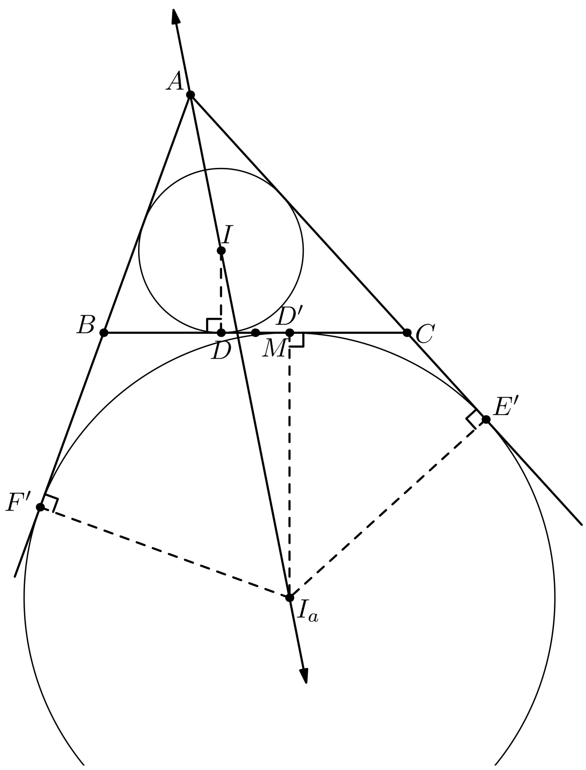
Geometry Configurations

Aaron Lin

3 Incircle and Excircle

This week's configuration covers basic properties of incircles and excircles. We covered a few of these properties in some of the classes, so it'd be a good time to put a lot of this together. To review, the **incircle** of a triangle is the circle inside of the triangle that is tangent to all three of its sides. An **excircle** of a triangle is a circle outside of the triangle that is tangent to the extensions of the sides of the triangle. Each triangle has one incircle and three excircles, so we refer to each excircle by the vertex it opposes. For example, the excircle in the diagram below is the A-excircle. The B-excircle and the C-excircle are not shown.

Often, a property that holds for a triangle's incircle usually also holds for its excircles. Thus, proofs that apply to incircles tend to have similar extensions for excircles. However, when writing up a proof, it is insufficient to say "A similar proof holds for the excircle as the incircle," but it is okay to say "A similar proof holds for the B-excircle as the A-excircle."



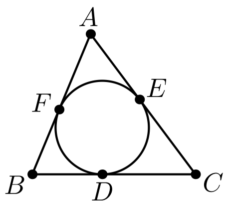
3.1 Basic Properties

1. Show that there is a homothety about A mapping the incircle into the A-excircle.

   Solution: All pairs of circles are homothetic. Their common external tangents intersect at point A, so the homothety about A that maps tangency points to each other also maps one circle into the other.

2. Let D be the tangency point of the incircle with side BC. Show that $BD = s - b$ and $CD = s - c$, where $BC = a$, $CA = b$, $AB = c$, and $s = (a + b + c)/2$.

   Solution:

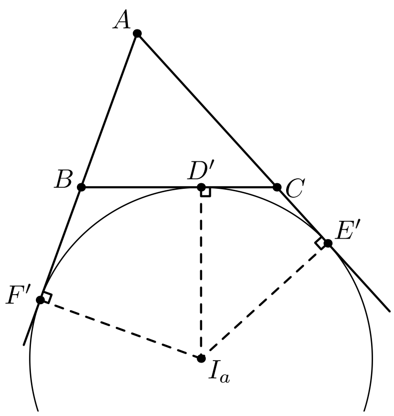
   Let E and F be the tangency points of the incircle to AC and AB respectively. Because $AE = AF$, $BF = BD$, and $CE = CD$, we can write the following system of equations:

   $$BD + CD = a$$
   $$CD + AE = b$$
   $$AE + BD = c$$

   Adding the three equations and solving for each variable using standard procedures yields the desired expressions for BD and CD.

3. Let D', E', and F' be the tangency points of the A-excircle with the lines BC, CA, and AB respectively. Show that $AE' = AF' = s$.

   Solution:

   Because $AE' = AF'$, $BD' = BF'$, and $CE' = CD'$, we can write the following system of equations:

$$BD' + CD' = a$$
$$AE' - CD' = b$$
$$AE' - BD' = c$$

Adding the three equations together yields the desired expression.

4. Prove that $BD' = s - c$ and $CD' = s - b$.

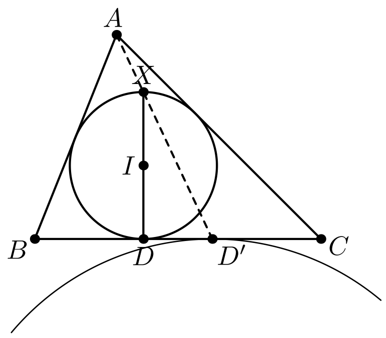
   Solution: One can easily solve for the other two lengths BD' and CD' from the system of equations in the solution above.

5. Show that $BD = CD'$, or that segments DD' and BC have the same midpoint.

   Solution: Because $BD = s - b = CD'$, then D and D' are symmetric about the midpoint of BC.

6. Let X be the point on the incircle such that DX is a diameter. Prove that points A, X, and D' are collinear.

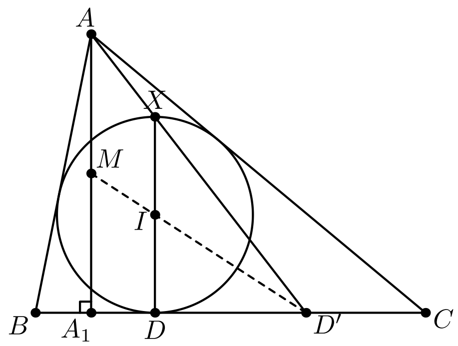
   Solution:



   Consider the homothety that maps the incircle into the A-excircle. The line tangent to the incircle at point X is parallel to BC, so D' is the corresponding point on the excircle. Thus, A, X, and D' are collinear.

7. Let A_1 be the foot of the altitude from A onto BC. Show that the line $D'I$ passes through the midpoint of AA_1.

   Solution:



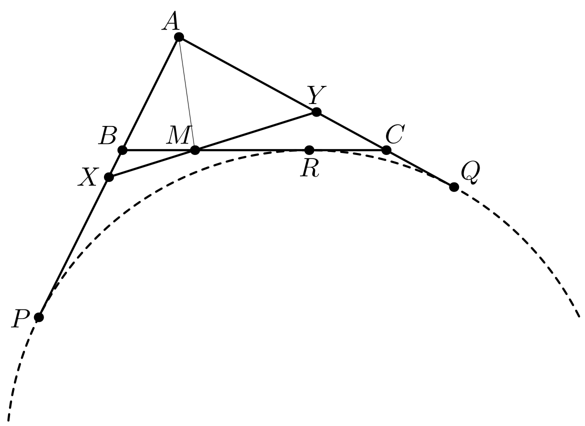
Consider the homothety centered at point D' that maps X into A. Since D', D, and A_1 are collinear and AA_1 and XD are parallel (as they are both perpendicular to BC), then this homothety also maps point D into A_1. Thus, the midpoint of XD, point I, is mapped into the midpoint of the altitude AA_1, implying that MID' is a line.

3.2 Practice Problems

Included below is one practice problem. For additional practice, refer to Yufei Zhao's handout (`http://yufeizhao.com/olympiad/three_geometry_lemmas.pdf`).

1. (All-Russian MO 2010) Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively, such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

   Solution:

   Without loss of generality, assume that $AB \leq AC$. Construct points P and Q to be on rays AB and AC such that $AP = AQ = 2$ and point R on side BC such that $AB+BR = AC+CR = 2$. By Statements 3 and 4, points P, Q, and R are the tangency points of Ω, the A-excircle, to the sides of triangle ABC.

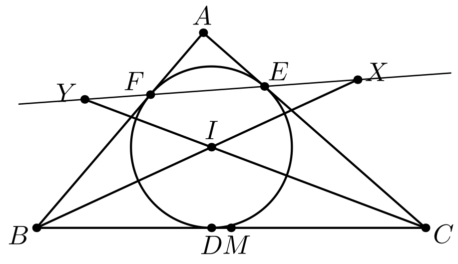
   Since $XA^2 = XP^2 = 1^2$ and $YA^2 = YQ^2 = 1^2$, then points X and Y have equal power with respect to Ω and the circle with radius 0 centered at point A. Thus, XY is the radical axis of the two circles, implying that M also has an equal power with respect to both circles. Since MR is tangent to Ω, then $MA^2 = MR^2$, or $MA = MR$.

   Thus, the perimeter of triangle ABM is $AB+BM+MA = AB+BM+MR = AB+BR = 2$.

3.3 The ABCDE Configuration

Let ABC be a triangle with incenter I.

1. Let X be the intersection of the angle bisector at B and the line EF. Show that $CDIEX$ is cyclic.



Solution: Since $\angle IDC = \angle IEC = 90^\circ$ by definition, quadrilateral $CDIE$ is cyclic. Additionally, with directed angles, we have that $\angle CIX = \angle ICB + \angle CBI = (\angle ACB + \angle CBA)/2 = (180^\circ - \angle BAC)/2 = \angle AEF = \angle CEX$, which implies that $CIEX$ is cyclic as well. The conclusion follows.

2. Points M, N, and P are defined to be the midpoints of sides BC, CA, and AB respectively. (Points N and P are not included in the diagram above.) Prove that X lies on MN and Y lies on MP.

   Solution: From the previous part, $\triangle BXC$ is right, so the length of median XM is equal to the lengths of MB and MC. Thus, $\angle XMC = 2\angle XBM = \angle ABC$, which implies that $XM \parallel AB$. The line through point M parallel to side AB is the midline, so points M, N, and X are collinear. Similarly, one can show that M, P, and Y are collinear.

3. Show that $BCXY$ is cyclic.

   Solution: From the previous part, $\angle BYC = \angle BYI = 90^\circ$ and $\angle BXC = \angle IXC = 90^\circ$.

4. (USAJMO 2014) Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides BC, CA, AB and let E, F be the tangency points of γ with CA and AB, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ. Prove that:

   (a) I lies on ray CV.

   (b) Line XI bisects UV.

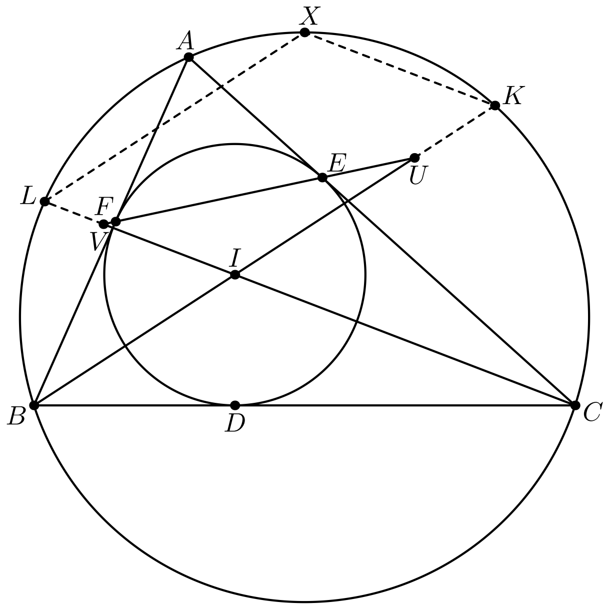
   Solution: From previous statements, part (a) is true.

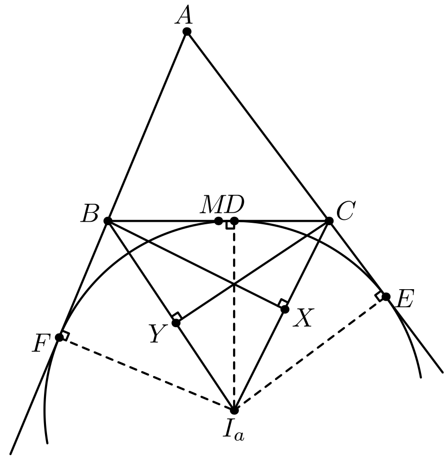
   For part (b), we let points K and L be the midpoints of minor arcs AC and AB respectively. Note that points K and L lie on BI and CI respectively. Since $\angle XKL = \angle XCL = \angle XCB - \angle LCB = (90 - \angle A/2) - \angle C/2 = \angle B/2 = \angle CBK = \angle CLK$, then $XK \parallel IL$. Similarly, $XL \parallel IK$, so $XKIL$ is a parallelogram, which implies that line XI bisects KL.

   From the previous part, $BVUC$ is cyclic, so $\angle IUV = \angle BCL = \angle BKL$, which implies that $KL \parallel UV$. Thus, there is a homothety $\mathcal{H}$ about point I mapping segment KL to UV, so IX also bisects segment UV.

3.4 Extra Excircles

1. In the above diagram, we define D, E, and F to be the tangency points of the A-excircle with the lines BC, CA, and AB respectively. If X and Y are the feet of the altitudes from B and C onto I_aC and I_aB respectively, show that E, F, X, and Y all lie on a line.





Solution: Let X_1 and Y_1 be the intersections of line EF with lines BI_a and CI_a respectively. Since $\angle FX_1I_a = CX_1E = 180° - \angle X_1EC - \angle ECX_1 = 180° - (180 - \angle EAF)/2 - (180 - \angle BCA)/2 = \angle CBF/2 = \angle I_aBF$, which implies that quadrilateral X_1I_aFB is cyclic. Since $\angle I_aFB = 90°$, then X and X_1 are the same point. Similarly, Y and Y_1 are the same point, so points E, F, X, and Y all lie on the same line.

2. Points M, N, and P are defined to be the midpoints of sides BC, CA, and AB respectively. (Points N and P not shown in the diagram above.) Prove that X lies on MP and Y lies on MN.

   Solution: Notice that $\angle YMB + \angle NMB = 2\angle BCY + 180° - \angle ABC = 2(90° - \angle I_aBC) + \angle CBF = 180° + (\angle CBF - \angle CBF) = 180°$, so Y lies on line MN. Similarly, X lies on line MP.

3. Construct the analogous points for X and Y for the B- and C-excircles. Show that these six points are concyclic.

   Solution 1: Define W and Z to be the analogous points to X and Y for the B-excircle such that W is the foot of A onto the external angle bisector of $\angle C$. Additionally, let U and V be the two analogous points for the C-excircle.

From the previous part, points Y and Z both lie on line MN, so $\angle ZYX = \angle MYX = \angle AFE = 90^\circ - \angle BAC/2$. Furthermore, since points X, C, and W are collinear, then $\angle XWZ = \angle CWZ = 90^\circ + \angle AWZ = 90^\circ + \angle ACZ = 90^\circ + (90^\circ - \angle ZAC) = 180^\circ - (180^\circ - \angle BAC)/2 = 90^\circ + \angle BAC/2$. Thus, $\angle ZYX + \angle XWZ = 180^\circ$, so the four points are concyclic.

Analogously, one can show that $WZUV$ and $UVXY$ are also cyclic quadrilaterals. Using the Radical Axis Theorem on the three circles, one can see that the mutual intersections of the radical axes do not concur at a single point, so two of the circumcircles must coincide. This implies that all three circumcircles are indeed the same circle.

Solution 2: Let I be the incenter of the medial triangle MNP and D_1, E_1, and F_1 be the feet of I onto sides NP, MP, and MN respectively. Additionally, let r be the inradius of triangle ABC. Since $\angle IF_1Y = 90\circ$, then by the Pythagorean theorem and the medial triangle similarity, we have:

$$\begin{aligned}
IY^2 &= ID_1^2 + D_1Y^2 \\
&= (r/2)^2 + (D_1M + MY)^2 \\
&= (r/2)^2 + \left(\frac{s-a}{2} + MB\right)^2 \\
&= (r/2)^2 + \left(\frac{-a+b+c}{4} + \frac{BC}{2}\right)^2 \\
&= (r/2)^2 + \left(\frac{-a+b+c}{4} + \frac{2a}{4}\right)^2 \\
&= (r/2)^2 + \left(\frac{a+b+c}{4}\right)^2
\end{aligned}$$

Computing the distance from I to each of the other five analogous points yields a similar expression because the expression above is symmetric with respect to each side of the triangle. Thus, the six points all lie on a circle with center I and radius of $\sqrt{\frac{r^2}{4} + \frac{(a+b+c)^2}{16}}$.

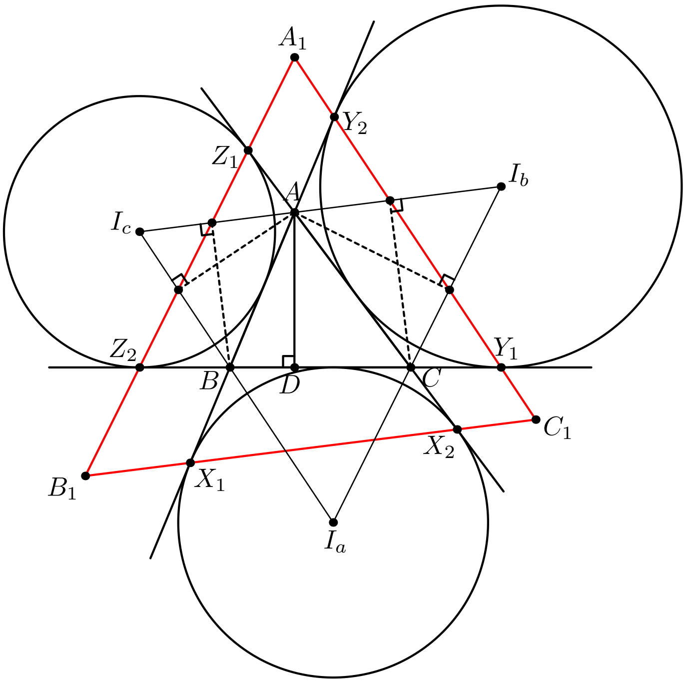
4. How does this relate to the previous section?

   Solution: The points X and Y in this section are defined for the excircle in the same way that the points X and Y in the previous section were defined for the incircle.

3.5 Altitudes and Excircles

1. * Let A_1, B_1, and C_1 be the mutual pairwise intersections of the lines connecting the tangency points as shown in the diagram. Additionally, let D be the foot of the altitude from A to BC. Prove that A_1 lies on the altitude AD.

   Hint: Use the Radical Axis Theorem.



Solution: Let N_1 and N_2 be the intersections of line Y_1Y_2 with lines I_bC and I_bA and P_1 and P_2 be the intersections of line Z_1Z_2 with lines I_cA and I_cB. Notice that since $\angle ADB = \angle AP_1B = \angle AP_2B = 90^\circ$, then quadrilateral $ADBP_2P_1$ is cyclic. Similarly, quadrilateral $ADCN_1N_2$ is cyclic. From the previous section, $N_1N_2P_1P_2$ is also a cyclic. Using the radical axis theorem on these three circles, we find that the lines AD, N_1N_2 and P_1P_2 all concur. The intersection of the latter two lines is A_1, so A_1 lies on line AD.

2. Let H be the orthocenter of the triangle ABC. Show that H is the circumcenter of the triangle $A_1B_1C_1$.

Solution: We have that:

$$\begin{aligned}
\angle B_1A_1H &= 180^\circ - \angle A_1AZ_1 - \angle AZ_1A_1 \\
&= 180^\circ - \angle DAC - (180^\circ - \angle Z_2Z_1A) \\
&= \angle ACB - 90^\circ + \left(90^\circ - \frac{\angle Z_1CZ_2}{2}\right) \\
&= \angle ACB/2
\end{aligned}$$

Additionally, $\angle A_1C_1B_1 = 180^\circ - \angle X_1Y_2C_1 - \angle C_1X_1Y_2 = 180^\circ - \left(90^\circ - \frac{\angle CBA}{2}\right) - \left(90^\circ - \frac{\angle BAC}{2}\right) = 90^\circ - \angle ACB/2$. Thus, $\angle B_1A_1H = 90^\circ - \angle A_1C_1B_1$. Similarly, one can show that $\angle A_1C_1H = 90^\circ - \angle C_1B_1A_1$ and $\angle C_1B_1H = 90^\circ - \angle B_1A_1C_1$. These angle relations uniquely define the circumcenter of triangle $A_1B_1C_1$, so the conclusion follows.

3. Prove that the length of AA_1 is r_a, the radius of the A-excircle.

Solution: By the Law of Sines, we have that $AA_1 = AY_2 \cdot \sin\angle A_1Y_2A / \sin\angle AA_1Y_2$. Using the angle chasing from the previous part, we have that $\angle A_1Y_2A = 90^\circ + \angle B/2$ and $\angle AA_1Y_2 =$

$\angle B/2$. Additionally, the first section tells us that $AY_2 = s - c$. This simplifies to $AA_1 = (s-c)/\tan\beta$, where $\beta = \angle B/2$.

Notice that $BX_1 = s-c$ and that $\angle BI_aX_1 = 90° - (90° - \beta) = \beta$. We have that $(s-c)/\tan\beta = \dfrac{BX_1}{BX_1/X_1I_a} = X_1I_a = r_a$, which finishes the proof.

Credits to Liubomir Chiriac.