# Homework 10

Aaron Wang

December 6 2024

1. $Y_1$ and $Y_2$ denoted the proportions of time during which employees $I$ and $II$ actually performed their assigned tasks during a workday. The joint density of $Y_1$ and $Y_2$ is given by

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 & 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the marginal density functions for $Y_1$ and $Y_2$.

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2$$

$$\int_0^1 (y_1 + y_2) dy_2 = y_1 y_2 + \frac{y_2^2}{2}\Big|_0^1 = y_1 + \frac{1}{2}$$

$$f_{Y_1}(y_1) = y_1 + \frac{1}{2} \text{ for } 0 \leq y_1 \leq 1$$

   Follow the same process to get:

$$f_{Y_2}(y_2) = y_2 + \frac{1}{2} \text{ for } 0 \leq y_2 \leq 1$$

   (b) Find $P(Y_1 \geq 0.5 | Y_2 \geq 0.5)$.

$$P(Y_1 \geq 0.5 \cap Y_2 \geq 0.5) = \int_{0.5}^1 \int_{0.5}^1 (y_1 + y_2) dy_1 dy_2 = 0.375$$

$$P(Y_2 \geq 0.5) = \int_{0.5}^1 y_2 + \frac{1}{2} = 0.625$$

$$P(Y_1 \geq 0.5 | Y_2 \geq 0.5) = \frac{P(Y_1 \geq 0.5 \cap Y_2 \geq 0.5)}{P(Y_2 \geq 0.5)} = \frac{0.375}{0.625} = 0.6$$

   (c) If employee II spends exactly 50% of the dayworking on assigned duties, find the probability that employee I spends more than 75% of the day working on similar duties.

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{y_1 + y_2}{0.5 + y_2}$$

$$P(Y_1 \geq 0.75 | Y_2 = 0.5) = \int_{0.75}^1 f_{Y_1|Y_2}(y_1|0.5) dy_1 = \int_{0.75}^1 (y_1 + 0.5) dy_1 = 0.34375$$

2. Assume that $Y$ denotes the number of bacteria per cubic centimeter in a particular liquid and that $Y$ has a Poisson distribution with parameter $Z$. Further assume that Z varies from location to location and has a gamma distribution with parameters $\alpha$ and $\beta$, where $\alpha$ is a positive integer. If we randomly select a location, what is the

   (a) expected number of bacteria per cubic centimeter?

$$E[Y] = E[E[Y|Z]] = E[Z] = \frac{\alpha}{\beta}$$

   (b) standard deviation of the number of bacteria per cubic centimeter?

$$Var[Y] = Var[E[Y|Z]] + E[Var[Y|Z]] = Var[Z] + E[Z] = \frac{\alpha}{\beta^2} + \frac{\alpha}{\beta}$$

$$STDEV[Y] = \sqrt{Var[Y]} = \sqrt{\frac{\alpha}{\beta^2} + \frac{\alpha}{\beta}}$$

3. A random variable $Y$ has the density function $f(y) = e^y$ if $y < 0$, and $f(y) = 0$ elsewhere.

   (a) Find $E[e^{3Y/2}]$.

$$E[e^{3Y/2}] = \int_{-\infty}^{\infty} e^{3y/2} \cdot f(y)dy = \int_{-\infty}^{0} e^{3y/2} \cdot e^y dy = \int_{-\infty}^{0} e^{5y/2} dy = \frac{2}{5}$$

   (b) Find the moment-generating function for $Y$.

$$M(t) = \int_{-\infty}^{\infty} e^{ty} \cdot f(y)dy = \int_{-\infty}^{0} e^{ty} \cdot e^y dy = \int_{-\infty}^{0} e^{(t+1)y} dy = \left.\frac{e^{(t+1)y}dy}{t+1}\right|_{-\infty}^{0} = \frac{1}{t+1}$$

   (c) Use the moment-generating function to find $Var(Y)$.

$$E[Y] = \frac{d}{dt}M(t)\Big|_{t=0} = \frac{d}{dt}\frac{1}{t+1}\Big|_{t=0} = \frac{-1}{(t+1)^2}\Big|_{t=0} = -1$$

$$E[Y^2] = \frac{d^2}{dt^2}M(t)\Big|_{t=0} = \frac{d^2}{dt^2}\frac{1}{t+1}\Big|_{t=0} = \frac{d}{dt}\frac{-1}{(t+1)^2}\Big|_{t=0} = \frac{2}{(t+1)^3}\Big|_{t=0} = 2$$

$$Var[Y] = E[Y^2] - E[Y]^2 = 2 - (-1)^2 = 1$$

4. Consider the bivariate random variable with density

$$f(y_1, y_2) = \begin{cases} 2 & 0 \le y_1 \le 1, 0 \le y_2 \le 1, 0 \le y_1 + y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(Y_1 + Y_2)$.

$$f_{Y_1}(y_1) = \int_0^{1-y_1} 2dy_2 = 2\Big|_0^{1-y_1} = 2 - 2y_1$$

$$E[Y_1] = \int_0^1 \big(y_1(2-2y_1)\big)dy_1 = \int_0^1 \big(2y_1 - 2y_1^2\big)dy_1 = \frac{1}{3}$$

Do the same for $Y_2$ to discover $E[Y_2] = \frac{1}{3}$

$$E[Y_1 + Y_2] = E[Y_1] + E[Y_2] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

(b) Find $Var(Y_1 + Y_2)$.

$$E[Y_1^2] = \int_0^1 \big(y_1^2(2-2y_1)\big)dy_1 = \int_0^1 \big(2y_1^2 - 2y_1^3\big)dy_1 = \frac{1}{6}$$

$$Var[Y_1] = E[Y_1^2] - E[Y_1]^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

Do the same for $Y_2$ to discover $Var[Y_2] = \frac{1}{18}$

$$E[XY] = \int_0^1 \int_0^{1-y_1} 2y_1y_2dy_2dy_1 = \frac{1}{12}$$

$$Cov[Y_1, Y_2] = E[XY] - E[X] \cdot E[Y] = \frac{1}{12} - \frac{1}{3} \cdot \frac{1}{3} = \frac{-1}{36}$$

$$Var[Y_1 + Y_2] = Var[Y_1] + Var[Y_2] + 2Cov[X, Y] = \frac{1}{18} + \frac{1}{18} + 2\Big(\frac{-1}{36}\Big) = \frac{1}{18}$$

5. You begin with a stick of length 1 and break it at point, chosen uniformly at random. You then take the left piece and break it once again at a uniformly random chosen point. What is the expectation and variance of the length of left piece after the breaking.

Let $Y_1 \sim \text{Unif}[0,1]$ represent the first breaking and $Y_2 \sim \text{Unif}[0, y_1]$ represent the second.

$$E[Y_2] = E[E[Y_2|Y_1]] = E[\frac{Y_1}{2}] = \frac{1}{2}E[Y_1] = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$E[Y_1^2] = \int_0^1 y_1^2 dy_1 = \frac{1}{3}$$

$$Var[Y_2] = Var[E[Y_1|Y_2]] + E[Var[Y_1|Y_2]] = Var[\frac{Y_1}{2}] + E[\frac{Y_1^2}{12}]$$

$$= \frac{1}{4}Var[Y_1] + \frac{1}{12}E[Y_1^2] = \frac{1}{4}\left(\frac{1}{12}\right) + \frac{1}{12}\left(\frac{1}{3}\right) = \frac{7}{144}$$