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A FAST NUMERICAL METHOD FOR THE BLACK–SCHOLES EQUATION OF AMERICAN OPTIONS*

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Abstract. This paper introduces a fast numerical method for computing American option pricing problems governed by the Black–Scholes equation. The treatment of the free boundary is based on some properties of the solution of the Black–Scholes equation. An artificial boundary condition is also used at the other end of the domain. The finite difference method is used to solve the resulting problem. Computational results are given for some American call option problems. The results show that the new treatment is very efficient and gives better accuracy than the normal finite difference method.

Key words. American option, free boundary, artificial boundary condition, finite difference method

AMS subject classifications. 35A35, 35A40, 65N99

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1. Introduction.

In option pricing theory, the Black–Scholes equation is one of the effective models for option pricing [2]. For European options, the Black–Scholes equation results in a boundary value problem of a diffusion equation. For American options, the Black–Scholes equation results in a free boundary value problem. There are usually two ways to solve the option pricing problem—the analytic and numerical approaches. For European options, the analytic solution is relatively easier to obtain. For the analytic approach, efforts have been mainly on the American options. Johnson [16] and MacMillan [18] use analytical approximation for American puts on a nondividend paying stock. For American options on dividend paying stock, Geske and Johnson [10] give an analytic solution in a series form. When closed form solutions cannot be obtained, or when the formulas for the exact solutions are too difficult to be practically usable, numerical solution is a natural way to solve the problem. The binomial method is a simple and very effective method for solving American options; this is introduced by Cox, Ross, and Rubinstein [7], and the convergence of the binomial method for American options is proved by Amin and Khanna [1]. Brennan and Schwartz [3], [4] and Schwartz [19] introduced finite difference methods for solving American options. Jaillet, Lamberton, and Lapeyre [15] show the convergence of the finite difference method. A comparison of different numerical methods for option pricing can be found in [5], [11].

In solving the Black–Scholes equation for American options, a natural approach is to transform the original equation to a standard forward diffusion equation over an infinite domain. The finite difference method is applied to the equation over a truncated finite domain, and the original asymptotic infinite boundary conditions are shifted to the ends of the truncated finite domain. To avoid generating large errors

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in the solution due to this approximation of the boundary conditions, the truncated domain must be large enough, which results in a large cost. Obviously, a large part of the finite difference solution is actually useless, and the reason to compute these is only to guarantee the accuracy of the rest of the solutions. Kangro and Nicolaides [17] give error estimates of the numerical solutions with far field boundary conditions. Artificial boundary conditions have been applied to different problems on infinite domains; see, for examples, [8], [12], [13], [14]. In this paper, we find the accurate boundary conditions on the far boundary for the American option problem, which is actually a relation between the function and its partial derivatives. Then this boundary condition is discretized and combined with the finite difference discretization for the partial differential equation. With these boundary conditions, we can make the computational domain small and obtain accurate solutions. For the free boundary, we give some properties of the solution of the Black–Scholes equations. Using these properties, we design a simple numerical method to determine the location of the free boundary. Some computational results are given for the American options with dividend paying, and the results are compared with approximations using standard finite difference methods.

The computational results show that these algorithms give more accurate numerical results than the standard finite difference approximation. With a relatively small truncated domain, the standard finite difference method usually cannot give satisfactory results. Our algorithms give more accurate numerical results, and the option price can be obtained for all the asset prices.

2. Some properties of the solution of the Black–Scholes equation.

Assume that S is the asset price, t is the time, and C is the call option value. Let r denote the risk-free interest rate, let σ denote the volatility of the asset price, and let D_0 denote the continuous dividend yield. Then the call value of the American option is given by the free boundary value problem of the Black–Scholes equation [20],

$$\begin{aligned}&\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+(r-D_0)S\frac{\partial C}{\partial S}-rC=0,\\ &0<S<S_f(t),\qquad 0\le t<T,\end{aligned}\tag{1}$$

where $S_f(t)$ is the free boundary of early exercise. The final and boundary conditions are given by

$$C(S,T)=h(S),\qquad 0\le S\le S_f(T)=S_0,\tag{2}$$

$$C(S_f(t),t)=h(S_f(t)),\quad \frac{\partial C}{\partial S}(S_f(t),t)=1,\qquad 0\le t\le T,\tag{3}$$

$$C(S,t)\to 0\quad\text{as}\quad S\to 0,\qquad 0\le t\le T,\tag{4}$$

where $h(S)=\max(S-E,0)$, with $E>0$ and $S_0=\max(E,rE/D_0)$.

We introduce the change of variable for t:

$$t=T-\frac{2\tau}{\sigma^2}.$$

Denote

$$\begin{aligned}
&C^*(S,\tau) = C(S,t) = C(S, T - 2\tau/\sigma^2),\\
&S_f^*(\tau) = S_f(T - 2\tau/\sigma^2),\\
&r^* = \frac{2r}{\sigma^2},\\
&D^* = \frac{2D_0}{\sigma^2},\\
&\tau^* = \frac{\sigma^2 T}{2}.
\end{aligned}$$

Then the free boundary value problem (1)–(4) is equivalent to the following problem:

$$LC^* \equiv -\frac{\partial C^*}{\partial \tau} + S^2 \frac{\partial^2 C^*}{\partial S^2} + (r^* - D^*) S \frac{\partial C^*}{\partial S} - r^* C^* = 0, \quad 0 < S < S_f^*(\tau), \quad 0 < \tau < \tau^*, \tag{5}$$

$$C^*(S,0) = h(S), \quad 0 \le S \le S_f^*(0), \tag{6}$$

$$C^*(S_f^*(\tau),\tau) = h(S_f^*(\tau)), \quad \frac{\partial C^*}{\partial S}(S_f^*(\tau),\tau) = 1, \quad 0 \le \tau \le \tau^*, \tag{7}$$

$$C^*(S,\tau) \to 0 \quad \text{as} \quad S \to 0. \tag{8}$$

Let

$$\begin{aligned}
&k' = r^* - D^*,\\
&\alpha = \frac{-(k'-1)}{2}, \qquad \beta = \frac{-(k'-1)^2}{4} - r^*.
\end{aligned}$$

Furthermore, we introduce the change of variables

$$\begin{aligned}
&S = Ee^x,\\
&C^*(S,\tau) = Ee^{\alpha x + \beta\tau} u(x,\tau).
\end{aligned}$$

Then the free boundary value problem (5)–(8) is equivalent to the following problem:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < x_f(\tau), \tag{9}$$

$$u(x,0) = g(x,0), \quad -\infty < x \le x_f(0), \tag{10}$$

$$u(x_f(\tau),\tau) = g(x_f(\tau),\tau), \ \alpha u(x_f(\tau),\tau) + \frac{\partial u(x_f(\tau),\tau)}{\partial x} = e^{(1-\alpha)x_f(\tau) - \beta\tau}, \ 0 \le \tau \le \tau^*, \tag{11}$$

$$u(x,\tau) \to 0 \text{ as } x \to -\infty, \tag{12}$$

where

$$g(x,\tau) = e^{-\alpha x - \beta\tau} \max(e^x - 1, 0).$$

The free boundary $x_f(\tau) = \ln(S_f^*(\tau)/E)$. It is known that $x_f(\tau) > 0$ for $\tau > 0$.

We now consider the problem (5)–(8). Let

$$W = \frac{\partial C^*(S,\tau)}{\partial S};$$

then W satisfies

$$-\frac{\partial W}{\partial \tau} + S^2 \frac{\partial^2 W}{\partial S^2} + (2 + r^* - D^*) S \frac{\partial W}{\partial S} - D^* W = 0, \tag{13}$$
$$0 < S < S_f^*(\tau), \quad 0 < \tau \le \tau^*,$$

$$W(S,0) = 0, \quad 0 < S < E, \quad \text{and } W(S,0) = 1, \quad E < S < S_f^*(0), \tag{14}$$

$$W(S_f^*(\tau), \tau) = 1, \quad 0 < \tau \le \tau^*. \tag{15}$$

For $W(S,\tau)$, we have the following lemma.

Lemma 1.

$$W(S,\tau) = \frac{\partial C^*(S,\tau)}{\partial S} \to 0 \text{ when } S \to 0^+.$$

Proof. Since

$$\begin{aligned} W(S,\tau) &= \frac{\partial C^*(S,\tau)}{\partial S} \\ &= \frac{\partial}{\partial S}\left(E e^{\alpha x + \beta \tau} u(x,\tau)\right) \\ &= \frac{\partial}{\partial x}\left(E e^{\alpha x + \beta \tau} u(x,\tau)\right) \frac{dx}{dS} \\ &= e^{(\alpha - 1)x + \beta \tau}\left(\frac{\partial u(x,\tau)}{\partial x} + \alpha u(x,\tau)\right), \end{aligned} \tag{16}$$

where $u(x,\tau)$ satisfies (9)–(12), let $\phi(\tau) = u(0,\tau)$, $\phi(0) = 0$, and then [6]

$$\begin{aligned} u(x,\tau) &= \frac{-x}{2\sqrt{\pi}} \int_0^\tau e^{-\frac{x^2}{4(\tau-\lambda)}} \frac{\phi(\lambda) d\lambda}{(\tau-\lambda)^{3/2}}, \quad x < 0, \\ |u(x,\tau)| &\le \frac{|x|}{2\sqrt{\pi}} \Phi \int_0^\tau e^{-\frac{x^2}{4(\tau-\lambda)}} \frac{d\lambda}{(\tau-\lambda)^{3/2}} \\ &\le \frac{4\Phi}{\sqrt{\pi}} \int_0^\tau e^{-\frac{x^2}{8(\tau-\lambda)}} \frac{d\lambda}{\sqrt{\tau-\lambda}} \\ &= \frac{4\Phi}{\sqrt{\pi}} e^{-\frac{x^2}{8\tau}} \int_0^\tau e^{-\frac{x^2}{8}\left(\frac{1}{\tau-\lambda} - \frac{1}{\tau}\right)} \frac{d\lambda}{\sqrt{\tau-\lambda}} \\ &\le \frac{8\Phi}{\sqrt{\pi}} \sqrt{\tau} e^{-\frac{x^2}{8\tau}}, \quad x \le -1, \quad 0 \le \tau \le \tau^*, \end{aligned} \tag{17}$$

where

$$\Phi = \max_{0 \le \lambda \le \tau^*} |\phi(\lambda)|.$$

Similarly,

$$\left|\frac{\partial u(x,\tau)}{\partial x}\right| \le C_0 \Phi \sqrt{\tau} e^{-\frac{x^2}{8\tau}}, \quad x \le -1, \quad 0 \le \tau \le \tau^*, \tag{18}$$

where C_0 is a constant. Combining (16)–(18), we obtain

$$\lim_{S \to 0^+} W(S,\tau) = \lim_{S \to 0^+} \frac{\partial C^*(S,\tau)}{\partial S} = 0. \quad \square \tag{19}$$

Finally, we know that $W(S,\tau)$ is the solution of problem (13)–(15) and (19). By the strong maximum principle of the parabolic equation [9] we have the following theorem.

THEOREM 1. *$W(S,\tau)$ satisfies the following inequality:*

$$0 < W(S,\tau) < 1, \quad 0 < S < S_f^*(\tau), \quad 0 < \tau \le \tau^*.$$

Namely,

$$0 < \frac{\partial C^*(S,\tau)}{\partial S} < 1, \quad 0 < S < S_f^*(\tau), \quad 0 < \tau \le \tau^*,$$

and

$$0 < e^{(\alpha-1)x+\beta\tau}\left(\frac{\partial u(x,\tau)}{\partial x} + \alpha u(x,\tau)\right) < 1, \quad 0 < x < x_f(\tau), \quad 0 < \tau \le \tau^*. \tag{20}$$

For the solution $\{C^*(S,\tau), S_f^*(\tau)\}$ of the problem (5)–(8), we extend $C^*(S,\tau)$ to the domain

$$S_f^*(\tau) < S < +\infty, \quad 0 \le \tau \le \tau^*,$$

by

$$C^*(S,\tau) = h(S), \quad S_f^*(\tau) < S < +\infty, \quad 0 \le \tau \le \tau^*.$$

For a given smooth boundary $S = \hat{S}(\tau)$ and a given τ_j, $0 < \tau_j < \tau^*$, satisfying

$$S_f^*(\tau) < \hat{S}(\tau), \quad \tau_j < \tau \le \tau^*,$$

consider the following auxiliary problem:

$$L\hat{C}(S,\tau) = 0, \quad 0 < S < \hat{S}(\tau), \quad \tau_j < \tau \le \tau^*, \tag{21}$$

$$\hat{C}(S,\tau) \to 0, \quad S \to 0, \tag{22}$$

$$\hat{C}(\hat{S}(\tau),\tau) = h(\hat{S}(\tau)), \quad \tau_j < \tau \le \tau^*, \tag{23}$$

$$\hat{C}(S,\tau_j) = C^*(S,\tau_j), \quad 0 \le S \le \hat{S}(\tau_j). \tag{24}$$

Problem (21)–(24) has a solution $\hat{C}(S,\tau)$ on

$$\Omega = \{(S,\tau) \mid 0 < S < \hat{S}(\tau),\ \tau_j \le \tau \le \tau^*\}.$$

Let

$$\varepsilon(S,\tau) = C^*(S,\tau) - \hat{C}(S,\tau).$$

A computation shows that

$$L\varepsilon(S,\tau) = \begin{cases} 0, & 0 < S < S_f^*(\tau),\ \tau_j < \tau \le \tau^*, \\ -D^*S + r^*E \le 0, & S_f^*(\tau) \le S \le \hat{S}(\tau),\ \tau_j < \tau \le \tau^*, \end{cases}$$

and

$$\frac{\partial \varepsilon(S,\tau)}{\partial S} \text{ is continuous on } S = S_f^*(\tau), \quad \tau_j < \tau \le \tau^*.$$

From the strong maximum principle we get

$$\varepsilon(S,\tau) > 0, \quad 0 < S < \hat{S}(\tau), \quad \tau_j < \tau \leq \tau^*.$$

When $S_f^*(\tau) < S < \hat{S}(\tau)$, we have

$$\varepsilon(S,\tau) = C^*(S,\tau) - \hat{C}(S,\tau) = h(S) - \hat{C}(S,\tau) > 0,$$

namely,

$$\hat{C}(S,\tau) < h(S), \quad S_f^*(\tau) < S < \hat{S}(\tau), \quad \tau_j < \tau \leq \tau^*. \tag{25}$$

Let

$$\hat{C}(S,\tau) = Ee^{\alpha x + \beta\tau}\hat{u}(x,\tau)$$

with $S = Ee^x$. Then the auxiliary problem (21)–(24) is equivalent to the problem

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{\partial^2 \hat{u}}{\partial x^2}, \quad -\infty < x < \hat{x}_f(\tau), \tag{26}$$

$$\hat{u}(x,0) = u(x,0), \quad -\infty < x \leq \hat{x}_f(\tau), \tag{27}$$

$$\hat{u}(\hat{x}_f(\tau),\tau) = g(\hat{x}_f(\tau),\tau), \tag{28}$$

$$\hat{u}(x,\tau) \to 0 \text{ as } x \to -\infty, \tag{29}$$

where $\hat{x}_f(\tau) = \ln(\hat{S}(\tau)/E) \geq x_f(\tau)$, $\tau_j < \tau \leq \tau^*$.

From inequality (25) we obtain the following theorem.

THEOREM 2. *For the solution $\hat{u}(x,\tau)$ of the auxiliary problem* (26)–(29) *the following inequality holds:*

$$\hat{u}(x,\tau) < g(x,\tau), \quad x_f(\tau) < x < \hat{x}_f(\tau), \quad \tau_j < \tau \leq \tau^*. \tag{30}$$

The inequality (30) is very useful for determining the location of the free boundary in the numerical schemes.

3. An exact boundary condition on the artificial boundary. We now return to the problem (9)–(12), which is defined on an unknown unbounded domain $\bar{\Omega}$:

$$\bar{\Omega} = \{(x,\tau) \mid -\infty < x < x_f(\tau),\ 0 < \tau \leq \tau^*\}.$$

It is known that the free boundary $x_f(\tau) > 0$, $0 < \tau \leq \tau^*$. Let $a < 0$ be a real number. We introduce an artificial boundary Γ_a:

$$\Gamma_a = \{(x,\tau) \mid x = a,\ 0 < \tau \leq \tau^*\}.$$

The artificial boundary Γ_a divides the domain Ω into two parts, the bounded part Ω_i and the unbounded part Ω_e:

$$\Omega_i = \{(x,\tau) \mid a < x < x_f(\tau),\ 0 < \tau \leq \tau^*\},$$
$$\Omega_e = \{(x,\tau) \mid -\infty < x < a,\ 0 < \tau \leq \tau^*\}.$$

If we can find a suitable boundary condition on the artificial boundary Γ_a, then the problem (9)–(12) can be reduced on the bounded domain Ω_i. On Ω_e, the solution of (9)–(12), $u(x,\tau)$, satisfies

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < a, \quad 0 < \tau \leq \tau^*, \tag{31}$$

$$u(x,0) = 0, \quad -\infty < x < a. \tag{32}$$

The problem (31)–(32) is an incompletely posed problem. If we know the value of $u(x,\tau)$ on the boundary Γ_a,

$$u(a,\tau)=\phi(\tau) \tag{33}$$

with $\phi(0)=0$, then the solution of (31)–(33) is given by [6]

$$u(x,\tau)=\frac{-(x-a)}{2\sqrt{\pi}}\int_0^{\tau} e^{\frac{-(x-a)^2}{4(\tau-\lambda)}}\frac{\phi(\lambda)d\lambda}{(\tau-\lambda)^{3/2}}. \tag{34}$$

Introducing the new variable

$$\mu=\frac{x-a}{2\sqrt{\tau-\lambda}},$$

we get

$$u(x,\tau)=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x-a}{2\sqrt{\tau}}}\phi\left(\tau-\frac{(x-a)^2}{4\mu^2}\right)e^{-\mu^2}d\mu. \tag{35}$$

Then we have

$$\begin{aligned}
\frac{\partial u(x,\tau)}{\partial x} &= \frac{2}{\sqrt{\pi}}\phi(0)e^{-\frac{(x-a)^2}{4\tau}}\cdot\frac{1}{2\sqrt{\tau}} \\
&\quad+\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x-a}{2\sqrt{\tau}}}\phi'\left(\tau-\frac{(x-a)^2}{4\mu^2}\right)\left(-\frac{x-a}{2\mu^2}\right)e^{-\mu^2}d\mu \\
&= -\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x-a}{2\sqrt{\tau}}}\phi'\left(\tau-\frac{(x-a)^2}{4\mu^2}\right)\left(\frac{x-a}{2\mu^2}\right)e^{-\mu^2}d\mu \\
&= \frac{1}{\sqrt{\pi}}\int_0^{\tau} e^{-\frac{(x-a)^2}{4(\tau-\lambda)}}\frac{\phi'(\lambda)d\lambda}{\sqrt{\tau-\lambda}},\quad x<a.
\end{aligned}$$

Thus we have

$$\left.\frac{\partial u}{\partial x}\right|_{x=a}=\frac{1}{\sqrt{\pi}}\int_0^{\tau}\frac{\partial u(a,\lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{t-\lambda}}. \tag{36}$$

The relationship in (36) is an exact boundary condition satisfied by $u(x,\tau)$, the solution of problem (9)–(12), on the artificial boundary Γ_a. By the exact boundary condition (36), the free boundary value problem for an American call option with dividend paying in an unbounded domain Ω is reduced to a problem in a bounded domain Ω_i:

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial x^2},\qquad a<x<x_f(\tau),\quad 0<\tau\le\tau^*, \tag{37}$$

$$u(x,0)=g(x,0),\qquad a<x<x_f(0), \tag{38}$$

$$u(x_f(\tau),\tau)=g(x_f(\tau),\tau),\qquad 0<\tau\le\tau^*, \tag{39}$$

$$e^{(\alpha-1)x_f(\tau)+\beta\tau}\left[\frac{\partial u(x_f(\tau),\tau)}{\partial x}+\alpha u(x_f(\tau),\tau)\right]=1,\qquad 0\le\tau\le\tau^*, \tag{40}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=a}=\frac{1}{\sqrt{\pi}}\int_0^{\tau}\frac{\partial u(a,\lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{t-\lambda}}. \tag{41}$$

It is straightforward to check that the problem (9)–(12) is equivalent to the problem (37)–(41) in the following sense: If $\{u(x,\tau), x_f(\tau)\}$ is the solution of problem (9)–(12), then $\{u(x,\tau), x_f(\tau)\}$ is the solution of problem (37)–(41). If $\{u^*(x,\tau), x_f^*(\tau)\}$ is the solution of problem (37)–(41), let

$$x_f(\tau) = x_f^*(\tau),$$
$$u(x,\tau) = \begin{cases} u^*(x,\tau), & a \le x \le x_f(\tau), \quad 0 \le \tau \le \tau^*, \\ -\frac{(x-a)}{2\sqrt{\pi}} \int_0^\tau \frac{u^*(a,\lambda) e^{-(x-a)^2/4(\tau-\lambda)}}{(\tau-\lambda)^{3/2}} d\lambda, & x < a, \quad 0 \le \tau \le \tau^*, \end{cases}$$

and then $\{u(x,\tau), x_f(\tau)\}$ is the solution of problem (9)–(12).

4. Finite difference approximation. In this section, we consider the numerical approximation of the problem (37)–(41). Let $\delta\tau$ and δx denote the step sizes of the finite difference approximation. Let $x_n = a + n\delta x$ and $\tau_m = m\delta\tau$, and denote the approximate solution of $u(x_n, \tau_m)$ by u_n^m. Using the Crank–Nicolson finite difference for (37), we get

$$\frac{u_n^m - u_n^{m-1}}{\delta\tau} = \frac{1}{2}\left(\frac{u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{(\delta x)^2} + \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}\right),$$
$$n = 1, 2, \dots, \qquad m = 0, 1, \dots.$$

Letting $\rho = \delta t/(\delta x)^2$, we have

$$(1+\rho)u_1^m - \frac{\rho}{2}u_2^m = b_1, \tag{42}$$

$$-\frac{\rho}{2}u_{n-1}^m + (1+\rho)u_n^m - \frac{\rho}{2}u_{n+1}^m = b_n, \qquad n = 2, 3, \dots, \tag{43}$$

where

$$b_1 = \frac{\rho}{2}(u_0^{m-1} + u_2^{m-1}) + (1-\rho)u_1^{m-1} + \frac{\rho}{2}u_0^m,$$

$$b_n = \frac{\rho}{2}(u_{n-1}^{m-1} + u_{n+1}^{m-1}) + (1-\rho)u_n^{m-1}, \qquad n = 2, 3, \dots.$$

The solution u_n^m can be obtained as follows. Let $s_1 = 1 + \rho$ and $y_1 = b_1$; then we have

$$s_1 u_1^m - \frac{\rho}{2}u_2^m = y_1.$$

Solving for u_1^m and substituting into (43), we get

$$s_2 u_2^m - \frac{\rho}{2}u_2^m = y_2,$$

where

$$s_2 = 1 + \rho - \frac{\rho^2}{4s_1}, \qquad y_2 = b_2 + \frac{\rho y_1}{2s_1}.$$

In general, we have

$$s_n u_n^m - \frac{\rho}{3}u_{n+1}^m = y_n,$$

where

$$s_n = 1 + \rho - \frac{\rho^2}{4s_{n-1}}, \qquad y_n = b_n + \frac{\rho y_{n-1}}{2s_{n-1}}.$$

If the boundary condition

$$u^m_{N_e+1} = g_{N_e+1}$$

is given at certain point, then u^m_n, $n \leq N_e$, can be obtained by back substitution. From Theorems 1 and 2 we know that for a given τ the free boundary is the only point satisfying the partial differential equation and the condition

$$e^{(\alpha-1)x+\beta\tau}\left(\frac{\partial u(x,\tau)}{\partial x} + \alpha u(x,\tau)\right) = 1$$

or, equivalently,

$$\frac{\partial C(S,t)}{\partial S} = 1,$$

and if the boundary condition $u(x,\tau) = g(x,\tau)$ is given at $x > x_f(\tau)$, then $u(x,\tau) < g(x,\tau)$ will occur on the left of the boundary. Let N_e be the largest number such that

$$u^m_{N_e} \geq g_{N_e};$$

then we have

$$\begin{aligned} u^m_{N_e} &= \frac{1}{s_{N_e}}\left(b_{N_e} + \frac{\rho}{2}g_{N_e+1} + \frac{\rho y_{N_e-1}}{2s_{N_e-1}}\right), \\ u^m_n &= \frac{1}{s_n}\left(y_n + \frac{\rho}{2}u^m_{n+1}\right) \qquad \text{for } n = N_e-1, N_e-2, \ldots, 1. \end{aligned}$$

For the artificial boundary condition, since

$$\begin{aligned} \int_0^{\tau_m} \frac{\partial u(a,\lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{\tau_m-\lambda}} &= \sum_{j=1}^m \int_{\tau_{j-1}}^{\tau_j} \frac{\partial u(a,\lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{\tau_m-\lambda}} \\ &= \sum_{j=1}^m \frac{\partial u(a,\xi_j)}{\partial\tau}\int_{\tau_{j-1}}^{\tau_j}\frac{d\lambda}{\sqrt{\tau_m-\lambda}} \\ &= \sum_{j=1}^m \frac{2(\tau_j-\tau_{j-1})u_\tau(a,\xi_j)}{\sqrt{\tau_m-\tau_j}+\sqrt{\tau_m-\tau_{j-1}}}, \end{aligned}$$

equation (41) is approximated by

$$\frac{u^m_1 - u^m_{-1}}{2\delta x} = \frac{1}{\sqrt{\pi}}\sum_{j=1}^m \frac{2(u^j_0 - u^{j-1}_0)}{\sqrt{\tau_m-\tau_j}+\sqrt{\tau_m-\tau_{j-1}}}. \tag{44}$$

Approximating (37) at $x = a$, we get

$$\frac{u^m_0 - u^{m-1}_0}{\delta\tau} = \frac{u^m_1 - 2u^m_0 + u^m_{-1}}{(\delta x)^2}. \tag{45}$$

From (44) and (45) we obtain the boundary condition

$$u_0^m = \frac{\theta H_1 + \sqrt{\pi} H_2/4}{\theta + \sqrt{\pi}(1+2\theta^2)/4}, \tag{46}$$

where $\theta = \sqrt{\delta\tau}/\delta x$, $H_2 = u_0^{m-1} + \theta^2 u_1^m$, and

$$H_1 = u_0^{m-1} + \sqrt{\pi}\theta u_1^m/4 - \sum_{j=1}^{m-1} \frac{u_0^j - u_0^{j-1}}{\sqrt{m-j} + \sqrt{m-j+1}}.$$

Thus we have the following algorithm.

Algorithm.

At each time step, do the following:

1. Set up the linear system using the Crank–Nicolson finite difference method for $x \geq a$.
2. Combine the artificial boundary condition (46) and (42) to eliminate u_0^m.
3. Use the elimination for the linear system in the interval $[a, b]$. Move the right boundary b until the free boundary is found.
4. Use back substitution to find all solutions in $[a, b]$.

At the end τ^*, use (34) to find all solutions to the left of a.

5. Computational results. To compare the above algorithm with the standard finite difference approximations, we computed two examples of call options. The second example was also computed by Broadie and Detemple [5]. The comparisons are based on the accuracy of the approximate option values and the total computation cost, i.e., the CPU time. Since the exact option values are unknown, we use the binomial method with large steps (15000) to find the option values. The results of the binomial method with large steps are considered very accurate. Thus we take these values as the exact option values for the purpose of comparison. All the algorithms are implemented using MATLAB for testing purposes, and the computations are carried out on an IBM RS/6000 43P Model 260 workstation.

In both examples, ABF stands for the numerical method given in the previous section, artificial boundary condition with free boundary treatment. FDP stands for the Crank–Nicolson finite difference approximation with projected SOR iteration to impose the free boundary condition. FDE stands for the Crank–Nicolson finite difference approximation with elimination-backsubstitution. In both FDP and FDE methods, the systems are set up in the interval $[x_m, x_p]$, where $x_m = a < 0$ and $x_p > x_f(\tau) > 0$ for all $\tau > 0$. The asymptotic boundary conditions are applied at both ends $x = x_m$ and $x = x_p$.

Example 1. Consider a six-month American call option with a dividend rate $D_0 = 0.03$. The exercise price is \$100, the risk-free interest rate is $r = 0.03$, and the volatility is 40% per annum. The right boundary is set to $x_p = 0.8$. (The largest value of $x_f(\tau)$ is about 0.62.) A step size $m = 400$ with ratio $\rho = 1$ is taken for all methods. Table 1 shows the results. When $a = -0.2$, the corresponding asset price is about 81.87. Thus the option values corresponding to $S \leq 80$ are not shown for FDP and FDE methods. Similarly, when $a = -0.6$, the corresponding asset price is about 54.88, and the option values corresponding to $S \leq 50$ are not shown. However, the ABF method can give the option values corresponding to any asset prices.

From the computational results shown in Table 1, it is clear that the accuracy of the option values are largely affected by the choice of the left boundary $x = a$. To obtain an accurate option value for the asset price $S = 80$, a must be smaller than

TABLE 1
American call option value (maturity T= 0.5, M= 400).

a	Asset price	FDP	FDE	ABF	True value
-0.2	40			0.0028	0.002792
	50			0.0456	0.045594
	60			0.3013	0.301387
	70			1.1459	1.145799
	80			3.0435	3.041536
	90	4.3058	4.3058	6.3643	6.328677
	100	10.1228	10.1228	11.1267	11.108407
	110	16.7980	16.7980	17.2772	17.266726
	120	24.3457	24.3458	24.5710	24.565972
CPU		21.2000	8.8300	6.2900	
-0.6	40			0.0028	0.002792
	50			0.0455	0.045594
	60	0.2493	0.2492	0.3011	0.301387
	70	1.1365	1.1366	1.1464	1.145799
	80	3.0396	3.0398	3.0416	3.041536
	90	6.3282	6.3283	6.3287	6.328677
	100	11.1066	11.1067	11.1068	11.108407
	110	17.2664	17.2665	17.2665	17.266726
	120	24.5654	24.5655	24.5655	24.565972
CPU		29.4000	12.2600	8.9100	
-1.0	40	0.0025	0.0025	0.0028	0.002792
	50	0.0457	0.0457	0.0457	0.045594
	60	0.3014	0.3015	0.3015	0.301387
	70	1.1459	1.1461	1.1461	1.145799
	80	3.0414	3.0415	3.0415	3.041536
	90	6.3285	6.3287	6.3287	6.328677
	100	11.1066	11.1068	11.1068	11.108407
	110	17.2664	17.2665	17.2665	17.266726
	120	24.5654	24.5655	24.5655	24.565972
CPU		37.1300	15.7600	11.5000	

-0.4, and for $S = 70$, a must be smaller than -0.5. The ABF method gives much more accurate solutions. Compared with the FDP and FDE methods, to obtain an accurate option value for $S = 80$, $a = -0.2$ is enough, and for $S = 70$, $a = -0.4$ is enough.

Figure 1 shows the comparison of error and CPU for the FDE and ABF methods. The error e is measured by

$$e = \left[\frac{1}{K} \sum_{i=1}^{K} (C_i - \bar{C}_i)^2 \right]^{1/2},$$

where C_i is the binomial value, $\bar{C}_i$ is the value of the FDE method, or of the value of the ABF method, and K is the total number of option values taken. The figure shows clearly that the ABF method is much more efficient than the FDE method.

When the maturity time is longer, the error generated due to the rough approximation at the left boundary can be more serious. In example 2, we compare the different algorithms for option value with longer maturity time.

Example 2. Consider a three-year American call option with a dividend rate $D_0 = 0.07$. The exercise price is \$100, the risk-free interest rate is $r = 0.03$, and the volatility is 40% per annum. The right boundary is set to $x_p = 0.8$. (The largest value of $x_f(\tau)$ is about 0.7.) A step size $m = 400$ with ratio $\rho = 1$ is taken for all methods. Table 2 shows the results. The corresponding asset price for $a = -0.4$ is

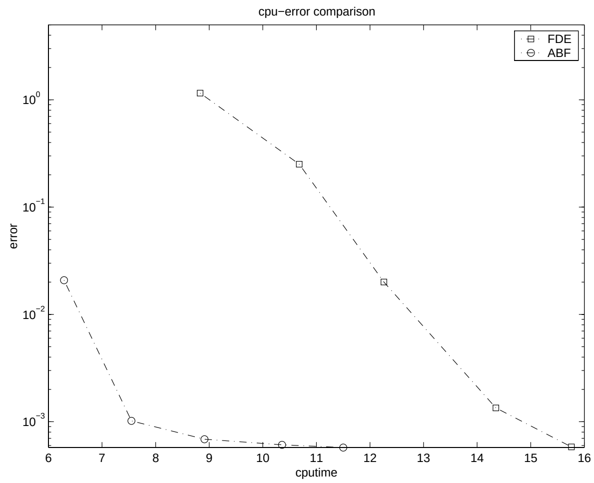

FIG. 1.

TABLE 2
American call option value (maturity T= 3.0, M= 400).

a	Asset price	FDP	FDE	ABF	True value
-0.4	20			0.053	0.053
	40			1.129	1.127
	60			4.729	4.719
	80	6.378	6.378	11.354	11.326
	100	17.639	17.639	20.853	20.793
	120	30.744	30.744	32.824	32.781
CPU		13.32	4.34	5.22	
-0.8	20			0.053	0.053
	40			1.127	1.127
	60	3.845	3.845	4.720	4.719
	80	10.942	10.942	11.329	11.326
	100	20.610	20.610	20.801	20.793
	120	32.686	32.686	32.784	32.781
CPU		18.64	6.19	6.32	
-1.2	20			0.053	0.053
	40	0.977	0.977	1.126	1.127
	60	4.684	4.684	4.717	4.719
	80	11.314	11.314	11.323	11.326
	100	20.786	20.786	20.790	20.793
	120	32.781	32.781	32.783	32.781
CPU		22.53	7.72	7.36	
-1.6	20			0.053	0.053
	40	1.124	1.124	1.127	1.127
	60	4.720	4.720	4.720	4.719
	80	11.327	11.327	11.327	11.326
	100	20.796	20.795	20.796	20.793
	120	32.783	32.783	32.783	32.781
CPU		27.53	9.29	8.44	
-2.0	20	0.052	0.052	0.053	0.053
	40	1.128	1.128	1.127	1.127
	60	4.720	4.720	4.720	4.719
	80	11.328	11.328	11.327	11.326
	100	20.796	20.796	20.796	20.793
	120	32.781	32.781	32.781	32.781
CPU		31.21	10.70	9.64	

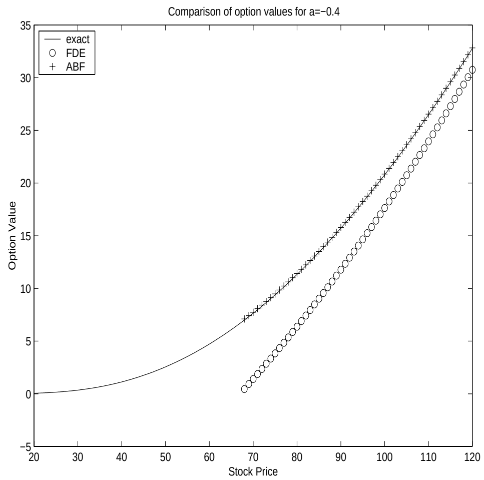

Fig. 2.

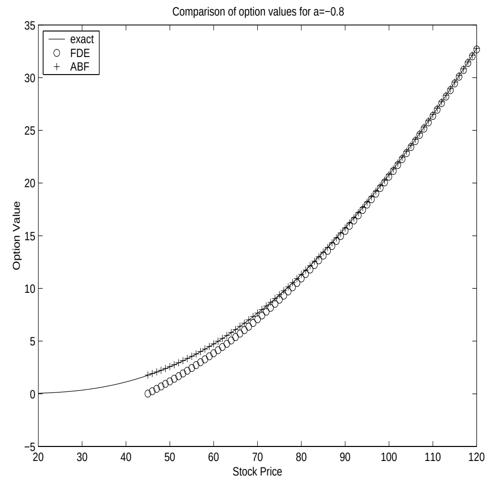

Fig. 3.

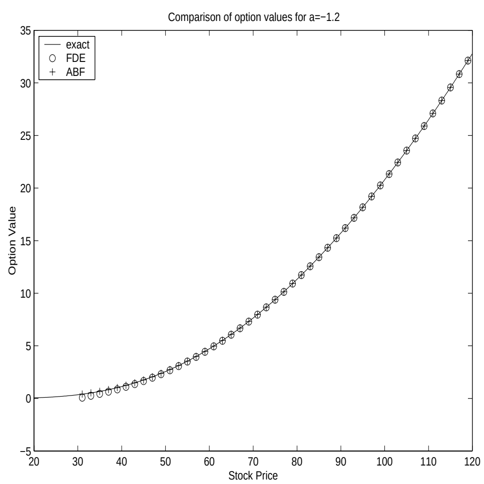

Fig. 4.

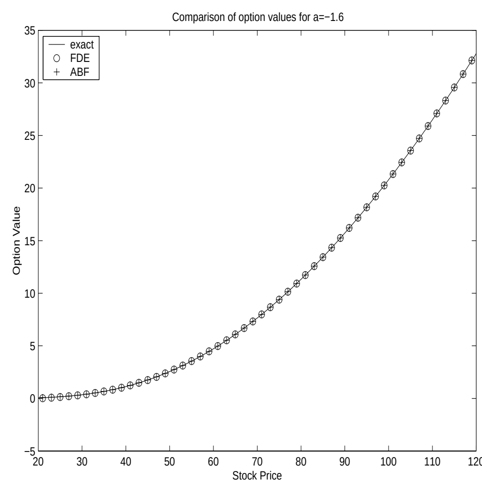

Fig. 5.

about 67.03, that for $a = -0.8$ is about 44.93, that for $a = -1.2$ is about 30.12, and that for $a = -1.6$ is about 20.19. Option values corresponding to asset prices smaller than these values for the FDP and FDE methods are not shown.

For the FDP and FDE methods, the option values are totally wrong for $a = -0.4$. When $a = -0.8$, the option values are still not accurate for asset prices up to $S = 90$, although the left boundary is about $S = 44.93$. To obtain accurate option values, the left boundary needs to be $a = -2.0$. The ABF method improved the results greatly. Even for $a = -0.4$, the option values are close to the true values.

In comparison of the efficiency of all algorithms, we can see from the table that if more option values are needed, then a large saving can be resulted by using artificial boundary conditions. For example, if the option values for $S = 20$ to $S = 120$ are needed, then for the FDP and FDE methods, a must be at least -2.0, and the corresponding CPU time is about 31.21 seconds for the FDP method and 10.7

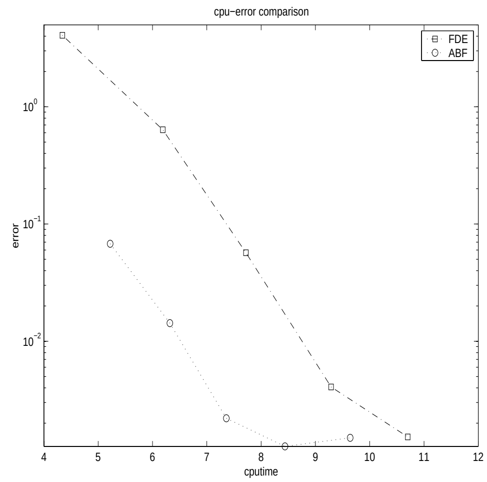

Fig. 6.

seconds for the FDE method. However, the ABF method with $a = -0.4$ can give more accurate option values, while the CPU time is only 5.22 seconds. For this example, the savings in CPU time is nearly 50%. The savings in CPU time will be different for different cases, but it is clear that the savings are significant.

Figures 2–5 show close comparisons of the option values for the FDE method and the ABF method, where the exact option values are obtained by the ABF method with $m = 2000$ on a large interval, $x \geq -2$; the results are very accurate. The four figures show the comparison results for $a = -0.4$, $a = -0.8$, $a = -1.2$, and $a = -1.6$. It is clear from the figures that the results of the FDE method are not acceptable for $a = -0.4$ and $a = -0.8$. When $a = -1.2$, the error can still be seen for the FDE method, while the ABF method gives accurate values for all the cases.

Figure 6 shows the comparison of error and CPU for the FDE and ABF methods. Again, we can see that the ABF method is much more efficient than the FDE method.

6. Conclusion. The artificial boundary conditions give accurate relations of the solutions at the boundary. By using the artificial boundary conditions in the finite difference approximation, we obtained the solution in a small truncated domain. Numerical results show that the solution is very accurate. The treatment for the free boundary is also very efficient. The computational cost is greatly reduced.

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REFERENCES

[1] K. Amin and A. Khanna, *Convergence of American option values from discrete- to continuous-time financial models*, Math. Finance, 4 (1994), pp. 289–304.

[2] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Pol. Econ., 81 (1973), pp. 637–659.

[3] M. Brennan and E. Schwartz, *The valuation of American put options*, J. Fin., 32 (1977), pp. 449–462.

[4] M. Brennan and E. Schwartz, *Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis*, J. Fin. Quan. Anal., 13 (1978), pp. 461–474.

[5] M. Broadie and J. Detemple, *American option valuation: New bounds, approximations, and a comparison of existing methods*, Rev. Fin. Stud., 9 (1996), pp. 1211–1250.

[6] H. S. Carslaw and J. C. Jaeqer, *Conduction of Heat in Solids*, Clarendon Press, Oxford, UK, 1959.

[7] J. C. Cox, S. A. Ross, and M. Rubinstein, *Option pricing: A simplified approach*, J. Fin. Econ., 7 (1979), pp. 229–263.

[8] B. Enguist and A. Majda, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp., 31 (1977), pp. 629–651.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Robert E. Krieger Publishing, Huntington, NY, 1983.

[10] R. Geske and H. E. Johnson, *The American put options valued analytically*, J. Fin., 39 (1984), pp. 1511–1524.

[11] R. Geske and K. Shastri, *Valuation by approximation: A comparison of alternative option valuation techniques*, J. Fin. Quan. Anal., 20 (1985), pp. 45–71.

[12] H. Han and Z. Huang, *A class of artificial boundary conditions for heat equation in unbounded domains*, Comput. Math. Appl., 43 (2002), pp. 889–900.

[13] H. Han and X. Wu, *Approximation of infinite boundary conditions and its application to finite element method*, J. Comput. Math., 3 (1985), pp. 179–192.

[14] T. M. Hagstrom and H. B. Keller, *Exact boundary conditions at an artificial boundary for partial differential equations in cylinders*, SIAM J. Math. Anal., 17 (1986), pp. 322–341.

[15] P. Jaillet, D. Lamberton, and B. Lapeyre, *Variational inequalities and the pricing of American options*, Acta Appl. Math., 21 (1990), pp. 263–289.

[16] H. Johnson, *An analytic approximation for the American put price*, J. Fin. Quan. Anal., 18 (1983), pp. 141–148.

[17] R. Kangro and R. Nicolaides, *Far field boundary conditions for Black–Scholes equations*, SIAM J. Numer. Anal., 38 (2000), pp. 1357–1368.

[18] L. W. MacMillan, *An analytic approximation for the American put price*, Advances in Futures and Options Research, 1 (1986), pp. 119–139.

[19] E. S. Schwartz, *The valuation of warrants: Implementing a new approach*, J. Fin. Econ., 4 (1977), pp. 79–93.

[20] P. Wilmott, J. Dewynne, and S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, Oxford, UK, 1993.