Comparing Accuracy and Efficiency of Finite Difference Methods for Solving the Black-Scholes Equation

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Although PDE models have limited applications in solving financial problems compared to time-series analyses and stochastic/probabilistic models, they are still a powerful and widely used method while pricing derivatives, especially options. PDE models have certain advantages; their derivations are often either known, relatively simple, or computationally non-intensive for a single case. Furthermore, for certain specific situations, deriving the model can be straightforward. Taking ordinary European options as an example, based on the no-arbitrage principle, the PDE of European options can be quickly deduced.

Also, the programming of a PDE numerical solving method, such as Finite Difference Method, is relatively easy to implement using python or other desired language. Moreover, solutions of PDE problems can be also evaluated by consistency, stability, convergence, etc. easily. For a real world problem, the calculation speed can then be improved as much as possible on the premise of ensuring stability because of the advantages above.

One of the most common financial PDE models is the Black-Scholes model, which is an effective and efficient way of estimating the price of an option. Financial engineers use this model to determine the initial capital required to perfectly hedge a short position in the option. Black-Scholes is modeled as a geometric Brownian motion which holds for all $x \geq 0$ and $t \in [0, T)$. Thus, the Black-Scholes equation can be written as below:

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \textit{ for all } t \in [0,T), x \geq 0$$

After observation, we can see from the equation that a Black-Scholes is a two-dimensional linear parabolic PDE. Since it does not involve probability, we can solve it using standard PDE methods by placing boundary conditions at $x = 0$ and $x = \infty$ and so it resembles the heat equation.

As the Black-Scholes model can differ widely depending on its given parameters, it is interesting to compare various numerical methods for solving the Black-Scholes model to determine the best solver for a given use case. Furthermore, one method may provide an advantage in accuracy but converge slowly, or vice versa. Considering these trade-offs is an important part of using the model in practice.

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We will evaluate the Black-Scholes model for a European call option with constant interest rate and volatility (so that an exact solution can be calculated and compared with numerical method-based solutions). After discretizing, we will apply and compare iterative methods (Gauss-Seidel, Successive Over-Relaxation, and Crank-Nicolson) and compare their convergence with the true solution in time and accuracy. We will furthermore compare the solution derived from PDE methods with a Monte Carlo simulation and a stochastic volatility model.