

# Iterative Learning Control with Measurements

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## Setup

Given a nominal state trajectory  $\hat{x}(t)$  and control trajectory  $u(t)$ , we apply the controls to the experimental system and retrieve a set of measurements (abusing notation)  $y(t)$  from the (possibly hidden) experimental trajectory  $\bar{x}(t)$ . Schematically we have:

$$u(t) \longrightarrow \bar{x}(u(t), t) \longrightarrow g(\bar{x}(t), t) = \bar{y}(t)$$

which coincides with the simplified model situation:

$$u(t) \longrightarrow \hat{x}(u(t), t) \longrightarrow g(\hat{x}(t), t) = \hat{y}(t)$$

We now have two sets of measurements:

- $\hat{y}(t)$  : the nominal measurement
- $\bar{y}(t)$  : the experimental measurement

## Problem Formulation

Let us write

$$\bar{x}(t) = \hat{x}(t) + \epsilon(t)$$

where  $\epsilon(t)$  is the error in the experimental trajectory. To correct for this error, we can find a correction term  $\Delta x(t)$  s.t.

$$g(\bar{x} + \Delta x) = g(\hat{x} + \epsilon + \Delta x) = \hat{y}$$

For example, if  $g(x) = x$  is the identity function, i.e. we are trying to track the trajectory:

$$\Delta x = -\epsilon$$

The real problem involves finding the corresponding correction to the controls:  $\Delta u(t)$ . This involves setting up a quadratic optimization problem.

## Quadratic Correction Problem

The goal is now to go from the measurement error  $\Delta y$  to a state correction  $\Delta x$  and a control correction  $\Delta u$  by simultaneously solving two linear systems. Schematically:

$$\Delta y \xrightarrow[g]{M \cdot \Delta x = \Delta y} \Delta x \xrightarrow[f]{D \cdot \Delta z = 0} \Delta u$$

### Measurement Correction to State Correction

With  $\Delta y \equiv \bar{y} - \hat{y}$ , we have

$$\begin{aligned} \bar{y} &= g(\bar{x}) \\ &= g(\hat{x} + \epsilon) \\ &\approx g(\hat{x}) + \nabla g(\hat{x}) \cdot \epsilon \\ &= \hat{y} + \nabla g(\hat{x}) \cdot \epsilon \end{aligned}$$

which, with writing  $\nabla \hat{g} = \nabla g(\hat{x})$  yields

$$\Delta y \approx \nabla \hat{g} \cdot \epsilon$$

and since  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  where  $m \leq n$ ,  $\nabla g$  is not necessarily invertible, but we can use the Moore-Penrose pseudoinverse here to get a guess for  $\epsilon$ :

$$\boxed{\epsilon \approx (\nabla \hat{g})^+ \cdot \Delta y \equiv \hat{\epsilon}}$$

To tie the experimental measurements to the model measurements we require

$$\begin{aligned} \hat{y} &= g(\bar{x} + \Delta x) \\ &\approx \bar{y} + \nabla g(\bar{x}) \cdot \Delta x \end{aligned}$$

which yields the condition

$$\boxed{\nabla g(\bar{x}) \cdot \Delta x = -\Delta y} \tag{1}$$

where

$$\begin{aligned} \nabla g(\bar{x})_i^j &= \nabla g(\hat{x} + \epsilon)_i^j \\ &\approx \nabla g(\hat{x} + \hat{\epsilon})_i^j \\ &= \nabla g(\hat{x})_i^j + \sum_k (\nabla^2 g(\hat{x}))_i^{jk} \hat{\epsilon}_k \\ &= \nabla \hat{g}_i^j + \sum_{kl} (\nabla^2 \hat{g})_i^{jk} \left( (\nabla \hat{g})^+ \right)_k^l \Delta y_l \end{aligned}$$

### State Correction to Control Correction

To propagate the state correction to the control correction, we utilize the dynamics constraint,  $f(z_t, z_{t+1}) = 0$ , where we define the *knot point*

$$z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}$$

Let's write  $\mathbf{z}_t = \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix}$ . Then we have

$$\begin{aligned} 0 &= f(\hat{\mathbf{z}}_t + \Delta \mathbf{z}_t) \\ &\approx f(\hat{\mathbf{z}}_t) + \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t \end{aligned}$$

which yields

$$\boxed{\nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0} \tag{2}$$

### Putting it all together

We seek to find the solution to

$$\begin{aligned} &\underset{\Delta x_{1:T}, \Delta u_{1:T}}{\text{minimize}} && \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\ &\text{subject to} && \nabla g(\bar{x}_\tau) \cdot \Delta x_\tau = -\Delta y_\tau \quad \forall \tau \\ &&& \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0 \quad \forall t \end{aligned}$$

where the  $\tau$ s are the measurement times.

Building the KKT matrix from this problem, we can solve the system and extract  $\Delta u(t)$  and repeat the procedure until convergence.

This problem, which returns  $\Delta Z$  is referred to as

$$\boxed{\Delta Z = \text{QuadraticProblem}(\hat{Z}, \Delta Y)}$$

### KKT Matrix (for just single quantum state and controls)

Below we use:

- $n = \dim z_t = \dim x_t + \dim u_t$
- $d = \dim x_t = \dim f(z_t, z_{t+1})$
- $c = \dim u_t$
- $m = \dim y_t$
- $M = \#$  of measurements

For a trajectory  $Z = \text{vec}(z_{1:T})$ , we need to construct the matrix

$$\begin{pmatrix} H & A^\top \\ A & 0 \end{pmatrix}$$

where  $H$  is the Hessian of the cost function:

$$H = \bigoplus_{t=1}^T (Q \oplus R) = I^{T \times T} \otimes (Q \oplus R)$$

and  $A$  is the constraint Jacobian:

$$A = \begin{pmatrix} \nabla F \\ \nabla G \end{pmatrix}$$

with

$$\nabla F = \begin{pmatrix} \nabla f(\hat{\mathbf{z}}_1) & & \\ & \ddots & \\ & & \nabla f(\hat{\mathbf{z}}_{T-1}) \end{pmatrix} \in \mathbb{R}^{d(T-1) \times nT}$$

and

$$\nabla G = \begin{pmatrix} \ddots & & \\ & \nabla g(\bar{x}_\tau) \mathbf{0}^{m \times (a+c)} & \\ & & \ddots \end{pmatrix} \in \mathbb{R}^{mM \times nT}$$

where  $\tau = t_1, \dots, t_M$  are the measurement times.

For the constraints we then have

$$\nabla F \cdot \Delta Z = 0 \quad \text{and} \quad \nabla G \cdot \Delta Z = -\Delta Y$$

where again

$$\Delta Y = \bar{Y} - \hat{Y}$$

## ILC Algorithm

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**Algorithm 1:** Iterative Control Learning

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**Data:**  $\hat{Z}^{\text{goal}}$ ,  $\text{tol} > 0$ ,  $\text{max\_iter} \gg 0$   
**Result:**  $U$   
 $Y^{\text{goal}} \leftarrow \text{measure}(\hat{Z}^{\text{goal}}) = \text{vec}(y_{\tau_1:\tau_M})$   
 $\hat{Z} \leftarrow \hat{Z}^{\text{goal}}$   
 $U \leftarrow \text{controls}(\hat{Z}) = \text{vec}(u_{1:T})$   
 $\bar{Y} \leftarrow \text{experiment}(U) = \text{vec}(\bar{y}_{\tau_1:\tau_M})$   
 $\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$   
 $k \leftarrow 1$   
**while**  $|\Delta Y| > \text{tol}$  **do**  
    **if**  $k > \text{max\_iter}$  **then**  
        **break**  
    **end**  
     $\Delta Z \leftarrow \text{QuadraticProblem}(\hat{Z}, \Delta Y)$   
     $\hat{Z} \leftarrow \hat{Z} + \Delta Z$   
     $U \leftarrow \text{controls}(\hat{Z})$   
     $\bar{Y} \leftarrow \text{experiment}(U)$   
     $\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$   
     $k \leftarrow k + 1$   
**end**  
**return**  $U$ 

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