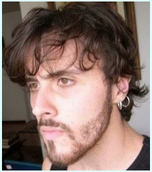
Level Set Teleportation: an Optimization Perspective

Aaron Mishkin   Alberto Bietti   Robert M. Gower

Introduction: Plateau's and Sudden Drops

Basic Problem: deep-learning objectives have many flat regions where the gradient is small and optimization is slow [FA00].

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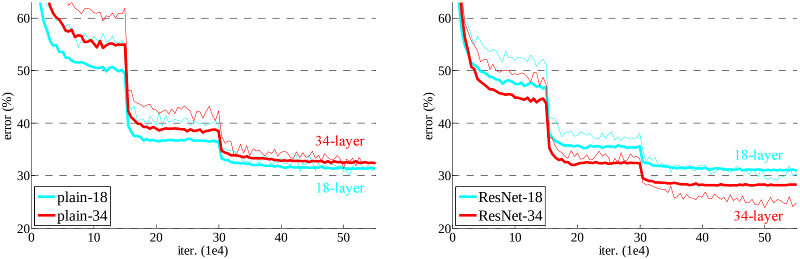

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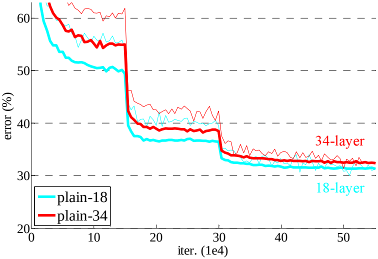

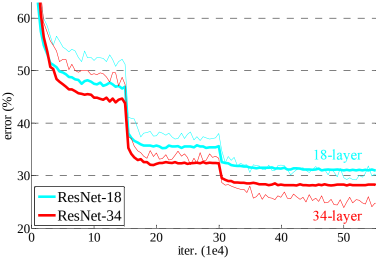

Flat regions can cause plateaus in training loss and then sudden drops when iterates finally escape.

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Faster optimization requires escaping flat regions quickly.

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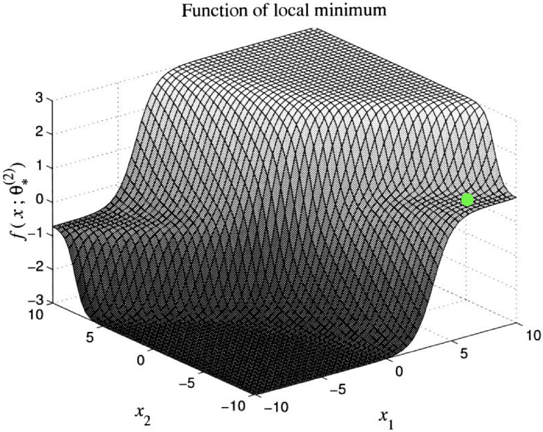

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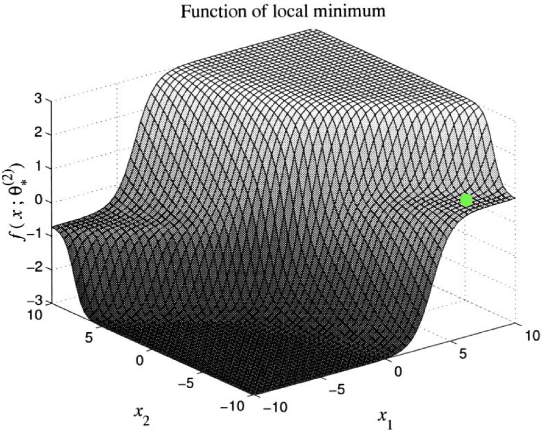

- Newton's method could do this, but Newton doesn't work for non-convex objectives due to negative curvature.

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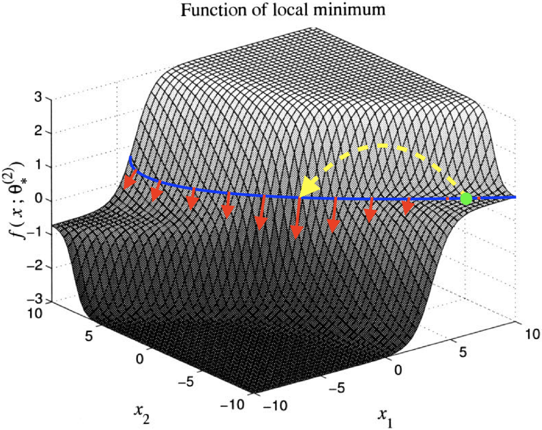

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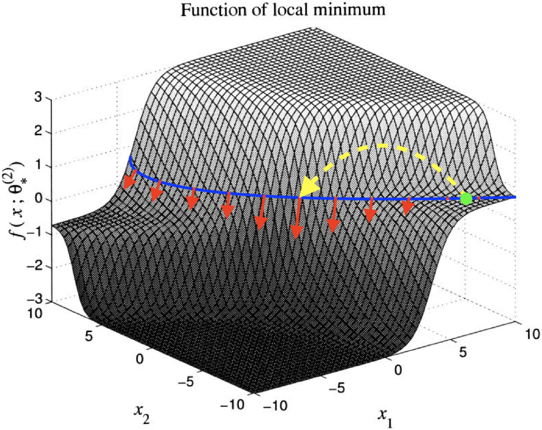

This is the picture behind **level set teleportation** [Zha+23b].

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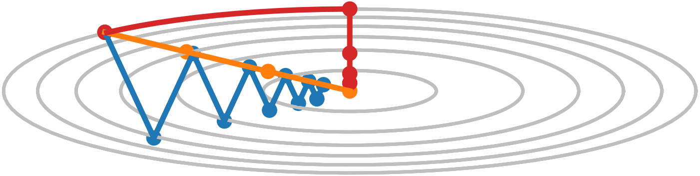

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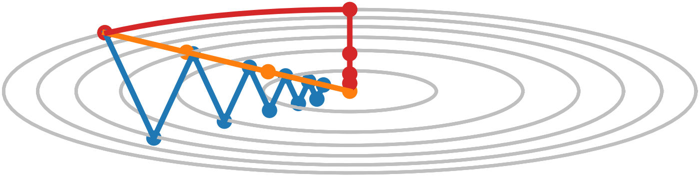

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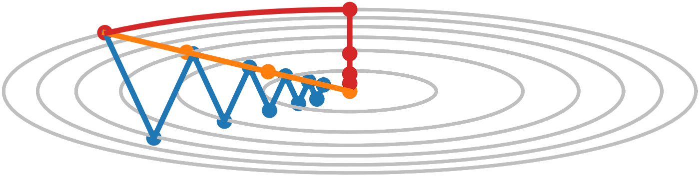

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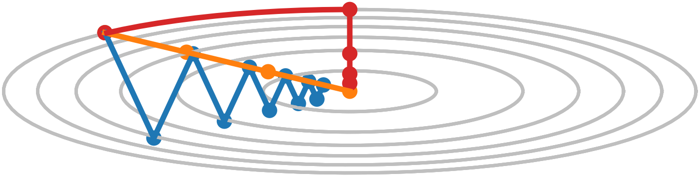

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- We show teleportation speeds-up optimization under Hessian stability (rates faster than $O(1/K)$!).
- We develop a fast, parameter-free algorithm for solving teleportation problems.

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Goal: minimize an objective $f : \mathbb{R}^d \to \mathbb{R}$,

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  - ∇f doesn't change too fast...

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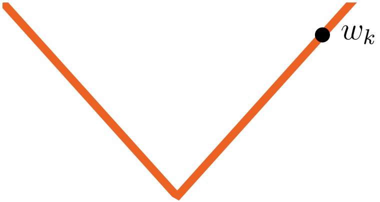

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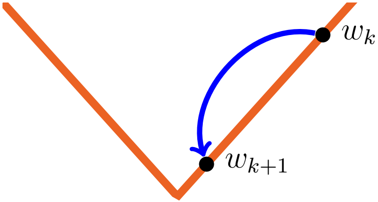

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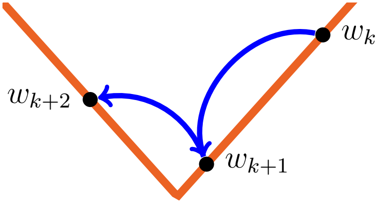

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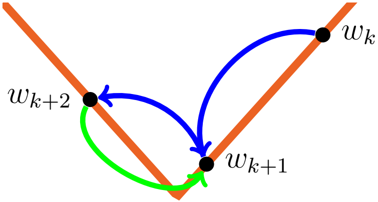

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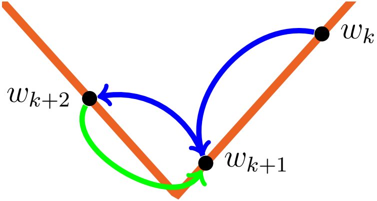

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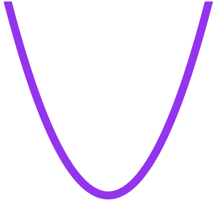

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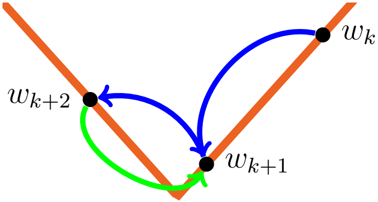

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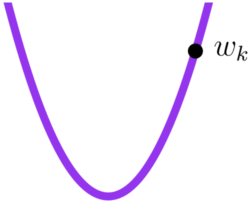

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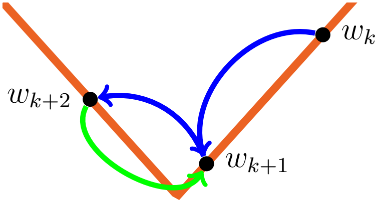

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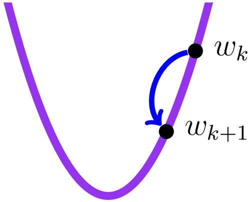

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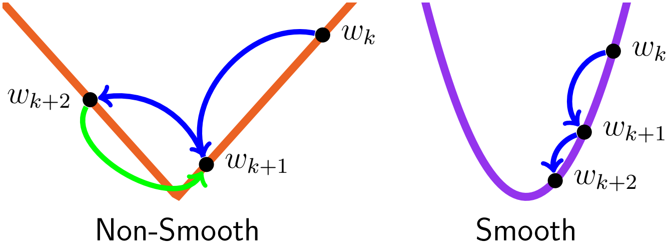

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The objective is upper-bounded by a quadratic function!

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|_2^2.$$

Lipschitz Gradients: L-Smoothness

We say f is L-smooth if for every $x, y \in \mathbb{R}^d$,

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Gradient descent is a majorization-minimization algorithm!

Lipschitz Gradients: Majorization-Minimization

Using $w_{k+1} = w_k - \frac{1}{L}\nabla f(x)$ in Q_{w_k} gives guaranteed progress.

$$\textbf{Descent Lemma}: f(w_{k+1}) \leq f(w_k) - \frac{1}{L}\|\nabla f(w_k)\|_2^2.$$

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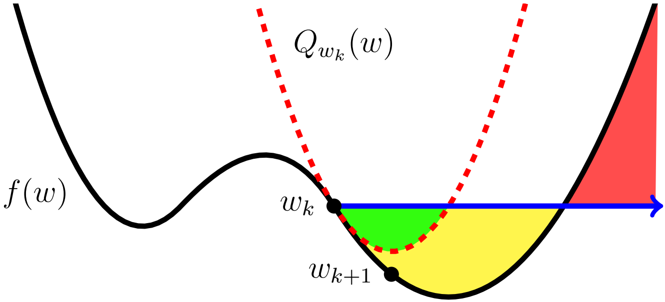

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Basic Idea: L-smoothness relates gradient magnitude to descent in function values,

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This lets us formalize our picture version of **level set teleportation**,

$$w_k^+ \in \arg\max_{w} \frac{1}{2}\|\nabla f(w)\|_2^2 \quad \text{s.t.} \quad f(w) \leq f(w_k).$$

Level Set Teleportation.

Level Set Teleportation: Algorithm

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- Each teleportation block $i \in \mathcal{B}$ consists of $b_i \geq 1$ steps.
- Complete teleportation schedule is $\mathcal{T}$.

Algorithm GD with Teleportation

Inputs: w_0; step-sizes η_k; block indices $\mathcal{B}$, sizes b_k.
$\mathcal{T} \leftarrow \bigcup_{k \in \mathcal{B}} \{k, k+1, \ldots, k + b_k - 1\}$
for $k \in \{0, \ldots, K\}$ **do**
  if $k \in \mathcal{T}$ **then**
    $w_k^+ \in \arg\max \{\|\nabla f(w)\|_2 : f(w) \leq f(w_k)\}$
  else
    $w_k^+ \leftarrow w_k$
  end if
  $w_{k+1} \leftarrow w_k^+ - \eta_k \nabla f(w_k^+)$
end for
Output: w_K

Level Set Teleportation: Test Functions

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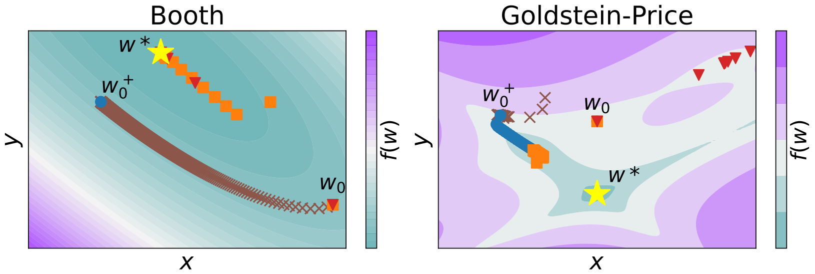

Teleportation in action on two test functions.

Level Set Teleportation: Newton's Method

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- The gradient direction is the Newton direction with scale λ_k!

Level Set Teleportation: Strong Convexity

Strong Convexity: f is μ-SC if for all $x, y \in \mathbb{R}^d$,

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- This approach leads to standard, super-linear rates using Newton-type analyses.
- But it requires teleporting before every iteration of GD and strong convexity is key.

Level Set Teleportation: More Problems

And there are some more problems. . .

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3. No efficient algorithms for teleporting in practice!

Analysis and Algorithms

Convergence Analysis: Convex Functions

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- But, we can't use this because $\lambda_k = 0$ may hold...
- Even worse, $\|w_k^+ - w^*\|_2^2 > C\|w_k - w^*\|_2^2$ breaks the standard GD recursion:

$$\begin{aligned}\|w_{k+1} - w^*\|_2^2 = \|w_k^+ - w^*\|_2^2 \\ - 2\eta_k \left\langle \nabla f(w_k^+), w_k^+ - w^* \right\rangle + \eta_k^2 \|\nabla f(w_k^+)\|_2^2,\end{aligned}$$

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Solution: Teleportation is non-expansive in function values,

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Theorem (Informal)

Let $R = \sup\{\|w - w^*\|_2 : w \in \mathcal{S}_0\}$. *If* f *is* L*-smooth and convex, then GD with* $\eta < 2/L$ *and teleportation schedule* $\mathcal{T}$ *satisfies,*

$$f(w_K) - f(w^*) \leq \frac{2R^2}{K\eta(2 - L\eta)}.$$

Moreover, there exists a function for which the convergence of GD with and without teleportation are identical.

Convergence: A Negative Tightness Result

Denote $\delta_K = f(w_K) - f(w^*)$. Comparing to standard GD [Bub15],

$$\underbrace{\delta_K \leq \frac{2R^2}{K\eta\,(2 - L\eta)}}_{\text{With Teleportation}} \quad \textbf{vs} \quad \underbrace{\delta_K \leq \frac{2\|w_0 - w^*\|_2^2}{K\eta\,(2 - L\eta)}}_{\text{Without Teleportation}}.$$

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Unfortunately, the dependence on the diameter is tight.

Theorem (Informal)

There exists an L-smooth and convex function such that teleporting from the initialization guarantees,

$$\|w_0^+ - w^*\|_2 \geq R/4.$$

Convergence: Hessian Stability

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Definition (Hessian Stability)

We say f has $(\tilde{L}, \tilde{\mu})$-stable Hessian over $\mathcal{Q} \subseteq \mathbb{R}^d$ if for every $x, y \in \mathcal{Q}$, $\nabla^2 f(x)(y - x) \neq 0$ and,

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\tilde{L}}{2} \|y - x\|_{\nabla^2 f(x)}^2,$$

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Holds for practical problems, including logistic regression [Bac10].

Convergence: Fast Rates under Hessian Stability

Recall that $\delta_k = f(w_k) - f(w^*)$ is the optimality gap.

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Suppose f is L-smooth, convex, and satisfies Hessian stability on $\mathcal{S}_0$. Let $M = K - |\mathcal{T}|$. Then GD with line-search satisfies,

$$\delta_K \leq \frac{2R^2 L}{M + 2R^2 L \sum_{k \in \mathcal{B}} \left[\left(\frac{\tilde{L}}{\tilde{L} - \tilde{\mu}} \right)^{b_k} - 1 \right] \frac{1}{\delta_{k-1}}}.$$

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Convergence is faster than GD when δ_k is small!

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This is a linear rate without strong convexity!

Level Set Teleportation: Towards an Algorithm

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Restrictions on an efficient teleportation algorithm.

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These constraints suggest first-order methods.

Level Set Teleportation: Practical Teleportation

We can't project onto $\mathcal{L}_k := \{w : f(w) = f(w_k)\}$, but we can project onto the linearization at x_k:

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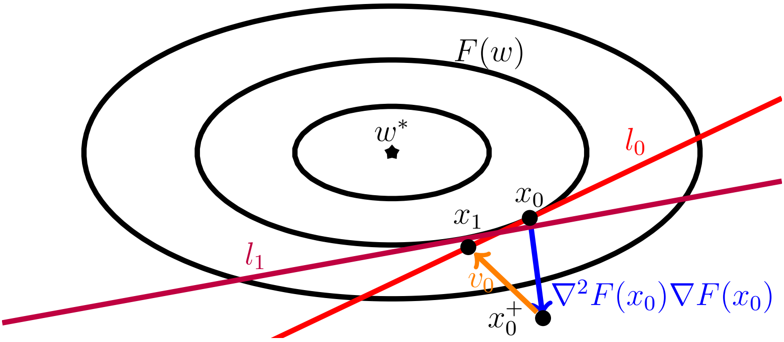

Level Set Teleportation: Practical Teleportation

Algorithm Sub-level Set Teleportation

$x_0 \leftarrow w_k$
$q_0 \leftarrow \nabla^2 f(x_0)\nabla f(x_0)$
while $\|\mathbf{P}_t q_t\|_2 > \epsilon$ or $f(x_k) - f(w_k) > \delta$ **do**
  $g_t \leftarrow \|\nabla f(x_k)\|_2^2$
  $v_t \leftarrow -\left(\rho \langle q_t, \nabla f(x_k)\rangle + f(x_k) - f(w_k)\right)_+ \nabla f(x_k)$
  $x_{k+1} \leftarrow x_k + (\rho \cdot q_t + v_t)/g_t$
  while $\phi_{\gamma_t}(x_{k+1}) > \frac{1}{2}g_t + (\langle q_t, v_t\rangle - \rho\|q_t\|_2^2)/g_t$ **do**
    $\rho \leftarrow \rho/2$
    $v_t \leftarrow -(\rho \langle q_t,\nabla f(x_k)\rangle + f(x_k) - f(w_k))_+ \nabla f(x_k)$
    $x_{k+1} \leftarrow x_k + (\rho \cdot q_t + v_t)/g_t$
  end while
  $q_{t+1} \leftarrow \nabla^2 f(x_k)\nabla f(x_k)$
  $t \leftarrow t + 1$
end while
Output: x_{k+1}

Experiments

Experiments: Evaluating the Algorithm

Consider teleportation for a non-convex neural network.

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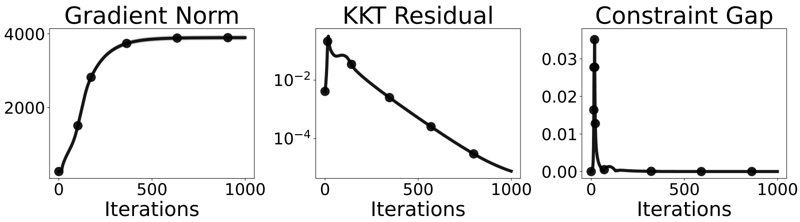

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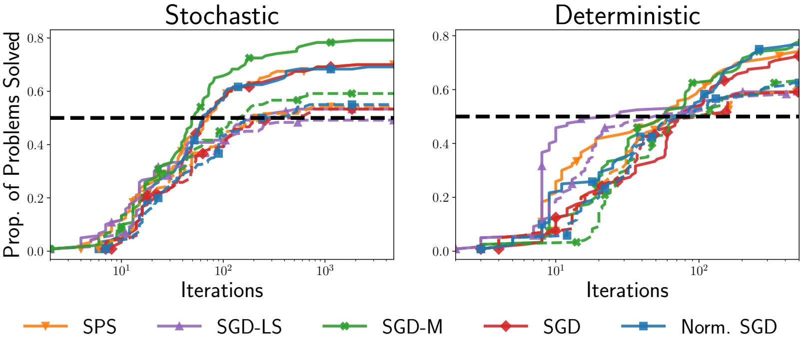

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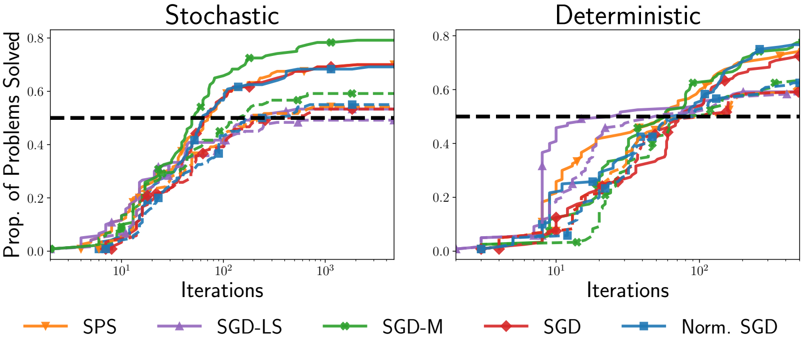

Teleportation uniformly improves speed and success rates!

Experiments: Convex Newton

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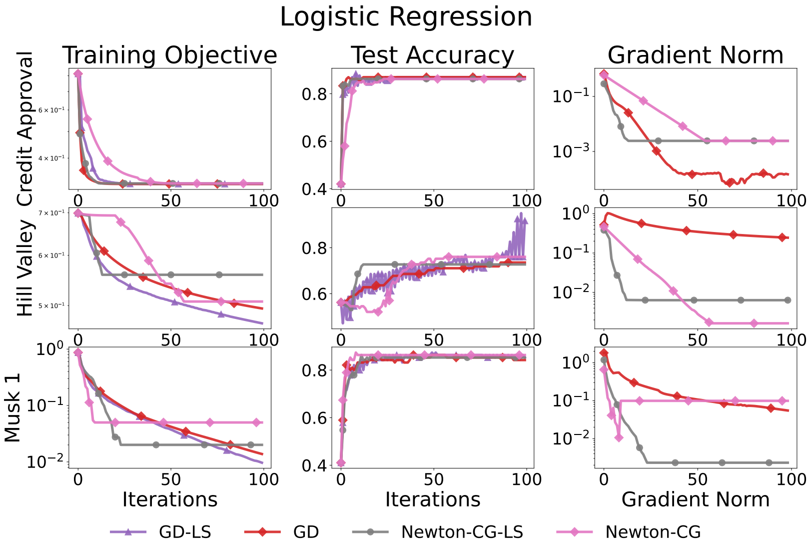

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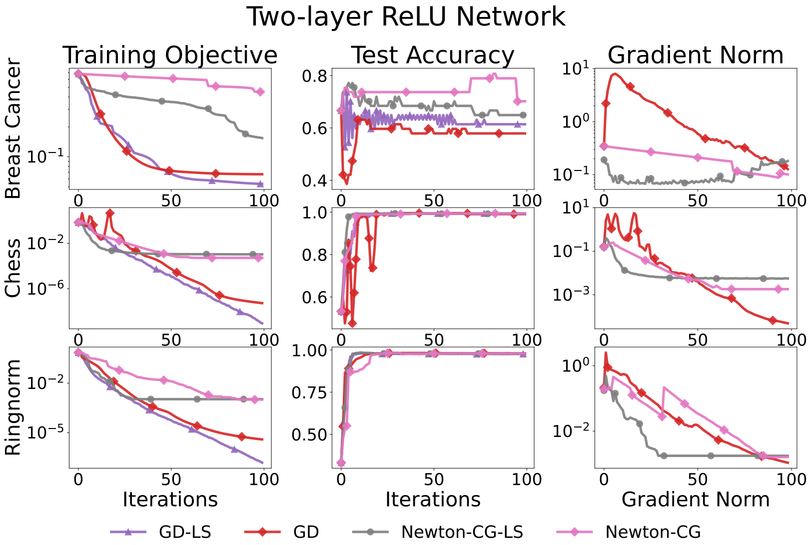

Questions?

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