Ex 1.

(i) Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.

(ii) Consider the additive group of integers $G = (\mathbb{Z}, +)$. Let V be a fixed finite-dimensional $\mathbb{C}$-vector space. Show that every linear transformation $\phi \in \mathrm{GL}(V)$ defines a unique (but possibly isomorphic) $\mathbb{C}$-linear G-representation.

Ex 2. Fix any $n \geq 2$ and take $G = \mathfrak{S}_n$ the symmetric group of rank n.

(i) Let V be the 1-dimensional subspace spanned by $\sum_{g \in G} g \in KG$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.

(ii) Find a generator $v \in KG$ such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g \in G} g$ of the trivial representation.)

(iii) Show that $\mathrm{Hom}_{KG}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{KG}(\mathrm{sgn}, \mathrm{triv})$ when $\mathrm{char}\, K \neq 2$; otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

Ex 3.

(i) Let X, Y be two G-sets. Determine the condition(s) on a map $f : X \to Y$ so that f induces a homomorphism of permutation representations from π_X to π_Y. Do the same for isomorphism in place of homomorphism.

(ii) Consider $G = C_3 = \langle g \mid g^3 = 1 \langle$ action on three letters $X = \{x_1, x_2, x_3\}$ by cyclic permutation. Recall the representations $R^{(k)} : G \to \mathrm{GL}_n(\mathbb{C})$ given by $R^{(k)}_g = \omega^k$ with $\omega := \exp(2\pi i/3)$, with $k \in \mathbb{Z}/3\mathbb{Z}$. Determine (with explanation) $a, b, c \in \mathbb{Z}/3\mathbb{Z}$ so that $\mathbb{C}X \cong R^{(a)} \oplus R^{(b)} \oplus R^{(c)}$.

Ex 4.

(i) Show that $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space.

(ii) Show that the composition of homomorphisms between representations is also a homomorphism of representations.

(iii) Find an injective ring homomorphism $K \to Z(KG) := \{x \in KG \mid xy = yx \ \forall y \in KG\}$.

(iv) Show that $f : V \to W$ is a homomorphism of K-linear G-representations if, and only if, it is a homomorphism of left KG-modules.

Deadline: 27th October, 2024

Submission / Enquiry: E-mail to (replace at by @) aaron.kychan at gmail.com