Problem Set 8

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1. Including the instructor, there are 32 people in our class. Prove that two of these people were born on the same day of the month.

   Proof. Consider the set P which represents the people in our class and the set D which represents the days of the month. Notice that $|P| = 32$ and $|D| = 31$. Let $f : P \to D$ given by $f(p_i) :=$ the day of the month that they p_i was born. Since $|P| = 32 > 31 = |D|$. we know that f is not injective by PHP. So, by definition, there exists $p_i, p_j \in P$ such that $f(p_i) = f(p_j)$ meaning there are two people who were born on the same day. Q.E.D.

2. As of the 28th of March, 2024, there are over 8.1 billion people living on Earth. A person's heart will beat no more than 7×10^9 times over their lifespan. Show that there are two currently-living people on Earth whose hearts have beat the exact same number of times.

   Proof. Consider the set P which represents the people in the world and the set H which represents the number of heartbeats they've had. Notice that $|P| = 8.1 \times 10^9$ and $|H| = 7 \times 10^9$. Let $f : P \to D$ given by $f(p_i) :=$ the number of heartbeats p_i has had. Since $|P| = 8.1 \times 10^9 > 7 \times 10^9 = |D|$. we know that f is not injective by PHP. So, by definition, there exists $p_i, p_j \in P$ such that $f(p_i) = f(p_j)$ meaning there are two people who have had the same heartbeats. Q.E.D.

3. Let $n \in \mathbb{N}_+$ and consider $\mathcal{A} \subseteq \mathbb{N}$ such that $|\mathcal{A}| = n + 1$. Prove there exist $x, y \in \mathcal{A}$ with $x \neq y$ such that $n \mid x - y$.

   Proof. Let $n \in \mathbb{N}_+$ and consider $\mathcal{A} \subseteq \mathbb{N}$ such that $|\mathcal{A}| = n + 1$. Define $f : \mathcal{A} \to n$ where

   $$f(a) = r \text{ such that } (\exists q, r \in \mathbb{Z})(a = qn + r) \text{ and } 0 \leq r < n$$

   Observe that $|\mathcal{A}| = n + 1 > n = |n|$ so we know that f is not injective by PHP. So, by definition, there exists $a_i, a_j \in \mathcal{A}$ such that $f(a_i) = f(a_j)$. Let $q_i, q_j, r_i, r_j \in \mathbb{Z}$ and let them satisfy $a_i = q_i n + r_i$ such that $0 \leq r_i < n$ and $a_j = q_j n + r_j$ such that $0 \leq r_j < n$. We know $r_i = r_j$ because $f(a_i) = f(a_j)$. Consequently, $a_i - a_j = (q_i n + r_i) - (q_j n + r_j) = (q_i - q_j)n$. We know that $(q_i - q_j) \in \mathbb{Z}$ because $q_i, q_j \in \mathbb{Z}$ and $a_i - a_j = (q_i - q_j)n$, so we know that $n | a_i - a_j$. Consequently, there exist $x, y \in \mathcal{A}$ with $x \neq y$ such that $n \mid x - y$. Q.E.D.

4. Consider $\mathcal{S} \coloneqq \{3, 4, 7, 8, 9, 10, 12, 15, 18, 19, 27, 28\}$ and $\mathcal{X} \subseteq \mathcal{S}$ with $|\mathcal{X}| \geq 9$. Show that there exist three *distinct* elements $x_1, x_2, x_3 \in \mathcal{X}$ such that $x_1 + x_2 + x_3 = 40$.

   Proof. Consider $\mathcal{S} \coloneqq \{3, 4, 7, 8, 9, 10, 12, 15, 18, 19, 27, 28\}$. Let $X \subseteq S$ with $|X| \geq 9$. Since X is a subset of S, X will only contain elements that are already in S. Observe that there are 4 sets, all with distinct elements, that sum up to 40: 3,10,27; 4,8,28; 7,15,18; 9,12,19. Thus if we show that one of these triples always exists in X, we have shown that there exists three distinct elements $x_1, x_2, x_3 \in \mathcal{X}$ such that $x_1 + x_2 + x_3 = 40$. Using this information, let us define $A \coloneqq \{\{3, 10, 27\}, \{4, 8, 28\}, \{7, 15, 18\}, \{9, 12, 19\}\}$ and define $f : X \to A$ where

   $$f(x) = a \text{ such that } x \in a$$

   Note that f is a function because every input has a unique output. Also notice that by PHP, since $|X| \geq 9$ and $|A| = 4$, there exists always exists three distinct elements of $\mathcal{X}$ which map to the same element of A. In other words, there exists $x_1, x_2, x_3 \in \mathcal{X}$, such that $f(x_1) = f(x_2) = f(x_3)$. Finally, by definition of f, since three distinct elements map to the same value, $x_1 + x_2 + x_3 = 40$. Q.E.D.

5. Recall that $\binom{n}{0} = \binom{n}{n} = 1$ for all $n, k \in \mathbb{N}$ when $k \leq n$.

   (a) Show $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ where $k \leq n$.

   Proof. Assume $n, k \in \mathbb{N}$ where $k \leq n$. Let $A \coloneqq \{z | z \subseteq n \wedge |z| = k\}$ and $B \coloneqq \{z | z \subseteq n \wedge |z| = n - k\}$. Let $f : A \to B$ given by $f(a) \coloneqq n \backslash a$.

   Observe that f is injective. Assume $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$. This implies that $n \backslash a_1 = n \backslash a_2$. Towards a contradiction suppose $a_1 \neq a_2$. This means that there exists $z \in a_1$ such that $z \notin a_2$. Because $a_1 \subseteq n$, $z \in n$. Since $z \in n \wedge z \notin a_2$, $z \in n \backslash a_2$ which means that $z \in n \backslash a_1$. However, $z \in a_1$ so $z \notin n \backslash a_1$. $\lightning$ Thus $a_1 = a_2$ and f is injective.

   Let $g : B \to A$ given by $g(b) \coloneqq n \backslash b$. Observe that b is injective by the same logic that f is injective.

   Since f is injective $|A| \leq |B|$ and since g is injective $|B| \leq |A|$. Thus we know $|A| = |B|$ by Cantor-Schöder-Bernstein. Consequently as $\binom{n}{k} = |A|$ and $\binom{n}{n-k} = |B|$, we know that $\binom{n}{k} = \binom{n}{n-k}$. Q.E.D.

(b) Show $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ for all $n, k \in \mathbb{N}$ where $k \leq n$.

Proof. Assume $n, k \in \mathbb{N}$ where $k \leq n$. Let $A \coloneqq \{z | z \subseteq n+1 \wedge |z| = k+1\}$ and $B \coloneqq \{z | z \subseteq n \wedge (|z| = k+1 \vee |z| = k)\}$. Let $f : A \to B$ given by

$$f(a) \coloneqq \begin{cases} a & \text{if } (n \notin a) \\ a \backslash \{n\} & \text{if } (n \in a) \end{cases}$$

Let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$.

$|f(a_1)| = k+1$*:*

If $|f(a_1)| = k+1$, $|f(a_2)| = k+1$. Suppose towards a contradiction that $n \in a_1$. This means that $f(a_1) = a_1 \backslash \{n\}$ which means that $|f(a_1)| = k$. $\lightning$. The same logic applies for a_2. As a result, we know that $n \notin a_1, a_2$ so $f(a_1) = a_1$ and $f(a_2) = a_2$. Consequently, $a_1 = a_2$.

$|f(a_1)| = k$*:*

If $|f(a_1)| = k$, $|f(a_2)| = k$. Suppose towards a contradiction that $n \notin a_1$. This means that $f(a_1) = a_1$ which means that $|f(a_1)| = k+1$. $\lightning$. The same logic applies for a_2. As a result, we know that $n \in a_1, a_2$ so $f(a_1) = a_1 \backslash \{n\}$ and $f(a_2) = a_2 \backslash \{n\}$. Furthermore, since we know $a_1 \backslash \{n\} = a_2 \backslash \{n\}$ and $n \in a_1, a_2$, we conclude that $a_1 = a_2$.

Consequently, f is injective.

Furthermore, we need to show that f is surjective. Let $b \in B$. $|b| = k+1$ or $|b| = k$ by definition of B.

$|b| = k+1$*:*

Because $b \subseteq A$ and $|b| = k+1$, $b \in A$ and $f(b) = b$. Therefore, there exists an input in A for $b \in B$ such that $|b| = k+1$.

$|b| = k$*:*

Because $b \subseteq A$ and $|b| = k$, $b \cup \{n\} \in A$ and $f(b \cup \{n\}) = b$. Therefore, there exists an input in A for $b \in B$ such that $|b| = k$.

Thus, for every element of B, there is an input in A which maps to that element of B. Consequently, f is surjective.

Observe that f is injective and surjective so f is bijective. Thus we know $|A| = |B|$. Observe that $\binom{n+1}{k+1} = |A|$. Let $C \coloneqq \{z | z \subseteq n \wedge |z| = k+1\}$ and $D \coloneqq \{z | z \subseteq n \wedge |z| = k\}$. $\binom{n}{k+1} + \binom{n}{k} = |C| + |D|$. Additionally, $|C| + |D| = |C \cup D|$ because $|C \cap D| = 0$ as C only contains sets of cardinality $k+1$ and D only contains sets of cardinality k. Additionaly observe that $B = C \cup D$ by definition so $\binom{n}{k+1} + \binom{n}{k} = |B|$. Consequently as $\binom{n+1}{k+1} = |A|$ and $\binom{n}{k+1} + \binom{n}{k} = |B|$, we know that $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$. Q.E.D.

6. Prove that $|\mathbb{P}(X)| = 2^{|X|}$ for any finite set X.

Proof. Basis Step:
Let X be a set such that $|X| = 0$ so $X = \varnothing$.
Show that $|\mathbb{P}(\varnothing)| = 2^{|\varnothing|}$
$|\mathbb{P}(\varnothing)| = |\{\varnothing\}| = 1$.
$2^{|\varnothing|} = 2^0 = 1$

Inductive Step:
Let X be a set such that $|X| = n$.
Inductive Hypothesis: $|\mathbb{P}(X)| = 2^{|X|}$.
Let a, Y be sets such that $a \notin X$.
Let's define $Y \coloneqq X \cup \{a\}$.
Observe that $|Y| = n + 1$.
Show that $|\mathbb{P}(Y)| = 2^{|Y|}$
Define $Z \coloneqq \{z | (\exists w \in \mathbb{P}(X))(z = w \cup \{a\})\}|$
Observe that $\mathbb{P}(Y) = \mathbb{P}(X) \cup Z$ as $\mathbb{P}(X)$ contains all the subsets of Y which don't contain a and Z contains the subsets which contain a.
Intersection of the two sets is empty because $(\forall z \in Z)(a \in z)$ and $(\forall x \in \mathbb{P}(X))(a \notin x)$.
Consequently, $|\mathbb{P}(Y)| = |\mathbb{P}(X)| + |Z|$.
Let's define a function $f : \mathbb{P}(X) \to Z$ where $f(x) = x \cup a$.
Let $x_1, x_2 \in \mathbb{P}(X)$. Assume $f(x_1) = f(x_2)$. We know that $x_1 \cup a = x_2 \cup a$. Since $a \notin x_1, x_2$, we know $x_1 = x_2$. Thus, f is injective.
Furthermore, let $z \in Z$. By definition of Z, $\exists x \in \mathbb{P}(X)$ such that $z = x \cup \{a\}$. Therefore f is surjective.
Since f is surjective and injective, it is bijective. Consequently $|\mathbb{P}(X)| = |Z|$.
Thus, $|\mathbb{P}(X)| + |Z| = |\mathbb{P}(X)| + |\mathbb{P}(X)| = 2 \cdot |\mathbb{P}(X)|$.
Using the IH, we know that $2 \cdot |\mathbb{P}(X)| = 2 \cdot 2^{|X|}$.
$2^{|Y|} = 2^{|X|+1} = 2 \cdot 2^{|X|}$
Therefore $|\mathbb{P}(Y)| = 2^{|Y|}$.
By induction, we have shown that $\forall X(|\mathbb{P}(X)| = 2^{|X|})$. Q.E.D.