Ex 1. Let A be a K-algebra.

1. Show that $Z(A) \cong \mathrm{End}_{A\otimes_K A^{\mathrm{op}}}(A)$ as rings.
2. Suppose $A = B \times B'$ for some K-algebras B, B' (in this case, call B and B' the direct factors of A). Show that the identity 1_B is a central idempotent of A, i.e. idempotent in $Z(A)$.
3. Show that B is a direct factor of A if, and only if, B is a direct summand of A as $A \otimes_K A^{\mathrm{op}}$-module.
4. Show that B is a direct factor of KG if, and only if, B is a direct summand of $K(G \times G)$.

Ex 2. For $V, W \in KG\,\mathsf{mod}$, show that there are the following isomorphisms.

1. $(V \otimes_K W)^* \cong V^* \otimes_K W^*$ as KG-modules.
2. $V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$ as KG-modules.

Ex 3. Recall that $\mathrm{End}_K(V)$ is a ring where multiplication is composition.

1. Find the multiplication on $V^* \otimes_K V$ that makes $\mathrm{End}_K(V) \cong V^* \otimes_K V$ an isomorphism of rings.
2. Suppose $V \in KG\,\mathsf{mod}$. Show that triv_G is a direct summand of $V^* \otimes_K V$ when $\dim_K M$ is invertible in K. (Hint: fix a basis of V and consider also the dual basis.)

Ex 4. Suppose $H \le G$ is a subgroup. For $g \in G$, let ${}^gH := gHg^{-1} = \{{}^gh := ghg^{-1} \mid h \in H\}$, and define for each KH-module W a $K({}^gH)$-module ${}^gW := \{{}^gw \mid w \in W\}$ (${}^gw$ is just a formal symbol) with ${}^gh \cdot {}^gw := {}^g(hw)$. Show that

1. $\mathrm{Ind}_{{}^gH}^G({}^gW) \cong \mathrm{Ind}_H^G(W)$ as KG-module;
2. ${}^gV \cong V$ as KG-module for all $V \in KG\,\mathsf{mod}$.

Ex 5. Let $H \le G$, $V \in KG\,\mathsf{mod}$ and $W \in KH\,\mathsf{mod}$. Show that

1. $V \otimes_K \mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes_K W)$;
2. $\mathrm{Ind}_H^G(W^*) \cong (\mathrm{Ind}_H^G(W))^*$;
3. use (2) to give an alternative proof of permutation module being self-dual.

Ex 6. Consider an integer $n \ge 1$ and an integer $r \le n/2$. Let Ω_r be the set of r-subsets (=subsets of size r) of $\{1, 2, \ldots, n\}$.

1. Find (and prove) a subgroup $H \le \mathfrak{S}_n$ such that $K\Omega_r \cong \mathrm{Ind}_H^{\mathfrak{S}_n} \mathrm{triv}_H$.
2. Let π_r be the character of $K\Omega_r$. Find $\pi_r(1)$. (Note: $\chi_{K\Omega}(g) = |\{\omega \in \Omega \mid g\omega = \omega\}$ for any G-set Ω.)

Deadline: 22nd November, 2022