Ex 1. Fix an integer $n \geq 1$. Let Ω_r be the set of r-subsets (=subsets of size r) of $[n] := \{1, 2, \dots, n\}$ for an integer $n \geq 1$.

(1) Consider the r-subset $[r] := \{1, 2, \dots, r\} \subset [n]$. What is the stabiliser subgroup $\mathrm{Stab}_{\mathfrak{S}_n}([r])$?
Useful tips: An r-subset ω is equivalent to a partition of $\{1, 2, \dots, n\}$ into $\omega \sqcup (\Omega_n \setminus \omega)$. If $\sigma \in \mathfrak{S}_n$ fixes $\omega = \{1, \dots, r\}$, then it also fixes $\{r+1, \dots, n\}$.

(2) Find a subgroup $H \leq \mathfrak{S}_n$ (and the reason) such that $K\Omega_r \cong \mathrm{Ind}_H^{\mathfrak{S}_n} \mathrm{triv}_H$.
Useful tips: Lemma 5.5, Prop 5.6 of the course notes.

(3) Let π_r be the character of $K\Omega_r$. Find $\pi_r(1)$ for any r.

(4) Take $n = 4$ and find $\pi_2(g)$ for all $g \in \mathfrak{S}_4$. (c.f. Lem 5.16 (6) of the course notes.)

Ex 2. Consider the dihedral group $D_{2n} = \langle a, b \mid a^n = 1 = b^2, bab = a^{-1} \rangle$ of order $2n$ for $n = 2m$ even. Let $\rho : D_{2n} \to \mathbb{C}^\times$ be a one-dimensional $\mathbb{C}$-linear representation of D_{2n}.

(1) Write down all conjugacy classes of D_{2n}. (Hint: There are $m + 3$ of them, of which $m - 1$ of them has size 2.)

(2) Show that $\rho(a) = \rho(a^{-1})$, and hence, $\rho(a) \in \{\pm 1\}$.

(3) By writing down the character χ of ρ (note that $\chi(g) = \rho(g)$ for one-dimensional representation), show that there are precisely 4 one-dimensional representations of D_{2n} by writing down their characters. (Hint: $\rho(b) = \rho(b^{-1})$)
Here, by 'writing down the character', we meant that writing down $\chi(g)$ for each conjugacy class representative g.

You can do this exercise by specialising to $n = 4$ if you prefer.

Ex 3. Use the answer from the previous question to compute the character table of the diheral group of order 8. Write down the reasoning for each step of your deduction.
Useful tips: There are 5 conjugacy classes, and the previous question says 4 of them are one-dimensional, so the remaining row can be deduced by applying column orthogonality repeatedly.

Ex 4. Consider the quaternion group

$$Q_8 := \langle -1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1, (-1)^2 = 1, ij = k, jk = i, ki = j \rangle$$

of order 8.

(1) Write down all 5 conjugacy classes of Q_8.

(2) By considering arbitrary one-dimensional representation $\rho : Q_8 \to \mathbb{C}^\times$ evaluated at i and j, write down 4 irreducible characters of degree 1. (Hint: this is basically the same exercise as Question 3)

(3) Compute the character table of Q_8. Write down the reasoning for each step of your deduction. (Hint: this is basically the same exercise as Question 4)

(4) Show that two non-isomorphic groups can have identical character table ('identical' here means that the two matrices, without the labelling of rows and columns, are the same up to permutations of rows and columns).

Ex 5. A certain group G has two columns of its character table as follows:

g_i	g_1	g_2
$\lvert C_G(g_i)\rvert$	21	7
χ_1	1	1
χ_2	1	1
χ_3	1	1
χ_4	3	ζ
χ_5	3	$\overline{\zeta}$

where $g_1 = 1$ and $\zeta \in \mathbb{C}$.

(1) Find ζ.

(2) Find one other column of the character table.
Hint: (a) Recall that if g, g^{-1} are in the same conjugacy class, then $\chi(g) \in \mathbb{R}$.
(b) Recall that if χ_i irreducible, then so is $\overline{\chi_i}$.
(b) $|C_G(g)| = |C_G(g^{-1})|$
(c) You can use $\zeta \notin \mathbb{R}$ if you cannot complete (1).
(d) Part (2) is not about using orthogonality relation.

Ex 6. Suppose $H \leq G$ is a subgroup. For any $g \in G$, let

$$^gH := gHg^{-1} = \{^gh := ghg^{-1} \mid h \in H\} \leq G.$$

Define, for each KH-module W a new set

$$^gW := \{^gw \mid w \in W\};$$

note that here ^gw is just a formal symbol, not a conjugation.

(1) Show that $^gh \cdot {}^gw := {}^g(hw)$ defines a $K(^gH)$-module structure on ^gW.

(2) Take $H = G$ and $V \in KG\,\mathsf{mod}$, find an isomorphism $^gV \cong V$ of KG-module.
Hint: You will find that $^gv \mapsto v$ does not work, but only one more letter is needed in this line.

Deadline: 16th January, 2023