The Finite EI Categories of Cartan Type, joint with Ren Wang

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§4. Finite EI categories of Cartan type

§1. Background / Overview

k - a field

$\mathcal{C}$ - a finite category

$$k\mathcal{C} = \bigoplus_{\alpha \in \mathrm{Mor}(\mathcal{C})} k\alpha$$ - the category algebra

Def: $\mathcal{C}$ is EI if every endomorphism is an iso. $\square$

Example: (1) a finite group G

(2) a finite acyclic quiver Q, its path category $\mathcal{P}_Q$

(3) Assume $G \curvearrowright Q$ admissibly. The its skew group category $\mathcal{P}_Q \rtimes G$.

RMK: finite EI categories in modular rep. theory of finite groups.

Li 2011: $k\mathcal{C}$ is hereditary $\Longleftrightarrow$ $\mathcal{C}$ is free and

$\forall x \in \mathrm{Obj}(\mathcal{C})$
$\mathrm{char}(k) \nmid |\mathrm{Aut}_{\mathcal{C}}(x)|$

Wang 2016: $k\mathcal{C}$ is 1-Gorenstein $\iff$ $\mathcal{C}$ is free and
$\forall\ x, y \in \mathrm{Obj}(\mathcal{C})$,
${}_{k\mathrm{Aut}_{\mathcal{C}}(y)}k\mathrm{Hom}_{\mathcal{C}}(x,y)_{k\mathrm{Aut}_{\mathcal{C}}(x)}$
is projective on each side.

Geiss-Leclerc-Schröer 2017: for each Cartan triple
(C, D, Ω)

the GLS algebra $H(C, D, \Omega)$ is 1-Gorenstein, used in a categorification of root system of (C, D).

Question: how the two 1-Gorenstein algebras
$k\mathcal{C}$ and $H(C, D, \Omega)$ are related?

Theorem: For each (C, D, Ω)

a free EI category
$\mathcal{C}(C, D, \Omega)$

depending on char k
(C', D', Ω')
GLS
$H(C', D', \Omega')$

Then $\exists$ an iso. of algebras
$$k\mathcal{C}(C, D, \Omega) \simeq H(C', D', \Omega').$$

RMK: Assume char $k = 0$ or coprime to entries of D.

Then $(C', \underset{I_{n'}}{\underset{\|}{D'}}, \Omega')$ is a finite acyclic quiver Γ (with an admissible auto. σ)

and $H(C', D', \Omega') = k\Gamma$

$$\therefore \quad k\ell(C, D, \Omega) \simeq k\Gamma \quad \cdots\cdots \quad (*)$$

Recall $(C, D) \xleftrightarrow{\text{unfolding}} (|\Gamma|, \sigma)$

(cf. Lustig 1991)

$\Longrightarrow$ $(*)$ is an algebric "enrichment" of the unfolding !

§2. The GLS algebra associatiated to (C, D, Ω)

$C = (c_{ij})_{n\times n} \in M_n(\mathbb{Z})$ a Carton matrix

$$\begin{cases} c_{ii} = 2 \\ c_{ij} \le 0 \quad i \ne j \\ c_{ij} < 0 \Longleftrightarrow c_{ji} < 0 \end{cases}$$

$D = \operatorname{diag}(c_1, c_2, \cdots, c_n)$, $\quad c_i \in \mathbb{Z}_+$

s.t. DC is symmetric

an <u>acyclic orientation</u> $\Omega \subseteq \{1, 2, \cdots n\} \times \{1, 2, \cdots, n\}$

$\{(i,j), (j,i)\} \cap \Omega \neq \phi \Longleftrightarrow c_{ij} < 0$

For each sequence $(i_1, i_2), (i_2, i_3), \cdots, (i_t, i_{t+1})$

With each $(i_s, i_{s+1}) \in \Omega$, then $i_1 \neq i_{t+1}$

$Q = Q(C, \Omega)$ a finite quiver

$Q_0 = \{1, 2, \cdots, n\}$

$$Q_1 = \left\{ \alpha_{ij}^{(g)} : j \longrightarrow i \;\middle|\; \begin{array}{l} (i,j) \in \Omega \\ 1 \le g \le \gcd(c_{ij}, c_{ji}) \end{array} \right\}$$

$\quad \cup \; \{ \varepsilon_i : i \to i \mid i \in Q_0 \}$

$1 \le g \le \gcd(c_{ij}, c_{ji}) \}$
$\cup \{ \varepsilon_i : i \to i \mid i \in Q_0 \}$

Q^0 = the acyclic quiver obtained from Q by removing all ε_i's.

<u>Def</u> (GLS 2017)

$$H = H(C, D, \Omega) = kQ(C, \Omega) \Big/ \left\langle \begin{array}{l} \varepsilon_i^{c_i}, \\ \forall (i,j) \in \Omega,\ 1 \le g \le \gcd(c_{ij}, c_{ji}) \\ \varepsilon_i^{\frac{c_i}{\gcd(c_i, c_j)}} \alpha_{ij}^{(g)} - \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{\gcd(c_i, c_j)}} \end{array} \right\rangle$$

$\square$

<u>Fact</u> ① If $D = I_n \Rightarrow C$ symmetric
$\Rightarrow H = kQ^0$ is hereditary

If $D = cI_n$,
$$H = kQ^0 \otimes k[\varepsilon]/(\varepsilon^c)$$

② H is 1-Gorenstein

③ Set $H_i = k[\varepsilon_i]/(\varepsilon_i^{c_i})$ $\quad i \in Q_0$

$$H_{ij} = k[\varepsilon_{ij}]/(\varepsilon_{ij}^{\gcd(c_i, c_j)}) \qquad (i,j) \in \Omega$$

$$\begin{tikzcd} & H_i \\ H_{ij} \arrow[ur, hook] \arrow[dr, hook] & \\ & H_j \end{tikzcd}$$

$${}_iH_j^{(g)} \otimes_{H_{ij}} H_j$$

as an H_i-H_j-bimodule

$$B = \prod_{i \in Q_0} H_i, \qquad W = \bigoplus_{(i,j) \in \Omega} \bigoplus_{g=1}^{\gcd(c_{ij}, c_{ji})} {}_iH_j^{(g)}$$

$$\Rightarrow \quad H \simeq T_B(W)$$

§3. Finite EI quivers

Notation: G, H, K groups

a G-H-biset ${}_G X_H$

an H-K-biset ${}_H Y_K$. the <u>biset product</u>

$$X \times_H Y = X \times Y /\sim$$

<u>Def</u> (Li 2011) a finite EI quiver (Γ, U)

① $\Gamma = (\Gamma_0, \Gamma_1; s, t)$ a finite acyclic quiver

② $U = (U(i), U(\alpha))_{\substack{i \in \Gamma_0 \\ \alpha \in \Gamma_1}}$ an <u>assignment</u>

i.e., each $U(i)$ a finite group

each $U(\alpha)$ a finite $U(t\alpha)$-$U(s\alpha)$-biset

<u>RMK</u>: a finite acyclic quiver = a finite EI quiver with trivial assignment.

Given a finite EI quiver (Γ, U), we define for each path $p = \alpha_n \cdots \alpha_2\alpha_1$ in U

$$U(p) = U(\alpha_n) \times_{U(t\alpha_{n-1})} U(\alpha_{n-1}) \times \cdots \times_{U(t\alpha_1)} U(\alpha_1)$$

Identify $U(e_i) = U(i)$

Assume $s(p) = t(q)$.

$$U(p) \times_{U(tq)} U(q) \xrightarrow{\sim} U(pq)$$

$$U(p) \times_{U(q)} U(q) \xrightarrow{\sim} U(pq)$$

Def (Li 2011) Each EI quiver (Γ, U) defines a finite EI category $\mathcal{C} = \mathcal{C}(\Gamma, U)$ as follows:

$$\left|\begin{array}{l} \mathrm{Obj}\,\mathcal{C} = \Gamma_0 \\ \mathrm{Hom}_{\mathcal{C}}(i,j) = \dot{\bigsqcup}_{\substack{p \text{ a path} \\ \text{from } i \text{ to } j}} U(p) \end{array}\right.$$

$\square$

Note $\mathrm{Aut}_{\mathcal{C}}(i,i) = U(i)$

Def (Li 2011) a finite EI category $\mathcal{C}$ is <u>free</u> if it is equivalent to $\mathcal{C}(\Gamma, U)$ for some (Γ, U). $\square$

Fact: $\mathcal{C} = \mathcal{C}(\Gamma, U)$

(1) $A = \prod_{i \in \Gamma_0} kU(i)$, $\quad V = \bigoplus_{\alpha \in \Gamma_1} kU(\alpha)$

Then $k\mathcal{C} \simeq T_A(V)$.

(2) (Wang 2016) Assume for each $\alpha \in \Gamma_1$,

${}_{kU(t\alpha)} kU(\alpha)_{kU(s\alpha)}$ is projective on each side.

Then $k\mathcal{C}$ is 1-Gorenstein.

§4. Finite EI categories of Cartan type

(C, D, Ω) a Cartan triple

$Q = Q(C, \Omega)$, $\quad Q^0 = Q \setminus \{\varepsilon_i \mid i \in Q_0\}$

define a finite EI quiver

(Q^0, X)

s.t. $\quad X(i) = \langle \eta_i \mid \eta_i^{c_i} = 1 \rangle$

$\forall (i,j) \in \Omega, \quad G_{ij} = \langle \eta_{ij} \mid \eta_{ij}^{\gcd(c_i, c_j)} = 1 \rangle$

$$\begin{tikzcd} & X(i) \\ G_{ij} \arrow[ur, hook] \arrow[dr, hook] & \\ & X(j) \end{tikzcd}$$

$\forall\ 1 \le g \le \gcd(c_{ij}, c_{ji})$

$$X(\alpha_{ij}^{(g)}) = X(i) \times_{G_{ij}} X(j)$$

as a $X(i)$–$X(j)$-biset.

<u>Def</u>: $\mathcal{C} = \mathcal{C}(C, D, \Omega)$
$= \mathcal{C}(Q^0, X)$ the free EI category given by (Q^0, X),

a <u>finite EI category of Cartan type</u> $\square$

<u>Theorem</u>: Assume k has <u>enough roots of unity</u>

Then $\exists\ (C', D', \Omega')$, depending on char$k$, s.t.

$$k\mathcal{C}(C, D, \Omega) \simeq H(C', D', \Omega').$$

§4.1 The construction of (C', D', Ω') from (C, D, Ω)

<u>Case char$k = p > 0$</u>: $\quad D = \mathrm{diag}(c_1, c_2, \dots, c_n)$

$c_i = p^{r_i} \cdot d_i \qquad r_i \ge 0,\ p \nmid d_i$

$$M = \bigsqcup_{1 \le i \le n} \{(i, l_i) \mid 0 \le l_i < d_i\}$$

$\forall\ 1 \le i, j \le n$

$\forall\, 1 \le i, j \le n$

$$\Sigma_{ij} = \left\{ (l_i, l_j) \;\middle|\; \begin{array}{l} 0 \le l_i < d_i, \\ 0 \le l_j < d_j, \end{array} \quad l_i p^{r_i} \equiv l_j p^{r_j} \mod \gcd(d_i, d_j) \right\}$$

Set $\quad C' \in M_m(\mathbb{Z}), \quad m = |M|$

$$C'_{(i,l_i),(j,l_j)} = \begin{cases} -\gcd(c_{ij}, c_{ji})\, p^{r_j - \min\{r_i, r_j\}} & \text{if } (l_i, l_j) \in \Sigma_{ij} \\ 0 & \text{otherwise} \end{cases}$$

$$D' = \mathrm{diag}\,(p^{r_i})$$

$$\Omega' = \left\{ ((i, l_i), (j, l_j)) \;\middle|\; \begin{array}{l} (i,j) \in \Omega \\ (l_i, l_j) \in \Sigma_{ij} \end{array} \right\}$$

$$\Rightarrow (C', D', \Omega')$$

<u>Case char $k = 0$</u>:

$$M = \bigcup_{1 \le i \le n} \{ (i, l_i) \mid 0 \le l_i < c_i \}$$

$$C'_{(i,l_i),(j,l_j)} = \begin{cases} -\gcd(c_{ij}, c_{ji}) & \text{if } (l_i, l_j) \in \Sigma_{ij} \\ 0 & \text{otherwise} \end{cases}$$

$$D' = I_m$$

<u>Example</u>:

$$C = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \qquad D = \mathrm{diag}(2, 1)$$

$$\Omega = \{(2,1)\}$$

$$\ell(C, D, \Omega) = \begin{tikzcd} 1 \arrow[loop left, "\eta_1"] \arrow[r] & 2 \end{tikzcd} \qquad \eta_2 = 1d_2$$

$$\eta_1^2 = 1d_1$$

$$X(2) \times_{G_{21}} X(1) =$$

$\boxed{\text{char } k = 2}$ $\qquad (C', D', \Omega') = (C, D, \Omega)$

$$k\ell(C,D,\Omega) \simeq H(C,D,\Omega)$$

$\boxed{\mathrm{char}\, k \neq 2}$ $\qquad M = \{(1,0), (1,1), (2,0)\}$, $D' = I_3$

$$\begin{tikzcd}[row sep=small] (1,0) \arrow[dr] & & \\ & (2,0) & = \Gamma \\ (1,1) \arrow[ur] & & \end{tikzcd}$$

$$k\ell(C,D,\Omega) \simeq k\Gamma$$

§4.2 The proof of Theorem

$$k\ell(C,D,\Omega) = T_A(V) \qquad A = \prod kX(i)$$

$$H(C',D',\Omega') = T_B(W) \qquad B = \prod_{(i,\ell_i)\in M} H_{(i,\ell_i)}$$

$$\begin{array}{ccc} A & \simeq & B \\ \cup\!\!\mid & & \cup\!\!\mid \\ kX(i) & & \prod\limits_{\ell_i} H_{(i,\ell_i)} \\ \parallel & & \parallel \\ k\langle \eta_i \rangle / (\eta_i^{c_i} - 1) & \xrightarrow{\ \sim\ } & \prod\limits_{0\le \ell_i < d_i} k[\varepsilon_{(i,\ell_i)}] / (\varepsilon_{(i,\ell_i)}^{p^{r_i}}) \end{array}$$

$\ulcorner$ Recall $c_i = p^{r_i} \cdot d_i \lrcorner$

Moreover, ${}_A V_A \xrightarrow{\ \sim\ } {}_B W_B$.

§4.3 An application (work in progress)

$\mathrm{char}\, k = p$

$G = \langle \sigma \rangle$ cyclic p-group

$G \curvearrowright$ a finite acyclic quiver Q

$\mathcal{P}_Q \rtimes G$ skew group category

$$k(\mathcal{P}_Q \rtimes G) \simeq kQ\#G$$

<u>Theorem</u>: $\exists$ a Morita equivalence

$$kQ\#G \simeq H(C,D,\Omega)$$

where $(|Q|,\sigma) \longleftrightarrow (C,D)$

Moreover, $\forall\, M \in kQ\text{-ind}$

$$M\#G \in H(C,D,\Omega)\text{-ind}.$$

<u>RMK</u>: The proof is via

$$\mathcal{P}_Q \rtimes G \simeq \mathcal{C}(C,D,\Omega)$$

<u>In Dynkin case</u>:

$$\begin{tikzcd}
kQ\text{-ind} \arrow[r, dashed, "-\#G"] \arrow[d, "\text{Gabriel 1972}\quad \underline{\dim}"'] & H(C,D,\Omega)\text{-}(\tau\text{-lf})\text{ind} \arrow[d, "\underline{\mathrm{rank}}\quad \text{GLS2017}"] \\
\Phi^+(Q) \arrow[r, two heads, "\text{folding}"] & \Phi^+(C)
\end{tikzcd}$$

$\therefore$ The functor $-\#G$ "categorifies" the folding!