Problem Set 2

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1. Consider the following proof of $p \to (q \to r) \equiv (p \to q) \to r$.

Proof. Observe the following chain of reasoning.

$$
\begin{aligned}
p \to (q \to r) &\equiv p \lor \neg(q \to r) && \text{by } \textit{conditional disintegration} && (1)\\
&\equiv p \lor \neg(q \lor \neg r) && \text{by } \textit{conditional disintegration} && (2)\\
&\equiv p \lor \neg q \lor \neg r && \text{by } \textit{associativity} && (3)\\
&\equiv (p \lor \neg q) \lor \neg r && \text{by } \textit{associativity} && (4)\\
&\equiv (p \to q) \lor \neg r && \text{by } \textit{conditional disintegration} && (5)\\
&\equiv (p \to q) \to r && \text{by } \textit{conditional disintegration} && (6)
\end{aligned}
$$

Therefore, $p \to (q \to r) \equiv (p \to q) \to r$. Q.E.D.

Find all of the mistakes, if any, in this proof, and explain why.

p,q and r are not declared as propositions.

In lines 1, 2, 5, and 6, conditional disintegration is incorrectly used. The axiom states "$p \to q \equiv \neg p \lor q$" yet the proof incorrectly uses "$p \to q \equiv p \lor \neg q$"

From (2) to (3) the proof incorrectly uses associativity. First, it incorrectly distributes $\neg$ and then it gets rid of the parenthesis, two things that should not happen.

2. Prove the claims below *without truth tables* for all propositions p, q, r.

   (a) $p \to q \equiv \neg q \to \neg p$.

   Proof. Let p and q be propositions. Observe the following chain of reasoning.

$$\begin{aligned} p \to q &\equiv \neg p \lor q && \text{by } \textit{conditional disintegration} \\ &\equiv q \lor \neg p && \text{by } \textit{commutativity} \\ &\equiv \neg(\neg q) \lor \neg p && \text{by } \textit{double negation} \\ &\equiv \neg q \to \neg p && \text{by } \textit{conditional disintegration} \end{aligned}$$

   Therefore, $p \to q \equiv \neg q \to \neg p$. Q.E.D.

   (b) $(p \land (p \to q)) \to q$ is a tautology.

   Proof. Let p and q be propositions. Observe the following chain of reasoning.

$$\begin{aligned} (p \land (p \to q)) \to q &\equiv (p \land (\neg p \lor q)) \to q && \text{by } \textit{conditional disintegration} \\ &\equiv ((p \land \neg p) \lor (p \land q)) \to q && \text{by } \textit{distributivity} \\ &\equiv (\bot \lor (p \land q)) \to q && \text{by } \textit{complement} \\ &\equiv (p \land q) \to q && \text{by } \textit{identity} \\ &\equiv \neg(p \land q) \lor q && \text{by } \textit{conditional disintegration} \\ &\equiv (\neg p \lor \neg q) \lor q && \text{by } \textit{de morgan's laws} \\ &\equiv \neg p \lor (\neg q \lor q) && \text{by } \textit{associativity} \\ &\equiv \neg p \lor \top && \text{by } \textit{complement} \\ &\equiv \top \lor \neg p && \text{by } \textit{commutativity} \\ &\equiv \top && \text{by } \textit{domination} \end{aligned}$$

   Therefore, $(p \land (p \to q)) \to q$ is a tautology. Q.E.D.

(c) $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

Proof. Let p and q be propositions. Observe the following chain of reasoning.

$$
\begin{aligned}
(\neg q \wedge (p \to q)) \to \neg p &\equiv (\neg q \wedge (\neg p \vee q)) \to \neg p && \text{by \textit{conditional disintegration}} \\
&\equiv ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \to \neg p && \text{by \textit{distributivity}} \\
&\equiv ((\neg q \wedge \neg p) \vee \bot) \to \neg p && \text{by \textit{complement}} \\
&\equiv (\bot \vee (\neg q \wedge \neg p)) \to \neg p && \text{by \textit{commutativity}} \\
&\equiv (\neg q \wedge \neg p) \to \neg p && \text{by \textit{identity}} \\
&\equiv \neg(\neg q \wedge \neg p) \vee \neg p && \text{by \textit{conditional disintegration}} \\
&\equiv (q \vee p) \vee \neg p && \text{by \textit{de morgans laws}} \\
&\equiv q \vee (p \vee \neg p) && \text{by \textit{associativity}} \\
&\equiv q \vee (\neg p \vee p) && \text{by \textit{commutativity}} \\
&\equiv q \vee \top && \text{by \textit{complement}} \\
&\equiv \top \vee q && \text{by \textit{commutativity}} \\
&\equiv \top && \text{by \textit{domination}}
\end{aligned}
$$

Therefore, $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology. Q.E.D.

(d) $(p \to q) \to ((p \to \neg q) \to \neg p)$ is a tautology.

Proof. Let p and q be propositions. Observe the following chain of reasoning.

$$
\begin{aligned}
(p \to q) \to ((p \to \neg q) \to \neg p) &\equiv (p \to q) \to ((\neg p \vee \neg q) \to \neg p) && \text{by \textit{conditional disintegration}} \\
&\equiv (p \to q) \to (\neg(\neg p \vee \neg q) \vee \neg p) && \text{by \textit{conditional disintegration}} \\
&\equiv (p \to q) \to (\neg(\neg p) \wedge \neg(\neg q)) \vee \neg p) && \text{by \textit{de morgans laws}} \\
&\equiv (p \to q) \to ((p \wedge q) \vee \neg p) && \text{by \textit{double negation}} \\
&\equiv (p \to q) \to (\neg p \vee (p \wedge q)) && \text{by \textit{double negation}} \\
&\equiv (p \to q) \to ((\neg p \vee p) \wedge (\neg p \vee q)) && \text{by \textit{distributivity}} \\
&\equiv (p \to q) \to (\top \wedge (\neg p \vee q)) && \text{by \textit{complement}} \\
&\equiv (p \to q) \to (\neg p \vee q) && \text{by \textit{identity}} \\
&\equiv (\neg p \vee q) \to (\neg p \vee q) && \text{by \textit{conditional disintegration}} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee q) && \text{by \textit{conditional disintegration}} \\
&\equiv \top && \text{by \textit{complement}}
\end{aligned}
$$

Therefore, $(p \to q) \to ((p \to \neg q) \to \neg p)$ is a tautology. Q.E.D.

3. In this problem, we will progressively establish that the alternative axioms Hilbert proposed are all tautologies *without truth tables*. Here, the variables p, q, and r all represent arbitrary propositions.

(a) Show $p \to p$ is a tautology.

Proof. Let p be a proposition. Observe the following chain of reasoning.

$$\begin{aligned} p \to p &\equiv \neg p \vee p && \text{by } \textit{conditional disintegration} \\ &\equiv \top && \text{by } \textit{complement} \end{aligned}$$

Therefore, $p \to p$ is a tautology. Q.E.D.

(b) Show $(p \to q) \to (\neg q \to \neg p)$ is a tautology.

Proof. Let p and q be propositions. Observe the following chain of reasoning.

$$\begin{aligned} (p \to q) \to (\neg q \to \neg p) &\equiv (\neg p \vee q) \to (\neg(\neg q) \vee \neg p) && \text{by } \textit{conditional disintegration} \times 2 \\ &\equiv (\neg p \vee q) \to (q \vee \neg p) && \text{by } \textit{double negation} \\ &\equiv (\neg p \vee q) \to (\neg p \vee q) && \text{by } \textit{commutativity} \\ &\equiv \neg(\neg p \vee q) \vee (\neg p \vee q) && \text{by } \textit{conditional disintegration} \\ &\equiv \top && \text{by } \textit{complement} \end{aligned}$$

Therefore, $(p \to q) \to (\neg q \to \neg p)$ is a tautology. Q.E.D.

(c) Show $p \to (q \to p)$ is a tautology.

Proof. Let p and q be propositions. Observe the following chain of reasoning.

$$\begin{aligned}
p \to (q \to p) &\equiv p \to (\neg q \lor p) && \text{by } \textit{conditional disintegration} \\
&\equiv \neg p \lor (\neg q \lor p) && \text{by } \textit{conditional disintegration} \\
&\equiv \neg p \lor (p \lor \neg q) && \text{by } \textit{commutativity} \\
&\equiv (\neg p \lor p) \lor \neg q && \text{by } \textit{associativity} \\
&\equiv \top \lor \neg q && \text{by } \textit{complement} \\
&\equiv \top && \text{by } \textit{domination}
\end{aligned}$$

Therefore, $p \to (q \to p)$ is a tautology. Q.E.D.

(d) Show $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is a tautology.

Proof. Let p, q, and r be propositions. Observe the following chain of reasoning.

$$\begin{aligned}
&(p \to (q \to r)) \to ((p \to q) \to (p \to r)) \\
&\equiv (p \to (\neg q \lor r)) \to ((\neg p \lor q) \to (\neg p \lor r)) && \text{by } \textit{conditional disintegration} \times 3 \\
&\equiv (\neg p \lor (\neg q \lor r)) \to (\neg(\neg p \lor q) \lor (\neg p \lor r)) && \text{by } \textit{conditional disintegration} \times 2 \\
&\equiv (\neg p \lor (\neg q \lor r)) \to ((\neg(\neg p) \land \neg q) \lor (\neg p \lor r)) && \text{by } \textit{de morgans law} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to ((p \land \neg q) \lor (\neg p \lor r)) && \text{by } \textit{double negation} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to ((p \lor (\neg p \lor r)) \land (\neg q \lor (\neg p \lor r))) && \text{by } \textit{distributivity} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to (((p \lor \neg p) \lor r) \land (\neg q \lor (\neg p \lor r))) && \text{by } \textit{associativity} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to (((\neg p \lor p) \lor r) \land (\neg q \lor (\neg p \lor r))) && \text{by } \textit{commutativity} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to ((\top \lor r) \land (\neg q \lor (\neg p \lor r))) && \text{by } \textit{complement} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to (\top \land (\neg q \lor (\neg p \lor r))) && \text{by } \textit{domination} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to (\neg q \lor (\neg p \lor r)) && \text{by } \textit{identity} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to ((\neg q \lor \neg p) \lor r) && \text{by } \textit{associativity} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to ((\neg p \lor \neg q) \lor r) && \text{by } \textit{commutativity} \\
&\equiv (\neg p \lor (\neg q \lor r)) \to (\neg p \lor (\neg q \lor r)) && \text{by } \textit{associativity} \\
&\equiv \neg(\neg p \lor (\neg q \lor r)) \lor (\neg p \lor (\neg q \lor r)) && \text{by } \textit{conditional disintegration} \\
&\equiv \top && \text{by } \textit{complement}
\end{aligned}$$

Therefore, $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is a tautology. Q.E.D.

4. Show that $\neg$ and $\wedge$ are sufficient to express *any* proposition.

   Proof. Observe the following chain of reasoning starting with the formal definition of a proposition.

   We say that r is a proposition if r satisfies the following recurrence.

   1. $r = \top$ or $r = \bot$.
   2. $r = \neg p$, where p is a proposition.
   3. $r = (p) \wedge (q)$ where p and q are propositions.
   4. $r = (p) \vee (q)$ where p and q are propositions.
   5. $r = (p) \rightarrow (q)$ where p and q are propositions.
   6. $r = (p) \leftrightarrow (q)$ where p and q are propositions.

   From this definition, five logical connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$) are sufficient to express any proposition. Thus if we can express $\vee$, $\rightarrow$, and $\leftrightarrow$ with $\neg$ and $\wedge$ then $\neg$ and $\wedge$ are sufficient to express any proposition

   Let p and q represent arbitrary propositions. Observe the following chain of reasoning.

   (a) $\top$ or $\bot$
   This is the base case of the recursive definition and will always be a proposition.

   (b) $\neg$
   This only uses $\neg$ and thus does not need to be altered to for this question.

   (c) $\wedge$
   This only uses $\wedge$ and thus does not need to be altered to for this question.

   (d) $\vee$

$$\begin{aligned}
\neg(\neg p \wedge \neg q) &\equiv \neg(\neg p) \vee \neg(\neg q) && \text{by } \textit{de morgans laws} \\
&\equiv p \vee q && \text{by } \textit{double negation} \times 2
\end{aligned}$$

   (e) $\rightarrow$

$$\begin{aligned}
\neg(p \wedge \neg q) &\equiv \neg p \vee \neg(\neg q) && \text{by } \textit{de morgans laws} \\
&\equiv \neg p \vee q && \text{by } \textit{double negation} \\
&\equiv p \rightarrow q && \text{by } \textit{conditional disintegration}
\end{aligned}$$

   (f) $\leftrightarrow$

$$\begin{aligned}
\neg(q \wedge \neg p) \wedge \neg(p \wedge \neg q) &\equiv (\neg q \vee \neg(\neg p)) \wedge (\neg p \vee \neg(\neg q)) && \text{by } \textit{de morgans laws} \times 2 \\
&\equiv (\neg q \vee p) \wedge (\neg p \vee q) && \text{by } \textit{double negation} \times 2 \\
&\equiv (q \rightarrow p) \wedge (p \rightarrow q) && \text{by } \textit{conditional disintegration} \times 2 \\
&\equiv p \leftrightarrow q && \text{by } \textit{biconditional disintegration}
\end{aligned}$$

Using these 6 premises, we can additionally break down all sub-propositions to only contain $\neg$ and $\land$ as long as they are of finite length. If they are not of finite length, then they are not propositions.

Therefore, as $\neg$ and $\land$ can express the three other logical connectives, $\neg$ and $\land$ are sufficient to express *any* proposition with this new definition.

We say that r is a proposition if r satisfies the following recurrence.

1. $r = \top$ or $r = \bot$.
2. $r = \neg p$, where p is a proposition.
3. $r = (p) \land (q)$ where p and q are propositions.
4. $r = (p) \lor (q)$ which can be rewritten as $\neg(\neg p \land \neg q)$ where p and q are propositions.
5. $r = (p) \to (q)$ which can be rewritten as $\neg(p \land \neg q)$ where p and q are propositions.
6. $r = (p) \leftrightarrow (q)$ which can be rewritten as $\neg(q \land \neg p) \land \neg(p \land \neg q)$ where p and q are propositions.

Q.E.D.

5. Is there a *single connective* capable of expressing *any* proposition?

   Yes, there is a single connective capable of expressing any proposition. Looking at this question, all we need is an expression that is able to express $\neg$ and $\wedge$ because of the logic shown in question 4. This new logical connective I will call "negand" with the symbol being $\neg\!\!\wedge$ and it will act like this: $p \neg\!\!\wedge q \equiv \neg p \wedge q$. Now to prove that this single connective can express any proposition, it has to be able to do the function of $\neg$ and the function of $\wedge$.

   Proof. Let p and q represent arbitrary propositions. Observe the following chain of reasoning.

   (a) $\neg$

$$
\begin{aligned}
p \neg\!\!\wedge \top &\equiv \neg p \wedge \top && \text{by } \textit{definition of negand} \\
&\equiv \neg p && \text{by } \textit{identity}
\end{aligned}
$$

   (b) $\wedge$

$$
\begin{aligned}
(p \neg\!\!\wedge q) \neg\!\!\wedge q &\equiv \neg(\neg p \wedge q) \wedge q && \text{by } \textit{definition of negand} \times 2 \\
&\equiv (\neg(\neg p) \vee \neg q) \wedge q && \text{by } \textit{de morgans laws} \\
&\equiv (p \vee \neg q) \wedge q && \text{by } \textit{double negation} \\
&\equiv (p \wedge q) \vee (\neg q \wedge q) && \text{by } \textit{definition of negation} \\
&\equiv (p \wedge q) \vee \bot && \text{by } \textit{complement} \\
&\equiv p \wedge q && \text{by } \textit{identity}
\end{aligned}
$$

   Therefore, because every proposition can be expressed by $\wedge$ and $\neg$, and $\neg\!\!\wedge$ can form these two logical connectives, every proposition can be expressed by a single connective. Q.E.D.