

# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 16

Posted Wednesday 25 September 2013. Due 5pm Wednesday 2 October 2013.

16. [25 points]

Suppose  $N \geq 1$  is an integer and define  $h = 1/(N + 1)$  and  $x_j = jh$  for  $j = 0, 1, \dots, N + 1$ . Consider the  $N$  hat functions  $\phi_k \in C[0, 1]$ , defined as

$$\phi_k(x) = \begin{cases} \frac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k]; \\ \frac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for  $k = 1, \dots, N$ . Let the inner product  $(\cdot, \cdot) : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$(u, v) = \int_0^1 u(x)v(x) dx$$

and let the norm  $\|\cdot\| : C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$\|u\| = \sqrt{(u, u)}.$$

(a) For  $j = 1, \dots, N$ , what is  $\phi_j(x_k)$  for  $k = 0, 1, \dots, N + 1$ ? Simplify your answer as much as possible.

(b) Show that  $\{\phi_1, \dots, \phi_N\}$  is linearly independent by showing that if  $c_k \in \mathbb{R}$  and  $\sum_{k=1}^N c_k \phi_k(x) = 0$  for all  $x \in [0, 1]$  then  $c_k = 0$  for  $k = 1, \dots, N$ .

(c) For  $f(x) = \sin(\pi x)$ , compute by hand the inner products  $(f, \phi_j)$  for  $j = 1, \dots, N$ .

(d) Compute by hand the inner products  $(\phi_j, \phi_k)$  for  $j, k = 1, \dots, N$ . Your final answers should be simplified as much as possible and in your formulas  $h$  should be left as  $h$  and not be replaced with  $1/(N + 1)$ . You must clearly state which values of  $j$  and  $k$  each formula you obtain is valid for. An acceptable way to present the final answer would be:

For  $j, k = 1, \dots, N$ ,

$$(\phi_j, \phi_k) = \begin{cases} ? & \text{if } k = j, \\ ? & \text{if } |j - k| = 1, \\ ? & \text{otherwise.} \end{cases}$$

with the question marks replaced with the correct values. Hint:

$$\int_{x_{j-1}}^{x_j} \left( \frac{x - x_{j-1}}{h} \right)^2 dx = \frac{1}{h^2} \int_{x_{j-1} - x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2 ds = \frac{1}{h^2} \int_0^h s^2 ds$$

where  $s = x - x_{j-1}$ .

(e) Use your solutions to (c) and (d) to set up a linear system (in MATLAB) and solve it to compute the best approximations  $f_N(x)$  to  $f(x) = \sin(\pi x)$  from  $\text{span}\{\phi_1, \dots, \phi_N\}$  with respect to the norm  $\|\cdot\|$  for  $N = 3$  and  $N = 9$ . For each of these  $N$ , produce one plot that compares  $f_N(x)$  to  $f(x)$ , and a second plot that shows the error  $f(x) - f_N(x)$ . The `hat.m` code (from Homework 2, either your code or the code from the solutions) should help you to produce these plots.