Homework 9

Aaron Wang

November 25 2024

1. If U_1, ..., U_n are independent uniform random variables on $[0, 1]$, find $E(U_{(n)} - U_{(1)})$, where $U_{(n)} = \max\{U_1, ..., U_n\}$ and $U_{(1)} = \min\{U_1, ..., U_n\}$. (Hint: From beta distribution, we have $\int_0^1 x^{a-1}(1-x)^{b-1}dx = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, for any a, b > 0.)

$$f_{U_{(n)}}(x) = nx^{n-1}, \quad 0 \le x \le 1$$

$$f_{U_{(1)}}(x) = n(1-x)^{n-1}, \quad 0 \le x \le 1$$

$$E(U_{(n)}) = \int_0^1 x f_{U_{(n)}}(x)\,dx = n\int_0^1 x^n\,dx = n \cdot \frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{n}{n+1}.$$

$$E(U_{(1)}) = \int_0^1 x f_{U_{(1)}}(x)\,dx = n\int_0^1 x^{2-1}(1-x)^{n-1}\,dx = n\frac{\Gamma(2)\Gamma(n)}{\Gamma(2+n)} = n\frac{1!(n-1)!}{(n+1)!} = \frac{1}{n+1}.$$

$$E(U_{(n)} - U_{(1)}) = E(U_{(n)}) - E(U_{(1)}) = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}.$$

2. You have two dice, one with three sides labeled $0, 1, 2$ and one with 4 sides, labeled $0, 1, 2, 3$. Let X_1 be the outcome of rolling the first die, and X_2 the outcome of rolling the second. The rolls are independent.

   (a) What is the joint p.m.f. of (X_1, X_2)?

$$P[X_1 = x_1, X_2 = x_2] = \frac{1}{12} \text{ for } x_1 \in \{0, 1, 2\} \text{ and } x_2 \in \{0, 1, 2, 3\}$$

   (b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of (Y_1, Y_2).

$Y_1 \backslash Y_2$	0	1	2	3
0	1/12	1/6	1/6	1/12
1	0	1/12	0	0
2	0	0	1/6	0
3	0	0	0	1/12
4	0	0	1/12	0
6	0	0	0	1/12

   (c) Are Y_1, Y_2 independent?
   $P(Y_1 = 0) = 1/2$, $P(Y_2 = 0) = 1/12$, $P(Y_1 = 0, Y_2 = 0) = 1/12$.
   $P(Y_1 = 0)P(Y_2 = 0) \neq P(Y_1 = 0, Y_2 = 0)$ so NOT independent.

3. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

   (a) What are $E(Y_1)$ and $E(Y_2)$? Standard Normal Random Variable means $\mu = 0$ and $\sigma = 1$.

   $$E(Y_1) = E(Z) = \mu = 0$$

   $$E(Y_2) = E(Z^2) = Var(Z) - [E(Z)]^2 = \sigma - \mu^2 = 1$$

   (b) What is $E(Y_1Y_2)$?

   $$E(Y_1Y_2) = E(Z^3) = 0 \text{ because of the symmetric distribution.}$$

   (c) What is $Cov(Y_1, Y_2)$?

   $$Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = (0) - (0)(1) = 0$$

   (d) Notice that $P(Y_2 > 1|Y_1 > 1) = 1$. Are Y_1 and Y_2 independent?
   If independent, then $P(Y_2 > 1|Y_1 > 1) = P(Y_2 > 1)$. However we know that $P(Y_2 > 1) < 1$. Therefore, Y_1 and Y_2 are NOT independent.

   (e) What can you learn from part (c) and (d)?
   From (c) and (d) we learn that the Covariance of two variables being 0 does not imply independence.

5. Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week and suppose that Y_1 has a uniform distribution over the interval $0 \le y_1 \le 1$. Let Y_2 denote the amount (by weight) of this item sold by the supplier during the week and suppose that Y_2 has a uniform distribution over the interval $0 \le y_2 \le y_1$, where y_1 is a specific value of Y_1. If the supplier stocked 3/4 ton, what amount could be expected to be sold during the week?

   $$E[Y_2|Y_1 = y_1] = \frac{y_1}{2} = \frac{3/4}{2} = \frac{3}{8}$$

8. Let $X \sim \text{Exp}(2)$ (rate parameter 2), $Y \sim \text{Unif}[1, 3]$, and assume that X and Y are independent. Calculate $P(Y - X \ge 1/2)$.

   $$f(x) = 2e^{-2x} \text{ for } 0 \le x$$

   $$f(y) = 1/2 \text{ for } 1 \le y \le 3$$

   $$f(x, y) = e^{-2x} \text{ for } 0 \le x \text{ and } 1 \le y \le 3$$

   $$P(Y - X \ge 1/2) = \int_1^3 \int_0^{y-1/2} f(x, y)dxdy = 0.9097.$$