Introduction to Computational Finance 2021, Homework 3

Due April 28, 2021

1/22/2021

Introduction

In this homework you will use R to

- simulate data and compute probabilities from bivariate normal distribution
- practice some basic matrix algebra
- explore some basic time series concepts

Use this notebook as a template for the assignment. Insert code chunks (use the R studio short-cut `ctrl-Alt-i`) with R to answer each question and write comments where appropriate. As you work, make sure to preview your notebook often. The output in the Rstudio viewer will update automatically each time you save the notebook (this is one of the advantages of a notebook vs. a standard R markdown document).

Readings

- EZ's book chapters on probability review, matrix algebra review and time series concepts.
- Ruppert and Matteson Chapter 12 (Time Series Models: Basics)
- R Cookbook, chapter 5 (data structures)
- Introduction to R (pdf document on webpage), Chapter 5 (Arrays and Matrices)
- Beginners Guide to R, Chapter 3, Section 5 (Manipulating objects)

Part I: Bivariate Normal Distribution

In this part you will use the R package **mvtnorm** to simulate data and compute probabilities from a bivariate normal distribution. Make sure you have downloaded and installed the package **mvtnorm**:

```
library("mvtnorm")
```

It it useful to read over the first part of the **mvtnorm** package vignette. To do this, click on the **Packages** tab, navigate down to the **mvtnorm** package and click on it (not, the check box but the name of the package). This will open the help tab for **mvtnorm**. Then click on `User guides, package vignettes and other documentation`, and then click on `mvtnorm::MVT_Rnews`. (the package vignette).

Let X and Y be distributed bivariate normal with

$$\mu_X = 0.05, \mu_Y = 0.025, \sigma_X = 0.1, \sigma_Y = 0.05$$

Let's set up some of the bivariate normal parameters:

```
mu.x = 0.05
sig.x = 0.10
mu.y = 0.025
sig.y = 0.05
```

1. Using R package **mvtnorm** function `rmvnorm()`, simulate 100 observations from the bivariate distribution with $\rho_{XY} = 0.9$. Using the `plot()` function create a scatterplot of the observations and comment on the direction and strength of the linear association. Using the **mvtnorm** function `pmvnorm()`, compute the joint probability $\Pr(X \leq 0, Y \leq 0)$.

Here is some code with $\rho_{XY} = 0.5$ to illustrate. Provide your own code for the case with $\rho_{XY} = 0.9$. First, compute the covariance matrix for the bivariate normal.

```
rho.xy = 0.5
sig.xy = rho.xy*sig.x*sig.y
Sigma.xy = matrix(c(sig.x^2, sig.xy, sig.xy, sig.y^2), 2, 2, byrow=TRUE)
Sigma.xy
```

```
##        [,1]   [,2]
## [1,] 0.0100 0.0025
## [2,] 0.0025 0.0025
```

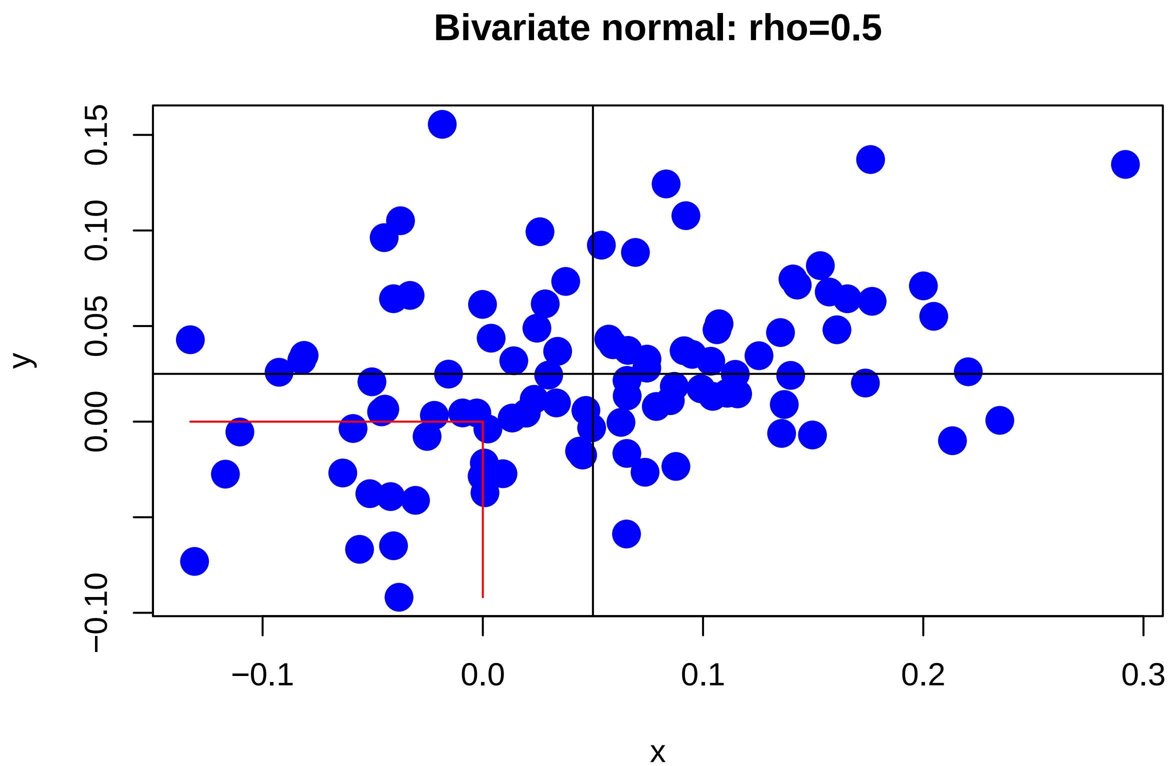
Here, we use $\sigma_{XY} = \rho_{XY} \times \sigma_X \times \sigma_Y$. Then, use the `rmvnorm()` function to simulate 100 observations from bivariate normal distribution:

```
n = 100
set.seed(123)
xy.vals = rmvnorm(n, mean=c(mu.x, mu.y), sigma=Sigma.xy)
head(xy.vals)
```

```
##          [,1]     [,2]
## [1,] -0.00917  0.00456
## [2,]  0.20476  0.05510
## [3,]  0.09221  0.10775
## [4,]  0.07366 -0.02648
## [5,] -0.02532 -0.00773
## [6,]  0.17677  0.06293
```

Create a scatter plot to show the dependence between the random variables

```
plot(xy.vals[,1], xy.vals[,2], pch=16, cex=2, col="blue",
     xlab="x", ylab="y")
title("Bivariate normal: rho=0.5")
abline(h=mu.y, v=mu.x)
segments(x0=0, y0=min(xy.vals[,2]), x1=0, y1=0, col="red")
segments(x0=min(xy.vals[,1]), y0=0, x1=0, y1=0, col="red")
```



The red segment in the lower quadrant represents the fraction of observations that satisfy $x \leq 0, y \leq 0$.

Finally, use the `pmvnorm()` function to compue $\Pr(X \leq 0, Y \leq 0)$

```
pmvnorm(lower=c(-Inf, -Inf), upper=c(0, 0), mean=c(mu.x, mu.y), sigma=Sigma.xy)
```

```
## [1] 0.163
## attr(,"error")
## [1] 1e-15
## attr(,"msg")
## [1] "Normal Completion"
```

Now, repeat the above analysis for $\rho_{XY} = 0.9$, and for the rest of the problems.

2. simulate 100 observations from the bivariate distribution with $\rho_{XY} = -0.9$. Create a scatterplot of the observations and comment on the direction and strength of the linear association. Compute the joint probability $\Pr(X \leq 0, Y \leq 0)$
3. Simulate 100 observations from the bivariate distribution with $\rho_{XY} = 0$. Create a scatterplot of the observations and comment on the direction and strength of the linear association. compute the joint probability $\Pr(X \leq 0, Y \leq 0)$.

Part II: Matrix Algebra

1. Create and print out the following matrices and vectors in R

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 4 & 8 \\ 6 & 1 & 3 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 4 & 4 & 0 \\ 5 & 9 & 1 \\ 2 & 2 & 5 \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; \mathbf{y} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

2. Compute the transposes of the matrices and vectors
3. Compute $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $2 * \mathbf{A}$, $\mathbf{Ax}$, $\mathbf{x}'\mathbf{Ay}$, and $\mathbf{y}'\mathbf{Ax}$

Consider the system of equations

$$x + y = 1 \tag{1}$$
$$2x + 4y = 2 \tag{2}$$

4. Plot the two lines and note the solution to the system of equations (hint: use the R function `abline()`). Write the system using matrix notation as $\mathbf{Az} = \mathbf{b}$ and solve for z.

Consider creating a portfolio of three assets denoted A, B and C. Assume the following information

$$\mu = \begin{bmatrix} 0.01 \\ 0.04 \\ 0.02 \end{bmatrix}, \; \Sigma = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.3 & 0.15 & -0.2 \\ 0.1 & -0.2 & 0.08 \end{bmatrix}$$

5. Compute the expected return and variance for an equally weighted portfolio portfolio (i.e., $x_A = x_B = x_C = 1/3$).

Let R_i denote the simple return on asset i, $(i = 1, \cdots N)$ with $E[R_i] = \mu_i, \text{var}(R_i) = \sigma_i^2$ and $\text{cov}(R_i, R_j) = \sigma_{ij}$. Define the $N \times 1$ vectors $\mathbf{R} = (R_1, \cdots, R_N)'$, $\mu = (\mu_1, \cdots, \mu_N)'$, $\mathbf{x} = (x_1, \cdots, x_N)'$, $\mathbf{y} = (y_1, \cdots, y_N)'$ and $\mathbf{1} = (1, \cdots, 1)'$, and the $N \times N$ covariance matrix

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ldots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{bmatrix}$$

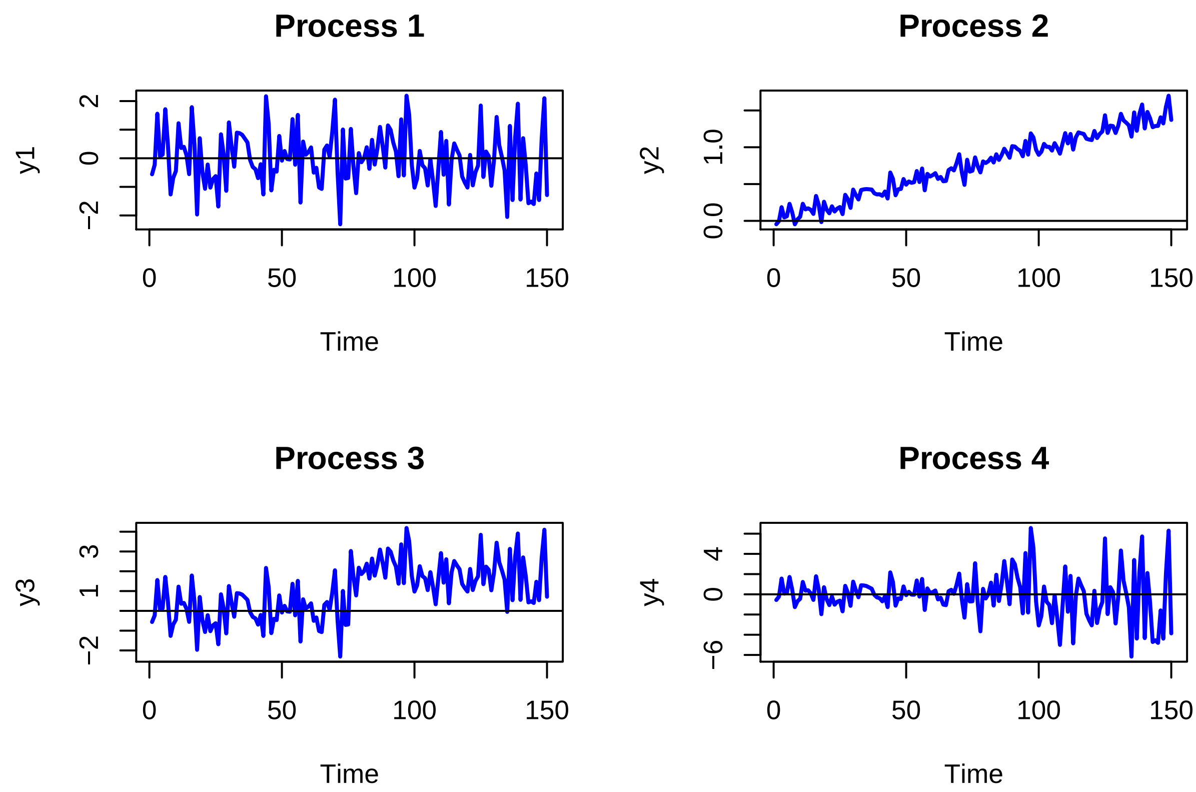
The vectors x and y contain portfolio weights (investment shares) that sum to one. Using simple matrix algebra, answer the following questions

6. For the portfolios defined by the vectors x and y give the matrix algebra expression for the portfolio returns, $R_{p,x}$ and $R_{p,y}$, and the portfolio expected returns, $\mu_{p,x}$ and $\mu_{p,y}$.
7. For the portfolios defined by the vectors x and y give the matrix algebra expression for the constraint that the portfolio weights sum to one.
8. For the portfolios defined by the vectors x and y give the matrix algebra expression for the portfolio variances, $\sigma_{p,x}^2$ and $\sigma_{p,y}^2$, and the covariance between $R_{p,x}$ and $R_{p,y}$, σ_{xy}.

Time Series Concepts

1. Let Y_t represent a stochastic process. Under what conditions is Y_t covariance stationary?

Realizations from four stochastic processes are given in the Figures below:



2. Which processes appear to be covariance stationary and which processes appear to be non-stationary? For those processes that you think are non-stationary, explain why the process is non-stationary.

3. Consider the following model:

$$Y_t = 10 - 0.67 Y_{t-1} + \epsilon_t, \quad \epsilon \sim N(0, 1)$$

- Is it stationary? Why or why not?
- Find the mean and the variance of this process.