Iterative Learning Control with Measurements

Aaron Trowbridge

Setup

Given a nominal state trajectory $\hat{x}(t)$ and control trajectory $u(t)$, we apply the controls to the experimental system and retrieve a set of measurements (abusing notation) $y(t)$ from the (possibly hidden) experimental trajectory $\bar{x}(t)$. Schematically we have:

$$u(t) \longrightarrow \bar{x}(u(t), t) \longrightarrow g(\bar{x}(t), t) = \bar{y}(t)$$

which coincides with the simplified model situation:

$$u(t) \longrightarrow \hat{x}(u(t), t) \longrightarrow g(\hat{x}(t), t) = \hat{y}(t)$$

We now have two sets of measurements:

- $\hat{y}(t)$: the nominal measurement
- $\bar{y}(t)$: the experimental measurement

Problem Formulation

Let us write

$$\bar{x}(t) = \hat{x}(t) + \epsilon(t)$$

where $\epsilon(t)$ is the error in the experimental trajectory. To correct for this error, we can find a correction term $\Delta x(t)$ s.t.

$$g(\bar{x} + \Delta x) = g(\hat{x} + \epsilon + \Delta x) = \hat{y}$$

For example, if $g(x) = x$ is the identity function, i.e. we are trying to track the trajectory:

$$\Delta x = -\epsilon$$

The real problem involves finding the corresponding correction to the controls: $\Delta u(t)$. This involves setting up a quadratic optimization problem.

Quadratic Correction Problem

The goal is now to go from the measurement error Δy to a state correction Δx and a control correction Δu by simultaneously solving two linear systems. Schematically:

$$\Delta y \xrightarrow[g]{M\cdot\Delta x=\Delta y} \Delta x \xrightarrow[f]{D\cdot\Delta z=0} \Delta u$$

Measurement Correction to State Correction

With $\Delta y \equiv \bar{y} - \hat{y}$, we have

$$\begin{aligned}
\bar{y} &= g(\bar{x}) \\
&= g(\hat{x} + \epsilon) \\
&\approx g(\hat{x}) + \nabla g(\hat{x}) \cdot \epsilon \\
&= \hat{y} + \nabla g(\hat{x}) \cdot \epsilon
\end{aligned}$$

which, with writing $\nabla \hat{g} = \nabla g(\hat{x})$ yields

$$\Delta y \approx \nabla \hat{g} \cdot \epsilon$$

and since $g : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$, ∇g is not necessarily invertible, but we can use the Moore-Penrose pseudoinverse here to get a guess for ϵ:

$$\boxed{\epsilon \approx (\nabla \hat{g})^+ \cdot \Delta y \equiv \hat{\epsilon}}$$

To tie the experimental measurements to the model measurements we require

$$\begin{aligned}
\hat{y} &= g(\bar{x} + \Delta x) \\
&\approx \bar{y} + \nabla g(\bar{x}) \cdot \Delta x
\end{aligned}$$

which yields the condition

$$\boxed{\nabla g(\bar{x}) \cdot \Delta x = -\Delta y} \tag{1}$$

where

$$\begin{aligned}
\nabla g(\bar{x})_i^j &= \nabla g(\hat{x} + \epsilon)_i^j \\
&\approx \nabla g(\hat{x} + \hat{\epsilon})_i^j \\
&= \nabla g(\hat{x})_i^j + \sum_k \left(\nabla^2 g(\hat{x})\right)_i^{jk} \hat{\epsilon}_k \\
&= \nabla \hat{g}_i^j + \sum_{kl} \left(\nabla^2 \hat{g}\right)_i^{jk} \left((\nabla \hat{g})^+\right)_k^l \Delta y_l
\end{aligned}$$

State Correction to Control Correction

To propagate the state correction to the control correction, we utilize the dynamics constraint, $f(z_t, z_{t+1}) = 0$, where we define the *knot point*

$$z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}$$

Let's write $\mathbf{z}_t = \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix}$. Then we have

$$\begin{aligned} 0 &= f(\hat{\mathbf{z}}_t + \Delta \mathbf{z}_t) \\ &\approx f(\hat{\mathbf{z}}_t) + \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t \end{aligned}$$

which yields

$$\boxed{\nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0} \tag{2}$$

Putting it all together

We seek to find the solution to

$$\begin{aligned} \underset{\Delta x_{1:T},\ \Delta u_{1:T}}{\text{minimize}} \quad & \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\ \text{subject to} \quad & \nabla g(\bar{x}_\tau) \cdot \Delta x_\tau = -\Delta y_\tau \quad \forall \tau \\ & \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0 \quad \forall t \end{aligned}$$

where the τs are the measurement times.

Building the KKT matrix from this problem, we can solve the system and extract $\Delta u(t)$ and repeat the procedure until convergence.

KKT Matrix (for just single quantum state and controls)

Below we use:

- $n = \dim z_t = \dim x_t + \dim u_t$
- $d = \dim x_t = \dim f(z_t, z_{t+1})$
- $c = \dim u_t$
- $m = \dim y_t$
- $M = \#$ of measurements

For a trajectory $Z = \mathrm{vec}(z_{1:T})$, we need to construct the matrix

$$\begin{pmatrix} H & A^\top \\ A & 0 \end{pmatrix}$$

where H is the Hessian of the cost function:

$$H = \bigoplus_{t=1}^{T} (Q \oplus R) = I^{T\times T} \otimes (Q \oplus R)$$

and A is the constraint Jacobian:

$$A = \begin{pmatrix} \nabla F \\ \nabla G \end{pmatrix}$$

with

$$\nabla F = \begin{pmatrix} \nabla f(\hat{\mathbf{z}}_1) & & \\ & \ddots & \\ & & \nabla f(\hat{\mathbf{z}}_{T-1}) \end{pmatrix} \in \mathbb{R}^{d(T-1)\times nT}$$

and

$$\nabla G = \begin{pmatrix} \ddots & & & \\ & \nabla g(\bar{x}_\tau) & \mathbf{0}^{m\times(a+c)} & \\ & & & \ddots \end{pmatrix} \in \mathbb{R}^{mM\times nT}$$

where $\tau = t_1, \ldots, t_M$ are the measurement times.

For the constraints we then have

$$\nabla F \cdot \Delta Z = 0 \quad \text{and} \quad \nabla G \cdot \Delta Z = -\Delta Y$$

where again

$$\Delta Y = \bar{Y} - \hat{Y}$$