# Geometry Configurations

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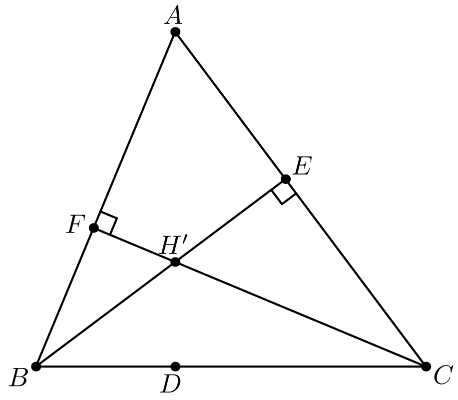
## 1 Orthocenter

### 1.1 Theory




1. The orthocenter is the point at which all of the altitudes concur. In the diagram above, $H$ is the orthocenter of triangle $ABC$. However, we have not yet proved that all three altitudes always concur. We will prove that all three altitudes $AD$, $BE$, and $CF$ concur at a single point, using what we learned about angle chasing in class today.

   We start by letting $H'$ be the intersection of altitudes $BE$ and $CF$. We want to show that $A$, $H'$ and $D$ are collinear.

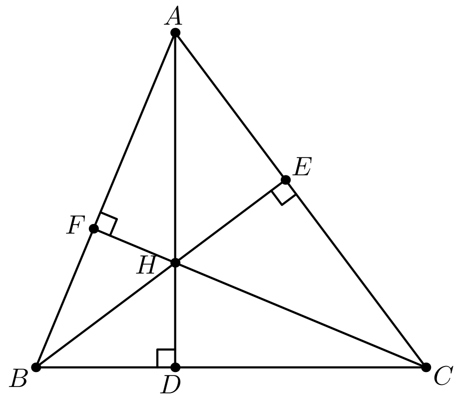
   a) Find two cyclic quadrilaterals in the above diagram. Prove that they are cyclic.

   b) Prove that $\angle H'AC = \angle EBC$. (There are multiple ways to use the two cyclic quadrilaterals from part (a) to show the desired, but we will stick with this route.)

   c) Use part (b) to show that $\angle H'AC = DAC$. Make sure you understand why this result implies that $A$, $H'$, and $D$ are collinear.

   **Solution:** (a) Since $\angle BFC = \angle BEC = 90°$, quadrilateral $BFEC$ is cyclic. Furthermore, from the same two right angles, we can also deduce that $AFH'E$ is cyclic. (b) Thus, $\angle H'AC = \angle H'FE = \angle CBE = 90 - \angle ACB = \angle DAC$. (c) This implies that points $H'$, $A$, and $D$ are collinear, so altitude $AD$ passes through the intersection of the other two altitudes $BE$ and $CF$.

2. Find six cyclic quadrilaterals. At this point, you should be able to express every angle using the letters $A$, $B$, $C$, $D$, $E$, $F$, $H$ in terms of the three angles of the original triangle. For a quick exercise, express angles $\angle BHC$, $\angle HDE$, and $\angle EFH$ in terms of the angles of the triangle.




**Solution:** We can conclude that $AFHE$, $BDHF$, and $CEHD$ are all cyclic. Additionally, $BFEC$, $CDFA$, and $AEDB$ are cyclic as well. There are many ways to arrive at the requested angle measures. One can find that $\angle BHC = 180 - \angle A$, $\angle HDE = 90 - \angle A$, and $\angle EFH = \angle 90 - \angle C$.

3. Prove that within the set of four points $H$, $A$, $B$, $C$, each point is the orthocenter of the triangle formed by the other three points.

   **Solution:** By definition $H$ is the orthocenter of triangle $ABC$. For triangle $HBC$, the three altitudes are $HD$, $BF$, and $CE$, which concur at point $A$. Similar arguments can be made for triangles $HCA$ and $HAB$.

4. Let point $H_A$ be the reflection of $H$ about side $BC$. Show that points $H_A$, $A$, $B$, and $C$ all lie on a circle.

   **Solution:** Since $\angle BH_AC = \angle BHC = \angle EHF = 180 - \angle BAC$, quadrilateral $ABH_AC$ is cyclic.

   **Remark:** This is result is a very handy lemma to know.

5. Show that point $H$ is the incenter of $\triangle DEF$.

   **Solution:** Since $\angle EDH = \angle ECH = 90 - \angle BAC = \angle HBF = \angle HDF$, line $DH$ is the angle bisector of $\angle EDF$. By similar reasoning, lines $EH$ and $FH$ are the angle bisectors of $\angle DEF$ and $\angle EFD$ respectively, so $H$ is the incenter of triangle $DEF$.

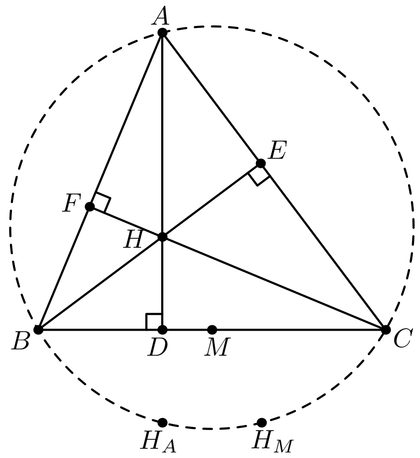
   **Remark:** Points $A$, $B$, and $C$ are the excenters of triangle $DEF$.

6. Let $M$ be the midpoint of side $BC$. Prove that $M$ is the circumcenter of cyclic quadrilateral $BCEF$.

   **Solution:** Both $\triangle BFC$ and $\triangle BEC$ are right triangles, so the circumcenter of $BEFC$ is the midpoint of the hypotenuse, which is $M$.

7. Show that lines $ME$ and $MF$ are both tangent to the circumcircle of cyclic quadrilateral $AEHF$.

   **Solution:** With some angle chasing, one can find that $\angle MEH = 90 - \angle MEC = 90 - \angle MCE = \angle DAC$, so the measure of $\angle MEH$ is equal to the measure of arc $EH$ of circle $AFHE$. This is sufficient for showing that $ME$ is tangent to the circle. A similar argument can be made for $MF$.




8. Let $H_M$ be the reflection of $H$ about the midpoint $M$. Show that the points $H_{A1}$, $A$, $B$, and $C$ all lie on a circle.

   **Solution:** Since $H_M$ is the reflection of $H$ about the midpoint of $BC$, $HBH_MC$ is a parallelogram. Thus, $\angle BH_MC = \angle BHC = 180 - \angle A$, with the latter angle condition established earlier. Thus, $ABH_MC$ is cyclic.

   **Remark:** Sometimes midpoints can be very difficult to work with in angle-heavy diagrams. Reflecting the correct point about a midpoint to create a parallelogram can sometimes be the key to a solution!

9. Let $O$ be the circumcenter of triangle $ABC$. Show that lines $AO$ and $AD$ are isogonal with respect to angle $BAC$. In other words, show that $\angle OAC = \angle DAB$ or $\angle OAB = \angle DAC$.

   **Solution:** Since $OA = OC$, $\angle OAC = \angle OCA$, so $\angle OAC = (180 - \angle AOC)/2 = 90 - (2 \cdot \angle ABC)/2 = 90 - \angle ABC = \angle BAD$, by subtending arc angle relations. The conclusion follows.

10. Show that $AH_M$ is the diameter of the circumcircle of triangle $ABC$.

    **Solution:** From the previous two parts, we have that $\angle H_MAC = \angle H_MBC = \angle HCB = \angle BAD = \angle OAC$, which implies that points $O$, $A$, and $H_M$ are collinear. Since $O$ is the center of the circle and both $A$ and $H_M$ are on the circle, then $AH_M$ is a diameter.

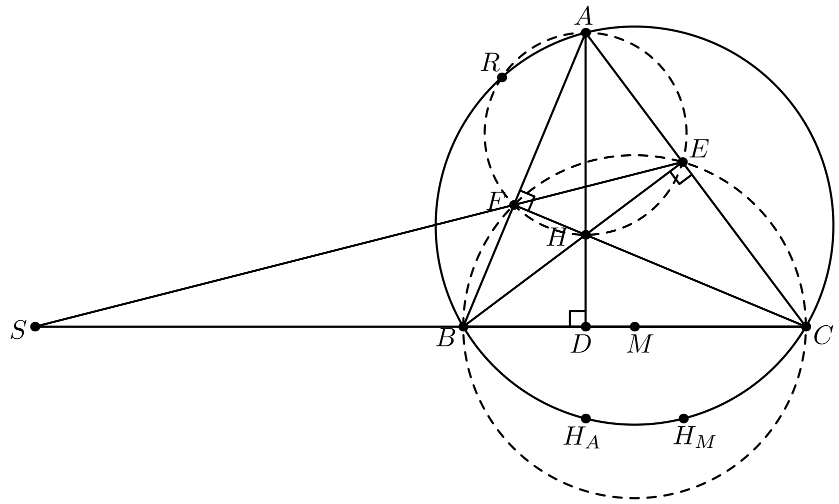
    **Remark:** Because $O$ is the midpoint of $AH_M$, with similar triangles, we can conclude that $AH = 2 \cdot OM$.

11. Let $R$ be the intersection of the circumcircles of $AEHF$ and $ABC$. Show that points $H_M$, $M$, $H$, and $R$ are collinear.

    **Solution:** By definition, $H_M$, $M$, and $H$ are collinear. Since $\angle AEH = \angle AFH = 90°$, $AH$ is the diameter of circle $AFHE$, so $\angle HRA = 90°$. Furthermore, since $AH_M$ is the diameter of circle $ABC$, we also have that $\angle H_MRA = 90°$. Thus, points $H_M$, $M$, $H$, and $R$ are all collinear.

12. Let $S$ be the intersection of lines $BC$ and $EF$. Show that points $A$, $R$, and $S$ are collinear.

    **Hint:** Power of a Point.




**Solution:** Suppose that line $AS$ intersects circle $ABC$ again at point $R_1$ and circle $AFHE$ at $R_2$. By Power of a Point on circle $BFEC$, we have that $SF \cdot SE = \angle SB \cdot SC$. The left hand side represents the power of $S$ with respect to circle $AFHE$ and the right hand side represents the power of $S$ with respect to circle $ABC$. Since the two powers are equal, we have that $SA \cdot SR_2 = SA \cdot SR_1$, so $SR_1 = SR_2$. This implies that $R_1$ and $R_2$ are the same point, so $R_1$ and $R_2$ must be the intersection of the two circles, which is point $R$ by definition.

**Remark:** The above proof was essentially a sketch of the proof of the radical axis theorem.

### 1.2 Extra Practice

1. (Class) $ABC$ is an acute triangle with $O$ as its circumcenter. Let $S$ be the circle through $C$, $O$, and $B$. The lines $AB$ and $AC$ meet circle $S$ again at $P$ and $Q$, respectively. Prove that the lines $AO$ and $PQ$ are perpendicular.

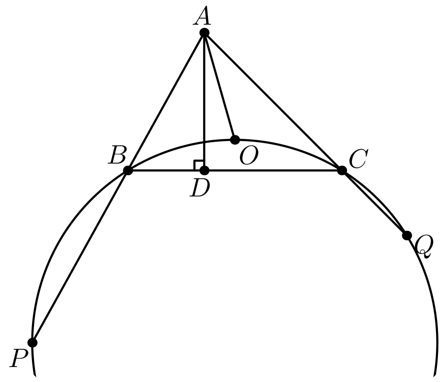
   **Solution:** Let $D$ be the foot of $A$ onto side $BC$. Since $BCQP$ is cyclic, $\angle ABC = \angle PQA$ and $\angle ACB = \angle QPA$, $\triangle ABC \sim \triangle AQP$. In this pair of triangles, sides $BC$ and $QP$ correspond to each other. Since $AD$ and $AO$ are isogonal with respect to $\angle BAC$, lines $AD$ and $AO$ also correspond to each other in this set of similar triangles. Since $AD$ is the altitude of $\triangle ABC$, line $AO$ coincides with the altitude of $\triangle AQP$ as well, so $AO \perp QP$.

   **Remark:** The angle chasing solution covered in class is logically equivalent to the reasoning to the proof presented above. Note that this problem can be generalized beyond just $AD$ and $AO$ to any pair of isogonal lines.

2. (Class) Let $ABCD$ be a convex quadrilateral inscribed in a semicircle with diameter $AB$. The lines $AC$ and $BD$ intersect at $E$ and the lines $AD$ and $BC$ meet at $F$. The line $EF$ meets the semicircle at $G$ and $AB$ at $H$. Prove that $E$ is the midpoint of $GH$ if and only if $G$ is the midpoint of the line segment $FH$.

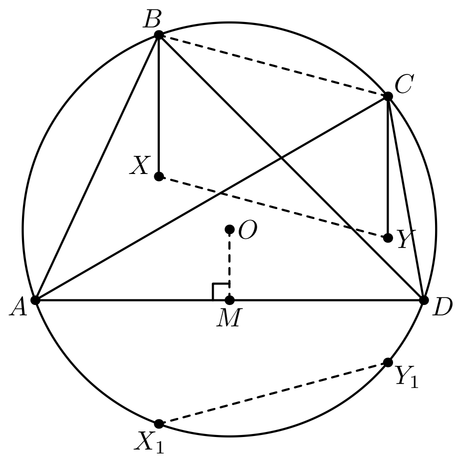
   **Solution:** The cleanest solution was presented in class. However, take note that the configuration is very similar.

3. Let $ABCD$ be a cyclic quadrilateral, and $X$ and $Y$ the orthocenters of triangles $ABD$ and $ACD$ respectively. Show that $XY$ is parallel to $BC$.




**Solution 1:** Let $X_1$ and $Y_1$ be the reflections of $X$ and $Y$ about line $AD$. From an earlier lemma, $X_1$ and $Y_1$ must lie on the circle. Since $BX_1 \perp AD$ and $AD \perp CY_1$, lines $BX_1$ and $CY_1$ are parallel, so $BX_1Y_1C$ is an isosceles trapezoid. To finish, we combine the fact that $BX \parallel CY$ with either (a) that $XYY_1X_1$ is also an isosceles trapezoid so $XY \parallel BC$ or (b) that $XY = X_1Y_1 = BC$ to conclude that $BXYC$ is a parallelogram.

**Solution 2:** Let $O$ be the center of the circle and $M$ be the midpoint of $AD$. From an earlier lemma, looking at triangle $ABD$, we find that $BX = 2 \cdot OM$. Similarly, $CY = 2 \cdot OM$, so $BX = CY$. Since $BX \perp AD$ and $AD \perp CY$, $BX \parallel CY$, so $BXYC$ is a parallelogram.




4. Prove that points $S$, $D$, $B$, and $C$ form a harmonic bundle; that is, show that $BS/SC = BD/DC$.

**Solution 1:** This ratio seems to be expressed in terms of lengths that do not seem too difficult to fetch. Thus, we will aim to equate each length in the desired equation into more primitive lengths. We start with lengths $SB$ and $SC$ because point $S$ is more distantly tied to triangle $ABC$ than are the lengths $BD$ and $CD$.

Immediately, from cyclic quadrilateral $BCEF$, we see spot two pairs of similar triangles: $\triangle SBF \sim \triangle SEC$ and $\triangle SBE \sim \triangle SFC$. From these two, we get two relations:

$$\frac{FS}{SC} = \frac{BF}{CE}$$
$$\frac{BS}{SF} = \frac{BE}{CF}$$

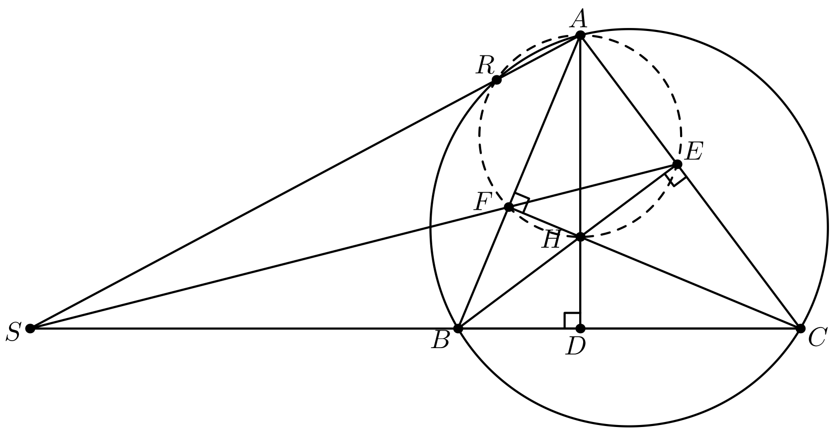
Multiplying the two equations, all lengths involving point $S$ disappear except the desired terms, leaving:

$$\frac{BS}{SC} = \frac{BF}{CF} \cdot \frac{BE}{CE}$$

Since $\triangle BFC \sim \triangle BDA$ and $\triangle CEB \sim \triangle CDA$, the right hand side becomes:

$$\frac{BS}{SC} = \frac{BD}{DA} \cdot \frac{AD}{DC} = \frac{BD}{DC}.$$

**Solution 2:**




Define point $R$ as in the configurations. Consider cyclic quadrilateral $RFHE$. From previous configurations, we know that points $M$, $H$, and $R$ are collinear and that lines $ME$ and $MF$ are tangent to the circumcircle. Thus, $\angle MFH = \angle MRF$ and $\angle MEH = \angle MRE$, which implies that $\triangle MFH \sim \triangle MRF$ and $\triangle MEH \sim \triangle MRE$. From these, we can write the ratios:

$$\frac{HF}{FR} = \frac{MH}{MF} \text{ and } \frac{HE}{ER} = \frac{MH}{ME}$$

Since the right hand sides of the two expressions are equal, we know that $\frac{HF}{FR} = \frac{HE}{ER}$. We call cyclic quadrilaterals that satisfy this length relation *harmonic*.

Next, note that all four of $SRFB$, $BFHD$, $DHEC$, and $CERS$ are cyclic, which can each be shown with quick angle chasing. Additionally, note that mutually intersecting sides (the radical axes) all intersect at point $A$. With this, we can write many similar triangle relations:

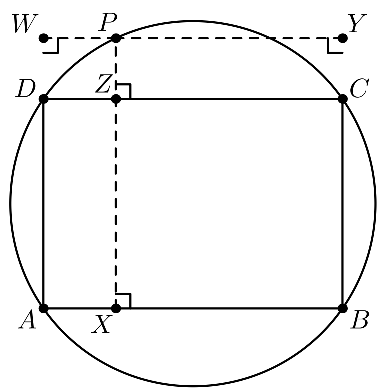
$$\left.\begin{aligned}\frac{RF}{SB}&=\frac{AR}{AB}\\ \frac{BD}{FH}&=\frac{AD}{AF}\\ \frac{EH}{CD}&=\frac{AH}{AC}\\ \frac{SC}{ER}&=\frac{AS}{AE}\end{aligned}\right\}\xrightarrow{\text{multiply}}\left(\frac{RF}{FH}\cdot\frac{HE}{ER}\right)\left(\frac{SC}{SB}\cdot\frac{BD}{CD}\right)=\frac{AR\cdot AS}{AF\cdot AB}\cdot\frac{AD\cdot AH}{AE\cdot AC}$$

By Power of a Point, the right hand side is equal to 1. Since the left argument on the left hand side is also equal to 1, we obtain our desired expression.

**Remark:** The second solution traces the reasoning used in projective geometry and cross ratios. Notice that line segment $SBDC$ can be mapped into cyclic quadrilateral $RFHE$ by projecting each point onto the circle through point $A$. We can also project line segment $SBDC$ into $SFXE$ through point $A$, where $X$ is the intersection of $AD$ and $EF$. Although the solution provided for this problem is insufficient for proving that the *cross ratio* is preserved through projections such as these, such invariant is the foundation of the study of projective geometry.

5. Let $ABCD$ be a rectangle and let $P$ be a point on its circumcircle, different from any vertex. Let $X$,$Y$,$Z$, and $W$ be the projections of $P$ onto the lines $AB$, $BC$, $CD$, and $DA$, respectively. Prove that one of the points $X$,$Y$,$Z$, and $W$ is the orthocenter of the triangle formed by the other three.

   **Solution:**

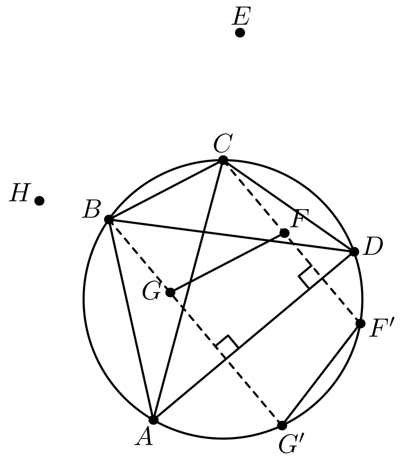



   Without loss of generality, assume that point $P$ is on arc $CD$ of the circle. From cyclic quadrilateral $ADPC$, we have $\angle WYZ = \angle ZCP = \angle DAP = \angle WXP = 90 - \angle XWY$, so $YZ \perp XW$. Furthermore, $XZ \perp WY$ is a given, so $Z$ is the orthocenter of $WXY$.

6. (MOT) Let $ABCD$ be a cyclic quadrilateral. Prove that the orthocenters of the triangles $ABC$, $BCD$, $CDA$, and $DAB$ are the vertices of a quadrilateral congruent to $ABCD$.

   **Solution:**

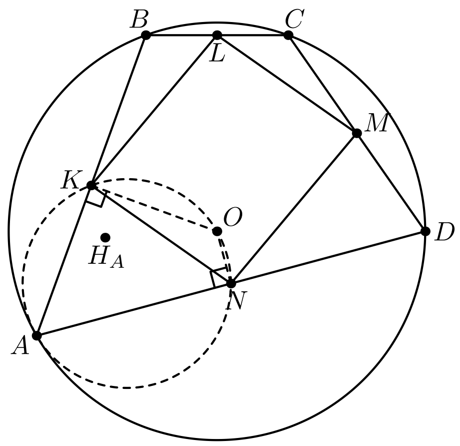
   Let $E, F, G, H$ to be the orthocenters of triangles $BCD$, $CDA$, $DAB$, $ABC$ respectively. Let $F'$ and $G'$ be the reflections of $F$ and $G$ across side $AD$. We know that $F'$ and $G'$ are




on the circumcircle of $ABCD$. Since $F'C \parallel G'B$, $F'G'BC$ is an isosceles trapezoid. Thus, $FG = F'G' = CB$ and $FG \parallel BC$, so $FCBG$ is a parallelogram. Hence, points $B$ and $C$ are reflections of points $F$ and $G$ respectively about the parallelogram's center $M_1$. Similarly, by constructing the analogous parallelogram with center $M_2$ from reflecting $G$ and $H$ about $AB$, one sees that $C$ and $D$ are the reflections of $G$ and $H$ about $M_2$. Since reflections are 1-to-1 transformations, $M_1$ and $M_2$ must be the same center of reflection. Completing the logic for $A$ and $E$, we conclude that $EFGH$ is congruent to $ABCD$.

7. (MOT) Let $K$,$L$,$M$, and $N$ be the midpoints of the sides $AB$,$BC$,$CD$, and $DA$, respectively, of a cyclic quadrilateral $ABCD$. Prove that the orthocenters of the triangles $AKN$,$BKL$,$CLM$, and $DMN$ are the vertices of a parallelogram.

**Solution:**




Let $O$ be the center of the circle and $H_A$, $H_B$, $H_C$, and $H_D$ be the orthocenters of triangles $AKN$, $BKL$, $CLM$, and $DMN$ respectively. Because $\angle ONA + \angle OKA = 90° + 90° = 180°$, point $O$ is the point diametrically opposite of $A$ on the circumcircle of $AKN$.

From an earlier configuration, the reflection of $H_A$ about the midpoint of side $KN$ coincides with point $O$. Analogously, $H_B$, $H_C$, and $H_D$ are the reflections of $O$ about the midpoints of sides $KL$, $LM$, and $MN$ respectively. Since $KN \parallel BD \parallel LM$ and $LK \parallel CA \parallel MN$, quadrilateral $KLMN$ is a parallelogram. By similar reasoning, the midpoints of quadrilateral

$KLMN$ also form a parallelogram. Since a homothety of ratio 2 maps this quadrilateral to$H_A H_B H_C H_D$, the four orthocenters are the vertices of a parallelogram.

8. In scalene triangle $ABC$, let the feet of the altitudes from $A$, $B$, $C$ to their respective opposite sides be points $D$, $E$, $F$ respectively. Let $M$ be the midpoint of side $BC$, and $S$ the intersection of lines $BC$ and $EF$. Prove that line $SH$ is perpendicular to line $AM$.

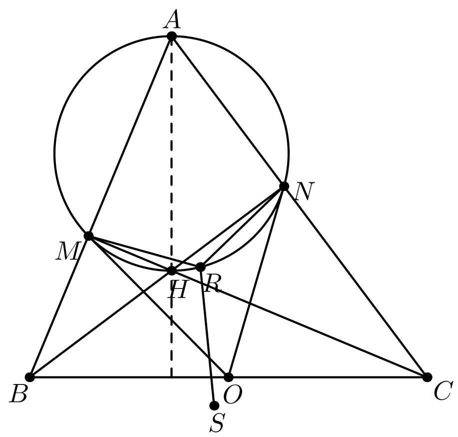
   **Solution:** Consider triangle $ASM$. Segment $AD$ is the altitude to side $SM$. Since $AH$ is the diameter of circle $AEHF$ and $H_M MHR$ is a line, $\angle MRA = \angle HRA = 90°$, so segment $MH$ is the altitude to side $AS$. Thus, $H$ is the intersection of two altitudes of triangle $ASM$, so $SH$ is the third altitude, implying that $SH \perp AM$.

9. (TSTST 2012) In scalene triangle $ABC$, let the feet of the perpendiculars from $A$ to $BC$, $B$ to $CA$, $C$ to $AB$ be $A_1, B_1, C_1$, respectively. Denote by $A_2$ the intersection of lines $BC$ and $B_1C_1$. Define $B_2$ and $C_2$ analogously. Let $D, E, F$ be the respective midpoints of sides $BC, CA, AB$. Show that the perpendiculars from $D$ to $AA_2$, $E$ to $BB_2$ and $F$ to $CC_2$ are concurrent.

   **Solution:** From the problem above, the three perpendiculars from $D$ to $AA_2$, $E$ to $BB_2$ and $F$ to $CC_2$ all pass through $H$, the orthocenter of $\triangle ABC$.

10. (IMO 2004) Let $ABC$ be an acute-angled triangle with $AB \neq AC$. The circle with diameter $BC$ intersects the sides $AB$ and $AC$ at $M$ and $N$ respectively. Denote by $O$ the midpoint of the side $BC$. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at $R$. Prove that the circumcircles of the triangles $BMR$ and $CNR$ have a common point lying on the side $BC$.

    **Solution:** Note: diagram not drawn to scale.

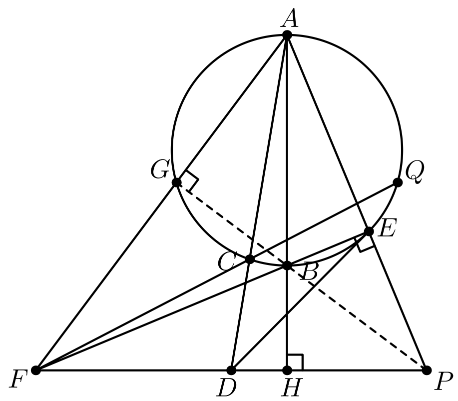
    Let $H$ be the orthocenter of triangle $ABC$. We have that $\angle ONB = \angle OBN = 90 - \angle ACB = \angle HAN$, so $ON$ (and $OM$, by symmetry) is tangent to the circumcircle of $AMHN$. Thus, the intersection of the angle bisectors of $\angle MON$ and $\angle MAN$ is the midpoint of arc $MN$ of circle $AMHN$.

    Let $S$ be the intersection of the circumcircles of $BMR$ and $CNR$. Notice that $\angle RSB + \angle RSC = \angle RMA + \angle RNA = 180°$, so $S$ lies on $BC$.

    **Remark:** Working backwards, one might note that the two circles meet at a common point on side $BC$ if and only if $AMRN$ is cyclic, by Miquel's Theorem. As an additional challenge, prove that $ARS$ is a line and the internal angle bisector of $\angle BAC$. Curiously, as discovered by Olympiad Geometry 908 students, the problem statement still holds if one misread the problem and swapped points $M$ and $N$.

11. * (ISL 2004) Let $\Gamma$ be a circle and let $d$ be a line such that $\Gamma$ and $d$ have no common points. Further, let $AB$ be a diameter of the circle $\Gamma$; assume that this diameter $AB$ is perpendicular to the line $d$, and the point $B$ is nearer to the line $d$ than the point $A$. Let $C$ be an arbitrary point on the circle $\Gamma$, different from the points $A$ and $B$. Let $D$ be the point of intersection of the lines $AC$ and $d$. One of the two tangents from the point $D$ to the circle $\Gamma$ touches this circle $\Gamma$ at a point $E$; hereby, we assume that the points $B$ and $E$ lie in the same halfplane with respect to the line $AC$. Denote by $F$ the point of intersection of the lines $BE$ and $d$. Let the line $AF$ intersect the circle $\Gamma$ at a point $G$, different from $A$. Prove that the reflection of the point $G$ in the line $AB$ lies on the line $CF$.

    **Solution:**


    Let $H$ and $P$ be the intersections of lines $AB$ and $AE$ respectively with line $DF$. Additionally, let $Q$ be the intersection of $FC$ with the circle. Since $FE \perp AP$ and $AB \perp FP$, point $B$ is the orthocenter of triangle $AFP$. If $M$ is the midpoint of side $FP$, we can show that $\angle MEF = \angle MFE = \angle HAP$, so $ME$ is tangent to the circle, implying that $D$ is the midpoint of side $FP$. Thus, we can write that $DF^2 = DE^2 = DC \cdot DA$, so $\triangle DCF \sim \triangle DFA$. This similarity relation yields that $\angle CAQ = \angle DCF = \angle DFA = \angle AEG$ since $FPEG$ is cyclic, so $G$ and $Q$ are reflections of each other across the diameter $AB$ of the circle.

12. * (IMO 1985) A circle with center $O$ passes through the vertices $A$ and $C$ of the triangle $ABC$ and intersects the segments $AB$ and $BC$ again at distinct points $K$ and $N$ respectively. Let $M$ be the point of intersection of the circumcircles of triangles $ABC$ and $KBN$ (apart from $B$). Prove that $\angle OMB = 90°$.

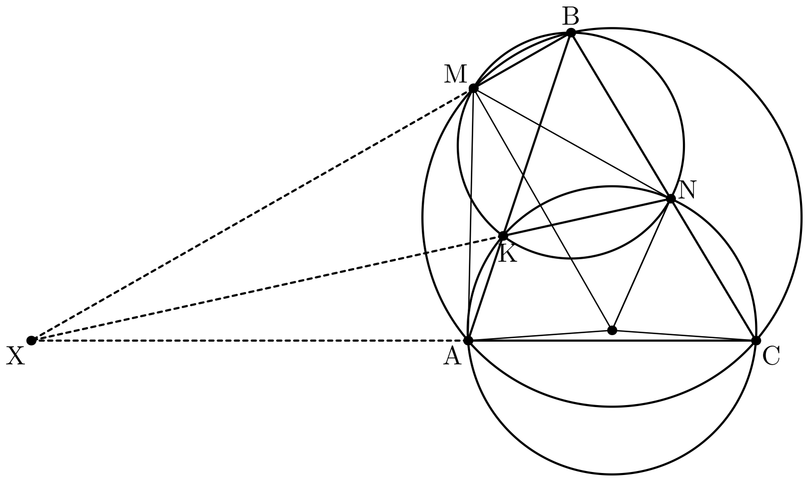
    **Solution:**

    Let $X$ be the radical center of the three circles.

    It will be shown that quadrilateral $XMKA$ is cyclic. From angles, we have:

    $$\angle XMA = \angle BCA = \angle BMN = \angle BKN = \angle XKA$$

    Note that because $\angle XMA = \angle BMN$, it suffices to show that $\angle OMN = \angle OMA$. We claim that quadrilateral $MNOA$ is cyclic. We chase angles:




$$
\begin{aligned}
180 - \angle MNO &= \angle BNM + \angle ONC \\
&= \angle MXA + \angle OCN \\
&= \angle MXK + (\angle NXC + \angle NCX) - \angle OCA \\
&= \angle MAK + \angle BNK - \angle OAC \\
&= \angle MAK + (\angle BAC - \angle OAC) \\
&= \angle MAK + \angle KAO \\
&= \angle MAO
\end{aligned}
$$

Let $R$ be the circumradius of $MNOA$. By the Law of Sines, we can write:

$$\frac{ON}{\sin \angle OMN} = 2R = \frac{OA}{\sin \angle OMA}$$

Since $ON = OA$, we have that $\sin \angle OMN = \sin \angle OMA$. Since the two angles add up to less than 180 degrees, $\angle OMN = \angle OMA$, as desired.

Hence, $\angle BMO = \angle OMN + \angle NMB = \angle OMA + \angle XMA = \angle XMO$. As $\angle XMO$ and $\angle BMO$ are supplementary, $OM \perp MB$.

**Remark:** This problem generalizes Statement 11.

### 1.3 Final Notes

- The orthocenter configurations stated above come up in several surprising scenarios, so try to know these configurations forwards and backwards.
- Statements 4 and 8 are especially useful constructions in certain problems, because they relate the orthocenter to the circumcircle of a triangle. Keep your eyes open for moments when reflecting an orthocenter would be a useful construction.