Homework 04

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1 Monday 2/3

Section 6

3. $y'' + y' - 2y = e^{2x}$

   First Find General Solution to homogeneous equation:

$$y_c'' + y_c' - 2y_c = 0 \implies (D^2 + D - 2)y_c = 0$$
$$\text{Auxiliary Equation: } (D^2 + D - 2) = 0$$
$$\implies (D+2)(D-1) = 0 \implies \text{root(s): } -2, 1$$
$$\text{General Solution: } y_c = Ae^{-2x} + Be^{x}$$

   Now Find particular Solution: Using a simple way to solve for the particular solution we know $y_p = Ce^{2x}$

$$y_p = Ce^{2x};\ y_p' = C2e^{2x};\ y_p'' = C4e^{2x}$$
$$y_p'' + y_p' - 2y_p = C\big((4e^{2x}) + (2e^{2x}) - 2(e^{2x})\big)$$
$$y_p'' - y_p' - 2y_p = 4Ce^{2x} = e^{2x} \text{ so } C = \frac{1}{4}$$
$$y_p = \frac{1}{4}e^{2x}$$

   Combine:

$$y = y_c + y_p$$
$$y = Ae^{-2x} + Be^{x} + \frac{1}{4}e^{2x}$$

4. $(D+1)(D-3)y = 24e^{-3x}$

   First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+1)(D-3) = 0 \implies \text{root(s): } -1, 3$$
$$\text{General Solution: } y_c = Ae^{-x} + Be^{3x}$$

   Now Find particular Solution: Using a simple way to solve for the particular solution we know $y_p = Ce^{-3x}$

$$y_p = Ce^{-3x};\ y_p' = C(-3)e^{-3x};\ y_p'' = C9e^{-3x}$$
$$y_p'' - 2y_p' - 3y_p = C\big((9e^{-3x}) - 2((-3)e^{-3x}) - 3(e^{-3x})\big)$$
$$y_p'' - 2y_p' - 3y_p = 12Ce^{-3x} = 24e^{-3x} \text{ so } C = 2$$
$$y_p = 2e^{-3x}$$

   Combine:

$$y = y_c + y_p$$
$$y = Ae^{-x} + Be^{3x} + 2e^{-3x}$$

5. $(D^2+1)y = 2e^x$

   First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+i)(D-i) = 0 \implies \text{root(s): } \pm i$$
$$\text{General Solution: } y_c = Ae^{-ix} + Be^{ix}$$

   Now Find particular Solution: Using a simple way to solve for the particular solution we know $y_p = Ce^x$

$$y_p = Ce^x;\ y_p' = Ce^x;\ y_p'' = Ce^x$$
$$y_p'' + y_p = C\big((e^x) + (e^x)\big)$$
$$y_p'' + y_p = 2Ce^x = 2e^x \text{ so } C = 1$$
$$y_p = e^x$$

   Combine:

$$y = y_c + y_p$$
$$y = Ae^{-ix} + Be^{ix} + e^x$$

6. $y'' + 6y' + 9y = 12e^{-x}$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+3)^2 = 0 \implies \text{root(s): } -3$$
$$\text{General Solution: } y_c = (Ax+B)e^{-3x}$$

Now Find particular Solution: Using a simple way to solve for the particular solution we know $y_p = Ce^{-x}$

$$y_p = Ce^{-x};\ y_p' = -Ce^{-x};\ y_p'' = Ce^{-x}$$
$$y_p'' + 6y_p' + 9y_p = C\big((e^{-x}) + 6(-e^{-x}) + 9(e^{x})\big)$$
$$y_p'' + 6y_p' + 9y_p = 4Ce^{-x} = 12e^{-x} \text{ so } C = 3$$
$$y_p = 3e^{-x}$$

Combine:

$$y = y_c + y_p$$
$$y = (Ax+B)e^{-3x} + 3e^{-x}$$

7. $y'' - y' - 2y = 3e^{2x}$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+1)(D-2) = 0 \implies \text{root(s): } -1, 2$$
$$\text{General Solution: } y_c = Ae^{-x} + Be^{2x}$$

Now Find particular Solution:
Because exponent on RHS is one of the roots of the auxiliary equation, particular solution in the form of Cxe^{2x}

$$y_p = Cxe^{2x};\ y_p' = C(e^{2x} + 2xe^{2x});\ y_p'' = C(4e^{2x} + 4xe^{2x})$$
$$y_p'' - y_p' - 2y_p = C\big((4e^{2x} + \cancel{4xe^{2x}}) - (e^{2x} + \cancel{2xe^{2x}}) - \cancel{2(xe^{2x})}\big)$$
$$y_p'' - y_p' - 2y_p = 3Ce^{2x} = 3e^{2x} \text{ so } C = 1$$
$$y_p = xe^{2x}$$

Combine:

$$y = y_c + y_p$$
$$y = Ae^{-x} + Be^{2x} + xe^{2x}$$

8. $y'' - 16y = 40e^{4x}$

   First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-4)(D+4) = 0 \implies \text{root(s): } \pm 4$$
$$\text{General Solution: } y_c = Ae^{-4x} + Be^{4x}$$

   Now Find particular Solution:
   Because exponent on RHS is one of the roots of the auxiliary equation, particular solution in the form of Cxe^{4x}

$$y_p = Cxe^{4x};\ y_p' = C(e^{4x} + 4xe^{4x});\ y_p'' = C(8e^{4x} + 16xe^{4x})$$
$$y_p'' - 16y_p = C\big((8e^{4x} + \cancel{16xe^{4x}}) - \cancel{16(xe^{4x})}\big)$$
$$y_p'' - 16y_p = 8Ce^{4x} = 40e^{4x} \text{ so } C = 5$$
$$y_p = 5xe^{4x}$$

   Combine:

$$y = y_c + y_p$$
$$y = Ae^{-4x} + Be^{4x} + 5xe^{4x}$$

9. $(D^2 + 2D + 1)y = 2e^{-x}$

   First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+1)^2 = 0 \implies \text{root(s): } -1$$
$$\text{General Solution: } y_c = (Ax + B)e^{-x}$$

   Now Find particular Solution:
   Because exponent on RHS is the same as both roots of the auxiliary equation, particular solution in the form of Cx^2e^{-x}

$$y_p = Cx^2e^{-x};\ y_p' = C(2xe^{-x} - x^2e^{-x});\ y_p'' = C(2e^{-x} - 4xe^{-x} + x^2e^{-x})$$
$$y_p'' + 2y_p' + y_p$$
$$= C\big((2e^{-x} - \cancel{4xe^{-x}} + \cancel{x^2e^{-x}}) + \cancel{2(2xe^{-x} - x^2e^{-x})} + \cancel{(x^2e^{-x})}\big)$$
$$= 2Ce^{-x} = 2e^{-x} \text{ so } C = 1$$
$$y_p = x^2e^{-x}$$

   Combine:

$$y = y_c + y_p$$
$$y = (Ax + B + x^2)e^{-x}$$

10. $(D-3)^2 y = 6e^{3x}$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-3)^2 = 0 \implies \text{root(s): } 3$$
$$\text{General Solution: } y_c = (Ax+B)e^{3x}$$

Now Find particular Solution:
Because exponent on RHS is the same as both roots of the auxiliary equation, particular solution in the form of Cx^2e^{3x}

$$y_p = Cx^2e^{3x};\ y_p' = C(2xe^{3x} + 3x^2e^{3x});\ y_p'' = C(2e^{3x} + 12xe^{3x} + 9x^2e^{3x})$$
$$y_p'' - 6y_p' + 9y_p$$
$$= C\big((2e^{3x} + \cancel{12xe^{3x}} + \cancel{9x^2e^{3x}}) - \cancel{6(2xe^{3x} + 3x^2e^{3x})} + \cancel{9(x^2e^{3x})}\big)$$
$$= 2Ce^{3x} = 6e^{3x} \text{ so } C = 3$$
$$y_p = 3x^2e^{3x}$$

Combine:

$$y = y_c + y_p$$
$$y = (Ax + B + 3x^2)e^{3x}$$

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Section 6

11. $y'' + 2y' + 10y = 100\cos 4x$

    First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+1+3i)(D+1-3i) = 0$$
$$\implies \text{root(s): } -1 \pm 3i$$
$$\text{General Solution: } y_c = e^{-x}(A\cos 3x + B\sin 3x)$$

    Now Find particular Solution:
    Because RHS is cos, we will use $Y = Ce^{4ix} = Y_R + iY_I$ s.t. $y_p = Y_R$.

$$Y = Ce^{4ix}; Y' = C(4ie^{4ix}); Y'' = C(-16e^{4ix})$$
$$Y'' + 2Y' + 10Y = C\big((-16e^{4ix}) + 2(4ie^{4ix}) + 10(e^{4ix})\big)$$
$$Y'' + 2Y' + 10Y = (-6+8i)Ce^{4ix} = 100e^{4ix}$$
$$\text{so } C = \frac{50}{-3+4i} = \frac{-150-200i}{25} = -6-8i$$
$$Ce^{4ix} = (-6-8i)(\cos 4x + i\sin 4x)$$
$$= -6\cos 4x - 6i\sin 4x - 8i\cos 4x - 8i^2\sin 4x$$
$$= -6\cos 4x + 8\sin 4x - 8i\cos 4x - 6i\sin 4x$$
$$= (-6\cos 4x + 8\sin 4x) + i(-8\cos 4x - 6\sin 4x)$$
$$y_p = Y_R = -6\cos 4x + 8\sin 4x$$

    Combine:

$$y = y_c + y_p$$
$$y = e^{-x}(A\cos 3x + B\sin 3x) - 6\cos 4x + 8\sin 4x$$

12. $(D^2 + 4D + 12)y = 80\sin 2x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D + 2 + \sqrt{8}i)(D + 2 - \sqrt{8}i) = 0$$
$$\implies \text{root(s): } -2 \pm \sqrt{8}i$$
$$\text{General Solution: } y_c = e^{-2x}(A\cos\sqrt{8}x + B\sin\sqrt{8}x)$$

Now Find particular Solution:
Because RHS is sin, we will use $Y = Ce^{2ix} = Y_R + iY_I$ s.t. $y_p = Y_I$.

$$Y = Ce^{2ix}; Y' = C(2ie^{2ix}); Y'' = C(-4e^{2ix})$$
$$Y'' + 4Y' + 12Y = C\big((-4e^{2ix}) + 4(2ie^{2ix}) + 12(e^{2ix})\big)$$
$$Y'' + 2Y' + 10Y = (8 + 8i)Ce^{2ix} = 80e^{2ix}$$
$$\text{so } C = \frac{10}{1+i} = \frac{10 - 10i}{2} = 5 - 5i$$
$$Ce^{2ix} = (5 - 5i)(\cos 2x + i\sin 2x)$$
$$= 5\cos 2x + 5i\sin 2x - 5i\cos 2x - 5i^2\sin 2x$$
$$= 5\cos 2x + 5\sin 2x - 5i\cos 2x + 5i\sin 2x$$
$$= (5\cos 2x + 5\sin 2x) + i(-5\cos 2x + 5\sin 2x)$$
$$y_p = Y_I = -5\cos 2x + 5\sin 2x$$

Combine:

$$y = y_c + y_p$$
$$y = e^{-2x}(A\cos\sqrt{8}x + B\sin\sqrt{8}x) - 5\cos 2x + 5\sin 2x$$

13. $(D^2 - 2D + 1)y = 2\cos x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D-1)^2 = 0 \implies \text{root(s): } 1$$
$$\text{General Solution: } y_c = (Ax + B)e^x$$

Now Find particular Solution:
Because RHS is cos, we will use $Y = Ce^{ix} = Y_R + iY_I$ s.t. $y_p = Y_R$.

$$Y = Ce^{ix}; Y' = C(ie^{ix}); Y'' = C(-e^{ix})$$
$$Y'' - 2Y' + Y = C\big((-e^{ix}) - 2(ie^{ix}) + (e^{ix})\big)$$
$$Y'' - 2Y' + Y = (-2i)Ce^{ix} = 2e^{ix}$$
$$\text{so } C = -\frac{1}{i} = i$$
$$Ce^{ix} = (i)(\cos x + i\sin x)$$
$$= i\cos x + i^2 \sin x$$
$$= (-\sin x) + i(\cos x)$$
$$y_p = Y_R = -\sin x$$

Combine:

$$y = y_c + y_p$$
$$y = (Ax + B)e^x - \sin x$$

17. $y'' + 16y = 16\cos 4x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+4i)(D-4i) = 0 \implies \text{root(s): } \pm 4i$$
$$\text{General Solution: } y_c = A\cos 4x + B\sin 4x$$

Now Find particular Solution:
Because RHS is cos, and $4i$ is a root we will use $Y = Cxe^{4ix} = Y_R + iY_I$ s.t. $y_p = Y_R$.

$$Y = Cxe^{4ix}; Y' = C(e^{4ix} + 4ixe^{4ix})$$
$$Y'' = C(8ie^{4ix} - 16xe^{4ix})$$
$$Y'' + 16Y = C\big((8ie^{4ix} - \cancel{16xe^{4ix}}) + \cancel{16(xe^{4ix}})\big)$$
$$Y'' + 16Y = (8i)Ce^{4ix} = 16e^{4ix}$$
$$\text{so } C = \frac{2}{i} = -2i$$
$$Cxe^{ix} = (-2i)x(\cos 4x + i\sin 4x)$$
$$= -2ix\cos 4x - 2i^2 x\sin 4x$$
$$= (2x\sin 4x) + i(-2x\cos 4x)$$
$$y_p = Y_R = 2x\sin 4x$$

Combine:

$$y = y_c + y_p$$
$$y = A\cos 4x + B\sin 4x + 2x\sin 4x$$

18. $(D^2+2D+17)y = 60e^{-4x}\sin 5x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+1+4i)(D+1-4i)=0$$
$$\implies \text{root(s): } -1\pm 4i$$
$$\text{General Solution: } y_c = e^{-x}(A\cos 4x + B\sin 4x)$$

Now Find particular Solution:
Because RHS is $e^{-4x}\sin 5x$, we will use $Y = Ce^{(-4+5i)x} = Y_R + iY_I$ s.t. $y_p = Y_I$.

$$Y = Ce^{(-4+5i)x}; Y' = C(-4+5i)e^{(-4+5i)x}$$
$$Y'' = C(-40i-9)e^{(-4+5i)x}$$
$$Y''+2Y'+17Y =$$
$$C\big((-40i-\cancel{9})e^{(-4+5i)x} + 2(\cancel{-4}+5i)e^{(-4+5i)x} + \cancel{17e^{(-4+5i)x}}\big)$$
$$Y''+2Y'+17Y = -30iCe^{(-4+5i)x} = 60e^{(-4+5i)x}$$
$$\text{so } C = -\frac{2}{i} = 2i$$
$$Ce^{(-4+5i)x} = (2i)e^{-4x}(\cos 5x + i\sin 5x)$$
$$= e^{-4x}(2i\cos 5x + 2i^2\sin 5x)$$
$$= -(2e^{-4x}\sin 5x) + i(2e^{-4x}\cos 5x)$$
$$y_p = Y_I = 2e^{-4x}\cos 5x$$

Combine:

$$y = y_c + y_p$$
$$y = e^{-x}(A\cos 4x + B\sin 4x) + 2e^{-4x}\cos 5x$$

20. $y'' + 4y' + 8y = 30e^{-x/2}\cos(5x/2)$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+2+2i)(D+2-2i) = 0$$
$$\implies \text{root(s): } -2 \pm 2i$$
$$\text{General Solution: } y_c = e^{-2x}(A\cos 2x + B\sin 2x)$$

Now Find particular Solution:
Because RHS is $e^{-x/2}\cos(5x/2)$, we will use $Y = Ce^{(-1/2+5i/2)x} = Y_R + iY_I$ s.t. $y_p = Y_R$.

$$Y = Ce^{(-1/2+5i/2)x}$$
$$Y' = C(-1/2+5i/2)e^{(-1/2+5i/2)x}$$
$$Y'' = C(-6-5i/2)e^{(-1/2+5i/2)x}$$

$$Y'' + 4Y' + 8Y = C\Big(((\cancel{-6} - 5i/2)e^{(-1/2+5i/2)x})$$
$$+4((\cancel{-1/2} + 5i/2)e^{(-1/2+5i/2)x}) + \cancel{8e^{(-1/2+5i/2)x}}\Big)$$
$$Y'' + 2Y' + 17Y = \frac{15}{2}iCe^{(-1/2+5i/2)x} = 30e^{(-1/2+5i/2)x}$$
$$\text{so } C = \frac{60}{15i} = -4i$$

$$Ce^{(-1/2+5i/2)x} = (-4i)e^{-x/2}(\cos(5x/2) + i\sin(5x/2))$$
$$= e^{-x/2}(-4i\cos(5x/2) - 4i^2\sin(5x/2))$$
$$= (4e^{-x/2}\sin(5x/2)) - i(4e^{-x/2}\cos(5x/2))$$
$$y_p = Y_R = 4e^{-x/2}\sin(5x/2)$$

Combine:

$$y = y_c + y_p$$
$$y = e^{-2x}(A\cos 2x + B\sin 2x) + 4e^{-x/2}\sin(5x/2)$$

21. $5y'' + 6y' + 2y = x^2 + 6x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (5D^2 + 6D + 2) = 0$$
$$\text{Quadratic Formula: } \frac{-6 \pm \sqrt{6^2 - 4(5)(2)}}{2(5)}$$
$$\implies \text{root(s): } \frac{-3 \pm i}{5}$$
$$\text{General Solution: } y_c = e^{-3x/5}(A\cos(x/5) + B\sin(x/5))$$

Now Find particular Solution:
Because RHS is $x^2 + 6x$, we will use $y_p = Ax^2 + Bx + C$

$$y_p = Ax^2 + Bx + C;\ y_p' = 2Ax + B;\ y_p'' = 2A$$
$$5y_p'' + 6y_p' + 2y_p = 5(2A) + 6(2Ax + B) + 2(Ax^2 + Bx + C)$$
$$5y_p'' + 6y_p' + 2y_p = 2Ax^2 + 12Ax + 2Bx + 10A + 6B + 2C = x^2 + 6x$$
$$x^2 + 6x = 2Ax^2 + 12Ax + 2Bx + 10A + 6B + 2C$$
$$2A = 1 \text{ so } A = \frac{1}{2}$$
$$6x = 6x + 2Bx + 5 + 6B + 2C$$
$$6x = 6x + 2Bx \text{ so } B = 0$$
$$0 = 5 + 2C \text{ so } C = -\frac{5}{2}$$
$$y_p = \frac{1}{2}x^2 - \frac{5}{2}$$

Combine:

$$y = y_c + y_p$$
$$y = e^{-3x/5}(A\cos(x/5) + B\sin(x/5)) + \frac{1}{2}x^2 - \frac{5}{2}$$

23. $y'' + y = 2xe^x$

First Find General Solution to homogeneous equation:

$$\text{Auxiliary Equation: } (D+i)(D-i) = 0$$
$$\implies \text{root(s): } \pm i$$
$$\text{General Solution: } y_c = A\cos(x) + B\sin(x)$$

Now Find particular Solution:
Because RHS is $2xe^x$, we know solution will be in the form of $Q_1(x)e^x$ where $Q_1(x) = a_1x + a_0$

$$y_p = (a_1x + a_0)e^x$$
$$y_p' = (a_1)e^x + (a_1x + a_0)e^x$$
$$y_p'' = 2(a_1)e^x + (a_1x + a_0)e^x$$
$$y_p'' + y_p = 2(a_1)e^x + (a_1x + a_0)e^x + (a_1x + a_0)e^x$$
$$y_p'' + y_p = 2a_1xe^x + 2(a_1 + a_0)e^x$$
$$2xe^x = 2a_1xe^x + 2(a_1 + a_0)e^x \text{ so } a_1 = 1$$
$$0 = 2(1 + a_0)e^x$$
$$0 = (1 + a_0) \text{ so } a_0 = -1$$
$$y_p = (x-1)e^x$$

Combine:

$$y = y_c + y_p$$
$$y = A\cos(x) + B\sin(x) + (x-1)e^x$$