Ex 1. Suppose $A = \Bbbk Q/I$ is a bounded path algebra.

1. Find the bounded quiver (Q', I') so that $\Bbbk Q'/I' \cong A^{\mathrm{op}}$.
2. Let e_x be a primitive idempotent. Find the bounded quiver (Q'', I'') so that $\Bbbk Q''/I'' \cong A/Ae_xA$.
3. Find an example of A so that
   - every indecomposable projective A-module is uniserial, but
   - there exists a non-uniserial indecomposable injective A-module.

   Note/Hint: Such an example can be found with $I = 0$ and Q acyclic.

Ex 2.

1. Consider the following representation M of the linearly oriented $\vec{\mathbb{A}}_5$-quiver:

$$\Bbbk \xrightarrow{1} \Bbbk \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \Bbbk^2 \xrightarrow{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)} \Bbbk^2 \xrightarrow{(1,1)} \Bbbk \xrightarrow{0} 0 \to \cdots \to$$

   Find the indecomposable decompositions of M in the cases when the matrix $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ is of rank 1 and of rank 2. You may use the fact that there are indecomposable modules of the form $U_{i,j}$ for $1 \le i \le j \le 5$ such that

$$U_{i,j}e_x = \begin{cases} \Bbbk & \text{if } i \le x \le j; \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } U_{i,j}\alpha_k = \begin{cases} \mathrm{id} & \text{if } i \le k < j; \\ 0 & \text{otherwise.} \end{cases}$$

2. Consider $A = \Bbbk Q/I$ given by

$$Q:\ \begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\delta"'] & 2 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4 \end{tikzcd}, \quad I = \langle \alpha\beta\gamma - \delta\gamma \rangle$$

   (a) Find a basis for the 2-dimensional socle of the indecomposable projective P_1.

   (b) Show that $\mathrm{rad} P_1$ is not indecomposable.

3. Consider the following quiver

$$Q:\quad \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & 2 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

   Let $I_1 := \langle \alpha^2 - \beta\gamma, \gamma\beta - \gamma\alpha\beta, \alpha^4 \rangle$ and $I_2 := \langle \alpha^2 - \beta\gamma, \gamma\beta, \alpha^2 \rangle$. Show that $\Bbbk Q/I_1 \cong \Bbbk Q/I_2$ as $\Bbbk$-algebra when the characteristic of $\Bbbk$ is not 2.

Ex 3. Consider $A = \Bbbk Q/I$ and $A' = \Bbbk Q/I'$ given by

$$Q : 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2, \quad I = \langle \alpha\beta, \beta\alpha \rangle, \quad I' = \langle \alpha\beta \rangle.$$

Recall that the *projective cover* of a module M is a projective module P_M equipped with a surjective homomorphism $p_M : P_M \to M$ such that $p_M|_P \neq 0$ for all direct summands P of P_M. Recall also that the *syzygy* $\Omega(M)$ of a module M the kernel $\mathrm{Ker}(p_M : P \to M)$. The n-th syzygy $\Omega^n(M)$ of a module M is the syzygy of $\Omega^{n-1}(M)$ for all $n \geq 1$ (with the convention $\Omega^0(M) := M$).

1. Show that A is self-injective, i.e. every indecomposable projective module is also an injective module.
2. Describe the $\Omega^k(S_x)$ of each simple module S_x and $k = 1, 2$, for both algebra A and A'.
3. Show that the global dimension of A is infinite (or equivalently, that the k-th syzygy of any simple is non-zero for all $k \geq 0$).
4. Show that the global dimension of A' is 2, i.e. $\Omega^3(A'/\mathrm{rad}A') = 0$ and $\Omega^2(A'/\mathrm{rad}A') \neq 0$.

Ex 4.

1. Let $A = \Bbbk Q$ for $Q = \vec{\mathbb{A}}_n$. Show that for every $1 \leq x \leq n$, Ae_xA is projective as a right A-module.
2. In the setting of the previous part, describe the quotient algebra A/Ae_xA.
3. Consider $A = \Bbbk\vec{\mathbb{A}}_4$. Consider the sequence $(x_1, x_2, x_3, x_4) := (3, 4, 1, 2)$. Let $x_5 := 0$ and $f_k := e_{x_4} + e_{x_3} + \cdots e_{x_{k+1}}$. Describe the right A-modules

$$\Delta(k) := P_k/f_kP_k, \text{ and } \nabla(k) := D(\frac{Ae_k}{Af_kAe_k})$$

for all $k = 1, 2, 3, 4$, in Loewy diagram or in quiver representation form.

4. For each $k = 1, 2, 3, 4$, find an indecomposable A-module $T(k)$ such that there is a surjective A-module homomorphism $T(k) \to \nabla(k)$ and an injective A-module homomorphism $\Delta(k) \to T(k)$.

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