You may assume all algebras are finite-dimensional over a field $\Bbbk$. You may attempt the exercises with the additional assumption of $\Bbbk$ being algebraically closed.

Throughout, unadorned tensor product over assumed to be taken over a field, i.e. $\otimes = \otimes_{\Bbbk}$. For a module X over some algebra, denote by $\mathsf{add}(X)$ the full subcategory of the module category consisting of finite direct sums of direct summands of X (up to isomorphism).

Ex 1. Let e be an idempotent of an algebra A. Note that indecomposable projective eAe-modules are of the form (up to isomorphism) fAe for some primitive idempotent $f \in A$ with $fe \neq 0$.

(1) Show that if $(AeA)_A$ is projective, then Ae is a projective (right) eAe-module.
Hint (i): Assumption implies that $AeA \cong (eA)^{\oplus m}$ (since $eA^{\oplus A} \twoheadrightarrow AeA$ splits).
Hint (ii): Show $Ae = AeAe$ and use description of projective eAe-modules.

(2) Show that if $(Ae)_{eAe}$ is projective, then $\mathrm{gldim}(eAe) \leq \mathrm{gldim} A$ for any simple A-module S.
Hint: $(-)e = - \otimes_A Ae$ takes simple module to simple module or zero.

Let I be a two-sided ideal of A such that I_A is projective, and take $B := A/I$.

(3) Show that $\mathrm{pdim}(B_A) \leq 1$.

(4) Show that $\mathrm{pdim}(M_A) \leq 1 + \mathrm{pdim}(M_B)$.
Hint (i): Prove by induction.
Hint (ii): Construct a short exact sequence in $\mathsf{mod}\, A$ involving M and a projective B-module.

Ex 2. Let e be an idempotent of a finite-dimensional algebra A. Consider the functors

$$\begin{tikzcd}
\mathsf{mod}\, A \arrow[rr, "\mathtt{j}:=-\otimes_A Ae"] & & \mathsf{mod}\, eAe \arrow[ll, bend right, "\mathtt{l}:=-\otimes_{eAe} eA"'] \arrow[ll, bend left, "\mathtt{r}:=\mathrm{Hom}_{eAe}(Ae,-)"]
\end{tikzcd}$$

(1) Show that, for any $M \in \mathsf{mod}\, A$, we have two isomorphisms $\mathrm{Hom}_A(eA, \mathrm{Hom}_{eAe}(Ae, M)) \cong M \cong Me \otimes_{eAe} eA$ of eAe-module. In particular, show that there are natural isomorphisms $\mathtt{jr} \cong \mathrm{Id}_{\mathsf{mod}\, eAe} \cong \mathtt{jl}$.

(2) Show that $\mathtt{l}(P) \in \mathsf{add}(eA)$ for all projective A-module P.
Hint: Consider first the case when $P = fA$ is an indecomposable projective A-module, where f is some primitive idempotent.
*** If you only present this as a consequence of property of adjointness, no mark will be awarded.*

(3) For $M \in \mathsf{mod}\, eAe$, show that there is a projective resolution of $M \otimes_{eAe} eA \in \mathsf{mod}\, A$ whose first two terms are in $\mathsf{add}(eA)$.

(4) Suppose that $M \in \mathsf{mod}\, A$ has a projective resolution $\cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$ such that $P_i \in \mathsf{add}(eA)$ for $i \leq 1$. Show that $Me \otimes_{eAe} eA \cong M$.
Hint: Use (2) and find an appropriate commutative diagram.

(5) Show that $\mathtt{l}, \mathtt{j}$ defines an equivalence of categories $\mathsf{pres}(eA) \simeq \mathsf{mod}\, eAe$, where $\mathsf{pres}(eA)$ is the full subcategory of $\mathsf{mod}\, A$ consisting of modules M satisfying the condition of part (4).

Ex 3.

(1) Show that $\mathrm{Hom}_A(M, N) \cong D(M \otimes_A DN)$ as vector spaces.

(2) Let $P_\bullet = (P_i, d_i : P_i \to P_{i-1})_{i \geq 0}$ be a projective resolution of an A-module M, and define

$$\mathrm{Tor}_1^A(M, N) := H_1(P_\bullet \otimes_A N) = \frac{\mathrm{Ker}(d_1 \otimes_A N)}{\mathrm{Im}(d_2 \otimes_A N)}$$

the first homology group of the complex $P_\bullet \otimes_A N$. Show that $\mathrm{Ext}_A^1(M, N) \cong D\,\mathrm{Tor}_1^A(M, DN)$ as $\Bbbk$-vector spaces.

(3) Show that $D\,\mathrm{Hom}_A(M, A) \cong M \otimes_A DA$ as right A-modules.

(4) Let ${}_AX_B$ be an A-B-bimodule. If M is a C-A-bimodule and N is a C-B-bimodule. Show that $\mathrm{Hom}_{C^{\mathrm{op}} \otimes B}(M \otimes_A X, N) \cong \mathrm{Hom}_{C^{\mathrm{op}} \otimes A}(M, \mathrm{Hom}_B(X, N))$ as vector spaces.

(5) Let $B := A^{\mathrm{op}} \otimes A$. Show that $\mathrm{Hom}_B(A, B) \cong \mathrm{Hom}_A(DA, A)$ as A-A-bimodules.
Hint: $B \cong (DDA) \otimes A \cong \mathrm{Hom}_{\Bbbk}(DA, A)$ as B-modules.

(6) In the setup of (5), show that $\mathrm{Ext}_B^1(A, B) \cong \mathrm{Ext}_A^1(DA, A)$.

Ex 4. Consider the quiver algebra $A = \Bbbk Q/I$ given by

$$Q: \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] & 4 \arrow[l, bend left, "\beta_3"] \end{tikzcd}, \quad I = (\beta_3\alpha_3,\ \alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid i = 1, 2)$$

For $i \in \{1, 2, 3, 4\}$, let $\Delta(i) := P_i/\alpha_i A$ (with $\alpha_4 := 0$ as a convention).

(1) Describe the Loewy filtration of each indecomposable projective A-module $P(i)$ with $1 \leq i \leq 4$. In particular, show that each of these has a simple socle, i.e. $\mathrm{soc} P(i) \cong S(j_i)$ for some $1 \leq j_i \leq 4$.

(2) Write down the minimal projective resolution of $\Delta(1)$ and determine its projective dimension.

(3) Show that $\mathrm{Ext}_A^k(\Delta(i), \Delta(j)) = 0$ whenever $i > j$ for any $k \geq 0$.

(4) Show that $\mathrm{Ext}_A^k(\Delta(i), \Delta(j)) = 0$ whenever $k > 3$ for any i, j.

(5) Compute $\dim_{\Bbbk} \mathrm{Ext}_A^k(\Delta(i), \Delta(j))$ for all possible i, j, k. Show your working.

Deadline: 19th December, 2025
Submission: In lecture or E-mail to (replace at by @) aaron.chan at math.nagoya-u.ac.jp