Problem 1

(a)

Since this is not an infinite series, we cannot use identities derived in the book. Instead, we must do our own. We must solve for c in $\sum_{k=1}^{10} ck^2 = 1.0$.

$$\begin{aligned}\sum_{k=1}^{10} ck^2 &= c \cdot \frac{10(10+1)(2\cdot 10+1)}{6} \\ &= c \cdot 385 \\ &\Rightarrow c = \frac{1}{385}\end{aligned}$$

(b)

We can find the expected value of X by computing $\sum_{k=1}^{10} k \cdot p_X(k)$. We know $p_X(k) = c \cdot k^2$ and $c = \frac{1}{385}$ so,

$$\begin{aligned}\frac{1}{385} \cdot \sum_{k=1}^{10} k \cdot k^2 &= \frac{1}{385} \sum_{k=1}^{10} k^3 \\ &= \frac{1}{385} \cdot \left[\frac{1}{4} \cdot 10^4 + \frac{1}{2} \cdot 10^3 + \frac{1}{4} \cdot 10^2\right] \\ &\approx 7.86\end{aligned}$$

(c)

By (3.30), we can compute the variance of X by finding

$$\begin{aligned}\left[\sum_{k=1}^{10} k^2 \cdot p_X(k)\right] - [EX]^2 &= \frac{1}{385} \cdot \sum_{k=1}^{10} k^3 - 7.86^2 \\ &= \frac{1}{385}\left[\frac{1}{5} \cdot 10^5 + \frac{1}{2} \cdot 10^4 + \frac{1}{3} \cdot 10^3 - \frac{1}{30} \cdot 10\right] - 7.86^2 \\ &\approx 4.02\end{aligned}$$

Problem 2

First, we take note that each set of coin tosses follows a binomial distribution. Therefore, we can express $P(X_2 = i), i = 0, 1, 2$ as $P(X_2 = i|C_1 \cup X_2 = i|C_2) = P(X_2 = i|C_1) + P(X_2 = i|C_2) = P(C_1)P(X_2 = i|C_1) + P(C_2)P(X_2 = i|C_2)$. This follows intuitively, since there are two possibilities: one where you pick the head-weighted coin and one where you pick the tail-weigted coin, after which the set of tosses is modeled with a binomial distribution. Using this information, we can construct a pmf for the coin tosses:

$$p_{X_2}(k) = 0.5 \cdot \binom{2}{k}(0.9)^k(0.1)^{2-k} + 0.5 \cdot \binom{2}{k}(0.1)^k(0.9)^{2-k}$$

Plugging this into the handy formula $EX_2 = \sum_{k=0}^{k=2} k \cdot p_{X_2}(k)$ yields

$$\begin{aligned}
EX_2 &= \sum_{k=0}^{k=2} k \cdot \left[0.5 \cdot \binom{2}{k}(0.9)^k(0.1)^{2-k} + 0.5 \cdot \binom{2}{k}(0.1)^k(0.9)^{2-k}\right] \\
&= 0 + [(0.5)(2)(0.9)(0.1) + (0.5)(2)(0.1)(0.9)] + 2 \cdot \left[(0.5)(0.9)^2 + (0.5)(0.1)^2\right] \\
&= 0.18 + 0.82 \\
&= 1
\end{aligned}$$

Finding variance is trivial at this point, since the final term of EX_2 can simply be doubled to give us $E({X_2}^2) = 0.18 + 1.64 = 1.82$. Then $Var(X_2) = E({X_2}^2) - (EX)^2 = 1.82 - 1^2 = 0.82$.

Problem 3

(a)

We can split this problem into a few special cases. First, we know that any time the state transitions from any number of packets in the buffer to one packet in the buffer, the container can pass through with at least as many open spaces as packets currently in the buffer. This can be generalized to the following: if there is no new packet, the container must pass through holding no more than $c - i$ packets or if a new packet is generated, it must pass through with no more than $c - (i + 1)$ packets. In equation form, this is

$$p_{i0} = (1 - p) \cdot P(N \leq c - i) + p \cdot P(N \leq c - (i + 1)), \quad i = 0, 1, \ldots, b - 1 \tag{1}$$

Second, we also know that when the buffer is full, whether or not a new packet is generated, the state can only make a successful transition if the container has exactly enough space for the number of packets to be removed. This means we only need to take into account the probability that the container has exactly that much room. Mathematically, this reduces to

$$p_{bj} = P(N = c - (b - j)), \quad j = 1, 2, \ldots, b \tag{2}$$

Furthermore, if $b = c$, then we can include $j = 0$ in the range of this function.

Every remaining case we can generalize much like our first case, with the exception that the number of space available in the container must be *exactly* the amount necessary to switch from the current state to the next. Once again, as an equation this is

$$p_{ij} = (1 - p) \cdot P(N = c - (i - j)) + p \cdot P(N = c - [(i + 1) - j]), \quad i = 0, 1, \ldots, b - 1 \quad j = 1, 2, \ldots, b \tag{3}$$

Note that $P(N > c) = 0$.

Knowing that N is uniformly distributed on the range (0,5) allows us to reduce equation (2) to

$$p_{bj} = \frac{1}{c + 1} \tag{4}$$

and equation (3) to

$$\begin{aligned}
p_{ij} &= (1 - p) \cdot \frac{1}{c + 1} + p \cdot \frac{1}{c + 1} \\
&= \frac{1}{c + 1}
\end{aligned} \tag{5}$$

However, equation (5) only applies where both probabilities exist.

Now we can begin to plug in values. Setting $p = 0.6$, $c = b = 5$ and simplifying a bit, we get the matrix

$$P = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 \\ \frac{11}{15} & \frac{1}{6} & \frac{1}{10} & 0 & 0 & 0 \\ \frac{17}{30} & \frac{1}{6} & \frac{1}{6} & \frac{1}{10} & 0 & 0 \\ \frac{2}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{10} & 0 \\ \frac{7}{30} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Solving the equation $(I - P')\pi = 0$ was made easy thanks to the function `findpi1()` from the book. Plugging our matrix into the function gave us the vector

$$\pi = \begin{pmatrix} 0.877 \\ 0.108 \\ 0.0134 \\ 0.00165 \\ 0.000203 \\ 0.0000243 \end{pmatrix}$$

To verify the correctness of our answer, we also found that $\sum_i \pi_i = 1$.

(b)

Finding the long-run average number of packets which are discarded is trivial, now that we have our stationary probabilities. We merely need to find the probability that the buffer is full and one more packet is generated. This is

$$\begin{aligned} \pi_b p &= \pi_5 \cdot 0.6 \\ &= 2.43 \times 10^{-5} \cdot 0.6 \\ &= 1.46 \times 10^{-5} \end{aligned}$$