Homework 10

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1 Wednesday 4/2

Section 2 Find the series solutions of the following problems.

1. Find the steady-state temperature distribution for the semi-infinite plate problem if the temperature of the bottom edge is $T = f(x) = x$ (in degrees; that is, the temperature at x cm is x degrees), the temperature of the other sides is $0°$, and the width of the plate is 10 cm.

   Define the boundary constraints

$$\begin{aligned} T(x, y = 0) &= x \\ T(x, y = \infty) &= 0 \\ T(x = 0, y) &= 0 \\ T(x = 10, y) &= 0 \end{aligned}$$

   Use $X(x)'' + k^2 X(x) = 0$ to get $X(x) = C_1 \sin(kx) + C_2 \cos(kx)$. Using the boundary constraints of $X(0) = 0$, $C_2 = 0$. Further from the constraint $X(10) = 0$, $k = \frac{n\pi}{10}$ for all positive integers n so

$$X(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right)$$

   Next, use $Y(y)'' - k^2 Y(y) = 0$ to get $Y(y) = Ae^{-ky} + Be^{ky}$. Because of the boundary constraint that as $y \to \infty$, $Y(y) = 0$, we know that $B = 0$ so $Y(y) = Ce^{-ky} = Ce^{-\frac{n\pi}{10}y}$.

   Thus, as $T(x, y) = X(x)Y(y)$ we now have $T(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right) e^{-\frac{n\pi}{10}y}$

   To solve for C_n, we use

$$\begin{aligned} C_n &= \frac{2}{10} \int_0^{10} x \sin\left(\frac{n\pi}{10}x\right) dx \\ &= \frac{2}{10} \left[\frac{x(-\cos(\frac{n\pi}{10}x))}{\frac{n\pi}{10}} + \frac{\sin(\frac{n\pi}{10}x)}{(\frac{n\pi}{10})^2} \right]_0^{10} \\ &= \frac{2}{10} \left[\frac{-10\cos(n\pi)}{n\pi/10} + \frac{\sin(n\pi)}{(n\pi/10)^2} - 0 \right] \\ &= \frac{2}{10} \left[\frac{-10(-1)^n}{n\pi/10} \right] \\ &= \frac{20}{\pi} \frac{(-1)^{n+1}}{n} \end{aligned}$$

   Substituting this in we have

$$T(x, y) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{10}x\right) e^{-\frac{n\pi}{10}y}$$

3. Solve the semi-infinite plate problem if the bottom edge of width π is held at $T = \cos x$ and the other sides are at $0°$.

Define the boundary constraints

$$\begin{aligned} T(x, y = 0) &= \cos x \\ T(x, y = \infty) &= 0 \\ T(x = 0, y) &= 0 \\ T(x = \pi, y) &= 0 \end{aligned}$$

Use $X(x)'' + k^2 X(x) = 0$ to get $X(x) = C_1 \sin(kx) + C_2 \cos(kx)$. Using the boundary constraints of $X(0) = 0$, $C_2 = 0$. Further from the constraint $X(\pi) = 0$, $k = n$ for all positive integers n so

$$X(x) = \sum_{n=1}^{\infty} C_n \sin(nx)$$

Next, use $Y(y)'' - k^2 Y(y) = 0$ to get $Y(y) = Ae^{-ky} + Be^{ky}$. Because of the boundary constraint that as $y \to \infty$, $Y(y) = 0$, we know that $B = 0$ so $Y(y) = Ce^{-ky} = Ce^{-ny}$.

Thus, as $T(x, y) = X(x)Y(y)$ we now have $T(x, y) = \displaystyle\sum_{n=1}^{\infty} C_n \sin(nx)\, e^{-ny}$

To solve for C_n, we use

$$\begin{aligned} C_n &= \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(nx)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(nx + x) + \sin(nx - x)] dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin\left((n+1)x\right) + \sin\left((n-1)x\right) dx \\ &= \frac{1}{\pi} \left[\frac{-\cos\left((n+1)x\right)}{n+1} + \frac{-\cos\left((n-1)x\right)}{n-1} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[-\frac{\cos\left((n+1)x\right)}{n+1} - \frac{\cos\left((n-1)x\right)}{n-1} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[-\frac{\cos\left((n+1)\pi\right)}{n+1} - \frac{\cos\left((n-1)\pi\right)}{n-1} + \frac{\cos\left((n+1)0\right)}{n+1} + \frac{\cos\left((n-1)0\right)}{n-1} \right] \\ &= \frac{1}{\pi} \left[-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right] \\ &= \frac{1}{\pi} \left[-\frac{(-1)^{n+1}(n-1)}{(n+1)(n-1)} - \frac{(-1)^{n-1}(n+1)}{(n+1)(n-1)} + \frac{n-1}{(n+1)(n-1)} + \frac{n+1}{(n+1)(n-1)} \right] \\ &= \frac{1}{\pi} \left[-\frac{(-1)^{n-1} 2n}{(n-1)(n+1)} + \frac{2n}{(n-1)(n+1)} \right] \\ &= \frac{2n}{\pi} \left[\frac{(-1)^n + 1}{n^2 - 1} \right] \end{aligned}$$

Thus, when n is even $C_n = \frac{4n}{\pi(n^2-1)}$ and when n is odd $C_n = 0$. Substituting this in we have

$$T(x, y) = \frac{4}{\pi} \sum_{\substack{n=2 \\ \text{even } n}}^{\infty} \frac{n}{n^2 - 1} \sin(nx)\, e^{-ny}$$

4. Solve the semi-infinite plate problem if the bottom edge of width 30 is held at

$$T = \begin{cases} x, & 0 < x < 15 \\ 30 - x, & 15 < x < 30 \end{cases}$$

and the other sides are at 0°. Define the boundary constraints

$$\begin{aligned} T(x, y = 0) &= \begin{cases} x, & 0 < x < 15 \\ 30 - x, & 15 < x < 30 \end{cases} \\ T(x, y = \infty) &= 0 \\ T(x = 0, y) &= 0 \\ T(x = 30, y) &= 0 \end{aligned}$$

Use $X(x)'' + k^2 X(x) = 0$ to get $X(x) = C_1 \sin(kx) + C_2 \cos(kx)$. Using the boundary constraints of $X(0) = 0$, $C_2 = 0$. Further from the constraint $X(30) = 0$, $k = \frac{n\pi}{30}$ for all positive integers n so

$$X(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{30}x\right)$$

Next, use $Y(y)'' - k^2 Y(y) = 0$ to get $Y(y) = Ae^{-ky} + Be^{ky}$. Because of the boundary constraint that as $y \to \infty$, $Y(y) = 0$, we know that $B = 0$ so $Y(y) = Ce^{-ky} = Ce^{-\frac{n\pi}{30}y}$.

Thus, as $T(x, y) = X(x)Y(y)$ we now have $T(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{30}x\right) e^{-\frac{n\pi}{30}y}$

To solve for C_n, we use

$$\begin{aligned} C_n &= \frac{2}{30} \int_0^{30} f(x) \sin\left(\frac{n\pi}{30}x\right) dx \\ &= \frac{2}{30} \left[\int_0^{15} x \sin\left(\frac{n\pi}{30}x\right) dx + \int_{15}^{30} (30 - x) \sin\left(\frac{n\pi}{30}x\right) dx \right] \\ &= 60 \left[\int_0^{1/2} u \sin(n\pi u)\, du + \int_{1/2}^{1} (1 - u) \sin(n\pi u)\, du \right] \\ &= 60 \left[\int_0^{1/2} u \sin(n\pi u)\, du + \int_{1/2}^{1} \sin(n\pi u)\, du + \int_1^{1/2} u \sin(n\pi u)\, du \right] \end{aligned}$$

Solve the first integral. $\int_0^{1/2} u \sin(n\pi u)\, du$

$$\begin{aligned} \int_0^{1/2} u \sin(n\pi u)\, du &= \left(\frac{-x \cos(\pi n x)}{\pi n} + \frac{\sin(\pi n x)}{\pi^2 n^2} \right) \Bigg|_0^{1/2} \\ &= \frac{-(1/2)\cos(\pi n(1/2))}{\pi n} + \frac{\sin(\pi n(1/2))}{\pi^2 n^2} - \frac{-(0)\cos(\pi n(0))}{\pi n} - \frac{\sin(\pi n(0))}{\pi^2 n^2} \\ &= \frac{\sin(\pi n(1/2))}{\pi^2 n^2} \end{aligned}$$

Solve the next integral. $\int_{1/2}^{1} \sin(n\pi u)\, du$

$$\begin{aligned} \int_{1/2}^{1} \sin(n\pi u)\, du &= \frac{-\cos(\pi n x)}{\pi n} \Bigg|_{1/2}^{1} \\ &= \frac{-\cos(\pi n(1))}{\pi n} - \frac{-\cos(\pi n(1/2))}{\pi n} \\ &= \frac{-\cos(\pi n)}{\pi n} \end{aligned}$$

Solve the final integral. $\displaystyle\int_1^{1/2} u \sin(n\pi u)\, du$

$$\begin{aligned}
\int_1^{1/2} u \sin(n\pi u)\, du &= \left(\frac{-x\cos(\pi n x)}{\pi n} + \frac{\sin(\pi n x)}{\pi^2 n^2} \right) \Bigg|_1^{1/2} \\
&= \frac{-(1/2)\cos(\pi n(1/2))}{\pi n} + \frac{\sin(\pi n(1/2))}{\pi^2 n^2} - \frac{-(1)\cos(\pi n(1))}{\pi n} - \frac{\sin(\pi n(1))}{\pi^2 n^2} \\
&= \frac{\sin(\pi n(1/2))}{\pi^2 n^2} + \frac{\cos(\pi n)}{\pi n}
\end{aligned}$$

Continuing from before for C_n

$$\begin{aligned}
C_n &= 60 \left[\frac{\sin(\pi n(1/2))}{\pi^2 n^2} + \frac{-\cos(\pi n)}{\pi n} + \frac{\sin(\pi n(1/2))}{\pi^2 n^2} + \frac{\cos(\pi n)}{\pi n} \right] \\
&= \frac{120 \sin(\pi n(1/2))}{\pi^2 n^2}
\end{aligned}$$

Thus, when n is even $C_n = 0$ and when n is odd $C_n = \frac{120(-1)^{(n-1)/2}}{\pi^2 n^2}$. Substituting this in we have

$$T(x,y) = \frac{120}{\pi^2} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi}{30} x\right) e^{-\frac{n\pi}{30} y}$$