Homework 11

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1 Monday 4/14

Chapter 13: Section 3 Find the series solutions of the following problems.

2. A bar 10 cm long with insulated sides is initially at 100°. Starting at $t = 0$, the ends are held at 0°. Find the temperature distribution in the bar at time t.

   Define the boundary constraints.

$$u(x, 0) = u_0(x) = 100$$
$$u(0, t) = u(10, t) = 0$$

   we know that

$$T(t) = Ce^{-k^2\alpha^2 t}$$
$$F(x) = C_1 \sin(kx) + C_2 \cos(kx)$$

   Since

$$u(0, t) = u(10, t) = 0$$

   we know that

$$F(0) = F(10) = 0$$

   Thus

$$F(x) = C_1 \sin(kx) \text{ where } k = \frac{n\pi}{10}$$

   So

$$u = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2\alpha^2 t/(10)^2} \sin(n\pi x/10)$$

   where

$$\begin{aligned} b_n &= \frac{2}{10} \int_0^{10} 100 \sin(n\pi x/10) dx \\ &= 20 \int_0^{10} \sin(n\pi x/10) dx \\ &= -\frac{200}{n\pi} \cos(n\pi x/10) \Big|_0^{10} \\ &= -\frac{200}{n\pi} (\cos(n\pi) - \cos(0)) \\ &= -\frac{200}{n\pi} ((-1)^n - 1) \\ &= \begin{cases} \frac{400}{n\pi}(-1)^n & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases} \end{aligned}$$

   Finally

$$u = \frac{400}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n} e^{-n^2\pi^2\alpha^2 t/(10)^2} \sin(n\pi x/10)$$

Chapter 14: Section 1 Find the real and imaginary parts $u(x,y)$ and $v(x,y)$ of the following functions.

1. z^3

$$\begin{aligned} f(z) &= z^3 \\ &= (x+iy)^3 \\ &= \Big(x^2 + 2xiy + (iy)^2\Big)(x+iy) \\ &= \Big(x^2 + 2xiy - y^2\Big)(x+iy) \\ &= x^3 + 2x^2iy - xy^2 + x^2iy + 2x(iy)^2 - iy^3 \\ &= x^3 + 2x^2iy - xy^2 + x^2iy - 2xy^2 - iy^3 \\ &= x^3 - 3xy^2 + 3xiy^2 - iy^3 \\ &= (x^3 - 3xy^2) + i(3xy^2 - y^3) \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= x^3 - 3xy^2 \\ v(x,y) &= 3xy^2 - y^3 \end{aligned}}$$

2. z

$$\begin{aligned} f(z) &= z \\ &= x + iy \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= x \\ v(x,y) &= y \end{aligned}}$$

3. $\overline{z}$

$$\begin{aligned} f(z) &= \overline{z} \\ &= \overline{x+iy} \\ &= x - iy \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= x \\ v(x,y) &= -y \end{aligned}}$$

4. $|z|$

$$\begin{aligned} f(z) &= |z| \\ &= |x+iy| \\ &= \sqrt{x^2+y^2} \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= \sqrt{x^2+y^2} \\ v(x,y) &= 0 \end{aligned}}$$

5. $\mathrm{Re}(z)$

$$\begin{aligned} f(z) &= \mathrm{Re}(z) \\ &= \mathrm{Re}(x+iy) \\ &= x \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= x \\ v(x,y) &= 0 \end{aligned}}$$

6. e^z

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x(\cos(y) + i\sin(y)) \qquad\qquad = e^x\cos(y) + ie^x\sin(y) \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= e^x\cos(y) \\ v(x,y) &= e^x\sin(y) \end{aligned}}$$

7. $\cosh(z)$

$$\begin{aligned} f(z) &= \cosh(z) \\ &= \cosh(x+iy) \\ &= \cosh(x)\cos(y) + i\sinh(x)\sin(y) \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= \cosh(x)\cos(y) \\ v(x,y) &= \sinh(x)\sin(y) \end{aligned}}$$

9. $\frac{1}{z}$

$$\begin{aligned} f(z) &= \frac{1}{z} \\ &= \frac{1}{x+iy} \\ &= \frac{1}{x+iy}\frac{x-iy}{x-iy} \\ &= \frac{x-iy}{x^2-(iy)^2} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= \frac{x}{x^2+y^2} \\ v(x,y) &= \frac{-y}{x^2+y^2} \end{aligned}}$$

11. $\frac{2z-i}{iz+2}$

$$\begin{aligned}
f(z) &= \frac{2z-i}{iz+2} \\
&= \frac{2(x+iy)-i}{i(x+iy)+2} \\
&= \frac{2x+2iy-i}{ix+i^2y+2} \\
&= \frac{2x+2iy-i}{ix-y+2} \\
&= \frac{2x+2iy-i}{ix-y+2}\frac{ix+y-2}{ix+y-2} \\
&= \frac{(2x+2iy-i)(ix+y-2)}{(ix-y+2)(ix+y-2)} \\
&= \frac{2ix^2+2i^2xy-i^2x+2xy+2iy^2-iy-4x-4iy+2i}{(ix)^2-(y-2)^2} \\
&= \frac{-3x+2ix^2+2iy^2-5iy+2i}{-x^2-(y-2)^2} \\
&= \frac{3x-2ix^2-2iy^2+5iy-2i}{x^2+(y-2)^2} \\
&= \frac{3x}{x^2+(y-2)^2}+i\frac{-2x^2-2y^2+5y-2}{x^2+(y-2)^2}
\end{aligned}$$

$$\boxed{\begin{aligned}
u(x,y) &= \frac{3x}{x^2+(y-2)^2} \\
v(x,y) &= \frac{-2x^2-2y^2+5y-2}{x^2+(y-2)^2}
\end{aligned}}$$

12. $\frac{z}{z^2+1}$

$$\begin{aligned}
f(z) &= \frac{z}{z^2+1} \\
&= \frac{x+iy}{(x+iy)^2+1} \\
&= \frac{x+iy}{x^2+2xiy+(iy)^2+1} \\
&= \frac{x+iy}{x^2-y^2+1+2xiy} \\
&= \frac{x+iy}{x^2-y^2+1+2xiy}\frac{x^2-y^2+1-2xiy}{x^2-y^2+1-2xiy} \\
&= \frac{x^3-xy^2+x-2x^2iy+x^2yi-iy^3+iy-2x(iy)^2}{(x^2-y^2+1)^2-(2xiy)^2} \\
&= \frac{x^3+x+xy^2-x^2iy-iy^3+iy}{(x^2-y^2+1)^2+(2xy)^2} \\
&= \frac{x^3+x+xy^2}{(x^2-y^2+1)^2+(2xy)^2}+i\frac{-x^2y-y^3+y}{(x^2-y^2+1)^2+(2xy)^2}
\end{aligned}$$

$$\boxed{\begin{aligned}
u(x,y) &= \frac{x^3+x+xy^2}{(x^2-y^2+1)^2+(2xy)^2} \\
v(x,y) &= \frac{-x^2y-y^3+y}{(x^2-y^2+1)^2+(2xy)^2}
\end{aligned}}$$

15. $\overline{e^z}$

$$\begin{aligned} f(z) &= \overline{e^z} \\ &= \overline{e^{x+iy}} \\ &= \overline{e^x(\cos(y) + i\sin(y))} \\ &= \overline{e^x\cos(y) + ie^x\sin(y)} \\ &= e^x\cos(y) - ie^x\sin(y) \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= e^x\cos(y) \\ v(x,y) &= -e^x\sin(y) \end{aligned}}$$

16. $z^2 - \overline{z^2}$

$$\begin{aligned} f(z) &= z^2 - \overline{z^2} \\ &= (x+iy)^2 - \overline{(x+iy)^2} \\ &= x^2 + 2xiy + (iy)^2 - \overline{x^2 + 2xiy + (iy)^2} \\ &= x^2 - y^2 + 2xiy - \overline{x^2 - y^2 + 2xiy} \\ &= x^2 - y^2 + 2xiy - (x^2 - y^2 - 2xiy) \\ &= i4xy \end{aligned}$$

$$\boxed{\begin{aligned} u(x,y) &= 0 \\ v(x,y) &= 4xy \end{aligned}}$$

2 Wednesday 4/16

Chapter 14: Section 2 Use the Cauchy-Riemann conditions to find out whether the functions are analytic.

1. z^3

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 \\ v(x,y) &= 3xy^2 - y^3 \end{aligned}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial x} = 6xy$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = z^3$ is analytical.

2. z

$$\begin{aligned} u(x,y) &= x \\ v(x,y) &= y \end{aligned}$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = z$ is analytical.

3. $\overline{z}$

$$u(x,y) = x$$
$$v(x,y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z) = \overline{z}$ is NOT analytical.

4. $|z|$

$$u(x,y) = \sqrt{x^2 + y^2}$$
$$v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = 2x\sqrt{x^2+y^2} \qquad \frac{\partial v}{\partial y} = 0$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z) = |z|$ is NOT analytical.

5. $\mathrm{Re}(z)$

$$u(x,y) = x$$
$$v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 0$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z) = \mathrm{Re}(z)$ is NOT analytical.

6. e^z

$$u(x,y) = e^x \cos(y)$$
$$v(x,y) = e^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) \qquad \frac{\partial v}{\partial y} = e^x \cos(y)$$
$$\frac{\partial u}{\partial y} = -e^x \sin(y) \qquad \frac{\partial v}{\partial x} = e^x \sin(y)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = e^z$ is analytical.

7. $\cosh(z)$

$$u(x,y) = \cosh(x)\cos(y)$$
$$v(x,y) = \sinh(x)\sin(y)$$

$$\frac{\partial u}{\partial x} = \sinh(x)\cos(y) \qquad \frac{\partial v}{\partial y} = \sinh(x)\cos(y)$$
$$\frac{\partial u}{\partial y} = -\cosh(x)\sin(y) \qquad \frac{\partial v}{\partial x} = \cosh(x)\sin(y)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = \cosh(z)$ is analytical.

9. $\frac{1}{z}$

$$u(x,y) = \frac{x}{x^2+y^2}$$
$$v(x,y) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$
$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = \frac{1}{z}$ is analytical (assuming $z \neq 0$).

11. $\frac{2z-i}{iz+2}$

$$u(x,y) = \frac{3x}{x^2+(y-2)^2}$$
$$v(x,y) = \frac{-2x^2-2y^2+5y-2}{x^2+(y-2)^2}$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{(3)(x^2+(y-2)^2)-(3x)(2x)}{(x^2+(y-2)^2)^2} \\
&= \frac{3x^2+3(y-2)^2-6x^2}{(x^2+(y-2)^2)^2} \\
&= \frac{-3x^2+3y^2-12y+12}{(x^2+(y-2)^2)^2} \\
\frac{\partial v}{\partial y} &= \frac{(-4y+5)(x^2+(y-2)^2)-(-2x^2-2y^2+5y-2)(2(y-2))}{(x^2+(y-2)^2)^2} \\
&= \frac{(-4y+5)(x^2+(y-2)^2)-(-4x^2y-4y^3+10y^2-4y)+(-8x^2-8y^2+20y-8)}{(x^2+(y-2)^2)^2} \\
&= \frac{(-4y+5)(x^2+(y-2)^2)+4x^2y+4y^3-10y^2+4y-8x^2-8y^2+20y-8}{(x^2+(y-2)^2)^2} \\
&= \frac{(-4y+5)(x^2+(y-2)^2)+4x^2y+4y^3-18y^2+24y-8x^2-8}{(x^2+(y-2)^2)^2} \\
&= \frac{-4x^2y+-4y(y-2)^2+5x^2+5(y-2)^2+4x^2y+4y^3-18y^2+24y-8x^2-8}{(x^2+(y-2)^2)^2} \\
&= \frac{-4y(y-2)^2+5(y-2)^2+4y^3-18y^2+24y-3x^2-8}{(x^2+(y-2)^2)^2} \\
&= \frac{-4y(y-2)^2+4y^3-13y^2+4y-3x^2+12}{(x^2+(y-2)^2)^2} \\
&= \frac{-3x^2+3y^2-12y+12}{(x^2+(y-2)^2)^2} \\
\frac{\partial u}{\partial y} &= \frac{(0)(x^2+(y-2)^2)-(3x)(2(y-2))}{(x^2+(y-2)^2)^2} \\
&= \frac{-6xy+12x}{(x^2+(y-2)^2)^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{(-4x)(x^2+(y-2)^2-(-2x^2-2y^2+5y-2)(2x)}{(x^2+y^2)^2} \\
&= \frac{-4x^3-4xy^2+16xy-16x+4x^3+4xy^2-10xy+4x}{(x^2+y^2)^2} \\
&= \frac{16xy-16x-10xy+4x}{(x^2+y^2)^2} \\
&= \frac{6xy-12x}{(x^2+y^2)^2}
\end{aligned}
$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = \frac{2z-i}{iz+2}$ is analytical (assuming $z \neq 2i$).

12. $\frac{z}{z^2+1}$

$$
\begin{aligned}
u(x,y) &= \frac{x^3+x+xy^2}{(x^2-y^2+1)^2+4x^2y^2} \\
v(x,y) &= \frac{-x^2y-y^3+y}{(x^2-y^2+1)^2+4x^2y^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{-x^6-x^4y^2-x^4+x^2y^4-10x^2y^2+x^2+y^6-y^4-y^2+1}{\left(x^4+2x^2y^2+2x^2+y^4-2y^2+1\right)^2} \\
\frac{\partial v}{\partial y} &= \frac{y^6+y^4x^2-y^4-y^2x^4-10y^2x^2-y^2-x^6-x^4+x^2+1}{\left(y^4+2y^2x^2-2y^2+x^4+2x^2+1\right)^2} \\
\frac{\partial u}{\partial y} &= \frac{2y^5x+4y^3x^3+4y^3x+2yx^5-4yx^3-6yx}{\left(y^4+2y^2x^2-2y^2+x^4+2x^2+1\right)^2} \\
\frac{\partial v}{\partial x} &= \frac{-2x^5y-4x^3y^3+4x^3y-2xy^5-4xy^3+6xy}{\left(x^4+2x^2y^2+2x^2+y^4-2y^2+1\right)^2}
\end{aligned}
$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $f(z) = \frac{2z-i}{iz+2}$ is analytical (assuming $z \neq \pm i$).

15. $\overline{e^z}$

$$
\begin{aligned}
u(x,y) &= e^x \cos(y) \\
v(x,y) &= -e^x \sin(y)
\end{aligned}
$$

$$
\frac{\partial u}{\partial x} = e^x \cos(y) \qquad\qquad \frac{\partial v}{\partial y} = -e^x \cos(y)
$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z) = \overline{e^z}$ is NOT analytical.

16. $z^2 - \overline{z^2}$

$$
\begin{aligned}
u(x,y) &= 0 \\
v(x,y) &= 4xy
\end{aligned}
$$

$$
\frac{\partial u}{\partial x} = 0 \qquad\qquad \frac{\partial v}{\partial y} = 4x
$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z) = z^2 - \overline{z^2}$ is NOT analytical.

22. $y + ix$

$$u(x,y) = y$$
$$v(x,y) = x$$

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = 1$$

Since $-\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$, $f(x,y) = y + ix$ is NOT analytical.

24. $\frac{y-ix}{x^2+y^2}$

$$u(x,y) = \frac{y}{x^2+y^2}$$
$$v(x,y) = \frac{-x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = -\frac{2yx}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{2xy}{(x^2+y^2)^2}$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z) = \frac{y-ix}{x^2+y^2}$ is NOT analytical.

54. y

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \Delta u = 0$$

Thus, u is harmonic.

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = 1 = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v = v(x), \quad \frac{dv}{dx} = -1 \Rightarrow v(x) = -x + C$$

Thus, For this question we choose $C = 0$

$$f(z) = y - ix$$

$$\frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0 \quad \Rightarrow \quad \Delta v = 0$$

Thus, v is harmonic.

55. $3x^2y - y^3$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(6xy) = 6y, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-3y^2) = -6y \quad \Rightarrow \quad \Delta u = 6y - 6y = 0$$

Thus, u is harmonic.

$$\frac{\partial u}{\partial x} = 6xy = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2 = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = 6xy \Rightarrow v = 3xy^2 + g(x), \quad -\frac{\partial v}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial v}{\partial x} = -3x^2 + 3y^2$$

Take the derivative of $v = 3xy^2 + g(x)$ with respect to x:

$$\frac{\partial v}{\partial x} = 3y^2 + g'(x) = -3x^2 + 3y^2 \Rightarrow g'(x) = -3x^2 \Rightarrow g(x) = -x^3 + C$$

Thus,

$$v(x, y) = 3xy^2 - x^3 + C$$

For this question we choose $C = 0$

$$f(z) = 3x^2y - y^3 + i(3xy^2 - x^3)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}(-3x^2 + 3y^2) = -6x, \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}(6xy) = 6x \quad \Rightarrow \quad \Delta v = -6x + 6x = 0$$

Thus, v is harmonic.