Varieties for Modules over Elementary Abelian p-Groups

Fergus Reid

Institute of Mathematics
University of Aberdeen

11/03/2011 / Postgraduate Seminar

Setting the Stage

- k is an algebraically closed field.
- G is a finite group.
- E is an elementary abelian p-group.
- If I use words such as algebraic set, algebraic variety etc,please insert an implicit prefix of ’affine’.
- Assume basic familiarity with varieties and cohomology.

Some Group Cohomology

Let E be an elementary abelian p-group of rank r. Then

- If char $k = 2$ then $H(E, k) \cong k[x_1, \cdots, x_r]$
- If char k is odd then $H(E, k) \cong \Lambda[u_1, \cdots, u_r] \otimes k[v_1, \cdots, v_r]$

Let k be a field of characteristic p, $H^{ev}(G, k)$ denote the subring of $H(G, k)$ generated by elements of even degree. Then

- characteristic $k = 2 \Rightarrow H(G, k)$ is commutative
- characteristic k odd $\Rightarrow H(G, k)$ is graded-commutative.

Some Varieties

We define

$$\begin{aligned} H^{\cdot}(G,k) &= H(G,k) \text{ if } p = 2 \\ &= H^{ev}(G,k) \text{ if } p \text{ odd.} \end{aligned}$$

So $H^{\cdot}(G,k)$ is a finitely generated commutative graded ring over k.
We then define $V_G = \max H^{\cdot}(G,k)$.

Varieties for Modules

Let $M \in kG - Mod$. We define a subvariety $V_G(M)$ of V_G as follows: There is a natural map

$$\Sigma : H^{\cdot}(G,k) = Ext_{kG}(k,k) \rightarrow^{\otimes M} Ext_{kG}(M,M)$$

We denote the kernel of Σ by $I_G(M)$. Then $V_G(M) = \max$ $H^{\cdot}(G,k)/I_G(M)$.

Rank Varieties

Let $E = (\mathbb{Z}/p\mathbb{Z})^r$. Let k be an algebraically closed field of characteristic p. Then the linear subspace $V_E^\sharp$ of $J(kE)$ spanned by $(g_1 - 1), \cdots, (g_r - 1)$ has dimension r and is isomorphic to $J(kE)/J(kE)^2$

Now let $v_1, \cdots, v_r \in V_E^\sharp$ be linearly independent and $E^{'} = \langle 1 + v_1, \cdots, 1 + v_s \rangle \subseteq (kE)^\times$. Then the group algebra $kE^{'}$ is a subalgebra of kE over which kE is free as a module. We call such a subgroup $E^{'}$ a shifted subgroup of E.

Rank Varieties (Continued)

Let M be a finitely-generated kE-module. Define $V_E^\sharp(M)$, the rank variety of M by

$$V_E^\sharp(M) = \{0\} \cup \{v \in V_E^\sharp : M_{\downarrow\langle 1+v\rangle} \text{ not free}\}$$

$$V_E \cong V_E^\sharp$$

$$V_E(M) \cong V_E^\sharp(M)$$

Example

Let $E = (\mathbb{Z}/2\mathbb{Z})^4$, k be a field of characteristic 2, M be the module defined by the following matrices:

$$g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, g_4 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then $\{(g_1 - 1), (g_2 - 1), (g_3 - 1), (g_4 - 1)\}$ gives a basis for $V_E^\sharp$.

Example (Continued)

The element

$$v = \lambda_1(g_1 - 1) + \lambda_2(g_2 - 1) + \lambda_3(g_3 - 1) + \lambda_4(g_4 - 1)$$

is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ \lambda_3 & \lambda_4 & 0 & 0 \end{pmatrix}$$

So the restriction $M_{\downarrow\langle 1+v \rangle}$ is free if and only if the matrix has rank two. Therefore $V_E^\sharp(M)$ is given by the following equation in $\mathbb{A}^4(k)$

$$X_1 X_4 + X_2 X_3 = 0$$

Projective covers and $\Omega(M)$

Before we proceed it will be useful to have a few additional definitions under our belt.

Projective Cover: Given a kE-module M a projective cover P is a kE-module of minimal dimension such that we have a surjection $f : P \to M$. It can be shown that any two projective covers will be isomorphic.

We then define $\Omega(M)$ to be the kernel of such a map f. Note that this is only well-defined upto isomorphism in the stable category.

Examples of $\Omega(k)$ - Diagrams for Modules

See blackboard.

Varieties for Modules of Small Dimension

Given a polynomial equation there is a standard construction producing a module whose rank variety is given by that equation. This uses the fact that $Ext^n_{kG}(k,k) \cong Hom(\Omega^n(k),k)$. Unfortunately the modules produced are in general very large.

Quadratics in Characteristic 2

Firstly, let characteristic $k = 2$ and choose some $n \in \mathbb{N}$. Let $E_n = (\mathbb{Z}/2n)$. Then, given polynomial $f(x) = x_1 x_{(n+1)} + \cdots + x_n x_{2n}$ there exists a kE_n-module of dimension $2n$ with variety given by $f(x) = 0$.

Quadratics in Characteristic 2 (Continued)

This module has a kE_n basis given by:

$$\{u_1 \cdots u_n : u_i^2 = 0, \sum_{i=1}^{n} u_i > 0\} \cup \{v\}$$

The action is defined as follows:

- If $1 \leq i \leq n$:

$$u_i(u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}) = u_1^{a_1} u_2^{a_2} \cdots u_i^{a_i+1} \cdots u_n^{a_n}$$
$$u_i(v) = 0$$

- If $n+1 \leq i \leq 2n$:

$$u_i(u_j) = v \text{ for } 1 \leq j \leq n.$$
$$u_i(b) = 0 \text{ for all other basis elements } b.$$

Example - n = 3

See blackboard.

Other Questions

- Characteristic $k = 2$, quadratics, have modules L_n of dimension 2^n
- Characteristic $k = 2$, homogeneous with two terms, have modules of dimension $2n$.
- Characteristic k is odd?
- Higher degree polynomials?
- Bounds on the dimensions of such modules?

For More

- D.J. Benson, Representations and Cohomology, volume 2, Cambridge University Press, 1991.
- L. Evens, Cohomology of Groups, Oxford University Press, 2002
- R. Hartshorne, Algebraic Geometry, Springer-Verlag, 1997