CAAM 336 Midterm Review

1/15/12

what have we done thus far?

1. classification of diff eqns
   - ODEs vs PDEs, order, linear vs nonlinear, homogeneous vs not
     constant vs variable coefficient

$u''(x) + \sin(x)u(x) = e^x \Rightarrow$ linear
u appears linearly

$u''(x) + \sin(u(x))u(x) = e^x \Rightarrow$ non-linear

*check exam for Fall 2008 → Embree wrote this one

2. derived the heat equation

$u_t(x,t) = (k(x)\,u_x(x,t))_x + f(x,t)$
→ no details of the physical derivation on the exam

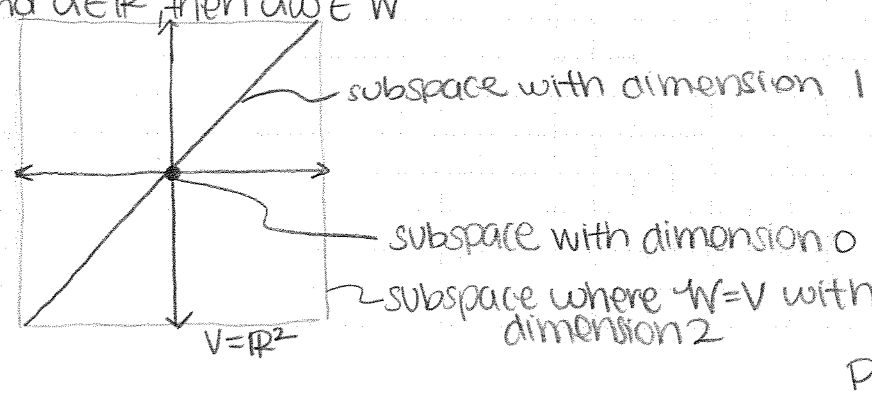
At steady state: $u_t(x,t) = 0$
$-(k(x)u'(x))' = f(x) \Rightarrow$ steady state heat equation

3. analogy between linear algebra and linear differential eqns
   → $Ax=b$ → $-\frac{d^2}{dx^2}u = f$

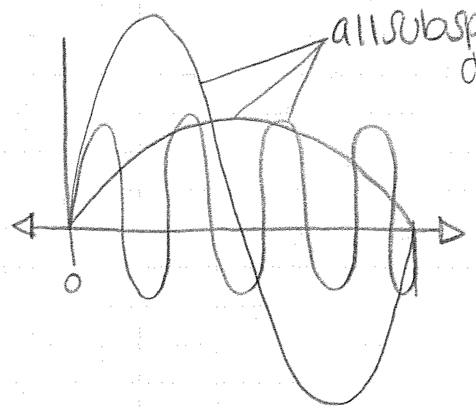
   - Theme: generalize notions of linear algebra to the diff eq. setting
     - crude approach: finite differences $\sim \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$
     - better approach requires more mathematical machinery

4. Definitions
   - linear spaces (vector spaces)
     A set of vectors V is a vectorspace provided
     - if $u, v \in V$, then $u+v \in V$
     - if $u \in V$ and $\alpha \in \mathbb{R}$, then $\alpha u \in V$

     A subset W of V ($W \subseteq V$) if a subspace if:
     - $w_1, w_2 \in W$, then $w_1 + w_2 \in W$
     - $w \in W$ and $\alpha \in \mathbb{R}$, then $\alpha w \in W$
     - $0 \in W$



$C[0,1]$ $\qquad C_0[0,1]$



all subspaces w/ dirichlet boundary conditions

* one way to make subspaces:

$\phi_1, \phi_2, \ldots, \phi_n \in V$

$W = \text{span}\{\phi_1, \phi_2, \ldots \phi_n\}$

$= \{c_1\phi_1 + c_2\phi_2 + \ldots + c_n\phi_n : c_1, c_2 \ldots c_n \in \mathbb{R}\}$

- <u>linear independence</u>: a set $\phi_1, \ldots, \phi_n$ is linearly independent provided none of these vectors can be written as the linear combination of the others. Equivolently, if $0 = c_1\phi_1 + \ldots + c_n\phi_n$, then $c_1 = c_2 = \ldots = c_n = 0$

  - <u>Basis</u>: $\phi_1, \ldots, \phi_n$ is a basis for W provided:
    - $\phi_1, \ldots, \phi_n$ are linearly independent
    - $\text{span}\{\phi_1, \ldots, \phi_n\} = W$

  - <u>dimension</u>: the number of vectors in the basis for W

- <u>inner products</u>: a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$ (V is a vectorspace) is an inner product provided:

  - $(u,v) = (v,u) \quad \forall u,v \in V$
  - $(u,u) > 0 \quad \forall u \in V, u \neq 0$
  - $(u,u) = 0 \quad$ if $u = 0$
  - $(\alpha u + w, v) = \alpha(u,v) + (w,v) \quad \forall u,v,w \in V, \forall \alpha \in \mathbb{R}$ (linearity)

  - norm: $\|u\| = (u,u)^{1/2} = \sqrt{(u,u)}$

- <u>Linear operator</u>: a map $L: V \to W$ (V & W are vectorspaces) is a linear operator provided:

$$L(\alpha u + w) = \alpha Lu + Lw \quad \forall u,w \in V, \forall \alpha \in \mathbb{R}$$

- <u>Eigenvalues and Eigenfunctions</u>

let $L: V \to V$ be a linear operator

A number $\lambda \in \mathbb{R}$ is an <u>eigenvalue</u> of L provided there exists a nonzero $v \in V$ such that

$$Lv = \lambda v$$

↳ vector space w/ inner product

- Symmetric Operator: a linear operator $L: V \to V$ is symmetric if $(Lu, v) = (u, Lv)$ $\forall u, v \in V$

  - the eigenvalues of symmetric matrices are real, and eigenvectors associated with distinct eigenvalues are orthogonal
  - Two vectors $u, v \in V$ are orthogonal provided $(u, v) = 0$ and are orthonormal provided $(u, v) = 0$ and $\|u\|$ and $\|v\|$ are 1
  - a basis $\{\phi_1, \ldots, \phi_n\}$ is orthonormal if $(\phi_j, \phi_k) \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ $\quad (\phi_j, \phi_j) = \|\phi_j\|^2$

5. Best approximation

- From a vector space w/ an inner product given $f \in V$ and a subspace W of V, find $\hat{\omega} \in W$ that minimizes $\|f - \hat{\omega}\|$ over all approximations of f from W:

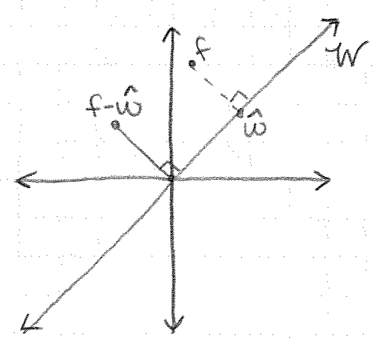
$$\|f - \hat{\omega}\| = \min_{\omega \in W} \|f - \omega\|$$

$$W = \text{span}\{\phi_1, \ldots, \phi_n\}$$

write $\hat{\omega} = \sum_{j=1}^{n} c_j \phi_j$. To find the best approximation find $c_1, \ldots, c_n$.

- Fact: $(f - \hat{\omega}, \omega) = 0$ $\forall \omega \in W$

  means: the error of the best approximation is orthogonal to the approx subspace.



- orthogonality of the error $\Rightarrow$

$$(f - \hat{\omega}, \phi_k) = 0 \quad k = 1, \ldots, n$$

$$\Rightarrow \sum_{j=1}^{n} c_j (\phi_j, \phi_k) = (f, \phi_k) \quad k = 1, \ldots, n$$

$$\begin{pmatrix} (\phi_1, \phi_1) & \cdots & (\phi_n, \phi_1) \\ \vdots & \ddots & \vdots \\ (\phi_1, \phi_n) & \cdots & (\phi_n, \phi_n) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} (f, \phi_1) \\ \vdots \\ (f, \phi_n) \end{pmatrix}$$

solve for $c_1, \ldots, c_n$

↳ this matrix is always invertible

if $\Phi_1, \dots, \Phi_n$ is an orthogonal basis,

$$c_j = \frac{(f,\Phi_j)}{(\Phi_j,\Phi_j)} \Rightarrow \hat{w} = \sum_{j=1}^{n} \frac{(f,\Phi_j)}{(\Phi_j,\Phi_j)} \Phi_j$$

the best approximation from $\mathcal{W}$ is unique

6. <u>Spectral method</u>

- For matrices:

  if $A \in \mathbb{R}^{n \times n}$ is symmetric ($A = A^T$)

  then $A = V \Lambda V^T$

  $V = [v_1 \dots\dots v_n]$ $\quad Av_j = \lambda_j v_j$, $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$V^T V = I \quad (\|v_j\| = 1)$$

to solve $Ax = b$, write $x = \sum_{j=1}^{n} d_j v_j$ coefficients $d_1, \dots, d_n$

find $d_1, \dots, d_n$. $b = \sum_{j=1}^{n} c_j v_j$ coefficients $c_1, \dots, c_n$ known

$$Ax = A\sum_{j=1}^{n} d_j v_j = \sum d_j A v_j = \sum d_j \lambda_j v_j$$

$$b = \sum c_j v_j = \sum_{j=1}^{n} \frac{(b,v_j)}{(v_j,v_j)} v_j$$

(b, v_k) except $j=k$

$= \sum c_j (v_j, v_k)$

$= c_k$

$= c_k (v_k, v_k)$

$$c_k = \frac{(b,v_k)}{(v_k,v_k)}$$

$$(Ax, v_k) = (\sum d_j \lambda_j v_j, v_k) = \sum d_j \lambda_j (v_j, v_k) \quad 0 \text{ if } j \neq k$$

$$\| \qquad = d_k \lambda_k (v_k, v_k)$$

$$(b, v_k) = c_k (v_k, v_k)$$

so: $d_k \lambda_k (v_k, v_k) = c_k (v_k, v_k) \Rightarrow d_k = \frac{c_k}{\lambda_k}$

$$x = \sum_{k=1}^{n} d_k v_k = \sum_{k=1}^{n} \frac{1}{\lambda_k} \frac{(b,v_k)}{(v_k,v_k)} v_k$$

- For linear operators

  if L is a symmetric linear operator w/ eigenvalues $\lambda_1, \lambda_2, \dots$ on eigenfunctions $\psi_1, \psi_2, \dots$ then the solution u to $Lu = f$ can be expressed as

$$u = \sum_{k=1}^{n} \frac{1}{\lambda_k} \frac{(f,\psi_k)}{(\psi_k,\psi_k)} \psi_k$$

* be able to calculate eigenvalues and eigenfunctions

- For a linear operator:
  - be able to check symmetry
  - compute eigenvalues and eigenfunctions (by hand!)

$$A = V\Lambda V^T \quad A^{-1} = V\Lambda^{-1}V^T \quad x = A^{-1}b = V\Lambda^{-1}V^T b = V\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} V_1^T b \\ \vdots \\ V_n^T b \end{bmatrix}$$

$$= \sum \frac{1}{\lambda_k} \frac{V_k^T b}{\underbrace{V_k^T V_k}_{=1}} V_k$$

- What if the operator has a zero eigenvalue?

ex: $L_\alpha U = -U'' + \alpha U \quad L: C_0^2[0,1] \to C[0,1]$

eigs: $\lambda_k = k^2\pi^2 + \alpha \quad , \psi_k(x) = \sqrt{2}\sin(k\pi x)$

the spectral method gives:

$$U = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2+\alpha} \frac{(f,\psi_k)}{(\psi_k,\psi_k)} \psi_k$$

As $\alpha \to -\pi^2$ the $\frac{1}{\pi^2-\alpha}\frac{(f,\psi_1)}{(\psi_1,\psi_1)}\psi_1$ term dominates provided $(f,\psi_1) \neq 0$

At $\alpha = -\pi^2$
- $LU = f$ has no solution if $(f,\psi_1) \neq 0$
- $LU = f$ has infinite many solutions if $(f,\psi_1) = 0$

$$L\psi_1 = \underbrace{(\pi^2+\alpha)}_{=0}\psi_1 = 0 \qquad U = \sum_{k=2}^{\infty} \frac{1}{k^2\pi^2+\alpha} \frac{(f,\psi_k)}{(\psi_k,\psi_k)} \psi_k + \gamma\psi_1$$

↳ Fredholm alternative

$$LU = \sum_{k=2}^{\infty} \frac{1}{k^2\pi^2+\alpha} \frac{(f,\psi_k)}{(\psi_k,\psi_k)} L\psi_k + \gamma\psi_1$$

$$= \sum_{k=2}^{\infty} \frac{1}{k^2\pi^2+\alpha} \frac{(f,\psi_k)}{(\psi_k,\psi_k)} (k^2\pi^2+\alpha)\psi_k$$

$$= \sum_{k=2}^{\infty} \frac{1}{k^2\pi^2+\alpha} \frac{(f,\psi_k)}{(\psi_k,\psi_k)} \psi_k = f$$

$A = A^T$ (symmetric) $\quad Ax = b$
- has a solution that is unique if $\mathcal{N}(A) = \{0\}$
  ↳ nullspace of A

nullspace: $\mathcal{N}(A) = \{x : Ax = 0\}$
= span of all zero eigenvectors

- has no solution if $(b,v) \neq 0$ for some $v \in \mathcal{N}(A)$

- has infinitely many solutions if $(b,v)=0 \ \forall v \in \mathcal{N}(A)$ and $\mathcal{N}(A) \neq \{0\}$

7. finite elements

- $-(K(x)\,u'(x))' = f(x)$ ← strong form, $u(0)=u(1)=0$

- weak form: $a(u,v)=(f,v) \ \forall v \in V = C_D^2[0,1]$

$a(u,v) = \int_0^1 u'(x)\,v'(x)\,K(x)\,dx \qquad (f,v) = \int_0^1 f(x)\,v(x)\,dx$

- strong form $\Leftrightarrow$ weak form

- gallerkin approximation: only impose the weak form on a subspace

$V_n = \text{span}\{\phi_1, \ldots, \phi_n\} \subseteq V$

find $u_n \in V_n$ s.t. $a(u_n,v)=(f,v) \ \forall v \in V_n$

$\Rightarrow a(u-u_n,v)=0 \ \forall v \in V_n$ gallerkin orthogonality

$\Rightarrow u_n$ is the best approximation to u from V_n in the energy norm $\|w_E\| = (a(w,w))^{1/2}$

gal. approx $\Rightarrow \quad u_n = \sum_{k=1}^{n} c_k \phi_k \ , \ a(u_n,\phi_j) = (f,\phi_j) \quad j=1,\ldots,n$

$$\Rightarrow \sum_{k=1}^{n} c_k\, a(\phi_k,\phi_j) = (f,\phi_j) \quad j=1,\ldots,n$$

$$\Rightarrow \underbrace{\begin{pmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_n,\phi_1) \\ \vdots & \ddots & \vdots \\ a(\phi_1,\phi_n) & \cdots & a(\phi_n,\phi_n) \end{pmatrix}}_{\substack{K \\ \text{(stiffness matrix)}}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{c} = \underbrace{\begin{pmatrix} (f,\phi_1) \\ \vdots \\ (f,\phi_n) \end{pmatrix}}_{\substack{f \\ \text{(load vector)}}}$$

- eigenfunctions:

$$a(\phi_j,\phi_k) = \begin{cases} \lambda_j (\phi_j,\phi_k) & j=k \\ 0 & j\neq k \end{cases}$$

where $\phi_1, \ldots, \phi_n =$ eigenfunctions of L

$$\Rightarrow u_n = \sum_{k=1}^{n} \frac{1}{\lambda_k} \frac{(f,\phi_k)}{(\phi_k,\phi_k)} \phi_k$$

· hat functions: $\phi_1, \dots, \phi_n$ = hat functions

for $K(x)=1$:

$$K = \frac{1}{h}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \qquad \text{solve } c = K^{-1}f$$

· inhomogeneous boundary conditions: spectral method & gallerkin / FEM

<u>NOTES</u>:

- test may ask for some simple derivations (similar to the derivations done in the homeworks)
- no calculator. know how to integrate by parts, etc.
- all questions won't be stuff we have done exactly before