Geometry Configurations

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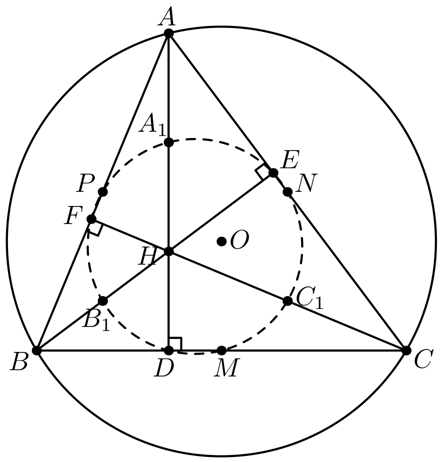
6 The Feuerbach Point

6.1 Nine-Point Circle (Feuerbach Circle)

For any triangle ABC, the three feet of its altitudes (labeled D, E, and F), the three midpoints of the sides (labeled M, N, and P), and the midpoints of segments connecting the orthocenter H to each vertex (not labeled, A_1, B_1, and C_1) are always concyclic. This circle is known as the **nine-point circle**, or the Feuerbach circle, of a triangle.



1. By angle chasing, prove that the nine points are concyclic.
2. Find a homothety that sends triangle $A_1B_1C_1$ to ABC. Where do M, N, P, D, E, F go under this homothety? Show that all nine points are concyclic.
3. Prove that the radius of the nine-point circle is half of that of the circumradius of the triangle.
4. Prove that the center of the nine-point circle is the midpoint of OH, where O is the circumcenter of the triangle.

6.2 Practice Problems

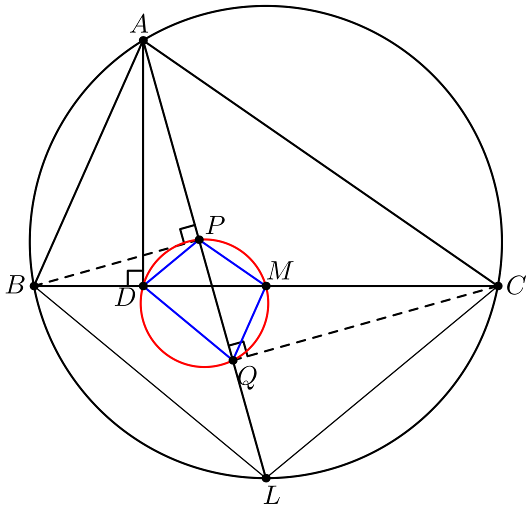
1. Let I_b and I_c be the B and C excenters of triangle ABC. Let M be the midpoint of I_bI_c. Show that $MABC$ is cyclic.
2. (NIMO 2014) Let ABC be a triangle with circumcenter O and let X, Y, Z be the midpoints of arcs BAC, ABC, ACB on its circumcircle. Let G and I denote the centroid of $\triangle XYZ$

and the incenter of $\triangle ABC$. Given that $AB = 13$, $BC = 14$, $CA = 15$, compute the ratio $\frac{GO}{GI}$.

3. Let H be the orthocenter of triangle ABC. Prove that the Euler lines of triangles ABC, HAB, HBC, and HCA all concur at a single point.
4. (Russia 1999) Let ABC be a triangle and A_1, B_1, and C_1 be the tangency points of the incircle to BC, AC, and AB respectively. Define K to be the point diametrically opposite of C_1 on the incircle. If lines A_1K and C_1B_1 intersect at point D, show that $CD = CB_1$.
5. (ISL 1959) Let $ABCD$ be a cyclic quadrilateral. Show that the centroids of the triangles ABC, CDA, BCD, DAB lie on a circle.

6.3 The "Grinberg" Configuration

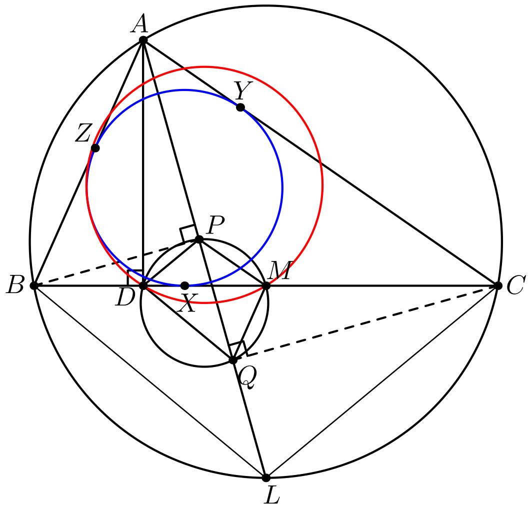
In the diagram below, let ABC be a triangle with D as the foot of the altitude from A onto BC and M the midpoint of the side BC. Consider L, the midpoint of the arc BC of the circumcircle of triangle ABC. Let P and Q be the feet of B and C respectively onto the angle bisector AL.



1. Prove that $DPMQ$ is cyclic.
2. Prove that quadrilateral $DPMQ \sim ABLC$. (Be sure to prove that your solution is sufficient for showing that the quadrilaterals are similar.)
3. Prove that $PM \parallel AC$ and $QM \parallel AB$.

6.4 Incircle and Nine-Point Circle

1. The incircle of triangle ABC is tangent to the sides at points X, Y, and Z as shown above. Prove that X is the incenter of triangle PDQ.
2. Prove that QXZ and XPY are lines.
3. Prove that the center of the circle about $DPMQ$ lies on the nine-point circle of triangle ABC.



4. (USA TST 2015) Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to BC, CA, AB at D, E, F, respectively. Denote by M the midpoint of BC. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

Credits to Darij Grinberg.

6.5 Feuerbach's Theorem

1. Prove that the incircle of $\triangle ABC$ and circle PXQ are orthogonal.
2. Using the previous part, prove that the incircle of $\triangle ABC$ and circle $DPMQ$ are orthogonal.
3. Prove **Feuerbach's theorem**, which states that the nine-point circle and the incircle are internally tangent.
4. Another proof of Feuerbach's Theorem takes advantage of inversive invariants. Locate a center and a radius at which one can invert about to preserve both the incircle and the A-excircle's respective locations.
5. Let ℓ be the common internal tangent of the incircle and the A-excircle that does not pass through point X. Prove that the inversion $\mathcal{I}$ from the previous part maps the nine-point circle into ℓ.
6. ($\sim$IMO 1982) Non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3. For each i, M_i is the midpoint of a_i, T_i is where a_i meets the incircle of $A_1A_2A_3$, and S_i is the reflection of T_i over the angle bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent at the Feuerbach point of $\triangle A_1A_2A_3$.

Credits to Jean-Louis Ayme.