Homework 09

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Section 8 Find the norm of each of the following functions on the given interval and state the normalized function.

1. $\cos nx$ on $(0, \pi)$

$$\|\cos nx\| = \sqrt{\langle \cos nx, \cos nx \rangle} = \sqrt{\int_0^\pi \cos^2 nx dx} = \sqrt{\frac{1}{2}\int_0^\pi 1 + \cos 2nx dx} = \sqrt{\frac{1}{2}\left(x + \frac{1}{2n}\sin 2nx\right)\Big|_0^\pi}$$
$$= \sqrt{\frac{1}{2}\left[\left(\pi + \frac{1}{2n}\sin(2n\pi)\right) - \left(0 + \frac{1}{2n}\sin(2n(0))\right)\right]} = \sqrt{\frac{1}{2}\left[\left(\pi + 0\right) - \left(0 + 0\right)\right]} = \sqrt{\frac{\pi}{2}}$$

$$\frac{\cos nx}{\|\cos nx\|} = \frac{\cos nx}{\sqrt{\pi/2}} = \sqrt{\frac{2}{\pi}} \cdot \cos nx$$

2. $P_2(x)$ on $(-1, 1)$

$$\|P_2(x)\| = \sqrt{\langle P_2(x), P_2(x) \rangle} = \sqrt{\int_{-1}^1 (P_2(x))^2 dx} = \sqrt{\int_{-1}^1 \left(\frac{1}{2}(3x^2 - 1)\right)^2 dx}$$
$$= \sqrt{\frac{1}{4}\int_{-1}^1 (9x^4 - 6x^2 + 1) dx} = \sqrt{\frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x\right)\Big|_{-1}^1}$$
$$= \sqrt{\frac{1}{4}\left[\left(\frac{9}{5}(1)^5 - 2(1)^3 + (1)\right) - \left(\frac{9}{5}(-1)^5 - 2(-1)^3 + (-1)\right)\right]}$$
$$= \sqrt{\frac{1}{2}\left(\frac{9}{5} - 2 + 1\right)} = \sqrt{\frac{1}{2}\left(\frac{4}{5}\right)} = \sqrt{\frac{2}{5}}$$

$$\frac{P_2(x)}{\|P_2(x)\|} = \frac{\frac{1}{2}(3x^2 - 1)}{\sqrt{2/5}} = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$$

3. $xe^{-x/2}$ on $(0, \infty)$

Sign	$u = x^2$ (Derivative)	$dv = e^{-x}$ (Integral)
$+$	x^2	$-e^{-x}$
$-$	$2x$	e^{-x}
$+$	2	$-e^{-x}$

$$\left\|xe^{-x/2}\right\| = \sqrt{\langle xe^{-x/2}, xe^{-x^2/2} \rangle} = \sqrt{\int_0^\infty (xe^{-x/2})^2 dx} = \sqrt{\int_0^\infty x^2 e^{-x} dx}$$
$$= \sqrt{-x^2e^{-x} - 2x^{-x} - 2e^{-x}\Big|_0^\infty} = \sqrt{(0 - 0 - 0) - (0 - 0 - 2)} = \sqrt{2}$$

$$\frac{xe^{-x/2}}{\|xe^{-x/2}\|} = \frac{xe^{-x/2}}{\sqrt{2}}$$

Section 9 Expand the following functions in Legendre series up to C_4

1. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

   Observe that

$$\int_{-1}^{1} f(x)P_l(x)dx = \begin{cases} 0, & l \text{ is even} \\ 2\int_0^1 P_l(x)dx, & l \text{ is odd} \end{cases}$$

$$\int_{-1}^{1} f(x)P_0(x)dx = C_0 \cdot \frac{2}{2(0)+1} = 2C_0$$
$$\int_{-1}^{1} f(x)P_0(x)dx = 0$$
$$2C_0 = 0 \implies C_0 = 0$$

$$\int_{-1}^{1} f(x)P_1(x)dx = C_1 \cdot \frac{2}{2(1)+1} = \frac{2}{3}C_1$$
$$\int_{-1}^{1} f(x)P_1(x)dx = 2\int_0^1 xdx = x^2\Big|_0^1 = 1$$
$$\frac{2}{3}C_1 = 1 \implies C_1 = \frac{3}{2}$$

$$\int_{-1}^{1} f(x)P_2(x)dx = C_2 \cdot \frac{2}{2(2)+1} = \frac{2}{5}C_2$$
$$\int_{-1}^{1} f(x)P_2(x)dx = 0$$
$$\frac{2}{5}C_2 = 0 \implies C_2 = 0$$

$$\int_{-1}^{1} f(x)P_3(x)dx = C_0 \cdot \frac{2}{2(3)+1} = \frac{2}{7}C_3$$
$$\int_{-1}^{1} f(x)P_3(x)dx = 2\int_0^1 \frac{1}{2}(5x^3 - 3x)dx = \int_0^1 (5x^3 - 3x)dx = \frac{5}{4}x^4 - \frac{3}{2}x^2\Big|_0^1 = -\frac{1}{4}$$
$$\frac{2}{7}C_3 = -\frac{1}{4} \implies C_3 = -\frac{7}{8}$$

$$\int_{-1}^{1} f(x)P_4(x)dx = C_4 \cdot \frac{2}{2(4)+1} = \frac{2}{9}C_4$$
$$\int_{-1}^{1} f(x)P_4(x)dx = 0$$
$$\frac{2}{9}C_4 = 0 \implies C_4 = 0$$

   The Legendre series for this function is:

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + ...$$

2. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$

$$\int_{-1}^{1} f(x)P_0(x)dx = C_0 \cdot \frac{2}{2(0)+1} = 2C_0$$
$$\int_{-1}^{1} f(x)P_0(x)dx = \int_{0}^{1} xdx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$
$$2C_0 = \frac{1}{2} \implies C_0 = \frac{1}{4}$$

$$\int_{-1}^{1} f(x)P_1(x)dx = C_1 \cdot \frac{2}{2(1)+1} = \frac{2}{3}C_1$$
$$\int_{-1}^{1} f(x)P_1(x)dx = \int_{0}^{1} x^2dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$
$$\frac{2}{3}C_1 = \frac{1}{3} \implies C_1 = \frac{1}{2}$$

$$\int_{-1}^{1} f(x)P_2(x)dx = C_2 \cdot \frac{2}{2(2)+1} = \frac{2}{5}C_2$$
$$\int_{-1}^{1} f(x)P_2(x)dx = \int_{0}^{1} \frac{3}{2}x^3 - \frac{1}{2}xdx = \frac{3}{8}x^4 - \frac{1}{4}x^2\Big|_0^1 = \frac{1}{8}$$
$$\frac{2}{5}C_2 = \frac{1}{8} \implies C_2 = \frac{5}{16}$$

$$\int_{-1}^{1} f(x)P_3(x)dx = C_0 \cdot \frac{2}{2(3)+1} = \frac{2}{7}C_3$$
$$\int_{-1}^{1} f(x)P_3(x)dx = \int_{0}^{1} \frac{5}{2}x^4 - \frac{3}{2}x^2dx = \frac{1}{2}x^5 - \frac{1}{2}x^3\Big|_0^1 = 0$$
$$\frac{2}{7}C_3 = 0 \implies C_3 = 0$$

$$\int_{-1}^{1} f(x)P_4(x)dx = C_4 \cdot \frac{2}{2(4)+1} = \frac{2}{9}C_4$$
$$\int_{-1}^{1} f(x)P_4(x)dx = \int_{0}^{1} \frac{35}{8}x^5 - \frac{30}{8}x^3 + \frac{3}{8}xdx = \frac{35}{48}x^6 - \frac{15}{16}x^4 + \frac{3}{16}x^2\Big|_0^1 = -\frac{1}{48}$$
$$\frac{2}{9}C_4 = -\frac{1}{48} \implies C_4 = -\frac{3}{32}$$

The Legendre series for this function is:

$$\frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + ...$$

3. $f(x) = P_3'(x)$

$$P_3'(x) = (\frac{5}{2}x^3 - \frac{3}{2}x)' = \frac{15}{2}x^2 - \frac{3}{2}$$

Observe that

$$\int_{-1}^{1} f(x)P_l(x)dx = \begin{cases} 0, & l \text{ is odd} \\ 2\int_0^1 (\frac{15}{2}x^2 - \frac{3}{2})P_l(x)dx, & l \text{ is even} \end{cases}$$

$$\int_{-1}^{1} f(x)P_0(x)dx = C_0 \cdot \frac{2}{2(0)+1} = 2C_0$$

$$\int_{-1}^{1} f(x)P_0(x)dx = 2\int_0^1 \frac{15}{2}x^2 - \frac{3}{2}dx = 5x^3 - 3x\Big|_0^1 = 2$$

$$2C_0 = 2 \implies C_0 = 1$$

$$\int_{-1}^{1} f(x)P_1(x)dx = C_1 \cdot \frac{2}{2(1)+1} = \frac{2}{3}C_1$$

$$\int_{-1}^{1} f(x)P_1(x)dx = 0$$

$$\frac{2}{3}C_1 = 0 \implies C_1 = 0$$

$$\int_{-1}^{1} f(x)P_2(x)dx = C_2 \cdot \frac{2}{2(2)+1} = \frac{2}{5}C_2$$

$$\int_{-1}^{1} f(x)P_2(x)dx = 2\int_0^1 \Big(\frac{15}{2}x^2 - \frac{3}{2}\Big)\Big(\frac{3}{2}x^2 - \frac{1}{2}\Big)dx$$

$$= \int_0^1 \frac{45}{2}x^4 - 12x^2 + \frac{3}{2}dx = \frac{9}{2}x^5 - 4x^3 + \frac{3}{2}x\Big|_0^1 = 2$$

$$\frac{2}{5}C_2 = 2 \implies C_2 = 5$$

$$\int_{-1}^{1} f(x)P_3(x)dx = C_0 \cdot \frac{2}{2(3)+1} = \frac{2}{7}C_3$$

$$\int_{-1}^{1} f(x)P_3(x)dx = 0$$

$$\frac{2}{7}C_3 = 0 \implies C_3 = 0$$

$$\int_{-1}^{1} f(x)P_4(x)dx = C_4 \cdot \frac{2}{2(4)+1} = \frac{2}{9}C_4$$

$$\int_{-1}^{1} f(x)P_4(x)dx = 2\int_0^1 \Big(\frac{15}{2}x^2 - \frac{3}{2}\Big)\Big(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}\Big)dx$$

$$= \frac{1}{8}\int_0^1 525x^6 - 555x^4 + 135x^2 - 9dx = \frac{1}{8}\Big(75x^7 - 111x^5 + 45x^3 - 9\Big)\Big|_0^1 = 0$$

$$\frac{2}{9}C_4 = 0 \implies C_4 = 0$$

The Legendre series for this function is:

$$P_0(x) + 5P_2(x) + ...$$

5. $f(x) = \begin{cases} x+1 & -1 < x < 0 \\ -x+1 & 0 < x < 1 \end{cases}$

Observe:

$$\begin{aligned} \int_{-1}^{1} f(x)P_l(x)dx &= \int_{-1}^{0} (x+1)P_l(x)dx + \int_{0}^{1} (-x+1)P_l(x)dx \\ &= \int_{-1}^{0} xP_l(x)dx - \int_{0}^{1} xP_l(x)dx + \int_{-1}^{1} P_l(x)dx \\ &= \int_{-1}^{0} xP_l(x)dx + \int_{1}^{0} xP_l(x)dx + \int_{-1}^{1} P_l(x)dx \end{aligned}$$

Observe: when $l \neq 0$

$$\int_{-1}^{1} P_l(x)dx = \int_{-1}^{1} P_0(x)P_l(x)dx = 0$$

and when l is even

$$\int_{-1}^{0} xP_l(x)dx + \int_{1}^{0} xP_l(x)dx = 0$$

and when l is odd

$$\int_{-1}^{0} xP_l(x)dx + \int_{1}^{0} xP_l(x)dx = 2\int_{-1}^{0} xP_l(x)dx$$

so for $l > 0$

$$\int_{-1}^{1} f(x)P_l(x)dx = \begin{cases} 0, & l \text{ is odd} \\ 2\int_{-1}^{0} xP_l(x)dx, & l \text{ is even} \end{cases}$$

$$\int_{-1}^{1} f(x)P_0(x)dx = C_0 \cdot \frac{2}{2(0)+1} = 2C_0$$

$$\int_{-1}^{1} f(x)P_0(x)dx = 2\int_{-1}^{0} xdx + \int_{-1}^{1} dx = -1 + 2 = 1$$

$$2C_0 = 1 \implies C_0 = \frac{1}{2}$$

$$\int_{-1}^{1} f(x)P_2(x)dx = C_2 \cdot \frac{2}{2(2)+1} = \frac{2}{5}C_2$$

$$\int_{-1}^{1} f(x)P_2(x)dx = 2\int_{-1}^{0} \frac{3}{2}x^3 - \frac{1}{2}xdx = \frac{3}{4}x^4 - \frac{1}{2}x^2\Big|_{-1}^{0} dx = -\frac{1}{4}$$

$$\frac{2}{5}C_2 = -\frac{1}{4} \implies C_2 = -\frac{5}{8}$$

$$\int_{-1}^{1} f(x)P_4(x)dx = C_4 \cdot \frac{2}{2(4)+1} = \frac{2}{9}C_4$$

$$\int_{-1}^{1} f(x)P_4(x)dx = 2\int_{-1}^{0} \frac{35}{8}x^5 - \frac{30}{8}x^3 + \frac{3}{8}xdx = \frac{35}{24}x^6 - \frac{15}{8}x^4 + \frac{3}{8}x^2\Big|x_{-1}^{0} = \frac{1}{24}$$

$$\frac{2}{9}C_4 = \frac{1}{24} \implies C_4 = \frac{3}{16}$$

The Legendre series for this function[1] is:

$$\frac{1}{2}P_0(x) - \frac{5}{8}P_2(x) + \frac{3}{16}P_4(x) + ...$$

[1] $C_1, C_3 = 0$ because when l is odd, the above integral is 0

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Section 16 Find the solutions of the following differential equations in terms of Bessel functions.

2. $y'' + 4x^2y = 0$

$$y'' + 4x^2y = y'' + \frac{1-2a}{x}y' + \Big[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\Big]y$$

$$\begin{aligned}
1 - 2a = 0 \implies 2a = 1 \implies a = \frac{1}{2}\\
(bcx^{c-1})^2 = 4x \implies b^2c^2x^{2c-2} = 4x^2 \implies 2c - 2 = 2 \implies c = 2\\
(bcx^{c-1})^2 = 4x \implies b^2c^2 = 4 \implies b^2(2)^2 = 4 \implies b = 1\\
x^2 = 0 \implies \frac{1}{2}^2 - p^2(2)^2 = 0 \implies 16p^2 = 1 \implies p = \frac{1}{4}
\end{aligned}$$

$$y = x^{1/2}Z_{1/4}(x^2)$$

3. $xy'' + 2y' + 4y = 0$

$$\frac{1}{x}\Big(xy'' + 2y' + 4y\Big) = y'' + \frac{2y'}{x} + \frac{4y}{x}$$

$$y'' + \frac{2y'}{x} + \frac{4y}{x} = y'' + \frac{1-2a}{x}y' + \Big[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\Big]y$$

$$\begin{aligned}
1 - 2a = 2 \implies 2a = -1 \implies a = -\frac{1}{2}\\
(bcx^{c-1})^2 = 4x^{-1} \implies b^2c^2x^{2c-2} = 4x^{-1} \implies 2c - 2 = -1 \implies c = \frac{1}{2}\\
(bcx^{c-1})^2 = 4x^{-1} \implies b^2c^2 = 4 \implies b^2(\frac{1}{2})^2 = 4 \implies b = 4\\
a^2 - p^2c^2 = 0 \implies \Big(-\frac{1}{2}\Big)^2 - p^2\Big(\frac{1}{2}\Big)^2 = 0 \implies p^2 = 1 \implies p = 1
\end{aligned}$$

$$y = x^{-1/2}Z_1(4x^{1/2})$$

4. $3xy'' + 2y' + 12y = 0$

$$\frac{1}{3x}\Big(3xy'' + 2y' + 12y\Big) = y'' + \frac{2y'}{3x} + \frac{4y}{x}$$

$$y'' + \frac{2y'}{3x} + \frac{4y}{x} = y'' + \frac{1-2a}{x}y' + \Big[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\Big]y$$

$$\begin{aligned}
1 - 2a = \frac{2}{3} \implies 2a = 1 - \frac{2}{3} \implies a = \frac{1}{6}\\
(bcx^{c-1})^2 = 4x^{-1} \implies b^2c^2x^{2c-2} = 4x^{-1} \implies 2c - 2 = -1 \implies c = \frac{1}{2}\\
(bcx^{c-1})^2 = 4x^{-1} \implies b^2c^2 = 4 \implies b^2(\frac{1}{2})^2 = 4 \implies b = 4\\
a^2 - p^2c^2 = 0 \implies \Big(\frac{1}{6}\Big)^2 - p^2\Big(\frac{1}{2}\Big)^2 = 0 \implies p^2 = \frac{1}{9} \implies p = \frac{1}{3}
\end{aligned}$$

$$y = x^{1/6}Z_{1/3}(4x^{1/2})$$

5. $y'' - \frac{1}{x}y' + \left(4 + \frac{1}{x^2}\right)y = 0$

$$y'' - \frac{1}{x}y' + 4y + \frac{1}{x^2}y = y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right]y$$

$$1 - 2a = -1 \implies 2a = 2 \implies a = 1$$
$$(bcx^{c-1})^2 = 4 \implies b^2c^2x^{2c-2} = 4 \implies 2c - 2 = 0 \implies c = 1$$
$$(bcx^{c-1})^2 = 4 \implies b^2c^2 = 4 \implies b^2(1)^2 = 4 \implies b = 2$$
$$a^2 - p^2c^2 = 1 \implies (1)^2 - p^2(1)^2 = 1 \implies 1 - p^2 = 1 \implies p = 0$$

$$y = xZ_0(2x)$$

6. $4xy'' + y = 0$

$$\frac{1}{4x}\left(4xy'' + y\right) = y'' + \frac{y}{4x}$$
$$y'' + \frac{y}{4x} = y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right]y$$

$$1 - 2a = 0 \implies 2a = 1 \implies a = \frac{1}{2}$$
$$(bcx^{c-1})^2 = \frac{1}{4x} \implies b^2c^2x^{2c-2} = \frac{1}{4x} \implies 2c - 2 = -1 \implies c = \frac{1}{2}$$
$$(bcx^{c-1})^2 = \frac{1}{4x} \implies b^2c^2 = \frac{1}{4} \implies b^2(\frac{1}{2})^2 = \frac{1}{4x} \implies b = 1$$
$$a^2 - p^2c^2 = 0 \implies \left(\frac{1}{2}\right)^2 - p^2\left(\frac{1}{2}\right)^2 = 0 \implies p^2 = 1 \implies p = 1$$

$$y = x^{1/2}Z_1(x^{1/2})$$

7. $xy'' + 3y' + x^3y = 0$

$$\frac{1}{x}\left(xy'' + 3y' + x^3y\right) = y'' + \frac{3}{x}y' + x^2y$$
$$y'' + \frac{3}{x}y' + x^2y = y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right]y$$

$$1 - 2a = 3 \implies 2a = -2 \implies a = -1$$
$$(bcx^{c-1})^2 = x^2 \implies b^2c^2x^{2c-2} = x^2 \implies 2c - 2 = 2 \implies c = 2$$
$$(bcx^{c-1})^2 = x^2 \implies b^2c^2 = 1 \implies b^2(2)^2 = 1 \implies b = \frac{1}{2}$$
$$a^2 - p^2c^2 = 0 \implies (-1)^2 - p^2(2)^2 = 0 \implies 4p^2 = 1 \implies p = \frac{1}{2}$$

$$y = x^{-1}Z_{1/2}(x^2/2)$$