\lambda_g = \lambda_{hgh^{-1}}$ for all $g, h \in G$. In other words, λ_g is constant over the conjugacy class containing g. This means that v is in the span of $\{\overline{C}_i\}_{i=1,\dots,r}$. □

Lemma 10.9. *Let $\operatorname{Mat}_n(K)$ be the ring of $n \times n$-matrices. Then we have a ring isomorphism $Z(\operatorname{Mat}_n(K)) \cong K$.*

Proof There is a map $K \to Z(\operatorname{Mat}_n(K))$ given by $\lambda \mapsto \lambda \operatorname{id}$; it is routine to check that this is a ring isomorphism (Exercise). □

Definition 10.10. *For rings A, B, we have a new ring $A \times B$ called the direct product of A and B given by the usual Cartesian product on the underlying set with multiplication $(a, b)(a', b') := (aa', bb')$.*

Exercise 10.11. (i) *Show that there is a ring isomorphism $\operatorname{End}_A(A)^{\mathrm{op}} \cong A$ for any ring A, where, for a ring Λ*
- $\operatorname{End}_\Lambda(X) := \operatorname{Hom}_\Lambda(X, X)$ *is the endomoprhism ring of Λ-module X with multiplication given by composition of maps, and*
- Λ^{op} *is the opposite ring of a ring Λ with multiplication $a \cdot_{\mathrm{op}} b := b \cdot a$.*

(ii) *Show that $Z(A \times B) = Z(A) \times Z(B)$.*

(iii) *Suppose M, N are A-modules with $\operatorname{Hom}_A(M, N) = 0 = \operatorname{Hom}_A(N, M)$. Show that $\operatorname{End}_A(M \oplus N) = \operatorname{End}_A(M) \times \operatorname{End}_A(N)$.*

(iv) *Suppose S is a simple KG-module over an algebraically closed field K. Show that there is a ring isomorphism $\operatorname{End}_{KG}(S^{\oplus m})^{\mathrm{op}} \cong \operatorname{Mat}_m(K)$.*

Theorem 10.12. *Over an algebraically closed field K with $\operatorname{char} K \nmid |G|$, we have a ring isomorphism*

$$KG \cong \operatorname{Mat}_{d_1}(K) \times \cdots \times \operatorname{Mat}_{d_r}(K),$$

where $\operatorname{Mat}_n(K)$ is the ring of $n \times n$-matrices over K, and r is the number of conjugacy classes of G.

Proof We have ring isomorphisms

$$\begin{aligned}
(KG)^{\mathrm{op}} &\cong \mathrm{End}_{KG}(KG) && \text{by Ex 10.11(i)}\\
&\cong \mathrm{End}_{KG}(S_1^{d_1} \oplus \cdots \oplus S_r^{d_r}) && \text{by Cor 10.6}\\
&\cong \mathrm{End}_{KG}(S_1^{d_1}) \times \cdots \times \mathrm{End}_{KG}(S^{d_r}) && \text{by Schur's lemma + Ex 10.11(iii)}\\
&\cong \mathrm{Mat}_{d_1}(K) \times \cdots \times \mathrm{Mat}_{d_r}(K) && \text{by Exercise 10.11(iv).}
\end{aligned}$$

Note that $\mathrm{Mat}_d(K)^{\mathrm{op}} \cong \mathrm{Mat}_d(K)$. Hence, we can apply Lemma 10.9 and Exercise 10.11 (ii) to get the following ring isomorphisms

$$Z(KG) \cong Z\big(\mathrm{Mat}_{d_1}(K) \times \cdots \mathrm{Mat}_{d_r}(K)\big) \cong Z(\mathrm{Mat}_{d_1}(K)) \times \cdots \times Z(\mathrm{Mat}_{d_r}(K)) \cong K \times \cdots \times K.$$

In particular, we have $r = \dim_K Z(KG)$, which is the same as the number of conjugacy classes in G by Proposition 10.8. □

Remark 10.13. For K algebraically closed with $\mathrm{char}\, K = p > 0$, the number of isoclasses of simple KG-modules coincides with the p'-conjugacy classes, i.e. conjugacy class C such that p does not divides $|C|$. The proof is much more involved and require closer comparison bewteen $KG/\operatorname{rad} KG$ and $Z(KG)$, where $\operatorname{rad} KG$ is the Jacobson radical of KG.

Exercise 10.14. *Recall from Example 6.3 that there is a 2-dimensional irreducible representation V_2 of $G = D_6 = \langle a, b \mid a^3 = 1 = b^2\rangle$.*

(1) *Find $u, v \in KG$ so that the $K\{u, v\}$ is the subrepresentation of KG that is isomorphic to V_2.*

(2) *Find a basis $\{v_1, v_2, \ldots, v_6\}$ of KG so that*

$$KG \cong K\{v_1\} \oplus K\{v_2\} \oplus K\{v_3, v_4\} \oplus K\{v_5, v_6\}$$

as KG-module. Describe each of these subrepresentations (by their name/action).

11 Dual space

Recall that the (K-)dual space V^* of a K-vector space V is the vector space given by linear 1-form

$$V^* := \mathrm{Hom}_K(V, K) = \{\text{linear map } f : V \to K\}.$$

Let $\rho : G \to \mathrm{GL}(V)$ be a K-linear G-representation. For any $g \in G$ and K-linear map $\alpha \in V^* := \mathrm{Hom}_K(V, K)$, consider the following map

$$\rho_g^*(\alpha) : V \to K, \qquad v \mapsto \alpha \circ \rho_{g^{-1}}(v) = \alpha(g^{-1}v).$$

Clearly, $\rho_g^* : V^* \to V^*$ given by $\alpha \mapsto \rho_g^*(\alpha)$ is a K-linear map.

Lemma 11.1. *For a representation $\rho : G \to \mathrm{GL}(V)$. Then $\rho^* : G \to \mathrm{GL}(V^*)$ given by $g \mapsto \rho_g^*$ is also a G-representation.*

Proof (1) $\rho_g^* \in \mathrm{GL}(V^*)$: We have

$$\rho_{g^{-1}}^*\rho_g^*(\alpha) = \rho_{g^{-1}}^*(\alpha \circ \rho_{g^{-1}}) = (\alpha \circ \rho_{g^{-1}}) \circ \rho_g = \alpha \circ (\rho_{g^{-1}} \circ \rho_g) = \alpha.$$

Note that, in particular, we have $(\rho_g^*)^{-1} = \rho_{g^{-1}}^*$.

(2) ρ^* is a group homomorphism: Clearly $\rho^*_{1_G} = \mathrm{id}_{V^*}$. We check $\rho^*_{gh^{-1}} = \rho^*_g \rho^*_{h^{-1}}$. Take $\alpha \in V^*$, then we have

$$\begin{aligned}\rho_{gh^{-1}} * (\alpha) = \alpha \circ \rho_{(gh^{-1})^{-1}} &= \alpha \circ \rho_{hg^{-1}} \\ &= \alpha \circ (\rho_h \rho_{g^{-1}}) = (\alpha \circ \rho_h) \circ \rho_{g^{-1}} \\ &= \left(\rho^*_{h^{-1}}(\alpha)\right) \circ \rho_{g^{-1}} \\ &= \rho^*_g(\rho^*_{h^{-1}}(\alpha)) \\ &= (\rho^*_g \circ \rho^*_{h^{-1}})(\alpha)\end{aligned}$$

□

Remark 11.2. Consider any matrix representation $R : G \to \mathrm{GL}_n(K)$ associated to $\rho : G \to \mathrm{GL}(V)$ with respect to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V. Let $\mathcal{B}^*$ be the dual basis of V^*, i.e. $\mathcal{B}^* = \{\alpha_1, \dots, \alpha_n\}$ with $\alpha_i(v_j) = \delta_{i,j}$. Then the matrix representation R^* associated to ρ^* with respect to $\mathcal{B}^*$ has action matrix R^*_g given by the transpose $R^t_{g^{-1}}$ of $R_{g^{-1}}$.

Although $V^* \cong V$ for any (finite-dimensional) K-vector space, this generally does not lift to an isomorphism of KG-modules.

Example 11.3. *Consider the* 1*-dimensional representation* $R^{(k)}$ *of* C_3 *where the generator* g *acts as (multiplying)*$\omega^k = \exp(2k\pi i/3)$*. Then* $(R^{(1)})* \cong R^{(2)}$ *and* $(R^{(0)})^* \cong R^{(0)}$.

Definition 11.4. *A* KG*-module* V *is self-dual if* $V^* \cong V$ *as* KG*-modules.*

Exercise. *Trivial representation and sign representation are both self-dual.*

Proposition 11.5. *The regular representation is self-dual.*

Proof KG has K-linear basis G. The canonical (dual) basis of $(KG)^*$ is given by $\{\alpha_g \mid g \in G\}$ where $\alpha_g(h) := \delta_{g,h}$, i.e. $\alpha_g(g) = 1$ and $\alpha_g(h) = 0$ for all $h \in G \setminus \{g\}$.

Consider the K-linear map $\alpha : KG \to (KG)^*$ given by linearly extending $g \mapsto \alpha_g$. This is clearly a K-vector space isomorphism. So we only need to show that α is a KG-module homomorphism. For any $g, h, k \in G$, we have

$$(h\alpha(g))(k) = (h \cdot \alpha_g)(k) = \alpha_g(h^{-1}k) = \delta_{g,h^{-1}k} = \delta_{hg,k} = \alpha_{hg}(k) = (\alpha(hg))(k).$$

This shows the claim. □

Remark. In ring theory, this is the same as saying that KG is self-injective. In fact, KG is a symmetric Frobenius algebra, meaning that $(KG)^* \cong KG$ as a KG-KG-bimodule.

Definition 11.6. *Let* $f : V \to W$ *be a homomorphism of* KG*-modules. Define* $f^* : W^* \to V^*$ *by*

$$f^*(\alpha)(v) := \alpha(f(v))$$

for all $\alpha \in W^*$ *and* $v \in V$.

Lemma 11.7. f^* *is a homomorphism of* KG*-modules. Moreover, it maps surjective homomorphism to injective ones, and vice versa.*

Proof Exercise. □

Lemma 11.8. *If* V *is a simple* KG*-module, then so is* V^*.

Proof Take the smallest non-trivial quotient KG-module U of V^*, then U is necessary simple and we have a non-zero surjective homomorphism $V^* \twoheadrightarrow U$. Dualising yield a non-zero injective homomorphism $U^* \hookrightarrow V$. Since V is simple, we have $U^* \cong V$, which means that $V^* \cong U$ is simple. □

Proposition 11.9. *Every irreducible representation is, up to isomorphism, a subrepresentation of the regular representation.*

Proof Combine Proposition 10.3 and Lemma 11.8. □

12 Tensor product

Definition 12.1. *Let V, W be finite-dimensional K-vector space with bases, say, $\mathcal{B}, \mathcal{C}$ respectively. Then the tensor product $V \otimes_K W$ (or simplifies to $V \otimes W$ if context is clear) is the finite-dimensional K-vector space with basis given by*

$$\{v \otimes w \mid v \in \mathcal{B}, w \in \mathcal{C}\}.$$

Suppose $\mathcal{B} = \{v_1, \ldots, v_m\}$ and $\mathcal{B} = \{w_1, \ldots, w_n\}$. Then for $v = \sum_i \lambda_i v_i$ and $w = \sum_j \lambda_j w_j$, we can use the short-hand

$$v \otimes w := \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j) \in V \otimes W.$$

Lemma 12.2. *Consider $\lambda \in K$, $v, v' \in V$ and $w, w' \in W$. Then we have the following.*

(1) $(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$.

(2) $(v + v') \otimes w = v \otimes w + v' \otimes w$.

(3) $v \otimes (w + w') = v \otimes w + v \otimes w'$.

Proof These are simple algebraic rewriting of symbols. For example, taking basis $\mathcal{B}, \mathcal{C}$ as before, the first equality of (1) is just

$$(\lambda v) \otimes w = \lambda(\sum_i \lambda_i v_i) \otimes (\sum_j \mu_j w_j) = \sum_{i,j} \lambda \lambda_i \mu_j (v_i \otimes w_j) = \lambda \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j) = \lambda(v \otimes w).$$

etc. □

Be very careful that there are elements $V \otimes W$ that can **not** be written in the form of $v \otimes w$ for $v \in V$ and $w \in W$. In particular, one common newbie mistake is to regard the following distinct elements as the same:

$$v_1 \otimes w_1 + v_2 \otimes w_2 \neq (v_1 + v_2) \otimes (w_1 + w_2).$$

The right-hand side is really $v_1 \otimes w_1 + v_1 \otimes w_2 + v_2 \otimes w_1 + v_2 \otimes w_2$.

Lemma 12.3. *The space $V \otimes_K W$ does not depend on the choice of basis on V and W.*

Proof Take any other basis $\{v'_1, \ldots, v'_m\}$ of V and $\{w'_1, \ldots, w'_n\}$ of W, with change of basis

$$v_i = \sum_k \alpha_{k,i} v'_k \qquad \text{and} \qquad w_j = \sum_l \beta_{l,j} w'_l.$$

Then

$$v_i \otimes w_j = \sum_{k,l} \alpha_{k,i} \beta l, j v'_k \otimes w'_l.$$

Hence, $\{v'_k \otimes w'_l\}_{k,l}$ spans $V \otimes_K W$, and this spanning set has size the mn; thus, it is a basis. □

One can define $V \otimes_K W$ in a basis-free way. Notice that if we write $v \otimes w$ as $\langle v, w \rangle$, then the 'relations' in Lemma 12.2 says that $\langle -, ? \rangle$ is like a "bilinear form without value". This can be phrased more precisely as follows.

Lemma 12.4. *Given any bilinear form $b := \langle -, ? \rangle : V \times W \to K$, there is always a* unique *K-linear map $\theta_b : V \otimes_K W \to K$ so that the following diagram commutes:*

$$\begin{tikzcd} V \times W \arrow[d] \arrow[dr, "{\forall b = \langle -,? \rangle}"] & \\ V \otimes_K W \arrow[r, dashed, "\exists ! \theta_b"'] & K \end{tikzcd}$$

where the vertical map is given by $(v, w) \mapsto v \otimes w$.

More generally, we can replace K by any vector space U in the statement above, and 'bilinear form' replaced by *bilinear map*, i.e. map that is linear in both the V-component and W-component of $V \times W$.

Proof Clear from Lemma 12.2 and the definition of $v \otimes w$ that $\theta_b(v \otimes w) := \langle v, w \rangle$ is the desired (K-linear) map. $\square$

The universal property of tensor product says that for any vector space U that satisfies the property:

- suppose there is a bilinear map $V \times W \to T$ such that, for all bilinear map $b : V \times W \to U$, there is a K-linear map $f : T \to U$ so that $b = fa$:

$$\begin{tikzcd} V \times W \arrow[d, "\text{bilinear}"'] \arrow[dr, "\forall b\text{: bilinear}"] & \\ T \arrow[r, dashed, "\exists \theta_b\text{: linear}"'] & K \end{tikzcd}$$

then $T \cong V \otimes_K W$.

In more advanced texts, tensor products are most probably defined using universal property, and one shows that it does exists and is unique (up to unique(!) isomorphism). Since we concerns only finite-dimensional vector spaces, a more practical approach via basis is (likely) easier to understand.

The following innocent looking isomorphisms are arguably the most used isomorphisms in homological algebra.

Lemma 12.5. *For any finite-dimensional K-vector spaces U, V, W, the following hold.*

(1) $V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$.

(2) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W))$.

Proof (1) Let $\mathcal{B} = \{v_1, \ldots, v_m\}, \mathcal{C} = \{w_1, \ldots, w_n\}$ be bases of V, W respectively. Let $\mathcal{B}^* = \{f_1, \ldots, f_m\}$ be the canonical dual basis, i.e. $f_i(v_j) = \delta_{i,j}$ for all $1 \le i, j \le m$.

Define $\theta(f_i \otimes w_j)$ to be the K-linear map that extends $v_k \mapsto f_i(v_k) w_j \in W$ and check that θ is K-linear.

Conversely, for $\alpha \in \mathrm{Hom}_K(V, W)$, let $\phi(\alpha) := \sum_i f_i \otimes \alpha(v_i)$. Check that ϕ and θ are inverse to each other.

(2) Define
$$\theta : \mathrm{Hom}_K(U \otimes V, W) \to \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)), \quad f \mapsto \theta_f,$$
where $\theta_f(u) : V \to W$ is the map that sends $v \in V$ to $f(u \otimes v) \in W$.

Define also
$$\phi : \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)) \to \mathrm{Hom}_K(U \otimes V, W), \quad f \mapsto \phi_f,$$
where $\phi_f(u \otimes v) := (f(u))(v)$. Check that ϕ and θ are inverse to each other. $\square$

Remark 12.6. The isomorphism (1) absolutely require finite-dimensionality. This property also provides a way to interpret the tensor product space as the space of linear transformation (matrices). The isomorphism (2) is called 'currying' in computer science, coined from Curry-Howard correspondence. This isomorphism is actually natural, and yields an adjoint pair $(- \otimes_K V, \mathrm{Hom}_K(V, -))$ of functors.

Example 12.7. *Consider $A = (a_{i,j})_{1\le i,j\le m} \in \mathrm{Mat}_m(K)$ and $B \in \mathrm{Mat}_n(K)$ and defines (what is sometimes called Kronecker product of matrices)*

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & \ddots & & a_{2,m}B \\ \vdots & & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,m}B \end{pmatrix}.$$

Then we have an isomorphism of algebras

$$\mathrm{Mat}_m(K) \otimes_K \mathrm{Mat}_n(K) \to \mathrm{Mat}_{mn}(K), \quad (A, B) \mapsto A \otimes B.$$

From this, we can see that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, if (and only if) both A, B are invertible. Thus, the isomorphism restricts to a group isomorphism $\mathrm{GL}(K^{\oplus m}) \otimes_K \mathrm{GL}(K^{\oplus n}) \cong \mathrm{GL}(K^{\oplus mn})$.

Exercise 12.8. (1) *Show that for finite groups G, H, $KG \otimes_K KH$ has a canonical ring structure so that $KG \otimes_K KH \cong K(G \times H)$ as rings.*

(2) *Show that $KG \otimes_K (KG)^{\mathrm{op}}$ has a canonical ring structure so that $KG \otimes_K (KG)^{\mathrm{op}} \cong K(G \times G)$ as rings. Here R^{op} denotes the opposite ring of a ring R whose underlying set is the same as R but has multiplication $a \cdot_{\mathrm{op}} b := ba$.*

One thing that makes group algebras special is that we can always 'tensor within the category of G-representations':

Proposition 12.9. *For any KG-modules V, W, we have a KG-module $V \otimes_K W$ where the action of g is given by $v \otimes w \mapsto gv \otimes gw$.*

Proof Let $\mathcal{B}, \mathcal{C}$ be the K-linear bases of V, W respectively and consider their respective representations $\rho : G \to \mathrm{GL}(V)$ and $\phi : G \to \mathrm{GL}(W)$. Consider the associated matrix representations $[\rho]_{\mathcal{B}} : G \to \mathrm{Mat}_m(K)$ and $[\phi]_{\mathcal{C}} : G \to \mathrm{Mat}_n(K)$. Define a map

$$\Psi : G \to \mathrm{Mat}_{mn}(K) = \mathrm{Mat}_m(K) \otimes \mathrm{Mat}_n(K), \quad g \mapsto [\rho_g]_{\mathcal{B}} \otimes [\phi_g]_{\mathcal{C}},$$

where we are using the Kronecker product of Example 12.7 to define $\Psi(g)$. One can check that Ψ is a group homomorphism (hence, a matrix representation of G); by construction, we have $g \in G$ acts on $V \otimes W$ by $v \otimes w \mapsto gv \otimes gw$. □

Exercise 12.10. *Show that* $\mathrm{triv}_G \otimes_K V \cong V \cong V \otimes_K \mathrm{triv}_G$ *for all KG-module V.*

Detour: Even in good characteristics, tensor products of group (or Hopf algebra in general) representations is still active theme of researches - one typical theme of problem is: For KG-modules V, W, describes the indecomposable direct summands of $V \otimes_K W$.

For example, in the representation theory of symmetric groups (its generalisations such as the Hecke algebra), the Mullineux problem asks for the description of $V \otimes_K \mathrm{sgn}$ for each irreducible V. Another example is McKay correspondence (which has deep implications in algebraic geometry) which comes from looking at representations of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ and relate them under tensoring with the natural representation (SL_2 matrix multiplying on vectors).

Exercise 12.11. *For KG-module V, W, show that there are the following isomorphisms.*

(1) $(V \otimes_K W)^* \cong V^* \otimes_K W^*$ *as KG-modules.*

(2) $V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$ *as* KG*-modules.*

Exercise 12.12. *Suppose* X *is a* G*-set (i.e.* G *acts by permuting elements of* X*) or a* KG*-module, denote by* X^G *the invariant subspace* $\{x \in X \mid gx = x \,\forall g \in G\}$ *of* X*. Let* $U, V, W \in KG\,\mathsf{mod}$*.*

(1) *Show that* $(V^* \otimes_K V)^G \cong \mathrm{End}_{KG}(V)$*.*

(2) *Show that* $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W)$

Exercise 12.13. *Show that, for* G*-representations* V, W*, there is an isomorphism* $\mathrm{Hom}_K(V, W) \cong V^* \otimes_K W$ *of* G*-representations.*

13 Character

From now on until further notice, we take $K = \mathbb{C}$.

Definition 13.1. *Let* ρ *be a representation of* G *over* $\mathbb{C}$*, and* V *be its corresponding* $\mathbb{C}G$*-module. Then the (ordinary) character of* ρ *(or of* V*) is the map*

$$\chi_\rho = \chi_V : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(\rho(g)),$$

where Tr *is the trace function (i.e. sum of all eigenvalues/'diagonal entries'). A character* χ_ρ *is irreducible if the associated representation* ρ *is irreducible.*

In the literature, when χ is the character of ρ, then one often says that ρ or V *affords* χ; we will just use 'associated to' instead for simplicity.

Note that the character of a 1-dimensional representation is just itself. In some geometry-oriented texts, a character is used as a synonym for 1-dimensional representation. The term 'character' has a different definition when considered for representation of Lie groups or Lie algebras; but the essential idea is still somewhat the same - it is a gadget that records the eigenvalues of action.

Definition 13.2. *The degree of a character* χ_V *is* $\dim_{\mathbb{C}} V$*.*

In some literature, degree 1 character are also called *linear character*; we will avoid this terminology.

Example 13.3. *When* $\rho = \mathrm{triv}_G$*, write* $\mathbf{1}_G$ *for its character and call it the trivial character. This is a degree 1 irreducible character.*

In the following, for $z = a + ib \in \mathbb{C}$, denote by $\overline{z}$ its conjugate $a - ib$.

Lemma 13.4. *Let* $\chi = \chi_V$ *be the character of* $\mathbb{C}G$*-module* V*.*

(1) $\deg \chi := \dim_{\mathbb{C}} V = \chi(1)$.

(2) χ_V *is constant on each conjugacy class of* G*.*

(3) $\chi(g)$ *is a sum of* m*-th roots of unity if* $g \in G$ *is of order* m*.*

(4) $\chi(g^{-1}) = \overline{\chi(g)}$ *for any* $g \in G$ *of finite order.*

(5) $\chi(g) \in \mathbb{R}$ *if* g *and* g^{-1} *is in the same conjugacy class.*

(6) $\chi_V = \chi_W$ *if* $V \cong W$ *are isomorphic* $\mathbb{C}G$*-modules.*

Proof (1) Clear since $\chi(1) = \mathrm{Tr}(\mathrm{id}_V)$.

(2) Since $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$ for any linear transformations f, g. We have $\mathrm{Tr}(\rho_{hgh^{-1}}) = \mathrm{Tr}((\rho_h\rho_g)\rho_h^{-1}) = \mathrm{Tr}(\rho_h^{-1}\rho_h\rho_g) = \mathrm{Tr}(\rho_g)$.

(3) $g^m = 1_G$ implies that $\rho_g^m = \mathrm{id}_V$, and so $\lambda^m = 1$ for every eigenvalue λ of ρ_g.

(4) Suppose $\lambda_1, \dots, \lambda_n$ are the eigenvalues (counted with multiplicity, i.e. $n = \dim_{\mathbb{C}} V$) of ρ_g. Since these are roots of unity, we have $\lambda_i^{-1} = \overline{\lambda}$. Hence,

$$\chi_V(g^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\chi_V(g)}.$$

(5) Consequence of (2) and (4).

(6) Suppose $f : V \to W$ is a $\mathbb{C}G$-module isomorphism. Then we have $f\rho_g f^{-1} = \phi_g$ for ρ, ϕ the representations corresponding to V, W respectively. Now we have

$$\chi_W(g) = \mathrm{Tr}(\phi_g) = \mathrm{Tr}(f\rho_g f^{-1}) = \mathrm{Tr}(\rho_g) = \chi_V(g).$$

□

Exercise 13.5. *Show that for a character $\chi = \chi_V$, $\mathrm{Ker}\,\chi := \{g \in G \mid \chi(g) = \chi(1)\}$ is a normal subgroup of G.*

Exercise 13.6. *Show that $\sum_i \chi_i(1)^2 = |G|$ where the sum is over all irreducible characters.*

14 Characters of various constructions

Recall that we can take direct sum and tensor products of representations, which behaves like addition and multiplication respectively. Indeed, this is the case for K-vector spaces, namely, that $\dim K\,\mathsf{mod} \to \mathbb{Z}$ 'sends' $\oplus$ to $+$ and $\otimes$ to $\times$. Note that $\mathbb{C} = \mathbb{C}1$ is the group algebra of the trivial group, and so character of $\mathbb{C}1$ is nothing but just the degree of the character, i.e. $\dim_{\mathbb{C}}$ by Lemma 13.4 (3). Hence, it makes sense to view characters as a generalisation of $\dim_{\mathbb{C}}$.

Lemma 14.1 (Character of direct sum). *For two $\mathbb{C}G$-modules V, W, we have $\chi_{V\oplus W} = \chi_V + \chi_W$.*

Proof Consequence of Lemma 6.9. □

If $\rho = \pi_X$ is a permutation representation associated to G-set X, then χ_ρ is called permutation character; in this case, by abuse of notation we write π_X for χ_{π_X}.

Lemma 14.2 (Permutation character). *For all $g \in G$ and any G-set X, we have $\pi_X(g) = \#X^g$, where $X^g := \{x \in X \mid gx = x\}$ is the set of g-fixed points.*

Proof Consider the matrix corresponding to g-action with respect to the basis X. Then a diagonal entry, say, corresponding to $x \in X$ is non-zero if, and only if, $gx = x$. Moreover, in such a case, the entry is exactly 1. □

Exercise 14.3. *Suppose $\mathbb{C}G$ has r conjugacy classes. Prove that $\pi_G = \sum_{i=1}^{r} \deg(\chi_i)\chi_i$, where $\chi_i = \chi_{S_i}$ is the character of a simple $\mathbb{C}G$-module such that $S_i \not\cong S_j$ for all $i \neq j$. Moreover, determine the value $\chi_V(g)$ for all $g \in G$.*

Recall that for a representation $\rho : G \to \mathrm{GL}(V)$, we have a dual representation $\rho^* : G \to \mathrm{GL}(V^*)$.

Lemma 14.4 (Character of dual representation). *For any $g \in G$, $\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$. In particular, we have the following:*

(1) If V is self-dual, then its character χ_V is real-valued.

(2) If $\chi = \chi_V$ is irreducible, then so is $\overline{\chi}$.

Proof Since $\rho^*(g) = (\rho(g^{-1}))^T$ by definition, the claim follows from Lemma 13.4 (4).

(1) now follows from the definition of self-dual and Lemma 13.4 (4): $V \cong V^*$ implies that $\chi_V(g) = \chi_{V^*}(g) = \overline{\chi_V(g)}$.

(2) follows from Lemma 11.8. $\square$

Lemma 14.5 (Character of tensor product). *Let V, W be two $\mathbb{C}G$-modules. For any $g \in G$, we have $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.*

Proof This follows from the fact that the matrix form of $\rho_{V\otimes W}(g)$ is the Kronecker product (Example 12.7) of those of $\rho_V(g)$ and $\rho_W(g)$. $\square$

15 Class functions

Definition 15.1. *A class function on G is a $\mathbb{C}$-valued function $\psi : G \to \mathbb{C}$ that is constant over each conjugacy class, i.e. $\psi(g) = \psi(h)$ whenever g and h are in the same conjugacy class. Denote by $\mathcal{C}(G)$ the set of all class functions on G.*

For $\psi, \phi \in \mathcal{C}(G)$ and $\lambda \in \mathbb{C}$, define:

(1) *$\lambda\phi$ the class function given by $(\lambda\phi)(g) := \lambda(\phi(g))$;*

(2) *$\psi + \phi$ the class function given by pointwise addition (i.e. $(\psi + \phi)(g) := \psi(g) + \phi(g)$);*

(3) *$\psi\phi$ the class function given by pointwise multiplication (i.e. $(\psi\phi)(g) := \psi(g)\phi(g)$).*

In particular, $\mathcal{C}(G)$ is a $\mathbb{C}$-vector space (and a $\mathbb{C}$-algebra).

From now on, unless otherwise specified, unadorned $\otimes$ means $\otimes_{\mathbb{C}}$.

Lemma 15.2. *A character is a class function on G.*

Proof Immediate from Lemma 13.4 (2). $\square$

Exercise 15.3. *Write $\overline{\chi_V}$ the function $g \mapsto \overline{\chi_V(g)}$. Show that $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \overline{\chi_V}\chi_W$.*

For ease of exposition, we take $G = C_1 \sqcup \ldots \sqcup C_r$ the decomposition of G into conjugacy classes. We also take representatives $g_1, \ldots, g_r$ with $g_i \in C_i$, and assume *always* that $g_1 = 1_G$.

Definition 15.4. *The characteristic function δ_j associated to conjugacy class C_j is the class function given by*

$$\delta_j(g) := \begin{cases} 1, & g \in C_j; \\ 0, & g \notin C_j. \end{cases}$$

Lemma 15.5. $\dim_{\mathbb{C}} \mathcal{C}(G)$ *is the number of conjugacy classes of G.*

Proof Suppose there are r conjugacy classes of G. Then it follows from Lemma 13.4 (2) that $\{\delta_1, \ldots, \delta_r\}$ form a basis of $\mathcal{C}(G)$. $\square$

Recall that there are exactly the number of (isomorphism classes of) irreducible representations also coincide with the number of conjugacy classes of G.

Definition 15.6. *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. The character table of G is the matrix $(\chi_i(g_j))_{1\le i,j\le r}$.*

In practice, we draw the character table with a heading row labelled by the conjugacy classes (or their representatives) and a heading column labelled by the irreducible characters.

The usual convention also takes the first row to be the trivial character $\chi_1 = \chi_{\mathrm{triv}}$ (and so the first row is just a row of 1's), and the first column to be the conjugacy class $C_1 = \{1\}$ (and so the first column tells us the dimension of each irreducible representation). In the symmetric group case, it is also usual to take the second row to be the character associated to the sign representation $\chi_2 = \chi_{\mathrm{sgn}}$.

Example 15.7 (Character table of C_n). *Each element of $C_n = \langle g \mid g^n = 1\rangle$ is a conjugacy classes of its own. From our previous study on irreducible representations of finite abelian group, we can take χ_k, with $1 \le k \le n$, to be the character of the irreducible representation where g acts by ξ^{k-1} for $\xi := \exp(2\pi i/n)$.*

Hence, the character table is of the form

	1	g	$g^j (1 \le j \le n)$
χ_1	1	1	1
χ_2	1	ξ	ξ^j
χ_k	1	ξ^k	ξ^{kj}

Example 15.8 (Character table of $D_6 \cong \mathfrak{S}_3$). *We have $D_6 = \langle a, b \mid a^3 = 1 = b^2, abab = 1\rangle$. There are three conjugacy classes*

$$C_1 = \{1\}, \quad C_2 = \{b, ab, a^2b\}, \quad C_3 = \{a, a^2\}.$$

We have also seen three irreducible representations: trivial, sign, and a 2-dimensional representation (Example 6.3(3)) given by

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have the following character table.

	1	b	a
$\chi_{(3)}$	1	1	1
$\chi_{(1^3)}$	1	-1	1
$\chi_{(2,1)}$	2	0	-1

Here we use a slightly weird labelling of the irreducible characters. They correspond to the partitions of the number 3.

16 Inner product on class functions

We now take a closer look to the space $\mathcal{C}(G)$ of class functions.

Recall that an inner product on a $\mathbb{C}$-vector space X is a non-degenerate Hermitian form $\langle -, -\rangle : X \times X \to \mathbb{C}$, i.e.

(1) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$;

(2) $\langle z, \lambda x + \mu y\rangle = \lambda\langle z, x\rangle + \mu\langle z, y\rangle$ for all $\lambda, \mu \in \mathbb{C}$ and all $x, y, z \in X$;

(3) $\langle x, x\rangle \in \mathbb{R}_{>0}$ for all non-zero $x \in X$.

Note that (1) and (2) combines to $\langle \lambda x + \mu y, z\rangle = \overline{\lambda}\langle x, z\rangle + \overline{\mu}\langle y, z\rangle$.

Definition 16.1. *For $\chi, \psi \in \mathcal{C}(G)$, define*

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \psi(g)$$

It is easy to check that this defines an inner product on $\mathcal{C}(G)$.

Exercise 16.2. *Show that $\langle \pi_X, \mathbf{1}_G \rangle$ is the number of G-orbits on the G-set X.*

Recall that for $g \in G$, its centraliser subgroup is $C_G(g) := \{h \in G \mid hgh^{-1} = g\}$, i.e. the stabiliser subgroup of $g \in G$ under conjugation (=adjoint) action of G on G itself. Recall that, by the orbit-stabiliser theorem, we have

$$|G| = |C_G(g_i)| \cdot |C_i|,$$

where C_i is a conjugacy class of G containing g_i.

Example 16.3. *We have*

$$\langle \delta_i, \delta_j \rangle = \frac{1}{|G|} \delta_{i,j} |C_i| = \frac{\delta_{i,j}}{|C_G(g_i)|}, \text{ and } \langle \delta_i, \chi \rangle = \frac{1}{|G|} \sum_{g \in C_i} \chi(g) = \frac{\chi(g)}{|C_G(g_i)|}.$$

Proposition 16.4. *Let $\chi, \psi \in \mathcal{C}(G)$.*

(1) *If χ, ψ are characters, then $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle \in \mathbb{R}$.*

(2) *If $g_1, \ldots, g_r$ are representatives of the conjugacy classes of G, then* $\langle \chi, \psi \rangle = \sum_{i=1}^{r} \frac{\overline{\chi(g_i)} \psi(g_i)}{|C_G(g_i)|}$.

Proof (1) Since $\overline{\chi(g)} = \chi(g^{-1})$ by Lemma 13.4 (4), we have

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) \psi(g) = \frac{1}{|G|} \sum_{h \in G} \chi(h) \psi(h^{-1}) = \langle \psi, \chi \rangle,$$

where the second equality follows from taking $h := g^{-1}$. But $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$ as $\langle -, - \rangle$ is an inner product, so $\langle \chi, \psi \rangle \in \mathbb{R}$.

(2) Similar to Example 16.3, we have

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{i=1}^{r} \frac{|G|}{|C_G(g_i)|} \overline{\chi(g_i)} \psi(g_i) = \sum_{i=1}^{r} \frac{\overline{\chi(g_i)} \psi(g_i)}{|C_G(g_i)|}$$

as required. □

17 Inner product vs homomorphisms

The aim of this section is the following result.

Theorem 17.1. *For any $\mathbb{C}G$-modules V, W, we have*

$$\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}G}(V, W).$$

In particular, any inner product of characters is always integer-valued.

To show this, we first consider how to extract homomorphism from the space of K-linear maps.

Note that, since $\operatorname{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$ and the right-hand side has $\mathbb{C}G$-module structure, the Hom-space is also a $\mathbb{C}G$-module. Carefully reading the isomorphism shows that g-action is given by $(g \cdot f)(v) = g(f(g^{-1}v))$ for all $v \in V$.

Lemma 17.2. $\mathrm{Hom}_{\mathbb{C}G}(V,W) = \mathrm{Hom}_{\mathbb{C}}(V,W)^G := \{f \mid g\cdot f = f\}$.

Proof For $f \in \mathrm{Hom}_{\mathbb{C}}(V,W)$, we have

$$\begin{aligned} f \in \mathrm{Hom}_{\mathbb{C}G}(V,W) &\Leftrightarrow g(f(v)) = f(gv) \ \ \forall g, v \\ &\Leftrightarrow (g\cdot f)(v) = gf(g^{-1}v) = g(g^{-1}f(v)) = f(v) \ \ \forall v. \end{aligned}$$

The claim now follows. □

Recall that $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is a G-representation, so we want to determine $\dim_{\mathbb{C}} U^G$ for a G-representation U.

Lemma 17.3. *For a $\mathbb{C}G$-module U, we have $\dim_{\mathbb{C}} U^G = \frac{1}{|G|}\sum_{g\in G}\chi_U(g)$.*

Proof Consider the element

$$x := \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{C}G.$$

Note that (see Homework 1) $|G|x$ is the generator of the trivial representation, and so $hx = x$ for all $h \in G$. Define a K-linear map $\pi : U \to U$ given by $v \mapsto xv = \frac{1}{|G|}\sum_{g\in G} gv$. Then we have

$$h(\pi(v)) = h(xv) = (hx)v = xv = \pi(v)$$

for all $h \in G$, and so $\pi(v) \in U^G$. Since $U^G \subset U$ and $\pi|_{U^G} = \mathrm{id}$, we have $\mathrm{Im}(\pi) = U^G$. In particular, we have

$$\dim_{\mathbb{C}} U^G = \mathrm{Tr}(\pi) = \mathrm{Tr}\left(\sum_{g\in G}\frac{1}{|G|}\rho_g\right) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\,\rho_g = \frac{1}{|G|}\sum_{g\in G}\chi_U(g)$$

as required. □

Proof of Theorem 17.1 Using Lemma 17.2 first, and then Lemma 17.3 (with $U = \mathrm{Hom}_{\mathbb{C}}(V,W)$ therein), we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V,W) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V,W)^G = \frac{1}{|G|}\sum_{g\in G}\chi(g),$$

where χ is the character of $\mathrm{Hom}_{\mathbb{C}}(V,W)$. Since $\mathrm{Hom}_{\mathbb{C}}(V,W) \cong V^* \otimes W$ as $\mathbb{C}G$-modules, we have

$$\chi(g) = \chi_{V^*\otimes W}(g) = \chi_{V^*}(g)\chi_W(g) = \overline{\chi_V(g)}\chi_W(g).$$

Substitute this back into the previous formula yields the claim. □

Corollary 17.4. *Suppose that $\mathbb{C}G$ has r simple modules $S_1, \dots, S_r$ with characters $\chi_1, \dots, \chi_r$ respectively. Then the following hold.*

(1) $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$.

(2) $\{\chi_i\}_{1\le i\le r}$ *is an orthonormal (with respect to $\langle -, - \rangle$) basis of $\mathcal{C}(G)$.*

(3) $[V : S_i] = \langle \chi_i, \chi_V \rangle$ *and* $\chi_V = \sum_{i=1}^r \langle \chi_i, \chi_V\rangle \chi_i$ *for all $\mathbb{C}G$-module V.*

(4) *We have*

$$\langle \chi_V, \chi_V \rangle = \sum_{i=1}^r \langle \chi_i, \chi_V \rangle^2$$

for all $\mathbb{C}G$-module V.

Proof (1) Combine Theorem 17.1 with Schur's lemma.

(2) By (1), we have $\{\chi_i\}_{1\leq i\leq r}$ is an orthonormal set of vectors in $\mathcal{C}(G)$. In particular, it is linear independent. By Lemma 15.5, we have $\dim_{\mathbb{C}} \mathcal{C}(G) = r$, and so $\{\chi_i\}_{1\leq i\leq r}$ is a maximal linear independent set. Now the claim follows.

(3) Apply Theorem 17.1 to Proposition 8.5.

(4) Combines (2) and (3). □

The following result which tells us that characters not only are representation-invariant, but can also tell apart non-isomorphic representations!, i.e. a *complete invariant* of representations.

Theorem 17.5. *For any $\mathbb{C}G$-module V, W, we have $V \cong W$ as $\mathbb{C}G$-module if and only if $\chi_V = \chi_W$.*

Proof Note that the $\Rightarrow$ direction is already shown in Lemma 13.4 (6). We can do both direction simultaneously now as follows:

$$\begin{aligned} V \cong W &\Leftrightarrow [V : S_i] = [W : S_i] \;\; \forall 1 \leq i \leq r \\ &\Leftrightarrow \langle \chi_i, \chi_V \rangle = \langle \chi_i, \chi_W \rangle \;\; \forall 1 \leq i \leq r \\ &\Leftrightarrow \chi_V = \chi_W \end{aligned}$$

by repeated use of Corollary 17.4 (3). □

Example 17.6. *Consider $G = C_3 = \langle g \mid g^3 = 1 \rangle$. Let ρ, ρ' be representations given by*

$$\rho_g = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \rho'_g = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then we have $\chi_\rho(1) = 2 = \chi_{\rho'}(1)$, $\chi_\rho(g) = -1 = \chi_{\rho'}(g)$, and $\chi_\rho(g^2) = -1 = \chi_{\rho'}(g^2)$. Hence, we have $\rho \cong \rho'$. This is much more difficult to see on the level of representation or $\mathbb{C}G$-module as one needs to find an appropriate change of basis.

We see one application of our above investigation.

Corollary 17.7. *Suppose V, W are simple $\mathbb{C}G$-modules with $\dim_{\mathbb{C}} W = 1$. Then $V \otimes W$ is also a simple $\mathbb{C}G$-module.*

Proof By Corollary 17.4, it suffices to show that $\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle = 1$. First note that, as W is 1-dimensional, the character χ_W of W is exactly the representation $\rho : G \to \mathbb{C}^\times$ associated to W. In particular, we have $\overline{\chi_W(g)} = \chi_W(g^{-1}) = \rho(g^{-1}) = \rho(g)^{-1}$, which implies that $\overline{\chi_W(g)}\chi_W(g) = \rho(g)^{-1}\rho(g) = 1$. Now we just need to compute the inner product

$$\begin{aligned} \langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle &= \langle \chi_V \chi_W, \chi_V \chi_W \rangle \\ &= \frac{1}{|G|} \sum_{g\in G} \overline{\chi_V(g)\chi_W(g)} \chi_V(g)\chi_W(g) \\ &= \frac{1}{|G|} \sum_{g\in G} |\chi_W(g)|^2 \overline{\chi_V(g)} \chi_V(g) \\ &= \frac{1}{|G|} \sum_{g\in G} \overline{\chi_V(g)} \chi_V(g) = \langle \chi_V, \chi_V \rangle = 1, \end{aligned}$$

as required. □

18 Orthogonality theorems

Theorem 18.1 (Row orthogonality). *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Then the following hold.*

$$\langle \chi_s, \chi_t \rangle = \sum_{i=1}^{r} \frac{\overline{\chi_s(g_i)}\chi_t(g_i)}{|C_G(g_i)|} = \delta_{s,t}$$

for any $1 \leq s, t \leq r$.

Proof Apply Proposition 16.4 (2) to Corollary 17.4 (1). □

Lemma 18.2. *The matrix $U := (u_{i,j})_{1 \leq i,j \leq r}$ given by*

$$u_{i,j} := \frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}$$

is a unitary matrix*, i.e. invertible with $U^{-1} = \overline{U^T}$. In particular, the character table of G is invertible.*

Proof By Theorem 18.1, we have

$$\delta_{i,j} = \langle \chi_i, \chi_j \rangle = \sum_{k=1}^{r} \frac{\overline{\chi_i(g_k)}\chi_j(g_k)}{|C_G(g_k)|} = \sum_{k=1}^{r} \overline{u_{k,i}} u_{k,j}.$$

This means that the identity matrix $I = (\delta_{i,j})_{1 \leq i,j \leq r}$ is given by $\overline{U^T}U$; the claim now follows. □

Theorem 18.3 (Column orthogonality). *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Then the following hold.*

$$\sum_{k=1}^{r} \overline{\chi_k(g_s)}\chi_k(g_t) = \delta_{s,t}|C_G(g_t)|$$

for any $1 \leq s, t \leq r$.

Proof Lemma 18.2 says that $\overline{U^T}U = I$, which is equivalent to

$$\delta_{s,t} = \sum_{k=1}^{r} \overline{u_{k,s}} u_{k,t} = \sum_{k=1}^{r} \frac{\overline{\chi_k(g_s)}\chi_k(g_t)}{|C_G(g_s)|},$$

as required. □

We can also refine Corollary 17.4 (3).

Proposition 18.4. *For any class function $\psi \in \mathcal{C}(G)$, we have $\psi = \sum_{i=1}^{r} \langle \psi, \chi_i \rangle \chi_i$.*

Proof Consider the character table matrix $X := (\chi_i(g_j))_{1 \leq i,j \leq r}$. This is the change of basis matrix from $\{\chi_i\}_i$ to $\{\delta_j\}_j$. By Lemma 18.2, the inverse of X is given by $M := (m_{i,j})_{1 \leq i,j \leq r}$ where

$$m_{i,j} := \langle \delta_j, \chi_i \rangle = \frac{\overline{\chi_i(g_j)}}{|C_G(g_j)|}.$$

Hence, M is the change of basis matrix from $\{\delta_j\}_j$ to $\{\chi_i\}_i$.

Since $\psi = \sum_{j=1}^{r} \psi(g_j)\delta_j$, applying M yields

$$\psi = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} \frac{\overline{\chi_i(g_j)}}{|C_G(g_j)|} \psi(g_j) \right) \chi_i$$

which yields $\sum_{i=1}^{r} \langle \psi, \chi_i \rangle \chi_i$ by Lemma 16.4 (2). □

19 Inflation from normal subgroup

In this section, we aim to *lift* characters of the quotient group G/N for some non-trivial normal subgroup $N \lhd G$ to characters of G. Thus giving more toolbox for us to figure out full character table.

Let us first look at it on the representation level. Since we have a canonical projection $p_N : G \twoheadrightarrow G/N$ (given by $g \mapsto gN$), a representation (group homomorphism) $\rho : G/N \to \mathrm{GL}(V)$ of G/N natural extends to a representation $\tilde{\rho} := (\rho \circ p_N) : G \to \mathrm{GL}(V)$ of G. We call $\tilde{\rho}$ the *inflation* of ρ (by N), or the *lift* of ρ. Same terminology applies to characters, for $\chi = \chi_\rho$, it is often to simply write $\tilde{\chi}$ for $\chi_{\tilde{\rho}}$.

Lemma 19.1. *For a non-trivial normal subgroup $N \lhd G$ and a representation $\rho : G/N \to \mathrm{GL}(V)$ of G/N with associated character $\chi = \chi_\rho$. The following hold.*

(1) *ρ is irreducible if and only if $\tilde{\rho}$ is irreducible.*

(2) *$\tilde{\chi}(g) = \chi(gN)$. In particular, we have $\deg \tilde{\chi} = \deg \chi$.*

(3) *There is a bijection of representation (up to isomorphism):*

$$\{\textit{irred. rep's of } G/N\} \leftrightarrow \{\textit{irred. rep's of } G \textit{ with kernel } N\}$$

given by $\rho \mapsto \tilde{\rho}$.

Proof (1) By Theorem 17.5, it is enough to show that $\langle \tilde{\chi}, \tilde{\chi} \rangle_G = 1$ if and only if $\langle \chi, \chi \rangle_{G/N} = 1$. Indeed,

$$\begin{aligned}
\langle \tilde{\chi}, \tilde{\chi} \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\tilde{\chi}(g)} \tilde{\chi}(g) \\
&= \frac{1}{|G|} \sum_{gN \in G/N} \sum_{n \in N} \overline{\tilde{\chi}(gn)} \tilde{\chi}(gn) \\
&= \frac{1}{|G|} \sum_{gN \in G/N} \sum_{n \in N} \overline{\chi(gN)} \chi(gN) \\
&= \frac{1}{|G|} \sum_{gN \in G/N} |N| \overline{\chi(gN)} \chi(gN) \\
&= \frac{1}{|G/N|} \sum_{gN \in G/N} \overline{\chi(gN)} \chi(gN) = \langle \chi, \chi \rangle_{G/N}
\end{aligned}$$

(2) Directly computation: $\tilde{\chi}(g) = \mathrm{Tr}(\tilde{\rho}_g) = \mathrm{Tr}(\rho_{gN}) = \chi(gN)$ for any $g \in G$.

(3) By definition, we have $\tilde{\rho}(n) = \rho(1_G N)$ for all $n \in N$, and so $\mathrm{Ker}\, \tilde{\rho} \geq N$. Hence, combining with (1), we have that $\rho \mapsto \tilde{\rho}$ is a well-defined map on the stated sets.

Suppose that $\theta : G \to \mathrm{GL}(V)$ is a representation with $\mathrm{Ker}\, \theta \geq N$. Consider the assignment $\rho : G/N \to \mathrm{GL}(V)$ given by $\rho_{gN} := \theta_g$. Let us check that ρ is a well-defined group homomorphism. Indeed, if $gN = g'N$, then $g^{-1}g' \in N$, and so $\theta_{g^{-1}g'} = \mathrm{id}$ by $\mathrm{Ker}\, \theta \geq N$. Since θ itself is a gorup homomorphism, we have

$$\theta_g^{-1} \theta_{g'} = \theta_{g^{-1}} \theta_{g'} = \theta_{g^{-1}g'} = \mathrm{id},$$

which means that

$$\rho_{gN} = \theta_g = \theta_{g'} = \rho_{g'N}.$$

It is routine to check that ρ is a group homomorphism. By direct computation, we have that χ_θ is the same as the lifted character $\chi_{\tilde{\rho}}$, and so $\theta \cong \tilde{\rho}$ by Theorem 17.5. In particular, the construction of ρ from θ is inverse to inflation, and vice versa. $\square$

It turns out that there is a normal subgroup of G allows us to obtain ALL 1-dimensional (irreducible) representations. Recall that the *derived subgroup* G', or *commutator subgroup*, of G is the generated by all elements of the form

$$[g,h] := ghg^{-1}h^{-1}$$

for $g,h \in G$. Note that this does not mean all elements of G' are of the form $[g,h]$, but rather the identity element 1_G or $[g_1,h_1][g_2,h_2]\cdots[g_n,h_n]$ for some $n \geq 1$. Note also that $[g,h]^{-1} = [h,g]$.

Lemma 19.2. *G' is the unique minimal normal subgroup of G whose quotient is abelian, i.e. G/N abelian $\Leftrightarrow N \geq G'$.*

Proof Take any $k \in G$. Then we have

$$\begin{aligned} k[g,h]k^{-1} &= k(ghg^{-1}h^{-1})k^{-1} = (kg)hg^{-1}(k^{-1}k)(h^{-1}k^{-1}) \\ &= \big((kg)h(kg)^{-1}h^{-1}\big)\big(hkh^{-1}k^{-1}\big) = [kg,h][h,k] \in G'. \end{aligned}$$

In particular, we have

$$k([g_1,h_1][g_2,h_2]\cdots[g_n,h_n])k^{-1} = \big(k[g_1,h_1]k^{-1}\big)\big(k[g_2,h_2]k^{-1}\big)\cdots\big(k[g_n,h_n]k^{-1}\big) \in G'.$$

Hence, G' is a normal subgroup of G.

Consider a normal subgroup $N \lhd G$ and $g,h \in G$. Then we have

$$[g,h] = ghg^{-1}h^{-1} \in N \Leftrightarrow ghN = hgN \Leftrightarrow (gN)(hN) = (hN)(gN).$$

Thus, $N \geq G'$ if and only if G/N is abelian. □

Proposition 19.3. *Let $\ell := |G|/|G'|$. Then G has precisely ℓ (irreducible) representations (up to isomorphism) of dimension 1, all of which are obtained by lifting the irreducible representations of G/G'.*

Proof By Proposition 9.6, we know there G/G' has exactly ℓ irreducible representations, all of which are of 1-dimensional. Thus, these lifts to 1-dimensional (irreducible) representations of G. By Lemma 19.1 (3), these representations all have kernel containing G'.

Suppose that $\rho : G \to \mathbb{C}^\times$ is a 1-dimensional representation of G. Then we have

$$\rho([g,h]) = \rho_g\rho_h\rho_g^{-1}\rho_h^{-1}.$$

But ρ is a group homomorphism and $\mathbb{C}^\times$ is abelian, and so the above formula evaluates to $1 \in \mathbb{C}$. Thus, we have $\operatorname{Ker}\rho \geq G'$, and so it follows from Lemma 19.1 (3) that ρ must be a lift of some representation of of G/G'. □

Example 19.4. *For all $n \geq 3$, the derived subgroup of the symmetric group $\mathfrak{S}_n$ of rank n is the alternating group $\mathfrak{A}_n$ of rank n. In particular, $\mathfrak{S}_n$ has exactly two characters of degree 1, namely, the trivial character and sign character.*

Proof Since $\mathfrak{S}_n/\mathfrak{A}_n \cong C_2$ is abelian, we have by Lemma 19.2 $\mathfrak{A}_n \geq \mathfrak{S}_n'$. For the reverse inclusion, recall that $\mathfrak{A}_n$ can be generated by 3-cycles of $\mathfrak{S}_n$. Notice that

$$(123) = (132)(12)(132)^{-1}(12)^{-1} = [(132),(12)] \in \mathfrak{S}_n',$$

and so $\mathfrak{A}_n \leq \mathfrak{S}_n'$. The final statement now follows from Proposition 19.3. □

Example 19.5 (Character table of $\mathfrak{S}_4$). *For symmetric groups, the conjugacy classes are determined by cycle-type, so for $G = \mathfrak{S}_4$ we can take the following representatives of its conjugacy classes:*

$$g_1 = 1, g_2 = (12), g_3 = (123), g_4 = (12)(34), g_5 = (1234).$$

Consider the following Klein 4-group $V_4 \cong C_2 \times C_2$.

$$N = V_4 = \{1, (12)(34), (13)(24), (14)(23)\}.$$

It is routine to check that $N \lhd G$. Let $a := (123)N$ and $b := (12)N$, then we have

$$G/N = \langle a, b \mid a^3 = N = b^2, abab = N \rangle,$$

i.e. $G/N \cong D_6$. Recall from Example 15.8 the character table of D_6. Lifting this to G yields the following character table:

C_i	1	6	8	3	6
g_i	1	(12)	(123)	(12)(34)	(1234)
$\chi_1 = \tilde{\chi}_1$	1	1	1	1	1
$\tilde{\chi}_{(1^3)}$	1	-1	1	1	-1
$\tilde{\chi}_{(2,1)}$	2	0	-1	2	0
χ	x_1	x_2	x_3	x_4	x_5
χ'	y_1	y_2	y_3	y_4	y_5

Note that $(1234) \cdot (12)(34) = (13)$, and so $\tilde{\chi}_{(2,1)}(1234) = \tilde{\chi}_{(2,1)}(13) = \tilde{\chi}_{(2,1)}(11) = 0$.

We can then calculate the rest of table using (only!) column orthogonality. We will do this column-by-column, so we will drop the subscripts in the following.

Col 1:

- *(Col 1 vs Col 1)* $\Rightarrow \quad x^2 + y^2 + 6 = 24$. *Since x, y must be positive integers, by slowly trying out $x = 1, 2, 3, 4$ one can see that $x = y = 3$ is the only possible solution.*

Col 2:

- *(Col 1 vs Col 2)* $\Rightarrow \quad 3x + 3y = 0 \quad \Rightarrow \quad x = -y$
- *(Col 2 vs Col 2)* $\Rightarrow \quad 2 + x^2 + y^2 = 4 \quad \Rightarrow \quad 2x^2 = 2 \quad \Rightarrow \quad (x, y) = (1, -1)$ *(at this stage, we can pick whichever sign without loss of generality)*

Col 3:

- *(Col 2 vs Col 3)* $\Rightarrow \quad x - y = 0 \quad \Rightarrow \quad x = y$
- *(Col 3 vs Col 3)* $\Rightarrow \quad 3 + x^2 + y^2 = 3 \quad \Rightarrow \quad 2x^2 = 0 \quad \Rightarrow \quad x = 0 = y$

Col 4:

- *(Col 2 vs Col 4)* $\Rightarrow \quad x - y = 0 \quad \Rightarrow \quad x = y$
- *(Col 1 vs Col 4)* $\Rightarrow \quad 6 + 3x + 3y = 0 \quad \Rightarrow \quad 2x = -2 \quad \Rightarrow \quad x = -1 = y$

Col 5:

- *(Col 2 vs Col 5)* $\Rightarrow \quad 2 + x - y = 0 \quad \Rightarrow \quad x = y - 2$
- *(Col 4 vs Col 5)* $\Rightarrow \quad x + y = 0 \quad \Rightarrow \quad 2x = 2 \quad \Rightarrow \quad (x, y) = (-1, 1)$

Thus, we have

$\lvert C_i \rvert$	1	6	8	3	6
g_i	1	(12)	(123)	(12)(34)	(1234)
$\chi_1 = \tilde{\chi}_1$	1	1	1	1	1
$\tilde{\chi}_{(1^3)}$	1	-1	1	1	-1
$\tilde{\chi}_{(2,1)}$	2	0	-1	2	0
χ	3	1	0	-1	-1
χ'	3	-1	0	-1	1

20 Fixed points, orbits, permutation character

We answer Exercise 16.2 here.

Lemma 20.1. *Let X be a G-set and π_X the associated permutation character. Then $\langle \pi_X, \mathbf{1}_G \rangle$ is the number of G-orbits on X. In particular,* triv_G *is always a direct summand of $\mathbb{C}X$.*

Proof Consider first the case when when G acts transitively on X. Now by Lemma 14.2 and exchange of summation we have

$$\begin{aligned}\langle \pi_X, \mathbf{1} \rangle &= \frac{1}{|G|} \sum_g \pi(g) = \frac{1}{|G|} \sum_g \# X^g = \frac{1}{|G|} \sum_g \# \{x \in X \mid gx = x\} \\ &= \frac{1}{|G|} \# \{(g, x) \in G \times X \mid gx = x\} \\ &= \frac{1}{|G|} \sum_{x \in X} |\mathrm{Stab}_G(x)|\end{aligned}$$

By the orbit-stabiliser theorem we have

$$\langle \pi_X, \mathbf{1} \rangle = \frac{1}{|G|} \sum_{x \in X} \frac{|G|}{|X|} = \frac{1}{|G|} \cdot |X| \cdot \frac{|G|}{|X|} = 1.$$

This proves the claim when G acts transitively. In general, partitioning X into orbits $X_1 \sqcup \cdots \sqcup X_m$ yields $\mathbb{C}X = \mathbb{C}X_1 \oplus \cdots \oplus \mathbb{C}X_m$ (details are left as Homework 2), and so the claim follows.

The final statement is immediate from Corollary 17.4 (3), which says that $\mathbb{C}X \cong \mathrm{triv}^{\oplus \langle \pi_X, \mathbf{1} \rangle} \oplus U$ for some $\mathbb{C}G$-module U. $\square$

Example 20.2. *The symmetric group $\mathfrak{S}_n$ acts on n letters $[n] := \{1, 2, \ldots, n\}$ transitively. Hence, we have $\langle \pi_{[n]}, \mathbf{1} \rangle = 1$. In particular, $\pi_{[n]} - \mathbf{1}$ is character of some $\mathbb{C}\,\mathfrak{S}_n$-module.*

In the case when $n = 3$, recall we have the following characters:

$\lvert C_i \rvert$	1	3	2
g_i	1	(12)	(123)
$\pi_{[3]}$	3	1	0
$\pi_{[3]} - \mathbf{1}$	2	0	-1

One can remember from Example 15.8 that this is the character associated to the 2-dimensional irreducible representation of $\mathfrak{S}_3 \cong D_6$.

For two G-sets X, Y, we can form a product $X \times Y$ that is naturally a G-set with diagonal G-action:

$$g(x, y) := (gx, gy)$$

for all $g \in G, x \in X, y \in Y$.

Proposition 20.3. *Suppose that X, Y are G-sets. Then $\langle \pi_X, \pi_Y \rangle$ is the number of G-orbits on the product G-set $X \times Y$.*

Proof For any $g \in G$, we have

$$(X \times Y)^g = \{(x, y) \in X \times Y \mid g(x, y) = (x, y)\} = X^g \times Y^g.$$

Thus, by Lemma 14.2 and Lemma 20.1, we have

$$\begin{aligned}\langle \pi_X, \pi_Y \rangle &= \frac{1}{|G|}\sum_g \overline{\pi_X(g)}\pi_Y(g) \\ &= \frac{1}{|G|}\sum_g \#X^g \cdot \#Y^g \\ &= \frac{1}{|G|}\sum_g \#(X \times Y)^g \\ &= \langle \pi_{X\times Y}, \mathbf{1}_G \rangle \\ &= \#(G\text{-orbits of } X \times Y),\end{aligned}$$

as required. □

Definition 20.4. *A G-action on a G-set X is called* 2-transitive *if the number of orbits of the diagonal G-action on $X \times X$ is precisely 2. Equivalently, $\forall x \neq y$ and $\forall x' \neq y'$ with $x, y, x', y' \in X$, $\exists g \in G$ such that $gx = x', gy = y'$.*

Note that 2-transitive implies transitive G-action on X.

Example 20.5. *$\mathfrak{S}_n$ acts 2-transitively on $[n]$ for any $n \geq 3$. However, when $n = 3$, the alternating group $\mathfrak{A}_3 = \{1, a := (123), b := (132)\}$ acts transitively on $[3]$ but not 2-transitively. Indeed, we have*

$$a(1,3) = (2,1) \text{ and } b(1,3) = (3,2),$$

which means that $(1,2)$ is not in the $\mathfrak{A}_3$-orbit of $(1,3)$.

Corollary 20.6. *If G-action on X is 2-transitive, then the character $\pi_X - \mathbf{1}_G$ is irreducible.*

Proof We have

$$\begin{aligned}\langle \pi_X - \mathbf{1}_G, \pi_X - \mathbf{1}_G \rangle &= \langle \pi_X, \pi_X \rangle - \langle \pi_X, \mathbf{1}_G \rangle - \langle \mathbf{1}_G, \pi_X \rangle + \langle \mathbf{1}_G, \mathbf{1}_G \rangle \\ &= \langle \pi_X, \pi_X \rangle - 2\langle \pi_X, \mathbf{1}_G \rangle + 1\end{aligned}$$

Thus, $\pi_X - \mathbf{1}_G$ is irreducible if and only if the last line evaluates to 1.

By Lemma 20.3, G-action on X is 2-transitive if and only if $\langle \pi_X, \pi_X \rangle = 2$. Since 2-transitive implies transitive, we also have $\langle \pi_X, \mathbf{1}_G \rangle = 1$ by Lemma 20.1. Substituting these values to the above calculation yields the $\langle \pi_X - \mathbf{1}, \pi_X - \mathbf{1} \rangle = 1$ as required. □

Example 20.7. *For any $n \geq 3$, $\pi_{[n]} - \mathbf{1}$ is an irreducible character of $\mathfrak{S}_n$.*

21 Restriction and restricted character

Note that the ground field K can be anything in the definition below, but we will take $K = \mathbb{C}$ whenever we talk about characters.

Definition 21.1. *Suppose that we have a subgroup $H \leq G$ and G-representation $\rho : G \to \mathrm{GL}(V)$ (equivalently, KG-module V). Then the* restriction $\mathrm{Res}^G_H(\rho)$ *(or $\rho \downarrow^G_H$) of ρ, is the H-representation given by the composition $H \hookrightarrow G \xrightarrow{\rho} \mathrm{GL}(V)$.*

Equivalently, the restriction of V is the KH-module $\mathrm{Res}^G_H(V)$ (or $V \downarrow^G_H$) given by same K-vector space V where we only remember the action of the elements in H.

We may omit the superscript G and subscript H if context is clear.

Lemma 21.2. *For any $\mathbb{C}G$-module V and subgroup $H \leq G$, we have restricted character $\chi_V \downarrow_H := \chi_{\mathrm{Res}^G_H(V)}$ given by $\chi_V \downarrow_H (h) := \chi_V(h)$ for all $h \in H$.*

Proof Clear from definition of $\mathrm{Res}^G_H(V)$. $\square$

We can also define *restricted class function* $\psi \downarrow_H \in \mathcal{C}(H)$ for any class function $\psi \in \mathcal{C}(G)$ given by

$$\psi \downarrow_H (h) = \psi(h) \ \ \forall h \in H.$$

By the above lemma and the fact that irreducible characters form a basis of the space of class function, we have

$$\psi = \sum_i a_i \chi_i \ \ \Rightarrow \ \ \psi \downarrow_H = \sum_i a_i \, (\chi_i \downarrow_H).$$

In general, restriction does not preserve simplicity (irreducibility). In the case, when $K = \mathbb{C}$, we know that $\mathrm{Res}(V)$ is a direct sum of simple $\mathbb{C}H$-modules. So one natural question is whether *all* simple $\mathbb{C}H$-module can appear as a direct summand of restriction.

Lemma 21.3. *For all irreducible character ψ of H, there exists an irreducible character χ of G such that $\langle \mathrm{Res}^G_H \chi, \psi \rangle_H \neq 0$. In other words, for the corresponding simple $\mathbb{C}H$-module $U = U_\psi$ and simple $\mathbb{C}G$-module $V = V_\chi$ we have $\mathrm{Res}(V) \cong U \oplus U'$ as $\mathbb{C}H$-modules.*

Proof Recall from Proposition 10.3 that every irreducible representation is a direct summand (up to isomorphism) of the regular representation. Hence, we have

$$\langle \pi_{\mathrm{reg}} \downarrow_H, \psi \rangle_H = \sum_{i=1}^r \langle (d_i \chi_i) \downarrow_H, \psi \rangle_H = \sum_{i=1}^r d_i \langle \chi_i \downarrow_H, \psi \rangle_H,$$

where π_{reg} is the character corresponding to the regular representation. Note that, in the last equality, we used the fact that $(\chi + \chi') \downarrow_H = \chi \downarrow_H + \chi' \downarrow_H$, or equivalently, $\mathrm{Res}(V \oplus V') \cong \mathrm{Res}(V) \oplus \mathrm{Res}(V')$; both of them are straightforward from the definition (albeit possibly not immediate at first glance).

Now note that the regular representation itself is a permutation representation $\mathbb{C}X$ associated to the G-set $X = G$. Hence, it follows from the formula for permutation character (Lemma 14.2) that

$$\pi_{\mathrm{reg}}(g) = \begin{cases} |G|, & \text{if } g = 1; \\ 0, & \text{if } g \neq 1. \end{cases}$$

Thus, we have

$$\langle \pi_{\mathrm{reg}} \downarrow_H, \psi \rangle_H = \frac{1}{|H|} \sum_{h \in H} \overline{\pi_{\mathrm{reg}}(h)} \psi(h) = \frac{1}{|H|} \pi_{\mathrm{reg}}(1) \psi(1) = \frac{|G|}{|H|} \psi(1) \neq 0$$

as required. $\square$

22 Clifford theory

Restriction to normal subgroups are often of particular interest; the theory around it (including the positive characteristic case) is called *Clifford theory*.

Lemma 22.1. *Consider a normal subgroup $H \lhd G$ and an element $g \in G$. For a KH-module U, denote by*

$${}^gU := \{ gu \mid u \in U \}$$

the set of symbols ug for $u \in U$. Then ${}^gU$ is a $\mathbb{C}H$-module with action

$$h(gu) := g(g^{-1}hgu)$$

and $\dim_K {}^gU = \dim_K U$.

In words, ${}^gU$ is the space U with H-action twisted by conjugation-by-g.

Proof Straightforward check for well-definedness of H-action (i.e. $(h'h)(gu) = h'(h(gu))$ for all $h, h' \in H$). For the dimension, just note that $u \mapsto gu$ is a linear map that is bijective. $\square$

Proposition 22.2. *Suppose that $H \lhd G$, V is a simple KG-module, and U is a simple direct summand of the $\mathbb{C}H$-module $\mathrm{Res}^G_H(V)$. The the following hold.*

(1) *There is some set $T \subset G$ such that $\mathrm{Res}^G_H(V) \cong \bigoplus_{g\in T} {}^gU$ as KH-module and each ${}^gU$ is simple.*

(2) *${}^{g_1}U \cong {}^{g_2}U$ as KG-module implies that ${}^{gg_1}U \cong {}^{gg_2}U$ as KG-module for all $g \in G$.*

Proof (1) By construction the space $\sum_{g\in G} {}^gU \subset V$ is a G-invariant, and so simplicity of V implies that $V = \sum_{g\in G} {}^gU$ as $\mathbb{C}G$-module, which in turn means that $\mathrm{Res}^G_H(V) = \sum_{g\in G} {}^gU$.

Let W be a simple submodule of ${}^gU$. Then ${}^{g^{-1}}W$ is a simple submodule of ${}^{g^{-1}}({}^gU) = U$. Simplicity of U then implies that ${}^{g^{-1}}W$ is either 0 or U, and so $W = 0$ or ${}^gU$.

For each pair $g, g' \in G$, irreducibility of ${}^gU$ and ${}^{g'}U$ as $\mathbb{C}H$-module implies that either they are isomorphic or ${}^gU \cap {}^{g'}U = 0$. So we can take one ${}^gU$ for each isomorphism class to get the desired decomposition $V \cong \bigoplus_{g\in T} {}^gU$.

(2) Exercise. $\square$

Theorem 22.3 (Clifford's theorem). *Suppose $H \lhd G$ is a normal subgroup and $\chi = \chi_V$ is an irreducible character for some simple $\mathbb{C}G$-module V. If U is a simple direct summand of the $\mathbb{C}H$-module $\mathrm{Res}^G_H(V)$, then there is some integer e such that*

$$\mathrm{Res}^G_H(V) \cong (U \oplus {}^{t_1}U \oplus \cdots {}^{t_k}U)^{\oplus e}$$

as $\mathbb{C}H$-modules.

Proof By Proposition 22.2, we already have $\mathrm{Res}^G_H(V)$ being isomorphic to a direct sum of ${}^gU$ for some set g. It remains to show that the multiplicity $[\mathrm{Res}^G_H(V) : {}^gU]$ is constant.

Denote by ψ_W the H-character corresponding to $\mathbb{C}H$-module W. Note that $\psi_{{}^gW}(h) = \psi_W(g^{-1}hg)$. In particular, we get that

$$\psi_{\mathrm{Res}(V)}(h) = \chi_V(h) = \chi_V(g^{-1}hg) = \psi_{{}^g\mathrm{Res}(V)}(h)$$

as χ_V is constant over any G-conjugacy classes.

By Corollary 17.4 (3), we then have

$$\begin{aligned}
[\mathrm{Res}^G_H(V) : {}^gU] &= \langle \chi_V \downarrow_H, \psi_{{}^gU}\rangle_H = \langle \psi_{\mathrm{Res}(V)}, \psi_{{}^gU}\rangle_H = \langle \psi_{{}^g\mathrm{Res}(V)}, \psi_{{}^gU}\rangle_H \\
&= \frac{1}{|H|}\sum_{h\in H} \overline{\chi_V(g^{-1}hg)}\psi_U(g^{-1}hg) \\
&= \frac{1}{|H|}\sum_{k(:=g^{-1}hg)\in H} \overline{\chi_V(k)}\psi_U(k) \\
&= \langle \chi_V \downarrow_H, \psi_U\rangle_H = [\mathrm{Res}^G_H(V) : U],
\end{aligned}$$

as required. $\square$

Remark 22.4. Clifford theory (Proposition 22.2 and Theorem 22.3) holds in arbitrary characteristic. But it is easier (only slightly, though) to explain the proof with character theory. By Proposition 22.2 (2), the elements g with ${}^gU$ isomorphic to U form a group (called the *inertia group* of U). The elements t_i's in Theorem 22.3 can be taken from the (right) coset representatives of this inertia group.

One can use Clifford theory to reveal a lot of information on the character table, especially in the case when $|G/N| = 2$; see Liebeck's book Chapter 20.

23 Induced module

Throughout this section, we relax our assumption on the ground field K to be any arbitrary field.

Definition 23.1. *Suppose that A is a K-algebra (=ring structure on a vector space, e.g. $A = KG$ the group algebra). Let M be a right A-module, and N be a left A-module. Then the tensor product $M \otimes_A N$ of M and N over A is the quotient K-vector space $M \otimes_K N/R$, where*

$$R = \{ma \otimes n - m \otimes an \mid m \in M, a \in A, n \in N\}.$$

Remark 23.2. Note that any A-module is automatically a K-vector space (since $K \hookrightarrow Z(A)$). Also be very careful that the K-vector space $M \otimes_A N$ is generally not an A-module has neither left nor right A-module structure (without additional assumptions)!

Definition 23.3. *Let A, B be K-algebras. An K-vector space M is an A-B-bimodule if it is a left A-module and right B-module with commuting A- and B-action, i.e. $r(ms) = (rm)s$ for all $r \in R, m \in M, s \in S$. In other words, it is a left module over $A \times B^{\mathrm{op}}$ (equivalently, right module over $B \times A^{\mathrm{op}}$).*

Lemma 23.4. *Consider rings A, B, C. Let M be an A-B-bimodule, N be an B-C-bimodule, and L be an A-C-bimodule.*

(1) *$M \otimes_B N$ is a A-C-bimodule given by $a \cdot (m \otimes n) := (am) \otimes n$ and $(m \otimes n) \cdot c := m \otimes (nc)$.*

(2) *$\mathrm{Hom}_A(L, M)$ is a C-B-bimodule given by $(c \cdot f)(l) := (f(lc))$ and $(f \cdot b)(l) := f(l)b$.*

Proof Exercise. □

The above lemma tells us that tensor and Hom can be used to 'transfer' modules (in fact, even homomorphisms) over different rings. Another consequence of Lemma 23.4 is that, if R is a commutative ring, then R-modules are the same as R-R-bimodules, and so $M \otimes_R N$ are automatically R-modules for R-modules M and N. More generally, a left (resp. right) modules over a K-algebra A is automatically an A-K-bimodules (resp. K-A-bimodules). Thus the tensor product $M \otimes_A N$ is automatically a K-vector space.

Example 23.5. *$A \otimes_A M \cong M$ as left A-module for all left A-module M.*

Recall the 'useful isomorphism' in Lemma 12.5 (adjoint property); it has the following enhanced version.

Lemma 23.6. *Suppose A, B are K-algebras, X is an A-B-bimodule. Then for any B-module M and A-module N, there is a K-vector space isomorphism $\mathrm{Hom}_A(X \otimes_B M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(X, N))$.*

Proof Verbatim to the proof of Lemma 12.5. □

Definition 23.7. *Suppose $H \leq G$. For a KH-module U, its induction to G, denoted by $\mathrm{Ind}_H^G(U)$ or $U \uparrow_H^G$, is the KG-module given by $KG \otimes_{KH} U$. For the representation ρ corresponding to U, we write $\mathrm{Ind}_H^G(\rho)$ or $\rho \uparrow_H^G$ for the induced representation corresponding to $\mathrm{Ind}_H^G(U)$.*

Remark 23.8. $KG \otimes_{KH} -$ is functorial (i.e. it can be applied to homomorphisms in a way that preserves axioms regarding compositions). Restriction can be made functorial by noticing that

$$\mathrm{Res}_H^G(V) = \mathrm{Hom}_{KG}({}_{KG}KG_{KH}, V)$$

where KG in the domain here is regarded as a KG-KH-bimodule.

Let us describe G-action in a slightly more explicit way. We need the following terminology for simpler exposition.

Definition 23.9. *A left transversal of H in G is a complete list $1 = t_1, t_2, \dots, t_k$ of H-cosets representatives, i.e. $G = \bigsqcup_{i=1}^{k} t_i H$.*

Lemma 23.10. *Consider a left transversal $1 = t_1, \dots, t_k$ of $H \leq G$.*

(1) *The right KH-module KG is free of rank k, namely,*

$$(KG)_{KH} = \bigoplus_{i=1}^{k} Kt_i \otimes_K KH \cong (KH)^{\oplus n}.$$

(2) *Let U be a KH-module. If U has a K-basis $\mathcal{B}$, then $\mathrm{Ind}_H^G(U)$ has a K-basis $\{t_i \otimes b \mid b \in \mathcal{B}, 1 \leq i \leq k\}$, i.e.*

$$\mathrm{Ind}_H^G(U) \overset{K\text{-v.sp.}}{\cong} \bigoplus_{i=1}^{k} Kt_i \otimes_K U.$$

In particular, we have $\dim_K \mathrm{Ind}_H^G(U) = |G/H| \dim_K(U)$.

Proof (1) Since every $g \in G$ can be written as $g = t_i h$ for some unique i and some $h \in H$, we have a K-vector space isomorphism

$$Kt_i \otimes KH \cong K(t_i H) \quad \text{given by } t_i \otimes h \mapsto t_i h \text{ for all } h \in H.$$

Since $t_i h h' \in t_i H$ for all $h, h' \in H$, each $K(t_i H)$ is isomorphic to KH as a right H-module.

(2) Now we have K-vector space isomorphisms:

$$\mathrm{Ind}_H^G(U) = KG \otimes_{KH} U \cong (\bigoplus_{i=1}^{k} Kt_i \otimes_K KH) \otimes_{KH} U \cong \bigoplus_{i=1}^{k} Kt_i \otimes_K (KH \otimes_{KH} U) \cong \bigoplus_{i=1}^{k} Kt_i \otimes U,$$

where the final isomorphism follows from Example 23.5. The claim follows from the right-hand side formulation. □

We can now describe G-action on $\mathrm{Ind}_H^G(U)$ more explicitly as follows. Take a left transversal $1_G = t_1, \dots, t_k$. Then by Lemma 23.10 (2) it is enough to describe G-action on $t_i \otimes u \in Kt_i \otimes U \subset \mathrm{Ind}_H^G(U)$. For $g \in G$, we have $g t_i H = t_j H$ for some j, i.e.

$$g t_i = t_j h \quad \text{for some } t \in H.$$

This yields, for any $u \in U$, the following g-action on $t_i \otimes u \in \mathrm{Ind}_H^G(U)$:

$$g(t_i \otimes u) = (g t_i) \otimes u = t_j h \otimes u = t_j \otimes h u = t_j \otimes (t_j^{-1} g t_i) u. \tag{23.1}$$

Exercise 23.11. *Check that* (23.1) *really defines a linear G-action on $\mathrm{Ind}_H^G(U)$, i.e. $(g'g)(t_i \otimes u) = g'(g(t_i \otimes u))$ for all t_i, u.*

Example 23.12. *Suppose $H \leq G$ is a subgroup. Consider the K-vector space $M_H := K(G/H)$ whose basis is given by the set of left G-cosets G/H. Then M_H is a KG-module. It follows from Lemma 23.10 (1) that $M_H \cong \operatorname{Ind}_H^G(\operatorname{triv}_H)$.*

Lemma 23.13. *Suppose we have subgroups $L \leq H \leq G$. Then $\operatorname{Ind}_H^G \operatorname{Ind}_L^H(U) = \operatorname{Ind}_L^G(U)$ for all $U \in KL\,\mathsf{mod}$.*

Proof This follows from the fact that $M \otimes_A (N \otimes_B L) \cong (M \otimes_A N) \otimes_B L$ as bimodules (check yourself). Namely, $KG \otimes_{KH} (KH \otimes_{KL} U) \cong (KG \otimes_{KH} KH) \otimes_{KL} U = KG \otimes_{KL} U$. □

Exercise 23.14. *Let $H \leq G$, V a KG-module and W a KH-module. Show that*

(1) $\operatorname{Ind}_H^G(W^*) \cong (\operatorname{Ind}_H^G(W))^*$.

(2) $V \otimes_K \operatorname{Ind}_H^G(W) \cong \operatorname{Ind}_H^G(\operatorname{Res}_H^G(V) \otimes_K W)$.

24 Induced class function and character

As before, we fix a subgroup $H \leq G$.

Definition 24.1. *For a class function $\psi \in \mathcal{C}(H)$, defined the induced class function $\operatorname{Ind}_H^G \psi = \psi\uparrow^G$ by*

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \psi(x^{-1}gx).$$

This is a class function of G by construction.

We can reformulate the definition in terms of left transversal only, instead of having to compute over all $x \in G$.

Lemma 24.2. *Let $t_1, \dots, t_k$ be a left transversal of H in G, then we have*

$$\psi\uparrow^G(g) = \sum_{t_i \text{ s.t. } t_i^{-1}gt_i \in H} \psi(t_i^{-1}gt_i).$$

Proof Every $x \in G$ can be written as $t_i h$ for some $h \in H$ and some *unique* (by Lagrange's theorem) i. Hence, we have

$$x^{-1}gx = (t_ih)^{-1}g(t_ih) = h^{-1}(t_i^{-1}gt_i)h.$$

If $t_i^{-1}gt_i \in H$, then the right-hand term in the formula above is in the same H-conjugacy class. Since ψ is a class function, we have $\psi(x^{-1}gx) = \psi(t_i^{-1}gt_i)$. Thus, we have

$$\begin{aligned}
\operatorname{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \psi(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{h \in H} \sum_{t_i : t_i^{-1}gt_i \in H} \psi(h^{-1}t_i^{-1}gt_ih) \\
&= \frac{|H|}{|H|} \sum_{t_i : t_i^{-1}gt_i \in H} \psi(t_i^{-1}gt_i)
\end{aligned}$$

as required. □

Lemma 24.3. *Suppose that W is a $\mathbb{C}H$-module for a subgroup $H \leq G$. Let $t_1, \dots, t_k$ be a left transversal of H in G. Then the induced character $\chi_W\uparrow^G$ is the character of the induced module $\operatorname{Ind}_H^G W$.*

Proof Suppose that W has a basis $\{w_i\}_{1\le i\le n}$. Then by Lemma 23.10 (2), $\operatorname{Ind} W = \operatorname{Ind}_H^G W$ has basis $\{t_a \otimes w_i \mid 1 \le a \le k, 1 \le i \le n\}$.

Fix some $g \in G$. By definition of character, we have

$$\chi_{\operatorname{Ind} W}(g) = \operatorname{Tr}(\operatorname{Ind}(\rho)_g) = \sum_{a,i} c_{a,i}$$

where $c_{a,i}$ is the coefficient of $t_a \otimes w_i$ in the vector $g(t_a \otimes w_i)$ expressed in terms of the basis $\{t_b \otimes w_j\}_{b,j}$. By from (23.1), we have

$$g(t_a \otimes w_i) = t_b \otimes (t_b^{-1} g t_a) w_i,$$

which means that $c_{a,i} \neq 0$ implies that $b = a$. Hence, we only care about the case when $t_a^{-1} g t_a \in H$, which means the induced character boils down to

$$\chi_{\operatorname{Ind} W}(g) = \sum_{a \text{ s.t. } t_a^{-1} g t_a \in H} \chi_W(t_a^{-1} g t_a).$$

The right-hand side is precisely the formula of induced character by Lemma 24.2. □

The above reduces computation to coset representatives, but we often only compute character evaluated at conjugacy class representatives, so it would be nice to have a formula that involves only conjugacy class representatives. Note that a conjugacy class in G may split to several conjugacy classes in H, and so we should expect $\chi_W \uparrow^G$ evaluated at g_i can be expressed in terms of $\chi_W(h_j)$'s where h_j are H-conjugacy classes representatives that are conjugate to g_i in H.

Proposition 24.4. *Let $H \le G$ be a subgroup and $\chi := \chi_W$ be the character for some $\mathbb{C}H$-module W. Suppose that $h_1, \ldots, h_m$ are H-conjugacy classes representatives such that h_i are G-conjugate to $g \in G$ for all $1 \le i \le m$. Then*

$$\chi_W \uparrow^G (g) = |C_G(g)| \sum_{i=1}^{m} \frac{\chi(h_i)}{|C_H(h_i)|}.$$

Proof Let $C_1, \ldots, C_m$ be the H-conjugacy classes containing $h_1, \ldots, h_m$ respectively. Then we have $\{xgx \mid x \in G\} \cap H = C_1 \sqcup \cdots \sqcup C_m$.

Let us write $g' \sim_G g$ if $g' = xgx^{-1}$ for some $x \in G$. Starting with Lemma 24.3, we have

$$\begin{aligned}\chi \uparrow^G (g) &= \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) = \frac{|C_G(g)|}{|H|} \sum_{g' \sim_G g} \hat{\chi}(g') = \frac{|C_G(g)|}{|H|} \sum_{i=1}^{m} \sum_{h \sim_H h_i} \chi(h) \\ &= \frac{|C_G(g)|}{|H|} \sum_{i=1}^{m} |C_i| \chi(h_i) = |C_G(g)| \sum_{i=1}^{m} \frac{\chi(h_i)}{|C_H(h_i)|},\end{aligned}$$

where the last equality follows from orbit-stabiliser theorem that $|H|/|C_i| = |C_H(h_i)|$. □

Example 24.5 (Character table of D_{2n} for n odd). *When n is odd, there are $(n+3)/2$ conjugacy classes of D_{2n} given by*

$$\{1\}, \{a^r, a^{-r}\}, \{a^s b \mid 0 \le s \le n-1\}$$

with $1 \le r \le (n-1)/2$. Their sizes are $1, 2, n$ respectively. We take transversal $1, a, a^2, \ldots, a^{(n-1)/2}, b$.

The derived subgroup is generated by rotations $H = \langle a \rangle \lhd G$ and has quotient $C_2 = \langle \beta H \rangle$. Hence, there are two 1-dimensional representation giving by lifting the two 1-dimensional irreducible representations of C_2, where βH acts by ± 1. Hence, we get

$\lvert C_G(g_i)\rvert$	$2n$	n	2
g_i	1	a^r	b
χ_1	1	1	1
χ_2	1	1	-1

On the other hand, we can also consider induction from H. Recall that, as $H \cong C_n$, the irreducible H-representations are 1-dimensional and given by $\psi_s : a \mapsto \xi^r$ for $1 \leq s \leq n$, where $\xi := \exp(2\pi i/n)$. Since $|G|/|H| = 2$, we have $\psi_r \uparrow^G (1) = 2 \times 1 = 2$. Let us apply Proposition 24.4 to get the remaining values of the induced characters $\psi_s \uparrow^G$. For a^r, it is G-conjugate to (the H-conjugacy class representatives) a^r and a^{-r}. Hence, we have

$$\psi_s \uparrow^G (a^r) = |C_G(a^r)|\Big(\frac{\psi_s(a^r)}{|C_H(a^r)|} + \frac{\psi_s(a^{-r})}{|C_H(a^{-r})|}\Big) = n(\frac{\xi^{sr}}{n} + \frac{\xi^{-sr}}{n}) = \xi^{sr} + \xi^{-sr}.$$

On the other hand, b is G-conjugate to none of the a^r's, and so $\psi_s \uparrow^G (b) = 0$. Thus, we have

$\lvert C_G(g_i)\rvert$	$2n$	n	2
g_i	1	a^r	b
$\psi_s \uparrow^G$	2	$\xi^{sr} + \xi^{-sr}$	0

We check that

$$\begin{aligned}
\langle \psi_s \uparrow^G, \psi_s \uparrow^G \rangle &= \frac{4}{2n} + \sum_{r=1}^{(n-1)/2} \frac{(\xi^{sr} + \xi^{-sr})^2}{n} + \frac{0}{2} \\
&= \frac{4}{2n} + \left(\frac{n-1}{2} \cdot \frac{2}{n} + \frac{1}{n} \sum_{r=1}^{(n-1)/2} \xi^{2sr} + \xi^{-2sr} \right) \\
&= \frac{2}{n} + 1 - \frac{1}{n} - \frac{1}{n} = 1,
\end{aligned}$$

and so $\psi_s \uparrow^G$ is irreducible as G-character. Thus, the full character table of D_{2n} is

$\lvert C_G(g_i)\rvert$	$2n$	n	2
g_i	1	a^r	b
χ_1	1	1	1
χ_2	1	1	-1
$\psi_s \uparrow^G$	2	$\xi^{sr} + \xi^{-sr}$	0

with $1 \leq r, s \leq (n-1)/2$

25 Induction-Restriction interaction

Lemma 25.1. *There are KH-module isomorphisms ${}_{KH}KG \otimes_{KG} V \cong \mathrm{Res}^G_H(V) \cong \mathrm{Hom}_{KG}(KG, V)$.*

Proof Exercise. □

The coinduction of a KH-module U is $\mathrm{Coind}^G_H(U) := \mathrm{Hom}_{KH}({}_{KH}KG_{KG}, U)$.

Proposition 25.2. *There is a KG-module isomorphism $\mathrm{Coind}^G_H(U) \cong \mathrm{Ind}^G_H(U)$.*

Proof Consider the map $\alpha : KG \otimes_{KH} U \to \mathrm{Hom}_{KH}(KG, U)$ given by

$$g \otimes u \mapsto \left(x \mapsto \begin{cases} (xg)u & \text{if } xg \in H \\ 0 & \text{otherwise.} \end{cases} \right)$$

Extend this linearly to a K-linear map and check that this is a KG-module homomorphism. (Exercise!)

On the other hand, we have $\beta : \mathrm{Hom}_{KH}(KG, U) \to KG \otimes_{KH} U$ given by

$$f \mapsto \sum_{i=1}^{k} t_i \otimes f(t_i^{-1}),$$

which can be easily checked to be KG-module homomorphism. (Exercise!)

Finally, by direct computation, we have $\alpha\beta = \mathrm{id} = \beta\alpha$. □

Corollary 25.3 (Eckmann-Shapiro lemma for Hom-spaces). *There are K-vector space isomorphisms:*

(1) (Frobenius reciprocity) $\mathrm{Hom}_{KG}(\mathrm{Ind}_H^G U, V) \cong \mathrm{Hom}_{KH}(U, \mathrm{Res}_H^G V)$.

(2) $\mathrm{Hom}_{KH}(\mathrm{Res}_H^G V, U) \cong \mathrm{Hom}_{KG}(V, \mathrm{Ind}_H^G U)$.

Proof Consequence of tensor-Hom adjunction Lemma 23.6. □

Remark 25.4. Both of these isomorphisms are (bi-)natural. In particular, this means that Ind_H^G and Res_H^G are biadjoint functors.

For time constraint, we omit the proof of the following theorem.

Theorem 25.5 (Mackey decomposition theorem). *For $H, L \leq G$. Let $U \in KL\,\mathsf{mod}$. Then there is the following KH-module isomorphism*

$$U \uparrow_L^G \downarrow_H^G \cong \bigoplus_{t \in H\backslash G/L} ({}^tU) \downarrow^L_{H \cap {}^tL} \uparrow^H_{H \cap {}^tL},$$

where $H\backslash G/L$ denotes the set of double cosets $\{HgL \mid g \in G\}$, and ${}^tL := \{t\ell t^{-1} \mid \ell \in L\}$ and ${}^tU \in K{}^tL\,\mathsf{mod}$ is given by $x \cdot u := txt^{-1}u$ for all $x \in L$ and $u \in U$.

Exercise 25.6. *Suppose $N \lhd G$ is a normal subgroup of G and $W \in KN\,\mathsf{mod}$. Show that*

$$\mathrm{Res}_N^G \mathrm{Ind}_N^G W \cong \bigoplus_{x \in G/N} {}^xW.$$

26 Permutation representations as induced representations

Recall that the stabiliser $\mathrm{Stab}_G(x)$ of $x \in \Omega$ is the subgroup $\{g \in G \mid gx = x\}$; for simplicity, we denote by $H_x := \mathrm{Stab}_G(x)$ whenever context is clear.

Lemma 26.1. *If G acts transitively on Ω and $x \in \Omega$, then the map*

$$\Omega \to G/H_x, \quad gx \mapsto gH_x,$$

is a bijection on G-sets that commutes with G-action, i.e. $\Omega \cong G/H_x$ are isomorphic as G-set. In particular, we have

(1) *$K\Omega \cong K(G/H_x) \cong \mathrm{Ind}_{H_x}^G(\mathrm{triv}_{H_x})$ as KG-modules.*

(2) *Any permutation module is a direct sum of induced modules.*

Proof Since $gx = hx \Leftrightarrow x = g^{-1}hx \Leftrightarrow g^{-1}h \in H_x \Leftrightarrow gH_x = hH_x$, the map is well-defined and injective. Surjective follows from orbit-stabiliser theorem and transitivity $|G/H_x| = |Gx| = |\Omega|$.

Finally, commutation with G-action follows from the assumption that Ω as $g(hx) = (gh)x$ for all $x \in \Omega$ and all $g, h \in G$.

(1) The first isomorphism follows immediately from the first part, namely, by linearly extending the bijection between Ω and G/H_x (see Homework2 Ex1).

The second isomorphism is Example 23.12. For the detailed explanation: note that by Lemma 23.10 we have $\mathrm{Ind}_{H_x}^G(\mathrm{triv}_{H_x})$ is a $|G/H_x| = |\Omega|$-dimensional K-vector space with basis given by $\{t_i \otimes u\}_i$ where $(t_i)_i$ is a transversal of $H_x \leq G$ and u is any spanning vector of the 1-dimensional triv_{H_x}.

By (23.1), if $gt_i \in t_j H$, then we have $g(t_i \otimes u) = t_j \otimes u$. This is equivalent to saying that the G-action on the induced modules coincides with the permutation action on cosets, which is what is claimed.

(2) For arbitrary G-set Ω, we can decompose it into G-orbits $\Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_m$. Then each Ω_i is a G-set for which G acts transitively, and the claim follows from (1). $\square$

Exercise 26.2. *Consider an integer $n \geq 1$ and an integer $r \leq n/2$. Let Ω_r be the set of r-subsets (=subsets of size r) of $\{1, 2, \ldots, n\}$. Find (and prove) a subgroup $H \leq \mathfrak{S}_n$ such that $K\Omega_r \cong \mathrm{Ind}_H^{\mathfrak{S}_n} \mathrm{triv}_H$.*

27 Partition, Young diagram, tableaux

Definition 27.1. *For a positive integer $n \geq 1$, a composition μ of n is a sequence $(\mu_i)_{i\geq 1}$ of non-negative integers such that $\sum_{i\geq 1} \mu_i = n$; we use the shorthand $\mu \vDash n$ to say that μ is a composition of n. A partition λ of n is a composition with*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots .$$

The shorthand to say that λ is a partition of n is written as $\lambda \vdash n$. It makes sense to cut off the trailing zeros when specifying a composition or partition.

We call each μ_i a component or a part of μ; likewise for partitions. An r-part partition λ is one with one with exactly r non-zero parts, i.e. $\lambda_r \neq 0$ and $\lambda_{>r} = 0$.

If we forget to write down the number of non-zero parts for a partition, then we assume r is such a number.

The importance of partition in the representation of symmetric groups are rooted in the following simple observation.

Lemma 27.2. *There is a one-to-one correspondence between the set of partitions of n and the conjugacy classes of $\mathfrak{S}_n$.*

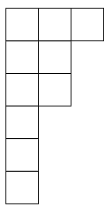
Proof A conjugacy class of $\mathfrak{S}_n$ is uniquely determined by the cycle-type of the elements it contains. We can represent the cycle-type in a well-defined way by writing down the length of each cycle in a decreasing sequence. Such a sequence necessarily sums to n and so is a partition of n. Conversely, a partition of n determines a cycle-type as its components gives the length of each cycle. $\square$

In particular, we know that the irreducible $K\mathfrak{S}_n$-modules (when K has good characteristics) are parameterised by partitions of n. These are called the *Specht modules*. Note that the modular representation version of Specht modules are not irreducible.

We use the English convention dealing with Young diagram and any tableaux combinatorics. There are also the French convention where rows are arranged in reversed order and the Russian convention where the diagram is the 45-degree counter-clockwise rotation of the French convention.

Definition 27.3. *The Young diagram of a partition $\lambda \vdash n$ is a array of boxes (also called nodes) where the i-th row has λ_i boxes, and the rows are left-justified.*

Example 27.4. *Usually, if there are k (consecutive) repeated entries of i r, then we write i^k instead of writing it k-times. For example, $\lambda = (3, 2, 2, 1, 1, 1) \vdash 10$ is written as $(3, 2^2, 1^3)$. The Young diagram of such λ is*



We coordinate the boxes in a Young diagram so that the box at (i, j) means it is placed at the i-th row at the j-th column (just like in matrix notation). In this way, the Young diagram of λ can be regarded as the set $[\lambda] := \{(i, j) \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$.

To construct an $\mathfrak{S}_n$-representation out of a partition λ, the natural way is to fill in $[n]$ to the Young diagram of λ.

Definition 27.5. *A Young tableau of shape $\lambda \vdash n$, or a λ-tableau for short, is a repeat-free filling of the boxes of the Young diagram of λ by elements of $[n]$, i.e. a bijective map $\mathfrak{t} : [\lambda] \to [n]$. We write $\mathfrak{t}_{i,j} := \mathfrak{t}(i,j) \in [n]$ the value placed in (i,j)-node.*

Note that the plural form of tableau is tableaux.

Example 27.6. *For $\lambda = (3, 2^2, 1^3)$, we have a λ-tableau $\mathfrak{t}$ given by*

$$\begin{array}{ccc} 8 & 2 & 10 \\ 1 & 4 & \\ 9 & 3 & \\ 6 & & \\ 5 & & \\ 7 & & \end{array}$$

This represents the map $\mathfrak{t}$ that maps $(1,1)$ to 8, $(1,2)$ to 2, $(1,3)$ to 10, $(2,1)$ to 1, etc. It is also customary to just draw a tableau without the grid lines of the underlying Young diagram.

For any given partition $\lambda \vdash n$, $\mathfrak{S}_n$ acts the set of λ-tableaux by applying the permutation on each entry of the nodes. However, this action is no different from the regular representation of $\mathfrak{S}_n$ itself. We need some modification.

Example 27.7. *For $\lambda = (3, 2^2, 1^3)$ and $\sigma = (156)(2397)(48)(10)$, we have*

$$\sigma \cdot \begin{array}{ccc} 8 & 2 & 10 \\ 1 & 4 & \\ 9 & 3 & \\ 6 & & \\ 5 & & \\ 7 & & \end{array} = \begin{array}{ccc} 4 & 3 & 10 \\ 5 & 8 & \\ 7 & 9 & \\ 1 & & \\ 6 & & \\ 2 & & \end{array}$$

Before we go on to explain the needed modification, let us introduce the following terminology.

Definition 27.8. *A λ-tableau is standard if the entries increase along each row and down each column, i.e. $\mathfrak{t}_{i,j} < \mathfrak{t}_{i,j'}$ for all $j < j'$ and $\mathfrak{t}_{i,j} < \mathfrak{t}_{i',j}$ for all $i < i'$.*

Example 27.9. *There is only one standard λ-tableau for $\lambda = (n)$ or $\lambda = (1^n)$. For example, when $n = 4$, we have*

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}.$$

Clearly, $\mathfrak{S}_n$ does not act on the set of standard λ-tableaux. But to spoil slightly, the *Specht module* S^λ associated to λ, which makes up the full list of irreducible $K\mathfrak{S}_n$-modules for good characteristic K, have a basis indexed by standard λ-tableaux. The first main aim of this introduction to symmetric group representation is to explain the construction of S^λ in the classical way (through tableaux combinatorics).

28 Young subgroup, tabloids, permutation modules

The modify $\mathfrak{S}_n$-action on tableaux, we can consider equivalence classes of tableaux.

Definition 28.1. *Two λ-tableaux $\mathfrak{t}, \mathfrak{t}'$ are said to be row equivalent if $\mathfrak{t}_{i,-} := \{\mathfrak{t}_{i,j} \mid j = 1, \ldots, \lambda_i\}$ and $\mathfrak{t}'_{i,-}$ coincides for all $i \geq 1$. This is an equivalence relation on the set of λ-tableaux, and the* *resulting equivalence class is called a λ-tabloid. Denote by* $\{\mathfrak{t}\}$ *the λ-tabloid induced by (containing) the λ-tableau* $\mathfrak{t}$.

Example 28.2. *λ-tabloids are conventionally drawn by removing the vertical gird lines in the tableaux diagram. For example,*

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 2 & 6 & 4 & 3 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} \quad \rightsquigarrow \quad \{\mathfrak{t}\} = \begin{array}{cccc}\hline 2 & 6 & 4 & 3 \\ \hline 5 & 1 & & \\ \hline\end{array}$$

Definition 28.3. *Let* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ *and* $\mathfrak{t}$ *be a λ-tableau. The row stabliser* $R_{\mathfrak{t}}$ *of* $\mathfrak{t}$ *is the stabliser subgroup associated to* $\mathfrak{S}_n$*-action on* $\{\mathfrak{t}\}$*, i.e.*

$$R_{\mathfrak{t}} := \{\sigma \in \mathfrak{S}_n \mid \sigma\{\mathfrak{t}\} = \{\mathfrak{t}\}\} = \mathfrak{S}(\mathfrak{t}_{1,-}) \times \mathfrak{S}(\mathfrak{t}_{2,-}) \times \cdots \times \mathfrak{S}(\mathfrak{t}_{k,-}),$$

where $\mathfrak{S}(X) \cong \mathfrak{S}_{|X|}$ *is the group of symmetries of* X.

Dually, we can consider column stabliser $C_{\mathfrak{t}}$ *of* $\mathfrak{t}$*; equivalently,* $C_{\mathfrak{t}} = R_{\mathfrak{t}'}$ *where* $\mathfrak{t}'$ *is the tranpose of* $\mathfrak{t}$*, i.e. reflecting the diagram* $\mathfrak{t}$ *along (north-west,south-east) diagonal.*

Example 28.4. *In Example 28.2, we have* $R_{\mathfrak{t}} = \mathfrak{S}(\{2,3,4,6\}) \times \mathfrak{S}(\{1,5\})$ *and* $C_{\mathfrak{t}} = \mathfrak{S}(\{2,5\}) \times \mathfrak{S}(\{1,6\}) \times \mathfrak{S}(\{4\}) \times \mathfrak{S}(\{3\})$.

Definition 28.5. *The (standard) Young subgroup* $\mathfrak{S}_\lambda \leq \mathfrak{S}_n$ *associated to a partition* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ *is the row stabiliser of the standard λ-tableau with first row* $1, 2, \ldots, \lambda_1$*, second row* $\lambda_1 + 1, \cdots, \lambda_2$*, etc.*

Remark 28.6. The correspondence between conjugacy classes and partitions implies that every $R_{\mathfrak{t}}$ is conjugate to a Young subgroup associated to the shape of $\mathfrak{t}$.

We should check that taking $\sigma\{\mathfrak{t}\} = \{\sigma\mathfrak{t}\}$ is a well-defined $\mathfrak{S}_n$-action, i.e. independent on the choice of $\mathfrak{t}$.

Lemma 28.7. *For a λ-tableau* $\mathfrak{t}$ *and* $\sigma \in \mathfrak{S}_n$*, we have* $R_{\sigma\mathfrak{t}} = \sigma R_{\mathfrak{t}} \sigma^{-1}$*; and likewise,* $C_{\sigma\mathfrak{t}} = \sigma C_{\mathfrak{t}} \sigma^{-1}$*. In particular, the set* Ω^λ *of λ-tabloids is a* G*-set for* $G = \mathfrak{S}_n$.

Proof For a subset $\Omega \subset [n]$, we have $\mathfrak{S}(\sigma\Omega) = \sigma\, \mathfrak{S}(\Omega)\sigma^{-1}$. Hence, we get that

$$R_{\sigma\mathfrak{t}} = \prod_{i=1}^{k} \mathfrak{S}(\sigma\mathfrak{t}_{i,-}) = \prod_{i=1}^{k} \sigma\, \mathfrak{S}(\mathfrak{t}_{i,-})\sigma^{-1} = \sigma\Big(\prod_{i=1}^{k} \mathfrak{S}(\mathfrak{t}_{i,-})\Big)\sigma^{-1} = \sigma R_{\mathfrak{t}} \sigma^{-1}.$$

Consequently, as $\{\mathfrak{t}\} = \{\mathfrak{t}'\}$ implies that $\mathfrak{t}' = \rho\mathfrak{t}$ for some $\rho \in R_{\mathfrak{t}}$, we have

$$\sigma\mathfrak{t}' = (\underbrace{\sigma\rho\sigma^{-1}}_{\in R_{\sigma\mathfrak{t}}})(\sigma\mathfrak{t}),$$

and so $\sigma\{\mathfrak{t}'\} = \{\sigma\mathfrak{t}'\} = \{\sigma\mathfrak{t}\} = \sigma\{\mathfrak{t}\}$, i.e. $\mathfrak{S}_n$ acts on Ω^λ. □

Definition 28.8. *Fix any field* K *(or even* $K = \mathbb{Z}$*). The (Young) permutation module associated to* $\lambda \vdash n$ *is the* $\mathfrak{S}_n$*-module given by* $M^\lambda := K\Omega^\lambda$.

Note that in a similar way we can define permutation module associated to compositions, too.

Example 28.9. (1) *For* $\lambda = (n)$*, we have* $M^\lambda = K\{\overline{1\,2\,\cdots\,n}\}$*, and so* M^λ *corresponds to the trivial representation.*

(2) *For* $\lambda = (1^n)$*, we have* $M^\lambda \cong K\,\mathfrak{S}_n$ *the regular representation of* $\mathfrak{S}_n$.

(3) *For* $\lambda = (n-k, k)$ *with* $0 < k < \lfloor n/2 \rfloor$*, the set* Ω^λ *is in bijection the set of* k*-subsets (meaning subsets of size* k*) of* $[n]$ *(given by taking only the second row in each tabloid). In the case when* $k = 1$*,* M^λ *is just the natural permutation module* $K[n]$ *corresponding the usual* $\mathfrak{S}_n$*-action on* $[n]$.

Lemma 28.10. *M^λ is generated by any single λ-tabloid (i.e. $M^\lambda = K\mathfrak{S}_n \cdot \{\mathfrak{t}\}$ for any $\mathfrak{t}$), and* $\dim_K M^\lambda = \frac{n!}{\lambda_1!\cdots\lambda_k!}$.

Proof The first part follows from the fact that $\mathfrak{S}_n$ acts transitively on the set of λ-tableaux – hence, on λ-tabloids, too. For the dimension, which is equal to the size of Ω^λ, we have

$$\#\Omega^\lambda = \frac{\#\,\mathfrak{S}_n \cdot \mathfrak{t}}{\# R_{\mathfrak{t}}} = \frac{n!}{\lambda_1! \cdots \lambda_r!},$$

as required. □

Exercise 28.11. *Show that $M^\lambda \cong \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\mathrm{triv})$ as $K\mathfrak{S}_n$-module.*

29 Specht module

Recall that any permutation representation always contains the trivial representation as a subrepresentation. Hence, M^λ is far from being irreducible. Nevertheless, one can still find one simple $K\mathfrak{S}_n$-module inside M^λ. We will not give the full detail explanation, but only explain the construction.

Definition 29.1. *Suppose that $\mathfrak{t}$ is a λ-tableau. Define the signed column sum associated to $\mathfrak{t}$ as*

$$\kappa_{\mathfrak{t}} := \sum_{\sigma \in C_{\mathfrak{t}}} \mathrm{sgn}(\sigma)\sigma \in K\mathfrak{S}_n .$$

Then we have a linear combination of λ-tabloids

$$\mathbf{e}_{\mathfrak{t}} := \kappa_{\mathfrak{t}}\{\mathfrak{t}\} = \sum_{\sigma \in C_{\mathfrak{t}}} \mathrm{sgn}(\sigma)\{\sigma\mathfrak{t}\} \in M^\lambda,$$

called the polytabloid associated with $\mathfrak{t}$. The Specht module associated to λ is given by

$$S^\lambda := K\text{-span}\{\mathbf{e}_{\mathfrak{t}} \mid \mathfrak{t} \text{ a } \lambda\text{-tableau}\}.$$

Remark 29.2. $\mathbf{e}_{\mathfrak{t}}$ *does* depend on the choice of $\mathfrak{t}$.

Lemma 29.3. *We have $\kappa_{\sigma\mathfrak{t}} = \sigma\kappa_{\mathfrak{t}}\sigma^{-1}$ and $\mathbf{e}_{\sigma\mathfrak{t}} = \sigma\mathbf{e}_{\mathfrak{t}}$. In particular, S^λ is a $K\mathfrak{S}_n$-module generated by any single polytabloid.*

Proof The claim on $\kappa_{\mathfrak{t}}$ is similar to Lemma 28.7 and we leave the reader to check it. For the polytabloid case, it then follows from the claim on $\kappa_{\mathfrak{t}}$ that

$$\mathbf{e}_{\sigma\mathfrak{t}} = \kappa_{\sigma\mathfrak{t}}\{\sigma\mathfrak{t}\} = (\sigma\kappa_{\mathfrak{t}}\sigma^{-1})(\sigma\{\mathfrak{t}\}) = \sigma\kappa_{\mathfrak{t}}\mathfrak{t} = \sigma\mathbf{e}_{\mathfrak{t}},$$

as claimed. □

Example 29.4. (1) *For $\lambda = (n)$, we have M^λ is 1-dimensional and S^λ is a non-zero submodule, and so $S^\lambda = M^\lambda \cong \mathrm{triv}$.*

(2) *For $\lambda = (1^n)$, take $\mathfrak{t}$ the unique standard λ-tableau (filled with i in the i-th row for all $1 \le i \le n$). Then $\kappa_{\mathfrak{t}} = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma$, and so $S^\lambda \cong K\kappa_{\mathfrak{t}} \subset \mathbb{C}\mathfrak{S}_n \cong M^\lambda$. This Specht module is precisely the sign representation of $\mathfrak{S}_n$ as we have seen in Homework 1. Alternatively, one can directly calculate*

$$\sigma\mathbf{e}_{\mathfrak{t}} = \mathbf{e}_{\sigma\mathbf{e}} = \sum_{\tau\in\mathfrak{S}_n} \mathrm{sgn}(\sigma^{-1}\tau)\tau\{\mathfrak{t}\} = \mathrm{sgn}(\sigma^{-1}) \sum_{\tau\in\mathfrak{S}_n} (\mathrm{sgn}\,\tau)\tau\{\mathfrak{t}\} = (\mathrm{sgn}\,\sigma)\mathbf{e}_{\mathfrak{t}}.$$

(3) *For $\lambda = (n-1,1)$, a λ-tableau (and λ-tabloid) can be identified with the entry in its second row:*

$$\mathfrak{t} = \begin{array}{|c|c|c|}\hline a & \cdots & c \\ \hline b \\ \cline{1-1}\end{array} =: \underline{b}.$$

Then, we have $\mathbf{e}_\mathfrak{t} = \mathbf{e}_{\underline{b}} = \underline{b} - \underline{a}$. Since $\underline{b} - \underline{a} = (\underline{b} - \underline{1}) - (\underline{a} - \underline{1})$, we have a basis of $S^{(n-1,1)}$ given by

$$\{\underline{b} - \underline{1} \mid 2 \leq b \leq n\},$$

and thus $\dim_K S^{(n-1,1)} = n-1$. In fact, it is easy to check that $S^{(n-1,1)}$ is the subspace of $M^{(n-1,1)}$ consisting of elements $v = \sum_{b=1}^{n} \alpha_b \underline{b}$ (where $\alpha_b \in K$) such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$. When $K = \mathbb{C}$, we can see that $S^{(n-1,1)}$ matches with the degree $n-1$ irreducible character of $\mathfrak{S}_n$ given by $\pi_{[n]} - \mathbf{1}$ (Example 20.2).

In order to "isolate" S^λ from M^λ, we need to a certain partial order on the set of partitions of n.

Definition 29.5. *For $\lambda, \mu \vdash n$, we say that λ dominates μ, denoted by $\lambda \trianglerighteq \mu$, if*

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad \forall k \geq 1.$$

We write $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$.

It is easy to check that $\trianglerighteq$ defines a partial order on the set of partitions of n. It is coarser than the lexicographic order $\lambda \geq_{\text{lex}} \mu$ (i.e. there is some $k \geq 1$ such that $\lambda_i = \mu_i$ for all $1 \leq i \leq k$ and $\lambda_{k+1} \geq \mu_{k+1}$), which is a total order; in other words, $\lambda \trianglerighteq \mu \;\Rightarrow\; \lambda \geq_{\text{lex}} \mu$.

Roughly, $\lambda \trianglerighteq \mu$ if λ is relatively "fat and short" and μ is relatively "thin and tall".

Lemma 29.6. *Suppose that $\mathfrak{t}$ is a λ-tableau and $\mathfrak{t}'$ is a μ-tableau for some $\lambda, \mu \vdash n$. Then the following hold.*

(1) *If $\lambda = \mu$, then $\kappa_\mathfrak{t}\{\mathfrak{t}'\} = \pm \kappa_\mathfrak{t}\{\mathfrak{t}\} = \pm \mathbf{e}_\mathfrak{t}$. In particular, for any $v \in M^\lambda$, we have $\kappa_\mathfrak{t} v \in K\mathbf{e}_\mathfrak{t}$.*

(2) *If $\kappa_\mathfrak{t}\{\mathfrak{t}'\} \neq 0$, then $\lambda \trianglerighteq \mu$.*

Proof (1) Notice that for $\sigma \in C_\mathfrak{t}$, we have

$$\kappa_\mathfrak{t}\sigma = \sum_{\tau \in C_\mathfrak{t}} \operatorname{sgn}(\tau)\tau\sigma = \sum_{\pi = \tau\sigma \in C_\mathfrak{t}} \operatorname{sgn}(\sigma^{-1}) \operatorname{sgn}(\pi)\tau\sigma = \operatorname{sgn}(\sigma)\kappa_\mathfrak{t}.$$

Since we have $\{\mathfrak{t}'\} = \sigma\{\mathfrak{t}\}$ for some $\sigma \in C_\mathfrak{t}$, and so

$$\kappa_\mathfrak{t}\{\mathfrak{t}'\} = \kappa_\mathfrak{t}\sigma\{\mathfrak{t}\} = \operatorname{sgn}(\sigma)\kappa_\mathfrak{t}\{\mathfrak{t}\} = \pm \mathbf{e}_\mathfrak{t}.$$

The last part follows from the above calculation by using the definition that v is a linear combination of λ-tabloids $\{\mathfrak{t}'\}$.

(2) We show that, for every $i \geq 1$, the numbers in the i-th row of $\mathfrak{t}'$ lie in different columns in $\mathfrak{t}$. In which case, the claim is a direct consequence of the "Basic Combinatorial Lemma" – see James' book 3.7 (it is possible to just think carefully about the combinatorics and convince yourself that this claim implies $\lambda \trianglelefteq \mu$).

Indeed, suppose on the contrary that there is some $a, b \in \mathfrak{t}'_{i,-}$ such that $a, b \in \mathfrak{t}_{-,j}$ for some j. This means that we have $(a,b) \in C_\mathfrak{t}$. Take a transversal $\{\sigma_1, \ldots, \sigma_m\}$ of $\langle (a,b) \rangle$ in $C_\mathfrak{t}$, then we have $C_\mathfrak{t} = \sqcup_{i=1}^{m} \sigma_i \langle (a,b) \rangle$. Hence, we get that

$$\begin{aligned}\kappa_\mathfrak{t} &= \sum_{i=1}^{m} \big(\operatorname{sgn}(\sigma_i)\sigma_i + \operatorname{sgn}(\sigma(a,b))\sigma_i(a,b)\big) \\ &= \sum_{i=1} \operatorname{sgn}(\sigma_i)\sigma\big(1 - (a,b)\big) = \Big(\sum_{i=1} \operatorname{sgn}(\sigma_i)\sigma\Big)\big(1 - (a,b)\big).\end{aligned}$$

The condition $a, b \in \mathfrak{t}'_{i,-}$ implies that

$$(1-(a,b))\{\mathfrak{t}'\} = \{\mathfrak{t}'\} - \{(a,b)\mathfrak{t}'\} = \{\mathfrak{t}'\} - \{\mathfrak{t}'\} = 0,$$

and so

$$\kappa_{\mathfrak{t}}\{\mathfrak{t}'\} = \Big(\sum_{i=1} \mathrm{sgn}(\sigma_i)\sigma\Big)(1-(a,b))\{\mathfrak{t}'\} = 0,$$

contradicting the assumption. □

Proposition 29.7. *The following hold for a non-zero homomorphism* $0 \neq f \in \mathrm{Hom}_{K\mathfrak{S}_n}(M^\lambda, M^\mu)$*.*

(1) *If* $\lambda = \mu$*, then* $f|_{S^\lambda} = \alpha\iota_\lambda$ *for some* $\alpha \in K$ *and* $\iota_\lambda : S^\lambda \to M^\lambda$ *the canonical inclusion.*

(2) *If* $S^\lambda \not\subseteq \mathrm{Ker}(f)$*, then* $\lambda \trianglerighteq \mu$*.*

Proof (1) Take any λ-tableau $\mathfrak{t}$. By Lemma 29.6 (1), we have $\kappa_{\mathfrak{t}} v \in K\mathbf{e}_{\mathfrak{t}}$ for any $v \in M^\lambda$. Now take $v = f(\{\mathfrak{t}\})$, then it follows that

$$f(\mathbf{e}_{\mathfrak{t}}) = f(\kappa_{\mathfrak{t}}\{\mathfrak{t}\}) = \kappa_{\mathfrak{t}} f(\{\mathfrak{t}\}) = \alpha\mathbf{e}_{\mathfrak{t}} \text{ for some } \alpha \in K.$$

Hence, we have $f(u) = \alpha u$ for all $u \in K\mathfrak{S}_n \cdot \mathbf{e}_{\mathfrak{t}}$, but $S^\lambda = K\mathfrak{S}_n \cdot \mathbf{e}_{\mathfrak{t}}$ by Lemma 29.3.

(2) Take $u \in S^\lambda$ such that $f(u) \neq 0$ (which exists by the assumption). Since S^λ is spanned by polytabloids $\mathbf{e}_{\mathfrak{t}} = \kappa_{\mathfrak{t}}\{\mathfrak{t}\}$, we can write

$$f(u) = f\big(\sum_{\mathfrak{t}} \alpha_{\mathfrak{t}}\mathbf{e}_{\mathfrak{t}}\big) = \sum_{\mathfrak{t}} \alpha_{\mathfrak{t}}\kappa_{\mathfrak{t}} f(\{\mathfrak{t}\})$$

for some $\alpha_{\mathfrak{t}} \in K$. Since M^μ is spanned by μ-tabloids, we can write

$$\kappa_{\mathfrak{t}} f(\{\mathfrak{t}\}) = \kappa_{\mathfrak{t}} \sum_{\mathfrak{t}'} \beta_{\mathfrak{t}'}\{\mathfrak{t}'\} = \sum_{\mathfrak{t}'} \beta_{\mathfrak{t}'}(\kappa_{\mathfrak{t}}\{\mathfrak{t}'\})$$

for some $\beta_{\mathfrak{t}'} \in K$. Hence, $f(u) \neq 0$ implies that there is some $\mathfrak{t}, \mathfrak{t}'$ with $\kappa_{\mathfrak{t}}\{\mathfrak{t}'\} \neq 0$. Now the claim follows from Lemma 29.6 (2). □

Theorem 29.8. *Suppose* $\mathrm{char}(K) \nmid |\mathfrak{S}_n|$*. Then the following holds for any* $\lambda, \mu \vdash n$*.*

(1) $S^\lambda \cong S^\mu$ *if, and only if,* $\lambda = \mu$*.*

(2) S^λ *is simple.*

(3) $\mathrm{Hom}_{K\mathfrak{S}_n}(S^\lambda, M^\lambda) \cong \mathrm{End}_{K\mathfrak{S}_n}(S^\lambda) \cong K$*.*

Proof (1) $\Rightarrow$: Trivial.

$\Leftarrow$: Take an isomorphism $f : S^\lambda \to S^\mu$ of $K\mathfrak{S}_n$-modules. Since we are under good characteristic, meaning that Maschke's theorem applies, we have $M^\lambda = S^\lambda \oplus U$ for some $K\mathfrak{S}_n$-module U. Then by composing with the canonical inclusion $\iota_\lambda : S^\lambda \hookrightarrow M^\lambda$ and projection $\pi_\lambda : M^\lambda \twoheadrightarrow S^\lambda$ onto direct summand, we have a homomorphism

$$\hat{f} : M^\lambda \twoheadrightarrow S^\lambda \xrightarrow[\cong]{f} S^\mu \hookrightarrow M^\mu$$

Now f being an isomorphism implies that $\mathrm{Ker}(\hat{f}) \not\supseteq S^\lambda$, and so we can apply Proposition 29.7 (2) to get that $\lambda \trianglerighteq \mu$.

Similarly, since we also have an isomorphism $S^\mu \to S^\lambda$, the same argument applies to get $\mu \trianglerighteq \lambda$.

(2) Suppose on the contrary that there is a non-zero submodule V of S^λ. By Maschke's theorem, we can then decompose $S^\lambda = V \oplus V'$ for some non-zero V' and $M^\lambda = V \oplus V' \oplus U$ for some U.

Now composing with canonical projection and inclusions (as in (1)), we can construct a non-zero endomorphism

$$f : M^\lambda \twoheadrightarrow V \hookrightarrow M^\lambda.$$

By Proposition 29.7 (1), we have that $f|_{S^\lambda} = \alpha\iota_\lambda$ for some $\alpha \in K$. This contradicts the construction which guarantees $f|_{V'} = 0$.

(3) For $f : S^\lambda \to M^\lambda$, precomposing with the canonical projection $\pi_\lambda : M^\lambda \twoheadrightarrow S^\lambda$ yields $\hat{f} := f \circ \pi_\lambda \in \mathrm{End}_{K\mathfrak{S}_n}(M^\lambda)$. It then follows from Proposition 29.7 (1) that

$$\alpha\iota_\lambda = \hat{f}|_{S^\lambda} = \hat{f} \circ \iota_\lambda = f \circ \pi_\lambda \circ \iota_\lambda = f \circ \mathrm{id}_{S^\lambda} = f$$

for some $\alpha \in K$. Thus we have the following (K-linear) bijections

$$\begin{array}{ccccc} \mathrm{Hom}_{K\mathfrak{S}_n}(S^\lambda, M^\lambda) & \xleftrightarrow{\sim} & \mathrm{Hom}_{K\mathfrak{S}_n}(S^\lambda, S^\lambda) & \xleftrightarrow{\sim} & K \\ f & \longleftrightarrow & \alpha\,\mathrm{id}_{S^\lambda} & \longleftrightarrow & \alpha, \end{array}$$

as required. □

Remark 29.9. The theorem fails in general characteristic:

(1) fails, for example, already in $\mathrm{char}(K) = 2$ (and $n \geq 2$), we have $S^{(n)} \cong S^{(1^n)}$ (as we have already seen in Homework 1 that trivial and sign representations are isomorphic in this setting).

(2) fails, for example, when $n = p = \mathrm{char}(K) > 0$, then for $\lambda = (p-k, 1^k)$ with $1 \leq k < p-1$, it turns out that S^λ is indecomposable and admits a filtration

$$S^{(p-k,1^k)} \supset D^{(p-k+1,1^{k-1})} \supset 0$$

with $D^{(p-k,1^k)} := S^{(p-k,1^k)}/D^{(p-k+1,1^{k-1}}$ a simple module for all $1 \leq k < p$.

Corollary 29.10. *When* $\mathrm{char}(K) \nmid n!$ *(or equivalently,* $\mathrm{char}\, K > n$ *or* $\mathrm{char}\, K = 0$*),* $\{S^\lambda \mid \lambda \vdash n\}$ *is the complete set of (isoclass representatives of) simple* $K\mathfrak{S}_n$*-modules. Moreover, we have*

$$M^\mu \cong \bigoplus_{\lambda \trianglerighteq \mu} (S^\lambda)^{\oplus K_{\lambda\mu}}$$

for some $K_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ *with* $K_{\mu\mu} = 1$.

Proof The first part follows from the fact that the number of partitions is the number of conjugacy classes of $\mathfrak{S}_n$, and that the Specht modules are pairwise non-isomorphic simple modules by Theorem 29.8.

For the second part, Mashcke's theorem applies and then we can get the decomposition with direct sum over all $\lambda \vdash n$ instead of $\lambda \trianglelefteq \mu$, and that

$$K_{\lambda\mu} = [M^\mu : S^\lambda] = \dim_K \mathrm{Hom}_{K\mathfrak{S}_n}(S^\lambda, M^\mu).$$

In particular, it follows from Theorem 29.8 (3) that $K_{\mu\mu} = 1$.

Suppose that $K_{\lambda\mu} \neq 0$. Then we have a non-zero homomorphism $S^\lambda \to M^\mu$, which precompose with the canonical projection $\pi_\lambda : M^\lambda \to S^\lambda$ to a non-zero homomorphism $f \in \mathrm{Hom}_{K\mathfrak{S}_n}(M^\lambda, M^\mu)$. Hence, it follows from Proposition 29.7 (2) that $\lambda \trianglerighteq \mu$. □

Remark 29.11. Although the proof above relies on Maschke's theorem, over arbitrary characteristic, M^μ still admits a filtration with subquotients in Specht that satisfy the same ordering property – in particular, one also has p-Kostka numbers for p the characteristic of the ground field.

The number $K_{\lambda\mu}$ is called *Kostka number*, and has a combinatorial interpretation, namely, it is the number of *semi*-standard λ-tableaux of type μ. We will not explain these for time reason; the point is just that this number can be combinatorially calculated.

Consider the character χ_λ of S^λ and the permutation character π_λ associated to M^λ. Note that the latter can be combinatorially calculated. By Corollary 29.10, we have

$$\pi_\lambda = \chi_\lambda + \sum_{\lambda \lhd \mu} K_{\mu\lambda}\chi_\mu.$$

and so one can inductively (along the dominance order starting from $\lambda = (n)$) calculate χ_λ so long as the Kostka numbers are known.

Of course, this way to calculate χ_λ is too slow. There is a more direct formula given by *Munaghan-Nakayama rule* which says that the value of character χ_λ at a cycle-type μ can be calculated by recursively removing rim hooks of length μ_i from λ and look at the heigh of such rim hooks. The character value is intimately related to *symmetric polynomials*, given by the *Frobenius character formula*. For time constraint, we will not explain any more details on these subjects – or perhaps leave them to a future course dedicated to symmetric group representation.

Finally, we just mention that we defined Specht module by a spanning set of polytabloids, but one can ask if this can be refined to a basis – the answer is affirmative.

Theorem 29.12. *The Specht module has a K-basis given by the set of standard λ-polytabloids, i.e. $\mathbf{e}_{\mathfrak{t}}$ for which $\mathfrak{t}$ is a standard λ-tableau.*

The strategy to prove this is to introduce a partial order on λ-tabloids, so that standard λ-tableaux form the maximal elements of this order with the property that $\{\mathfrak{t}'\}$ appears in $\mathbf{e}_{\mathfrak{t}}$ for standard $\mathfrak{t}$ implies $\{\mathfrak{t}\} \trianglerighteq \{\mathfrak{t}'\}$. With this, it is relatively easy to show linear independence. For spanning, one uses the so-called Garnir relation to alter (or "SugarCrash-ing" to) non-standard λ-tableaux to a standard one in a way that is "compatible" with the polytabloids.