Short Paper

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Abstract

Insert abstract...

TODO:

• Check consistency of Super Shot vs. Power 5 period (use Power 5 period)

Introduction

Netball is a court-based team sport played predominantly among Commonwealth nations, and has one of the highest participation rates for team sports in Australia (???). As in many court-based team sports, the goal of netball is to score more than the opposition. Netball is, however, unique in that goals may only be scored by two players on each team from within the 'shooting circle' (i.e. a half circle around the goal with a 4.9m radius) at their end of the court (???). Traditionally, goals scored from within this circle result in one 'goal' or 'point' for the team (???). In the 2020 season, Australia's national elite-level league (i.e. Suncorp Super Netball) made the decision to introduce the 'Super Shot' (???). The Super Shot period provided teams an opportunity to gain one-versus two-points for successful shots made from the 'inner' (i.e. 0m-3.0m) versus 'outer' (i.e. 3.0m-4.9m), respectively, within the final five minutes of each quarter (???). The league has confirmed that the Super Shot rule will continue through the 2021 season (???).

Our analysis prior to the 2020 season (???) suggested that the added value of the Super Shot (i.e. two-points) aligned well with the elevated risk of shooting from long range, and that teams may have been able to maximise their scoring by taking a high proportion of Super Shots. These findings were, however, based on shooting statistics from past seasons where the Super Shot rule was not in effect – and further investigation of leagues where a 'two-point rule' was in place (i.e. international Fast5) resulted in a much higher risk of missing long-range shots (???). We hypothesised that the elevated risk of missing long-range shots with a 'two-point rule' in place stems from situational factors, whereby defensive strategies were likely altered to place a heavier emphasis on defending long-range shots (???). Data from the first full season with the Super Shot

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in place provides an opportunity to re-evaluate the risk-reward value of taking Super Shots with more valid shooting statistics. Further, these data can provide a better foundation for simulating Super Shot periods as a means to identify optimal shooting strategies. In the present study we re-visited the question of whether the weighting of a 2:1 value is appropriate based on the relative risk of missing a shot from the outer versus inner circle during the Super Shot period using data from the 2020 season. Further – we ran a series of simulations of the five-minute Super Shot period, driven by shooting statistics from the 2020 season, in an attempt to identify optimal team-specific shooting strategies for the proportion of Super Shots to take.

TODO: add aim of competitive sims...

Methods

Participants

Participants for this study included all players across the eight teams from the 2020 season of the Australian national netball league (i.e. Suncorp Super Netball). Our study included publicly available, pre-existing data held on the Suncorp Super Netball match centre ($TODO: add\ link$). An exemption from ethics review (and subsequent waiver of individual consent) was granted by the Deakin University Human Research Ethics Committee ($TODO: add\ details$).

Data Collection

We used the {SuperNetballR} TODO: citation package to extract match data from all regular season games during the 2020 Super Netball Season via the Champion Data (official provider of competition statistics) match centre. Within the match centre data – all shots are labelled with identifiers that place them in the inner or outer circle, along with whether they were made or missed. Combined with the timestamp of these events within quarters, we extracted team-specific shooting statistics for: (i) the total number of shots taken; (ii) the number of shots taken from the inner and outer circle; and (iii) the number of made and missed shots from the inner and outer circle from each Super Shot period across the season.

Data Analysis

Our study required estimating the probability of making versus missing shots from the inner versus outer circle across the different teams. We achieved this by defining a beta distribution in a probability density function for the different circle zones, specified by:

$$f(x, a, b) = \frac{\Gamma(a+b)x^{a-1}(1-x)^{b-1}}{\Gamma(a)\Gamma(b)}$$

where a and b represent the number of missed and made shots within a circle zone, respectively; x is the probability of a relative to b; and Γ is the gamma function (???). Probability density functions were created for made versus

missed shots in the inner and outer circles for each team, as well as all teams combined, to be used in subsequent analyses.

To examine the relative value of the 2:1 point ratio, we replicated the approach from our previous work (???), but this time with data from the 2020 season. Specifically, we compared the average relative odds (\pm 95% confidence intervals [CI]) of missing from the outer versus inner circle during the Super Shot period. This was achieved by dividing randomly sampled values (n=100,000) from the probability density functions of the outer by those from the inner circle at each sample iteration. This analysis was run using shooting statistics from the entire league, as well as individual teams, to give overall and team-specific risk-reward values for attempting Super Shots. We also applied this analysis to opposition shooting statistics against each team, providing a risk-reward value for attempting Super Shots against an opponent. Theoretically, the relative odds of missing from the outer to inner circle should match the ratio of points awarded (i.e. 2:1) for the Super Shot to represent 'good value.'

Next, we ran a series of simulations (n = 1,000 each) of the Power 5 period for each team altering the proportion of Super Shots taken from 0% to 100%, at 10% increments. We used the previously calculated probabilities of making versus missing shots from within and outside the outer circle during the Power 5 period from each team to estimate the number of points the team may score during this period. ACross each individual simulation, the total number of shots the team would take and the proportion of these that were Super Shots was first determined. We determined the mean and standard deviation for the number of shots a team would expect during a Power 5 period based on overall season statistics — and the total number of shots for a team in an individual simulation was randomly sampled from a truncated normal distribution between the lower and upper 95% CI limits of the mean/standard deviation. The number of standard versus Super Shots being taken within the simulation was then determined based on the current Super Shot proportion (i.e. from 0% to 100%) being examined. The success (i.e. make vs. miss) of each individual standard or Super Shot within the simulation was then determined by generating a random value between 0 and 1, alongside a value sampled from the teams relevant probability density function of making a shot from the relevant location (i.e. inner or outer circle). If the value sampled from the probability density function was greater versus lower than the random value — the shot was considered successful versus unsuccessful, respectively. After all individual shots were simulated, the total team score was summed given the value of the made standard and Super shots. We examined the scores achieved, and calculated the relative frequency of when the minimum and maximum score was achieved for each team under the different Super Shot proportions.

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• Include some probability calculations of winning vs. losing the 5 minute period (i.e. X team's relative probability of winning vs. losing the super shot period under different proportions; or maybe even between proportions?)

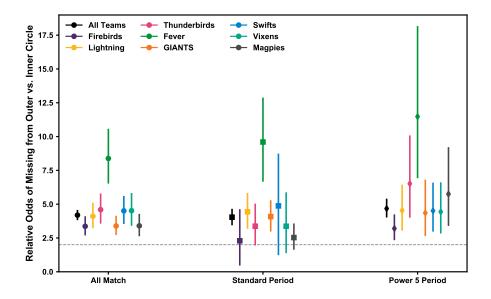


Figure 1: Relative odds (mean \pm 95% confidence intervals) of teams missing from the outer versus inner circle across the entire match, and during the standard and Super Shot scoring periods.

Results

The relative combined odds (\pm 95% CIs) from all teams of missing from the outer versus the inner circle across the entire match were 4.19 [3.88, 4.52], and 4.04 [3.5, 4.6] versus 4.68 [4.07, 5.35] in the standard and Super Shot periods, respectively. The relative odds of missing from the outer versus inner circle across the individual teams were relatively similar, with the exception of the Fever having higher odds than the majority of teams across the entire match period (see Figure 1). The relative odds (\pm 95% CIs) of missing from the outer versus inner circle during the Super Shot Period were greater than 2:1 across all teams (see Figure 1). No team appeared more or less effective in elevating the risk of missing from the outer versus inner circle, with mostly similar odds observed across all teams opponents in the various periods of the quarter (see Figure 2).

 \dots standard sims results...

Across all teams, as the proportion of Super Shots increased within simulations both the maximum and minimum scores achievable increased and decreased, respectively (see Figure 3). Both the highest and lowest achievable scores for each team therefore came from simulations using a high proportion of Super Shots (i.e. > 80%).

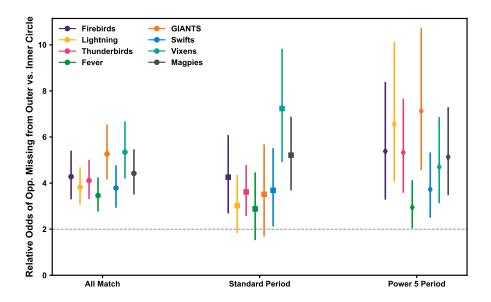


Figure 2: Relative odds (mean \pm 95% confidence intervals) of opposition teams missing from the outer versus inner circle across the entire match, and during the standard and Super Shot scoring periods.

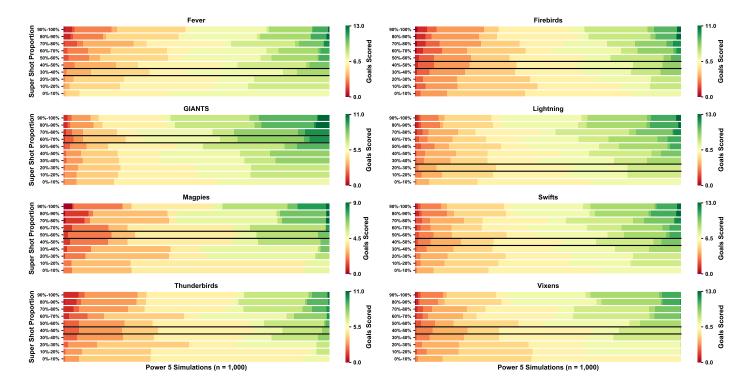


Figure 3: Distribution of simulated score outputs for each team using different Super Shot proportions. Black outline represents each teams actual proportion of Super Shots taken during Power 5 periods from 2020 season statistics.

Table 1: TODO Add caption

...standard sim max and min results...

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Table 2: TODO Add caption

| X | Fever | Firebirds | GIANTS | Lightning | Magpies | Swifts | Thunderbirds | Vixens |
|--|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| Team 0% / Opp. 0% Team 0% / Opp. 25% Team 0% / Opp. 50% Team 0% / Opp. 75% Team 0% / Opp. 100% | 0.25 [0.06, 0.45] | -0.51 [-0.70, -0.32] | -0.10 [-0.29, 0.09] | -0.01 [-0.20, 0.19] | 0.13 [-0.07, 0.32] | -0.01 [-0.21, 0.18] | 0.13 [-0.06, 0.33] | 0.12 [-0.07, 0.31] |
| | 0.21 [0.00, 0.41] | -0.29 [-0.50, -0.09] | 0.33 [0.12, 0.54] | 0.27 [0.06, 0.49] | 0.46 [0.24, 0.67] | 0.25 [0.04, 0.47] | 0.31 [0.10, 0.53] | 0.54 [0.32, 0.76] |
| | 0.19 [-0.03, 0.40] | -0.02 [-0.25, 0.20] | 0.71 [0.47, 0.94] | 0.56 [0.33, 0.79] | 0.78 [0.54, 1.01] | 0.49 [0.26, 0.73] | 0.51 [0.27, 0.74] | 0.97 [0.73, 1.21] |
| | 0.07 [-0.16, 0.30] | 0.13 [-0.11, 0.36] | 1.10 [0.86, 1.35] | 0.77 [0.53, 1.01] | 1.04 [0.80, 1.28] | 0.75 [0.51, 0.99] | 0.70 [0.46, 0.94] | 1.37 [1.12, 1.62] |
| | -0.06 [-0.30, 0.19] | 0.28 [0.03, 0.52] | 1.49 [1.22, 1.75] | 1.02 [0.76, 1.27] | 1.31 [1.05, 1.57] | 0.95 [0.69, 1.20] | 0.75 [0.50, 1.01] | 1.64 [1.38, 1.90] |
| Team 25% / Opp. 0% Team 25% / Opp. 25% Team 25% / Opp. 50% Team 25% / Opp. 75% Team 25% / Opp. 100% | -0.04 [-0.26, 0.18] | -0.77 [-0.98, -0.56] | -0.33 [-0.54, -0.12] | -0.27 [-0.48, -0.05] | -0.13 [-0.34, 0.09] | -0.28 [-0.49, -0.07] | -0.14 [-0.35, 0.07] | -0.12 [-0.33, 0.09] |
| | -0.09 [-0.31, 0.14] | -0.55 [-0.77, -0.32] | 0.10 [-0.13, 0.33] | 0.02 [-0.21, 0.25] | 0.20 [-0.03, 0.43] | -0.01 [-0.24, 0.22] | 0.03 [-0.19, 0.26] | 0.30 [0.07, 0.53] |
| | -0.11 [-0.35, 0.13] | -0.28 [-0.52, -0.04] | 0.48 [0.22, 0.73] | 0.30 [0.05, 0.55] | 0.52 [0.27, 0.77] | 0.23 [-0.02, 0.48] | 0.23 [-0.02, 0.48] | 0.73 [0.47, 0.98] |
| | -0.22 [-0.47, 0.03] | -0.13 [-0.38, 0.12] | 0.87 [0.61, 1.13] | 0.51 [0.26, 0.77] | 0.79 [0.53, 1.05] | 0.49 [0.23, 0.74] | 0.42 [0.17, 0.67] | 1.13 [0.87, 1.39] |
| | -0.35 [-0.62, -0.09] | 0.02 [-0.24, 0.28] | 1.25 [0.98, 1.53] | 0.76 [0.49, 1.03] | 1.05 [0.78, 1.33] | 0.68 [0.41, 0.95] | 0.48 [0.21, 0.74] | 1.40 [1.12, 1.67] |
| Team 50% / Opp. 0% Team 50% / Opp. 25% Team 50% / Opp. 50% Team 50% / Opp. 75% Team 50% / Opp. 100% | -0.34 [-0.58, -0.10] | -1.04 [-1.27, -0.80] | -0.57 [-0.81, -0.34] | -0.52 [-0.75, -0.29] | -0.37 [-0.60, -0.14] | -0.54 [-0.77, -0.31] | -0.42 [-0.65, -0.19] | -0.37 [-0.60, -0.15] |
| | -0.38 [-0.63, -0.13] | -0.81 [-1.06, -0.57] | -0.15 [-0.40, 0.10] | -0.24 [-0.49, 0.01] | -0.05 [-0.30, 0.20] | -0.28 [-0.52, -0.03] | -0.24 [-0.49, 0.01] | 0.05 [-0.20, 0.29] |
| | -0.40 [-0.67, -0.14] | -0.54 [-0.81, -0.28] | 0.23 [-0.04, 0.50] | 0.05 [-0.22, 0.31] | 0.27 [0.00, 0.54] | -0.03 [-0.30, 0.23] | -0.05 [-0.31, 0.22] | 0.48 [0.21, 0.75] |
| | -0.52 [-0.78, -0.26] | -0.40 [-0.66, -0.13] | 0.63 [0.35, 0.91] | 0.26 [-0.01, 0.53] | 0.54 [0.27, 0.81] | 0.22 [-0.05, 0.49] | 0.15 [-0.12, 0.41] | 0.88 [0.60, 1.15] |
| | -0.65 [-0.93, -0.37] | -0.24 [-0.52, 0.03] | 1.01 [0.72, 1.30] | 0.51 [0.22, 0.79] | 0.81 [0.52, 1.10] | 0.42 [0.13, 0.71] | 0.20 [-0.08, 0.48] | 1.15 [0.86, 1.44] |
| Team 75% / Opp. 0% Team 75% / Opp. 25% Team 75% / Opp. 50% Team 75% / Opp. 75% Team 75% / Opp. 100% | -0.62 [-0.86, -0.38] | -1.26 [-1.50, -1.03] | -0.77 [-1.01, -0.53] | -0.75 [-0.99, -0.50] | -0.59 [-0.83, -0.35] | -0.75 [-0.99, -0.51] | -0.63 [-0.87, -0.39] | -0.55 [-0.79, -0.31] |
| | -0.66 [-0.92, -0.41] | -1.04 [-1.29, -0.79] | -0.34 [-0.60, -0.09] | -0.46 [-0.72, -0.21] | -0.27 [-0.52, -0.01] | -0.49 [-0.75, -0.23] | -0.46 [-0.71, -0.20] | -0.13 [-0.39, 0.13] |
| | -0.69 [-0.95, -0.42] | -0.77 [-1.04, -0.51] | 0.04 [-0.24, 0.31] | -0.18 [-0.45, 0.09] | 0.05 [-0.22, 0.33] | -0.25 [-0.52, 0.03] | -0.26 [-0.54, 0.01] | 0.30 [0.02, 0.58] |
| | -0.80 [-1.07, -0.53] | -0.63 [-0.90, -0.35] | 0.43 [0.15, 0.72] | 0.03 [-0.25, 0.31] | 0.32 [0.04, 0.60] | 0.01 [-0.27, 0.29] | -0.07 [-0.35, 0.21] | 0.70 [0.41, 0.99] |
| | -0.93 [-1.21, -0.65] | -0.47 [-0.75, -0.19] | 0.81 [0.52, 1.11] | 0.28 [-0.01, 0.57] | 0.59 [0.29, 0.88] | 0.21 [-0.09, 0.50] | -0.01 [-0.30, 0.28] | 0.97 [0.67, 1.27] |
| Team 100% / Opp. 0% Team 100% / Opp. 25% Team 100% / Opp. 50% Team 100% / Opp. 75% Team 100% / Opp. 100% | -0.80 [-1.06, -0.53] | -1.41 [-1.67, -1.16] | -0.90 [-1.15, -0.64] | -0.91 [-1.16, -0.65] | -0.77 [-1.03, -0.51] | -0.96 [-1.21, -0.70] | -0.87 [-1.12, -0.61] | -0.77 [-1.02, -0.51] |
| | -0.84 [-1.11, -0.57] | -1.19 [-1.46, -0.92] | -0.47 [-0.74, -0.20] | -0.62 [-0.90, -0.35] | -0.44 [-0.71, -0.17] | -0.69 [-0.97, -0.42] | -0.69 [-0.96, -0.42] | -0.35 [-0.62, -0.07] |
| | -0.86 [-1.14, -0.58] | -0.92 [-1.20, -0.64] | -0.09 [-0.38, 0.20] | -0.34 [-0.63, -0.05] | -0.12 [-0.41, 0.17] | -0.45 [-0.74, -0.16] | -0.50 [-0.78, -0.21] | 0.08 [-0.21, 0.38] |
| | -0.98 [-1.27, -0.69] | -0.77 [-1.06, -0.48] | 0.30 [0.01, 0.60] | -0.13 [-0.42, 0.17] | 0.14 [-0.15, 0.44] | -0.19 [-0.48, 0.10] | -0.30 [-0.59, -0.01] | 0.48 [0.19, 0.78] |
| | -1.11 [-1.41, -0.81] | -0.62 [-0.92, -0.32] | 0.69 [0.38, 1.00] | 0.12 [-0.19, 0.43] | 0.41 [0.10, 0.72] | 0.00 [-0.30, 0.31] | -0.25 [-0.55, 0.06] | 0.75 [0.44, 1.07] |

Discussion

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Our analysis of shot statistics from the first season of the Super Shot demonstrated an approximate four-times higher likelihood of missing from the outer versus inner circle, irrespective of the match period (i.e. standard vs. Super Shot period). While the 95% CIs did overlap between the two periods, a slightly higher risk of missing from the outer versus inner circle was present in the Super Shot period. Our present findings conflict with our earlier work from the 2018 Super Netball season where we observed an approximate two-times higher risk of missing from the outer versus inner circle — and suggested that the 2:1 point ratio of the Super Shot represented 'good' value relative to the elevated risk of missing (???). Our present analysis perhaps suggests this is no longer the case as the risk of missing from the outer circle has approximately doubled, and now outweighs the additional point value on offer. Across the different teams and periods, We found in all but one instance that the risk of missing from the outer versus inner circle was greater than two-times that of the inner circle. The only instance where the 95% CIs overlapped the 2:1 ratio was for the Firebirds during the standard period. This specific case is irrelevant, however, as the additional points were not available during this time. The obvious difference between the present study and our previous work (???) is the actual presence of the Super Shot being on offer during the 2020 season. The elevated risk of missing from the outer versus inner circle from the 2018 to 2020 Super Netball season is similar to what we observed in originally comparing the 2018 Super Netball to Fast5 (i.e. where long range shots were also rewarded with additional points) (???). Therefore, it seems apparent that adding value to long-range shots in netball induces an adjustment in the way either attacking and/or defensive teams play that elevates the risk of missing long- versus close-range shots. There are a number of factors that could be contributing to this elevated risk. The most likely reason is an additional emphasis from defensive players on contesting longrange shots given their added value, while the presence if the Super Shot may also introduce psychological pressure on the shooting player. A more thorough analysis of defensive strategies and understanding shooting players perspectives around the Super Shot can assist in understanding the mechanisms behind the elevated risk.

Remaining discussion points...

Our approach in the present paper to simulate Super Shot scoring periods differs to our original work (???). Previously, we allocated an overall success rate to shots from the inner versus outer circle (i.e. if the sampled probability was 50%, a total of 50% of shots were counted as successful). This contrasts to our present work, where we sample and apply the probability of shot success to simulated individual shots (i.e. if the sampled probability is 50%, the individual shot being simulated has a 50% probability of success). This approach likely reflects an improvement on our analysis, better representing the independent nature of shots in a netball match.

NOTES...{

The relative odds of missing from the outer versus inner circle suggests that, on average, the value of the Super shot was outweighed by the elevated risk of missing. We found that in all but one instance, the risk of missing from the outer circle was greater than 2 times that of the inner circle. The only case where the 95% CIs overlapped 2 was for the Firebirds during the standard period. This specific case is, however, somewhat irrelevant as the additional points were not available during this time. The mean relative odds across the dataset for our versus inner circle ranged from X to X. These values are much higher than our original analysis from a previous season, however align with the fast 5 data which also included additional points for long range shots. Together, these findings suggest that the presence of additional points is in some way making these shots more difficult. It is still difficult to infer what is causing this, with altered defensive strategies and added psychological pressure remaining as valid potential reasons. The risk of missing from the outer versus inner circle was also elevated in standard scoring times compared to previous seasons. This potentially suggests opposition teams may have adopted a general or overall shift in defending long range shots, even when the additional points were not available. Further analysis of gameplay or investigation of coach/player strategy and perceptions is likely necessary to confirm the factors contributing.

General approach and considerations around this. There are likely still situations where the Super shot holds appropriate value. Despite the average relative risk outweighing the additional point value in offer, there may be situations where teams identify this as an appropriate risk. For example, when trailing by a large margin with minimal time remaining, the 1 point on offer for a standard shot may have very little value to the trailing team. In this instance the Super shot becomes the only option for a team to remain competitive. Our analysis also considered overall team shooting statistics. The relative risk may change for an individual specialist long range shooter, hence having the ball in their hands in Super shot range may actually present a relatively valuable opportunity. Hot hand too...

Defensive relative risk data.... The opposition also appears to be a factor in the relative value of the Super shot. X teams appeared to defend the shot better by defending this better than the inner circle, particularly in the Super shot period. This further indicates an emphasis on certain teams defensive strategy for the Super shot, which could have made the long range harder, or by virtue of defending long range positions making it easier for teams to get easier shots closer to the post. Conversely, certain teams oppositions reduced their risk of outer vs. inner miss during the super shot period – perhaps suggesting that their defensive strategies were not as effective as their oppositions strategy in generating 'good' shots during the super shot period. Despite the mean differences, the confidence intervals typically overlapped. Again, these inferences can only be confirmed with more detailed analysis of team strategy.

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- General/overall value with respect to different proportions?
- Team specific values, any obvious differences? For example, one team may

- have had better success and therefore using a higher proportion in general may have led to higher percentage of 'won' periods
- Team vs. team specific values and if they are different across various opponents. For example, better shooting success with high super shot proportions vs. one team but not another? This may be more relevant if we use opponent specific probabilities of super shot success. Important that the lack of 'defensive' presence within simulations is acknowledged as a limitation, in that we applied the same super shot probability rates for each team from their entire season, rather than individually vs. their oposition team. Given we might have some relative risk of missing against different defensive opponents, this may actually reveal that this should be a consideration if one team is more effective with their defense
- Practical considerations of work include strategising around super shot, with respect to how many to take perhaps depending on margin along with opposition, as well as own teams success in this realm

Discussion notes...check paragraphs here

Our simulation data does, however, demonstrate potential value in using the Super shot for certain teams and in certain scenarios. Times where higher proportion of super shot was valuable? We incorporated variable shot opportunity numbers based on league data, and hence the number of shot opportunities varied across individual simulations for teams. This was balanced, in that when teams received more shot opportunities, their opposition received fewer. Across all simulations, the wining team received more shots on x% of times. This factor became more/less evident across scenarios where a team took a higher proportion of super shots, whereby the winning team had more shots in x% of these simulations. This firstly suggests that generating more shot opportunities than your opponent is obviously beneficial, but potentially awards you more flexibility when considering taking a greater proportion of super shots.

Similarly, teams who were better with the Super shot fared better in simulations with greater proportions of super shots, and vice versa for teams that are worse. For example, the fever lost x% of simulations when they went heavy on the Super shot, vs. X team who won x% of simulations when using a high % proportion of super shots. This is not surprising as the fever had the highest risk of missing from the outer vs inner circle, particularly during Super shot periods. These findings likely demonstrate a need for teams to play to their shooting strengths.

Looking at the margin summaries (mean +/- 95% CI's) from each teams competitive simulations across all opponents, certain strategies appeared more or less favourable across different teams. For example, where the Fever extended above 50% of their shots as Super Shots, it was typical for them to score less than their opponent in the Super Shot period; whereas when they used no Super Shots they typically outscored their opponents. Other trends...?

Nonetheless, the simulated margins within the Super Shot period, irrrespective of the strategies used, were typically low - rarely exceeding 1.5 to 2 points in a typical simulation. At most we suggest that optimising the Super Shot

proportions for a given team and opponent may yield (on average) a 1 or 2 point gain each quarter. This may still be beneficial for teams, as this could equate to 4 to 8 points across an entire match. However, it is important to note that this is the average \pm 1 yellow CIs, and hence it will not occur the same each time.

Conclusion

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Bibliography styles

There are various bibliography styles available. You can select the style of your choice in the preamble of this document. These styles are Elsevier styles based on standard styles like Harvard and Vancouver. Please use BibTeX to generate your bibliography and include DOIs whenever available.

Here are two sample references: Feynman and Vernon Jr. (1963; Dirac, 1953).

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