Topics in Mathematical Science VI
Autumn 2022
Group Representations and Character Theory

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Last update: October 27, 2022

Contents

Lecture 1

Throughout, 'group' means 'finite group', unless otherwise stated. K will always be a field.

Definition 1.1. *A finite-dimensional (resp. n-dimensional) K-linear representation of a group G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action of g on V.

Often, the symbol ρ is suppressed and we write $G \curvearrowright V$ instead, and say 'G acts on V'. In particular, instead of $\rho_g(v)$ for $v \in V$, we write $g(v)$ instead.

Example 1.2. *(1) The trivial representation of G is the one-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}\,.$$

(2) $G = \mathfrak{S}_n$ the symmetric group of rank n. The sign representation of $\mathfrak{S}_n$ is the one-dimensional representation

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is the parity (or sign) of the permutation σ.

Exercise 1.3. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.*

Definition 1.4. *Let KG be the K-vector space with basis G, i.e. $x \in KG \Leftrightarrow x = \sum_{g\in G} \lambda_g g$ with $\lambda_g \in K$ for all $g \in G$.*

Define a map

$$KG \times KG \to KG, \qquad (\sum_{g\in G} \lambda_g g, \sum_{h\in G} \mu_h h) \mapsto \sum_{g,h\in G} \lambda_g \mu_h (gh).$$

It is routine to check that this defines a ring structure on KG with identity given by that of G. We call this ring the group algebra of G over K.

Clearly, $G \curvearrowright KG$ naturally; this is called the regular representation.

Exercise. *Show that there is an injective ring homomorphism* $K \to Z(KG) := \{x \in KG \mid xy = yx \ \forall y \in KG\}$*. In other words, the group algebra* KG *is a* K*-algebra.*

Lemma 1.5. $\rho : G \to \mathrm{GL}(V)$ *is a (finite-dimensional)* K*-linear representation of* G *if, and only if,* V *has the structure of a (finite-dimensional) left* KG*-module.*

Proof $\underline{\Rightarrow:}$ For $x = \sum_g \lambda_g g \in KG, v \in V$. It is routine to check that $x \cdot v := \sum_g \lambda_g \rho_g(v)$ defines a left KG-module structure.

$\underline{\Leftarrow:}$ Define a map $\rho_g : V \to V$ by $v \mapsto gv$. Since $g^{-1}g(v) = v$, we have $\rho_{g^{-1}}\rho_g = \mathrm{id}$, and so $\rho_g \in \mathrm{GL}(V)$. It is routine to check that $g \mapsto \rho_g$ is a group homomorphism. □

Remark 1.6. One may find in older textbooks that use terminologies like 'the KG-module V is afforded by ρ' in the setting of this lemma.

Definition 1.7. $V = (V, \rho), W = (W, \theta)$ *be* K*-linear representations of* G*. A* homomorphism *from* V *to* W *is a* K*-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$*, i.e.* $f\rho_g = \theta_g f$ *for all* $g \in G$*.*

An isomorphism *from* V *to* W *is a homomorphism that is invertible, i.e.* $\exists g$ *s.t.* $gf = \mathrm{id}_V$ *and* $fg = \mathrm{id}_W$*. In this case,* V *and* W *are* equivalent *representations, and write* $V \cong W$*.*

Write $\mathrm{Hom}_G(V, W)$ *to be the (*K*-vector) space of all homomorphisms from* V *to* W*.*

Lemma 1.8. $f : V \to W$ *is a homomorphism of* K*-linear* G*-representations if, and only if, it is a homomorphism of left* KG*-modules; in other words,* $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{KG}(V, W)$*. Consequently,* $\mathrm{Ker}(f)$*,* $\mathrm{Im}(f)$*,* $W/\mathrm{Im}(f)$ *are naturally* K*-linear* G*-representations.*

Proof This first part is clear (if not, think through it).

For the second part, just recall that the kernel, image, and quotient of image of any homomorphism of modules are also modules. □

Remark. In the language of category theory, Lemma 1.5 and 1.8 together says that the category of finite-dimensional K-linear G-representations (where morphisms are homomorphisms) and the category of finitely generated left KG-modules are isomorphic (note that this is stronger than just equivalence of categories).

Exercise 1.9. *Let* V *be the 1-dimensional subspace spanned by* $\sum_{g \in G} g \in KG$*. Show that* V *is a* KG*-module and that* $\mathrm{triv}_G \cong V$*.*

Recall that for a ring R with identity 1, either 1 has infinite order (under addition) or has prime, say p, order. The *characteristic* of R, denoted by $\mathrm{char}\, R$, is 0 in the former case, p in the latter.

Exercise. *Fix any* $n \geq 2$*.*

(i) *Find a generator* v *such that* $\mathrm{sgn} = Kv$*. (Hint: Modify the generator* $\sum_{g \in G} g$ *of the trivial representation.)*

(ii) *Show that* $\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ *when* $\mathrm{char}\, K = 2$*, otherwise,* $\mathrm{triv} \cong \mathrm{sgn}$*.*

Two classes of group representations. In the literature, by *ordinary representations* we mean K-linear representations with $\mathrm{char}\, K = 0$; by *modular representations* we mean K-linear representations with $\mathrm{char}\, K \mid |G|$.

The *Maschke's theorem* (and its consequence) justifies that ordinary representation theory is (significantly) easier to understand than modular ones - this will be our next goal. The material we will use is based on a more ring theoretic approach (from Benson's book Chapter 1) to the subject, which has the advantage of shedding some light on what happen on the modular side too. The proof of Maschke's theorem will follow the exposition of James and Liebecks.

Interlude on terminology and notation. For a field K, recall that a *K-algebra* is a ring R equipped with a ring homomorphism $K \to Z(R) := \{x \in R \mid xy = yx \; \forall y \in R\}$. This is equivalent to saying that R is a K-vector space equipped with a ring structure.

For a K-algebra A, let $A\,\mathsf{mod}$ be the category of finitely generated left A-modules. So by $M \in A\,\mathsf{mod}$ we mean that M is an A-module, and by $(f : M \to N) \in A\,\mathsf{mod}$ we mean that f is an A-module homomorphism. We will use 0 to denote either the zero homomorphism, or the zero element in a vector space, or the vector space with only the zero element; this should be clear from context.

Like numbers, we like to break down modules into simpler 'components'. The first candidate is via the notion of direct sum. Recall that an A-module M is a *direct sum*, say $M = M_1 \oplus M_2$, if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$. We will come back to this next lecture. In this lecture, we consider a more refined way to break down M into smaller modules.

Definition 1.10. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$.*

(1) *M is simple if for any submodule L of M, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 1.11. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Example 1.12. (1) *Trivial module and sign module are both simple. In general, any 1-dimensional representation of a group G will be simple for dimension reason.*

(2) *Consider the matrix ring $A := \mathrm{Mat}_n(K) := \{n \times n$ matrices with entries in $K\}$. Let V be the 'column space', i.e. $V = \{(v_j)_{1\le j\le n} \mid v_j \in K\}$ where $X \in \mathrm{Mat}_n(K)$ acts on $v \in V$ by $v \mapsto Xv$ (matrix multiplication from the left). Then V is an n-dimensional simple module. The regular representation A is semisimple as it is isomorphic to the direct sum of n column spaces (corresponding to the n choices of column we can cut matrix into V).*

(3) *The ring of dual numbers is $A := K[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as (x) is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by Kv where $v = 1 + (x)$. If A is semisimple, then Kv is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in K$. If $a \neq 0$, then $A(a + bx) = A$. So $a = 0$, and $Kv \cong (x)$, a contradiction.*

The following easy yet fundamental lemma describes the relation between simple modules.

Lemma 1.13 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division } K\text{-algebra}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Proof For $f \in \mathrm{Hom}_A(S,T)$, $\mathrm{Im}(f)$ is a submodule of T, and so f is either zero or a K-vector space isomorphism, and the latter case only happens when $S \cong T$. □

Remark 1.14. If K is algebraically closed, then any division K-algebra is just K itself. The complication with the divison K-algebra appearing is the reason why most literature consider only the case when K is algebraically closed. In particular, for ordinary representation one usually just consider

$K = \mathbb{C}$. In this course, this will also often be the case - perhaps the only exception is when we consider general K-algebra instead of group algebra.

Lemma 1.15. *Consider $M = S_1 \oplus \cdots S_r$ with simples $S_i \cong S_j$ for all i, j. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_r(D)$ as K-algebras, where $D := \mathrm{End}_A(S_i)$.*

Note that $\mathrm{End}_A(M)$ is a ring where multiplication is given by composition. Since A is a K-algebra, $\mathrm{End}_A(M)$ is also a K-algebra as K acts by scalar multiplications and commutes with homomorphisms, i.e. $(\lambda \cdot f)(m) := \lambda f(m) = f(\lambda m) = (f \cdot \lambda)(m)$ for all $(f : M \to M) \in A\,\mathsf{mod}$ and $m \in M$.

Proof We have canonical homomorphisms $\iota_j : S_j \hookrightarrow M$ and $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this can be identified with an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(\lambda_{i,j})_{1 \le i,j \le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \xrightarrow{\lambda_{i,j}} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. $\square$

Lecture 2

Definition 2.1. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \leq i \leq \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \leq i \leq \ell$ are called the composition factors of the series.

Composition series allows us to understand the structure of a module by simple modules. It is desirable to have a rigidity result - that composition factors do not change.

Lemma 2.2. *Let M be a finite-dimensional left A-module. Then composition series of M exists.*

Proof This is by induction on $\dim_K M$. For $\dim_K M = 0$ this is trivial. For $\dim_K M > 0$, if M is simple, then we are done. Otherwise, M proper non-zero submodule, and we pick N such a submodule so that M/N is simple. Clearly $\dim_K N < \dim_K M$ and so we can apply induction hypothesis. □

Theorem 2.3 (Jordan-Hölder Theorem). *Any two composition series have the same length and their composition factors are the same up to permutations.*

Proof Suppose we have two composition series

$$\begin{array}{ccccccc} 0 = M_0 & \subset & M_1 & \subset & \cdots & \subset & M_\ell = M, \\ 0 = N_0 & \subset & N_1 & \subset & \cdots & \subset & N_n = M. \end{array}$$

Without loss of generality, we can assume $n > \ell$. We claim that $N_\ell = M$. Indeed, we can do this by induction on ℓ. If $\ell = 0$, then clearly $M_0 = 0 = N_0$ and we are done; likewise, when $\ell = 1$, then M is simple and we have $M_1 = M = N_1$. For $\ell > 1$, we have

$$0 = M_1 \cap N_0 \subset M_1 \cap N_1 \subset \cdots \subset M_1 \cap N_n = M_1 \cap M = M_1.$$

So as M_1 simple, there is some n_0 such that $N_{n_0} \cap M_1 = M_1$ and $N_i \cap M_1 = 0$ for all $i < n_0$.

We now consider two new chains

$$\begin{array}{ccccccccccccccc} 0 & \subset & \frac{M_2}{M_1} & \subset & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \subset & \frac{M_\ell}{M_1} = \frac{M}{M_1}, \\ 0 & \subset & N_1 & \subset & \cdots & \subset & N_{n_0-1} & \subset & \frac{N_{n_0+1}}{M_1} & \subset & \frac{N_{n_0+2}}{M_1} & \subset & \cdots & \subset & \frac{N_n}{M_1} = \frac{M}{M_1}, \end{array}$$

which are both composition series of M/M_1. By induction hypothesis, we thus have $n-1 = \ell-1$ and the composition factors of these two series coincide up to permutation. □

Remark. This (simpler) version of proof relies on M having composition series of finite length. One can expect similar more careful argument applies for modules that are both noetherian and artinian. In fact, for general K-algebra, M admits a composition series of finite length if and only if it is noetherian and artinian. In this case, Jordan-Hölder theorem also holds.

Exercise 2.4. *Let A be the algebra of upper triangular $n \times n$-matrices:*

$$A := \begin{pmatrix} K & K & \cdots & K \\ 0 & K & \cdots & K \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & K \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i,j \leq n} \,\middle|\, \begin{array}{l} a_{i,j} \in K \;\forall i,j \\ a_{i,j} = 0 \;\forall i > j \end{array} \right\}$$

For $1 \leq i \leq j \leq n$, let $M_{i,j} \subset K^{\oplus n}$ be the vector space given by column vectors $v = (v_k)_{1 \leq k \leq n}$ where $v_k = 0$ for $k \notin \{i, i+1, \ldots, j\}$.

(i) *Determine which $M_{i,j}$'s are simple.*

(ii) *Describe the composition series of $M_{i,j}$.*

Jordan-Hölder theorem effectively says that the notion of length and composition factor of a module is well-defined without any reference to a chosen composition series.

Now that we no longer worries about building blocks (composition factors) of a module is non-well-defined, we can move on to understand the simplest form of algebra - where every module is semisimple.

Definition 2.5. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$.*

(1) *The (Jacobson) radical of A is the (two-sided) ideal*

$$J(A) := \{a \in A \mid aM = 0 \,\forall simple\ M\}.$$

This is equivalent to saying that $J(A)$ is the intersection of all maximal left ideals of A, as well as the intersection of all maximal right ideals of A.

(2) *A is semisimple if $J(A) = 0$. This is equivalent to saying that left (equivalently, right) regular A-module ${}_AA$ is semisimple.*

Example 2.6. (1) *A field K on its own is a semisimple K-algebra.*

(2) *Suppose D is a division K-algebra, then $\mathrm{Mat}_n(D) := \{n \times n$ matrices with entries in $D\}$ is a semisimple K-algebra.*

(3) *A finite product of semisimple algebras is semisimple.*

(4) *The ring of dual numbers $A := K[x]/(x^2)$ is not semisimple since it has a non-trivial maximal ideal $J(A) = (x)$. More generally, the truncated polynomial ring $K[x]/(x^n)$ for any $n \geq 2$ is also non-semisimple.*

Theorem 2.7. (see [Benson, Lemma 1.2.4] or [Erdmann-Holm, Theorem 4.11, 4.23]) *The following are equivalent for a K-algebra A.*

(i) *A is a semisimple algebra.*

(ii) *The regular representation ${}_AA$ is a semisimple module.*

(iii) *Every A-module is semisimple.*

A natural question is whether all semisimple is always a product of matrix rings over division rings. To answer this question, we need some elementary (but fundamental) properties of simple modules first.

Lemma 2.8. *Let $e \in A$ be an idempotent, i.e. $e = e^2 \in A$. Then the following hold.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(Ae, M) \cong eM$ *as a K-vector space for all $M \in A\,\mathsf{mod}$.*

(2) *There is an isomorphism of rings $\mathrm{End}_A(Ae)^{\mathrm{op}} \cong eAe$.*

Proof (1): Check $\mathrm{Hom}_A(Ae, M) \ni f \mapsto f(e) \in eM$ defines a K-linear map with inverse $em \mapsto (ae \mapsto aem)$.

(2): Take $M = Ae$ in (1) and notice that order of multiplication in reverse that of homomorphism composition. □

Exercise. *Recall (or check any reference book) the notion of free module and the rank of it. Check that for an idempotent $e \in A$, Ae is a direct summand of A. In ring/module theory terms, (by definition) Ae is thus a projective module since it is a direct summand of a free module.*

Theorem 2.9 (Artin-Wedderburn's theorem). *Let A be a finite-dimensional K-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \ldots, S_r$. Let $D_i := \mathrm{End}_A(S_i)^{\mathrm{op}}$ be the division K-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of K-algebras*

$$A/J(A) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r).$$

Proof Let $B := A/J(A)$. By definition of $J(A)$, the A-module $A/J(A)$ is semisimple, and any A-submodule M of $A/J(A)$ satisfies $J(A)M = 0$. Hence, $M = M/J(A)M$ is naturally a B-module and $\mathrm{End}_B(M) \cong \mathrm{End}_A(M)$ (even as rings!).

By Lemma 2.8, we have $B \cong \mathrm{End}_B(B)^{\mathrm{op}}$. Since B is semisimple, the regular representation B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Lemma 1.13 and Lemma 1.15 that

$$B \cong \mathrm{End}_B(B)^{\mathrm{op}} \cong \big(\mathrm{Mat}_{n_1}(E_1) \times \cdots \times \mathrm{Mat}_{n_r}(E_r)\big)^{\mathrm{op}} \cong \mathrm{Mat}_{n_1}(E_1^{\mathrm{op}}) \times \cdots \times \mathrm{Mat}_{n_r}(E_r^{\mathrm{op}}),$$

where $E_i := \mathrm{End}_B(S_i)$ for all $1 \le i \le r$. This completes the proof. □

Theorem 2.10 (Maschke's theorem). *If $\mathrm{char}\, K \nmid |G|$, then for any KG-module V and submodule $U \subset V$, there is a KG-module W such that $V = U \oplus W$.*

Proof Let W_0 be any K-vector space complement of U in V, and $\pi : V \to U$ be the K-linear projection map. If π is a homomorphism, then W_0 is a KG-module and we are done by Lemma 1.8 – unfortunately this is not true in general. So our goal is to modify π into an idempotent homomorphism. The clever trick is to consider

$$p : V \to V, \quad v \mapsto \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(v).$$

Let us now show that $p \in \mathrm{End}_{KG}(V)$. Indeed, for any $g \in G$, we have

$$p(gv) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(gv) = \frac{1}{|G|} \sum_{h \in G} g(g^{-1}h^{-1}) \pi(hg) v = g \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h v = gp(v).$$

Now we check that $p^2 = p$. It is easy to see that, as $\mathrm{Im}(\pi) = U$, we have $\mathrm{Im}(p) \subset U$. Hence, it remains to show that $p(u) = u$ for all $u \in U$. Indeed, we have

$$p(u) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi \underbrace{h(u)}_{\in U} = \frac{1}{|G|} \sum_{h \in G} h^{-1} h(u) = \frac{1}{|G|} \sum_{h \in G} u = u.$$

This completes the proof. □

Corollary 2.11. *KG is semisimple if, and only if, $\mathrm{char}\, K \nmid |G|$.*

Proof $\underline{\Leftarrow:}$ Consequence of iteratively applying Maschke's theorem (Theorem 2.10) starting with $V = KG$.

$\underline{\Rightarrow:}$ Suppose on the contrary that KG is semisimple. Let $a := \sum_g g \in KG$. Recall that $\mathrm{triv}_G \cong V := Ka \subset KG$. So we must have $KG \cong V \oplus W$ for some left ideal W of KG.

Consider $w = \sum_h \lambda_h h \in KG$. Since W is a left ideal of KG, we have $aw \in W$. On the other hand, we also have

$$aw = (\sum_g g)(\sum_h \lambda_h h) = \sum_h \lambda_h (\sum_g gh) = \sum_h \lambda_h a,$$

which means that $aw \in V$. But $V \cap W = 0$ and so we must have $\sum_h \lambda_h = 0$, which means that

$$W \subset W' := \left\{ \sum_g \mu_g g \in KG \,\middle|\, \sum_g \mu_g = 0 \right\}.$$

The space W' can be rewritten as the kernel of the map (a.k.a. the augmentation map)

$$\epsilon : KG \to K \text{ given by } \sum_g \mu_g g \mapsto \sum_g \mu_g.$$

Thus, $\dim_K W' = |G| - 1 = \dim_K W$ which means that $W = W'$.

However, we can also see that $\epsilon(a) = 0$, and so $V \subset W$, a contradiction. □

Remark. Note that the proof of this result (both directions) relies neither on Jordan-Hölder nor Artin-Wedderburn. From ring theory perspective, it makes more sense to first talk about unicity of composition factors and structure theory for semisimple algebras, so that we know semisimple modules (and algebras) can be completely understood once we know their composition factors.

Lecture 3

We have seen Jordan-Hölder theorem, which tells us that the decomposition of a module into composition factors ('irreducible constituents' in the language of classical representation theory) does not 'change'. One could have also considered the decomposition of a module into direct sum of smaller ones, and ask whether such a decomposition is unique (up to permutation of the direct summands).

Definition 3.1. *Let A be a K-algebra and M be an A-module.*

(1) *M is indecomposable if $M = L \oplus N$ implies that either L or N is zero.*

(2) *We say that $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be unique if for any other decomposition $M = \bigoplus_{j=1}^{n} N_j$, we have $n = m$ and the N_j's are permutation of the M_i's.*

(3) *$A\,\mathsf{mod}$ is said to be Krull-Schmidt if every finitely generated A-module M admits a unique indecomposable decomposition.*

(4) *A ring R is local if it has a unique maximal left (equivalently, right) ideal.*

Theorem 3.2. *Suppose $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \le i \le m$, then the decomposition of M is unique.*

Proof We proceed by induction on m. This is clear if $m = 0, 1$. Suppose that $m > 1$ and we have another decomposition $M = \bigoplus_{j=1}^{n} N_j$. Consider the homomorphisms given by composing canonical inclusions and projections

$$\begin{tikzcd} & M \arrow[dr, two heads] & \\ N_j \arrow[ur, hook] \arrow[rr, "\alpha_j"] & & M_1, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & M \arrow[dr, two heads] & \\ M_1 \arrow[ur, hook] \arrow[rr, "\beta_j"] & & N_j. \end{tikzcd}$$

Then we have $\sum_j \alpha_j\beta_j = \mathrm{id}_{M_1}$. Since $\mathrm{End}_A(M_1)$ is local and each $\alpha_j\beta_j \in \mathrm{End}_A(M_1)$, there is some j such that $\alpha_j\beta_j$ is a unit. Without loss of generality, we can take $j = 1$, and so $M_1 \cong N_1$.

In order to apply induction hypothesis, we need isomorphism $f : \bigoplus_{i=2}^{m} M_i = M/M_1 \to M/N_1 = \bigoplus_{j=2}^{n} N_j$. This amounts to finding an isomorphism $\hat{f} : M \to M$ such that $\hat{f}(M_1) = N_1$. Let $\hat{f} := \mathrm{id}_M - p + qp \in \mathrm{End}_A(M)$, where p and q are given by

$$\begin{tikzcd} & M_1 \arrow[dr, hook] & \\ M \arrow[ur, two heads] \arrow[rr, "p"] & & M, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & N_1 \arrow[dr, hook] & \\ M \arrow[ur, two heads] \arrow[rr, "q"] & & M \end{tikzcd}$$

respectively.

We first show that $\hat{f}$ is an isomorphism; it suffices to show that this is injective by dimension of the domain and the range. Indeed, if $\hat{f}(m) = 0$, then as $p^2 = p$, we have

$$0 = (p\hat{f})(m) = p(m) - p^2(m) + pqp(m) = pqp(m)$$

Observe from the definition of pqp that we have the following commutative diagram:

$$\begin{tikzcd} & M_1 \arrow[rr, "\beta_1"] \arrow[dr, hook] & & N_1 \arrow[rr, "\alpha_1"] \arrow[dr, hook] & & M_1 \arrow[dr, hook, "\iota_1"] & \\ M \arrow[ur, two heads, "\pi_1"] \arrow[rr, "p"] & & M \arrow[ur, two heads] \arrow[rr, "q"] & & M \arrow[ur, two heads] \arrow[rr, "p"] & & M \end{tikzcd}$$

Since $\alpha_1\beta_1$ is a unit and ι_1 is an injection, $\iota_1\alpha_1\beta_1$ is injective. Hence, $pqp(m) = \iota_1\alpha_1\beta_1(\pi_1(m)) = 0$ implies that $\pi_1(m) = 0$. But $p = \iota_1\pi_1$, and so $p(m) = \iota_1(\pi_1(m)) = 0$, which then implies that $\hat{f}(m) = m - p(m) + qp(m) = m$. Hence, $\hat{f}(m) = 0$ implies that $m = 0$ as required.

Let us now consider $\hat{f}(M_1)$. Since $qp = \iota_1\alpha_1\pi_1$ and we have shown that α_1 is an isomorphism, $\hat{f}(m_1) = m_1 - m_1 + \iota(\alpha_1(m_1)) = \iota(\alpha_1(m_1))$ for all $m_1 \in M_1$. Hence, $\hat{f}(M_1) = N_1$ as required. $\square$

Tensor and dual

Let us now come back to the setting of group algebra (group representation) and look at various natural way to construct new representations from old.

Definition 3.3. *Let V, W be finite-dimensional K-vector space with bases, say, $\mathcal{B}, \mathcal{C}$ respectively. Then the tensor product $V \otimes_K W$ (or simplifies to $V \otimes W$ if context is clear) is the finite-dimension K-vector space with bases given by*

$$\{v \otimes w \mid v \in \mathcal{B}, w \in \mathcal{C}\}.$$

Notation. For $V \in K\,\mathsf{mod}$, $V^* := \mathrm{Hom}_K(V, K)$ denotes the dual vector space.

The following innocent looking isomorphisms are arguably the most used isomorphisms in homological algebra.

Lemma 3.4. *For any finite-dimensional K-vector spaces U, V, W, the following hold.*

(1) $V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$.

(2) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W))$.

Proof (1) Let $\mathcal{B} = \{v_1, \ldots, v_m\}, \mathcal{C} = \{w_1, \ldots, w_n\}$ be bases of V, W respectively. Let $\mathcal{B}^* = \{f_1, \ldots, f_m\}$ be the canonical dual basis, i.e. $f_i(v_j) = \delta_{i,j}$ for all $1 \leq i, j \leq m$.

Define $\theta(f_i \otimes w_j)$ to be the K-linear map that extends $v_k \mapsto f_i(v_k)w_j \in W$ and check that θ is K-linear.

Conversely, for $\alpha \in \mathrm{Hom}_K(V, W)$, let $\phi(\alpha) := \sum_i f_i \otimes \alpha(v_i)$. Check that ϕ and θ are inverse to each other.

(2) Define

$$\theta : \mathrm{Hom}_K(U \otimes V, W) \to \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)), \quad f \mapsto \theta_f,$$

where $\theta_f(u) : V \to W$ is the map that sends $v \in V$ to $f(u \otimes v) \in W$.

Define also

$$\phi : \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)) \to \mathrm{Hom}_K(U \otimes V, W), \quad f \mapsto \phi_f,$$

where $\phi_f(u \otimes v) := (f(u))(v)$. Check that ϕ and θ are inverse to each other. $\square$

Remark 3.5. The isomorphism (1) absolutely require finite-dimensionality. The isomorphism (2) is called ‘currying’ in computer science, coined from Curry-Howard correspondence. This isomorphism is actually natural, and yields an adjoint pair $(- \otimes_K V, \mathrm{Hom}_K(V, -))$ of functors.

Proposition 3.6. *Let A, B be K-algebras. Then $A \otimes_K B$ is also a K-algebra with multiplication given by extending $(a \otimes b)(a' \otimes b') \mapsto aa' \otimes bb'$ linearly. For $M \in A\,\mathsf{mod}$ and $N \in B\,\mathsf{mod}$, we have $M \otimes_K N \in A \otimes_K B\,\mathsf{mod}$.*

Proof Routine checking. $\square$

Example 3.7. *Consider $A = (a_{i,j})_{1 \leq i,j \leq m} \in \mathrm{Mat}_m(K)$ and $B \in \mathrm{Mat}_n(K)$ and defines (what is* *sometimes called Kronecker product of matrices)*

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & \ddots & & a_{2,m}B \\ \vdots & & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,m}B \end{pmatrix}.$$

Then we have an isomorphism of algebras

$$\mathrm{Mat}_m(K) \otimes_K \mathrm{Mat}_n(K) \to \mathrm{Mat}_{mn}(K), \quad (A, B) \mapsto A \otimes B.$$

From this, we can see that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, if (and only if) both A, B are invertible. Thus, the isomorphism restricts to a group isomorphism $\mathrm{GL}(K^{\oplus m}) \otimes_K \mathrm{GL}(K^{\oplus n}) \cong \mathrm{GL}(K^{\oplus mn})$.

Exercise 3.8. *Show that the tensor product algebra $KG \otimes_K (KG)^{\mathrm{op}}$ is isomorphic to the group algebra $K(G \times G)$ of the direct product $G \times G$ as K-algebras.*

One thing that makes group algebras special is that we can always 'tensor within the category of G-representations':

Proposition 3.9. *For any $V, W \in KG\,\mathsf{mod}$, we have $V \otimes_K W \in KG\,\mathsf{mod}$ where the action of g is given by $v \otimes w \mapsto gv \otimes gw$.*

Proof Let $\mathcal{B}, \mathcal{C}$ be the K-linear bases of V, W respectively and consider their respective representations $\rho : G \to \mathrm{GL}(V)$ and $\phi : G \to \mathrm{GL}(W)$. Then ρ and ϕ extends to a group homomorphism $G \to \mathrm{Mat}_r(K)$ for $r = m := |\mathcal{B}|$ and $r = n := |\mathcal{C}|$ respectively. Define

$$\rho \otimes \phi : G \to \mathrm{Mat}_{mn}(K) = \mathrm{Mat}_m(K) \otimes \mathrm{Mat}_n(K), \quad g \mapsto \rho_g \otimes \phi_g,$$

where ρ_g, ϕ_g are regarded as matrices. By the discussion in Example 3.7, this map factors through $\mathrm{GL}(V \otimes W)$. Hence, $\rho \otimes \phi$ is a representation of G, and it is clear by construction $\rho_g \otimes \phi_g$ corresponds to the given action of g on the vector space $V \otimes W$. □

Remark 3.10. Proposition 3.9 holds for any *Hopf algebra* in place of KG. Otherwise, for $M \in A\,\mathsf{mod}$ and $N \in B\,\mathsf{mod}$ with A, B algebras, then $M \otimes_K N$ is only a $A \otimes_K B$-module. In the case when $B = A$, we need a ring homomorphism $A \to A \otimes_K A$ in order to induce an A-module structure on $M \otimes_K N$; when A is a Hopf algebra, then such a ring homomorphism is given by the comultiplication map.

Exercise 3.11. *Let A be the ring of upper 2×2-triangular matrices. Let V_1 be the column space $\binom{K}{0}$ and V_2 be the column space $\binom{K}{K}$; i.e. the modules $M_{1,1}$ and $M_{1,2}$, respectively, in the notation of Example 2.4. Consider the identity element $1_A = e_1 + e_2$ where e_i is the matrix with (i,i)-entry 1 and zero everywhere else. Use this decomposition of 1_A to show that $V_1 \otimes_K V_2$ cannot be an A-module if we define a candidate A-action by $v_1 \otimes v_2 \mapsto av_1 \otimes av_2$ for all $a \in A$.*

Exercise 3.12. *Show that $\mathrm{triv}_G \otimes_K V \cong V$ for all $V \in KG\,\mathsf{mod}$.*

Detour: Even in good characteristics, tensor products of group (or Hopf algebra in general) representations is still active theme of researches that falls under the realm of *categorification* - the more precise problem is: For $V, W \in KG\,\mathsf{mod}$, describes the indecomposable direct summands of $V \otimes_K W$.

For example, in the representation theory of symmetric groups (its generalisations such as the Hecke algebra), the Mullineux problem asks for the description of $V \otimes_K \mathrm{sgn}$ for each irreducible V. Another example is McKay correspondence (which has deep implications in algebraic geometry) which comes from looking at tensor product representation of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Let us move on to the next construction.

Definition 3.13. *Let $V, W \in KG\,\mathsf{mod}$ and $g \in G$.*

(1) *For any K-linear map f in the (K-linear) dual space $V^* := \mathrm{Hom}_K(V,K)$, define $g \cdot f$ to be the K-linear map $v \mapsto f(g^{-1}v)$ for all $v \in V$.*

(2) *For any K-linear map $f \in \mathrm{Hom}_K(V,W)$, define $g \cdot f$ to be the K-linear map $v \mapsto gf(g^{-1}v)$ for all $v \in V$.*

Exercise. *Check that the two maps in the above definition yield two representations of G.*

Remark 3.14. Let ρ be the representation corresponding to $V \in KG\,\mathsf{mod}$, and ρ^* be the representation corresponding to V^*. Then $(\rho^*)_g = (\rho_{g^{-1}})^\top$ (the transpose of $\rho_{g^{-1}}$).

Although $V^* \cong V$ for any (finite-dimensional) K-vector space, this generally does not lift to an isomorphism of KG-modules.

Definition 3.15. *$V \in KG\,\mathsf{mod}$ is self-dual if $V^* \cong V$ as KG-modules.*

Exercise. *Trivial representation is clearly self-dual. Check that $\mathrm{sgn} \in K\mathfrak{S}_n\,\mathsf{mod}$ is self-dual.*

Proposition 3.16. *The regular representation is self-dual.*

Proof KG has K-linear basis G. The canonical (dual) basis of $(KG)^*$ is given by $\{f_g \mid g \in G\}$ where $f_g(h) := \delta_{g,h}$, i.e. $f_g(g) = 1$ and $f_g(h) = 0$ for all $h \in G \setminus \{g\}$.

Consider the K-linear map $\alpha : KG \to (KG)^*$ given by linearly extending $g \mapsto f_g$. This is clearly a K-vector space isomorphism. So we only need to show that $\alpha \in KG\,\mathsf{mod}$. For any $g,h,k \in G$, we have

$$(h\alpha(g))(k) = (h \cdot f_g)(k) = f_g(h^{-1}k) = \delta_{g,h^{-1}k} = \delta_{hg,k} = f_{hg}(k) = (\alpha(gh))(k).$$

This shows the claim. □

Remark. In ring theory, this is the same as saying that KG is self-injective (and in fact, Frobenius and symmetric).

In general, finding self-dual representations amounts to finding a 'G-invariant bilinear form'.

Proposition 3.17. *Suppose $\langle -,- \rangle : U \times V \to K$ is a G-invariant non-degenerate bilinear pairing of $U,V \in KG\,\mathsf{mod}$, i.e. $\langle gu, gv \rangle = \langle u,v \rangle$ for all $g \in G$ and all $u \in U, v \in V$. Then $U \cong V^*$ as KG-module.*

Proof Recall that for finite-dimensional K-vector spaces U,V, a non-degenerate bilinear pairing $\langle -,- \rangle : U \otimes V \to K$ yields an isomorphism $U \cong V^*$ via $u \mapsto \langle u, - \rangle$. One just needs to check that when $\langle -,- \rangle$ is G-invariant, then this K-vector space isomorphism lifts to a KG-module homomorphism. Indeed, if we write $f_u := \langle u, - \rangle$, then we have

$$f_{gu}(v) = \langle gu, v \rangle = \langle gu, g(g^{-1}(v)) \rangle = \langle u, g^{-1}(v) \rangle = f_u(g^{-1}v) = (g \cdot f_u)(v).$$

This shows the claim. □

Exercise 3.18. *For $V,W \in KG\,\mathsf{mod}$, show that there are the following isomorphisms.*

(1) *$(V \otimes_K W)^* \cong V^* \otimes_K W^*$ as KG-modules.*

(2) *$V^* \otimes_K W \cong \mathrm{Hom}_K(V,W)$ as KG-modules.*

Exercise 3.19. *Suppose X is a G-set (i.e. G acts by permuting elements of X) or a KG-module, denote by X^G the invariant subspace $\{x \in X \mid gx = x\,\forall g \in G\}$ of X. Let $U,V,W \in KG\,\mathsf{mod}$.*

(1) *Show that $(V^* \otimes_K V)^G \cong \mathrm{End}_{KG}(V)$.*

(2) *Show that $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W)$*

Lecture 4

To understand operation on a representation, it is natural to start looking at its effect on the simples. Naively, one may guess that being simple is preserved under taking the dual representation. This is our next aim. To this end, we want to construct submodule of the dual representation from a submodule of the original. Since duality swaps injective map with surjective map, simply taking the dual of a submodule will not gives us the submodule of the dual. But we may consider its complement in the following sense.

Definition 4.1. *Let $V \in K\,\mathsf{mod}$. For a K-linear subspace $U \subset V$, define a K-vector subspace*

$$U^\circ := \{f \in V^* \mid f(u) = 0, \forall u \in U\} \subset V^*.$$

For a K-linear subspace $L \subset V^*$, define a K-vector subspace

$$L^\perp := \{v \in V \mid f(v) = 0 \,\forall f \in L\} \subset V.$$

Lemma 4.2. *Consider $V \in K\,\mathsf{mod}$, $U \subset V$ and $L \subset V^*$ are K-linear subspaces.*

(1) $\dim_K L^\perp = \dim_K V - \dim_K L$

(2) $\dim_K U^\circ = \dim_K V - \dim_K U$.

Proof We show the first one; the other one is analogous. Pick a basis $\{f_1, \ldots, f_m\}$ of L and extends it to a basis $\{f_1, \ldots, f_n\}$ of V^*. Let $\{e_1, \ldots, e_n\}$ be the dual basis, i.e. $f_i(e_j) = \delta_{i,j}$. Then by definition $e_j \in L^\perp$ if and only if $m < j \leq n$. □

Lemma 4.3. *Consider $V \in KG\,\mathsf{mod}$.*

(1) *If $L \subset V^*$ a KG-submodule, then $L^\perp$ is a KG-submodule of V.*

(2) *If $U \subset V$ is a KG-submodule, then U° is a KG-submodule of V^*.*

Proof (1) For any $g \in G$ and any $w \in L^\perp$, since $(g^{-1} \cdot f)(w) = f(g(w))$ and $g^{-1} \cdot f \in L$, we have $f(g(w)) = 0$, and so $L^\perp$ is closed under G-action.

(2) For any $g \in G$ and any $f \in {}^\perp U$, since $(g \cdot f)(u) = f(g^{-1}(u))$ and $g^{-1}(u) \in U$, we have $(g \cdot f)(u) = 0$, and so ${}^\perp U$ is closed under G-action. □

Proposition 4.4. *For $V \in KG\,\mathsf{mod}$, V is simple if and only if so is V^*.*

Proof Consequence of Lemma 4.2 and Lemma 4.3. □

In general, simple KG-module is not always self-dual, not even when $K = \mathbb{C}$, but ordinary character theory provides a simple way to check whether a simple $\mathbb{C}G$-module is self-dual.

Definition 4.5. *Let ρ be a representation of G over $\mathbb{C}$, and V be its corresponding $\mathbb{C}G$-module. Then the (ordinary) character of ρ (or of V) is the map*

$$\chi_\rho = \chi_V : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(\rho(g)),$$

where Tr is the trace function (i.e. sum of all eigenvalues).

We will explore more on characters later in the course. For now, we just note that character is a representation-invariant, i.e. $V \cong W$ as $\mathbb{C}G\,\mathsf{mod}$ implies that $\chi_V = \chi_W$.

Lemma 4.6. *For any $g \in G$, $\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$, where $\overline{z}$ denotes the conjugate of $z \in \mathbb{C}$. In particular, if V is self-dual, then its character χ_V is real-valued.*

Proof Recall that $\rho^*(g) = (\rho(g)^{-1})^\top$. Suppose $\lambda_1, \dots, \lambda_n$ are the eigenvalues (counted with multiplicity, i.e. $n = \dim_{\mathbb{C}} V$) of $\rho(g)$. Since G is finite, $\rho(g)$ has finite order, and so every eigenvalue is a root of unity. So we have

$$\chi_{V^*}(g) = \mathrm{Tr}(\rho_V(g^{-1})^\top) = \mathrm{Tr}(\rho_V(g^{-1})) = \chi_V(g^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\chi_V(g)}$$

for all $g \in G$. □

Remark 4.7. This requires G being finite.

Induction and Restriction.

Definition 4.8. *Let A be a K-algebra, M be a right A-module, and N be a left A-module. Then the tensor product $M \otimes_A N$ of M and N over A is the quotient K-vector space $M \otimes_K N/R$, where*

$$R = \{ma \otimes n - m \otimes an \mid m \in M, a \in A, n \in N\}.$$

WARNING: $M \otimes_A N$ is generally not an A-module.

Definition 4.9. *Let A, B be K-algebras. An K-vector space M is an A-B-bimodule if it is a left A-module and right B-module with commuting A- and B-action, i.e. $r(ms) = (rm)s$ for all $r \in R, m \in M, s \in S$. In other words, it is a left module over $A \times B^{\mathrm{op}}$ (equivalently, right module over $B \times A^{\mathrm{op}}$).*

Lemma 4.10. *Consider rings A, B, C. Let M be an A-B-bimodule, N be an B-C-bimodule, and L be an A-C-bimodule.*

(1) *$M \otimes_B N$ is a A-C-bimodule given by $a \cdot (m \otimes n) := (am) \otimes n$ and $(m \otimes n) \cdot c := m \otimes (nc)$.*

(2) *$\mathrm{Hom}_A(L, M)$ is a C-B-bimodule given by $(c \cdot f)(l) := (f(lc))$ and $(f \cdot b)(l) := f(l)b$.*

Proof Exercise. □

The above lemma tells us that tensor and Hom can be used to transfer modules (in fact, even homomorphisms) between different rings. Another consequence of Lemma 4.10 is that, if R is a commutative ring, then R-modules are the same as R-R-bimodules, and so $M \otimes_R N$ are automatically R-modules for R-modules M and N. Similarly, as left (resp. right) modules over a K-algebra, say A, are really A-K-bimodules (resp. K-A-bimodules), and so $M \otimes_A N$ is automatically a K-vector space.

Example 4.11. (1) *$A \otimes_A M \cong M$ as left A-module for all $M \in A\,\mathsf{mod}$.*

(2) *Suppose $\phi \in \mathrm{Aut}_K(A)$ is a K-linear (ring) automorphism of A. For $M \in A\,\mathsf{mod}$, let ${}_\phi M$ be the left A-module where the left A-action is twisted by ϕ, i.e. am on ${}_\phi M$ is given by $\phi(a)m$ on the original M. Consider A as an A-A-bimodule (action being multiplication), and write ${}_\phi A_1$ the A-A-bimodule with left action twisted by ϕ. Then ${}_\phi A \otimes_A M \cong {}_\phi M$.*

Recall the 'useful isomorphism' in Lemma 3.4; it has the following enhanced version.

Lemma 4.12. *Suppose A, B are K-algebras, X is an A-B-bimodule. Then for any $M \in B\,\mathsf{mod}, N \in A\,\mathsf{mod}$, there is a K-vector space isomorphism $\mathrm{Hom}_A(X \otimes_B M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(X, N))$.*

Proof Verbatim to the proof of Lemma 3.4. □

Definition 4.13. *Suppose $H \leq G$.*

(1) *For $V \in KG\,\mathsf{mod}$, its restriction to H, denoted by $\mathrm{Res}_H^G(V)$ or $V \downarrow_H^G$, is KH-module given by the same K-vector space where H-action is inherited from G-action.*

(2) *For $U \in KH\,\mathsf{mod}$, its* induction *to G (a.k.a. induced representation, induced module), denoted by $\mathrm{Ind}_H^G(U)$ or $U\uparrow_H^G$, is the KG-module given by $KG \otimes_{KH} U$.*

Remark 4.14. G-action on $\mathrm{Ind}_H^G(U)$ can be described as follows. Take coset representatives $g_1, \ldots, g_n$, i.e. $G/H = \{g_1H, \ldots, g_nH\}$. For $g \in G$, we have $gg_iH = g_jH$ for some j, i.e. $gg_i = g_jh$ for some $h \in H$. This yields, for any $m \in M$, the following g-action on $g_i \otimes m \in \mathrm{Ind}_H^G(U)$:

$$g(g_i \otimes u) = (gg_i) \otimes u = g_jh \otimes u = g_j \otimes hu.$$

Remark 4.15. $KG \otimes_{KH} -$ is functorial (i.e. it can be applied to homomorphisms in a way that preserves axioms regarding compositions). Restriction can be made functorial by noticing that

$$\mathrm{Res}_H^G(V) = \mathrm{Hom}_{KG}({}_{KG}KG_{KH}, V)$$

where KG in the domain here is regarded as a KG-KH-bimodule.

Lemma 4.16. *Consider subgroup $H \le G$ with coset representatives $g_1, \ldots, g_n$.*

(1) *The right KH-module KG is free of rank n, namely, $(KG)_{KH} \cong (KH)^{\oplus n}$ in $\mathsf{mod}\,KH$.*

(2) *If $U \in KH\,\mathsf{mod}$ has K-basis $\mathcal{B}$, then $\mathrm{Ind}_H^G(U)$ has K-basis $\{g_i \otimes b \mid b \in \mathcal{B}, 1 \le i \le n\}$, i.e. $\dim_K \mathrm{Ind}_H^G(U) = |G/H| \dim_K(U)$.*

Proof (1) Clearly, as K-vector space we have decomposition $KG = \bigoplus_{i=1}^n K(g_iH)$. Since $g_ihh' \in g_iH$ for all $h, h' \in H$, each $K(g_iH)$ is isomorphic to KH as a right H-module.

(2) Now we have K-vector space isomorphisms:

$$\mathrm{Ind}_H^G(U) = KG \otimes_{KH} U \cong (\bigoplus_{i=1}^n g_i \cdot KH) \otimes_{KH} U \cong \bigoplus_{i=1}^n g_i \cdot U,$$

and the claim follows. □

Example 4.17. *Suppose $H \le G$ is a subgroup. Consider the K-vector space $M_H := K(G/H)$ whose basis is given by the set of left G-cosets G/H. Then M_H is a KG-module. It follows from Lemma 4.16 (1) that $M_H \cong \mathrm{Ind}_H^G(\mathrm{triv}_H)$.*

Lemma 4.18. *Suppose we have subgroups $L \le H \le G$. Then $\mathrm{Ind}_H^G \mathrm{Ind}_L^H(U) = \mathrm{Ind}_L^G(U)$ for all $U \in KL\,\mathsf{mod}$.*

Proof This follows from the fact that $M \otimes_A (N \otimes_B L) \cong (M \otimes_A N) \otimes_B L$ as bimodules (check yourself). Namely, $KG \otimes_{KH} (KH \otimes_{KL} U) \cong (KG \otimes_{KH} KH) \otimes_{KL} U = KG \otimes_{KL} U$. □

Exercise 4.19. *Let $H \le G$, $V \in KG\,\mathsf{mod}$ and $W \in KH\,\mathsf{mod}$. Show that*

(1) $\mathrm{Ind}_H^G(W^*) \cong (\mathrm{Ind}_H^G(W))^*$.

(2) $V \otimes_K \mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes_K W)$.

Lemma 4.20 (Eckmann-Shapiro lemma). *There are K-vector space isomorphisms:*

(1) (Frobenius reciprocity) $\mathrm{Hom}_{KG}(\mathrm{Ind}_H^G U, V) \cong \mathrm{Hom}_{KH}(U, \mathrm{Res}_H^G V)$.

(2) $\mathrm{Hom}_{KG}(V, \mathrm{Ind}_H^G U) \cong \mathrm{Hom}_{KH}(\mathrm{Res}_H^G V, U)$.

Proof (1) Special case of Lemma 4.12.

(2) For $f : \mathrm{Res}_H^G V \to U$, define $\theta_f : V \to \mathrm{Ind}_H^G(U)$ to be the map $v \mapsto \sum_{g_i \in G/H} g_i \otimes f(g_i^{-1}v)$. It is routine to check that θ_f is a KG-module homomorphism and so we have a map $\theta : \mathrm{Hom}_{KH}(\mathrm{Res}_H^G V, U) \to \mathrm{Hom}_{KG}(V, \mathrm{Ind}_H^G U)$. It is clear that θ is K-linear and injective.

To show surjective, take homomorphism $f : V \to \mathrm{Ind}_H^G(U)$ and write $f(v) = \sum_{g_i \in G/H} g_i \otimes f_i(v)$. We have $h(f(v)) = h \sum_i g_i \otimes f_i(v) = \sum_i hg_i \otimes f_i(v)$, and $f(hv) = \sum_i g_i \otimes f_i(hv)$ for all $h \in H$. Since f is a KG-module homomorphism, we have $\sum_i hg_i \otimes f_i(v) = \sum_i g_i \otimes f_i(hv)$. Note that we can take g_1 to be the identity element of G, and so using $h \otimes f_1(v) = g_1 h \otimes f_1(v) = g_1 \otimes hf_1(v)$ we have

$$g_1 \otimes hf_1(v) + \sum_{i \neq 1} hg_i \otimes f_i(v) = g_1 \otimes f_1(v) + \sum_{i \neq 1} g_i \otimes f_i(hv).$$

This means that $v \mapsto f_1(v)$ is a KH-module homomorphism.

On the other hand, if we consider $g_j^{-1} f(v) = f(g_j^{-1} v)$, then we have

$$\sum_i g_j^{-1} g_i \otimes f_i(v) = \sum_i g_i \otimes f_i(g_j^{-1} v),$$

which yields

$$g_1 \otimes f_j(v) + \sum_{i \neq j} g_j^{-1} g_i \otimes f_i(v) = g_1 \otimes f_1(g_j^{-1} v) + \sum_{i \neq j} g_i \otimes f_i(g_j^{-1} v),$$

meaning that $f_1(g_j^{-1} v) = f_j(v)$. Hence, we have the map θ_{f_1} is given by

$$\sum_i g_i \otimes f_1(g_i^{-1} v) = \sum_i g_i \otimes f_i(v) = f(v).$$

This proves the required surjection. □

Remark 4.21. Both of these isomorphisms are (bi-)natural. In particular, this means that Ind_H^G and Res_H^G are biadjoint functors.

For time constraint, we omit the proof of the following theorem.

Theorem 4.22 (Mackey decomposition theorem). *For $H, L \leq G$. Let $U \in KL\,\mathsf{mod}$. Then there is the following KH-module isomorphism*

$$U \uparrow_L^G \downarrow_H^G \cong \bigoplus_{t \in H \backslash G / L} ({}^t U) \downarrow^L_{H \cap {}^t L} \uparrow^H_{H \cap {}^t L},$$

where $H \backslash G / L$ denotes the set of double cosets $\{HgL \mid g \in G\}$, and ${}^t L := \{t \ell t^{-1} \mid \ell \in L\}$ and ${}^t U \in K {}^t L\,\mathsf{mod}$ is given by $x \cdot u := txt^{-1} u$ for all $x \in L$ and $u \in U$.