Homework 8

Aaron Wang

November 15 2024

1. The joint probability mass (frequency) function of two discrete random variables, X and Y , is given in the following table:

$X\backslash Y$	1	2	3	4
1	0.10	0.05	0.02	0.02
2	0.05	0.20	0.05	0.02
3	0.02	0.05	0.20	0.04
4	0.02	0.02	0.04	0.10

Find the marginal probability mass functions of X and Y.

$$P(X = 1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$$
$$P(X = 2) = 0.05 + 0.20 + 0.05 + 0.02 = 0.32$$
$$P(X = 3) = 0.02 + 0.05 + 0.20 + 0.04 = 0.31$$
$$P(X = 4) = 0.02 + 0.02 + 0.04 + 0.10 = 0.18$$

Thus, the marginal probability mass function of X is:

$$P(X = 1) = 0.19, \quad P(X = 2) = 0.32, \quad P(X = 3) = 0.31, \quad P(X = 4) = 0.18$$

$$P(Y = 1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$$
$$P(Y = 2) = 0.05 + 0.20 + 0.05 + 0.02 = 0.32$$
$$P(Y = 3) = 0.02 + 0.05 + 0.20 + 0.04 = 0.31$$
$$P(Y = 4) = 0.02 + 0.02 + 0.04 + 0.10 = 0.18$$

Thus, the marginal probability mass function of Y is:

$$P(Y = 1) = 0.19, \quad P(Y = 2) = 0.32, \quad P(Y = 3) = 0.31, \quad P(Y = 4) = 0.18$$

2. Find the joint and marginal densities corresponding to the CDF:

$$F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y}), \quad x \geq 0,\ y \geq 0,\ \alpha > 0,\ \beta > 0.$$

Joint Density

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

$$\frac{\partial}{\partial x} F(x, y) = (1 - e^{-\beta y}) \cdot \frac{\partial}{\partial x}(1 - e^{-\alpha x}) = (1 - e^{-\beta y}) \cdot \alpha e^{-\alpha x}.$$

$$\frac{\partial}{\partial y}\left((1 - e^{-\beta y})\alpha e^{-\alpha x}\right) = \alpha e^{-\alpha x} \cdot \frac{\partial}{\partial y}(1 - e^{-\beta y}) = \alpha e^{-\alpha x} \beta e^{-\beta y}.$$

$$f(x, y) = \alpha\beta e^{-\alpha x} e^{-\beta y}, \quad x \geq 0,\ y \geq 0,\ \alpha > 0,\ \beta > 0.$$

Marginal Density of X:

$$f_X(x) = \int_0^\infty f(x, y)\, dy = \int_0^\infty \alpha\beta e^{-\alpha x} e^{-\beta y}\, dy.$$

$$f_X(x) = \alpha e^{-\alpha x} \int_0^\infty \beta e^{-\beta y}\, dy.$$

$$\int_0^\infty \beta e^{-\beta y}\, dy = 1.$$

$$f_X(x) = \alpha e^{-\alpha x}, \quad x \geq 0, \alpha > 0.$$

Marginal Density of Y:

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty \alpha\beta e^{-\alpha x} e^{-\beta y}\, dx.$$

$$f_Y(y) = \beta e^{-\beta y} \int_0^\infty \alpha e^{-\alpha x}\, dx.$$

$$\int_0^\infty \alpha e^{-\alpha x}\, dx = 1.$$

$$f_Y(y) = \beta e^{-\beta y}, \quad y \geq 0, \beta > 0.$$

3. The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & 0 \leq y_2 \leq y_1 < \infty \\ 0 & \text{otherwise} \end{cases}$$

with time measured in minutes. Find

(a) $P(Y_1 < 2, Y_2 > 1)$

$$P(Y_1 < 2, Y_2 > 1) = \int_1^2 \int_1^{y_1} e^{-y_1}\, dy_2 dy_1 = 0.0972$$

(b) $P(Y_1 \geq 2Y_2)$

$$P(Y_1 \geq 2Y_2) = \int_0^{\infty} \int_0^{\frac{y_1}{2}} e^{-y_1}\, dy_2 dy_1 = 0.5$$

(c) $P(Y_1 - Y_2 > 1)$. Notice that $Y_1 - Y_2$ denotes the time spent at the service window.

$$P(Y_1 \geq 2Y_2) = \int_1^{\infty} \int_{y_1 - 1}^{y_1} e^{-y_1}\, dy_2 dy_1 = 0.3679$$

4. Three players play 10 independent rounds of a game, and each player has probability $\frac{1}{3}$ of winning each round. Find the joint distribution of the numbers of games won by each of the three players.

Observe that this follows a Multinomial Experiment. Thus, the expression for the joint distribution of the number of games won by each of the three players is

$$P(X_A = k_A, X_B = k_B, X_C = k_C) = \frac{10!}{k_A!\, k_B!\, k_C!} \left(\frac{1}{3}\right)^{k_A} \left(\frac{1}{3}\right)^{k_B} \left(\frac{1}{3}\right)^{k_C}$$

5. Let $F(x, y)$ be the cumulative distribution function (CDF) for a bivariate random variable (X, Y). For any $a_1 < a_2$ and $b_1 < b_2$, prove that

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1).$$

$$\begin{aligned} P(X \leq a_2, Y \leq b_2) &= F(a_2, b_2) \\ P(X \leq a_1, Y \leq b_2) &= F(a_1, b_2) \\ P(X \leq a_2, Y \leq b_1) &= F(a_2, b_1) \\ P(X \leq a_1, Y \leq b_1) &= F(a_1, b_1) \end{aligned}$$

$$\begin{aligned} P(a_1 < X \leq a_2, Y \leq b_2) &= F(a_2, b_2) - F(a_1, b_2) \\ P(a_1 < X \leq a_2, Y \leq b_1) &= F(a_2, b_1) - F(a_1, b_1) \end{aligned}$$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \big(F(a_2, b_2) - F(a_1, b_2)\big) - \big(F(a_2, b_1) - F(a_1, b_1)\big)$$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)$$