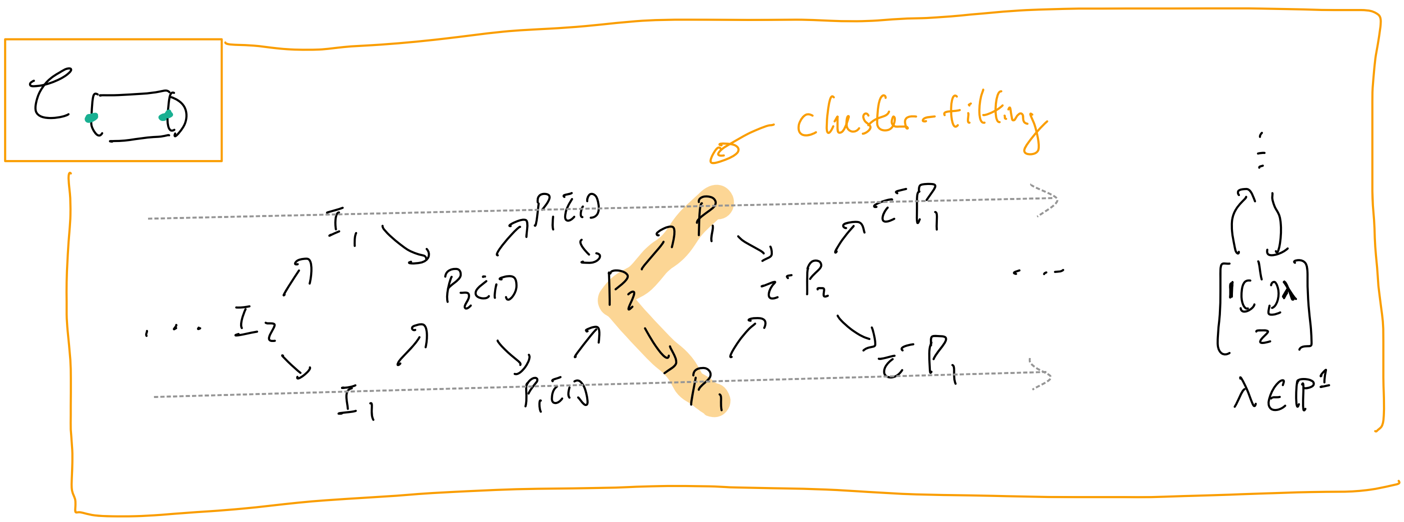
work in progress, jt. with V. Bazier-Matte, K. Wright



Summary

(Known): **Topology** | **Category**

- Orientable surface w/ triangulation $\overset{\mathrm{I}}{\overset{1:1}{\longleftrightarrow}}$ gentle Jacobian algebra (and cluster category)
- curves (+ ...) $\overset{\mathrm{II}}{\overset{1:1}{\longleftrightarrow}}$ indec. (families of) obj's
- triangulations $\overset{\mathrm{III}}{\overset{1:1}{\longleftrightarrow}}$ cluster-tilting obj.'s

$\Downarrow$ IV

Categorification of mutation (= flip) and associated cluster algebra [Fomin-Shapiro-Thurston, Fock-Goncharov]

Question : Non-orientable surfaces (NoS) ?

[Dupont-Palesi] : $\exists$ analogue of cluster algebra (+ corresp. Teichmüller space)

So, what about the category side ?

Today: Goal I, II, III

§§1 | Surface topology

Setup

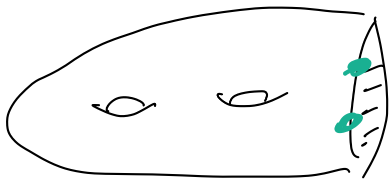
$\partial \mathbb{S} \neq \emptyset$

$\mathbb{S}$: Surface = compact 2-dim'l$/\mathbb{R}$ w/ non-empty boundary

$\mathbb{M}$: finite set of marked points in $\partial \mathbb{S}$ (∴ unpunctured)

s.t. each boundary component contains ≥ 1 marked pt.

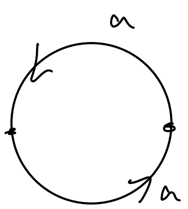
and $(\mathbb{S}, \mathbb{M}) \neq 1,2,3$-gon



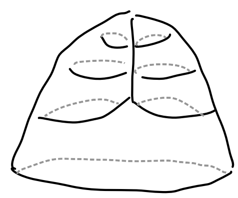
$|\mathbb{M}| = 2.$

§1.1 | Working with non-orientable surfaces (NoS)

• $\mathbb{RP}^2$ =



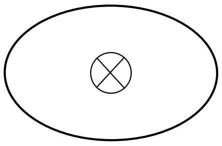
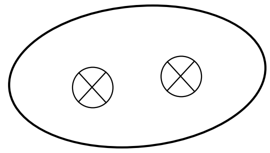
• crosscap $:= \mathbb{RP}^2 \setminus$ disc

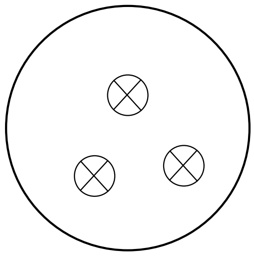
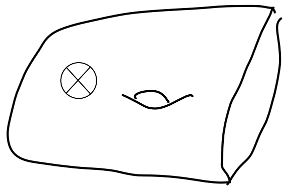


$\overset{\text{homeo}}{\simeq}$ Möbius strip,

In practice, represent by the symbol $\otimes$ (or , , etc.)

Rmk: Some literature call $\mathbb{RP}^2$ the crosscap instead.

E.g.  = Möbius strip,  = Klein bottle \ disc,

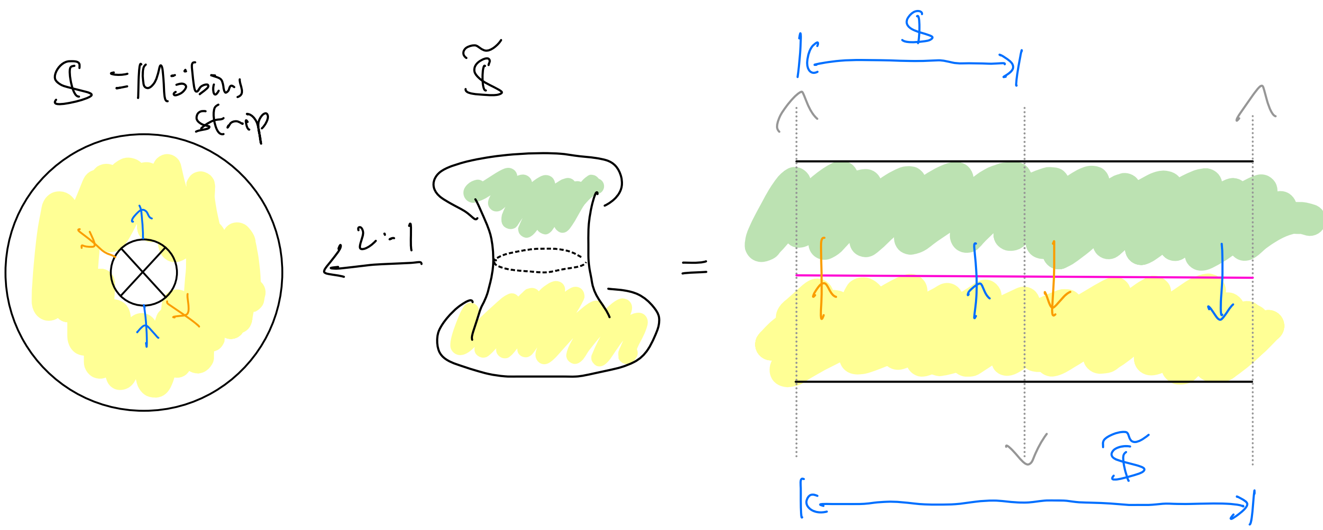
 =  = Dyck's surface \ disc.

§1.2 Orientable double cover of NOS

$$\widetilde{(\mathbb{S},\mathbb{M})} = (\tilde{\mathbb{S}},\tilde{\mathbb{M}}) \xrightarrow{2:1} (\mathbb{S},\mathbb{M})$$

$\circlearrowright$

σ: orientation-reversing auto. of order 2.

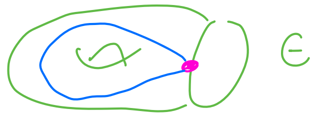


☆ no intersection between the 2 curves & .

§1.2 Objects of interest

γ: **Curve** on $(\mathcal{S}, M)$ $:\Longleftrightarrow$ either $\begin{cases} \textbf{closed} & \gamma \simeq S^1 \\ \text{s.t.} & \gamma \cap M = \emptyset \\ & \text{non-contractible} \end{cases}$

or $\begin{cases} \textbf{non-closed} & \gamma : [0,1] \to \mathcal{S} \\ & \gamma(0), \gamma(1) \in M \\ & \gamma(0,1) \subset \mathcal{S} \setminus \partial\mathcal{S} \end{cases}$

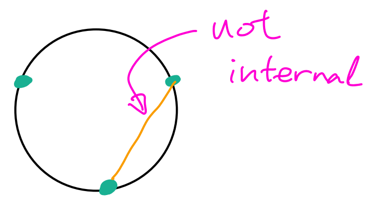


* Always considered up to isotopies that fix $\partial\mathcal{S}$ pointwise

Non-crossing (set of) curves $:\Longleftrightarrow$ no intersection except possibly at endpoints.
$\cup$
NC

- **Arc** $:\Longleftrightarrow$ NC non-closed curves

- **Internal arc** $:\Longleftrightarrow$ arc $\not\simeq$ boundary inter

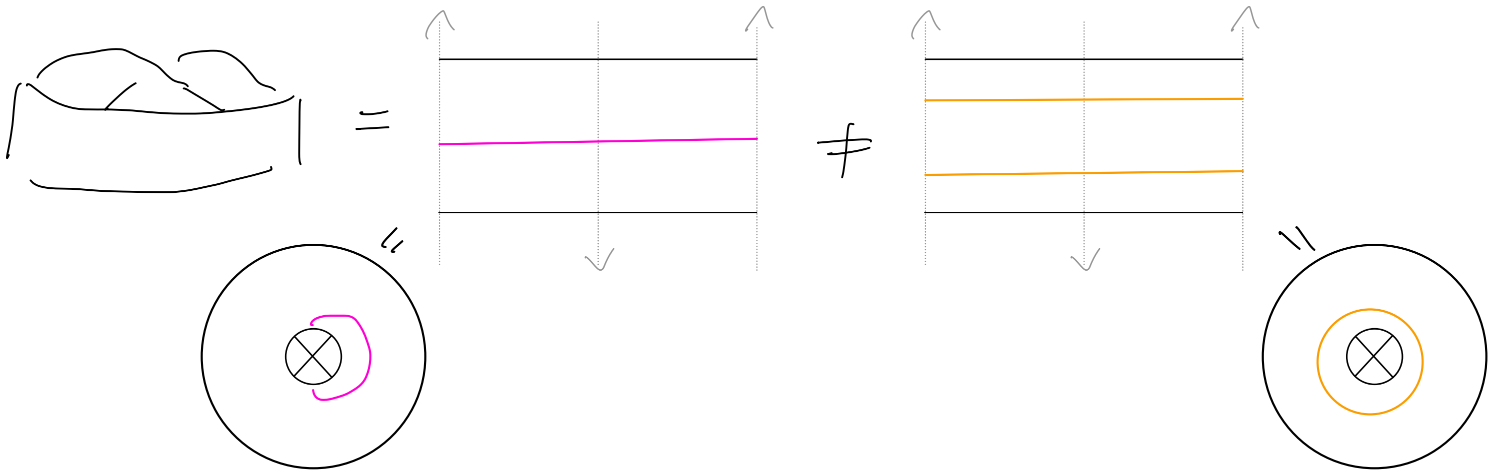


- **1-sided** closed curve $:\Longleftrightarrow$ non-orientable closed curve

  (If simple, then equiv. to $\exists$ regular nbhd $\underset{\text{homeo}}{\simeq}$ Möbius strip)

- **2-sided** closed curve $:\Longleftrightarrow$ not 1-sided

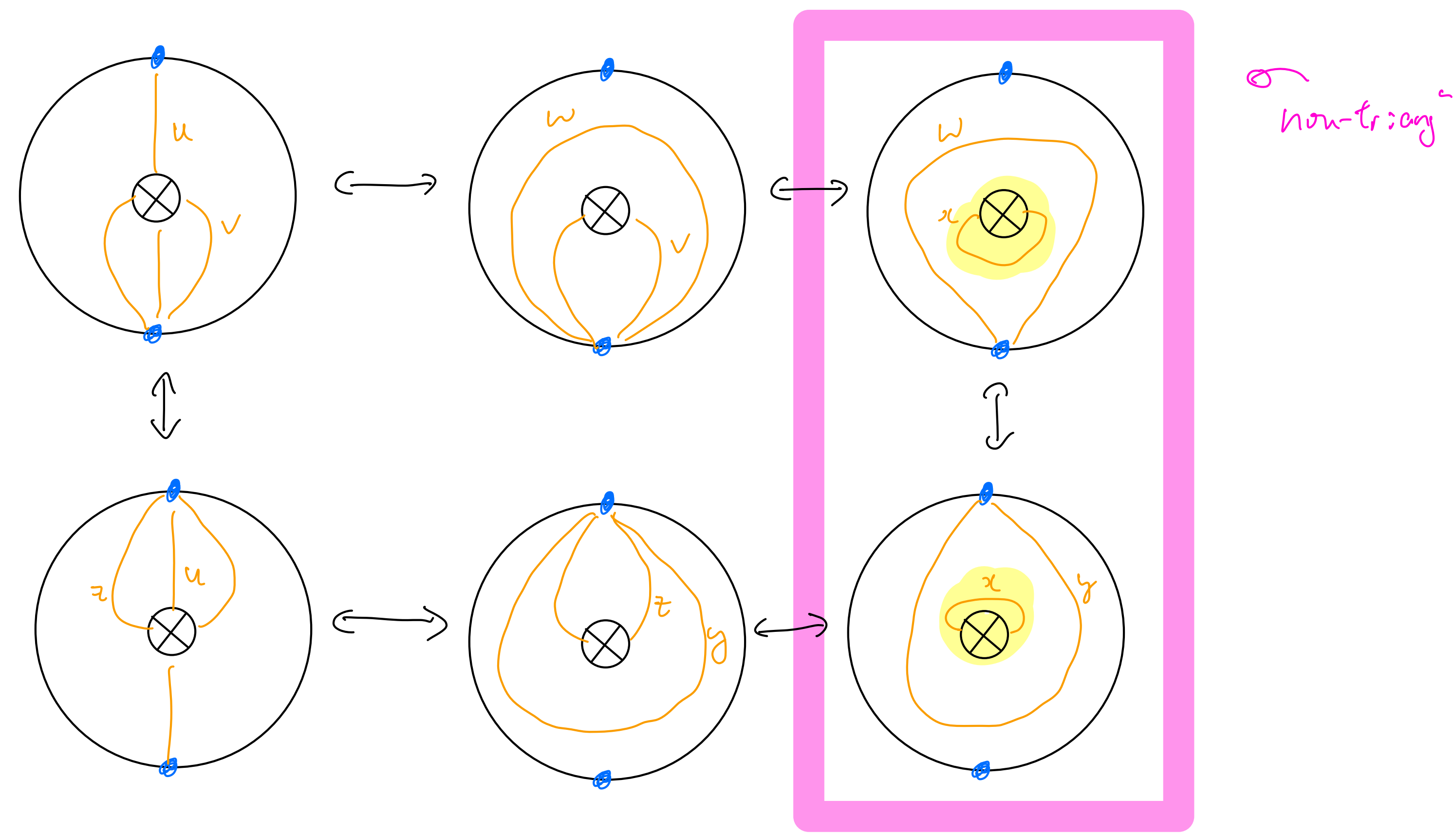
  (If simple, then equiv. to $\exists$ regular nbhd $\underset{\text{homeo}}{\simeq}$ annulus)



- **Quasi-arc** :⟺ either internal arc,
  or 1-sided simple closed curve
  (simple: no self-intersection)

- **(quasi-)triangulation** :⟺ maximal NC set of (quasi-)arcs

E.g. M_2 (:= Möbius strip w/ 2 marked pt's) has 6 quasi-triang''s.

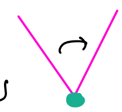


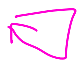
§§2) Triangulation vs QP

2.1) Orientable case $(\tilde{S},\tilde{M})$, $\tilde{T}$

$\rightsquigarrow$ (Q,W) : QP (Quiver with Potential)

where
$$\begin{cases} Q_0 = \{\text{internal arcs of } \tilde{T}\} \\ Q_1 = \text{CW oriented angles between internal arcs} \\ W = \displaystyle\sum_{\substack{\text{internal}\\ \text{triangles}}} \end{cases}$$



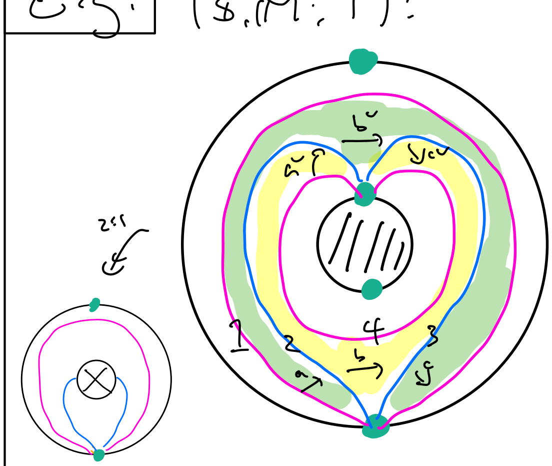


$\rightsquigarrow$ Jacobian algebra $J_{\tilde{T}} = \mathrm{Jac}(Q,W) := \Bbbk Q/(\partial_a W) = \Bbbk Q/\left(\begin{smallmatrix}\text{length 2 paths}\\ \text{in int.-}\Delta\text{'s}\end{smallmatrix}\right)$

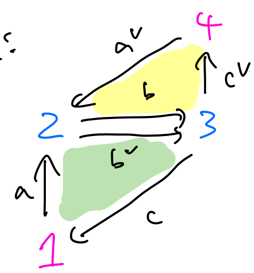
N.B. $J_{\tilde{T}}$ is f.d. and a gentle algebra

Convention/Definition QP is gentle if induced by a triang^n.

E.g. $(\tilde{S},\tilde{M},\tilde{T})$:



Q:



$W = a^\vee b c^\vee + a b^\vee c$

$\partial_* W$:

$a^\vee b,\ b c^\vee,\ c^\vee a^\vee$

$a b^\vee,\ b^\vee c,\ ca$

Thm [Assem, Brüstle, Charbonneau-Jodoin, Plamondon]

$$\left\{ (\tilde{S}, \tilde{M}; \tilde{T}) : \begin{array}{c}\text{triangulated} \\ \text{ori. surfaces}\end{array} \right\} \overset{1:1}{\longleftrightarrow} \{ (Q, W) : \text{gentle QP} \}$$

2.2) NoS case $\quad (S, M) \xleftarrow{2:1} (\tilde{S}, \tilde{M}) \circlearrowleft \sigma$

Def ① Q : quiver

An involution $\sigma : Q \to Q$

$:\Leftrightarrow \quad \sigma = (\sigma_0 : Q_0 \to Q_0,\ \sigma_1 : Q_1 \to Q_1)$

s.t. $\begin{cases} \sigma^2 = 1 \\ \sigma(v \xrightarrow{\alpha} w) = \left(\sigma(v) \xleftarrow{\sigma(\alpha)} \sigma(w)\right) \end{cases}$

In [Derksen-Weyman] language: (Q, σ) is a symmetric quiver

N.B. Specifying involution $\sigma \Rightarrow \exists \left(kQ \overset{\text{alg}}{\xrightarrow{\sim}} kQ^{op} \right)$

② An involution on a QP (Q, W).

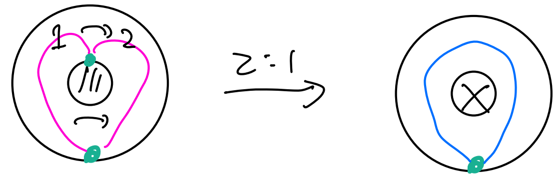
$:\Leftrightarrow$ an involution σ on Q s.t. $\sigma W = W$

$\Leftrightarrow$ —— " —— s.t. $\sigma(\partial_a W) = (\partial_a W) \quad \forall a \in Q_1$

Eg. ① $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \qquad \sigma : \begin{array}{c} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \alpha \end{array}$

② $\overset{1}{\bullet} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \overset{2}{\bullet} \qquad \sigma : \begin{array}{c} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \alpha \\ \beta \longleftrightarrow \beta \end{array}$

(3) $\overset{1}{\circ} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \overset{2}{\circ}$ σ ; $1 \longleftrightarrow 2$, $\alpha \longleftrightarrow \beta$



(4) Exercise: $A_{n,n}$-quiver
$$\begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & & & n{+}1 \\ & 2n \arrow[r] & \cdots \arrow[r] & n{+}2 \arrow[ur] & \end{tikzcd}$$
has an involution

Obs: $\sigma \circlearrowright (\widetilde{S, M}) \xrightarrow{2:1} (S, M)$

$\rightsquigarrow$ involution σ on (Q, W)

Moreover, σ is fixed-point free (FPF)

$\sigma(v) \neq v \quad \forall v \in Q_0$

$\sigma(\alpha) \neq \alpha \quad \forall v \in Q_1$

Prop [BM-C-W] (Goal I)

$$\left\{ \begin{array}{l} \text{Connected NoS} \\ \text{with triangulation} \end{array} \right\} / \text{homeo} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} (Q, W; \sigma) \text{ s.t. } (Q, W) \text{ gentle,} \\ Q \text{ connected, } \sigma \text{ is a FPF involution} \end{array} \right\} / \text{“}\cong\text{”}$$

$$(S, M; T) \longmapsto (Q, W; \sigma)$$

§3 Triangulation vs CTO (cluster-tilting obj)

Setup: From now on, $\boxed{\mathbb{k} = \text{complex numbers}}$

Orientable case

See [Brüstle-Zhang]

TOPOLOGY	CATEGORY
$(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}})$	cluster category $\mathcal{C} = \mathcal{C}_{(\widetilde{\mathbb{S}}, \widetilde{\mathbb{M}})}$ $\circlearrowleft$ $[1] \cong \tau$
non-closed curves	"string objects"
$\begin{pmatrix}\text{closed} \\ \text{curves}\end{pmatrix} \times \mathbb{k}^\times$ $\parallel$ $\begin{pmatrix}\text{primitive} \\ \text{closed curves}\end{pmatrix} \times \mathbb{N} \times \mathbb{k}^\times$	"band objects"

N.B. This is KS Hom-fin. 2-CY tri. cat.

all indec's of $\mathcal{C}_{(\mathbb{S}, \mathbb{M})}$

$\rotatebox{90}{$\in$}$ (ω^n, λ)

ω: primitive $:\Leftrightarrow$ $\omega \neq \rho^k$ in $\pi_1(\widetilde{\mathbb{S}})$ with $k > 1$

Moreover,

$$\boxed{\text{Crossings} = \text{Non-split extensions}}$$:

TOPOLOGY	CATEGORY
Crossing between curves (+ loc. sys.)	Canonical basis (+ local sys. hom.) of $\mathrm{Ext}^1_{\mathcal{C}}(-,-) := \mathrm{Hom}_{\mathcal{C}}(-,-[1])$
arcs	rigid objects := no self-ext^n ($\mathrm{Ext}^1=0$)
triangulation	cluster-tilting object (CTO) := maximal rigid object. (i.e. $M \oplus N$ rigid $\Rightarrow N \in \mathrm{add}\, M$)

Prop [BM-C-W]

$\exists$ exact $\underline{\underline{\text{contravariant}}}$ $\boxed{\text{duality}}$ functors ∇ — i.e. $\nabla^2 \cong \mathrm{Id}$

$$\mathcal{C} \xrightarrow[\sim]{\nabla} \mathcal{C}$$

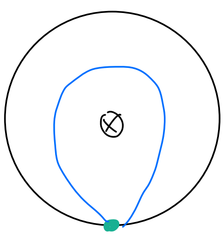
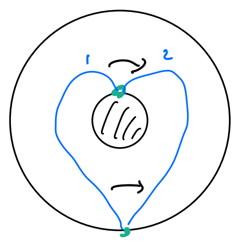
such that

$\overbrace{}^{\text{Topology}}$ $\overbrace{}^{\text{Category}}$

① $\forall \gamma$: non-closed on $(\widetilde{S,M})$, $\quad \sigma(\gamma) \longleftrightarrow \nabla(\gamma)$

② $\forall (\omega^n, \lambda) \in \left\{ \begin{array}{c} \text{c.c.'s} \\ \text{on } (\widetilde{S,M}) \end{array} \right\} \times k^\times$, $\quad (\sigma(\omega)^n, \lambda^{-1}) \longleftrightarrow \nabla(\omega^n, \lambda)$

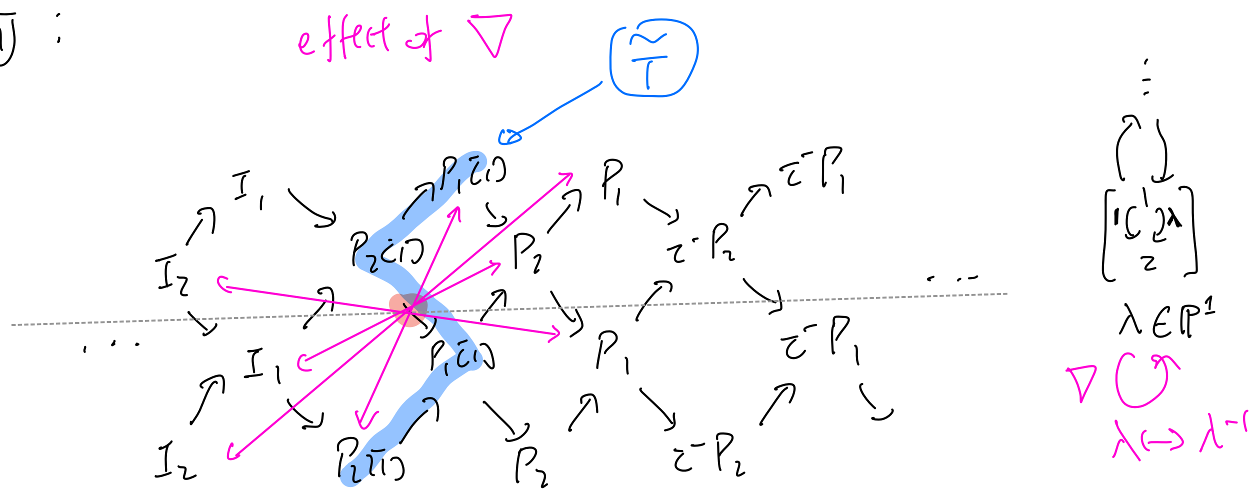
E.g.

$(S,M;T) =$  $\xleftarrow{2:1}$ $(\widetilde{S},\widetilde{M},\widetilde{T}) =$ 

$Q = \left(1 \overset{a}{\underset{b}{\rightrightarrows}} 2 \right)$, $W = 0$

$\sigma: 1 \longleftrightarrow 2$, $a \longleftrightarrow b$

$\mathcal{C}_{(\widetilde{S,M})}$:



§4 Symmetric representations (= ε-representations)

$\varepsilon = \{\pm 1\}$

[Derksen-Weyman] orthogonal ($\varepsilon = +1$)

symplectic ($\varepsilon = -1$)

[Boos-Cerulli Irelli]

Throughout this section,

- $(A = \Bbbk Q/I, \sigma)$: σ involution on Q fixing I.
- modules = right f.d. modules

Def 1) $\underline{\varepsilon\text{-form}}$ $:\Leftrightarrow$ $\begin{cases} \cdot \text{ symmetric bilinear form if } \varepsilon = +1, \\ \cdot \text{ skew-symm. bilinear form if } \varepsilon = -1. \end{cases}$

2) An $\underline{\varepsilon\text{-representation}}$ over (A, σ)

$:\Leftrightarrow$
- M : ordinary rep^n.
- $\langle -, - \rangle : M \times M \to \Bbbk$ non-degen. ε-form

s.t. $\begin{cases} \cdot \ \langle m e_i, n e_j \rangle = 0 \quad \forall j \neq \sigma(i) \ ; \ e_i, e_j : \text{primitive idem's.} \\ \cdot \ \langle m\alpha, n \rangle + \langle m, n\sigma(\alpha) \rangle = 0 \\ \quad (\text{i.e. } \sigma(\alpha) \text{ is adjoint of } \alpha) \end{cases}$

FACT [DW, BCI] M : A-module

- $M : \varepsilon\text{-rep}^n \Leftrightarrow \exists \psi_M : M \xrightarrow{\sim} \nabla M$ as A-module s.t. $\nabla(\psi_M) = \varepsilon \psi_M$

☆ Indecomposability makes sense for ε-rep^n's.

Prop [D-W, B-CI] (Characterisation of indec ε-rep^n's)

M: indec. ε-rep^n / (A, σ)

$\Rightarrow \exists \overline{M}$: indec. A-module

s.t. exactly one of the following holds.

a) $\nabla \overline{M} \not\cong \overline{M}$, $\quad M = \overline{M} \oplus \nabla \overline{M}$ → call M split

b) $\nabla \overline{M} \cong \overline{M}$, $\quad M = \overline{M} \oplus \nabla \overline{M}$ → call M ramified

c) $\nabla \overline{M} \cong \overline{I}$, $\quad M = \overline{M}$ → call M ~~type I~~ 1-sided

Notation: ω: primitive cc, $\ell(\omega) := \# \omega \cap \tilde{\tau}$.

Thm [BM-C-W] (Goal II)

M: indec ε-rep^n / $Jac(Q,W)$, (Q,W): gentle FPF-Symm. QP

$\Rightarrow M \cong$ exactly one of the following.

split
- $M(\gamma) \oplus M(\sigma(\gamma))$ ← all strings
- $M_\lambda(\omega^n) \oplus M_{\lambda^{-1}}(\sigma(\omega^n))$ s.t. $\begin{cases} \omega \neq \sigma(\omega) \text{ or } \lambda \notin \{\pm 1\}, \\ \text{any } n \geq 1. \end{cases}$

1-sided
- $M_\lambda(\omega)$ s.t. $\sigma(\omega) = \omega$ & $\varepsilon = (-1)^{\ell(\omega)/2} \lambda$
- $M_\lambda(\omega^2)$ s.t. $\sigma(\omega) = \omega$ & $\varepsilon = (-1)^{\frac{\ell(\omega)}{2}+1} \lambda$

ramified
- $M_\lambda(\omega^n)^{\oplus 2}$ s.t. all remaining cases

all band

Hence

"Indec" top. obj. | Indec ε-rep^n's

$\{$1-sided c.c. ω on $(S,M)\}$ $\xleftrightarrow{1:1}$ $\{$1-sided ε-indec $M_{\varepsilon_\omega}(\omega)\}$

$\updownarrow$ 1:1

$\{$2-sided c.c. ω on $(S,M)\}$ $\xleftrightarrow{1:1}$ $\{$ramified ε-indec $M_{-\varepsilon_\omega}(\omega)^{\oplus 2}\}$

$\{$non-closed curve γ on $(S,M)\} \setminus \tilde{\tau}$ $\xleftrightarrow{1:1}$ $\{$split ε-indec "of strong type" $M(\gamma) \oplus \nabla M(\gamma)\}$

→ Goal II ✓

§§5 | ε-extension, ε-rigid

Setup $\tilde{S} \xrightarrow{2:1} S$
orientable nos.

Fix $\varepsilon \in \{\pm 1\}$, $\tilde{T}$: triang^n. on $\tilde{S}$ that is σ-stable

$\rightsquigarrow$ $\pi : \mathcal{C}_{\tilde{S}} \twoheadrightarrow \mathrm{mod}\, J_{\tilde{T}}$ exact

Def :

- $X \in \mathcal{C}$ is an ε-object $:\Longleftrightarrow$ $\exists\, \psi_X : \nabla X \xrightarrow{\sim} X$ s.t. $\nabla(\psi_X) = \varepsilon \psi_X$

- $X \in \mathcal{C}$ is an indec. ε-object $:\Longleftrightarrow$ · πX an indec. ε-rep$^n/\Lambda$
  or
  · $X = \alpha \oplus \underset{\nabla(\alpha)}{\sigma(\alpha)}$, some $\alpha \in \tilde{T}$

- In this case, πX is indec ε-rep^n

$\rightsquigarrow$ Case 1: $\pi X = \pi \boxed{\overline{X}} \oplus \pi \boxed{D\overline{X}}$

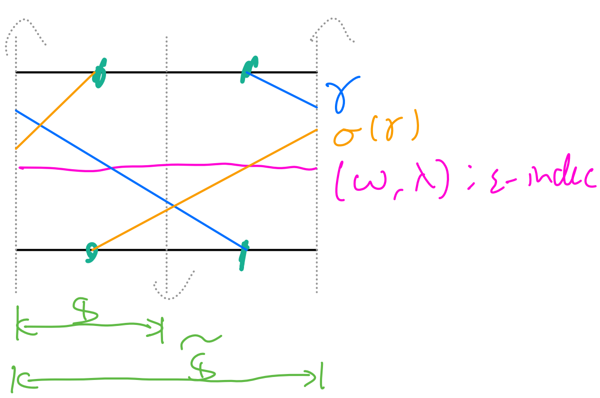
Def: ε-factors of X.

Case 2: $\pi X = M_\lambda(\omega)$ or $M_\lambda(\omega^2)$

$\rightsquigarrow$ $\exists$ non-split
$0 \to DM \to \pi X \to M \to 0$
with M indec.

call $\overline{X}$: indec obj in $\mathcal{C}$
s.t. $\pi \overline{X} = M$
as ε-factors of X.

e.g. $S = M_2$:



· X, Y : $\mathcal{E}$-objects

$$f : X \to \bar{Y}[1] \quad \text{s.t.} \quad \begin{cases} \bar{Y} : \mathcal{E}\text{-factor of } Y \\ f \circ \psi_X \circ \nabla f = 0 \end{cases} \qquad \text{"} \begin{tikzcd} \nabla\bar{Y}[-1] \arrow[d, "\nabla f"'] \arrow[dr, dotted] & \\ X \arrow[r, "f"'] & \bar{Y}[1] \end{tikzcd} \text{"}$$

In this case, $\exists$ comm. dgm :

$$\begin{tikzcd} & & \nabla\bar{Y}[-1] \arrow[d, "\psi_X \nabla f"'] & \\ \bar{Y} \arrow[r] \arrow[d, equal] & C^f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & \bar{Y}[1] \text{ s.t.} \\ \bar{Y} \arrow[r] & E \arrow[r] \arrow[d] & C_{\psi_X \nabla f} \arrow[d] & \\ & \nabla\bar{Y} \arrow[r, equal] & \nabla\bar{Y} & \end{tikzcd}$$

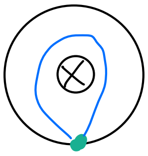
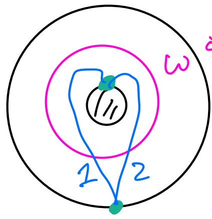
all rows & all col^{n}'s are $\triangle$'s of $\mathcal{C}$.

Call this an $\mathcal{E}$-extension from X to Y.

· An $\mathcal{E}$-extension splits $:\Leftrightarrow$ $E \cong X \oplus \bar{Y} \oplus \nabla\bar{Y}$

· X is indec. $\mathcal{E}$-rigid $:\Leftrightarrow$ X indec $\mathcal{E}$-obj
all $\mathcal{E}$-extension $X \to X$ splits.

· X is $\mathcal{E}$-rigid $:\Leftrightarrow$ $\begin{cases} \cdot\ X = \bigoplus \text{indec } \mathcal{E}\text{-rigid.} \\ \cdot\ \forall\ \mathcal{E}\text{-indec's } Y, Z \subset \oplus X, \\ \quad Y \not\cong Z \Rightarrow \operatorname{Ext}^1_{\mathcal{C}}(Y, Z) = 0. \end{cases}$

Eg. $\mathbb{S} =$  , $\tilde{\mathbb{S}} =$ 

ω ← lift of the unique quasi-arc on $\mathbb{S}$

$\tilde{\gamma} \leftrightarrow (Q, W) = (1 \rightrightarrows 2, 0)$

$\Rightarrow \quad \pi\omega = 1\overset{1}{\underset{2}{(}}2\,\lambda \qquad \varepsilon := -\lambda \in \{\pm 1\}$

$\pi\gamma = S_1$

$\pi\nabla\gamma = S_2$

$(\nabla\gamma)[-1] \to \omega \to \gamma[1]$

$\pi \circlearrowleft$

$\begin{smallmatrix} 1 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \xrightarrow{\nabla f} 1\overset{1}{\underset{2}{(}}2 \xrightarrow{f} \begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 \end{smallmatrix}$

If $f \neq 0$, this composition is non-zero.

$\Rightarrow$ only split extensions everywhere.

$\Rightarrow$ ε-rigid.

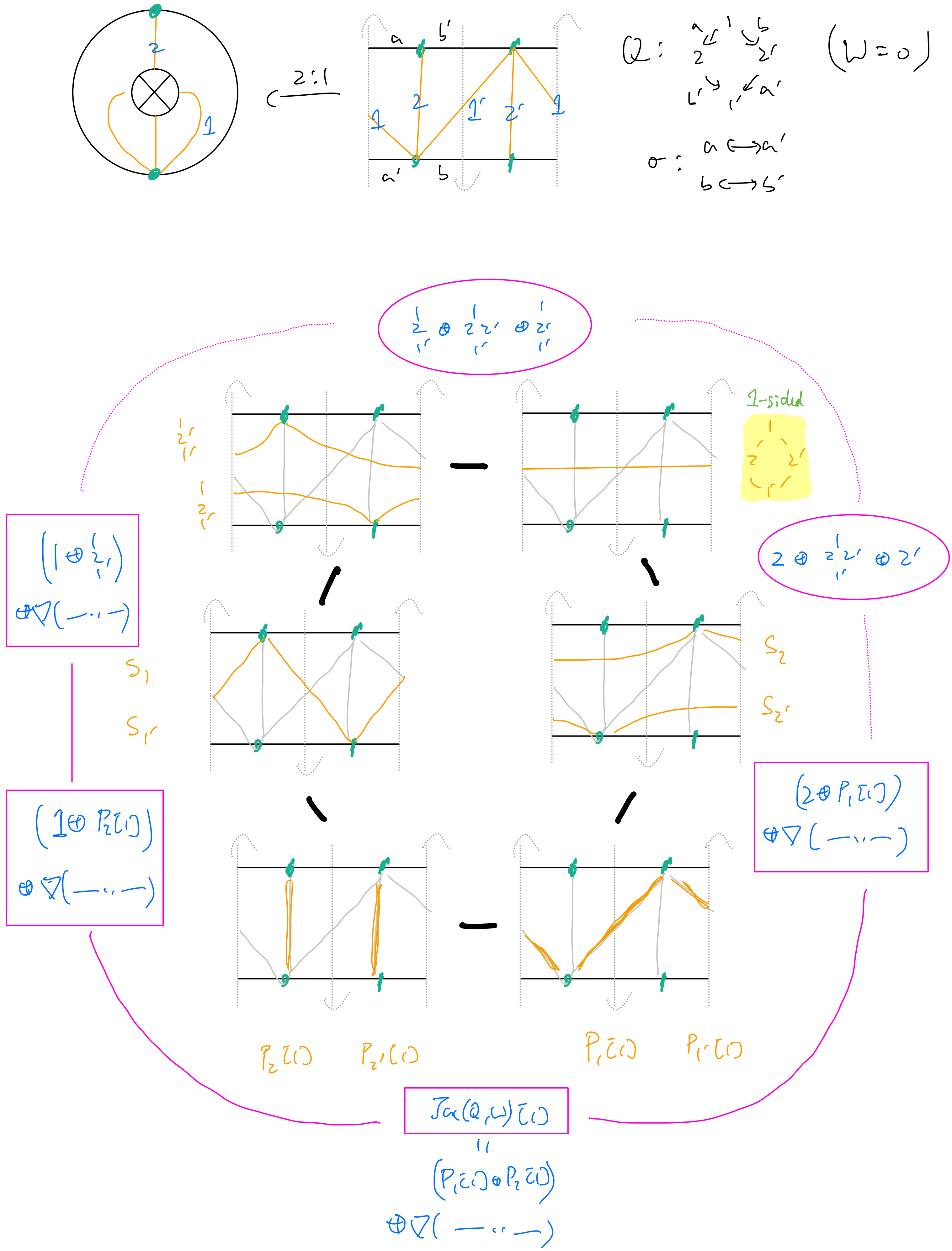
Goal III

Thm [BM-C-W] ("Baby" categorification of quasi-triang")

$\{\text{indec } \varepsilon\text{-rigid}\} \xleftrightarrow{1:1} \{\text{quasi-arc of } (\mathbb{S}, M)\}$

which then induces

$\{\text{maximal } \varepsilon\text{-rigid obj}\} \xleftrightarrow{1:1} \{\text{quasi-triangulations of } (\mathbb{S}, M)\}$



2
1
2:1
a
b'
a'
b
2
1
1'
2'
Q:
(W=0)
σ: a ⟷ a'
b ⟷ b'
1-sided
2 ⊕ … ⊕ 2'
S_1
S_1'
S_2
S_2'
(1 ⊕ P_2[1])
⊕ ∇(— .. —)
(2 ⊕ P_1[1])
⊕ ∇(— .. —)
P_2[1]
P_2'[1]
P_1[1]
P_1'[1]
Jac(Q,W)[1]
=
(P_1[1] ⊕ P_2[1])
⊕ ∇(— .. —)