Topics in Mathematical Science V
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From quiver to quasi-hereditary algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M,+)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have*

- $m \cdot 1 = m$,
- $(m+n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r+s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i \in I}$ such that for all $x \in F$, $x = \sum_{i \in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $f(R)$ being in the center $Z(A) := \{z \in A \mid za = az \ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_{\Bbbk} A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the *characteristic* of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the* direct product *of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Definition 1.7. *A map $f : M \to N$ between right R-modules M, N is a* homomorphism *if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\operatorname{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$.*

Lemma 1.8. *$\operatorname{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \operatorname{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\operatorname{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.9. *$\operatorname{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the* endomorphism ring *of M.*

Lemma 1.10. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\operatorname{Hom}_A(M, N)$ is also an R-module (hence, $\operatorname{End}_R(M)$ is an R-algebra).*

Example 1.11. *$A \cong \operatorname{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra).*

Exercise 1.12. *Recall that R^{op} is the opposite ring of R, whose underlying set is the same as that of R with multiplication $(a \cdot^{\mathrm{op}} b) := b \cdot a$. A* representation *of R is a ring homomorphism*

$$\rho : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(M), \qquad r \mapsto \rho_r,$$

for some abelian group $(M, +)$. A homomorphism $f : \rho_M \to \rho_N$ of representations $\rho_M : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(M), \rho_N : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(N)$ given by an abelian group homomorphism $f : M \to N$ that intertwines R-action, i.e. $\rho_N(r) \circ f = f \circ \rho_M(r)$ for all $r \in R$.

Eplain why a representation of R is equivalent to a right R-module; and why homomorphisms correspond.

Indecomposable modules and Krull-Schmidt theorem

Recall that an R-module M is finitely generated if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x \in X} x r_x$ for some $r_x \in R$.

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \ldots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + \hat{N_j} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be unique if for any other decomposition $M = \bigoplus_{j=1}^n N_j$, we have $n = m$ and the N_j's are permutation of the M_i's.

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose R_R is indecomposable as an R-module. Then the free module $R \oplus R \oplus \cdots \oplus R$ with R copies of R is a decomposition of R^n.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1 \le j \le n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Then V is an indecomposable A-module. One can see this by showing that any submodule of V must be zero or V itself (Exercise).*

The n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \geq 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

For example, $V = \Bbbk\{v_1, v_2\}$ be a 2-dimensional space with $\rho_x = \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$. If V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. By definition $(av_1 + bv_2)x = (a+b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules; a contradiction. Hence, V must be indecomposable.

A natural question is to ask when a decomposition of modules, if it exists, is unique up to permuting the direct summands.

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent.

Lemma 2.5. *A non-zero A-module M is indecomposable if, and only if, the endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*

Proof $\underline{\Leftarrow}$: Suppose $M = U \oplus V$. Then we have

$$\text{a projection map } \pi_W : M \twoheadrightarrow W,$$
$$\text{and an inclusion map } \iota_W : W \hookrightarrow M,$$

for $W \in \{U, V\}$. Both of these are clearly A-module homomorphisms. Now $e_W := \iota_W \pi_W$ is an endomorphism of M with $e_V = \mathrm{id}_M - e_U$. Since any $m \in M$ can be written as $u + v$ for $u \in U$ and $v \in V$, we have

$$e_V^2(m) = e_V^2(u + v) = e_V^2(v) = v = e_V(m);$$

and likewise for e_W, so we have idempotents different from 0 and id_M when both U and V are non-zero.

$\underline{\Rightarrow}$: Suppose that M is indecomposable, and $e \in \mathrm{End}_A(M)$ is an idempotent. Note that

$$(\mathrm{id}_M - e)^2 = \mathrm{id}_M - e \cdot \mathrm{id}_M - \mathrm{id}_M \cdot e + e^2 = \mathrm{id}_M - 2e + e = \mathrm{id}_M - e$$

is also an idempotent and $\mathrm{id}_M = e + (\mathrm{id}_M - e)$. So we have $M = e(M) + (\mathrm{id}_M - e)(M)$. We want to show that $M = e(M) \oplus (\mathrm{id}_M - e)(M)$, i.e. $e(M) \cap (\mathrm{id}_M - e)(M) = 0$. Indeed, $x \in e(M) \cap (\mathrm{id}_M - e)(M)$ means that we have $e(m) = x = (\mathrm{id}_M - e)(m')$ for some $m, m' \in M$, and so

$$x = e(m) = e^2(m) = e((\mathrm{id}_M - e)(m')) = (e(\mathrm{id}_M - e))(m') = (e - e^2)(m') = 0(m') = 0,$$

as required.

Since M is indecomposable, one of $e(M)$ or $(\mathrm{id}_M - e)(M)$ is zero. In the former case, we get $e = 0$; whereas the latter case yields $\mathrm{id}_M = e$; as required. $\square$

The following is one of the main reasons why we like to consider finite-dimensional (or finite generated) modules over finite-dimensional $\Bbbk$-algebras.

Lemma 2.6 (Fitting's lemma (special version)). *Let M be a finite-dimensional A-module of a finite-dimensional $\Bbbk$-algebra, and $f \in \mathrm{End}_A(M)$. Then there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

Remark 2.7. The general version for rings requires M to be artinian and noetherian (i.e. ascending and descending chains of submodules stabilises).

We omit the proof to save time. The point is really just take n large enough so that the chains of submodules given by $(\mathrm{Ker}(f^k))_k$ and $(\mathrm{Im}(f^k))_k$ stabilises.

Corollary 2.8. *Let M be a non-zero finite-dimensional A-module. Then M is indecomposable if, and only if, every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*

Proof By Fitting's lemma, for any $f \in \mathrm{End}_A(M)$, we have $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$ for some $n \geq 1$. So indecomposability means that one of these direct summands is is zero. If $\mathrm{Ker}(f^n) = 0$, then f^n is an isomorphism and so is f. If $\mathrm{Im}(f^n) = 0$, then $f^n = 0$ and so f is nilpotent.

Conversely, consider an idempotent endomorphism $e \in \mathrm{End}_A(M)$. The assumption says that e is either an isomorphism or nilpotent.

If e is an isomorphism, then we have $\mathrm{Im}(e) = M$, which means that for every $m \in M$, there is some $m' \in M$ with $e(m) = e^2(m') = e(m') = m$, i.e. $e = \mathrm{id}_M$.

If e is nilpotent, then $e^n = 0$ for some $n \geq 1$, but $e = e^2 = e^3 = \cdots = e^n$, and so $e = 0$.

Hence, an idempotent endomorphism of M is either 0 or id_M, which means that M is indecomposable by Lemma 2.5. $\square$

Definition 2.9. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal. Equivalently, the non-invertible elements of R form a maximal right (equivalently, left; equivalently; two-sided) ideal.*

Remark 2.10. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit right inverses.

Definition 2.11. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^{m} M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^{n} N_j$ implies that $m = n$ and there is a permutation σ such that $M_i \cong N_{\sigma(i)}$ for all $1 \le i \le m$.*

$\mathsf{mod}\, A$ *is said to be Krull-Schmidt if every finitely generated A-module M admits a unique indecomposable decomposition.*

Theorem 2.12 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \le i \le m$, then the decomposition of M is unique.*

Remark 2.13. Some people refer to this result as Krull-Remak-Schmidt theorem.

We omit the proof to save time. Interested reader can see lecture notes from last year.

Corollary 2.14. *For a finite-dimensional algebra A,* $\mathsf{mod}\, A$ *is Krull-Schmidt.*

Convention. *For simplicity, from now on, an algebra means a $\Bbbk$-algebra (for a field $\Bbbk$). We take A to be an arbitrary $\Bbbk$-algebra. Again, all modules are finitely generated right modules unless otherwise specified. We write* $\mathsf{mod}\, A$ *for the collection of all finitely generated right A-modules.*