Gompertz Processes: A Theory of Ageing

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Abstract

Motivated by considering infinitesimal stochastic accelerations of time, we outline a theory of Gompertz processes, Poisson processes subordinated by integrated Geometric Brownian motion.

1 Preliminaries

Basic science experiments in biology have ubiquitously observed that organisms respond to environmental stresses, including communicable diseases and exposures to toxins, with an acceleration a in their failure time $ah\left(at\right)$, where h is the bare hazard rate of the failure event. Outside of the controlled setting of a laboratory the environmental stresses occur stochastically, resulting in a sequence of accelerations a_n of the underlying metabolic time. The accelerations can either increase the rate of failure $a_n > 1$, an exacerbation of the stresses, or decrease the rate of failure $a_n < 1$, an alleviation from the stresses. However, even in the controlled setting of a laboratory it is experimentally challenging to directly measure the underlying accelerated metabolic time, instead we only have access to the failure time in bare units. Thus a theory of ageing must be one the studies the subordination of the failure time by an elapsed metabolic time that is stochastically incremented.

Integrated geometric Brownian motion occurs in the asymptotic limit of infinitesimal stochastic accelerations.

If we assume as a first approximation that the stresses, and their respective accelerations, are independent and stationary over time then by the Lévy-Khintchine characterization integrated Brownian motion is the only representation of elapsed metabolic time that is Lévy and continuous, *jump free*.

Definition 1 (Gompertz Process)**.** A Gompertz process G_t is a subordinated Poisson process N_t, with rate λ, where the subordinating process is integrated geometric Brownian motion Y_t, with drift μ and diffusion σ:

$$G_t = N_{Y_t} \tag{1}$$

given:

$$Y_t = \int_0^t X_s ds \tag{2}$$
$$= \int_0^t e^{\mu s+\sigma W_s} ds \tag{3}$$

Phenomenologically the finite real stochastic process $\mu t+\sigma W_s$ is the infinitesimal acceleration at time t that generates a non-negative stochastic process X_t of accumulated accelerations up to time t and whose integral Y_t is a strictly increasing stochastic process of elapsed metabolic time up to time t.

To start our exploration of the rich and subtletie of Gompertz processes we will briefly review of the properties of integrated geometric Brownian motion which are salient to developing our theory. This is by no means a comprehensive compendium. Much of the material I will cover has been explored deeply and thoroughly in the quantitative finance literature within the context of pricing Asian options.

Our first observation is that the increments of Y_t can be factored by its carrier process X_t, for times $t > s$:

$$Y_t - Y_s \sim X_s Y_{t-s} \tag{4}$$

where the process X_s is independent of the process Y_{t-s}. For example, this allows us to immediately observe that for times $t > s$:

$$\mathbb{E}\left[(Y_t - Y_s)^n \| X_s\right] = X_s^n \, \mathbb{E}\left[Y_{t-s}^n\right] \tag{5}$$

We will liberally exploit this technique of arbitraging the elapsed metabolic time Y_t against the accumulated acceleration X_t to reduce expectations down to the well know standard terms for X_t and Y_t

$$\mathbb{E}\left[X_t\right] = e^{\left(\mu+\frac{\sigma^2}{2}\right)t} \tag{6}$$
$$\mathbb{E}\left[Y_t\right] = \frac{e^{\left(\mu+\frac{\sigma^2}{2}\right)t} - 1}{\mu + \frac{\sigma^2}{2}} \tag{7}$$

Lemma 1 (Acceleration Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of the carrier process of geometric Brownian motion X_t conditioned on the increment of integrated geometric Brownian motion $Y_t - Y_s$, where $t > s$, is given by:*

$$\mathbb{E}\left[X_t^n \| Y_t - Y_s\right] = \frac{\mathbb{E}\left[X_t^n\right]}{\mathbb{E}\left[(Y_t - Y_s)^n\right]} \left(Y_t - Y_s\right)^n \tag{8}$$

Proof. Carry out induction on n. $\square$

A simple corollary follows from a nearly trivial derivation.

Corollary 1 (Counterpoint Corollary)**.** *For a triplet of times $t_1 > t_0 > t_{-1}$ we have the following conditional expectation:*

$$\mathbb{E}\left[(Y_{t_1} - Y_{t_0})^n \| \, Y_{t_0} - Y_{t_{-1}}\right] \\ = e^{\left(n\mu + \frac{n^2}{2}\sigma\right)t_0} \frac{\mathbb{E}\left[Y^n_{t_1 - t_0}\right]}{\mathbb{E}\left[\left(Y_{t_0} - Y_{t_{-1}}\right)^n\right]} \left(Y_{t_0} - Y_{t_{-1}}\right)^n \qquad (9)$$

Proof. Factor and apply the previous lemma. □

Even with the factorization observation we are still in need of a means of reducing the expectation of the powers Y_t^n. A small lemma suffices to provide the means of finding powers:

Lemma 2 (Recursion-Convolution Lemma)**.** *The expectation of non-negative integer powers $n \geq 0$ of integrated geometric Brownian motion Y_t is given by:*

$$\mathbb{E}\left[Y_t^n\right] = n \int_0^t e^{\left(n\mu + \frac{n^2}{2}\sigma^2\right)u}\, \mathbb{E}\left[Y^{n-1}_{t-u}\right] du \qquad (10)$$

Proof. Carry out a routine factorization. □

With the preceding lemma in hand we have the sufficient tools required to estimate all the usual statistics involving powers of Y_t, including the expectation, variance, and covariances.

2 Martingale

The scaled increments of integrated geometric Brownian motion Y_t form a two point martingale, for a triplet of times $t_1 > t_0 > t_{-1}$:

$$\mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{e^{\left(\mu + \frac{\sigma^2}{2}\right)t_1} - e^{\left(\mu + \frac{\sigma^2}{2}\right)t_0}} \middle\| \frac{Y_{t_0} - Y_{t_{-1}}}{e^{\left(\mu + \frac{\sigma^2}{2}\right)t_0} - e^{\left(\mu + \frac{\sigma^2}{2}\right)t_{-1}}}\right] \\ = \frac{Y_{t_0} - Y_{t_{-1}}}{e^{\left(\mu + \frac{\sigma^2}{2}\right)t_0} - e^{\left(\mu + \frac{\sigma^2}{2}\right)t_{-1}}} \qquad (11)$$

From this martingale property of Y_t any triple of stopping times T_{n+1}, T_n, T_{n-1} of consecutive passages of G_t will have a 2 Markov dependence. This amounts to a concrete prediction that ageing alone introduces a statistical dependence between the intervals of consecutive admissions for healthcare services.

3 Lévy Process

Integrated geometric Brownian motion Y_t and its carrier process of geometric Brownian motion X_t can be embedded into the Lie algebra of 2×2 upper triangular matrices $\mathfrak{h}_2$ so that the increments are independent under matrix multiplication, for times $t > s$:

$$\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} \sim \begin{bmatrix} 1 & \tilde{Y}_{t-s} \\ 0 & \tilde{X}_{t-s} \end{bmatrix} \begin{bmatrix} 1 & Y_s \\ 0 & X_s \end{bmatrix} \tag{12}$$

where the increment processes of $\tilde{Y}_{t-s}$ and $\tilde{X}_{t-s}$ are independent of the processes Y_s and X_s.

4 Fokker-Planck

The upper triangular Lévy process of integrated Brownian motion satisfies the stochastic differential equation:

$$d\begin{bmatrix} 1 & Y_t \\ 0 & X_t \end{bmatrix} = \begin{bmatrix} 0 & X_t \\ 0 & \left(\mu + \frac{\sigma^2}{2}\right) X_t \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma X_t \end{bmatrix} dW_t \tag{13}$$

It follows from Fokker-Planck that the probability density of the joint process $p = \mathbb{P}\left[Y_t = y, X_t = x\right]$ satisfies the partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\mu + \frac{\sigma^2}{2}\right) xp - \frac{\partial}{\partial y} xp + \frac{\partial^2}{\partial x^2}\frac{\sigma^2}{2} x^2 p \tag{14}$$

We can restate this as an Eigen evolution equation:

$$\frac{\partial p}{\partial t} + \left(\mu + \frac{3}{2}\sigma^2\right) x\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 p}{\partial x^2} = \left(-\mu + \frac{\sigma^2}{2}\right) p \tag{15}$$

5 Hazard Rate

Consider the first passage stopping time T_1 of the Gompertz process G_t, its cumulative distribution is the characteristic functions of Y_t:

$$\mathbb{P}\left[T_1 \geq t\right] = \mathbb{E}\left[e^{-\lambda Y_t}\right] \tag{16}$$

Thus the hazard rate h of T:

$$h = \mathbb{P}\left[T_1 = t \| \, T_1 \geq t\right] \tag{17}$$

satisfies the partial differential Eigen equation:

$$\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\frac{\partial h}{\partial t} - \lambda\frac{\partial h}{\partial \lambda} = \left(\frac{\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[X_t\right]}\right)^2 \left(\frac{\partial}{\partial t}\frac{\mathbb{E}\left[X_t\right]}{\mathbb{E}\left[Y_t\right]}\right) h \tag{18}$$

Which by trial solution of separation of variables has the general solution:

$$h = \lambda^2\, \mathbb{E}\left[X_t\right] \mathbb{E}\left[Y_t\right] f_{\mu,\sigma}\left(\lambda\, \mathbb{E}\left[Y_t\right]\right) \tag{19}$$

for any analytic $f_{\mu,\sigma}$ dependent on the drift and diffusion of the carrier process X_t. Taking the limit to deterministic subordination yields the constraints on $f_{\mu,\sigma}$ by L'Hôpital's rule:

Table 1: Boundary conditions on the hazard rate h

Boundary	Condition	Constrains
$\lambda = 0$	$h = 0$	$\frac{\partial}{\partial \lambda} f_{\mu,\sigma}\left(\lambda\, \mathbb{E}\left[Y_t\right]\right)$
$t = 0$	$h = \lambda$	$\frac{\partial}{\partial t} f_{\mu,\sigma}\left(\lambda\, \mathbb{E}\left[Y_t\right]\right)$
$\mu = \sigma = 0$	$h = \lambda$	$\left[\frac{\partial}{\partial \mu} \cdot + \frac{\partial}{\partial \sigma} \cdot\right]\left(f_{\mu,\sigma}\left(\lambda\, \mathbb{E}\left[Y_t\right]\right)\right)$
$\sigma = 0$	$h = \lambda \mu e^{\mu t}$	$\frac{\partial}{\partial \sigma} f_{\mu,\sigma}\left(\lambda\, \mathbb{E}\left[Y_t\right]\right)$

Boundary conditions on the hazard rate derived from the limits to deterministic subordination.