

# Gompertz Processes: A Theory of Ageing

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## Abstract

Motivated by considering infinitesimal stochastic accelerations of time, we outline a theory of Gompertz processes, Poisson processes subordinated by integrated Geometric Brownian motion.

## 1 Preliminaries

Basic science experiments in biology have ubiquitously observed that organisms respond to environmental stresses, including communicable diseases and exposures to toxins, with an acceleration  $a$  in their failure time  $ah(at)$ , where  $h$  is the bare hazard rate of the failure event. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations  $a_n$  of the underlying metabolic time. The accelerations can either increase the rate of failure  $a_n > 1$ , an exacerbation of the stresses, or decrease the rate of failure  $a_n < 1$ , an alleviation from the stresses. However, even in the controlled setting of a laboratory it is experimentally challenging to directly measure the underlying accelerated metabolic time, instead we only have access to the failure time in bare units. Thus a theory of ageing must be one that studies the subordination of the failure time by an elapsed metabolic time that is stochastically incremented.

Over the course of an organism's lifetime it will encounter exacerbations and alleviations that accelerate  $a_n, \dots, a_0 = 1$  metabolic time at times  $t_n, \dots, t_0 = 0$ . Furthermore, because the accelerations are all positive  $a_i > 0$  for each acceleration we can find a finite real valued generator  $b_n, \dots, b_0 = 0$  such that  $a_i = e^{b_i}$ . It follows that the elapsed metabolic time  $\tilde{t}_{t_n}$  at time  $t_n$  is the sum of the products of the accelerations up to time  $t_{n-1}$  and the elapsed bare time steps  $t_i - t_{i-1}$ :

$$\tilde{t}_{t_n} = \sum_{i=1}^n e^{\sum_{j=0}^{i-1} b_j} (t_i - t_{i-1}) \quad (1)$$

Provided no great explosions of acceleration occur in any small time scale, like say when an actual explosion occurs, the generators  $b_i$  will become infinitesimal on the same order as  $t_i - t_{i-1} \rightarrow 0$ :

$$\mathcal{O}(b_i) = \mathcal{O}(t_i - t_{i-1}) \quad (2)$$

The sum of products then becomes a stochastic process  $\tilde{T}_t$  that is an integral of a geometric random infinitesimal generator process  $e^{B_u}$ :

$$\tilde{T}_t = \int_0^t e^{B_u} du \quad (3)$$

This is essentially the continuous part of the Kolmogorov's characterization of stochastic processes as being composed of either a finite number of discrete jumps or an infinite number of continuous changes in a span of time. If we assume as a first approximation that the exacerbations and alleviations, and their respective accelerations, are independent and stationary over time then by the Lévy-Khintchine characterization the only infinitesimal generator of elapsed metabolic time that is Lévy and continuous, *jump free*, is Brownian motion  $B_u$ . The stochastic process of elapsed metabolic time is better known as integrated geometric Brownian motion.

**Definition 1** (Gompertz Process). A Gompertz process  $G_t$  is a subordinated Poisson process  $N_t$ , with rate  $\lambda$ , where the subordinating process is integrated geometric Brownian motion  $Y_t$ , with drift  $\mu$  and diffusion  $\sigma$ :

$$G_t = N_{Y_t} \quad (4)$$

given:

$$Y_t = \int_0^t X_s ds \quad (5)$$

$$= \int_0^t e^{\mu s + \sigma W_s} ds \quad (6)$$

Phenomenologically the finite real stochastic process  $\mu t + \sigma W_s$  is the infinitesimal acceleration at time  $t$  that generates a non-negative geometric stochastic process  $X_t$  of accumulated accelerations up to time  $t$  and whose integral  $Y_t$  is a strictly increasing stochastic process of elapsed metabolic time up to time  $t$ .

To start our exploration of the rich and subtleties of Gompertz processes we will briefly review of the properties of integrated geometric Brownian motion which are salient to developing our theory. This is by no means a comprehensive compendium. Much of the material I will cover has been deeply and thoroughly explored in the quantitative finance literature in the theory of pricing Asian options.

Our first observation is that the increments of  $Y_t$  can be factored by its carrier process  $X_t$ , for times  $t > s$ :

$$Y_t - Y_s \sim X_s Y_{t-s} \quad (7)$$

where the process  $X_s$  is independent of the process  $Y_{t-s}$ . For example, this allows us to immediately observe that for times  $t > s$ :

$$\mathbb{E}[(Y_t - Y_s)^n | X_s] = X_s^n \mathbb{E}[Y_{t-s}^n] \quad (8)$$

We will liberally exploit this technique of arbitraging the elapsed metabolic time  $Y_t$  against the accumulated acceleration  $X_t$  to reduce expectations down to the well know standard terms for  $X_t$  and  $Y_t$ :

$$\mathbb{E}[X_t] = e^{\left(\mu + \frac{\sigma^2}{2}\right)t} \quad (9)$$

$$\mathbb{E}[Y_t] = \frac{e^{\left(\mu + \frac{\sigma^2}{2}\right)t} - 1}{\mu + \frac{\sigma^2}{2}} \quad (10)$$

Note the in the last equation we have implicitly invoked Fubini's theorem to switch the order of integration, and will broadly continue to throughout this work.

**Lemma 1** (Acceleration Lemma). *The expectation of non-negative integer powers  $n \geq 0$  of the carrier process of geometric Brownian motion  $X_t$  conditioned on the increment of integrated geometric Brownian motion  $Y_t - Y_s$ , where  $t > s$ , is given by:*

$$\mathbb{E}[X_t^n \mid Y_t - Y_s] = \frac{\mathbb{E}[X_t^n]}{\mathbb{E}[(Y_t - Y_s)^n]} (Y_t - Y_s)^n \quad (11)$$

*Proof.* Carry out induction on  $n$ . □

A simple corollary follows from a nearly trivial derivation.

**Corollary 1** (Counterpoint Corollary). *For a triplet of times  $t_1 > t_0 > t_{-1}$  we have the following conditional expectation:*

$$\begin{aligned} \mathbb{E}[(Y_{t_1} - Y_{t_0})^n \mid Y_{t_0} - Y_{t_{-1}}] \\ = e^{\left(n\mu + \frac{n^2}{2}\sigma^2\right)t_0} \frac{\mathbb{E}[Y_{t_1-t_0}^n]}{\mathbb{E}[(Y_{t_0} - Y_{t_{-1}})^n]} (Y_{t_0} - Y_{t_{-1}})^n \end{aligned} \quad (12)$$

*Proof.* Factor and apply the previous lemma. □

Even with the factorization observation we are still in need of a means of reducing the expectation of the powers  $Y_t^n$ . A small lemma suffices to provide the means of finding powers:

**Lemma 2** (Recursion-Convolution Lemma). *The expectation of non-negative integer powers  $n \geq 0$  of integrated geometric Brownian motion  $Y_t$  is given by:*

$$\mathbb{E}[Y_t^n] = n \int_0^t e^{\left(n\mu + \frac{n^2}{2}\sigma^2\right)u} \mathbb{E}[Y_{t-u}^{n-1}] du \quad (13)$$

*Proof.* Carry out a routine factorization. □

With the preceding lemma in hand we have the sufficient tools required to estimate all the usual statistics involving powers of  $Y_t$ , including the expectation, variance, and covariances.

## 2 Martingale Increments

By our counterpoint corollary the scaled increments of integrated geometric Brownian motion  $Y_t$  form a two point martingale, for a triplet of times  $t_1 > t_0 > t_{-1}$ :

$$\mathbb{E} \left[ \frac{Y_{t_1} - Y_{t_0}}{e^{(\mu + \frac{\sigma^2}{2})t_1} - e^{(\mu + \frac{\sigma^2}{2})t_0}} \middle| \frac{Y_{t_0} - Y_{t_{-1}}}{e^{(\mu + \frac{\sigma^2}{2})t_0} - e^{(\mu + \frac{\sigma^2}{2})t_{-1}}} \right] = \frac{Y_{t_0} - Y_{t_{-1}}}{e^{(\mu + \frac{\sigma^2}{2})t_0} - e^{(\mu + \frac{\sigma^2}{2})t_{-1}}} \quad (14)$$

From this martingale property of  $Y_t$  any triple of stopping times  $T_{n+1}, T_n, T_{n-1}$  of consecutive passages of  $G_t$  will have a 2 Markov dependence. This amounts to a concrete prediction that ageing alone introduces a statistical dependence between the intervals of consecutive admissions for healthcare services.

## 3 Lévy Process

Integrated geometric Brownian motion  $Y_t$  and its carrier process of geometric Brownian motion  $X_t$  can be embedded into the Lie algebra of  $2 \times 2$  upper triangular matrices  $\mathfrak{h}_2$  by means of the factorization observed earlier, so that the increments are independent under matrix multiplication, for times  $t > s$ :

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \sim \begin{bmatrix} 1 & \tilde{Y}_{t-s} \\ 0 & \tilde{X}_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \quad (15)$$

where the increment processes of  $\tilde{Y}_{t-s}$  and  $\tilde{X}_{t-s}$  are independent of the processes  $Y_s$  and  $X_s$ .

## 4 Fokker-Planck

The upper triangular Lévy process of integrated Brownian motion satisfies the stochastic differential equation:

$$d \begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu + \frac{\sigma^2}{2}\right) X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \quad (16)$$

It follows from Fokker-Planck that the probability density of the joint process  $p = \mathbb{P}[Y_t = y, X_t = x]$  satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} \left( \mu + \frac{\sigma^2}{2} \right) xp - \frac{\partial}{\partial y} xp + \frac{\partial^2}{\partial x^2} \frac{\sigma^2}{2} x^2 p \quad (17)$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left( \mu + \frac{3}{2}\sigma^2 \right) x \frac{\partial p}{\partial x} + x \frac{\partial p}{\partial y} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 p}{\partial x^2} = \left( -\mu + \frac{\sigma^2}{2} \right) p \quad (18)$$

## 5 Hazard Rate

Consider the first passage stopping time  $T_1$  of the Gompertz process  $G_t$ , its cumulative distribution is the characteristic function of  $Y_t$ :

$$\mathbb{P}[T_1 \geq t] = \mathbb{E}[e^{-\lambda Y_t}] \quad (19)$$

Thus the hazard rate  $h$  of  $T$ :

$$h = \mathbb{P}[T_1 = t | T_1 \geq t] \quad (20)$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}[Y_t]}{\mathbb{E}[X_t]} \frac{\partial h}{\partial t} - \lambda \frac{\partial h}{\partial \lambda} = \left( \frac{\mathbb{E}[Y_t]}{\mathbb{E}[X_t]} \right)^2 \left( \frac{\partial}{\partial t} \frac{\mathbb{E}[X_t]}{\mathbb{E}[Y_t]} \right) h \quad (21)$$

Which by trial solution of separation of variables has the general solution:

$$h = \lambda^2 \mathbb{E}[X_t] \mathbb{E}[Y_t] f_{\mu,\sigma}(\lambda \mathbb{E}[Y_t]) \quad (22)$$

for any analytic  $f_{\mu,\sigma}$  dependent on the drift and diffusion of the carrier process  $X_t$ . Taking the limit to deterministic subordination yields the constraints on  $f_{\mu,\sigma}$  by L'Hôpital's rule:

Table 1: Boundary conditions on the hazard rate  $h$

Boundary	Condition	Constrains
$\lambda = 0$	$h = 0$	$\frac{\partial}{\partial \lambda} f_{\mu,\sigma}(\lambda \mathbb{E}[Y_t])$
$t = 0$	$h = \lambda$	$\frac{\partial}{\partial t} f_{\mu,\sigma}(\lambda \mathbb{E}[Y_t])$
$\mu = \sigma = 0$	$h = \lambda$	$\left[ \frac{\partial}{\partial \mu} \cdot + \frac{\partial}{\partial \sigma} \cdot \right] (f_{\mu,\sigma}(\lambda \mathbb{E}[Y_t]))$
$\sigma = 0$	$h = \lambda e^{\mu t}$	$\frac{\partial}{\partial \sigma} f_{\mu,\sigma}(\lambda \mathbb{E}[Y_t])$

Boundary conditions on the hazard rate derived from the limits to deterministic subordination.

From the boundary conditions we can immediately deduce that in the deterministic limit of  $\sigma \rightarrow 0$  we have:

$$f_{\mu,\sigma}(x) \xrightarrow{\sigma=0} \frac{1}{x} \quad (23)$$

However this alone cannot be the solutions as it is the characteristic function in  $\lambda$  of a purely deterministic  $Y_t$ . Until more constraints can be found the following proposition remains open.

**Proposition 1** (Gompertz Hazard). *The analytic function  $f_{\mu,\sigma}$  is simply the exponential divide by its argument, so that the hazard  $h$  is given by:*

$$h = \lambda \mathbb{E}[X_t] e^{\frac{\sigma^2}{2} \lambda \mathbb{E}[Y_t]} \quad (24)$$

In most circumstances  $\lambda$  is new born infant mortality due to ageing alone and is less than one in ten thousand person years. As such the hazard is very close to the original hazard observed by Gompertz.