CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Examination 3

Instructions:

1. Time limit: **3 uninterrupted hours**.
2. There are three questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You are allowed one cheat sheet to refer to during the exam.
   You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions throughly (but succintly!) and justify all your answers.
   Show your work for partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

   ____

   ____

   ____

7. Staple this page to the front of your exam.

1. [33 points: 5 points for (a), 10 points for (b)-(c)] Consider the time-dependent heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x)$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \psi(x).$$

The eigenfunctions for operator $Lu = -\frac{\partial^2 u}{\partial x^2}$ under the above boundary conditions are

$$\phi_j(x) = \sin(j\pi x), \qquad \lambda_j = (j\pi)^2.$$

(a) Let $f(x) = 0$. Consider the discontinuous initial condition

$$\psi(x) = \begin{cases} 1, & 1/3 < x < 2/3 \\ 0, & \text{otherwise} \end{cases}$$

Compute the coefficients $\alpha_j(0)$ such that

$$\psi(x) = \sum_{j=1}^{\infty} \alpha_j(0)\phi_j(x).$$

Use these computed values to give an expression for $u(x,t)$. Your final formula should not have integrals in it (you may leave it in terms of trigonometric functions).

(b) Let now $\psi(x) = 0$ and $f(x)$ be the same discontinuous function.

$$f(x) = \begin{cases} 1, & 1/3 < x < 2/3 \\ 0, & \text{otherwise} \end{cases}$$

Give an expression for the solution $u(x,t)$ and for the steady state solution (i.e. $u(x,t)$ as $t \to \infty$). You may use information from part (a), and you may use the fact that the solution to the differential equation

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} + A\alpha(t) = B$$

(where A and B are constant in t) is given by

$$\alpha(t) = e^{-At}\alpha(0) + \frac{B}{A}\left(1 - e^{-At}\right).$$

(c) Consider now the case when $\psi(x) = f(x) = 0$, but where we have *time-dependent* Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = t.$$

We may shift the data by setting $u(x,t) = w(x,t) + g(x,t)$, where $g(x,t) = xt$. Show that $u(x,t)$ satisfies boundary conditions, and that the part $w(x,t)$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = -x.$$

Derive an expression for $u(x,t)$ by solving for $w(x,t)$. You may use that

$$\int_0^1 x\sin(j\pi x) = \frac{(-1)^{j+1}}{j\pi}.$$

2. [33 points: 10 points for (a), 5 points for each of (b)-(c)]

   In the last part of the semester we have seen the finite element method applied to the time-dependent heat equation; the most fundamental being the case of homogeneous Dirichlet boundary conditions as in (1).

$$\frac{\partial u(x,t)}{\partial t} - \kappa \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) \tag{1}$$
$$u(x,0) = \psi(x) \tag{2}$$
$$u(0,t) = u(1,t) = 0 \tag{3}$$

   We discretized equation using the finite element method and the space of linear hat functions $S_0^N = \text{span}\{\phi_1, \phi_2, \dots, \phi_N\}$ where ϕ_i is the hat function corresponding to the internal mesh point x_i on a uniformly spaced mesh of the interval $[0,1]$. The discrete solution for (1) can be represented as

$$u_h(x,t) = \sum_{i=1}^{N} \alpha_i(t)\phi_i(x)$$

   and the initial condition can be approximated as

$$u_h(x,0) = \sum_{i=1}^{N} \psi(x_i)\phi_i(x).$$

   Let $\vec{\alpha}$, $\vec{\psi}$, and $\vec{b}$ denote coefficient vectors with $\vec{\alpha}_i = \alpha_i(t)$, $\vec{\psi}_i = \psi(x_i)$.

   (a) The finite element method applied to (1) results in a first-order system of ordinary differential equations for $\vec{\alpha}(t)$. Write down this system. Specify the entries of the right-hand side vector $\vec{b}$ in addition to each matrix in the formulation. That is, clearly indicate formulas for the components $\vec{b}_i$ and individual entries of every matrix involved in your answer.

   (b) Write down the numerical scheme for the Backward Euler method applied to the linear system of ordinary differential equations

$$M\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = A\vec{y}(t) + \vec{g}(t),$$

   where M is the mass matrix. You should specify what the matrix A is, and give a formula for $\vec{y}(t_{n+1})$ in terms of $\vec{y}(t_n)$

   (c) Explain how you would deal with *inhomogeneous* (and possibly time-dependent) Dirichlet boundary conditions $u(0,t) = u_L(t)$, $u(1,t) = u_R(t)$. If you need to introduce any additional functions to deal with these non-zero boundary conditions, please draw pictures or give expressions for them. *Hint: Recall that the problem can be split into two parts; what are these parts and how do you deal with them?*

3. [34 points: 10 points for (a), 5 points for each of (b)-(c)] When studying new numerical techniques for modeling fluid phenomena periodic boundary conditions are often employed as they simulate an 'infinite' space in a finite interval $[-1,1]$. As we saw in exam two, the operator L defined by

$$Lu = -\frac{\partial^2 u}{\partial x^2}$$

with periodic boundary conditions in space

$$u(x,-1) = u(x,1), \qquad \frac{\partial u}{\partial x}(x,-1) = \frac{\partial u}{\partial x}(x,1)$$

has eigenvalues $\lambda_0 = 0$ with eigenvector $\Psi_0 = 1$ and $\lambda_n = (n\pi)^2$ corresponding to the pair of eigenvectors $\Psi_{1,n} = \sin(n\pi x)$ and $\Psi_{2,n} = \cos(n\pi x)$. All of the eigenvectors are mutually orthogonal with respect to the inner product $(u,v) = \int_{-1}^{1} uv\,dx$. In this problem we will consider a modified wave equation with periodic boundary conditions on $[-1,1]$.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \pi^2 u = 0 \tag{4}$$

$$u(-1,t) = u(1,t), \quad \frac{\partial u}{\partial x}(-1,t) = \frac{\partial u}{\partial x}(1,t) \tag{5}$$

$$u(x,0) = G(x), \quad \frac{\partial u}{\partial t}(x,0) = H(x) \tag{6}$$

(a) Show that the modified spatial operator $-\frac{\partial^2 u}{\partial x^2} + \pi^2 u$ with periodic boundary conditions has eigenvalue $\tilde{\lambda}_0 = \left(\lambda_0 + \pi^2\right)$ with eigenvector $\tilde{\Psi}_0 = \Psi_0$ and that $\tilde{\Psi}_{1,n} = \Psi_{1,n}$, $\tilde{\Psi}_{2,n} = \Psi_{2,n}$ are eigenvectors which both have eigenvalue $\tilde{\lambda}_j = \left(\lambda_j + \pi^2\right)$.

(b) As a result of part (a) we write the solution to the system (4) as in equation (7) and expand the initial conditions as given in equations (8) and (9).

$$u(x,t) = \alpha_0(t) + \sum_{k=1}^{\infty} \Big(\alpha_k(t)\tilde{\Psi}_{1,n}(x) + \beta_k(t)\tilde{\Psi}_{2,n}(x)\Big) \tag{7}$$

$$G(x) = \alpha_0(0) + \sum_{k=1}^{\infty} \Big(\alpha_k(0)\Psi_{1,n}(x) + \beta_k(0)\tilde{\Psi}_{2,n}(x)\Big) \tag{8}$$

$$H(x) = \frac{\mathrm{d}\alpha_0(0)}{\mathrm{d}t} + \sum_{k=1}^{\infty} \left(\frac{\mathrm{d}\alpha_k(0)}{\mathrm{d}t}\Psi_{1,n}(x) + \frac{\mathrm{d}\beta_k(0)}{\mathrm{d}t}\tilde{\Psi}_{2,n}(x)\right) \tag{9}$$

Use these expansions in the modified damped wave equation (4) and write down the series of initial boundary value problems (ordinary differential equations) determining the unknown coefficients $\alpha_0(t)$ and $\alpha_k(t)$, $\beta_k(t)$ for $k = 1, 2, \ldots$.

(c) Solve the modified damped wave equation (4) with $G(x) = 1 + \sin(\pi x)$ and $H(x) = 1 + \cos(\pi x)$. You will need the fact that the solution to the differential equation (10)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \theta^2 y = 0 \tag{10}$$

$$y(0) = A, \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = B$$

is given by the equation

$$y(t) = A\cos(\theta t) + \frac{B}{\theta}\sin(\theta t).$$