# Iterative Learning Control with Measurements

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## Setup

Given a nominal state trajectory $\hat{x}(t)$ and control trajectory $u(t)$, we apply the controls to the experimental system and retrieve a set of measurements (abusing notation) $y(t)$ from the (possibly hidden) experimental trajectory $\bar{x}(t)$. Schematically we have:

$$u(t) \longrightarrow \bar{x}(u(t), t) \longrightarrow g(\bar{x}(t), t) = \bar{y}(t)$$

which coincides with the simplified model situation:

$$u(t) \longrightarrow \hat{x}(u(t), t) \longrightarrow g(\hat{x}(t), t) = \hat{y}(t)$$

We now have two sets of measurements:

- $\hat{y}(t)$ : the nominal measurement
- $\bar{y}(t)$ : the experimental measurement

## Problem Formulation

Let us write

$$\bar{x}(t) = \hat{x}(t) + e(t)$$

where $e(t)$ is the error in the experimental trajectory. To correct for this error, we can find a correction term $\Delta x(t)$ s.t.

$$g(\bar{x} + \Delta x) = g(\hat{x} + e + \Delta x) = \hat{y}$$

For example, if $g(x) = x$ is the identity function, i.e. we are trying to track the trajectory:

$$\Delta x = -$$

The real problem involves finding the corresponding correction to the controls: $\Delta u(t)$. This involves setting up a quadratic optimization problem.

## Quadratic Correction Problem

The goal is now to go from the measurement error $\Delta y$ to a state correction $\Delta x$ and a control correction $\Delta u$ by simultaneously solving two linear systems. Schematically:

$$\Delta y \xrightarrow[g]{M\cdot\Delta x=\Delta y} \Delta x \xrightarrow[f]{D\cdot\Delta z=0} \Delta u$$

### Measurement Correction to State Correction

With $\Delta y \equiv \bar{y} - \hat{y}$, we have

$$\begin{aligned}
\bar{y} &= g(\bar{x}) \\
&= g(\hat{x} + e) \\
&\approx g(\hat{x}) + \nabla g(\hat{x}) \cdot e \\
&= \hat{y} + \nabla g(\hat{x}) \cdot e
\end{aligned}$$

which, with writing $\nabla\hat{g} = \nabla g(\hat{x})$ yields

$$\Delta y \approx \nabla\hat{g} \cdot e$$

and since $g : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$, $\nabla g$ is not necessarily invertible, but we can use the Moore-Penrose pseudoinverse here to get a guess for $e$:

$$\boxed{e \approx (\nabla\hat{g})^{+} \cdot \Delta y \equiv \hat{e}}$$

To tie the experimental measurements to the model measurements we require

$$\begin{aligned}
\hat{y} &= g(\bar{x} + \Delta x) \\
&\approx \bar{y} + \nabla g(\bar{x}) \cdot \Delta x
\end{aligned}$$

which yields the condition

$$\boxed{\nabla g(\bar{x}) \cdot \Delta x = -\Delta y} \tag{1}$$

where

$$\begin{aligned}
\nabla g(\bar{x})_i^j &= \nabla g(\hat{x} + e)_i^j \\
&\approx \nabla g(\hat{x} + \hat{e})_i^j \\
&= \nabla g(\hat{x})_i^j + \sum_k \left(\nabla^2 g(\hat{x})\right)_i^{jk} \hat{e}_k \\
&= \nabla\hat{g}_i^j + \sum_{kl} \left(\nabla^2\hat{g}\right)_i^{jk} \left((\nabla\hat{g})^{+}\right)_k^l \Delta y_l
\end{aligned}$$

### State Correction to Control Correction

To propagate the state correction to the control correction, we utilize the dynamics constraint, $f(z_t, z_{t+1}) = 0$, where we define the *knot point*

$$z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}$$

Let's write $\mathbf{z}_t = \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix}$. Then we have

$$\begin{aligned} 0 &= f(\hat{\mathbf{z}}_t + \Delta \mathbf{z}_t) \\ &\approx f(\hat{\mathbf{z}}_t) + \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t \end{aligned}$$

which yields

$$\boxed{\nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0} \tag{2}$$

### Putting it all together

We seek to find the solution to

$$\begin{aligned} \underset{\Delta x_{1:T},\ \Delta u_{1:T}}{\text{minimize}} \quad & \frac{1}{2} \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\ \text{subject to} \quad & \nabla g(\bar{x}_\tau) \cdot \Delta x_\tau = -\Delta y_\tau \quad \forall \tau \\ & \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0 \quad \forall t \end{aligned}$$

where the $\tau$s are the measurement times.

Building the KKT matrix from this problem, we can solve the system and extract $\Delta u(t)$ and repeat the procedure until convergence.

This problem, which returns $\Delta Z$ is referred to as

$$\boxed{\Delta Z = \mathsf{QuadraticProblem}(\hat{Z}, \Delta Y)}$$

**KKT Matrix (for just single quantum state and controls)**

Below we use:

- $n = \dim z_t = \dim x_t + \dim u_t$
- $d = \dim x_t = \dim f(z_t, z_{t+1})$
- $c = \dim u_t$
- $m = \dim y_t$
- $M = \#$ of measurements

For a trajectory $Z = \mathrm{vec}(z_{1:T})$, we need to construct the matrix

$$\begin{pmatrix} H & A^\top \\ A & 0 \end{pmatrix}$$

where $H$ is the Hessian of the cost function:

$$H = \bigoplus_{t=1}^{T} (Q \oplus R) = I^{T\times T} \otimes (Q \oplus R)$$

and $A$ is the constraint Jacobian:

$$A = \begin{pmatrix} \nabla F \\ \nabla G \end{pmatrix}$$

with

$$\nabla F = \begin{pmatrix} \nabla f(\hat{\mathbf{z}}_1) & & \\ & \ddots & \\ & & \nabla f(\hat{\mathbf{z}}_{T-1}) \end{pmatrix} \in \mathbb{R}^{d(T-1)\times nT}$$

and

$$\nabla G = \begin{pmatrix} \ddots & & & \\ & \nabla g(\bar{x}_\tau) & \mathbf{0}^{m\times(a+c)} & \\ & & & \ddots \end{pmatrix} \in \mathbb{R}^{mM\times nT}$$

where $\tau = t_1, \ldots, t_M$ are the measurement times.

For the constraints we then have

$$\nabla F \cdot \Delta Z = 0 \quad \text{and} \quad \nabla G \cdot \Delta Z = -\Delta Y$$

where again

$$\Delta Y = \bar{Y} - \hat{Y}$$

## An Alternative Quadratic Problem

In the regime of noisy measurements, satisfying both the dynamics constraints and the measurement constraints becomes infeasible. To overcome this we we can relax the measurement into a maximum likelihood problem. To see this let's write

$$\begin{aligned}\bar{y} = \hat{y} - \nabla g \cdot \Delta x &\implies \bar{y} \sim \mathcal{N}(\hat{y} - \nabla g \cdot \Delta x, \Sigma) \\ &\implies \Delta y \sim \mathcal{N}(-\nabla g \cdot \Delta x, \Sigma)\end{aligned}$$

where $\Sigma$ is the covariance matrix of the measurement noise, which we can get from the experiment. To make the following clearer, let's define the parameterized distribution over $\Delta y$ as

$$p(\Delta x) = \mathcal{N}(-\nabla g \cdot \Delta x, \Sigma)$$

then, given an observation $\bar{y}$, we can find the MLE for the parameter $\Delta x$ as the solution to the following optimization problem:

$$\begin{aligned}\max_{\Delta x} \ \log p(\Delta x) \quad &\implies \quad \min_{\Delta x} \ \frac{1}{2}(\Delta y + \nabla g \cdot \Delta x)^\top \Sigma^{-1}(\Delta y + \nabla g \cdot \Delta x) \\ &\implies \quad \min_{\Delta x} \ \frac{1}{2}\Delta x^\top \nabla g^\top \Sigma^{-1} \nabla g \Delta x + \Delta y^\top \Sigma^{-1} \nabla g \Delta x\end{aligned}$$

We can then augment our initial problem with this objective term and remove the measurement constraint. This yields the following problem:

$$\begin{aligned}\underset{\Delta x_{1:T},\ \Delta u_{1:T}}{\text{minimize}} \quad & \frac{1}{2}\sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\ & + \sum_\tau \frac{1}{2}\Delta x_\tau^\top \big(\nabla g^\top \Sigma^{-1} \nabla g\big)\Delta x_\tau + \big(\Delta y_\tau^\top \Sigma^{-1} \nabla g\big)\Delta x_\tau \\ \text{subject to} \quad & \nabla f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t = 0 \quad \forall t \\ & - u_{\max} - \hat{u}_t < \Delta u_t < u_{\max} - \hat{u}_t \\ & \Delta u_1 = \Delta u_T = 0\end{aligned}$$

# ILC Algorithm

**Algorithm 1:** Iterative Control Learning

**Data:** $\hat{Z}^{\text{goal}}$, $\mathsf{tol} > 0$, $\mathsf{max\_iter} \gg 0$
**Result:** $U$
$Y^{\text{goal}} \leftarrow \mathsf{measure}(\hat{Z}^{\text{goal}}) = \text{vec}(y_{\tau_1:\tau_M})$
$\hat{Z} \leftarrow \hat{Z}^{\text{goal}}$
$U \leftarrow \mathsf{controls}(\hat{Z}) = \text{vec}(u_{1:T})$
$\bar{Y} \leftarrow \mathsf{experiment}(U) = \text{vec}(\bar{y}_{\tau_1:\tau_M})$
$\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$
$k \leftarrow 1$
**while** $|\Delta Y| > \mathit{tol}$ **do**
  **if** $k > \mathit{max\_iter}$ **then**
    **break**
  **end**
  $\Delta Z \leftarrow \mathsf{QuadraticProblem}(\hat{Z}, \Delta Y)$
  $\hat{Z} \leftarrow \hat{Z} + \Delta Z$
  $U \leftarrow \mathsf{controls}(\hat{Z})$
  $\bar{Y} \leftarrow \mathsf{experiment}(U)$
  $\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$
  $k \leftarrow k + 1$
**end**
**return** $U$