A GUIDE TO 2-REPRESENTATION THEORY

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Disclaimer: This is not intended to be a formal mathematical article. The aim is to explain ideas and motivations, rather than the actual mathematics. Hence, rigour of mathematics (and grammar) is not guaranteed.

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Notation: Follows MM most of the time, unless otherwise specified. Short recap:
2-categories: $\mathscr{C}, \mathscr{A}, \mathscr{S}, \dots$
2-ideal (left/right/two-sided): $\mathscr{I}, \mathscr{J}, \dots$
objects: $\mathtt{i}, \mathtt{j}, \dots$
1-morphisms: $\mathrm{F}, \mathrm{G}, \dots$ (identity $\mathbb{1}_{\mathtt{i}}, \mathbb{1}_{\mathtt{j}}, \dots$)
2-morphisms: $\alpha, \beta, \dots$ (identity $\mathrm{id}_{\mathrm{F}}, \mathrm{id}_{\mathrm{G}}, \dots$)
1-representations: $M, N, \dots$
2-representations: $\mathbf{M}, \mathbf{N}, \dots$

1. Algebra to 1-category to 2-category

1.1. **Algebra.** A: f.d. associative unital $\Bbbk$-algebras, then we have identity $1_A = e_1 + \cdots + e_n$ with e_i's primitive idempotent (pim for short)
For convenience, assume A basic (dim of simple = 1)
In particular, e_i's are pairwise orthogonal.

Although using right modules gives nicer compatibility between arrows of quivers and maps of modules, we will follow MM's convention to use left modules.
Indecomposable projective A-modules (also short for pim) are $P_1, \dots, P_n$, with $P_i = Ae_i$.
Injective hasn't play a part in the MM's study so far, but just in case, injectives are $D(e_iA)$ with $D(-)$ the $\Bbbk$-linear dual.
$L_1, \dots, L_n$ are corresponding simple top of P_i's.

Special properties of projective A-module (these will be level-up later):
(a) $e^2 = e \in A$, then there is a map $\iota : Ae \to A$, $\pi : A \to Ae$ with $\iota\pi = e$ and $\pi\iota = 1_{Ae}$.
– we should view e as the map π
(b) $\mathrm{Hom}_A(P_i, M) = e_iM$ (as $\Bbbk$-v.s.), so dimension of this space is composition multiplicity of L_i in M, conceptually:
- viewing e_i as projection map, e_iM effectively extracts the a subspace of M acted on by "the e_i-part of A"

1.2. **1-category.** Associated to A are two (1-)categories:
(1) finitary additive 1-category A-proj = add(A)
– indec objects: $\mathtt{1}, \dots, \mathtt{n}$; identified with $P_1, \dots, P_n$ resp.

– morphisms: $\mathrm{Hom}(P_i, P_j) = e_i A e_j$
(2) abelian category A-mod

When thinking about finitary 2-representation, (almost) always think of them as either of the above two, depending on the type (additive or abelian) of 2-representation you look at. In fact, (I believe) the definition of finitary 1-category is defined using the above properties.

Note
(a) finitary (additive) 1-cat = idem split + fin. many indec + f.d. $\Bbbk$-linear 1-morphism space
–idem split $\Leftarrow$ properties of pims $e_1, \dots, e_n$
–fin. many indec $\Leftarrow$ A f.d. so fin. many proj. indec
–f.d. 1-mor $\Leftarrow$ $e_i A e_j$ f.d.
(b) abelianisation of A-proj is A-mod, where the diagram $(F \to G)$ is simply representing a projective presentation of a module, because A-mod's are determined by projective presentations, and projective presentation is unqiue up to homotopy.
(c) on the other hand, A-proj can be obtained from A-mod by taking the full subcategory of indecomposable projective modules.
(d) classically, the category associated to an algebra is the one-object category where morphisms are elements of A. Nevertheless, if A is local, the classical interpretation is the same as A-proj.

1.3. **2-category.** The slogan here is:
1-category $\leftrightarrow$ (1-)algebra
2-category $\leftrightarrow$ 2-algebra

Recall the decategorification process: for a 2-category $\mathscr{C}$ so that 1-morphism spaces are $\Bbbk$-linear abelian (resp. additive) category. Decategorify $\mathscr{C}$ is the 1-category $[\mathscr{C}]$ with $[\mathscr{C}]_0 = \mathscr{C}_0$, $[\mathscr{C}](\mathtt{i}, \mathtt{j})$ the (resp. split) Grothendieck group.

Let us look at one object case first. One-object 2-category $\mathscr{C}$ decategorifies to a one-object 1-category $[\mathscr{C}]$ with (the only) morphism space being a free $\mathbb{Z}$-module. Recall the classical algebra-category translation in note (d) above, then $[\mathscr{C}]$ can be identified as a $\mathbb{Z}$-algebra (i.e. a ring). Note $[\mathrm{F}][\mathrm{G}] = [\mathrm{F} \circ \mathrm{G}]$ for this algebra.

The reason for not using translation (1) is (?) to align with classical categorification theory, where the single object is A-mod and 1-morphisms are functors, 2-morphisms are natural transformations.

Problem: in Maz's note on algebraic categorification. If we take a genuine categorification $\mathscr{C}$ of a $\Bbbk$-linear 1-category (e.g. A-proj), decategorification should use translation (1)...? So far, in my study, it seems we need to use classical translation:

For example, $\mathscr{C}_D$ (I call this projection functor 2-category):
one object $\mathtt{i}$ being (a small category equivalent to) D-mod with $D = \Bbbk[x]/(x^2)$
$\mathscr{C}_1 = \mathscr{C}(\mathtt{i}, \mathtt{i}) = \mathrm{add}(D, D \otimes_{\Bbbk} D)$
$\mathscr{C}_2$ = all nat. transf.
$\Rightarrow$ $[\mathscr{C}_D] \cong \mathbb{C}\mathfrak{S}_2 = \mathbb{C}\mathrm{triv} \oplus \mathbb{C}\mathrm{sign}$
$\Rightarrow$ not a local algebra.
Note: this is also the 2-category of Soergel bimodules of type A_2

Note:
1-morphisms control relations between objects.
2-morphisms control relations between 1-morphisms.
Although decategorification "forgets" 2-morphisms, the information is encoded in the multiplication rule/formula for 1-morphisms, most notably it controls indecomposability/splitness of 1-morphisms.

Summarising again
Algebra $A \rightsquigarrow$ additive (resp. abelian) 1-category A-proj (resp. A-mod)
2-category $\mathscr{C} \rightsquigarrow$ 1-category $[\mathscr{C}]$ identified as an algebra, but *NOT* A-proj!

When $\mathscr{C}$ has more than one object, then decategorifying we get an algebra for each object in $\mathscr{C}$. Moreover, $[\mathscr{C}](\mathtt{i},\mathtt{k})$ gives mathematical structure relating the two associated to $\mathtt{i}$ and $\mathtt{k}$. (so...algebra or bimodule??) So, a 2-category (with good enough properties) gives some "hyper structure" describing different algebras at the same time. This is called a 2-algebra.

2. Module to 1-functor to 2-functor

We only consider finite dimensional modules (i.e. finitely generated modules for f.d algebras).

We explain below why a (1-)representation of A (i.e. an A-module) is a (1-)functor A-proj $\to \Bbbk$-mod.

A left A-module is a vector-space M with:
(1) $am \in M$
(2) $(a+a')m = am + a'm$
(3) $(aa')m = a(a'm)$
(4) $1_A m = m$

A representation of A = a $\Bbbk$-linear map $\rho : A \to \mathrm{End}_\Bbbk(M)$
$\Rightarrow$ a representation ρ = a left A-module M

Since $(A\text{-proj})_1 \leftrightarrow A$ (as set)
$\Rightarrow$ ρ is a 1-functor:

$$\begin{array}{rcl} \rho : A\text{-proj} & \to & \Bbbk\text{-mod} \\ \mathtt{i} & \mapsto & e_i M \\ f : \mathtt{i} \to \mathtt{k} & \mapsto & \rho(f) : e_i M \to e_j M \end{array}$$

Note: To ensure (2), need ρ additive; to ensure (3), need ρ covariant.
In particular, for right modules, take contravariant functor instead.

The convention is to denote this functor by M itself.

2.1. **2-representation.**

Detour to categorification philosophy (again):
$V \in \Bbbk$-mod with basis $\mathcal{B}$
$\Rightarrow$ $\#\mathcal{B}$ uniquely determine V in $\Bbbk$-mod up to isom
$\rightsquigarrow$ a vector-space "categorifies" natural number

A natural number = cardinality of a set
A set is uniquely determined by its cardinality inside the category of sets
$\Rightarrow$ vector-space categorifies sets

Philosophically, there is no structural relations between vectors in a vector-space (in the sense that we cannot multiply two vectors "very canonically"). Above discussion says that:

(finitary) 0-category	decats to	(finitary)(-1)-category
vector-space	$\rightsquigarrow$	set
basis vector	$\rightsquigarrow$	object

Levelling up:
$[A\text{-mod}]^{\Bbbk} = \oplus_i \Bbbk[S_i]$ where S_i are (isoclass representatives of) simple A-modules.
$[A\text{-proj}]^{\Bbbk}_{\oplus} = \oplus_i \Bbbk[P_i]$, where P_i is proj. cover of S_i.

So, the picture:

(finitary) 1-category	decats to	(finitary) 0-category
A-proj ; A-mod ; $K^b(A\text{-proj})$; $D^b(A\text{-mod})$	$\rightsquigarrow$	vector-space
PIM; simple; stalk PIM; stalk simple	$\rightsquigarrow$	basis vector

An A-module is a vector-space M with blah...
$\therefore$ a 1-algebra A acts on a 0-category M, where "acts" means
a 1-functor from a 1-category (1-algebra) to the 1-category ($\Bbbk$-mod) of some 0-categories (f.d. vector-spaces)

Levelling up this picture, a 2-representation means that we have:
a 2-algebra acts on a 1-category, where "acts" means
a 2-functor from a 2-category (2-algebra) to the 2-category of some 1-categories (finitary $\Bbbk$-categories).

As in 1-representation case, we use the same symbol $\mathbf{M}$ for the 2-functor (2-rep) and the 1-category that $\mathscr{C}$ acts on.

$$\begin{array}{rcll}
\mathbf{M} : \mathscr{C} & \to & \text{(some "nice" 2-category)} & \\
\mathtt{i} & \mapsto & \text{a 1-category} & \mathcal{C}_{\mathtt{i}} \\
\mathrm{F} : \mathtt{i} \to \mathtt{k} & \mapsto & \text{functor} & F : \mathcal{C}_{\mathtt{i}} \to \mathcal{C}_{\mathtt{j}} \\
\alpha : \mathrm{F} \Rightarrow \mathrm{G} & \mapsto & \text{nat. transf.} & F \Rightarrow G
\end{array}$$

What is a nice 2-category on the RHS? The usual three choices are:
(i) finitary additive $\Bbbk$-categories $\approx$ 1-cat's of form A-proj
(ii) "nice" abelian categories $\approx$ 1-cat's of form A-mod
(iii) "nice" triangulated categories $\approx$ $D^b(A\text{-mod})$ or $K^b(A\text{-proj})$

1-mor's and 2-mor's: (i) projective functors (=left proj. right proj. bimodules) and their morphisms

(ii) exact functors ($\approx$ "nice" bimodules) and their morphisms
(iii) triangulated functors ($\approx$ "nice" two-sided complex) and their morphisms

So to define a 2-rep. We specify some 1-categories and specify how 1-mor's and 2-mor's of $\mathscr{C}$ acts on them. To be slightly more precise, we stick to finitary additive 2-rep's (or rather 2-rep's of form A-proj) for now.

(0-mor): $\mathbf{M}(\mathtt{i})$:
— a 1-category $A_{\mathtt{i}}$-proj for each $\mathtt{i} \in \mathscr{C}_0$

(1-mor): functors $\mathbf{M}(\mathrm{F}) : A_{\mathtt{i}}\text{-proj} \to A_{\mathtt{j}}\text{-proj}\ \ \forall \mathrm{F} : \mathtt{i} \to \mathtt{j}$ such that
— $\mathbf{M}(\mathrm{F} \circ \mathrm{G}) \cong \mathbf{M}(\mathrm{F}) \circ \mathbf{M}(\mathrm{G})\ \ \forall \mathrm{F}, \mathrm{G} \in \mathscr{C}_1$
— $\mathbf{M}(\mathbb{1}_{\mathtt{i}}) \cong \mathbb{1}_{A_{\mathtt{i}}\text{-proj}}$ (i.e. $\mathbf{M}(\mathbb{1}_{\mathtt{i}})M \cong M\ \forall M$)
(question: I ain't sure if we need strict equality or just nat. isom. $\cong$)

(2-mor): Nat.Transf.'s $\mathbf{M}(\alpha) : \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{G})$ such that
— $\mathbf{M}(\alpha) \circ_x \mathbf{M}(\beta) = \mathbf{M}(\alpha \circ_x \beta)\ \ \forall \alpha, \beta \in \mathscr{C}_1;\ x \in \{0, 1\}$
— $\mathbf{M}(\mathrm{id}_{\mathrm{F}}) = \mathrm{id}_{\mathbf{M}(F)}$

In practice, we will not write $\mathbf{M}(\mathrm{F})$ and $\mathbf{M}(\alpha)$ every time - just like when specifying a algebra-rep's, we only write $a.m$ instead of $\rho(a)m$.

2.2. **Minor notes about (de)categorification.** When people say "(weak) categorification of the A-module M" (A can be a Kac-Moody algebra if you like), they mean:
$\exists$ a 1-category $\mathbf{M}$ s.t. there is a map $A \to \mathrm{End}_{\mathrm{Func}}(\mathbf{M})$
(the RHS is the class of endo-functors on $\mathbf{M}$)
We call this "categorical action" (of A on $\mathbf{M}$)

So 2-representation theory is the formalism of "categorifying both algebras and its modules". This picture best explains the whole philosophy:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, squiggly, "\text{acts on}"] \arrow[d, "\text{decat}"'] & \text{2-rep } \mathbf{M} \arrow[d, "\text{decat}"] \\
{[\mathscr{C}] = A} \arrow[r, squiggly, "\text{acts on}"'] \arrow[ur, squiggly, "\text{categorical action}"] & M = [\mathbf{M}]
\end{tikzcd}$$

Obviously, there is no guarantee a priori that all the 1-representation theoretic phenomenon can be obtained from 2-representation theory. So, we need to set up 2-representation theory so that we can get a really good approximation to this goal.

I don't want to say too much about why we should do this. This is the same question as why we should study categorification. The major achievements so far that I knew:
(1) Khovanov homology (which categorifies Jones polynomial) can detect the unknot, while it is still unknown if the same can be achieved for Jones polynomial.
(2) Broúe's conjecture for Hecke algebras, Schur algebras, finite GL_n, etc.
(3) Identifying apparently-really-different algebras/categories using uniqueness theorem of categorification of Lie theoretical representations (this really need the theory of 2-algebra, or more specifically, 2-Kac Moody algebra).
Please ask experts more on "why categorification"

3. Finitary and fiatness

Studying representations or structure of an arbitrary algebra is impossible.
$\rightsquigarrow$ Studying purely abstract 2-representations or 2-categories is impossible.
$\Rightarrow$ need structures, i.e. conditions on 2-categories (2-algebras).

MM's theory are built for *finitary* and *fiat* 2-categories.

3.1. Finitary.
Philosophy: resemblance of *finite dimensional* algebras.
Question: how about generalising to locally f.d. locally unital algebra?

(Fin1) $\mathscr{C}_0$ is a finite set
$\rightsquigarrow$ study only finitely many algebras
- if we use translation (1) $\rightsquigarrow$ algebra with finitely many PIMs
- in KLR-theory setting $\rightsquigarrow$ finitely many weights
Question: what if "$\mathscr{C}_0$ is countable"?

(Fin2) $\mathscr{C}(\mathtt{i},\mathtt{j})$ is finitary additive 1-category such that both horizontal and vertical compositions are additive and linear
$\rightsquigarrow$ $\mathscr{C}(\mathtt{i},\mathtt{j}) = U_{\mathtt{i},\mathtt{j}}$-proj so that
- $U_{\mathtt{i},\mathtt{j}}$ is a finite dimensional algebra
- $U_{\mathtt{i},\mathtt{j}}$ is an $U_{\mathtt{k},\mathtt{i}}$-$U_{\mathtt{j},\mathtt{k}'}$ bimodule for any $\mathtt{k},\mathtt{k}'$

I will say keep the notation $U_{\mathtt{i},\mathtt{j}}$ from now on. This is the algebra $\mathcal{C}_{\mathtt{i},\mathtt{j}}$ in [MM1] (...to be confirmed...).

(Fin3) $\mathbb{1}_{\mathtt{i}}$ is indecomposable $\forall \mathtt{i}$
This is needed because (?) for any f.d. algebra $U = U_{\mathtt{i},\mathtt{i}}$, U is an indecomposable U-U-bimodule

Now, fiat-ness. I think fiat-ness is modelled on a technical reason: the best understood functors between abelian/triangulated categories are the exact functors - even better biadjoint pair of exact functors.

Questions (may not make sense):
Most of the (multiplicative) categorification are on (modules of) algebras which have some duality. So....
- How close is fiat-ness to symmetric algebra ($D(A_A) \cong {}_AA$ as left A-modules)?
- How close is fiat-ness to algebra with "BGG" duality (having involutive contravariant functor which fixes simple modules)?

Weakly fiat-axiom:
(WF1) there is weak equivalence $* : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$
Note: $\mathscr{C}^{\mathrm{op}}$ inverted both 1-mor's and 2-mor's

(WF2) $\forall \mathtt{i},\mathtt{j}; \forall \mathtt{i} \xrightarrow{\mathrm{F}} \mathtt{j} \in \mathscr{C}_1$
$\exists\, \alpha : \mathrm{F} \circ \mathrm{F}^* \to \mathbb{1}_{\mathtt{i}},\ \ \beta : \mathbb{1}_{\mathtt{j}} \to \mathrm{F}^* \circ \mathrm{F}$ s.t.
$\alpha_{\mathrm{F}} \circ_1 \mathrm{F}(\beta) = \mathrm{id}_{\mathrm{F}}$ and $\mathrm{F}^*(\alpha) \circ_1 \beta_{\mathrm{F}^*} = \mathrm{id}_{\mathrm{F}^*}$

fiat = weakly fiat and $*$ involutive (fixes objects)

The extremely important feature (technical reason for studying fiat?):
this forces (F, F^*) to acts as biadjoint functors on 2-representations (1-category)

in particular, all 1-morphisms are represented by ***exact*** functors on (additve/abelian/triangulated) 1-category.

Question: weakly fiat $\rightsquigarrow$ adjoint; fiat $\rightsquigarrow$ biadjoint?

Important example: [MM2, 7.1]

4. Principal and cell 2-representation

4.1. **Principal.** Recall $\mathrm{Hom}_A(P_i, M) = e_i M$, so a PIM effectively extract isotopic composition factors out of M. Note: $e_i M = M(\mathtt{i})$ if you view M as a 1-functor.

Yoneda lemma is equivalent to $P_i = (A\text{-proj})(\mathtt{i}, -)$ (and equivalent to the above property). So just level up this.

Formal definition (2-functor interpretation): $\mathbb{P}_{\mathtt{i}} := \mathscr{C}(\mathtt{i}, -)$
Note: Abelianising this gives the abelian principal 2-rep in [MM1].

$\mathrm{Hom}_A(P_i, M) = e_i M = M(\mathtt{i}) \rightsquigarrow \mathrm{Hom}(\mathbb{P}_{\mathtt{i}}, \mathbf{M}) = \mathbf{M}(\mathtt{i})$ [MM2, Lem9]

Warning:
- principal 2-reps $\not\Rightarrow$ projective objects in the category of (additive) 2-reps
- no guarantee that $[\mathbb{P}] \in [\mathscr{C}]$-proj even for one-object case.

Recall f.d. algebra $U_{\mathtt{i},\mathtt{j}}$ in a previous section:

$$U_{\mathtt{i},\mathtt{j}} = \mathrm{End}_{\mathscr{C}_2}\left(\bigoplus_{\mathrm{F}\in\mathrm{ind}\text{-}\mathscr{C}(\mathtt{i},\mathtt{j})} \mathrm{F}\right)^{\mathrm{op}}$$

$\Rightarrow$ proj presentation of a $U_{\mathtt{i},\mathtt{j}}$-module $\leftrightarrow$ diagram $\mathrm{F} \xrightarrow{\alpha\in\mathscr{C}_2} \mathrm{G}$

(2-algebra)-action interpretation of $\mathbb{P}_{\mathtt{i}}$:
0: algebras: $U_{\mathtt{i},\mathtt{j}}$ [Note: elements of this algebra are 2-mor's of $\mathscr{C}$]
1: action $\mathrm{G} : \mathtt{j} \to \mathtt{k}; M \in U_{\mathtt{i},\mathtt{j}}$-mod with proj pres $F_M^{-1} \xrightarrow{\alpha} F_M^0$,

$$\mathrm{G} \cdot M = \mathrm{G} \cdot (F_M^{-1} \xrightarrow{\alpha} F_M^0) = \left(\mathrm{G} \circ F_M^{-1} \xrightarrow{G(\alpha)} \mathrm{G} \circ F_M^0\right) \in U_{\mathtt{i},\mathtt{k}}\text{-mod}$$

[i.e. $G \cdot (\alpha) = \mathrm{id}_{\mathrm{G}} \circ_0 \alpha = G(\alpha)$]
2: horizontal multiplication $\beta \cdot (\alpha) = \beta \circ_0 \alpha$

Example: $\mathscr{C}_D$ as before.
$(\mathscr{C}_D)_1 = \mathrm{add}(D, D \otimes_{\Bbbk} D) \subset D\text{-mod-}D = D \otimes_{\Bbbk} D^{\mathrm{op}}$-mod.
$\mathbb{P}_{\mathtt{i}} = \mathrm{End}_{D\otimes D^{\mathrm{op}}}(D \oplus D \otimes D)^{\mathrm{op}}$-proj $= U_{\mathtt{i},\mathtt{i}}$-proj
Calculating $U_{\mathtt{i},\mathtt{j}}$ for general 2-category can be difficult.
This algebra is given in [MM1, Ex3]. Denote $\mathrm{F} := D \otimes D$, also obviously $\mathbb{1}_{\mathtt{i}} := D$.
The quiver of $U := U_{\mathtt{i},\mathtt{i}}$ is:

$$\begin{tikzcd} \mathrm{F} \arrow[loop left, "\gamma"] \arrow[r, bend left, "\alpha"] & \mathbb{1}_{\mathtt{i}} \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and relations: $\gamma^2 = -(\beta\alpha)^2$, $(\alpha\beta)^2 = 0$, $\alpha\gamma = 0 = \gamma\beta$.
Explicitly, these arrows are 2-mor's:

$$\begin{array}{rcl}
\alpha : D \otimes D & \to & D \\
1 \otimes 1 & \mapsto & 1 \\ \hline
\beta : \quad D & \to & D \otimes D \\
1 & \mapsto & 1 \otimes x + x \otimes 1 \\ \hline
\gamma : D \otimes D & \to & D \otimes D \\
1 \otimes 1 & \mapsto & 1 \otimes x - x \otimes 1
\end{array}$$

Loewy structure of $U = P_{\mathbb{1}_{\mathtt{i}}} \oplus P_{\mathrm{F}}$:

$$\begin{tikzcd}[row sep=tiny, column sep=small]
 & & & \mathrm{F} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{1}_{\mathtt{i}} & & & & \mathbb{1}_{\mathtt{i}} \\
\mathrm{F} & \oplus & \mathrm{F} \arrow[dr, dash] & & \mathrm{F} \\
\mathbb{1}_{\mathtt{i}} & & & & \mathbb{1}_{\mathtt{i}} \arrow[dl, dash] \\
\mathrm{F} & & & \mathrm{F} &
\end{tikzcd}$$

$\mathbb{1}_{\mathtt{i}}$ obviously acts as identity (functor) on U-mod (and proj)
F-action:
$\mathrm{F}P_{\mathbb{1}_{\mathtt{i}}} = \mathrm{F}(0 \to \mathbb{1}_{\mathtt{i}}) = (0 \to \mathrm{F}) = P_{\mathrm{F}}$
$\mathrm{F}^2 \cong 2\mathrm{F} \quad \Rightarrow \quad \mathrm{F}P_{\mathrm{F}} = \mathrm{F}(0 \to \mathrm{F}) \cong (0 \to \mathrm{F} \oplus \mathrm{F}) = P_{\mathrm{F}} \oplus P_{\mathrm{F}}$
Seeing explicit F-action on simple modules are more difficult. Also 2-mor-action?

4.2. **Cell.** Princpal 2-reps are too big (just like projective modules, they are the most complicated ones for an algebra). We need to restrict to smaller 2-reps
$\rightsquigarrow$ find smaller but not-so-trivial algebra (i.e. bigger than $\Bbbk$)

$[\mathbb{P}] \approx$ left regular representation with basis $[P_{\mathrm{F}}]$
$\rightsquigarrow$ want to group together nice subset of P_{F}'s to form a (sub/quot.) 2-rep
$\rightsquigarrow$ want to group together nice subset of 1-mor's
$\rightsquigarrow$ Cell = generalisation of Green's relation for multi-semigroup.

Note:
- Cell $\approx$ Kazhdan-Lusztig cell (example of Green's relation in semigroup theory)
- Unless otherwise specified, cell means left cell.
- multi-set = set-like thing which account for multiplicities

Why cell 2-rep should/could be candidate for analogue of simple modules:
- KL-cells in type A corresponds to simples
- current state of affair: under some (really strong?) conditions, this is a good analogue ([MM1,MM3,MM5])

Recall 2-rep's $\approx$ algebras
$\Rightarrow$ cell 2-rep's corresp. to $\mathcal{L}$ $\approx$ "subquotient" algebras of $U_{\mathtt{i},\mathtt{j}}$
Note: all 1-mor in a chosen cell has a common domain which is $\mathtt{i}$ here.

By subquotient we mean the algebra is of the form

$$(\mathrm{id}_{\mathrm{F}} U_{\mathtt{i},\mathtt{j}} \mathrm{id}_{\mathrm{F}})/I$$

where F = sum of all (indec) 1-mor $\mathrm{G} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ (?!?) with $\mathrm{G} \geq_L \mathcal{L}$,
(i.e. G summand of a 1-mor from left-composing 1-mor in $\mathcal{L}$)

and I = maximal $\mathscr{C}_1$-stable two-sided ideal of $\mathrm{id}_{\mathrm{F}} U_{\mathtt{i},\mathtt{j}} \mathrm{id}_{\mathrm{F}}$ s.t. $\mathrm{id}_{\mathrm{G}} \notin I,\ \ \forall \mathrm{G} \in \mathcal{L}$.
Note also id_{F} is an idem (2-mor) element so that $P_{\mathrm{F}} = U_{\mathtt{i},\mathtt{j}} \mathrm{id}_{\mathrm{F}}$

Question:
(1) Can choose F to be sum of all 1-mor in $\mathcal{L}$ with range $\mathtt{j}$?
(2) is this algebra isom to $\mathrm{id}_{\mathrm{F}}(U_{\mathtt{i},\mathtt{j}}/I)\mathrm{id}_{\mathrm{F}}$ with some tweak on F and I?

Note: left/right/two-sided (1- or 2-) ideals (of a category) used in [MM4,5,6] is defined to be compatible with this construction on algebras. i.e. left 2-ideal gives you left ideal $U_{\mathtt{i},\mathtt{j}}$ etc. But it has more restriction than simply an ideal. - It needs to respect action. This also means that we cannot quotient out $U_{\mathtt{i},\mathtt{j}}$ by arbitrary 2-sided ideal to obtain a 2-rep. (c.f. [MM5, 2.6])

In particular, I in general is more than just $\langle \mathrm{id}_{\sum \mathrm{H}} \rangle$, where the sum is over $(\text{ind-}\mathscr{C}(\mathtt{i},\mathtt{j})) \setminus \mathcal{L}$.
This smaller thing consist of all 2-mor's which factors through a 1-mor lying outside the cell.
(For $\mathscr{C}_D$, this is enough. I don't know if this is fluke or there is a good reason.)

e.g.:
$\mathscr{C} = \mathscr{C}_A$ for A a commutative local non-simple algebra. i.e.
$\mathscr{C}_0 = \{\mathtt{i} \cong A\text{-mod}\}$
$\mathscr{C}_1 = \mathrm{add}(\mathbb{1}_{\mathtt{i}}) = \mathrm{add}_{A\otimes A^{\mathrm{op}}}(A)$
$\mathscr{C}_2$ = all NT = all $A \otimes A^{\mathrm{op}}$-modules morphisms
$\Rightarrow\ \ \mathbb{P}_{\mathtt{i}} = \mathrm{End}_{A\otimes A^{\mathrm{op}}}(A)\text{-proj} = Z(A)\text{-proj} = A\text{-proj}$
since there is only one projective, the smaller ideal is 0, hence $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}} = A\text{-proj}$?
FALSE! Formal definition says I is maximal ideal not containing $\mathrm{id}_{\mathbb{1}_{\mathtt{i}}} = 1_A$
$\Rightarrow\ \ I$ =Jacobson radical
$\Rightarrow\ \ \mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}} = \Bbbk\text{-proj}$

More generally,
Suppose $\mathscr{C}_0 = \{\mathtt{i} = A\text{-mod}\}$ with $\mathcal{L} = \{\mathbb{1}_{\mathtt{i}}\}$ being a left cell.
(Q: always true for such $\mathscr{C}$?)
$\mathbb{P}_{\mathtt{i}} = U$-proj
"sub"-algebra: $\mathrm{id}_{\mathrm{F}} U \mathrm{id}_{\mathrm{F}}$ with F being sum of 1-mor in $\mathcal{L}$
$\Rightarrow\ \ U' = \mathrm{id}_{\mathrm{F}} U \mathrm{id}_{\mathrm{F}} = \mathrm{id}_{\mathbb{1}_{\mathtt{i}}} U \mathrm{id}_{\mathbb{1}_{\mathtt{i}}}$ is a local algebra
$\Rightarrow\ \ U'$ has simple top
$\Rightarrow$ maximal ideal I of U' not containing $\mathrm{id}_{\mathbb{1}_{\mathtt{i}}}$ is $\mathrm{rad} U'$
$\Rightarrow\ \ U'/I = U'/\mathrm{rad}U' \cong \Bbbk$
$\Rightarrow\ \ \mathbf{C}_{\mathcal{L}} = \Bbbk\text{-proj}$

Q: OK? This phenomenon does not depend on how large $\mathscr{C}_2$ is!

4.3. **Checking action-stability of an ideal.** *Concentrate in one object case.*

Formal definition: $\mathbf{I}$ is an ideal of $\mathbf{M} = X$-proj means
We have ideal $\mathscr{C}_1$-stable ideal I of X, i.e. $\mathbf{M}(\mathrm{F})\mathrm{add}(I) \subset \mathrm{add}(I)\ \forall \mathrm{F} \in \mathscr{C}_1$

Do this by example: $\mathscr{C} = \mathscr{C}_D$.
Recall $\mathrm{F} = D \otimes D$ and $\mathbb{1}_{\mathtt{i}} = D$ (as D-D-bimodule)
$\mathbb{P}_{\mathtt{i}} = U$-proj
Let $X = \mathrm{id}_{\mathrm{F}} U \mathrm{id}_{\mathrm{F}}$ (this gives a sub 2-rep $\mathbf{N}$ of $\mathbb{P}_{\mathtt{i}}$)

Goal: Find the (2-rep) ideal of $\mathbf{N} = X$-proj defining cell 2-rep.
i.e. find the maximal "F-stable ideal" of X

The algebra X is given by basis:

$$\begin{aligned} X = \mathrm{id}_{\mathrm{F}} U \mathrm{id}_{\mathrm{F}} = \mathrm{End}_U(P_{\mathrm{F}})^{\mathrm{op}} &\cong \mathrm{End}_{D\otimes D^{\mathrm{op}}}(D\otimes D)^{\mathrm{op}} = \mathrm{End}_{\mathscr{C}_2}(\mathrm{F})^{\mathrm{op}} \\ \text{basis: } \{\mathrm{id}_{\mathrm{F}}, \gamma, \gamma^2, \beta\alpha\} &\cong \{\mathrm{id}_{\mathrm{F}}, \phi_l, \phi_r, \phi_l\phi_r\} \end{aligned}$$

where $\phi_l =$ left multiplying x, and $\phi_r =$ right multiplying x
(so relation in the RHS basis is $\phi_l\phi_r = \phi_r\phi_l$ and $\phi_r^2 = \phi_l^2 = 0$)

$F \circ F = D \otimes D \otimes_D D \otimes D \cong (D \otimes D) \oplus (D \otimes D) = F \oplus F$, explicitly:

$$\begin{tikzcd}
& 1\otimes1\otimes1\otimes1 \arrow[dl, dash] \arrow[dr, dash] & & & & & \\
x\otimes1\otimes1\otimes1 \arrow[dr, dash] & & 1\otimes1\otimes1\otimes x \arrow[dl, dash] & \oplus & & 1\otimes x\otimes1\otimes1 \arrow[dl, dash] \arrow[dr, dash] & \\
& x\otimes1\otimes1\otimes x & & & x\otimes x\otimes1\otimes1 \arrow[dr, dash] & & 1\otimes x\otimes1\otimes x \arrow[dl, dash] \\
& & & & & x\otimes x\otimes1\otimes x &
\end{tikzcd}$$

Let I be a two-sided ideal of X, and $\theta \in I \subset \mathrm{End}(\mathrm{F})$.
$\mathbf{N}$ sub 2-rep of $\mathbb{P}_{\mathtt{i}}$, so $\theta \in (\mathscr{C}_D)_2$, and action is horizontal multiplication
$\Rightarrow$ $\mathbf{N}(\mathrm{F})\theta = \mathrm{id}_{\mathrm{F}} \circ_0 \theta \in \mathrm{End}(\mathrm{F}\circ\mathrm{F}) = \mathrm{End}(\mathrm{F}\oplus\mathrm{F}) = \mathrm{Mat}_2(X)$
$\Rightarrow$ stability means that we need $\mathrm{id}_{\mathrm{F}} \circ_0 \theta \in \mathrm{Mat}_2(I)$
Also note that using bimodule interpretation $\mathrm{id}_{\mathrm{F}} \circ_0 \theta = \mathrm{id}_{\mathrm{F}} \otimes \theta$.

Suppose $\phi_l \in I$, then $\mathrm{id}_{\mathrm{F}} \circ_0 \phi_l$ can be graphically presented as

$$\begin{tikzcd}
& 1\otimes1\otimes1\otimes1 \arrow[dl, dotted, dash] \arrow[dr, dotted, dash] \arrow[drrrr, "\phi_l"] & & & & & \\
x\otimes1\otimes1\otimes1 \arrow[dr, dotted, dash] & & 1\otimes1\otimes1\otimes x \arrow[dl, dotted, dash] & \oplus & & 1\otimes x\otimes1\otimes1 \arrow[loop above, "\phi_l"] \arrow[dl, dotted, dash] \arrow[dr, dotted, dash] & \\
& x\otimes1\otimes1\otimes x & & & x\otimes x\otimes1\otimes1 \arrow[dr, dotted, dash] & & 1\otimes x\otimes1\otimes x \arrow[dl, dotted, dash] \\
& & & & & x\otimes x\otimes1\otimes x &
\end{tikzcd}$$

$\therefore$ $\mathrm{id}_{\mathrm{F}} \circ_0 \phi_l = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{\mathrm{F}} & \mathrm{id}_{\mathrm{F}} \end{pmatrix}$
If I is proper, then $\mathrm{id}_{\mathrm{F}} \notin I$
$\Rightarrow$ ideal containing ϕ_l is *not* F-stable.

Suppose $\phi_r \in I$, then $\mathrm{id}_{\mathrm{F}} \circ_0 \phi_r$ can be graphically presented as

$$\begin{tikzcd}
& 1\otimes1\otimes1\otimes1 \arrow[dl, dotted, dash] \arrow[dr, "\phi_r"] & & & & & \\
x\otimes1\otimes1\otimes1 \arrow[dr, "\phi_r"] & & 1\otimes1\otimes1\otimes x \arrow[dl, dotted, dash] & \oplus & & 1\otimes x\otimes1\otimes1 \arrow[dl, dotted, dash] \arrow[dr, "\phi_r"] & \\
& x\otimes1\otimes1\otimes x & & & x\otimes x\otimes1\otimes1 \arrow[dr, "\phi_r"] & & 1\otimes x\otimes1\otimes x \arrow[dl, dotted, dash] \\
& & & & & x\otimes x\otimes1\otimes x &
\end{tikzcd}$$

$\therefore$ $\mathrm{id}_{\mathrm{F}} \circ_0 \phi_r = \begin{pmatrix} \phi_r & 0 \\ 0 & \phi_r \end{pmatrix} \in \mathrm{Mat}_2(I)$
Similarly, $\phi_l\phi_r$ is also F-stable
In particular $\langle \phi_r \rangle$ defines an ideal of $\mathbf{N}$.
Moreover, combining with previous calculation for ϕ_l, $\langle \phi_r \rangle$ is the unique maximal F-stable ideal of X.
$\therefore$ $\mathbf{C}_{\{\mathrm{F}\}} = (X/\langle\phi_r\rangle)$-proj $= D$-proj