Topics in Mathematical Science VIII

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Introduction to quiver representations and homological algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M,+)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r,s \in R$ and $m,n \in M$, we have*

- $m \cdot 1 = m$,
- $(m+n)\cdot r = m\cdot r + n\cdot r$,
- $m\cdot(r+s) = m\cdot r + m\cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $\theta(R)$ being in the center $Z(A) := \{z \in A \mid za = az\ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_\Bbbk A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. Most of the time, we will also assume $\Bbbk$ is algebraically closed for simplicity. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the *characteristic* of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the direct product of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Example 1.7. *Suppose that A is a $\Bbbk$-algebra and B is a $\Bbbk$-subspace of A containing 1_A and closed under multiplication. Then B is also a $\Bbbk$-algebra. We call such a B a subalgebra of A. For a concrete example, the space of diagonal matrices forms a subalgebra of $M_n(\Bbbk)$.*

Definition 1.8. *A map $f : M \to N$ between right R-modules M, N is a homomorphism if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\mathrm{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$.*

Lemma 1.9. *$\mathrm{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \mathrm{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\mathrm{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.10. *$\mathrm{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the endomorphism ring of M.*

Lemma 1.11. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\mathrm{Hom}_A(M, N)$ is also an R-module (hence, $\mathrm{End}_R(M)$ is an R-algebra).*

Example 1.12. *$A \cong \mathrm{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra). Note that if we work with left modules, then $A \cong \mathrm{End}_A({}_AA)^{\mathrm{op}}$, where $(-)^{\mathrm{op}}$ denotes the opposite ring given by the same underlying set with reverse direction of multiplication, i.e. $a \cdot_{\mathrm{op}} b := b \cdot a$.*

Recall that an R-module M is *finitely generated* if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x\in X} x r_x$ for some $r_x \in R$.

Notation. *We write $\mathsf{mod}\,A$ for the collection of all finitely generated right A-modules.*

2 Indecomposable modules and Krull-Schmidt property

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \ldots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + N_{\hat{j}} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable.

Convention. *We write $(n_1, \ldots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose that R_R is indecomposable as an R-module. If F is a free R-module of rank n, then $R^{\oplus n} := R \oplus R \oplus \cdots \oplus R$ (with n copies of R) is a decomposition of F.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1 \le j \le n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \ge 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By definition $(av_1 + bv_2)x = (a + b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules. If, on the contrary, V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. But v_2 must be contained in any submodule of V, hence, we have $v_2 \in U_1 \cap U_2$, i.e. $U_1 \cap U_2 \neq 0$ – a contradiction not decomposability.

Proposition 2.5. *There is a canonical R-module isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) & \xrightarrow{\cong} & \bigoplus_{i,j} \mathrm{Hom}_A(M_j, N_i) \\ f & \longmapsto & (\pi_i f \iota_j)_{i,j} \end{array}$$

where $\iota_j : N_j \to \bigoplus_j N_j$ is the canonical inclusion for all j and $\pi_i : \bigoplus_i M_i \to M_i$ is the canonical projection for all i.

One can think of the right-hand space above as the space of m-by-n matrix with entries in each corresponding Hom-space.

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent. From the previous proposition, we see that for a decomposition $M = N_1 \oplus N_2$, we have idempotents

$$e_i : M \xrightarrow{\pi_i} N_i \xrightarrow{\iota_i} M$$

for both $i = 1, 2$. Hence, being decomposable implies existence of multiple idempotents; this turns out characterise indecomposability completely.

Proposition 2.6. *Let A be a finite-dimensional algebra and M be a finite-dimensional non-zero A-module. Then the following hold.*

(1) *(Fitting's lemma) For any $f \in \mathrm{End}_A(M)$, there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

(2) *The following are equivalent.*
   - *M is indecomposable.*
   - *The endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*
   - *Every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*
   - *$\mathrm{End}_A(M)$ is local (see below).*

Remark 2.7. It is known that if M is only artinian or only noetherian, then Fitting's lemma (and hence part (2)) fails. Nevertheless, in general, the proposition still hold for M that is both artinian and noetherian.

Let us briefly recall various characterisation of local rings.

Definition 2.8. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 2.9. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit (right) inverses.

Lemma 2.10. *The following are equivalent for a finite-dimensional algebra A.*

- *A is local (i.e. has a unique maximal right ideal).*
- *Non-invertible elements of A form a two-sided ideal.*
- *For any $a \in A$, one of a or $1 - a$ is invertible.*
- *0 and 1_A are the only idempotents of A.*
- *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

Example 2.11. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq 2} \;\middle|\; \begin{matrix} a_{i,j} \in \Bbbk \;\forall i \leq j \\ a_{i,j} = 0 \;\forall i > j \end{matrix} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

A natural question is to ask when is a decomposition of modules, if it exists, unique up to permuting the direct summands.

Definition 2.12. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^{m} M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^{n} N_j$ implies that $m = n$ and there is a permutation σ such that $M_i \cong N_{\sigma(i)}$ for all $1 \le i \le m$. $\mathsf{mod}\, A$ is said to be Krull-Schmidt if every (finitely generated) A-module M admits a unique indecomposable decomposition.*

Theorem 2.13. *For a finite-dimensional algebra A, $\mathsf{mod}\, A$ is Krull-Schmidt.*

Remark 2.14. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Theorem 2.15 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \le i \le m$, then the decomposition of M is unique.*

Remark 2.16. Some people refer to this result as Krull-Remak-Schmidt theorem.

3 Simple modules, Schur's lemma

Definition 3.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 3.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 3.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 3.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 3.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 3.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Let us see how one can find a simple module.

Definition 3.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 3.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module. In particular, if A is finite-dimensional, then every simple A-module is also finite-dimensional.*

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 3.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. □

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 3.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S, T) = \begin{cases} \text{a division ring,} & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.