Study highlights: 01/23/2015

<u>Introduction to vector spaces and linear operators:</u> (Chap 3.1)

Linear Algebra is the study of <u>linear functions</u> (also called linear operators) defined on a <u>finite dimensional vector space</u>.

Defn: A vector space, V, is a set (whose elements we call 'vectors') on which two operations are defined. These operations are called "addition" and "scalar multiplication". These operations must be defined in such a way that the following properties are satisfied:

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}, \vec{v}$ in V
2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$
3) There is a "zero vector", denoted $\vec{0}$, in V with $\vec{u} + \vec{0} = \vec{u}$ for all $\vec{u}$ in V
4) For each $\vec{u}$ in V there is a vector "$-\vec{u}$" with $\vec{u} + (-\vec{u}) = \vec{0}$
5) $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ for all $\vec{u}, \vec{v}$ in V and all scalars α.
6) $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$ for all $\vec{u}$ in V and all scalars α, β.
7) $\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}$ for all $\vec{u}$ in V a all scalars α, β.
8) $1\vec{u} = \vec{u}$ for all $\vec{u}$ in V.

<u>Examples</u>:

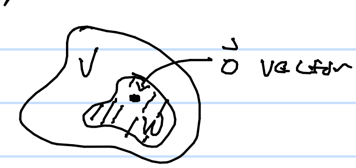
- n-dimensional real space $\mathbb{R}^n$
- polynomials of degree at most n
- Set of all real-valued functions on $[0,1]$

Q: Are "polynomials of degree <u>exactly</u> $n>0$" a vector space?

Defn: Let V be a vector space and let W be a subset of V. (e.g. if $f \in W$ then $f \in V$) Suppose that W has the following properties:

1) The zero vector $\vec{0} \in W$
2) If $\vec{u}, \vec{v} \in W$ then $\alpha\vec{u} + \beta\vec{v} \in W$ for any scalars α, β.

Then W satisfies all of the criteria of a vector space in its own right and we say that <u>W is a subspace of V</u>.



* Although the term "vector space" typically encourages one to think in terms of the "vectors" of multivariable Calculus the term "vector" is an abstract concept which is used encourage <u>analogies</u> with the more intuitive and well understood "physical" spaces such as $\mathbb{R}^n$.

<u>Example</u>:

Define $C_D^2[a,b] = \{ u \in C^2[a,b] : u(a) = u(b) = 0 \}$

<u>Recall</u>: that $C^2[a,b]$ is the set of all functions whose second derivative is continuous on $[a,b]$.

▷ $C_D^2[a,b]$ is a vector space.

To see this is true verify each of the requirements of the definition of a vector space.

<u>Remark</u>: The set of functions $C^2[a,b]$ is <u>also</u> a vector space. Since every f in $C_D^2[a,b]$ is also in $C^2[a,b]$ then $C_D^2[a,b]$ is a sub-vector space of $C^2[a,b]$

▷ <u>Question</u>: Consider $\hat{C}_D^2[a,b] = \{ f \text{ in } C^2[a,b] \mid f(a) = 1 \ f(b) = 2 \}$ is $\hat{C}_D^2[a,b]$ a vector space? Why or why not?

<u>Hint</u>: if f and g are in $\hat{C}_D^2[a,b]$ is $f+g$?

<u>Example</u>: Define $C_N^2[a,b] = \{ f \text{ in } C^2[a,b] \mid \frac{\partial f}{\partial x}(b) = \frac{\partial f}{\partial x}(a) = 0 \}$

Then $C_N^2[a,b]$ is a vector space.

<u>Definition</u> (linear operators defined on vector spaces): Let X and Y be vector spaces and let $f: X \to Y$ be a function assigning to each $x \in X$ a $y \in Y$. Then we say f is a <u>linear operator</u> if for all scalars α, β and every x, z in X we have

$$f(\alpha x + \beta z) = \alpha f(x) + \beta f(z)$$

As a corollary one can verify that an operator f is linear if and only if Both of the following hold:

1) $f(\alpha x) = \alpha f(x)$
2) $f(x+z) = f(x) + f(z)$

<u>Example:</u> $\mathbb{R}$ is a vector space and $f(x) = \sqrt{x}$ is an operator from $\mathbb{R}$ to $\mathbb{R}$. However f is <u>not</u> a linear operator since

$$f(3x) = \sqrt{3x} = \sqrt{3}\sqrt{x} = \sqrt{3} f(x) \neq 3f(x) = 3\sqrt{x}$$

<u>Example:</u> Matrix multiplication $f: \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(x) = Ax$ <u>is</u> a linear operator. This can be shown by writing down the definition of matrix multiplication and multiplication of a matrix by a scalar and comparing each term.

<u>Question:</u> Show that <u>differentiation</u> is a linear operator. $\frac{\partial}{\partial x}: C_D^2[a,b] \to C^1[a,b]$

<u>Note:</u> if we take a function f in $C_D^2[a,b]$ we know that f has a first and second derivative which are continuous. We also know that $f(a) = f(b) = 0$. So we know that $\frac{\partial}{\partial x} f$ has at least one more continuous derivative; e.g. $\frac{\partial}{\partial x} f$ is in $C^1[a,b]$. Do we necessarily know that $\frac{\partial}{\partial x} f(a) = \frac{\partial}{\partial x} f(b) = 0$? To answer this think about $\sin(x)$ where $[a,b] = [0,\pi]$.

<u>Key IDEA:</u> WE CAN think of linear differential equations as linear operators on appropriate vector spaces!

<u>Example:</u> Think of the steady state heat eqn with homogeneous boundary conditions:

$$-K \frac{\partial^2}{\partial x^2} u = f, \quad u(0) = 0 \quad u(l) = 0$$

- Know that $\frac{\partial^2}{\partial x^2}$ is a linear operator
- Define L_D as $L_D[g] = \left(-K \frac{\partial^2}{\partial x^2}\right) g$ Then $L_D: C_D^2[0,l] \to C[0,l]$ is a linear operator
- Solving the differential equation means finding a "vector" u in $C_D^2[0,l]$ such that $L_D[u] = f$
  - → Very similar to solving the problem "$Ax = b$" in a typical linear algebra course!