Homework 01

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1. **Proof Practice**

   (a) Convert this paragraph proof to a statement–reason proof. Please be sure to write which statement(s) each statement depends on.

   To show: If s is a string, every substring of a substring of s is a substring of s.

   Proof: Let y be a substring of s, that is, $s = xyz$ for some x, z; and let v be a substring of y, that is, $y = uvw$ for some u, w. Then $s = xuvwz$, so v is a substring of s.

1. y is a substring of s	Given
2. v is a substring of y	Given
3. $\exists x, z$ s.t. $s = xyz$	(1) definition of substring
4. $\exists u, w$ s.t. $y = uvw$	(2) definition of substring
5. $s = xuvwz$	(3), (4), substitution
6. v is a substring of s	(5) definition of substring

   (b) Convert this statement-reason proof to a paragraph proof.

   To show: If w is a string, every prefix of a suffix of w is a suffix of a prefix of w.

1. v is a suffix of w	Given
2. y is a prefix of v	Given
3. $\exists x$ s.t. $xv = w$	(1), definition of suffix
4. $\exists z$ s.t. $yz = v$	(2), definition of prefix
5. $xyz = w$	(3), (4), substitution
6. xy is a prefix of w	(5), definition of prefix
7. y is a suffix of xy	(6), definition of suffix

   Proof: Let v be a suffix of w, that is, $w = xv$ for some x. Let y be a prefix of v, that is, $v = yz$ for some x. Then $w = xyz$, so xy is a prefix of w and thus y is a suffix of xy.

2. **String homomorphisms.** If Σ and Γ are finite alphabets, define a *string homomorphism* to be a function $f : \Sigma^* \to \Gamma^*$ that has the property that for any $u, v \in \Sigma^*$,

$$f(uv) = f(u)f(v)$$

Prove that, in general, every string homomorphism operates by replacing each symbol with a (possibly empty) string. That is, prove that if f is a string homomorphism, then for any $w = w_1...w_n$ (where $n \geq 0$ and, for $j = 1, ..., n, w_j \in \Sigma$), we have

$$f(w) = f(w_1)...f(w_n). \tag{*}$$

Use induction on n.

(a) State and prove the base case ($n = 0$).

To show: if f is a string homomorphism, then for ε, we have $f(\varepsilon) = \varepsilon$.

Proof: Let $f(\varepsilon) = x$. By def. of *string homomorphism*, $f(\varepsilon\varepsilon) = f(\varepsilon)f(\varepsilon) = xx$. Because $\varepsilon = \varepsilon\varepsilon$, $f(\varepsilon\varepsilon) = f(\varepsilon) = x$. Then $xx = x$ so $x = \varepsilon$ (see footnote)[1]. Thus, $f(\varepsilon) = \varepsilon$.

(b) Assume that (*) is true for $n = i$ and prove (*) for $n = i + 1$.

To show: if f is a string homomorphism, then for $w \in \Sigma^*$ and $w_{i+1} \in \Sigma$, $f(w) = f(w_1)...f(w_i) \to f(ww_{i+1}) = f(w_1)...f(w_i)f(w_{i+1})$.

Proof: Let $w \in \Sigma^*$ such that $f(w) = f(w_1)...f(w_i)$. Let $w_{i+1} \in \Sigma$. By def. of *string homomorphism*, $f(ww_{i+1}) = f(w)f(w_{i+1})$. By I.H.[2], $f(w)f(w_{i+1}) = f(w_1)...f(w_i)f(w_{i+1})$. Thus, $f(ww_{i+1}) = f(w_1)...f(w_i)f(w_{i+1})$

3. **Finite and cofinite.** Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Define FINITE to be the set of all finite languages over Σ, and let coFINITE be the set of all languages over Σ whose *complement* is finite:

$$\text{co}\mathsf{FINITE} = \{L \subseteq \Sigma^* | \overline{L} \in \mathsf{FINITE}\}$$

(where $\overline{L} = \Sigma^* \setminus L$). For example, Σ^* is in coFINITE because its complement is $\emptyset$, which is finite. (Please think carefully about this definition, and note that coFINITE isn't the same thing as $\overline{\mathsf{FINITE}}$.)

(a) If $L \in \mathsf{FINITE}$, what data structure could you use to represent L, and given a string w, how would you decide whether $w \in L$?

I would use a Python set/dictionary/hashmap to represent L. To decided whether $w \in L$, I would check if w is in the chosen data structure.

Example: $L = \{\varepsilon, \mathtt{a}, \mathtt{b}\}$ would be represented by

L = set([‘’, ‘a’, ‘b’])

decide whether $w \in L$

w in L

(b) If $L \in$ coFINITE, what data structure could you use to represent L, and given a string w, how would you decide whether $w \in L$?

I would do the same as (a) but for $\overline{L}$ and check that w is not in $\overline{L}$.[3]

Example: $L = \{w \in \Sigma^* | w \notin \{\varepsilon, \mathtt{a}, \mathtt{b}\}\}$ a.k.a. $\overline{L} = \{\varepsilon, \mathtt{a}, \mathtt{b}\}$ would be represented by

L_complement = set([‘’, ‘a’, ‘b’])

decide whether $w \in L$

w not in L_complement

[1] $xx = x \to |xx| = |x| \to 2|x| = |x| \to |x| = 0 \to x = \varepsilon$

[2] Inductive Hypothesis: $f(w) = f(w_1)...f(w_i)$

[3] In Class Professor Chiang clarified that $w \in \Sigma^*$ so $w \in L$ or $w \in \overline{L}$

(c) Are there any languages in $\mathsf{FINITE} \cap \mathsf{coFINITE}$? Prove your answer.

No there are not any languages in $\mathsf{FINITE} \cap \mathsf{coFINITE}$.

To show: $\forall L \subseteq \Sigma^*(L \notin \mathsf{FINITE} \cap \mathsf{coFINITE})$

TAC, assume $\exists L \subseteq \Sigma^*(L \in \mathsf{FINITE} \cap \mathsf{coFINITE})$. Let $L \subseteq \Sigma^*$ s.t. $(L \in \mathsf{FINITE} \cap \mathsf{coFINITE})$. This means that $L \in \mathsf{FINITE} \wedge L \in \mathsf{coFINITE}$.

Observe that $L \in \mathsf{FINITE} \implies |L| = n$ s.t. $n \in \mathbb{N}$.
$L \in \mathsf{coFINITE}$ means $\overline{L} \in \mathsf{FINITE} \implies |\overline{L}| = m$ s.t. $m \in \mathbb{N}$.

We know that $|\Sigma^*| = |L \cup \overline{L}| = |L| + |\overline{L}|$ because L and $\overline{L}$ are disjoint and both $L, \overline{L} \subseteq \Sigma^*$.[4] From substitution, $|L| + |\overline{L}| = n + m$ s.t. $(n+m) \in \mathbb{N}$ so Σ^* is finite. $\lightning$. Thus, by contradiction we know that $\forall L \subseteq \Sigma^*(L \notin \mathsf{FINITE} \cap \mathsf{coFINITE})$

(d) Are there languages *not* in $\mathsf{FINITE} \cup \mathsf{coFINITE}$? Prove your answer.

Yes, there are languages not in *not* in $\mathsf{FINITE} \cup \mathsf{coFINITE}$.

To show: $\exists L \subseteq \Sigma^*(L \notin \mathsf{FINITE} \cup \mathsf{coFINITE})$

Let $L := \{w \in \Sigma^* | w_1 = \mathtt{a}\}$. Observe, $\overline{L} = \{w \in \Sigma^* | w = \varepsilon \vee w_1 = \mathtt{b}\}$. By def. of L, L is infinite.[5] By the same logic, $\overline{L}$ is also infinite. Since L is infinite, $L \notin \mathsf{FINITE}$. Since $\overline{L}$ is infinite, $\overline{L} \notin \mathsf{FINTIE}$ so $L \notin \mathsf{coFINITE}$ by definition of $\mathsf{coFINITE}$. Since $L \notin \mathsf{FINITE} \wedge L \notin \mathsf{coFINITE}$, $L \notin \mathsf{FINITE} \cup \mathsf{coFINITE}$. Thus, we know that $\exists L \subseteq \Sigma^*(L \notin \mathsf{FINITE} \cup \mathsf{coFINITE})$

[4] This is derived from $\overline{L} = \Sigma^* \setminus L$

[5] TA said this was sufficient and did not need to prove that L is infinite.