Topics in Mathematical Science V
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From quiver to quasi-hereditary algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M, +)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have*

- $m \cdot 1 = m$,
- $(m + n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r + s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $f(R)$ being in the center $Z(A) := \{z \in A \mid za = az \ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_{\Bbbk} A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the characteristic of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the direct product of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Definition 1.7. *A map $f : M \to N$ between right R-modules M, N is a homomorphism if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\operatorname{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$.*

Lemma 1.8. *$\operatorname{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \operatorname{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\operatorname{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.9. *$\operatorname{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the endomorphism ring of M.*

Lemma 1.10. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\operatorname{Hom}_A(M, N)$ is also an R-module (hence, $\operatorname{End}_R(M)$ is an R-algebra).*

Example 1.11. *$A \cong \operatorname{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra).*

Exercise 1.12. *Recall that R^{op} is the opposite ring of R, whose underlying set is the same as that of R with multiplication $(a \cdot^{\mathrm{op}} b) := b \cdot a$. A representation of R is a ring homomorphism*

$$\rho : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(M), \qquad r \mapsto \rho_r,$$

for some abelian group $(M, +)$. A homomorphism $f : \rho_M \to \rho_N$ of representations $\rho_M : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(M), \rho_N : R^{\mathrm{op}} \to \operatorname{End}_{\mathbb{Z}}(N)$ given by an abelian group homomorphism $f : M \to N$ that intertwines R-action, i.e. $\rho_N(r) \circ f = f \circ \rho_M(r)$ for all $r \in R$.

Eplain why a representation of R is equivalent to a right R-module; and why homomorphisms correspond.

Indecomposable modules and Krull-Schmidt property

Recall that an R-module M is finitely generated if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x\in X} x r_x$ for some $r_x \in R$.

Notation. *We write* $\operatorname{mod} A$ *for the collection of all finitely generated right A-modules.*

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \dots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + \hat{N_j} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be unique if for any other decomposition $M = \bigoplus_{j=1}^n N_j$, we have $n = m$ and the N_j's are permutation of the M_i's.

Convention. *We write $(n_1, \dots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose R_R is indecomposable as an R-module. Then the free module $R \oplus R \oplus \cdots \oplus R$ with R copies of R is a decomposition of R^n.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1\leq j\leq n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \geq 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. By definition $(av_1 + bv_2)x = (a+b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules; a contradiction. Hence, V must be indecomposable.

A natural question is to ask when a decomposition of modules, if it exists, is unique up to permuting the direct summands.

Definition 2.5. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^m M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^n N_j$ implies that $m = n$ and there is a permutation σ such* *that* $M_i \cong N_{\sigma(i)}$ *for all* $1 \leq i \leq m$*.* $\mathsf{mod}\, A$ *is said to be Krull-Schmidt if every finitely generated* A*-module* M *admits a unique indecomposable decomposition.*

Theorem 2.6. *For a finite-dimensional algebra* A*,* $\mathsf{mod}\, A$ *is Krull-Schmidt.*

Remark 2.7. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Extra: Krull-Schmidt theorem

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent.

Lemma 3.1. *A non-zero A-module M is indecomposable if, and only if, the endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*

Proof $\Leftarrow$: Suppose $M = U \oplus V$. Then we have

$$\begin{aligned}\text{a projection map } \pi_W : M \twoheadrightarrow W,\\ \text{and an inclusion map } \iota_W : W \hookrightarrow M,\end{aligned}$$

for $W \in \{U, V\}$. Both of these are clearly A-module homomorphisms. Now $e_W := \iota_W \pi_W$ is an endomorphism of M with $e_V = \mathrm{id}_M - e_U$. Since any $m \in M$ can be written as $u + v$ for $u \in U$ and $v \in V$, we have

$$e_V^2(m) = e_V^2(u + v) = e_V^2(v) = v = e_V(m);$$

and likewise for e_W, so we have idempotents different from 0 and id_M when both U and V are non-zero.

$\Rightarrow$: Suppose that M is indecomposable, and $e \in \mathrm{End}_A(M)$ is an idempotent. Note that

$$(\mathrm{id}_M - e)^2 = \mathrm{id}_M - e \cdot \mathrm{id}_M - \mathrm{id}_M \cdot e + e^2 = \mathrm{id}_M - 2e + e = \mathrm{id}_M - e$$

is also an idempotent and $\mathrm{id}_M = e + (\mathrm{id}_M - e)$. So we have $M = e(M) + (\mathrm{id}_M - e)(M)$. We want to show that $M = e(M) \oplus (\mathrm{id}_M - e)(M)$, i.e. $e(M) \cap (\mathrm{id}_M - e)(M) = 0$. Indeed, $x \in e(M) \cap (\mathrm{id}_M - e)(M)$ means that we have $e(m) = x = (\mathrm{id}_M - e)(m')$ for some $m, m' \in M$, and so

$$x = e(m) = e^2(m) = e((\mathrm{id}_M - e)(m')) = (e(\mathrm{id}_M - e))(m') = (e - e^2)(m') = 0(m') = 0,$$

as required.

Since M is indecomposable, one of $e(M)$ or $(\mathrm{id}_M - e)(M)$ is zero. In the former case, we get $e = 0$; whereas the latter case yields $\mathrm{id}_M = e$; as required. □

The following is one of the main reasons why we like to consider finite-dimensional (or finite generated) modules over finite-dimensional $\Bbbk$-algebras.

Lemma 3.2 (Fitting's lemma (special version)). *Let M be a finite-dimensional A-module of a finite-dimensional $\Bbbk$-algebra, and $f \in \mathrm{End}_A(M)$. Then there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

Remark 3.3. The general version for rings requires M to be artinian and noetherian (i.e. ascending and descending chains of submodules stabilises).

We omit the proof to save time. The point is really just take n large enough so that the chains of submodules given by $(\mathrm{Ker}(f^k))_k$ and $(\mathrm{Im}(f^k))_k$ stabilises.

Corollary 3.4. *Let M be a non-zero finite-dimensional A-module. Then M is indecomposable if, and only if, every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*

Proof By Fitting's lemma, for any $f \in \mathrm{End}_A(M)$, we have $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$ for some $n \geq 1$. So indecomposability means that one of these direct summands is is zero. If $\mathrm{Ker}(f^n) = 0$, then f^n is an isomorphism and so is f. If $\mathrm{Im}(f^n) = 0$, then $f^n = 0$ and so f is nilpotent.

Conversely, consider an idempotent endomorphism $e \in \mathrm{End}_A(M)$. The assumption says that e is either an isomorphism or nilpotent.

If e is an isomorphism, then we have $\mathrm{Im}(e) = M$, which means that for every $m \in M$, there is some $m' \in M$ with $e(m) = e^2(m') = e(m') = m$, i.e. $e = \mathrm{id}_M$.

If e is nilpotent, then $e^n = 0$ for some $n \geq 1$, but $e = e^2 = e^3 = \cdots = e^n$, and so $e = 0$.

Hence, an idempotent endomorphism of M is either 0 or id_M, which means that M is indecomposable by Lemma 3.1. $\square$

Definition 3.5. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 3.6. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit right inverses.

Lemma 3.7. *Let A be a finite-dimensional algebra and M be a finite-dimensional A-module. Then the following hold.*

(1) *The following are equivalent.*
  - *A is local (i.e. has a unique maximal right ideal).*
  - *Non-invertible elements of A form a two-sided ideal.*
  - *For any $a \in A$, one of a or $1 - a$ is invertible.*
  - *0 and 1_A are the only idempotents of A.*
  - *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

(2) *M is indecomposable $\Leftrightarrow \mathrm{End}_A(M)$ is local.*

We omit the proof to save time.

Example 3.8. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1\leq i \leq j \leq 2} \,\middle|\, \begin{matrix} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{matrix} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

Theorem 3.9 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.*

Remark 3.10. Some people refer to this result as Krull-Remak-Schmidt theorem.

For proof, interested reader can see lecture notes from last year.

Simple modules, Schur's lemma

Definition 4.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 4.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 4.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 4.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 4.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 4.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Proof $\Rightarrow$: $mA \subset S$ is a submodule and contains a non-zero element m, so by simplicity of S we must have $mA = S$.

$\Leftarrow$: Suppose that there is a non-zero submodule $L \subset S$. For a non-zero element $m \in L$, the assumption says that we have $mA \subset L \subset S = mA$, and so $L = S$. □

Let us see how one can find a simple module.

Definition 4.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 4.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module.*

Proof Since $S = mA$, the element m defines a surjective A-module homomorphism $f : A_A \to S$ given by $a \mapsto ma$. On the other hand, we have $\mathrm{Ker}(f) = \mathrm{Ann}_A(m)$, and so $A/\mathrm{Ann}_A(m) \cong S$. □

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 4.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. □

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 4.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S,T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T, simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S, so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. □

Example 4.12. *In Example 3.8, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x,y) \mid x,y \in \Bbbk^2\}$ given by 'row vectors'. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0,y) \mid y \in \Bbbk^2\}$.*

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A-action given by

$$w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0,y)$ for some $y \in \Bbbk$ and has to satisfy

$$(0, ya) = (0,y)a = f(wa) = f(w\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = f(w)\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0,y)c = (0,yc)$$

for any $a,b,c \in \Bbbk$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 4.13. *Consider a semisimple A-module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$ for some i.*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this is an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1\le i,j\le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \xrightarrow{a_{i,j}} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. □

Example 4.14. *For a tautological example, take $A = \Bbbk$ to be just a field. Then we have a 1-dimensional simple A-module $S = \Bbbk$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\Bbbk)) = \mathrm{Mat}_n(\Bbbk)$. Note that now we have an n-dimensional simple $\mathrm{Mat}_n(\Bbbk)$-module (given by the row vectors).*

Quiver and path algebra

Definition 5.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 5.2. *Let Q be a quiver.*

- *A trivial path on Q is a "stationary walk at i", denoted by e_i for some $i \in Q_0$.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2 \cdots \alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.*

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right.*

Definition 5.3. *The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying vector space is given by $\bigoplus_{p:\text{paths of } Q} \Bbbk p$, with multiplication given by path concatenation. That is $x \in \Bbbk Q$ is a formal linear combinations of paths on Q.*

Note that $e_i e_j = \delta_{i,j} e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an *idempotent* of the path algebra $\Bbbk Q$. Moreover, we have an idempotent decomposition

$$1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\Bbbk Q$.

Example 5.4. *Consider the one-looped quiver, a.k.a. Jordan quiver,*

$$Q = \left(\begin{tikzcd} \bullet \arrow[loop above, "\alpha"] \end{tikzcd} \right)$$

Then $\Bbbk Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\Bbbk Q \cong \Bbbk[x]$.

An *oriented cycle* is a path of the form $v_1 \to v_2 \to \cdots v_r \to v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is *acyclic*.

Proposition 5.5. *$\Bbbk Q$ is finite-dimensional if, and only if, Q is finite acyclic.*

Proof If there is an oriented cycle c, then $c^k \in \Bbbk Q$ for all $k \geq 0$, and so $\Bbbk Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q. $\square$

Example 5.6. *Consider the linear $\mathbb{A}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\Bbbk Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j \alpha_{j+1} \cdots \alpha_k$.

Consider the upper triangular n-by-n matrix ring

$$\begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \,\middle|\, \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\Bbbk Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \leq j < k \leq n$.

Representations of quiver

Definition 6.1. *A $\Bbbk$-linear representation of Q is a datum $(\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ where M_i is a $\Bbbk$-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is K-linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional if $\dim_{\Bbbk} M_i < \infty$ for all $i \in Q_0$.

Notation. *For a representation M of Q, we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

Example 6.2. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to \Bbbk \xrightarrow{\mathrm{id}} \to \cdots \xrightarrow{\mathrm{id}} \Bbbk \to 0 \to \cdots \to 0$$

with a copy of $\Bbbk$ on vertices $i, i+1, \ldots, j$ is the uniserial $\Bbbk Q$-module corresponding to the column space (under the isomorphism of $\Bbbk Q$ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \leq k \leq j$.

Example 6.3. *Let Q be the Jordan quiver with unique arrow α. Then a representation of Q is nothing but an n-dimensional vector space equipped with a linear endomorphism, equivalently, an n-by-n matrix.*

Definition 6.4. *A homomorphism $f : M \to N$ of ($\Bbbk$-linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram*

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "M_\alpha"'] & N_i \arrow[d, "N_\alpha"] \\ M_j \arrow[r, "f_j"'] & N_j \end{tikzcd}$$

for all arrows $\alpha : i \to j$ in Q.

A homomorphism $f = (f_i)_{i\in Q_0} : M \to N$ of quiver representations is injective, resp. surjective, resp. an isomorphism, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 6.5. *Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n-by-n matrix M_α. By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α. If we assume $\Bbbk$ is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α. That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \ldots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m-by-m Jordan block with eigenvalue $\lambda \in \Bbbk$.*

Proposition 6.6. *There is an isomorphism between the category of representations of Q and $\mathsf{mod}\,\Bbbk Q$, where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i\in Q_0} M_i$ with $\Bbbk Q$-action given by (linear combinations of compositions of) M_α's, and isomorphism classes of Q-representations correspond to isomorphism classes of $\Bbbk Q$-modules.*