Homework 02

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1 Wednesday 1/22

Section 2

27. According to Newton's law of cooling, the rate at which the temperature of an object changes is proportional to the difference between its temperature and that of its surroundings. A cup of coffee at 200° in a room of temperature 70° is stirred continually and reaches 100° after 10 min. At what time was it at 120°?

$$T(t) = T_s + (T_0 - T_s)e^{-kt} = 70 + 130e^{-kt}$$

$$T(10) = 100 = 70 + 130e^{-k(10)} \text{ so } k = -\frac{\ln(\frac{3}{13})}{10}$$

$$T(t_1) = 120 = 70 + 130e^{-k(t_1)} \text{ so } t_1 = -\frac{\ln(\frac{5}{13})}{k}$$

$$t_1 = -\frac{\ln(\frac{5}{13})}{-\frac{\ln(\frac{3}{13})}{10}} = 10\frac{\ln(\frac{5}{13})}{\ln(\frac{3}{13})} = 6.52 \text{ min}$$

Section 3

1. $y' + y = e^x$

$$P(x) = 1; \quad Q(x) = e^x$$

$$I(x) = \int P(x)dx = \int 1dx = x$$

$$e^{-I(x)} = e^{-x}$$

$$\int e^{I(x)}Q(x)dx = \int e^x e^x = \int e^{2x} = \frac{1}{2}e^{2x} + C$$

$$y = e^{-I(x)} \int e^{I(x)}Q(x)dx = e^{-x}(\frac{1}{2}e^{2x} + C)$$

$$y = \frac{e^x}{2} + \frac{C}{e^x}$$

2. $x^2y' + 3xy = 1$

$$y' + 3x^{-1}y = x^{-2}$$
$$P(x) = 3x^{-1}; \quad Q(x) = x^{-2}$$
$$I(x) = \int P(x)dx = \int 3x^{-1}dx = 3\ln|x|$$
$$e^{-I(x)} = e^{-3\ln|x|} = x^{-3}$$
$$\int e^{I(x)}Q(x)dx = \int e^{3\ln|x|}x^{-2}dx = \int xdx = \frac{1}{2}x^2 + C$$
$$y = e^{-I(x)}\int e^{I(x)}Q(x)dx = x^{-3}(\frac{1}{2}x^2 + C)$$
$$y = \frac{1}{2x} + \frac{C}{x^3}$$

3. $dy + (2xy - xe^{-x^2})dx = 0$

$$y' + 2xy = xe^{-x^2}$$
$$P(x) = 2x; \quad Q(x) = xe^{-x^2}$$
$$I(x) = \int P(x)dx = \int 2xdx = x^2$$
$$e^{-I(x)} = e^{-x^2} = e^{-x^2}$$
$$\int e^{I(x)}Q(x)dx = \int e^{x^2}xe^{-x^2}dx = \int xdx = \frac{1}{2}x^2 + C$$
$$y = e^{-I(x)}\int e^{I(x)}Q(x)dx = e^{-x^2}(\frac{1}{2}x^2 + C)$$
$$y = \frac{x^2}{2e^{x^2}} + \frac{C}{e^{x^2}}$$

4. $2xy' + y = 2x^{5/2}$

$$y' + \frac{y}{2x} = x^{3/2}$$
$$P(x) = \frac{1}{2x}; \quad Q(x) = x^{3/2}$$
$$I(x) = \int P(x)dx = \int 2\frac{1}{2x}dx = \frac{1}{2}\ln|x|$$
$$e^{-I(x)} = e^{-\frac{1}{2}\ln|x|} = x^{-1/2}$$
$$\int e^{I(x)}Q(x)dx = \int e^{\frac{1}{2}\ln|x|}x^{3/2}dx = \int x^2dx = \frac{1}{3}x^3 + C$$
$$y = e^{-I(x)}\int e^{I(x)}Q(x)dx = x^{-1/2}(\frac{1}{3}x^3 + C)$$
$$y = \frac{1}{3}x^{5/2} + \frac{C}{x^{1/2}}$$

7. $(1+e^x)y' + 2e^xy = (1+e^x)e^x$

$$y' + \frac{2e^x}{1+e^x}y = e^x$$
$$P(x) = \frac{2e^x}{1+e^x}; \quad Q(x) = e^x$$
$$I(x) = \int P(x)dx = \int \frac{2e^x}{1+e^x}dx = 2\ln|1+e^x|$$
$$e^{-I(x)} = e^{-2\ln|1+e^x|} = (1+e^x)^{-2}$$
$$\int e^{I(x)}Q(x)dx = \int e^{2\ln|1+e^x|}e^xdx = \int e^x(1+e^x)^2dx = \frac{1}{3}(1+e^x)^3 + C$$
$$y = e^{-I(x)}\int e^{I(x)}Q(x)dx = (1+e^x)^{-2}(\frac{1}{3}(1+e^x)^3 + C)$$
$$y = \frac{1}{3}(1+e^x) + C(1+e^x)^{-2}$$

Section 4

1. $y' + y = xy^{2/3}$

$$n = \frac{2}{3}; \quad z = y^{1-n} = y^{1/3}; \quad z' = \frac{1}{3}y^{-2/3}y'$$
$$\frac{1}{3}y^{-2/3}y' + \frac{1}{3}y^{1/3} = \frac{1}{3}x$$
$$z' + \frac{1}{3}z = \frac{1}{3}x$$
$$P(x) = \frac{1}{3}; \quad Q(x) = \frac{1}{3}x$$
$$I(x) = \int P(x)dxdx = \int \frac{1}{3}dx = \frac{1}{3}x$$
$$e^{-I(x)} = e^{-x/3}$$
$$\int e^{I(x)}Q(x) = \int e^{x/3}\frac{1}{3}xdx = \int xe^{x/3}\frac{1}{3}dx$$
$$\text{Let } u = \frac{x}{3}; \quad du = \frac{1}{3}dx$$
$$\int ue^u du = ue^u - \int e^u du = ue^u - e^u + C$$
$$\int xe^{x/3}\frac{1}{3}dx = 3\int ue^u du = xe^{x/3} - 3e^{x/3} + C$$
$$z = e^{-I(x)}\int e^{I(x)}Q(x)dx = e^{-x/3}(xe^{x/3} - 3e^{x/3} + C)$$
$$z = x - 3 + \frac{C}{e^{x/3}}$$
$$y = \left(x - 3 + \frac{C}{e^{x/3}}\right)^3$$

2. $y' + \frac{1}{x}y = 2x^{3/2}y^{1/2}$

$$n = \frac{1}{2}; \quad z = y^{1-n} = y^{1/2}; \quad z' = \frac{1}{2}y^{-1/2}y'$$

$$\frac{1}{2}y^{-1/2}y' + \frac{1}{2x}y^{1/2} = x^{3/2}$$

$$z' + \frac{1}{2x}z = x^{3/2}$$

$$P(x) = \frac{1}{2x}; \quad Q(x) = x^{3/2}$$

$$I(x) = \int P(x)dx = \int \frac{1}{2x}dx = \frac{1}{2}\ln|x|$$

$$e^{-I(x)} = e^{-\ln|x|/2} = x^{-1/2}$$

$$\int e^{I(x)}Q(x)dx = \int x^{1/2}x^{3/2}dx = \int x^2 dx = \frac{x^3}{3} + C$$

$$z = e^{-I(x)}\int e^{I(x)}Q(x)dx = x^{-1/2}(\frac{x^3}{3} + C)$$

$$z = \frac{x^{5/2}}{3} + \frac{C}{x^{1/2}}$$

$$y = \left(\frac{x^{5/2}}{3} + \frac{C}{x^{1/2}}\right)^2$$