Geometry Configurations

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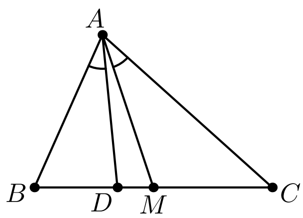
4 Symmedians

This week's configuration is related to symmedians. It's difficult to call this a "configuration" because there are so many ways that a symmedian can show up, but there are a few basic ways that you can recognize one if they come up.

4.1 Definition of Symmedian



As the name implies, a symmedian is related to the median of a triangle. In the diagram above, M is the midpoint of side BC. We choose a point D on side BC such that $\angle BAD = \angle MAC$. (Cevians such as AD and AM that satisfy this angle relationship are also called "isogonal".) The segment AD is referred to as the A-symmedian of triangle ABC.

If you've ever tried to angle chase either $\angle BAM$ or $\angle CAM$ of a triangle, you'd realize that there isn't an elegant way to express either angle in terms of other angles. For this reason, the angle condition above is not always a very useful definition to use (although it does help to relate angles tied to the median), but the fact that the symmedian can be constructed in so many ways makes it an especially useful tool.

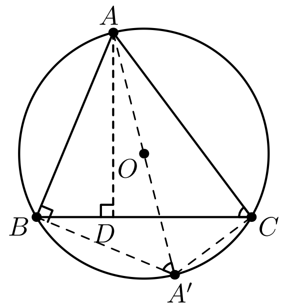
Here are some quick problems:

1. (Extended Law of Sines) In triangle ABC, show that $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C} = \dfrac{BC}{\sin A} = 2R$, where R is the circumradius of the triangle.

   Solution:

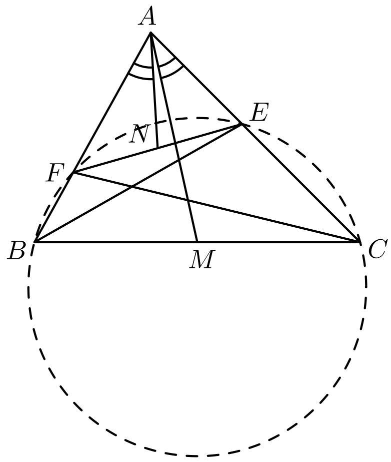
   We create a triangle ABC such that D is the foot of A onto side BC and A' is the point diametrically opposite to A on the circumcircle of ABC.

   By the definition of sines, we have that $AD = AB \sin B$ and $AD = AC \sin C$. Equating these two expressions yields $\frac{AC}{\sin B} = \frac{AB}{\sin C}$. Doing a similar computation with the altitude from point B yields the Law of Sines.



2. Points E and F are on sides AC and AB respectively of triangle ABC. Show that if $BCEF$ is cyclic, then the A-symmedian of triangle ABC passes through the midpoint of EF.

   Solution:

   Let M and N be the midpoints of segments BC and EF respectively. We will show that the A-symmedian of triangle ABC passes through point N.

   Because quadrilateral $BCEF$ is cyclic, we have that $\angle AFE = \angle ACB$, and consequently $\Delta AFE \sim \Delta ACB$ by AA. By similar triangles and the fact that M and N are midpoints, we can write:

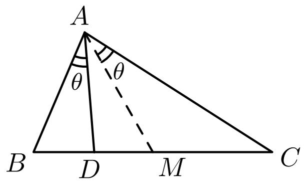
   $$\frac{AF}{FE} = \frac{AC}{CB} \Longrightarrow \frac{AF}{FN} = \frac{AC}{CM}$$

   Thus, we have that $\Delta AFN \sim \Delta ACM$ by SAS. It follows that $\angle BAN = \angle CAM$, so line AN is the A-symmedian of triangle ABC, as desired.

3. The A-symmedian of the triangle ABC intersects side BC at point D. Show that $BD : DC = c^2/b^2$, where b and c are the side lengths of AC and AB respectively.

   Solution:

   Let M be the midpoint of side BC. With a few applications of the Law of Sines, we get:



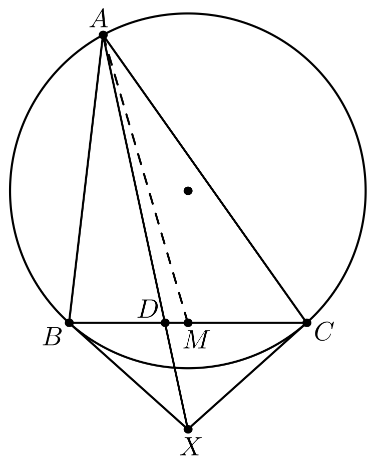
$$\begin{aligned}\frac{BD}{CD} &= \frac{\sin\angle BAD \cdot AD/\sin\angle ABC}{\sin\angle CAD \cdot AD/\sin\angle ACB}\\ &= \left(\frac{\sin\angle ACB}{\sin\angle ABC}\right)\left(\frac{\sin\angle MAC}{\sin\angle MAB}\right)\\ &= \left(\frac{AB}{AC}\right)\left(\frac{MC\cdot\sin\angle AMC/AC}{MB\cdot\sin\angle AMB/AB}\right)\end{aligned}$$

Since $MB = MC$ and $\sin\angle AMC = \sin\angle AMB$, we get the desired ratio.

4. Show that the three symmedians of a triangle concur at a point in the triangle.

   Solution: Using the ratio from the previous problem, a direct application of Ceva's Theorem shows that the three symmedians are concurrent at the *symmedian point* or the *Lemoine point*.

4.2 Symmedians and Tangents

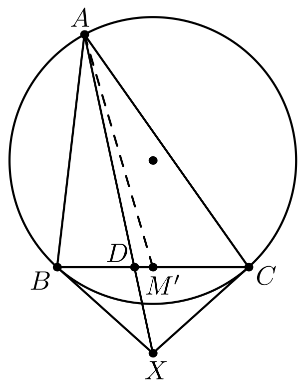


Perhaps one of the most well-known constructions of the symmedian is shown above. Consider the circumcircle of the triangle ABC. Let the tangents to this circle at points B and C meet at a point X. Then line AX coincides with the symmedian of triangle ABC. There are a few proofs of this, as you can find on the first page of Yufei Zhao's handout (`http://yufeizhao.com/olympiad/geolemmas.pdf`). The simplest one, a Law of Sines computation, is not particularly enlightening, although a synthetic and project proof have both been provided.

This lemma is powerful. If you see a symmedian appear in a problem, often a nice way to approach it is to construct the corresponding tangencies to the circumcircle, as this kind of configuration is conducive to Power of a Point and other angle chasing possibilities. Furthermore, if you study poles and polars, symmedians tie in very well with certain projective ideas. (I'd still recommend having a solid synthetic foundation first before learning to abuse projective geometry.) I'm not sure if you can cite this directly on a proof; it's certainly well known, but the statement is nontrivial and the proof isn't terribly difficult either.

1. Prove that the symmedian coincides with the line AX by using Law of Sines. (You can check your work in the link above when you're done.)

   Solution:

   Let M' be the point on side BC such that $\angle M'AC = \angle PAB$. By the Law of Sines, we have:

$$\begin{aligned}\frac{BM'}{CM'} &= \frac{AM' \sin\angle BAM' / \sin\angle ABC}{AM' \sin\angle CAM' / \sin\angle ACB} \\ &= \frac{\sin\angle BAM' \sin\angle ABP}{\sin\angle ACP \sin\angle CAM'} \\ &= \frac{\sin\angle CAP \sin\angle ABP}{\sin\angle ACP \sin\angle BAP} \\ &= \frac{AP \cdot BP}{CP \cdot AP} \\ &= 1\end{aligned}$$

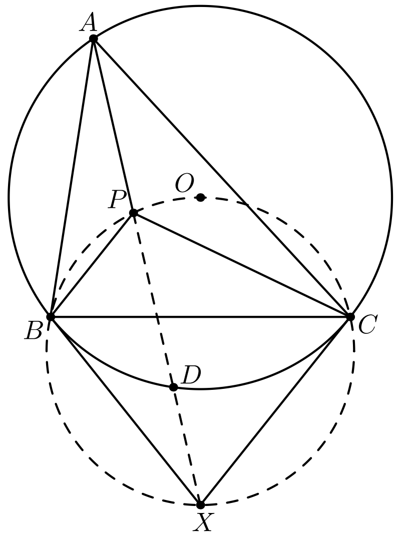
   Thus, AM' is the median of the triangle, so AP coincides with the symmedian.

2. (Harmonic Quadrilaterals) Points B and D are on circle ω, and point P is a point outside of ω such that PB and PD are tangent to the circle. A line through P intersects the circle again at two points A and C. Show that $AB/BC = AD/DC$.

   Solution: No symmedians required! Notice that $\angle CDP = \angle PAD$, so by similar triangles, $AD/DC = PA/PD$. Similarly, we can obtain the expression $AB/BC = PA/PB$. Since PB and PD are tangents, $PB = PD$, so the expressions are equal.

   Remark: You should note that line AC coincides with the symmedian of triangle ABD. Additionally, note the duality of the configuration: AC also coincides with the symmedian of triangle CBD.

4.3 Similarity Definition



Here is another very common definition of the symmedian.

1. Let P be a point in triangle ABC such that $\triangle PBA \sim \triangle PAC$. Show that AP coincides with the A symmedian.

   Hint: Let O be the circumcenter of the triangle, and X be the point where the tangents to the circumcircle at B and C meet. Show that $BPOCX$ is cyclic.

   Solution:

   Let O be the circumcenter of $\triangle ABC$ and X be the intersection of the tangents to the circumcircle at points B and C.

   Notice that $\angle BPA = 180 - \angle PAB - \angle ABP = 180 - \angle PAB - \angle CAP = 180 - \angle CAB$. Since $\Delta PBA \sim \Delta PCA$, we have that $\angle BPC = 360 - \angle BPA - \angle APC = 360 - 2(180 - \angle BAC) = 2\angle BAC$. Since $\angle BOC$ faces the same arc that $\angle BAC$ subtends to, $\angle BOC = 2\angle BAC$, so $BPOC$ is cyclic.

   Now consider point X such that XB and XC are tangents to the circumcircle of triangle ABC. Because $BPOC$ is cyclic and $\angle OCX + \angle XBO = 90 + 90 = 180°$, points B, C, X, P, and O all lie on the same circle.

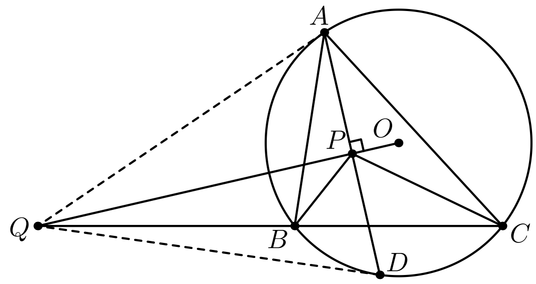
   Since $\angle BPA + \angle BPX = (180 - \angle BAC) + (\angle BCX) = 180 - \angle BAC + \angle BAC = 180°$, points A, P, and X are collinear, so AP coincides with the A-symmedian of triangle ABC.

2. Suppose that the line AP hits the circumcircle of ABC again at point D. Show that $DP = PA$.

   Solution: Building upon the previous proof, we can see, by definition of tangents, that $\angle OCX = \angle OBX = 90°$, so OX is the diameter of circle $BPOCX$. Thus, $\angle OPX = 90°$, so P is the midpoint of chord AD of circle ABC.

3. Let line OP meet line BC again at point Q. Prove that QA and QD are tangents to the circumcircle of $ABDC$.

   Solution:



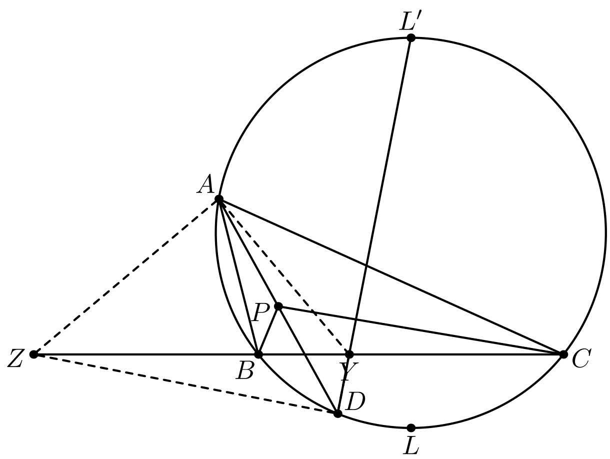
Consider circles APO, ABC, and $BPOC$. Since $\angle APO$, then the circumcenter of triangle APO lies on line AO, so circles APO and ABC are tangent. By the Radical Axis Theorem, the line tangent to the circumcircle of ABC at point A, OP, and BC all concur at the same point Q. Because AD is a chord of circle ABC and $OQ \perp AD$, then QD is also tangent to the circumcircle.

Remark: Point Q in this figure also has several important properties explored in the Orthocenter configuration handout.

Additionally, it is worth noting that point Q is the pole of line AD with respect to circle ABC. This implies that line CQ is the C-symmedian of triangle CAD.

4. Let Y and Z be the points where the internal and external bisectors of angle A meet line BC respectively. (Not pictured.) Let L' be the midpoint of arc BAC (the point diametrically opposite to the Fact 5 point). Show that D, Y, and L' are collinear.

   Solution:

   From the second section above, it follows that $ABDC$ is a harmonic quadrilateral. Thus, by the angle bisector theorem, $BY/YC = AB/AC = BD/DC$, so the line DY is the angle bisector of $\angle BDC$. Thus, DY intersects the midpoint of arc BAC, by Configuration 2.

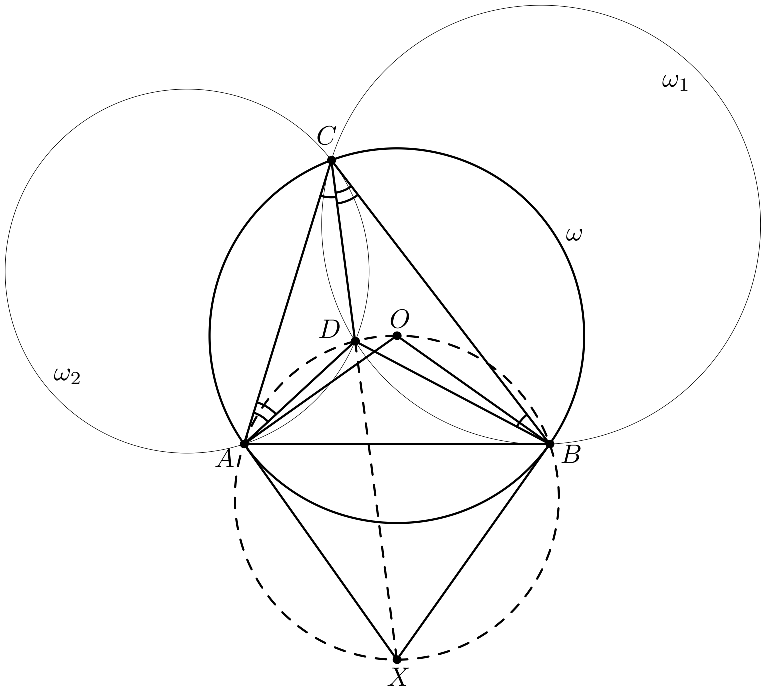
5. Show that $AYDZ$ is cyclic. (Note: This circle is the Apollonius circle of triangle ABC with respect to point A.)

   Solution: Let L be the midpoint of minor arc BC. By Configuration 2, points A, Y, and L are collinear. We have that $\angle AZY = 90 - \angle AYZ = 90 - (\angle YAC + \angle YCA) = 90 - (\angle YAB + \angle YCA) = 90 - (\angle LCB + \angle BCA) = 90 - \angle ACL = \angle ACL' = \angle ADL'$.

Remark: The Apollonius circle with respect to A is the locus of points J such that BJ/JC equals to a fixed ratio BA/AC. From the second section above, we could have directly concluded that all four of the points A, Y, Z, and D satisfy this condition, and therefore, lie on the same circle.

6. (Vietnam 2005) On the circle ω with center O and radius R, consider two fixed points A ad B, and a variable point C. Let ω_1 be the circle through A tangent to BC at C. Similarly, let ω_2 be the circle passing through B, which is tangent to AC at C. Let D be the second point of intersection (other than C) of ω_1 and ω_2. (a) Show that line CD passes through a fixed point. (b) Show that $CD \leq R$.

   Solution: Notice that because ω_1 and ω_2 are tangent to sides BC and AC respectively, we



   have that $\angle DBC = \angle DCA$ and $\angle DAC = \angle DCB$, so $\Delta CAD \sim \Delta BCD$ by AA.

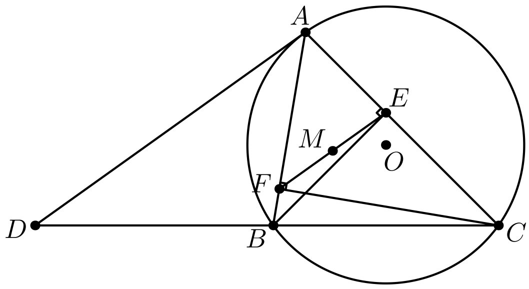
   We let X be the intersection point of the tangents to ω at points A and B. From the previous problem, it is clear that CD passes through point X, which is fixed.

   Since $AODBX$ is cyclic with OX as the diameter, we have that $Area(ADB) \leq Area(AOB)$, or $\frac{1}{2}AD \cdot BD \sin\angle ADB \leq \frac{1}{2}OA \cdot OB \sin\angle AOB$. Because $\angle ADB = \angle AOB$, we are left with the inequality $AD \cdot BD \leq R^2$. By similar triangles ADC and CDB, we have that $CD^2 = AD \cdot BD$, so $CD \leq R$.

7. (Source: Unknown) In triangle ABC, let E and F be the feet of the altitudes from B and C respectively. Additionally, let D be the intersection of BC and the tangent line to the circumcircle O at point A. If M is the midpoint of EF, show that $DO \perp AM$.

   Solution:

   Because lines EF and BC are antiparallel (in other words, $BCEF$ is cyclic), by 5.1.2, AM coincides with the A-symmedian of triangle ABC. From 5.3.1 and 5.3.2, we consider the point



P inside of the triangle such that $\triangle PBA \sim \triangle PAC$. By the Radical Axis Theorem applied on the circles of APO, $BPOC$, and ABC, the lines AA (the line tangent to the circumcircle of ABC at A), OP, and BC are concurrent at point D. Since 5.3.2 tells us that OP is perpendicular to the symmedian AM, so $DO \perp AM$ as desired.

4.4 Practice with Symmedians

You can find more practice problems in Yufei Zhao's handout (`http://yufeizhao.com/olympiad/geolemmas.pdf`).

1. Let M be the midpoint of side BC of triangle ABC with $AB < AC$. Let D be the point inside of triangle ABC such that $\angle BAD = \angle MAC$ and $\angle DBA = \angle BCA$. Prove that $DM \parallel AC$.

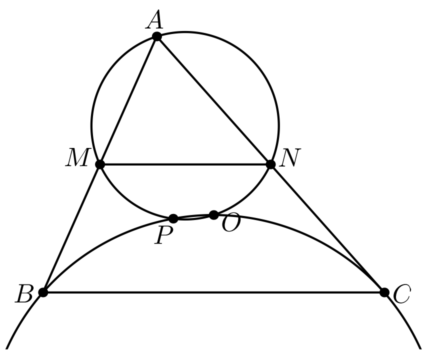
   Solution: Let N and P be the midpoints of AC and AB respectively. Since $\triangle DAB \sim \triangle MAC$, then $\angle MNA = \angle DPA$. Since $\angle MNA = 180 - \angle A = \angle MPA$, this implies that MDP is a line, and the conclusion follows.

2. Let P be the point in triangle ABC such that $\triangle PBA \sim \triangle PAC$. Let O be the circumcenter of triangle ABC. Show that the lines AA, BC, and OP concur. (Here, AA is the line that is tangent to the circumcircle of ABC.)

   Solution: See 5.3.7.

3. Let M and N be the midpoints of sides AB and AC of triangle ABC. Additionally, let O is the circumcenter of the triangle, and P be the intersection of the circumcircles of OMN and OBC. Show that AP is a symmedian.

   Solution:



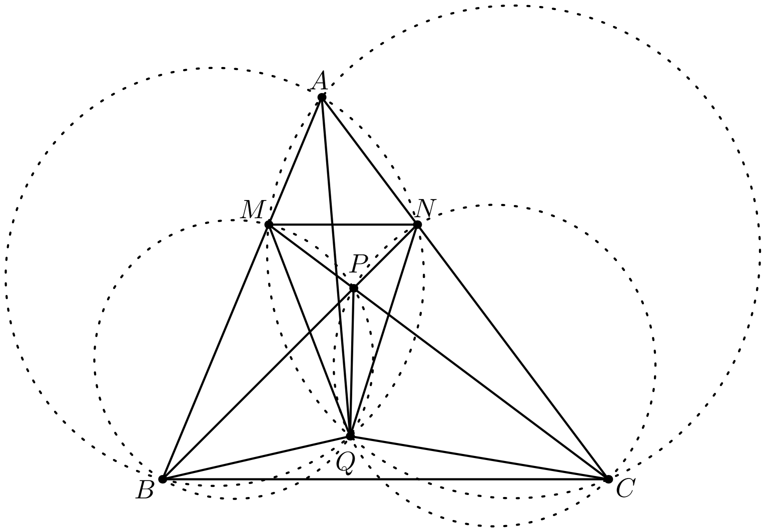
Let P' be the point inside of triangle ABC such that $\triangle P'BA \sim \triangle P'AC$. From 5.3.1 and

5.3.2, the point P' satisfies both $\angle AP'O = 90°$ and $BP'OC$ is cyclic. Thus, the construction of P' is uniquely defined by considering the second intersection of the circle with diameter AO and the circumcircle of BOC.

Since $\angle ONA = \angle OMA = 90°$, points M and N lie on the circle with diameter AO. Since P and P' are defined identically, then AP must be a symmedian of the triangle.

4. Let M and N be points on sides AB and AC of triangle ABC such that $MN \parallel BC$. Let P be the intersection of lines BN and CM. The circumcircles of BMP and CNP intersect again at point Q. Show that $\angle QAB = \angle PAC$.

Solution:



From angle chasing, we have that $\angle QBA = \angle QPC = \angle QNC$, and $\angle NCQ = \angle BPQ = \angle BMQ$, so quadrilaterals $AMQC$ and $ANQB$ are cyclic. Thus, $\angle BAQ = \angle MCQ$ and $\angle ABQ = \angle CPQ$, so $\Delta ABQ \sim \Delta CPQ$. Similarly, $\Delta QNB \sim \Delta QCM$.

Notice that because $MN \parallel BC$, we have two sets of similar triangles: $\Delta PNM \sim \Delta PBC$ and $\Delta AMN \sim \Delta ABC$. With this and that $\Delta BAQ \sim \Delta PCQ$, we can write the following ratios:

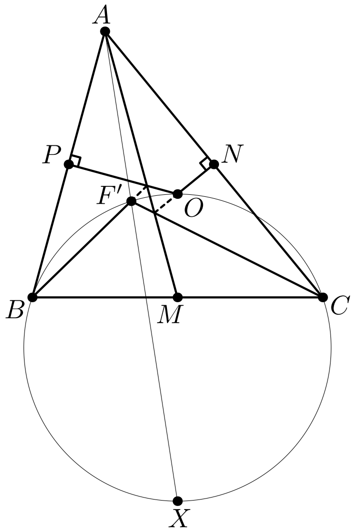
$$\frac{PC}{MP} = \frac{BC}{MN} = \frac{AB}{AM}$$
$$\frac{AM}{MP} = \frac{AB}{PC} = \frac{AQ}{QC}$$

Since $\angle AMP = \angle AQC$ by cyclic quadrilaterals, we have that $\Delta AMP \sim \Delta AQC$, so $\angle MAP = \angle QAC$. By Ceva's Theorem on point P with respect to triangle ABC, we have that AP is the median of triangle ABC, so AQ must coincide with the symmedian of the triangle.

5. (USAMO 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Solution:

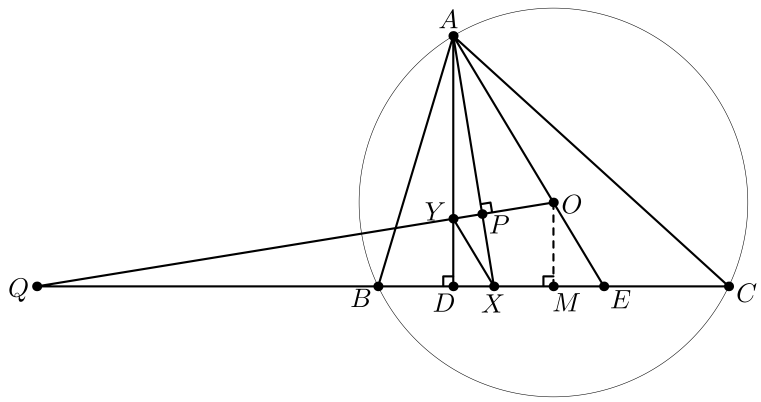
Label $\angle BAM = \alpha$ and $\angle MAC = \beta$. Notice that $\angle BOC = 2\angle BAC = 2(\alpha+\beta)$ and $\angle BFC = \angle FDE + \angle DEF = (\angle BAD + \angle DBA) + (\angle EAC + \angle ECA) = 2\angle BAD + 2\angle EAC = 2(\alpha+\beta)$, so $BFOC$ is cyclic.



Next, construct point X to be the intersection of the two tangents to the circumcircle of ABC at points B and C and construct point F' to be the second intersection of AX with circle BOC. From previous sections, AF' is a symmedian, so $\angle BAX = \beta$ and $\angle XAC = \alpha$. Since point X also lies on circle BOC, $\angle BF'X = \angle BCX = \angle OCX - \angle OCB = 90° - (90° - \angle BAC) = \alpha + \beta$. Thus, $\angle ABF' = \angle BF'X - \angle BAF' = (\alpha+\beta) - \beta = \alpha = \angle BAD = \angle ABD$. Hence, points B, F', and D are collinear, which implies that F and F' are the same point.

6. ($\sim$HMMT 2014) Let ABC be an acute triangle with circumcenter O. Let D be the foot of the altitude from A to BC, and E be the intersection of AO with BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Prove that AX is a symmedian of triangle ABC.

   Solution:



Let P be the intersection of lines AX and OY and let Q be the intersection of the lines OY and BC. Since $\angle APO = \angle ADC = 90°$, quadrilaterals $PYDX$ and $APDQ$ are both cyclic. Using the fact that $AO \parallel XY$, we have that $\angle QAD = \angle QPD = \angle YXD = \angle AED = 90° - \angle DAO$, which implies that $\angle QAO = 90°$ or that AQ is tangent to the circumcircle of ABC.

By 5.3.3, point P is the unique point in triangle ABC such that $\triangle PBA \sim \triangle PAC$, so AX is a symmedian.

Remark: There are several remarkable properties untouched in this solution. First, one can show that quadrilateral $POED$ is cyclic with basic angle chasing. Second, quadrilaterals $POMX$, $POED$, and $POCB$ are all cyclic and all coaxial, which implies that $QX \cdot QM = QD \cdot QE = QB \cdot QC$. Note the pairs of isogonal cevians: AD and AE are isogonal to each other as well as AX and AM. The coaxiality of these circles is one motivation for the construction for point Q.