Homework 01

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1 Monday 1/13

1. Verify the statement of Example 2. Also verify that $y = \cosh(x)$ and $y = \sinh(x)x$ are solutions of $y'' = y$.

   (a) e^x

   $$y = e^x;\ y' = e^x;\ y'' = e^x \text{ so } y = y''$$

   (b) e^{-x}

   $$y = e^{-x};\ y' = -e^{-x};\ y'' = e^{-x} \text{ so } y = y''$$

   (c) $Ae^x + Be^{-x}$

   $$y = Ae^x + Be^{-x};\ y' = Ae^x - Be^{-x};\ y'' = Ae^x + Be^{-x} \text{ so } y = y''$$

   (d) $\cosh(x)$

   $$y = \cosh(x);\ y' = \sinh(x);\ y'' = \cosh(x) \text{ so } y = y''$$

   (e) $\sinh(x)$

   $$y = \sinh(x);\ y' = \cosh(x);\ y'' = \sinh(x) \text{ so } y = y''$$

2. Find the solution of $y'' = y$ which passes through the origin and through the point $(\ln 2, \frac{3}{4})$.

   The general solution of the differential equation is

   $$y = a \sinh x + b \cosh x$$

   $$0 = a \sinh 0 + b \cosh 0 = a(0) + b(1) = b$$
   $$b = 0$$
   $$\frac{3}{4} = a \sinh(\ln 2) + b \cosh(\ln 2) = a(\frac{3}{4}) + b(\frac{5}{4}) = \frac{3}{4}a$$
   $$a = 1$$

   The desired particular solution is

   $$y = \sinh x$$

3. Verify that $y = \sin x$, $y = \cos x$, $y = e^{ix}$, and $y = e^{-ix}$ are all solutions of $y'' = -y$.

   (a) $y = \sin x$

$$y = \sin x;\ y' = \cos x;\ y'' = -\sin x \text{ so } y'' = -y$$

   (b) $y = \cos x$

$$y = \cos x;\ y' = -\sin x;\ y'' = -\cos x \text{ so } y'' = -y$$

   (c) e^{ix}

$$y = e^{ix};\ y' = ie^{ix};\ y'' = i^2 e^{-x} = -e^{ix} \text{ so } y = y''$$

   (d) e^{-ix}

$$y = e^{-ix};\ y' = -ie^{-ix};\ y'' = i^2 e^{-x} = -e^{-ix} \text{ so } y = y''$$

4. Find the distance which an object moves in time t if it starts from rest and has an acceleration $\frac{d^2x}{dt^2} = ge^{-kt}$.

$$\frac{dx}{dt} = \int \frac{d^2x}{dt^2} dt = \int ge^{-kt} dt = -\frac{g}{k} e^{-kt} + C_1$$

$$\left.\frac{dx}{dt}\right|_{x=0} = -\frac{g}{k} e^{-k(0)} + C_1 = 0 \text{ so } C_1 = \frac{g}{k} e^{-k(0)} = \frac{g}{k}$$

$$x = \int \frac{dx}{dt} dt = \int -\frac{g}{k} e^{-kt} + C_1 dt = \frac{g}{k^2} e^{-kt} + C_1 t + C_2$$

$$x(0) = \frac{g}{k^2} e^{-k(0)} + C_1(0) + C_2 = 0 \text{ so } C_2 = -\Big(\frac{g}{k^2} e^{-k(0)} + C_1(0)\Big) = -\frac{g}{k^2}$$

$$x(t) = \frac{g}{k^2} e^{-kt} + \frac{g}{k} t - \frac{g}{k^2}$$

Show that for small t the result is approximately (1.10)($x = \frac{1}{2} gt^2$)

$$\lim_{t \to 0} a = \lim_{t \to 0} ge^{-kt} = g.$$

Thus, when t is small:

$$v(t) \approx \int g = gt + C \text{ where } C = 0 \text{ because } v(0) = 0$$

$$x(t) \approx \int gt = \frac{1}{2} gt^2 + C \text{ where } C = 0 \text{ because } x(0) = 0$$

Thus,

$$x(t) \approx \frac{1}{2} gt^2 \text{ for small t}$$

Show for very large t, the speed $\frac{dx}{dt}$ is approximately constant.

$$\lim_{t \to \infty} \frac{dx}{dt} = \lim_{t \to \infty} \Big(-\frac{g}{k} e^{-kt} + \frac{g}{k}\Big) = \frac{g}{k}$$

5. Find the position x of a particle at time t if its acceleration is $\frac{d^2x}{dt^2} = A\sin(\omega t)$.

$$\frac{dx}{dt} = \int \frac{d^2x}{dt^2}dt = \int A\sin(\omega t)dt = -A\omega^{-1}\cos(\omega t) + C_1$$

$$x = \int \frac{dx}{dt}dt = -\int A\omega\cos(\omega t) + C_1 dt = -A\omega^{-2}\sin(\omega t) + C_1 t + C_2$$

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For each of the following differential equations, separate variables and find a solution containing one arbitrary constant. Then find the value of the constant to give a particular solution satisfying the given boundary condition.

1. $xy' = y$, $y = 3$ when $x = 2$

General Solution:

$$x\frac{dy}{dx} = y$$
$$\frac{1}{y}dy = \frac{1}{x}dx$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$
$$\ln|y| = \ln|x| + C$$
$$e^{\ln|y|} = e^{\ln|x|+C}$$
$$y = Cx$$

Particular Solution:

$$(3) = C(2)$$
$$C = \frac{3}{2}$$
$$y = \frac{3}{2}x$$

2. $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$, $y = \frac{1}{2}$ when $x = \frac{1}{2}$

General Solution:

$$x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$$
$$y\sqrt{1-x^2}dy = -x\sqrt{1-y^2}dx$$
$$\frac{y}{\sqrt{1-y^2}}dy = -\frac{x}{\sqrt{1-x^2}}dx$$
$$\int \frac{y}{\sqrt{1-y^2}}dy = -\int \frac{x}{\sqrt{1-x^2}}dx$$
$$-\frac{1}{2}\sqrt{1-y^2} = \frac{1}{2}\sqrt{1-x^2} + C$$
$$\frac{1}{2}\sqrt{1-y^2} + \frac{1}{2}\sqrt{1-x^2} = C$$
$$\sqrt{1-y^2} + \sqrt{1-x^2} = C$$

Integration:
$u = 1 - t^2$; $du = -2tdt$; $dt = -\frac{1}{2t}du$

$$\int \frac{t}{\sqrt{1-t^2}}dt = \int \frac{t}{\sqrt{u}}\frac{-1}{2t}du$$
$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -\frac{1}{2}(\sqrt{u}) + C$$
$$= -\frac{1}{2}\sqrt{1-t^2} + C$$

Particular Solution:

$$\sqrt{1-(1/2)^2} + \sqrt{1-(1/2)^2} = C$$
$$C = \sqrt{3/4} + \sqrt{3/4} = \sqrt{3}$$
$$\sqrt{1-y^2} + \sqrt{1-x^2} = \sqrt{3}$$

3. $y' \sin x = y \ln y$, $y = e$ when $x = \pi/3$

General Solution:

$$\frac{dy}{dx}\sin x = y \ln y$$
$$\frac{dy}{y \ln y} = \csc x dx$$
$$\int \frac{dy}{y \ln y} = \int \csc x dx$$
$$\ln|\ln(y)| = \ln|\csc x - \cot x| + C$$
$$e^{\ln|\ln(y)|} = e^{\ln|\csc x - \cot x| + C}$$
$$\ln(y) = C(\csc x - \cot x)$$

Particular Solution:

$$\ln(e) = C(\csc(\pi/3) - \cot(\pi/3))$$
$$1 = C\Big(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\Big)$$
$$1 = C\Big(\frac{1}{\sqrt{3}}\Big)$$
$$\sqrt{3} = C$$
$$\ln(y) = \sqrt{3}(\csc x - \cot x)$$

Integration for LHS:
$u = \ln(y)$; $du = \frac{1}{y}dy$; $dy = ydu$

$$\int \frac{dy}{y \ln y} = \int \frac{1}{u} du$$
$$= \ln|u| + C = \ln|\ln(y)| + C$$

Integration for RHS:

$$\int \csc x dx$$
$$= \int \csc x \Big(\frac{\csc x - \cot x}{\csc x - \cot x}\Big) dx$$
$$= \int \Big(\frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\Big) dx$$

$u = \csc x - \cot x$
$du = (\csc^2 x - \csc x \cot x)dx$

$$= \int \frac{1}{u} du = \ln|u| + C$$
$$= \ln|\csc x - \cot x| + C$$

4. $(1 + y^2)dx + xydy = 0$, $y = 0$ when $x = 5$.

General Solution:

$$(1 + y^2)dx + xydy = 0$$
$$(1 + y^2)dx = -xydy$$
$$\frac{y}{1 + y^2}dy = -\frac{1}{x}dx$$
$$\int \frac{y}{1 + y^2}dy = -\int \frac{1}{x}dx$$
$$\frac{1}{2}\ln|1 + y^2| = -\ln|x| + C$$
$$e^{\frac{1}{2}\ln|1+y^2|} = e^{-\ln|x| + C}$$
$$\sqrt{1 + y^2} = \frac{C}{x}$$
$$1 + y^2 = \frac{C}{x^2}$$

Integration for LHS:
$u = 1 + y^2$; $du = 2ydy$

$$\int \frac{y}{1 + y^2}dy = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1 + y^2| + C$$

Particular Solution:

$$1 + (0)^2 = \frac{C}{(5)^2}$$
$$C = 25$$
$$1 + y^2 = \frac{25}{x^2}$$

5. $xy' - xy = y$, $y = 1$ when $x = 1$

General Solution:

$$x\frac{dy}{dx} - xy = y$$
$$x\frac{dy}{dx} = y + xy$$
$$x\frac{dy}{dx} = y(1+x)$$
$$\frac{dy}{y} = (\frac{1}{x} + 1)dx$$
$$\int \frac{dy}{y} = \int (\frac{1}{x} + 1)dx$$
$$\ln|y| = \ln|x| + x + C$$
$$e^{\ln|y|} = e^{\ln|x| + x + C}$$
$$y = Cxe^x$$

Particular Solution:

$$(1) = C(1)e^{(1)}$$
$$C = \frac{1}{e}$$
$$y = \frac{1}{e}xe^x$$

6. $y' = \frac{2xy^2+x}{x^2y-y}$, $y = 0$ when $x = \sqrt{2}$

General Solution:

$$\frac{dy}{dx} = \frac{x(2y^2+1)}{y(x^2-1)}$$
$$\frac{y}{2y^2+1}dy = \frac{x}{x^2-1}dx$$
$$\int \frac{y}{2y^2+1}dy = \int \frac{x}{x^2-1}dx$$
$$\frac{1}{4}\ln|2y^2+1| = \frac{1}{2}\ln|x^2-1| + C$$
$$\ln|2y^2+1| = 2\ln|x^2-1| + C$$
$$e^{\ln|2y^2+1|} = e^{2\ln|x^2-1|+C}$$
$$2y^2 + 1 = C(x^2-1)^2$$

Integration for LHS:
$u = 2y^2 + 1$; $du = 4ydy$

$$\int \frac{y}{2y^2+1}dy = \frac{1}{4}\int \frac{1}{u}du$$
$$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|2y^2+1| + C$$

Integration for RHS:
$u = x^2 - 1$; $du = 2xdx$

$$\int \frac{x}{x^2-1}dx = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-1| + C$$

Particular Solution:

$$2(0)^2 + 1 = C((\sqrt{2})^2 - 1)^2$$
$$1 = C(2-1)^2$$
$$C = 1$$
$$2y^2 + 1 = (x^2-1)^2$$

7. $ydy + (xy^2 - 8x)dx = 0$, $y = 3$ when $x = 1$

General Solution:

$$ydy + (xy^2 - 8x)dx = 0$$
$$-ydy = (xy^2 - 8x)dx$$
$$-ydy = x(y^2 - 8)dx$$
$$-\frac{y}{y^2 - 8}dy = xdx$$
$$-\int \frac{y}{y^2 - 8}dy = \int xdx$$
$$-\frac{1}{2}\ln|y^2 - 8| = \frac{x^2}{2} + C$$
$$\ln|y^2 - 8| = -x^2 + C$$
$$e^{\ln|y^2-8|} = e^{-x^2+C}$$
$$y^2 - 8 = Ce^{-x^2}$$

Integration for LHS:
$u = y^2 - 8$; $du = 2ydy$

$$\int \frac{y}{y^2 - 8}dy = \frac{1}{2}\int \frac{1}{u}dy$$
$$\frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|y^2 - 8| + C$$

Particular Solution:

$$(3)^2 - 8 = Ce^{-(1)^2}$$
$$9 - 8 = C/e$$
$$C = e$$
$$y^2 - 8 = e \cdot e^{-x^2}$$

8. $y' + 2xy^2 = 0$, $y = 1$ when $x = 2$

General Solution:

$$\frac{dy}{dx} + 2xy^2 = 0$$
$$\frac{dy}{dx} = -2xy^2$$
$$-\frac{dy}{y^2} = 2xdx$$
$$\int -\frac{dy}{y^2} = \int 2xdx$$
$$\frac{1}{y} = x^2 + C$$
$$y = \frac{1}{x^2 + C}$$

Particular Solution:

$$1 = \frac{1}{(2)^2 + C}$$
$$4 + C = 1$$
$$C = -1$$
$$y = \frac{1}{x^2 - 3}$$

9. $(1+y)y' = y$, $y = 1$ when $x = 1$

General Solution:

$$(1+y)\frac{dy}{dx} = y$$
$$\left(\frac{1}{y}+1\right)dy = dx$$
$$\int \left(\frac{1}{y}+1\right)dy = \int dx$$
$$\ln|y| + y = x + C$$
$$e^{\ln|y|+y} = e^{x+C}$$
$$ye^{y} = Ce^{x}$$

Particular Solution:

$$(1)e^(1) = Ce^(1)$$
$$C = 1$$
$$ye^{y} = e^{x}$$

10. $y' - xy = x$, $y = 1$ when $x = 0$

General Solution:

$$\frac{dy}{dx} - xy = x$$
$$\frac{dy}{dx} = xy + x$$
$$\frac{dy}{dx} = x(y+1)$$
$$\frac{dy}{y+1} = xdx$$
$$\int \frac{dy}{y+1} = \int xdx$$
$$\ln|y+1| = \frac{x^2}{2} + C$$
$$e^{\ln|y+1|} = e^{x^2/2+C}$$
$$y + 1 = Ce^{x^2/2}$$
$$y = Ce^{x^2/2} - 1$$

Particular Solution:

$$(1) = Ce^{(0)^2/2} - 1$$
$$C = 2$$
$$y = 2e^{x^2/2} - 1$$

11. $2y' = 3(y-2)^{1/3}$, $y = 3$ when $x = 1$

General Solution:

$$\begin{aligned}
2\frac{dy}{dx} &= 3(y-2)^{1/3} \\
\frac{2}{3}\frac{1}{(y-2)^{1/3}}dy &= dx \\
\frac{2}{3}\int \frac{1}{(y-2)^{1/3}}dy &= \int dx \\
\frac{2}{3}\frac{3}{2}(y-2)^{2/3} &= x + C \\
(y-2)^{2/3} &= x + C
\end{aligned}$$

Integration for LHS:
$u = y - 2$; $du = dy$

$$\begin{aligned}
\int \frac{1}{(y-2)^{1/3}}dy &= \int (u)^{-1/3}du \\
\frac{3}{2}u^{2/3} + C &= \frac{3}{2}(y-2)^{2/3} + C
\end{aligned}$$

Particular Solution:

$$\begin{aligned}
((3)-2)^{2/3} &= (1) + C \\
1 &= 1 + C \\
C &= 0 \\
(y-2)^{2/3} &= x
\end{aligned}$$

12. $(x + xy)y' + y = 0$

General Solution:

$$\begin{aligned}
(x + xy)\frac{dy}{dx} &= -y \\
x(1+y)\frac{dy}{dx} &= -y \\
\frac{1+y}{y}dy &= -xdx \\
\left(\frac{1}{y} + 1\right)dy &= -\frac{1}{x}dx \\
\int \left(\frac{1}{y} + 1\right)dy &= -\int \frac{1}{x}dx \\
\ln|y| + y &= -\ln|x| + C \\
e^{\ln|y|+y} &= e^{-\ln|x|+C} \\
ye^y &= \frac{C}{x}
\end{aligned}$$

Particular Solution:

$$\begin{aligned}
(1)e^{(1)} &= \frac{C}{(1)} \\
C &= e \\
ye^y &= \frac{e}{x}
\end{aligned}$$