Geometry Configurations

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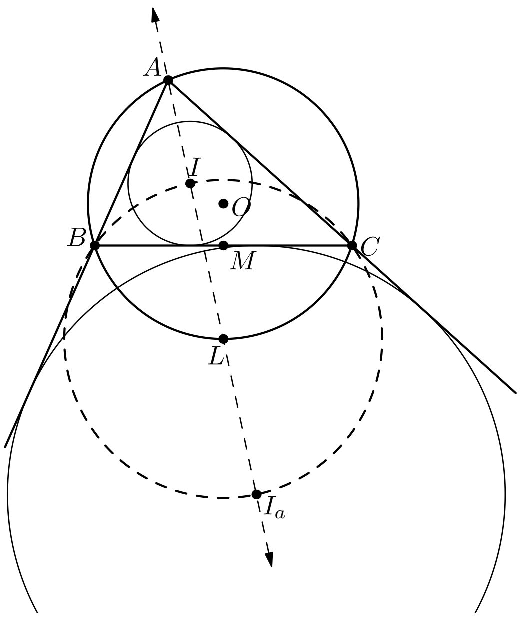
2 Midpoint of Arc (Fact 5)

2.1 Theory



In triangle ABC, let I be the incenter. Let L be the intersection of line AI and the circumcircle of ABC, such that L and A are distinct points. Define I_a to be the A-excenter of triangle ABC. (The A-excenter is where the external angle bisectors of angles B and C of triangle ABC and the internal angle bisector of angle A concur. This is also the center of the A-excircle of triangle ABC, as depicted above.) Point L has some pretty remarkable properties.

1. Prove that points A, I, L, and I_a are collinear.

2. Prove that $LB = LC$. Note that this is equivalent to points O, M, and L being on the same line, where O is the circumcenter and M is the midpoint of side BC.

3. Prove that L is the circumcenter of triangle BIC, or that $LB = LC = LI$.

4. Prove that L is the midpoint of II_a. Use this to show that $BICI_a$ is cyclic.

2.2 Practice Problems (Computational)

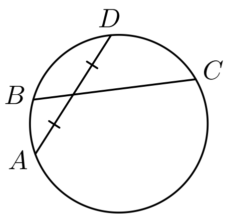
1. (HMMT 2011) Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

2. In cyclic quadrilateral $ABCD$, AC bisects angle BAD. Point F is on AB such that CF and AB are perpendicular. If $AF = 2005$ and $AB = 2006$, find AD.

3. (CHMMC Spring 2012) In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5^\circ$, $\angle BEF = 10^\circ$, and $AC = 3$, find the length of DF.

4. (SMT 2014) In cyclic quadrilateral $ABCD$, $AB = AD$. If $AC = 6$ and $AB/BD = 3/5$, find the maximum possible area of $ABCD$.

5. (AIME 1983) Chords AD and BC of the same circle intersect. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. Find the sine of the minor arc AB. (Assume that A is closer to point B than to point C.)



6. (NIMO 2012) In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then find the length XI.

7. (OMO 2012) Let ABC be a triangle with circumcircle ω. Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

2.3 Practice Problems (Olympiad)

1. (CGMO 2012) The incircle of ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.

2. In cyclic quadrilateral $ABCD$, let I_1 and I_2 be the incenters of triangles ABC and DBC respectively. Show that I_1I_2CB is a cyclic quadrilateral.

3. In triangle ABC, the angle bisector AD (with D on side BC) hits the circumcircle of ABC at point L. Show that $\triangle LAB \sim \triangle LBD \sim \triangle CAD$.

4. (HMMT 2013) Let triangle ABC satisfy $2BC = AB+AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.

5. (ISL 2006) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$
Show that $AP \geq AI$, and that equality holds if and only if $P = I$.