The Finite EI Categories of Cartan Type, joint with Ren Wang

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§4. Finite EI categories of Cartan type

§1. Background / Overview

k – a field

$\mathcal{C}$ – a <u>finite</u> category $\big(|\mathrm{Mor}(\mathcal{C})| < \infty \Rightarrow |\mathrm{Obj}(\mathcal{C})| < \infty\big)$

$k\mathcal{C} = \bigoplus_{\alpha \in \mathrm{Mor}(\mathcal{C})} k\alpha$ – the <u>category algebra</u>

$\alpha\beta = \begin{cases} \alpha\circ\beta & \text{if composable} \\ 0 & \text{otherwise} \end{cases}$ $\quad 1_{k\mathcal{C}} = \sum_{x \in \mathrm{Obj}(\mathcal{C})} \mathrm{Id}_x$

<u>Def</u>: $\mathcal{C}$ is <u>EI</u> if every <u>e</u>ndomorphism is an <u>i</u>so. □

$\Rightarrow \forall x \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{End}_{\mathcal{C}}(x) = \mathrm{Aut}_{\mathcal{C}}(x)$ is a finite group.

<u>Example</u>: (1) a finite group G ← a category with one object

(2) a finite acyclic quiver Q, its <u>path category</u> $\mathcal{P}_Q$ is EI

(3) Assume $G \curvearrowright Q$. Then its <u>skew group category</u> $\mathcal{P}_Q \rtimes G$. $\mathrm{obj} = Q_0$

$\mathrm{Hom}(i,j) = \left\{ (p, g) \;\middle|\; \begin{array}{l} g \in G \\ p: \text{a path from } g(i) \text{ to } j \end{array} \right\}$.

<u>RMK</u>: ① finite EI categories in modular rep. theory of finite groups (the orbit category, Webb, Linckelmann, Xu...)

② related to graphs of groups (Bass–Serre 1980?)

Li 2011: $k\mathcal{C}$ is hereditary $\Longleftrightarrow$ $\mathcal{C}$ is free and

$\forall x \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{char}(k) \nmid |\mathrm{Aut}(x)|$

Wang 2016: $k\mathcal{C}$ is 1-Gorenstein $\Longleftrightarrow$ $\mathcal{C}$ is free and

(self inj.dim ≤ 1)

$\forall\ x, y \in \mathrm{Obj}(\mathcal{C})$,

$${}_{k\mathrm{Aut}_\mathcal{C}(y)}\, k\mathrm{Hom}_\mathcal{C}(x,y)\, {}_{k\mathrm{Aut}_\mathcal{C}(x)}$$

is projective on each side.

Geiss-Leclerc-Schröer 2017: for each Cartan triple

(C, D, Ω)

generalized symmetric Cartan matrix — its symmetric — acyclic orientation.

the GLS algebra $H(C, D, \Omega)$ is 1-Gorenstein, used in a categorification of root system of C, over any field k

Question: how the two 1-Gorenstein algebras $k\mathcal{C}$ and $H(C, D, \Omega)$ are related?

Theorem: For each (C, D, Ω)

a free EI category $\mathcal{C}(C, D, \Omega)$

depending on char k

(C', D', Ω')

GLS

$H(C', D', \Omega')$

$\mathcal{C}(C,D,\Omega)$

(of Cartan type)

$H(C',D',\Omega')$

Then $\exists$ an iso. of algebras

$$k\mathcal{C}(C,D,\Omega) \simeq H(C',D',\Omega').$$

RMK: Assume char $k = 0$ or coprime to entries of D.

Then $(C', \underset{\underset{I_{n'}}{\|}}{D'}, \Omega')$ is a finite acyclic quiver Γ (with an admissible auto. σ)

and $H(C', D', \Omega') = k\Gamma$ ← path alg of Γ

$$\therefore \quad k\mathcal{C}(C,D,\Omega) \simeq k\Gamma \quad \cdots\cdots (*)$$

Recall $(C,D) \xleftrightarrow{\text{unfolding}} (|\Gamma|, \sigma)$ Steinberg 1967

(cf. Lustig 1993)

$\Longrightarrow$ $(*)$ is an algebric "enrichment" of the unfolding !

§2. The GLS algebra associatiated to (C, D, Ω)

$C = (c_{ij})_{n\times n} \in M_n(\mathbb{Z})$ a Cartan matrix

$$\begin{cases} c_{ii} = 2 \\ c_{ij} \leq 0 & i \neq j \\ c_{ij} < 0 \Longleftrightarrow c_{ji} < 0 \end{cases}$$

$D = \operatorname{diag}(c_1, c_2, \cdots c_n), \quad c_i \in \mathbb{Z}_+$

s.t. DC is symmetric

symmetrizer of C

an acyclic orientation $\Omega \subseteq \{1,2,\cdots n\} \times \{1,2,\cdots,n\}$

- $\{(i,j),(j,i)\} \cap \Omega \neq \phi \iff c_{ij} < 0$
- each sequence $(i_1,i_2),(i_2,i_3),\cdots,(i_t,i_{t+1})$ with each $(i_s,i_{s+1}) \in \Omega$, then $\boxed{i_1 \neq i_{t+1}}$

$Q = Q(C,\Omega)$ a finite quiver

$Q_0 = \{1,2,\cdots,n\}$

$Q_1 = \left\{ \alpha_{ij}^{(g)} : j \longrightarrow i \;\middle|\; \begin{array}{l} (i,j) \in \Omega \\ 1 \le g \le \underline{\gcd(c_{ij},c_{ji})} \end{array} \right\}$

$\cup \; \{ \varepsilon_i : i \to i \mid i \in Q_0 \}$

loop!

Q° = the acyclic quiver obtained from Q by removing all ε_i's.

Def (GLS 2017)

$H = H(C,D,\Omega)$

$= kQ(C,\Omega) \big/ \; \varepsilon_i^{c_i},$ nilpotency relation

$\forall (i,j) \in \Omega, \quad 1 \le g \le \gcd(c_{ij},c_{ji})$

$\varepsilon_i^{\frac{c_i}{\gcd(c_i,c_j)}} \alpha_{ij}^{(g)} - \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{\gcd(c_i,c_j)}}$ $\square$

commutativity relations.

Fact ① If $D = I_n \Rightarrow C$ symmetric

$\Rightarrow H = kQ^\circ$ is hereditary $(\varepsilon_i = 0)$

If $D = cI_n$,

$H = kQ^\circ \otimes k[\varepsilon]/(\varepsilon^c)$ $\left[\begin{array}{l} \varepsilon_i^c = 0 \\ \varepsilon_i \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j \end{array}\right]$

② H is 1-Gorenstein

③ Set $H_i = k[\varepsilon_i]/(\varepsilon_i^{c_i})$ $\underline{i \in Q_0}$

$\quad$ " $\quad k[\varepsilon_i]/$

$$H_{ij} = k[\varepsilon_{ij}] \big/ \left(\varepsilon_{ij}^{\gcd(c_i,c_j)}\right) \qquad (i,j)\in\Omega$$

$$\begin{tikzcd} & H_i \\ H_{ij} \arrow[ur, hook] \arrow[dr, hook] & \\ & H_j \end{tikzcd} \qquad \begin{tikzcd} & \varepsilon_i^{\frac{c_i}{\gcd(c_i,c_j)}} \\ \varepsilon_{ij} \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & \varepsilon_j^{\frac{c_j}{\gcd(c_i,c_j)}} \end{tikzcd}$$

$${}_iH_j^{(g)} = H_i \otimes_{H_{ij}} H_j, \quad \forall\ 1 \le g \le \gcd(c_i,c_j) \qquad \text{an } H_i\text{-}H_j\text{-bimodule}$$

$$B = \prod_{i\in Q_0} H_i, \qquad {}_BW_B = \bigoplus_{(i,j)\in\Omega} \bigoplus_{g=1}^{\gcd(c_j,c_i)} {}_iH_j^{(g)}$$

[GLS]

$$\Rightarrow \quad H \simeq T_B(W) = B \oplus W \oplus W\otimes_B W \oplus \cdots$$

§3. Finite EI quivers

Notation: G, H, K groups

a G-H-biset ${}_GX_H$

an H-K-biset ${}_HY_K$, the <u>biset product</u>

$$X\times_H Y = X\times Y \big/ \sim \qquad (xh, y)\sim(x, hy)$$

$$\therefore\ {}_G(X\times_H Y)_K$$

<u>Def</u> (Li 2011) a finite EI quiver (Γ, U)

① $\Gamma = (\Gamma_0, \Gamma_1; s, t)$ a finite acyclic quiver

② $U = (U(i), U(\alpha))_{i\in\Gamma_0,\ \alpha\in\Gamma_1}$ an <u>assignment</u>

i.e., each $U(i)$ a finite group

each $U(i)$ a finite group

each $U(\alpha)$ a finite $U(t\alpha)$-$U(s\alpha)$-biset

$\alpha: s\alpha \longrightarrow t\alpha$

RMK: a finite acyclic quiver = a finite EI quiver with trivial assignment.

($U(i)$ = trivial, $U(\alpha)$ trivial)

Given a finite EI quiver (Γ, U), we define for each path

$$p = \alpha_n \cdots \alpha_2\alpha_1 \quad \text{in } \Gamma$$

$$U(p) = U(\alpha_n) \times_{U(t\alpha_{n-1})} U(\alpha_{n-1}) \times \cdots \times U(\alpha_2) \times_{U(t\alpha_1)} U(\alpha_1)$$

Identify $U(e_i) = U(i)$

trivial path

Assume $s(p) = t(q)$.

$$U(p) \times_{U(tq)} U(q) \xrightarrow{\sim} U(pq)$$

Def (Li 2011) Each EI quiver (Γ, U) defines a finite EI category $\mathcal{C} = \mathcal{C}(\Gamma, U)$ as follows:

$\mathrm{Obj}\,\mathcal{C} = \Gamma_0$

disjoint union

$$\mathrm{Hom}_{\mathcal{C}}(i,j) = \bigsqcup_{\substack{p \text{ a path} \\ \text{from } i \text{ to } j}} U(p)$$

$\square$

Note $\mathrm{Aut}_{\mathcal{C}}(i) = U(i)$

Def (Li 2011) a finite EI category $\mathcal{C}$ is free if it is equivalent to $\mathcal{C}(\Gamma, U)$ for some (Γ, U).

EI quiver

$\square$

Fact: $\mathcal{C} = \mathcal{C}(\Gamma, U)$

Fact: $\mathcal{C} = \mathcal{C}(\Gamma, U)$

(1) $A = \prod_{i \in \Gamma_0} k U(i)$, $\quad {}_A V_A = \bigoplus_{\alpha \in \Gamma_1} k U(\alpha)$

$U(t\alpha) \quad U(\alpha) \quad U(s\alpha)$

Then $k\mathcal{C} \simeq T_A(V) = A \oplus V \oplus V \otimes_A V \oplus \cdots$

(2) (Wang 2016) Assume for each $\alpha \in \Gamma_1$,

${}_{kU(t\alpha)} kU(\alpha)_{kU(s\alpha)}$ is projective on each side.

Then $k\mathcal{C}$ is 1-Gorenstein.

$\| \; k\mathcal{C}(\Gamma, U)$

§4. Finite EI categories of Cartan type

(C, D, Ω) a Cartan triple

$D = (c_1, c_2, \ldots, c_n)$

$Q = Q(C, \Omega)$, $\quad Q^0 = Q \setminus \{\varepsilon_i \mid i \in Q_0\}$

Q^0: acyclic; ε_i: loops at i; $Q_0 = \{1, 2, \ldots, n\}$

define a finite EI quiver

(Q^0, X)

s.t. $X(i) = \langle \eta_i \mid \eta_i^{c_i} = 1 \rangle$ $\quad \forall 1 \leq i \leq n$ cyclic group of order c_i

$\forall (i,j) \in \Omega$, $\quad G_{ij} = \langle \eta_{ij} \mid \eta_{ij}^{\gcd(c_i, c_j)} = 1 \rangle \to$ smaller cyclic group

$$\begin{tikzcd} & X(i) \\ G_{ij} \arrow[ur, hook] \arrow[dr, hook] & \\ & X(j) \end{tikzcd} \qquad \begin{tikzcd} & \eta_i^{\frac{c_i}{\gcd(c_i,c_j)}} \\ \eta_{ij} \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & \eta_j^{\frac{c_j}{\gcd(c_i,c_j)}} \end{tikzcd}$$

$\forall 1 \leq g \leq \gcd(c_{ij}, c_{ji})$

$$X(\alpha_{ij}^{(g)}) = {}_{X(i)}\big(X(i) \times_{G_{ij}} X(j)\big)_{X(j)}$$

as an $X(i)$-$X(j)$-biset.

Def: $\mathcal{C} = \boxed{\mathcal{C}(C, D, \Omega)}$
$= \mathcal{C}(Q^\circ, X)$ the free EI category given by (Q°, X),

a <u>finite EI category of Cartan type</u> $\square$

Theorem: Assume k has <u>enough roots of unity</u> ($\forall a \geq 1$ $t^a - 1 \in k[t]$ splits)

Then $\exists$ (C', D', Ω'), depending on char k, s.t.

$$k\mathcal{C}(C, D, \Omega) \simeq H(C', D', \Omega').$$

category alg GLS-alg.

§4.1 The construction of (C', D', Ω') from (C, D, Ω) (similar to [Lusztig's book 1993])

<u>Case char $k = p > 0$</u>: $D = \mathrm{diag}(c_1, c_2, \ldots, c_n)$

$c_i = p^{r_i} \cdot d_i$ $r_i \geq 0$, $p \nmid d_i$

$$M = \dot{\bigcup_{1 \leq i \leq n}} \{(i, l_i) \mid 0 \leq l_i < d_i\} \qquad |M| = \sum_{i=1}^{n} d_i \geq n$$

$\forall 1 \leq i, j \leq n$

$$\Sigma_{ij} = \left\{ (l_i, l_j) \;\middle|\; \begin{array}{l} 0 \leq l_i < d_i, \\ 0 \leq l_j < d_j, \end{array} \; l_i p^{r_i} \equiv l_j p^{r_j} \mod \gcd(d_i, d_j) \right\}$$

Set $C' \in M_m(\mathbb{Z})$, $m = |M|$

$$c'_{(i,l_i),(j,l_j)} = \begin{cases} -\gcd(c_j, c_{ji}) \cdot p^{r_j - \min\{r_i, r_j\}} & \text{if } (l_i, l_j) \in \Sigma_{ij} \\ 0 & \text{otherwise} \end{cases}$$

$D'_{m \times m} = \mathrm{diag}(p^{r_i})$ the (i, l_i)-th entry is p^{r_i}

$\Omega' = \{((i, l_i), (j, l_j)) \mid (i,j) \in \Omega\}$

$$\Omega' = \left\{ ((i,l_i),(j,l_j)) \;\middle|\; \begin{array}{l} (i,j)\in\Omega \\ \boxed{(l_i,l_j)\in\Sigma_{ij}} \end{array} \right\}$$

$$\Rightarrow (C', D', \Omega')$$

Case char $k = 0$:

$$M = \bigcup_{1\le i\le n} \{(i,l_i) \mid 0 \le l_i < c_i\}$$

$$C'_{(i,l_i),(j,l_j)} = \begin{cases} -\gcd(c_{ij}, c_{ji}) & \text{if } (l_i,l_j)\in\Sigma_{ij} \\ 0 & \text{otherwise} \end{cases}$$

$$D' = I_m, \qquad C' \text{ is symmetric}$$

Example:

$$C = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \qquad D = \mathrm{diag}(2,1)$$

$$\Omega = \{(2,1)\}$$

$$\ell(C,D,\Omega) = \begin{tikzcd} 1 \arrow[loop left, "\eta_1"] \arrow[r] & 2 \end{tikzcd}$$

$\eta_2 = Id_2$ $\qquad \eta_1^2 = Id_2$

$G_{21} = \{*\}$

$$X(2) \times_{G_{21}} X(1) = X(2) \times X(1) \text{ (has 2 elements)}$$

$\boxed{\text{char } k = 2}$

$$(C', D', \Omega') = (C, D, \Omega)$$

$$\boxed{k\ell(C,D,\Omega) \simeq H(C,D,\Omega)}$$

$\boxed{\text{char } k \neq 2}$

$$M = \{\underbrace{(1,0), (1,1)}_{d_i = c_i = 2}, (2,0)\}, \qquad D' = I_3$$

C' symmetric $\downarrow$ $|\Gamma|$

$$\begin{tikzcd} (1,0) \arrow[dr] & \\ & (2,0) \\ (1,1) \arrow[ur] & \end{tikzcd} = \Gamma$$

$$k\ell(C,D,\Omega) \simeq k\Gamma' \overset{\text{red}}{=} H(C',D',\Omega')$$

$$k\Gamma' = H(C',D',\Omega')$$

§4.2 The proof of Theorem

$$k\ell(C,D,\Omega) = T_A(V) \qquad A = \prod kX(i)$$

a product of group alg.s

$$H(C',D',\Omega') = T_B(W) \qquad B = \prod_{(i,l_i)\in M} H_{(i,l_i)}$$

a product of truncated polynomials.

$$\begin{array}{ccc} A & \simeq & B \\ \cup\!\!| & & \cup\!\!| \\ kX(i) & & \prod_{0\le l_i<d_i} H_{(i,l_i)} \\ \| & & \| \\ k\langle\eta_i\rangle/(\eta_i^{c_i}-1) & \xrightarrow{\sim} & \prod_{0\le l_i<d_i} k[\varepsilon_{(i,l_i)}]/(\varepsilon_{(i,l_i)}^{p^{r_i}}) \end{array}$$

$$\boxed{\eta_i^{c_i}-1 = (\eta_i^{d_i}-1)^{p^{r_i}}}$$

[Recall $c_i = p^{r_i}\cdot d_i$]

Moreover, ${}_AV_A \xrightarrow{\sim} {}_BW_B$, iso. as A-A-bimodules

a little bit involved!!

§4.3 An application (work in progress)

char $k = p > 0$

$G = \langle\sigma\rangle$ cyclic p-group

$G \curvearrowright$ a finite acyclic quiver Q

$\mathcal{P}_Q \rtimes G$ skew group category

$$k(\mathcal{P}_Q \rtimes G) \simeq kQ\# G$$

$\to$ skew group algebra

Theorem: $\exists$ a Morita equivalence

$$kQ\#G \simeq H(C,D,\Omega)$$

where $(|Q|,\sigma) \overset{\text{unfolding}}{\longleftrightarrow} (C,D)$

Moreover, $\forall\ M \in kQ\text{-ind}$,

$$M\#G \in H(C,D,\Omega)\text{-ind}. \quad (k=\bar{k})$$

as module over $kQ\#G$

<u>RMK</u>: The proof is via

$$\mathcal{P}_Q \rtimes G \simeq \mathcal{C}(C,D,\Omega)$$

$$\Downarrow$$

Morita equivalence

$$k(\mathcal{P}_Q \rtimes G) \simeq k\mathcal{C}(C,D,\Omega)$$

$kQ\#G = k(\mathcal{P}_Q \rtimes G)$

CW 2.5.0.
$k\mathcal{C}(C,D,\Omega) \simeq H(C,D,\Omega)$

<u>In Dynkin case</u>:

τ-locally free

$$\begin{array}{ccc}
kQ\text{-ind} & \overset{-\#G}{\dashrightarrow} & H(C,D,\Omega)\text{-}(\tau\text{-lf})\text{ind} \\
M & \longmapsto & M\#G \\
{\scriptstyle \text{dim}}\ \big\downarrow {\scriptstyle 1:1} & \circlearrowleft & {\scriptstyle 1:1}\ \big\downarrow\ {\scriptstyle \text{rank}} \\
\Phi^+(Q) & \xrightarrow{\text{folding}\ 1967} & \Phi^+(C) \\
\text{simple} & \longmapsto & \text{simples}
\end{array}$$

Gabriel 1972 GLS 2017

[Tanisaki 1980, Hubery 2004 ...]

$\therefore$ The functor $-\#G$ "categorifies" the folding!

<u>RMK</u>: "folding" is surjective, so any indec. τ-locally free H-module is of the form $M\#G$, $M \in kQ\text{-ind}$.