Example: Gram matrix applied to approximation of a "best fit" function.

Consider the vector space $C[0,1]$ of continuous functions on $[0,1]$ with the L^2 inner product.

In class we discussed how the functions defined by:
$f_n = \sin(n(2\pi)x)$ and $g_m = \cos(m(2\pi)x)$ satisfy the following:

$$(f_{n_1}, f_{n_2}) = \begin{cases} 0 & \text{if } n_1 \neq n_2 \\ 1/2 & \text{if } n_1 = n_2 \end{cases} \qquad (g_{m_1}, g_{m_2}) = \begin{cases} 0 & \text{if } m_1 \neq m_2 \\ 1/2 & \text{if } m_1 = m_2 \end{cases}$$

$$\text{and} \quad (f_n, g_m) = 0.$$

Note that this implies that the functions (f_n, g_m) are all orthogonal and therefore they are <u>linearly independent</u>. Now this means we have an infinite set of linearly independent vectors in $C[0,1]$ so that $C[0,1]$ is (at least) <u>infinite dimensional</u>.

Lets consider the vectors $w_1 = 1$ $w_2 = x$ and $W = \operatorname{Span}\{w_1, w_2\}$ the vector subspace of all linear functions. This space is well known in applied mathematics and is typically denoted as $\mathbb{P}_1(C[0,1])$

Lets compute the best approximation to the function $f(x) = \sin(2\pi x) + e^x$ in W with respect to the inner product $(f,g)_v = \int fg\,dx$. That is we are looking for $m = x_1 w_1 + x_2 w_2$ such that $\|f - m\| \leq \|f - u\|$ for every $u \in W$.

To find the coefficients x_1, x_2 we need to solve $Gx = b$ where G is the Gram matrix $G = \begin{bmatrix} (w_1, w_1) & (w_2, w_1) \\ (w_1, w_2) & (w_2, w_2) \end{bmatrix}$

and $b = \begin{bmatrix} (f, w_1) \\ (f, w_2) \end{bmatrix}$

we have:

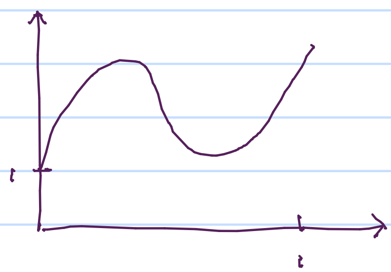
$(w_1, w_1) = \int_0^1 1 \cdot 1 = 1$ $(w_2, w_1) = \int_0^1 x \cdot 1 = 1/2$

$(w_2, w_1) = \int_0^1 x \cdot 1 = 1/2$ $(w_2, w_2) = \int_0^1 x \cdot x = 1/3$

$(f, w_1) = \int_0^1 (\sin(2\pi x) + e^x) \simeq 1.7183$

$(f, w_2) = \int_0^1 x(\sin(2\pi x) + e^x) \simeq 0.84085$

$f(x) = \sin(2\pi x) + e^x$ on $[0,1]$



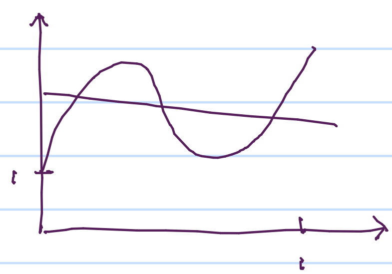
So we want to solve:

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.7183 \\ 0.84085 \end{bmatrix}$$

From matlab: $G^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$ so $\vec{x} = G^{-1} b$ is:

$$\vec{x} = \begin{bmatrix} 1.8281 \\ -0.2196 \end{bmatrix}$$

So the best approximation to $f(x) = \sin(2\pi x) + e^x$ from $\mathbb{P}_1([0,1]) = \mathrm{Span}\{1, x\}$ is $1.8281 - 0.2196x$



E.g. it is the line minimizing the error $\|f - m\|$ in the L^2 norm.