Topics in Mathematical Science VI

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Group Representations and Character Theory

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Contents

Lecture 1

Throughout, 'group' means 'finite group', unless otherwise stated. K will always be a field.

Definition 1.1. *A finite-dimensional (resp. n-dimensional) K-linear representation of a group G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action of g on V.

Often, the symbol ρ is suppressed and we write $G \curvearrowright V$ instead, and say 'G acts on V'. In particular, instead of $\rho_g(v)$ for $v \in V$, we write $g(v)$ instead.

Example 1.2. *(1) The trivial representation of G is the one-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}\,.$$

(2) $G = \mathfrak{S}_n$ *the symmetric group of rank* n*. The sign representation of* $\mathfrak{S}_n$ *is the one-dimensional representation*

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ *is the parity (or sign) of the permutation* σ*.*

Exercise 1.3. *Suppose* $\rho : G \to \mathrm{GL}(V)$ *is a representation. Show that* $\det \rho$ *is also a representation.*

Definition 1.4. *Let* KG *be the* K*-vector space with basis* G*, i.e.* $x \in KG \Leftrightarrow x = \sum_{g \in G} \lambda_g g$ *with* $\lambda_g \in K$ *for all* $g \in G$*.*

Define a map

$$KG \times KG \to KG, \qquad (\sum_{g \in G} \lambda_g g, \sum_{h \in G} \mu_h h) \mapsto \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

It is routine to check that this defines a ring structure on KG *with identity given by that of* G*. We call this ring the group algebra of* G *over* K*.*

Clearly, $G \curvearrowright KG$ naturally; this is called the regular representation.

Exercise. *Show that there is an injective ring homomorphism* $K \to Z(KG) := \{x \in KG \mid xy = yx \ \forall y \in KG\}$*. In other words, the group algebra* KG *is a* K*-algebra.*

Lemma 1.5. $\rho : G \to \mathrm{GL}(V)$ *is a (finite-dimensional)* K*-linear representation of* G *if, and only if,* V *has the structure of a (finite-dimensional) left* KG*-module.*

Proof
$\underline{\Rightarrow}$: For $x = \sum_g \lambda_g g \in KG, v \in V$. It is routine to check that $x \cdot v := \sum_g \lambda_g \rho_g(v)$ defines a left KG-module structure.

$\underline{\Leftarrow}$: Define a map $\rho_g : V \to V$ by $v \mapsto gv$. Since $g^{-1}g(v) = v$, we have $\rho_{g^{-1}}\rho_g = \mathrm{id}$, and so $\rho_g \in \mathrm{GL}(V)$. It is routine to check that $g \mapsto \rho_g$ is a group homomorphism. □

Remark 1.6. One may find in older textbooks that use terminologies like 'the KG-module V is afforded by ρ' in the setting of this lemma.

Definition 1.7. $V = (V, \rho), W = (W, \theta)$ *be* K*-linear representations of* G*. A homomorphism from* V *to* W *is a* K*-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$*, i.e.* $f\rho_g = \theta_g f$ *for all* $g \in G$*.*

An isomorphism from V *to* W *is a homomorphism that is invertible, i.e.* $\exists g$ *s.t.* $gf = \mathrm{id}_V$ *and* $fg = \mathrm{id}_W$*. In this case,* V *and* W *are equivalent representations, and write* $V \cong W$*.*

Write $\mathrm{Hom}_G(V, W)$ *to be the (*K*-vector) space of all homomorphisms from* V *to* W*.*

Lemma 1.8. $f : V \to W$ *is a homomorphism of* K*-linear* G*-representations if, and only if, it is a homomorphism of left* KG*-modules; in other words,* $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{KG}(V, W)$*. Consequently,* $\mathrm{Ker}(f)$*,* $\mathrm{Im}(f)$*,* $W/\mathrm{Im}(f)$ *are naturally* K*-linear* G*-representations.*

Proof
This first part is clear (if not, think through it).

For the second part, just recall that the kernel, image, and quotient of image of any homomorphism of modules are also modules. □

Remark. In the language of category theory, Lemma 1.5 and 1.8 together says that the category of finite-dimensional K-linear G-representations (where morphisms are homomorphisms) and the category of finitely generated left KG-modules are isomorphic (note that this is stronger than just equivalence of categories).

Exercise 1.9. *Let V be the 1-dimensional subspace spanned by $\sum_{g\in G} g \in KG$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.*

Recall that for a ring R with identity 1, either 1 has infinite order (under addition) or has prime, say p, order. The *characteristic* of R, denoted by $\operatorname{char} R$, is 0 in the former case, p in the latter.

Exercise. *Fix any $n \geq 2$.*

(i) Find a generator v such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g\in G} g$ of the trivial representation.)

(ii) Show that $\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ when $\operatorname{char} K = 2$, otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

Two classes of group representations. In the literature, by *ordinary representations* we mean K-linear representations with $\operatorname{char} K = 0$; by *modular representations* we mean K-linear representations with $\operatorname{char} K \mid |G|$.

The *Maschke's theorem* (and its consequence) justifies that ordinary representation theory is (significantly) easier to understand than modular ones - this will be our next goal. The material we will use is based on a more ring theoretic approach (from Benson's book Chapter 1) to the subject, which has the advantage of shedding some light on what happen on the modular side too. The proof of Maschke's theorem will follow the exposition of James and Liebecks.

Interlude on terminology and notation. For a field K, recall that a *K-algebra* is a ring R equipped with an injective ring homomorphism $K \to Z(R) := \{x \in R \mid xy = yx \ \forall y \in R\}$. This is equivalent to saying that R is a K-vector space equipped with a ring structure.

Notation. For a K-algebra A, let $A\,\mathsf{mod}$ be the category of finitely generated left A-modules. So by $M \in A\,\mathsf{mod}$ we mean that M is an A-module, and by $(f : M \to N) \in A\,\mathsf{mod}$ we mean that f is an A-module homomorphism. We will use 0 to denote either the zero homomorphism, or the zero element in a vector space, or the vector space with only the zero element; this should be clear from context.

Like numbers, we like to break down modules into simpler 'components'. The first candidate is via the notion of direct sum. Recall that an A-module M is a *direct sum*, say $M = M_1 \oplus M_2$, if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$. We will come back to this next lecture. In this lecture, we consider a more refined way to break down M into smaller modules.

Definition 1.10. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$.*

(1) *M is simple if for any submodule L of M, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 1.11. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Example 1.12. (1) *Trivial module and sign module are both simple. In general, any 1-dimensional representation of a group G will be simple for dimension reason.*

(2) *Consider the matrix ring $A := \mathrm{Mat}_n(K) := \{n \times n$ matrices with entries in $K\}$. Let V be the 'column space', i.e. $V = \{(v_j)_{1\leq j\leq n} \mid v_j \in K\}$ where $X \in \mathrm{Mat}_n(K)$ acts on $v \in V$ by $v \mapsto Xv$ (matrix multiplication from the left). Then V is an n-dimensional simple module. The*

regular representation A is semisimple as it is isomorphic to the direct sum of n column spaces (corresponding to the n choices of column we can cut matrix into V).

(3) *The ring of dual numbers is $A := K[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as (x) is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by Kv where $v = 1 + (x)$. If A is semisimple, then Kv is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in K$. If $a \neq 0$, then $A(a + bx) = A$. So $a = 0$, and $Kv \cong (x)$, a contradiction.*

The following easy yet fundamental lemma describes the relation between simple modules.

Lemma 1.13 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \textit{a division } K\textit{-algebra}, & \textit{if } S \cong T; \\ 0, \textit{ otherwise.} \end{cases}$$

Proof
For $f \in \mathrm{Hom}_A(S,T)$, $\mathrm{Im}(f)$ is a submodule of T, and so f is either zero or a K-vector space isomorphism, and the latter case only happens when $S \cong T$. □

Remark 1.14. If K is algebraically closed, then any division K-algebra is just K itself. The complication with the divison K-algebra appearing is the reason why most literature consider only the case when K is algebraically closed. In particular, for ordinary representation one usually just consider $K = \mathbb{C}$. In this course, this will also often be the case - perhaps the only exception is when we consider general K-algebra instead of group algebra.