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A new direct method for solving the Black–Scholes equation[☆]

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Abstract

Using the Mellin transform a new method for solving the Black–Scholes equation is proposed. Our approach does not require either variable transformations or solving diffusion equations.

Keywords: Black–Scholes equation; Mellin transform

1. Introduction and preliminaries

The Black–Scholes model (BS) for pricing stock options has been applied to many different commodities and payoff structures. In spite of the market crash of 1987, in practice simple BS models are widely used because they are very easy to use [1,2]. In this paper we consider the final-value problem, [3, p. 188],

$$\left.\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0,\ \ 0 < S < \infty, \quad 0 \le t < T \\ C(S,T) = f(S) \end{aligned}\right\} \tag{1}$$

where T is the maturity date, and σ, r are positive constants. It is well-known that problem (1) has a closed-form solution obtained after several changes of variables and solving certain related diffusion

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equations [4]. However such a technique is not applicable in the vector framework where C is a vector and σ, r are matrices. The aim of this note is to give an exact solution of problem (1) using the Mellin transform and that will permit in a future work to solve problem (1) in a more general framework.

Throughout this paper the set of all absolute Lebesgue integrable functions in a set J of the real line will be denoted by $L^1(J)$. A function $f(x)$ is Mellin transformable if the function $f(x)x^{k-1}$ is in $L^1([0,\infty[)$ for some $k > 0$. Then the Mellin transform of f, denoted by $\mathcal{M}[f(x)](z)$ is defined by

$$f^*(z) = \mathcal{M}[f(x)](z) = \int_0^\infty f(x)x^{z-1}\mathrm{d}x, \quad \mathrm{Re}(z) \le k. \tag{2}$$

An account of properties of the Mellin transform may be found in [4]. In particular, $\mathcal{M}$ is linear and assuming that $\lim_{x\to 0} x^{k-1} f(x) = 0 = \lim_{x\to 0} x^{k-2} f'(x)$, then

$$\mathcal{M}[xf'(x)](z) = -zf^*(z), \tag{3}$$

$$\mathcal{M}[x^2 f''(x)](z) = (z^2 + z)f^*(z), \tag{4}$$

see [4 (p. 363),5] for details. The inverse Mellin transform of $f^*(z)$ is given by

$$\mathcal{M}^{-1}[f^*(z)](x) = \frac{1}{2\pi\mathrm{i}}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} f^*(z)x^{-z}\mathrm{d}z, \quad \alpha > k. \tag{5}$$

2. Solution of Black–Scholes equation

Let us assume for the moment that problem (1) admits a solution $C(S,t)$ that is regarded as a function of the active variable S, i.e., $C(\cdot,t)$ as well as $\frac{\partial C(\cdot,t)}{\partial t}$, $\frac{\partial^2 C(\cdot,t)}{\partial S^2}$, $\frac{\partial C(\cdot,t)}{\partial S}$ are Mellin transformable, and let

$$c(t)(z) = \mathcal{M}[C(\cdot,t)](z) = \int_0^\infty S^{z-1}C(S,t)\mathrm{d}S. \tag{6}$$

By (3), (4) and (6) one gets

$$\mathcal{M}\left[S\frac{\partial C}{\partial S}(\cdot,t)\right](z) = -zc(t)(z); \quad \mathcal{M}\left[S^2\frac{\partial^2 C}{\partial S^2}(\cdot,t)\right](z) = (z+z^2)c(t)(z). \tag{7}$$

Let us assume that function $f(S)$ appearing in (1) is Mellin transformable and let

$$f^*(z) = \mathcal{M}[f(S)](z) = \int_0^\infty S^{z-1}f(S)\mathrm{d}S. \tag{8}$$

Taking into account (6)–(8) and applying the Mellin transform to problem (1) one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}(c(t)(z)) = -p(z)c(t)(z), \quad 0 \le t < T; \quad c(T)(z) = f^*(z), \tag{9}$$

where

$$p(z) = \frac{1}{2}\sigma^2 z^2 + \left(\frac{1}{2}\sigma^2 - r\right)z - r. \tag{10}$$

The solution of (9) is given by

$$c(t)(z) = f^*(z)\mathrm{e}^{-p(z)(t-T)}, \quad 0 \le t \le T \tag{11}$$

and by the Mellin inversion formula (5) and (11) it follows that

$$
\begin{aligned}
C(S,t) &= \mathcal{M}^{-1}[c(t)(z)](S) \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} S^{-z} f^*(\mathrm{z}) \mathrm{e}^{-p(z)(t-T)} \mathrm{dz} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S^{-(\alpha+\mathrm{i}\tau)} f^*(\alpha+\mathrm{i}\tau) \mathrm{e}^{-p(\alpha+\mathrm{i}\tau)(t-T)} \mathrm{d}\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S^{-(\alpha+\mathrm{i}\tau)} f^*(\alpha+\mathrm{i}\tau) \mathrm{e}^{\tilde{p}(\tau)(t-T)} \mathrm{d}\tau,
\end{aligned} \tag{12}
$$

where

$$
\tilde{p}(\tau) = -p(\alpha+\mathrm{i}\tau) = \frac{1}{2}\sigma^2\tau^2 + \left[r - \left(\alpha+\frac{1}{2}\right)\sigma^2\right]\tau\mathrm{i} - \left[\frac{1}{2}\sigma^2\alpha^2 + \alpha\left(\frac{1}{2}\sigma^2 - r\right) - r\right].
$$

Now we prove that expression (12) is a rigorous solution of problem (1). Note that by (12), if $f(S)$ is Mellin transformable and continuous one gets

$$
C(S,T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S^{-(\alpha+\mathrm{i}\tau)} f^*(\alpha+\mathrm{i}\tau)\mathrm{d}\tau = f(S). \tag{13}
$$

Note that by (2)

$$
|f^*(\alpha+\mathrm{i}\tau)| \le M(\alpha) = \int_0^{\infty} |f(x)|x^{\alpha-1}\mathrm{d}x, \quad \text{for all } \tau \in \mathbb{R},
$$

and for $0 \le t < T$ one gets

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left|S^{-(\alpha+\mathrm{i}\tau)}\right| |f^*(\alpha+\mathrm{i}\tau)| \left|\mathrm{e}^{\tilde{p}(\tau)(t-T)}\right| \mathrm{d}\tau \\
&\quad \le M(\alpha) S^{-\alpha} \mathrm{e}^{(t-T)\left(-\frac{1}{2}\sigma^2(\alpha^2+\alpha)+r(\alpha+1)\right)} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{1}{2}\sigma^2\tau^2(t-T)}\mathrm{d}\tau < \infty.
\end{aligned}
$$

Thus expression (12) is well defined and satisfies (13).

Taking into account that $\int_{-\infty}^{\infty} \tau^j \mathrm{e}^{\frac{1}{2}\sigma^2\tau^2(t-T)}\mathrm{d}\tau < \infty$ for $j = 0, 1, 2$ and $0 \le t < T$, and differentiation theorem of parametric integrals, th. 14.23 of [6], for $0 \le t < T$ it follows that

$$
\frac{\partial C}{\partial t}(S,t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} S^{-(\alpha+\mathrm{i}\tau)} f^*(\alpha+\mathrm{i}\tau) p(\alpha+\mathrm{i}\tau) \mathrm{e}^{-p(\alpha+\mathrm{i}\tau)(t-T)} \mathrm{d}\tau, \tag{14}
$$

$$
\frac{\partial C}{\partial S}(S,t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha+\mathrm{i}\tau) S^{-(\alpha+\mathrm{i}\tau)-1} f^*(\alpha+\mathrm{i}\tau) \mathrm{e}^{-p(\alpha+\mathrm{i}\tau)(t-T)} \mathrm{d}\tau, \tag{15}
$$

$$
\frac{\partial^2 C}{\partial S^2}(S,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha+\mathrm{i}\tau)[(\alpha+\mathrm{i}\tau)+1] S^{-(\alpha+\mathrm{i}\tau)-2} f^*(\alpha+\mathrm{i}\tau) \mathrm{e}^{-p(\alpha+\mathrm{i}\tau)(t-T)} \mathrm{d}\tau. \tag{16}
$$

By (12)–(16) for $0 \le t \le T$ and $S > 0$ it follows that

$$
\begin{aligned}
&\frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) + rS\frac{\partial C}{\partial S}(S,t) - rC(S,t) \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[-p(\alpha+\mathrm{i}\tau) + \frac{\sigma^2}{2} S^2 (\alpha+\mathrm{i}\tau)(\alpha+1+\mathrm{i}\tau) S^{-2} - rS(\alpha+\mathrm{i}\tau)S^{-1} - r\right] \\
&\qquad \times f^*(\alpha+\mathrm{i}\tau) \mathrm{e}^{-p(\alpha+\mathrm{i}\tau)(t-T)} \mathrm{d}\tau = 0
\end{aligned}
$$

because

$$-p(\alpha + \mathrm{i}\tau) + \frac{1}{2}\sigma^2(\alpha + \mathrm{i}\tau)(\alpha + 1 + \mathrm{i}\tau) - r(\alpha + \mathrm{i}\tau) - r = 0.$$

Thus $C(S, t)$ defined by (12) is a rigorous solution of problem (1) and the following result has been established:

Theorem 1. *Let $f(S)$ be Mellin transformable and continuous, then $C(S, t)$ defined by* (12) *for $S > 0$, $0 \leq t < T$ is a solution of problem* (1).

Remark. It is important to point out that although the explicit expression (12) usually is not expressible in a closed form, an efficient numerical approximation is available by truncating firstly the integral in a bounded interval and further numerical integration using, for instance, the composite Simpson rule.

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