Topics in Mathematical Science VIII

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Introduction to quiver representations and homological algebras

Aaron Chan

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M, +)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have*

- $m \cdot 1 = m$,
- $(m + n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r + s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $\theta(R)$ being in the center $Z(A) := \{z \in A \mid za = az\ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_{\Bbbk} A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a* $\mathbb{Z}$*-algebra.*

The matrix ring $M_n(R)$ *given by* n*-by-*n *matrices with entries in* R *is an* R*-algebra.*

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. Most of the time, we will also assume $\Bbbk$ is algebraically closed for simplicity. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the *characteristic* of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let* G *be a finite (semi)group and* R *a commutative ring. Let* $A := R[G]$ *be the free* R*-module with basis* G*, i.e. every* $a \in A$ *can be written as the formal* R*-linear combination* $\sum_{g\in G} \lambda_g g$ *with* $\lambda_g \in R$*. Then group multiplication extends (*R*-linearly) to a ring multiplication on* $R[G]$*, making* A *an* R*-algebra.*

Example 1.6. *Recall that the* direct product *of two rings* A, B *is the ring* $A \times B = \{(a, b) \mid a \in A, b \in B\}$ *with unit* $1_{A\times B} = (1_A, 1_B)$*. It is straightforward to check that if* A, B *are* R*-algebras, then* $A \times B$ *is also an* R*-algebra.*

Example 1.7. *Suppose that* A *is a* $\Bbbk$*-algebra and* B *is a* $\Bbbk$*-subspace of* A *containing* 1_A *and closed under multiplication. Then* B *is also a* $\Bbbk$*-algebra. We call such a* B *a* subalgebra *of* A*. For a concrete example, the space of diagonal matrices forms a subalgebra of* $M_n(\Bbbk)$*.*

Definition 1.8. *A map* $f : M \to N$ *between right* R*-modules* M, N *is a* homomorphism *if it is a homomorphism of abelian groups (i.e.* $f(m + n) = f(m) + f(n)$ *for all* $m, n \in M$*) that intertwines* R*-action (i.e.* $f(mr) = f(m)r$ *for all* $m \in M$ *and* $r \in R$*). Denote by* $\operatorname{Hom}_R(M, N)$ *the set of all* R*-module homomorphisms from* M *to* N*. We also write* $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$*.*

Lemma 1.9. $\operatorname{Hom}_R(M, N)$ *is an abelian group with* $(f + g)(m) = f(m) + g(m)$ *for all* $f, g \in \operatorname{Hom}_R(M, N)$ *and all* $m \in M$*. If* R *is commutative, then* $\operatorname{Hom}_R(M, N)$ *is an* R*-module, namely, for a homomorphism* $f : M \to N$ *and* $r \in R$*, the homomorphism* fr *is given by* $m \mapsto f(mr)$*.*

Definition 1.10. $\operatorname{End}_R(M)$ *is an associative ring where multiplication is given by composition and identity element being* id_M*. We call this the* endomorphism ring *of* M*.*

Lemma 1.11. *If* A *is an* R*-algebra over a commutative ring* R*, then any right* A*-module is also an* R*-module, and* $\operatorname{Hom}_A(M, N)$ *is also an* R*-module (hence,* $\operatorname{End}_R(M)$ *is an* R*-algebra).*

Example 1.12. $A \cong \operatorname{End}_A(A)$ *given by* $a \mapsto (1_A \mapsto a)$ *is an isomorphism of rings (or of* R*-algebras if* A *is an* R*-algebra). Note that if we work with left modules, then* $A \cong \operatorname{End}_A({}_AA)^{\mathrm{op}}$*, where* $(-)^{\mathrm{op}}$ *denotes the* opposite ring *given by the same underlying set with reverse direction of multiplication, i.e.* $a \cdot_{\mathrm{op}} b := b \cdot a$*.*

Recall that an R-module M is *finitely generated* if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x\in X} x r_x$ for some $r_x \in R$.

Notation. *We write* $\mathsf{mod}\, A$ *for the collection of all finitely generated right* A*-modules.*

2 Indecomposable modules and Krull-Schmidt property

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \ldots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + \hat{N_j} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable.

Convention. *We write $(n_1, \ldots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose that R_R is indecomposable as an R-module. If F is a free R-module of rank n, then $R^{\oplus n} := R \oplus R \oplus \cdots \oplus R$ (with n copies of R) is a decomposition of F.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1 \le j \le n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \ge 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By definition $(av_1 + bv_2)x = (a + b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules. If, on the contrary, V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. But v_2 must be contained in any submodule of V, hence, we have $v_2 \in U_1 \cap U_2$, i.e. $U_1 \cap U_2 \neq 0$ – a contradiction not decomposability.

Proposition 2.5. *There is a canonical R-module isomorphism*

$$\begin{tikzcd} \mathrm{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) \arrow[r, "\cong"] & \bigoplus_{i,j} \mathrm{Hom}_A(M_j, N_i) \\ f \arrow[r, mapsto] & (\pi_i f \iota_j)_{i,j} \end{tikzcd}$$

where $\iota_j : N_j \to \bigoplus_j N_j$ is the canonical inclusion for all j and $\pi_i : \bigoplus_i M_i \to M_i$ is the canonical projection for all i.

One can think of the right-hand space above as the space of m-by-n matrix with entries in each corresponding Hom-space.

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent. From the previous proposition, we see that for a decomposition $M = N_1 \oplus N_2$, we have idempotents

$$e_i : M \xrightarrow{\pi_i} N_i \xrightarrow{\iota_i} M$$

for both $i = 1, 2$. Hence, being decomposable implies existence of multiple idempotents; this turns out characterise indecomposability completely.

Proposition 2.6. *Let A be a finite-dimensional algebra and M be a finite-dimensional non-zero A-module. Then the following hold.*

(1) *(Fitting's lemma) For any $f \in \mathrm{End}_A(M)$, there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

(2) *The following are equivalent.*
- *M is indecomposable.*
- *The endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*
- *Every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*
- *$\mathrm{End}_A(M)$ is local (see below).*

Remark 2.7. It is known that if M is only artinian or only noetherian, then Fitting's lemma (and hence part (2)) fails. Nevertheless, in general, the proposition still hold for M that is both artinian and noetherian.

Let us briefly recall various characterisation of local rings.

Definition 2.8. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 2.9. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit (right) inverses.

Lemma 2.10. *The following are equivalent for a finite-dimensional algebra A.*

- *A is local (i.e. has a unique maximal right ideal).*
- *Non-invertible elements of A form a two-sided ideal.*
- *For any $a \in A$, one of a or $1 - a$ is invertible.*
- *0 and 1_A are the only idempotents of A.*
- *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

Example 2.11. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq 2} \,\middle|\, \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

A natural question is to ask when is a decomposition of modules, if it exists, unique up to permuting the direct summands.

Definition 2.12. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^m M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^n N_j$ implies that $m = n$ and there is a permutation σ such that $M_i \cong N_{\sigma(i)}$ for all $1 \leq i \leq m$. $\mathsf{mod}\, A$ is said to be Krull-Schmidt if every (finitely generated) A-module M admits a unique indecomposable decomposition.*

Theorem 2.13. *For a finite-dimensional algebra A, $\mathsf{mod}\, A$ is Krull-Schmidt.*

Remark 2.14. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Theorem 2.15 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.*

Remark 2.16. Some people refer to this result as Krull-Remak-Schmidt theorem.

3 Simple modules, Schur's lemma

Definition 3.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 3.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 3.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 3.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 3.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 3.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Let us see how one can find a simple module.

Definition 3.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 3.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module. In particular, if A is finite-dimensional, then every simple A-module is also finite-dimensional.*

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 3.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. □

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 3.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S, T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S,T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T, simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S, so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. $\square$

Example 3.12. *In Example 2.11, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x,y) \mid x,y \in \Bbbk^2\}$ given by 'row vectors'. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0,y) \mid y \in \Bbbk^2\}$.*

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A-action given by

$$w\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0,y)$ for some $y \in \Bbbk$ and has to satisfy

$$(0, ya) = (0,y)a = f(wa) = f(w\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = f(w)\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0,y)c = (0,yc)$$

for any $a,b,c \in \Bbbk$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 3.13. *Suppose that S is a simple A-module. Consider a semisimple A-module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$.*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this is an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1\le i,j\le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \xrightarrow{\pi_j} S_j \xrightarrow{a_{i,j}} S_i \xhookrightarrow{\iota_i} M$, which yields the required surjection. $\square$

Example 3.14. *For a tautological example, take $A = \Bbbk$ to be just a field. Then we have a 1-dimensional simple A-module $S = \Bbbk$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\Bbbk)) = \mathrm{Mat}_n(\Bbbk)$. Note that now we have an n-dimensional simple $\mathrm{Mat}_n(\Bbbk)$-module (given by the row vectors).*

4 Quiver and path algebra

Definition 4.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 4.2. *Let Q be a quiver.*

- *A trivial path on Q is a "stationary walk at i", denoted by e_i for some $i \in Q_0$.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2\cdots\alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.*

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right*.

Definition 4.3. *The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying vector space is given by $\bigoplus_{p:\text{paths of } Q} \Bbbk p$, with multiplication given by path concatenation. That is $x \in \Bbbk Q$ is a formal linear combinations of paths on Q.*

Note that $e_i e_j = \delta_{i,j} e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an *idempotent* of the path algebra $\Bbbk Q$. Moreover, we have an idempotent decomposition

$$1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\Bbbk Q$.

Example 4.4. *Consider the one-looped quiver, a.k.a. Jordan quiver,*

$$Q = \left(\begin{tikzcd} \bullet \arrow[loop above, "\alpha"] \end{tikzcd} \right)$$

Then $\Bbbk Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\Bbbk Q \cong \Bbbk[x]$.

An *oriented cycle* is a path of the form $v_1 \to v_2 \to \cdots v_r \to v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is acyclic.

Proposition 4.5. *$\Bbbk Q$ is finite-dimensional if, and only if, Q is finite acyclic.*

Proof If there is an oriented cycle c, then $c^k \in \Bbbk Q$ for all $k \geq 0$, and so $\Bbbk Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q. □

Example 4.6. *Consider the linearly oriented $\vec{\mathbb{A}}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\Bbbk Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j\alpha_{j+1}\cdots\alpha_k$.

Consider the upper triangular n-by-n matrix ring

$$\begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \;\middle|\; \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\Bbbk Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \le j < k \le n$.

From now on, we will focus in the following setting.

Assumption 4.7. (1) *Quivers are finite (i.e. finitely many vertices and arrows).*

(2) *Representations (equivalently, modules) are finite-dimensional.*

5 Duality

For a quiver Q, the opposite quiver Q^{op} has the same set of vertices with the reverse direction of arrows, i.e. $Q_0^{\mathrm{op}} = Q_0, Q_1^{\mathrm{op}} = Q_1, s_{Q^{\mathrm{op}}} = t_Q$, and $t_{Q^{\mathrm{op}}} = s_Q$.

Exercise 5.1. *Show that there is a canonical isomorphism $(\Bbbk Q)^{\mathrm{op}} \cong \Bbbk(Q^{\mathrm{op}})$.*

Let M be a finite-dimensional A-module. Then we have a dual space

$$D(M) := M^* := \mathrm{Hom}_{\Bbbk}(M, \Bbbk),$$

which has a natural A^{op}-module structure, namely, $(a\cdot f)(m) := f(ma)$ for any $a \in A, f \in M^*, m \in M$. Moreover, for an A-module homomorphism $\theta : M \to N$, we have also an A^{op}-module homomorphism $\theta^* : N^* \to M^*$ with $\theta^*(f)(m) = f(\theta(m))$.

Lemma 5.2. *There is a $\Bbbk$-vector space isomorphism $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$.*

Proof Just a straightforward check that $(\theta^*)^* = \theta$. □

We note as a fact that D preserves indecomposability of (finite-dimensional) modules. This can be seen using the fact that $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$ and can be upgraded to an algebra isomorphism for the case when $N = M$; then uses characterisation of indecomposable module by local endomorphism ring.

Example 5.3. *The* left *A-module ${}_AA$ yields a right A-module structure on $D(A)$. More generally, suppose we have a left ideal Ae of A for some element $e \in A$, then $D(Ae)$ is a right ideal of A.*

Remark 5.4. There is another natural duality, which we will not used, between $\mathsf{mod}\, A$ and $\mathsf{mod}\, A^{\mathrm{op}}$ given by sending M to $\mathrm{Hom}_A(M, A)$. In general, this duality is different from the $\Bbbk$-linear dual unless A is a so-called *symmetric algebra*, meaning that $A \cong DA$ as bimodule; in which case, $\mathrm{Hom}_A(-, A)$ dual is naturally isomorphic to D (as functors).

6 Representations of quiver

Definition 6.1. *A $\Bbbk$-linear* representation *of Q is a datum $(\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ where M_i is a $\Bbbk$-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is K-linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional *if $\dim_{\Bbbk} M_i < \infty$ for all $i \in Q_0$.*

Notation. *For a representation M of Q, we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

It is easy to notice that every representation of Q is equivalent to a $\Bbbk Q$-module, namely,

$$\text{representation } (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1}) \leftrightarrow \begin{array}{c} \Bbbk Q\text{-module } \prod_{i\in Q_0} M_i \\ \text{s.t. } \sum_{p:\text{path}} \lambda_p p \text{ acts as } \sum_p \lambda_p M_p. \end{array}$$

Example 6.2 (Simple). *For $x \in Q_0$, denote by S_x (or $S(x)$) the representation given by putting a 1-dimensional space on x, zero on all other vertices, and zero on all arrows. This corresponds to a 1-dimensional $\Bbbk Q$-module and so we call it the simple at x.*

Note: at this stage, it is not clear if these are all the simple $\Bbbk Q$-modules (up to isomorphism) yet.

Example 6.3 (Projective). *For $x \in Q_0$, denote by P_x (or $P(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=x,\\ t(p)=y}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p\alpha=q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the projective at x. This corresponds to the right ideal $e_x\Bbbk Q$ of $\Bbbk Q$.

Example 6.4 (Injective). *Dual to the projective module construction, for $x \in Q_0$, denote by I_x (or $I(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=y,\\ t(p)=x}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p=\alpha q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the injective at x. This corresponds to the dual of the left ideal generated by e_x, i.e. $D(\Bbbk Q e_x)$.

Example 6.5. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to \Bbbk \xrightarrow{\text{id}}\to \cdots \xrightarrow{\text{id}} \Bbbk \to 0 \to \cdots \to 0$$

with a copy of $\Bbbk$ on vertices $i, i+1, \ldots, j$ is the uniserial $\Bbbk Q$-module corresponding to the column space (under the isomorphism of $\Bbbk Q$ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \le k \le j$.

Example 6.6. *Let Q be the Jordan quiver with unique arrow α. Then a representation of Q is nothing but an n-dimensional vector space equipped with a linear endomorphism, equivalently, an n-by-n matrix.*

Definition 6.7. *A homomorphism $f : M \to N$ of ($\Bbbk$-linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram*

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "M_\alpha"'] & N_i \arrow[d, "N_\alpha"] \\ M_j \arrow[r, "f_j"'] & N_j \end{tikzcd}$$

for all arrows $\alpha : i \to j$ in Q.

A homomorphism $f = (f_i)_{i\in Q_0} : M \to N$ of quiver representations is injective, resp. surjective, resp. an isomorphism, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 6.8. *Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n-by-n matrix M_α. By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α. If we assume $\Bbbk$ is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α. That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \ldots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m-by-m Jordan block with eigenvalue $\lambda \in \Bbbk$.*

Proposition 6.9. *There is an isomorphism between the category of representations of Q and* $\mathsf{mod}\,\Bbbk Q$*, where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i \in Q_0} M_i$ with $\Bbbk Q$-action given by (linear combinations of compositions of) M_α's, and isomorphism classes of Q-representations correspond to isomorphism classes of $\Bbbk Q$-modules.*