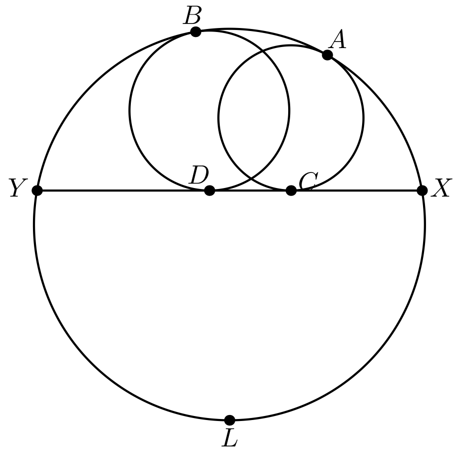
Geometry Configurations

Aaron Lin

5 Mannheim's Theorem

5.1 Semi-Inscribed Circles



1. Let Ω be a circle, and XY be a chord of Ω. A circle ω_1 is tangent to XY and Ω, as shown in the diagram above. Show that L, the midpoint of the arc XY opposite of ω_1, lies on the line connecting the tangency points A and C.

   Solution: Consider the homothety centered at point A mapping one circle to the other. The tangent to Ω at point L is parallel to the tangent XY to ω_1, so C and L are images of each other. Thus, A, C, and L are collinear.

2. Prove that $LX^2 = LC \cdot LA$.

   Solution: By the midpoint of arc configuration, $\angle YXL = \angle YAL = \angle LAX$. Since ACL is a line, triangles LCX and LXA are similar by AA. Thus, $LX/LC = LA/LX$.

   Remark: This result implies that the inversion at point L with radius LX maps ω_1 to itself.

3. Consider a second circle ω_2 that has tangency points B and D. Show that $ABDC$ is always cyclic.

   Solution: Because $LA \cdot LC = LX^2 = LB \cdot LD$, the power of point L with respect to both circles is same, so L lies on the radical axis of the two circles. Because BDL and ACL are both lines, by the Radical Axis Theorem, $ABDC$ is also cyclic.

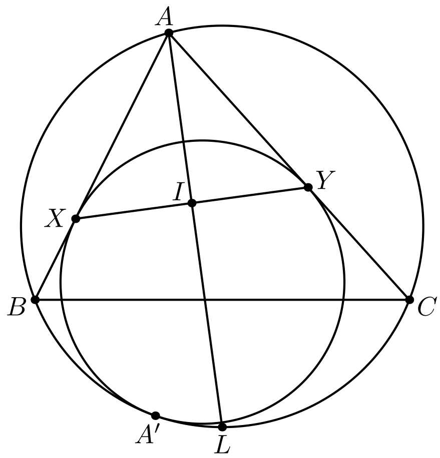
5.2 Semi-Inscribed Circles: Practice

I'm sure there are more, but I can't think of any right now.

1. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.

   Solution: Incomplete.

5.3 Mannheim's Theorem

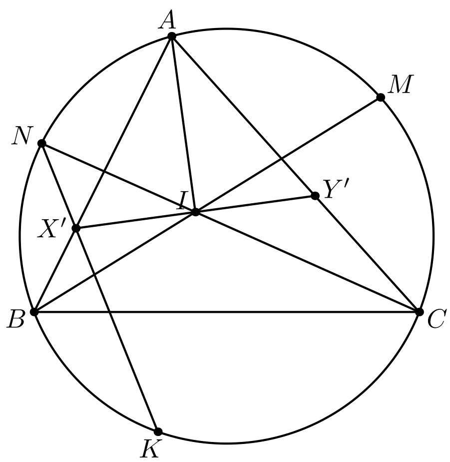


1. * Let ω_A be the circle tangent to AB and AC at X and Y respectively, and internally tangent to the circumcircle of ABC at A'. Prove that I, the incenter of triangle ABC, is the midpoint of XY.

   Hints: 1, 3

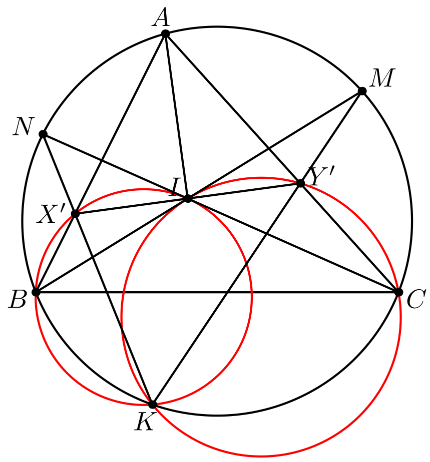
   Solution:

   Approach 1: Elementary



Denote points X' and Y' to be points on AB and AC such that I is the midpoint of $X'Y'$. Additionally, let M and N be the midpoints of the minor arcs AC and AB respectively. By the midpoint of arc configuration, BIM and CIN are both lines. Let K be the second point of intersection of line NX' with the circumcircle of ABC.

It can easily be shown that $MN \perp AI$ with angle chasing, so $X'Y' \parallel MN$. Thus, $\angle KX'Y' = \angle KNM = \angle KBM$, which implies that $KBX'I$ is cyclic. Thus, $\angle Y'IK = 180 - \angle KIX' = \angle X'BK = 180 - \angle KCA$, which implies that $IY'CK$ is cyclic as well. Thus, $\angle KY'I = \angle KCN = \angle KMN$, implying that K, Y', and M are collinear.



Consider the homothety $\mathcal{H}$ about point K that maps $X'Y'$ to NM. Because N is the midpoint of arc AB of the circumcircle of ABC, by the semi-inscribed circles lemma, there exists a circle ω tangent to the circumcircle at point K and to AB at point X'. Since $\mathcal{H}$ takes ω into the circumcircle of ABC, then $\mathcal{H}^{-1}$ sends M to point Y'. By the semi-inscribed circle lemma, ω is tangent to AC at point Y', so ω and ω_A are the same circle.

Approach 2: Pascal's Theorem

Let M and N be the midpoints of minor arcs AC and AB respectively. Consider the hexagon $ABMA'NC$. By Pascal's Theorem and semi-inscribed circles, the points $AB \cap A'N = X$, $BM \cap NC = I$, and $MA' \cap CA = Y$ are collinear. Because ω_A is tangent to sides AB and AC, $AX = AY$. As AI is the angle bisector of $\angle XAY$, I is the midpoint of XY.

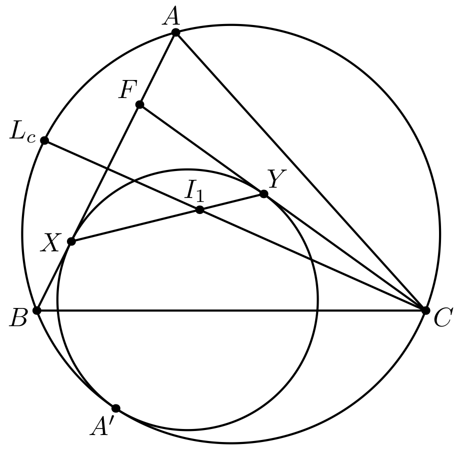
5.4 Generalization of Mannheim's Theorem

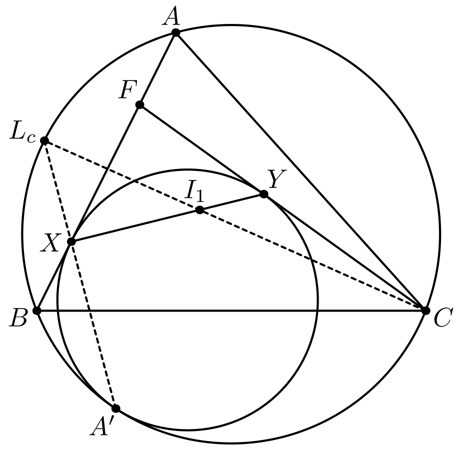
1. Take an arbitrary point F on side AB, and construct a circle tangent to sides FB and FC at X and Y respectively and the circumcircle of ABC at point A'. Let L_c be the midpoint of arc AB. Additionally, set I_1 to be the intersection of lines CL_c and XY. Prove that CYI_1A' is cyclic.

   Solution:

   There is a homothety about point A' mapping circle AXY to circle ABC. Thus, $\angle I_1YA' = \widehat{XA'}/2 = \widehat{L_cA'}/2 = \angle L_cCA'$, so CYI_1A' is cyclic.

2. Prove that $\triangle A'XY \sim \triangle A'I_1C$.





Solution: Note that $\angle I_1CA' = \angle XYA'$ and $\angle A'I_1C = \angle A'YC = \angle A'XY$, so $\triangle A'XY \sim \triangle A'I_1C$.

Remark: Note that this similarity also implies that $\triangle AXI_1 \sim \triangle AYC$. (Why?) Sometimes this duality of similar triangle pairs is helpful for chasing lengths that are otherwise difficult to access.

3. Prove that $\triangle L_cI_1X \sim \triangle LA'I_1$.

   Solution: We have $\angle L_cI_1X = \angle YI_1C = \angle YA'C = \angle I_1A'C - \angle I_1A'Y = \angle XA'Y - \angle I_1A'Y = \angle L_cA'I_1$. Because both triangles share $\angle I_1L_cA'$, by AA, the two desired triangles are similar.

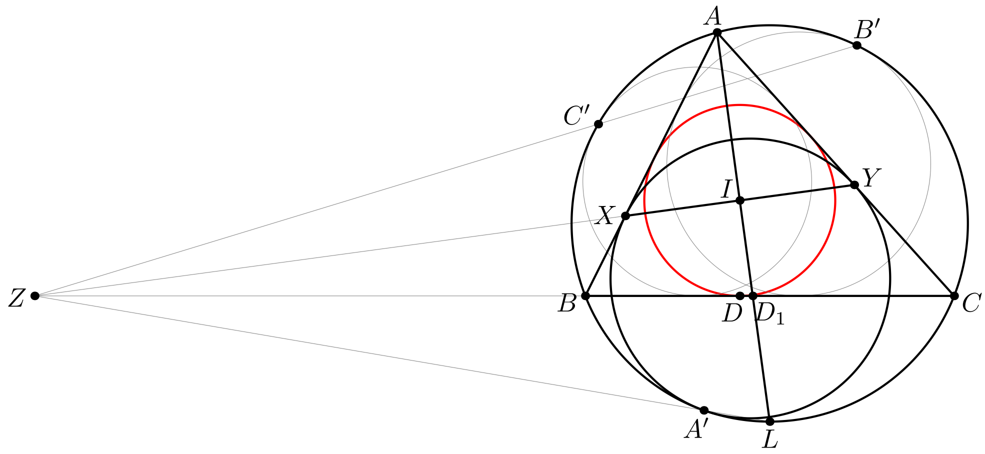
4. Prove that I, the incenter of triangle ABC, lies on XY.

   Solution: We have $L_cI_1^2 = L_cX \cdot L_cA' = L_cB^2$, which implies that $L_cI_1 = L_cB$. From the midpoint of arc configuration, I is the unique point on the angle bisector CL_c that satisfies $L_cI = L_cB$, so I and I_1 are the same point.

5.5 Consequences of Mannheim's Theorem

1. Prove that $\triangle A'BI \sim \triangle A'XY \sim \triangle A'IC$.

   Solution: See the first proof of Mannheim's Theorem.

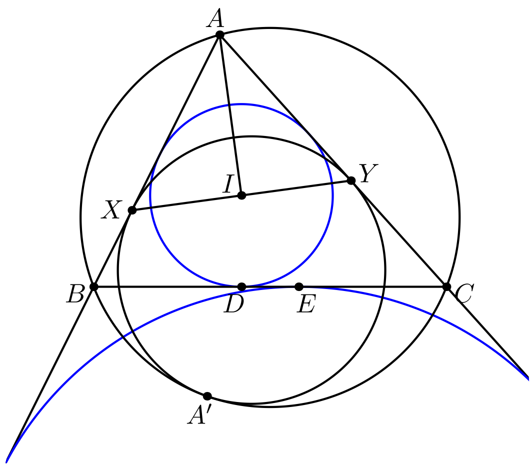


2. Prove that line $A'I$ passes through the midpoint of arc BAC.

   Solution: $\angle BA'I = \angle IA'C$, so $A'I$ is the angle bisector of $\angle BA'C$. Thus, $A'I$ intersects the midpoint of arc BAC.

3. Let D be the tangency point of the incircle with side BC and E be the tangency point of the A-excircle with side BC. Prove that $\triangle AEC \sim \triangle ABA'$ and $\triangle DA'C \sim \triangle BA'A$.

   Solution:



4. Prove that the circumcircle of $\triangle DA'C$ is tangent to side AC at point C.

5. Define points B' and C' analogously to A'. Prove that the lines AA', BB', and CC' concur at a point on the line IO, where I and O are the incenter and circumcenter of the triangle ABC respectively.

   Hints: 2

6. Let D_1 be the intersection point of the angle bisector of $\angle BAC$ with the side BC. Prove that DD_1LA', DD_1YX, and $DD_1B'C'$ are all cyclic.

7. Prove that the lines $B'C'$, XY, BC, and $A'L$ all concur at the same point Z.

8. Prove that $AZA'D_1$ is cyclic.

5.6 Hints

1. Complete the diagram.
2. Prove the following lemma: Let $\mathcal{H}_1$ be the homothety centered at point A with ratio k_1, and $\mathcal{H}_2$ be the homothety centered at point B with ratio k_2. If F is some figure, then show that $\mathcal{H}_1(\mathcal{H}_2(F))$ is also equivalent to a homothety centered at a point C on line AB with ratio of k_1k_2. (Unrelated note: You can use this to prove Menelaus' Theorem.)
3. First, prove the generalized version in the next section.