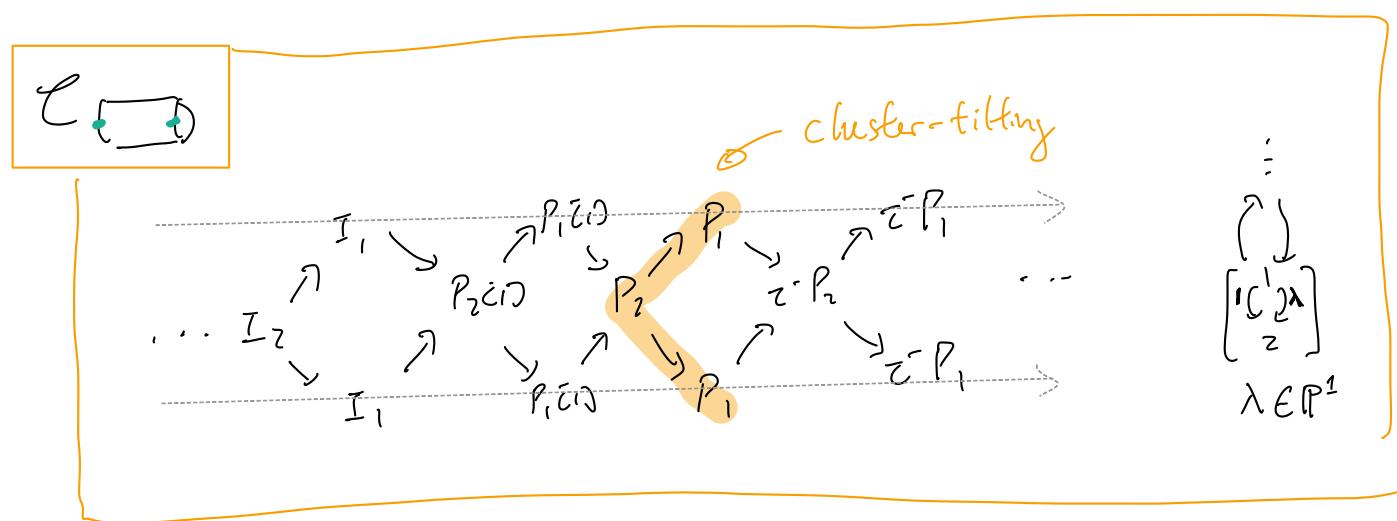
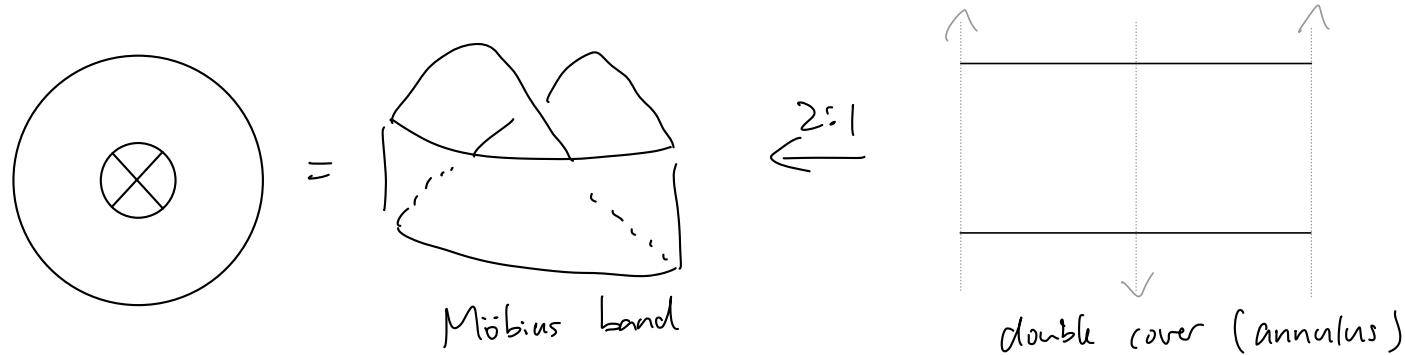


Categorification of unpunctured non-orientable marked surfaces

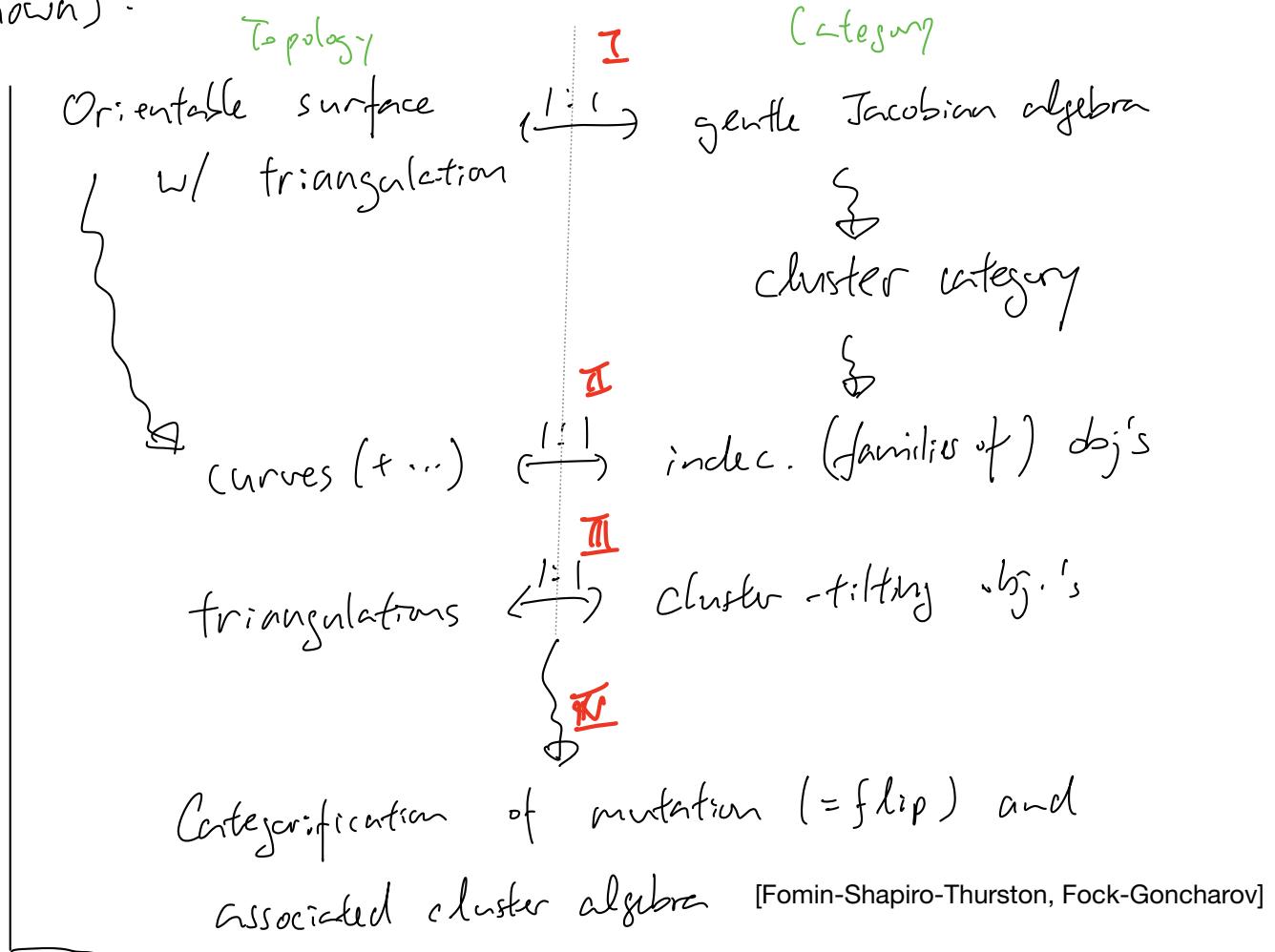
work in progress, jt. with V. Bazier-Matte, K. Wright



* Green texts = extra info

Summary

(Known) :



Question : Non-orientable surfaces (NoS) ?

Idea : \exists orientable double cover $\tilde{\mathbb{S}}$ for an NoS \mathbb{S} .

i.e., $\mathbb{S} = \tilde{\mathbb{S}} / \sigma$ σ : orientation-reversing auto.
 $\sigma^2 = 1$

Today : Goal I, II, "III"

§§1] Surface topology

Setup

$\partial S \neq \emptyset$

S : Surface = compact 2-dim/ \mathbb{R} w/ non-empty boundary

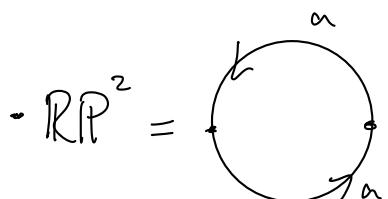
M : finite set of marked points in ∂S (\vdash : unpunctured)

s.t. each boundary component contains ≥ 1 marked pt.

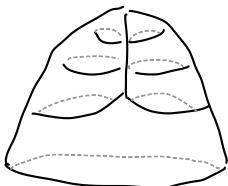
and $(S, M) \neq 1, 2, 3$ -gon



§1.1] Working with non-orientable surfaces (NoS)



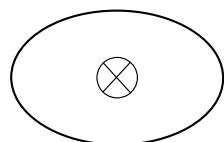
YouTube
“The cross-cap”

• crosscap := $\mathbb{RP}^2 \setminus \text{disc}$  $\xrightarrow{\text{homeo}} \cong$ Möbius strip,

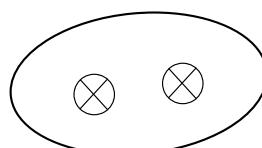
In practice, represent by the symbol  (or , etc.)

Cont: Some literature call \mathbb{RP}^2 the crosscap instead.

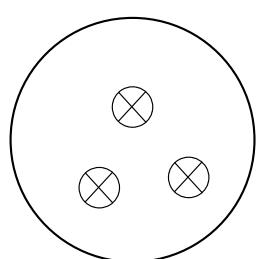
E.S.



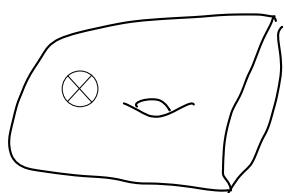
= Möbius strip ,



= Klein bottle \ disc .



=



= Dyck's surface \ disc .

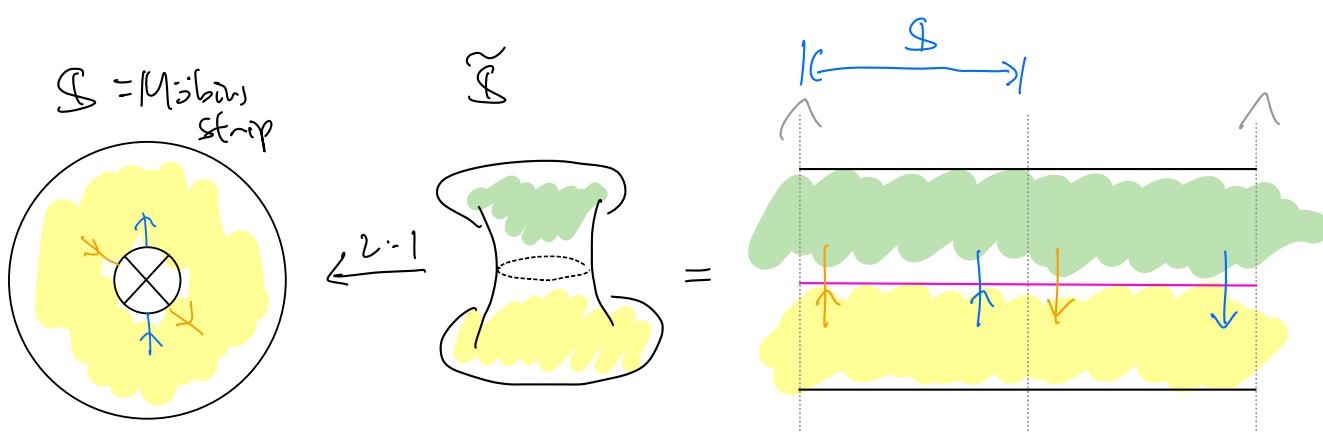
§1.2] Double cover

$$(N) \circ S \quad (S, M) \Leftrightarrow (\tilde{S}, \tilde{M}) / \sigma$$

where

$(\tilde{S}, \tilde{M}) :=$ Orientable double cover of $\overset{\text{marked}}{a} N$ os. (S, M)

σ : orientation-reversing auto. of order 2 .



* no intersection

between the 2 curves shown !

§ 1.2] Objects of interest

γ : Curve on (S, M) : \Leftrightarrow either

$\left\{ \begin{array}{l} \text{closed} \\ \text{i.f.} \end{array} \right. \quad \begin{array}{l} \gamma \cong S^1 \\ \gamma \cap M = \emptyset \\ \text{non-contractible} \end{array}$	or	$\left\{ \begin{array}{l} \text{non-closed} \\ \gamma: [0, 1] \rightarrow S \\ \gamma(0), \gamma(1) \in M \\ \gamma(0, 1) \subset S \setminus \partial S \end{array} \right.$
---	-------------	---



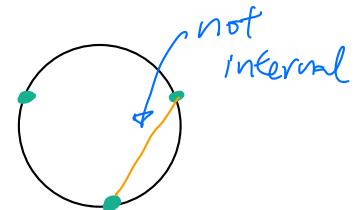
* Always considered up to isotopies that fix ∂S pointwise

Non-crossing (set of) curves : \Leftrightarrow no intersection except possibly at endpoints.

NC

* Arc : \Leftrightarrow NC non-closed curves

* Internal arc : \Leftrightarrow arc $\not\subset$ boundary interval

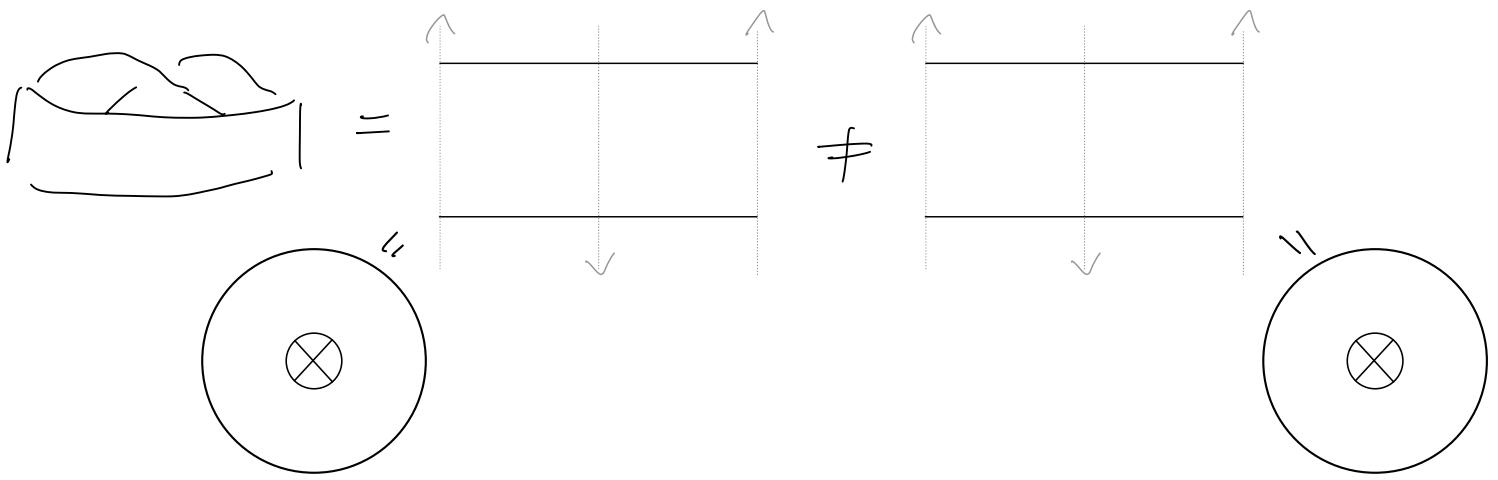


- 1-sided closed curve : \Leftrightarrow non-orientable closed curve

(If simple, then equiv. to \exists regular nbhd $\xrightarrow{\text{homeo}}$ Möbius strip)

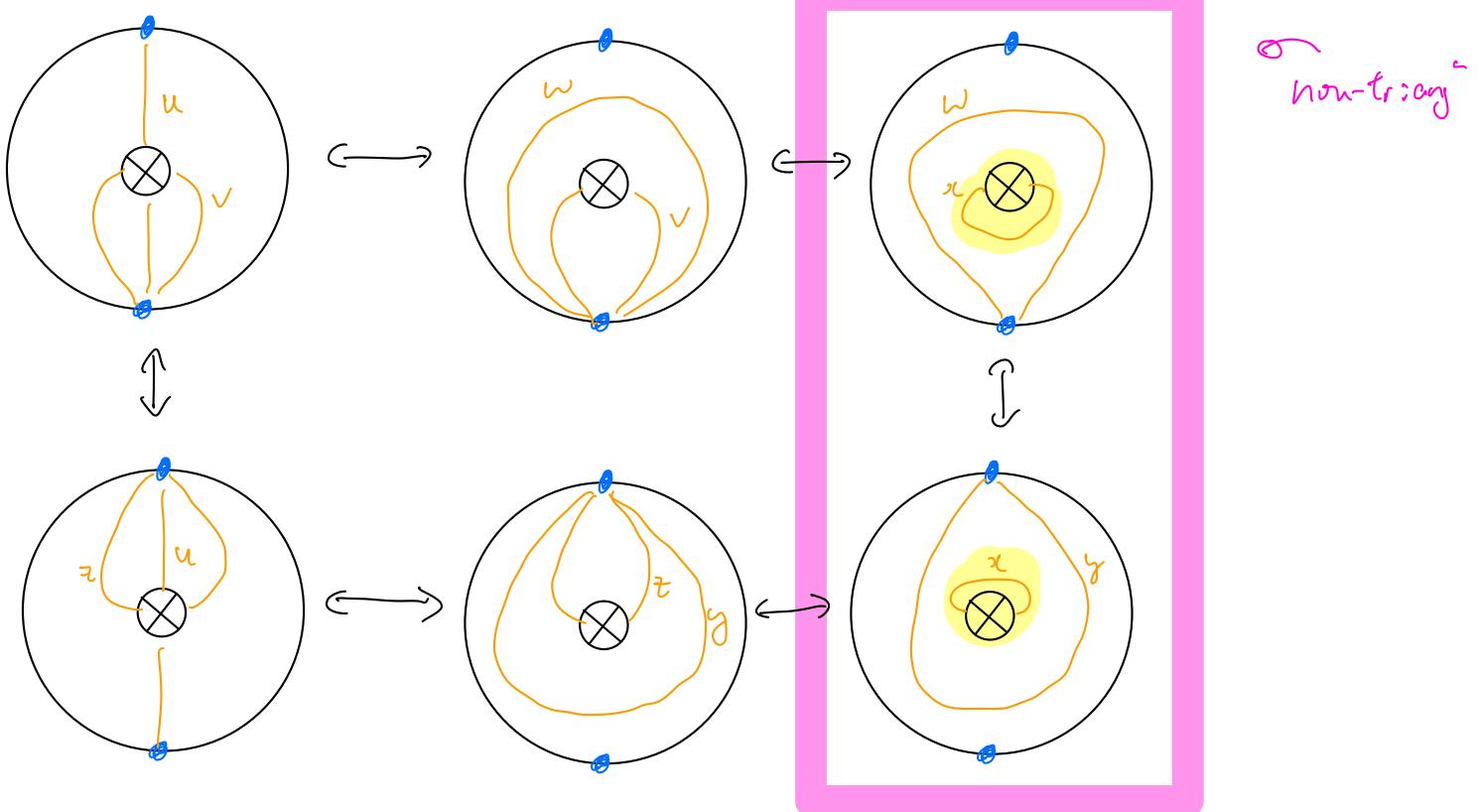
* 2-sided closed curve : \Leftrightarrow not 1-sided

(If simple, then equiv. to \exists regular nbhd $\xrightarrow{\text{homeo}}$ annulus)



- Quasi-arc \Leftrightarrow either internal arc,
or 1-sided simple closed curve
 \nwarrow no self-intersection
- (quasi-)triangulation \Leftrightarrow maximal NC set of (quasi-)arcs

\square M_2 ($=$ Möbius strip w/ 2 marked pt's) has 6 quasi-triang["]'s.



SS2] Triangulation vs QP

2.1) Orientable case (\tilde{S}, \tilde{M}) , \tilde{T}

$\rightsquigarrow (Q, W)$: QP (Quiver with Potential)

where $\begin{cases} Q_0 = \{\text{internal arcs of } \tilde{T}\} \\ Q_1 = \text{cw oriented angles of triangles of } \tilde{T} \\ W = \sum_{\text{internal triangles}} \end{cases}$

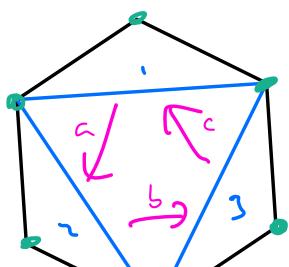
\rightsquigarrow Jacobian algebra $\mathcal{J}^{\tilde{T}} \cong \frac{kQ}{\text{Jac}(Q, W)}$:= $kQ / (\partial_a W) = kQ / (\text{length 2 paths in int. } \Delta's)$

N.B. . $\mathcal{J}^{\tilde{T}}$ is f.d. and a gentle algebra (cf. next talk)

Convention/Definition QP is gentle. if induced by a triangⁿ.

E.g.

(\tilde{S}, \tilde{M}) , \tilde{T} :



$$Q = \begin{matrix} & 1 & \\ & \swarrow & \searrow \\ 2 & & 3 \end{matrix}, \quad W = abc$$

$$\mathcal{J}^{\tilde{T}} = \begin{matrix} & 1 & \\ & \swarrow & \searrow \\ 2 & & 3 \end{matrix} / (ab, bc, ca)$$

Thm

[Assem, Brüstle, Charbonneau-Jodoin, Plamondon]

$$\left\{ (\tilde{S}, \tilde{M}; \tilde{T}) : \text{triangulated or. surfaces} \right\} \xleftrightarrow{1:1} \left\{ (Q, W) : \text{gentle QP} \right\}$$

2.2) NoS case $(S, M) \xrightarrow{\text{2:1}} (\widetilde{S}, \widetilde{M}) \supset \sigma$

Def ① Q : inner

An involution $\sigma: Q \rightarrow Q$

: $\Leftrightarrow \sigma = (\sigma_0: Q_0 \rightarrow Q_0, \sigma_1: Q_1 \rightarrow Q_1)$

s.t. $\begin{cases} \sigma^2 = 1 \\ \sigma(v \xrightarrow{\alpha} w) = (\sigma(v) \xleftarrow{\sigma(\alpha)} \sigma(w)) \end{cases}$

N.B. Specifying involution $\sigma \Leftrightarrow$ Specifying $1kQ \xrightarrow{\text{alg}} 1kQ^\sigma$

② An involution on a QP (Q, W) .

: \Leftrightarrow an involution σ on Q s.t. $\sigma W = W$

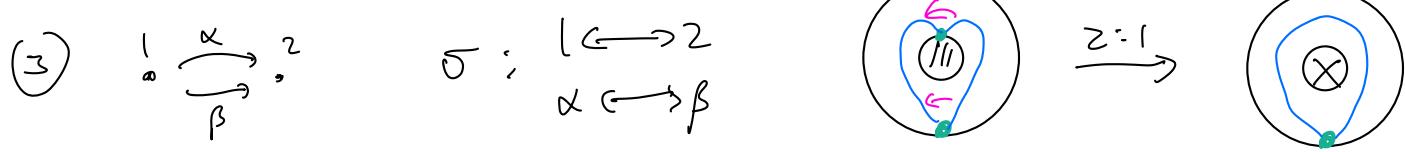
\Leftrightarrow $\sigma(\partial_\alpha W) = (\partial_\alpha W)$

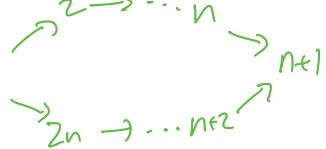
$\sigma W = \pm W \text{ OK}$

E.g.

① $! \xrightarrow{\alpha} ?$ $\sigma: \begin{array}{l} 1 \xrightarrow{\quad} 2 \\ \alpha \xrightarrow{\quad} \alpha \end{array}$

② $! \xrightarrow[\beta]{\alpha} ?$ $\sigma: \begin{array}{l} 1 \xrightarrow{\quad} 2 \\ \alpha \xrightarrow{\quad} \alpha \\ \beta \xrightarrow{\quad} \beta \end{array}$



(4) Exercise: An n,n -quiver  has an involution

Obs: The setup $\sigma \in (\mathbb{S}, \widetilde{\mathcal{M}}) \xrightarrow{z=1} (\mathbb{S}, \mathcal{M})$

induces an involution σ on (Q, W)

Moreover,  is fixed-point free (FPF)

$$\sigma(v) \neq v \quad \forall v \in Q_0$$

$$\sigma(\alpha) \neq \alpha \quad \forall \alpha \in Q,$$

Prop [BM-C-W] (Goal I)

$$\left\{ \begin{array}{l} \text{connected Nos} \\ \text{with triangulation} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} (Q, W; \sigma) \text{ s.t. } (Q, W) \text{ gentle,} \\ Q \text{ connected, } \sigma \text{ is a FPF involution} \end{array} \right\}$$

$$(\mathbb{S}, \mathcal{M}; T) \mapsto (Q, W; \sigma)$$

Rank Same for locally gentle quivers version

& "punctured surface" (orbifold surface)

§§3

Triangulation vs CTO (cluster-fitting obj)

Setup: From now on, $\mathbb{H} = \text{complex numbers}$

Orientable case

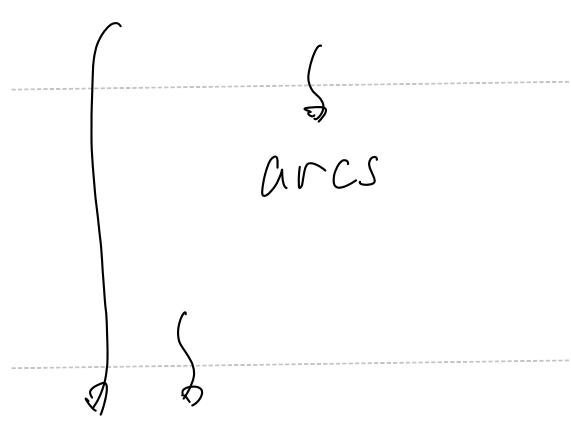
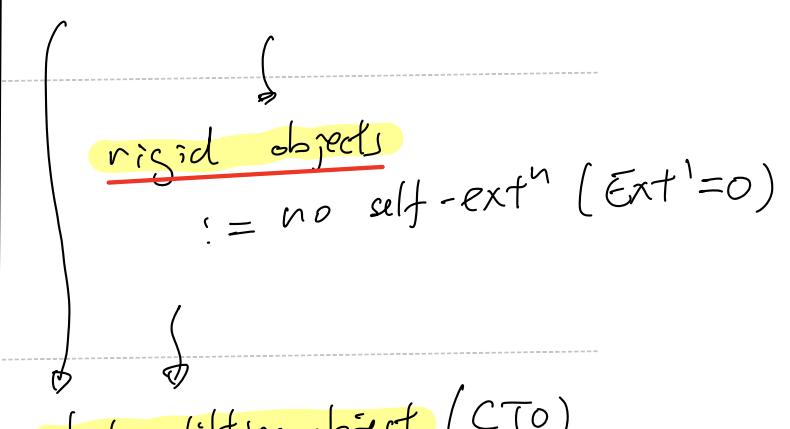
See [Brustle-Zhang]

TOPOLOGY	CATEGORY
(\tilde{S}, \tilde{M}) non-closed curves	cluster category $\mathcal{C} = \mathcal{C}_{(\tilde{S}, \tilde{M})}$ \cup $[,] \cong \tau$ <u>N.B.</u> This is ks Hon-fin. 2-CY tri. cat.
$(\text{closed curves}) \times \mathbb{H}^X$ Elements are	"string objects" "brane objects" all index's of $\mathcal{C}_{(S, M)}$

(ω^n, λ) , where ω : primitive,

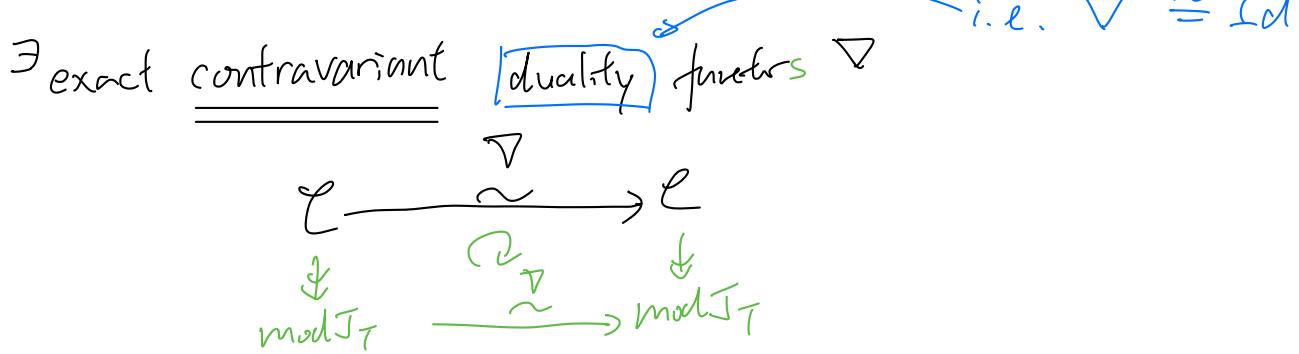
i.e. $\omega \neq p^k$ in $\pi_1(\tilde{S})$
with $k > 1$

Moreover,

TOPOLOGY	CATEGORY
<p>Crossing between curves (+ loc. sys.)</p>  <p>triangulation</p>	<p>Canonical basis (+ local sys. hom.)</p> <p>of $\tilde{\text{Ext}}^1_{\mathcal{C}}(-, -) := \text{Hom}_{\mathcal{C}}(-, -[1])$</p>  <p><u>rigid objects</u> := no self-extⁿ ($\text{Ext}^1 = 0$)</p> <p><u>cluster-tilting object</u> (CTO) := maximal rigid object. (i.e. $M \otimes N$ rigid $\Rightarrow N \in \text{add } M$)</p>

Slogan: Crossings = Non-split extensions

Prop [IM-C-W]



such that

- ① $\forall \gamma: \text{non-closed on } (\widetilde{\mathbb{S}, M}), \underbrace{\sigma(\gamma)}_{\text{Topology}} \longleftrightarrow \nabla(\gamma)$
- ② $\forall (\omega^n, \lambda) \in \left\{ \begin{smallmatrix} \text{c.c.'s} \\ \text{on } (\widetilde{\mathbb{S}, M}) \end{smallmatrix} \right\} \times \mathbb{H}^\times, (\sigma(\omega^n), \lambda^{-1}) \longleftrightarrow \nabla(\omega^n, \lambda)$

E.g.

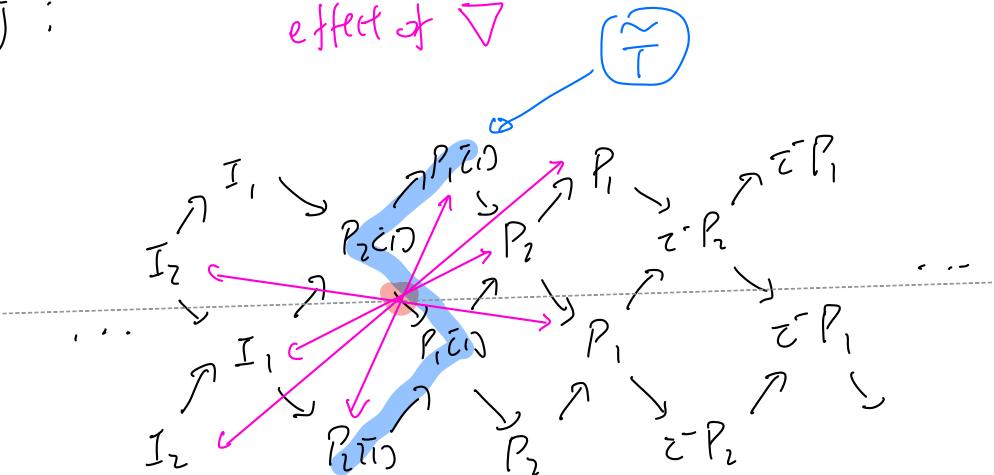
$$(\mathbb{S}, M; T) = \text{circle with a blue loop} \xleftarrow{\cong} (\widetilde{\mathbb{S}, M}, \widetilde{T}) = \text{circle with a blue loop and a green loop}$$

$$Q = \begin{pmatrix} 1 & a \\ b & 2 \end{pmatrix}, \quad \omega = 0$$

$$\sigma: \begin{matrix} 1 & \leftrightarrow & 2 \\ a & \rightarrow & b \end{matrix}$$

$\mathcal{L}(\widetilde{\mathbb{S}, M})$:

effect of ∇



$$\begin{aligned} &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \\ &\lambda \in \mathbb{P}^1 \\ &\nabla \circ \lambda \mapsto \lambda^{-1} \end{aligned}$$

Thm [BM-C-W] (Partial Goal II, III)

HNS (\mathbb{S}, M), have

$$\left\{ \text{non-closed curves on } (\mathbb{S}, M) \right\} \xleftrightarrow{1:1} \left\{ \nabla Y \oplus Y \mid Y: \text{strong obj} \right\}$$

which induces

$$\left\{ \begin{array}{l} \text{triangulations} \\ \text{of } (\mathbb{S}, M) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{basic CTO } M \in \mathcal{C} \\ \text{s.t. } \nabla M \cong M \end{array} \right\}$$

self-dual

Rmk • This is mutation compatible (as long as it is possible.)

- Mutation send corresp. s.r-tilting pair
to incomparable s.r-tilting pair.

Ex. $(\mathbb{S}, M) = M_2$

$$\Rightarrow \text{LHS} = \left\{ \begin{array}{c} \text{circle} \\ \text{with a blue arc} \\ \text{and a green dot} \end{array} \right\}, \text{RHS} = \left\{ \text{initial CTO} \right\}$$

$$\xleftrightarrow{1:1} \left\{ \Lambda(\mathbb{C}) \right\} \subset \text{sc-tilt}(\Lambda)$$

What about quasi-triangulations?
and their mutations?

§SF] Symmetric representations (= Σ -representations)

\checkmark
orthogonal
($\Sigma = +1$)

[Derksen-Weyman]

\checkmark
symplectic
($\Sigma = -1$)

$$\Sigma = \{\pm 1\}$$

[Boos-Cerulli Irelli]

Throughout this section,

- $(A = k[G], \sigma)$: σ involution on G fixing I .
- modules = right f.d. modules

Def $\exists \Sigma$ -form : \Leftrightarrow $\begin{cases} \text{symmetric bilinear form if } \Sigma = +1, \\ \text{skew-symm. bilinear form if } \Sigma = -1. \end{cases}$

2) An Σ -representation over (A, σ)

- $\therefore \Leftrightarrow$
- M : ordinary rep.
 - $\langle -, - \rangle: M \times M \rightarrow k$ non-degen. Σ -form

s.t. $\left\{ \begin{array}{l} \cdot \langle m e_i, n e_j \rangle = 0 \text{ if } j \neq \sigma(i); e_i, e_j = \text{primitive idem's.} \\ \cdot \langle m \alpha, n \rangle + \langle m, n \sigma(\alpha) \rangle = 0 \\ \quad (\text{i.e., } \sigma(\alpha) \text{ is adjoint of } \alpha) \end{array} \right.$

* $M: \Sigma\text{-rep}^n \Rightarrow \exists \psi_M: M \xrightarrow{\sim} DM$ as A -module s.t. $\nabla(\psi_M) = \Sigma \psi_M$

* Indecomposability makes sense for ε -rep's.

"Slogan"

ε -rep \approx anti-version of
repⁿ over skew-group ring $A^* \rtimes \mathbb{Z}_2$

Prop [D-W, B-CI] (Characterisation of indec ε -rep's)

M : indec. ε -rep / (A, σ)

$\Rightarrow \exists \bar{M}$: indec. A -module

s.t. exactly one of the following holds.

a) $\nabla \bar{M} \not\cong \bar{M}$, $M = \bar{M} \oplus \nabla \bar{M}$ \rightarrow call M split

b) $\nabla \bar{M} \cong \bar{M}$, $M = \bar{M} \oplus \nabla \bar{M}$ \rightarrow call M ramified

c) $\nabla \bar{M} \cong \bar{M}$, $M = \bar{M}$ \rightarrow call M ~~Type I~~ 1-sided

Notation: ω : primitive cc, $\ell(\omega) := \#\omega \cap \overline{\Gamma}$.

Theorem [IM-C-W]

M : indec ε -rep n / $\text{Jac}(\mathbb{Q}, \omega)$, (\mathbb{Q}, ω) : gentle FPF-Symm. CP

$\Rightarrow M \cong$ exactly one of the following.

- split $\left[\begin{array}{l} \cdot M(\gamma) \oplus M(\sigma(\gamma)) \quad \text{+ all strong} \\ \cdot M_\lambda(\omega^n) \oplus M_{\lambda^{-1}}(\sigma(\omega^n)) \text{ s.t. } \begin{cases} \omega \neq \sigma(\omega) \text{ or } \lambda \notin \{\pm 1\}, \\ \text{any } n \geq 1. \end{cases} \end{array} \right]$
- 1-sided $\left[\begin{array}{l} \cdot M_\lambda(\omega) \text{ s.t. } \sigma(\omega) = \omega \quad \varepsilon = (-1)^{\frac{\ell(\omega)}{2}} \lambda \\ \cdot M_\lambda(\omega^2) \text{ s.t. } \sigma(\omega) = \omega \quad \varepsilon = (-1)^{\frac{\ell(\omega)}{2} + 1} \lambda \end{array} \right]$
- ramified $\left[\begin{array}{l} \cdot M_\lambda(\omega^n)^{\oplus 2} \text{ s.t. all remaining case} \end{array} \right]$

Hence

$$\left\{ \begin{array}{l} \text{"Indec" top. obj.} \\ \{ \text{1-sided c.c. } \omega \text{ on } (\mathbb{S}, M) \} \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} \text{Index } \varepsilon\text{-rep}'s \\ \{ \text{1-sided indec } M_{\varepsilon\omega}(\omega) \} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{2-sided c.c. } \omega \text{ on } (\mathbb{S}, M) \end{array} \right\} \xrightarrow{1:1} \left\{ \text{ramified indec } M_{-\varepsilon\omega}(\omega)^{\oplus 2} \right\}$$

$$\left\{ \begin{array}{l} \text{non-closed curve } Y \\ \text{on } (\mathbb{S}, M) \end{array} \right\} \setminus \overline{\Gamma} \xrightarrow{1:1} \left\{ \begin{array}{l} \text{split indec} \\ \text{"of strong type"} \quad M(\gamma) \oplus M(\gamma) \end{array} \right\}$$

\rightarrow Goal II ✓ 1935

§§5] ε -extension, ε -rigid

Go back to $\mathcal{C} = \mathcal{C}_{(\mathbb{Q}, \mathbb{M})}$.

Fact : $\begin{cases} \tilde{T} : \text{triang. on } (\mathbb{S}, \mathbb{M}), \quad \Lambda := \text{Jac}(\mathbb{Q}, \mathbb{M}) \\ \Rightarrow C := \tilde{T}[\mathbb{I}] \text{ is a CTD} \\ + \pi := \text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C}/[\tilde{T}] \xrightarrow{\sim} \text{mod } \Lambda \end{cases}$

Fix $\varepsilon \in \{\pm 1\}$, T : triang.

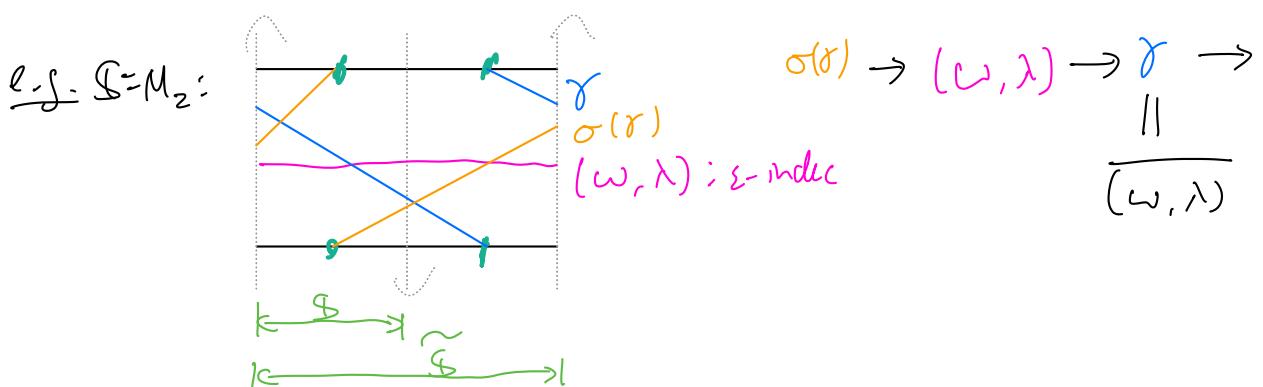
Def :

- $X \in \mathcal{C}$ is an indec. ε -object: \Leftrightarrow
 - πX an indec. ε -repⁿ/1
 - or $\nabla^{(\alpha)}$
- $X = \alpha \oplus \sigma(\alpha)$, some $\alpha \in \tilde{T}$

- In this case,

(1) $\exists (\nabla \bar{X} \rightarrow X \rightarrow \bar{X} \rightarrow) : \Delta \text{ in } \mathcal{C} \text{ with } \bar{X} : \text{indec}$

we call any such \bar{X} ε -factor N.B. not unique!



(2) Isom. $X \xleftarrow{\cong} \nabla X$ s.t. $\nabla(\psi_X) = \varepsilon \psi_X$

$X \in \mathcal{C}$ is an ε -object : $\Leftrightarrow X = \bigoplus$ indec ε -obj's.

X, Y : ε -objects

$$f: X \rightarrow \widehat{Y}[\square] \text{ s.t. } \begin{cases} \bar{Y}: \varepsilon\text{-factor of } Y \\ f \circ \psi_X \circ \nabla f = 0 \end{cases} \quad (\nabla \bar{Y})[E]\square \xrightarrow{\nabla f} X \xrightarrow{f} \widehat{Y}[\square]$$

In this case, \exists comm. diag:

$$\begin{array}{ccccc} & & \bar{Y} & \xrightarrow{\nabla \bar{Y}[E]} & \\ & & \downarrow \psi_X \circ \nabla f & \uparrow & \\ \bar{Y} & \xrightarrow{\square} & C^f & \xrightarrow{\quad} & X \xrightarrow{\quad} \widehat{Y}[\square] \text{ s.t. } \\ \parallel & & \downarrow & & \text{all rows} \\ \bar{Y} & \xrightarrow{\square} & E & \xrightarrow{\quad} & \text{all columns} \\ & & \downarrow & & \\ & & \nabla \bar{Y} = \nabla Y & & \text{are } \Delta's \text{ of } \mathcal{C}. \end{array}$$

Fact: This is self-dual obj. in \mathcal{C}

We call this E an ε -extension of $[X, Y]$.

An ε -extension splits : $\Leftrightarrow E \cong X \oplus \bar{Y} \oplus \nabla \bar{Y}$

While $\varepsilon \text{Ext}^1(X, Y) = 0$ if all ε -ext of (X, Y) splits

X is indec. ε -rigid : $\Leftrightarrow \begin{cases} X \text{ indec } \varepsilon\text{-obj} \\ \varepsilon \text{Ext}^1(X, X) = 0. \end{cases}$

X is ε -rigid : $\Leftrightarrow \begin{cases} - X = \bigoplus \text{indec } \varepsilon\text{-rigid} \\ - \forall \varepsilon\text{-indecs } Y, Z \not\subset X, \\ Y \neq Z \Rightarrow \text{Ext}_{\mathcal{C}}^1(Y, Z) = 0. \end{cases}$

Eg:

$$S = M_1, \quad \tilde{S} =$$

$$\tilde{\gamma} \hookrightarrow (\varphi, \omega) = (1 \xrightarrow{\omega} 2, \circ)$$

$$\Rightarrow \pi \omega = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \varepsilon = -\lambda \in \{-1\}$$

$$\pi \gamma = s_1$$

$$(\nabla \delta_{\varepsilon=1}) \rightarrow \omega \rightarrow \gamma(1)$$

$$\pi \gamma = s_2$$

$$\pi \circ$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \xrightarrow{\nabla f} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

If $f \neq 0$, this composition is nonzero.

\Rightarrow only split extensions everywhere.

\Rightarrow ε -rigid.

Goal III

Conj / "Thm" [IM-C-W] ("Baby" categorification of "quasi-triang")

$$\left\{ \begin{array}{l} \text{indec} \\ \varepsilon\text{-rigid} \end{array} \right\} \xrightarrow{1:1} \left\{ \text{quasicore of } (S, M) \right\}$$

which then induces

$$\left\{ \begin{array}{l} \text{maximal} \\ \varepsilon\text{-rigid obj} \end{array} \right\} \xrightarrow{1:1} \left\{ \text{quasi-triangulations of } (S, M) \right\}$$

Expectations

1) Define strictly Σ -rigid obj $X : \Leftrightarrow \mathbb{E} \text{Ext}^1(X, X) = 0$
 $\Rightarrow (\text{maximal } \Sigma\text{-rigid} \Leftrightarrow \text{maximal } \underline{\text{strictly }} \Sigma\text{-rigid})$

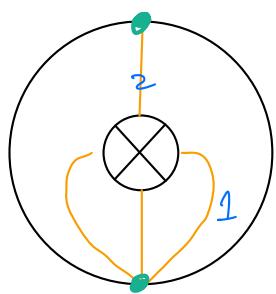
2) Σ -ext's count crossings on (S, M)
(egw. ??)

3) Exchange relation/Mutation of quasi-triangulation:

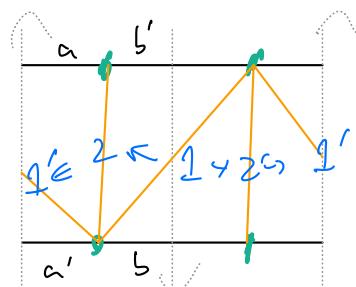
$$\alpha \in T \xleftarrow{M} (T \setminus \{\alpha\}) \cup \{\alpha'\}$$

is categorified by existence of some "special" Σ -extension
(egw. ??)

Part of Goal IV.



$\xleftarrow{2:1}$



$$Q: \begin{matrix} \nearrow \swarrow \\ 2 & 2' \\ \downarrow & \downarrow \\ 1' & 1 \end{matrix} \quad (\omega=0)$$

$$\sigma: \begin{matrix} a \leftrightarrow a' \\ b \leftrightarrow b' \end{matrix}$$

