Geometry Configurations

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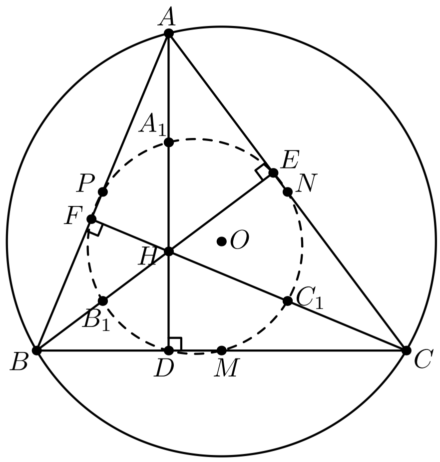
6 The Feuerbach Point

6.1 Nine-Point Circle (Feuerbach Circle)

For any triangle ABC, the three feet of its altitudes (labeled D, E, and F), the three midpoints of the sides (labeled M, N, and P), and the midpoints of segments connecting the orthocenter H to each vertex (not labeled, A_1, B_1, and C_1) are always concyclic. This circle is known as the **nine-point circle**, or the Feuerbach circle, of a triangle.



1. By angle chasing, prove that the nine points are concyclic.

   Solution: We have that $\angle PMN = \angle PAN = \angle PAD + \angle NAC = \angle PDA + \angle NDA = \angle PND$, so quadrilateral $MNPD$ is cyclic. By similar reasoning, $MNPE$ and $MNPF$ are also cyclic. We also have that $\angle FA_1E = \angle FA_1H + \angle HA_1E = 2\angle FAH + 2\angle HAE = 2\angle BAC = (90^\circ - \angle FBH) + (90^\circ - \angle HCE) = 180^\circ - \angle FDH - \angle HDE = 180^\circ - \angle FDE$, which proves that $DEFA_1$ is cyclic. Similarly, $DEFB_1$ and $DEFC_1$ are also cyclic.

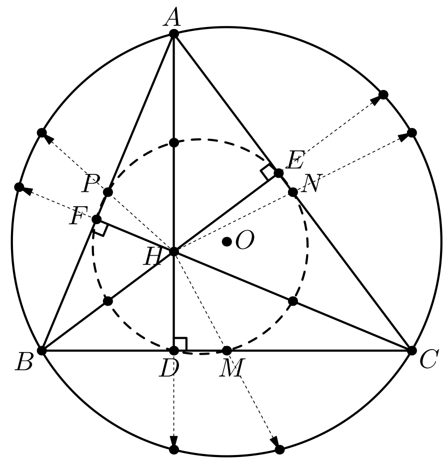
   Remark: There are many ways to arrive at the final result with angle chasing. The key is to use the triangle's symmetry to arrive at conclusions quickly. Because the cyclic condition can be proved with angle chasing, there are no problems that require the use of the nine-point circle that cannot be solved with angle chasing.

   There are several other interesting properties about the points on the nine-point circle:

   - $MDNP$ and MDB_1C_1 are isosceles trapezoids.
   - MEA_1F is a kite.
   - NC_1B_1P is a rectangle.

2. Find a homothety that sends triangle $A_1B_1C_1$ to ABC. Where do M, N, P, D, E, F go under this homothety? Show that all nine points are concyclic.

   Solution: Because A_1, B_1, and C_1 are the midpoints of AH, BH, and CH respectively,



   there exists a homothety $\mathcal{H}$ centered at point H with a scaling factor of 2 that maps $\triangle A_1B_1C_1$ to $\triangle ABC$. From orthocenter configurations, we know that the reflection of H about the sides of the triangle lie on the circumcircle, so the images of points D, E, and F under $\mathcal{H}$ map to points on the circumcircle of $\triangle ABC$. Similarly, the reflections of H about the midpoints of the sides of a triangle also lie on the circumcircle, so the images of M, N, and P under $\mathcal{H}$ also lie on the circumcircle of $\triangle ABC$. Since homotheties preserve circles, then all nine points lie on the same circle.

3. Prove that the radius of the nine-point circle is half of that of the circumradius of the triangle.

   Solution: From the homothety in the previous problem, because the scaling factor is 2, the radius of the nine-point circle is half the circumradius of triangle ABC.

4. Prove that the center of the nine-point circle is the midpoint of OH, where O is the circumcenter of the triangle.

   Solution: From the homothety in the previous problem, the center of the nine-point circle N_9 maps to the circumcenter O of $\triangle ABC$, so N_9 lies on HO. Since the scaling factor is 2 and the center of the homothety is H, then $HN_9 = HO/2$. The conclusion follows.

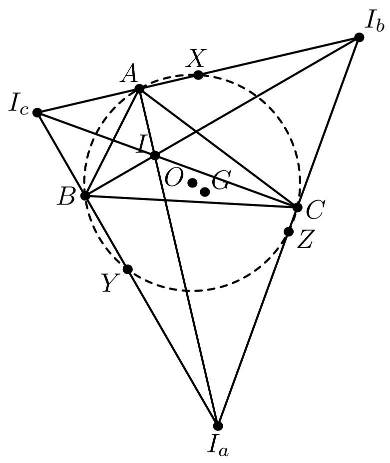
6.2 Practice Problems

1. Let I_b and I_c be the B and C excenters of triangle ABC. Let M be the midpoint of I_bI_c. Show that $MABC$ is cyclic.

   Solution: Let I_a be the A-excenter of the triangle. Notice that A, B, and C are the feet of the altitudes of triangle $I_aI_bI_c$. Thus, the circumcircle of ABC is the nine-point circle of $\triangle I_aI_bI_c$, so $MABC$ is cyclic.

2. (NIMO 2014) Let ABC be a triangle with circumcenter O and let X, Y, Z be the midpoints of arcs BAC, ABC, ACB on its circumcircle. Let G and I denote the centroid of $\triangle XYZ$ and the incenter of $\triangle ABC$. Given that $AB = 13$, $BC = 14$, $CA = 15$, compute the ratio $\frac{GO}{GI}$.

Solution:



Let I_a, I_b, and I_c be the A, B, and C excenters of triangle ABC. If X' is the midpoint of I_bI_c, from the previous problem, we know that X' lies on circle ABC. Since $\angle I_cBI_b = \angle I_cCI_b = 90^\circ$, we also have that $X'B = X'C$, so X and X' are the same points. Similarly, Y and Z are the midpoints of I_aI_c and I_aI_b respectively.

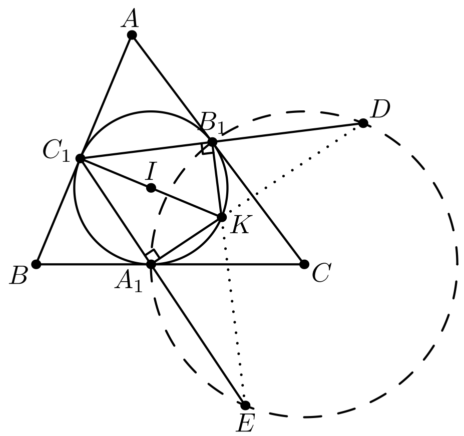
From the perspective of triangle $I_aI_bI_c$, O is its nine-point center, G is the centroid, and I is the orthocenter. We let O_1 be its circumcenter. These four points lie on the Euler line of $\triangle I_aI_bI_c$. Since $IO/OO_1 = 1$ and $IG/GO_1 = 2$, then $GO/GI = 1/4$.

3. Let H be the orthocenter of triangle ABC. Prove that the Euler lines of triangles ABC, HAB, HBC, and HCA all concur at a single point.

   Solution: One can confirm that each of the four triangles share the same nine-point circle. Thus, all four lines concur at the nine-point center.

4. (Russia 1999) Let ABC be a triangle and A_1, B_1, and C_1 be the tangency points of the incircle to BC, AC, and AB respectively. Define K to be the point diametrically opposite of C_1 on the incircle. If lines A_1K and C_1B_1 intersect at point D, show that $CD = CB_1$.

   Solution 1:



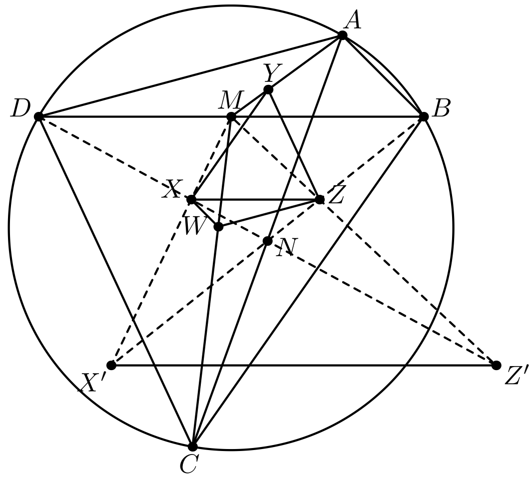
Let point E be the intersection of C_1A_1 and B_1K and I be the center of the incircle. We claim that D, C, and E are collinear. Since $A_1D \perp C_1E$ and $B_1E \perp C_1D$, K is the orthocenter of $\triangle C_1DE$. From the Orthocenter configuration handout, the intersection of the tangents to the circle with a diameter of C_1K at points A_1 and B_1 intersects at the midpoint C of DE. Thus, C is the circumcenter of quadrilateral DB_1A_1E, so $CD = CB_1$.

Solution 2: (Solution by ABCDE) Let I be the incenter. We need to show that C is the circumcenter of A_1B_1D. We have that C lies on the perpendicular bisector of A_1B_1. Thus, it suffices to show that $\angle A_1CB_1 = 2\angle A_1DB_1$.

We have that $\angle DA_1C_1$ is a right angle. Since IA_1CB_1 is cyclic, $\angle A_1DB_1 = 90° - \angle A_1C_1B_1 = 90° - \frac{1}{2}\angle A_1IB_1 = \angle 90° - \frac{1}{2}(180° - \angle ACB) = \frac{1}{2}\angle A_1CB_1$.

5. (ISL 1959) Let $ABCD$ be a cyclic quadrilateral. Show that the centroids of the triangles ABC, CDA, BCD, DAB lie on a circle.

   Solution:



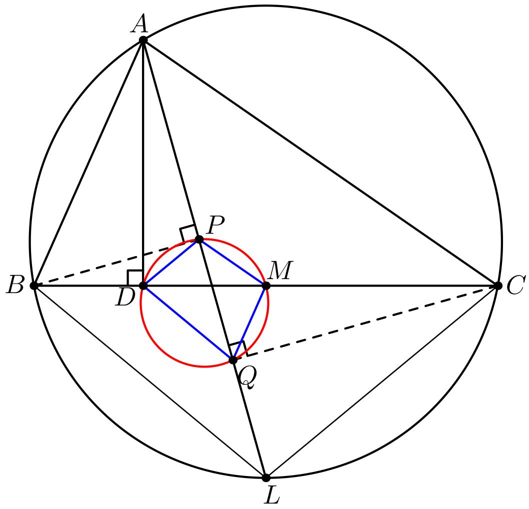
Let W, X, Y, and Z be the centroids of the triangles BCD, ACD, BAD, and BCA respectively. Furthermore, let M and N be midpoints of the diagonals BD and AC respectively.

Consider the homothety $\mathcal{M}$ about M with a ratio of 3. Let X' and Z' be the images of X and Z under this homothety. Since $NX/ND = NZ/NB = 1/3$, we have that $DB \parallel XZ \parallel X'Z'$ and $DB = 3 \cdot XZ = X'Z'$. Since $XZ/MB = 2/3$ and $X'X/X'M = 2/3$, there is a homothety about X' mapping XZ to MB, so X', Z, and B are collinear. By similar reasoning, Z', X, and D are collinear as well, so $X'Z'$ is the reflection of BD about N. Similarly, points W and Y are mapped to C and A under $\mathcal{M}$, which are reflections of each other about point N. Thus, quadrilaterals $CX'AZ'$ and $ABCD$ are congruent. Since $ABCD$, $CX'AZ'$, and $WXYZ$ are similar, $WXYZ$ is cyclic.

6.3 The "Grinberg" Configuration

In the diagram below, let ABC be a triangle with D as the foot of the altitude from A onto BC and M the midpoint of the side BC. Consider L, the midpoint of the arc BC of the circumcircle of triangle ABC. Let P and Q be the feet of B and C respectively onto the angle bisector AL.

1. Prove that $DPMQ$ is cyclic.



Solution: Notice that $ABDP$ is cyclic. Additionally, Because L is the midpoint of arc BC and M is the midpoint of side BC, $LM \perp BC$ also, so $MQLC$ is cyclic. We have $\angle MDP = \angle BAL = \angle MCL = \angle MQP$, so $DPMQ$ is cyclic.

Remark: The abundance of right angles yields many cyclic quadrilateral, several of which have not been noted in the solution above. This property can be generalized to the following problem: Let $ABCD$ be a cyclic quadrilateral. Let W and Y be the feet of points A and C respectively onto diagonal BD. Let X and Z be the feet of points B and D respectively onto diagonal AC. Prove that W, X, Y, and Z are concyclic.

2. Prove that quadrilateral $DPMQ \sim ABLC$. (Be sure to prove that your solution is sufficient for showing that the quadrilaterals are similar.)

   Solution: Note that $\angle DPQ = \angle DBA$ and $\angle DQP = \angle DCA$, implying that $\triangle DPQ \sim \triangle ABC$ by AA. Furthermore, because $\angle MPL = \angle MBL = \angle LCM = \angle PQM$, by the midpoint of arc configuration, M must be the midpoint of arc PQ of circle DPQ, which is uniquely defined. Because L is the corresponding midpoint of arc on circle ABC, the two quadrilaterals are similar.

3. Prove that $PM \parallel AC$ and $QM \parallel AB$.

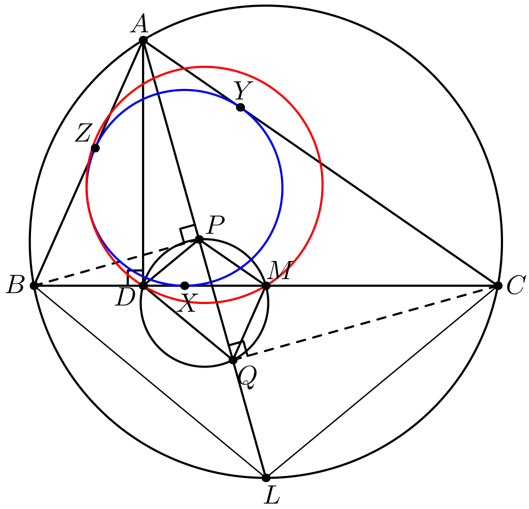
   Solution: We have $\angle PMD = \angle AQD = \angle ACD$, which implies that $PM \parallel AC$. By symmetry (or similar reasoning), one can obtain that $QM \parallel AB$.

6.4 Incircle and Nine-Point Circle

1. The incircle of triangle ABC is tangent to the sides at points X, Y, and Z as shown above. Prove that X is the incenter of triangle PDQ.

   Solution 1: Because $DQMP$ and $ACLB$ are similar, we may conclude that $\angle MDP = \angle QDM$, so the tangency point X lies on the angle bisector of $\angle PDQ$.

   Next, we compute the lengths of MP and MX separately. Without loss of generality, suppose that $AC \geq AB$. Then $XM = XC - MC = (s-c) - \frac{a}{2} = \frac{a}{2} + \frac{b}{2} - \frac{c}{2} - \frac{a}{2} = \frac{b-c}{2}$, where a, b,



and c are the side lengths of the triangle ABC and s is the semi-perimeter. To compute the length of MP, we use that the two quadrilaterals are similar:

$$\frac{MP}{PQ} = \frac{LC}{BC}$$

$$MP = \frac{(AQ - AP) \cdot LC}{2 \cdot MC}$$

$$MP = (AC \cos \angle CAL - AB \cos \angle BAL) \cdot \frac{1}{2} \cdot \frac{LC}{MC}$$

Since $\angle BCL = \angle CAL = \angle BAL$, $\cos \angle CAL = \cos \angle BAL = MC/LC$, so the final expression becomes $MX = (b - c)/2$.

By the midpoint of arc configuration, because M, the midpoint of arc PQ, is equidistant from points P and X, then X is the unique incenter of triangle DPQ.

Solution 2: (Solution by El_Ectric) Let $E = AL \cap BC$ and I be the incenter of $\triangle ABC$. Then $\triangle IEX \sim \triangle AED$ so $\frac{IE}{AE} = \frac{XE}{DE}$. Note that AE and DE are corresponding segments in the similar triangles ABC and DPQ. Since I divides AE in the same ratio that X divides DE, we may conclude that I and X are corresponding points in triangles ABC and DPQ. That is, X is the incenter of $\triangle DPQ$.

Remark: The motivation for employing the length condition from the midpoint of arc configuration to prove that X is the incenter rather than using angle-chasing methods is twofold. First, the tangency points of an incircle to the sides of its triangle often do not yield nice angles. Second, the point X is better suited for length-chasing. One can conduct a quick check before any computations to see that the lengths MP and MX are not too difficult to express in terms of fundamental triangle lengths.

2. Prove that QXZ and XPY are lines.

Solution: Notice that $\angle PXC = \angle XPD + \angle XDP = (\angle DPQ + \angle PDQ)/2 = (\angle BAC + \angle ABC)/2 = \angle YXC$. The last step of the angle chasing can be verified by expressing each of the angles involving points A, B, C, X, Y, and Z only in terms of the three angles of the triangle.

Remark: This property was explored in the Incircles/Excircles configuration.

3. Prove that the center of the circle about $DPMQ$ lies on the nine-point circle of triangle ABC.

   Solution: By definition, the nine-point circle passes through points D and M. Let O be the circumcenter of quadrilateral $DPMQ$. We have that $\angle DOM = 2\angle DQM = 2\angle ACL = 2\angle C + \angle A$.

   Let N be the midpoint of AC. Without loss of generality, again assume that $AC \geq AB$. We have that $\angle DNM = \angle DNC - \angle MNC = 2\angle DAC - \angle BAC = 180 - 2\angle C - \angle A$. Since $\angle DNM + \angle DOM = 180$ and N lies on the nine-point circle, then $DNMO$ is cyclic, so the circumcenter is also on the nine-point circle.

4. (USA TST 2015) Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to BC, CA, AB at D, E, F, respectively. Denote by M the midpoint of BC. Let Q be a point on the incircle such that $\angle AQD = 90°$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90°$ or $\angle PQF = 90°$.

   Solution: Without loss of generality, let $AB < AC$. Additionally, define D' to be the point of tangency of the A-excircle to BC and R to be the point on the incircle diametrically opposite to D. From previous statements, because $\angle DQD' = \angle DPD' = 90°$, point Q lies on circle DPD'. Furthermore, P is on line DE as shown earlier. Thus, $\angle PQE = \angle PQR + \angle RQE = \angle EDC + \angle RDE = 90°$.

Credits to Darij Grinberg.

6.5 Feuerbach's Theorem

1. Prove that the incircle of $\triangle ABC$ and circle PXQ are orthogonal.

   Solution: The two circles are centered at points I and M, and intersect at point X. Since $\angle IXM = 90°$, the two circles are orthogonal.

2. Using the previous part, prove that the incircle of $\triangle ABC$ and circle $DPMQ$ are orthogonal.

   Hint: Think about the power of I with respect to circle $DPMQ$.

   Solution:

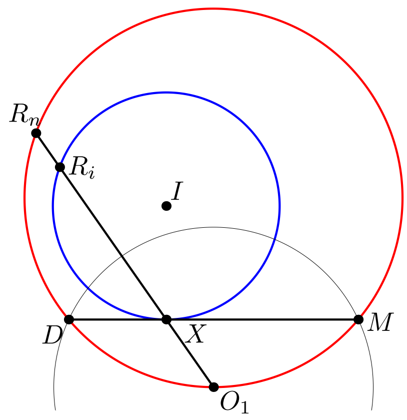
3. Point I lies on the radical axis of circles $DPMQ$ and PXQ. Let X' be a point on circle $DPMQ$ such that IX' is tangent to it. By power of a point, we have that $r^2 = XI^2 = IP \cdot IQ = X'I^2$, so the two circles are orthogonal.

4. Prove **Feuerbach's theorem**, which states that the nine-point circle and the incircle are internally tangent.

   Hint 1: Use the fact that the center of circle $DPMQ$ lies on the nine-point circle.

   Hint 2: Think about the semi-inscribed circles lemma from the Mannheim's Theorem configuration handout.

   Solution:



Let O_1 be the center of circle $DPMQ$. From a previous problem, O_1 lies on the nine-point circle, so it is the midpoint of arc DM. Let O_1X intersect the incircle again at point R_i and the nine-point circle at point R_n.

Since O_1 is the midpoint of arc DM, then $\angle DR_nO_1 = \angle DMO_1 = \angle MDO_1$, so $\triangle O_1XD \sim \triangle O_1DR_n$, which implies that $O_1X \cdot O_1R_n = O_1D^2$. Since circle $DPMQ$ and the incircle are orthogonal, then we also have that the power of O_1 with respect to the incircle is the square of its radius, or $O_1D^2 = O_1X \cdot O_1R_i$. Thus, $O_1R_n = O_1R_i$, so R_n and R_i are the same point R. Since X and O_1 are corresponding points on the incircle and circle $DPMQ$ respectively, then the homothety about point R must map one circle into the other, implying that the nine-point circle and the incircle are tangent.

Remark: Similarly, the nine-point circle is also externally tangent to each of the three excircles.

5. Another proof of Feuerbach's Theorem takes advantage of inversive invariants. Locate a center and a radius at which one can invert about to preserve both the incircle and the A-excircle's respective locations.

   Solution: Let X_1 be the tangency point of the excircle with center I_a to side BC. By the Incircle/Excircle configuration handout, M is the midpoint of XX_1. Since $\angle MXI = \angle MX_1I_a = 90°$, the circle with center of M and radius of $XX_1/2$ (also the circumcircle of PXQ) is orthogonal to both the incircle and the excircle, so this is our desired inversive circle.

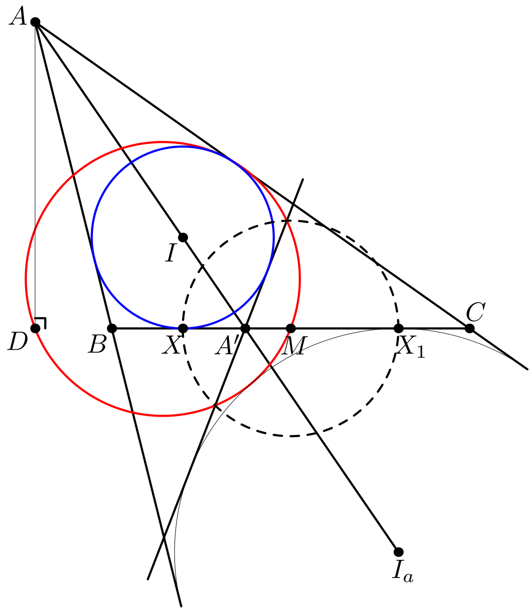
6. Let ℓ be the common internal tangent of the incircle and the A-excircle that does not pass through point X. Prove that the inversion $\mathcal{I}$ from the previous part maps the nine-point circle into ℓ.

   Hint: Let A' be the intersection of the angle bisector of $\angle BAC$ with side BC. Prove that ℓ passes through A'.

   Solution:

   Let A' be the intersection of the angle bisector of $\angle BAC$ with side BC. Since A' lies on one of the internal angle bisectors XX_1 and also lies on the angle bisector AII_a connecting the centers of the incircle and the A-excircle, then A' lies on ℓ.

   Claim 1: For a general triangle, $MA' \cdot MD = MX^2$.



Proof: The radical axis of $DPMQ$ and $PXQX_1$ is the angle bisector AII_a, so A' has an equal power with respect to both circles. This implies that $A'X \cdot A'X_1 = A'M \cdot A'D$. which can be reduced to $MX_1/MA' = XD/XA'$.

Now let X_d be the point diametrically across from X on the incircle. From a homothety at point A mapping the incircle to the excircle, points A, X_d, and X_1 are collinear. Since $\triangle X_1XX_d \sim \triangle X_1DA$ and $\triangle A'XI \sim \triangle A'DA$, then we can write the following relations:

$$\frac{XX_d}{AD} = \frac{XX_1}{DX_1} \text{ and } \frac{IX}{AD} = \frac{A'X}{A'D}$$

Since $XX_d = 2 \cdot IX$ and $XX_1 = 2 \cdot MX$, the above can be rewritten as:

$$\frac{IX}{AD} = \frac{MX}{DX_1} = \frac{A'X}{A'D} = \frac{MX}{DM + MX_1}$$

Reciprocating and subtracting 1 from each of the two rightmost fractions yields $MD/MX_1 = XD/XA'$. Thus, $MD/MX_1 = MX_1/MA'$, and since $MX = MX_1$, we get that $MA' \cdot MD = MX^2$, as claimed.

Claim 2: ℓ_1, the line through point M tangent to the nine-point circle, is parallel to ℓ.

Proof: Without loss of generality, assume that $AB \leq AC$. Let M_1 and M_2 be a points on ℓ_1 and ℓ respectively, each on the same side of BC as the A-excircle. Then $\angle M_1MB = \frac{1}{2}\angle MN_9D = 180° - 2\angle C - \angle A$ and $\angle M_2A'B = 180° - 2\angle BA'A = 180 - 2\angle C - \angle A$. Thus, $\ell \parallel \ell_1$.

Putting this together, we know that $\mathcal{I}$ must map the nine-point circle into a line parallel to ℓ_1. Furthermore, since the radius of inversion is MX, from Claim 1, D maps to point A'. Thus, the image of the nine-point circle is the line through A' parallel to ℓ_1, which by Claim 2, must be ℓ.

Inversions preserve tangencies, so since ℓ is tangent to both the incircle and the excircle, then the nine-point circle is tangent to both as well.

7. ($\sim$IMO 1982) Non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3. For each i, M_i is the midpoint of a_i, T_i is where a_i meets the incircle of $A_1A_2A_3$, and S_i is the reflection of T_i over the angle bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent at the Feuerbach point of $\triangle A_1A_2A_3$.

   Solution: Because this problem was presented in *Week 3: Homothety* class, the full solution for proving that the three lines are concurrent will not be written out in detail here. The idea is to show that $S_iS_j \parallel M_iM_j$ for all $i \neq j \in \{1, 2, 3\}$, showing that there is a homothety mapping $\triangle S_1S_2S_3$ into $\triangle M_1M_2M_3$.

   Since the circumcircle of $\triangle S_1S_2S_3$ is the incircle and the circumcircle of $\triangle M_1M_2M_3$ is the nine-point circle, the three lines are concurrent at the center of homothety mapping the incircle to the nine-point circle. Since the two circles are tangent, the center of homothety is the Feuerbach point.

   Remark: One can combine the ideas of the two proofs of Feuerbach's theorem with the setup in this problem to get a third proof. The point S_1 defined in this problem lies on line ℓ from the second proof. From Claims 1 and 2 from the second proof, we can also deduce that M is the midpoint of the arc of the nine-point circle cut off by line ℓ. Since the incircle is tangent to ℓ and is orthogonal to the circle with center M and radius of MX, the idea from the first proof can be applied to show that the two circles are tangent.

Credits to Jean-Louis Ayme.