Homework 5

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1. Let $F(x) = 1 - exp(-\alpha x^{\beta})$ for $x \geq 0$, $\alpha > 0$, $\beta > 0$, and $F(x) = 0$ for $x < 0$. Show that F is a cdf, and find the corresponding density.

$$F(x) = \begin{cases} 1 - e^{-\alpha x^{\beta}} & 0 \leq x \\ 0 & x < 0 \end{cases}$$

   Property 1: $F(-\infty) \equiv \lim_{x \to -\infty} F(x) = 0$

   Property 2: $F(\infty) \equiv \lim_{x \to \infty} F(x) = 1$

   Property 3: $F(x)$ is a nondecreasing function of x $F'(x)$ is always positive. Look below at density function.

   As $F(x)$ satisified these three properties, it is a cdf.

   Density: f(x)=F'(x)

$$f(x) = \begin{cases} e^{-\alpha x^{\beta}} (\alpha \beta x^{\beta - 1}) & 0 \leq x \\ 0 & x < 0 \end{cases}$$

2. Suppose that X has the density function

$$f(x) = \begin{cases} cx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus } \int_{-\infty}^{x} f(t) = F(x) = \begin{cases} 0 & x < 0 \\ \frac{cx^3}{3} & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

(a) Find c

Since $F(x)$ must be continuous, $F(1) = 1$. Thus, $\frac{c(1)^3}{3} = 1$ so $c = 3$

(b) Find the cdf

Plug in c from above and we get:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

(c) What is $P(0.1 \le X \le 0.5)$

$P(0.1 \le X \le 0.5) = F(0.5) - F(0.1) = 0.125 - 0.001 = 0.124$

(d) Find $E(X)$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_0^1 x(3x^2)dx = \int_0^1 (3x^3)dx = \frac{3x^4}{4}\Big|_0^1 = \frac{3}{4}$$

(e) Find $Var(X)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_0^1 x(3x^2)dx = \int_0^1 (3x^4)dx = \frac{3x^5}{5}\Big|_0^1 = \frac{3}{5}$$

$$Y(X) = E(X^2) - E(X)^2 = \frac{3}{5} - \frac{3}{4}^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

3. Suppose that Y has density function

$$f(y) = \begin{cases} ky(1-y) & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of k that makes $f(y)$ a probability density function

Property 1 $f(y) \geq 0$ for all y, $-\infty < y < \infty$
Property 2 $\int_{-\infty}^{\infty} f(y)dy = 1$

Observe:

$$\int_{-\infty}^{\infty} f(y)dy = \int_0^1 ky(1-y)dy = k\int_0^1 y - y^2 dy = k(\frac{y^2}{2} - \frac{y^3}{3})\Big|_0^1 = \frac{k}{6}$$

To be a probability density function, both properties must be satisfied. Property 1 is always satisfied by this equation. Property 2 is satisfied when $k = 6$. Observe:

$$F(y) = \int_{-\infty}^{y} f(t)dt = \int_0^y 6y(1-y)dy = 6\int_0^y y - y^2 dy = 6(\frac{y^2}{2} - \frac{y^3}{3})\Big|_0^y = 3y^2 - 2y^3$$

$$F(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \leq y \leq 1 \\ 0 & 1 < y \end{cases}$$

(b) Find $P(0.4 \leq Y \leq 1)$

$$P(0.4 \leq Y \leq 1) = F(1) - F(0.4) = (3(1)^2 - 2(1)^3) - (3(0.4)^2 - 2(0.4)^3) = 0.648$$

(c) Find $P(0.4 \leq Y < 1)$

By theorem 4.3 $P(0.4 \leq Y \leq 1) = P(0.4 \leq Y < 1) = 0.648$

(d) Find $P(Y \leq 0.4 | Y \leq 0.8)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(Y \leq 0.4 \cap Y \leq 0.8) = P(Y \leq 0.4)$$

$$P(Y \leq 0.4) = F(0.4) = 0.352$$

$$P(Y \leq 0.4) = F(0.8) = 0.896$$

$$P(Y \leq 0.4 | Y \leq 0.8) = \frac{P(Y \leq 0.4)}{P(Y \leq 0.8)} = \frac{0.352}{0.896} \approx .393$$

(e) Find $P(Y \leq 0.4 | Y < 0.8)$

Same as (d) $P(Y \leq 0.8) = P(Y < 0.8)$ so $P(Y \leq 0.4 | Y < 0.8) \approx .393$

4. Let $f(x) = (1 + \alpha x)/2$ for $-1 \le x \le 1$ and $f(x) = 0$ otherwise, where $-1 \le \alpha \le 1$.

   (a) Show that f is a density.

      Property 1 $f(x) \ge 0$ for all x, $-\infty < x < \infty$

      This property is satisfied as the range of f(x) for $-1 \le x \le 1$ is from $(1-\alpha)/2$ to $(1+\alpha)/2$ and since $-1 \le \alpha \le 1$, for any $\alpha, x \in [0,1]$, $f(x)$ is between $[0,1]$ and $f(x) = 0$ otherwise so $f(x) \ge 0$ for all $-\infty < x < \infty$

      Property 2 $\int_{-\infty}^{\infty} f(x)dx = 1$

      $$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} \frac{1+\alpha x}{2} dx = \left(\frac{x}{2} + \frac{\alpha x^2}{4}\right)\Big|_{-1}^{1} = \frac{(1)}{2} - \frac{(-1)}{2} = 1$$

      Therefore, no matter what α is, property 2 is always satisfied.

   Since both properties are always satisfied, f is a density function.

   (b) Find the corresponding cdf.

   $$\int_{-\infty}^{x} f(t)dt = \int_{-1}^{x} \frac{1+\alpha t}{2} dt = \left(\frac{t}{2}+\frac{\alpha t^2}{4}\right)\Big|_{-1}^{x} = \frac{x}{2}+\frac{\alpha x^2}{4}-\left(\frac{-1}{2}+\frac{\alpha(-1)^2}{4}\right) = \frac{\alpha x^2}{4}+\frac{x}{2}+\frac{1-\alpha}{4}$$

   $$F(x) = \begin{cases} 0 & x < -1 \\ \frac{\alpha x^2}{4} + \frac{x}{2} + \frac{1-\alpha}{4} & -1 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

5. Define the function

   $$f(x) = \begin{cases} 9x^2 - 4x^3 + b & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

   Show that there is no value of b for which this is the p.d.f. of some continuous random variable.

   Property 1 $f(x) \ge 0$ for all x, $-\infty < x < \infty$

   This property is satisfied if and only if $b \ge 0$ because $f(0) = b$.

   Property 2 $\int_{-\infty}^{\infty} f(x)dx = 1$

   $$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{1} (9x^2 - 4x^3 + b)dx = 3x^3 - x^4 + bx\Big|_{0}^{1}$$

   $$= 3(1)^3 - (1)^4 + b(1) - (3(0)^3 - (0)^4 + b(0)) = 3 - 1 + b = b + 2$$

   Therefore, only $b = -1$ satisfies this requirement.

   As the two properties can not be satisfied simultaneously, there is no value b for which this is the p.d.f. of some continuous random variable.