Gompertz Processes: A Theory of Ageing

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Abstract

Ageing is a process of accelerating failures that is universal to all biological systems. Motivated by considering infinitesimal stochastic accelerations of time, I hypothesize a general explanation for the emergent determinism of ageing processes in the theory of Gompertz processes: Poisson processes subordinated by integrated geometric Brownian motion.

1 Preliminaries

Basic science experiments in biology have ubiquitously observed that organisms respond to environmental stresses, including communicable diseases and exposures to toxins, with an acceleration a in their failure time $ah\left(at\right)$, where $h\left(t\right)$ is the unperturbed hazard rate of the failure event. Outside of the controlled setting of a laboratory, the environmental stresses are random resulting in a stochastic sequence of accelerations a_n of the organism's metabolic time at actual times t_n. The stochastic accelerations can either increase the rate of failure $a_n > 1$, an exacerbation of the stresses, or decrease the rate of failure $a_n < 1$, an alleviation from the stresses. However, even in the controlled setting of a laboratory it is experimentally challenging to directly measure the stochastically accelerated metabolic time of the organism, instead we only have access to the failure events measured in bare time units. Thus, a theory of ageing must study the subordination of failure times by a stochastically accelerated metabolic time.

Over the course of an organism's lifetime it will encounter exacerbations and alleviations that stochastically accelerate $a_n, \ldots, a_0 = 1$ metabolic time at times $t_n > \cdots > t_0 = 0$. Furthermore, because the stochastic accelerations are all positive $a_i > 0$ for each stochastic acceleration we can find a finite real valued generator $w_n, \ldots, w_0 = 0$ such that $a_i = e^{w_i}$. It follows that the stochastically accelerated metabolic time y_{t_n} at time t_n is the sum of the products of the stochastic accelerations up to time t_{n-1} and the elapsed bare time steps $t_i - t_{i-1}$:

$$y_{t_n} = \sum_{i=1}^{n} e^{\sum_{j=0}^{i-1} w_j} \left(t_i - t_{i-1}\right) \tag{1}$$

Provided no great explosions of acceleration occur in any small time scale, like say when an actual explosion occurs, the generators will become infinitesimal

$w_i \to 0$ on the same order as the bare time steps become infinitesimal $t_i - t_{i-1} \to 0$:

$$\mathcal{O}(w_i) = \mathcal{O}(t_i - t_{i-1}) \tag{2}$$

The sum of products then becomes a stochastic process Y_t that is an integral of a random geometric infinitesimal generator process e^{W_t}:

$$Y_t = \int_0^t e^{W_u} du \tag{3}$$

This asymptotic argument is essentially the continuous part of Kolmorgorov's characterization of stochastic processes as being composed of, in bounded time, either a finite number of discrete jumps or an infinite number of continuous changes.

If we reasonably assume as a first approximation that the exacerbations and alleviations, and their respective stochastic accelerations are independent and stationary over time then by the Lévy-Khintchine characterization the only infinitesimal generator of stochastically accelerated metabolic time that is Lévy and continuous *"jump free"* is Brownian motion W_t. The stochastically accelerated metabolic time Y_t is better known as integrated geometric Brownian motion.

2 Gompertz

Motivated by the preceding heuristic derivation we formally define the Gompertz process on time $t \geq 0$.

Definition 1 (Gompertz Process)**.** A Gompertz process G_t is a subordinated Poisson process N_t, with rate λ, where the subordinating process is integrated geometric Brownian motion Y_t, with drift μ and diffusion σ:

$$G_t = N_{Y_t} \tag{4}$$

given:

$$\begin{aligned} Y_t &= \int_0^t X_s ds \qquad (5) \\ &= \int_0^t e^{\mu s + \sigma W_s} ds \qquad (6) \end{aligned}$$

Phenomenologically the finite real stochastic process $\mu t + \sigma W_s$ is the infinitesimal acceleration at time t that generates a non-negative geometric stochastic process X_t of accumulated accelerations up to time t and whose integral Y_t is a strictly increasing stochastically accelerated metabolic time up to time t.

From the definition of the Gompertz process G_t conditioning on the history of a sample path of integrated geometric Brownian motion Y_t yields a conditional

Poisson process, where $\mathrm{d}\mathbb{P}$ represents the Radon-Nikodym density:

$$\mathrm{d}\mathbb{P}\left[G_t = n \| \, Y_t\right] = \frac{\left(\lambda Y_t\right)^n}{n!} e^{-\lambda Y_t} \tag{7}$$

$$\mathbb{E}\left[G_t^n \| \, Y_t\right] = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left(\lambda Y_t\right)^k \tag{8}$$

Where the sum in the expectation is over Stirling's numbers of the second kind. Applying Bayes' theorem we can condition on the Gompertz process to estimate the elapsed stochastically accelerated metabolic time:

$$\mathrm{d}\mathbb{P}\left[Y_t \| \, G_t = n\right] = \frac{Y_t^n e^{-\lambda Y_t}}{\mathbb{E}\left[Y_t^n e^{-\lambda Y_t}\right]} \mathrm{d}\mathbb{P}\left[Y_t\right] \tag{9}$$

$$\mathbb{E}\left[Y_t^m \| \, G_t = n\right] = \frac{\mathbb{E}\left[Y_t^{n+m} e^{-\lambda Y_t}\right]}{\mathbb{E}\left[Y_t^n e^{-\lambda Y_t}\right]} \tag{10}$$

Reflecting that integrated geometric Brownian motion Y_t measures the elapsed stochastically accelerated metabolic time.

3 Factorization

To start our exploration of the rich subtleties of the Gompertz process we will briefly review the properties of integrated geometric Brownian motion which are salient to developing our theory. This is by no means a comprehensive compendium. Much of the material I will cover has been deeply and thoroughly explored in the quantitative finance literature in the theory of pricing Asian options, and in the graduate syllabus of stochastic processes covering subordinated Poisson processes, usually in the context of deriving the characteristic function of the subordinating process.

Our first observation is that the increments of Y_t can be factored by its carrier process X_t, for times $t > s$ we have in law:

$$Y_t - Y_s = X_s Y_{t-s} \tag{11}$$

where the process X_s is independent of the process Y_{t-s}. Phenomenologically the increments of $Y_t - Y_s$ are equivalent to a process Y_{t-s} that starts with acceleration X_s.

Analogous to the Fundamental Theorem of Arithmetic carrier factorization provides for immediate shallow inferences. For example we can immediately observe that for times $t > s$:

$$\mathbb{E}\left[\left(Y_t - Y_s\right)^n \| \, X_s\right] = X_s^n \, \mathbb{E}\left[Y_{t-s}^n\right] \tag{12}$$

I will liberally exploit this technique of arbitraging the stochastically accelerated metabolic time Y_t against the accumulated stochastic accelerations X_t to reduce expectations down to the well known standard terms for X_t and Y_t:

$$\mathbb{E}\left[X_t^n\right] = e^{\left(n\mu + n^2\sigma^2/2\right)t} \tag{13}$$

$$\mathbb{E}\left[Y_t\right] = \frac{e^{\left(\mu+\sigma^2/2\right)t} - 1}{\mu + \sigma^2/2} \tag{14}$$

Note that in the last equation I have implicitly invoked the Fubini-Tonelli Theorem to switch the order of integration, and will broadly continue to use this theorem throughout this work to evaluate expectations, such as the covariance between Y_t and X_t:

$$\mathbb{C}\mathrm{ov}\left[X_t, Y_t\right] = \frac{\mathbb{V}\mathrm{ar}\left[X_t\right]}{\mu + 3\sigma^2/2} - \mathbb{E}\left[X_t\right]\mathbb{E}\left[Y_t\right] \tag{15}$$

Carrier factorization implies that $Y_{t+\Delta t}$ can be recovered through marginalization over Y_t and $Y_{\Delta t}$. Applying this observation to the definition of the derivative we obtain the hazard rate relationship:

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\left[X_t \| \, Y_t \geq y\right] = \left(\mu + \sigma^2/2\right)\mathbb{E}\left[X_t \| \, Y_t \geq y\right] \\ + \mathbb{E}\left[X_t \| \, Y_t = y\right] \mathrm{d}\mathbb{H}\left[Y_t = y\right]\end{aligned} \tag{16}$$

where $\mathrm{d}\mathbb{H}\left[Y_t = y\right] = \mathrm{d}\mathbb{P}\left[Y_t = y \| \, Y_t \geq y\right]$ is the hazard rate of the value of Y_t. From the integral of this differential equation evaluated at $Y_t = 0$ we deduce the boundary condition:

$$\mathrm{d}\mathbb{P}\left[X_t = x, Y_t = 0\right] = \frac{\delta\left(x\right)}{t} \tag{17}$$

There are limits to what can be learned from carrier factorization alone. By the application of Itô calculus conditioning on X_t generates a measure space that is the product of two σ-finite measure spaces. We cannot say the same for conditioning on Y_t. To illustrate the difficulty consider the conditional probability:

$$\mathbb{P}\left[X_t \| \, Y_t\right] = \lim_{n\to\infty} \frac{1}{n!}\underset{\substack{\frac{t}{n}\sum\prod u_i = Y_t, \\ \prod u_i = X_t}}{\int \cdots \int} \begin{aligned} &\mathrm{d}\mathbb{P}\left[X_{\frac{t}{n}} = u_1\right]\cdots \\ &\cdots \mathrm{d}\mathbb{P}\left[X_{\frac{t}{n}} = u_n\right] du_1 \cdots du_n \end{aligned} \tag{18}$$

The conditional probability irreducibly mixes the time $s < t$ and accelerations X_s, so that the measure space of the conditional probability is not the product measure space of two σ-finite measure spaces. As such we cannot apply the Fubini-Tonelli theorem to integrate processes conditioned on their integrals. The crux of the failure is that the conditioning X_t on it's integral Y_t introduces a reciprocal constraint between the size of the excursions of X_s for $s < t$ and the duration of the excursions of X_s, because by the construction of the integral the area under the excursions of X_s must be less than the integral Y_t.

4 Engelbert-Schmidt

At outset of this research project contemplation of the bridged conditioning $\mathbb{P}[X_s \| Y_t]$ of a continuous stochastic process $X_s > 0$ on it's future integral Y_t for times $s < t$ lead me to have deep reservations that the central concepts of my investigations were not well formed. Specifically, from the factorization:

$$\mathbb{P}[X_s \leq x \| Y_t \leq y] = \int_0^x \int_0^y \frac{\mathrm{d}\mathbb{P}[X_s = u, Y_s = v]\, \mathbb{P}\left[Y_{t-s} \leq \frac{y-v}{u}\right]}{\mathbb{P}[Y_t \leq y]} dv du \quad (19)$$

we can see that $\mathbb{P}[X_s \| Y_t]$ being well formed hinges on $\mathbb{P}[X_t, Y_t]$ being well formed. As such I will undertake a certain amount of measure theoretic *"worrying"* and *"hand wringing"* to establish that sets of the joint processes of X_t and Y_t are adapted to the filtration $\mathscr{F}_t$ of the carrier Brownian motion W_t. Largely for my own pedantic edification I will informally derive the central result of the Engelbert-Schmidt zero-one law, from elementary first principles; namely that the integral of X_t specifies $\mathscr{F}_t$ measurable sets.

The concern of the Engelbert-Schmidt zero-one law is that probability measures of the integral Y_t of a continuous stochastic process X_t are implicitly probability measures of the limit of sets of X_s with $s < t$:

$$\mathbb{P}[Y_t, \dots] = \mathbb{P}\left[Y_t = \lim_{\pi \subset [0,t)} \sum_{t_i \in \pi} X_{t_i} \Delta t_i, \dots\right] \quad (20)$$

The heart of the result is to show that the limit can be countably constructed from measurable sets in $\mathscr{F}_s \subset \mathscr{F}_t$, the filtration of X_t. As such, consider the sets of successively closer Lorentz factor approximations of the lower bounded integral $Y_t > y$, with the Riemann sum indexed by integers $n \geq 2$:

$$A_n = \left\{ \sum_{m=0}^{n-2} X_{\frac{n}{m}t} \vee X_{\frac{n+1}{m}t} > \left(n - \sqrt{n}\right) \frac{y}{t} \right\} \quad (21)$$

I have approximated the area using the maximum of consecutive evaluations of X_t to ensure all the stragglers of Y_t that just barely cross y near t are captured in the sets. To define the joint probability of $X_t > x$ and $Y_t > y$ we constrain the approximations along the direction x:

$$B_n = \left\{ \sum_{m=0}^{n-2} X_{\frac{n}{m}t} \vee X_{\frac{n+1}{m}t} > \left(n\frac{y}{t} - x\right) \vee 0 \right\} \quad (22)$$

The sets push the approximations of $Y_{\frac{n-1}{n}t}$ into the future Y_t so that the stopped indicator processes of:

$$A_{s\uparrow t} = \begin{cases} \mathbb{I}\left[A_{\lceil \frac{t}{t-s} \rceil}\right] & s < t, \\ 0 & \text{else} \end{cases} \quad (23)$$

$$B_{s\uparrow t} = \begin{cases} \mathbb{I}\left[B_{\lceil \frac{t}{t-s} \rceil}\right] & s < t, \\ 0 & \text{else} \end{cases} \quad (24)$$

$$(25)$$

are adapted to the filtration $\mathscr{F}_s \subset \mathscr{F}_t$ of the carrier Brownian motion W_s for $s < t$. Liberally invoking the continuity of X_t we define the integral process $Y_t > y$ by requiring that the approximations eventually hold in the limit inferior sense:

$$\begin{aligned}
\liminf A_n &= \{X_t \in A_n \text{ eventually } \forall n\} && (26)\\
&= \left\{ X_t < \infty, \int_0^t X_u du > y \right\} && (27)\\
\liminf B_n &= \{X_t \in B_n \text{ eventually } \forall n\} && (28)\\
&= \left\{ X_t < x, \int_0^t X_u du > y \right\} && (29)
\end{aligned}$$

The danger is that the containing limit superior asymptotically induces oscillations in Y_t of increasing frequency and decreasing magnitude so that X_t becomes *"fuzzy"* at t, and thus fails to be a set in the filtration $\mathscr{F}_t$. Essentially we are faced with the risk of a path integral formulation of the Heisenberg uncertainty principle. However, the limit superior and limit inferior differ only by negligible sets in $\mathscr{F}_t$:

$$\begin{aligned}
\limsup A_n - \liminf A_n &= \left\{ X_t \to \infty, \int_0^t X_u du = y \right\} && (30)\\
&\subset \{X_t \to \infty\} && (31)\\
\limsup B_n - \liminf B_n &= \left\{ X_t = x, \int_0^t X_u du = y \right\} && (32)\\
&\subset \{X_t = x\} && (33)
\end{aligned}$$

Although the limit inferiors are strict subsets of the limit superiors they are nonetheless equivalent in probability in $\mathscr{F}_t$, and all the undefined *"fuzziness"* of X_t induced by the limit superior is pruned out by the restriction of the filtration to continuous processes.

The difference between the two limit sets characterizes the exceedence functions of the approximations, and can be interpreted as the directional derivatives of the marginal and joint probabilities along $Y_t = y$ and $X_t = x, Y_t = y$ respectively. The difference sets have negligible measure because we are implicitly working within the integral of the probability measure and not the Radon-Nikodym probability density.

For clarity I have used Riemann sums. If one were to tackle this argument rigorously they would be required to work with a countable net of finite partitions of time. Nonetheless through the countable construction of the limit inferiors we reach the future constraints by carefully pruning the paths in the of present $\mathscr{F}_s \subset \mathscr{F}_t$, without ever actually peaking into the future of $\mathscr{F}_t$ to condition on a future event. Thus the marginal probability $\mathbb{P}[Y_t]$ and joint probability $\mathbb{P}[X_t, Y_t]$ exist and are finite.

5 Identities

Returning from our digression, even with the factorization observation we are still in need of a means of reducing the expectation of the powers Y_t^n. A small lemma suffices to provide the means of finding powers:

Lemma 1 (Recursion-Convolution Lemma). *The expectation of non-negative integer powers $m, n \geq 0$ of geometric Brownian motion X_t and integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}\left[X_t^m Y_t^n\right] = n \int_0^t \mathbb{E}\left[X_{t-u}^{m+n}\right] \mathbb{E}\left[X_u^m Y_u^{n-1}\right] du \tag{34}$$

Proof. Expanding the expectation as a multi-variable integral, applying the Fubini-Tonelli Theorem, factoring, and a final change of variables we have:

$$\begin{aligned}
\mathbb{E}\left[X_t^m Y_t^n\right] &= \mathbb{E}\left[\int_0^t \cdots \int_0^t X_t^m X_{u_1} \cdots X_{u_n} du_1 \ldots du_n\right] &(35)\\
&= \binom{n}{1} \mathbb{E}\left[\int_0^t X_t^m X_u \left(Y_t - Y_u\right)^{n-1} du\right] &(36)\\
&= n \int_0^t \mathbb{E}\left[X_{t-u}^{m+n}\right] \mathbb{E}\left[X_u^m Y_u^{n-1}\right] du &(37)
\end{aligned}$$

□

The recursion-convolution lemma relates the moments of integrated geometric Brownian motion through a linear operator, and like all good linear operators this relationship deserves a uniqueness constraint.

Corollary 1 (Uniqueness Corollary). *If two sequences of functions $f_t^{(n)}$ and $g_t^{(n)}$ of time t satisfy the recursion-convolution relation and $f_t^{(0)} = g_t^{(0)}$ then $f_t^{(n)} = g_t^{(n)}$ for all n.*

Proof. We proceed with induction on n

1. By assumption for $n = 0$ functions are equal.
2. Now assume that up to n the functions are equal.
3. Taking the difference between the functions at $n + 1$ we have

$$\begin{aligned}
f_t^{(n+1)} - g_t^{(n+1)} &= \int_0^t \mathbb{E}\left[X_{t-u}^{m+n}\right] \left(f_u^{(n)} - g_u^{(n)}\right) du &(38)\\
&= 0 &(39)
\end{aligned}$$

□

Convolution equations are dual to differential equations, and as such the products of powers of X_t and Y_t satisfy the recursive differential equation:

$$\begin{aligned}\frac{d}{dt}\,\mathbb{E}\left[X_t^m Y_t^n\right] = n\,\mathbb{E}\left[X_t^m Y_t^{n-1}\right] \\ + \left((m+n)\,\mu + (m+n)^2\,\sigma^2/2\right) n\,\mathbb{E}\left[X_t^m Y_t^n\right]\end{aligned} \tag{40}$$

With the recursion-convolution lemma in hand we have the sufficient tools required to estimate all the usual statistics involving powers of Y_t, including the expectation, variance, and covariances. For example a straightforward, if rather tedious integration gives the expected value of the square of integrated geometric Brownian motion:

$$\mathbb{E}\left[Y_t^2\right] = \frac{\mathbb{V}\mathrm{ar}\left[X_t\right]}{\left(\mu+\sigma^2\right)\left(\mu+3\sigma^2/2\right)} - \frac{\mathbb{E}\left[Y_t\right]}{\left(\mu+\sigma^2\right)} \tag{41}$$

Which by differentiation yields the expectation of the product of X_t and Y_t:

$$\mathbb{E}\left[X_t Y_t\right] = \frac{\mathbb{V}\mathrm{ar}\left[X_t\right]}{\mu+3\sigma^2/2} \tag{42}$$

Arriving round trip at a formula that exactly agrees with the previously derived covariance.

6 Stopping Times

In practice we cannot directly measure the stochastically accelerated metabolic time Y_t instead we have access to the stopping times T_n of the passages $G_{T_n} = n$ of the Gompertz process, which we construct in the usual manner. As before, we condition the the stopping time on the history of stochastic accelerations to recover the familiar forms of the Poisson process, by differentiating the subordinated cumulative distribution of a single stopping time:

$$\mathrm{d}\mathbb{P}\left[T_n = t \|\, X_{T_n}, Y_{T_n}\right] = \begin{cases} \delta\left(t\right) & n = 0, \\ \dfrac{\lambda^n X_t Y_t^{(n-1)} e^{-\lambda Y_t}}{(n-1)!} & \text{otherwise} \end{cases} \tag{43}$$

Applying Bayes' Theorem we can condition on the stopping times to estimate the underlying stochastically accelerated metabolic time:

$$\mathrm{d}\mathbb{P}\left[X_{T_n}, Y_{T_n} \|\, T_n = t\right] = \begin{cases} \delta\left(X_t\right)\delta\left(Y_t\right) & n = 0, \\ \dfrac{X_t Y_t^{n-1} e^{-\lambda Y_t}}{\mathbb{E}\left[X_t Y_t^{n-1} e^{-\lambda Y_t}\right]}\,\mathrm{d}\mathbb{P}\left[X_t, Y_t\right] & \text{otherwise} \end{cases} \tag{44}$$

From this re-weighting of the probabilities we can immediately deduce the elegant expectations conditioned on the stopping time:

$$\mathbb{E}\left[Y_{T_n}^m \| T_n = t\right] = \begin{cases} 0 & n = 0, \\ \dfrac{(n+m-1)!}{\lambda^m (n-1)!} \dfrac{\mathrm{d}\mathbb{P}\left[T_{n+m} = t\right]}{\mathrm{d}\mathbb{P}\left[T_n = t\right]} & \text{otherwise} \end{cases} \tag{45}$$

$$\mathbb{E}\left[Y_{T_n}^m\right] = \begin{cases} 0 & n = 0, \\ \dfrac{(n+m-1)!}{\lambda^m (n-1)!} & \text{otherwise} \end{cases} \tag{46}$$

$$\mathbb{E}\left[X_{T_n}^m \| T_n = t\right] = \begin{cases} 0 & n = 0, \\ \dfrac{\mathbb{E}\left[X_t^{m+1} Y_t^{n-1} e^{-\lambda Y_t}\right]}{\mathbb{E}\left[X_t Y_t^{n-1} e^{-\lambda Y_t}\right]} & \text{otherwise} \end{cases} \tag{47}$$

Finding a closed form for the distribution of the stopping times of the passages of the Gompertz process will require a deeper understanding of the characteristic function of integrated geometric Brownian motion, which we will develop later. In the meantime there are many fruits to be plucked from a study of stopping times of the passages of the Gompertz process.

The central statistic of study in the longitudinal analysis of biological systems is the latency $T_{1+G_t} - T_{G_t}$ between consecutive stopping times of the passages of the Gompertz process. We have subordinated the stopping times T_{n+G_t} of the passages of the Gompertz process by the increments of the Gompertz process $n + G_t$ from a sentinel event G_t due to immortal time bias *"we cannot see indefinitely into the past"*. In practice observational studies, particularly in clinical research and epidemiology, are only able to observe consecutive passages of the Gompertz process from a fixed sentinel event without the knowledge of how many events have occurred before the sentinel event.

From the preceding probability density we can immediately deduce the tail probability and hence the expectation of the latency $s > 0$ conditioned on a sentinel event at time $t \geq 0$:

$$\begin{aligned} &\mathbb{P}\left[T_{1+G_t} - T_{G_t} \geq s \| T_{G_t} = t\right] \\ &\quad = \begin{cases} \mathbb{E}\left[e^{-\lambda Y_s}\right] & t = 0, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{\mathbb{E}\left[X_t Y_t^{n-1} e^{-\lambda Y_{t+s}}\right]}{\mathbb{E}\left[X_t Y_t^{n-1} e^{-\lambda Y_t}\right]} \frac{\mathbb{E}\left[Y_t^n e^{-\lambda Y_t}\right]}{1 - \mathbb{E}\left[e^{-\lambda Y_t}\right]} & \text{otherwise} \end{cases} \end{aligned} \tag{48}$$

$$\begin{aligned} &\mathbb{E}\left[T_{1+G_t} - T_{G_t} \| T_{G_t} = t\right] \\ &\quad = \begin{cases} \mathbb{E}\left[\int_0^\infty e^{-\lambda Y_u} du\right] & t = 0, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{\mathbb{E}\left[X_t Y_t^{n-1} \int_0^\infty e^{-\lambda Y_{t+u}} du\right]}{\mathbb{E}\left[X_t Y_t^{n-1} e^{-\lambda Y_t}\right]} \frac{\mathbb{E}\left[Y_t^n e^{-\lambda Y_t}\right]}{1 - \mathbb{E}\left[e^{-\lambda Y_t}\right]} & \text{otherwise} \end{cases} \end{aligned} \tag{49}$$

Clearly if we have available to us the additional information of the full history of event counts G_t this difficult expectation becomes much easier as we can elide the marginalization over all cardinalities of events.

Developing this line of reasoning further, consider the covariance of the latency between consecutive events $T_{2+G_t}, T_{1+G_t}, T_{G_t}$ conditioned on the sentinel event $T_{G_t} = t$:

$$\begin{aligned}
&\mathbb{C}\mathrm{ov}\left[T_{2+G_t} - T_{1+G_t}, T_{1+G_t} - T_{G_t} \| \, T_{G_t} = t\right] \\
&\quad = \int_0^\infty \int_0^\infty \mathbb{E}\left[e^{-\lambda X_{t+v} Y_u}\right] \mathbb{E}\left[\lambda v X_v e^{-\lambda X_t Y_v}\right] dv du \\
&\qquad - \int_0^\infty \mathbb{E}\left[\lambda X_t Y_u e^{-\lambda X_t Y_u}\right] du \int_0^\infty \mathbb{E}\left[e^{-\lambda X_t Y_u}\right] du
\end{aligned} \tag{50}$$

where all the stochastic processes are independent except the final acceleration X_v and the increment Y_v.

The Gompertz process introduces an irreducible exponential dependence on age *"older organisms are red-shifted with respect to younger organisms"* that cannot be removed or linearized by a coordinate transform, analogous to the Hubble constant which describes the intrinsic expansion of space-time and for which no coordinate transform can remove the intrinsic red-shift of space-time. We can concretely predict that the age of an organism introduces a correlation within the latencies between consecutive events, and that this correlation can be fully accounted for by conditioning on the age of the organism at the sentinel event.

Our observations of the Gompertz process have a serious consequence. Every single longitudinal experiment that has studied outcomes whose latencies are on the same timescale as the lifespan of the investigated organism has introduced spurious correlations into their longitudinal analysis of event latency, which are purely an artefact of the failing to account for the exponential age dependence of the Gompertz process. Fortunately, the majority of the studies of biological systems have either been cross-sectional or of short enough duration that the hazard of the Gompertz process is approximately constant over the timescale of the experiment on the organism.

7 Markov

Integrated geometric Brownian motion Y_t is a 2 step Markov process, and thus the increments $Y_t - Y_s$, with $t > s$ are Markov. To verify this consider the sequence of bare times $0 = t_0 < \cdots < t_{n+1} = t$, and accelerated times $0 = y_0 < \cdots < y_{n+1} = y$, working through the conditional probability we have:

$$\begin{aligned}
&\mathbb{P}\left[Y_{t_{n+1}} - Y_{t_n} \geq y_{n+1} - y_n \middle\| \, Y_{t_n} = y_n, \ldots, Y_{t_0} = y_0\right] \\
&\quad = \mathbb{P}\left[X_{t_n - t_{n-1}} Y_{t_{n+1} - t_n} \geq \frac{y_{n+1} - y_n}{y_n - y_{n-1}} Y_{t_n - t_{n-1}}\right] && (51) \\
&\quad = \mathbb{P}\left[Y_{t_{n+1}} - Y_{t_n} \geq y_{n+1} - y_n \middle\| \, Y_{t_n} - Y_{t_{n-1}} = y_n - y_{n-1}\right] && (52)
\end{aligned}$$

for process $Y_{t_{n+1} - t_n}$ independent of the processes $X_{t_n - t_{n-1}}$ and $Y_{t_n - t_{n-1}}$.

Remarkably the stopping times of the passages of the Gompertz process are Markov, even though the stochastically accelerated metabolic time of integrated geometric Brownian motion is only Markov in its increments. Specifically knowledge of the stopping T_n is sufficient to determine the distribution of the latency $t > 0$ to next event T_{n+1}. To see this consider the stopping times $T_n = t_n, \ldots, T_0 = t_0$, the cumulative probability of the latency conditioned of the previous events is:

$$\begin{aligned}
&\mathbb{P}\left[T_{n+1} - T_n \geq t \| \, T_n = t_n, \ldots, T_0 = t_0\right] \\
&\quad = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \mathbb{E}\left[Y_s^k\right] \mathbb{E}\left[X_{t_n}^k \| \, T_n = t_n, \ldots, T_0 = t_0\right] && (53) \\
&\quad = \frac{\mathbb{E}\left[X_{t_n} Y_{t_n}^{n-1} e^{-\lambda Y_{t+t_n}}\right]}{\mathbb{E}\left[X_{t_n} Y_{t_n}^{n-1} e^{-\lambda Y_{t_n}}\right]} && (54) \\
&\quad = \mathbb{P}\left[T_{n+1} - T_n \geq t \| \, T_n = t_n\right] && (55)
\end{aligned}$$

8 Lévy

Integrated geometric Brownian motion Y_t and its carrier process of geometric Brownian motion X_t can be embedded into the Lie group of 2×2 upper triangular matrices H_2 by means of the factorization observed earlier, so that the increments are independent under matrix multiplication, for times $t > s$ we have in law:

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 1 & Y_{t-s} \\ 0 & X_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{56}$$

where the increment processes of Y_{t-s} and X_{t-s} are independent of the processes Y_s and X_s. It follows that the infinitesimal generator of the matrix exponential map into the group is the sub-algebra of the Lie algebra of upper triangular matrices $\mathfrak{h}_2$ consisting of first column zero matrices. For example, one branch of the logarithm is given by:

$$\exp\left(\frac{\mu t + \sigma W_t}{X_t - 1} \begin{bmatrix} 0 & Y_t \\ 0 & X_t - 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \tag{57}$$

In the next section we will find use for the drift free auxillary stochastic process U_t; which we motivate by considering the expectation of Y_t:

$$U_t = Y_t - \frac{X_t - 1}{\mu + \sigma^2/2} \tag{58}$$

Conveniently the joint process of X_t and U_t enjoys the same embedding into the Lie algebra H_2 as the original joint process, for times $t > s$ we have in law:

$$\begin{bmatrix} 1 & U_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 1 & U_{t-s} \\ 0 & X_{t-s} \end{bmatrix} \begin{bmatrix} 1 & U_s \\ 0 & X_s \end{bmatrix} \tag{59}$$

A word of caution about the generators in $\mathfrak{h}_2$, because the basis elements of the Lie sub-algebra do not commute:

$$\left[\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right] = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \tag{60}$$

the generator of the product of two increments in H_2 is not the sum of the generators of each increment. The representation of multiplicative Lévy processes in Lie groups raises the prospect of a deep connection between the adjoint derivative of the exponential map and stochastic differential equations.

9 Fokker-Planck

The upper triangular Lévy process of integrated geometric Brownian motion satisfies the stochastic differential equation:

$$d\begin{bmatrix}1 & Y_t\\0 & X_t\end{bmatrix} = \begin{bmatrix}0 & X_t\\0 & \left(\mu+\sigma^2/2\right)X_t\end{bmatrix}dt + \begin{bmatrix}0 & 0\\0 & \sigma X_t\end{bmatrix}dW_t \tag{61}$$

$$= \left(\begin{bmatrix}0 & 1\\0 & \mu+\sigma^2/2\end{bmatrix}dt + \begin{bmatrix}0 & 0\\0 & \sigma\end{bmatrix}dW_t\right)\begin{bmatrix}1 & Y_t\\0 & X_t\end{bmatrix} \tag{62}$$

It follows from Fokker-Planck that the Radon-Nikodym probability density of the joint process $p = \mathrm{d}\mathbb{P}\left[Y_t = y, X_t = x\right]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}xp - \frac{\partial}{\partial x}\left(\mu + \frac{\sigma^2}{2}\right)xp + \frac{\partial^2}{\partial x^2}\frac{\sigma^2}{2}x^2p \tag{63}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + x\frac{\partial p}{\partial y} + \left(\mu - \frac{3}{2}\sigma^2\right)x\frac{\partial p}{\partial x} - \frac{\sigma^2}{2}x^2\frac{\partial^2 p}{\partial x^2} = \left(-\mu + \frac{\sigma^2}{2}\right)p \tag{64}$$

Rather than directly integrating this equation we instead consider the stochastic differential equation of the auxillary process:

$$d\begin{bmatrix}1 & U_t\\0 & X_t\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & \left(\mu+\sigma^2/2\right)X_t\end{bmatrix}dt + \begin{bmatrix}0 & -\frac{\sigma}{\mu+\sigma^2/2}X_t\\0 & \sigma X_t\end{bmatrix}dW_t \tag{65}$$

$$= \left(\begin{bmatrix}0 & 0\\0 & \mu+\sigma^2/2\end{bmatrix}dt + \begin{bmatrix}0 & -\frac{\sigma}{\mu+\sigma^2/2}\\0 & \sigma\end{bmatrix}dW_t\right)\begin{bmatrix}1 & U_t\\0 & X_t\end{bmatrix} \tag{66}$$

The distribution of the increments can be derived from the factorization of integrated geometric Brownian motion Y_t by geometric Brownian motion X_t:

$$\begin{aligned}&\mathbb{P}\left[X_t = x, Y_t = y \| X_{t_0} = x_0, Y_{t_0} = y_0\right]\\&\quad = \mathbb{P}\left[X_{t-t_0} = \frac{x}{x_0}, Y_{t-t_0} = \frac{y-y_0}{x_0}\right]\end{aligned} \tag{67}$$

10 Hazard Rate

Consider the first passage stopping time T_1 of the Gompertz process G_t, its tail distribution is the characteristic function of Y_t:

$$\mathbb{P}\left[T_1 \geq t\right] = \mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{68}$$

and in turn yields the hazard rate of T_1:

$$\mathrm{d}\mathbb{H}\left[T_1 = t\right] = \frac{\lambda\,\mathbb{E}\left[X_t e^{-\lambda Y_t}\right]}{\mathbb{E}\left[e^{-\lambda Y_t}\right]} \tag{69}$$

From the application of change of variables of integration:

$$\lambda Y_t = \int_0^{\lambda t} e^{\frac{\mu}{\lambda}s + \frac{\sigma}{\sqrt{\lambda}}W_s} ds \tag{70}$$

we have the following rescaling of the hazard rate:

$$\mathrm{d}\mathbb{H}_{\mu,\sigma,\lambda}\left[T_1 = t\right] = \lambda\,\mathrm{d}\mathbb{H}_{\frac{\mu}{\lambda},\frac{\sigma}{\sqrt{\lambda}},1}\left[T_1 = \lambda t\right] \tag{71}$$

Which we can take as the definition of the hazard rate for every value of λ. As such we need only determine $\mathbb{E}\left[e^{-Y_t}\right]$. Taking the limit to deterministic subordination yields the constraints on the hazard rate:

Table 1: Sequential Boundary Conditions

Boundary	Condition	Removes
$\sigma = 0$	$\mathrm{d}\mathbb{H} = \lambda e^{\mu t}$	diffusion
$\mu = 0$	$\mathrm{d}\mathbb{H} = \lambda$	then drift
$\lambda = 0$	$\mathrm{d}\mathbb{H} = 0$	finally jumps

Sequential boundary conditions on the hazard rate h derived from the limits to deterministic subordination.

Dimensional analysis provides an inference for a solution, which remains an open problem:

Proposition 1 (Gompertz Anomaly)**.** *The hazard rate is given by:*

$$\mathrm{d}\mathbb{H} = \lambda\,\mathbb{E}\left[X_t\right] e^{\frac{\lambda\sigma^2/2}{\mu+\sigma^2/2}\mathbb{E}[Y_t]} \tag{72}$$

Proof. The parity multiplied derivatives of the exponential of the integral of the hazard rate satisfies the recursion-convolution lemma, generating the moments of Y_t, and hence is the characteristic function of Y_t. □

11 Discussion

In most circumstances λ is the new born infant mortality due to ageing alone, and is less than 1 in 32000 person-years. As such the hazard rate is very close to the original hazard rate observed by Gompertz. Intuitively, when $\mu \ggg \sigma^2/2$ the process becomes approximately deterministic due to the large impact of the drift.

Conversely, the central conjecture in the preliminary material is that ageing is driven by stochastic accelerations, requiring that the drift vanish $\mu = 0$. In this case the anomalous Gompertz hazard rate simplifies to:

$$h = \lambda e^{t\sigma^2/2} e^{\lambda \frac{e^{t\sigma^2/2} - 1}{\sigma^2/2}} \tag{73}$$

We have observed the anomalous Gompertz hazard rate in mortality rates in Alberta while conducting the Hazard Rate Zoo experiment. Specifically when we remove the dominate exponential process of a doubling of mortality every 7 years from an infant mortality of 1 in 32000 person-years:

$$h = \frac{2^{t/7}}{32000} e^{\frac{7\left(2^{t/7} - 1\right)}{32000 \ln 2}} \tag{74}$$

there remains a residual anomalous growth in mortality with ageing, reflecting the higher order affect of the stochastic accelerations.