Problem 1

(a)

Since this is not an infinite series, we cannot use identities derived in the book. Instead, we must do our own. We must solve for c in $\displaystyle\sum_{k=1}^{10} ck^2 = 1.0$.

$$\begin{aligned}
\sum_{k=1}^{10} ck^2 &= c \cdot \frac{10(10+1)(2\cdot 10+1)}{6} \\
&= c \cdot 385 \\
&\Rightarrow c = \frac{1}{385}
\end{aligned}$$

(b)

We can find the expected value of X by computing $\sum_{k=1}^{10} k \cdot p_X(k)$. We know $p_X(k) = c \cdot k^2$ and $c = \frac{1}{385}$ so,

$$\begin{aligned}
\frac{1}{385} \cdot \sum_{k=1}^{10} k \cdot k^2 &= \frac{1}{385} \sum_{k=1}^{10} k^3 \\
&= \frac{1}{385} \cdot \left[\frac{1}{4}\cdot 10^4 + \frac{1}{2}\cdot 10^3 + \frac{1}{4}\cdot 10^2\right] \\
&\approx 7.86
\end{aligned}$$

By (3.30), we can compute the variance of X by finding

$$\begin{aligned}
\left[\sum_{k=1}^{10} k^2 \cdot p_X(k)\right] - [EX]^2 &= \frac{1}{385} \cdot \sum_{k=1}^{10} k^3 - 7.86^2 \\
&= \frac{1}{385}\left[\frac{1}{5}\cdot 10^5 + \frac{1}{2}\cdot 10^4 + \frac{1}{3}\cdot 10^3 - \frac{1}{30}\cdot 10\right] - 7.86^2 \\
&\approx 4.02
\end{aligned}$$

Problem 2

First, we take note that each set of coin tosses follows a binomial distribution. Therefore, we can express $P(X_2 = i), i = 0,1,2$ as $P(X_2 = i|C_1 \cup X_2 = i|C_2) = P(X_2 = i|C_1) + P(X_2 = i|C_2) = P(C_1)P(X_2 = i|C_1) + P(C_2)P(X_2 = i|C_2)$. This follows intuitively, since there are two possibilities: one where you pick the head-weighted coin and one where you pick the tail-weigted coin, after which the set of tosses is modeled with a binomial distribution. Using this information, we can construct a pmf for the coin tosses:

$$p_{X_2}(k) = 0.5 \cdot \binom{2}{k}(0.9)^k(0.1)^{2-k} + 0.5\cdot\binom{2}{k}(0.1)^k(0.9)^{2-k}$$

Plugging this into the handy formula $EX_2 = \sum_{k=0}^{k=2} k \cdot p_{X_2}(k)$ yields

$$\begin{aligned}
EX_2 &= \sum_{k=0}^{k=2} k \cdot \left[0.5\cdot\binom{2}{k}(0.9)^k(0.1)^{2-k} + 0.5\cdot\binom{2}{k}(0.1)^k(0.9)^{2-k}\right] \\
&= 0 + [(0.5)(2)(0.9)(0.1) + (0.5)(2)(0.1)(0.9)] + 2\cdot\left[(0.5)(0.9)^2 + (0.5)(0.1)^2\right] \\
&= 0.18 + 0.82 \\
&= 1
\end{aligned}$$

Finding variance is trivial at this point, since the final term of EX_2 can simply be doubled to give us $E({X_2}^2) = 0.18 + 1.64 = 1.82$. Then $Var(X_2) = E({X_2}^2) - (EX)^2 = 1.82 - 1^2 = 0.82$.