Homework 6

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October 18 2024

1. Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = 0.1$.
$$F(x) = 1 - e^{-\lambda x} = 1 - e^{-0.1x}$$

   (a) Find the probability that the lifetime is less than 10.
$$F(10) = 1 - e^{-0.1(10)} = 1 - e^{-1} \approx 0.6321$$

   (b) Find the probability that the lifetime is between 5 and 15.
$$F(15) = 1 - e^{-0.1(15)} = 1 - e^{-1.5} \approx 0.7769$$
$$F(5) = 1 - e^{-0.1(5)} = 1 - e^{-0.5} \approx 0.3935$$
$$F(15) - F(5) \approx 0.3834$$

   (c) Find t such that the probability that the lifetime is greater than t is 0.01.
$$F(t) = 1 - 0.01 =$$
$$1 - e^{-0.1(t)} = 1 - 0.01$$
$$e^{-0.1(t)} = 0.01$$
$$-0.1(t) = ln(0.01)$$
$$t = -\frac{ln(0.01)}{0.01} \approx 46.05$$

2. The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 480 with standard deviation 100. The average and standard deviation for ACT mathematics scores are 18 and 6, respectively.

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(-y-\mu)^2}{2\sigma^2}}$$

   (a) An engineering school sets 550 as the minimum SAT math score for new students. What percentage of students will score below 550 in a typical year?

$$\mu = 480 \text{ and } \sigma = 100$$

$$f(y) = \frac{1}{(100)\sqrt{2\pi}} e^{\frac{(-y-(480))^2}{2(100)^2}}$$

$$\int_0^{550} f(y)dy \approx 0.7580$$

   About 75.80% students will score below 550 in a typical year.

   (b) What score should the engineering school set as a comparable standard on the ACT math test?

$$\mu = 18 \text{ and } \sigma = 6$$

$$f(y) = \frac{1}{(6)\sqrt{2\pi}} e^{\frac{(-y-(18))^2}{2(6)^2}}$$

$$\int_0^{s} f(y)dy = 0.7580$$

$$s \approx 22.2$$

   Thus, a comparable score on the ACT is a 22.

3. Let T be an exponential random variable with parameter λ. Let X be a discrete random variable defined as $X = k$ if $k \leq T < k+1$, $k = 0, 1, ...$ Find the probability mass function of X.

$$\text{Exponential Random Variable: } F(x) = 1 - e^{-\lambda x}$$

$$\begin{aligned} F(X = k) &= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda k}) \\ &= e^{-\lambda(k)} - e^{-\lambda(k+1)} \\ &= e^{-\lambda k} - e^{-\lambda k - \lambda} \\ &= e^{-\lambda k} - e^{-\lambda k} \cdot e^{-\lambda} \\ &= e^{-\lambda k} \cdot (1 - e^{-\lambda}) \end{aligned}$$

Thus the PMF is $F(X = k) = e^{-\lambda k} \cdot (1 - e^{-\lambda})$.

4. The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Find the probability that an earthquake striking this region will

$$2.4 = \frac{1}{\lambda} \text{ so } \lambda = \frac{5}{12}$$

$$F(x) = 1 - e^{-\lambda x} = 1 - e^{-\frac{5}{12}x}$$

   (a) exceed 3.0 on a Richter scale.

$$F(3) = 0.7135$$

$$1 - F(3) = 1 - 0.7135 = 0.2865$$

   (b) fall between 2.0 and 3.0 on a Richter scale.

$$F(2) = 0.5654$$

$$F(3) - F(2) = 0.7135 - 0.5654 = 0.1481$$

5. If Y has an exponential distribution and $P(Y > 2) = 0.0821$, what is

   (a) $E(Y)$

$$1 - F(2) = 0.0821$$

$$1 - (1 - e^{-2\lambda}) = 0.0821$$

$$e^{-2\lambda} = 0.0821$$

$$\lambda = -\frac{\ln(0.0821)}{2} = 1.2499$$

$$E(Y) = \frac{1}{1.2499} = 0.8001$$

   (b) $P(Y \leq 1.7)$

$$P(Y \leq 1.7) = F(1.7) = 1 - e^{-1.2499(1.7)} = 0.8805$$