Set theory

If X is a set (class), then $r \in X$ means that r belongs to X or r is a member of X.

Two sets (classes) X and Y are called *equipotent* if there is a one-to-one correspondence between X and Y.

If X and Y are sets (classes), then $Y \subseteq X$ means that Y is a *subset* (subclass) of X, i.e., Y is a set such that all elements of Y belong to X, and X is a *superset* of Y. A subset is *proper* if it does coincide with the whole set. It is denoted by $Y \subset X$.

The *union* $Y \cup X$ of two sets (classes) Y and X is the set (class) that consists of all elements from Y and from X. The union $Y \cup X$ is called disjoint if $Y \cap X = \varnothing$.

The *intersection* $Y \cap X$ of two sets (classes) Y and X is the set (class) that consists of all elements that belong both to Y and to X.

The *union* $\bigcup_{i\in I} X_i$ of sets (classes) X_i is the set (class) that consists of all elements from all sets (classes) X_i , $i\in I$.

The *intersection* $\bigcap_{i\in I} X_i$ of sets (classes) X_i is the set (class) that consists of all elements that belong to each set (class) X_i , $i\in I$.

The *difference* $Y \setminus X$ of two sets (classes) Y and X is the set (class) that consists of all elements that belong to Y but does not belong to X.

If a set (class) X is a subset of a set (class) Y, i.e., $X \subseteq Y$, then the difference $Y \setminus X$ is called the *complement* of the set (class) X in the set (class) Y and is denoted by $C_Y X$.

$\varnothing$ is the *empty set*, i.e., the set that has no elements.

If X is a set, then 2^X is the *power set* of X, which consists of all subsets of X. The *power set* of X is also denoted by PX.

If X and Y are sets (classes), then $X \times Y = \{(x, y); x \in X, y \in Y\}$ is the *Cartesian product* of X and Y, in other words, $X \times Y$ is the set (class) of all pairs (x, y), in which x belongs to X and y belongs to Y.

A mapping (function) f from a set X into a set Y is denoted by $f: X \to Y$ and a mapping of an element x into an element y is denoted by $x \mapsto y$.

Y^X denotes the set of all mappings from X into Y.

$$X^n = \underbrace{X \times X \times \ldots X \times X}_{n}$$

Elements of the set X^n have the form $(x_1, x_2, \ldots , x_n)$ with all $x_i \in X$ and are called n-tuples, or simply, tuples.

A fundamental structure of mathematics is *function*. However, functions are special kinds of binary relations between two sets, which are defined below.

A *binary relation* T between sets X and Y, also called *correspondence* from X to Y, is a subset of the Cartesian product $X\times Y$. The set X is called the *domain* of T ($X = \mathrm{Dom}(T)$) and Y is called the *codomain* of T ($Y = \mathrm{Codom}(T)$). The *range* of the relation T is $\mathrm{Rg}(T) = \{ y ; \exists x \in X ((x, y) \in T) \}$. The *domain of definition* also called the *definability domain* of the relation T is $\mathrm{DDom}(T) = \{ x ; \exists y \in Y ((x, y) \in T) \}$. If $(x, y) \in T$, then one says that the elements x and y are in relation T, and one also writes $T(x, y)$.

The image $T(x)$ of an element x from X is the set $\{y; (x, y) \in T\}$ and the coimage $T^{-1}(y)$ of an element y from Y is the set $\{x; (x, y) \in T\}$.

The *graph* of binary relation T between sets of real numbers is the set of points in the two dimensional vector space (a plane), the coordinates of which satisfy this relation.

Binary relations are also called *multivalued functions* (mappings or maps).

Taking binary relations $T \subseteq X\times Y$ and $R \subseteq Y\times Z$, it is possible to build a new binary relation $RT \subseteq X\times Z$ that is called the (*sequential*) *composition* or *superposition* of binary relations T and R and is defined as

$$R \circ T = \{(x, z); x \in X, z \in Z; \text{where } (x, y) \in T \text{ and } (y, z) \in R \text{ for some } y \in Y\}.$$

A *preorder* (also called *quasiorder*) on a set (class) X is a binary relation Q on X that satisfies the following axioms:

O1. Q is *reflexive*, i.e. $x\mathrm{Q}x$ for all x from X.

O2. Q is *transitive*, i.e., $x\mathrm{Q}y$ and $y\mathrm{Q}z$ imply $x\mathrm{Q}z$ for all $x, y, z \in X$.

A preorder can be *partial* or *total* when for all $x, y \in X$, we have either $x\mathrm{Q}y$ or $y\mathrm{Q}x$.

A *partial order* is a preorder that satisfies the following additional axiom:

O3. Q is *antisymmetric*, i.e., $x\mathrm{Q}y$ and $y\mathrm{Q}x$ imply $x = y$ for all $x, y \in X$.

A *strict* also called *sharp partial order* is a preorder that is not reflexive, is transitive and satisfies the following additional axiom:

O4. Q is *asymmetric*, i.e., only one relation $x\mathrm{Q}y$ or $y\mathrm{Q}x$ is true for all $x, y \in X$.

A *linear* or *total order* is a strict partial order that satisfies the following additional axiom:

O5. We have either $x\mathrm{Q}y$ or $y\mathrm{Q}x$ for all $x, y \in X$.

A set (class) X is *well-ordered* if there is a partial order on X such that any its non-empty subset has the least element. Such a partial order is called *well-ordering*.

An *equivalence* on a set (class) X is a binary relation Q on X that is reflexive, transitive and satisfies the following additional axiom:

O6. Q is *symmetric*, i.e., xQy implies yQx for all x and y from X.

Set-theoretical symbols

$>$ larger than

$<$ less than

$\geq$ larger than or equal to

$\leq$ less than or equal to

$=$ equal

$\approx$ approximately equal

$\neq$ not equal

$\in$ belongs

$\notin$ does not belong

$\subseteq$ is a subset

$\subset$ is a proper subset

$\not\subset$ is not a proper subset

Traditionally, a *function* (also called a *mapping* or *map* or *total function* or *total mapping* or *everywhere defined function*) f from X to Y is defined as a binary relation between sets X and Y in which there are no elements from X which are corresponded to more than one element from Y and to any element from X, some element from Y is corresponded. At the same time, the function f is also denoted by $f: X \to Y$ or by $f(x)$. In the latter formula, x is a variable and not a definite element from X. The *support*, or *carrier*, of a function f is the closure of the set where $f(x) \neq 0$. Usually the element $f(a)$ is called the *image* of the element a and denotes the value of f on the element a from X. The *coimage* $f^{-1}(y)$ of an element y from Y is the set $\{x; f(x) = y\}$.

However, the traditional definition does not include all kinds of functions and their representations.

There are three basic forms of function representation (definition):

1. (The *set-theoretical*, e.g., *table*, *representation*) A function f is given as a subset R_f of the direct product $X \times Y$ such that the first element if each pair from R_f uniquely defines the second element in this pair, e.g., in a form of a table or of a list of pairs (x, y) where the first element x is taken from X, while the second element y is the image $f(x)$ of the first one. The set R_f is called the *graph* of the function f. When X and Y are sets of points in a geometrical space, e.g., their elements are real numbers, the graph of the function f is called the *geometrical graph* of f.
2. (The *analytic representation*) A function f is described by a formula, i.e., a relevant expression in a mathematical language, e.g., $f(x) = \sin(e^{x + \cos x})$.
3. (The *algorithmic representation*) A function f is given as an algorithm that computes $f(x)$ given x.

$f(x) \equiv a$ means that the function $f(x)$ is equal to a at all points where $f(x)$ is defined.

A function (mapping) f from X to Y is an *injection* if the equality $f(x) = f(y)$ implies the equality $x = y$ for any elements x and y from X, i.e., different elements from X are mapped into different elements from Y.

A function (mapping) f from X to Y is a *projection* also called *surjection* if for any y from Y there is x from X such that $f(x) = y$.

A function (mapping) f from X to Y is a *bijection* if it is both a projection and injection.

A function (mapping) f from X to Y is an *inclusion* if the equality $f(x) = x$ holds for any element x from X.

Two important concepts of mathematics are the domain and range of a function. However, there is some ambiguity for the first of them. Namely, there are two distinct meanings in current mathematical usage for this concept. In the majority of mathematical areas, including the calculus and analysis, the term "domain of f" is used for the set of all values x such that $f(x)$ is defined. However, some mathematicians (in particular, category theorists), consider the domain of a function f: $X \rightarrow Y$ to be X, irrespective of whether $f(x)$ is defined for all x in X. To eliminate this ambiguity, we suggest the following terminology consistent with the current practice in mathematics.

If f is a function from X into Y, then the set X is called the *domain* of f (it is denoted by Dom f) and Y is called the *codomain* of T (it is denoted by Codom f). The *range* Rg f of the function f is the set of all elements from Y assigned by f to, at least, one element from X, or formally, Rg f $= \{y; \exists\, x \in X\, (f(x) = y)\}$. The *domain of definition* also called the *definability domain*, DDom f, of the function f is the set of all elements from X that related by f to, at least, one element from Y is or formally, DDom $f = \{x; \exists\, y \in Y\, (f(x) = y)\}$. Thus, for a partial function f, its domain of definition DDom f is the set of all elements for which $f(x)$ is defined.

f is a *total function* if DDom f = Dom f .

f is a *partial function* if DDom $f \neq$ Dom f .

Taking two mappings (functions) f: $X \rightarrow Y$ and g: $Y \rightarrow Z$, it is possible to build a new mapping (function) gf: $X \rightarrow Z$ that is called the (*sequential*) *composition* or *superposition* of mappings (functions) f and g and defined by the rule $gf(x) = g(f(x))$ for all x from X.

For any set S, $\chi_S(x)$ is its *characteristic function*, also called *set indicator function*, if $\chi_S(x)$ is equal to 1 when $x \in S$ and is equal to 0 when $x \notin S$, and $C_S(x)$ is its partial characteristic function if $C_S(x)$ is equal to 1 when $x \in S$ and is undefined when $x \notin S$.

If f: $X \rightarrow Y$ is a function and $Z \subseteq X$, then the restriction $f|_Z$ of f on Z is the function defined only for elements from Z and $f|_Z(z) = f(z)$ for all elements z from Z.

Named sets as the most encompassing and fundamental mathematical construction include all generalizations of ordinary sets and provide unified foundations for the whole mathematics (Burgin, 2004b). Functions, relations, graphs, fiber bundles, enumerations and many other mathematical structures are named sets. Moreover, all mathematical structures are built of named sets (Burgin, 2011).

A *named set* (also called a *fundamental triad*) has the following graphic representation (Burgin, 1990; 1995; 2011):

$$\text{Entity 1} \xrightarrow{\text{connection}} \text{Entity 2} \qquad (1)$$

or

$$\text{Essence 1} \xrightarrow{\text{correspondence}} \text{Essence 2} \qquad (2)$$

In the fundamental triad (named set) (1) or (2), Entity 1 (Essence 1) is called the *support*, the Entity 2 (Essence 2) is called the *reflector* (also called the *set* or *component of names*) and the connection (correspondence) between Entity 1 (Essence 1) and connection (correspondence) is called the *reflection* (also called the *naming correspondence*) of the fundamental triad (1) (respectively, (2)).

In the symbolic form, a *named set* (*fundamental triad*) **X** is a triad (X, f, I) where X is the *support* of **X** and is denoted by S(**X**), I is the *component of names* (also called *set of names* or *reflector*) of **X** and is denoted by N(**X**), and f is the *naming correspondence* (also called *reflection*) of the named set **X** and is denoted by n(**X**). The most popular type of named sets is a named set **X** = (X, f, I) in which X and I are sets and f consists of connections between their elements. When these connections are set theoretical, i.e., each connection is represented by a pair (x, a) where x is an element from X and a is its name from I, we have a *set theoretical named set*, which is binary relation. Even before the concept of a fundamental triad was introduced, Bourbaki in their fundamental monograph (Bourbaki, 1960) had also represented binary relations in a form of a triad (named set).

Named sets as the most encompassing and fundamental mathematical construction is pervasively used in computer science and information technology.

It’s possible to read much more about fundamental triads (named sets) in the book

Burgin, M. *Theory of Named Sets*, Nova Science Publishers, New York, 2011

Logical concepts and structures

If P and Q are two statements, then $P \rightarrow Q$ means that P implies Q and $P \leftrightarrow Q$ means that P is equivalent to Q.

Logical operations:

negation is denoted by $\neg$ or by $\sim$,

conjunction also called logical “and” is denoted by $\wedge$ or by & or by $\cdot$,

disjunction also called logical “or” is denoted by $\vee$,

implication is denoted by $\rightarrow$ or by $\Rightarrow$ or by $\supset$,

equivalence is denoted by $\leftrightarrow$ or by $\equiv$ or by $\Leftrightarrow$.

The logical symbol $\forall$ is called the *universal quantifier* and means “for any”.

The logical symbol $\exists$ is called the *existential quantifier* and means “there exists”.