Gompertz Processes

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Abstract

Motivated by considering infinitesimal stochastic accelerations of time, we outline a theory of Gompertz processes, Poisson processes subordinated by integrated Geometric Brownian motion.

1 Preliminaries

Basic science experiments in biology have observed that organisms respond to environmental stress, including communicable diseases and exposures to toxins, with an acceleration in their failure time. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations. The accelerations can either increase the rate of failure, an exacerbation, or decrease the rate of failure, a recovery. Even in the controlled setting of a laboratory it is very difficult to measure the underlying metabolic time that is being accelerated, instead we have access to the failure time in bare time units.

Integrated geometric Brownian motion occurs in the asymptotic limit of infinitesimal stochastic accelerations. By Lévy-Khintchine this is the only representation that is Lévy and jump free.

Definition 1 (Gompertz Process)**.** A Gompertz process G_t is a subordinated Poisson process N_t, with rate λ, where the subordinating process is integrated geometric Brownian motion Y_t, with drift μ and diffusion σ:

$$G_t = N_{Y_t} \tag{1}$$

given:

$$\begin{aligned} Y_t &= \int_0^t X_s ds && (2) \\ &= \int_0^t e^{\mu s + \sigma W_s} ds && (3) \end{aligned}$$

We will make great use of the increment relationship between Y_t and its carrier process X_t, for times $t > s$:

$$Y_t - Y_s \sim X_s Y_{t-s} \tag{4}$$

where the process X_s is independent of the process Y_{t-s}.

Lemma 1 (Acceleration Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of the carrier process of geometric Brownian motion X_t conditioned on integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}\left[X_t^n \| \, Y_t\right] = \frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[Y_t^n\right]} Y_t^n \tag{5}$$

Lemma 2 (Recursion Convolution Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}\left[Y_t^n\right] = n \int_0^t e^{\left(n\mu + \frac{n^2}{2}\sigma^2\right)u} \, \mathbb{E}\left[Y_{t-u}^{n-1}\right] du \tag{6}$$

2 Martingale

The scaled two point increments of integrated geometric Brownian motion Y_t form a martingale, for a triplet of times $t_1 > t_0 > t_{-1}$:

$$\mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{e^{\left(\mu + \frac{\sigma^2}{2}\right)t_1} - e^{\left(\mu + \frac{\sigma^2}{2}\right)t_0}} \middle\| \frac{Y_{t_0} - Y_{t_{-1}}}{e^{\left(\mu + \frac{\sigma^2}{2}\right)t_0} - e^{\left(\mu + \frac{\sigma^2}{2}\right)t_{-1}}}\right] = \frac{Y_{t_0} - Y_{t_{-1}}}{e^{\left(\mu + \frac{\sigma^2}{2}\right)t_0} - e^{\left(\mu + \frac{\sigma^2}{2}\right)t_{-1}}} \tag{7}$$

3 Lévy Process

Integrated geometric Brownian motion Y_t and its carrier process of geometric Brownian motion X_t can be embedded into the Lie algebra of 2×2 upper triangular matrices $\mathfrak{h}_2$ so that the increments are independent under matrix multiplication, for times $t > s$:

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \sim \begin{bmatrix} 1 & \tilde{Y}_{t-s} \\ 0 & \tilde{X}_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{8}$$

where the increment processes of $\tilde{Y}_{t-s}$ and $\tilde{X}_{t-s}$ are independent of the processes Y_s and X_s.

4 Fokker-Planck

The upper triangular Lévy process of integrated Brownian motion satisfies the stochastic differential equation:

$$d\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu + \frac{\sigma^2}{2}\right) X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \tag{9}$$

It follows from Fokker-Planck that the probability density of the joint process $p = \mathbb{P}[Y_t = y, X_t = x]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\mu + \frac{\sigma^2}{2}\right) xp - \frac{\partial}{\partial y} xp + \frac{\partial^2}{\partial x^2}\frac{\sigma^2}{2} x^2 p \tag{10}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left(\mu + \frac{3}{2}\sigma^2\right) x\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 p}{\partial x^2} = \left(-\mu + \frac{\sigma^2}{2}\right) p \tag{11}$$

5 Hazard Rate

Consider the first passage stopping time T of the Gompertz process G_t, its cumulative distribution is the characteristic functions of Y_t:

$$\mathbb{P}[T \geq t] = \mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{12}$$

Thus the hazard rate h of T:

$$h = \mathbb{P}[T = t \| \, T \geq t] \tag{13}$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}[Y_t]}{\mathbb{E}[X_t]}\frac{\partial h}{\partial t} - \lambda\frac{\partial h}{\partial \lambda} = \left(\frac{\mathbb{E}[Y_t]}{\mathbb{E}[X_t]}\right)^2 \left(\frac{\partial}{\partial t}\frac{\mathbb{E}[X_t]}{\mathbb{E}[Y_t]}\right) h \tag{14}$$

Which by trial solution of separation of variables has the general solution:

$$h = \lambda^2 \, \mathbb{E}[X_t] \, \mathbb{E}[Y_t] \, \mathrm{f}\left(\lambda \, \mathbb{E}[Y_t]\right) \tag{15}$$

for any f analytic.