Homework 07

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1 Monday 3/3

Section 3

2. $\Gamma(2/3)/\Gamma(5/3)$

$$\frac{\Gamma(2/3)}{\Gamma(5/3)} = \frac{\Gamma(2/3)}{2/3\Gamma(2/3)} = \frac{1}{2/3} = \frac{3}{2}$$

4. $\Gamma(2/5)/\Gamma(12/5)$

$$\frac{\Gamma(2/5)}{\Gamma(12/5)} = \frac{\Gamma(2/3)}{7/5\Gamma(7/5)} = \frac{\Gamma(2/3)}{7/5 \cdot 2/5\Gamma(2/5)} = \frac{1}{7/5 \cdot 2/5} = \frac{25}{14}$$

6. $\Gamma(10)/\Gamma(8)$

$$\frac{\Gamma(10)}{\Gamma(8)} = \frac{9!}{7!} = 9 \cdot 8 = 72$$

7. $\Gamma(4)\Gamma(3/4)/\Gamma(7/4)$

$$\frac{\Gamma(4)\Gamma(3/4)}{\Gamma(7/4)} = \frac{\Gamma(4)\Gamma(3/4)}{3/4\Gamma(3/4)} = \frac{\Gamma(4)}{3/4} = \frac{3!}{3/4} = 8$$

8. $\int_0^\infty x^{2/3}e^{-x}dx$

$$\int_0^\infty x^{2/3}e^{-x}dx = \Gamma(2/3+1) = \Gamma(5/3)$$

9. $\int_0^\infty e^{-x^4}dx$

   Using u-sub where $u = x^4$ and $du = 4x^3dx$

$$\int_0^\infty e^{-x^4}dx = \int_0^\infty \frac{4x^3}{4x^3}e^{-x^4}dx = \frac{1}{4}\int_0^\infty u^{-3/4}e^{-u}du = \frac{1}{4}\Gamma(-3/4+1) = \frac{1}{4}\Gamma(1/4) = \Gamma(5/4)$$

10. $\int_0^\infty x^{-2/5}e^{-x}dx$

$$\int_0^\infty x^{-2/5}e^{-x}dx = \Gamma(-2/5+1) = \Gamma(3/5)$$

11. $\int_0^\infty x^5e^{-x^2}dx$

    Using u-sub where $u = x^2$ and $du = 2xdx$

$$\int_0^\infty x^5e^{-x^2}dx = \frac{1}{2}\int_0^\infty u^2e^{-u}du = \frac{1}{2}\Gamma(2+1) = \frac{1}{2}\Gamma(3)$$

2 Wednesday 3/5

Section 6

2. Prove $B(p,q) = \int_0^1 \frac{y^{p-1}dy}{(1+y)^{p+q}}$

   Let $x = \frac{y}{1+y}$ and $dx = \frac{1}{(1+y)^2}dy$

$$
\begin{aligned}
B(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}dx \\
&= \int_0^1 \frac{y}{1+y}^{p-1}\left(1-\frac{y}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}dy \\
&= \int_0^1 \frac{y}{1+y}^{p-1}\left(\frac{1+y}{1+y}-\frac{y}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}dy \\
&= \int_0^1 \frac{y}{1+y}^{p-1}\left(\frac{1}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}dy \\
&= \int_0^1 y^{p-1}\frac{1}{(1+y)^{p-1}}\frac{1}{(1+y)^{q-1}}\frac{1}{(1+y)^2}dy \\
&= \int_0^1 \frac{y^{p-1}dy}{(1+y)^{(p-1)+(q-1)+2}} \\
&= \int_0^1 \frac{y^{p-1}dy}{(1+y)^{p+q}}
\end{aligned}
$$

Section 7

3. $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

   Using u-sub let $u = x^3$ and $dx = \frac{1}{3}u^{-2/3}du$

$$
\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \int_0^1 \frac{1}{\sqrt{1-u}}\frac{1}{3}u^{-2/3}du = \frac{1}{3}\int_0^1 (1-u)^{-2/3}u^{-1/2}du
$$

$$
= \frac{1}{3}B(1-2/3, 1-1/2) = \frac{1}{3}B(1/3, 1/2)
$$

4. $\int_0^1 x^2(1-x^2)^{3/2}dx$

   Using u-sub let $u = x^2$ and $dx = \frac{1}{2}u^{-1/2}du$

$$
\int_0^1 x^2(1-x^2)^{3/2}dx = \int_0^1 u(1-u)^{3/2}\frac{1}{2}u^{-1/2}du = \frac{1}{2}\int_0^1 u^{1/2}(1-u)^{3/2}du
$$

$$
= \frac{1}{2}B(1+1/2, 1+3/2) = \frac{1}{2}B(3/2, 5/2) = \frac{\Gamma(3/2)\Gamma(5/2)}{2\Gamma(4)} = \frac{3\Gamma(1/2)^2}{2^4\cdot 3!} = \frac{\pi}{32}
$$

5. $\int_0^1 \frac{y^2dy}{(1+y)^6}$

$$
\int_0^1 x^2(1-x^2)^{3/2}dx = B(3,3) = \frac{\Gamma(3)\Gamma(3)}{\Gamma(3+3)} = \frac{1}{30}
$$

7. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin(\theta)}}$

$$
\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin(\theta)}} = \int_0^{\pi/2} (\sin(\theta))^{-1/2}(\cos(\theta))^0 d\theta = \frac{1}{2}B(1/4, 1/2)
$$