Gompertz Processes: A Theory of Ageing

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Abstract

Ageing is a accelerating failure process that is universal to all biological systems. Motivated by considering infinitesimal stochastic accelerations of time, we hypothesize a general explanation for the emergent determinism of ageing processes in the theory of Gompertz processes: Poisson processes subordinated by integrated Geometric Brownian motion.

1 Preliminaries

Basic science experiments in biology have ubiquitously observed that organisms respond to environmental stresses, including communicable diseases and exposures to toxins, with an acceleration a in their failure time $ah\left(at\right)$, where h is the bare hazard rate of the failure event. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations a_n of the organism's metabolic time. The accelerations can either increase the rate of failure $a_n > 1$, an exacerbation of the stresses, or decrease the rate of failure $a_n < 1$, an alleviation from the stresses. However, even in the controlled setting of a laboratory it is experimentally challenging to directly measure the organism's accelerated metabolic time of the organism, instead we only have access to the failure times in bare units. Thus a theory of ageing must be one that studies the subordination of the failure time by an accelerated metabolic time that is stochastically incremented.

Over the course of an organism's lifetime it will encounter exacerbations and alleviations that accelerate $a_n, \ldots, a_0 = 1$ metabolic time at times $t_n, \ldots, t_0 = 0$. Furthermore, because the accelerations are all positive $a_i > 0$ for each acceleration we can find a finite real valued generator $w_n, \ldots, w_0 = 0$ such that $a_i = e^{w_i}$. It follows that the accelerated metabolic time y_{t_n} at time t_n is the sum of the products of the accelerations up to time t_{n-1} and the elapsed bare time steps $t_i - t_{i-1}$:

$$y_{t_n} = \sum_{i=1}^{n} e^{\sum_{j=0}^{i-1} w_j} \left(t_i - t_{i-1}\right) \tag{1}$$

Provided no great explosions of acceleration occur in any small time scale, like say when an actual explosion occurs, the generators w_i will become infinitesimal on the same order as $t_i - t_{i-1} \to 0$:

$$\mathcal{O}(w_i) = \mathcal{O}(t_i - t_{i-1}) \tag{2}$$

The sum of products then becomes a stochastic process Y_t that is an integral of a random geometric infinitesimal generator process e^{W_u}:

$$Y_t = \int_0^t e^{W_u} du \tag{3}$$

This is essentially the continuous part of the Kolmorgorov's characterization of stochastic processes as being composed of either a finite number of discrete jumps or an infinite number of continuous changes in a span of time. If we assume as a first approximation that the exacerbations and alleviations, and their respective accelerations, are independent and stationary over time then by the Lévy-Khintchine characterization the only infinitesimal generator of accelerated metabolic time that is Lévy and continuous *"jump free"* is Brownian motion W_u. The stochastic process of accelerated metabolic time is better known as integrated geometric Brownian motion.

2 Gompertz

Motivated by the preceding heuristic derivation we formally define the Gompertz process on time $t \geq 0$.

Definition 1 (Gompertz Process)**.** A Gompertz process G_t is a subordinated Poisson process N_t, with rate λ, where the subordinating process is integrated geometric Brownian motion Y_t, with drift μ and diffusion σ:

$$G_t = N_{Y_t} \tag{4}$$

given:

$$Y_t = \int_0^t X_s ds \tag{5}$$

$$= \int_0^t e^{\mu s + \sigma W_s} ds \tag{6}$$

Phenomenologically the finite real stochastic process $\mu t + \sigma W_s$ is the infinitesimal acceleration at time t that generates a non-negative geometric stochastic process X_t of accumulated accelerations up to time t and whose integral Y_t is a strictly increasing stochastic process of accelerated metabolic time up to time t.

From the definition of the Gompertz process G_t conditioning on the history of a sample path of integrated geometric Brownian motion Y_t yields a conditional Poisson process:

$$\mathbb{P}\left[G_t = n \| \, Y_t\right] = \frac{(\lambda Y_t)^n}{n!} e^{-\lambda Y_t} \tag{7}$$

$$\mathbb{E}\left[G_t \| \, Y_t\right] = \lambda Y_t \tag{8}$$

Reflecting that integrated geometric Brownian motion Y_t is truly measuring the elapsed stochastically accelerated metabolic time.

To start our exploration of the rich and subtleties of Gompertz processes we will briefly review of the properties of integrated geometric Brownian motion which are salient to developing our theory. This is by no means a comprehensive compendium. Much of the material I will cover has been deeply and thoroughly explored in the quantitative finance literature in the theory of pricing Asian options.

Our first observation is that the increments of Y_t can be factored by its carrier process X_t, for times $t > s$:

$$\mathbb{P}\left[Y_t - Y_s\right] = \mathbb{P}\left[X_s\right] \mathbb{P}\left[Y_{t-s}\right] \tag{9}$$

where the process X_s is independent of the process Y_{t-s}. Phenomenologically the increments of $Y_t - Y_s$ are equivalent to a process that starts with acceleration X_s. Analogous to the Fundamental Theorem of Arithmetic this factorization allows us to reach many useful inferences; for example we can immediately observe that for times $t > s$:

$$\mathbb{E}\left[(Y_t - Y_s)^n \| X_s\right] = X_s^n \, \mathbb{E}\left[Y_{t-s}^n\right] \tag{10}$$

We will liberally exploit this technique of arbitraging the accelerated metabolic time Y_t against the accumulated acceleration X_t to reduce expectations down to the well known standard terms for X_t and Y_t:

$$\mathbb{E}\left[X_t^n\right] = e^{\left(n\mu + n^2\sigma^2/2\right)t} \tag{11}$$

$$\mathbb{E}\left[Y_t\right] = \frac{e^{\left(\mu+\sigma^2/2\right)t} - 1}{\mu + \sigma^2/2} \tag{12}$$

Note the in the last equation we have implicitly invoked the Fubini-Tonelli Theorem to switch the order of integration, and will broadly continue to use this theorem throughout this work.

Lemma 1 (Acceleration Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of the carrier process of geometric Brownian motion X_t conditioned on the increment of integrated geometric Brownian motion $Y_t - Y_s$, where $t > s$, is given by:*

$$\mathbb{E}\left[X_t^n \| Y_t - Y_s\right] = \frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^n\right]} \left(Y_t - Y_s\right)^n \tag{13}$$

Proof. We proceed with induction on n.

1. For $n = 0$ clearly $\mathbb{E}\left[X_t^0 \| Y_t - Y_s\right] = 1$

2. Now assume that up to n the relation holds.

3. For $u \in [s,t]$ consider the integrable function

$$f(s,u,t,y) = \mathbb{E}\left[X_u X_t^n \| \, Y_t - Y_s = y\right] \tag{14}$$

4. It follows by the Fubini-Tonelli Theorem that and the induction assumption that for n the relation holds

$$\int_s^t f(s,u,t,y)\, du = y^{n+1} \frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^n\right]} \tag{15}$$

5. Thus by the Fundamental Theorem of Calculus the integrand has the decomposition $f(s,u,t,y) = y^{n+1} f(s,u,t)$

6. By the Law of Conditional Expectations we have

$$\begin{aligned} \mathbb{E}\left[X_u X_t^n\right] &= \mathbb{E}\left[\mathbb{E}\left[X_u X_t^n \| \, Y_t - Y_s\right]\right] && (16)\\ &= f(s,u,t)\, \mathbb{E}\left[(Y_t - Y_s)^{n+1}\right] && (17) \end{aligned}$$

7. Which immediately yields the decomposed integrand as

$$f(s,u,t) = \frac{\mathbb{E}\left[X_u X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^{n+1}\right]} \tag{18}$$

8. Finally taking $u = t$ yields the desired result

$$\begin{aligned} \mathbb{E}\left[X_t^{n+1} \| \, Y_t - Y_s\right] &= f(s,t,t)\,(Y_t - Y_s)^{n+1} && (19)\\ &= \frac{\mathbb{E}\left[X_t^{n+1}\right]}{\mathbb{E}\left[(Y_t - Y_s)^{n+1}\right]} (Y_t - Y_s)^{n+1} && (20) \end{aligned}$$

□

A simple corollary follows from a nearly trivial derivation.

Corollary 1 (Counterpoint Corollary)**.** *For a triplet of times $t_1 > t_0 > t_{-1}$ we have the following conditional expectation:*

$$\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}\right] = \frac{\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]} \left(Y_{t_0} - Y_{t_{-1}}\right)^n \tag{21}$$

Proof. Factoring and applying the previous lemma:

$$\begin{aligned} \mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}\right] &= \mathbb{E}\left[X_{t_0}^n \| \, Y_{t_0} - Y_{t_{-1}}\right] \mathbb{E}\left[Y_{t_1 - t_0}^n\right] && (22)\\ &= \frac{\mathbb{E}\left[Y_{t_1 - t_0}^n\right] \mathbb{E}\left[X_{t_0}^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]} \left(Y_{t_0} - Y_{t_{-1}}\right)^n && (23)\\ &= \frac{\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]} \left(Y_{t_0} - Y_{t_{-1}}\right)^n && (24) \end{aligned}$$

□

Even with the factorization observation we are still in need of a means of reducing the expectation of the powers Y_t^n. A small lemma suffices to provide the means of finding powers:

Lemma 2 (Recursion-Convolution Lemma). *The expectation of non-negative integer powers $n \geq 0$ of integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}\left[Y_t^n\right] = n \int_0^t \mathbb{E}\left[X_u^n\right] \mathbb{E}\left[Y_{t-u}^{n-1}\right] du \tag{25}$$

Proof. Expanding the expectant as a multi-variable integral we have, applying th Fubini-Tonelli Theorem we have, and then factoring we have:

$$\begin{aligned} \mathbb{E}\left[Y_t^n\right] &= \int_0^t \cdots \int_0^t \mathbb{E}\left[X_t^n\right] du_1 \ldots du_n && (26) \\ &= \binom{n}{1} \int_0^t \mathbb{E}\left[X_u \left(Y_t - Y_u\right)^{n-1}\right] du && (27) \\ &= n \int_0^t \mathbb{E}\left[X_u^n\right] \mathbb{E}\left[Y_{t-u}^{n-1}\right] du && (28) \end{aligned}$$

□

With the preceding lemma in hand we have the sufficient tools required to estimate all the usual statistics involving powers of Y_t, including the expectation, variance, and covariances.

3 Martingale

By our counterpoint corollary the scaled increments of integrated geometric Brownian motion $Y_{s\uparrow t}$, with $t > s$:

$$Y_{s\uparrow t} = \frac{Y_t - Y_s}{\mathbb{E}\left[Y_t - Y_s\right]} \tag{29}$$

form a two point martingale, so that for a triplet of times $t_1 > t_0 > t_{-1}$:

$$\mathbb{E}\left[Y_{t_0\uparrow t_1} \| \, Y_{t_{-1}\uparrow t_0}\right] = Y_{t_{-1}\uparrow t_0} \tag{30}$$

This allows us to leverage optional stopping time theorems to evaluate expectations of stopped versions of Gompertz processes.

4 Markov

From the Martingale $Y_{s\uparrow t}$ we can infer that Y_t is a 2 step Markov process, and that the increments $Y_t - Y_s$, with $t > s$ are Markov. To verify this consider the sequence of bare times $0 = t_0 < \cdots < t_{n+1} = t$, and accelerated times $0 = y_0 < \cdots < y_{n+1} = y$, working through the conditional probability we have:

$$\begin{aligned}
\mathbb{P}\left[Y_{t_{n+1}} - Y_{t_n} = y_{n+1} - y_n \| \, Y_{t_n} - Y_{t_{n-1}} = y_n - y_{n-1}, \ldots, Y_{t_1} - Y_{t_0} = y_1 - y_0\right] \\
= \int_0^\infty \int_0^\infty \mathbb{P}\left[X_{t_n - t_{n-1}} = \frac{u}{v}\frac{y_{n+1} - y_n}{y_n - y_{n-1}}\right] \\
\cdot \mathbb{P}\left[Y_{t_n - t_{n-1}} = u\right] \\
\cdot \mathbb{P}\left[Y_{t_{n+1} - t_n} = v\right] dudv \quad (31) \\
= \mathbb{P}\left[Y_{t_{n+1}} - Y_{t_n} = y_{n+1} - y_n \| \, Y_{t_n} - Y_{t_{n-1}} = y_n - y_{n-1}\right] \quad (32)
\end{aligned}$$

In practice we cannot directly measure the accelerated metabolic time Y_t instead we have access to the stopping times T_n of the passage $G_{T_n} = n$ of the Gompertz process.

To study the Markov properties of the stopping times we return to our ability to recover the Poisson process from the Gompertz process by conditioning on the history of the sample path of integrated geometric Brownian motion. By differentiating the subordinated cumulative distribution of a single stopping time we have the conditional expectation:

$$\mathbb{P}\left[T_n = t \| \, Y_{T_n}\right] = \frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]} \frac{(\lambda Y_t)^n}{(n-1)!} e^{-\lambda Y_t} \tag{33}$$

Of central interest in the study of the biological systems is the latency $T_n - T_{n-1}$ between consecutive stopping times. From the preceding derivation we can immediately deduce that the expectation and the tail probability of the latency are given by:

$$\mathbb{E}\left[T_n - T_{n-1}\right] = \int_0^\infty \mathbb{E}\left[e^{-\lambda X_{T_{n-1}} Y_t}\right] dt \tag{34}$$

$$\mathbb{P}\left[T_n - T_{n-1} \geq t\right] = \mathbb{E}\left[e^{-\lambda X_{T_{n-1}} Y_t}\right] \tag{35}$$

where X_t and Y_t are the increment process independent of $X_{T_{n-1}}$, which in turn sets the initial acceleration at the start of the increment. We have presented the unconditional expectation over T_{n-1} because in many experiments, due to immortal time bias *"we cannot see indefinitely into the past"*, we are only able to observe consecutive passages of the Gompertz process, without knowledge of how many had occurred before the consecutive passages were observed. Further exploration of the unconditional expectations requires a deeper understanding of the characteristic function of integrated geometric Brownian motion, which we will develop later.

While the consecutive increments of integrated geometric Brownian motion are Markov, because knowledge of just the most recent stopping time and it's passage count is sufficient. This is because conditioning on a sample path of integrated geometric Brownian motion is sufficient to recovery a Poisson process from a Gompertz process. It is thus more important to understand the conditional dependence of consecutive stopping times while marginalizing over a lack of knowledge of the past. To do so, consider triple of stopping times of consecutive passages of the Gompertz process $T_{2+G_t}, T_{1+G_t}, T_{G_t}$ from the Law of Total Expectation we then have:

$$\mathbb{P}\left[T_{2+G_t} - T_{1+G_t} \geq r \| \, T_{1+G_t} - T_{G_t} = s\right] = \mathbb{E}\left[e^{-\lambda X_{s+t} Y_r}\right] \tag{36}$$

where the initial acceleration X_{s+t} and the increment Y_r are independent. Furthermore the conditional expectation marginalized of lack of knowledge of the past is given by the integral:

$$\mathbb{E}\left[T_{2+G_t} - T_{1+G_t} \| \, T_{1+G_t} - T_{G_t} = s\right] = \int_0^\infty \mathbb{E}\left[e^{-\lambda X_{s+t} Y_u}\right] du \tag{37}$$

again where the initial acceleration X_{s+t} and the increment Y_u are independent. This amounts to a concrete prediction that ageing alone introduces a statistical dependence between the intervals of consecutive admissions for healthcare services.

5 Lévy

Integrated geometric Brownian motion Y_t and its carrier process of geometric Brownian motion X_t can be embedded into the Lie algebra of 2×2 upper triangular matrices $\mathfrak{h}_2$ by means of the factorization observed earlier, so that the increments are independent under matrix multiplication, for times $t > s$:

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 1 & Y_{t-s} \\ 0 & X_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{38}$$

where the increment processes of Y_{t-s} and X_{t-s} are independent of the processes Y_s and X_s.

6 Fokker-Planck

The upper triangular Lévy process of integrated geometric Brownian motion satisfies the stochastic differential equation:

$$d \begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu + \frac{\sigma^2}{2}\right) X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \tag{39}$$

It follows from Fokker-Planck that the probability density of the joint process $p = \mathbb{P}\left[Y_t = y, X_t = x\right]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\mu + \frac{\sigma^2}{2}\right) xp - \frac{\partial}{\partial y} xp + \frac{\partial^2}{\partial x^2} \frac{\sigma^2}{2} x^2 p \tag{40}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left(\mu + \frac{3}{2}\sigma^2\right) x \frac{\partial p}{\partial x} + x \frac{\partial p}{\partial y} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 p}{\partial x^2} = \left(-\mu + \frac{\sigma^2}{2}\right) p \tag{41}$$

Marginalizing over the probability of X_t and applying the acceleration lemma yields the first order partial differential equation for the distribution of Y_t:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\frac{\partial p}{\partial t} - y\frac{\partial p}{\partial y} = p \tag{42}$$

Which by trial solution of separation of variables has the general solution:

$$p = \mathbb{E}\left[Y_t\right] f_{\mu,\sigma}\left(y\,\mathbb{E}\left[Y_t\right]\right) \tag{43}$$

for any analytic $f_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process X_t. Note that we are offloading the dimensional analysis into the analytic function.

7 Hazard Rate

Consider the first passage stopping time T_1 of the Gompertz process G_t, its cumulative distribution is the characteristic function of Y_t:

$$\mathbb{P}\left[T_1 \geq t\right] = \mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{44}$$

Thus the hazard rate h of T_1:

$$h = \mathbb{P}\left[T_1 = t \|\, T_1 \geq t\right] \tag{45}$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\frac{\partial h}{\partial t} - \lambda\frac{\partial h}{\partial \lambda} = \left(\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\right)^2 \left(\frac{\partial}{\partial t}\frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]}\right) h \tag{46}$$

Which by trial solution of separation of variables has the general solution:

$$h = \lambda^2\,\mathbb{E}\left[X_t\right]\mathbb{E}\left[Y_t\right] g_{\mu,\sigma}\left(\lambda\,\mathbb{E}\left[Y_t\right]\right) \tag{47}$$

for any analytic $g_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process X_t. Taking the limit to deterministic subordination yields the constraints on $g_{\mu,\sigma}$:

Table 1: Sequential Boundary Conditions

Boundary	Condition	Removes
$\sigma = 0$	$h = \lambda e^{\mu t}$	diffusion
$\mu = 0$	$h = \lambda$	then drift
$\lambda = 0$	$h = 0$	finally jumps

Sequential boundary conditions on the hazard rate h derived from the limits to deterministic subordination.

From the boundary conditions we can immediately deduce that in the deterministic limit of $\sigma \to 0$ we have:

$$g_{\mu,\sigma}(x) \xrightarrow{\sigma=0} \frac{1}{x} \tag{48}$$

However this alone cannot be the solution as it results in the characteristic function in λ of a purely deterministic Y_t. Equating the general solution for the hazard rate to the Laplace of the Fokker-Planck solution yields the implicit equation in $f_{\mu,\sigma}$ and $g_{\mu,\sigma}$:

$$-\frac{\partial}{\partial t}\ln\int_0^\infty e^{\frac{\lambda u}{\mathbb{E}[Y_t]}} f_{\mu,\sigma}(u)\,du = \lambda^2\,\mathbb{E}[X_t]\,\mathbb{E}[Y_t]\,g_{\mu,\sigma}(\lambda\,\mathbb{E}[Y_t]) \tag{49}$$

Dimensional analysis provides an inference for a solution, which remains an open problem:

Proposition 1 (Gompertz Anomaly)**.** *The analytic function $g_{\mu,\sigma}$ is simply the exponential divided by its argument, so that the hazard h is given by:*

$$h = \lambda\,\mathbb{E}[X_t]\,e^{\frac{\sigma^2/2}{\mu+\sigma^2/2}\lambda\,\mathbb{E}[Y_t]} \tag{50}$$

Proof. The parity multiplied derivatives of the exponential of the integral of the hazard rate satisfies the recursion-convolution lemma, generating the moments of Y_t, and hence is the characteristic function of Y_t. $\square$

8 Discussion

In most circumstances λ is the new born infant mortality due to ageing alone, and is less than 1 in 32000 person-years. As such the hazard is very close to the original hazard observed by Gompertz. Intuitively, when $\mu \ggg \sigma^2/2$ the process becomes approximately deterministic due to the large impact of the drift.

Conversely, the central conjecture in the preliminary material is that ageing is driven by stochastic accelerations, hence requiring that the drift vanish $\mu = 0$. In this case the anomalous Gompertz hazard rate simplifies to:

$$h = \lambda e^{t\sigma^2/2} e^{\lambda\frac{e^{t\sigma^2/2}-1}{\sigma^2/2}} \tag{51}$$

We have observed the anomalous Gompertz hazard rate in mortality rates in Alberta in the Hazard Rate Zoo experiment. Specifically when we remove the dominate exponential process of a doubling of mortality every 7 years from an infant mortality of 1 in 32000 person-years:

$$h = \frac{2^{t/7}}{32000} e^{\frac{7\left(2^{t/7}-1\right)}{32000\ln 2}} \tag{52}$$

there remains a residual anomalous growth in mortality with ageing, reflecting the higher order affect of the stochastic accelerations.