Problem Set 4

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1. Show the following

   (a) Show $\forall x(\varnothing \subseteq x)$.

   Proof. Let x be a set. Towards a contradiction, suppose $\varnothing \not\subseteq x$. Consequently, there must exist some z such that $z \in \varnothing \land z \notin x$ by definition. This implies $z \in \varnothing$; however, we know $\forall w(w \notin \varnothing)$. Therefore, because we achieved a contradiction we conclude $\varnothing \subseteq x$.

   Q.E.D.

   (b) Show $\forall x(x \subseteq x)$.

   Proof. Let x and z be sets. Recall that $z \in x \implies z \in x$. By definition $x \subseteq x$.

   Q.E.D.

   (c) Show $\forall x(\varnothing \in \mathbb{P}(x))$.

   Proof. Let x be a set. By definition, $\mathbb{P}(x) = \{w | w \subseteq x\}$. Therefore, since $\varnothing \subseteq x$ (1a), we can conclude $\forall x(\varnothing \in \mathbb{P}(x))$.

   Q.E.D.

   (d) Show $\forall x(x \in \mathbb{P}(x))$.

   Proof. Let x be a set. By definition, $\mathbb{P}(x) = \{w | w \subseteq x\}$. Therefore, since $x \subseteq x$ (1b), we can conclude $\forall x(x \in \mathbb{P}(x))$.

   Q.E.D.

   (e) Show $\forall x \forall y \forall z((x \subseteq y) \land (y \subseteq z) \implies x \subseteq z)$.

   Proof. Let x, y, and z be sets. Assume $(x \subseteq y) \land (y \subseteq z)$. Towards a contradiction, assume $x \not\subseteq z$. Consequently, there must exist some w such that $w \in x \land w \notin z$ by definition of a subset. Using $w \in x$ and $x \subseteq y$, we conclude $w \in y$ by definition of a subset. Following the same logic, from $w \in y$ and $y \subseteq z$ we conclude $w \in z$. Consequently, since we assumed $w \notin z$ and arrived at $w \in z$, we have reached a contradiction and can conclude that $\forall x \forall y \forall z((x \subseteq y) \land (y \subseteq z) \implies x \subseteq z)$.

   Q.E.D.

2. We define the $\cap$ and $\setminus$ of any two sets x and y below.

$$x \cap y \coloneqq \{z | z \in x \wedge z \in y\}$$
$$x \setminus y \coloneqq \{z | z \in x \wedge z \notin y\}$$

(a) Show $\forall x \forall y \exists z (z = x \cap y)$.

Proof. Let x and y be arbitrary sets. Let $\varphi(w) = w \in y$. By the Schema of Separation, $\{w | w \in x \wedge \varphi(w)\}$ exists. Let $z \coloneqq \{w | w \in x \wedge \varphi(w)\}$. Substituting in φ we can get $z = \{w | w \in x \wedge w \in y\}$. Thus by the definition of intersection, $z = x \cap y$. Thus, we know that $\forall x \forall y \exists z (z = x \cap y)$.

Q.E.D.

(b) Show $\forall x \forall y \exists z (z = x \setminus y)$.

Proof. Let x and y be arbitrary sets. Let $\varphi(w) = w \notin y$. By the Schema of Separation, $\{w | w \in x \wedge \varphi(w)\}$ exists. Let $z \coloneqq \{w | w \in x \wedge \varphi(w)\}$. Substituting in φ we can get $z = \{w | w \in x \wedge w \notin y\}$. Thus by the definition of difference, $z = x \setminus y$. Thus, we know that $\forall x \forall y \exists z (z = x \setminus y)$.

Q.E.D.

3. We define the $\cup$ of any two sets x and y below.

$$x \cup y \coloneqq \{z | z \in x \vee z \in y\}$$

(a) Show $\forall x \forall y (x \cap y \subseteq x)$.

Proof. Let x and y be arbitrary sets. To show that $x \cap y \subseteq x$, we must show that $\forall z (z \in x \cap y \implies z \in x)$. Assume $z \in x \cap y$. By definition of intersection, we can conclude that $z \in x \wedge z \in y$. By conjunction elimination, we get $z \in x$. Thus, we get $\forall z (z \in x \cap y \implies z \in x)$, and consequently, by the definition of subsets, we can conclude $x \cap y \subseteq x$. Furthermore, since x and y are arbitrary sets using universal introduction we get $\forall x \forall y (x \subseteq x \cap y \subseteq x)$.

Q.E.D.

(b) Show $\forall x \forall y (x \subseteq x \cup y)$.

Proof. Let x and y be arbitrary sets. To show that $x \subseteq x \cup y$, we must show that $\forall z (z \in x \implies z \in x \cup y)$. Assume $z \in x$. By disjunction introduction, we get $z \in x \vee z \in y$, and consequently, by the definition of union, we can conclude $z \in x \cup y$. Thus, we get $\forall z (z \in x \implies z \in x \cup y)$, and consequently, by the definition of subsets, we can conclude $x \subseteq x \cup y$. Furthermore, since x and y are arbitrary sets using universal introduction we get $\forall x \forall y (x \subseteq x \cup y)$.

Q.E.D.

(c) Show $\forall x \forall y(\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y))$.

Proof. Let x and y be arbitrary sets. To show that $\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y)$, we must show that $\forall z(z \in \mathbb{P}(x) \cup \mathbb{P}(y) \implies z \in \mathbb{P}(x \cup y))$. Assume $z \in \mathbb{P}(x) \cup \mathbb{P}(y)$. By definition of union, we get $z \in \mathbb{P}(x) \vee z \in \mathbb{P}(y)$. Let's consider these two cases separately.

i. $z \in \mathbb{P}(x)$ means that $z \subseteq x$ by definition of a power set. We proved $x \subseteq x \cup y$ in 3b. Consequently, by 1e, $z \subseteq x \cup y$ which by definition of power sets states that $z \in \mathbb{P}(x \cup y)$.

ii. $z \in \mathbb{P}(y)$ means that $z \subseteq y$ by definition of a power set. We proved $y \subseteq x \cup y$ in 3b. Consequently, by 1e, $z \subseteq x \cup y$ which by definition of power sets states that $z \in \mathbb{P}(x \cup y)$.

Thus, $\forall z(z \in \mathbb{P}(x) \cup \mathbb{P}(y) \implies z \in \mathbb{P}(x \cup y))$. Furthermore, since x and y are arbitrary sets, by universal introduction, we conclude $\forall x \forall y(\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y))$.

Q.E.D.

(d) Show $\forall x \forall y(x \cap y = x \iff x \in \mathbb{P}(y))$.

Proof. Let x and y be arbitrary sets. To show $x \cap y = x \iff x \in \mathbb{P}(y)$ we must show $x \cap y = x \implies x \in \mathbb{P}(y)$ and $x \in \mathbb{P}(y) \implies x \cap y = x$.

i. $x \cap y = x \implies x \in \mathbb{P}(y)$
Assume $x \cap y = x$. By extensionality, we know that $z \in x \iff (z \in x \wedge z \in y)$. Thus breaking up the biconditional we get $z \in x \implies (z \in x \wedge z \in y)$. Assume $z \in x$. Thus, $(z \in x \wedge z \in y)$. Consequently, by conjunction elimination, we get $z \in y$. Since we derived $z \in x \implies z \in y$, $x \subseteq y$ by definition of subsets. Since $x \subseteq y$, by definition of a power set, $x \in \mathbb{P}(y)$.

ii. $x \in \mathbb{P}(y) \implies x \cap y = x$
Assume $x \in \mathbb{P}(y)$. To show that $x \cap y = x$ we must show $(z \in x \implies (z \in x \wedge z \in y)) \wedge ((z \in x \wedge z \in y) \implies z \in x)$ (extensionality).

A. $z \in x \implies (z \in x \wedge z \in y)$
Assume $z \in x$. Thus we have $z \in x$. By definition of power sets, $x \subseteq y$ from the assumption $x \in \mathbb{P}(y)$. Consequently, the definition of subsets states that $z \in x \implies z \in y$ so from $z \in x$ we can conclude $(z \in x \wedge z \in y)$.

B. $(z \in x \wedge z \in y) \implies z \in x$
Assume $(z \in x \wedge z \in y)$. Using conjunction elimination we conclude $z \in x$.

Thus, we have shown that $x \in \mathbb{P}(y) \implies x \cap y = x$.

Since we have shown $x \cap y = x \implies x \in \mathbb{P}(y) \wedge x \in \mathbb{P}(y) \implies x \cap y = x$, we have $x \cap y = x \iff x \in \mathbb{P}(y)$. Furthermore, because x and y are arbitrary sets, $\forall x \forall y(x \cap y = x \iff x \in \mathbb{P}(y))$.

Q.E.D.

4. We define the $\cup x$ and $\cap x$ for any set x below.

$$\cup x := \{z | \exists y(y \in x \land z \in y)\}$$
$$\cap x := \{z | \forall y(y \in x \implies z \in y)\}$$

(a) Show that $\forall x(\cup \mathbb{P}(x) = x)$.

Proof. Let x be an arbitrary set. By 1d, $x \in \mathbb{P}(x)$. As such, by existential elimination $\cup \mathbb{P}(x) = \{z | \exists y(y \in \mathbb{P}(x) \land z \in y)\} = \{z | x \in \mathbb{P}(x) \land z \in x\}$. Because $x \in \mathbb{P}(x)$ we can conclude $\{z | x \in \mathbb{P}(x) \land z \in x\} = \{z | z \in x\}$ and since all the elements of $\{z | z \in x\}$ are in x and all the elements of x are in $\{z | z \in x\}$ by extensionality, $\{z | z \in x\} = x$ (see 4d for concrete proof).

Q.E.D.

(b) What is $\cup \varnothing$? Justify your answer with a proof.

Proof. We know from the definition of union that $\cup \varnothing = \{z | \exists y(y \in \varnothing \land z \in y\}$. Since we know that the empty set is empty, there exists no y in the context of that predicate. Thus, the predicate for $\cup \varnothing$ is always false. Consequently, no element satisfies the predicate and $\cup \varnothing = \varnothing$.

Q.E.D.

(c) What is $\cap \varnothing$? Justify your answer with a proof.

Proof. Towards a contradiction assume that $\cap \varnothing$ exists. From the definition of intersection, we know that $\cap \varnothing = \{z | \forall y(y \in \varnothing \implies z \in y\}$. Using logic we know that the predicate $\forall y(y \in \varnothing \implies z \in y) \equiv \neg \exists y(y \in \varnothing \land z \notin y)$. From this, we can see that the predicate will always be True and as such every element will be in $\cap \varnothing$. As such, because the theorem of Well-Foundedness of Elementhood states that a set can not contain itself, the $\cap \varnothing$ can not exist.

Q.E.D.

(d) Is $\varnothing = \{z | z \in \varnothing\}$? Justify your answer with a proof.

Proof. Let x be a set. To show that $x = \{z | z \in x\}$ we must show that $\forall w(w \in x \iff w \in \{z | z \in x\})$ (extensionality).

i. Let w be a set. Suppose $w \in x$. By definition of set comprehension notation, $w \in \{z | z \in x\}$.
ii. Let w be a set. Suppose $w \in \{z | z \in x\}$. By definition of set comprehension notation, $w \in x$.

Thus, as $x = \{z | z \in x\}$ for any arbitrary x, we know that $\varnothing = \{z | z \in \varnothing\}$.

Q.E.D.

(e) Is $\varnothing = \{z | z \notin \varnothing\}$? Justify your answer with a proof.

Proof. Let $x := \{z | z \notin \varnothing\}$. In any universe of discourse, $x \in x$ because $x \notin \varnothing$. However the theorem of Well-Foundedness of Elementhood states $x \notin x$, Thus, $\{z | z \notin \varnothing\}$ does not exist as it leads to a contradiction and as such $\varnothing \neq \{z | z \notin \varnothing\}$.

Q.E.D.