

# On (Large) ( $\omega$ )sifting Bijections

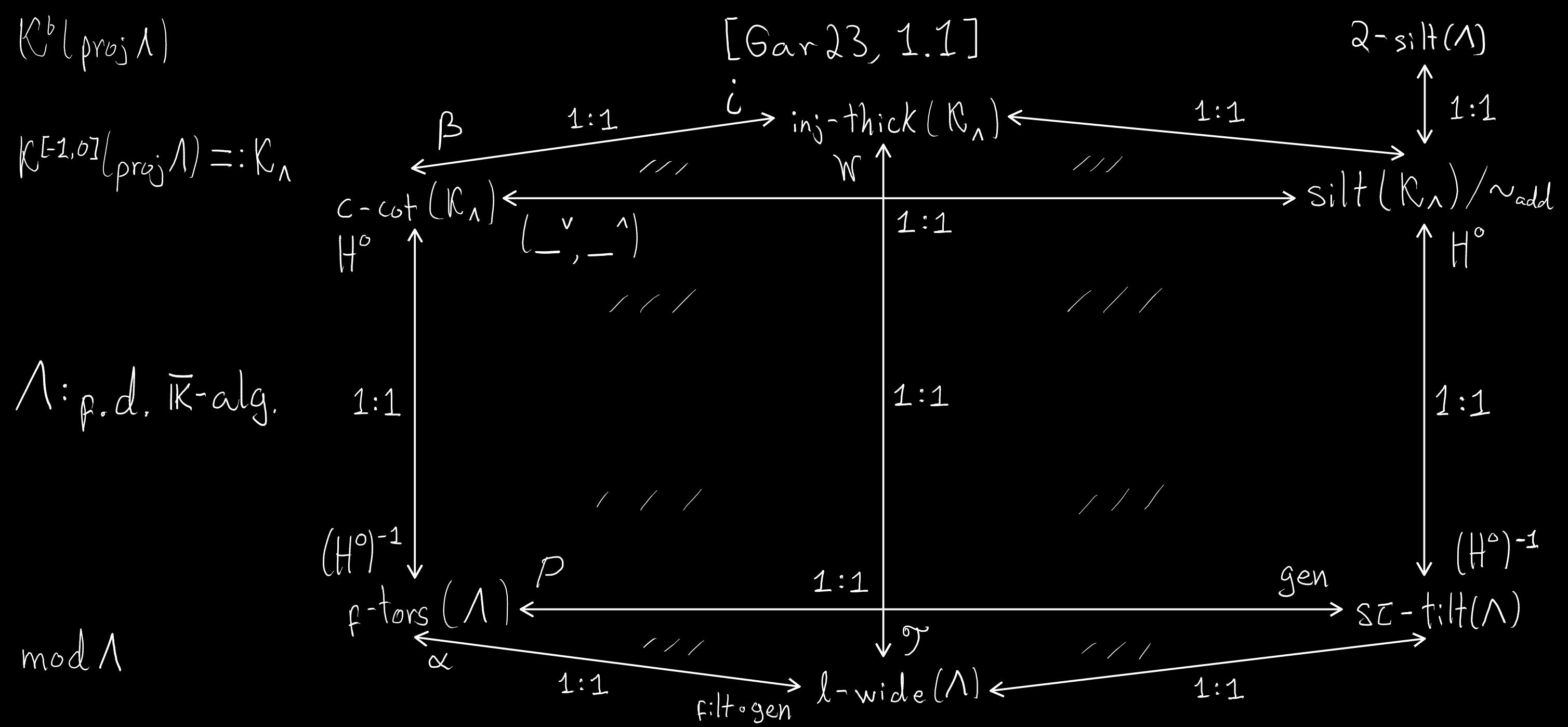
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Joint work in progress w/ Lidia Angeleri Hugel

Perspectives in Tilting Theory and Related Topics

Kyoto, 18/2/2025

S M M o t i v a t i o n



$\mathcal{K}^b(\text{proj } \Lambda)$  $[\text{AIR14, 2.7 \& 3.2}]$  $\Lambda: \text{f.d. } \overline{\mathbb{K}}\text{-alg.}$  $\text{mod } \Lambda$  $f\text{-tors } (\Lambda) \xleftarrow{P}$  $1:1$  $\text{gen}$  $(H^\diamond)^{-1}$  $2\text{-silt } (\Lambda)$  $H^\diamond$  $1:1$

$\mathcal{K}^b(\text{proj } \Lambda)$  $[\check{\text{MS}}17, 3.10]$  $\mathcal{Q}\text{-silt } (\Lambda)$  $\Lambda: \text{f.d. } \overline{\mathbb{K}}\text{-alg.}$  $\text{mod } \Lambda$  $f\text{-tors } (\Lambda)$  $1:1$  $1:1$  $\cdots$  $\text{filt. } \circ \text{gen}$  $\ell\text{-wide } (\Lambda)$  $1:1$  $\text{gen}$  $(H^\diamond)^{-1}$  $1:1$  $H^\diamond$

$\mathcal{K}^b(\text{proj } \Lambda)$ 

[PZ23, 3.6]

2-silt( $\Lambda$ ) $\mathcal{K}^{[-1,0]}(\text{proj } \Lambda) =: \mathcal{K}_\Lambda$  $\Lambda: \text{f.d. } \overline{\mathbb{K}}\text{-alg.}$  $\text{mod } \Lambda$  $c\text{-cot}(\mathcal{K}_\Lambda)$  $H^0$ 

1:1

 $(H^0)^{-1}$  $f\text{-tors}(\Lambda) \xleftarrow{\alpha} P$ 

1:1

1:1

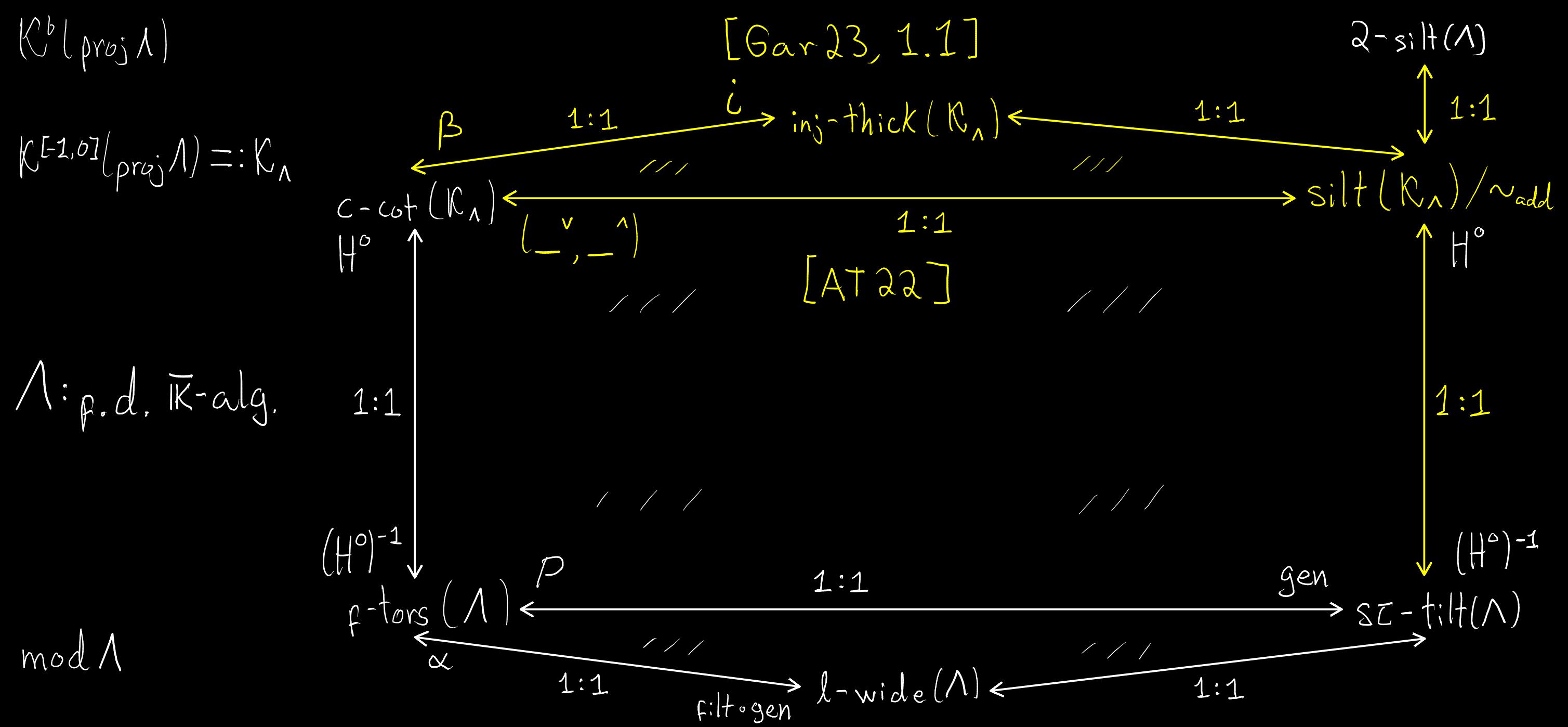
filt.  $\circ$  gen $\ell\text{-wide}(\Lambda) \xleftarrow{\quad}$ 

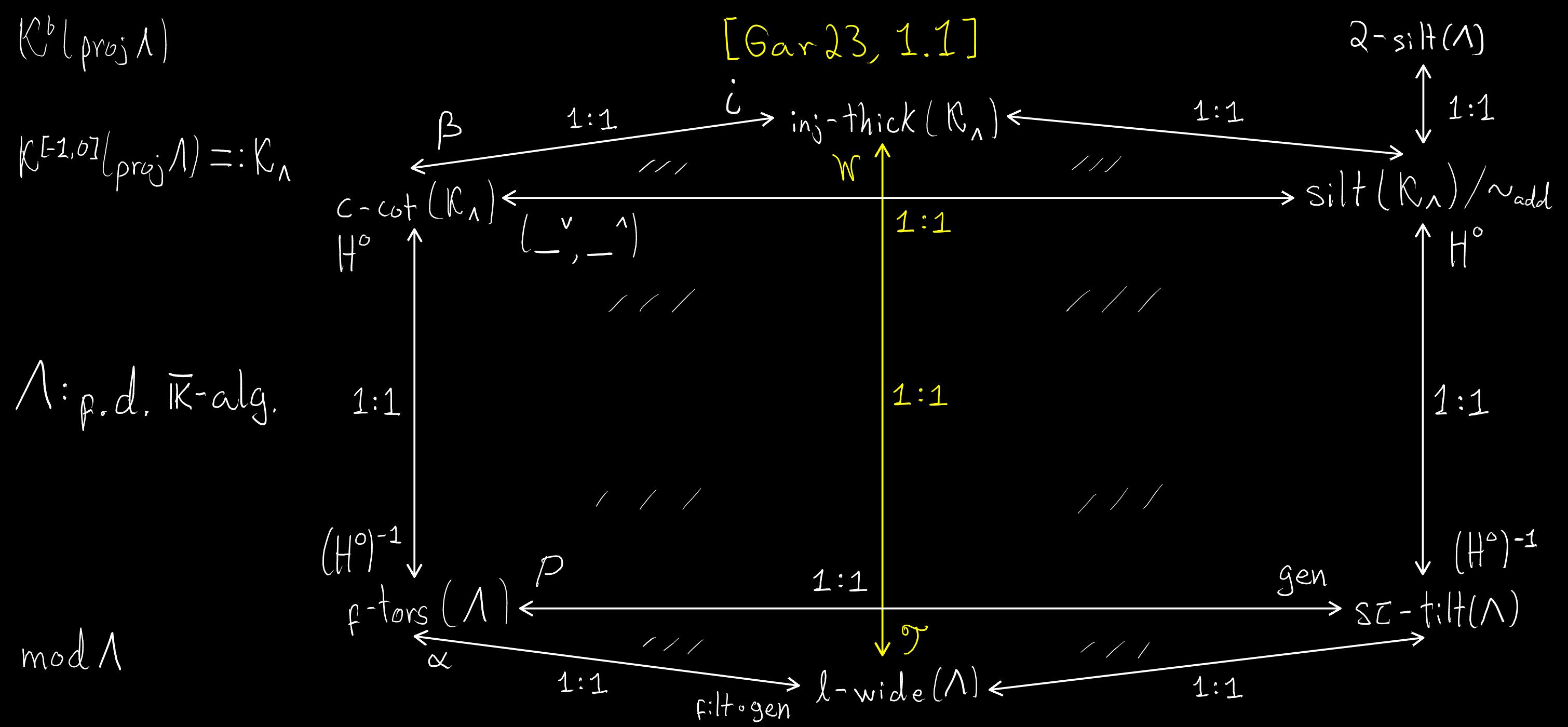
1:1

gen

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 $(H^0)^{-1}$  $\text{sc-tilt}(\Lambda)$





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- (1)  $\mathcal{K}_\lambda := \mathcal{K}^{[-1, 0]}(\text{proj } \Lambda) \subseteq \mathcal{K}^b(\text{proj } \Lambda)$  is a reduced  $\mathcal{O}$ -Auslander extriangulated category:
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- $\text{Proj}_{IE}(\mathcal{K}_\lambda) \cap \text{Inj}_{IE}(\mathcal{K}_\lambda) = \{0\}$ ;
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(2) Moreover, for any ring  $A$ , the following are also red.  $O$ -Auslander:

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- the category of large injective copresentations  $\mathcal{K}_I^2 := \mathcal{K}^{[0, 1]}(\text{Inj } A) \subseteq \mathcal{K}^b(\text{Inj } A)$ .

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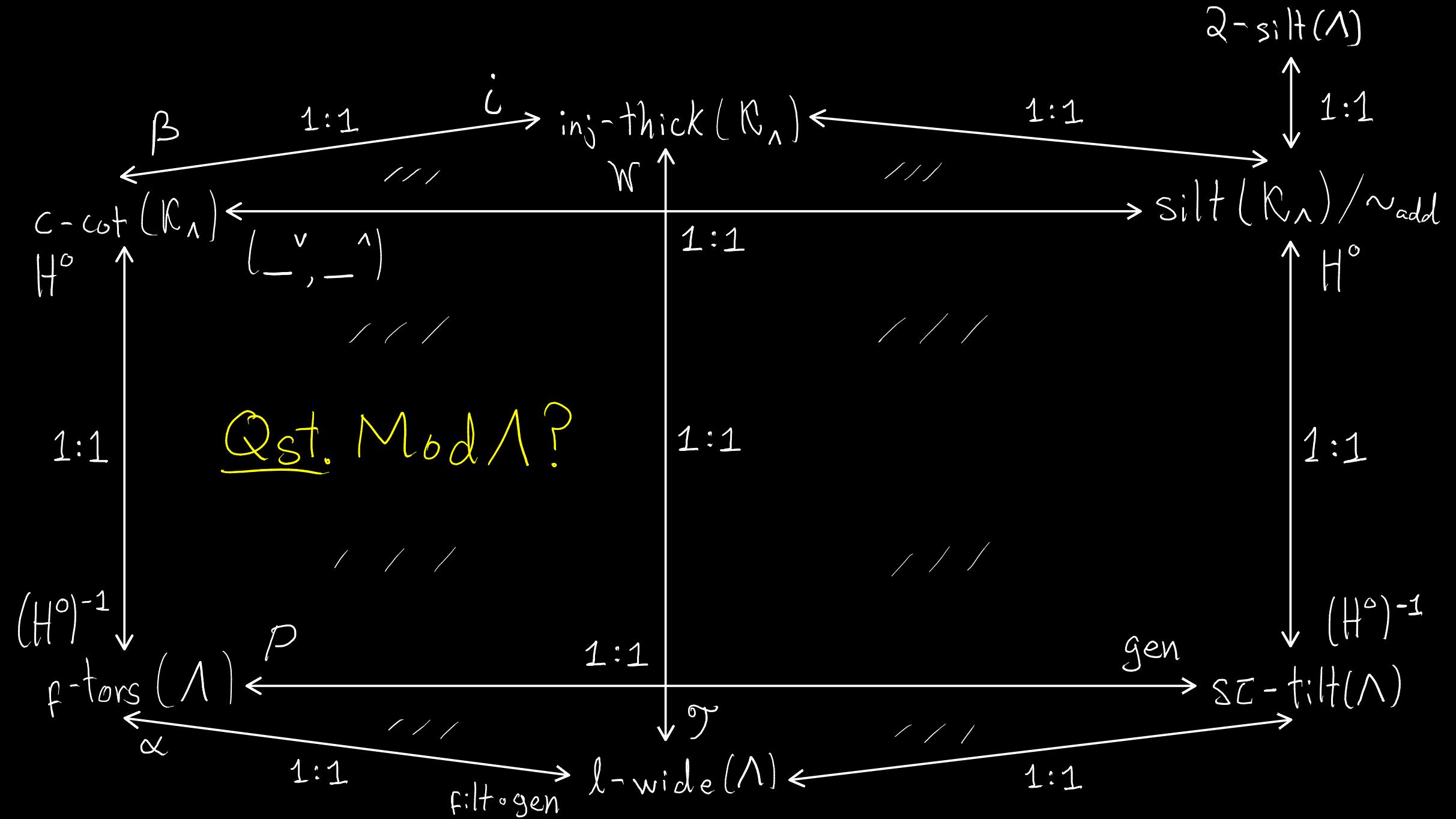
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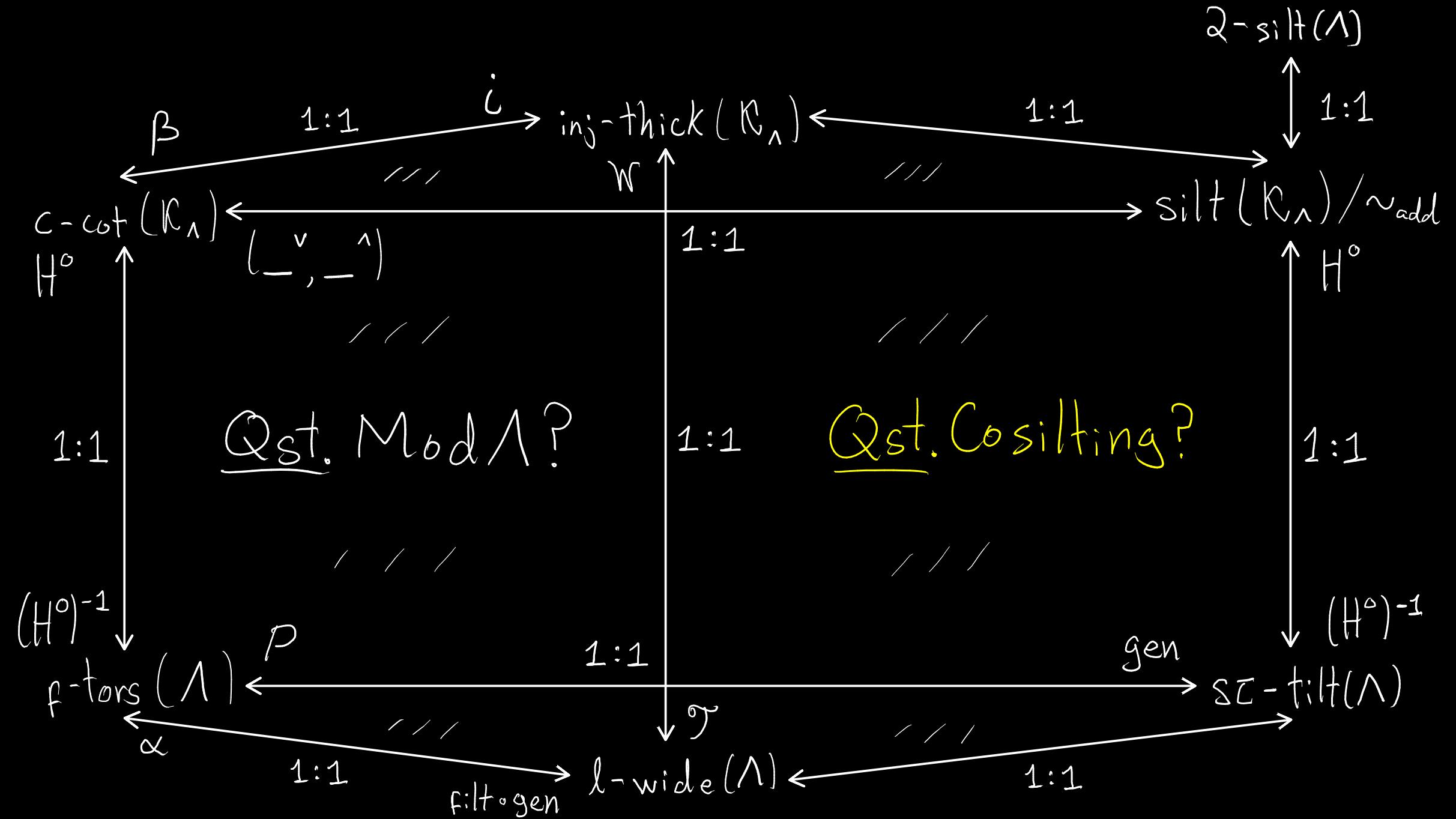
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5. Restrict the torsion pair  $({}^{\perp_0} C', \text{Cogen}(C')) \in \text{tor}(\text{Mod}\Lambda)$  to obtain the mutation  $(\mathcal{T}', \mathcal{F}') := ({}^{\perp_0} C' \cap \text{mod}\Lambda, \text{Cogen}(C') \cap \text{mod}\Lambda) \in \text{tor}(\text{mod}\Lambda)$  of  $(\mathcal{T}, \mathcal{F}) \in \text{tor}(\text{mod}\Lambda)$ .

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\* However, important questions regarding cosilting mutation remain. We hope to find some answers in  $\mathbb{R}_I^2$ .





§Q. Large ( $\omega$ )Sifting Theory

Dfn. [AMV15, 4.1] A (large) silting complex is a complex  $\bar{\sigma} \in \mathcal{K}^b(\text{Proj } A)$  such that

(i)  $\text{Hom}_{\mathcal{K}^b(\text{Proj } A)}(\bar{\sigma}, \bar{\sigma}^{(I)}[i]) = 0$  for all sets  $I$  and  $i > 0$ ;

(ii) the smallest triangulated subcategory of  $\mathcal{K}^b(\text{Proj } A)$  which contains  $\text{Add}(\bar{\sigma})$  is  $\mathcal{K}^b(\text{Proj } A)$ .

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Dfn. [AMV15, 3.7 & 4.9] Let  $T \in \text{Mod } A$  and  $\sigma \in \mathcal{K}_P^2$  be such that  $H^0(\sigma) = T$ . Then  $T$  is a silting module (w.r.t.  $\sigma$ ) and  $\text{Gen } T$  is a silting (torsion) class if

- $\sigma \in 2\text{-Silt}(A)$ ;
- equivalently,  $(\text{Gen } T, T^{\perp_0}) \in \text{tor}(\text{Mod } A)$ .

Rmk. [AMV15, 3.7, 3.11 & 4.9] We have the following bijections:

$\mathcal{K}^b(\text{Proj } A)$

$A$ : ring



St<sub>E</sub>. Let  $(\mathcal{C}, E, \leq)$  be extriangulated and assume its subcategories are full, strict and additive.

Dfn.  $S \subseteq \mathcal{C}$  is *silting* if  $E^{>0}(S, S) = 0$  and  $\text{thick}(S) = \mathcal{C}$ .

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Thm. [AT22, 5.7] There exist mutually inverse bijections  $\text{bddhc-cot}(\mathcal{C}) \xleftrightarrow{\Phi} \text{silt}(\mathcal{C})$  given by

$$\Phi(X, Y) := X \cap Y \text{ and } \Psi(M) = (M^{\vee}, M^{\perp}).$$

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Rmk. For  $(\mathcal{C}, \mathbb{E}, \mathbb{S})$  reduced 0-Auslander, all cotorsion pairs are bounded and hereditary.

Defn. An object  $\sigma \in \mathcal{K}_P^2$  is *silting* if  $\text{Add}(\sigma) \subseteq \mathcal{K}_P^2$  is a silting subcategory; we denote their collection by  $\text{Silt}(\mathcal{K}_P^2)$ .

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Rmk. Let  $\sigma \in \text{Silt}(\mathcal{K}_P^2)$  and  $T := H^0(\sigma)$ .

(1) By [AT22],  $(\text{Add}(\sigma)^\vee, \text{Add}(\sigma)^\wedge) \in (\text{bddh})_{\text{c-cot}}(\mathcal{K}_P^2)$ .

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Prp. Let  $\sigma \in \text{Silt}(\mathcal{K}_P^2)$ ,  $T := H^\circ(\sigma)$  and  $\Sigma := (H^\circ)^{-1}(\text{Gen } T)$ . Then  ${}^\perp_1 \Sigma \cap \Sigma = \text{Add}(\sigma)$  and  $({}^\perp_1 \Sigma, \Sigma) = (\text{Add}(\sigma)^\vee, \text{Add}(\sigma)^\wedge)$ . Thus, there is an injection

$$\text{Silt}(\mathcal{K}_P^2)/\sim_{\text{Add}} \hookrightarrow \text{c-cot}(\mathcal{K}_P^2).$$

Qst. For which  $(X, Y) \in C\text{-cot}(R^2)$  does there exist a  $\sigma \in X \cap Y$  such that  $X \cap Y = \text{Add}(\sigma)$ ?

Qst. For which  $(X, Y) \in c\text{-cot}(\mathcal{C}_P^2)$  does there exist a  $\sigma \in X \cap Y$  such that  $X \cap Y = \text{Add}(\sigma)$ ?

Lmm. Let  $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$  have (arbitrary) coproducts, enough  $\mathbb{E}$ -projectives and  $\mathbb{E}$ -injectives, and let  $(X, Y) \in c\text{-cot}(\mathcal{C})$  be such that  $X$  is closed under coproducts and  $\text{pdim}(X) < \infty$  for all  $X \in X$ . Then, for any  $X' \in X$ , there exists a finite family  $\{K_i\}_{i=0}^n \subseteq X \cap Y$  such that  $X' \in \text{add}(\bigoplus_{i=0}^n K_i)$ .

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Proof sketch: 1. Apply the previous Lemma to  $A \in \text{Proje}(\mathcal{K}_P^2) \subseteq X$  and define  $\sigma := \bigoplus_{i=0}^n K_i$ .  
2. Show that  $\text{Add}(\sigma)$  is a silting subcategory of  $\mathcal{K}_P^2$ .  
3. Note that  $\text{Add}(\sigma) \subseteq X \cap Y$  and apply [AT22, Lemma 5.3]. □

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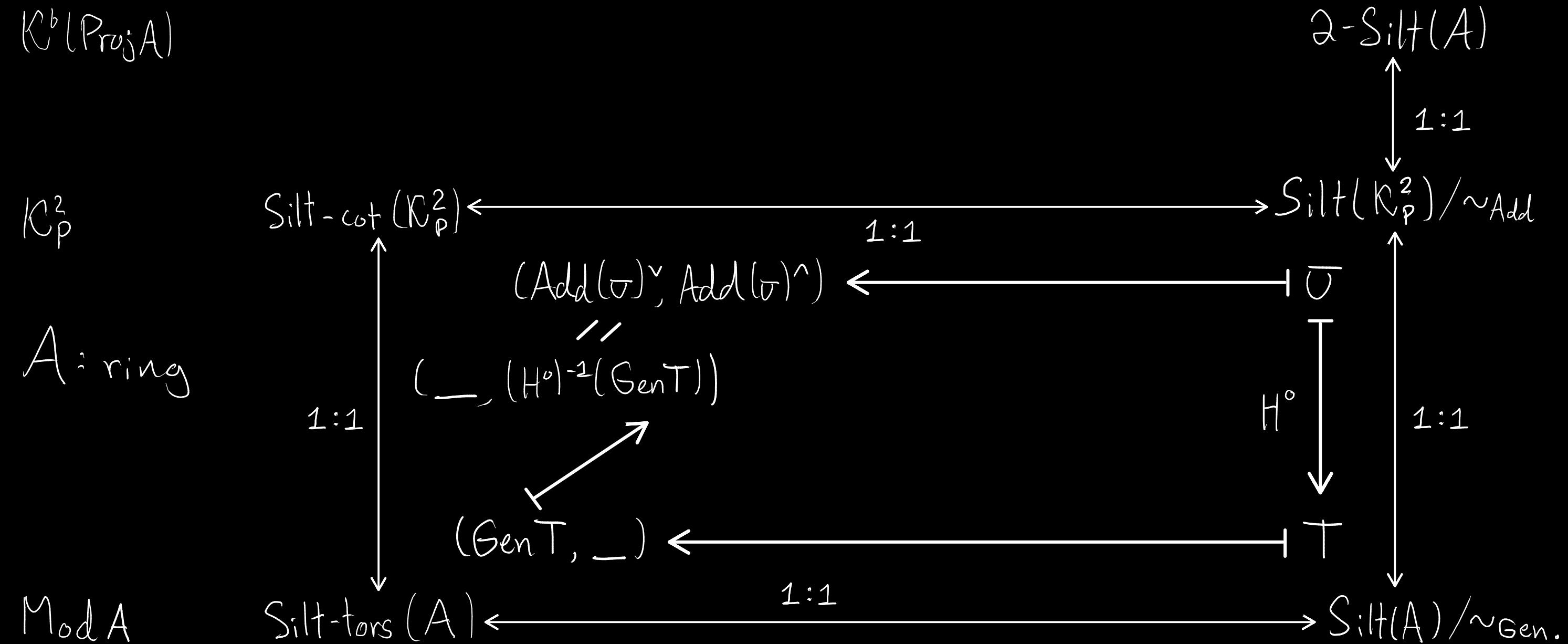
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Dfn. A cotorsion pair  $(X, Y)$  in  $\mathbb{K}_P^2$  is *silting* if it is complete and its kernel is closed under coproducts.

Cor. We have the following commutative diagram of bijections:



Rmk. [BP17, 3.5(2)] We have the following bijections:

$K^b(\text{Inj } A)$

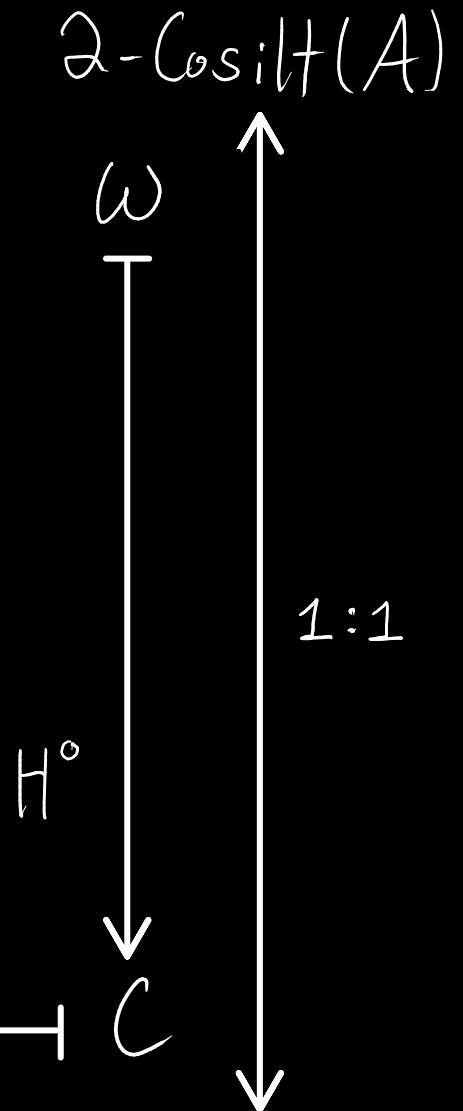
$A$ : ring

$(\_, \text{Cogen } C)$

$\text{Mod } A$

$\text{Cosilt-tor}_f(A)$

$1:1$



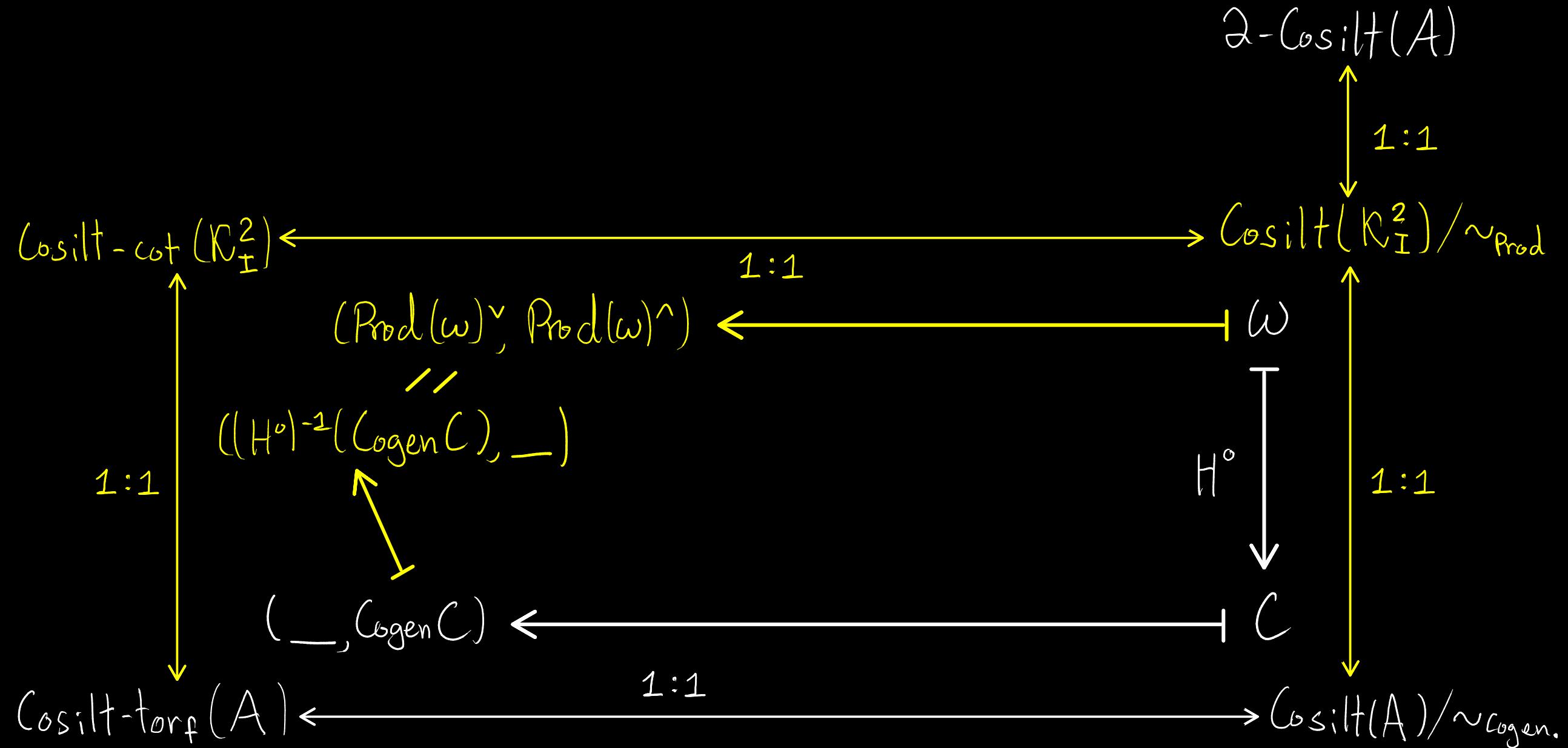
Rmk. By dual definitions and results, we obtain the commutative diagram of bijections

$K^b(\text{Inj } A)$

$K_I^2$

$A : \text{ring}$

$\text{Mod } A$



§3. Inverting Ingalls-Thomas  
maps

Rmk. In an extriangulated category  $(\mathcal{C}, \mathbb{E}, \mathbb{S})$ , the common generalization of kernel-cokernel pairs and distinguished triangles are called  $\mathbb{S}$ -conflations, and they are denoted by

$$A \rightarrow B \rightarrow C \dashrightarrow .$$

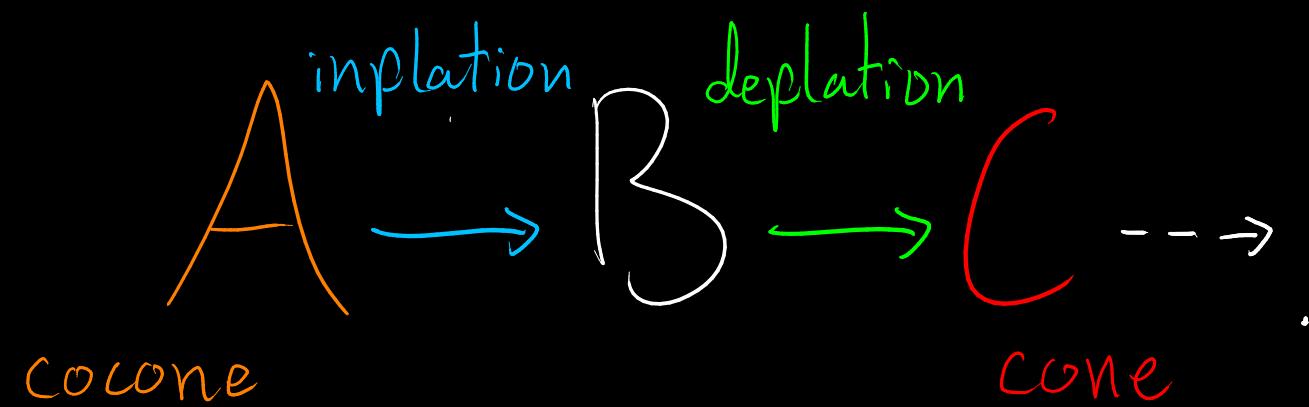
Rmk. In an extriangulated category  $(\mathcal{C}, \mathbb{E}, \mathbb{S})$ , the common generalization of kernel-cokernel pairs and distinguished triangles are called  $\mathbb{S}$ -conflations, and they are denoted by

$$\begin{array}{ccccc} & & \text{inflation} & & \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \dashrightarrow \\ & & & & \text{cone} \end{array}$$

Def. Let  $\mathcal{R}, \mathcal{S} \subseteq \mathcal{C}$ .

(a)  $\text{Cone}(\mathcal{R}, \mathcal{S})$  denotes the class of cones of inflations from objects in  $\mathcal{R}$  to objects in  $\mathcal{S}$ .

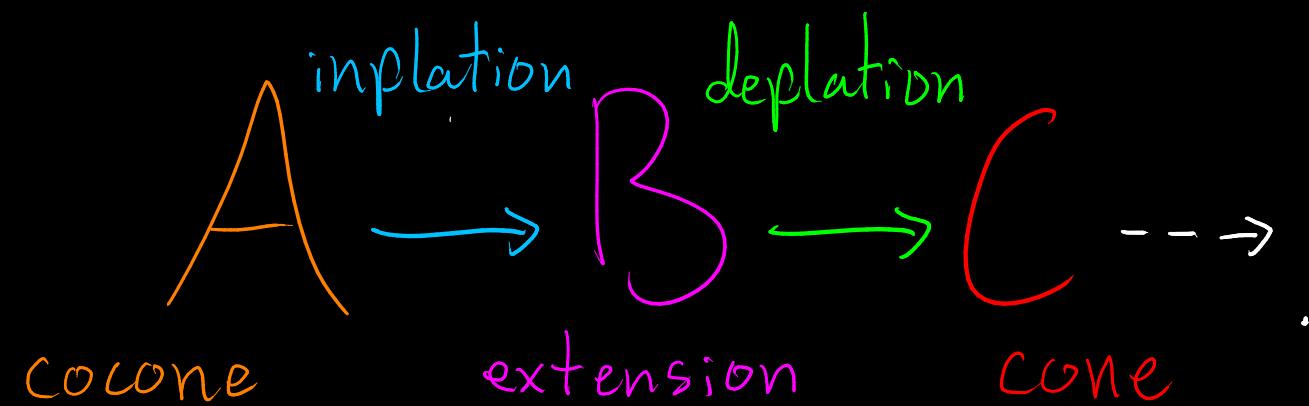
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- (c)  $\mathcal{R} * \mathcal{S}$  denotes the class of extensions of objects in  $\mathcal{S}$  by objects in  $\mathcal{R}$ .

St<sub>p</sub>. Let  $(\mathcal{C}, \mathbb{E}, \mathbb{S})$  be extriangulated and assume its subcategories are full, strict and additive.

Dfn. For  $S \in \mathcal{C}$ , let  $\beta(S) := \{S' \in \mathcal{S} \mid \exists S \rightarrow S' \rightarrow S'' \dashrightarrow, S' \perp S \Rightarrow S'' \in \mathcal{S}\}$  and  $i(S) := \{C \in \mathcal{C} \mid \exists C \rightarrow S \rightarrow C \dashrightarrow \text{ with } S \in \mathcal{S}\}$ .

Rmk. Let  $S \in \mathcal{C}$ .

- (1)  $\beta(S)$  is the biggest subcategory contained in  $S$  and closed under cones.
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Dfn. We say  $S \subseteq \mathcal{C}$  is

- (a) *torsionfree* if  $S * S \subseteq S$  and  $i(S) \subseteq S$ ;
- (b) *thick* if  $S * S \subseteq S$ ,  $\text{smd}(S) \subseteq S$ ,  $\text{Cone}(S, S) \subseteq S$  and  $S = \text{Cocone}(S, S)$ ;

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- (c) *resolving* if  $S^*S \subseteq S$ ,  $\text{smd}(S) \subseteq S$ ,  $S = \text{Cocone}(S, S)$  and  $\text{Cone}(\mathcal{C}, S) = \mathcal{C}$ .

Str. Suppose  $(\mathcal{C}, \mathbb{E}, \mathbb{S})$  is hereditary.

Prp. Let  $S \subseteq \mathcal{C}$ .

(a) If  $S^* S \subseteq S$ , then  $i(S)$  is the smallest torsionfree class containing  $S$ .

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- (b) If  $S * S \subseteq S$ ,  $\text{smd}(S) \subseteq S$  and  $S \subseteq \text{Cocone}(S, S)$ , then  $B(S)$  is the biggest thick subcategory contained in  $S$ .
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Cor. Let  $S \in \text{torf}(\mathcal{C})$ . Then  $S$  is thickly generated if and only if,  $i(B(S)) = S$ . In particular, the maps  $\text{thick}(\mathcal{C}) \xleftrightarrow[\beta]{i} \{\text{Setorfp}(\mathcal{C}) \mid S \text{ is thickly generated}\}$  are mutually inverse.

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Rmk. If  $(x, y) \in \text{cot}(\mathcal{C})$ , then  $x$  is torsionfree.

Str. Suppose  $(\mathcal{C}, \mathbb{E}, \mathbb{S})$  is reduced O-Auslander.

Dfn. Let  $S \subseteq \mathcal{C}$  be such that  $S^* S \subseteq S$ . Then  $S$  has enough injectives if the induced extriangulated category  $(S, \mathbb{E}|_S, \mathbb{S}|_S)$  has enough  $(\mathbb{E}|_S)^-$  injectives (see [NP19]).

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Prp. [Sak] The maps  $\{S \in \text{torp}(\mathcal{C}) \mid S \text{ has enough injectives}\} \xleftrightarrow[G_1]{F_1} c\text{-cot}(\mathcal{C})$  given by  $F_1(S) := (S, S^{+1})$  and  $G_1(x, y) := x$  are mutually inverse.

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Prf. [Sak] The maps  $\{S \in \text{top}(\mathcal{C}) \mid S \text{ has enough injectives}\} \xleftrightarrow[G_1]{F_1} \text{c-cot}(\mathcal{C})$  given by  $F_1(S) := (S, S^{+1})$  and  $G_1(x, y) := x$  are mutually inverse.

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Rmk. [Gar23, 3.10] A thick subcategory  $\mathcal{P}$  of a hereditary extriangulated category  $\mathcal{C}$  has enough injectives if, and only if,  $\mathcal{P} = \text{thick}(\mathcal{U})$  for some presilting subcategory  $\mathcal{U} \subseteq \mathcal{C}$  such that  $\text{cone}(\mathcal{U}, \mathcal{U}) \subseteq \mathcal{U}$ .

Prp. Let  $(X, Y) \in \text{Cosilt-cot}(\mathbb{K}^{[0,1]}(\text{Inj } \Lambda))$  for  $\Lambda$  a f.d.  $\bar{\mathbb{K}}$ -alg. Then there exist  $\mathbb{S}$ -conflations

$$w_1 \rightarrow w_0 \xrightarrow{\psi} D(\Lambda) \dashrightarrow \quad \text{and} \quad D(\Lambda)[-1] \xrightarrow{\phi} w_1 \rightarrow w_0 \dashrightarrow$$

such that

(a)  $\text{Prod}(w_0 \oplus w_1) = X \cap Y$ ;

(b)  $\psi$  is a minimal right  $X \cap Y$ -approximation of  $D(\Lambda)$  (+ dual statement).

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Moreover,  $i(\beta(X)) = X$ . In particular,  $X$  is thickly generated.

Rmk. In the silting case, the minimality of the approximations - and thus, the equality - do not hold in general.

Qst. Is there an "intrinsic" characterization for the subcategories  $S \subseteq \mathbb{K}_I^2$  such that  $S = \beta(X)$  for  $(X, Y) \in \text{Cosilt}(\mathbb{K}_I^2)$ ?

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