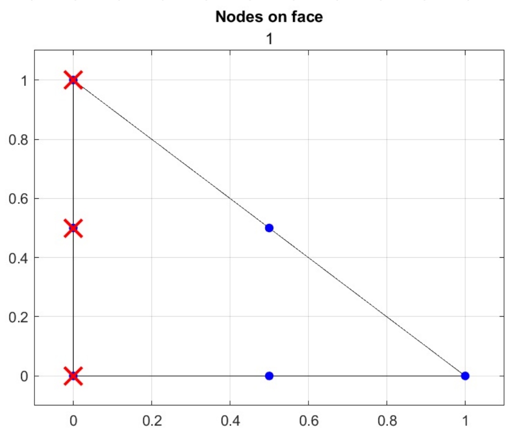
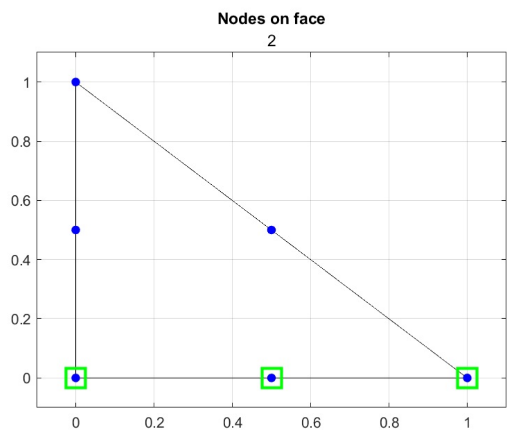
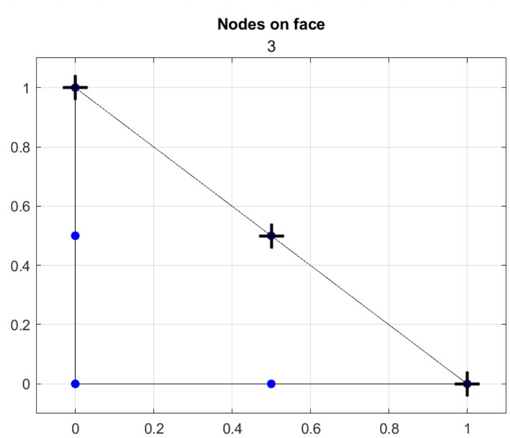
Tasks for part 1.1

2)







Tasks for Part 1.2.

3) A function was written to check if the properties are being satisfied. The function has the number of dimensions, the polynomial order and some evaluation nodes as the inputs. The 3 tests are done, and an error message is returned in case one of the tests fails. For the one dimensional case, the Lagrangian functions are plotted.

4) The derivatives of the basis functions were compared to a finite element approximation in order to check if they were being evaluated correctly.

All tests were run for dimensions 1, 2, 3 and polynomial orders 1, 2, 3, 4 and 5, with tolerances chosen as 10^{-8} for the properties of the Lagrangian basis, and 10^{-6} for the finite difference test (with a $\varepsilon = 10^{-7}$, where

$$\text{finite difference approximation} = \frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon}$$

Tasks for Part 1.3

1) $\mathcal{G}_e(\xi) := \sum_{i=1}^{n_v} \hat{x}_i^e \, \psi_i(\xi)$

$$G_e(\xi) := \frac{\partial}{\partial \xi} \mathcal{G}(\xi) \Rightarrow [G_e(\xi)]_j = \frac{\partial}{\partial \xi_j} \quad \hat{x}_i^e \, \psi_i(\xi)$$

$$\Rightarrow \boxed{[G_e(\xi)]_j = \quad \hat{x}_i^e \, \psi_{i,j}(\xi)} \qquad j = 1, \dots, d, \quad i = 1, \dots, n_v$$

$$\Rightarrow \boxed{\underset{d \times d}{G_e(\xi)} = \underset{d \times n_v}{\hat{x}^e} \cdot \underset{n_v \times d}{\frac{\partial \psi}{\partial \xi}(\xi)}}, \quad \text{where} \quad \hat{x}^e = [\hat{x}_1^e \; \dots \; \hat{x}_{n_v}^e]$$

$$\mathcal{F}_{ef}(r) := \mathcal{G}_e(\gamma_f(r)) = \quad \hat{x}_i^e \, \psi_i(\gamma_f(r))$$

$$F_{ef}(r) := \frac{\partial}{\partial r} \mathcal{F}_{ef}(r) \Rightarrow [F_{ef}(r)]_j = \frac{\partial}{\partial r_j} \left(\hat{x}_i^e \, \psi_i(\gamma_f(r)) \right)$$

$$\Rightarrow [F_{ef}(r)]_j = \hat{x}_i^e \cdot \frac{\partial \psi_i(\gamma_f(r))}{\partial \xi_k} \cdot \frac{\partial (\gamma_f(r))_k}{\partial r_j} \qquad \begin{aligned} i &= 1, \dots, n_v \\ j &= 1, \dots, d-1 \\ k &= 1, \dots, d-1, d \end{aligned}$$

$$\Rightarrow \boxed{[F_{ef}(r)]_j = \hat{x}_i^e \cdot \left[\frac{\partial \psi(\gamma_f(r))}{\partial \xi} \cdot \frac{\partial \gamma_f(r)}{\partial r} \right]_{ij}}$$

$$\Rightarrow \boxed{\underset{d \times d-1}{F_{ef}(r)} = \underset{d \times n_v}{\hat{x}^e} \cdot \underset{n_v \times d}{\frac{\partial \psi(\gamma_f(r))}{\partial \xi}} \cdot \underset{d \times d-1}{\frac{\partial \gamma_f(r)}{\partial r}}}$$

2) $\Gamma_{\square} := [0,1]$ (1-dimensional reference simplex element)

$\Omega_{\square} := \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 + \xi_2 \leq 1 \text{ and } \xi_1, \xi_2 \geq 0\}$

$\gamma_1 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_1 = 0\} \Rightarrow \boxed{\gamma_1(r) = (0, r)}$

$\gamma_2 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_2 = 0\} \Rightarrow \boxed{\gamma_2(r) = (r, 0)}$

$\gamma_3 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_1 + \xi_2 = 1\} \Rightarrow \boxed{\gamma_3(r) = (1-r, r)}$

3) $\Gamma_{\square} := \{r \in \mathbb{R}^2 \mid r_1 + r_2 \leq 1 \text{ and } r_1, r_2 \geq 0\}$

$\Omega_{\square} := \{\xi \in \mathbb{R}^3 \mid \xi_1 + \xi_2 + \xi_3 \leq 1 \text{ and } \xi_1, \xi_2, \xi_3 \geq 0\}$

$\gamma_1 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_1 = 0\} \Rightarrow \boxed{\gamma_1(r) = (0, r_1, r_2)}$

$\gamma_2 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_2 = 0\} \Rightarrow \boxed{\gamma_2(r) = (r_1, 0, r_2)}$

$\gamma_3 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_3 = 0\} \Rightarrow \boxed{\gamma_3(r) = (r_1, r_2, 0)}$

$\gamma_4 : \Gamma_{\square} \to \{\xi \in \Omega_{\square} \mid \xi_1 + \xi_2 + \xi_3 = 1\} \Rightarrow \boxed{\gamma_4(r) = (r_1, r_2, 1 - r_1 - r_2)}$

<u>Number of quadrature points per dimension:</u>

For the <u>master hypercube element</u>, the polynomial degrees of the integrand of the volume, centroid and surface area are, respectively 0, 1, 0, meaning that 1 quadrature point is enough to evaluate the integrals. The <u>master simplicial element</u>, on the other hand, has its quadrature rule obtained from the quadrature rule for a hypercube mapped on the simplex, which modifies the polynomial degrees of the integrands. If $G(\xi) = x$ maps the hypercube on the simplex,

$$V(\Omega_\square) = \int_{\Omega_\square} \det\left(\frac{dG(\xi)}{d\xi}\right) d\xi$$

$$c(\Omega_\square) = \frac{1}{V(\Omega_\square)} \cdot \int_{\Omega_\square} \underset{2}{G(\xi)} \cdot \underbrace{\det\left(\frac{dG}{d\xi}(\xi)\right)}_{1} d\xi$$

The polynomial degree of the integrand is at most 3, meaning that $\frac{3+1}{2} = 2$ quadrature points are necessary per dimension.

When dealing with physical elements, the same reasoning can be applied to conclude that the polynomial degrees will increase (for the integrands) due to the composition of mappings. In some cases, the integrand might not even be a polynomial function, and more quadrature points might be necessary as well.

Tasks for Part 1.4

2) Two functions were written to check if the moments are correct: one for hypercubes and and for simplices. The input is only the dimension, and the functions check:

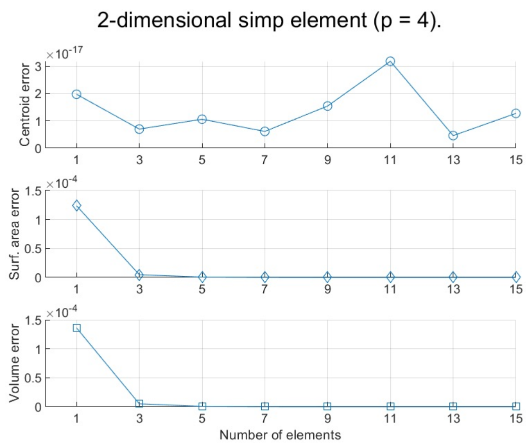
* Hypercube domain $[0,1]^d$ (both hcube and simplex elements):
  - If the volume is 1
  - If the centroid c satisfies $c_i = 0{,}5$ for $i = 1 \dots d$
  - If the surface area is $2 \cdot d$.

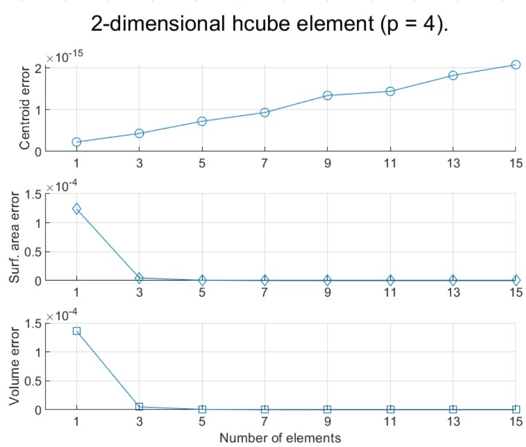
3) One function was written to check the volume, centroid and surface area of a d-dimensional sphere domain ($d = 2$ or 3) for a mesh composed of both hypercubical and simplicial elements. Different polynomial degrees and number of elements were used in the tests. The errors relative to the analytical expressions were plotted to show the influence of the number of elements and the polynomial degree. The analytical expressions for the 2d case are:

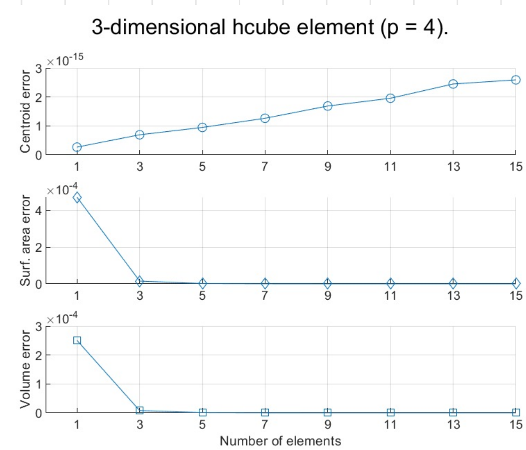
$$V = \pi R^2 = \pi \qquad SA = 2\pi R = 2\pi,$$

$$C = [0, 0]^T$$

where $R = 1$ is the radius.





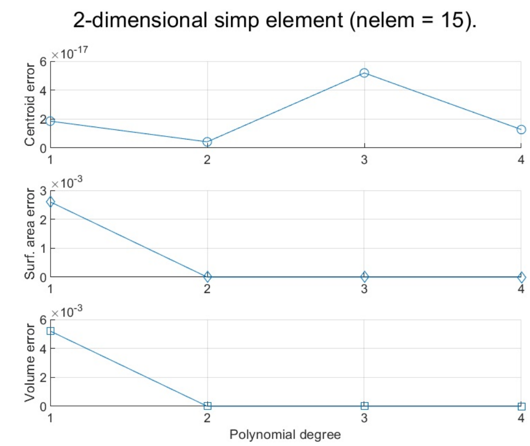


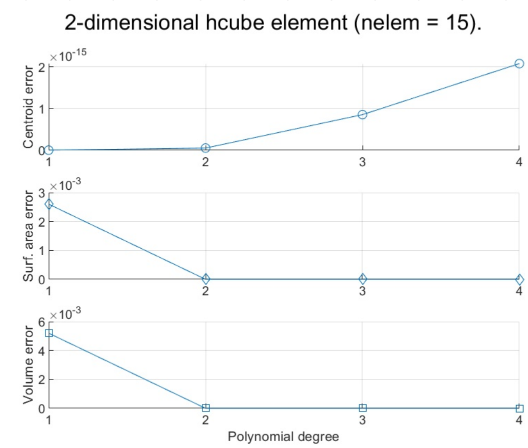
For the 3d case (only hypercube elements), the values are:

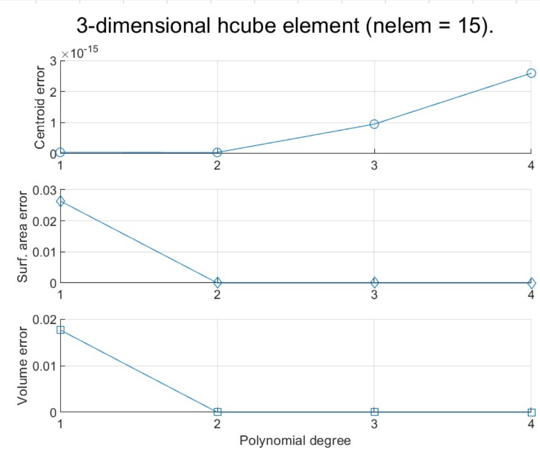
$$V = \frac{4\pi R^3}{3} = 4\pi/3$$

$$c = [0, 0, 0]^T, \quad SA = 4\pi R^2 = 4\pi.$$

From the graphs at the left, one concludes that the error decreases with the increase of the number of elements. The centroid error is very close to zero, and its behavior is more related to machine error (noise).



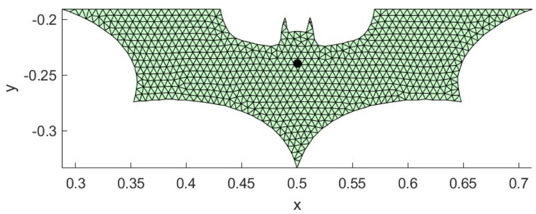




In this case, the number of elements is kept constant and the polynomial degree (related to the number of nodes per element) is increased. This decreases the error for both volume and surface area, while the centroid assumes low errors for any case.

Note: the number of quadrature nodes per dimension was chosen as a function of the polynomial degree in the code for checking. This means that the error is ACTUALLY decreasing with the increase of the number of quadrature points.

4) The black circle indicates the centroid of the domain.

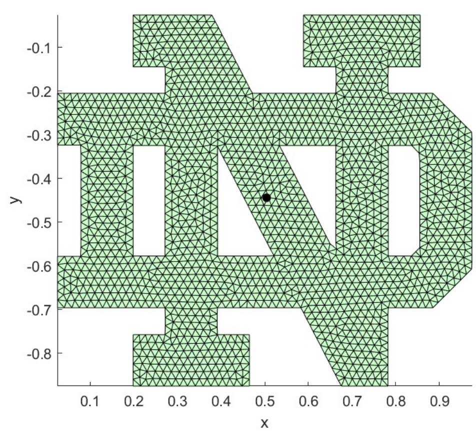


"Batman" domain :

Volume $\cong$ 0,0259

Centroid $\cong \begin{bmatrix} 0,5004 \\ -0.2396 \end{bmatrix}$

Surface area = 1,1021



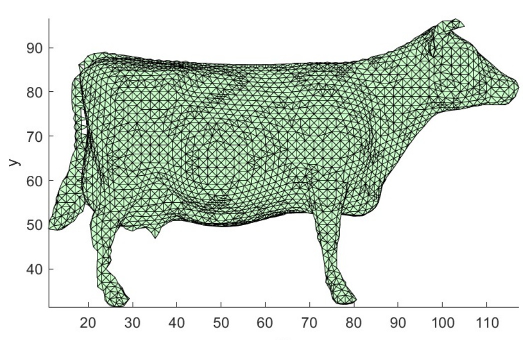
"ND" domain :

Volume $\cong$ 0.5171

Centroid $\cong \begin{bmatrix} 0.5042 \\ -0.4440 \end{bmatrix}$

Surface area $\cong$ 7.4856

5)

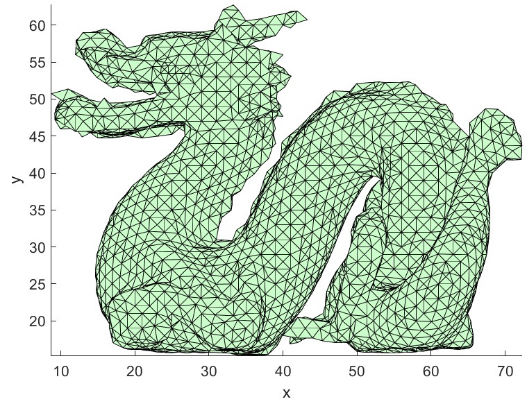


"Cow" domain :

Volume $\cong 5.6400 \cdot 10^4$

Centroid $\cong \begin{bmatrix} 54.8510 \\ 68.6509 \\ 63.9978 \end{bmatrix}$

Surface area $\cong 1.1313 \cdot 10^4$

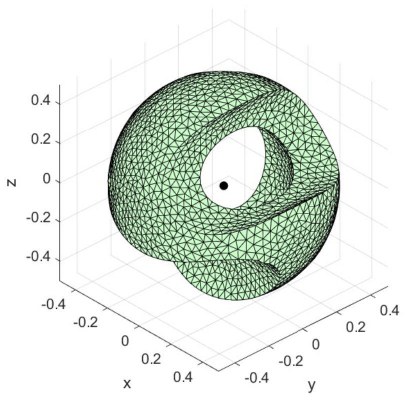


"Dragon" domain:

Volume $\cong 1.9586 \cdot 10^4$

$$\text{Centroid} \cong \begin{bmatrix} 39.6524 \\ 32.5786 \\ 38.1160 \end{bmatrix}$$

Surface area $\cong 7.9771 \cdot 10^3$



"Sculpt10KV" domain:

Volume $\cong 0.2116$

$$\text{Centroid} \cong \begin{bmatrix} 0.0244 \\ -0.1458 \\ 0.1246 \end{bmatrix} \cdot 10^{-3}$$

Surface area $\cong 3.5590$

Tasks for Part 2.2

1) $F_{ij} = -K_{js} \cdot u_{,s} = -K_{js} \cdot Q_s \Rightarrow \boxed{\frac{\partial F_{ij}}{\partial U_1} = 0 \; , \; \frac{\partial F_{ij}}{\partial U_{1,s}} = -K_{js}}$

$S_1(x)$ does not depend on U or ∇U : $\boxed{\frac{\partial S_1}{\partial U_1} = 0 \; , \; \frac{\partial S_1}{\partial U_{1,s}} = 0}$ $s = 1, 2$

Tasks for Part 2.3

1) $S_i = f_i$ (constant) $\Rightarrow \boxed{\frac{\partial S_i}{\partial U_j} = 0 \; , \; \frac{\partial S_i}{\partial U_{j,k}} = 0}$

$-F_{ij} = +\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$

$\Rightarrow -F_{ij} = \lambda \cdot \frac{1}{2}\left(u_{k,k} + u_{k,k}\right)\delta_{ij} + 2\mu \cdot \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

$\Rightarrow F_{ij} = -\lambda\, u_{k,k}\,\delta_{ij} - \mu\left(u_{i,j} + u_{j,i}\right) \quad , \quad i, j = 1, \dots, d$

$\Rightarrow \boxed{\frac{\partial F_{ij}}{\partial U_l} = 0 \quad , \quad \frac{\partial F_{ij}}{\partial U_{l,r}} = -\lambda\,\delta_{lr}\,\delta_{ij} - \mu\left(\delta_{il}\,\delta_{jr} + \delta_{jl}\,\delta_{ir}\right)}$

Tasks for Part 2.4

1) If $i < d+1$:

$F_{ij} = -\rho\nu\, v_{i,j} + P\,\delta_{ij}$

$\frac{\partial F_{ij}}{\partial v_k} = 0 \; , \; \frac{\partial F_{ij}}{\partial P} = \delta_{ij} \Rightarrow \boxed{\frac{\partial F_{ij}}{\partial U_k} = \begin{cases} 0 & k < d+1 \\ \delta_{ij} & k = d+1 \end{cases}}$

$\frac{\partial F_{ij}}{\partial v_{k,r}} = -\rho\nu \cdot \delta_{ki} \cdot \delta_{rj} \; , \; \frac{\partial F_{ij}}{\partial P_{,r}} = 0 \Rightarrow \boxed{\frac{\partial F_{ij}}{\partial U_{k,r}} = \begin{cases} -\rho\nu\,\delta_{ki}\,\delta_{rj} & k < d+1 \\ 0 & k = d+1 \end{cases}}$

$$S_i = -\rho\, v_j\, v_{i,j}$$

$$\frac{\partial S_i}{\partial v_k} = -\rho\, v_{i,j} \cdot \delta_{kj} \quad , \quad \frac{\partial S_i}{\partial P} = 0 \Rightarrow \boxed{\frac{\partial S_i}{\partial U_k} = \begin{cases} -\rho v_{i,k} & k < d+1 \\ 0 & k = d+1 \end{cases}}$$

$$\frac{\partial S_i}{\partial U_{k,r}} = -\rho v_j\, \delta_{ik}\, \delta_{rj} \quad , \quad \frac{\partial S_i}{\partial P_{,r}} = 0 \Rightarrow \boxed{\frac{\partial S_i}{\partial U_{k,r}} = \begin{cases} -\rho v_r \delta_{ik} & k < d+1 \\ 0 & k = d+1 \end{cases}}$$

If $i = d+1$:

$$F_{ij} = \frac{\partial F_{ij}}{\partial U_k} = \frac{\partial F_{ij}}{\partial U_{k,r}} = 0$$

$$S_i = -v_{s,s}$$

$$\frac{\partial S_i}{\partial U_k} = 0 \quad , \quad \frac{\partial S_i}{\partial U_{k,r}} = \begin{cases} -\delta_{kr} & k < d+1 \\ 0 & k = d+1 \end{cases}$$

Matricial forms:

$$S = -\begin{bmatrix} \rho\, \nabla v^T \cdot v \\ \mathrm{trace}(\nabla v) \end{bmatrix}_{d+1 \times 1} \qquad F = \begin{bmatrix} -\rho\, \nu\, \nabla v^T + P \cdot I_{d \times d} \\ 0 \cdot v^T \end{bmatrix}_{d+1 \times d}$$

Tasks for Part 4

6) Solving the finite element code with PDE0:

$p \Rightarrow$ optimal convergence rate $= p+1$

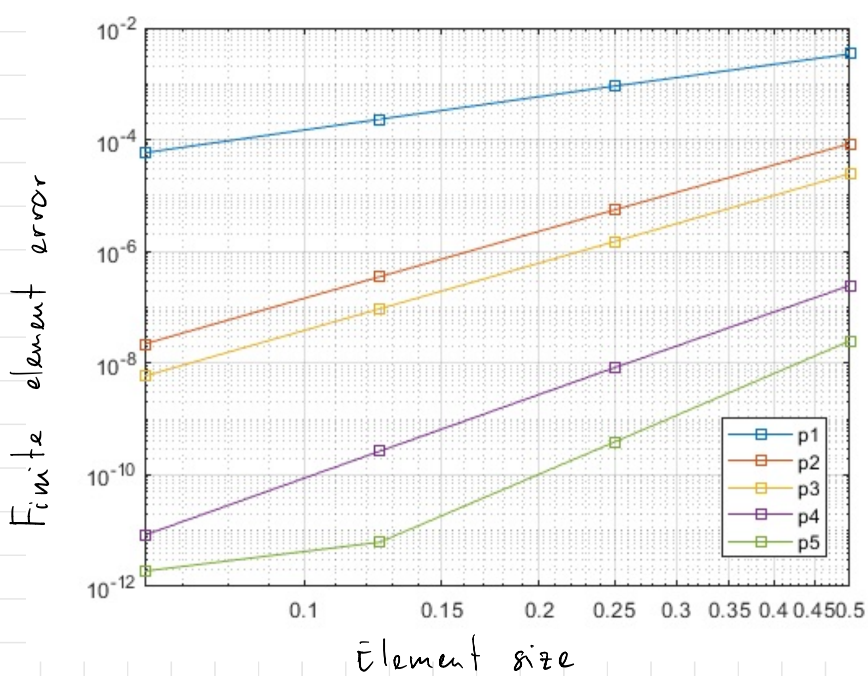
$p=1 \Rightarrow$ convergence rate $= 1.9574$

$p=2 \Rightarrow$ " " $= 3.9711$

$p=3 \Rightarrow$ " " $= 4.014$

$p=4 \Rightarrow$ " " $= 4.9482$

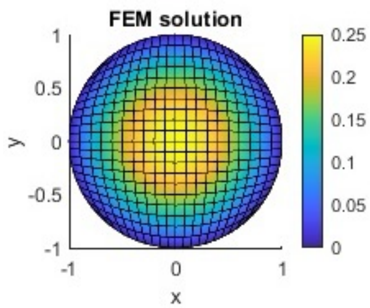
$p=5 \Rightarrow$ " " $= 5.9875$

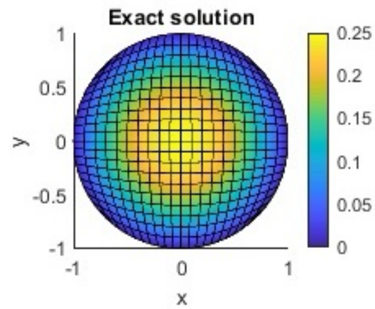


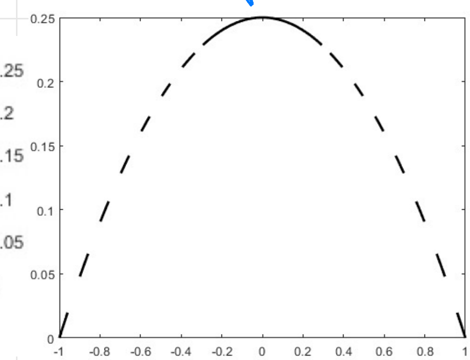
Tasks for Part 5

SOLUTION ALONG ANY LINE PASSING THROUGH THE CENTER

1) Poisson equation on the unit disk:

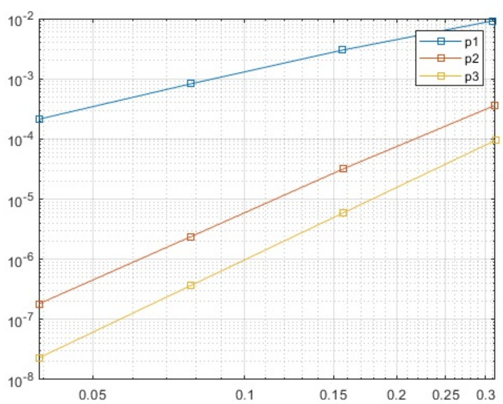






SIMPLEX ELEMENTS

FE error



Convergence rates (simplex):

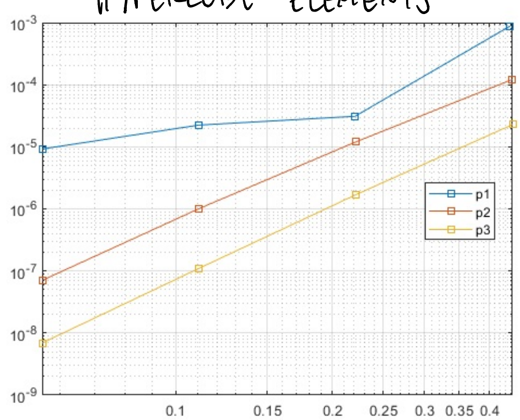
$p=1 \Rightarrow$ conv. rate $= 1.8194$

$p=2 \Rightarrow$ " " $= 3,6555$

$p=3 \Rightarrow$ " " $= 4.004$

HYPERCUBE ELEMENTS

FE error



Convergence rates (hypercube):

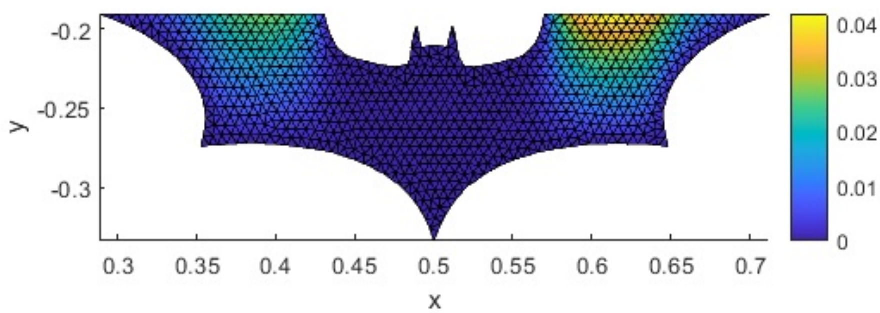
$p=1 \Rightarrow$ conv. rate $= 2.2011$

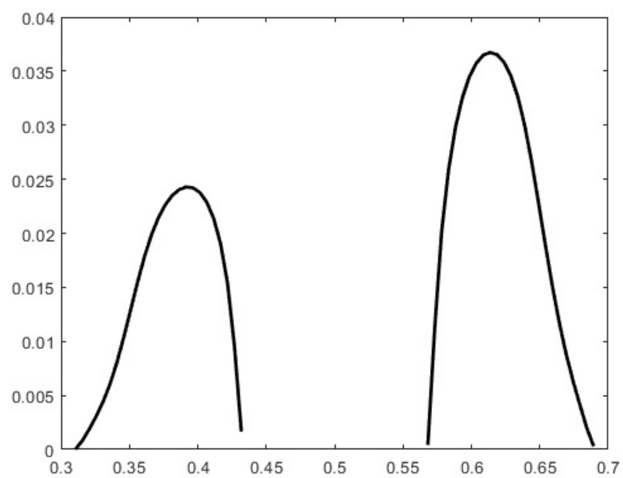
$p=2 \Rightarrow$ " " $= 3.5855$

$p=3 \Rightarrow$ " " $= 3.8962$

Element size

2) Second-order linear PDE on the Batman domain:



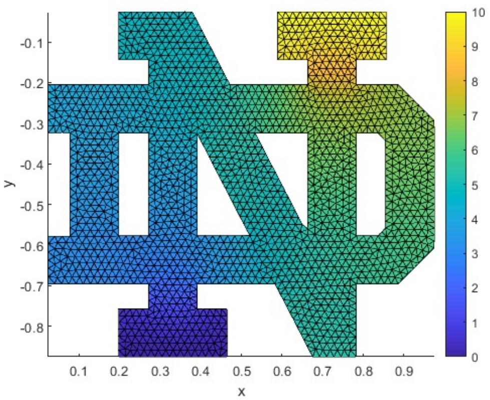


Solution along the line $\Gamma \subset \Omega$ such that $(x, y) \in \Gamma$

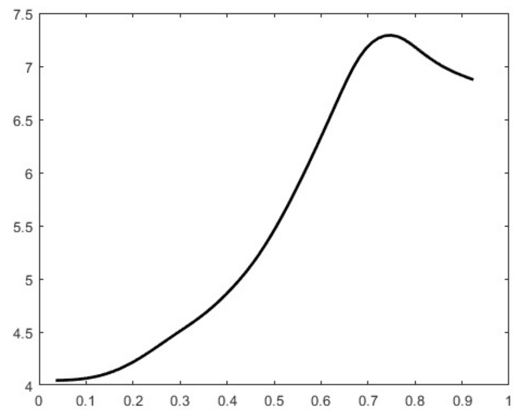
$$\Downarrow$$

$y = -0.2$.

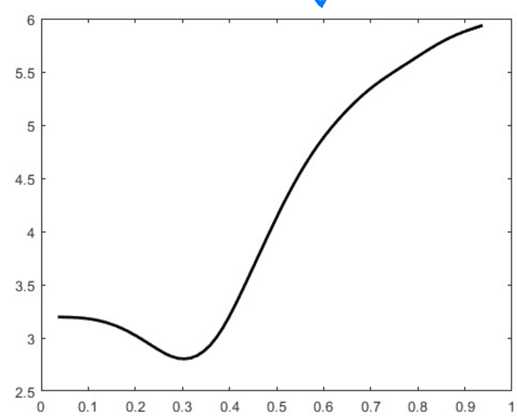
3) Poison equation on the ND logo domain:



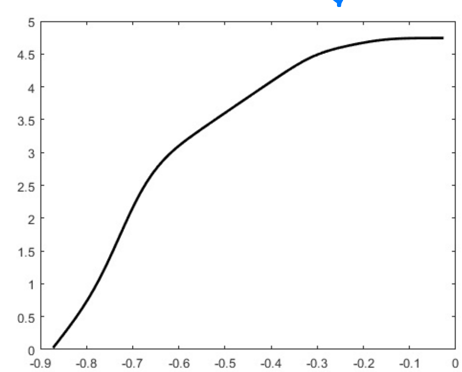
SOLUTION OVER Γ_1



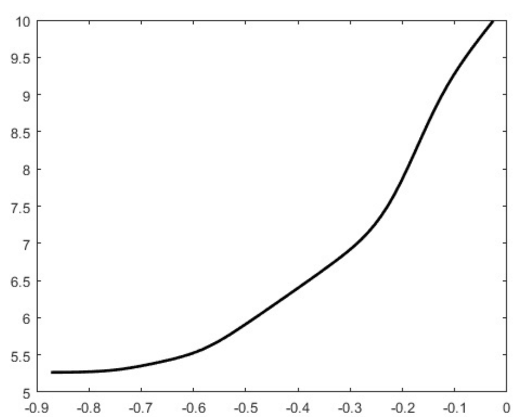
SOLUTION OVER Γ_2



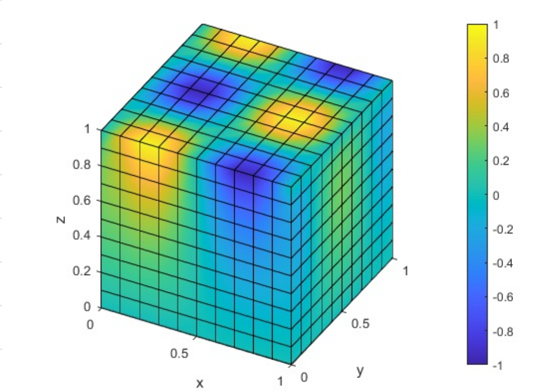
SOLUTION OVER Γ_3

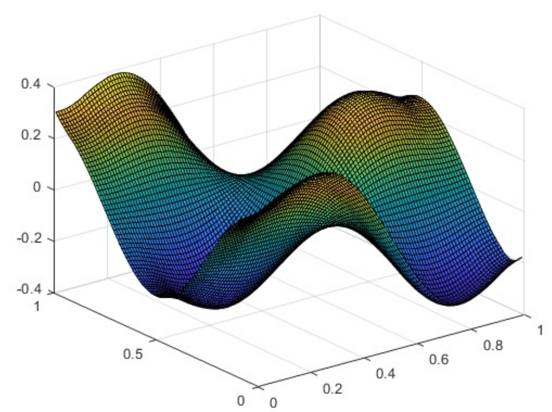


SOLUTION OVER Γ_4



4) Second-order linear PDE over the unit cube:

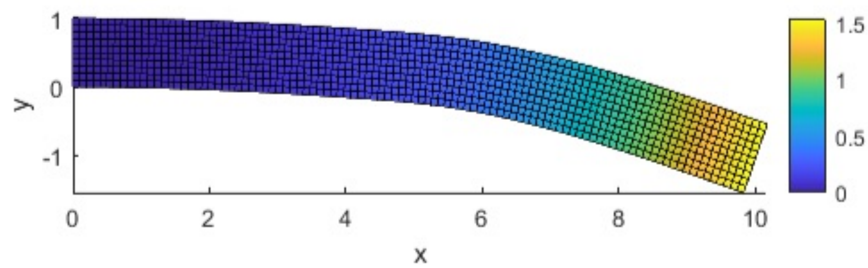


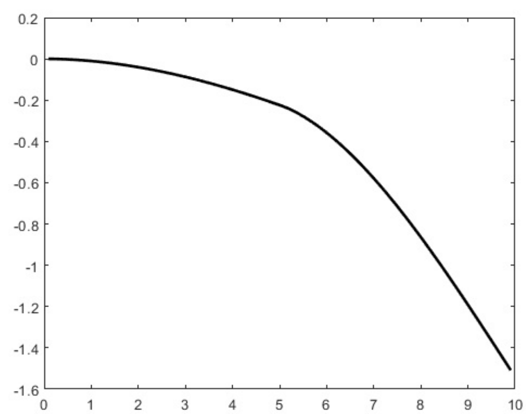


SOLUTION ALONG THE PLANE $\Gamma \subset \Omega$ SUCH THAT $z = 0.5$.

Tasks for Part 6

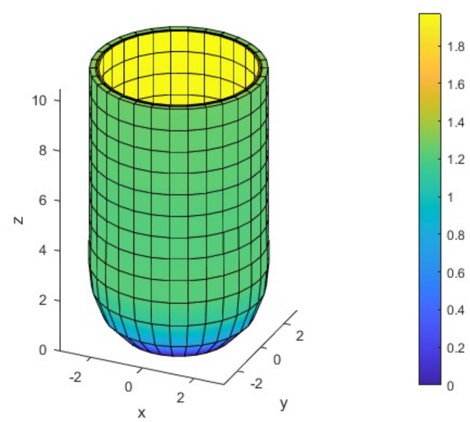
1) Mutimaterial beam:

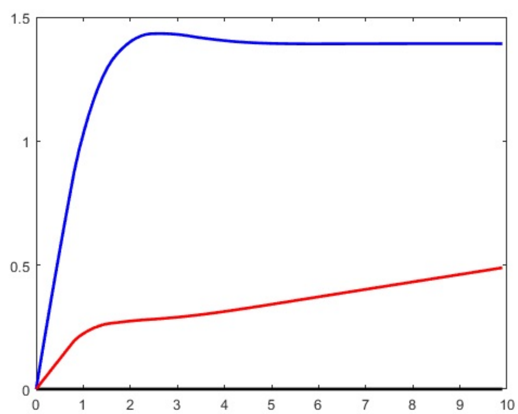




→ Magnitude of the displacement along the horizontal line through the center of the beam on the deformed geometry.

2) Hollow cylinder:





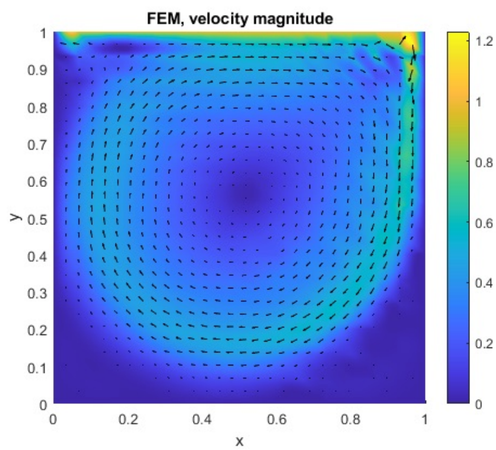
Displacement on direction x along Γ

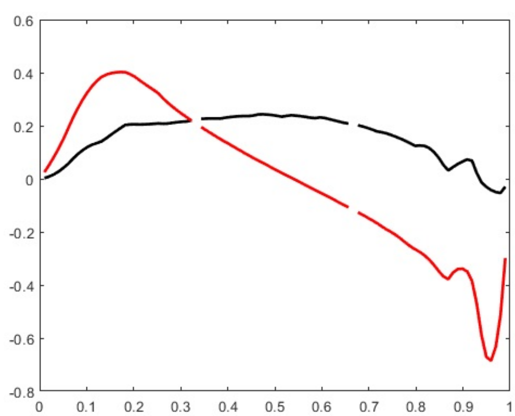
Displacement on direction z along Γ

Displacement on direction y along Γ

Tasks for Part 7

1) Lid-driven cavity problem:





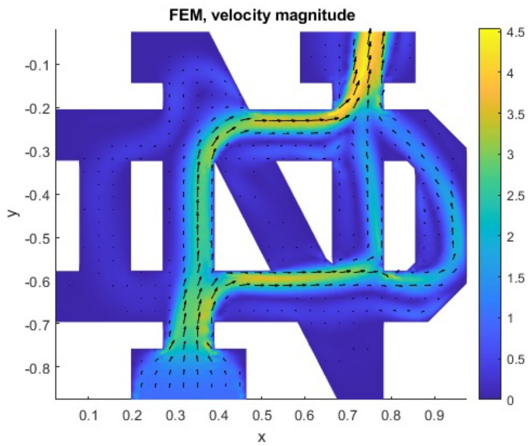
Velocity over Γ

$(x, +0.75)$

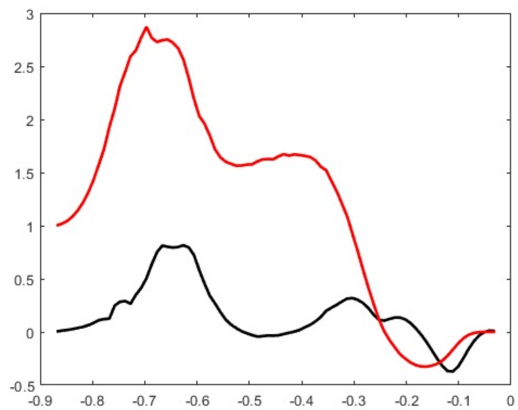
u_x (horizontal component of velocity)

u_y (vertical component of velocity)

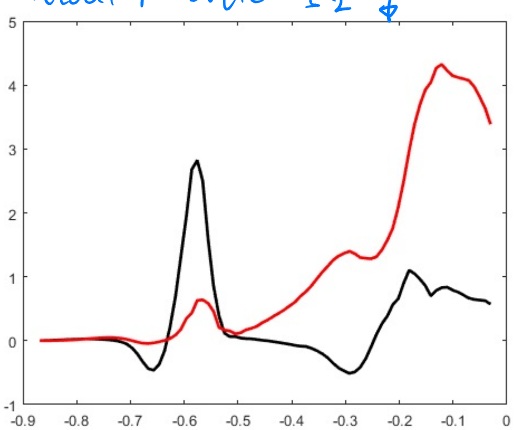
2) Flow through the ND logo:



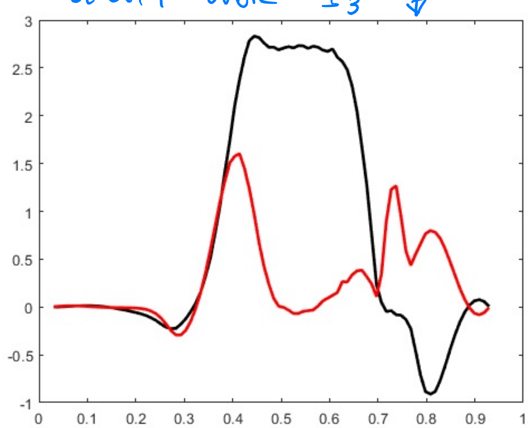
VELOCITY OVER Γ_1



VELOCITY OVER Γ_2



VELOCITY OVER Γ_3



u_x

u_y

VELOCITY OVER Γ_4