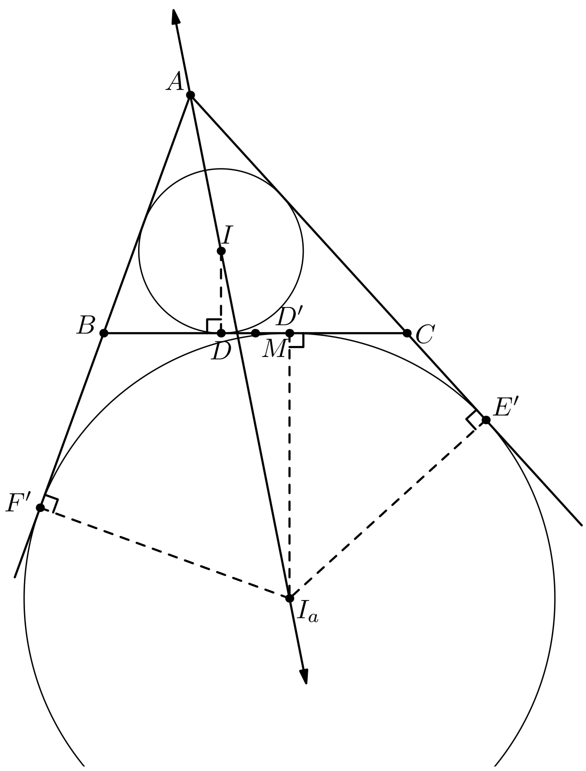
Geometry Configurations

Aaron Lin

4 Symmedians

This week's configuration is related to symmedians. It's difficult to call this a "configuration" because there are so many ways that a symmedian can show up, but there are a few basic ways that you can recognize one if they come up.

4.1 Definition of Symmedian



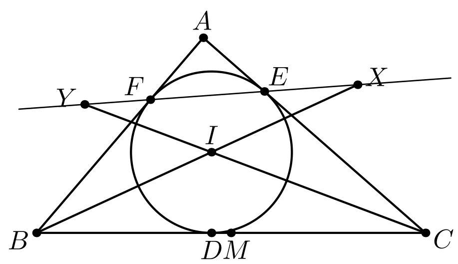
As the name implies, a symmedian is related to the median of a triangle. In the diagram above, M is the midpoint of side BC. We choose a point D on side BC such that $\angle BAD = \angle MAC$. (Cevians such as AD and AM that satisfy this angle relationship are also called "isogonal".) The segment AD is referred to as the A-symmedian of triangle ABC.

If you've ever tried to angle chase either $\angle BAM$ or $\angle CAM$ of a triangle, you'd realize that there isn't an elegant way to express either angle in terms of other angles. For this reason, the angle condition above is not always a very useful definition to use (although it does help to relate angles tied to the median), but the fact that the symmedian can be constructed in so many ways makes it an especially useful tool.

Here are some quick problems:

1. (Extended Law of Sines) In triangle ABC, show that $\frac{AC}{\sin B} = \frac{AB}{\sin C} = \frac{BC}{\sin A} = 2R$, where R is the circumradius of the triangle.
2. Points E and F are on sides AC and AB respectively of triangle ABC. Show that if $BCEF$ is cyclic, then the A-symmedian of triangle ABC passes through the midpoint of EF.
3. The A-symmedian of the triangle ABC intersects side BC at point D. Show that $BD : DC = c^2/b^2$, where b and c are the side lengths of AC and AB respectively.
4. Show that the three symmedians of a triangle concur at a point in the triangle.

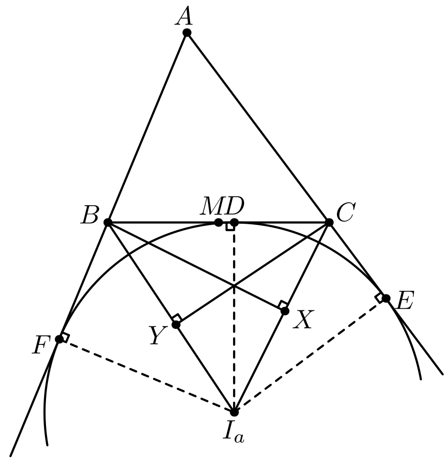
4.2 Symmedians and Tangents



Perhaps one of the most well-known constructions of the symmedian is shown above. Consider the circumcircle of the triangle ABC. Let the tangents to this circle at points B and C meet at a point X. Then line AX coincides with the symmedian of triangle ABC. There are a few proofs of this, as you can find on the first page of Yufei Zhao's handout (`http://yufeizhao.com/olympiad/geolemmas.pdf`). The simplest one, a Law of Sines computation, is not particularly enlightening, although a synthetic and project proof have both been provided.

This lemma is powerful. If you see a symmedian appear in a problem, often a nice way to approach it is to construct the corresponding tangencies to the circumcircle, as this kind of configuration is conducive to Power of a Point and other angle chasing possibilities. Furthermore, if you study poles and polars, symmedians tie in very well with certain projective ideas. (I'd still recommend having a solid synthetic foundation first before learning to abuse projective geometry.) I'm not sure if you can cite this directly on a proof; it's certainly well known, but the statement is nontrivial and the proof isn't terribly difficult either.

1. Prove that the symmedian coincides with the line AX by using Law of Sines. (You can check your work in the link above when you're done.)
2. (Harmonic Quadrilaterals) Points B and D are on circle ω, and point P is a point outside of ω such that PB and PD are tangent to the circle. A line through P intersects the circle again at two points A and C. Show that $AB/BC = AD/DC$.



4.3 Similarity Definition

Here is another very common definition of the symmedian.

1. Let P be a point in triangle ABC such that $\triangle PBA \sim \triangle PAC$. Show that AP coincides with the A symmedian.
2. Suppose that the line AP hits the circumcircle of ABC again at point D. Show that $DP = PA$.
3. Let line OP meet line BC again at point Q. Prove that QA and QD are tangents to the circumcircle of $ABDC$.
4. Let Y and Z be the points where the internal and external bisectors of angle A meet line BC respectively. (Not pictured.) Let L' be the midpoint of arc BAC (the point diametrically opposite to the Fact 5 point). Show that D, Y, and L' are collinear.
5. Show that $AYDZ$ is cyclic. (Note: This circle is the Apollonius circle of triangle ABC with respect to point A.)
6. (Vietnam 2005) On the circle ω with center O and radius R, consider two fixed points A ad B, and a variable point C. Let ω_1 be the circle through A tangent to BC at C. Similarly, let ω_2 be the circle passing through B, which is tangent to AC at C. Let D be the second point of intersection (other than C) of ω_1 and ω_2. (a) Show that line CD passes through a fixed point. (b) Show that $CD \leq R$.
7. (Source: Unknown) In triangle ABC, let E and F be the feet of the altitudes from B and C respectively. Additionally, let D be the intersection of BC and the tangent line to the circumcircle O at point A. If M is the midpoint of EF, show that $DO \perp AM$.

4.4 Practice with Symmedians

You can find more practice problems in Yufei Zhao's handout (`http://yufeizhao.com/olympiad/geolemmas.pdf`).

1. Let M be the midpoint of side BC of triangle ABC with $AB < AC$. Let D be the point inside of triangle ABC such that $\angle BAD = \angle MAC$ and $\angle DBA = \angle BCA$. Prove that $DM \parallel AC$.

2. Let P be the point in triangle ABC such that $\triangle PBA \sim \triangle PAC$. Let O be the circumcenter of triangle ABC. Show that the lines AA, BC, and OP concur. (Here, AA is the line that is tangent to the circumcircle of ABC.)

3. Let M and N be the midpoints of sides AB and AC of triangle ABC. Additionally, let O is the circumcenter of the triangle, and P be the intersection of the circumcircles of OMN and OBC. Show that AP is a symmedian.

4. Let M and N be points on sides AB and AC of triangle ABC such that $MN \parallel BC$. Let P be the intersection of lines BN and CM. The circumcircles of BMP and CNP intersect again at point Q. Show that $\angle QAB = \angle PAC$.

5. (USAMO 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

6. ($\sim$HMMT 2014) Let ABC be an acute triangle with circumcenter O. Let D be the foot of the altitude from A to BC, and E be the intersection of AO with BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Prove that AX is a symmedian of triangle ABC.