Homework 06

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Section 9

7. $y'' - 4y' + 4y = 4$, $y_0 = 0$, $y'_0 = -2$

$$L(y'') = p^2Y - py_0 - y'_0;\ L(y') = pY - y_0;\ L(y) = Y;\ L(4) = 4/p$$

$$\begin{aligned} y'' - 4y' + 4y &= p^2Y - py_0 - y'_0 - 4(pY - y_0) + 4(Y) \\ &= p^2Y - p(0) - (-2) - 4(pY - (0)) + 4Y \\ &= p^2Y + 2 - 4pY + 4Y \\ &= p^2Y - 4pY + 4Y + 2 \\ &= (p-2)^2Y + 2 \end{aligned}$$

$$\begin{aligned} (p-2)^2Y + 2 &= \frac{4}{p} \\ Y &= \frac{4-2p}{p(p-2)^2} = \frac{-2}{p(p-2)} \end{aligned}$$

Using partial fraction decomposition

$$\begin{aligned} Y &= \frac{A}{p} + \frac{B}{p-2} \text{ s.t. } A(p-2) + Bp = -2 \\ &\implies (A+B)p - 2A = -2 \text{ so } A = 1 \\ &\implies (1+B)p = 0 \text{ so } B = -1 \\ Y &= \frac{1}{p} - \frac{1}{p-2} \end{aligned}$$

$$L^{-1}\left(\frac{1}{p}\right) = 1 \qquad \text{(By } L1\text{)}$$

$$L^{-1}\left(-\frac{1}{p-2}\right) = -L^{-1}\left(\frac{1}{p+(-2)}\right) = -e^{-(-2)t} = -e^{2t} \qquad \text{(By } L2\text{)}$$

$$y = 1 - e^{2t}$$

9. $y'' + 16y = 8\cos(4t)$, $y_0 = 0$, $y_0' = 8$

$$L(y'') = p^2Y - py_0 - y_0';\ L(y') = pY - y_0;\ L(y) = Y;\ L(8\cos(4t)) = \frac{8p}{p^2+16}$$

$$\begin{aligned} y'' - 4y' + 4y &= p^2Y - py_0 - y_0' + 16(Y) \\ &= p^2Y - p(0) - (8) + 16Y \\ &= p^2Y - 8 + 16Y \\ &= p^2Y + 16Y - 8 \\ &= (p^2+16)Y - 8 \end{aligned}$$

$$\begin{aligned} (p^2+16)Y - 8 &= \frac{8p}{p^2+16} \\ Y &= \frac{8p}{(p^2+16)^2} + \frac{8}{p^2+16} \end{aligned}$$

$$L^{-1}\left(\frac{8p}{(p^2+16)^2}\right) = -L^{-1}\left(\frac{2(4)p}{(p^2+(4)^2)^2}\right) = t\sin((4)t) = t\sin(4t) \qquad \text{(By } L11\text{)}$$

$$L^{-1}\left(\frac{8}{p^2+16}\right) = 2L^{-1}\left(\frac{(4)}{p^2+(4)^2}\right) = 2\sin((4)t) = 2\sin(4t) \qquad \text{(By } L3\text{)}$$

$$y = t\sin(4t) + 2\sin(4t)$$

12. $y'' - y = e^{-t} - 2te^{-t}$, $y_0 = 1$, $y_0' = 2$

$$L(y'') = p^2Y - py_0 - y_0';\ L(y') = pY - y_0;\ L(y) = Y;\ L(e^{-t} - 2te^{-t}) = \frac{1}{p+1} - \frac{2}{(p+1)^2}$$

$$\begin{aligned} y'' - y &= p^2Y - py_0 - y_0' - (Y) \\ &= p^2Y - p(1) - (2) - Y \\ &= p^2Y - p - 2 - Y \\ &= p^2Y - Y - p - 2 \\ &= (p+1)(p-1)Y - p - 2 \end{aligned}$$

$$(p+1)(p-1)Y - p - 2 = \frac{1}{p+1} - \frac{2}{(p+1)^2}$$

$$Y = \frac{\frac{1}{p+1} - \frac{2}{(p+1)^2} + p + 2}{(p+1)(p-1)} = \frac{\frac{p-1}{(p+1)^2} + p + 2}{(p+1)(p-1)} = \frac{1}{(p+1)^3} + \frac{p+2}{(p+1)(p-1)}$$

Using Partial Fraction Decomp:

$$\frac{p+2}{(p+1)(p-1)} = \frac{A}{p+1} + \frac{B}{p-1}$$

$$\implies A(p-1) + B(p+1) = p + 2 \implies (A+B)p + (-A+B) = p + 2$$

$$\implies A + B = 1 \text{ and } -A + B = 2 \implies A + A + 2 = 1 \implies A = -1/2 \implies B = 3/2$$

$$\frac{p+2}{(p+1)(p-1)} = \frac{-1/2}{p+1} + \frac{3/2}{p-1}$$

$$Y = \frac{1}{(p+1)^3} + \frac{-1/2}{p+1} + \frac{3/2}{p-1}$$

$$L^{-1}\left(\frac{1}{(p+1)^3}\right) = \frac{1}{2}L^{-1}\left(\frac{(2)!}{(p+(1))^{(2)+1}}\right) = \frac{1}{2}t^{(2)}e^{-(1)t} = \frac{1}{2}t^2e^{-t} \qquad \text{(By } L6\text{)}$$

$$L^{-1}\left(\frac{-1/2}{p+1}\right) = -\frac{1}{2}L^{-1}\left(\frac{1}{p+(1)}\right) = -\frac{1}{2}e^{-(1)t} = -\frac{1}{2}e^{-t} \qquad \text{(By } L2\text{)}$$

$$L^{-1}\left(\frac{3/2}{p-1}\right) = \frac{3}{2}L^{-1}\left(\frac{1}{p+(-1)}\right) = \frac{3}{2}e^{-(-1)t} = \frac{3}{2}e^{t} \qquad \text{(By } L2\text{)}$$

$$y = \frac{1}{2}t^2e^{-t} - \frac{1}{2}e^{-t} + \frac{3}{2}e^{t}$$

Section 10 Use the convolution integral to find the inverse transforms of:

3. $\frac{p}{p^2-1} = \frac{p}{p^2-1} \cdot \frac{1}{p^2-1}$

$$L^{-1}\left(\frac{p}{p^2-1}\right) = \cosh(t) \qquad \text{(By } L10\text{)}$$

$$L^{-1}\left(\frac{1}{p^2-1}\right) = \sinh(t) \qquad \text{(By } L9\text{)}$$

$$\begin{aligned}
L^{-1}\left(\frac{p}{p^2-1}\right) &= \int_0^t \cosh(t-\tau)\sinh(\tau)d\tau \\
&= \frac{1}{4}\int_0^t (e^{t-\tau}+e^{\tau-t})(e^{\tau}-e^{-\tau})d\tau \\
&= \frac{1}{4}\int_0^t e^{t-\tau}e^{\tau}+e^{\tau-t}e^{\tau}-e^{t-\tau}e^{-\tau}-e^{\tau-t}e^{-\tau}d\tau \\
&= \frac{1}{4}\int_0^t e^{t}+e^{2\tau-t}-e^{t-2\tau}-e^{-t}d\tau \\
&= \frac{1}{4}\int_0^t e^{t}-e^{-t}+e^{2\tau-t}-e^{t-2\tau}d\tau \\
&= \frac{1}{4}\left(\tau e^{t}-\tau e^{-t}+\frac{e^{2\tau-t}}{2}+\frac{e^{t-2\tau}}{2}\right)\Big|_0^t \\
&= \frac{1}{4}\left(\left(te^{t}-te^{-t}+\frac{e^{2t-t}}{2}+\frac{e^{t-2t}}{2}\right)-\left(0-0+\frac{e^{2(0)-t}}{2}+\frac{e^{t-2(0)}}{2}\right)\right) \\
&= \frac{1}{4}\left(te^{t}-te^{-t}\right) \\
&= \frac{t}{4}\left(e^{t}-e^{-t}\right) \\
&= \frac{1}{2}t\sinh(t)
\end{aligned}$$

4. $\frac{1}{(p+a)(p+b)^2}$

$$\frac{1}{(p+a)(p+b)^2} = \frac{1}{(p+a)} \cdot \frac{1}{(p+b)^2}$$

$$L^{-1}\left(\frac{1}{(p+a)}\right) = e^{-at} \qquad \text{(By } L2\text{)}$$

$$L^{-1}\left(\frac{1}{(p+b)^2}\right) = te^{-bt} \qquad \text{(By } L6\text{)}$$

$$\begin{aligned}
L^{-1}\left(\frac{1}{(p+a)(p+b)^2}\right) &= \int_0^t e^{-a(t-\tau)}\tau e^{-b\tau}d\tau \\
&= \int_0^t \tau e^{-a(t-\tau)-b\tau}d\tau \\
&= \int_0^t \tau e^{-at+(a-b)\tau}d\tau \\
&= e^{-at}\int_0^t \tau e^{(a-b)\tau}d\tau \\
&= e^{-at}\left(\left(\frac{\tau e^{(a-b)\tau}}{(a-b)}\right)\Bigg|_0^t - \frac{1}{(a-b)}\int_0^t e^{(a-b)\tau}d\tau\right) \\
&= e^{-at}\left(\frac{\tau e^{(a-b)\tau}}{(a-b)} - \frac{e^{(a-b)\tau}}{(a-b)^2}\right)\Bigg|_0^t \\
&= e^{-at}\left(\left(\frac{te^{(a-b)t}}{(a-b)} - \frac{e^{(a-b)t}}{(a-b)^2}\right) - \left(0 - \frac{e^{(a-b)0}}{(a-b)^2}\right)\right) \\
&= e^{-at}\left(\frac{(a-b)te^{(a-b)t} - e^{(a-b)t} + 1}{(a-b)^2}\right) \\
&= \frac{(a-b)te^{-bt} - e^{-bt} + e^{-at}}{(a-b)^2}
\end{aligned}$$

5. $\frac{p}{(p+a)(p+b)^2}$

$$\frac{1}{(p+a)(p+b)^2} = \frac{1}{(p+b)} \cdot \frac{p}{(p+a)(p+b)}$$

$$L^{-1}\left(\frac{1}{(p+b)}\right) = e^{-bt} \qquad \text{(By } L2\text{)}$$

$$L^{-1}\left(\frac{p}{(p+a)(p+b)}\right) = \frac{ae^{-at} - be^{-bt}}{a-b} \qquad \text{(By } L8\text{)}$$

$$\begin{aligned}
L^{-1}\left(\frac{p}{(p+a)(p+b)^2}\right) &= \int_0^t e^{-b(t-\tau)} \frac{ae^{-a\tau} - be^{-b\tau}}{a-b} d\tau \\
&= \frac{e^{-bt}}{a-b} \int_0^t e^{b\tau}(ae^{-a\tau} - be^{-b\tau}) d\tau \\
&= \frac{e^{-bt}}{a-b} \int_0^t ae^{(b-a)\tau} - be^{(b-b)\tau} d\tau \\
&= \frac{e^{-bt}}{a-b} \int_0^t ae^{(b-a)\tau} - b d\tau \\
&= \frac{e^{-bt}}{a-b} \left(\frac{ae^{(b-a)\tau}}{(b-a)} - b\tau\right)\Big|_0^t \\
&= \frac{e^{-bt}}{a-b} \left(\left(\frac{ae^{(b-a)t}}{(b-a)} - bt\right) - \left(\frac{ae^{(b-a)(0)}}{(b-a)} - b(0)\right)\right) \\
&= \frac{e^{-bt}}{a-b} \left(\frac{ae^{(b-a)t}}{(b-a)} - bt - \frac{a}{(b-a)}\right) \\
&= \frac{e^{-bt}}{a-b} \left(-\frac{ae^{(b-a)t}}{a-b} + \frac{(b-a)bt}{a-b} + \frac{a}{a-b}\right) \\
&= \frac{e^{-bt}}{(a-b)^2} \left(-ae^{(b-a)t} + (b-a)bt + a\right) \\
&= \frac{1}{(a-b)^2} \left(-ae^{-bt}e^{(b-a)t} + (b-a)bte^{-bt} + ae^{-bt}\right) \\
&= \frac{1}{(a-b)^2} \left(-ae^{-at} + (b-a)bte^{-bt} + ae^{-bt}\right) \\
&= \frac{-ae^{-at} + (b-a)bte^{-bt} + ae^{-bt}}{(a-b)^2}
\end{aligned}$$

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Section 10 Use the convolution integral to find the inverse transforms of:

9. $\frac{2}{p^3(p+2)}$

$$\frac{2}{p^3(p+2)} = \frac{2}{p^3} \cdot \frac{1}{p+2}$$

$$L^{-1}\left(\frac{2}{p^3}\right) = t^2 \qquad \text{(By } L5\text{)}$$

$$L^{-1}\left(\frac{1}{p+2}\right) = e^{-2t} \qquad \text{(By } L2\text{)}$$

$$\begin{aligned}
L^{-1}\left(\frac{2}{p^3(p+2)}\right) &= \int_0^t \tau^2 e^{-2(t-\tau)} d\tau \\
&= e^{-2t} \int_0^t \tau^2 e^{2\tau} d\tau \\
&= e^{-2t}\left(\frac{\tau^2 e^{2\tau}}{2}\Big|_0^t - \int_0^t \tau e^{2\tau} d\tau\right) \\
&= e^{-2t}\left(\left(\frac{\tau^2 e^{2\tau}}{2} - \frac{\tau e^{2\tau}}{2}\right)\Big|_0^t + \frac{1}{2}\int_0^t e^{2\tau} d\tau\right) \\
&= e^{-2t}\left(\frac{\tau^2 e^{2\tau}}{2} - \frac{\tau e^{2\tau}}{2} + \frac{e^{2\tau}}{4}\right)\Bigg|_0^t \\
&= e^{-2t}\left(\left(\frac{t^2 e^{2t}}{2} - \frac{t e^{2t}}{2} + \frac{e^{2t}}{4}\right) - \left(\frac{(0)^2 e^{2(0)}}{2} - \frac{(0) e^{2(0)}}{2} + \frac{e^{2(0)}}{4}\right)\right) \\
&= e^{-2t}\left(\frac{t^2 e^{2t}}{2} - \frac{t e^{2t}}{2} + \frac{e^{2t}}{4} - \frac{1}{4}\right) \\
&= \frac{t^2}{2} - \frac{t}{2} + \frac{1}{4} - \frac{e^{-2t}}{4}
\end{aligned}$$

10. $\frac{1}{p(p^2+a^2)^2}$

$$\frac{1}{p(p^2+a^2)^2} = \frac{1}{a^3} \cdot \frac{a^2}{p(p^2+a^2)} \cdot \frac{a}{p^2+a^2}$$

$$L^{-1}\left(\frac{a^2}{p(p^2+a^2)}\right) = 1 - \cos(at) \qquad \text{(By } L15\text{)}$$

$$L^{-1}\left(\frac{a}{p^2+a^2}\right) = \sin(at) \qquad \text{(By } L3\text{)}$$

$$\begin{aligned}
L^{-1}\left(\frac{1}{p(p^2+a^2)^2}\right) &= \frac{1}{a^3}\int_0^t \big(1-\cos(a(t-\tau))\big)\sin(a\tau)d\tau \\
&= \frac{1}{a^3}\int_0^t \sin(a\tau) - \sin(a\tau)\cos(at-a\tau)d\tau \\
&= \frac{1}{a^3}\left(\int_0^t \sin(a\tau)d\tau - \int_0^t \sin(a\tau)\cos(at-a\tau)d\tau\right)
\end{aligned}$$

$$\int_0^t \sin(a\tau)d\tau = -\frac{\cos(a\tau)}{a}\Big|_0^t = -\Big(\frac{\cos(at)}{a} - \frac{\cos(a(0))}{a}\Big) = \frac{1}{a} - \frac{\cos(at)}{a}$$

$$\begin{aligned}
\int_0^t \sin(a\tau)\cos(at-a\tau)d\tau &= \int_0^t \frac{1}{2}\left[\sin\Big(a\tau+(at-a\tau)\Big) + \sin\Big(a\tau-(at-a\tau)\Big)\right]d\tau \\
&= \frac{1}{2}\int_0^t \sin(at) + \sin(2a\tau - at)d\tau \\
&= \frac{1}{2}\Big(\tau\sin(at) - \frac{1}{2a}\cos(2a\tau - at)\Big)\Big|_0^t \\
&= \frac{1}{2}\tau\sin(at) - \frac{1}{4a}\cos(2a\tau - at)\Big|_0^t \\
&= \Big(\frac{1}{2}t\sin(at) - \frac{1}{4a}\cos(2at-at)\Big) - \Big(\frac{1}{2}(0)\sin(at) - \frac{1}{4a}\cos(2a(0)-at)\Big) \\
&= \frac{1}{2}t\sin(at) - \frac{1}{4a}\cos(at) - 0 + \frac{1}{4a}\cos(-at) \\
&= \frac{1}{2}t\sin(at) - \frac{1}{4a}\cos(at) + \frac{1}{4a}\cos(at) \\
&= \frac{1}{2}t\sin(at)
\end{aligned}$$

$$\begin{aligned}
L^{-1}\left(\frac{1}{p(p^2+a^2)^2}\right) &= \frac{1}{a^3}\left(\frac{1}{a} - \frac{\cos(at)}{a} - \frac{t\sin(at)}{2}\right) \\
&= \frac{1}{a^4} - \frac{\cos(at)}{a^4} - \frac{t\sin(at)}{2a^3}
\end{aligned}$$

11. $\frac{p}{(p^2+a^2)(p^2+b^2)}$

$$\frac{p}{(p^2+a^2)(p^2+b^2)} = \frac{1}{a} \cdot \frac{a}{p^2+a^2} \cdot \frac{p}{p^2+b^2}$$

$$L^{-1}\left(\frac{a}{p^2+a^2}\right) = \sin(at) \qquad \text{(By } L3\text{)}$$

$$L^{-1}\left(\frac{p}{p^2+b^2}\right) = \cos(bt) \qquad \text{(By } L4\text{)}$$

$$\begin{aligned}
L^{-1}\left(\frac{p}{(p^2+a^2)(p^2+b^2)}\right) &= \frac{1}{a}\int_0^t \sin(a\tau)\cos(b(t-\tau))d\tau \\
&= \frac{1}{a}\int_0^t \sin(a\tau)\cos(bt-b\tau))d\tau \\
&= \frac{1}{a}\int_0^t \frac{1}{2}\Bigg[\sin(a\tau+(bt-b\tau))+\sin(a\tau-(bt-b\tau))\Bigg]d\tau \\
&= \frac{1}{2a}\int_0^t \sin((a-b)\tau+bt)+\sin((a+b)\tau-bt)d\tau \\
&= \frac{1}{2a}\left(\frac{-\cos((a-b)\tau+bt)}{a-b}+\frac{-\cos((a+b)\tau-bt)}{a+b}\right)\Bigg|_0^t \\
&= \left(\frac{\cos((a-b)\tau+bt)}{2a(b-a)}-\frac{\cos((a+b)\tau-bt)}{2a(a+b)}\right)\Bigg|_0^t \\
&= \left(\frac{\cos((a-b)t+bt)}{2a(b-a)}-\frac{\cos((a+b)t-bt)}{2a(a+b)}\right) \\
&- \left(\frac{\cos((a-b)0+bt)}{2a(b-a)}-\frac{\cos((a+b)0-bt)}{2a(a+b)}\right) \\
&= \left(\frac{\cos(at)}{2a(b-a)}-\frac{\cos(at)}{2a(a+b)}\right)-\left(\frac{\cos(bt)}{2a(b-a)}-\frac{\cos(bt)}{2a(a+b)}\right) \\
&= \frac{((a+b)-(b-a))\cos(at)}{2a(b-a)(a+b)}-\left(\frac{\cos(bt)}{2a(b-a)}-\frac{\cos(bt)}{2a(a+b)}\right) \\
&= \frac{2a\cos(at)}{2a(b-a)(a+b)}-\left(\frac{\cos(bt)}{2a(b-a)}-\frac{\cos(bt)}{2a(a+b)}\right) \\
&= \frac{2a\cos(at)}{2a(b-a)(a+b)}-\frac{((a+b)-(b-a))\cos(bt)}{2a(b-a)(a+b)} \\
&= \frac{2a\cos(at)}{2a(b-a)(a+b)}-\frac{2a\cos(bt)}{2a(b-a)(a+b)} \\
&= \frac{\cos(at)}{(b-a)(a+b)}-\frac{\cos(bt)}{(b-a)(a+b)} \\
&= \frac{\cos(at)}{b^2-a^2}-\frac{\cos(bt)}{b^2-a^2}
\end{aligned}$$

14. $y'' + 5y' + 6y = e^{-2t}, y_0 = y_0' = 0$

$$L(y'') = p^2Y - py_0 - y_0';\ L(y') = pY - y_0;\ L(y) = Y;$$
$$L(y'' + 5y' + 6y) = (p^2Y - py_0 - y_0') + 5(pY - y_0) + 6(Y)$$
$$L(y'' + 5y' + 6y) = p^2Y + 5pY + 6Y = (p+2)(p+3)Y$$
$$(p+2)(p+3)Y = L(e^{2t}) \implies Y = \frac{1}{(p+2)(p+3)} \cdot L(e^{2t})$$

$$L^{-1}\left(\frac{1}{(p+2)(p+3)}\right) = \frac{e^{-(3)t} - e^{-(2)t}}{2-(3)} = e^{-2t} - e^{-3t} \qquad \text{(By } L7\text{)}$$

$$\begin{aligned}
y &= \int_0^t (e^{-2\tau} - e^{-3\tau})e^{-2(t-\tau)}d\tau \\
&= e^{-2t}\int_0^t (e^{-2\tau} - e^{-3\tau})e^{2\tau}d\tau \\
&= e^{-2t}\int_0^t 1 - e^{-\tau}d\tau \\
&= e^{-2t}(\tau + e^{-\tau})\Big|_0^t \\
&= e^{-2t}\Big((t + e^{-t}) - (0 + e^{-0}\Big) \\
&= e^{-2t}(t + e^{-t} - 1) \\
&= te^{-2t} + e^{-3t} - e^{-2t}
\end{aligned}$$

15. $y'' + 3y' - 4y = e^{3t}, y_0 = y_0' = 0$

$$L(y'') = p^2Y - py_0 - y_0';\ L(y') = pY - y_0;\ L(y) = Y;$$
$$L(y'' + 3y' - 4y) = (p^2Y - py_0 - y_0') + 3(pY - y_0) - 4(Y)$$
$$L(y'' + 3y' - 4y) = p^2Y + 3pY - 4Y = (p+4)(p-1)Y$$
$$(p+4)(p-1)Y = L(e^{3t}) \implies Y = \frac{1}{(p+4)(p-1)} \cdot L(e^{3t})$$

$$L^{-1}\left(\frac{1}{(p+4)(p-1)}\right) = \frac{e^{-(-1)t} - e^{-(4)t}}{4-(-1)} = \frac{e^t - e^{-4t}}{5} \qquad \text{(By } L7\text{)}$$

$$\begin{aligned}
y &= \int_0^t \frac{e^{\tau} - e^{-4\tau}}{5} e^{3(t-\tau)} d\tau \\
&= \frac{e^{3t}}{5} \int_0^t (e^{\tau} - e^{-4\tau}) e^{-3\tau} d\tau \\
&= \frac{e^{3t}}{5} \int_0^t e^{-2\tau} - e^{-7\tau} d\tau \\
&= \frac{e^{3t}}{5} \left(\frac{1}{7}e^{-7\tau} - \frac{1}{2}e^{-2\tau}\right)\Big|_0^t \\
&= \frac{e^{3t}}{5} \left(\left(\frac{1}{7}e^{-7t} - \frac{1}{2}e^{-2t}\right) - \left(\frac{1}{7}e^{-7(0)} - \frac{1}{2}e^{-2(0)}\right)\right) \\
&= \frac{e^{3t}}{5} \left(\frac{1}{7}e^{-7t} - \frac{1}{2}e^{-2t} + \frac{5}{14}\right) \\
&= \frac{1}{35}e^{-4t} - \frac{1}{10}e^{t} + \frac{1}{14}e^{3t}
\end{aligned}$$