Problem Set 3

Aaron Wang

February 11 2024

1. Prove each of the following statements for any propositions φ, ψ, ξ.

   (a) $(\varphi \to \psi), (\psi \to \xi) \vdash (\varphi \to \xi)$

   Proof. Let φ, ψ, and ξ. be arbitrary propositions, and suppose $\varphi \to \psi$ and $\psi \to \xi$. We will first show that $\varphi \vdash \xi$. Assume φ. Since we have $\varphi \to \psi$, we get ψ by modus ponens. Further, since we have $\psi \to \xi$, we get ξ by modus ponens. Thus, $\varphi \vdash \xi$. Therefore, by applying the deduction rule, we can conclude $\varphi \to \xi$ Q.E.D.

   (b) $\varphi, \psi \vdash \varphi \wedge \psi$

   Proof. Let φ and ψ be arbitrary propositions. Assume φ, and also separately assume ψ. Towards a contradiction, suppose $\neg(\varphi \wedge \psi)$. We can see that

   $$\begin{aligned} \neg(\varphi \wedge \psi) &\equiv \neg\varphi \vee \neg\psi && \text{by } \textit{De Morgan's laws} \\ &\equiv \varphi \to \neg\psi && \text{by } \textit{conditional disintegration} \end{aligned}$$

   So we have φ and $\varphi \to \neg\psi$, which gives us $\neg\psi$ by modus ponens. However, since we had ψ by assumption, we get a contradiction.
   Therefore, we can conclude $\varphi \wedge \psi$ by Reductio ad absurdum.

   Q.E.D.

2. Prove each of the following statements for any propositions φ, ψ, ξ.

   (a) $\vdash \varphi \to \varphi$

   Proof. Let φ be an arbitrary proposition. Assume φ. Now observe that φ follows from this assumption. Therefore, $\varphi \vdash \varphi$. Now by deduction rule, we can conclude $\varphi \to \varphi$.

   Q.E.D.

   (b) $\vdash (\neg\varphi \to \varphi) \to \varphi$

   Proof. Let φ be an arbitrary proposition. Let's first show that $\varphi \equiv \neg\varphi \to \varphi$

$$
\begin{aligned}
\varphi &\equiv \varphi \lor \varphi && \text{by } \textit{idempotence} \\
&\equiv \neg(\neg\varphi) \lor \varphi && \text{by } \textit{double negation} \\
&\equiv \neg\varphi \to \varphi && \text{by } \textit{conditional disintegration}
\end{aligned}
$$

   Now we can substitute φ for $\neg\varphi \to \varphi$ which turns $(\neg\varphi \to \varphi) \to \varphi$ into an equivalent expression $\varphi \to \varphi$, something we have already proved in (a). Therefore we can conclude $(\neg\varphi \to \varphi) \to \varphi$.

   Q.E.D.

   (c) $\vdash \neg\varphi \to (\varphi \to \neg\psi)$

   Proof. Let φ and ψ be arbitrary propositions. Let's first show that $\psi \to \neg\varphi \equiv \varphi \to \neg\psi$.

$$
\begin{aligned}
\psi \to \neg\varphi &\equiv \neg\psi \lor \neg\varphi && \text{by } \textit{conditional disintegration} \\
&\equiv \neg\varphi \lor \neg\psi && \text{by } \textit{commutativity} \\
&\equiv \varphi \to \neg\psi && \text{by } \textit{conditional disintegration}
\end{aligned}
$$

   Using this equivalence, we can substitute $\psi \to \neg\varphi$ for $\varphi \to \neg\psi$ in $\neg\varphi \to (\varphi \to \neg\psi)$ to create an equivalent expression $\neg\varphi \to (\psi \to \neg\varphi)$ which is in the form of Hilbert's first axiom. Consequently, by Hilbert's first axiom, we can conclude $\neg\varphi \to (\varphi \to \neg\psi)$.

   Q.E.D.

(d) $\varphi \wedge \psi \vdash \varphi$

Proof. Let φ and ψ be arbitrary propositions. Assume $\varphi \wedge \psi$.
Assume towards a contradiction $\neg\varphi$. Using conjunction introduction,
$\varphi \wedge \psi, \neg\varphi \vdash (\varphi \wedge \psi) \wedge (\neg\varphi)$. Observe:

$$
\begin{aligned}
(\varphi \wedge \psi) \wedge \neg\varphi &\equiv (\psi \wedge \varphi) \wedge \neg\varphi && \text{by } \textit{commutativity} \\
&\equiv \psi \wedge (\varphi \wedge \neg\varphi) && \text{by } \textit{associativity} \\
&\equiv \psi \wedge (\bot) && \text{by } \textit{complement} \\
&\equiv \bot && \text{by } \textit{domination}
\end{aligned}
$$

So, we have $\bot$. However, we also have $\top$ which is proven by the Truth theorem (proven by the fact that we proved (a) and (a) is a tautology). Therefore, by Reductio ad Absurdum we can conclude φ.

Q.E.D.

(e) $\vdash \top$

Proof. From (a) we concluded $\varphi \to \varphi$. Since $\varphi \to \varphi \equiv \top$ (proved in PSet 2), we can also conclude $\top$.

Q.E.D.

3. Prove each of the following statements for any propositions φ, ψ, ξ, χ

   (a) $\varphi \vdash (\varphi \vee \psi)$

   Proof. Let φ and ψ be arbitrary propositions. Assume φ. Towards a contradiction, suppose $\neg(\varphi \vee \psi)$. Using conjunction introduction,
   $\varphi, \neg(\varphi \vee \psi) \vdash \varphi \wedge \neg(\varphi \vee \psi)$ Observe:

$$
\begin{aligned}
\varphi \wedge \neg(\varphi \vee \psi) &\equiv \varphi \wedge (\neg\varphi \wedge \neg\psi) && \text{by \textit{De Morgan's laws}}\\
&\equiv (\varphi \wedge \neg\varphi) \wedge \neg\psi && \text{by \textit{associativity}}\\
&\equiv \bot \wedge \neg\psi && \text{by \textit{complement}}\\
&\equiv \bot && \text{by \textit{domination}}
\end{aligned}
$$

   So, we have $\neg(\varphi \vee \psi) \vdash \bot$. However, we also have $\neg(\varphi \vee \psi) \vdash \top$ because we assumed $\neg(\varphi \vee \psi)$. Therefore, by Reductio ad Absurdum we can conclude $\neg(\neg(\varphi \vee \psi))$ or $\varphi \vee \psi$.

   Q.E.D.

   (b) $(\varphi \to \xi), (\psi \to \xi), (\varphi \vee \psi) \vdash \xi$

   Proof. Let φ, ψ and ξ be arbitrary propositions. Assume $(\varphi \to \xi)$, $(\psi \to \xi)$, and $(\varphi \vee \psi)$. Assume towards a contradiction $\neg\xi$.
   Using conjunction introduction:

$$
(\varphi \to \xi), (\psi \to \xi), \vdash (\varphi \to \xi) \wedge (\psi \to \xi)
$$

   Observe:
   $(\varphi \to \xi) \wedge (\psi \to \xi)$

$$
\begin{aligned}
&\equiv (\neg\varphi \vee \xi) \wedge (\neg\psi \vee \xi) && \text{by \textit{conditional disintegration}} \times 2\\
&\equiv (\neg\varphi \wedge \neg\psi) \vee \xi && \text{by \textit{distributivity}}\\
&\equiv \neg(\varphi \vee \psi) \vee \xi && \text{by \textit{De Morgan's laws}}\\
&\equiv (\varphi \vee \psi) \to \xi && \text{by \textit{conditional disintegration}}
\end{aligned}
$$

   So, we have $\varphi \vee \psi$ and$(\varphi \vee \psi) \to \xi$. By *modus ponens* we can conclude ξ.

   Q.E.D.

(c) $\varphi, \neg\varphi \vdash \psi$

Proof. Let φ and ψ be arbitrary propositions. Assume φ as a premise. By disjunction introduction, $\varphi \vdash \varphi \vee \psi$. Observe:

$$\begin{aligned} \varphi \vee \psi &\equiv \neg(\neg\varphi) \vee \psi && \text{by } \textit{double negation} \\ &\equiv \neg\varphi \to \psi && \text{by } \textit{conditional disintegration} \end{aligned}$$

Now assume $\neg\varphi$ as another premise. By modus ponens, we conclude ψ.

Q.E.D.

(d) $(\varphi \vee \psi), \neg\varphi \vdash \psi$

Proof. Let φ and ψ be arbitrary propositions. Assume $(\varphi \vee \psi)$ and $\neg\varphi$. Observe:

$$\begin{aligned} \varphi \vee \psi &\equiv \neg(\neg\varphi) \vee \psi && \text{by } \textit{double negation} \\ &\equiv \neg\varphi \to \psi && \text{by } \textit{conditional disintegration} \end{aligned}$$

So, we have $\neg\varphi$ and $\neg\varphi \to \psi$. By *modus ponens* we can conclude ψ.

Q.E.D.

(e) $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi) \vdash \xi \vee \chi$

Proof. Let φ, ψ, ξ, and χ be arbitrary propositions.
Assume $\varphi \to \xi$, $\psi \to \chi$, and $\varphi \vee \psi$. Assume towards a contradiction $\neg(\xi \vee \chi)$. Observe that by De Morgan's laws $\neg(\xi \vee \chi) \equiv \neg\xi \vee \neg\chi$ from which we can use conjunction elimination to conclude $\neg\xi$ and $\neg\chi$. Now, by modus tollens, we can use $\neg\xi$ and $\varphi \to \xi$ to conclude $\neg\varphi$. Similarly, by modus tollens, we can use $\neg\chi$ and $\psi \to \chi$ to conclude $\neg\psi$. Since we have $\neg\psi$ and $\neg\varphi$ we can use conjuction introduction to conclude $\neg\psi \wedge \neg\varphi$ which is equivalent to $\neg(\varphi \vee \psi)$ by De Morgan's laws. Since we proved $\neg(\varphi \vee \psi)$ and assumed $\varphi \vee \psi$, by Reductio Ad Absurdum we can conclude $\neg(\neg(\xi \vee \chi))$ or $\xi \vee \chi$.

Q.E.D.

4. Let $\mathcal{L}$ be a binary predicate. Prove the following statement.

$$\vdash \neg\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$$

Proof. Let $\mathcal{L}$ be a binary predicate. Towards a contradiction, assume $\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$ which by existential elimination says $\forall y(\mathcal{L}(t, y) \leftrightarrow \neg\mathcal{L}(y, y))$ for a new term t. By universal elimination, $\forall y(\mathcal{L}(t, y) \leftrightarrow \neg\mathcal{L}(y, y))$ is true for any value y so let y = t. In this case, $\mathcal{L}(t, t) \leftrightarrow \neg\mathcal{L}(t, t) \equiv \bot$ so $\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y)) \vdash \bot$ (Look at a truth table). However, we assumed $\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$ so by the truth theorem $\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y)) \vdash \top$. Since $\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$ leads to a contradiction, we can conclude $\neg\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$ by Reductio Ad Absurdum.

Q.E.D.

5. Consider a universe of discourse consisting of every natural number. Recall that a positive integer is *prime* when it has *exactly two* positive divisors: one and itself.

Let $\omega(x) :=$ *"x is an odd number."*
Let $\pi(x) :=$ *"x is a prime number."*

Further, suppose the following statements only contain propositions.

(a) Prove φ, where φ is the statement $\varphi \vdash \forall x\,(\omega(x) \to \pi(x))$.

Proof. Let $\varphi :=$"$\varphi \vdash \forall x\,(\omega(x) \to \pi(x))$". Assume φ. Therefore we have φ which says $\varphi \vdash \forall x\,(\omega(x) \to \pi(x))$. Observe that $\varphi \vdash \forall x\,(\omega(x) \to \pi(x))$ is $\varphi \to \forall x\,(\omega(x) \to \pi(x))$ by deduction rule. Thus by modus ponens, we use φ and $\varphi \to \forall x\,(\omega(x) \to \pi(x))$ to get $\forall x\,(\omega(x) \to \pi(x))$. Therefore by modus ponens $\varphi \vdash \forall x\,(\omega(x) \to \pi(x))$

Q.E.D.

(b) Prove $\forall x\,(\omega(x) \to \pi(x))$.

Proof. From (a) we found $\vdash \varphi$. Thus, we have $\varphi \vdash \forall x\,(\omega(x) \to \pi(x))$ from which deduction rule gives us $\varphi \to \forall x\,(\omega(x) \to \pi(x))$. By Modus ponens, since we have φ and $\varphi \to \forall x\,(\omega(x) \to \pi(x))$, we can conclude $\forall x\,(\omega(x) \to \pi(x))$.//

Q.E.D.