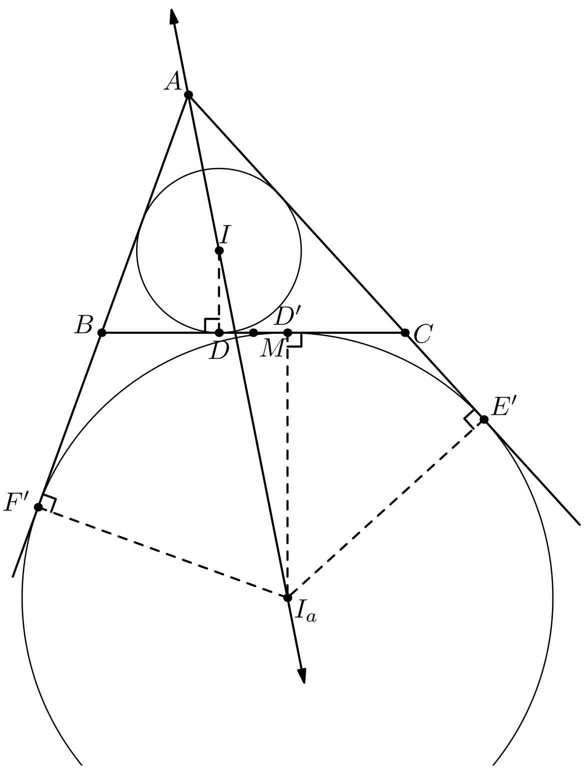
Geometry Configurations

Aaron Lin

3 Incircle and Excircle

This week's configuration covers basic properties of incircles and excircles. We covered a few of these properties in some of the classes, so it'd be a good time to put a lot of this together. To review, the **incircle** of a triangle is the circle inside of the triangle that is tangent to all three of its sides. An **excircle** of a triangle is a circle outside of the triangle that is tangent to the extensions of the sides of the triangle. Each triangle has one incircle and three excircles, so we refer to each excircle by the vertex it opposes. For example, the excircle in the diagram below is the A-excircle. The B-excircle and the C-excircle are not shown.

Often, a property that holds for a triangle's incircle usually also holds for its excircles. Thus, proofs that apply to incircles tend to have similar extensions for excircles. However, when writing up a proof, it is insufficient to say "A similar proof holds for the excircle as the incircle," but it is okay to say "A similar proof holds for the B-excircle as the A-excircle."



3.1 Basic Properties

1. Show that there is a homothety about A mapping the incircle into the A-excircle.
2. Let D be the tangency point of the incircle with side BC. Show that $BD = s - b$ and $CD = s - c$, where $BC = a$, $CA = b$, $AB = c$, and $s = (a + b + c)/2$.
3. Let D', E', and F' be the tangency points of the A-excircle with the lines BC, CA, and AB respectively. Show that $AE' = AF' = s$.
4. Prove that $BD' = s - c$ and $CD' = s - b$.
5. Show that $BD = CD'$, or that segments DD' and BC have the same midpoint.
6. Let X be the point on the incircle such that DX is a diameter. Prove that points A, X, and D' are collinear.
7. Let A_1 be the foot of the altitude from A onto BC. Show that the line $D'I$ passes through the midpoint of AA_1.

   Consider the homothety centered at point D' that maps X into A. Since D', D, and A_1 are collinear and AA_1 and XD are parallel (as they are both perpendicular to BC), then this homothety also maps point D into A_1. Thus, the midpoint of XD, point I, is mapped into the midpoint of the altitude AA_1, implying that MID' is a line.

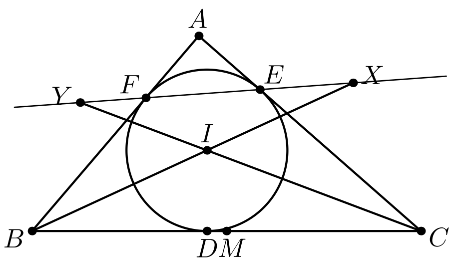
3.2 Practice Problems

Included below is one practice problem. For additional practice, refer to Yufei Zhao's handout (`http://yufeizhao.com/olympiad/three_geometry_lemmas.pdf`).

1. (All-Russian MO 2010) Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively, such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

3.3 The ABCDE Configuration

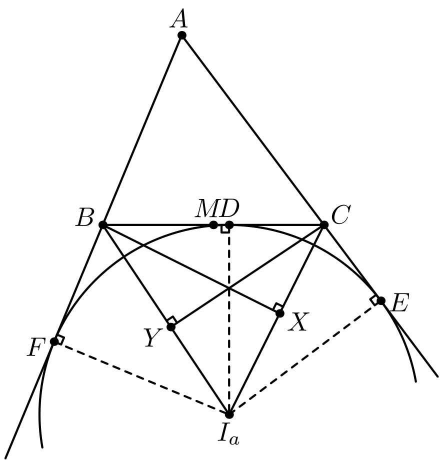
Let ABC be a triangle with incenter I.



1. Let X be the intersection of the angle bisector at B and the line EF. Show that $CDIEX$ is cyclic.

2. Points M, N, and P are defined to be the midpoints of sides BC, CA, and AB respectively. (Points N and P are not included in the diagram above.) Prove that X lies on MN and Y lies on MP.
3. Show that $BCXY$ is cyclic.
4. (USAJMO 2014) Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides BC, CA, AB and let E, F be the tangency points of γ with CA and AB, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ. Prove that:
   (a) I lies on ray CV.
   (b) Line XI bisects UV.

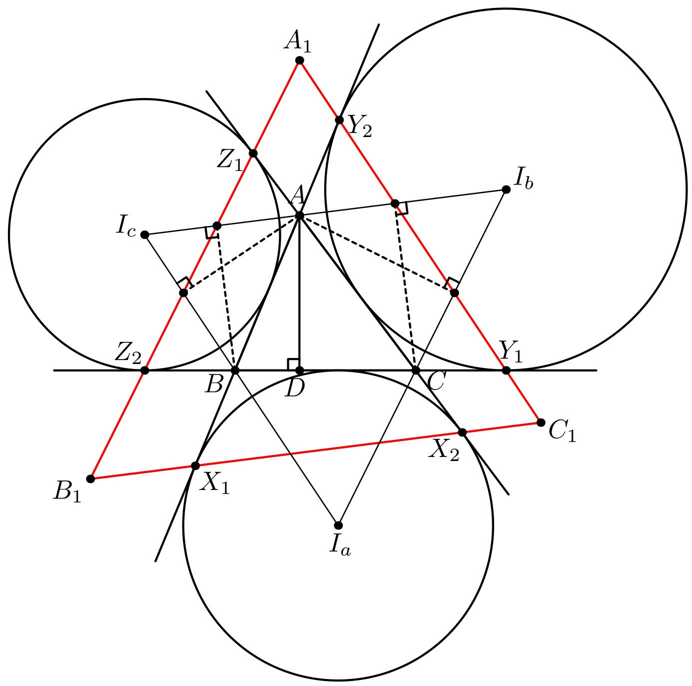
3.4 Extra Excircles



1. In the above diagram, we define D, E, and F to be the tangency points of the A-excircle with the lines BC, CA, and AB respectively. If X and Y are the feet of the altitudes from B and C onto I_aC and I_aB respectively, show that E, F, X, and Y all lie on a line.
2. Points M, N, and P are defined to be the midpoints of sides BC, CA, and AB respectively. (Points N and P not shown in the diagram above.) Prove that X lies on MP and Y lies on MN.
3. Construct the analogous points for X and Y for the B- and C-excircles. Show that these six points are concyclic.
4. How does this relate to the previous section?

3.5 Altitudes and Excircles

1. * Let A_1, B_1, and C_1 be the mutual pairwise intersections of the lines connecting the tangency points as shown in the diagram. Additionally, let D be the foot of the altitude from A to BC. Prove that A_1 lies on the altitude AD.



2. Let H be the orthocenter of the triangle ABC. Show that H is the circumcenter of the triangle $A_1B_1C_1$.

3. Prove that the length of AA_1 is r_a, the radius of the A-excircle.

Credits to Liubomir Chiriac.