Problem Set 10

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1. Construct—*with proof*—the *explicit* functions requested below.

   (a) A bijection from $\{x \in \mathbb{R} | -1 < x < 1\}$ to $\{x \in \mathbb{R} | -\pi < x < \pi\}$.

   Proof. Let $A \coloneqq \{x \in \mathbb{R} | -1 < x < 1\}$ and let $B \coloneqq \{x \in \mathbb{R} | -\pi < x < \pi\}$. Let $f : A \to B$ where $f(a) = \pi \cdot a$.

   - Injective
     Let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$. By definition of f, we get $\pi \cdot a_1 = \pi \cdot a_2$. Since $\pi \neq 0$, we know by multiplicative cancellation $a_1 = a_2$. Thus, f is injective.
   - Surjective
     Let $b \in B$. Let $x \coloneqq \frac{b}{\pi}$. Observe that $x > -1$ because $b > -\pi \implies \frac{b}{-\pi} > \frac{-\pi}{\pi} = -1$. Similarly, observe that $x < 1$ because $b < \pi \implies \frac{b}{\pi} < \frac{\pi}{\pi} = 1$. Since $x \in \mathbb{R} \wedge -1 < x < 1$, $x \in A$ so $f(x) = b$. Therefore, f is surjective.

   As f is surjective and injective, it is bijective. Q.E.D.

   (b) A surjection from $\mathbb{N}$ to $\{p | p \text{ is prime}\}$.

   Proof. Let $P \coloneqq \{p | p \text{ is prime}\}$. Let $f : \mathbb{N} \to P$ where

   $$f(n) = \begin{cases} n & \text{when } n \text{ is prime} \\ 2 & \text{otherwise} \end{cases}$$

   Let $p \in P$. Let $x = p$. Observe that $x \in \mathbb{N}$ and x is prime so $f(x) = p$. Thus, f is a surjection. Q.E.D.

   (c) An injection from X to $\mathbb{P}(X)$ for *every* set X.

   Proof. Let X be an arbitrary set. Let $f : X \to \mathbb{P}(X)$ where $f(x) = \{x\}$. Let $x_1, x_2 \in X$ and assume $f(x_1) = f(x_2)$. Thus, we know that $\{x_1\} = \{x_2\}$. By extensionality, we know that $x_1 \in \{x_1\} \iff x_1 \in \{x_2\}$. Since x_2 is the only element in $\{x_2\}$, we know $x_1 = x_2$. Thus, f is injective. Q.E.D.

(d) A surjection from $\mathbb{P}(X)$ to X for *every* set $X \neq \varnothing$.

Proof. Let X be an arbitrary set and let $x_0 \in X$. Let $f : \mathbb{P}(X) \to X$ where

$$f(y) = \begin{cases} \cup y & \text{if } |y| = 1 \\ x_0 & \text{otherwise} \end{cases}$$

Let $x \in X$. Let $y = \{x\}$. Observe that $y \subseteq X$ and $|y| = 1$ so $f(y) = \cup y = x$. Thus, f is surjective. Q.E.D.

2. Let A be an arbitrary finite set of cardinality $|A| = n$, where $n \in \mathbb{N}$. How many finite strings over A are there?

Proof. Let $k \in \mathbb{N}$ and let $s : k \to A$. Observe that there are $k^{|A|}$ distinct strings s. Let $S_k \coloneqq \{s | s : k \to A\}$ so as shown above $|S_k| = k^{|A|}$. Also let $S \coloneqq \{S_k | k \in \mathbb{N}\}$. By theorem 8.8, since $|S| \leq |\mathbb{N}|$ and $(\forall k \in \mathbb{N})(|S_k| \leq |\mathbb{N}|)$, we know that $|\cup S| \leq |\mathbb{N}|$.
Towards a contradiction, assume that $|\cup S| = n$ for some $n \in \mathbb{N}$. So there must be $f : n \to \cup S$ such that f is surjective. Let $a \in A$ and let $h \coloneqq f(0) \mathbin{+\!\!+} f(1) \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} f(n-1) \mathbin{+\!\!+} a$. Observe $|h| = \big(\sum_{i=0}^{n-1} |f(i)|\big) + 1$. Thus we know that $(\forall k \in n)(|f(k)| \leq \sum_{i=0}^{n-1} |f(i)| < \big(\sum_{i=0}^{n-1} |f(i)|\big) + 1 = |h|$. Since $(\forall k \in n)(|f(k)| < |h|)$ we know that $(\forall k \in n)(f(k) \neq h)$. Thus we have achieved a contradiction as h is a finite string over A (aka $h \in \cup S$) and f does not map to h so f can not be surjective.
Consequently, $\cup S \neq n$ for any $n \in \mathbb{N}$ so $|\cup S| \geq |\mathbb{N}|$.
From $|\cup S| \leq |\mathbb{N}|$ and $|\cup S| \geq |\mathbb{N}|$, we know $|\cup S| = |\mathbb{N}| = \aleph_0$.
There are $\aleph_0$ many finite strings over A. Q.E.D.

3. The Library

Proof. Let $S \coloneqq$ the set of all books in the library. Let $\beta \coloneqq$ the set of all sentences. Observe that there are a countable infinite amount of sentences so $|\beta| = |\mathbb{N}|$. Also observe that since β is numbered, we can use $\mathbb{N}$ to represent the sentences in β. Now, let $i : \mathbb{N} \to \mathbb{N}$ (a string of infinite length over alphabet $\mathbb{N}$ which represents the sentences in β) represent a book of infinite length. Let $I \coloneqq \{i | i : \mathbb{N} \to \mathbb{N}\}$ (the set of all infinite books).
Towards a contradiction, assume that $|\mathbb{N}| \geq |I|$. This means that there exists an $f : \mathbb{N} \to I$ where f is surjective. Let us define $g : \mathbb{N} \to \mathbb{N}$ where:

$$g(i) = \begin{cases} 1 & f(i)(i) \neq 1 \\ 2 & f(i)(i) = 1 \end{cases}$$

Notice that since f is surjective so for some value $t \in \mathbb{N}$, $f(t) = g$. As such, $(\forall i \in \mathbb{N})(f(t)(i) = g(i))$ so $f(t)(t) = g(t)$. Note that there are two cases: $f(t)(t) = 1$ or $f(t)(t) \neq 1$
$f(t)(t) = 1$ implies that $g(t) = 2 \neq 1$ so $f(t)(t) \neq g(t) \lightning$.
$f(t)(t) \neq 1$ implies that $g(t) = 1$ so $f(t)(t) \neq g(t) \lightning$.
In both cases, we achieve a contradiction and since f was arbitrary, we know that there are no injective functions from $\mathbb{N} \to I$. As such, $|\mathbb{N}| < |I|$. Notice that $I \subseteq S$. Consequently, $|\mathbb{N}| < |S|$ as well so we know that S is uncountable. Therefore, as there are an uncountable amount of books, I will not be set free. Q.E.D.