Geometry Configurations

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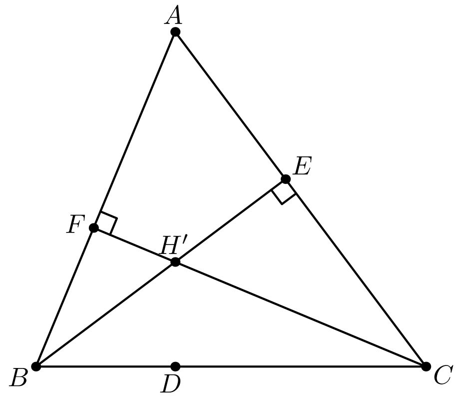
1 Orthocenter

1.1 Theory



1. The orthocenter is the point at which all of the altitudes concur. In the diagram above, H is the orthocenter of triangle ABC. However, we have not yet proved that all three altitudes always concur. We will prove that all three altitudes AD, BE, and CF concur at a single point, using what we learned about angle chasing in class today.

   We start by letting H' be the intersection of altitudes BE and CF. We want to show that A, H' and D are collinear.

   a) Find two cyclic quadrilaterals in the above diagram. Prove that they are cyclic.

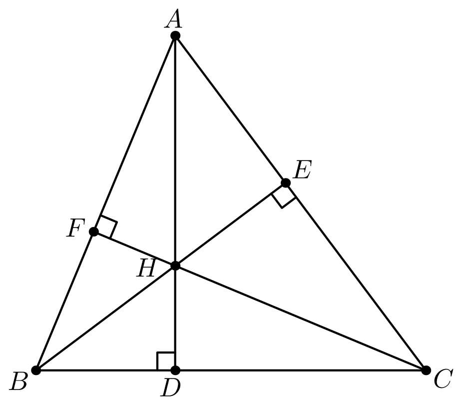
   b) Prove that $\angle H'AC = \angle EBC$. (There are multiple ways to use the two cyclic quadrilaterals from part (a) to show the desired, but we will stick with this route.)

   c) Use part (b) to show that $\angle H'AC = DAC$. Make sure you understand why this result implies that A, H', and D are collinear.

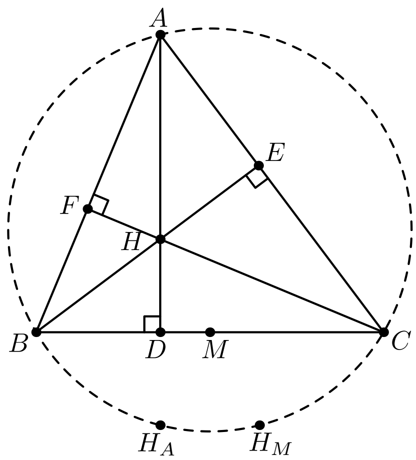
2. Find six cyclic quadrilaterals. At this point, you should be able to express every angle using the letters A, B, C, D, E, F, H in terms of the three angles of the original triangle. For a quick exercise, express angles $\angle BHC$, $\angle HDE$, and $\angle EFH$ in terms of the angles of the triangle.

3. Prove that within the set of four points H, A, B, C, each point is the orthocenter of the triangle formed by the other three points.

4. Let point H_A be the reflection of H about side BC. Show that points H_A, A, B, and C all lie on a circle.



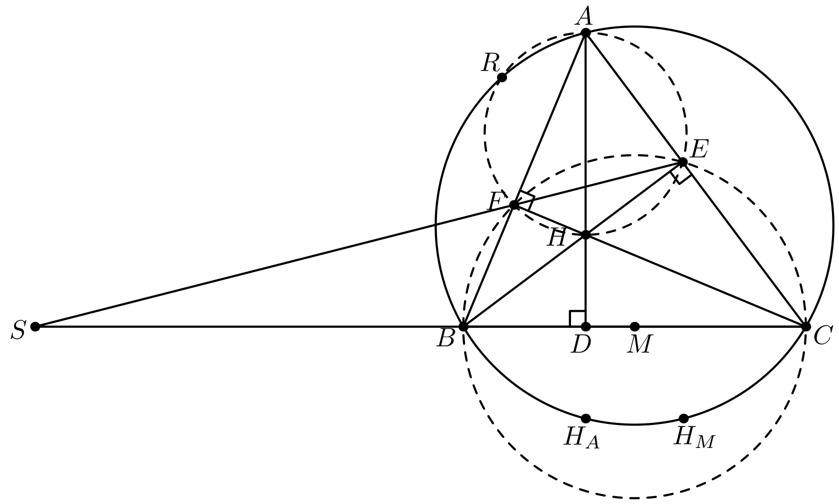
5. Show that point H is the incenter of $\triangle DEF$.



6. Let M be the midpoint of side BC. Prove that M is the circumcenter of cyclic quadrilateral $BCEF$.
7. Show that lines ME and MF are both tangent to the circumcircle of cyclic quadrilateral $AEHF$.
8. Let H_M be the reflection of H about the midpoint M. Show that the points H_{A1}, A, B, and C all lie on a circle.
9. Let O be the circumcenter of triangle ABC. Show that lines AO and AD are isogonal with respect to angle BAC. In other words, show that $\angle OAC = \angle DAB$ or $\angle OAB = \angle DAC$.
10. Show that AH_M is the diameter of the circumcircle of triangle ABC.
11. Let R be the intersection of the circumcircles of $AEHF$ and ABC. Show that points H_M, M, H, and R are collinear.
12. Let S be the intersection of lines BC and EF. Show that points A, R, and S are collinear.

1.2 Extra Practice

1. (Class) ABC is an acute triangle with O as its circumcenter. Let S be the circle through C, O, and B. The lines AB and AC meet circle S again at P and Q, respectively. Prove that



the lines AO and PQ are perpendicular.

2. (Class) Let $ABCD$ be a convex quadrilateral inscribed in a semicircle with diameter AB. The lines AC and BD intersect at E and the lines AD and BC meet at F. The line EF meets the semicircle at G and AB at H. Prove that E is the midpoint of GH if and only if G is the midpoint of the line segment FH.

3. Let $ABCD$ be a cyclic quadrilateral, and X and Y the orthocenters of triangles ABD and ACD respectively. Show that XY is parallel to BC.

4. Prove that points S, D, B, and C form a harmonic bundle; that is, show that $BS/SC = BD/DC$.

5. Let $ABCD$ be a rectangle and let P be a point on its circumcircle, different from any vertex. Let X,Y,Z, and W be the projections of P onto the lines AB, BC, CD, and DA, respectively. Prove that one of the points X,Y,Z, and W is the orthocenter of the triangle formed by the other three.

6. (MOT) Let $ABCD$ be a cyclic quadrilateral. Prove that the orthocenters of the triangles ABC, BCD, CDA, and DAB are the vertices of a quadrilateral congruent to $ABCD$.

7. (MOT) Let K,L,M, and N be the midpoints of the sides AB,BC,CD, and DA, respectively, of a cyclic quadrilateral $ABCD$. Prove that the orthocenters of the triangles AKN,BKL,CLM, and DMN are the vertices of a parallelogram.

8. In scalene triangle ABC, let the feet of the altitudes from A, B, C to their respective opposite sides be points D, E, F respectively. Let M be the midpoint of side BC, and S the intersection of lines BC and EF. Prove that line SH is perpendicular to line AM.

9. (TSTST 2012) In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.

10. (IMO 2004) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of

the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

11. * (ISL 2004) Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A. Prove that the reflection of the point G in the line AB lies on the line CF.

12. * (IMO 1985) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.

1.3 Final Notes

- The orthocenter configurations stated above come up in several surprising scenarios, so try to know these configurations forwards and backwards.
- Statements 4 and 8 are especially useful constructions in certain problems, because they relate the orthocenter to the circumcircle of a triangle. Keep your eyes open for moments when reflecting an orthocenter would be a useful construction.