Problem Set 6

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1. (a) Show that $(c \neq 0 \land ac \mid bc) \implies (a \mid b)$ for all $a, b, c \in \mathbb{Z}$.

   Proof. Let $a, b, c \in \mathbb{Z}$ and assume $c \neq 0 \land ac \mid bc$. From $ac \mid bc$, we know there is a k such that $(k \in \mathbb{Z})$ $(ack = bc)$ by definition of divides. Using commutativity, we get $akc = bc$. Since we know $c \neq 0 \land akc = bc$, by multiplicative cancellation, we can conclude $ak = b$ which implies $a \mid b$.

   Q.E.D.

   (b) Show that $(n \mid x \land n \mid y) \implies (n \mid ax + by)$ for all $n, x, y, a, b \in \mathbb{Z}$.

   Proof. Let $a, b, n, x, y \in \mathbb{Z}$ and assume $n \mid x \land n \mid y$. By definition of division, we know there is a g such that $(g \in \mathbb{Z})$ $(ng = x)$ and some value h for which $nh = y$. Now let $k = ag + bh$. Thus, we know that $n \cdot k = n \cdot (ag + bh)$. By the definition of division (and distribution and commutativity), we know that $n|ang + bnh$. Finally, by substitution, we know that $n|ax + by$.

   Q.E.D.

2. For all $z \in \mathbb{Z}$, show that z is even implies z is not odd.

   Proof. Let $z \in \mathbb{Z}$ and assume that z is even. By definition of even $2 \mid z$ and by definition of divides there is a k such that $(k \in \mathbb{Z})(2k = z)$. Towards a contradiction, assume that z is odd. By definition of odd, $2 \mid z - 1$ and as such, there is a h such that $(h \in \mathbb{Z})(2h = z - 1)$. Using substitution, we get $2h = 2k - 1$. Subtract $2k$ from and multiply by -1 for both sides and we get $2k - 2h = 1$. Furthermore, by distributivity, we get $2(k - h) = 1$. Since $(k - h) \in \mathbb{Z}$, we conclude that $2 \mid 1$ which by the Absolute Monotonicity of Divisibility implies $|2| \leq |1|$ ↯. Consequently, we conclude that z is even implies z is not odd.

   Q.E.D.

3. (a) For all $n \in \mathbb{N}$, show that n is even implies $n+1$ is odd.

Proof. Let $n \in \mathbb{N}$ and assume that n is even. This means that $2 \mid n$. Since $n = (n+1) - 1$ we can conclude that $2 \mid (n+1) - 1$ and consequently we know that $n+1$ is odd.

Q.E.D.

(b) For all $n \in \mathbb{N}$, show that n is odd implies $n+1$ is even.

Proof. Let $n \in \mathbb{N}$ and assume that n is odd. This means that $2 \mid n-1$. Note that $2 \mid 2$ ($2 \cdot k = 2$ when $k = 1$). Thus, $2 \mid n - 1 + 2$ by (1b). Also, observe that $n - 1 + 2 = n + 1$. Consequently, $2 \mid n + 1$. Therefore, $n + 1$ is even.

Q.E.D.

4. Show that $3 \mid n^3 - n$ for all $n \in \mathbb{N}$.

Proof. Proof by mathematical induction

Basis Step:
Need to show $3 \mid 0^3 - 0$.
Since $(0^3 - 0 = 0) \wedge (3 \mid 0)$ we can conclude $3 \mid 0^3 - 0$.

Inductive Step:
Let $k \in \mathbb{N}$ and assume that $3 \mid k^3 - k$.
Need to show $3 \mid \mathbb{S}(k)^3 - \mathbb{S}(k)$.
$\mathbb{S}(k)^3 - \mathbb{S}(k) = (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 - k + 3k^2 + 3k = (k^3 - k) + 3(k^2 + k)$
Let $b = k^2 + k$ and let $a = 1$. Now $(k^3 - k) + 3(k^2 + k) = a(k^3 - k) + 3b$.
Since $(3 \mid k^3 - k) \wedge (3 \mid 3) \wedge a, b \in \mathbb{Z}$ we can conclude $3 \mid a(k^3 - k) + 3b$ by (Problem 1b).
Since $(\mathbb{S}(k)^3 - \mathbb{S}(k) = a(k^3 - k) + 3b) \wedge (3 \mid a(k^3 - k) + 3b)$, we can conclude $3 \mid \mathbb{S}(k)^3 - \mathbb{S}(k)$.
Therefore $3 \mid \mathbb{S}(k)^3 - \mathbb{S}(k)$.

Thus by mathematical induction, we can conclude $3 \mid n^3 - n$ for all $n \in \mathbb{N}$. Q.E.D.

5. The Fibonacci sequence is the recursive function $\mathcal{F} : \mathbb{N} \to \mathbb{N}$ below.

$$\begin{aligned}
\mathcal{F}(0) &\coloneqq 0 \\
\mathcal{F}(1) &\coloneqq 1 \\
\mathcal{F}(n+2) &\coloneqq \mathcal{F}(n+1) + \mathcal{F}(n)
\end{aligned}$$

Show that $1 + \sum_{i=0}^{n} \mathcal{F}(i) = \mathcal{F}(n+2)$ for all $n \in \mathbb{N}$.

Proof. Proof by mathematical induction.

Basis Step:

Need to show $1 + \sum_{i=0}^{0} \mathcal{F}(i) = \mathcal{F}(0+2)$

$$\begin{aligned}
1 + \sum_{i=0}^{0} \mathcal{F}(i) &= 1 + \mathcal{F}(0) && \text{By } \textstyle\sum \text{ Rule 1} \\
&= \mathcal{F}(1) + \mathcal{F}(0) && \text{By definition of } \mathcal{F}(1) \\
&= \mathcal{F}(2) && \text{By definition of } \mathcal{F}(n+2) \\
&= \mathcal{F}(0+2) && \text{By addition}
\end{aligned}$$

Inductive Step:

Let $k \in \mathbb{N}$ and assume that $1 + \sum_{i=0}^{k} \mathcal{F}(i) = \mathcal{F}(k+2)$.

Need to show $1 + \sum_{i=0}^{\mathbb{S}(k)} \mathcal{F}(i) = \mathcal{F}(\mathbb{S}(k)+2)$.

$$\begin{aligned}
1 + \sum_{i=0}^{\mathbb{S}(k)} \mathcal{F}(i) &= 1 + \sum_{i=0}^{k} \mathcal{F}(i) + \mathcal{F}(\mathbb{S}(k)) && \text{By } \textstyle\sum \text{ Rule 1} \\
&= \mathcal{F}(k+2) + \mathcal{F}(\mathbb{S}(k)) && \text{By IH} \\
&= \mathcal{F}(k+\mathbb{S}(1)) + \mathcal{F}(\mathbb{S}(k)) && \text{By definition of } \mathbb{S}(n) \\
&= \mathcal{F}(k+1+1) + \mathcal{F}(\mathbb{S}(k)) && \text{By Theorem} \\
&= \mathcal{F}(\mathbb{S}(k)+1) + \mathcal{F}(\mathbb{S}(k)) && \text{By definition of } \mathbb{S}(n) \\
&= \mathcal{F}(\mathbb{S}(k)+2) && \text{By definition of } \mathcal{F}(n+2)
\end{aligned}$$

Therefore we have concluded $1 + \sum_{i=0}^{\mathbb{S}(k)} \mathcal{F}(i) = \mathcal{F}(\mathbb{S}(k)+2)$.

Thus by mathematical induction, we can conclude $1 + \sum_{i=0}^{n} \mathcal{F}(i) = \mathcal{F}(n+2)$ for all $n \in \mathbb{N}$.

Q.E.D.