Homework 7

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1. Suppose that a random variable Y has a probability density function given by

$$f(y) = \begin{cases} ky^3e^{-y/2} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the values of k that make $f(y)$ a density function.
   The pdf of a Gamma random variable is

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   Thus, for f(y) to be a pdf, $k = \frac{\lambda^\alpha}{\Gamma(\alpha)}$.
   $\alpha - 1 = 3 \to \alpha = 4$; $\lambda = \frac{1}{2}$; $\Gamma(4) = 3! = 6$; $k = \frac{1/2^4}{6} = \frac{1}{96}$

   (b) What are the mean and Standard Deviation of Y?
   $E(Y) = \frac{\alpha}{\lambda} = \frac{4}{1/2} = 8$
   $\sigma(Y) = \sqrt{\frac{\alpha}{\lambda^2}} = \sqrt{\frac{4}{1/2^2}} = \sqrt{16} = 4$

2. Suppose that a random variable Y has a probability density function given by

$$f(y) = \begin{cases} 6y(1-y) & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find $F(y)$

$$\int_0^y 6t - 6t^2 = 3t^2 - 2t^3\Big|_0^y = 3y^2 - 2y^3$$

$$F(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

   (b) Find $P(0.5 \leq Y \leq 0.8)$

$$F(0.8) - F(0.5) = (3(0.8)^2 - 2(0.8)^3) - (3(0.5)^2 - 2(0.5)^3)$$
$$= (1.92 - 1.024) - (0.75 - 0.25) = 0.896 - 0.5 = 0.396$$

3. The Weibull cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(x/\alpha)^\beta} & x \geq 0 \end{cases}$$

for α, $\beta > 0$.

(a) Find the density function

$$\frac{d}{dx}\left[1 - e^{-(x/\alpha)^\beta}\right] = -e^{-(x/\alpha)^\beta}(-\frac{x}{\alpha}^{\beta-1})\beta\frac{1}{\alpha} = \frac{\beta}{\alpha}\frac{x}{\alpha}^{\beta-1}e^{-(x/\alpha)^\beta}$$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{\beta}{\alpha}\frac{x}{\alpha}^{\beta-1}e^{-(x/\alpha)^\beta} & x \geq 0 \end{cases}$$

(b) Show that if W follows a Weibull distribution, then $X = (W/\alpha)^\beta$ follows an exponential distribution.
The cdf of an exponential distribution is

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & 0 \leq x \\ 0 & x < 0 \end{cases}$$

Assume:

$$F(w) = \begin{cases} 0 & w < 0 \\ 1 - e^{-(w/\alpha)^\beta} & w \geq 0 \end{cases}$$

Observe that $w = 0 \rightarrow x = 0$ and substitute $w = \alpha(x^{-\beta})$:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(\alpha(x^{-\beta})/\alpha)^\beta} & x \geq 0 \end{cases}$$

Thus:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

This follows the form of an exponential cdf where $\lambda = 1$

4. A supplier of kerosene has a weekly demand Y possessing a probability density function given by $f(y)$ with measurements in hundreds of gallons. The supplier's profit is given by $U = 10Y - 4$.

$$f(y) = \begin{cases} y & 0 \leq y \leq 1 \\ 1 & 1 < y \leq 1.5 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the probability density function for U.
Assume: $F_Y(y) = F_U(u)$. Thus $\frac{d}{du}F_Y(y) = \frac{d}{du}F_U(u)$ which means that $\frac{dy}{du}f_Y(y) = f_U(u)$.
$Y = \frac{U+4}{10}$; $\frac{dy}{du} = \frac{1}{10}$; $f_U(u) = \frac{1}{10}f_Y(y)$
Now we must consider the boundaries.

$$U(Y=0) = -4;\ U(Y=1) = 6;\ U(Y=1.5) = 11$$

Thus,

$$f(u) = \frac{1}{10}\begin{cases} \frac{u+4}{10} & -4 \leq u \leq 6 \\ 1 & 4 < u \leq 11 \\ 0 & \text{elsewhere} \end{cases}$$

or

$$f(u) = \begin{cases} \frac{u+4}{100} & -4 \leq u \leq 6 \\ \frac{1}{10} & 4 < u \leq 11 \\ 0 & \text{elsewhere} \end{cases}$$

(b) Find $E(U)$

$$E(U) = \int_{-\infty}^{\infty} u \cdot f(u)du = \int_{-4}^{11} u \cdot f(u)du = \int_{-4}^{6} u \cdot f(u)du + \int_{6}^{11} u \cdot f(u)du$$

$$\int_{-4}^{6} u \cdot f(u)du = \frac{1}{100}\int_{-4}^{6} u^2 + 4udu = \frac{1}{100}\left(\frac{u^3}{3} + 2u^2\right)\Bigg|_{-4}^{6} = \frac{4}{3}$$

$$\int_{6}^{11} u \cdot f(u)du = \frac{1}{10}\int_{6}^{11} udu = \frac{u^2}{20}\Bigg|_{6}^{11} = \frac{121-36}{20} = \frac{17}{4}$$

$$\frac{4}{3} + \frac{17}{4} = \frac{67}{12}$$

5. Find the density of cX when X follows a gamma distribution. Show that only λ is affected by such a transformation, which justifies calling λ a rate (or scale) parameter.
The pdf of a Gamma random variable is

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Let $Y = cX$ and observe $X = \frac{Y}{c}$ and $\frac{dx}{dy} = \frac{1}{c}$
Assume: $F_X(x) = F_Y(y)$. Thus $\frac{d}{dy}F_X(x) = \frac{d}{dy}F_Y(y)$ gives us $\frac{dx}{dy}f_X(x) = f_Y(y)$ which means $\frac{1}{c}f_X(x) = f_Y(y)$.
So $\frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} = \frac{1}{c}\frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda \frac{y}{c}} (\frac{\lambda}{c})^{\alpha-1} = \frac{(\frac{\lambda}{c})^\alpha}{\Gamma(\alpha)} e^{-(\frac{\lambda}{c})y} (y)^{\alpha-1}$.
From this example, we can see that c only affects λ as λ now is λ/c which justifies calling λ a rate (or scale) parameter.