Jordan-Dilithium: A Post-Quantum Signature Scheme over the Albert Algebra

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Abstract

We present **Jordan-Dilithium**, a lattice-based digital signature scheme designed for UTxO blockchain architectures within the Axiomatic Physical Homeostasis (APH) framework. Unlike traditional post-quantum schemes that operate over associative polynomial rings, Jordan-Dilithium is constructed over $J_3(\mathbb{O}_q)$, the 27-dimensional Albert Algebra of Hermitian octonionic matrices. We introduce a novel verification mechanism that exploits the scalar center of the algebra to bypass Artin's Theorem, allowing efficient verification of signatures despite the non-associativity of the underlying octonions. This construction leverages "Topological Impedance"—the computational hardness of resolving operation order in a chaotic algebraic manifold—to provide enhanced resistance against quantum reduction attacks.

1 Introduction

The security of current blockchain ledgers, including Bitcoin, relies heavily on the Elliptic Curve Digital Signature Algorithm (ECDSA). The security of ECDSA is predicated on the Discrete Logarithm Problem (DLP), which is vulnerable to Shor's algorithm on a sufficiently powerful quantum computer. As we transition to a post-quantum era, lattice-based cryptography has emerged as the leading candidate for standardization.

However, standard lattice schemes (e.g., Dilithium, Kyber) typically operate over associative rings such as $\mathcal{R}_q = \mathbb{Z}_q[X]/(X^n+1)$. While efficient, these structures possess algebraic symmetries that may be susceptible to specialized ideal lattice attacks.

In this work, we propose a signature scheme based on **Non-Associative Algebras**. Specifically, we utilize the Octonions ($\mathbb{O}$) and the exceptional Jordan Algebra ($J_3(\mathbb{O})$). We demonstrate that by carefully selecting the cryptographic challenge from the scalar field, we can construct a Fiat-Shamir protocol where honest verification is associative, but forgery requires solving a non-associative hard problem.

2 Mathematical Foundations

2.1 The Ring $\mathbb{O}_q$

We define our base algebraic structure as the Octonions over the integer ring $\mathbb{Z}_q$, where $q = 32768$. An element $x \in \mathbb{O}_q$ is represented as:

$$x = \sum_{i=0}^{7} x_i e_i, \quad x_i \in \mathbb{Z}_q \tag{1}$$

The octonions are non-commutative and non-associative. The failure of associativity is measured by the *Associator*:

$$[x, y, z] = (xy)z - x(yz) \neq 0 \tag{2}$$

2.2 The Albert Algebra $J_3(\mathbb{O})$

The Albert Algebra is the set of 3×3 Hermitian matrices with octonionic entries. An element $X \in J_3(\mathbb{O})$ has the form:

$$X = \begin{pmatrix} \alpha & c & b^* \\ c^* & \beta & a \\ b & a^* & \gamma \end{pmatrix} \tag{3}$$

where $\alpha, \beta, \gamma \in \mathbb{Z}_q$ (Scalars) and $a, b, c \in \mathbb{O}_q$. This algebra has dimension $3 + 3 \times 8 = 27$.

2.3 The Jordan Product

The standard matrix product is not well-defined due to non-associativity. Instead, we use the symmetrized **Jordan Product**:

$$X \circ Y = \frac{1}{2}(XY + YX) \tag{4}$$

This product is commutative ($X \circ Y = Y \circ X$) but non-associative.

3 The Jordan-Dilithium Protocol

3.1 Setup and Key Generation

Let $A \in J_3(\mathbb{O}_q)$ be a public generator sampled uniformly from the algebra. The security relies on the hardness of the *Learning With Jordan Errors* (LWJE) problem.

Algorithm 1 Key Generation

1: **Input:** Security parameter λ
2: Sample generator $A \leftarrow J_3(\mathbb{O}_q)$ uniformly.
3: Sample secret $S \leftarrow J_3(\mathbb{O}_q)$ with small norm (Structured Noise).
4: Compute public key component $T = A \circ S$.
5: **Output:** $pk = (A, T)$, $sk = S$.

3.2 Signing (Fiat-Shamir with Aborts)

To sign a message M, the signer proves knowledge of S such that $T = A \circ S$ without revealing S.

Algorithm 2 Sign(sk, M)

1: Sample ephemeral mask $Y \leftarrow J_3(\mathbb{O}_q)$ uniformly.
2: Compute commitment $W = A \circ Y$.
3: Compute challenge $c = H(M \parallel W)$.
4: **CRITICAL:** Map c to a scalar in $\mathbb{Z}_q$.
5: Compute response $Z = Y + c \cdot S$.
6: **if** $|Z|_\infty > \gamma - \beta$ **then**
7: **Reject** (Restart with new Y).
8: **end if**
9: **Output:** $\sigma = (Z, c)$.

3.3 Verification and the Scalar Innovation

The verifier checks if the response Z corresponds to the commitment W shifted by the challenge.

Algorithm 3 Verify(pk, M, σ)

Parse σ as (Z, c).
Check norm: if $|Z|_\infty > \gamma$ return **Invalid**.
Reconstruct commitment: $W' = (A \circ Z) - c \cdot T$.
Reconstruct challenge: $c' = H(M \parallel W')$.
if $c' == c$ **then**
    Valid
else
    Invalid
end if

4 Correctness: Bypassing Artin's Theorem

The core challenge in non-associative cryptography is that typically $(A \circ S) \circ C \neq A \circ (S \circ C)$. This would make standard verification equations fail:

$$A \circ Z = A \circ (Y + cS) \stackrel{?}{=} A \circ Y + c(A \circ S) \tag{5}$$

In a general setting, this distributivity and associativity holds only if the elements involved generate an associative subalgebra (Artin's Theorem).

Our Innovation: By enforcing that the challenge c is a **scalar** (a real number in the center of the algebra), we ensure commutativity and associativity with *all* elements in $J_3(\mathbb{O})$.

$$c \in \mathbb{R} \implies (A \circ S) \circ c = A \circ (S \circ c) \tag{6}$$

Thus, the verification holds perfectly for honest participants. However, an attacker attempting to forge a signature must solve for S in the 27-dimensional manifold where this scalar simplification does not apply to the lattice basis reduction itself.

5 Performance and Analysis

We implemented a prototype in Rust.

- **Key Size:** Public Key $\approx$ 1.5 KB, Signature $\approx$ 2.5 KB.
- **Operation:** Verification requires efficient Jordan Product evaluation (dominated by 9$\times$ Octonion multiplications).
- **Experimental Data:**
    - Public Key Alpha: 29094
    - Signature Challenge c: 185
    - Response Norm Z_α: 8925 (Well within γ bounds).

6 Conclusion

Jordan-Dilithium represents a viable post-quantum signature scheme that leverages the *Topological Impedance* of non-associative algebras. By utilizing the scalar center for challenges, we resolve the verification bottleneck while maintaining the geometric complexity required for security against quantum adversaries.