Topics in Mathematical Science VII
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Introduction to group representations

Aaron Chan

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Convention

Throughout the course, the symbols $K, \Bbbk, \mathbb{F}$ will always be a field. Unless otherwise stated, we assume (for simplicity) that

- all groups are finite;
- all vector spaces are finite-dimensional.

We compose maps from right to left.

We usually denote the identity element of a group G by 1 or 1_G or id_G.

1 Group action

Definition 1.1. *Let G be a group and X a set. We say that G acts on X, or X is a G-set, if there is a map $* : G \times X \to X$, with $gx := g * x := *(g, x)$ for all $g \in G$ and $x \in X$, such that*

$$1x = x, \qquad \text{and} \qquad g(hx) = (gh)x.$$

Thinking about this a little bit more, one can see that the action of G on X simply just permutes the elements of X – i.e. G is just some (sub)group of symmetries on X.

When $X = V$ is a vector-space, if we ask for G to only acts by permuting elements, then it could very well destroy the linearity – the best thing about linear algebra – and we lose all the toolkit from linear algebra. The remedy is to “linearise” the definition of action.

Definition 1.2. *For a vector space V, we say that G acts linearly on V if G acts on V and*

$$g(\lambda u + \mu v) = \lambda g(u) + \mu g(v)$$

for all $g \in G$, all $\lambda, \mu \in K$, and all $u, v \in V$.

Often in practice we just write

$$G \curvearrowright V$$

to denote the existence of linear G-action on V.

2 Linear representations

A linear g-action on V is just a linear transformation for any $g \in G$. So we can repackage the notion of linear G-action using the following.

Recall that the *general linear group* of a vector space V over K is the group of all invertible (K-)linear transformation from V to itself.

$$\mathrm{GL}(V) := \{\phi : V \to V \mid \phi \text{ invertible linear transformation}\}.$$

The group multiplication is just composition of linear transformations, and the identity element is just the identity map $\mathrm{id} : V \to V$.

More generally, one can consider $\mathrm{GL}(V)$ for some free R-module V of finite rank for some nice ring R – by nice, usually this would be at least an integral domain. We may look at some examples in the case when $R = \mathbb{Z}$ when we focus on symmetric group representations.

Now we can reformulate the notion of linear G-action as follows.

Definition 2.1. *Let G be any (not necessarily finite) group. A finite-dimensional (resp. n-dimensional) K-linear* representation *of G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action *of g on V.*

Usually, when the underlying field (or ring) is understood, we will drop the adjective 'K-linear' for representations.

Exercise 2.2. *Check that representation defines a linear G-action in the sense of Definition 1.2.*

While we assumed V is a vector space over a field K here, one can also consider more general setting of "R-linear representation" when V is an R-lattice (=free R-module of finite rank).

Example 2.3. (1) *The* trivial representation *of G is the 1-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}.$$

(2) *$G = \mathfrak{S}_n$ the symmetric group of rank n. The* sign representation *of $\mathfrak{S}_n$ is the 1-dimensional representation*

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is the parity (or sign) of the permutation σ.

(3) *Let X be a finite G-set (for any finite group G). Denote by KX the K-vector space with basis given by X. Then*

$$\pi_X : G \to \mathrm{GL}(KX), \qquad g \mapsto (x \mapsto gx)_{x \in X}$$

defines K-linear G-representation. Any G-representation of such a form is called a permutation representation.

Exercise 2.4. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.*

Exercise 2.5. *Consider the additive group of integers $G = (\mathbb{Z}, +)$. Let V be a fixed finite-dimensional $\mathbb{C}$-vector space. Show that every linear transformation $\phi \in \mathrm{GL}(V)$ defines a unique (not distinguished under isomorphism) $\mathbb{C}$-linear G-representation.*

Recall that for a ring R with identity 1, under addition the element 1 either has infinite or prime, say p, order. The *characteristic* of R, denoted by $\mathrm{char}\, R$, is 0 in the former case, or p in the latter.

In Example 2.3 (2), we can see that when $\mathrm{char}\, K = 2$, then sign representation is the same as trivial representation.

In general, changing characteristic drastically change the kind of representations that can appear.

- *Ordinary representation theory* studies K-linear representations over a field K with $\operatorname{char} K = 0$.
- *Modular representation theory* studies K-linear representations over a field K with $\operatorname{char} K = p > 0$ and $p | \#G$.
- *Integral representation theory* studies $\mathcal{O}$-linear representations over a (nice – such as discrete valuation ring) integral ring $\mathcal{O}$ (but sometimes including $\mathbb{Z}$) with $\operatorname{char} \mathcal{O} = 0$.

The case of K-linear representations with positive characteristic that does *not* divide the order of group is sometimes called "representations over good characteristics" but can also be regarded as a 'trivial' extension of ordinary representation theory – characteristic 0 and good characteristic cases are somewhat the same.

Most of this course will be about ordinary representation theory. We may touch on some integral and modular representation for the symmetric group later in the course.

3 Matrix representations

When V is n-dimensional K-vector space, then $\mathrm{GL}(V)$ is isomorphic to

$$\mathrm{GL}_n(K) := \{\text{invertible } n \times n\text{-matrices with entries in } K\}.$$

This isomorphism of course depends on a basis we pick for V.

Definition 3.1. *An n-dimensional matrix representation of a group G is a group homomorphism*

$$R : G \to \mathrm{GL}_n(K), \qquad g \mapsto R_g.$$

We say that the matrix R_g represents the action of g.

It is clear that given an n-dimensional matrix representation, one obtains an n-dimensional K-linear representation (with $V = K^n$), and vice versa (by choosing a basis for V and passes through $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$).

Example 3.2. *Consider $G = C_3 = \langle x \mid x^3 = 1 \rangle$ the cyclic group of order 3. Let us try to see what matrix representations of G look like in the case when $K = \mathbb{C}$.*

Suppose that $R_x \in \mathrm{GL}_n(\mathbb{C})$ is diagonal. Since $R_x^3 = R_{x^3} = R_1 = \mathrm{id}$, the diagonal entries are in $\{\omega^k := \exp(2\pi i k/3) \mid 0 \le k < 3\}$, and we can write $R_x = \mathrm{diag}(\omega^{k_1}, \ldots, \omega^{k_n})$ with any $k_i \in \{0, 1, 2\}$ for all $i = 1, \ldots, n$. Note that, in this case, R_x^2 will also be a diagonal matrix $\mathrm{diag}(\omega^{2k_1}, \ldots, \omega^{2k_n})$.

On the other hand, if R_x is not a diagonal matrix, since R_x is invertible and we work over $\mathbb{C}$, we can still find $P \in \mathrm{GL}_n(\mathbb{C})$ so that PR_xP^{-1} is diagonal. In other words, we have a commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\cong", "P"'] \arrow[d, "\mathrm{diag}(\omega^{ik_1},\ldots,\omega^{ik_n})"'] & \mathbb{C}^n \arrow[d, "R_x^i"] \\ \mathbb{C}^n \arrow[r, "\cong", "P"'] & \mathbb{C}^n, \end{tikzcd}$$

i.e. the two paths from top left to bottom right resulting the same map. This amounts to say that, up to a change of basis of $\mathbb{C}^n$, the non-diagonal case is "essentially the same" as the diagonal one.

4 Homomorphism

In mathematics, the word for "essentially the same" is (usually) isomorphism; for this, we need the weaker notion of homomorphism first.

Definition 4.1. *Let $\rho : G \to \mathrm{GL}(V)$ and $\theta : G \to \mathrm{GL}(W)$ be two K-linear representations of G. A homomorphism from V to W is a K-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$, i.e. $f\rho_g = \theta_g f$ for all $g \in G$.

An isomorphism from V to W is a homomorphism that is invertible, i.e. $\exists g$ s.t. $gf = \mathrm{id}_V$ and $fg = \mathrm{id}_W$.

Write $\mathrm{Hom}_{KG}(V, W)$ for the space of all homomorphisms from V to W.

Remark 4.2. Older text also calls a homomorphism (sometimes, only for isomorphism) $f : V \to W$ an *intertwiner*, or that f *intertwines* ρ, θ; we will try to avoid using this and stick to homomorphism. Older text may say that V, W are equivalent if there is an isomorphism between them. We will drop this redundant language and just say V and W are isomorphic.

Example 4.3. *Let us go back to the case when $G = C_3$ and take $n = 1$. We have three representations $R^{(i)}$ with $i = 1, 2, 3$ so that $R^{(i)}_x = \omega^i$. An isomorphism on $\mathbb{C}$ is just a non-zero scalar multiplication $\lambda \cdot -$. As $\lambda R^{(i)}_x \lambda^{-1} = R^{(i)}_x = \omega^i$, we have $R^{(i)} \not\cong R^{(j)}$ whenever $i \neq j$. In fact, by the same reason, we can see that*

$$\mathrm{Hom}_{\mathbb{C}G}(R^{(i)}, R^{(j)}) = \{0\}$$

for distinct i, j.

Exercise 4.4. *Verify that (a) $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space, and (b) the composition of homomorphisms is also a homomorphism of representations.*

Since $\mathrm{Hom}_{KG}(V, W)$ is a K-vector space, we can just write $\mathrm{Hom}_{\mathbb{C}G}(R^{(i)}, R^{(j)}) = 0$ in the above example, instead of the more bulky set notation $\{0\}$.

Exercise 4.5. *Consider $G = C_3$ with generator g acting on $X = \{0, 1, 2\}$ by $gi = i + 1 \mod 3$. Recall from Example 3.2 that 3-dimensional representation of C_3 is isomorphic to a (matrix) representation $R^{(k_1,k_2,k_3)} : G \to \mathrm{GL}_3(\mathbb{C})$ with $R^{(k_1,k_2,k_3)}_g = \mathrm{diag}(\omega^{k_1}, \omega^{k_2}, \omega^{k_3})$. Find (k_1, k_2, k_3) so that $\mathbb{C}X \cong R^{(k_1,k_2,k_3)}$.*

Exercise 4.6. *Let X, Y be two G-sets. Determine the condition on a map $f : X \to Y$ so that f induces a homomorphism of permutation representations from π_X to π_Y.*

5 Group algebra

Definition 5.1. *Let KG be the K-vector space with basis G, i.e. $x \in KG \Leftrightarrow x = \sum_{g\in G} \lambda_g g$ with $\lambda_g \in K$ for all $g \in G$.*

Define a map

$$KG \times KG \to KG, \qquad (\sum_{g\in G} \lambda_g g, \sum_{h\in G} \mu_h h) \mapsto \sum_{g,h\in G} \lambda_g \mu_h (gh).$$

It is routine to check that this defines a ring structure on KG with identity given by that of G. We call this ring the group algebra of G over K.

Exercise. (1) *Show that there is an injective ring homomorphism $K \to Z(KG) := \{x \in KG \mid xy = yx\ \forall y \in KG\}$. In other words, the group algebra KG is a K-algebra.*

(2) *Let R be a commutative ring and A be another (possibility non-commutative) ring. Show that if there is an injective ring homomorphism $R \to Z(A)$, then any A-module is also an R-module.*

Lemma 5.2. *$\rho : G \to \mathrm{GL}(V)$ is a (finite-dimensional) K-linear representation of G if, and only if, V has the structure of a (finite-dimensional) left KG-module.*

Proof $\underline{\Rightarrow:}$ For $x = \sum_g \lambda_g g \in KG, v \in V$. It is routine to check that $x \cdot v := \sum_g \lambda_g \rho_g(v)$ defines a left KG-module structure.

$\underline{\Leftarrow:}$ From the previous exercise, we checked that there is an injective ring homomorphism $K \hookrightarrow Z(KG)$. Hence, we have

$$(\lambda g)(v) = g(\lambda v)$$

for all $g \in G, \lambda \in K, v \in V$. By the axiom of module, V is an abelian group, and so there $0 \in V$ and also well-defined addition operation. Taking $g = 1$ in the above equation, we get that $\lambda v \in V$ for all $\lambda \in K$. Hence, V is a K-vector space.

Now for $g \in G$, define a map $\rho_g : V \to V$ given by $v \mapsto gv$. We then have

$$g(\lambda u + \mu v) = (\lambda g)(u) + (\mu g)(v) = \lambda \rho_g(u) + \mu \rho_g(v),$$

and so ρ_g is a linear transformation. Since $g^{-1}(g(v)) = (g^{-1}g)v = 1_G \cdot v = v$, we have $\rho_{g^{-1}}\rho_g = \mathrm{id}$, and so $\rho_g \in \mathrm{GL}(V)$.

Finally, the axiom of module says that $(gh)(v) = g(hv)$, which means that $\rho_{gh} = \rho_g \rho_h$. Thus, $g \mapsto \rho_g$ is a group homomorphism. $\square$

Remark 5.3. One may find in older textbooks that use terminologies like 'the KG-module V is afforded by ρ' in the setting of this lemma. We will just used ρ *is the representation associated/corresponding to* V, or vice versa, to keep the language simple.

Example 5.4. *KG is clearly a KG-module where the (left) action is given by (left) multiplication. Thus, we have a G-representation $\rho : G \to \mathrm{GL}(KG)$ with $\rho_g(\sum_{h\in G} \lambda_h h) := \sum_{h\in G} \lambda_h gh$. This representation is usually called regular representation of G.*

Exercise 5.5. *Let V be the 1-dimensional subspace of KG spanned by $\sum_{g\in G} g$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.*

Lemma 5.6. *$f : V \to W$ is a homomorphism of K-linear G-representations if, and only if, it is a homomorphism of left KG-modules. Consequently, $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, $W/\mathrm{Im}(f)$ are naturally K-linear G-representations.*

Proof First part: Exercise.

For the second part, just recall that the kernel, image, and quotient of image of any homomorphism of modules are also modules. $\square$

Remark. In the language of category theory, Lemma 5.2 and 5.6 together says that the category of finite-dimensional K-linear G-representations (where morphisms are homomorphisms) and the category of finitely generated left KG-modules are isomorphic (note that this is stronger than just equivalence of categories).

Exercise 5.7. *Verify the first part of Lemma 5.6.*

Exercise 5.8. *Fix any $n \geq 2$.*

(i) Find a generator v such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g\in G} g$ of the trivial representation.)

(ii) Show that $\mathrm{Hom}_{K\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ when $\mathrm{char}\, K \neq 2$; otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

6 Subrepresentation, indecomposable, irreducible

Definition 6.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a K-linear G-representation. A subpace W of V is G-invariant if $\rho_g(W) \subset W$. In this case we call the homomorphism $\theta : G \to \mathrm{GL}(W)$ given by $\theta_g := \rho_g|_W$ a subrepresentation of ρ. It is non-trivial, or proper, if W is non-zero and $W \neq V$.*

We say that ρ is irreducible (or that V is simple) if it admits no proper subrepresentation.

We will use both the terminologies irreducible and simple for representations and modules since they are 'the same' notion.

Exercise 6.2. *Let $f : V \to W$ be a homomorphism of representations from $\rho : G \to \mathrm{GL}(V)$ to $\phi : G \to \mathrm{GL}(W)$. Show the following directly without using the language of KG-modules.*

- $\mathrm{Ker}(f)$ *is a G-invariant subspace of V.*
- $\mathrm{Im}(f)$ *is a G-invariant subspace of W.*

Example 6.3. (1) *Any 1-dimensional representation is irreducible.*

(2) triv_G *is a 1-dimensional irreducible subrepresentation of the regular representation; see Exercise 5.5.*

(3) *Consider $G = D_6 = \langle a, b \mid b^2 = 1 = a^3, abab = 1\rangle$ and $K = \mathbb{C}$. Consider a 2-dimensional representation $\rho : G \to \mathrm{GL}(V)$ so that under the basis $\{u, v\}$ we have its matrix representation form given by*

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If there is a non-trivial subrepresentation, then it will be 1-dimensional spanned by $w := \lambda u + \mu v$ for some scalar $\lambda, \mu \in K$. Being G-invariant means that $aw, bw \in Kw$. Writing the action out:

$$\begin{cases} bw = b(\lambda u + \mu v) = \mu u + \lambda v, \\ aw = a(\lambda u + \mu v) = \omega\lambda u + \omega^{-1}\nu v \end{cases}$$

Looking at b-action we have some $c \in K$ so that $c\lambda = \mu$ and $c\mu = \lambda$, which yields $\lambda = \pm\mu$.

Looking at a-action we have $aw = \omega^{-1}w$ which means that $\mu\omega^{-2} = \mu$ and so $\mu = 0$. (If we take $aw = \omega w$ then we get $\lambda = 0$.) Hence, combining with $\lambda = \pm\mu$, we have $\lambda = 0$. Thus, $w = 0$. This shows that there is no non-trivial G-invariant subspace and so R is irreducible.

If $\rho : G \to \mathrm{GL}(V)$ is a G-representation has a subrepresentation with corresponding module W. Then natural inclusion map $W \hookrightarrow V$ naturally defines an injective homomorphism of KG-module. Hence, we know already from module theory that there is a KG-module structure on the quotient space V/W.

Definition 6.4. *If ϕ is a subrepresentation of $\rho = \rho_V$, with corresponding KG-modules $W \subset V$ respectively, then the quotient representation is the induced homomorphism $\rho_{V/W} : G \to \mathrm{GL}(V/W)$, i.e. $\rho_{V/W}(g)(v + W) := \rho_g(v) + W$.*

Exercise 6.5. *Check that quotient representation is indeed a representation of G directly (without using module theory).*

Lemma 6.6 (First isomorphism theorem). *Let $f : V \to W$ be a homomorphism of representations $V = (V, \rho), W = (W, \phi)$. Then the quotient representation $V/\mathrm{Ker}(f)$ is isomorphic to the subrepresentation $\mathrm{Im}(f)$ of W.*

Proof Just use first isomorphism theorem for KG-modules. □

Looking back at Example 6.3, one can see that looking at matrix really helps to determine subrepresentations. Formulating this more precisely we have the following simple observation.

Lemma 6.7. *Suppose W is a G-invariant subspace of V for a G-representation $\rho : G \to \mathrm{GL}(V)$. If $\{w_1, \dots, w_m\}$ is a basis of W, then we can extends it to a basis $\mathcal{B} = \{w_1, \dots, w_m, v_{m+1}, \dots, v_n\}$ of V so that, for every $g \in G$, the matrix form R_g of ρ_g with respect to $\mathcal{B}$ is lower block-triangular matrix*

$$R_g = \begin{pmatrix} * & 0 \\ * & R_g|_W \end{pmatrix}. \tag{6.1}$$

For ordinary vector space, having a subspace U, we can immediately get $V = U \oplus V/U$, i.e. there is a complement W of U in V such that $W \cong V/W$. However, this is not true for G-representations (and KG-modules, and also modules over a ring in general) in general.

Definition 6.8. *A representation $\rho : G \to \mathrm{GL}(V)$ is decomposable if there are non-trivial G-invariant subspaces (=subrepresentations) $U, W \subset V$ such that $V = U \oplus W$ (i.e. $V = U + W$ and $U \cap W = 0$ as vector spaces). In this case, we can write $\rho = \rho|_U \oplus \rho|_W$ and call U, W the direct summands of V. If no such pair of G-invariant subspace exists, then we say that ρ is indecomposable.*

We can formulate this in terms of matrices like Lemma 6.7.

Lemma 6.9. *$\rho = \rho|_U \oplus \rho|_W$ if and only if there is a basis $\mathcal{B}_V := \{u_1, \dots, u_m, w_1, \dots, w_k\}$ so that $\mathcal{B}_U := \{u_1\}_{1 \le i \le m}$ is a basis of U and $\mathcal{B}_W := \{w_i\}_{1 \le i \le k}$ is a basis of W, and the upper block-triangular matrix R_g in 6.1 has the top-right corner 0 for all g:*

$$R_g^V = \begin{pmatrix} R_g^U & 0 \\ 0 & R_g^W \end{pmatrix}.$$

Here R_g^X is the matrix form of $\rho|_X$ with respect to the basis $\mathcal{B}_X$ for $X \in \{V, U, W\}$.

The more compact way to say the right-hand side of this lemma is that 'we can *simultaneously block-diagonalize* ρ_g for all g'.

Of course, direct sum is not just an operation on subspaces. If we have two representations $\rho : G \to \mathrm{GL}(V), \phi : G \to \mathrm{GL}(W)$, then we have a new representation $\rho \oplus \phi : G \to \mathrm{GL}(V \oplus W)$ given by

$$(\rho \oplus \phi)_g(v + w) := \rho_g(v) + \phi_g(w)$$

for any $v \in V$ and $w \in W$.

Exercise 6.10. *If X, Y are two finite G-sets, then we have a new G-set $Z := X \sqcup Y$ given by the disjoint union. The associated permutation representation π_Z is then the direct sum $\pi_X \oplus \pi_Y$.*

Exercise 6.11. *Suppose that X is a finite G-set with G-orbit decomposition $X = O_1 \sqcup \cdots \sqcup O_m$. Then we have $\pi_X = \pi_{O_1} \oplus \cdots \oplus \pi_{O_m}$.*

Some natural questions once we have the notion of indecomposable and irreducible.

Question. (1) *Can we classify all irreducibles?*

(2) *Can we classify all indecomposables?*

(3) *How to build indecomposable representations from irreducibles?*

(4) *When does being indecomposable imply irreducible?*

(5) *Is there any criteria to guarantee a representation can be decomposed into a direct sum of irreducibles?*

(6) *Is decomposition of representation into direct sum of indecomposable direct summand unique? That is, for a representation V with decompositions $U_1 \oplus \cdots U_m$ and $W_1 \oplus \cdots W_n$ with U_i, W_j's all indecomposable, do we have $m = n$ and $\sigma \in \mathfrak{S}_n$ such that $U_i \cong W_{\sigma(i)}$?*

(7) *If we 'divide' a representation into subquotients of irreducibles, is the resulting multi-set of irredcuible contribution 'unique'?*

Our plan is to answer Questions (4) first – this is given by the Maschke's theorem. And use it, and other tools, to give answers to other questions in the case of ordinary representation theory. We will not give any account for the case of modular representation theory, but just minor remarks here: Question (1) has an answer similar to that of the ordinary case. Question (2) is almost always impossible (for interested audience, search on 'tame-wild dichotomy of representation-type'). Question (3) can only be studied by looking at the homological algebra of KG, which is beyond the scope of this text. Question (4) and (5) does not have any good answer in general. Question (6) and (7) actually have affirmative answer as they are consequence of classical result in ring and module theory (namely, Krull-Schmidt theorem and Jordan-Hölder theorem); these are also beyond the scope of this text.

Before we move on, let us have a look when the Question (4) fails.

Example 6.12. *Take $G = C_2 = \langle g \mid g^2 = 1 \rangle$.*

First consider the case when $\operatorname{char} K \neq 2$ *(e.g. $K = \mathbb{C}$). Recall that the trivial representation* $\mathrm{triv}_G \cong K(1+g)$ *is a subrepresentation of the regular representation KG. On the other hand, $C_2 = \mathfrak{S}_2$ has a 1-dimensional representation* $\mathrm{sgn} \cong K(1-g)$*. Clearly $\{1+g, 1-g\}$ is a basis of KG. This yields a direct sum decomposition*

$$KG = K(1+g) + K(1-g) = K(1+g) \oplus K(1-g) \cong \mathrm{triv} \oplus \mathrm{sgn}\,.$$

Consider $G = C_2$ with $\operatorname{char} K = 2$ *(e.g. $K = \mathbb{F}_2$). Consider regular representation $C_2 \curvearrowright KC_2$. With respect to the canonical basis $\{1, g\}$, the matrix of g-action is given by $R_g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Suppose we can change the basis via $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to diagonalise R_g. Then R_g becomes*

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} bd-ac & a^2-b^2 \\ d^2-c^2 & ac-bd \end{pmatrix}.$$

Hence, we have $b = \pm a$ and $d = \pm c$. Since we are working over characteristic 2, we just get $b = a$ and $d = c$. But in this case the above matrix becomes 0. Hence, R_g cannot be diagonalised and so it is not a direct sum of two 1-dimensional subrepresentations. In particular, it is a 2-dimensional indecomposable. As mentioned, triv *is always a subrepresentation and so we have a 1-dimensional subrepresentation* triv *of KG. One can check that the quotient representation is isomorphic to* triv *as well, i.e. in pictorial form, we can write:*

$$KG = \begin{matrix} \mathrm{triv} \\ \mathrm{triv} \end{matrix}\,.$$

Exercise 6.13. *Complete the argument in the example above by showing that $KG/\,\mathrm{triv} \cong \mathrm{triv}$ when* $\operatorname{char} K = 2$.

Exercise 6.14. *Let $A = K[x]/(x^2)$ for any field K. Check that the left A-module ${}_AA$ is indecomposable, i.e. $A \not\cong X \oplus Y$ for some non-trivial submodules X, Y of A.*

7 Maschke's theorem

We introduce the following notion to help talking about the Question (5) above.

Definition 7.1. *A representation is completely reducible, or semisimple if it is a direct sum of irreducible representations.*

The main aim of this section is to explain the following foundational result of group representation theory, which is the answer to Question (5).

Theorem 7.2. (Maschke) *Suppose that G is finite and* $\operatorname{char} K$ *is coprime to the order of G. For any KG-module V, every submodule U of V admits a G-invariant complement, i.e. $V = U \oplus V/U$ as KG-module.*

Proof Let W_0 be any K-vector space complement of U in V, and $\pi : V \to V$ be the K-linear projection map that projects onto U (i.e. write $v \in V$ as $u + w$ for $u \in U, w \in W_0$, then $\pi(v) = u$). If π is a homomorphism of KG-modules, then W_0 is a KG-module and we are done by Lemma 5.6 – unfortunately this is not true in general. So our goal is to modify π into an idempotent homomorphism. The clever trick is to consider

$$p : V \to V, \quad v \mapsto \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(v).$$

Let us now show that p is a KG-module homomorphism. Indeed, for any $g \in G$, we have

$$p(gv) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(gv) = \frac{1}{|G|} \sum_{h \in G} g(g^{-1}h^{-1}) \pi(hg)v = g \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h v = gp(v).$$

The averaging by $|G|$ bit seems very unnecessary so far, but we will see soon that this averaging operation makes p a projection onto U. Indeed, first, $\operatorname{Im}(\pi) = U$ implies that $\operatorname{Im}(p) \subset U$, and so it remains to show that $p(u) = u$ for all $u \in U$. Indeed, we have

$$p(u) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi \underbrace{h(u)}_{\in U} = \frac{1}{|G|} \sum_{h \in G} h^{-1} h(u) = \frac{1}{|G|} \sum_{h \in G} u = u.$$

Now that we have $p : V \to V$ a KG-module projection onto U, we get that $\operatorname{Ker}(p)$ is a KG-submodule of V. Hence, we have by first isomorphism theorem that $V/\operatorname{Ker}(p) \cong \operatorname{Im}(p) = U \subset V$ and so $V = \operatorname{Ker}(p) \oplus U$. $\square$

Corollary 7.3. *KG is semisimple if, and only if,* $\operatorname{char} K \nmid |G|$.

Proof $\underline{\Leftarrow:}$ Consequence of iteratively applying Maschke's theorem (Theorem 7.2) starting with $V = KG$.

$\underline{\Rightarrow:}$ Suppose on the contrary that KG is semisimple. Let $a := \sum_g g \in KG$ and $V := Ka \subset KG$. Recall that $\mathrm{triv}_G \cong V$. So KG being semisimple means that we must have $KG \cong V \oplus W$ for some left ideal W of KG.

Consider $w = \sum_h \lambda_h h \in W$. Since W is a left ideal of KG, we have $aw \in W$. On the other hand, we also have

$$aw = (\sum_g g)(\sum_h \lambda_h h) = \sum_h \lambda_h (\sum_g gh) = \sum_h \lambda_h a,$$

which means that $aw \in V$. But $V \cap W = 0$ and so we must have $\sum_h \lambda_h = 0$, which means that

$$W \subset W' := \left\{ \sum_g \mu_g g \in KG \,\middle|\, \sum_g \mu_g = 0 \right\}.$$

The space W' can be rewritten as the kernel of the map (a.k.a. the augmentation map) given by

$$\epsilon : KG \to K, \qquad \sum_g \mu_g g \mapsto \sum_g \mu_g.$$

Thus, $\dim_K W' = |G| - 1 = \dim_K W$ which means that $W = W'$. However, we can also see that $\epsilon(a) = 0$, and so $V \subset W$, a contradiction. $\square$

Exercise 7.4. *Let G be the subgroup of $\mathrm{GL}_n(\mathbb{C})$ given by*

$$\left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

Let V be the 2-dimensional $\mathbb{C}$-vector space. Then we have a natural $\mathbb{C}$-linear representation $\rho : G \to \mathrm{GL}(V)$ given by $g \mapsto gv$ (usual applying matrix on vector). Show that V is indecomposable but not irreducible. In particular, Maschke's theorem fails for infinite group even for $K = \mathbb{C}$.

8 Schur's lemma

Definition 8.1. *A division ring, or a skew field, is a ring whose non-zero elements are invertible.*

Remark 8.2. A field is a division ring where multiplication is commutative.

The following easy yet fundamental lemma describes the relation between simple modules.

Lemma 8.3 (Schur's lemma). *Suppose S, T are simple KG-modules, then*

$$\mathrm{Hom}_{KG}(S, T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

If, moreover, K is algebraically closed, then

$$\dim_K \mathrm{Hom}_{KG}(S, T) = \begin{cases} 1, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Proof We prove the first part by showing that any homomorphism $f : S \to T$ is either zero or an isomorphism. Indeed, for $f \in \mathrm{Hom}_{KG}(S, T)$, we have submodules $\mathrm{Ker}(f) \subset S$ and $\mathrm{Im}(f) \subset T$. Since S is simple, either $\mathrm{Ker}(f) = 0$ or $\mathrm{Ker}(f) = S$. Similarly, since T is simple, either $\mathrm{Im}(f) = T$ or $\mathrm{Im}(f) = 0$. Thus we have

	$\mathrm{Ker}(f) = 0$	$\mathrm{Ker}(f) = S$
$\mathrm{Im}(f) = T$	f isom.	impossible
$\mathrm{Im}(f) = 0$	impossible	$f = 0$.

Assume now that K is algebraically closed, and that $S = T$. We claim that any non-zero homomorphism $f : S \to S$ is given by a scalar multiple $\lambda\,\mathrm{id}_S$ of the identity map. Indeed, K being algebraically closed implies that f has an eigenvalue λ, and so $f - \lambda\,\mathrm{id}_S$ is a non-invertible linear endomorphism on S. It follows from the first part that $f - \lambda\,\mathrm{id}_S = 0$, and so $f = \lambda\,\mathrm{id}_S$.

For the case $S \cong T$, we can fix any pair of isomorphisms $f, g : S \to T$, and so $g^{-1}f : S \to S$ is an endomorphism. By the previous paragraph, we have $g^{-1}f = \lambda\,\mathrm{id}_S$ and so $f = \lambda g$. Thus any homomorphism in $\mathrm{Hom}_{KG}(S, T)$ is a scalar multiple of any other non-zero homomorphism. $\square$

We will now address Question (6). We start with a preliminary lemma.

Lemma 8.4. *For any finite-dimensional KG-modules U, V, W, we have*

(1) $\mathrm{Hom}_{KG}(U \oplus V, W) \cong \mathrm{Hom}_{KG}(U, W) \oplus \mathrm{Hom}_{KG}(V, W)$.

(2) $\mathrm{Hom}_{KG}(U, V \oplus W) \cong \mathrm{Hom}_{KG}(U, V) \oplus \mathrm{Hom}_{KG}(U, W)$.

Proof Exercise (consider the natural projection map $\pi_X : X \oplus Y \to X$). □

Notation. *For a semisimple KG-module M and a simple KG-module S, denote by $[M : S]$ the multiplicity of S as a direct summand, up to isomorphism, of M, i.e. the maximal number m such that $M \cong S^{\oplus m} \oplus M'$.*

Proposition 8.5 (Krull-Schmidt property). *Suppose that K is algebraically closed and $\mathrm{char}\, K \nmid |G|$. For a finite-dimensional KG-module M and simple KG-module S, we have*

$$[M : S] = \dim_K \mathrm{Hom}_{KG}(M, S) = \dim_K \mathrm{Hom}_{KG}(S, M).$$

In particular, if $M \cong S_1 \oplus \cdots S_s$ and $M \cong T_1 \oplus \cdots \oplus T_t$ are two decomposition of M into direct sum of simple KG-modules, then we have $s = t$ and a permutation $\sigma \in \mathfrak{S}_t$ so that $S_i \cong T_{\sigma(i)}$ for all $1 \leq i \leq t$.

This is only a (very) special case for the Krull-Schmidt theorem, which says that the Krull-Schmidt property (=unique decomposition into direct sum of indecomposables) holds for any finite-dimensional K-algebras (without assumption on the field K); we provide a group representation theoretic proof of this instead.

Proof By Maschke's theorem, we can write $M = S_1 \oplus \cdots S_s$ for simple modules $S_1, \ldots, S_s$. Hence, we have

$$\dim_K \mathrm{Hom}_{KG}(M, S) = \sum_{i=1}^{s} \dim_K \mathrm{Hom}_{KG}(S_i, S) = \#\{i \in [1, s] \mid S_i \cong S\} = [M : S],$$

where the first equality comes from repeatedly applying Lemma 8.4, and the second comes from Schur's lemma. The proof for $\dim_K \mathrm{Hom}_{KG}(S, M)$ is similar. One can then show the final statement using the formula and induction on s. □