Homework 08

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1 Monday 3/17

Section 1

1. $xy' = xy + y$

$$\begin{aligned}
y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_nx^n \\
xy + y &= a_0 + (a_0 + a_1)x + (a_1 + a_2)x^2 + (a_2 + a_3)x^3 + (a_{n-1} + a_n)x^n \\
y' &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + na_nx^{n-1} \\
xy' &= a_1x + 2a_2x^2 + 3a_3x^3 + na_nx^n \\
a_0 &= 0 \\
a_1 &= a_0 + a_1 = a_1 = a_1/0! \\
2a_2 &= a_1 + a_2 \implies a_2 = a_1 = a_1/1! \\
3a_3 &= a_2 + a_3 \implies a_3 = a_2/2 = a_1/2! \\
4a_4 &= a_3 + a_4 \implies a_4 = a_3/3 = a_1/3!
\end{aligned}$$

$$a_n = \begin{cases} 0 & n = 0 \\ \frac{a_1}{(n-1)!} & \text{otherwise} \end{cases}$$

$$y = \sum_{n=0}^{\infty} a_nx^n = \sum_{n=1}^{\infty} \frac{a_1x^n}{(n-1)!} = a_1x\sum_{n=0}^{\infty} \frac{x^n}{n!} = a_1xe^x$$

2. $y' = 3x^2y$

$$\begin{aligned}
y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_nx^n \\
3x^2y &= 3a_0x^2 + 3a_1x^3 + 3a_2x^4 + 3a_3x^5 + 3a_nx^{n+2} \\
y' &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + na_nx^{n-1} \\
a_0 &= a_0 = a_0/0! \\
a_1 &= 0 \\
2a_2 &= 0 \\
3a_3 &= 3a_0 \implies a_3 = a_0 = a_0/1! \\
4a_4 &= 3a_1 \implies a_4 = 0 \\
5a_5 &= 3a_2 \implies a_5 = 0 \\
6a_6 &= 3a_3 \implies a_6 = a_3/2 = a_0/2!
\end{aligned}$$

$$a_n = \begin{cases} \frac{a_0}{(n/3)!} & \text{if 3 divides } n \\ 0 & \text{otherwise} \end{cases}$$

$$y = \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} \frac{a_0x^{3n}}{n!} = a_0\sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = a_0e^{x^3}$$

3. $xy' = y$

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_n x^n \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + n a_n x^{n-1} \\
xy' &= a_1 x + 2a_2 x^2 + 3a_3 x^3 + n a_n x^n \\
a_0 &= 0 \\
a_1 &= a_1 \\
2a_2 &= a_2 \implies a_2 = 0 \\
3a_3 &= a_3 \implies a_3 = 0 \\
4a_4 &= a_4 \implies a_4 = 0
\end{aligned}$$

$$a_n = \begin{cases} a_1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_1 x$$

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Section 3

2. $\dfrac{d^{10}}{dx^{10}}(xe^x)$

$$\begin{aligned}
\frac{d^{10}}{dx^{10}}(xe^x) &= \sum_{k=0}^{10} \binom{10}{k} \frac{d^k}{dx^k}(x) \frac{d^{10-k}}{dx^{10-k}}(e^x) \\
&= \binom{10}{0} \frac{d^0}{dx^0}(x) \frac{d^{10-0}}{dx^{10-0}}(e^x) + \binom{10}{1} \frac{d^1}{dx^1}(x) \frac{d^{10-1}}{dx^{10-1}}(e^x) + \sum_{k=2}^{10} \binom{10}{k} \frac{d^k}{dx^k}(x) \frac{d^{10-k}}{dx^{10-k}}(e^x) \\
&= xe^x + 10e^x + 0 \\
&= xe^x + 10e^x
\end{aligned}$$

3. $\dfrac{d^6}{dx^6}(x^2 \sin x)$

$$\begin{aligned}
\frac{d^6}{dx^6}(x^2 \sin x) &= \sum_{k=0}^{6} \binom{6}{k} \frac{d^k}{dx^k}(x^2) \frac{d^{6-k}}{dx^{6-k}}(\sin x) \\
&= \binom{6}{0} \frac{d^0}{dx^0}(x^2) \frac{d^{6-0}}{dx^{6-0}}(\sin x) + \sum_{k=1}^{6} \binom{6}{k} \frac{d^k}{dx^k}(x^2) \frac{d^{6-k}}{dx^{6-k}}(\sin x) \\
&= -x^2 \sin x + \binom{6}{1} \frac{d^1}{dx^1}(x^2) \frac{d^{6-1}}{dx^{6-1}}(\sin x) + \sum_{k=2}^{6} \binom{6}{k} \frac{d^k}{dx^k}(x^2) \frac{d^{6-k}}{dx^{6-k}}(\sin x) \\
&= -x^2 \sin x + 12x \cos x + \binom{6}{2} \frac{d^2}{dx^2}(x^2) \frac{d^{6-2}}{dx^{6-2}}(\sin x) + \sum_{k=3}^{6} \binom{6}{k} \frac{d^k}{dx^k}(x^2) \frac{d^{6-k}}{dx^{6-k}}(\sin x) \\
&= -x^2 \sin x + 12x \cos x + 30 \sin x + 0 \\
&= -x^2 \sin x + 12x \cos x + 30 \sin x
\end{aligned}$$

4. $\dfrac{d^{25}}{dx^{25}}(x\cos x)$

$$
\begin{aligned}
\frac{d^{25}}{dx^{25}}(x\cos x) &= \sum_{k=0}^{25}\binom{25}{k}\frac{d^k}{dx^k}(x)\frac{d^{25-k}}{dx^{25-k}}(e^x) \\
&= \binom{25}{0}\frac{d^0}{dx^0}(x)\frac{d^{25-0}}{dx^{25-0}}(\cos x) + \binom{25}{1}\frac{d^1}{dx^1}(x)\frac{d^{25-1}}{dx^{25-1}}(\cos x) + \sum_{k=2}^{25}\binom{25}{k}\frac{d^k}{dx^k}(x)\frac{d^{25-k}}{dx^{25-k}}(e^x) \\
&= -x\sin x + 25\cos x + 0 \\
&= -x\sin x + 25\cos x
\end{aligned}
$$

5. $\dfrac{d^{100}}{dx^{100}}(x^2e^{-x})$

$$
\begin{aligned}
\frac{d^{100}}{dx^{100}}(x^2e^{-x}) &= \sum_{k=0}^{100}\binom{100}{k}\frac{d^k}{dx^k}(x^2)\frac{d^{100-k}}{dx^{100-k}}(e^{-x}) \\
&= \binom{100}{0}\frac{d^0}{dx^0}(x^2)\frac{d^{100-0}}{dx^{100-0}}(e^{-x}) + \sum_{k=1}^{100}\binom{100}{k}\frac{d^k}{dx^k}(x^2)\frac{d^{100-k}}{dx^{100-k}}(e^{-x}) \\
&= x^2e^{-x} + \binom{100}{1}\frac{d^1}{dx^1}(x^2)\frac{d^{100-1}}{dx^{100-1}}(e^{-x}) + \sum_{k=2}^{100}\binom{100}{k}\frac{d^k}{dx^k}(x^2)\frac{d^{100-k}}{dx^{100-k}}(e^{-x}) \\
&= x^2e^{-x} - 200xe^{-x} + \binom{100}{2}\frac{d^2}{dx^2}(x^2)\frac{d^{100-2}}{dx^{100-2}}(e^{-x}) + \sum_{k=3}^{100}\binom{100}{k}\frac{d^k}{dx^k}(x^2)\frac{d^{100-k}}{dx^{100-k}}(e^{-x}) \\
&= x^2e^{-x} - 200xe^{-x} + 9900e^{-x} + 0 \\
&= x^2e^{-x} - 200xe^{-x} + 9900e^{-x}
\end{aligned}
$$

Section 4

3. Find $P_0(x)$, $P_1(x)$, $P_2(x)$,$P_3(x)$, and $P_4(x)$ from Rodrigues' formula.

$$
\begin{aligned}
P_0(x) &= \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = \frac{1}{1}1 = 1 \\
P_1(x) &= \frac{1}{2^1 1!}\frac{d^1}{dx^1}(x^2-1)^1 = \frac{1}{2}2x = x \\
P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{3}{2}x^2 - \frac{1}{2} \\
P_3(x) &= \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1) = \frac{5}{2}x^3 - \frac{3}{2}x \\
P_4(x) &= \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4 = \frac{1}{384}\frac{d^4}{dx^4}(x^8-4x^6+6x^4-4x^2+1) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}
\end{aligned}
$$

Section 5

3. Use the recursion relation and the values of $P_0(x)$ and $P_1(x)$ to find $P_2(x)$, $P_3(x)$, $P_4(x)$, $P_5(x)$, and $P_6(x)$.

$$2P_2(x) = (2(2)-1)xP_1(x) - (2-1)P_0(x)$$
$$2P_2(x) = 3x(x) - 1(1)$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$3P_3(x) = (2(3)-1)xP_2(x) - (3-1)P_1(x)$$
$$3P_3(x) = 5x(\frac{3}{2}x^2 - \frac{1}{2}) - 2(x)$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$4P_4(x) = (2(4)-1)xP_3(x) - (4-1)P_2(x)$$
$$4P_4(x) = 7x(\frac{5}{2}x^3 - \frac{3}{2}x) - 3(\frac{3}{2}x^2 - \frac{1}{2})$$
$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

$$5P_5(x) = (2(5)-1)xP_4(x) - (5-1)P_3(x)$$
$$5P_5(x) = 9x(\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}) - 4(\frac{5}{2}x^3 - \frac{3}{2}x)$$
$$P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$$

$$6P_6(x) = (2(6)-1)xP_5(x) - (6-1)P_4(x)$$
$$6P_6(x) = 11x(\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x) - 5(\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8})$$
$$P_6(x) = \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$$

Section 6

2. Show that the functions $e^{in\pi x/l}$, $n = 0, \pm 1, \pm 2, \cdots$, are a set of orthogonal functions on $(-l, l)$.

$$\int_{-l}^{l} f^*(x)g(x)dx = 0 \implies f \perp g \text{ on } (-l,l) \text{ where } f(x) = e^{in\pi x/l} \text{ and } g(x) = e^{-in\pi x/l}$$

$$f^*(x) = [e^{in\pi x/l}]^* = \Big[\cos(n\pi x/l) + i\sin(n\pi x/l)\Big]^* = \cos(n\pi x/l) - i\sin(n\pi x/l) = e^{-in\pi x/l}$$

$$\begin{aligned}
\int_{-l}^{l} f^*(x)g(x)dx &= \int_{-l}^{l} e^{-in\pi x/l}e^{-in\pi x/l}dx \\
&= \int_{-l}^{l} e^{-2in\pi x/l}dx \\
&= \frac{l}{-2in}e^{-2in\pi x/l}\Big|_{-l}^{l} \\
&= \frac{l}{-2in}e^{2in\pi} + \frac{l}{2in}e^{-2in\pi} \\
&= \frac{l}{2in}\left(e^{-2in\pi} - e^{2in\pi}\right) \\
&= \frac{l}{2in}(\cos 2n\pi - \cos 2n\pi)^{1} \\
&= \frac{l}{2in}(0) \\
&= 0
\end{aligned}$$

Thus, we have shown that the functions are orthogonal on interval $(-l, l)$

3. Show that the functions x^2 and $\sin(x)$ are orthogonal on $(-1, 1)$.

$$\int_{-1}^{1} x^2 \sin(x) = 0 \implies x^2 \perp \sin(x) \text{ on } (-1,1).$$

Sign	$u = x^2$ (Derivative)	$dv = \sin x$ (Integral)
$+$	x^2	$-\cos x$
$-$	$2x$	$-\sin x$
$+$	2	$\cos x$
$-$	0	

$$\begin{aligned}
\int_{-1}^{1} x^2\sin(x) &= x^2\cos(x) + 2x\sin(x) + 2\cos(x)\Big|_{-1}^{1} \\
&= \left((1)^2\cos(1) + 2(1)\sin(1) + 2\cos(1)\right) - \left((-1)^2\cos(-1) + 2(-1)\sin(-1) + 2\cos(-1)\right) \\
&= \cos(1) + 2\sin(1) + 2\cos(1) - \cos(-1) + 2\sin(-1) - 2\cos(-1) \\
&= \cos(1) + 2\sin(1) + 2\cos(1) - \cos(1) - 2\sin(1) - 2\cos(1) \\
&= 0
\end{aligned}$$

Thus, we have shown that the $x^2 \perp \sin(x)$ on interval $(-1, 1)$

[1] Observe: $e^{2in\pi} = \cos(2n\pi) + i\sin(2n\pi) = \cos(2n\pi)$ and $e^{-2in\pi} = \cos(-2n\pi) + i\sin(-2n\pi) = \cos(2n\pi)$ when n is an integer.