CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 4 · Solutions

Posted Wednesday 12 September 2012. Due Wednesday 19 September 2012, 5pm.

1. [20 points: 8 points for (a); 12 points for (b)]
   The equation $x_1 + x_2 + x_3 = 0$ defines a plane in $\mathbb{R}^3$ that passes through the origin.

   (a) Find two linearly independent vectors in $\mathbb{R}^3$ whose span is this plane.

   (b) Find the point in this plane closest (in the standard Euclidean norm, $\|\mathbf{z}\| = \sqrt{\mathbf{z}^T\mathbf{z}}$) to the vector

   $$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   by formulating this as a best approximation problem. (You may use MATLAB to invert a matrix.)

Solution.

(a) Since two linearly independent vectors determine a plane, we simply need to find two linearly independent vectors that satisfy $x_1 + x_2 + x_3 = 0$. One can do this by inspection, for example, and find

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

However, it would be nice to have an orthogonal basis for this space. To do that, pick one vector, say the first vector given above; set the second vector to be $(\alpha, \beta, \gamma)^T$. We would like the this vector to be in the plane:

$$\alpha + \beta + \gamma = 0$$

and to be orthogonal to the first vector:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha - \beta + 0 = 0.$$

This gives two equations in three unknowns, which will be satisfied if $\beta = \alpha$ and $\gamma = -2\alpha$ for any α, i.e., we have the vector

$$\begin{bmatrix} \alpha \\ \alpha \\ -2\alpha \end{bmatrix}.$$

With $\alpha = 1$, we have two orthogonal vectors whose span is the desired plane:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

(b) The closest point in the plane to the vector $\mathbf{v}$ is found solving the usual best-approximation problem matrix equation:

$$\begin{bmatrix} \mathbf{x}^T\mathbf{x} & \mathbf{x}^T\mathbf{y} \\ \mathbf{y}^T\mathbf{x} & \mathbf{y}^T\mathbf{y} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T\mathbf{v} \\ \mathbf{y}^T\mathbf{v} \end{bmatrix},$$

that is,

$$\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The orthogonality of the vectors $\mathbf{x}$ and $\mathbf{y}$ make this an easy problem to solve:

$$c_1 = 1/2, \qquad c_2 = -1/6.$$

Thus, the best approximation to $\mathbf{v} = (1, 0, 1)^T$ is the vector

$$\widehat{\mathbf{v}} = c_1\mathbf{x} + c_2\mathbf{y} = \begin{bmatrix} 1/2 - 1/6 \\ -1/2 - 1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

We can verify this answer by checking (1) that $\widehat{\mathbf{v}}$ is in the desired plane: $1/3 - 2/3 + 1/3 = 0$, and (2) verifying that the error

$$\mathbf{v} - \widehat{\mathbf{v}} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

is orthogonal to the two basis vectors $\mathbf{x}$ and $\mathbf{y}$ for the plane, $(\mathbf{v} - \widehat{\mathbf{v}})^T\mathbf{x} = (\mathbf{v} - \widehat{\mathbf{v}})^T\mathbf{y} = 0$.

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2. [25 points: 10 points each for (a) and (b); 5 points for (c)]
Recall that a linear operator P is a projection from the vector space V to the vector space V provided $P^2 = P$, that is, $P(Pf) = Pf$ for all $f \in V$. Consider $V = C[-1, 1]$ with the usual inner product

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx,$$

and the two linear operators P_e and P_o the project a function onto their even and odd parts. That is,

$$(P_e f)(x) = \frac{f(x) + f(-x)}{2}, \qquad (P_o f)(x) = \frac{f(x) - f(-x)}{2}.$$

(a) Show that P_e and P_o are projections.

(b) Verify that $P_e f$ and $P_o f$ are orthogonal for any $f \in C[-1, 1]$.

(c) Is $P_e + P_o$ a projection? Explain.

Solution.

(a) To check that P_e is a projection, we will apply P_e to some function $f \in V = C[-1, 1]$, then apply P_e to the result, i.e., we will check if $P_e f = P_e(P_e f)$ for any $f \in C[-1, 1]$. Note that

$$P_e f = \frac{f(x) + f(-x)}{2},$$

so

$$P_e(P_e f) = \frac{\left(\frac{f(x) + f(-x)}{2}\right) + \left(\frac{f(-x) + f(x)}{2}\right)}{2} = \frac{f(x) + f(-x)}{2} = P_e f.$$

Thus we conclude that $P_e^2 f = P_e f$ for all f, which means that $P_e^2 = P_2$, i.e., P_e is a projection. (We have just proved that “the even part of an even function is itself”.) In the same way, we have

$$P_o f = \frac{f(x) - f(-x)}{2},$$

and

$$P_o(P_o f) = \frac{\left(\frac{f(x)-f(-x)}{2}\right) - \left(\frac{f(-x)-f(x)}{2}\right)}{2} = \frac{f(x)-f(-x)}{2} = P_o f,$$

so P_o is also a projector.

(b) Notice that for any $f \in C[-1,1]$, $P_e f$ is even and $P_o f$ is odd. Consider the integrand in the inner product

$$(P_e f, P_o f) = \int_{-1}^{1} (P_e f)(x)(P_o f)(x)\,dx.$$

Since $P_e f$ is even and $P_o f$ is odd, the integrand is the product of even and odd functions, which must be odd. But the integral of an odd function from -1 to 1 is zero. Hence, $P_e f$ and $P_o f$ are orthogonal.

The above is an entirely acceptable mathematical argument, and is sufficient for full credit. One could have equivalently proceeded more algebraically:

$$\begin{aligned}
(P_e f, P_o f) = \int_{-1}^{1} (P_e f)(x)(P_o f)(x)\,dx &= \frac{1}{4}\int_{-1}^{1} f(x)^2 - f(x)f(-x) + f(x)f(-x) - f(-x)^2\,dx \\
&= \frac{1}{4}\int_{-1}^{1} f(x)^2 - f(-x)^2\,dx \\
&= \frac{1}{4}\Big(\int_{-1}^{1} f(x)^2\,dx - \int_{-1}^{1} f(-x)^2\,dx \\
&= \frac{1}{4}\Big(\int_{-1}^{1} f(x)^2\,dx + \int_{1}^{-1} f(x)^2\,dx\Big) \\
&= \frac{1}{4}\Big(\int_{-1}^{1} f(x)^2\,dx - \int_{-1}^{1} f(x)^2\,dx\Big) \\
&= 0.
\end{aligned}$$

(c) Yes, $P_e + P_o$ is a projection; there are several ways to see this.

Notice that $P_e(P_o f) = P_o(P_e f) = 0$ for all $f \in C[-1,1]$, which implies that $P_e P_o = P_o P_e = 0$, hence

$$(P_e + P_o)^2 = P_e^2 + P_e P_o + P_o P_e + P_o^2 = P_e^2 + P_o^2 = P_e + P_o.$$

More simply, notice that $(P_e + P_o)f = P_e f + P_o f = f$, so $P_e + P_o = I$, the identity operator, which is trivially a projection.

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3. [25 points: 12 points for (a); 5 points for (b); 8 points for (c)]
Suppose that

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) Compute *by hand* the eigenvalues and eigenvectors of this matrix.

(b) Verify *by hand* that these eigenvectors are orthogonal.

(c) Solve the linear system $\mathbf{Ax} = \mathbf{b}$ using the spectral method, where

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Solution.

(a) For this matrix $\mathbf{A}$ we have

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix},$$

and hence the characteristic polynomial is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - 3)(\lambda^2 + 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial, which we label

$$\lambda_1 = -1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3.$$

[**GRADERS:** Below it is fine if students leave the eigenvectors in a general form (e.g., $\mathbf{u}_1 = [0, u_2, -u_2]^T$) at this stage, and wait until part (c) to pick concrete entries.]

To compute the eigenvectors associated with each eigenvalue, we look for a nonzero vector in the null space of $\lambda_j\mathbf{I} - \mathbf{A}$.

$\lambda_1 = -1$: We seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -(u_2 + u_3) \\ -(u_2 + u_3) \end{bmatrix}.$$

The only way to make this vector zero is to set $u_1 = 0$ and $u_2 = -u_3$. Thus any vector of the form

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ u_2 \\ -u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalues $\lambda_1 = -1$. We select

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

so that $\|\mathbf{u}_1\| = 1$.

$\lambda_2 = 1$: We now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_2\mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ u_2 - u_3 \\ u_3 - u_2 \end{bmatrix}.$$

Any vector of the form

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalues $\lambda_2 = 1$. We select

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$\lambda_3 = 3$: We now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_3\mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2 - u_3 \\ 3u_3 - u_2 \end{bmatrix}.$$

To make the second component zero we need $u_2 = u_3/3$, while to make the third component zero we need $u_3 = u_2/3$. The only way to accomplish both is to set $u_2 = u_3 = 0$. Thus any vector of the form

$$\mathbf{u}_3 = \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, \quad u_1 \neq 0$$

is an eigenvector associated with the eigenvalues $\lambda_3 = 3$. We select

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) One can quickly check that

$$\mathbf{u}_1^T\mathbf{u}_2 = \mathbf{u}_2^T\mathbf{u}_1 = (1/\sqrt{2})(0\cdot 0 + 1\cdot 1 + (-1)\cdot 1) = 0.$$

$$\mathbf{u}_1^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_1 = (0\cdot 1 + (1/\sqrt{2})\cdot 0 + (-1/\sqrt{2})\cdot 0) = 0.$$

$$\mathbf{u}_2^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_2 = (0\cdot 1 + (1/\sqrt{2})\cdot 0 + (1/\sqrt{2})\cdot 0) = 0.$$

(c) The spectral method gives $\mathbf{x}$ as a linear combination of the eigenvectors:

$$\mathbf{x} = \sum_{j=1}^{3} \frac{\mathbf{u}_j^T\mathbf{b}}{\lambda_j}\mathbf{u}_j.$$

We compute

$$\begin{aligned} \mathbf{u}_1^T\mathbf{b} &= 0\cdot 2 + (1/\sqrt{2})\cdot(-1) + (-1/\sqrt{2})\cdot 3 = -2\sqrt{2} \\ \mathbf{u}_2^T\mathbf{b} &= 0\cdot 2 + (1/\sqrt{2})\cdot(-1) + (1/\sqrt{2})\cdot 3 = \sqrt{2} \\ \mathbf{u}_3^T\mathbf{b} &= 1\cdot 2 + 0\cdot(-1) + 0\cdot 3 = 2, \end{aligned}$$

and hence

$$\mathbf{x} = \frac{-2\sqrt{2}}{-1}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \frac{\sqrt{2}}{1}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 3 \\ -1 \end{bmatrix}.$$

We can multiply $\mathbf{Ax}$ out to verify that the desired $\mathbf{b}$ is obtained.

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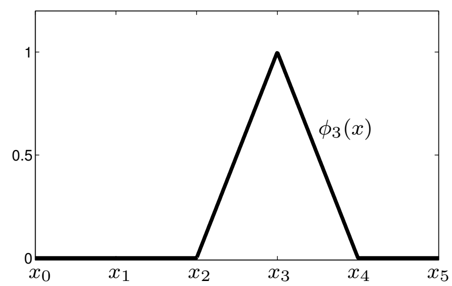
4. [30 points: 12 points for (a); 8 points for (b); 10 points for (c)]
Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_k = kh$ for $k = 0, \ldots, N+1$. Consider the N *hat functions*, defined as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$



The plot to the right shows $\phi_3(x)$ for $N = 4$.

Consider the standard inner product on $C[0,1]$,

$$(u,v) = \int_0^1 u(x)v(x)\,dx.$$

(a) Compute the inner products (ϕ_j, ϕ_k) for $k = 1, \ldots, N$, obtaining answers that depend on N (or h) only. Consider the following cases individually:

- (ϕ_j, ϕ_j) for $j = 1, \ldots, N$;
- (ϕ_j, ϕ_{j+1}) for $j = 1, \ldots, N-1$;
- (ϕ_j, ϕ_k) for $|j-k| > 1$.

(b) For $f(x) = \sin(\pi x)$, compute the inner products (ϕ_k, f) for $k = 1, \ldots, N$.

(c) Use your solutions to (a) and (b) to set up a linear system (in MATLAB) and solve it to compute the best approximations $f_N(x)$ from $\text{span}\{\phi_1, \ldots, \phi_N\}$ to $f(x) = \sin(\pi x)$ for $N = 3$ and $N = 9$ over the interval $[0,1]$ with the standard inner product.

For each of these N, use the `hat.m` code (from Problem Set 1, either your code or from the solutions) to plot your best approximations. For each N, produce one plot that compares $f_N(x)$ to $f(x)$, and a second plot that shows the error $f(x) - f_N(x)$.

[Be careful: Are the basis functions used for the best approximation orthogonal?]

Solution.

(a) To find (ϕ_j, ϕ_k), we integrate only over the *support* of the integrand, that is, we only consider the region of $[0,1]$ on which $\phi_j(x)\phi_k(x)$ is nonzero.

- First we consider $j = k$. Note that the answer (the area under the square of a hat function) is independent of placement on the x axis, so we can pick the interval of integration as convenient:

$$\begin{aligned}
(\phi_j, \phi_j) &= \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h}\right)^2 dx \\
&= \int_0^h \frac{x^2}{h^2}\,dx + \int_{-h}^0 \frac{(-x)^2}{h^2}\,dx \\
&= \frac{h^3}{3h^2} + \frac{h^3}{3h^2} = \frac{2h}{3}.
\end{aligned}$$

- Next we consider the inner product of two adjacent hat functions, (ϕ_j, ϕ_{j+1}), noting that the support of the two functions ϕ_j and ϕ_{j+1} only overlaps on (x_j, x_{j+1}). Again shifting to integration to a convenient region, we have:

$$\begin{aligned}
(\phi_j, \phi_{j+1}) &= \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h}\right)\left(\frac{x - x_j}{h}\right) dx \\
&= \int_0^h \left(\frac{h - x}{h}\right)\left(\frac{x}{h}\right) dx \\
&= \frac{h^3}{2h^2} - \frac{h^3}{3h^2} = \frac{h}{6}.
\end{aligned}$$

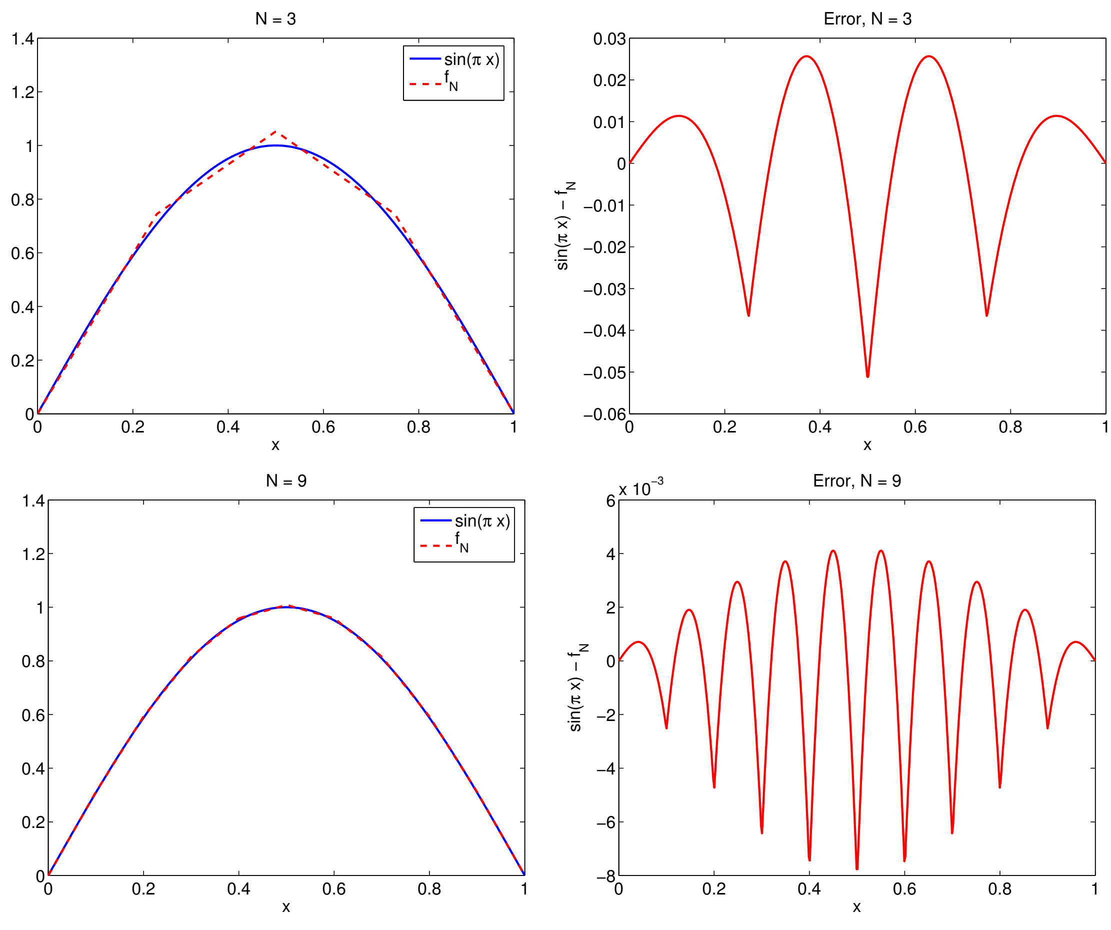
- Finally, we note that $(\phi_j, \phi_k) = 0$ when $|j-k| > 1$, as the supports of ϕ_j and ϕ_k do not overlap and hence $\phi_j(x)\phi_k(x) = 0$ for all $x \in [0,1]$.

(b) The inner products $(\phi_j, \sin(\pi x))$ are tedious to compute by hand; one requires one integration by parts and a considerable amount of tedious algebra to arrive at the formula

$$\begin{aligned}(\phi_j, f) &= \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \sin(\pi x)\,dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \sin(\pi x)\,dx \\ &= \frac{2\sin(\pi x_j) - \sin(\pi x_{j-1}) - \sin(\pi x_{j+1})}{\pi^2 h} \\ &= \frac{2\sin(\pi x_j)}{\pi^2 h}(1 - \cos(h\pi)).\end{aligned}$$

(c) The requested plots are shown below, followed by the MATLAB code that generated them.

[**GRADERS:** Please deduct 10 points for solutions that treat the hat functions as if they are orthogonal, and thus don't set up a Gram matrix to determine the coefficients of the best approximation.]




```
xx = linspace(0,1,500)';
for N = [3 9]
   h = 1/(N+1);
   x = [0:N+1]*h;

% set up the matrix from the inner products computed in part (a)
   A = 2*h/3*eye(N) + h/6*diag(ones(N-1,1),1) + h/6*diag(ones(N-1,1),-1);
```

```
% set up the right-hand side vector from the inner products in part (b)
    b = 2/(h*pi^2)*(1-cos(h*pi))*sin(h*pi*[1:N]');

% solve for the coefficients
    c = A\b;

% compute the approximation on fine grid on [0,1]
    fN = zeros(length(xx),1);
    for j=1:N
       fN = fN + c(j)*hat(xx,j,N);
    end

% plot the function f and the approximation
    figure(2), clf
    plot(xx, sin(pi*xx), 'b-','linewidth',2), hold on
    plot(xx, fN, 'r--','linewidth',2)
    legend('sin(\pi x)', 'f_N')
    set(gca,'fontsize',16)
    xlabel('x'), title(sprintf('N = %d', N))

% plot the error
    figure(3), clf
    plot(xx, sin(pi*xx)-fN, 'r-','linewidth',2)
    set(gca,'fontsize',16)
    xlabel('x'), title(sprintf('Error, N = %d', N))
    ylabel('sin(\pi x) - f_N')
end



 function phi_k = hat(x,k,n)

% function phi_k = hat(x,k,n)
%
% evaluates the hat function phi_k(x), where n denotes the
% size of the mesh, so that phi_k is non-zero on ((k-1)*h,(k+1)*h)
% with h = 1/(n+1).

 h = 1/(n+1);
 xk = [0:n+1]*h;

 if k==0,       phi_k =  ((x>=xk(1))&(x<xk(2))).*((xk(2)-x)/h);
 elseif k==n+1, phi_k =  ((x>=xk(n+1))&(x<=xk(n+2))).*((x-xk(n+1))/h);
 else,          phi_k =  ((x>=xk(k))&(x<xk(k+1))).*((x-xk(k))/h) ...
                       + ((x>=xk(k+1))&(x<xk(k+2))).*((xk(k+2)-x)/h);
 end
```