Geometry Configurations

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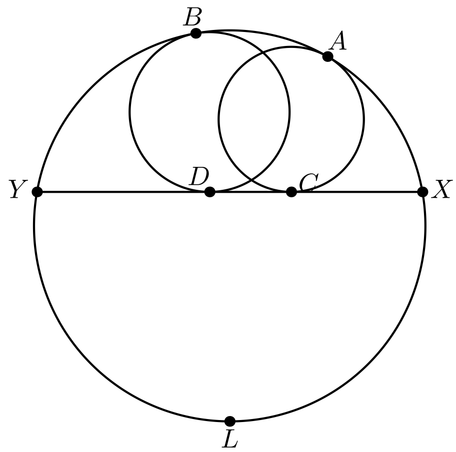
5 Mannheim's Theorem

5.1 Semi-Inscribed Circles



1. Let Ω be a circle, and XY be a chord of Ω. A circle ω_1 is tangent to XY and Ω, as shown in the diagram above. Show that L, the midpoint of the arc XY opposite of ω_1, lies on the line connecting the tangency points A and C.
2. Prove that $LX^2 = LC \cdot LA$.
3. Consider a second circle ω_2 that has tangency points B and D. Show that $ABDC$ is always cyclic.

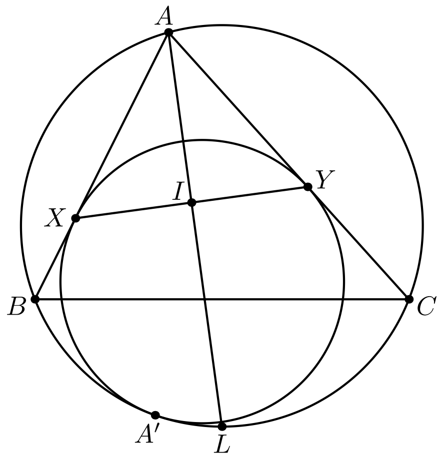
5.2 Semi-Inscribed Circles: Practice

I'm sure there are more, but I can't think of any right now.

1. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.

5.3 Mannheim's Theorem

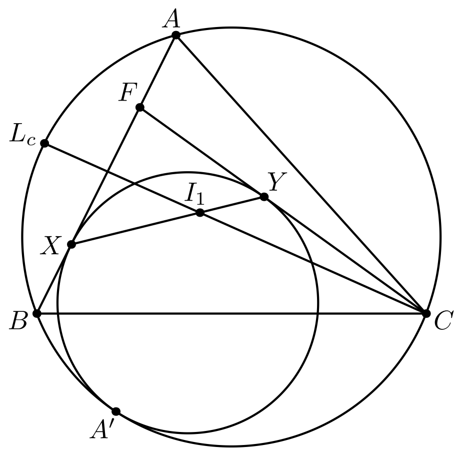
1. * Let ω_A be the circle tangent to AB and AC at X and Y respectively, and internally tangent to the circumcircle of ABC at A'. Prove that I, the incenter of triangle ABC, is the midpoint



of XY.

Hints: 1, 3

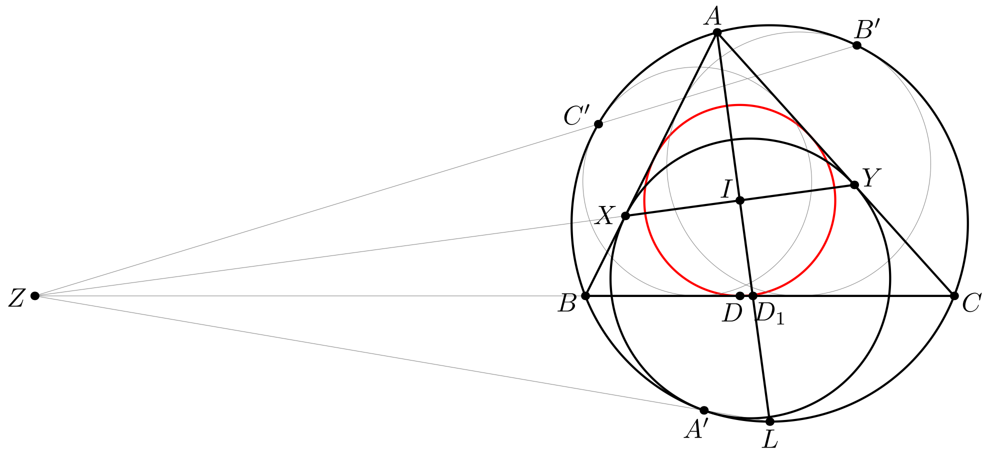
5.4 Generalization of Mannheim's Theorem



1. Take an arbitrary point F on side AB, and construct a circle tangent to sides FB and FC at X and Y respectively and the circumcircle of ABC at point A'. Let L_c be the midpoint of arc AB. Additionally, set I_1 to be the intersection of lines CL_c and XY. Prove that CYI_1A' is cyclic.
2. Prove that $\triangle A'XY \sim \triangle A'I_1C$.
3. Prove that $\triangle L_cI_1X \sim \triangle LA'I_1$.
4. Prove that I, the incenter of triangle ABC, lies on XY.

5.5 Consequences of Mannheim's Theorem

1. Prove that $\triangle A'BI \sim \triangle A'XY \sim \triangle A'IC$.



2. Prove that line $A'I$ passes through the midpoint of arc BAC.
3. Let D be the tangency point of the incircle with side BC and E be the tangency point of the A-excircle with side BC. Prove that $\triangle AEC \sim \triangle ABA'$ and $\triangle DA'C \sim \triangle BA'A$.
4. Prove that the circumcircle of $\triangle DA'C$ is tangent to side AC at point C.
5. Define points B' and C' analogously to A'. Prove that the lines AA', BB', and CC' concur at a point on the line IO, where I and O are the incenter and circumcenter of the triangle ABC respectively.

   Hints: 2
6. Let D_1 be the intersection point of the angle bisector of $\angle BAC$ with the side BC. Prove that DD_1LA', DD_1YX, and $DD_1B'C'$ are all cyclic.
7. Prove that the lines $B'C'$, XY, BC, and $A'L$ all concur at the same point Z.
8. Prove that $AZA'D_1$ is cyclic.

5.6 Hints

1. Complete the diagram.
2. Prove the following lemma: Let $\mathcal{H}_1$ be the homothety centered at point A with ratio k_1, and $\mathcal{H}_2$ be the homothety centered at point B with ratio k_2. If F is some figure, then show that $\mathcal{H}_1(\mathcal{H}_2(F))$ is also equivalent to a homothety centered at a point C on line AB with ratio of k_1k_2. (Unrelated note: You can use this to prove Menelaus' Theorem.)
3. First, prove the generalized version in the next section.