Ex 1. Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. Show that $\det \rho$ is also a representation.

Ex 2. Let V be the 1-dimensional subspace spanned by $\sum_{g\in G} g \in KG$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.

Ex 3. Fix any $n \geq 2$.

(i) Find a generator v such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g\in G} g$ of the trivial representation.)

(ii) Show that $\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ when $\mathrm{char}\, K = 2$, otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

Ex 4. Classify (with reason) all simple $\mathbb{C}\mathfrak{S}_3$-modules (up to isomorphism). (Hint: there are only three of them.)

Ex 5. Recall that the center of $GL_n(K)$ is just $K^\times \mathrm{Id} = \{\lambda \mathrm{Id} \mid \lambda \in K^\times = K \setminus \{0\}\}$.

(i) Classify (with reason) all simple $\mathbb{C}C_n$-modules (up to isomorphism) of the cyclic group C_n of order $n \geq 2$. (Hint: To find, notice that $\mathbb{C}$ has all roots of 1. To show you have all, consider dimensionality and Artin-Wedderburn.)

(ii) Classify all simple $\mathbb{C}G$-modules (up to isom.) of the abelian group $G = C_m \times C_n$ for $m, n \geq 2$.

(iii) Classify all simple $\mathbb{C}G$-modules (up to isom.) of a finite abelian group G (recall that a finite abelian group is a finite direct product of finite cyclic groups, i.e. $G = C_{n_1} \times \cdots \times C_{n_r}$).

Ex 6. Let A be the algebra of upper triangular $n \times n$-matrices:

$$A := \begin{pmatrix} K & K & \cdots & K \\ 0 & K & \cdots & K \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & K \end{pmatrix} = \left\{ (a_{i,j})_{1\leq i,j\leq n} \,\middle|\, \begin{array}{l} a_{i,j} \in K \ \forall i,j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}$$

For $1 \leq i \leq j \leq n$, let $M_{i,j} \subset K^{\oplus n}$ be the vector space given by column vectors $v = (v_k)_{1\leq k\leq n}$ where $v_k = 0$ for $k \notin \{i, i+1, \ldots, j\}$.

(i) Determine which $M_{i,j}$'s are simple.

(ii) Describe the composition series of $M_{i,j}$.