

Problem Set 1

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1. Determine the truth value of each sentence below.

(a) Madrid is the capital of Spain.

True: Here, we have a declarative sentence. Since the aforementioned declarative sentence is accurate, (a) is true.

(b) Santa Claus lives on the North Pole.

False: Here we have a declarative sentence. Looking at the North Pole, we can see that Santa Claus does not live there. Subsequently, (b) is false.

(c) This sentence is *false*.

No Truth Value: Let's assume that (c) is true. Asserting (c) as true would cause the self-referential statement to assert (c) as false. Therefore (c) can not be true. Now, let's assume that (c) is false. Asserting (c) as false would cause us to believe that the self-referential statement is not false. Consequently, as something can not be both false and not false, (c) can not be false. Since (c) can not be true or false, there is no truth value.

(d) The set of all sets that don't contain themselves contains itself.

No Truth value: Let's declare $S :=$ the set of all sets that don't contain themselves. Assuming $(d) \equiv \top$, S contains itself. Yet, if S contains itself, then it must not contain itself as S is a set of all sets that don't contain themselves. As such, we have achieved a contradiction. Assuming $S \equiv \perp$, S does not contain itself. Yet, if S does not contain itself, then it must contain itself. As such, we have achieved a contradiction because in both assumptions, $(d) \equiv \top$ and $(d) \equiv \perp$, a contradiction is returned. Thus, (d) has no truth value.

(e) Red is a beautiful color.

False: This is a declarative statement I do not believe to be true. Red is not a beautiful color, therefore (e) is false.

(f) Every declarative sentence is either *true* or *false* but not both.

False: Although the axiom (principle of Bivalence) states that any sentence expressing a truth value is either true or false but not both, not every well-formed, declarative sentence expresses a truth value. Therefore, (f) is false.

- (g) If this sentence is *false*, then 7 is a prime number.

True: (g) is a conditional and can thus be separated into two atomic statements. Let $p :=$ “this sentence is false” where this sentence refers to $p \rightarrow q$ and let $q :=$ “7 is a prime number.” First, it is evident that $q \equiv \top$ as it is an accurate declarative statement. Therefore only two conditions are now possible

$$p \equiv \top \text{ and } q \equiv \top \text{ meaning } (p \rightarrow q) \equiv \top \quad (1)$$

$$p \equiv \perp \text{ and } q \equiv \top \text{ meaning } (p \rightarrow q) \equiv \top \quad (2)$$

Considering (1), a contradiction occurs as $p \rightarrow q \equiv \top$ and $p \equiv \top$ declares that $p \rightarrow q \equiv \perp$. Considering (2), no contradictions occur: $p \rightarrow q \equiv \top$ and $p \equiv \perp$ declares that $p \rightarrow q \equiv \neg \perp$. Thus, (g) is true.

- (h) The set of all sets contains itself.

True: The statement claims that the set contains all sets. Therefore, if the set contains all sets, it must include the set that is itself. Thus, (h) is true.

- (i) This sentence is *true*.

True or False: Let’s assume that (i) is true. Asserting (i) as true would cause the self-referential statement to assert (i) itself as true. Therefore (i) can be true. Now, let’s assume that (i) is false. Asserting (i) as false would cause us to believe that the self-referential statement is not true. Since both trains of thought do not end in a contradiction, (i) can be both true or false.

- (j) If this sentence is *true*, then 2 is an odd number.

No Truth Value: (j) is a conditional and can thus be separated into two atomic statements. Let $p :=$ “this sentence is true” where this sentence refers to $p \rightarrow q$ and let $q :=$ “2 is an odd number.” First, it is evident that $q \equiv \perp$ as it is an inaccurate declarative statement. Therefore only two conditions are now possible

$$p \equiv \top \text{ and } q \equiv \perp \text{ meaning } (p \rightarrow q) \equiv \perp \quad (3)$$

$$p \equiv \perp \text{ and } q \equiv \perp \text{ meaning } (p \rightarrow q) \equiv \top \quad (4)$$

Considering (3), a contradiction occurs as $p \rightarrow q \equiv \perp$ and $p \equiv \top$ declares that $p \rightarrow q \equiv \top$. Considering (4), a contradiction occurs as $p \rightarrow q \equiv \top$ and $p \equiv \perp$ declares that $p \rightarrow q \equiv \neg \top$. Thus, as the only two possible conditions are contradictions, there is no truth value.

2. Suppose we have an infinite sequence of sentences

$$S_0, S_1, S_2, \dots S_i, \dots \quad (5)$$

where each sentence asserts that every sentence following it is *false*.

$$S_i := "S_j \text{ is false for all } j > i."$$

What are the truth values of the sentences in this sequence?

S_i has No Truth Value for all $i \geq 0$: Let us separate this question into two separate assumptions. The first where $S_0 \equiv \top$ and the second where $S_0 \equiv \perp$.

- (a) Case 1: $S_0 \equiv \top$

In the first case, we assume that $S_0 \equiv \top$. From this, we learn that $S_n \equiv \perp$ for $n > 0$. Moving to S_1 we learn at least one $S_m \equiv \neg\perp$ for $m > 1$ because we do not believe what S_1 declares. Consequently, there is at least one statement S_k that is contradictory for $k > 1$ as S_0 implies that every $S_k \equiv \perp$ and S_1 declares that at least one $S_k \equiv \neg\perp$. \top

- (b) Case 2: $S_0 \equiv \perp$

In the second case, we assume that $S_0 \equiv \perp$. From this, we learn $S_n \equiv \neg\perp \equiv \top$ for a value $n > 0$. Looking at this we fall into the same logic that led Case 1 to a contradiction as S_n acts like S_0

After analyzing these two cases, we then realize that S_0 has no truth value. We also realize that once S_0 has no truth value, S_1 acts like S_0 , and then the same set of contradictions occur. Consequently, no sentence in this infinite series has any truth value.

3. In the sentence below, “*you*” refers to *you*, the student reading these sentences and solving this problem set. Determine the truth value of the following sentence.

“You have finitely many beliefs.”

True: I have lived for a finite amount of time. With that, neural signals in the brain travel slower than the speed of light, and neural signals have to travel over at least 1 micron of neurons. Because of these two facts, I can not think faster than the time it takes for my brain to send a neural signal over the neurons in my head. Thus, I can not form a belief faster than the time it takes my brain to send a neural signal. Assuming that every thought and as such every belief forms at the speed of light, only a finite amount of beliefs would be able to be formed in a finite amount of time. Consequently, as I have lived for a finite amount of time, “I have finitely many beliefs” is a true statement.