Geometry Configurations

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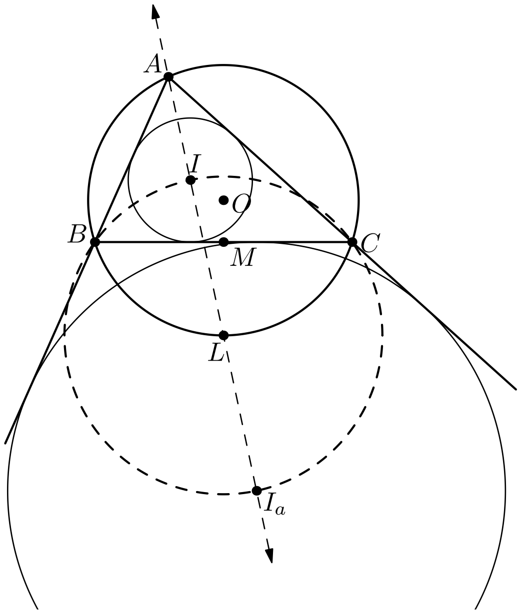
2 Midpoint of Arc (Fact 5)

2.1 Theory



In triangle ABC, let I be the incenter. Let L be the intersection of line AI and the circumcircle of ABC, such that L and A are distinct points. Define I_a to be the A-excenter of triangle ABC. (The A-excenter is where the external angle bisectors of angles B and C of triangle ABC and the internal angle bisector of angle A concur. This is also the center of the A-excircle of triangle ABC, as depicted above.) Point L has some pretty remarkable properties.

1. Prove that points A, I, L, and I_a are collinear.

   Solution: By definition, points I and I_a lie on the internal angle bisector of $\angle BAC$. Since L is the midpoint of arc BC, then $\angle BAL = \angle LAC$, so L also lies on the internal angle bisector.

2. Prove that $LB = LC$. Note that this is equivalent to points O, M, and L being on the same line, where O is the circumcenter and M is the midpoint of side BC.

   Solution: L is the midpoint of arc BC, so the lengths of arcs BL and CL are equal. Thus, the $LB = LC$.

3. Prove that L is the circumcenter of triangle BIC, or that $LB = LC = LI$.

Solution: Notice that $\angle BIL = \angle BAI + \angle IBA = \angle BCL + \frac{1}{2}\angle CBA = \angle LBC + \angle CBI = \angle LBI$. Thus, $LI = LB$. From the previous part, we can conclude that $LI = LB = LC$.

Remark: This implies that L is the circumcenter of triangle BCI.

4. Prove that L is the midpoint of II_a. Use this to show that $BICI_a$ is cyclic.

   Solution: Let X be a point on ray AB such that B is between A and X. Since $\angle IBI_a = \angle IBC + \angle CBI_a = \frac{1}{2}\angle ABC + \frac{1}{2}\angle CBX = \frac{1}{2}\angle ABX = 90°$. Similarly, $\angle ICI_a = 90°$, so IBI_aC is cyclic. Since II_a is the diameter, L is the midpoint so $LI = LI_a$.

2.2 Practice Problems (Computational)

1. (HMMT 2011) Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

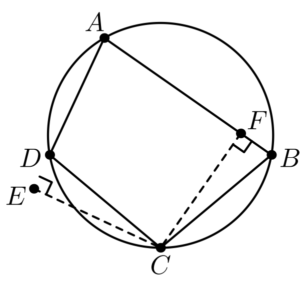
   Solution: Let E be the intersection of AC and BD. Since $BC = CD$, point C is the midpoint of arc BD. From the configuration theory, we find that $AC = AI + IC = 2 + 2 = 4$. Notice that $\angle ACB = \angle ADB$ and $\angle DBC = \angle DAC = \angle CAB$. This gives us $\triangle CEB \sim \triangle CBA$, so $CE/CB = CB/CA$, which implies that $AE = 3$ and $CE = 1$. Furthermore, the same angle relation gives us $\triangle ABC \sim \triangle AED$, so $AB/AC = AE/AD$, which implies that $AB = \frac{12}{AD}$.

   By Ptolemy's Theorem, we have that $BD \cdot AC = CD \cdot AB + CB \cdot AD$, or $BD = \frac{AB+AD}{2}$. By AM-GM inequality, $BD = \frac{AB+AD}{2} = \frac{AB+12/AB}{2} \geq \sqrt{AB \cdot 12/AB} = \boxed{2\sqrt{3}}$. This length is achievable when we let $AB = AD = 2\sqrt{3}$.

   Remark: This solution captures many of the common techniques in computational problems such as these. Both the set of three similar triangles and Ptolemy's Theorem can reveal information about the lengths of the triangle very efficiently.

2. In cyclic quadrilateral $ABCD$, AC bisects angle BAD. Point F is on AB such that CF and AB are perpendicular. If $AF = 2005$ and $AB = 2006$, find AD.

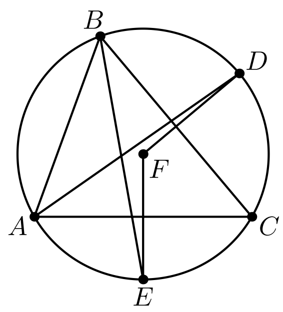
   Solution:



   Let E be the foot of C onto side AD. Since $CB = CD$, $\angle ABC = \angle EDC$, and $\angle CFB = \angle CED = 90°$, $\triangle CFB \cong \triangle CED$. Additionally, since AC is the angle bisector of $\angle EAF$, $\triangle EAC \cong \triangle FAC$. Thus, $AD = AE - DE = AF - FB = AF - (AB - AF) = 2005 - 1 = \boxed{2004}$.

3. (CHMMC Spring 2012) In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5°$, $\angle BEF = 10°$, and $AC = 3$, find the length of DF.

   Solution:

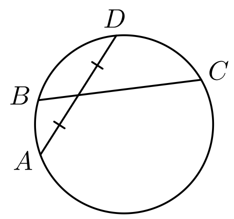
   Point F is the circumcenter of the triangle. Since points D and E lie on the triangle's respective angle bisectors and opposite perpendicular bisectors, they must be the midpoints of the arcs BC and AC respectively. Thus, we draw in the circumcircle.

   Next, we angle chase the given measures in terms of the angles of $\triangle ABC$: $\angle BEF = \angle AEF - \angle AEB = \frac{1}{2}\angle AEC - \angle ACB = 90° - \frac{1}{2}\angle B - \angle C = 10°$. Similarly, $\angle ADF = 90 - \frac{1}{2}\angle A - \angle C = 5°$. Solving these two equations with $\angle A + \angle B + \angle C = 180°$, we get that $(\angle A, \angle B, \angle C) = (70°, 60°, 50°)$. Since DF is the circumradius of $\triangle ABC$, with the Law of Sines, we find that $DF = \frac{AC}{2\sin\angle ABC} = \frac{3}{2\cdot\sqrt{3}/2} = \boxed{\sqrt{3}}$.

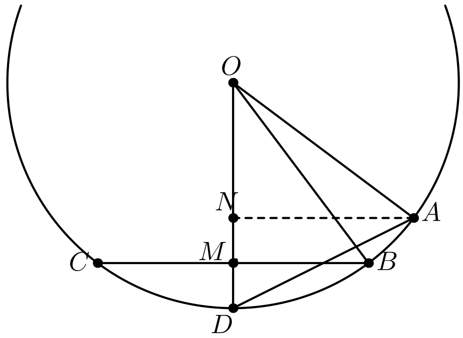
4. (SMT 2014) In cyclic quadrilateral $ABCD$, $AB = AD$. If $AC = 6$ and $AB/BD = 3/5$, find the maximum possible area of $ABCD$.

   Solution: From Ptolemy's Theorem, we have that $AB \cdot CD + AD \cdot BC = AC \cdot BD$. Since $AB/BD = AD/BD = 3/5$ and $AC = 6$, this equation simplifies to $BC + CD = 10$. By the Law of Sines, $\sin\angle BCA = \sin\angle ACD = \sin\angle ABD = AM/AB = (\sqrt{3^2 - (5/2)^2}/3) = \sqrt{11}/6$, where M is the midpoint of side BD. Thus, $[ABCD] = [ABC] + [ADC] = \frac{1}{2}BC \cdot CA\sin\angle BCA + \frac{1}{2}CD \cdot CA\sin\angle ACD = \frac{1}{2}CA \cdot \frac{\sqrt{11}}{6}(BC + CD) = \boxed{5\sqrt{11}}$. Note that the area is fixed.

5. (AIME 1983) Chords AD and BC of the same circle intersect. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. Find the sine of the minor arc AB. (Assume that A is closer to point B than to point C.)

   Solution:



Let D_1 be a point on arc BC and define another point D_2 on arc BC such that $D_1D_2 \parallel BC$. If AD_1 is bisected by chord BC, it is not difficult to see that AD_2 is also bisected by chord BC. Thus, point D must be the midpoint of arc BC and A such that the distance from D to BC the same as the distance from A to BC.

Now let O be the center of the circle, M the midpoint of BC, and N the foot of A onto OD. With Pythagorean Theorem, one can compute that $MB = 3$ and $OM = 4$. This implies that the length of MD, which is also the distance from A to BC, is $OD - OM = 1$. Thus, $ON = OM - MD = 3$, so both $\triangle ONA$ and OMB are $3-4-5$ triangles. By either using angle difference formulae or computing ratios in triangle OAB, we find that $\sin \angle AOB = \boxed{7/25}$.

6. (NIMO 2012) In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then find the length XI.

   Solution: Let E and F be the feet of I onto sides AC and AB respectively. By Pythagorean Theorem, since ID is the inradius, we have that $AE = AF = \sqrt{25^2 - 7^2} = 24$.

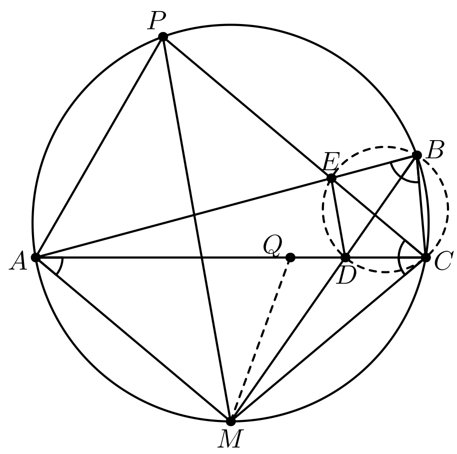
   Next, let $XI = XB = XC = x$. By Ptolemy's Theorem, we have that $AX \cdot BC = AB \cdot XC + AC \cdot XB$, or $14(x+25) = x(EC+24) + x(FB+24) = 48x + x(CD+BD) = 48x + 14x = 62x$. Thus, $x = \boxed{175/24}$.

7. (OMO 2012) Let ABC be a triangle with circumcircle ω. Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

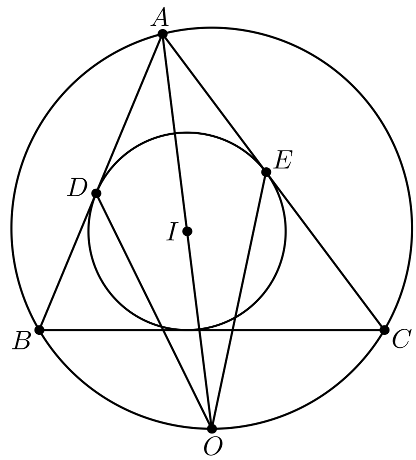
   Solution: Notice that $\angle CAM = \angle ACM = \angle MBC = \angle ABM = \angle PCA = \beta$ by cyclic quadrilateral properties, so QC is the angle bisector of $\angle PCM$. Since QM is the angle bisector of $\angle PMC$, Q is the incenter of triangle PCM. Since $\angle PCA = \angle ACM$, we have that $PA = AM = MC = PQ$, so $\angle PAQ = \angle PQA$. Additionally, we find that $\angle PQA = \angle QPC + \angle QCP = \angle MPC/2 + \angle ACP = 3\beta/2 = \angle QPM + \angle MPA = \angle QPA$. Since all three angles of $\triangle PAQ$ are equal, then it is equilateral, so $3\beta/2 = 60^\circ$ or $\angle ABC = 2\beta = \boxed{80^\circ}$.

2.3 Practice Problems (Olympiad)

1. (CGMO 2012) The incircle of ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.



Solution:



By Statement 3, O is the midpoint of arc BC on the circumcircle of ABC. Since $AD = AE$ and $\angle BAO = \angle OAC$, $\triangle OAD \cong \triangle OAE$. The conclusion follows.

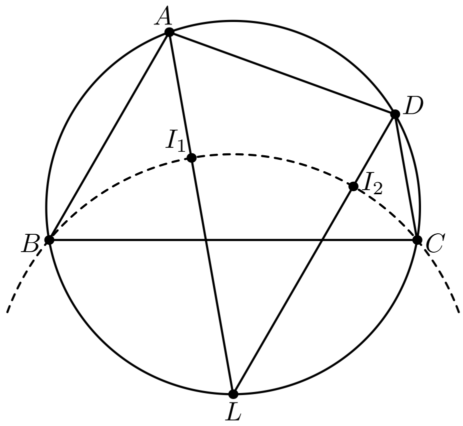
2. In cyclic quadrilateral $ABCD$, let I_1 and I_2 be the incenters of triangles ABC and DBC respectively. Show that I_1I_2CB is a cyclic quadrilateral.

   Solution 1: Construct L, the midpoint of arc BC. Note that L lies on both lines AI_1 and DI_2. From Statement 3, $LI_1 = LB = LC = LI_2$, so points B, C, I_1, and I_2 all lie on a circle with center L.

   Solution 2: Note that $\angle BI_1C = 180 - \frac{1}{2}(\angle ABC + \angle ACB) = 180 - \frac{1}{2}(180 - \angle BAC) = 180 - \frac{1}{2}(180 - \angle BDC) = 180 - \frac{1}{2}(\angle DBC + \angle DCB) = \angle BI_2C$. The motivation for this angle chase comes from knowing that for a triangle ABC with incenter I, the measure of $\angle BIC$ can be expressed solely in terms of $\angle A$.

3. In triangle ABC, the angle bisector AD (with D on side BC) hits the circumcircle of ABC at point L. Show that $\triangle LAB \sim \triangle LBD \sim \triangle CAD$.

   Solution: We have that $\angle LBD = \angle CAD = \angle LAB$ and $\angle DLB = \angle ALB = \angle ACB$. Thus, the three triangles are similar by AA.



4. (HMMT 2013) Let triangle ABC satisfy $2BC = AB+AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.

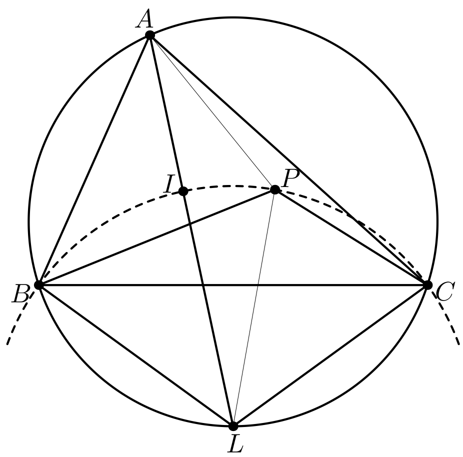
   Solution: By Ptolemy's Theorem, $AB \cdot DC + AC \cdot DB = AD \cdot BC$, or $DI(AB + AC) = AD(BC)$ since $DI = DB = DC$. Since $2BC = AB + AC$, $DI = AD/2$, so I is the midpoint of AD.

5. (ISL 2006) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

   Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

   Solution:



   Let L be the midpoint of arc BC, which is also the circumcenter of $\triangle BIC$. Adding the right hand side of the angle condition to both sides of the equation yields that $\angle ABC + \angle ACB = 2(\angle PBC + \angle PCB)$, which implies that $\angle BAC = 180 - \angle ABC - \angle ACB = 180 - 2(\angle PBC + \angle PCB) = 2\angle BPC - 180$ or $\angle BPC = 90 + \angle BAC/2$. One can easily confirm that this angle measure is equal to that of $\angle BIC$, so the angle condition restricts P to all points on arc BIC inside of $\triangle ABC$.

By the triangle inequality, we have that $AP + PL \geq AL$. Since L is the center of circle $BIPC$, $LI = LP$, so this inequality becomes $AP + PL \geq AI + IL$ or $AP \geq AI$. Equality holds when triangle LAP is degenerate, or when P lies on AL at point I.