

TOPICS IN MATHEMATICAL SCIENCE V

AUTUMN 2023

FROM QUIVER TO QUASI-HEREDITARY ALGEBRAS

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Last update: October 27, 2023

Convention

Throughout the course, \mathbb{k} will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. Let R be a ring. A **right R -module** M is an abelian group $(M, +)$ equipped with a (linear) **R -action on the right of M** $\cdot : M \times R \rightarrow M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have

- $m \cdot 1 = m$,
- $(m + n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r + s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a **left R -module** is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A -module and ${}_A M$ for left A -module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. R is naturally a left, and a right, R -module. Both are **free R -module** of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.

In general, a free R -module F is one where there is a basis $\{x_i\}_{i \in I}$ such that for all $x \in F$, $x = \sum_{i \in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n .

Convention. All modules are right modules unless otherwise specified.

Definition 1.3. Suppose R is a commutative ring. A ring A is called an **R -algebra** if there is a (unital) ring homomorphism $\theta : R \rightarrow A$ with image $f(R)$ being in the **center** $Z(A) := \{z \in A \mid za = az \forall a \in A\}$ of A . In such a case, A is an R -module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.

An (unital) **R -algebra homomorphism** $f : A \rightarrow A'$ is a (unital) ring homomorphism f that **intertwines R -action**, i.e. $f(ar) = f(a)r$.

The **dimension** of a \mathbb{k} -algebra A is the dimension of A as a \mathbb{k} -vector space; we say that A is **finite-dimensional** if $\dim_{\mathbb{k}} A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. Every ring is a \mathbb{Z} -algebra.

The matrix ring $M_n(R)$ given by n -by- n matrices with entries in R is an R -algebra.

We will only really work with \mathbb{k} -algebras, where \mathbb{k} is a field. But it worth reminding there are many interesting R -algebras for different R , such as group algebra. Recall that the [characteristic](#) of R , denoted by $\text{char } R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R -module with basis G , i.e. every $a \in A$ can be written as the formal R -linear combination $\sum_{g \in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R -linearly) to a ring multiplication on $R[G]$, making A an R -algebra.

Example 1.6. Recall that the [direct product](#) of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A \times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R -algebras, then $A \times B$ is also an R -algebra.

Definition 1.7. A map $f : M \rightarrow N$ between right R -modules M, N is a [homomorphism](#) if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R -action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\text{Hom}_R(M, N)$ the set of all R -module homomorphisms from M to N . We also write $\text{End}_R(M) := \text{Hom}_R(M, M)$.

Lemma 1.8. $\text{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \text{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\text{Hom}_R(M, N)$ is an R -module, namely, for a homomorphism $f : M \rightarrow N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.

Definition 1.9. $\text{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M . We call this the [endomorphism ring](#) of M .

Lemma 1.10. If A is an R -algebra over a commutative ring R , then any right A -module is also an R -module, and $\text{Hom}_A(M, N)$ is also an R -module (hence, $\text{End}_R(M)$ is an R -algebra).

Example 1.11. $A \cong \text{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R -algebras if A is an R -algebra).

Exercise 1.12. Recall that R^{op} is the opposite ring of R , whose underlying set is the same as that of R with multiplication $(a \cdot^{\text{op}} b) := b \cdot a$. A [representation](#) of R is a ring homomorphism

$$\rho : R^{\text{op}} \rightarrow \text{End}_{\mathbb{Z}}(M), \quad r \mapsto \rho_r,$$

for some abelian group $(M, +)$. A homomorphism $f : \rho_M \rightarrow \rho_N$ of representations $\rho_M : R^{\text{op}} \rightarrow \text{End}_{\mathbb{Z}}(M), \rho_N : R^{\text{op}} \rightarrow \text{End}_{\mathbb{Z}}(N)$ given by an abelian group homomorphism $f : M \rightarrow N$ that intertwines R -action, i.e. $\rho_N(r) \circ f = f \circ \rho_M(r)$ for all $r \in R$.

Explain why a representation of R is equivalent to a right R -module; and why homomorphisms correspond.

2 Indecomposable modules and Krull-Schmidt property

Recall that an R -module M is *finitely generated* if there exists a surjective homomorphism $R^n \rightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x \in X} x r_x$ for some $r_x \in R$.

Notation. We write $\text{mod } A$ for the collection of all finitely generated right A -modules.

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. Let M be a R -module and N_1, \dots, N_r be submodules. We say that M is the *direct sum* $N_1 \oplus \dots \oplus N_r$ of the N_i 's if $M = N_1 + \dots + N_r$ and $N_j \cap (N_1 + \dots + N_j + \dots + N_r) = 0$. Equivalently, every $m \in M$ can be written *uniquely* as $n_1 + n_2 + \dots + n_r$ with $n_i \in N_i$ for all i . In such a case, we write $M \cong N_1 \oplus \dots \oplus N_r$. Each N_i is called a *direct summand* of M .

M is called *indecomposable* if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an *indecomposable decomposition* (or just *decomposition* for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be *unique* if for any other decomposition $M = \bigoplus_{j=1}^n N_j$, we have $n = m$ and the N_j 's are permutation of the M_i 's.

Convention. We write (n_1, \dots, n_r) instead of $n_1 + \dots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \dots \oplus N_r$.

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. Suppose R_R is indecomposable as an R -module. Then the free module $R \oplus R \oplus \dots \oplus R$ with R copies of R is a decomposition of R^n .

Example 2.3. Consider the matrix ring $A := \text{Mat}_n(\mathbb{k})$ over a field \mathbb{k} . Let V be the 'row space', i.e. $V = \{(v_j)_{1 \leq j \leq n} \mid v_j \in \mathbb{k}\}$ where $X \in \text{Mat}_n(\mathbb{k})$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A -submodule of V other than 0 or V itself. Hence, V is an indecomposable A -module. In particular, the n different ways of embedding a row into an n -by- n -matrix yields an A -module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A -module A_A .

The above example shows indecomposability by showing that V is a *simple* A -module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. Let $A = \mathbb{k}[x]/(x^k)$ the *truncated polynomial ring* for some $k \geq 2$. This is an algebra generated by $(1_A \text{ and } x)$, and an A -module is just a \mathbb{k} -vector space V equipped with a linear transformation $\rho_x \in \text{End}_{\mathbb{k}}(V)$ (representing the action of x) such that $\rho_x^k = 0$.

Consider a 2-dimensional space $V = \mathbb{k}\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2 . By definition $(av_1 + bv_2)x = (a + b)v_2$, and so any submodules must contain $\mathbb{k}v_2$, i.e. v_2 spans a unique non-zero submodule; a contradiction. Hence, V must be indecomposable.

A natural question is to ask when a decomposition of modules, if it exists, is unique up to permuting the direct summands.

Definition 2.5. We say that an indecomposable decomposition $M = \bigoplus_{i=1}^m M_i$ is *unique* if any other indecomposable decomposition $M = \bigoplus_{j=1}^n N_j$ implies that $m = n$ and there is a permutation σ such

that $M_i \cong N_{\sigma(i)}$ for all $1 \leq i \leq m$. $\text{mod } A$ is said to be **Krull-Schmidt** if every finitely generated A -module M admits a unique indecomposable decomposition.

Theorem 2.6. *For a finite-dimensional algebra A , $\text{mod } A$ is Krull-Schmidt.*

Remark 2.7. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Proposition 2.8. *There is a canonical R -module isomorphism*

$$\begin{array}{ccc} \text{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) & \xrightarrow{\cong} & \bigoplus_{i,j} \text{Hom}_A(M_j, N_i) \\ f \mapsto & & (\pi_i f \iota_j)_{i,j} \end{array}$$

where $\iota_j : N_j \rightarrow \bigoplus_j N_j$ is the canonical inclusion for all j and $\pi_i : \bigoplus_i M_i \rightarrow M_i$ is the canonical projection for all i .

One can think of the right-hand space above as the space of m -by- n matrix with entries in each corresponding Hom-space.

3 Extra: Krull-Schmidt theorem

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\text{id}_M \in \text{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent.

Lemma 3.1. *A non-zero A -module M is indecomposable if, and only if, the endomorphism algebra $\text{End}_A(M)$ does not contain any idempotents except 0 and id_M .*

Proof \Leftarrow : Suppose $M = U \oplus V$. Then we have

$$\begin{aligned} & \text{a projection map } \pi_W : M \twoheadrightarrow W, \\ & \text{and an inclusion map } \iota_W : W \hookrightarrow M, \end{aligned}$$

for $W \in \{U, V\}$. Both of these are clearly A -module homomorphisms. Now $e_W := \iota_W \pi_W$ is an endomorphism of M with $e_V = \text{id}_M - e_U$. Since any $m \in M$ can be written as $u + v$ for $u \in U$ and $v \in V$, we have

$$e_V^2(m) = e_V^2(u + v) = e_V^2(v) = v = e_V(m);$$

and likewise for e_U , so we have idempotents different from 0 and id_M when both U and V are non-zero.

\Rightarrow : Suppose that M is indecomposable, and $e \in \text{End}_A(M)$ is an idempotent. Note that

$$(\text{id}_M - e)^2 = \text{id}_M - e \cdot \text{id}_M - \text{id}_M \cdot e + e^2 = \text{id}_M - 2e + e = \text{id}_M - e$$

is also an idempotent and $\text{id}_M = e + (\text{id}_M - e)$. So we have $M = e(M) + (\text{id}_M - e)(M)$. We want to show that $M = e(M) \oplus (\text{id}_M - e)(M)$, i.e. $e(M) \cap (\text{id}_M - e)(M) = 0$. Indeed, $x \in e(M) \cap (\text{id}_M - e)(M)$ means that we have $e(m) = x = (\text{id}_M - e)(m')$ for some $m, m' \in M$, and so

$$x = e(m) = e^2(m) = e((\text{id}_M - e)(m')) = (e(\text{id}_M - e))(m') = (e - e^2)(m') = 0(m') = 0,$$

as required.

Since M is indecomposable, one of $e(M)$ or $(\text{id}_M - e)(M)$ is zero. In the former case, we get $e = 0$; whereas the latter case yields $\text{id}_M = e$; as required. \square

The following is one of the main reasons why we like to consider finite-dimensional (or finite generated) modules over finite-dimensional \mathbb{k} -algebras.

Lemma 3.2 (Fitting's lemma (special version)). *Let M be a finite-dimensional A -module of a finite-dimensional \mathbb{k} -algebra, and $f \in \text{End}_A(M)$. Then there exists $n \geq 1$ such that $M \cong \text{Ker}(f^n) \oplus \text{Im}(f^n)$.*

Remark 3.3. The general version for rings requires M to be artinian and noetherian (i.e. ascending and descending chains of submodules stabilises).

We omit the proof to save time. The point is really just take n large enough so that the chains of submodules given by $(\text{Ker}(f^k))_k$ and $(\text{Im}(f^k))_k$ stabilises.

Corollary 3.4. *Let M be a non-zero finite-dimensional A -module. Then M is indecomposable if, and only if, every homomorphism $f \in \text{End}_A(M)$ is either an isomorphism or is nilpotent.*

Proof By Fitting's lemma, for any $f \in \text{End}_A(M)$, we have $M \cong \text{Ker}(f^n) \oplus \text{Im}(f^n)$ for some $n \geq 1$. So indecomposability means that one of these direct summands is zero. If $\text{Ker}(f^n) = 0$, then f^n is an isomorphism and so is f . If $\text{Im}(f^n) = 0$, then $f^n = 0$ and so f is nilpotent.

Conversely, consider an idempotent endomorphism $e \in \text{End}_A(M)$. The assumption says that e is either an isomorphism or nilpotent.

If e is an isomorphism, then we have $\text{Im}(e) = M$, which means that for every $m \in M$, there is some $m' \in M$ with $e(m) = e^2(m') = e(m') = m$, i.e. $e = \text{id}_M$.

If e is nilpotent, then $e^n = 0$ for some $n \geq 1$, but $e = e^2 = e^3 = \dots = e^n$, and so $e = 0$.

Hence, an idempotent endomorphism of M is either 0 or id_M , which means that M is indecomposable by Lemma 3.1. \square

Definition 3.5. A ring R is *local* if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.

Remark 3.6. When R is non-commutative, the ‘non-invertible elements’ are the ones that do not admit right inverses.

Lemma 3.7. Let A be a finite-dimensional algebra and M be a finite-dimensional A -module. Then the following hold.

- (1) The following are equivalent.
 - A is local (i.e. has a unique maximal right ideal).
 - Non-invertible elements of A form a two-sided ideal.
 - For any $a \in A$, one of a or $1 - a$ is invertible.
 - 0 and 1_A are the only idempotents of A .
 - $A/J(A) \cong \mathbb{k}$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.
- (2) M is indecomposable $\Leftrightarrow \text{End}_A(M)$ is local.

We omit the proof to save time.

Example 3.8. Consider the upper triangular 2-by-2 matrix ring

$$A = \begin{pmatrix} \mathbb{k} & \mathbb{k} \\ 0 & \mathbb{k} \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq 2} \mid \begin{array}{l} a_{i,j} \in \mathbb{k} \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Let $M = \{(x, y) \in \mathbb{k}^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \text{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \mathbb{k}$. Then being an A -module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f \left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \mathbb{k}$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\text{End}_A(M) \cong \mathbb{k}$ which is clearly a local algebra. Hence, M is indecomposable.

Theorem 3.9 (Krull-Schmidt). Suppose $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition of M . If $\text{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.

Remark 3.10. Some people refer to this result as Krull-Remak-Schmidt theorem.

For proof, interested reader can see lecture notes from last year.

4 Simple modules, Schur's lemma

Definition 4.1. Let M be an R -module.

- (1) M is **simple** if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.
- (2) M is **semisimple** if it is a direct sum of simples.

Remark 4.2. In the language of representations, simple modules are called **irreducible** representations, and semisimple modules are called **completely reducible** representations.

Remark 4.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 4.4. Consider the matrix ring $A := \text{Mat}_n(\mathbb{k})$ over a field \mathbb{k} . Then the row-space representation V is an n -dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.

Example 4.5. The **ring of dual numbers** is $A := \mathbb{k}[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = Ax$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A , and the quotient module can be described by $\mathbb{k}v$ where $v = 1 + (x)$. If A is semisimple, then the 1-dimensional space $\mathbb{k}v$ is isomorphic to a submodule of A . Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \mathbb{k}$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\mathbb{k}v \cong (x)$, a contradiction.

Lemma 4.6. S is a simple A -module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).

Proof \Rightarrow : $mA \subset S$ is a submodule and contains a non-zero element m , so by simplicity of S we must have $mA = S$.

\Leftarrow : Suppose that there is a non-zero submodule $L \subset S$. For a non-zero element $m \in L$, the assumption says that we have $mA \subset L \subset S = mA$, and so $L = S$. \square

Let us see how one can find a simple module.

Definition 4.7. Let M be an A -module and take any $m \in M$. The **annihilator** of m (in A) is the set $\text{Ann}_A(m) := \{a \in A \mid ma = 0\}$.

Note that $\text{Ann}_A(m)$ is a right ideal of A - hence, a right A -module.

Lemma 4.8. For a simple A -module S and any non-zero $m \in S$, we have $S \cong A/\text{Ann}_A(m)$ as A -module. In particular, if A is finite-dimensional, then every simple A -module is also finite-dimensional.

Proof Since $S = mA$, the element m defines a surjective A -module homomorphism $f : A_A \rightarrow S$ given by $a \mapsto ma$. On the other hand, we have $\text{Ker}(f) = \text{Ann}_A(m)$, and so $A/\text{Ann}_A(m) \cong S$. \square

Suppose I is a two-sided ideal of A . Then we have a quotient algebra $B := A/I$. For any B -module M , we have a canonical A -module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the **restriction of M along the algebra homomorphism $A \rightarrow A/I$** .

Lemma 4.9. Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \text{mod } B$ is simple, then S is also simple as A -module.

Proof This follows from the easy observation that any a B -submodule of S_B is also a A -submodule of S_A under restriction. \square

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 4.10 (Schur's lemma). *Suppose S, T are simple A -modules, then*

$$\mathrm{Hom}_A(S, T) = \begin{cases} \text{a division ring,} & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.11. Note that if A is an R -algebra, then the division ring appearing is also an R -algebra (since it is the endomorphism ring of an A -module). In particular, if R is an algebraically closed field $\mathbb{k} = \bar{\mathbb{k}}$, then any division \mathbb{k} -algebra is just \mathbb{k} itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S, T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T , simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S , so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. \square

Example 4.12. In Example 3.8, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x, y) \mid x, y \in \mathbb{k}^2\}$ given by ‘row vectors’. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0, y) \mid y \in \mathbb{k}^2\}$.

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A -action given by

$$w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0, y)$ for some $y \in \mathbb{k}$ and has to satisfy

$$(0, ya) = (0, y)a = f(wa) = f\left(w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = f(w) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0, y)c = (0, yc)$$

for any $a, b, c \in \mathbb{k}$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 4.13. *Consider a semisimple A -module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i . Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$ for some i .*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \rightarrow S_i$, and by Schur's lemma, this is an element of D . Now we have a ring homomorphism

$$\mathrm{End}_A(M) \rightarrow \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \leq i, j \leq r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1 \leq i, j \leq r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \xrightarrow{\pi_j} S_j \xrightarrow{a_{i,j}} S_i \xrightarrow{\iota_i} M$, which yields the required surjection. \square

Example 4.14. For a tautological example, take $A = \mathbb{k}$ to be just a field. Then we have a 1-dimensional simple A -module $S = \mathbb{k}$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\mathbb{k})) = \mathrm{Mat}_n(\mathbb{k})$. Note that now we have an n -dimensional simple $\mathrm{Mat}_n(\mathbb{k})$ -module (given by the row vectors).

5 Quiver and path algebra

Definition 5.1. A (finite) **quiver** is a datum $Q = (Q_0, Q_1, s, t : Q_1 \rightarrow Q_0)$ for finite sets Q_0, Q_1 . The elements of Q_0 are called **vertices** and those of Q_1 are called **arrows**. The **source** (resp. **target**) of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 5.2. Let Q be a quiver.

- A **trivial path** on Q is a “stationary walk at i ”, denoted by e_i for some $i \in Q_0$.
- A **path** of Q is either a trivial path or a word $\alpha_1 \alpha_2 \cdots \alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right*.

Definition 5.3. The **path algebra** $\mathbb{k}Q$ of a quiver Q is the \mathbb{k} -algebra whose underlying vector space is given by $\bigoplus_{p: \text{paths of } Q} \mathbb{k}p$, with multiplication given by path concatenation. That is $x \in \mathbb{k}Q$ is a formal linear combinations of paths on Q .

Note that $e_i e_j = \delta_{i,j} e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an **idempotent** of the path algebra $\mathbb{k}Q$. Moreover, we have an idempotent decomposition

$$1_{\mathbb{k}Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\mathbb{k}Q$.

Example 5.4. Consider the **one-looped quiver**, a.k.a. **Jordan quiver**,

$$Q = \left(\begin{array}{c} \alpha \\ \bullet \end{array} \right)$$

Then $\mathbb{k}Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\mathbb{k}Q \cong \mathbb{k}[x]$.

An **oriented cycle** is a path of the form $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_r \rightarrow v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is **acyclic**.

Proposition 5.5. $\mathbb{k}Q$ is finite-dimensional if, and only if, Q is finite acyclic.

Proof If there is an oriented cycle c , then $c^k \in \mathbb{k}Q$ for all $k \geq 0$, and so $\mathbb{k}Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q . \square

Example 5.6. Consider the linearly oriented $\vec{\mathbb{A}}_n$ -quiver

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\mathbb{k}Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j \alpha_{j+1} \cdots \alpha_k$.

Consider the upper triangular n -by- n matrix ring

$$\begin{pmatrix} \mathbb{k} & \mathbb{k} & \cdots & \mathbb{k} \\ 0 & \mathbb{k} & \cdots & \mathbb{k} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \mathbb{k} \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \mid \begin{array}{l} a_{i,j} \in \mathbb{k} \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\mathbb{k}Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \leq j < k \leq n$.

From now on, we will focus in the following setting.

Assumption 5.7. (1) *Quivers are always finite.*

(2) *Modules (and representations) are finitely generated (which is equivalent to finite-dimensional when the algebra is so).*

6 Duality

For a quiver Q , the *opposite quiver* Q^{op} has the same set of vertices with the reverse direction of arrows, i.e. $Q_0^{\text{op}} = Q_0, Q_1^{\text{op}} = Q_1, s_{Q^{\text{op}}} = t_Q$, and $t_{Q^{\text{op}}} = s_Q$.

Exercise 6.1. *Show that there is a canonical isomorphism $(\mathbb{k}Q)^{\text{op}} \cong \mathbb{k}(Q^{\text{op}})$.*

Let M be a finite-dimensional A -module. Then we have a dual space

$$D(M) := M^* := \text{Hom}_{\mathbb{k}}(M, \mathbb{k}),$$

which has a natural A^{op} -module structure, namely, $(a \cdot f)(m) := f(ma)$ for any $a \in A, f \in M^*, m \in M$. Moreover, for an A -module homomorphism $\theta : M \rightarrow N$, we have also an A^{op} -module homomorphism $\theta^* : N^* \rightarrow M^*$ with $\theta^*(f)(m) = f(\theta(m))$.

We note as a fact that D preserves indecomposability of (finite-dimensional) modules. This can be seen using the fact that $\text{Hom}_A(M, N) \cong \text{Hom}_{A^{\text{op}}}(DN, DM)$ and can be upgraded to an algebra isomorphism for the case when $N = M$; then uses characterisation of indecomposable module by local endomorphism ring.

Example 6.2. *The left A -module ${}_A A$ yields a right A -module structure on $D(A)$. More generally, suppose we have a left ideal Ae of A for some element $e \in A$, then $D(Ae)$ is a right ideal of A .*

Remark 6.3. There is another natural duality, which we will not use, between $\text{mod } A$ and $\text{mod } A^{\text{op}}$ given by sending M to $\text{Hom}_A(M, A)$. In general, this duality is different from the \mathbb{k} -linear dual unless A is a so-called *symmetric algebra*; interested reader can read lecture notes from last year.

7 Representations of quiver

Definition 7.1. *A \mathbb{k} -linear representation of Q is a datum $(\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$ where M_i is a \mathbb{k} -vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$ is \mathbb{k} -linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional if $\dim_{\mathbb{k}} M_i < \infty$ for all $i \in Q_0$.

Notation. *For a representation M of Q , we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

It is easy to notice that every representation of Q is equivalent to a $\mathbb{k}Q$ -module, namely,

$$\text{representation } (\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1}) \leftrightarrow \text{s.t. } \sum_{p: \text{path}} \lambda_p p \text{ acts as } \sum_p \lambda_p M_p.$$

Example 7.2 (Simple). *For $x \in Q_0$, denote by S_x (or $S(x)$) the representation given by putting a 1-dimensional space on x , zero on all other vertices, and zero on all arrows. This corresponds to a 1-dimensional $\mathbb{k}Q$ -module and so we call it the *simple at x* .*

Note: at this stage, it is not clear if these are all the simple $\mathbb{k}Q$ -modules (up to isomorphism) yet.

Example 7.3 (Projective). For $x \in Q_0$, denote by P_x (or $P(x)$) the representation given by $(\{M_y\}_{y \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$, where

$$M_y := \bigoplus_{\substack{p: \text{path with} \\ s(p)=x, \\ t(p)=y}} \mathbb{k}p, \quad \text{and} \quad (M_\alpha : M_y \rightarrow M_z) := \sum_{p\alpha=q} (M_y \twoheadrightarrow \mathbb{k}p \xrightarrow{\text{id}} \mathbb{k}q \hookrightarrow M_z).$$

This is called the **projective at x** . This corresponds to the right ideal $e_x \mathbb{k}Q$ of $\mathbb{k}Q$.

Example 7.4 (Injective). Dual to the projective module construction, for $x \in Q_0$, denote by I_x (or $I(x)$) the representation given by $(\{M_y\}_{y \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$, where

$$M_y := \bigoplus_{\substack{p: \text{path with} \\ s(p)=y, \\ t(p)=x}} \mathbb{k}p, \quad \text{and} \quad (M_\alpha : M_y \rightarrow M_z) := \sum_{p=\alpha q} (M_y \twoheadrightarrow \mathbb{k}p \xrightarrow{\text{id}} \mathbb{k}q \hookrightarrow M_z).$$

This is called the **injective at x** . This corresponds to the dual of the left ideal generated by e_x , i.e. $D(\mathbb{k}Qe_x)$.

Example 7.5. The representation of $Q = \vec{A}_n$ given by

$$U_{i,j} := 0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathbb{k} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{k} \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

with a copy of \mathbb{k} on vertices $i, i+1, \dots, j$ is the uniserial $\mathbb{k}Q$ -module corresponding to the column space (under the isomorphism of $\mathbb{k}Q$ with the lower triangular matrix ring) with non-zero entries in the k -th row for $i \leq k \leq j$.

Example 7.6. Let Q be the Jordan quiver with unique arrow α . Then a representation of Q is nothing but an n -dimensional vector space equipped with a linear endomorphism, equivalently, an n -by- n matrix.

Definition 7.7. A **homomorphism** $f : M \rightarrow N$ of (\mathbb{k} -linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \rightarrow N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & N_i \\ M_\alpha \downarrow & & \downarrow N_\alpha \\ M_j & \xrightarrow{f_j} & N_j \end{array}$$

for all arrows $\alpha : i \rightarrow j$ in Q .

A homomorphism $f = (f_i)_{i \in Q_0} : M \rightarrow N$ of quiver representations is **injective**, resp. **surjective**, resp. an **isomorphism**, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 7.8. Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n -by- n matrix M_α . By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α . If we assume \mathbb{k} is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α . That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \dots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m -by- m Jordan block with eigenvalue $\lambda \in \mathbb{k}$.

Proposition 7.9. There is an isomorphism between the category of representations of Q and $\text{mod } \mathbb{k}Q$, where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i \in Q_0} M_i$ with $\mathbb{k}Q$ -action given by (linear combinations of compositions of) M_α 's, and isomorphism classes of Q -representations correspond to isomorphism classes of $\mathbb{k}Q$ -modules.

8 Idempotents

Recall that an *idempotent* of an algebra A is an element x with $x^2 = x$.

The right A -modules of the form eA and $D(Ae)$ for an idempotent $e \in A$ are of central importance in representation theory and in homological algebra.

Lemma 8.1. *The following hold for any idempotent $e \in A$.*

- (1) (Yoneda's lemma) $\text{Hom}_A(eA, M) \cong Me$ as a \mathbb{k} -vector space for all $M \in A \text{ mod}$.
- (2) There is an isomorphism of rings $\text{End}_A(eA) \cong eAe$.

Proof For (1), check that $\text{Hom}_A(eA, M) \ni f \mapsto f(e) = f(1)e \in Me$ defines a \mathbb{k} -linear map with inverse $me \mapsto (ea \mapsto mea)$. (2) follows from (1) by putting $M = eA$ with straightforward check of correspondence of multiplication on both sides. \square

Remark 8.2. Under the isomorphism $A \cong \text{End}_A(A)$, an idempotent e of A corresponds to the ‘project to direct summand $P = eA$ endomorphism’, i.e. $A \twoheadrightarrow P \hookrightarrow A$. This is compatible with Yoneda lemma (think about this!) which says that there is a vector space isomorphism $fAe \cong \text{Hom}_A(eA, fA)$ for any idempotents e, f .

Lemma 8.3. *For idempotents $e, f \in A$, we have $eA \cong fA$ as right A -module if and only if $f = ueu^{-1}$ for some unit $u \in A^\times$.*

Proof \Leftarrow : By Yoneda lemma, an isomorphism $\phi \in \text{Hom}_A(fA, eA)$ corresponds to an element in $x \in eAf \subset A$; likewise an isomorphism $\psi \in \text{Hom}_A((1-f)A, (1-e)A)$ corresponds to $y \in (1-e)A(1-f) \subset A$. Let $x' \in fAe$ and $y' \in (1-f)A(1-e)$ be the elements corresponding to ϕ^{-1} and ψ^{-1} respectively. Since $\phi^{-1}\phi = \text{id}_{eA}$ corresponds to $e \in eAe$, we have

$$x'x = f, xx' = e, y'y = 1 - f, yy' = 1 - e.$$

Take $u := x + y$ and $v := x' + y'$. Then we have $vu = f + (1 - f) = 1$ and $uv = e + (1 - e) = 1$. Therefore, u, v are units such that $uf = x = eu$, i.e. $e = ufu^{-1}$ as required.

\Rightarrow : The required isomorphism $fA \rightarrow eA$ is given by $fa \mapsto eua$. \square

Given an idempotent $e = e^2 \in A$ in an algebra A , then eA and $(1 - e)A$ are both right ideal of A . Since $e(1 - e) = 0 = (1 - e)e$, we have $eA \cap (1 - e)A = 0$, which means that $A \cong eA \oplus (1 - e)A$ as right A -module. In particular, in the setting of the above lemma, we have that $eA \cong fA$ and $(1 - e)A \cong (1 - f)A$ by Krull-Schmidt property.

Definition 8.4. Two idempotents e, f are *orthogonal* if $ef = 0 = fe$. An idempotent e is *primitive* if $e \neq f + f'$ for some orthogonal (pair of) idempotents f, f' .

It follows from the definition of primitivity that

eA and $D(Ae)$ are indecomposable A -modules for a primitive idempotent e .

Example 8.5. The trivial paths e_x for $x \in Q_0$ is (by design) a primitive idempotent of the path algebra $\mathbb{k}Q$ (where Q is finite but not necessarily acyclic), and $1 = \sum_{x \in Q_0} e_x$ is an orthogonal decomposition of primitive idempotents. Hence, we have a decomposition

$$\mathbb{k}Q \cong \bigoplus_{x \in Q_0} e_x \mathbb{k}Q = \bigoplus_{x \in Q_0} P_x \text{ and } D(\mathbb{k}Q) \cong \bigoplus_{x \in Q_0} D(\mathbb{k}Q e_x) \cong \bigoplus_{x \in Q_0} I_x.$$

9 Composition series, Jordan-Hölder Theorem

Definition 9.1. Let A be a \mathbb{k} -algebra and $M \in A \text{ mod}$. A *composition series* of M is a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \leq i \leq \ell$. The number ℓ here is the *length* of the composition series. The module M_i/M_{i-1} for each $1 \leq i \leq \ell$ are called the *composition factors* of the series.

Theorem 9.2 (Jordan-Hölder Theorem). Any two composition series have the same length and their composition factors are the same up to permutations.

We omit the proof. The strategy is basically by induction on the length of series.

Remark 9.3. Jordan-Hölder theorem holds as long as a module, regardless of what kind of algebra, has a (finite) composition series; this condition is actually equivalent to saying that it is noetherian and artinian.

Remark 9.4. The Jordan-Hölder theorem may not hold if one relaxes the form of composition factors from simple modules to something else. There are a few active research themes, including one related to quasi-hereditary algebras, that are stemmed from this.

Lemma 9.5. Let M be a finite-dimensional left A -module. Then M has a composition series.

Proof Induction on $\dim_{\mathbb{k}} M$, at each step choose a maximal submodule (i.e. a submodule whose quotient is simple). \square

Example 9.6. Let $A = \mathbb{k}\vec{A}_n$. Then the module $U_{i,j}$ has a composition series

$$0 \subset U_{j,j} \subset U_{j-1,j} \subset \cdots \subset U_{i+1,j} \subset U_{i,j}$$

with composition factors $S_k = U_{k,j}/U_{k+1,j}$ for $i \leq k \leq j$. We note that this composition series is actually unique - such kind of modules are called *uniserial*.

Lemma 9.7. If $M \in \text{mod } A$ and $N \subset M$ is a submodule, then there is a composition series $(M_i)_{0 \leq i \leq \ell}$ so that $N = M_k$ for some $0 \leq k \leq \ell$.

Proof N has a composition series, say, of length k , so we take that as the first k terms of the required composition series of M . On the other hand, M/N also has a composition series, and since every submodule of M/N is of the form L/N (for a submodule U of M/N , take $L := \{m \in M \mid m+N \in U\}$; it is routine to check that this is an inverse operation as quotienting N on the submodules of M that contains N), a composition series of M/N is of the form $(L_i/N)_{0 \leq i \leq r}$. Now take $M_{k+i} = L_i$. \square

Proposition 9.8. Suppose A is a \mathbb{k} -algebra such that A_A has a composition series. Then there are only finitely many simple A -modules up to isomorphisms, and they all appear in the form A/I for some A -submodule I of A .

Note that while this does not require A to be finite-dimensional, it requires A_A to be of finite length (equivalently, noetherian and artinian).

Proof The final clause of the claim is just restating Lemma 4.8: any simple S is given by $A/\text{Ann}_A(m)$ for any non-zero $m \in S$. Now fix such an S and $I := \text{Ann}_A(m)$. Since A has a composition series, I also have one by Lemma 9.7 so that the series ends with $I \subset A$. Since this is possible for any simple S , it follows from Jordan-Hölder theorem that all simple modules other than S must appear as composition factors of I .

Since composition series is a finite chain, there must be finitely many composition factors - hence, the simple modules of A must be finite. \square

10 Semisimplicity and Artin-Wedderburn theorem

In order to obtain all (isomorphism classes of) simple A -modules - or equivalently maximal right A ideal (i.e. maximal submodules of A_A) - for a finite-dimensional \mathbb{k} -algebra A , we will use the following.

Definition 10.1. Let A be a \mathbb{k} -algebra and $M \in \text{mod } A$.

- (1) The **(Jacobson) radical** $\text{rad}(A)$ (sometimes also written as $J(A)$) of A is the intersection of all maximal right ideals (i.e. maximal A -submodules) of A .
- (2) A is **semisimple** if $\text{rad}(A) = 0$.

Example 10.2. For $A = \mathbb{k}Q$ of a finite quiver Q and $x \in Q_0$. The projective P_x at x contains a submodule spanned by all paths starting from x with length at least 1. This is a maximal submodule of P_x since the cokernel of the natural embedding to P_x is a one-dimensional module spanned by the coset of e_x - in particular, this simple module is isomorphic to S_x . Thus, we have $\text{rad}(A) = \mathbb{k}Q_{\geq 1}$ the submodule of A_A spanned by all paths of length at least 1.

Example 10.3. This example shows that we really need composition series on A_A for things to be well-behaved. Let $A = \mathbb{k}[x]$. Each irreducible polynomial f generates a maximal ideal $(f) \subset \mathbb{k}[x]$ and so $\text{rad}(A) \subset \bigcap_{f: \text{irred.}} (f)$. Note that there are infinitely many irreducible polynomials in $\mathbb{k}[x]$.

We claim that $\text{rad}(A) = 0$. If, on the contrary, there is some non-zero g in this intersection of ideals, then all irreducible polynomials are factors of g ; this is a contradiction as g can only have finite degree, i.e. finitely many irreducible factors.

Proposition 10.4. Suppose A_A has a composition series. Then the following holds for the Jacobson radical $\text{rad}(A)$.

- $\text{rad}(A)$ is the intersection of finitely many maximal right ideals.
- $\text{rad}(A)$ is the intersection of all two-sided ideals $\text{Ann}_A(S) := \{a \in A \mid ma = 0 \forall m \in S\}$, in other words

$$\text{rad}(A) = \{a \in A \mid Sa = 0 \text{ for all simple } S\}.$$

- $\text{rad}(A)$ is a two-sided ideal of A .
- $\text{rad}(A)^\ell = 0$ for ℓ at most the length of A_A .
- $(A/\text{rad}(A))_{A/\text{rad}(A)}$ is a semisimple (as a module).
- A_A is a semisimple (as a module) if, and only if, $\text{rad}(A) = 0$ (i.e. A semisimple as an algebra).

Proof omitted. We note that all of these claims do make use of the Jordan-Hölder theorem.

Example 10.5. (1) Direct product of two semisimple algebras is semisimple.

- (2) $A = \text{Mat}_n(D)$ with D a division \mathbb{k} -algebra is a semisimple \mathbb{k} -algebra. We have decomposition $A_A \cong V^{\oplus n}$ into n copies of n -dimensional simple module

$$V = \{(v_i)_{1 \leq i \leq n} \mid v_i \in D \forall i\}.$$

- (3) $A := \mathbb{k}[x]/(x^n)$ is not semisimple for any $n \geq 2$ as it has a non-trivial (unique) maximal ideal $\text{rad}(A) = (x)$.

Theorem 10.6 (Artin-Wedderburn theorem). Let A be a finite-dimensional \mathbb{k} -algebra and let r be the number of isoclasses of simple A -modules, say, with representatives S_1, \dots, S_r . Let $D_i := \text{End}_A(S_i)$ be the division \mathbb{k} -algebra given by endomorphism of the simple module S_i . Then there is an isomorphism of \mathbb{k} -algebras

$$A/\text{rad}(A) \cong \text{Mat}_{n_1}(D_1) \times \cdots \times \text{Mat}_{n_r}(D_r).$$

As before, if we work over algebraically closed field $\mathbb{k} = \bar{\mathbb{k}}$, then all the D_i 's are just \mathbb{k} .

Proof Let $B := A/\text{rad}(A)$. By definition of $\text{rad}(A)$, the A -module $A/\text{rad}(A)$ is semisimple, and any A -submodule M of $A/\text{rad}(A)$ satisfies $M\text{rad}(A) = 0$. Hence, $M = M/M\text{rad}(A)$ is naturally a B -module and $\text{End}_B(M) \cong \text{End}_A(M)$ (even as algebras!).

By Lemma 8.1, we have $B \cong \text{End}_B(B)$. Since B is semisimple, the B_B is a semisimple B -module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B -modules. Hence, it follows from Schur's lemma and its consequence (Lemma 4.10 and Lemma 4.13) that

$$B \cong \text{End}_B(B) \cong \text{Mat}_{n_1}(D_1) \times \cdots \times \text{Mat}_{n_r}(D_r),$$

where $D_i := \text{End}_B(S_i)$ for all $1 \leq i \leq r$. This completes the proof. \square

Corollary 10.7. *For any finite-dimensional \mathbb{k} -algebra A , let $\text{Sim}(A)$ be the set of isomorphism-class representatives of simple A -modules. Then there is a one-to-one correspondence*

$$\begin{array}{ccc} \text{Sim}(A) & \xleftarrow{1:1} & \text{Sim}(A/\text{rad}(A)) \\ S & \longmapsto & \bar{S} := S/S\text{rad}(A) \\ & & (= S \text{ as underlying vector space}) \\ \text{res}T & \longleftarrow & T \end{array}$$

where $\text{res}T$ is the restriction of T along $A \rightarrow A/\text{rad}(A)$.

Definition 10.8. The *radical* of an A -module M is $\text{rad}(M) := M\text{rad}(A)$. In general, take $\text{rad}^0(M) := M$ and denote by $\text{rad}^{k+1}(M) := \text{rad}(\text{rad}^k(M)) = \text{rad}^k(M)\text{rad}(A)$ for all $k \geq 0$.

Successively taking the radical yields a series:

$$0 \subset \text{rad}^\ell(M) \subset \cdots \subset \text{rad}(M) \subset M$$

This is called the *radical series*. The quotient $M/\text{rad}(M)$ is called the *top* of M , and is denoted by $\text{top}(M)$.

Proposition 10.9. *The following hold for $M \in \text{mod } A$.*

- (1) $\text{rad}(M)$ is the intersection of all maximal submodules of M .
- (2) $\text{top}(M) := M/\text{rad}(M)$ is the maximal semisimple quotient of M .
- (3) $\text{rad}(M \oplus N) = \text{rad}(M) \oplus \text{rad}(N)$.
- (4) (Nakayama's Lemma, special case) For a submodule $N \subset M$, $(N + \text{rad}(M) = M) \Rightarrow N = M$.

Proof omitted; this follows the same kind of arguments as in the case for $\text{rad}(A)$.

There is a construction dual to $\text{rad}(M)$.

Definition 10.10. The *socle* of an A -module M is $\text{soc}(M)$, which is defined as the maximal semisimple submodule of M . More generally, take $\text{soc}^0(M) = 0$ and for $k \geq 0$, let $\text{soc}^{k+1}(M)$ to be the submodule of M generated by the lift of $\text{soc}(M/\text{soc}^k(M)) \subset M/\text{soc}^k(M)$. This yields a series

$$0 \subset \text{soc}(M) \subset \text{soc}^2(M) \subset \cdots \subset \text{soc}^\ell(M) = M$$

called the *socle series* of M .

Example 10.11. Consider a path algebra $\mathbb{k}Q$ of a finite acyclic (for simplicity) quiver Q , and $x \in Q_0$. The indecomposable injective $I_x = D(\mathbb{k}Qe_x)$ has a simple socle isomorphic to S_x . Essentially this can be seen by a dual argument in showing $\text{top}(P_x) \cong S_x$.

Lemma 10.12. *For $M \in \text{mod } A$, the socle series and radical series has the same length, and this length is called the [Loewy length](#) of M .*

Note that the semisimple subquotients in (between the layers of) the socle series and the radical series of a module may not coincide.

Example 10.13. *Let Q be the quiver $1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and consider the projective P_2 which has the form*

$$\mathbb{k} \xleftarrow{1} \mathbb{k} \xrightarrow{1} \mathbb{k} \xrightarrow{1} \mathbb{k}$$

Then we have radical series

$$0 \subset S_4 = \mathbb{k}\beta\gamma \xrightarrow{S_1 \oplus S_3} \text{rad}(P_2) = \mathbb{k}\alpha + \mathbb{k}\beta + \mathbb{k}\beta\gamma \xrightarrow{S_2} P_2$$

and socle series

$$0 \subset S_2 \oplus S_4 = \mathbb{k}\alpha + \mathbb{k}\beta\gamma \xrightarrow{S_3} \text{rad}(P_2) \subset P_2.$$

11 Bounded path algebra

For general quiver, we lose finite-dimensionality, and so many nice things we explained do not hold any more. To retain finite-dimensionality, we need to consider nice quotients of path algebras.

Definition 11.1. An ideal $I \triangleleft \mathbb{k}Q$ is **admissible** if $(\mathbb{k}Q_1)^k \subset I \subset (\mathbb{k}Q_1)^2$ for some $k \geq 2$, i.e. I is generated by linear combinations of paths of finite length at least 2. The pair (Q, I) is sometimes called **bounded quiver**. A **bounded path algebra** or **quiver algebra** (with relations) is an algebra of the form $\mathbb{k}Q/I$ for some quiver Q and admissible ideal I .

Remark 11.2. Admissibility ensures there is no redundant arrows (which appears if there is a relation like, for example, $\alpha - \beta\gamma \in I$ for some $\alpha \neq \beta, \gamma \in Q_1$) and there is enough vertices (trivial paths may not be primitive if there is a loop x at a vertex with relation $x^2 - x \in I$).

Lemma 11.3. A bounded path algebra is finite-dimensional.

Proof There exists a surjective algebra homomorphism $\mathbb{k}Q/(\mathbb{k}Q_1)^k \twoheadrightarrow \mathbb{k}Q/I$; the former is finite-dimensional. \square

Example 11.4. Let Q be the Jordan quiver with unique arrow α . Let I be the ideal of $\mathbb{k}Q$ generated by α^k for some $k \geq 2$. Then I is an admissible ideal and $\mathbb{k}Q/I \cong \mathbb{k}[x]/(x^k)$ is a **truncated polynomial ring**.

Definition 11.5. A **representation** M of a bounded quiver (Q, I) is a representation $M = (M_i, M_\alpha)_{i, \alpha}$ of Q such that $M_a = 0$ for all $a \in I$; here $M_a := \sum_p \lambda_p M_p$ for $a = \sum_p \lambda_p p$ written as a linear combinations of paths p .

A **homomorphism** $f : M \rightarrow N$ of representations of (Q, I) is a collection of linear maps $f_i : M_i \rightarrow N_i$ that intertwines arrows' action.

As before, representations are really just synonyms of modules.

Lemma 11.6. A representation of a bounded quiver (Q, I) is equivalent to a $\mathbb{k}Q/I$ -module, and homomorphisms between representations are equivalent to those between $\mathbb{k}Q/I$ -modules.

We have seen that it is easy to write down the indecomposable decomposition of the free $\mathbb{k}Q$ -module $\mathbb{k}Q_{\mathbb{k}Q}$, we would like such nice thing to carry over to bounded path algebras.

Theorem 11.7. (Idempotent lifting) If I is a nilpotent ideal of A (i.e. $I^n = 0$ for some $n \geq 1$) and $\bar{e} = \bar{e}^2 \in A/I$, then there is a **lift** $e = e^2 \in A$ of \bar{e} , i.e. $\bar{e} = e + I$.

Proof omitted.

Corollary 11.8. Let I be an nilpotent ideal in A . Suppose that

$$1_{A/I} = f_1 + \cdots + f_n$$

for $f_i \in A/I$ are primitive orthogonal idempotents. Then we have

$$1_A = e_1 + \cdots + e_n$$

where each $e_i \in A$ is a primitive orthogonal idempotent that lifts f_i .

Corollary 11.9. For a bounded path algebra $A = \mathbb{k}Q/I$, the primitive idempotent $e_x \in \mathbb{k}Q$ lifts to one in A . In particular, we have decompositions

$$A_A \cong \bigoplus_{x \in Q_0} P_x, \text{ and } (DA)_A \cong \bigoplus_{x \in Q_0} I_x.$$

Proof I is an admissible ideal, so is nilpotent. Apply the previous corollary. \square

Primitive idempotent decomposition, say, $1 = \sum_{i=1}^n e_i$, allows us to write an algebra A in matrix form $(e_i A e_j)_{1 \leq i, j \leq n}$, where the ‘row spaces’ form the indecomposable direct summands $e_i A$ and the dual of the ‘column space’ form the indecomposable direct summands $D(A e_i)$. It could be a helpful mental exercise to think about the meaning of $e A e \cong \text{End}_A(eA)$ from Yoneda lemma - this will be useful to think about the next result.

Notation. As in the case of path algebra, denote by S_x or $S(x)$ the simple $\mathbb{k}Q/I$ -module given by placing a one-dimensional vector space at vertex $x \in Q_0$ and zero everywhere else.

Similarly, denote by P_x or $P(x)$ the indecomposable $\mathbb{k}Q/I$ -module $e_x \mathbb{k}Q/I$. Likewise, by I_x or $I(x)$ the indecomposable $D((\mathbb{k}Q/I)e_x)$.

Proposition 11.10. *There is a decomposition of A -modules*

$$A_A = \bigoplus_{x \in Q_0} P_x, \text{ and } (DA)_A = \bigoplus_{x \in Q_0} I_x.$$

Moreover, $\{S_x \cong \text{top}(P_x) \cong \text{soc}(I_x) \mid x \in Q_0\}$ form the complete set of isoclasses representatives of simple A -modules.

Proof Each arrow $\alpha \in Q_1$ generates a maximal right ideal of A with quotient S_x for $x = s(\alpha)$. So we have $A/\text{rad}(A) \cong \mathbb{k}Q_0 = \prod_{x \in Q_0} \mathbb{k}$. As primitive orthogonal decomposition of the identity element of A lifts to that of the identity element of $A/\text{rad}(A)$ by Corollary 11.8, we have e_x primitive, and so P_x and I_x are indecomposable.

The simple A -modules (up to isomorphisms) correspond to those over the semisimple quotient algebra $A/\text{rad}(A)$ by Corollary 10.7. Hence, there are precisely $|Q_0|$ simple modules (up to isomorphism), given by the simple top of P_x , which is also isomorphic to the simple socle of I_x . \square

We give a brief justification of why quiver representations provide a good way to construct lots of algebras.

Theorem 11.11. *Suppose \mathbb{k} is algebraically closed. Then every finite-dimensional \mathbb{k} -algebra A is Morita equivalent to a bounded path algebra $\mathbb{k}Q/I$. More precisely, $\mathbb{k}Q/I$ is given by $\text{End}_A(\bigoplus_e eA)$ where e varies over the set of representative of equivalence classes of primitive idempotents of A .*

We do not explain here the precise meaning of Morita equivalent; it roughly translates to saying that understanding A -modules and homomorphisms between them is equivalently (but not necessarily ‘equal to’) to understanding modules and homomorphisms between a Morita equivalent bounded path algebra.

Example 11.12. *Let $A = \text{Mat}_n(\mathbb{k})$ be a matrix ring. Then the elementary matrix $e := E_{1,1}$ is a primitive idempotent and $eA \cong E_{j,j}A$ for all $1 \leq j \leq n$. So A is Morita equivalent to $\mathbb{k} \cong \mathbb{k}Q \cong \text{End}_A(eA)$ where Q is a one-vertex-no-arrow quiver.*

Module diagram

It is convenient to display the structure of a module via a more combinatorial form (a diagram) – if possible.¹ This is (as of today technology) a better way to display module structure – at least compare to composition series, or lattice diagram of the submodule lattice, or even, quiver representations, in some cases.

Definition 11.13. Let $M \in \text{mod } A$ be a finite-dimensional A -module for $A = \mathbb{k}Q/I$ a bounded path algebra. The **module diagram** is a (directed) graph with vertices labelled by composition factors of M (in particular, there are $\dim_{\mathbb{k}} Me_x$ many vertices labelled by x), and arrows labelled by those in Q_1 in such a way that $x \xrightarrow{a} y$ if for an arrow $a \in Q$ that sends (the lift of) an element in the composition factor at x to (the lift of) an element in the composition factor at y .

Module diagram drawn in this way is not invariant under isomorphism. A connected diagram may not even implies indecomposability in general (c.f. Homework 2). Nevertheless, when the algebras or modules are well-behaved, then these diagram provide a very efficient combinatorial way to perform a lot of (linear algebra) calculation.

It is customary to draw the the module diagram flowing from top to bottom; in particular, the top (semisimple quotient) of M is placed on the top of the diagram and the socle at the bottom. We may omit a connecting line if there is no ambiguity.

Example 11.14. The indecomposable $U_{i,j}$ of $\mathbb{k}\vec{A}_n$ is just a column of numbers labelled from i down to j . For a concrete example, the module diagram of $U_{4,6}$ is just $\frac{4}{5}$.

Example 11.15. Consider the following bounded quiver:

$$Q : 1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} 3, \quad I = (\alpha_1\alpha_2, \beta_1\beta_2, \beta_1\alpha_1 - \alpha_2\beta_2).$$

Then we have

$$P_1 = \frac{\mathbb{k}e_1}{\mathbb{k}\alpha_1} = \frac{1}{1}, \quad P_2 = \frac{\mathbb{k}e_1}{\mathbb{k}\alpha_2 \oplus \mathbb{k}\beta_1} = \frac{2}{2}, \quad P_3 = \frac{\mathbb{k}e_3}{\mathbb{k}\beta_2} = \frac{3}{3}$$

Let us consider the two-sided ideal Ae_1A . This is spanned by all paths ('up to I ') that passes through the vertex 1. As a right module, we can find its manifestation in the module diagram by picking everything below any appearance of the label 1 – in this case, it is all of P_1 and the $\frac{1}{2}$ part submodule of P_2 . In particular, the quotient algebra $(A/Ae_1A)_{A/Ae_1A}$ has module diagram:

$$P_2^{A/Ae_1A} = e_2A/e_2Ae_1A = P_2/P_2e_1A = \frac{2}{3}, \quad P_3^{A/Ae_1A} = P_3/P_3e_1A = P_3^A = \frac{3}{3}$$

The bounded quiver presentation of A/Ae_1A is given by

$$Q : 2 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 3, \quad I = (\alpha\beta).$$

On the other hand, for eAe with $e = e_2 + e_3$, the module diagram is given by removing all composition factors that are not S_2, S_3 , i.e.

$$e_2Ae = \frac{2}{2}, \quad e_3Ae = \frac{3}{3}$$

and the bounded quiver presentation of eAe is given by

$$Q : 2 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 3, \quad I = (\alpha\beta\alpha, \beta\alpha\beta).$$

¹There is no widely agreed name to these diagrams; for convenience, we just call them 'module diagram' in this notes.