CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 12

Posted Friday 13 September 2013. Due 5pm Wednesday 25 September 2013.

12. [25 points]

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since f is linear, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

(b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

(c) Now we want to generalize further: Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

(Thus any linear function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix-vector product.)