# Gompertz Processes: A Theory of Ageing

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May 1, 2020

**Abstract**

Motivated by considering infinitesimal stochastic accelerations of time, we outline a theory of Gompertz processes, Poisson processes subordinated by integrated Geometric Brownian motion.

## 1 Preliminaries

Basic science experiments in biology have ubiquitously observed that organisms respond to environmental stresses, including communicable diseases and exposures to toxins, with an acceleration $a$ in their failure time $ah\left(at\right)$, where $h$ is the bare hazard rate of the failure event. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations $a_n$ of the underlying metabolic time. The accelerations can either increase the rate of failure $a_n > 1$, an exacerbation of the stresses, or decrease the rate of failure $a_n < 1$, an alleviation from the stresses. However, even in the controlled setting of a laboratory it is experimentally challenging to directly measure the underlying accelerated metabolic time, instead we only have access to the failure time in bare units. Thus a theory of ageing must be one that studies the subordination of the failure time by an elapsed metabolic time that is stochastically incremented.

Over the course of an organism's lifetime it will encounter exacerbations and alleviations that accelerate $a_n, \ldots, a_0 = 1$ metabolic time at times $t_n, \ldots, t_0 = 0$. Furthermore, because the accelerations are all positive $a_i > 0$ for each acceleration we can find a finite real valued generator $w_n, \ldots, w_0 = 0$ such that $a_i = e^{w_i}$. It follows that the elapsed metabolic time $y_{t_n}$ at time $t_n$ is the sum of the products of the accelerations up to time $t_{n-1}$ and the elapsed bare time steps $t_i - t_{i-1}$:

$$y_{t_n} = \sum_{i=1}^{n} e^{\sum_{j=0}^{i-1} w_j} \left(t_i - t_{i-1}\right) \tag{1}$$

Provided no great explosions of acceleration occur in any small time scale, like say when an actual explosion occurs, the generators $w_i$ will become infinitesimal on the same order as $t_i - t_{i-1} \to 0$:

$$\mathcal{O}\left(w_i\right) = \mathcal{O}\left(t_i - t_{i-1}\right) \tag{2}$$

The sum of products then becomes a stochastic process $Y_t$ that is an integral of a geometric random infinitesimal generator process $e^{W_u}$:

$$Y_t = \int_0^t e^{W_u} du \tag{3}$$

This is essentially the continuous part of the Kolmorgorov's characterization of stochastic processes as being composed of either a finite number of discrete jumps or an infinite number of continuous changes in a span of time. If we assume as a first approximation that the exacerbations and alleviations, and their respective accelerations, are independent and stationary over time then by the Lévy-Khintchine characterization the only infinitesimal generator of elapsed metabolic time that is Lévy and continuous, *"jump free"*, is Brownian motion $W_u$. The stochastic process of elapsed metabolic time is better known as integrated geometric Brownian motion.

## 2 Gompertz

Motivated by the preceding heuristic derivation we formally define the Gompertz process on time $t \geq 0$.

**Definition 1** (Gompertz Process)**.** A Gompertz process $G_t$ is a subordinated Poisson process $N_t$, with rate $\lambda$, where the subordinating process is integrated geometric Brownian motion $Y_t$, with drift $\mu$ and diffusion $\sigma$:

$$G_t = N_{Y_t} \tag{4}$$

given:

$$\begin{aligned} Y_t &= \int_0^t X_s ds && (5) \\ &= \int_0^t e^{\mu s + \sigma W_s} ds && (6) \end{aligned}$$

Phenomenologically the finite real stochastic process $\mu t + \sigma W_s$ is the infinitesimal acceleration at time $t$ that generates a non-negative geometric stochastic process $X_t$ of accumulated accelerations up to time $t$ and whose integral $Y_t$ is a strictly increasing stochastic process of elapsed metabolic time up to time $t$.

It is immediately clear from the definition that conditioning the Gompertz process $G_t$ on integrated geometric Brownian motion $Y_t$ yields the analogous relationships from the familiar standard Poisson process:

$$\mathbb{P}\left[G_t = n \| \, Y_t\right] = \frac{(\lambda Y_t)^n}{n!} e^{-\lambda Y_t} \tag{7}$$

$$\mathbb{E}\left[G_t \| \, Y_t\right] = \lambda Y_t \tag{8}$$

Reflecting that integrated geometric Brownian motion $Y_t$ is truly measuring the elapsed stochastically accelerated metabolic time.

To start our exploration of the rich and subtleties of Gompertz processes we will briefly review of the properties of integrated geometric Brownian motion which are salient to developing our theory. This is by no means a comprehensive compendium. Much of the material I will cover has been deeply and thoroughly explored in the quantitative finance literature in the theory of pricing Asian options.

Our first observation is that the increments of $Y_t$ can be factored by its carrier process $X_t$, for times $t > s$:

$$\mathbb{P}\left[Y_t - Y_s\right] = \mathbb{P}\left[X_s\right]\mathbb{P}\left[Y_{t-s}\right] \tag{9}$$

where the process $X_s$ is independent of the process $Y_{t-s}$. For example, this allows us to immediately observe that for times $t > s$:

$$\mathbb{E}\left[\left(Y_t - Y_s\right)^n \| \, X_s\right] = X_s^n \, \mathbb{E}\left[Y_{t-s}^n\right] \tag{10}$$

We will liberally exploit this technique of arbitraging the elapsed metabolic time $Y_t$ against the accumulated acceleration $X_t$ to reduce expectations down to the well know standard terms for $X_t$ and $Y_t$:

$$\mathbb{E}\left[X_t^n\right] = e^{\left(n\mu + n^2\sigma^2/2\right)t} \tag{11}$$

$$\mathbb{E}\left[Y_t\right] = \frac{e^{\left(\mu+\sigma^2/2\right)t} - 1}{\mu + \sigma^2/2} \tag{12}$$

Note the in the last equation we have implicitly invoked the Fubini-Tonelli Theorem to switch the order of integration, and will broadly continue to use this theorem throughout this work.

**Lemma 1** (Acceleration Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of the carrier process of geometric Brownian motion $X_t$ conditioned on the increment of integrated geometric Brownian motion $Y_t - Y_s$, where $t > s$, is given by:*

$$\mathbb{E}\left[X_t^n \| \, Y_t - Y_s\right] = \frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[\left(Y_t - Y_s\right)^n\right]} \left(Y_t - Y_s\right)^n \tag{13}$$

*Proof.* We proceed with induction on $n$.

1. For $n = 0$ clearly $\mathbb{E}\left[X_t^0 \| \, Y_t - Y_s\right] = 1$

2. Now assume that up to $n$ the relation holds.

3. For $u \in [s, t]$ consider the integrable function

$$f\left(s, u, t, y\right) = \mathbb{E}\left[X_u X_t^n \| \, Y_t - Y_s = y\right] \tag{14}$$

4. It follows by the Fubini-Tonelli Theorem that and the induction assumption that for $n$ the relation holds

$$\int_s^t f(s,u,t,y)\,du = y^{n+1}\frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^n\right]} \tag{15}$$

5. Thus by the Fundamental Theorem of Calculus the integrand has the decomposition $f(s,u,t,y) = y^{n+1} f(s,u,t)$

6. By the Law of Conditional Expectations we have

$$\begin{aligned} \mathbb{E}\left[X_u X_t^n\right] &= \mathbb{E}\left[\mathbb{E}\left[X_u X_t^n \| \, Y_t - Y_s\right]\right] && (16)\\ &= f(s,u,t)\,\mathbb{E}\left[(Y_t - Y_s)^{n+1}\right] && (17) \end{aligned}$$

7. Which immediately yields the decomposed integrand as

$$f(s,u,t) = \frac{\mathbb{E}\left[X_u X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^{n+1}\right]} \tag{18}$$

8. Finally taking $u = t$ yields the desired result

$$\begin{aligned} \mathbb{E}\left[X_t^{n+1} \| \, Y_t - Y_s\right] &= f(s,t,t)\,(Y_t - Y_s)^{n+1} && (19)\\ &= \frac{\mathbb{E}\left[X_t^{n+1}\right]}{\mathbb{E}\left[(Y_t - Y_s)^{n+1}\right]}(Y_t - Y_s)^{n+1} && (20) \end{aligned}$$

□

A simple corollary follows from a nearly trivial derivation.

**Corollary 1** (Counterpoint Corollary)**.** *For a triplet of times $t_1 > t_0 > t_{-1}$ we have the following conditional expectation:*

$$\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}\right] = \frac{\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]}\left(Y_{t_0} - Y_{t_{-1}}\right)^n \tag{21}$$

*Proof.* Factoring and applying the previous lemma:

$$\begin{aligned} \mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}\right] &= \mathbb{E}\left[X_{t_0}^n \| \, Y_{t_0} - Y_{t_{-1}}\right]\mathbb{E}\left[Y_{t_1 - t_0}^n\right] && (22)\\ &= \frac{\mathbb{E}\left[Y_{t_1-t_0}^n\right]\mathbb{E}\left[X_{t_0}^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]}\left(Y_{t_0} - Y_{t_{-1}}\right)^n && (23)\\ &= \frac{\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]}\left(Y_{t_0} - Y_{t_{-1}}\right)^n && (24) \end{aligned}$$

□

Even with the factorization observation we are still in need of a means of reducing the expectation of the powers $Y_t^n$. A small lemma suffices to provide the means of finding powers:

**Lemma 2** (Recursion-Convolution Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of integrated geometric Brownian motion $Y_t$ is given by:*

$$\mathbb{E}\left[Y_t^n\right] = n \int_0^t \mathbb{E}\left[X_u^n\right] \mathbb{E}\left[Y_{t-u}^{n-1}\right] du \tag{25}$$

*Proof.* Expanding the expectant as a multi-variable integral we have, applying th Fubini-Tonelli Theorem we have, and then factoring we have:

$$\begin{aligned} \mathbb{E}\left[Y_t^n\right] &= \int_0^t \cdots \int_0^t \mathbb{E}\left[X_t^n\right] du_1 \dots du_n && (26) \\ &= \binom{n}{1} \int_0^t \mathbb{E}\left[X_u \left(Y_t - Y_u\right)^{n-1}\right] du && (27) \\ &= n \int_0^t \mathbb{E}\left[X_u^n\right] \mathbb{E}\left[Y_{t-u}^{n-1}\right] du && (28) \end{aligned}$$

$\square$

With the preceding lemma in hand we have the sufficient tools required to estimate all the usual statistics involving powers of $Y_t$, including the expectation, variance, and covariances.

# 3 Martingale

By our counterpoint corollary the scaled increments of integrated geometric Brownian motion $Y_{s\uparrow t}$, with $t > s$:

$$Y_{s\uparrow t} = \frac{Y_t - Y_s}{\mathbb{E}\left[Y_t - Y_s\right]} \tag{29}$$

form a two point martingale, so that for a triplet of times $t_1 > t_0 > t_{-1}$:

$$\mathbb{E}\left[Y_{t_0\uparrow t_1} \| \, Y_{t_{-1}\uparrow t_0}\right] = Y_{t_{-1}\uparrow t_0} \tag{30}$$

This allows us to leverage optional stopping time theorems to evaluate expectations of stopped versions of Gompertz processes.

# 4 Markov

From the Martingale $Y_{s\uparrow t}$ we can infer that $Y_t$ is a 2 step Markov process, and that the increments $Y_t - Y_s$, with $t > s$ are Markov. To verify this consider the sequence of bare times $0 = t_0 < \cdots < t_{n+1} = t$, and accelerated times $0 = y_0 < \cdots < y_{n+1} = y$, working through the conditional probability we have:

$$
\begin{aligned}
\mathbb{P}\left[Y_{t_{n+1}} - Y_{t_n} = y_{n+1} - y_n \| \, Y_{t_n} - Y_{t_{n-1}} = y_n - y_{n-1}, \ldots, Y_{t_1} - Y_{t_0} = y_1 - y_0\right] & \\
= \int_0^\infty \int_0^\infty \mathbb{P}\left[X_{t_n - t_{n-1}} = \frac{u}{v} \frac{y_{n+1} - y_n}{y_n - y_{n-1}}\right] & \\
\cdot \mathbb{P}\left[Y_{t_n - t_{n-1}} = u\right] & \\
\cdot \mathbb{P}\left[Y_{t_{n+1} - t_n} = v\right] dudv & \quad (31) \\
= \mathbb{P}\left[Y_{t_{n+1}} - Y_{t_n} = y_{n+1} - y_n \| \, Y_{t_n} - Y_{t_{n-1}} = y_n - y_{n-1}\right] & \quad (32)
\end{aligned}
$$

For the stopping time $T_n$ of the passage of the Gompertz process $G_{T_n} = n$ we have the conditional expectation:

$$
\mathbb{P}\left[T_n \| \, Y_{T_n}\right] = \frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]} \frac{(\lambda Y_t)^n}{(n-1)!} e^{-\lambda Y_t} \tag{33}
$$

In practice we cannot measure the accelerated metabolic time $Y_t$ instead we have access to the stopping times $T_n, \ldots, T_0$. Thus any triple of stopping times $T_{n+1}, T_n, T_{n-1}$ of consecutive passages of $G_t$ will have a 2 step Markov dependence. This amounts to a concrete prediction that ageing alone introduces a statistical dependence between the intervals of consecutive admissions for healthcare services.

# 5 Lévy

Integrated geometric Brownian motion $Y_t$ and its carrier process of geometric Brownian motion $X_t$ can be embedded into the Lie algebra of $2 \times 2$ upper triangular matrices $\mathfrak{h}_2$ by means of the factorization observed earlier, so that the increments are independent under matrix multiplication, for times $t > s$:

$$
\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \sim \begin{bmatrix} 1 & \tilde{Y}_{t-s} \\ 0 & \tilde{X}_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{34}
$$

where the increment processes of $\tilde{Y}_{t-s}$ and $\tilde{X}_{t-s}$ are independent of the processes $Y_s$ and $X_s$.

# 6 Fokker-Planck

The upper triangular Lévy process of integrated geometric Brownian motion satisfies the stochastic differential equation:

$$
d \begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu + \frac{\sigma^2}{2}\right) X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \tag{35}
$$

It follows from Fokker-Planck that the probability density of the joint process $p = \mathbb{P}\left[Y_t = y, X_t = x\right]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\mu + \frac{\sigma^2}{2}\right) xp - \frac{\partial}{\partial y} xp + \frac{\partial^2}{\partial x^2}\frac{\sigma^2}{2} x^2 p \tag{36}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left(\mu + \frac{3}{2}\sigma^2\right) x\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 p}{\partial x^2} = \left(-\mu + \frac{\sigma^2}{2}\right) p \tag{37}$$

Marginalizing over the probability of $X_t$ and applying the acceleration lemma yields the first order partial differential equation for the distribution of $Y_t$:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\frac{\partial p}{\partial t} - y\frac{\partial p}{\partial y} = p \tag{38}$$

Which by trial solution of separation of variables has the general solution:

$$p = \mathbb{E}\left[Y_t\right] f_{\mu,\sigma}\left(y\,\mathbb{E}\left[Y_t\right]\right) \tag{39}$$

for any analytic $f_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process $X_t$. Note that we are offloading the dimensional analysis into the analytic function.

# 7 Hazard Rate

Consider the first passage stopping time $T_1$ of the Gompertz process $G_t$, its cumulative distribution is the characteristic function of $Y_t$:

$$\mathbb{P}\left[T_1 \geq t\right] = \mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{40}$$

Thus the hazard rate $h$ of $T_1$:

$$h = \mathbb{P}\left[T_1 = t \| \, T_1 \geq t\right] \tag{41}$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\frac{\partial h}{\partial t} - \lambda\frac{\partial h}{\partial \lambda} = \left(\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\right)^2 \left(\frac{\partial}{\partial t}\frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]}\right) h \tag{42}$$

Which by trial solution of separation of variables has the general solution:

$$h = \lambda^2\,\mathbb{E}\left[X_t\right]\mathbb{E}\left[Y_t\right] g_{\mu,\sigma}\left(\lambda\,\mathbb{E}\left[Y_t\right]\right) \tag{43}$$

for any analytic $g_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process $X_t$. Taking the limit to deterministic subordination yields the constraints on $g_{\mu,\sigma}$:

Table 1: Sequential Boundary Conditions

| Boundary | Condition | Removes |
|---|---|---|
| $\sigma = 0$ | $h = \lambda e^{\mu t}$ | diffusion |
| $\mu = 0$ | $h = \lambda$ | then drift |
| $\lambda = 0$ | $h = 0$ | finally jumps |

Sequential boundary conditions on the hazard rate $h$ derived from the limits to deterministic subordination.

From the boundary conditions we can immediately deduce that in the deterministic limit of $\sigma \to 0$ we have:

$$g_{\mu,\sigma}(x) \xrightarrow{\sigma=0} \frac{1}{x} \tag{44}$$

However this alone cannot be the solution as it results in the characteristic function in $\lambda$ of a purely deterministic $Y_t$. Equating the general solution for the hazard rate to the Laplace of the Fokker-Planck solution yields the implicit equation in $f_{\mu,\sigma}$ and $g_{\mu,\sigma}$:

$$-\frac{\partial}{\partial t} \ln \int_0^\infty e^{\frac{\lambda u}{\mathbb{E}[Y_t]}} f_{\mu,\sigma}(u)\, du = \lambda^2\, \mathbb{E}[X_t]\, \mathbb{E}[Y_t]\, g_{\mu,\sigma}(\lambda\, \mathbb{E}[Y_t]) \tag{45}$$

Dimensional analysis provides an inference for a solution, which remains an open problem:

**Proposition 1** (Gompertz Anomaly)**.** *The analytic function $g_{\mu,\sigma}$ is simply the exponential divided by its argument, so that the hazard $h$ is given by:*

$$h = \lambda\, \mathbb{E}[X_t]\, e^{\frac{\sigma^2/2}{\mu+\sigma^2/2} \lambda\, \mathbb{E}[Y_t]} \tag{46}$$

*Proof.* The parity multiplied derivatives of the exponential of the integral of the hazard rate satisfies the recursion-convolution lemma, generating the moments of $Y_t$, and hence is the characteristic function of $Y_t$. $\square$

# 8 Discussion

In most circumstances $\lambda$ is the new born infant mortality due to ageing alone, and is less than 1 in 32000 person-years. As such the hazard is very close to the original hazard observed by Gompertz. Intuitively, when $\mu \ggg \sigma^2/2$ the process becomes approximately deterministic due to the large impact of the drift.

Conversely, the central conjecture in the preliminary material is that ageing is driven by stochastic accelerations, hence requiring that the drift vanish $\mu = 0$.

In this case the anomalous Gompertz hazard rate simplifies to:

$$h = \lambda e^{t\sigma^2/2} e^{\lambda \frac{e^{t\sigma^2/2} - 1}{\sigma^2/2}} \tag{47}$$

We have observed the anomalous Gompertz hazard rate in mortality rates in Alberta in the Hazard Rate Zoo experiment. Specifically when we remove the dominate exponential process of a doubling of mortality every 7 years from an infant mortality of 1 in 32000 person-years:

$$h = \frac{2^{t/7}}{32000} e^{\frac{7\left(2^{t/7} - 1\right)}{32000 \ln 2}} \tag{48}$$

there remains a residual anomalous growth in mortality with ageing, reflecting the higher order affect of the stochastic accelerations.