CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6

Posted Wednesday 8 October, 2014. Due 5pm Wednesday 15 October, 2014.

*Please write your name and **residential college** on your homework.*

1. [28 points: 14 points each]

   All parts of this question should be done by hand.

   (a) Let

   $$\mathbf{D} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

   and

   $$\mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

   Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to

   $$\mathbf{Dc} = \mathbf{g}.$$

   (b) Let

   $$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

   and

   $$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

   Use the spectral method to obtain the solution $\mathbf{x} \in \mathbb{R}^3$ to

   $$\mathbf{Ax} = \mathbf{b}.$$

2. [24 points: 6 points each]

Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx.$$

Consider the linear operator $L : C^2_m[0,1] \to C[0,1]$ defined by

$$Lu = -u''$$

where

$$C^2_m[0,1] = \left\{u \in C^2[0,1] : u'(0) = u(1) = 0\right\}.$$

(a) Is L symmetric?

(b) What is the null space of L?

(c) Show that $(Lu, u) \geq 0$ for all $u \in C^2_m[0,1]$ and explain why this and the answer to part (b) mean that $\lambda > 0$ for all eigenvalues λ of L.

(d) Find the eigenvalues and eigenfunctions of L.

3. [32 points: 10 points for (a) and (b), 12 points for (c)]
Define the inner product (u, v) to be

$$(u, v) = \int_0^1 u(x)v(x)\,dx$$

and let the norm $\|v(x)\|$ be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let N be a positive integer and let $\phi_1, \ldots, \phi_N \in C[0, 1]$ be such that $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$. We wish to approximate a continuous function $f(x)$ with $f_N(x)$

$$f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

where

$$\phi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \ldots.$$

and where $\alpha_n = (f, \phi_n)$. (Note that f_N is the best approximation to g from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$.)

(a) Assume that $f_N \to f$ as $N \to \infty$. Show that, since $\phi_1, \ldots, \phi_N$ are orthonormal,

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2.$$

(b) The best approximation to $f(x) = x(1 - x)$ has coefficients α_n which satisfy

$$\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3}(1 - (-1)^n).$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 5$. On a separate figure, plot the error using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using `loglog` in MATLAB.

(c) Verify that the best approximation to the function $f(x) = 1 - x$ (which does not satisfy the same boundary conditions as $\phi_n(x)$!) has coefficients

$$\alpha_n = \frac{\sqrt{2}}{\pi n}.$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 100$. On a separate figure, plot the error using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using `loglog` in MATLAB.

You may have noticed that the rate at which the coefficients $\alpha_n \to 0$ determines how fast the error decreases — this is not coincidental!

4. [40 points: 8 points each]
This problem concerns the same operator from Problem 2, $L : C_D^2[0,1] \to C[0,1]$ defined by

$$L_D u = -\frac{d^2u}{dx^2},$$

with homogeneous Dirichlet boundary conditions imposed via

$$C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}.$$

The eigenvalues and (normalized) eigenfunctions remain as in Problem 2: $\lambda_n = n^2\pi^2$ and $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for $n = 1, 2, \ldots$. Now let $f(x) = x^2(1-x)$.

(a) For this f, compute the coefficients

$$c_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

in the expansion

$$f = \sum_{n=1}^{\infty} c_n \psi_n.$$

You may determine these by hand, by consulting a table of integrals, or by using a symbolic mathematics package like Mathematica or the Symbolic Toolbox in MATLAB.

(b) Produce a plot (or series of plots) comparing $f(x)$ to the partial sums

$$f_N(x) = \sum_{k=1}^{N} c_k \psi_k(x)$$

for $N = 1, \ldots, 10$.

(c) Plot the approximations u_N to the true solution u that you obtain using the spectral method:

$$u_N(x) = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \psi_k(x)$$

for $N = 1, \ldots, 10$.

(d) Now replace the homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ above with the inhomogeneous Dirichlet conditions $u(0) = -1/100$ and $u(1) = 1/100$. Describe how to adjust your solution from part (c) to account for these boundary conditions, and produce a plot of the solution with these inhomogeneous boundary conditions, based on u_{10} from part (c).

We have been able to obtain nice formulas for the eigenvalues of the operators that we have considered thus far. This problem illustrates that this is not always the case.

Let the inner product $(\cdot, \cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx.$$

Let the linear operator $L : V \to C[0,1]$ be defined by

$$Lu = -u''$$

where

$$V = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

Note that if $u \in V$ then u satisfies the homogeneous Robin boundary condition

$$u(0) - u'(0) = 0$$

and the homogeneous Dirichlet boundary condition

$$u(1) = 0.$$

(a) Prove that L is symmetric.

(b) Is zero an eigenvalue of L?

(c) Show that $(Lu, u) \geq 0$ for all $u \in V$. What does this and the answer to part (b) then allow us to say about the eigenvalues of L?

(d) Show that the eigenvalues λ of L must satisfy the equation $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$.

(e) Use MATLAB to plot $g(x) = -\tan(x)$ and $h(x) = x$ on the same figure. Use the command `axis([0 5*pi -5*pi 5*pi])` and make sure that your plot gives an accurate representation of these functions on the region shown on the figure when this command is used. By hand or using MATLAB, mark on your plot the points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Note that $g \notin C[0, 5\pi]$. How many eigenvalues λ does L have which are such that $\sqrt{\lambda} \leq 5\pi$?