Gompertz Processes

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Abstract

Motivated by considering infinitesimal stochastic accelerations of time, we outline a theory of Gompertz processes, Poisson processes subordinated by integrated Geometric Brownian motion.

1 Preliminaries

Basic science experiments in biology have observed that organisms respond to environmental stress, including communicable diseases and exposures to toxins, with an acceleration in their failure time. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations. The accelerations can either increase the rate of failure, an exacerbation, or decrease the rate of failure, a recovery. Even in the controlled setting of a laboratory it is very difficult to measure the underlying metabolic time that is being accelerated, instead we have access to the failure time in bare time units.

Integrated geometric Brownian motion occurs in the asymptotic limit of infinitesimal stochastic accelerations. By Lévy-Khintchine this is the only representation that is Lévy and jump free.

Definition 1 (Gompertz Process)**.** A Gompertz process G_t is a subordinated Poisson process N_t, with rate λ, where the subordinating process is integrated geometric Brownian motion Y_t, with drift μ and diffusion σ:

$$G_t = N_{Y_t} \tag{1}$$

given:

$$\begin{aligned} Y_t &= \int_0^t X_s ds \qquad (2) \\ &= \int_0^t e^{\mu s + \sigma W_s} ds \qquad (3) \end{aligned}$$

Phenomenologically the finite real stochastic process $\mu t + \sigma W_s$ is the infinitesimal acceleration at time t that generates a non-negative stochastic process X_t

which measures the cumulative acceleration up to time t and whose integral Y_t is a strictly increasing stochastic process that measures the elapsed metabolic time at up to time t.

We begin with brief review of the properties of integrated geometric Brownian motion which are salient to developing a theory of Gompertz processes. This is by no means a comprehensive compendium. Much of the material I will cover has been explored deeply and thoroughly in the quantitative finance literature within the context of pricing Asian options.

Our first observation is that the increments of Y_t can be factored by its carrier process X_t, for times $t > s$:

$$Y_t - Y_s \sim X_s Y_{t-s} \tag{4}$$

where the process X_s is independent of the process Y_{t-s}. For example, this allows us to immediately observe that for times $t > s$:

$$\mathbb{E}\left[(Y_t - Y_s)^n \| \, X_s\right] = X_s^n \, \mathbb{E}\left[Y_{t-s}^n\right] \tag{5}$$

Lemma 1 (Acceleration Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of the carrier process of geometric Brownian motion X_t conditioned on the increment of integrated geometric Brownian motion $Y_t - Y_s$, where $t > s$, is given by:*

$$\mathbb{E}\left[X_t^n \| \, Y_t - Y_s\right] = \frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^n\right]} \left(Y_t - Y_s\right)^n \tag{6}$$

A simple corollary follows from a nearly trivial derivation.

Corollary 1. *For a triplet of times $t_1 > t_0 > t_{-1}$ we have the following conditional expectation:*

$$\begin{aligned} &\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}\right] \\ &\qquad = e^{\left(n\mu + \frac{n^2}{2}\sigma\right)t_0} \frac{\mathbb{E}\left[Y_{t_1 - t_0}^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]} \left(Y_{t_0} - Y_{t_{-1}}\right)^n \end{aligned} \tag{7}$$

Even with the factorization observation we are still in need of a means of reducing the expectation of the powers Y_t^n. A small lemma suffices to provide the means of finding powers:

Lemma 2 (Recursion-Convolution Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}\left[Y_t^n\right] = n \int_0^t e^{\left(n\mu + \frac{n^2}{2}\sigma^2\right)u} \, \mathbb{E}\left[Y_{t-u}^{n-1}\right] du \tag{8}$$

2 Martingale

The scaled increments of integrated geometric Brownian motion Y_t form a two point martingale, for a triplet of times $t_1 > t_0 > t_{-1}$:

$$\mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{e^{\left(\mu+\frac{\sigma^2}{2}\right)t_1} - e^{\left(\mu+\frac{\sigma^2}{2}\right)t_0}} \middle\| \frac{Y_{t_0} - Y_{t_{-1}}}{e^{\left(\mu+\frac{\sigma^2}{2}\right)t_0} - e^{\left(\mu+\frac{\sigma^2}{2}\right)t_{-1}}}\right] = \frac{Y_{t_0} - Y_{t_{-1}}}{e^{\left(\mu+\frac{\sigma^2}{2}\right)t_0} - e^{\left(\mu+\frac{\sigma^2}{2}\right)t_{-1}}} \tag{9}$$

From this martingale property of Y_t any triple of stopping times T_{n+1}, T_n, T_{n-1} for consecutive passages of G_t will have a 2 Markov dependence. This amounts to a concrete prediction that ageing alone introduces a statistical dependence in the intervals between consecutive admissions for healthcare.

3 Lévy Process

Integrated geometric Brownian motion Y_t and its carrier process of geometric Brownian motion X_t can be embedded into the Lie algebra of 2×2 upper triangular matrices $\mathfrak{h}_2$ so that the increments are independent under matrix multiplication, for times $t > s$:

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \sim \begin{bmatrix} 1 & \tilde{Y}_{t-s} \\ 0 & \tilde{X}_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{10}$$

where the increment processes of $\tilde{Y}_{t-s}$ and $\tilde{X}_{t-s}$ are independent of the processes Y_s and X_s.

4 Fokker-Planck

The upper triangular Lévy process of integrated Brownian motion satisfies the stochastic differential equation:

$$d\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu + \frac{\sigma^2}{2}\right) X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \tag{11}$$

It follows from Fokker-Planck that the probability density of the joint process $p = \mathbb{P}\left[Y_t = y, X_t = x\right]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\mu + \frac{\sigma^2}{2}\right) xp - \frac{\partial}{\partial y} xp + \frac{\partial^2}{\partial x^2}\frac{\sigma^2}{2} x^2 p \tag{12}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left(\mu + \frac{3}{2}\sigma^2\right) x\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} - \frac{\sigma^2}{2}x^2\frac{\partial^2 p}{\partial x^2} = \left(-\mu + \frac{\sigma^2}{2}\right) p \tag{13}$$

5 Hazard Rate

Consider the first passage stopping time T_1 of the Gompertz process G_t, its cumulative distribution is the characteristic functions of Y_t:

$$\mathbb{P}\left[T_1 \geq t\right] = \mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{14}$$

Thus the hazard rate h of T:

$$h = \mathbb{P}\left[T_1 = t \| \, T_1 \geq t\right] \tag{15}$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]} \frac{\partial h}{\partial t} - \lambda \frac{\partial h}{\partial \lambda} = \left(\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\right)^2 \left(\frac{\partial}{\partial t} \frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]}\right) h \tag{16}$$

Which by trial solution of separation of variables has the general solution:

$$h = \lambda^2 \, \mathbb{E}\left[X_t\right] \mathbb{E}\left[Y_t\right] \mathrm{f}_{\mu,\sigma}\left(\lambda \, \mathbb{E}\left[Y_t\right]\right) \tag{17}$$

for any analytic $\mathrm{f}_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process X_t. Taking the limit to deterministic subordinations yield the] boundary conditions:

Table 1: Boundary constraints on the hazard rate h

Boundary	Constraint
$\lambda = 0$	$h = 0$
$t = 0$	$h = \lambda$
$\mu = \sigma = 0$	$h = \lambda$
$\sigma = 0$	$h = \lambda \mu e^{\mu t}$

Boundary constraints on the hazard rate derived from the limit to deterministic subordination.