1. **Prosecutor's Fallacy**

   Bow Valley College has purposed to implement random COVID screening of students using a rapid test. Currently 1 in 1000 Albertans have COVID. Using the Confusion Matrix for the rapid test, what is the probability a student has COVID given a positive test.

	Actual COVID Infection	
Test Result	Absent	Present
Positive	1/100 False Positive	2/3 True Positive Sensitivity
Negative	99/100 True Negative Specificity	$\frac{1}{3}$ False Negative

   Solution: We apply Bayes' Theorem to find the probability of having COVID given a positive test.

$$\begin{aligned}
\mathbb{P}\left[\text{Present}\|\text{Positive}\right] &= \mathbb{P}\left[\text{Positive}\|\text{Present}\right] \cdot \frac{\mathbb{P}\left[\text{Present}\right]}{\mathbb{P}\left[\text{Positive}\right]} \\
&= \frac{\mathbb{P}\left[\text{Positive}\|\text{Present}\right] \cdot \mathbb{P}\left[\text{Present}\right]}{\mathbb{P}\left[\text{Positive}\|\text{Present}\right] \cdot \mathbb{P}\left[\text{Present}\right] + \mathbb{P}\left[\text{Positive}\|\text{Absent}\right] \cdot \mathbb{P}\left[\text{Absent}\right]} \\
&= \frac{1}{1 + \frac{\mathbb{P}[\text{Positive}\|\text{Absent}] \cdot \mathbb{P}[\text{Absent}]}{\mathbb{P}[\text{Positive}\|\text{Present}] \cdot \mathbb{P}[\text{Present}]}} \\
&= \frac{1}{1 + \frac{\frac{1}{100} \cdot \frac{999}{1000}}{\frac{2}{3} \cdot \frac{1}{1000}}} \\
&= \frac{1}{1 + \frac{2997}{200}} \\
&= \frac{200}{3197}
\end{aligned}$$

   Thus there is only a $\approx$6.3% chance the randomly selected student who tested positive actually has COVID.

2. **Monty Hall Problem**

   The Monty Hall Problem is a game of chance with conditional information:

   1. The game begins with a prize randomly hidden behind one of three doors.
   2. The contestant chooses an initial door out of the three.
   3. A door that neither the contestant has chosen, nor containing a prize is then opened.
   4. The contestant can then optionally choose to switch their choice of doors to the remaining unopened door.
   5. Finally the prize door is revealed. If it is the final choice of the contestant then the contestant wins, otherwise they lose.

   The counterintuitive result of this game is that the optimum strategy is for the contestant to always switch their choice of doors, as that has a two thirds chance of winning.

Solution: We will use a tree diagram to calculate the theoretical probability of winning conditioned on the strategy of switching the choice of doors.

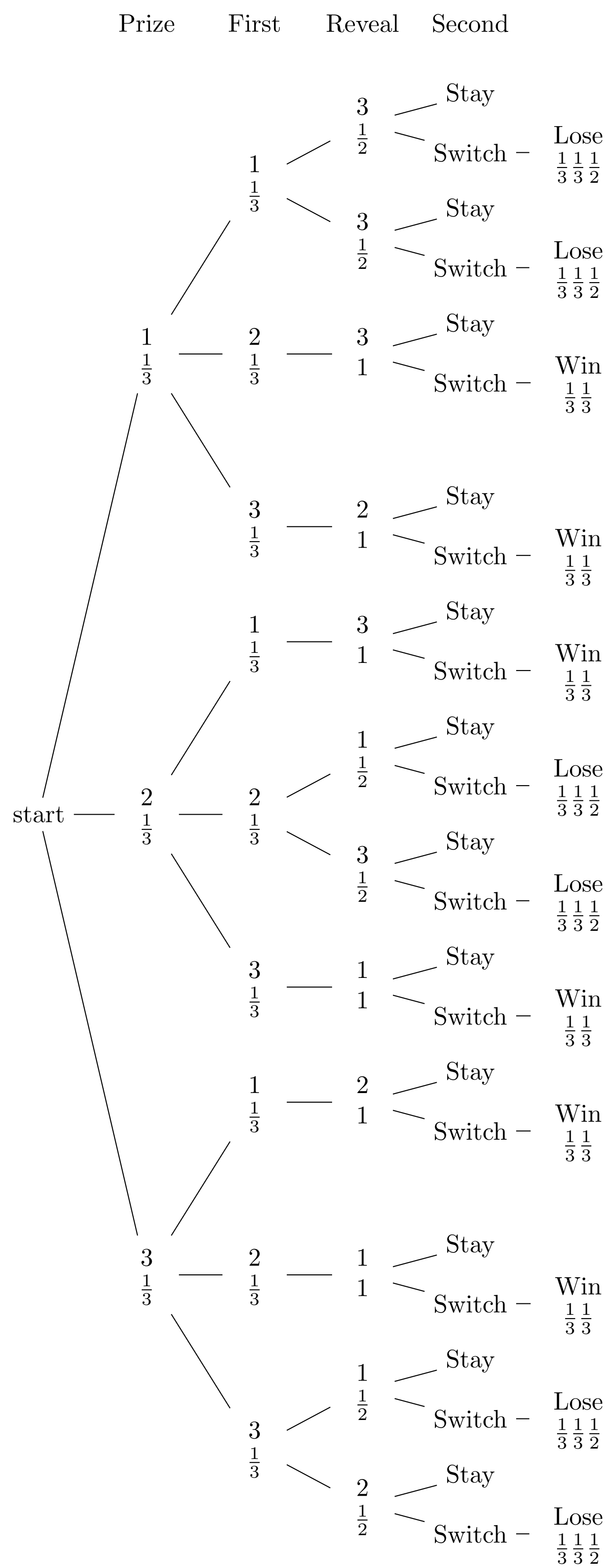
Tree diagrams allow for the calculation of discrete, or enumerable, probabilities by observing that the joint probability of any finite sequence of events $E_1, \ldots, E_n$ can be recursively factored into a sequence of conditional probabilities. We will start by writing the events in reverse order:

$$\begin{aligned}
\mathbb{P}[E_n, \ldots, E_1] &= \mathbb{P}[E_n \| E_{n-1} \ldots, E_1] \cdot \mathbb{P}[E_{n-1}, \ldots, E_1] \\
&= \mathbb{P}[E_n \| E_{n-1} \ldots, E_1] \cdot \mathbb{P}[E_{n-1} \| E_{n-2} \ldots, E_1] \cdot \mathbb{P}[E_{n-2}, \ldots, E_1] \\
&\vdots \\
&= \mathbb{P}[E_n \| E_{n-1} \ldots, E_1] \cdot \mathbb{P}[E_{n-1} \| E_{n-2} \ldots, E_1] \cdot \\
&\qquad \vdots \\
&\qquad \cdot \mathbb{P}[E_2 \| E_1] \cdot \mathbb{P}[E_1]
\end{aligned}$$

Each conditional probability is the probability of visiting the next branch given the current branch. In many circumstances the probability of the next event conditioned on the current event is much more easily modelled than the overall joint probability. In this example we will write our tree starting from the left. At each node of the tree we write the outcome and the conditional probability of the outcome.

For the Monty Hall Problem their are two important observations: first, the conditional probabilities of the branches within the preceding branch are uniform; second, the contestants choice of door is independent of the assignment of the prize. The last observation simply asserts that the contestant does not possess psychic abilities. The outcomes of each branch is simply one of the three possible doors. To calculate the probability of winning conditioned on switching we sum all the joint probabilities of winning and using the switching strategy and divide by all the joint probabilities of either winning or losing under the switching strategy. The denominator is the marginal probability of the switching strategy.

$$\begin{aligned}
\mathbb{P}[\text{Win} \| \text{Switch}] &= \frac{\mathbb{P}[\text{Win}, \text{Switch}]}{\mathbb{P}[\text{Win}, \text{Switch}] + \mathbb{P}[\text{Lose}, \text{Switch}]} \\
&= \frac{6 \cdot \frac{1}{3} \cdot \frac{1}{3}}{6 \cdot \frac{1}{3} \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} \cdot \frac{1}{3} \frac{1}{2}} \\
&= \frac{2}{3}
\end{aligned}$$


Prize
First
Reveal
Second
start
1 1/3
2 1/3
3 1/3
Stay
Switch
Lose 1/3 1/3 1/2
Win 1/3 1/3