You may assume all algebras are finite-dimensional over a field $\Bbbk$. You may attempt the exercises with the additional assumption of $\Bbbk$ being algebraically closed.

Throughout, unadorned tensor product over assumed to be taken over a field, i.e. $\otimes = \otimes_{\Bbbk}$.

Ex 1.

1. Let X, M be an A-module. The *reject* of X in M is the submodule
$$\mathrm{Rej}_X(M) := \bigcap_f \mathrm{Ker}(f) \subset M.$$
Show that $M/\mathrm{Tr}_X(M) \cong D\mathrm{Rej}_{DX}(DM)$ where $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ is the $\Bbbk$-linear duality functor.

2. Consider $A = \Bbbk Q/I$ with
$$Q = (\ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n\ ), \qquad I = (\alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i, \beta_{n-1}\alpha_{n-1})$$
and $\Lambda = \{1 \lhd 2 \lhd \cdots \lhd n\}$. Show that $\mathrm{gldim} A = 2n - 2$.

Ex 2. For a quasi-hereditary algebra $(A, (\Lambda, \unlhd))$, show the following.

1. Let $\mathcal{X}$ be a subset of $\{\Delta(\lambda) \mid \lambda \in \Lambda\}$. If $\mathrm{Ext}^1_A(\Delta(\lambda), N) = 0$ for all $\Delta(\lambda) \in \mathcal{X}$, then $\mathrm{Ext}^1_A(M, N) = 0$ for any $\mathcal{X}$-filtered module M.
Hint: Induction on Δ-length.

2. $\mathrm{Ext}^{>0}_A(\Delta(\lambda), \Delta(\mu)) = 0$ for all $\lambda \not\trianglelefteq \mu$.
Hint: Reverse induction on λ (i.e. starting from λ maximal) and consider $\mathrm{Hom}(-, \Delta(\mu))$.
Note: We already learnt that $\mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) = 0$ for all $\lambda \not\trianglelefteq \mu$.

Ex 3. For a quasi-hereditary algebra $(A, (\Lambda, \unlhd))$, show the following.

1. If X is Δ-filtered, then so is $\Omega(X)$, where $\Omega(X)$ is the kernel of the projective cover of X.

2. If $\mathrm{Ext}^1_A(M, N) = 0$ for all Δ-filtered module M, then $\mathrm{Ext}^{>0}_A(M, N) = 0$.
Hint: Consider dimension shifting $\mathrm{Ext}^k_A(X, Y) \cong \mathrm{Ext}^{k-1}_A(\Omega(X), Y)$ where $\Omega(X)$ is the kernel of the projective cover of X.

3. $\mathrm{Ext}^1_A(M, \nabla(\mu)) = 0$ for all $\mu \in \Lambda$ and all Δ-filtered module M.
Hint: Induction on Δ-length. (Or if you have done Exercise 2, you can quote from your solution from there.)

4. $\mathrm{Ext}^{>0}_A(M, \nabla(\mu)) = 0$ for all $\mu \in \Lambda$ and all Δ-filtered module M.

Ex 4. Consider the quiver algebra $A = \Bbbk Q/I$ given by

$$Q: \begin{tikzcd} & 1 \arrow[dr, "\alpha"] & \\ 3 \arrow[ur, "\gamma"] & & 2 \arrow[ll, "\beta"] \end{tikzcd} \quad , \quad I = (\gamma\alpha)$$

You can use the following information in the exercise: every indecomposable A-module M is uniserial of length at most 4, and $[M : S(i)] \leq 1$ for $i = 2, 3$ and $[M : S(1)] \leq 2$ with equality if and only if $M = P(1)$.

1. Write down all the standard and costandard modules of A.
2. Write down all indecomposable Δ-filtered modules.
3. Write down all indecomposable ∇-filtered modules.
4. There are 3 indecomposable modules. Show that we can label each of them by $T(i)$ so that the following are satisfied:
   - $[T(i) : S(i)] = 1$.
   - $\Delta(i)$ is a submodule of $T(i)$.
   - $\nabla(i)$ is a quotient of $T(i)$.
5. Write down the projective resolutions of each $T(i)$.
6. Show that $\mathrm{Ext}_A^{>0}(T(i), T(j)) = 0$ for any i, j.

Deadline: 2nd February, 2024
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