Homework 11

Aaron Wang

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1. Suppose X and Y are i.i.d. random variables, where $Var(X) < 1$. Show that

$$P(|X-Y|>2) \leq \frac{Var(X)}{2}.$$

Observe that X and Y are identical. Thus $E[X-Y]=0$

$$P(|X-Y|>2) = P(|(X-Y)-E[X-Y]|>2)$$

By Chebyshev's Inequality

$$P(|X-Y-E[X-Y]|>2) \leq \frac{Var(X-Y)}{2^2}$$

Observe that since X and Y are identical $Var(X)=Var(Y)$:

$$Var(X-Y) = Var(X)+Var(Y) = 2Var(X)$$

$$P(|X-Y-E[X-Y]|>2) \leq \frac{2Var(X)}{2^2}$$

$$P(|X-Y|>2) \leq \frac{Var(X)}{2}$$

2. Suppose that X_1, X_2, ..., X_n and Y_1, Y_2, ..., Y_n are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 , respectively. Let $\bar{X} = n^{-1}\Sigma_{i=1}^{n} X_i$ and $\bar{Y} = n^{-1}\Sigma_{i=1}^{n} Y_i$. Show that the random variable U_n, which is defined as

$$U_n = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}$$

converges to a standard normal distribution function as n $\to \infty$.

$$E[X] = \mu_1 \qquad E[Y] = \mu_2$$
$$Var[X] = \sigma_1^2 \quad Var[Y] = \sigma_2^2$$

$$E[\bar{X}] = \mu_1 \qquad E[\bar{Y}] = \mu_2 \qquad \text{Let } \bar{Z} = \bar{X} - \bar{Y}$$
$$Var[\bar{X}] = \tfrac{\sigma_1^2}{n} \quad Var[\bar{Y}] = \tfrac{\sigma_2^2}{n}$$

$$\text{Let } \mu_Z = E[\bar{Z}] = E[\bar{X} - \bar{Y}] = E[\bar{X}] - E[\bar{Y}] = \mu_1 - \mu_2$$

$$\text{Let } \frac{\sigma_Z^2}{n} = Var[\bar{Z}] = Var[\bar{X} - \bar{Y}] = Var[\bar{X}] + Var[\bar{Y}] = \frac{\sigma_1^2 + \sigma_2^2}{n}$$

$$U_n = \frac{\bar{Z} - \mu_Z}{\sqrt{\sigma_Z^2/n}}$$

Finally, by the Central Limit Theorem, as $n \to \infty$, $U_n \to N(0,1)$

3. Suppose $\{X_k\}_{k\geq 1}$ are i.i.d. Unif$(0,1)$ random variables, and for each integer $k \geq 1$, define $Y_n := min(X_1, ..., X_n)$. Show that $Y_n \xrightarrow{P} 0$ as $n \to \infty$.

$$F_{Y_n}(y) = P[min(X_1, ..., X_n) \leq y] = 1 - P[min(X_1, ..., X_n) > y]$$

$$= 1 - \Pi_{k=1}^{n} P(X_k > y) = 1 - \Pi_{k=1}^{n}(1 - y) = 1 - (1 - y)^n$$

$$\text{Observe } 0 < y < 1 \text{ so } 0 < 1 - y < 1. \text{ Additionally, observe } \lim_{n\to\infty}(1 - y)^n = 0$$

$$\text{Thus, } \lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} 1 - (1 - y)^n = 1$$

$$F_{Y_n}(y) = P[Y_n \leq y] = P[|Y_n - 0| \leq y] = 1 \text{ So } Y_n \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty$$

4. An experiment is designed to test whether operator A or operator B gets the job of operating a new machine. Each operator is timed on 50 independent trials involving the performance of a certain task using the machine. If the sample means for the 50 trials differ by more than 1 second, the operator with the smaller mean time gets the job. Otherwise, the experiment is considered to end in a tie. If the standard deviations of times for both operators are assumed to be 2 seconds, what is the probability that operator A will get the job even though both operators have equal ability?

$$\sigma_{diff} = \sqrt{(\sigma_1^2 + \sigma_2^2)/n} = 0.4$$

$$Z = \frac{1-0}{0.4} = 2.5$$

Looking to a normal distribution graph, $P(Z > 2.5) = 0.0062$

6. Suppose X_1, X_2, ..., X_n, are i.i.d. $Unif(0,1)$ random variables, let $Y_n = (\Pi_{k=1}^{n} X_k)^{1/n}$. Show that $Y_n \xrightarrow{\mathbb{P}} c$, where c is a constant. And find the value of c.

$$Y_n = (\prod_{k=1}^{n} X_k)^{1/n} \Rightarrow \ln Y_n = \frac{1}{n}\sum_{k=1}^{n} \ln X_k$$

$$E[\ln X_k] = \int_0^1 \ln x dx = -1$$

By weak law of large numbers, $\ln Y_n$ converges to $E[\ln X_k]$ so

$$\ln Y_n \xrightarrow{\mathbb{P}} E[\ln X_k] = -1$$

Now when we exponentiate both sides:

$$Y_n \xrightarrow{\mathbb{P}} \frac{1}{e} \text{ so } c = \frac{1}{e}$$