Problem Set 5

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1. Find and explain the flaw(s) in this argument.

> We prove every nonempty set of people all have the same age.
>
> *Proof.* We denote the age of a person p by $\alpha(p)$.
>
> *Basis Step:*
> Suppose $P = \{p\}$ is a set with one person in it. Clearly, all the people in P have the same age as each other.
>
> *Inductive Step:*
> Let $k \in \mathbb{N}_+$ and suppose any set of k-many people all have the same age. Let $P = \{p_1, p_2, \dots p_k, p_{k+1}\}$ be a set with $k+1$ people in it. Consider $L \coloneqq \{p_1, \dots p_k\}$ and $R \coloneqq \{p_2, \dots p_{k+1}\}$. Since L and R both have k people, we know everyone in these sets has the same age by the *inductive hypothesis.*
>
> Let $\ell, r \in P$. If $\ell \in L \wedge r \in L$, then $\alpha(\ell) = \alpha(r)$. Similarly, if $\ell \in R \wedge r \in R$, then $\alpha(\ell) = \alpha(r)$. Now, suppose $\ell \in L \wedge r \in R$.
>
> $$\alpha(\ell) = \alpha(p_1) = \alpha(p_2) = \alpha(p_{k+1}) = \alpha(r)$$
>
> So, all people in P have the same age.
>
> Therefore, everyone on Earth has the same age. Q.E.D.

The argument wants to prove $\forall n(\varphi(n))$ where n is a nonempty set. However, the inductive step iterates over the size of each set rather than each set. In other words, the inductive step considers a set of k people for all k, but not all sets of k people.

In addition to that, the inductive step falsely assumes that $p_2 \in L$ for every case ($p_2 \notin L$ when k=1). As such, it makes a false claim that $\alpha(p_1) = \alpha(p_2)$.

2. Show that $\forall x(x \neq x \cup \{x\})$.

Proof. Let x be an arbitrary set. Let $y \coloneqq x \cup \{x\}$. By definition of union, we know that $y = \{z | z \in x \vee z \in \{x\}\}$. To show that $x \neq y$, we must find an element that is in one set but not the other (extensionality). Looking at y, we see that $x \in y$ but looking at x, we see that $x \notin x$. Therefore, since there is an element in y that is not in x, $y \neq x$ and as such $x \neq x \cup x$. Q.E.D.

3. We will work up to a proof of the commutativity of addition on $\mathbb{N}$.

   (a) Show $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.

   Proof. Proof by mathematical induction.
   Basis Step:
   Need to show $0 + 0 = 0 + 0$
   $$0 + 0 = 0 + 0$$

   Inductive Step:
   Let $x \in \mathbb{N}$ and assume that $x + 0 = 0 + x$.
   Need to show $\mathbb{S}(x) + 0 = 0 + \mathbb{S}(x)$

   $$\begin{aligned} \mathbb{S}(x) + 0 &= \mathbb{S}(x) && \text{By + Rule 1} \\ &= \mathbb{S}(x + 0) && \text{By + Rule 1} \\ &= \mathbb{S}(0 + x) && \text{By IH} \\ &= 0 + \mathbb{S}(x) && \text{By + Rule 2} \end{aligned}$$

   Thus by mathematical induction, we can conclude $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.
   Q.E.D.

   (b) Show $(\forall x, y \in \mathbb{N})(x + \mathbb{S}(y) = \mathbb{S}(y) + x)$.

   Proof. Let $x, y \in \mathbb{N}$ and assume that $x + y = y + x$.
   Need to show $x + \mathbb{S}(y) = \mathbb{S}(y) + x$

   $$\begin{aligned} x + \mathbb{S}(y) &= \mathbb{S}(x + y) && \text{By + Rule 2} \\ &= \mathbb{S}(y + x) && \text{By IH} \\ &= y + \mathbb{S}(x) && \text{By + Rule 2} \\ &= y + 1 + x && \text{By Theorem 2} \\ &= \mathbb{S}(y) + x && \text{By Theorem 1} \end{aligned}$$

   Q.E.D.

   (c) Show $(\forall x, y \in \mathbb{N})(x + y = y + x)$.

   Proof. Proof by mathematical induction.

   Basis Step: We can conclude $x + 0 = 0 + x$ by 3a
   Inductive Step: We can conclude $x + \mathbb{S}(y) = \mathbb{S}(y) + x$ by 3b

   Thus by mathematical induction, we can conclude $(\forall x, y \in \mathbb{N})(x + y = y + x)$.
   Q.E.D.

4. Show $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.

Proof. Proof by mathematical induction

Basis Step:
Let $x, y \in \mathbb{N}$. Need to show $x \cdot (y + 0) = (x \cdot y) + (x \cdot 0)$.

$$\begin{aligned} x \cdot (y + 0) &= x \cdot y && \text{By + Rule 1} \\ &= (x \cdot y) + 0 && \text{By + Rule 1} \\ &= (x \cdot y) + (x \cdot 0) && \text{By} \cdot \text{Rule 1} \end{aligned}$$

Inductive Step:
Let $x, y, z \in \mathbb{N}$ and assume $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
Need to show $x \cdot (y + \mathbb{S}(z)) = (x \cdot y) + (x \cdot \mathbb{S}(z))$.

$$\begin{aligned} x \cdot (y + \mathbb{S}(z)) &= x \cdot \mathbb{S}(y + z) && \text{By + Rule 2} \\ &= x \cdot (y + z) + x && \text{By} \cdot \text{Rule 2} \\ &= (x \cdot y) + (x \cdot z) + x && \text{By IH} \\ &= (x \cdot y) + (x \cdot \mathbb{S}(z)) && \text{By} \cdot \text{Rule 2} \end{aligned}$$

Therefore by mathematical induction, we can conclude $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.
Q.E.D.

5. Prove the following statement for all $n \in \mathbb{N}$.

$$1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$$

Proof. Proof by mathematical induction

Basis Step:
Need to show $1 + \sum_{i=0}^{0} 2^i = 2^{0+1}$.

$$\begin{aligned}
1 + \sum_{i=0}^{0} 2^i &= 1 + 2^0 && \text{By } \sum \text{ Rule 1} \\
&= 1 + 1 && \text{By } n^m \text{ Rule 1} \\
&= \mathbb{S}(1) && \text{By Theorem} \\
&= 2 && \text{By Def of } \mathbb{S}(n) \\
&= 2 + 0 && \text{By + Rule 1} \\
&= 0 + 2 && \text{By Comm.} \\
&= (2 \cdot 0) + 2 && \text{By } \cdot \text{ Rule 1} \\
&= 2 \cdot \mathbb{S}(0) && \text{By } \cdot \text{ Rule 2} \\
&= 2 \cdot 1 && \text{By Def of } \mathbb{S}(n) \\
&= 2 \cdot 2^0 && \text{By } n^m \text{ Rule 1} \\
&= 2^{\mathbb{S}(0)} && \text{By } n^m \text{ Rule 2} \\
&= 2^1 && \text{By Def of } \mathbb{S}(n) \\
&= 2^{0+1} && \text{By Theorem}
\end{aligned}$$

Inductive Step:
Assume $1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$.
Need to show $1 + \sum_{i=0}^{\mathbb{S}(n)} 2^i = 2^{\mathbb{S}(n)+1}$.

$$\begin{aligned}
1 + \sum_{i=0}^{\mathbb{S}(n)} 2^i &= 1 + \sum_{i=0}^{n} 2^i + 2^{\mathbb{S}(n)} && \text{By } \sum \text{ Rule 2} \\
&= 2^{n+1} + 2^{\mathbb{S}(n)} && \text{By IH} \\
&= 2^{\mathbb{S}(n)} + 2^{\mathbb{S}(n)} && \text{By Def of } \mathbb{S}(n) \\
&= 2^{\mathbb{S}(n)} + 0 + 2^{\mathbb{S}(n)} && \text{By + Rule 1} \\
&= 2^{\mathbb{S}(n)} + (2^{\mathbb{S}(n)} \cdot 0) + 2^{\mathbb{S}(n)} && \text{By } \cdot \text{ Rule 1} \\
&= 2^{\mathbb{S}(n)} + (2^{\mathbb{S}(n)} \cdot \mathbb{S}(0)) && \text{By } \cdot \text{ Rule 2} \\
&= 2^{\mathbb{S}(n)} + (2^{\mathbb{S}(n)} \cdot 1) && \text{By Def of } \mathbb{S}(n) \\
&= (2^{\mathbb{S}(n)} \cdot 1) + 2^{\mathbb{S}(n)} && \text{By Comm.} \\
&= (2^{\mathbb{S}(n)} \cdot \mathbb{S}(1)) && \text{By } \cdot \text{ Rule 2} \\
&= 2^{\mathbb{S}(n)} \cdot 2 && \text{By Def of } \mathbb{S}(n) \\
&= 2 \cdot 2^{\mathbb{S}(n)} && \text{By Comm.} \\
&= 2^{\mathbb{S}(\mathbb{S}(n))} && \text{By } n^m \text{ Rule 2} \\
&= 2^{\mathbb{S}(n)+1} && \text{By Def of } \mathbb{S}(n)
\end{aligned}$$

Therefore by mathematical induction, we can conclude $1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$.

Q.E.D.

6. We say x is $\in$-transitive by definition when $(\forall y \in x)(\forall z \in y)(z \in x)$. Show that every natural number is $\in$-transitive.

Proof. Proof by mathematical induction.

Basis Step:
Let $y, z \in \mathbb{N}$
Need to show: 0 is $\in$-transitive or $(\forall y \in 0)(\forall z \in y)(z \in 0)$.

Assume that $y \in 0$ and $z \in y$. By the recursive definition of natural numbers, $0 \coloneqq \varnothing$. This means that $y \in \varnothing$. However, as the empty set is empty, $y \notin \varnothing$. Thus, we can conclude by the explosion theorem that $z \in \varnothing$. Consequently, 0 is $\in$-transitive.

Inductive Step:
Let $n, y, z \in \mathbb{N}$ and assume $(\forall y \in n)(\forall z \in y)(z \in n)$.
Need to show $(\forall y \in \mathbb{S}(n))(\forall z \in y)(z \in \mathbb{S}(n))$.
Let $y \in \mathbb{S}(n)$ and $z \in y$. First, observe that by definition, $\mathbb{S}(n) = n \cup \{n\}$. This means that $y \in n \vee y \in \{n\}$. Now let's consider these as two cases.

$y \in n$:
We can see that $z \in n$ by inductive hypothesis.

$y \in \{n\}$:
Since $y \in \{n\}$, we can see that $y = n$. Since we know that $z \in y$, we know that $z \in n$ by extensionality. Furthermore, since $\mathbb{S}(n) = n \cup \{n\}$, we know that $z \in \mathbb{S}(n)$.

As such, by mathematical induction, we can conclude that $(\forall y \in x)(\forall z \in y)(z \in x)$.

Q.E.D.