Homework 05

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1 Monday 2/10

Section 6

25. $(D-3)(D+1)y = 16x^2e^{-x}$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-3)(D+1) = 0$$
$$\implies \text{root(s): } -1, 3$$
$$\text{General Solution: } y_c = Ae^{-x} + Be^{3x}$$

Now Find particular Solution:
Because RHS is $16x^2e^{-x}$ and one of the roots are the same, we know solution will be in the form of $Q_2(x)xe^{-x}$ where $Q_2(x) = a_2x^2 + a_1x + a_0$

$$y_p = (a_2x^2 + a_1x + a_0)xe^{-x}$$
$$y_p = (a_2x^3 + a_1x^2 + a_0x)e^{-x}$$

$$y_p' = (3a_2x^2 + 2a_1x + a_0)e^{-x} - (a_2x^3 + a_1x^2 + a_0x)e^{-x}$$
$$y_p' = (-a_2x^3 + (3a_2 - a_1)x^2 + (2a_1 - a_0)x + a_0)e^{-x}$$

$$y_p'' = (-3a_2x^2 + 2(3a_2 - a_1)x + (2a_1 - a_0))e^{-x}$$
$$-(-a_2x^3 + (3a_2 - a_1)x^2 + (2a_1 - a_0)x + a_0)e^{-x}$$
$$y_p'' = (a_2x^3 + (-6a_2 + a_1)x^2 + (6a_2 - 4a_1 + a_0)x + (2a_1 - 2a_0))e^{-x}$$

$$y_p'' - 2y_p' - 3y_p = \Big((\cancel{a_2x^3} + (-6a_2 + \cancel{a_1})x^2 + (6a_2 - 4a_1 + \cancel{a_0})x + (2a_1 - 2a_0))$$
$$-2(\cancel{-a_2x^3} + (3a_2 - \cancel{a_1})x^2 + (2a_1 - \cancel{a_0})x + a_0) - 3(\cancel{a_2x^3} + \cancel{a_1x^2} + \cancel{a_0x})\Big)e^{-x}$$
$$= (-12a_2x^2 + (6a_2 - 8a_1)x + 2a_1 - 4a_0)e^{-x} = 16x^2e^{-x}$$
$$-12a_2x^2 + (6a_2 - 8a_1)x + 2a_1 - 4a_0 = 16x^2$$
$$-12a_2x^2 = 16x^2 \implies a_2 = -4/3$$
$$(6(-4/3) - 8a_1)x + 2a_1 - 4a_0 = 0$$
$$(6(-4/3) - 8a_1)x = 0 \implies a_1 = -1$$
$$2(-1) - 4a_0 = 0$$
$$2(-1) - 4a_0 = 0 \implies a_0 = -1/2$$

Finally we know $a_2 = -4/3$, $a_1 = -1$, and $a_0 = -1/2$ so

$$y_p = ((-4/3)x^3 + (-1)x^2 + (-1/2)x)e^{-x}$$
$$y_p = (-\frac{4}{3}x^3 - x^2 - \frac{1}{2}x)e^{-x}$$

Combine:

$$y = y_c + y_p$$
$$y = (-\frac{4}{3}x^3 - x^2 - \frac{1}{2}x + A)e^{-x} + Be^{3x}$$

26. $(D^2+1)y = 8x\sin x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-i)(D+i) = 0 \implies \text{root(s): } \pm i$$
$$\text{General Solution: } y_c = A\cos(x) + B\sin(x)$$

Now Find particular Solution:
Because RHS has sin we will use $Y = Ce^{ix} = Y_R + iY_I$ s.t. $y_p = Y_I$. However since one of the roots is the same, we actually will use $Y = Cxe^{ix} = Y_R + iY_I$ s.t. $y_p = Y_I$.

$$Y = (Ax+B)xe^{ix} = (Ax^2+Bx)e^{ix}$$

$$Y' = (2Ax+B)e^{ix} + i(Ax^2+Bx)e^{ix} = ((2Ax+B) + i(Ax^2+Bx))e^{ix}$$

$$Y'' = (2Aix + 2A + Bi)e^{ix} + i(Aix^2 + 2Ax + Bix + B)e^{ix}$$
$$Y'' = (2Aix + 2A + Bi + Ai^2x^2 + 2Aix + Bi^2x + Bi)e^{ix}$$
$$Y'' = (-Ax^2 + (4Ai - B)x + 2(A+Bi))e^{ix}$$

$$Y'' + Y = (\cancel{-Ax^2} + (4A - \cancel{B})x + 2(A+Bi) + \cancel{Ax^2} + \cancel{Bx})e^{ix}$$
$$Y'' + Y = (4Ax + 2(A+Bi))e^{ix} = 8xe^{ix}$$

$$(4Aix + 2(A+Bi)) = 8x \implies 4Aix = 8x \implies A = -2i$$
$$2(A+Bi) = 2(-2i - Bi) = 0 \implies B = 2$$

$$Y = (-2ix^2 + 2x)e^{ix}$$
$$Y = (-2ix^2 + 2x)(\cos(x) + i\sin(x))$$
$$Y = -2ix^2\cos(x) + 2x\cos(x) - 2i^2x^2\sin(x) + 2xi\sin(x)$$
$$Y = 2x\cos(x) + 2x^2\sin(x) + i(-2x^2\cos(x) + 2x\sin(x))$$

$$y_p = Y_I = -2x^2\cos(x) + 2x\sin(x)$$

Combine:

$$y = y_c + y_p$$
$$y = A\cos(x) + B\sin(x) - 2x^2\cos(x) + 2x\sin(x)$$

33. $y'' + y = [x^3 - 1] + [2\cos x] + [(2-4x)e^x]$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D^2+1) = (D+i)(D-i) = 0 \implies \text{root(s): } \pm i$$
$$\text{General Solution: } y_c = A\cos(x) + B\sin(x)$$

Solve $y''_{p_1} + y_{p_1} = x^3 - 1$. Solution will be in form $Q_3(x)e^0$ where $Q_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$y_{p_1} = a_3x^3 + a_2x^2 + a_1x + a_0;\ y'_{p_1} = 3a_3x^2 + 2a_2x + a_1;\ y''_{p_1} = 6a_3x + 2a_2$$
$$y''_{p_1} + y_{p_1} = 6a_3x + 2a_2 + a_3x^3 + a_2x^2 + a_1x + a_0$$
$$y''_{p_1} + y_{p_1} = a_3x^3 + a_2x^2 + (6a_3 + a_1)x + (2a_2 + a_0) = x^3 - 1 \implies a_3x^3 = x^3 \implies a_3 = 1$$
$$a_2x^2 + (6(1) + a_1)x + (2a_2 + a_0) = -1 \implies a_2x^2 = 0 \implies a_2 = 0$$
$$(6(1) + a_1)x + (2(0) + a_0) = -1 \implies (6(1) + a_1)x = 0 \implies a_1 = -6$$
$$(2(0) + a_0) = -1 \implies a_0 = -1$$
$$y_{p_1} = x^3 - 6x - 1$$

Solve $y''_{p_2} + y_{p_2} = 2\cos x$. Use $Y = Cxe^{ix} = Y_R + iY_I$ s.t. $y_{p_2} = Y_R$.

$$Y = Cxe^{ix};\ Y' = Ce^{ix} + iCxe^{ix};\ Y'' = 2iCe^{ix} - Cxe^{ix}$$
$$Y'' + Y = 2iCe^{ix} = 2e^{ix} \implies C = -i$$
$$Y = -ix(\cos x + i\sin x) = x\sin x + i(-x\cos x)$$
$$y_{p_2} = Y_R = x\sin x$$

Solve $y''_{p_3} + y_{p_3} = (2-4x)e^x$. Solution will be in the form of $y_{p_3} = Q_1(x)e^x$ where $Q_1(x) = a_1x + a_0$

$$y_{p_3} = (a_1x + a_0)e^x;\ y'_{p_3} = a_1e^x + (a_1x + a_0)e^x;\ y''_{p_3} = 2a_1e^x + (a_1x + a_0)e^x$$
$$y''_{p_3} + y_{p_3} = 2a_1e^x + (a_1x + a_0)e^x + (a_1x + a_0)e^x$$
$$= 2(a_1x + a_1 + a_0)e^x = (2-4x)e^x$$
$$2(a_1x + a_1 + a_0) = (2-4x) \implies 2a_1x = -4x \implies a_1 = -2$$
$$2((-2) + a_0) = 2 \implies a_0 = 3$$
$$y_{p_3} = (-2x + 3)e^x$$

Combine:

$$y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$$
$$y = A\cos(x) + B\sin(x) + x^3 - 6x - 1 + x\sin x + (-2x + 3)e^x$$

34. $y'' - 5y' + 6y = 2e^x + 6x - 5$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D^2 - 5D + 6) = (D-3)(D-2) = 0 \implies \text{root(s): } 2, 3$$
$$\text{General Solution: } y_c = Ae^{2x} + Be^{3x}$$

Solve $y''_{p_1} - 5y'_{p_1} + 6y_{p_1} = 2e^x$. Solution will be in the form Ce^x.

$$y_{p_1} = Ce^x;\ y'_{p_1} = Ce^x;\ y''_{p_1} = Ce^x$$
$$y''_{p_1} - 5y'_{p_1} + 6y_{p_1} = Ce^x - 5(Ce^x) + 6(Ce^x) = 2Ce^x = 2e^x \implies C = 1$$
$$y_{p_1} = e^x$$

Solve $y''_{p_2} - 5y'_{p_2} + 6y_{p_2} = 6x - 5$. Solution will be in the form $Q_1(x)e^0$ s.t. $Q_1(x) = a_1x + a_0$

$$y_{p_2} = a_1x + a_0; y'_{p_2} = a_1; y''_{p_2} = 0$$
$$y''_{p_2} - 5y'_{p_2} + 6y_{p_2} = -5(a_1) + 6(a_1x + a_0) = 6x - 5 \implies 6a_1x = 6x \implies a_1 = 1$$
$$-5(1) + 6a_0 = -5 \implies a_0 = 0$$
$$y_{p_2} = x$$

Combine:

$$y = y_c + y_{p_1} + y_{p_2}$$
$$y = Ae^{2x} + Be^{3x} + e^x + x$$

36. $(D^2+1)y = 2\sin x + 4x\cos x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-i)(D+i) = 0 \implies \text{root(s): } \pm i$$
$$\text{General Solution: } y_c = A\cos x + B\sin x$$

Solve $y''_{p_1} + y_{p_1} = 2\sin x$. Use $Y = Cxe^{ix}$ where $Y = Y_R + iY_I$ and $y_{p_1} = Y_I$.

$$Y = Cxe^{ix};\ Y' = Ce^{ix} + iCxe^{ix};\ Y'' = 2iCe^{ix} - Cxe^{ix}$$
$$Y'' + Y = 2iCe^{ix} = 2e^{ix} \implies C = -i$$
$$Y = -ix(\cos x + i\sin x) = x\sin x + i(-x\cos x)$$
$$y_{p_2} = Y_I = -x\cos x$$

Solve $y''_{p_2} + y_{p_2} = 4x\cos x$. Use $Y = (Ax+B)xe^{ix}$ where $Y = Y_R + iY_I$ and $y_{p_2} = Y_R$.

$$Y = (Ax+B)xe^{ix} = (Ax^2+Bx)e^{ix}$$

$$Y' = (2Ax+B)e^{ix} + i(Ax^2+Bx)e^{ix} = ((2Ax+B) + i(Ax^2+Bx))e^{ix}$$

$$Y'' = (2Aix + 2A + Bi)e^{ix} + i(Aix^2 + 2Ax + Bix + B)e^{ix}$$
$$Y'' = (2Aix + 2A + Bi + Ai^2x^2 + 2Aix + Bi^2x + Bi)e^{ix}$$
$$Y'' = (-Ax^2 + (4Ai - B)x + 2(A+Bi))e^{ix}$$

$$Y'' + Y = (-\cancel{Ax^2} + (4A - \cancel{B})x + 2(A+Bi) + \cancel{Ax^2} + \cancel{Bx})e^{ix}$$
$$Y'' + Y = (4Ax + 2(A+Bi))e^{ix} = 4xe^{ix}$$

$$(4Aix + 2(A+Bi)) = 4x \implies 4Aix = 4x \implies A = -i$$
$$2(A+Bi) = 2(-i - Bi) = 0 \implies B = 1$$

$$Y = (-ix^2 + x)e^{ix}$$
$$Y = (-ix^2 + x)(\cos(x) + i\sin(x))$$
$$Y = -ix^2\cos(x) + x\cos(x) - i^2x^2\sin(x) + xi\sin(x)$$
$$Y = x\cos(x) + x^2\sin(x) + i(-x^2\cos(x) + x\sin(x))$$

$$y_p = Y_R = x\cos(x) + x^2\sin(x)$$

Combine:

$$y = y_c + y_{p_1} + y_{p_2}$$
$$y = A\cos x + B\sin x - x\cos x + x\cos(x) + x^2\sin(x)$$
$$y = A\cos x + B\sin x + x^2\sin(x)$$

37. $(D-1)^2y = 4e^x + (1-x)(e^2x - 1)$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-1)^2 = 0 \implies \text{root(s): } 1$$
$$\text{General Solution: } y_c = (Ax+B)e^x$$

Solve $y''_{p_1} - 2y'_{p_1} + y_{p_1} = 4e^x$. Because exponent on RHS is the same as both roots of the auxiliary equation, particular solution in the form of Cx^2e^x

$$y_{p_1} = Cx^2e^x;\ y'_{p_1} = C(2xe^x + x^2e^x);\ y''_{p_1} = C(2e^x + 4xe^x + x^2e^x)$$
$$y''_{p_1} - 2y'_{p_1} + y_{p_1} = C\big((2e^x + \cancel{4xe^x} + \cancel{x^2e^x}) - \cancel{2(2xe^x + x^2e^x)} + \cancel{(x^2e^x)}\big)$$
$$= 2Ce^x = 4e^x \text{ so } C = 2$$
$$y_{p_1} = 2x^2e^x$$

Solve $y''_{p_2} - 2y'_{p_2} + y_{p_2} = (1-x)e^2x$. Solution will be in the form $Q_1(x)e^{2x}$ where $Q_1(x) = a_1x + a_0$

$$y_{p_2} = (a_1x + a_0)e^{2x};\ y'_{p_2} = (a_1 + 2a_1x + 2a_0)e^{2x};\ y''_{p_2} = 4(a_1x + a_1 + a_0)e^{2x}$$
$$y''_{p_2} - 2y'_{p_2} + y_{p_2} = 4(\cancel{a_1x} + a_1 + \cancel{a_0})e^{2x} - 2((\cancel{2a_1x} + a_1 + \cancel{2a_0})e^{2x}) + (a_1x + a_0)e^{2x}$$
$$y''_{p_2} - 2y'_{p_2} + y_{p_2} = (a_1x + 2a_1 + a_0)e^{2x} = (1-x)e^{2x}$$
$$(a_1x + 2a_1 + a_0) = (1-x) \implies a_1x = -x \implies a_1 = -1$$
$$(2(-1) + a_0) = 1 \implies a_0 = 3$$
$$y_{p_2} = (3-x)e^{2x}$$

Solve $y''_{p_3} - 2y'_{p_3} + y_{p_3} = x - 1$. Solution will be in the form $Q_1(x)e^0$ where $Q_1(x) = a_1x + a_0$

$$y_{p_3} = a_1x + a_0;\ y'_{p_3} = a_1;\ y_{p_3} = 0$$
$$y''_{p_3} - 2y'_{p_3} + y_{p_3} = -2a_1 + a_1x + a_0 = x - 1$$
$$a_1x - 2a_1 + a_0 = x - 1 \implies a_1x = x \implies a_1 = 1$$
$$-2(1) + a_0 = -1 \implies a_0 = 1$$
$$y_{p_3} = x + 1$$

Combine:

$$y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$$
$$y = (Ax+B)e^x + 2x^2e^x + (3-x)e^{2x} + x + 1$$

38. $y'' - 2y' = 9xe^{-x} - 6x^2 + 4e^{2x}$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-2)D = 0 \implies \text{root(s): } 0, 2$$
$$\text{General Solution: } y_c = A + Be^{2x}$$

Solve $y''_{p_1} - 2y'_{p_1} = 9xe^{-x}$. Solution will be in the form $Q_1(x)e^{-x}$ where $Q_1(x) = a_1x + a_0$

$$y_{p_1} = (a_1x + a_0)e^{-x};\ y'_{p_1} = (-a_1x + a_1 - a_0)e^{-x};\ y''_{p_1} = (a_1x - 2a_1 + a_0)e^{-x}$$
$$y''_{p_1} - 2y'_{p_1} = (a_1x - 2a_1 + a_0)e^{-x} - 2(-a_1x + a_1 - a_0)e^{-x}$$
$$= (3a_1x - 4a_1 + 3a_0)e^{-x} = 9xe^{-x}$$
$$3a_1x - 4a_1 + 3a_0 = 9x \implies 3a_1x = 9x \implies a_1 = 3$$
$$-4(3) + 3a_0 = 0 \implies a_0 = 4$$
$$y_{p_1} = (3x + 4)e^{-x}$$

Solve $y''_{p_2} - 2y'_{p_2} = -6x^2$. Solution will be in the form of $Q_2(x)xe^{0x}$ where $Q_2(x) = a_2x^2 + a_1x + a_0$

$$y_{p_2} = a_2x^3 + a_1x^2 + a_0x;\ y'_{p_2} = 3a_2x^2 + 2a_1x + a_0;\ y''_{p_2} = 6a_2x + 2a_1$$
$$y''_{p_2} - 2y'_{p_2} = 6a_2x + 2a_1 - 2(3a_2x^2 + 2a_1x + a_0)$$
$$= -6a_2x^2 + 6a_2x - 4a_1x + 2a_1 - 2a_0 = -6x^2 \implies -6a_2x^2 = -6x^2 \implies a_2 = 1$$
$$6(1)x - 4a_1x + 2a_1 - 2a_0 = 0 \implies 6(1)x - 4a_1x = 0 \implies a_1 = 3/2$$
$$2(3/2) - 2a_0 = 0 \implies a_0 = 3/2$$
$$y_{p_2} = x^3 + \frac{3}{2}x^2 + \frac{3}{2}x$$

Solve $y''_{p_3} - 2y'_{p_3} = 4e^{2x}$. Solution will be in the form Cxe^{2x}.

$$y_{p_3} = Cxe^{2x};\ y'_{p_3} = Ce^{2x} + C2xe^{2x};\ y_{p_3} = C4e^{2x} + C4xe^{2x}$$
$$y''_{p_3} - 2y'_{p_3} + y_{p_3} = C4e^{2x} + C4xe^{2x} - 2(Ce^{2x} + C2xe^{2x})$$
$$C2e^{2x} = 4e^{2x} \implies C = 2$$
$$y_{p_3} = 2xe^{2x}$$

Combine:

$$y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$$
$$y = A + Be^{2x} + (3x + 4)e^{-x} + x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + 2xe^{2x}$$

2 Wednesday 2/12

Section 8

2. By using $L2$, verify $L7$ and $L8$ in the Laplace transform table.

   $L7$

$$\begin{aligned}
L\left(\frac{e^{-at}-e^{-bt}}{b-a}\right) &= \frac{L(e^{-at})-L(e^{-bt})}{b-a} && \text{B/c } L \text{ is a linear transformation}\\
&= \frac{\frac{1}{p+a}-\frac{1}{p+b}}{b-a} && \text{By } L2\\
&= \frac{p+b-p-a}{(p+a)(p+b)(b-a)} && \text{Algebra}\\
&= \frac{1}{(p+a)(p+b)} && \text{Algebra}
\end{aligned}$$

   $L7$

$$\begin{aligned}
L\left(\frac{ae^{-at}-be^{-bt}}{a-b}\right) &= \frac{aL(e^{-at})-bL(e^{-bt})}{a-b} && \text{B/c } L \text{ is a linear transformation}\\
&= \frac{\frac{a}{p+a}-\frac{b}{p+b}}{a-b} && \text{By } L2\\
&= \frac{ap+ab-bp-ba}{(p+a)(p+b)(a-b)} && \text{Algebra}\\
&= \frac{p}{(p+a)(p+b)} && \text{Algebra}
\end{aligned}$$

3. Using either $L2$, or $L3$ and $L4$, verify $L9$ and $L10$.

   $L9$

$$\begin{aligned}
L\big(\sinh(at)\big) &= L\left(\frac{e^{at}-e^{-at}}{2}\right) && \text{By definition of } \sinh\\
&= \frac{L(e^{-(-a)t})-L(e^{-at})}{2} && \text{B/c } L \text{ is a linear transformation}\\
&= \frac{\frac{1}{p-a}-\frac{1}{p+a}}{2} && \text{By } L2\\
&= \frac{p+a-p+a}{2(p-a)(p+a)} && \text{algebra}\\
&= \frac{a}{p^2-a^2} && \text{algebra}
\end{aligned}$$

$L10$

$$\begin{aligned}
L\big(\cosh(at)\big) &= L\left(\frac{e^{at}+e^{-at}}{2}\right) && \text{By definition of } \cosh \\
&= \frac{L(e^{-(-a)t})+L(e^{-at})}{2} && \text{B/c } L \text{ is a linear transformation} \\
&= \frac{\frac{1}{p-a}+\frac{1}{p+a}}{2} && \text{By } L2 \\
&= \frac{p+a+p-a}{2(p-a)(p+a)} && \text{algebra} \\
&= \frac{p}{p^2-a^2} && \text{algebra}
\end{aligned}$$

4. By differentiating the appropriate formula with respect to a, verify $L12$.

$$\begin{aligned}
L\big(t\cos(at)\big) &= L\Big(\frac{\partial}{\partial a}\sin(at)\Big) && \text{By partial derivative} \\
&= \frac{\partial}{\partial a}L\big(\sin(at)\big) && \text{B/c } L \text{ is a linear transformation} \\
&= \frac{\partial}{\partial a}\frac{a}{p^2+a^2} && \text{By } L3 \\
&= \frac{p^2+a^2-2a^2}{(p^2+a^2)^2} && \text{By Quotient Rule} \\
&= \frac{p^2-a^2}{(p^2+a^2)^2} && \text{Algebra}
\end{aligned}$$

5. By integrating the appropriate formula with respect to a, verify $L19$.

$$\begin{aligned}
L\Big(\frac{\sin(at)}{t}\Big) &= L\Big(\int_0^a \cos(bt)db\Big) && \text{By Integral} \\
&= \int_0^a L(\cos(bt))db && \text{B/c } L \text{ is a linear transformation} \\
&= \int_0^a \frac{p}{p^2+b^2}db && \text{By } L4 \\
&= p\frac{1}{p}\arctan(\frac{a}{p}) && \text{By Integration} \\
&= \arctan(\frac{a}{p}) && \text{Algebra}
\end{aligned}$$

Section 9

2. $y' - y = 2e^t$, $y_0 = 3$

$$L(y') = pY - y_0;\ L(y) = Y;\ L(2e^t) = \frac{2}{p-1}$$

$$y' - y = pY - y_0 - Y = pY - 3 - Y = (p-1)Y - 3 = \frac{2}{p-1}$$

$$Y = \frac{2}{(p-1)^2} + \frac{3}{p-1}$$

$$L^{-1}\left(\frac{2}{(p-1)^2}\right) = 2L^{-1}\left(\frac{(1)!}{(p+(-1))^{(1)+1}}\right) = 2t^{(1)}e^{-(-1)t} = 2te^t \qquad \text{(By } L6\text{)}$$

$$L^{-1}\left(\frac{3}{p-1}\right) = 3L^{-1}\left(\frac{1}{p+(-1)}\right) = 3e^{-(-1)t} = 3e^t \qquad \text{(By } L2\text{)}$$

$$y = L^{-1}(Y) = L^{-1}\left(\frac{2}{(p-1)^2} + \frac{3}{p-1}\right) = 2te^t + 3e^t$$

3. $y'' + 4y' + 4y = e^{-2t}$, $y_0 = 0$, $y_0' = 4$

$$L(y'') = p^2Y - py_0 - y_0';\ L(y') = pY - y_0;\ L(y) = Y;\ L(e^{-2t}) = \frac{1}{p+2}$$

$$\begin{aligned} y'' + 4y' + 4y &= p^2Y - py_0 - y_0' + 4(pY - y_0) + 4Y \\ &= p^2Y - p(0) - (4) + 4(pY - (0)) + 4Y \\ &= p^2Y + 4pY + 4Y - 4 \\ &= (p+2)^2Y - 4 \end{aligned}$$

$$(p+2)^2Y - 4 = \frac{1}{p+2}$$

$$Y = \frac{1}{(p+2)^3} + \frac{4}{(p+2)^2}$$

$$L^{-1}\left(\frac{1}{(p+2)^3}\right) = \frac{1}{2}L^{-1}\left(\frac{(2)!}{(p+(2))^{(2)+1}}\right) = \frac{1}{2}t^{(2)}e^{-(2)t} = \frac{1}{2}t^2e^{-2t} \qquad \text{(By } L6\text{)}$$

$$L^{-1}\left(\frac{4}{(p+2)^2}\right) = 4L^{-1}\left(\frac{(1)!}{(p+(2))^{(1)+1}}\right) = 4t^{(1)}e^{-(2)t} = 4te^{-2t} \qquad \text{(By } L6\text{)}$$

$$y = L^{-1}(Y) = L^{-1}\left(\frac{1}{(p+2)^3} + \frac{4}{(p+2)^2}\right) = \frac{1}{2}t^2e^{-2t} + 4te^{-2t}$$

4. $y'' + y = \sin t$, $y_0 = 1$, $y_0' = 0$

$$L(y'') = p^2Y - py_0 - y_0';\; L(y') = pY - y_0;\; L(y) = Y;\; L(\sin t) = \frac{1}{p^2+1}$$

$$\begin{aligned} y'' + y &= p^2Y - py_0 - y_0' + Y \\ &= p^2Y - p(1) - (0) + Y \\ &= p^2Y - p + Y \\ &= (p^2+1)Y - p \end{aligned}$$

$$\begin{aligned} (p^2+1)Y - p &= \frac{1}{p^2+1} \\ Y &= \frac{1}{(p^2+1)^2} + \frac{p}{p^2+1} \end{aligned}$$

$$\begin{aligned} L^{-1}\left(\frac{1}{(p^2+1)^2}\right) &= \frac{1}{2}L^{-1}\left(\frac{2(1)^3}{(p^2+1^2)^2}\right) = \frac{1}{2}(\sin((1)t) - (1)t\cos((1)t)) \\ &= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) && \text{(By } L17\text{)} \\ L^{-1}\left(\frac{p}{p^2+1}\right) &= L^{-1}\left(\frac{p}{p^2+1^2}\right) = \cos((1)t) = \cos(t) && \text{(By } L4\text{)} \end{aligned}$$

$$y = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) + \cos(t)$$

5. $y'' + y = \sin t$, $y_0 = 0$, $y_0' = -\frac{1}{2}$

$$L(y'') = p^2Y - py_0 - y_0';\; L(y') = pY - y_0;\; L(y) = Y;\; L(\sin t) = \frac{1}{p^2+1}$$

$$\begin{aligned} y'' + y &= p^2Y - py_0 - y_0' + Y \\ &= p^2Y - p(0) - (-\frac{1}{2}) + Y \\ &= p^2Y + \frac{1}{2} + Y \\ &= (p^2+1)Y + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} (p^2+1)Y + \frac{1}{2} &= \frac{1}{p^2+1} \\ Y &= \frac{1}{(p^2+1)^2} - \frac{1/2}{p^2+1} \end{aligned}$$

$$\begin{aligned} L^{-1}\left(\frac{1}{(p^2+1)^2}\right) &= \frac{1}{2}L^{-1}\left(\frac{2(1)^3}{(p^2+1^2)^2}\right) = \frac{1}{2}(\sin((1)t) - (1)t\cos((1)t)) \\ &= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) && \text{(By } L17\text{)} \\ L^{-1}\left(\frac{-1/2}{p^2+1}\right) &= -\frac{1}{2}L^{-1}\left(\frac{1}{p^2+1^2}\right) = -\frac{1}{2}\sin((1)t) = -\frac{1}{2}\sin(t) && \text{(By } L3\text{)} \end{aligned}$$

$$y = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) - \frac{1}{2}\sin(t)$$

$$y = -\frac{1}{2}t\cos(t)$$

6. $y'' - 6y' + 9y = te^{3t}$, $y_0 = 0$, $y_0' = 5$

$$L(y'') = p^2Y - py_0 - y_0';\; L(y') = pY - y_0;\; L(y) = Y;\; L(te^{3t}) = \frac{1}{(p-3)^2}$$

$$\begin{aligned}
y'' - 6y' + 9y &= p^2Y - py_0 - y_0' - 6(pY - y_0) + 9(Y) \\
&= p^2Y - p(0) - (5) - 6(pY - (0)) + 9Y \\
&= p^2Y - 5 - 6pY + 9Y \\
&= p^2Y - 6pY + 9Y - 5 \\
&= (p-3)^2Y - 5
\end{aligned}$$

$$\begin{aligned}
(p-3)^2Y - 5 &= \frac{1}{(p-3)^2} \\
Y &= \frac{1}{(p-3)^4} + \frac{5}{(p-3)^2}
\end{aligned}$$

$$L^{-1}\left(\frac{1}{(p-3)^4}\right) = \frac{1}{6}L^{-1}\left(\frac{(3)!}{(p+(-3))^{(3)+1}}\right) = \frac{1}{6}t^{(3)}e^{-(-3)t} = \frac{1}{6}t^3e^{3t} \qquad \text{(By } L6\text{)}$$

$$L^{-1}\left(\frac{5}{(p-3)^3}\right) = 5L^{-1}\left(\frac{(1)!}{(p+(-3))^{(1)+1}}\right) = 5t^{(1)}e^{-(-3)t} = 5te^{3t} \qquad \text{(By } L6\text{)}$$

$$y = \frac{1}{6}t^3e^{3t} + 5te^{3t}$$