Gompertz Processes: A Theory of Ageing

Aaron Geoffrey Sheldon

April 24, 2020

Abstract

Motivated by considering infinitesimal stochastic accelerations of time, we outline a theory of Gompertz processes, Poisson processes subordinated by integrated Geometric Brownian motion.

1 Preliminaries

Basic science experiments in biology have ubiquitously observed that organisms respond to environmental stresses, including communicable diseases and exposures to toxins, with an acceleration a in their failure time $ah\left(at\right)$, where h is the bare hazard rate of the failure event. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations a_n of the underlying metabolic time. The accelerations can either increase the rate of failure $a_n > 1$, an exacerbation of the stresses, or decrease the rate of failure $a_n < 1$, an alleviation from the stresses. However, even in the controlled setting of a laboratory it is experimentally challenging to directly measure the underlying accelerated metabolic time, instead we only have access to the failure time in bare units. Thus a theory of ageing must be one that studies the subordination of the failure time by an elapsed metabolic time that is stochastically incremented.

Over the course of an organism's lifetime it will encounter exacerbations and alleviations that accelerate $a_n, \ldots, a_0 = 1$ metabolic time at times $t_n, \ldots, t_0 = 0$. Furthermore, because the accelerations are all positive $a_i > 0$ for each acceleration we can find a finite real valued generator $b_n, \ldots, b_0 = 0$ such that $a_i = e^{b_i}$. It follows that the elapsed metabolic time $\tilde{t}_{t_n}$ at time t_n is the sum of the products of the accelerations up to time t_{n-1} and the elapsed bare time steps $t_i - t_{i-1}$:

$$\tilde{t}_{t_n} = \sum_{i=1}^{n} e^{\sum_{j=0}^{i-1} b_j} \left(t_i - t_{i-1}\right) \tag{1}$$

Provided no great explosions of acceleration occur in any small time scale, like say when an actual explosion occurs, the generators b_i will become infinitesimal on the same order as $t_i - t_{i-1} \to 0$:

$$\mathcal{O}\left(b_i\right) = \mathcal{O}\left(t_i - t_{i-1}\right) \tag{2}$$

The sum of products then becomes a stochastic process $\tilde{T}_t$ that is an integral of a geometric random infinitesimal generator process e^{B_u}:

$$\tilde{T}_t = \int_0^t e^{B_u} du \tag{3}$$

This is essentially the continuous part of the Kolmorgorov's characterization of stochastic processes as being composed of either a finite number of discrete jumps or an infinite number of continuous changes in a span of time. If we assume as a first approximation that the exacerbations and alleviations, and their respective accelerations, are independent and stationary over time then by the Lévy-Khintchine characterization the only infinitesimal generator of elapsed metabolic time that is Lévy and continuous, *jump free*, is Brownian motion B_u. The stochastic process of elapsed metabolic time is better known as integrated geometric Brownian motion.

Definition 1 (Gompertz Process)**.** A Gompertz process G_t is a subordinated Poisson process N_t, with rate λ, where the subordinating process is integrated geometric Brownian motion Y_t, with drift μ and diffusion σ:

$$G_t = N_{Y_t} \tag{4}$$

given:

$$\begin{aligned} Y_t &= \int_0^t X_s ds && (5) \\ &= \int_0^t e^{\mu s + \sigma W_s} ds && (6) \end{aligned}$$

Phenomenologically the finite real stochastic process $\mu t + \sigma W_s$ is the infinitesimal acceleration at time t that generates a non-negative geometric stochastic process X_t of accumulated accelerations up to time t and whose integral Y_t is a strictly increasing stochastic process of elapsed metabolic time up to time t.

To start our exploration of the rich and subtleties of Gompertz processes we will briefly review of the properties of integrated geometric Brownian motion which are salient to developing our theory. This is by no means a comprehensive compendium. Much of the material I will cover has been deeply and thoroughly explored in the quantitative finance literature in the theory of pricing Asian options.

Our first observation is that the increments of Y_t can be factored by its carrier process X_t, for times $t > s$:

$$Y_t - Y_s \sim X_s Y_{t-s} \tag{7}$$

where the process X_s is independent of the process Y_{t-s}. For example, this allows us to immediately observe that for times $t > s$:

$$\mathbb{E}\left[(Y_t - Y_s)^n \| X_s\right] = X_s^n \, \mathbb{E}\left[Y_{t-s}^n\right] \tag{8}$$

We will liberally exploit this technique of arbitraging the elapsed metabolic time Y_t against the accumulated acceleration X_t to reduce expectations down to the well know standard terms for X_t and Y_t:

$$\mathbb{E}[X_t] = e^{\left(\mu+\frac{\sigma^2}{2}\right)t} \tag{9}$$

$$\mathbb{E}[Y_t] = \frac{e^{\left(\mu+\frac{\sigma^2}{2}\right)t} - 1}{\mu + \frac{\sigma^2}{2}} \tag{10}$$

Note the in the last equation we have implicitly invoked Fubini's theorem to switch the order of integration, and will broadly continue to throughout this work.

Lemma 1 (Acceleration Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of the carrier process of geometric Brownian motion X_t conditioned on the increment of integrated geometric Brownian motion $Y_t - Y_s$, where $t > s$, is given by:*

$$\mathbb{E}[X_t^n \| \, Y_t - Y_s] = \frac{\mathbb{E}[X_t^n]}{\mathbb{E}[(Y_t - Y_s)^n]} (Y_t - Y_s)^n \tag{11}$$

Proof. Carry out induction on n. ☐

A simple corollary follows from a nearly trivial derivation.

Corollary 1 (Counterpoint Corollary)**.** *For a triplet of times $t_1 > t_0 > t_{-1}$ we have the following conditional expectation:*

$$\begin{aligned} &\mathbb{E}[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}] \\ &\qquad = e^{\left(n\mu+\frac{n^2}{2}\sigma\right)t_0} \frac{\mathbb{E}\left[Y_{t_1-t_0}^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]} \left(Y_{t_0} - Y_{t_{-1}}\right)^n \end{aligned} \tag{12}$$

Proof. Factor and apply the previous lemma. ☐

Even with the factorization observation we are still in need of a means of reducing the expectation of the powers Y_t^n. A small lemma suffices to provide the means of finding powers:

Lemma 2 (Recursion-Convolution Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}[Y_t^n] = n \int_0^t e^{\left(n\mu+\frac{n^2}{2}\sigma^2\right)u} \, \mathbb{E}\left[Y_{t-u}^{n-1}\right] du \tag{13}$$

Proof. Carry out a routine factorization. ☐

With the preceding lemma in hand we have the sufficient tools required to estimate all the usual statistics involving powers of Y_t, including the expectation, variance, and covariances.

2 Martingale Increments

By our counterpoint corollary the scaled increments of integrated geometric Brownian motion Y_t form a two point martingale, for a triplet of times $t_1 > t_0 > t_{-1}$:

$$\mathbb{E}\left[\frac{Y_{t_1}-Y_{t_0}}{e^{\left(\mu+\frac{\sigma^2}{2}\right)t_1}-e^{\left(\mu+\frac{\sigma^2}{2}\right)t_0}}\middle\| \frac{Y_{t_0}-Y_{t_{-1}}}{e^{\left(\mu+\frac{\sigma^2}{2}\right)t_0}-e^{\left(\mu+\frac{\sigma^2}{2}\right)t_{-1}}}\right] = \frac{Y_{t_0}-Y_{t_{-1}}}{e^{\left(\mu+\frac{\sigma^2}{2}\right)t_0}-e^{\left(\mu+\frac{\sigma^2}{2}\right)t_{-1}}} \tag{14}$$

From this martingale property of Y_t any triple of stopping times T_{n+1}, T_n, T_{n-1} of consecutive passages of G_t will have a 2 Markov dependence. This amounts to a concrete prediction that ageing alone introduces a statistical dependence between the intervals of consecutive admissions for healthcare services.

3 Lévy Process

Integrated geometric Brownian motion Y_t and its carrier process of geometric Brownian motion X_t can be embedded into the Lie algebra of 2×2 upper triangular matrices $\mathfrak{h}_2$ by means of the factorization observed earlier, so that the increments are independent under matrix multiplication, for times $t > s$:

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \sim \begin{bmatrix} 1 & \tilde{Y}_{t-s} \\ 0 & \tilde{X}_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{15}$$

where the increment processes of $\tilde{Y}_{t-s}$ and $\tilde{X}_{t-s}$ are independent of the processes Y_s and X_s.

4 Fokker-Planck

The upper triangular Lévy process of integrated Brownian motion satisfies the stochastic differential equation:

$$d\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu+\frac{\sigma^2}{2}\right)X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \tag{16}$$

It follows from Fokker-Planck that the probability density of the joint process $p = \mathbb{P}\left[Y_t = y, X_t = x\right]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\mu+\frac{\sigma^2}{2}\right)xp - \frac{\partial}{\partial y}xp + \frac{\partial^2}{\partial x^2}\frac{\sigma^2}{2}x^2 p \tag{17}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left(\mu+\frac{3}{2}\sigma^2\right)x\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} - \frac{\sigma^2}{2}x^2\frac{\partial^2 p}{\partial x^2} = \left(-\mu+\frac{\sigma^2}{2}\right)p \tag{18}$$

Marginalizing over the probability of X_t and applying the acceleration lemma yields the first order partial differential equation for the distribution of Y_t:

$$\frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]}\frac{\partial p}{\partial t}-\frac{\partial p}{\partial y}=p \tag{19}$$

Which by trial solution of separation of variables has the general solution:

$$p=e^{\frac{1}{\mu+\frac{\sigma^2}{2}}\left(t-\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[E_t\right]}\right)}g_{\mu,\sigma}\left(y+\frac{1}{\mu+\frac{\sigma^2}{2}}\left(t-\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[E_t\right]}\right)\right) \tag{20}$$

for any analytic $g_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process X_t.

5 Hazard Rate

Consider the first passage stopping time T_1 of the Gompertz process G_t, its cumulative distribution is the characteristic function of Y_t:

$$\mathbb{P}\left[T_1\geq t\right]=\mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{21}$$

Thus the hazard rate h of T:

$$h=\mathbb{P}\left[T_1=t\|\,T_1\geq t\right] \tag{22}$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\frac{\partial h}{\partial t}-\lambda\frac{\partial h}{\partial\lambda}=\left(\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\right)^2\left(\frac{\partial}{\partial t}\frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]}\right)h \tag{23}$$

Which by trial solution of separation of variables has the general solution:

$$h=\lambda^2\,\mathbb{E}\left[X_t\right]\mathbb{E}\left[Y_t\right]f_{\mu,\sigma}\left(\lambda\,\mathbb{E}\left[Y_t\right]\right) \tag{24}$$

for any analytic $f_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process X_t. Taking the limit to deterministic subordination yields the constraints on $f_{\mu,\sigma}$ by L'Hôpital's rule:

Table 1: Boundary conditions on the hazard rate h

Boundary	Condition	Constrains
$\lambda=0$	$h=0$	$\frac{\partial}{\partial\lambda}f_{\mu,\sigma}\left(\lambda\,\mathbb{E}\left[Y_t\right]\right)$
$t=0$	$h=\lambda$	$\frac{\partial}{\partial t}f_{\mu,\sigma}\left(\lambda\,\mathbb{E}\left[Y_t\right]\right)$

Continued on next page.

Continued from previous page.

Boundary	Condition	Constrains
$\mu = \sigma = 0$	$h = \lambda$	$\left[\dfrac{\partial}{\partial \mu} \cdot + \dfrac{\partial}{\partial \sigma} \cdot\right] \left(f_{\mu,\sigma} \left(\lambda \, \mathbb{E}\left[Y_t\right]\right)\right)$
$\sigma = 0$	$h = \lambda e^{\mu t}$	$\dfrac{\partial}{\partial \sigma} f_{\mu,\sigma} \left(\lambda \, \mathbb{E}\left[Y_t\right]\right)$

Boundary conditions on the hazard rate derived from the limits to deterministic subordination.

From the boundary conditions we can immediately deduce that in the deterministic limit of $\sigma \to 0$ we have:

$$f_{\mu,\sigma}(x) \xrightarrow{\sigma=0} \frac{1}{x} \tag{25}$$

However this alone cannot be the solution as it is the characteristic function in λ of a purely deterministic Y_t. Equating the general solution for the hazard rate to the Laplace of the Fokker-Planck solution yields the implicit equation in $f_{\mu,\sigma}$ and $g_{\mu,\sigma}$:

$$\begin{aligned} -\frac{\partial}{\partial t} \ln \int_{\frac{1}{\mu+\frac{\sigma^2}{2}}\left(t - \frac{\mathbb{E}[Y_t]}{\mathbb{E}[E_t]}\right)}^{\infty} & e^{\lambda u} g_{\mu,\sigma}(u)\, du \\ &= (1-\lambda) \frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[E_t\right]} + \lambda^2 \, \mathbb{E}\left[X_t\right] \mathbb{E}\left[Y_t\right] f_{\mu,\sigma}\left(\lambda \, \mathbb{E}\left[Y_t\right]\right) \end{aligned} \tag{26}$$

Until more constraints can be found the following proposition remains open.

Proposition 1 (Gompertz Anomaly)**.** *The analytic function $f_{\mu,\sigma}$ is simply the exponential divided by its argument, so that the hazard h is given by:*

$$h = \lambda \, \mathbb{E}\left[X_t\right] e^{\frac{\sigma^2}{2} \lambda \, \mathbb{E}[Y_t]} \tag{27}$$

In most circumstances λ is the new born infant mortality due to ageing alone and is less than one in thirty thousand person years. As such the hazard is very close to the original hazard observed by Gompertz.