Iterative Learning Control with Nonlinear Measurements

Aaron Trowbridge

Setup

Given a nominal state trajectory $\hat{x}(t)$ and control trajectory $u(t)$, we apply the controls to the experimental system and retrieve a set of measurements (abusing notation) $y(t)$ from the (possibly hidden) experimental trajectory $\bar{x}(t)$. Schematically we have:

$$u(t) \longrightarrow \bar{x}(u(t), t) \longrightarrow g(\bar{x}(t), t) = \bar{y}(t)$$

$$\begin{array}{ccccc} u(t) & \longrightarrow & \overbrace{\hat{x}(t) \longrightarrow g(\hat{x}(t))}^{\text{model}} = \hat{y}(t) & & \\ & \searrow & & \searrow & \\ & & \underbrace{\bar{x}(t) \longrightarrow g(\bar{x}(t))}_{\text{experiment}} = \bar{y}(t) & \longrightarrow & \Delta y = \bar{y} - \hat{y} \end{array}$$

$$u^k \longrightarrow \bar{y}^k \longrightarrow \Delta y^k \longrightarrow \mathrm{QP}(\Delta y^k) \longrightarrow (\Delta x, \Delta u) \longrightarrow u^{k+1}$$

which coincides with the simplified model situation:

$$u(t) \longrightarrow \hat{x}(u(t), t) \longrightarrow g(\hat{x}(t), t) = \hat{y}(t)$$

We now have two sets of measurements:

- $\hat{y}(t)$: the nominal measurement
- $\bar{y}(t)$: the experimental measurement

Problem Formulation

Let us write

$$\bar{x}(t) = \hat{x}(t) + e(t)$$

where $e(t)$ is the error in the experimental trajectory. To correct for this error, we can find a correction term $\Delta x(t)$ s.t.

$$g(\bar{x} + \Delta x) = g(\hat{x} + e + \Delta x) = \hat{y}$$

For example, if $g(x) = x$ is the identity function, i.e. we are trying to track the trajectory:

$$\Delta x = -e$$

The real problem involves finding the corresponding correction to the controls: $\Delta u(t)$. This involves setting up a quadratic optimization problem.

Quadratic Correction Problem

The goal is now to go from the measurement error Δy to a state correction Δx and a control correction Δu by simultaneously solving two linear systems. Schematically:

$$\Delta y \xrightarrow[g]{M\cdot\Delta x=\Delta y} \Delta x \xrightarrow[f]{D\cdot\Delta z=0} \Delta u$$

Measurement Correction to State Correction

With $\Delta y \equiv \bar{y} - \hat{y}$, we have

$$\begin{aligned}
\bar{y} &= g(\bar{x}) \\
&= g(\hat{x} + e) \\
&\approx g(\hat{x}) + \frac{\partial g}{\partial \hat{x}} \cdot e \\
&= \hat{y} + \frac{\partial g}{\partial \hat{x}} \cdot e
\end{aligned}$$

which, with writing $\hat{M} = \partial g/\partial \hat{x}$ yields

$$\Delta y \approx \hat{M} \cdot e$$

and since $g : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$, $\hat{M} \in \mathbb{R}^{m\times n}$ is not necessarily invertible, but we can use the Moore-Penrose pseudoinverse here to get a guess for e:

$$\boxed{e \approx \hat{M}^+ \cdot \Delta y \equiv \hat{e}}$$

To tie the experimental measurements to the model measurements we require

$$\begin{aligned}
\hat{y} &= g(\bar{x} + \Delta x) \\
&\approx \bar{y} + \frac{\partial g}{\partial \bar{x}} \cdot \Delta x \\
&= \bar{y} + \bar{M} \cdot \Delta x
\end{aligned}$$

which yields the condition

$$\boxed{\bar{M} \cdot \Delta x = -\Delta y} \tag{1}$$

where

$$
\begin{aligned}
\bar{M}_i^j &= \partial g(\hat{x}+e)_i^j \\
&\approx \partial g(\hat{x}+\hat{e})_i^j \\
&\approx \partial g(\hat{x})_i^j + \sum_k \left(\partial^2 g(\hat{x})\right)_i^{jk} \ \hat{e}_k \\
&= \hat{M}_i^j + \sum_{kl} \left(\partial^2 g(\hat{x})\right)_i^{jk} \ \left(\hat{M}^+\right)_k^l \Delta y_l
\end{aligned}
$$

where

$$
\partial g(\cdot) = \left.\frac{\partial g}{\partial x}\right|_{x=\cdot}
$$

State Correction to Control Correction

To propagate the state correction to the control correction, we utilize the dynamics constraint, $f(z_t, z_{t+1}) = 0$, where we define the *knot point*

$$
z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}
$$

Let's write $\mathbf{z}_t = \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix}$. Then we have

$$
\begin{aligned}
0 &= f(\hat{\mathbf{z}}_t + \Delta \mathbf{z}_t) \\
&\approx f(\hat{\mathbf{z}}_t) + \partial f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t
\end{aligned}
$$

which yields, with $\hat{D} = \partial f(\hat{\mathbf{z}}_t)$

$$
\boxed{\hat{D} \cdot \Delta \mathbf{z}_t = 0} \tag{2}
$$

Putting it all together

We seek to find the solution to

$$
\begin{aligned}
\underset{\Delta x_{1:T},\ \Delta u_{1:T}}{\text{minimize}} \quad & \frac{1}{2} \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\
\text{subject to} \quad & \bar{M} \cdot \Delta x_\tau = -\Delta y_\tau \quad \forall \tau \\
& \hat{D} \cdot \Delta \mathbf{z}_t = 0 \quad \forall t
\end{aligned}
$$

where the τs are the measurement times.

Building the KKT matrix from this problem, we can solve the system and extract $\Delta u(t)$ and repeat the procedure until convergence.

This problem, which returns ΔZ is referred to as

$$\boxed{\Delta Z = \mathsf{QuadraticProblem}(\hat{Z}, \Delta Y)}$$

KKT Matrix (for just single quantum state and controls)

Below we use:

- $n = \dim z_t = \dim x_t + \dim u_t$
- $d = \dim x_t = \dim f(z_t, z_{t+1})$
- $c = \dim u_t$
- $m = \dim y_t$
- $M = \#$ of measurements

For a trajectory $Z = \mathrm{vec}(z_{1:T})$, we need to construct the matrix

$$\begin{pmatrix} H & A^\top \\ A & 0 \end{pmatrix}$$

where H is the Hessian of the cost function:

$$H = \bigoplus_{t=1}^{T} (Q \oplus R) = I^{T\times T} \otimes (Q \oplus R)$$

and A is the constraint Jacobian:

$$A = \begin{pmatrix} \partial F \\ \partial G \end{pmatrix}$$

with

$$\partial F = \begin{pmatrix} \partial f(\hat{\mathbf{z}}_1) & & \\ & \ddots & \\ & & \partial f(\hat{\mathbf{z}}_{T-1}) \end{pmatrix} \in \mathbb{R}^{d(T-1)\times nT}$$

and

$$\partial G = \begin{pmatrix} \ddots & & & \\ & \partial g(\bar{x}_\tau) & \mathbf{0}^{m\times(a+c)} & \\ & & & \ddots \end{pmatrix} \in \mathbb{R}^{mM\times nT}$$

where $\tau = t_1, \ldots, t_M$ are the measurement times.

For the constraints we then have

$$\partial F \cdot \Delta Z = 0 \quad \text{and} \quad \partial G \cdot \Delta Z = -\Delta Y$$

where again

$$\Delta Y = \bar{Y} - \hat{Y}$$

An Alternative Quadratic Problem

In the regime of noisy measurements, satisfying both the dynamics constraints and the measurement constraints becomes infeasible. To overcome this we we can relax the measurement into a maximum likelihood problem by assuming additive gaussian noise $w \sim \mathcal{N}(0, \Sigma)$. To see this let's write

$$\begin{aligned} \bar{y} = \hat{y} - M \cdot \Delta x + w &\implies \bar{y} \sim \mathcal{N}(\hat{y} - M \cdot \Delta x, \Sigma) \\ &\implies \Delta y \sim \mathcal{N}(-M \cdot \Delta x, \Sigma) \end{aligned}$$

where Σ is the covariance matrix of the measurement noise, which we can get from the experiment. To make the following clearer, let's define the parameterized distribution over Δy s.t.

$$\Delta y \sim p(\Delta x) = \mathcal{N}(-M \cdot \Delta x, \Sigma)$$

then, given an observation $\bar{y}$, we can find the MLE for the parameter Δx as the solution to the following optimization problem:

$$\begin{aligned} \max_{\Delta x} \ \log p(\Delta x) \quad &\implies \quad \min_{\Delta x} \ \frac{1}{2}(\Delta y + M \cdot \Delta x)^\top \Sigma^{-1} (\Delta y + M \cdot \Delta x) \\ &\implies \quad \min_{\Delta x} \ \frac{1}{2} \Delta x^\top \left(M^\top \Sigma^{-1} M \right) \Delta x + \left(\Delta y^\top \Sigma^{-1} M \right) \Delta x \end{aligned}$$

We can then augment our initial problem with this objective term and remove the measurement constraint. This yields the following problem:

$$\begin{aligned} \underset{\Delta x_{1:T},\ \Delta u_{1:T}}{\text{minimize}} \quad & \frac{1}{2} \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\ & + \sum_\tau \frac{1}{2} \Delta x_\tau^\top \left(M_\tau^\top \Sigma^{-1} M_\tau \right) \Delta x_\tau + \left(\Delta y_\tau^\top \Sigma^{-1} M_\tau \right) \Delta x_\tau \\ \text{subject to} \quad & \hat{D} \cdot \Delta \mathbf{z}_t = 0 \quad \forall t \\ & -u_{\max} - \hat{u}_t < \Delta u_t < u_{\max} - \hat{u}_t \\ & \Delta u_1 = \Delta u_T = 0 \end{aligned}$$

ILC Algorithm

Tying everything together, *iterative learning control* (ILC) solves the aforementioned quadratic problem and updates the trajectory iteratively until convergence. The following algorithm codifies this:

Algorithm 1: Iterative Control Learning

Data: $\hat{Z}^{\text{goal}}$, $\mathsf{tol} > 0$, $\alpha = 0.5$, $\beta = 0.1$,
Result: U
$Y^{\text{goal}} \leftarrow \mathsf{measure}(\hat{Z}^{\text{goal}}) = \text{vec}(y_{\tau_1:\tau_M})$
$\hat{Z} \leftarrow \hat{Z}^{\text{goal}}$
$U \leftarrow \mathsf{controls}(\hat{Z}) = \text{vec}(u_{1:T})$
$\bar{Y} \leftarrow \mathsf{experiment}(U) = \text{vec}(\bar{y}_{\tau_1:\tau_M})$
$\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$
$k \leftarrow 1$
while $|\Delta Y| > \mathit{tol}$ **do**
    $\Delta Z \leftarrow \beta \cdot \mathsf{QuadraticProblem}(\hat{Z}, \Delta Y)$
    $\hat{Z}_{\text{next}} \leftarrow \hat{Z} + \Delta Z$
    $\bar{y}_{T,\text{next}} \leftarrow \mathsf{measure\_final\_state}(\hat{Z}_{\text{next}})$
    $\Delta y_{T,\text{next}} \leftarrow \bar{y}_{T,\text{next}} - \bar{y}_{T,\text{goal}}$
    while $\|\Delta y_{T,next}\|_p > \|\Delta y_T\|$; `// Backtracking line search`
    do
        $\Delta Z \leftarrow \alpha \cdot \Delta Z$
        $\hat{Z}_{\text{next}} \leftarrow \hat{Z} + \Delta Z$
        $\bar{y}_{T,\text{next}} \leftarrow \mathsf{measure\_final\_state}(\hat{Z}_{\text{next}})$
        $\Delta y_{T,\text{next}} \leftarrow \bar{y}_{T,\text{next}} - \bar{y}_{T,\text{goal}}$
    end
    $\hat{Z} \leftarrow \hat{Z}_{\text{next}}$
    $U \leftarrow \mathsf{controls}(\hat{Z})$
    $\bar{Y} \leftarrow \mathsf{experiment}(U)$
    $\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$
    $k \leftarrow k + 1$
end
return U

1 Misc.

$$
\begin{aligned}
\underset{\Delta x, \Delta u}{\text{minimize}} \quad & \|\Delta y\|^2 = \|\partial_x g \cdot \Delta x\|^2 + \Delta y^\top \cdot \partial_x g \cdot \Delta x \\
\text{subject to} \quad & \partial_{\mathbf{z}} f \cdot \Delta \mathbf{z} = 0
\end{aligned}
$$

$$
\begin{aligned}
\Delta y &= g(\hat{x} + e + \Delta x) - \hat{y} \\
&\approx g(\hat{x}) + \partial_x g \cdot e + \partial_x g \cdot \Delta x - \hat{y}
\end{aligned}
$$

$$
\begin{aligned}
\underset{\Delta x, \Delta u}{\text{minimize}} \quad \frac{1}{2}\left\|\Delta y^{k+1}\right\|^2 &= \frac{1}{2}\left\|g(\bar{x}^k + \Delta x) - \hat{y}\right\|^2 \\
&\approx \frac{1}{2}\left\|g(\bar{x}^k) + M\Delta x - \hat{y}\right\|^2 \\
&= \frac{1}{2}\left\|\Delta y^k + M\Delta x\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
\underset{\Delta x, \Delta u}{\text{minimize}} \quad & \frac{1}{2}\Delta x^\top M^\top M \Delta x + \Delta y^\top M \Delta x \\
\text{subject to} \quad & \partial f \cdot (\Delta x_{t+1}, \Delta x_t, \Delta u_t)^\top = 0
\end{aligned}
$$