# SPIN CANONICAL DIVISORS OF LOG STACKY CURVES

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ABSTRACT. Arithmetic geometry group at the Emory University Number Theory REU.

## CONTENTS



## 1. INTRODUCTION

This is our project. Modular Forms. Edit this stuff

Example, stuff from Voight and Zureick-Brown [5] and O'Dorney's paper [3].

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We consider divisors of the form $D = \sum_{i=1}^{r} \frac{e_i - 1}{2e_i}$.

### 1.1. Main Results.

These are our results.

**Theorem 1.1.** *Let $(\mathcal{X}, \Delta, L)$ be a tame log spin curve over a perfect field $k$, so that $\mathcal{X}$ has signature $\sigma = (0; e_1, \ldots, e_r; \delta)$. Then the canonical ring*

$$R(\mathcal{X}, \Delta, L) = \bigoplus_{d=0}^{\infty} H^0(\mathcal{X}, \mathcal{O}(L)^{\otimes d})$$

*is generated as a $k$-algebra by elements of degree at most $e = \max(5, e_1, \ldots, e_r)$ with relations in degree at most $2e$, so long as $\sigma$ does not lie in a finite list of exceptions.*

Include reference to the table with this list of exceptions.

## 2. Background

Look at Spencer-Kodaira [1].

edit this

### 2.1. Definitions.

Here are some definitions.

**Definition 2.1.** **Saturations** of the canonical divisor, as defined in 7. 2 of VZB [5].

## 3. General Lemmata

First we present the general inductive lemmata that will be applied for all genera.

### 3.1. General: quadratic initial ideal minimal relations.

First we prove a nice result about minimal generation of ideals of relations:

Include one of the following, if the first one, include a remark for quotient

**Lemma 3.1.** *Suppose $k[x_1, \ldots, x_n]$ is a graded ring, not necessarily generated in degree 1, is equipped with a monomial ordering $\prec$. Let $\phi : k[x_1, \ldots, x_n] \to B$ be a map of graded rings with kernel $I$, such that $I$ is generated by elements $f_1, \ldots, f_j$, where $\operatorname{in}_\prec(f_i) = x_i x_j$, and no term of $f_i$ is of the form $x_k$. Then, the elements $f_i$ determine minimal generators for $I$.*

*Proof.* Regrade the ring $k[x_1, \ldots, x_n]$ so that all $x_i$ lie in degree 1. Then,

$$\phi : k[x_1, \ldots, x_n] \to B$$

determines a map of rings, no longer necessarily graded. Consider the grlex monomial ordering $<$ on $k[x_1, \ldots, x_n]$. By assumption, the kernel $I$ is generated by elements whose initial terms have degree 2 and are all distinct. Therefore, $\dim_k(k[x_1, \ldots, x_n])_2 = \binom{n}{2} - j$. This implies any set of generators for $I$ must have at least $j$ elements. Since $f_1, \ldots, f_j$ is such a set, it is minimal. $\square$

This ordering isn't quite right. We need to preserve the ordering on quadratic terms

*Remark* 3.2. It will be useful to use lemma 3.1 in the case $m = 0$, in which case we find explicitly that $f_1, \ldots, f_r$ are $k$-minimal generators for $I'$ (over the trivial base vector space).

TRANSITION

transition!

3.2. **Inductive Theorem: Increasing the Ramification Orders.** In this section, we aim to prove the inductive theorem, Theorem 3.6. Under fairly general conditions, this theorem will allow us to show that if Theorem 3.6 holds for a curve with signature $(0; e_1', \dots, e_r'; 0)$ then it also holds for a curve with signature $(0; e_1, \dots, e_r; 0)$ with $e_i' - e_i \in \{0, 2\}$.

**Definition 3.3.** Let $(\mathcal{X}', 0, L'), (\mathcal{X}, 0, L)$ be tame, separably rooted log stacky spin curves, both ramified over ramified at $Q_1, \dots, Q_r$. Suppose there is a birational map $\mathcal{X} \to \mathcal{X}'$, and let $J \subset \{1, \dots, r\}$. Suppose $e_i' + 2\chi_J(i) = e_i$ where

$$\chi_J(i) = \begin{cases} 1, & \text{if } i \in J \\ 0, & \text{otherwise.} \end{cases}$$

Let $R'$ be the canonical ring associated to $\mathcal{X}'$. Define $(\mathcal{X}', J)$ to be **admissible** if the $R'$ admits a presentation

$$R' \cong \left( k[x_1, \dots, x_m] \otimes k[y_{i,e_i'}]_{i \in J} \right) / I'$$

with $y_{i,e_i'} \in R'$ for each $i \in J$ such that the following three conditions hold.

Peter: We should make the definition of admissibility be a condition for the triple: $(\mathcal{X}', \mathcal{X}, J)$.

(Ad-i) For all $i \in J$, $\deg y_{e_i'} = e_i'$ and $-\mathrm{ord}_{Q_i}(y_{i,e_i'}) = \frac{e_i'-1}{2}$.

(Ad-ii) For all $i \in J$, any generator $z \neq y_{i,e_i'}$ of $R'$ satisfies

$$\frac{-\mathrm{ord}_{Q_i}(z)}{\deg z} < \frac{e_i' - 1}{2e_i'}.$$

(Ad-iii) For all $i \in J$ we have

$$\deg \lfloor e_i L \rfloor \geq 2g - 1 + \max_{d \geq 0} \# S_{\sigma,J}(i, d)$$

where

$$S_{\sigma,J}(i, d) := \{j \in J : j \neq i \text{ and } e_j' + 2d \mid e_i - e_j'\}$$

Check the 2g - 1, in particular, David seems to think we are usually 1 off and possibly 2 off when g = 1. I have a feeling this is a mistake, but don't understand why.

**Lemma 3.4.** *Condition (Ad-ii) of Definition 3.3 implies the stronger condition that*

$$-\mathrm{ord}_{Q_i}(z) \leq \deg(z) \frac{e_i' - 1}{2e_i'} - \frac{1}{e_i'}$$

*Proof.* We know by (Ad-ii) that

$$-\mathrm{ord}_{Q_i}(z) < \deg(z) \frac{e_i' - 1}{2e_i'}$$

If we write $\frac{\alpha}{\beta} = \deg(z)\frac{e_i'-1}{2e_i'}$ as a fraction in lowest terms, then we see $\beta \mid e_i'$ since $e_i' - 1$ is even. Therefore, since $-\mathrm{ord}_{Q_i}(z)$ is an integer, we must have

$$-\mathrm{ord}_{Q_i}(z) \leq \deg(z) \frac{e_i' - 1}{2e_i'} - \frac{1}{\beta} \leq \deg(z) \frac{e_i' - 1}{2e_i'} - \frac{1}{e_i'}.$$

□

**Lemma 3.5.** *If $(\mathcal{X}', J)$ is admissible and $W \subset J$ is any subset, then $(\mathcal{X}', W)$ is also admissible.*

*Proof.* Each of the conditions (Ad-i), (Ad-ii), and (Ad-iii) hold for $W$ if they hold for $J$. □

Peter: You should also instantiate $(\mathcal{X}, 0, L)$ as well

**Theorem 3.6.** *Suppose $(\mathcal{X}', 0, L')$ is a tame separably rooted log stacky spin curve and $(\mathcal{X}', J)$ is admissible with generators $y_{i,e_i'} \in R' = R_{L'}$ as in (Ad-i) of Definition 3.3. Then the following are true:*

(a) *For all $i \in J$, there exists $y_{i,e_i} \in H^0(\mathcal{X}, e_i(K_{\mathcal{X}}))$ so that*
$$-\mathrm{ord}_{Q_i}(y_{i,e_i}) = \frac{e_i - 1}{2}$$
*and*
$$\frac{-\mathrm{ord}_{Q_j}(y_{i,e_i})}{\deg(y_{i,e_i})} \leq \frac{e_j' - 1}{2e_j'} - \frac{1}{\deg(y_{i,e_i})e_j'}$$
*for all $j \in J$ with $j \neq i$.*

(b) *The elements $y_{i,e_i'}^a y_{i,e_i}^b$, with $a \geq 0, b > 0$ span $R$ over $R'$ and the elements $y_{i,e_i}$ minimally generate $R$ over $R'$.*

(c) *Equip $k[y] = k[y_{i,e_i}]_{i \in J}$ and $k[x]$ with any graded monomial order and $k[y, x] = k[y] \otimes k[x]$ with block order. Let $R = k[y, x]/I$. Then,*
$$\mathrm{in}_{\prec}(I) = \mathrm{in}_{\prec}(I')k[x, y] + \langle y_{i,e_i} z \mid z \neq y_{i,e_i}, y_{i,e_i'} \rangle$$
*as $z$ ranges over all generators of $R$.*

(d) *Suppose $\mathrm{in}_{\prec}(I')$ is minimally generated by products of two monomials and that for all $i \in J$, we have $e_i > \deg z$ for any generator $z$ of $R'$. Then any set of minimal generators for $I'$ together with any set of relations as in part (c) minimally generate $I$.*

(e) *For any $W \subset J$, the set $(\mathcal{X}, W)$ is admissible.*

possibly generalize these to arbitrary log stacky $\Delta \neq 0$ spin curves

*Proof.* Let $(\mathcal{X}, 0, L)$ and $(\mathcal{X}', 0, L')$ be the two log stacky spin curves in question, with coarse spaces $X = X'$. Define
$$S(i, 0) = \{j \in J : j \neq i \text{ and } e_j' \mid e_i\}$$
Define
$$E_i = \sum_{j \in S(i,0)} Q_j$$
Note that
$$\lfloor e_i L' \rfloor + Q_i \leq \lfloor e_i L \rfloor$$
and so we obtain an inclusion
$$H^0(\mathcal{X}', e_i L' - E_i + Q_j) \to H^0(\mathcal{X}, e_i L - E_i) \subset H^0(\mathcal{X}, e_i L).$$

Peter: should be $Q_i$?

The assumption (Ad-iii) implies
$$\deg\left(e_i L' - E_i\right) \geq 2g - 1$$

Peter: Or more for higher genus, by our hypothesis

implying that $H^0(\mathcal{X}', e_i L' - E_i + Q_i)$ is base point free by Riemann Roch , and so a general element

$$y_{i,e_i} \in H^0(\mathcal{X}', e_i L' - E_i + Q_i)$$

satisfies

$$-\operatorname{ord}_{Q_i}(y_{i,e_i}) = \lfloor e_i \frac{e_i' - 1}{2e_i'} \rfloor + 1 = \frac{e_i - 1}{2}$$

Therefore, $y_{i,e_i}$ satisfies the first part of the claim of $(a)$.

We next show $y_{i,e_i}$ also satisfies the second part of the claim of $(a)$, by considering separately the cases of whether $j \in S(i,0)$, and $j \notin S(i,0)$.

If $j \in S(i,0)$ then, since $E_i \geq Q_j$, and $y_{i,e_i} \in H^0(\mathcal{X}', e_i L' - E_i + Q_i)$,

$$-\operatorname{ord}_{Q_j}(y_{i,e_i}) \leq e_i \frac{e_j' - 1}{2e_j'} - 1 \leq e_i \frac{e_j' - 1}{2e_j'} - \frac{1}{e_j'}.$$

If instead $j \notin S(i,0)$, then since $e_j' \nmid e_i$, we know $e_i \frac{e_j'-1}{2e_j'} \notin \mathbb{Z}$, so

$$-\operatorname{ord}_{Q_j}(y_{i,e_i}) \leq \lfloor e_i \frac{e_j' - 1}{2e_j'} \rfloor \leq e_i \frac{e_j' - 1}{2e_j'} - \frac{1}{e_j'},$$

completing the proof of (a).

Next, define $R_0 = R'$ and inductively define

$$R_i = \begin{cases} R_{i-1} & \text{if } i \notin J \\ R_{i-1}[y_{i,e_i}] & \text{otherwise.} \end{cases}$$

To prove (b), it suffices to show that elements of the form $y_{i,e_i'}^a y_{i,e_i}^b$ with $a \geq 0, b > 0$ are linearly independent, and that these elements together with $R_{i-1}$ span $R_i$ as a k- vector space. These elements do not lie in $R_{i-1}$ because their order of pole at $Q_i$ is bigger than the order of any element in $R_{i-1}$, and are linearly independent amongst themselves because of injectivity of the linear map

$$(a,b) \mapsto \left(\deg\left(y_{i,e_i'}^a y_{i,e_i}^b\right), -\operatorname{ord}_{Q_i}\left(y_{i,e_i'}^a y_{i,e_i}^b\right)\right) = (a,b)\begin{pmatrix} e_i - 2 & \frac{e_i-3}{2} \\ e_i & \frac{e_i-1}{2} \end{pmatrix}$$

The fact the the elements $y_{i,e_i'}^a y_{i,e_i}^b$ with $a \geq 0, b > 0$ span $R_i$ over $Ri-1$ follows from the fact that the integer lattice points in the cone generated by the vectors $\left(e_i - 2, \frac{e_i-3}{2}\right)$ and $\left(e_i, \frac{e_i-1}{2}\right)$ is saturated, as the corresponding determinant is

$$(e_i - 2)\frac{e_i - 1}{2} - e_i \frac{e_i - 3}{2} = 1,$$

completing part (b).

To show (c), we wish to show $y_{i,e_i} z \in R'$ for $z \neq y_{i,e_i}, y_{i,e_i'}$. Note that $f \in R$ lies in $R'$ if and only if for all $j \in J$ we have

$$-\operatorname{ord}_{Q_j}(f) \leq \deg f \left(\frac{e_j' - 1}{2e_j'}\right)$$

We now check this in three cases, depending on whether $j \notin \{i\} \cup S(i,0), j = i$, or $j \in S(i,0)$.

Case 1: $j \notin \{i\} \cup S(i, 0)$.

Here, $-\operatorname{ord}_{Q_j}(L) = -\operatorname{ord}_{Q_j}(L')$, so

$$-\operatorname{ord}_{Q_j}(y_{i,e_i}) - \operatorname{ord}_{Q_j}(z) \leq e_i \frac{e'_j - 1}{2e'_j} + \deg z \frac{e'_j - 1}{2e'_j} = \deg(y_{i,e_i} z) \frac{e'_j - 1}{2e'_j}.$$

Case 2: $j = i$.

By part (a), condition (Ad-ii), and Lemma 3.4 we have

$$\begin{aligned} -\operatorname{ord}_{Q_j}(y_{i,e_i}) - \operatorname{ord}_{Q_j}(z) &\leq \frac{e_j - 1}{2} + \deg z \left( \frac{e'_j - 1}{2e'_j} \right) - \frac{1}{e'_j} \\ &= \frac{e_j - 1}{2} - \frac{1}{e'_j} + \deg z \left( \frac{e'_j - 1}{2e'_j} \right) \\ &= \frac{e_j(e_j - 3)}{2(e_j - 2)} + \deg z \left( \frac{e'_j - 1}{2e'_j} \right) \\ &= \deg(y_{i,e_i} z) \frac{e'_j - 1}{2e'_j}. \end{aligned}$$

Case 3: $j \in S(i, 0)$.

In this case, we may first assume $z \neq y_{j,e_j}$, as this is covered by case 2, with $i$ and $j$ reversed. Hence,

$$-\operatorname{ord}_{Q_j}(z) \leq \deg z \frac{e'_j - 1}{2e'_j}$$

implying

$$-\operatorname{ord}_{Q_j}(y_{i,e_i}) - \operatorname{ord}_{Q_j}(z) \leq e_i \frac{e'_j - 1}{2e'_j} + \deg z \frac{e'_j - 1}{2e'_j} = \deg(y_{i,e_i} z) \frac{e'_j - 1}{2e'_j},$$

completing part (c).

Next, note that the minimal generators of $I'$ together with the new generators of I given in (c) have independent initial terms by construction, and therefore are minimal by Lemma 3.1.

Finally, to check (e), we will first show that $(\mathcal{X}, J)$ is admissible. We know (Ad-i) is satisfied by part (b), taking the $y_{i,e_i}$ as the generators in degree $e_i$. Next, (Ad-ii) is strictly monotonic in the $e_i$, and hence also holds for $(\mathcal{X}, J)$. Finally, if (Ad-iii) holds for $e$ then it holds for $e + 2$ by definition. This is where we use that (Ad-iii) holds for $d > 0$ and not just for $d = 0$.

Finally, if $(\mathcal{X}, J)$ is admissible then so is $(\mathcal{X}, W)$ for any $W \subset J$ by Lemma 3.5. □

## 4. Genus one

We now consider the case when the genus is positive. The genus zero case is the most complicated and reserve that for Section 6 and Section 7. We begin with the genus one case.

description

4.1. **Inductive Theorem: Increasing the Number of Ramified Points for a 2-Saturated Divisor.** We prove an inductive theorem on adding 2-sat here.

**Proposition 4.1.** *Let* $(\mathcal{X}, \Delta, L)$ *and* $(\mathcal{X}', \Delta, L')$ *be tame separably rooted log stacky spin curves. Let* $\mathcal{X} \to \mathcal{X}'$ *be a birational map of tame, separably rooted log stacky curves so that* $\mathcal{X}$ *has signature* $\sigma = (g; e_1, \ldots, e_r; \delta)$ *with* $g > 0$*, and, if* $g = 1$ *then* $\deg 3L \geq 2$*. Let* $R_{L'} = R' = k[x]/I' = k[x_2, x_3, x_5, \ldots, x_m]/I'$ *and let* $L = L' + \frac{1}{3}P$*. Suppose* $L$ *and* $L'$ *satisfy* $\mathsf{sat}(\mathsf{Eff}(L')) = \mathsf{sat}(\mathsf{Eff}(L)) = 2$*. Suppose* $\deg x_i = i$ *for* $i \in \{2, 3\}$ *and that the ordering on* $k[x]$ *satisfies*

$$\operatorname{ord}_{x_2}(f) < \operatorname{ord}_{x_2}(g) \implies f \prec g.$$

*Then, the following statements hold.*

(a) *General elements* $y_i \in H^0(\mathcal{X}, iL)$ *for* $i \in \{3, 4\}$ *satisfy* $-\operatorname{ord}_P(y_i) = 1$ *and any such choice of elements* $y_3, y_4$*, minimally generate* $R$ *over* $R'$*.*

(b) *Equip* $k[y_3, y_4]$ *with* $\mathsf{grlex}$ *so that* $y_4 \prec y_3$ *and equip the ring* $k[y_3, y_4, x]$ *with the block order so that* $R = k[y_3, y_4, x]/I$*. Let,*

$$\begin{aligned} J = &\operatorname{gin}_\prec(I')k[x, y_3, y_4] \\ &+ \langle y_4 x_j \mid 2 \leq j \leq m\rangle \\ &+ \langle y_4^2 . \rangle \end{aligned}$$

*Then,* $\operatorname{gin}_\prec(I) = J$*.*

(c) *Any set of minimal generators for* $I'$ *together with any set of relations with leading terms as in (b) minimally generate* $I$*.*

*Proof.* The assumptions on $g, \sigma$ imply $H^0(\mathcal{X}, 3L), H^0(\mathcal{X}, 4L)$ are both base point free: If $g \geq 2$ then $\deg 3L > 2g-1$ and $\deg 4L > 2g-1$, so $H^0(\mathcal{X}, 3L)$ and $H^0(\mathcal{X}, 4L)$ are base point free. If $g = 1$, we assume $\deg 3L \geq 2 > 2g-1$, so we also have $\deg 4L \geq 2 > 2g-1$, so again $H^0(\mathcal{X}, 3L)$ and $H^0(\mathcal{X}, 4L)$ are base point free. Therefore, general elements $y_3$ and $y_4$ satisfy $-\operatorname{ord}_P(y_i) = 1$ by Riemann Roch, and the assumptions on $\sigma$. Understand why/whether we need to cite lemma 5.4.7.

In a fashion analogous to the proof of Proposition 7.1, we see the following is a $k$ basis for $R$ over $R'$:

$$\begin{aligned} &\{y_3^a x_2^b x_3^\epsilon \mid a \geq 0, b \geq 0, \epsilon \in \{0, 1\}\} \\ &\cup \{y_3^a y_4 \mid a \geq 0\}, \end{aligned} \tag{4.1}$$

completing part (a).

To show part (b), note that the generators in 4.1 are precisely a set of monomials which generate $k[x, y]/J$ over $k$. So, to show $J = \operatorname{in}_\prec(I)$, it suffices to show that all generators of $J$ lie in $\operatorname{in}_\prec(I)$.

This follows, since there exist constants $A_j \in k$ for $3 \leq j \leq m$, and a constnant $B \in k$ and elements $w_{i,j} \in R'$ so that the following linear combinations of elements lie in $R'$.

$$y_4 x_j - A_j y_3 w_j \qquad \text{so that } 2 \leq j \leq m \text{ and } \deg w_j = \deg x_j + 1$$
$$y_4^2 + B y_3^2 x_2$$

Of course, the initial terms of these elements are precisely the generators of $J$, completing (b).

Finally, (c) follows immediately from Lemma 3.1 □

4.2. **Genus 1 base cases.** Let $\mathcal{X}$ be a tame, seperably rooted stacky curve of genus 1. Let $P, Q$ are distinct hyperelliptic fixed points, and $R_1, \ldots, R_n$ be points on $\mathcal{X}$. In this section, will compute the canonical ring $R_L$ in the cases that

(1) $L = 0$
(2) $L = P - Q$
(3) $L = P - Q + \frac{1}{3}R$
(4) $L = P - Q + \frac{e-1}{2e}R, e > 3, e \equiv 1 \bmod 2$
(5) $L = P - Q + \frac{1}{3}R_1 + \frac{1}{3}R_3$

In addition to being illustrative examples, these will serve as the base cases of induction, in order to compute generators and relations for an arbitrary genus 1, stacky log half canonical divisor. In each of the following examples, we will give explicit generators and relations, and in case (3) and (4) we check the corresponding signature is admissible.

**Example 4.2.** In the case $L = 0$, we see immediately that $R_L \cong k[x]$ with $x$ a generator in degree 1.

**Example 4.3.** In the case $L = P - Q$. Then, we have $R_{(2k+1)L} \cong 0$ while $R_{2kL} \cong k$, with a single generator in degree 2. Therefore, $R_L \cong k[y]$, where $y$ is a generator in degree 2.

**Example 4.4.** Let $L = P - Q + \frac{1}{3}R$. Since $P \neq Q$, we see $h^0(L) = 0$, and we know $h^0(2L) = h^0(K) = 1$. For $n > 2$, we have $\deg nL > 0$ and so by Riemann–Roch, we have $h^0(nL) = \deg\lfloor nL/3 \rfloor$. Next, We find that $R_L \cong k[u, x, z]/(z^2 - ux^4)$, where $\deg u = 2, \deg x = 3, \deg z = 7$ so that $\operatorname{div} u|_L = -2P + 2Q, \operatorname{div} x|_L = -3P + 3Q - R, \operatorname{div} z|_L = -7P + 7Q - 2R$.

Peter: Why don't we have something in degree 4?

**Example 4.5.** Let $L = P - Q + \frac{e-1}{2e}R_1$. Here, we claim $R_L \cong k[u, y_3, y_5, \ldots, y_e]/I_e$, where we define $I_5 = y_3^4 - uy_5^2$ and $I_e$ is inductively defied via Theorem 3.6 so that $\mathrm{in}(I_e) = \mathrm{in}(I_{e-2})k[u, y_3, \ldots, y_e] + \langle y_e z \mid z \neq y_e, y_{e-2} \rangle$. Furthermore, $\deg u = 2, \deg y_i = i$.

Note that here there is a quartic generator of I in degree 12, providing exceptions to minimality by quadrics and an exception to the relation bound when $e = 5$.

Check admissibility

**Example 4.6.** Let $L = P - Q + \frac{1}{3}R_1 + \frac{1}{3}R_2$. Here, we claim $R_L \cong k[u, x, y_3, y_4]/(xy_3 - \alpha u y_4, y_4^2 - \beta x^2 u - \gamma y^2 u)$ where $\alpha, \beta, \gamma \in k$, and $\deg u = 2, \deg x = \deg y_3 = 3, \deg y_4 = 4$, where $u, x$ are as in Example 4.4 and $y_3, y_4$ are as in Proposition 4.1.

Explain why this forms a system of generators and relations, and also state more clearly what $y_i$ are.

check admissibility

4.3. **Main Theorem for Genus 1.**

## 5. High genus

5.1. **Inductive Theorem: Adding points for high genus.**

**Lemma 5.1.** *Let $(\mathcal{X}, \Delta, L)$ and $(\mathcal{X}, \Delta', L')$ be tame separably rooted log stacky spin curves with $0 \neq \Delta = \Delta' + 2P$ for some non-stacky point $P$ not in the support of $L'$, meaning that $L = L' + P$. Let $(g; e_1, \ldots, e_\tau; \delta + 2)$ and $(g; e_1, \ldots, e_\tau, \delta)$ with $g \geq 2$ be their respective signatures. Suppose $\mathrm{sat}(\mathrm{Eff}(L)) = 1$ and $\mathrm{sat}(\mathrm{Eff}(L')) = 2$. Further suppose $R_{L'} = k[x_2, x_3, \ldots, x_m]/I'$ with $\deg(x_2) = 2$ and $\deg(x_3) = 3$. Then*

(a) *Then generic elements* $y_1$ *and* $y_2$ *in* $(R_L)_1$ *and* $(R_L)_2$ *respectively, such that* $-\operatorname{ord}_P(y_2) > 0$ *and* $\{y_1^2, y_2\}$ *is linearly independent, generate* $R_L$ *over* $R_{L'}$*.*

(b) *Equip* $k[y_1, y_2]$ *with* grlex *so that* $y_2 \prec y_1$ *and equip* $k[y_1, y_2, x_2, \dots, x_m]$ *with the block order so that* $R_L = k[y_1, y_2, x_2, \dots, x_m]/I$*. Then*

$$\begin{aligned} \mathrm{in}_{\prec}(I) = \mathrm{in}_{\prec}(I')k[x_2, \dots, x_m, y_1, y_2] \\ + \langle y_2 x_j \mid 2 \le j \le m \rangle \\ + \langle y_2^2 \rangle. \end{aligned}$$

(c) *Any set of minimal generators for* $I'$ *together with any set of relations with leading terms as above minimally generate* $I$*.*

*Proof.* Since adding $P$ to $L'$, reduces saturation from 2 to 1, $h^0(X, \lfloor L \rfloor) = 1$. Choose any $y_1 \in H^0(X, \lfloor L \rfloor)$; we know by saturation arguments that $-\operatorname{ord}_P(y_1) = 1$. Furthermore, since $\delta > 0$, we use Riemann-Roch to deduce that for all $d > 1$ $h^0(X, \lfloor dL \rfloor) = h^0(dL') + d$, so $(R_L)_d$ is a $d$ dimension $k$-vector space over $(R_{L'})_d$. In particular, this means we can generically choose an element $y_2 \in H^0(X, \lfloor 2L \rfloor)$ linearly independent from $y_1^2$. By dimension counting, we see that

$$\langle y_1^a x_2^b x_3^c : a \ge 0, b \ge 0, c \in \{0, 1\} \rangle$$

$$\langle y_1^a y_2 : a \ge 0 \rangle$$

form a $k$-basis for $R_L$ over $R_{L'}$. This completes part (a).

To show part (b), we note that the monomials in the basis above generate $k[x, y]/\mathrm{in}_{\prec}(I)$. Thus, it is sufficient to show that $J \subseteq \mathrm{in}_{\prec}(I)$. For $j \ge 2$, we can choose $w_j$ in degree $\deg(x_j) + 1$ and $A_j \in k$ such that

$$y_2 x_j - A_j y_3 w_j$$

has no pole at $P$ at thus lies in $R_{L'}$. Similarly, we can choose $B$ and $C$ such that

$$y_4^2 + B y_1^2 x_2 + C y_1 x_3$$

has no pole at $P$, so it lies in $R_{L'}$. These give us relations with initial terms as desired, concluding part (b).

Finally, (c) follows immediately from Lemma 3.1.

□

**Lemma 5.2.** *Let* $(\mathcal{X}, 2P, L)$ *and* $(\mathcal{X}, 0, L')$ *be tame separably rooted log stacky spin curves with* $P$ *not in the support of* $L'$*, meaning that* $L = L' + P$*. Let* $(g; e_1, \dots, e_\tau; 2)$ *and* $(g; e_1, \dots, e_\tau, 0)$ *with* $g \ge 2$ *be their respective signatures. Suppose* $\mathrm{sat}(\mathrm{Eff}(L)) = 1$ *and* $\mathrm{sat}(\mathrm{Eff}(L')) = 2$*. Further suppose* $R_{L'} = k[x_2, x_3, \dots, x_m]/I'$ *with* $\deg(x_2) = 2$ *and* $\deg(x_3) = 3$*. Then*

(a) *Then generic elements* $y_1$ *and* $y_3$ *in* $(R_L)_1$ *and* $(R_L)_3$ *respectively, such that* $-\operatorname{ord}_P(y_3) > 1$ *and* $\{y_1^3, y_2\}$ *is linearly independent, generate* $R_L$ *over* $R_{L'}$*.*

(b) *Equip* $k[y_1, y_3]$ *with* grlex *so that* $y_3 \prec y_1$ *and equip* $k[y_1, y_3, x_2, \dots, x_m]$ *with the block order so that* $R_L = k[y_1, y_3, x_2, \dots, x_m]/I$*. Then*

$$\begin{aligned} \mathrm{in}_{\prec}(I) = \mathrm{in}_{\prec}(I')k[x_2, \dots, x_m, y_1, y_2] \\ + \langle y_3 x_j \mid 2 \le j \le m \rangle \\ + \langle y_3^2 \end{aligned}$$

(c) *Any set of minimal generators for $I'$ together with any set of relations with leading terms as above minimally generate $I$.*

*Proof.* The proof follows similarly to that of lemma 5.1, with the change that $(R_L)_2$ is only a degree 1 $k$-vector space over $(R_{L'})_2$ forcing a generator in degree 3 rather than degree 2. $\square$

5.2. **Bounds on Generators and Relations in high genus.**

**Lemma 5.3.** [2][*Proposition III.4*] *Let $X$ be a curve of genus $g \geq 2$ and let $L$ be a divisor with $2L \sim K$. Then, for $n \geq 5$, the map*

$$H^0(nL) \otimes H^0(2L) \to H^0((n+2)L)$$

*is surjective. In particular, $R_L$ is generated in degree at most 6.*

*Remark* 5.4. In [2][Proposition III.4], Neves claims to prove that $R_L$ is generated in degree at most 5. However, when the proof is carefully examined, it only shows generation in degree at most 6. For this reason, we only state Lemma 5.3 as generation in degree at most 6. We next proceed to show this can be reduced to generation in degree 5. The trickiest case is genus 2, which we address next.

**Proposition 5.5.** *Let $X$ be a smooth genus 2 curve with canonical divisor $K$ and let $L \in$ div $X$ be so that $2L \sim K$. Then, $R_L$ is generated in degree at most 5.*

*Proof.* By Lemma 5.3 it suffices to show there are no new generators in degree 6. Let $P_1, \ldots, P_6$ be the six hyperelliptic fixed points of $X$. Recall that all spin divisors on $X$ can be written as

$$L = P_1 + \sum_{i=2}^{6} \epsilon_i (P_1 - P_i),$$

cite this somewhere, maybe ask david for a place to cite?

with $\epsilon_i \in \{0, 1\}$, and some $i \in \{2, 3, 4, 5, 6\}$ for which $\epsilon_i = 0$. Note that $H^0(X, K)$ is spanned by rational functions $u, x$ with $\operatorname{div} u|_{2L} = -2L, \operatorname{div} x|_{2L} = -2L + 2P_1$. Therefore, the image of the multiplication map

$$\operatorname{Sym}^3 H^0(X, 2L) \to H^0(X, 6L)$$

is 4 dimensional by Riemann–Roch, with image spanned by $u^3, u^2x, ux^2, x^3$, which have $\operatorname{div} u^3|_{2L} = -2L, \operatorname{div} u^2x|_{2L} = -2L + 2P_1, \operatorname{div} ux^2|_{2L} = -2L + 4P_1, \operatorname{div} x^3|_{2L} = -2L + 6P_1$. Since $H^0(X, 6L)$ is 5 dimensional by Riemann–Roch, it suffices to produce one more independent function in the image of the multiplication map. There are now two cases, depending on whether $H^0(X, L) = 0$.

First, we deal with the case that $H^0(X, L) \neq 0$. Then, since $X \not\cong \mathbb{P}^1$, and $\deg L = 1$, we must have $H^0(X, L) = 1$. Hence, we know $L$ is effective, so we must have $L = P$ where $P$ is hyperelliptic fixed. In this case, it is shown in [2][Theorem III.19] that $R_L$ is generated in degree $\leq 3$.

Second, suppose $H^0(X, L) = 0$. Note that because $\deg 3L = 3$, we have by Riemann–Roch that $H^0(X, 3L) = 2$. Note additionally that $3L \sim K + L \sim 2P_1 + L$. Since $3L$ is base point free, by Riemann–Roch we have that there exists a rational function $f$ with $\div f|_{2P_1+L} = -2P_1 - L$. Additionally, there are no functions $g \in H^0(X, L + 2P_1)$ with $\operatorname{div} g|_{2P_1+L} = -L$, since $H^0(X, L) = 0$. Therefore, there must exist a function $h \in H^0(X, L + 2P_1)$ so that $\operatorname{div} h|_{L+2P_1} = -L - P_1$. Then, $h \cdot f \in H^0(X, 2L + 4P_1) \cong H^0(X, 6L)$ has a pole of odd order at $P_1$, and is therefore independent from the image of $\operatorname{Sym}^3 H^0(X, 3L)$. $\square$

**Lemma 5.6.** *Let $X$ be a curve of genus $g \geq 2$ and let $L$ be a divisor with $2L \sim K$. Then, $R_L$ is generated in degree at most 5.*

*Proof.* If $g = 2$ then the claim follows from Proposition 5.5. However, if $g > 2$, then by [5][Theorem 3.2.1], we obtain the map $\operatorname{Sym}^3 H^0(2L) \to \operatorname{Sym}^3 H^0(6L)$ is surjective, so again there are no new generators in degree 6. □

This should be $k[x_1, \ldots, x_n]$ instead of $\operatorname{Sym} H^0$.

**Lemma 5.7.** *Suppose $X$ is a curve of genus $g \geq 2$ and let $L$ be a divisor with $2L \sim K$. Choose generators $x_1, \ldots, x_n$ of $R_L$ so that we obtain a surjection $\phi : k[x_1, \ldots, x_n] \twoheadrightarrow R_L$ with kernel $I_L$. Let $I_{L,d}$ be the $d$th graded piece of $I_L$ and $J_{L,d} = \sum_{j=1}^{d-1} k[x_1, \ldots, x_n]_j \cdot I_{L,d-j}$. Let for any $f \in k[x_1, \ldots, x_n]$ with $\deg f \geq 12$, there exist $s_1, s_2 \in k[x_1, \ldots, x_n]_2$ and $g, h \in k[x_1, \ldots, x_n]_{d-2}$ so that $s_1 g + s_2 h = f \bmod J_{L,d}$.*

*Proof.* Choose $s_1, s_2 \in k[x_1, \ldots, x_n]_2$ so that $\phi(s_1), \phi(s_2) \in H^0(2L)$ are independent elements. By Lemma 5.6, $\deg x_i \leq 5$ for $1 \leq i \leq n$. Then, we may write $f = \sum_{i=1}^{n} a_i x_i$ with $a_i \in k[x_1, \ldots, x_n]_{d-\deg x_i}$. We will next show that for all $1 \leq i \leq n$ there exist $g_i, h_i \in k[x_1, \ldots, x_n]_{d-\deg x_i - 2}$ so that $a_i = s_1 g_i + s_2 h_i \bmod I_{L,\deg a_i}$. Let $V \subset H^0(K)$ be the vector subspace generated by $\phi(s_1)$ and $\phi(s_2)$. By Lemma 5.3,

$$V \otimes H^0((\deg f - \deg x_i - 2)L) \to H^0((\deg f - \deg x_i)L)$$

is surjective, since $\deg f - \deg a_i - 2 \geq 12 - 5 - 2 = 5$. In particular, we can write $\phi(a_i) = \phi(s_1)\alpha + \phi(s_1) \cdot \beta$. Chosing $g_i, h_i$ so that $\phi(g_i) = \alpha, \phi(h_i) = \beta$, we have $a_i = s_1 g_i + s_2 h_i \bmod I_{L,\deg a_i}$, as claimed.

Finally, we may then take $g = \sum_i g_i x_i, h = \sum_i h_i x_i$, so that

$$\begin{aligned} f &\equiv \sum_i a_i x_i \equiv \sum_i (s_1 g_i + s_2 h_i) x_i \equiv s_1 \left( \sum_i g_i x_i \right) + s_2 \left( \sum_i h_i x_i \right) \\ &\equiv s_1 g + s_2 h \bmod J_{L,d}. \end{aligned}$$

□

**Proposition 5.8.** *Let $X$ be a curve of genus $g \geq 2$ and let $L$ be a divisor with $2L \sim K$. Then $I_L$ is generated in degree at most 11.*

*Proof.* Suppose $f \in I_L$ with $\deg f \geq 12$. To complete the proof, it suffices to show $f \in I'_{L,d}$. By Lemma 5.7, it suffices to show $s_1 g + s_2 h \in I'_{L,d}$. Let $V \subset H^0(K)$ be the vector subspace generated by $\phi(s_1)$ and $\phi(s_2)$, and consider the map

$$V \otimes H^0((\deg f - 2)L) \xrightarrow{f} H^0((\deg f)L),$$

we know that $\phi(s_1)\phi(g) + \phi(s_2)\phi(h) \mapsto 0$. So, using the refined version of the basepoint free pencil trick, as shown in the proof of [4][Lemma 2.6], if $\phi(s_1)g + \phi(s_2)h \mapsto 0$, there exists some $\rho \in k[x_1, \ldots, x_n]$ so that $\phi(\rho) \in H^0((\deg f - 4)L)$ satisfies $\phi(g) = \phi(s_2)\phi(\rho)$ and $\phi(h) = -\phi(s_1)\phi(\rho)$. Therefore, $g \equiv s_2 \rho \bmod I_{k,d-2}$ and $h \equiv -s_1 \rho \bmod I_{k,d-2}$. Therefore,

$$s_1 g + s_2 h \equiv s_1 s_2 \rho + s_2(-s_1 \rho) \equiv 0 \bmod J_{L,d}.$$

□

*Remark* 5.9. In [2][Theorem III.19], Neves shows that if $L = P$ is a half canonical divisor on a genus 2 curve, then $R_L$ has a relation in degree 10. Therefore, the bound from Proposition 5.8, that $I_L$ is generated in degree at most 11 is close to sharp.

### 5.3. **Main theorem for high genus.**

## 6. Genus Zero Effective

write intro

Now we consider the genus zero case.

In this section, we examine the case $2L \sim K_{\mathcal{X}} + \Delta$ when $\Delta$ is a non- zero effective divisor. Using this, we can reduce to looking at effective stacky divisors. Then, applying a result for general effective **Q** divisors, we deduce that $R_L$ is generated in degrees up to $e = \max e_i$ with relations generated up to degree $2e$ with initial terms quadratic in the generators. We also demonstrate that these bounds are tight up to .

insert how tight they are

fill in hole here

We have Lemma 3.1.

We will now prove a lemma that will be used to inductively deal with divisors that are sufficiently saturated, by combining and extending methods of VZB and O'Dorney.

add reference

reference

reference this-Section 1 of Evan's paper, or preferably define it in the background section

**Lemma 6.1.** *Let $X$ be a genus $g$ curve with a **Q**-divisor $D$ (as defined by O'Dorney ) satisfying $\deg(\lfloor D \rfloor) \geq 0$, so $R_D$ has a generator $u$ in degree 1. Suppose $D$ is generated by $u, x_1, \ldots, x_m$ in degree at most $\bar{d}$ with relations generated in degree at most $\bar{\tau}$. Suppose $D' = D + \frac{\alpha}{\beta}P$ for some $P \notin Supp(D)$, $\alpha, \beta \in \mathbf{N}$ such that*

$$h^0(X, \lfloor dD + d\frac{\alpha}{\beta}P \rfloor) = h^0(X, \lfloor dD \rfloor) + \lfloor d\frac{\alpha}{\beta} \rfloor \forall d \in \mathbf{N} : d \geq \frac{\beta}{\alpha}. \tag{6.1}$$

*Let*

$$0 < \frac{c_1}{d_1} < \ldots < \frac{c_n}{d_n} = \frac{\alpha}{\beta}$$

*be the non-negative best lower approximation. Then*

- *$R_{D'}$ is generated over $R_D$ by elements $y_1, \ldots, y_n$ such that $y_i$ lies in degree $d_i$ and has a pole at $P$ of degree $c_i$*
- *If $I$ and $I'$ are the ideal of relations of $R_D$ and $R_{D'}$ respectively, then*

$$\begin{aligned} in_{\prec}(I') = {} & in_{\prec}(I)k[u, x_1, \ldots, x_m, y_1, \ldots, y_n] \\ & + \langle y_i x_j : 1 \leq i \leq n, 1 \leq j \leq m \rangle \\ & + \langle y_i y_j : \lfloor \frac{\alpha}{\beta} \rfloor \rangle \leq i \leq j \leq n-1 \\ & + \langle y_i y_j : 1 \leq i \leq j \lfloor \frac{\alpha}{\beta} \rfloor \\ & + \langle (y_i)^{\deg(y_{i+1}+1)} : 1 \leq i = \lfloor \frac{\alpha}{\beta} \rfloor \end{aligned}$$

  *In particular, if $\frac{\alpha}{\beta} \leq 1$, then $I$ is minimally generated by elements with leading terms of the form $y_i x_j$ and $y_i y_j$.*
- *If $gin_{\prec}(I) = in_{\prec}(I)$ then $gin_{\prec}(I') = in_{\prec}(I')$.*

- *A Grobner basis with the initial terms* $y_i x_j$, $y_i y_j$, *and possibly* $(y_{\lfloor \frac{\alpha}{\beta} \rfloor})^{y_{\lfloor \frac{\alpha}{\beta} \rfloor}+1}$ *as shown above minimally generates* $I'$.

*Proof.* Observe that $u_D, x_1, ..., x_m$ each of which has no pole at $P$. These can be chosen generically to not have a zero at $P$ .

check that we can actually do this, or eliminate if it turns out to be unnecessary

Begin by focusing on the $d^{th}$ component of $R_{D'}$: $(R_{D'})_d = H^0(X, \lfloor D + \frac{\alpha}{\beta} P \rfloor)$. By our hypothesis (Equation 6.1), for any $d \in \mathbf{N}$ such that $\lfloor d\frac{\alpha}{\beta} \rfloor > 0$,

$$h^0(X, d(D + \frac{\alpha}{\beta})) = h^0(X, dD) + \deg(d\frac{\alpha}{\beta})$$

so in $H^0(X, \lfloor dD' \rfloor)$ we can find elements with pole order $i$ at $P$ for any $i \in \{0, \dots, \lfloor d\frac{\alpha}{\beta} \rfloor\}$. We can use a method similar to O'Dorney to construct the $y_i$'s as desired:

cite where

Let

$$0 < \frac{c_1}{d_1} < \ldots < \frac{c_n}{d_n} = \frac{\alpha}{\beta}$$

be the non-negative best lower approximation of $\frac{\alpha}{\beta}$. Note that elements of $H^0(X, dD')$ have pole of orders ranging from 0 to $\lfloor d_i \frac{\alpha}{\beta} \rfloor = c_i$ at $P$. A set of these one for each pole degree 1 to $\lfloor d_i \frac{\alpha}{\beta} \rfloor$ are linearly independent from each other and $(R_D)_d$, so by degree considerations, they span $(R_{D'})_d$ over $(R_D)_d$ as k-vector spaces.

For any $j < i$, $\frac{c_i}{d_i} > \frac{c_j}{d_j}$ by construction, so any product of $\prod z_i$ of elements of degrees less than $d_i$ must have

$$\frac{-\operatorname{ord}(z_i)}{\deg(z_i)} < \frac{c_i}{d_i},$$

so we have the pole order of $\prod z_i$ is

$$\sum -\operatorname{ord}(z_i) = d_i(\frac{\sum -\operatorname{ord}(z_i)}{\sum \deg(z_i)}) < d_i \frac{c_i}{d_i} = c_i.$$

Thus, the elements of $(R_{D'})_{d_i}$ with pole order $c_i$ at $P$ are not generated by lower degrees. Generically choose some $y_i \in (R_{D'})_{d_i}$ to have maximal pole order at $P$ (note that if $d_i = 1$ then

$$-\operatorname{ord}_P(y_i) = \max(c_j : d_j = 1)$$

and if more than one $d_i$ is 1, we must also choose them generically so they generate $(R_{D'})_1$).

Suppose $d \leq \beta$ and positive $\frac{c}{d} \leq \frac{\alpha}{\beta}$ not a best lower approximation. Then choose best lower approximation $\frac{c_i}{d_i}$ such that $c_i \leq c$ is maximal; since $c < c_{i+1}$, $\frac{c_i}{d_i} \geq \frac{c}{d}$. From this we can deduce that

$$c_i + (d - d_i)\lfloor \frac{\alpha}{\beta} \rfloor \geq c.$$

Therefore, we can generate an element of $(R_{D'})_d$ as a product of lower degrees that has pole degree $c$ at $P$ by either $y_i u^b$ or $y_i z u^b$, or $z u^b$ satisfying $i > \lfloor \frac{\alpha}{\beta} \rfloor$ and $z = y_j (y_{\lfloor \frac{\alpha}{\beta} \rfloor})^b$ where $d_j = 1$ $(= d_{\lfloor \frac{a}{b} \rfloor})$ and $b < d_{\lfloor \frac{\alpha}{\beta} \rfloor + 1}$. Thus, for a general $d$ (not necessarily less than $\beta$), if $\frac{c}{d} \leq \frac{\alpha}{\beta}$ is not a best lower approximation than we can construct an element in degree $d$ with pole degree $c$ by either $y_n^a y_i u^b$, $y_n^a z y_i u^b$ or $y_n^a z u^b$ satisfying the same conditions. We notice that elements of this form are all linearly independent in $R_{D'}$. Thus, they form a k-basis for $R_{D'}$ over $R_D$. Thus $u, x_1, ..., x_m, y_1, ..., y_n$ generate $R_{D'}$.

Choose $x_i \in \{x_1, \dots, x_m\}$ and $y_j \in \{y_1, \dots, y_n\}$, where $y_j$ corresponds to the best lower approximation $\frac{c_j}{d_j}$. Recall from the previous paragraph that we can uniquely construct an element in degree $d$ with pole order $c$ at $P$ by elements of the $y_n^a y_l u^b$ and $y_n^a y_l y_{l'} u^b$ with $d_l = 1$ and if $d_{l'} = 1$ then $c_{l'} = \lfloor \frac{\alpha}{\beta} \rfloor$. Specifically, we can successively subtract some $\gamma_l y_l u^{b_l}$, $\gamma_l y_l z_l u^{b_l}$, or $\gamma_l z_l$ with $y_l, z_l, b_l$ satisfying the same conditions as above and $\gamma_l \in k$ the element ensuring that the difference decreases in pole degree at $P$ by 1. For convenience, we can combine these into the case $\gamma_l (y_l)^{s_l} (z_l)^{a_l} u^{b_l}$ with $(s_l, a_l) \in \{(1,0), (1,1), (0,1)\}$. In the cases when $\frac{\alpha}{\beta} < 1$ so $z_l$ is not well-defined, just pick an arbitrary element of $R_D$ for $z_l$ ensuring that the expression still makes sense; in such cases we will always choose $a_l = 0$ so this choice is irrelevant, and only makes writing out cases more convenient. Therefore, we can write

$$x_i y_j - \sum_l (y_l)^{s_l} (z_l)^{a_l} u^{b_l} \in R_D$$

Explicitly state ordering at the beginning

giving us a non-trivial relation with leading term $x_i y_j$.

Similarly, suppose $y_i, y_j \in \{y_1, \dots, y_{n-1}\}$ with $1 < d_i$ and $j \geq i$ (that is, so $y_i y_j$ does not appear as a basis element of the form described above: $y_n^0 y_j y_i u^0$). Choose the maximal $l$ such that $c_h \leq c_i + c_j$; since $1 < d_i \leq d_j$, $(c_i + c_j) - c_j > \lfloor \frac{\alpha}{\beta} \rfloor$ so $c_{j+1} < c_i + c_j$ meaning $c_h \geq c_{j+1} > c_j$. Furthermore, $c_i + c_j - c_h < \lfloor \frac{\alpha}{\beta} \rfloor$ so either $c_h = c_i + c_j$ or there is some $h'$ such that $c_{h'} = c_i + c_j - c_l$ and $d_{h'} = 1$. Since $\frac{c_h}{d_h}$ is a best lower approximation of $\frac{\alpha}{\beta}$ with denominator larger than those of $\frac{c_i}{d_i}$ and $\frac{c_j}{d_j}$, we know that since $c_h \leq c_i + c_j$ we must have $d_h < d_i + d_j$. Thus in the case with a $y_{l'}$ term we have $d_h + d_{h'} = d_h + 1 \leq d_i + d_j$. That is to say, by choosing $b - d_i + d_j - a d_h - 1$ we find (where $a \in \{0, 1\}$ corresponding to if a $y_{h'}$ appears) that $y_h (y_h')^a u^b$ and $y_i y_j$ both lie in degree $d_i + d_j$ with pole order $c_i + c_j$ at $P$. From here we can follow a similar technique as in the previous paragraph to cancel out poles of $y_i y_j$ at $P$ to find

$$y_i y_j - \gamma_h (y_h)^{s_h} (z_h)^{a_h} u^{b_h} - \sum_l (y_l)^{s_l} (z_l)^{a_l} u^{b_l} \in R_D$$

implying a relation with initial term $y_i y_j$.

Next, suppose $y_i, y_j \in \{y_1, \dots, y_{n-1}\}$ such that $i \leq j$, $d_j = d_i = 1$, and $d_{j+1} = 1$ (i.e. $c_i, c_j \leq \lfloor \frac{\alpha}{\beta} - 1 \rfloor$). If $i > 1$ and then $y_{j+1} y_{i-1}$ lies in the same degree as $y_i y_j$ (degree 2) with the same pole order; following the same method as the previous two paragraphs we find

$$y_i y_j - \gamma y_{i-1} y_{j+1} - \sum_l (y_l)^{s_l} (z_l)^{a_l} u^{b_l} \in R_D.$$

Similarly if $i = 1$, then we get the same result using

$$y_i y_j - \gamma u y_{j+1} - \sum_l y_l u^{b_l} \in R_D.$$

In both cases we deduce that $y_i y_j$ is an initial term of some relation.

Finally, we examine the case (for $\alpha \geq \beta$) of $(y_i)^{i+1}$ when $i = \lfloor \frac{\alpha}{\beta} \rfloor$. If $\beta = 1$, then there are no relations with initial term a power of $y_i$ and we are immediately done. Otherwise, assuming $\beta \neq 1$, we note that $c_i d_{i+1} = c_{i+1} d_i + 1 = c_{i+1} + 1$. Therefore, if $i > 1$ then $c_i^{d_{i+1}+1}$ and $c_{i-1} c_{i+1}$ both lie in the $d_{i+1}+1$ degree with pole order $c_{i+1}+1$. Following a similar technique to the previous paragraphs we find

reference to Evan's paper where he proves this

$$(y_i)^{c_{i+1}+1} - y_{i+1} y_{i-1} - \sum_l (y_l)^{s_l} (z_l)^{a_l} u^{b_l}.$$

Similarly if $i = 1$ then

$$(y_i)^{c_{i+1}+1} - y_{i+1} u - \sum_l (z_l)^{a_l} u^{b_l}.$$

In each case we demonstrate $(y_i)^{c_{i+1}+1}$ as the initial term of a relation.

Let $J'$ be the ideal of $k[u, x_1, \ldots, x_m, y_1, \ldots, y_n]$ generated by the initial terms described in the previous paragraphs together with $\mathrm{gin}_{\prec}(I) k[u, x_1, \ldots, x_m, y_1, \ldots, y_n]$. We show that $J$ is in fact all of the initial ideal. Suppose $f$ is a non-constant monomial that does not lie within $J$. Further suppose $f$ does not lie in $k[u, x_1, \ldots, x_m]$. Then there is some $y_i | f$. Since $f \notin j$, we cannot have some $x_j | f$ or else we would have $y_i x_j | f$ contradicting $f \notin J$. Furthermore, if $y_l | f$ for $l < i$, then we must have $d_l = 1$ or else $y_l y_i$ is an initial element of a relation and divides $f$. If $d_i = 1$ as well, then we must have $c_i = \lfloor \frac{\alpha}{\beta} \rfloor$, or else $y_i y_l \in I$; finally we can not have $(y_{\lfloor \frac{\alpha}{\beta} \rfloor})^{d_{\lfloor \frac{\alpha}{\beta} \rfloor + 1}+1} | f$. We observe that the remaining elements are precisely those of the form $y_n^a y_i u^b$, $y_n^a z y_i u^b$, and $y_n^a z u^b$ which are the elements of our chosen $k$-basis for $k[u, x_1, \ldots, x_m, y_1, \ldots, y_n]/I'$ over $k[u, x_1, \ldots, x_m]/I$. Therefore, by dimension counting (in each degree), we see that $J'$ is the initial ideal of $I'$. Furthermore, since our choice of $y_i$s was generic, this is in fact the generic initial ideal.

In the case when all $\frac{\alpha}{\beta} \leq 1$, lemma 3.1 immediately tells us that the Grobner basis defined above is minimal. □

What about $\frac{\alpha}{beta} > 1$?

*Remark* 6.2. Notice that in lemma 6.1, $D'$ is generated in degree at most $\max(\bar{d}, \beta)$ with relations generated in degree at most $\max(\bar{\tau}, \bar{d} + \beta, 2\beta)$

This yields some immediate corollaries:

**Corollary 6.3.** *If the genus of* $X$ *is 0 and* $D \sim \sum \frac{\alpha_i}{\beta_i} P_i$ *is linearly equivalent to an effective* $\mathbf{Q}$ *divisor of* $X$*, then* $R_D$ *is generated in degree at most* $\max(\beta_i, 1)$ *with relations generated in degree at most* $2\max(\beta_i, 1)$.

*Proof.* We can prove this by induction. As a base case, let $D \sim 0$. Then $R_D \cong k[x]$ which is generated in degree 1 with no relations, satisfying the inductive hypothesis. Now, suppose the result is proven for all effective $D$ with support at most $r$ points; then if $D' \sim \sum_{i=1}^{r+1} \frac{\alpha_i}{\beta_i}$, we set $D = \sum_i^r \frac{\alpha_i}{\beta_i}$. By hypothesis, $R_D$ is generated in degree at most $\max_{1 \leq i \leq r}(\beta_i)$ with relations generated in degree at most $2\max_{1 \leq i \leq r}(\beta_i)$. Since $D$ is effective, the condition of equation 6.1 is met, so by remark 6.2 of lemma 6.1, $R_{D'}$ is generated in degree at most $\max(\beta_i)$ with relations generated in degree at most $2\max(\beta_i)$. □

**Corollary 6.4.** *Let* $L$ *be a stacky log spin canonical divisor of* $(\mathcal{X}, \Delta)$ *with signature* $(g; e_1, \ldots, e_n; \delta)$ *such that* $2L = K_{\mathcal{X}} + \Delta$ *with* $\delta = \deg(\Delta) > 0$. *Then* $L$ *is effective so we can inductively apply lemma 6.1.*

*Proof.* Since $L$ is a half canonical divisor,

$$2L = D + \sum \frac{e_i - 1}{2e_i} P_i$$

where

$$D \sim K_X + \Delta \sim -2\infty + \Delta$$

is a divisor of $X$ (i.e. with possibly negative integer coefficients). Since $\Delta$ is non-zero effective, $\deg(D) \geq -1$. Noting that the coefficient of any point $P$ occurring in a stacky divisor of $\mathcal{X}$ must have coefficient lying in $\mathbf{Z}[\frac{1}{e_i}]$ (ranging over $e_i$ s appearing in the characteristic of $(\mathcal{X}, \Delta)$).

Since all $e_i$s are odd, 2 cannot appear in a denominator of the coefficient of a point $P$ in $L$, meaning that $\frac{D}{2}$ must be a $X$-divisor. Since $\deg(\frac{D}{2}) = \frac{1}{2}\deg(D) \geq -\frac{1}{2}$, we in fact have $\deg(D) \geq 0$. Thus $L$ is linearly equivalent to an effective $X$ divisor plus $\sum \frac{i-1}{2i} P_i$. We have thus shown $L$ is linearly equivalent to an effective divisor. Thus, by 6.3 the desired result follows. □

**Corollary 6.5.** *Let $D$ be a fractional $\frac{a}{b}$-canonical divisor of a genus 0 stacky curve with denominator $b > 2$ when $\frac{a}{b}$ is written in reduced form.*

*Proof.* For a similar reason as in corollary 6.4, $D$ must be equivalent to an effective divisor (in this case even when $\delta = 0$), so by corollary 6.3 we get the desired result. □

Move this lemma:

**Lemma 6.6.** *Let $(X, \Delta, L) \to (X, \Delta', L)$ be the natural map of genus $g$ log curves with $\Delta = \Delta' + P$, such that $2L \sim K_X + \Delta$, $2L' \sim K_X + \Delta'$, and $\Delta = \Delta' + P$ with $\Delta' \neq 0$ for some $P \notin \operatorname{supp}(L)$. Suppose $\deg(\lfloor L \rfloor) = \deg(\lfloor L' \rfloor) \geq 0$, so $R_L$ has a generator $u$ in degree 1, and $R_{L'}$ has no new generators in degree 0. Suppose $D$ is generated by $u, x_1, \ldots, x_m$ in degree at most $\bar{d}$ with relations generated in degree at most $\tau$. Then*

- *$R_{D'}$ is generated over $R_D$ by elements $y_1, ..., y_n$ such that $y_i$ lies in degree $d_i$ and has a pole at $P$ of degree $c_i$*
- *If $I$ and $I'$ are the ideal of relations of $R_D$ and $R_{D'}$ respectively, then*

$$\begin{aligned} in_{\prec}(I') = {} & in_{\prec}(I)k[u, x_1, \ldots, x_m, y_1, \ldots, y_n] \\ & + \langle y_i x_j : 1 \leq i \leq n, 1 \leq j \leq m \rangle \\ & + \langle y_i y_j : \lfloor \frac{\alpha}{\beta} \rfloor \rangle \leq i \leq j \leq n - 1 \\ & + \langle y_i y_j : 1 \leq i \leq j \lfloor \frac{\alpha}{\beta} \rfloor \\ & + \langle (y_i)^{\deg(y_{i+1}+1)} : 1 \leq i = \lfloor \frac{\alpha}{\beta} \rfloor \end{aligned}$$

  *In particular, if $\frac{\alpha}{\beta} \leq 1$, then $I$ is minimally generated by elements with leading terms of the form $y_i x_j$ and $y_i y_j$.*
- *If $gin_{\prec}(I) = in_{\prec}(I)$ then $gin_{\prec}(I') = in_{\prec}(I')$.*
- *A Grobner basis with the initial terms $y_i x_j$, $y_i y_j$, and possibly $(y_{\lfloor \frac{\alpha}{\beta} \rfloor})^{y_{\lfloor \frac{\alpha}{\beta} \rfloor}+1}$ as shown above minimally generates $I'$.*

## 7. Genus Zero Non-effective

In this section, we will prove that if $(\mathcal{X}, 0, L)$ is a spin curve, so that $\mathcal{X}$ has signature $\sigma = (0; e_1, \dots, e_r; 0)$, then $R(\mathcal{X}, 0, L)$ is generated in degree at most $e := \max(5, e_1, \dots, e_r)$ with relations generated in degree at most $2e$, so long as $\sigma$ does not lie in a finite list of exceptions. This is proven in Subsection 7.5. The proof is a rather involved induction. In Subsections 7.2, 3.2 we prove inductive theorems, and in Subsection 7.3 we check the base cases of the induction. Finally, in Subsection 7.4 we list out the exceptional cases.

Peter: throughout this section I will provide a few notes of how we can generalize this to arbitrary genus with the appropriate conditions

7.1. **Saturation.** First, we present the saturations $s$ of the canonical divisor for all cases where $g = 0, \delta = 0$ as defined in Definition 2.1. The saturation can be computed via the application of Riemann-Roch on degrees up to $s$. Exceptional cases are listed first and generic cases follow.

| Signature | Saturation |
|---|---|
| $(0;3,3,3;0)$ | $\infty$ |
| $(0;3,3,5;0)$ | 18 |
| $(0;3,3,7;0)$ | 12 |
| $(0;3,3,9;0)$ | 12 |
| $(0;3,5,5;0)$ | 8 |
| $(0;5,5,5;0)$ | 8 |
| $(0;3,3,3,3;0)$ | 6 |
| $(0;3,3,k;0)$ | 9 |
| $(0;a,b,c;0)$ | 5 |
| $(0;e_1,\dots,e_r;0)$ | 3 |

**Table (I)**: Genus 0 saturation

7.2. **Inductive Theorem: Increasing the Number of Ramified Points for a 3-saturated divisor.**

**Proposition 7.1.** *Let* $R_{L'} = R' = k[x]/I' = k[x_3, x_4, x_5, \dots, x_m]/I'$ *satisfy* $\mathrm{sat}(\mathrm{Eff}(D'))$ 3. *Let* $L = L' + \frac{1}{3}P$, *and let* $R = R_L$. *Suppose* $\deg x_i = i$ *for* $i \in \{3,4,5\}$ *and that the ordering on* $k[x]$ *satisfies*

$$\mathrm{ord}_{x_3}(f) < \mathrm{ord}_{x_3}(h) \implies f \prec h.$$

*Then, the following statements hold.*

.

(a) *General elements* $y_i \in H^0(\mathcal{X}, iL)$ *for* $i \in \{3,4,5\}$ *satisfy* $-\mathrm{ord}_P(y_i) = 1$ *and any such choice of elements* $y_3, y_4, and y_5$ *minimally generate* $R$ *over* $R'$.

(b) *Equip* $k[y_3, y_4, y_5]$ *with* grlex *so that* $y_5 \prec y_4 \prec y_3$ *and equip the ring* $k[y_3, y_4, y_5, x]$ *with the block order so that* $R = k[y_3, y_4, y_5, x]/I$. *Let,*

$$\begin{aligned} J = {} & \mathrm{gin}_{\prec}(I')k[x, y_3, y_4, y_5] \\ & + \langle y_i x_j \mid 4 \le i \le 5, 3 \le j \le m\rangle \\ & + \langle y_i y_k \mid 4 \le i \le j \le 5.\rangle \end{aligned}$$

*Then,* $\mathrm{gin}_{\prec}(I) = J$.

Make explicit assumptions on genus.

Should switch either $R$ and $R'$ or $\mathcal{X}$ and $\mathcal{X}'$ to make prime notation consistent (note that inclusion is reversed since this map adds a stacky point to $\mathcal{X}$; note that we should also say that this map is only ramified at P), Aaron: you're right, I wrote the inclusion on rings in the wrong direction. Actually, the map isn't ramified at P, it's just defined away from P. I decided to remove this from the proposition though.

Peter: Do you mean to have $D = \frac{1}{3}P + D'$ or is it correct as written?

Peter: We can generalize to arbitrary genus with the condition $\deg(D) > 2g - 2$ meaning $h^0(X, K - D) = 0$ so Riemann-Roch gives us everything we wanted from $\mathbb{P}^1$.

(c) *Any set of minimal generators for $I'$ together with any set of relations with leading terms as in (b) minimally generate $I$.*

Understand why/whether we need to cite lemma 5.4.7.

*Proof.* First, the elements $y_3, y_4$, and $y_5$ are general and have $-\operatorname{ord}_P(y_i) = 1$ by Riemann Roch. To complete the proof of (a) we only need show these elements generate $R$ over $R'$. We can first see that the following is a $k$ basis for $R$ over $R'$:

$$\begin{aligned}(7.1)\qquad &\langle y_3^a x_3^b x_4^{\epsilon} x_5^{\epsilon'} \mid a \geq 0, b \geq 0, (\epsilon, \epsilon') \in \{(0,0),(0,1),(1,0)\}\rangle \\ &+ \langle y_3^a y_4, y_3^b y_5 \mid a \geq 0, b \geq 0\rangle\end{aligned}$$

To see these generate $R'$ over $R$, note that $\dim R'_d - \dim_k R_d = \lfloor \frac{d}{3} \rfloor$, by Riemann–Roch. So, it suffices to show that we have precisely $\lfloor \frac{d}{3} \rfloor$, elements of degree $d$ in the claimed basis of 7.1. Indeed, letting $a = \lfloor \frac{d}{3} \rfloor$ and $b = d \bmod 3$, we have that the elements

$$x_3^{a-1} x_{3+b}, y_3 x_3^{a-2} x_{3+b}, \ldots, y_3^a x_{3+b}, y_3^a y_{3+b}$$

are precisely $a$ elements, which are all independent as they have distinct pole orders at $P$. This completes part (a).

To show part (b), note that the generators in 7.1 are precisely a set of monomials which generate $k[x,y]/J$ over $k$. So, to show $J = \operatorname{in}_{\prec}(I)$, it suffices to show that all generators of $J$ lie in $\operatorname{in}_{\prec}(I)$.

This follows, since there exist constants $A_{i,j} \in k$ for $4 \leq i \leq 5, 3 \leq j \leq m$, elements $B_1, B_2, B_3 \in k$ and elements $w_{i,j} \in R'$ so that the following linear combinations of elements lie in $R'$.

$$\begin{aligned} &y_i x_j - A_{i,j} y_{i-1} w_j \qquad \text{so that } 4 \leq i \leq 5, 3 \leq j \leq m \text{ and } \deg w_j = \deg x_j + 1 \\ &y_4^2 + B_1 y_3 y_5 \\ &y_4 y_5 + B_2 y_3^2 x_3 \\ &y_5^2 + B_3 y_3^2 x_4 \end{aligned}$$

Of course, the initial terms of these elements are precisely the generators of $J$, completing (b).

Finally, (c) follows immediately from Lemma 3.1 $\square$

fill out

Here we describe induction of raising degrees from Theorem 3.6

7.3. **Base Cases.** Now we have an inductive statement on the number of ramified points in Proposition 7.1 and an inductive statement on ramification orders in Theorem 3.6. After verifying that explicit pairs $(\mathcal{X}, J)$ of stacky curves $\mathcal{X}$ with particular signatures and particular choices of $J$ are admissible, the application of the two inductive statements demonstrate that the spin canonical ring $R(\mathcal{X}, 0, L)$ is generated in degree at most $e := \max(5, e_1, \ldots, e_r)$ with relations generated in degree at most $2e$ for $\mathcal{X}$ with any signature $\sigma = (0; e_1, \ldots, e_r; 0)$ outside of a finite set of exceptional cases.

**Lemma 7.2.** *Let $(\mathcal{X}', \Delta, L')$ be a tame log spin curve over a perfect field $k$, so that $\mathcal{X}$ has signature $\sigma = (0; e_1, \ldots, e_r; 0)$. Then $R' := R(\mathcal{X}', \Delta, L')$ is generated as a $k$-algebra by* *elements of degree at most* $e = \max(5, e_1, \ldots, e_r)$ *with relations in degree at most* $2e$ *and* $(\mathcal{X}', J)$ *is admissible for the following cases:*

(a) $\sigma_1 = (0; 3, 3, 11; 0), J_1 = \{3\}$
(b) $\sigma_2 = (0; 3, 5, 9; 0), J_2 = \{3\}$
(c) $\sigma_3 = (0; 3, 7, 7; 0), J_3 = \{2, 3\}$
(d) $\sigma_4 = (0; 5, 5, 7; 0), J_4 = \{3\}$
(e) $\sigma_5 = (0; 5, 7, 7; 0), J_5 = \{2, 3\}$
(f) $\sigma_6 = (0; 7, 7, 7; 0), J_6 = \{1, 2, 3\}$
(g) $\sigma_7 = (0; 3, 3, 3, 5; 0), J_7 = \{4\}$
(h) $\sigma_8 = (0; 3, 3, 5, 5; 0), J_8 = \{3, 4\}$
(i) $\sigma_9 = (0; 3, 5, 5, 5; 0), J_9 = \{2, 3, 4\}$
(j) $\sigma_{10} = (0; 5, 5, 5, 5; 0), J_{10} = \{1, 2, 3, 4\}$
(k) $\sigma_{11} = (0; 3, 3, 3, 3, 3; 0), J_{11} = \{1, 2, 3, 4, 5\}$

*Furthermore,* $R'$ *satisfies the hypothesis of Theorem 3.6(d), which asserts that for all* $i \in J$, $e_i > \deg z$ *for any generator* $z$ *of* $R'$ *and if* $R' = k[y, x]/I'$ *(as defined in Theorem 3.6(c)) then* $\mathrm{in}_{\prec}(I')$ *is minimally generated by products of two monomials.*

**Example 7.3.** Here we demonstrate, in detail, that case (b) with $(\sigma_2 = (0; 3, 7, 7; 0), J_2 = \{3, 4\})$ satisfies the conditions. We have that $L' \sim -(\infty) + \frac{1}{3}P_1 + \frac{3}{7}P_2 + \frac{3}{7}P_3$ for some points $P_i \neq \infty$. We exhibit a (minimal) presentation for $R'$.

Notice that $\deg\lfloor dK_{\mathcal{X}'} = -d + \lfloor \frac{d}{3} \rfloor + 2\lfloor \frac{3d}{7} \rfloor$.

By Max Noether's theorem over $\mathbb{P}^1$ (see Lemma 3.1.1 of Voight and Zureick-Brown [5]),

$$H^0(\mathcal{X}', \lfloor 21L' \rfloor) \otimes H^0(\mathcal{X}', \lfloor (d-21)L' \rfloor) \twoheadrightarrow H^0(\mathcal{X}', \lfloor dL' \rfloor)$$

is surjective whenever $\deg(\lfloor (d-21)K_{\mathcal{X}'} \rfloor) \geq 0$. Since $\sigma_2$ has saturation $s = 5$, as noted in Subsection 7.1, then this map is surjective when $d \geq 21 + s = 25$ (i.e. $R'$ is generated up to degree 24).

For $d = 0, 1, 2, \ldots$ we have

$$\dim H^0(\mathcal{X}', dK_{\mathcal{X}'}) = 1, 0, 0, 1, 0, 1, 1, 2, 1, 1, 2, 1, 3, 2, 3, \ldots$$

so $R'$ must have some generators $x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}$.

To show that these generate all of $R'$, we need to show that all $H^0(\mathcal{X}', dL')$ are generated by lower degrees for $d = 6$ and $7 < d \leq 24$. Note that we can again use Max Noether's theorem (taking saturation into account) to see that

$$H^0(\mathcal{X}', \lfloor 3L' \rfloor) \otimes H^0(\mathcal{X}', \lfloor (d-3)L' \rfloor) \twoheadrightarrow H^0(\mathcal{X}', \lfloor dL' \rfloor) \text{ if } d \geq 12 \text{ and } 3 \nmid d$$
$$H^0(\mathcal{X}', \lfloor 7L' \rfloor) \otimes H^0(\mathcal{X}', \lfloor (d-7)L' \rfloor) \twoheadrightarrow H^0(\mathcal{X}', \lfloor dL' \rfloor) \text{ if } d \geq 6, d \not\equiv 0, 5 \pmod 7$$

are surjective. This reduces the problem to checking generation in the degrees of the exceptions $d \in \{8, 12, 21\}$. Checking the remaining cases via pole degrees shows that these degrees are also generated in lower degrees, so $R'$ is generated in degrees $\{3, 5, 7, 7\}$.

Furthermore, we can directly see that the relations are given in degrees 10 and 14. In particular, we have a relation of the form

$$a_1 x_{5,1}^2 + a_2 x_{3,1} x_{7,1} + a_3 x_{3,1} x_{7,2} = 0$$

in degree 10 and a relation of the form

$$a_1 x_{3,1}^3 x_{5,1} + a_2 x_{7,1}^2 + a_3 x_{7,1} x_{7,2} + a_4 x_{7,2}^2 = 0$$

in degree 14.

Let I be the ideal generated by these relations in $k[x_{7,1}, x_{7,2}, x5, 1, x_{3,1}]$. Under grlex with $x_{i,j} \prec x_{k,l}$ if $i > j$ or $i = j$ and $j < l$, the initial ideal of I is

$$in_{\prec}(I) =< x_{5,1}^2, x_{7,1}^2 >$$

To demonstrate that these two relations generate all relations, we show that $\dim(R')_d = \dim(k[x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}]/I)_d$ for all degrees $d \geq 0$. In particular,

$$\begin{aligned} \dim(k[x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}]/I)_d &= \dim(k[x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}]/in_{\prec}(I))_d \\ &= \dim(k[x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}]/ < x_{5,1}^2, x_{7,1}^2 > \end{aligned}$$

for all $d \geq 0$. But the fact $\dim(R')_d = \dim(k[x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}]/ < x_{5,1}^2, x_{7,1}^2 >$ can be checked by looking at the k-basis at each degree d.

Therefore, the canonical ring $R'$ has presentation $R' = k[x_{7,1}, x_{7,2}, x_{5,1}, x_{3,1}]/I$ with initial ideal $in_{\prec}(I)$ generated by quadratics under grlex with $x_{i,j} \prec x_{k,l}$ if $i > j$ or $i = j$ and $j < l$. These quadratics must also determine minimal generators for I by Lemma 3.1. Thus, $R'$ is generated up to degree $e = 7$ with relations up to degree $2e = 14$, as desired. We can also see that $e_i > \deg z$ for any generator $z$ of $R'$ since $e_i = 9 > e$ for all $i \in J$.

Admissibility is the remaining condition to check. We can assume that $x_{7,1}$ corresponds to the generator with maximal pole order at $P_2$ and $x_{7,2}$ correspond to the generator with maximal pole order at $P_3$ (relabel otherwise). Use the presentation given above with $y_{2,e_2'} := x_{7,1}$ and $y_{3,e_3'} := x_{7,2}$. We see that it immediately satisfies (Ad-i) of Definition 3.3. We may also choose pole orders of the generators such that $y_{i,e_i'}$ is the only generator lying on the line $-\mathrm{ord}_{P_i}(z) = \deg z \frac{3d}{7}$ in the $(\deg z, -\mathrm{ord}_{P_i}(z))$ lattice and with the other generators lying below the line as seen in Figure 7.3 (e.g. the pole orders $(-\mathrm{ord}_{P_1}(z), -\mathrm{ord}_{P_2}(z), -\mathrm{ord}_{P_3}(z))$ may be chosen to be $(1,1,1)$, $(1,2,2)$, $(2,3,2)$, and $(2,2,3)$ for $x_{3,1}, x_{5,1}, x_{7,1}, x_{7,2}$ respectively).

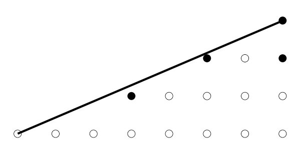

FIGURE 1. Generators in the $(\deg z, -\mathrm{ord}_{P_i}(z))$ lattice

Also note that $e_j' + 2d = 7 + 2d \nmid 2 = e_i - e_j'$ for all $i, j \in J_2 = \{2, 3\}$ such that $j \neq i$ and for all $d \geq 0$. Thus, $S_{\sigma_2, J_2}(i, d) = \emptyset$ for each $i \in J$. Furthermore, $\deg\lfloor e_i L \rfloor = \deg\lfloor 9L \rfloor = 2\lfloor \frac{4*9}{9} \rfloor + \lfloor \frac{9}{3} \rfloor - 9 = 2$, so (Ad-iii) is satisfied and $\sigma_2, J_2$ is admissible.

*Proof.* For the remaining cases, we can use a similar method to the one used in Example 7.3 to find a presentation the desired conditions.

The following table demonstrates that $R(\mathcal{X}', \Delta, L')$ is generated as a $k$-algebra by elements of degree at most $e$ with relations in degree at most $2e$ for each case. Also note that in these cases, $e_i = e + 2 > \deg z$ for all $i \in J$ and any generator $z$ of $R'$.

| Case | Generator Degrees | Minimal Relation Degrees | $e$ |
|---|---|---|---|
| (a) | $\{3, 7, 9, 11\}$ | $\{14, 18\}$ | 11 |
| (b) | $\{3, 5, 7, 9\}$ | $\{12, 14\}$ | 9 |
| (c) | $\{3, 5, 7, 7\}$ | $\{10, 14\}$ | 7 |
| (d) | $\{3, 5, 5, 7\}$ | $\{10, 12\}$ | 7 |
| (e) | $\{3, 5, 5, 7, 7\}$ | $\{10, 10, 12, 12, 14\}$ | 7 |
| (f) | $\{3, 5, 5, 7, 7, 7\}$ | $\{10, 10, 10, 12, 12, 12, 14, 14, 14\}$ | 7 |
| (g) | $\{3, 3, 4, 5\}$ | $\{8, 9\}$ | 5 |
| (h) | $\{3, 3, 4, 5, 5\}$ | $\{8, 8, 9, 9, 10\}$ | 5 |
| (i) | $\{3, 3, 4, 5, 5, 5\}$ | $\{8, 8, 8, 9, 9, 9, 10, 10, 10\}$ | 5 |
| (j) | $\{3, 3, 4, 5, 5, 5, 5\}$ | $\{8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 10, 10, 10\}$ | 5 |
| (k) | $\{3, 3, 3, 4, 4, 5\}$ | $\{6, 7, 7, 8, 8, 8, 9, 9, 10\}$ | 5 |

**Table (II)**: Genus 0 noneffective base cases generator/relations

We can also always find a presentation for these cases such that they satisfy (Ad-i) and (Ad-ii) and that $\mathrm{in}_{\prec}(I')$ is minimally generated by products of two monomials. The procedure for verifying these are similar to those for the given example.

Furthermore, each case always satisfies (Ad-iii) as demonstrated below. Notice that the $e_i$ and $\{e'_j : j \neq i\}$ are equivalent for any choice of $i \in J$ for these cases, so $\deg\lfloor e_i L \rfloor$ and $\max_{d \geq 0} \#S_{(\sigma,J)}(i)$ are independent of the choice of $i$.

| Case | $\sigma$ | $J$ | $\deg\lfloor e_i L\rfloor$ | $\max_{d\geq 0} \#S_{(\sigma,J)}(i)$ |
|---|---|---|---|---|
| (a) | $(0; 3, 3, 11; 0)$ | $\{3\}$ | 1 | 0 |
| (b) | $(0; 3, 5, 9; 0)$ | $\{3\}$ | 1 | 0 |
| (c) | $(0; 3, 7, 7; 0)$ | $\{3\}$ | 2 | 0 |
| (d) | $(0; 5, 5, 7; 0)$ | $\{3\}$ | 1 | 0 |
| (e) | $(0; 5, 7, 7; 0)$ | $\{2, 3\}$ | 2 | 0 |
| (f) | $(0; 7, 7, 7; 0)$ | $\{1, 2, 3\}$ | 3 | 0 |
| (g) | $(0; 3, 3, 3, 5; 0)$ | $\{4\}$ | 2 | 0 |
| (h) | $(0; 3, 3, 5, 5; 0)$ | $\{3, 4\}$ | 3 | 0 |
| (i) | $(0; 3, 5, 5, 5; 0)$ | $\{2, 3, 4\}$ | 4 | 0 |
| (j) | $(0; 5, 5, 5, 5; 0)$ | $\{1, 2, 3, 4\}$ | 5 | 0 |
| (k) | $(0; 3, 3, 3, 3, 3; 0)$ | $\{1, 2, 3, 4, 5\}$ | 5 | 0 |

**Table (III)**: Genus 0 noneffective base cases (Ad-iii)

Thus, all of the cases are admissible and satisfy the additional desired conditions. $\square$

Now we can induct on cases (a)-(e) of 7.2 to obtain bounds on the minimal generator and relation degrees for canonical rings of curves in the genus 0 case.

Using the results of Proposition 7.1 and Theorem 3.6, we can inductively raise the ramification orders and number of ramified points in all of the cases in Lemma 7.2 to obtain the following result.

**Corollary 7.4.** *Let $(\mathcal{X}, \Delta, L)$ be a tame log spin curve over a perfect field $k$, so that $\mathcal{X}$ has signature $\sigma = (0; e_1, \ldots, e_r; 0)$. Then the canonical ring*

$$R(\mathcal{X}, 0, L) = \bigoplus_{d=0}^{\infty} H^0(\mathcal{X}, \mathcal{O}(L)^{\otimes d})$$

*is generated as a $k$-algebra by elements of degree at most $e := \max(5, e_1, \ldots, e_r)$ with relations in degree at most $2e$, so long as $\sigma \notin S$ where*

$$\begin{aligned} S :=\ & \{(0;3,3,k;0) : 3 \leq k \leq 9 \text{ odd}\} \\ & \cup \{(0;3,5,5;0), (0;3,5,7;0), (0;5,5,5;0), (0;3,3,3,3;0)\} \end{aligned}$$

*is a finite set.*

*Proof.* A process of induction raising the ramification orders of the ramified points of the spin curves corresponding to Lemma 7.2 cases (a)-(c) via Theorem 3.6 will demonstrate the desired generator and relation degree results for curves with three ramified points. Similarly, this induction process on Lemma 7.2 case (d) will give the desired result for curves with four ramified points excluding signature $(0;3,3,3,3;0)$. A process of induction increasing the number of ramified points via Proposition 7.1 on Lemma 7.2 case (e) combined with the induction process on ramification orders will yield the result for curves with at least five ramified points. □

Detailed description of how the inductive lemmata yield e and 2e

7.4. **Exceptional Cases.** Here we present the explicit generators and relations of the remaining cases given by signatures in the finite set

$$\begin{aligned} S :=\ & \{(0;3,3,k;0) : 3 \leq k \leq 9 \text{ odd}\} \\ & \cup \{(0;3,5,5;0), (0;3,5,7;0), (0;5,5,5;0), (0;3,3,3,3;0)\} \end{aligned}$$

and in particular describe the only exceptions to the $e$ and $2e$ bounds on the generator degree and relation degree, respectively, in the genus 0 case.

| Signature | Generator Degrees | Minimal Relation Degrees | $e$ |
|---|---|---|---|
| $(0;3,3,3;0)$ | $\{3\}$ | $\emptyset$ | 3 |
| $(0;3,3,5;0)$ | $\{3,10,15\}$ | $\{30\}$ | 5 |
| $(0;3,3,7;0)$ | $\{3,7,12\}$ | $\{24\}$ | 7 |
| $(0;3,3,9;0)$ | $\{3,7,9\}$ | $\{21\}$ | 9 |
| $(0;3,5,5;0)$ | $\{3,5,10\}$ | $\{20\}$ | 5 |
| $(0;3,5,7;0)$ | $\{3,5,7\}$ | $\{17\}$ | 7 |
| $(0;5,5,5;0)$ | $\{3,5,5\}$ | $\{15\}$ | 5 |
| $(0;3,3,3,3;0)$ | $\{3,3,4\}$ | $\{12\}$ | 3 |

**Table (IV)**: Genus 0 exceptional cases

*Remark* 7.5. These cases give all of the exceptions to the $e$ and $2e$ bounds on the generator and relation degree. Notice that each of these exceptional cases, apart from $(0;3,5,7;0)$, corresponds to a signature with non-generic saturation (i.e. saturation not equal to 3, 5, or 9 as seen in the table in Subsection 7.1). Exceptional saturation can be viewed as "forcing" generators (and thus relations) in higher degrees than in the generic case.

7.5. **Application of induction.** Now we can combine this.

write this section

**Theorem 7.6.** *Let $(\mathcal{X}, \Delta, L)$ be a tame log spin curve over a perfect field $k$, so that $\mathcal{X}$ has signature $\sigma = (0; e_1, \ldots, e_r; \delta)$. Then the canonical ring*

$$R(\mathcal{X}, \Delta, L) = \bigoplus_{d=0}^{\infty} H^0(\mathcal{X}, \mathcal{O}(L)^{\otimes d})$$

*is generated as a $k$-algebra by elements of degree at most $e = \max(5, e_1, \ldots, e_r)$ with relations in degree at most $2e$, so long as $\sigma$ does not lie in a finite list of exceptions.*

Include reference to the table with this list of exceptions.

## 8. Further Research

- More on fractional modular forms
- Make explicit generators and relations for base case of non-stacky higher genus half-canonical divisors
- In the case $g \geq 2$, can the bound from Proposition **??** be reduced from 11 to 10 (which would then be sharp by Remark 5.9)?

Possibly think more about this

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## References

[1] Kunihiko Kodaira. *Complex manifolds and deformation of complex structures*, volume 283 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1986. Translated from the Japanese by Kazuo Akao, With an appendix by Daisuke Fujiwara.

[2] Jorge Manuel Sentieiro Neves. *Halfcanonical rings on algebraic curves and applications to surfaces of general type*. PhD thesis, University of Warwick, 2003.

[3] Evan O'Dorney. Canonical rings of $\mathbb{Q}$ divisors on $\mathbb{P}^1$. *Accepted by Annals of Combinatorics*, 2014.

[4] Bernard Saint-Donat. Projective models of $K-3$ surfaces. *Amer. J. Math.*, 96:602–639, 1974.

[5] John Voight and David Zureick-Brown. The canonical ring of a stacky curve. *arXiv:1501.04657*, 2015.

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