CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 10

Posted Wednesday 7 November 2012. Due Wednesday 14 November 2012, 5pm.

1. [50 points: 8 points each for (a), (b), (d), (e); 4 points for (c); 14 points for (f)]
   This problem and the next study the heat equation in two dimensions. We begin with the steady-state problem. In place of the one dimensional equation, $-u'' = f$, we now have

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

   with homogeneous Dirichlet boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$ for all $0 \le x \le 1$ and $0 \le y \le 1$. The associated operator L is defined as

$$Lu = -(u_{xx} + u_{yy}),$$

   acting on the space $C^2_D[0,1]^2$ consisting of twice continuously differentiable functions on $[0,1] \times [0,1]$ with homogeneous boundary conditions. We can solve the differential equation $Lu = f$ using the spectral method just as we have seen in class before. This problem will walk you through the process; you may consult Section 8.2 of the text for hints.

   (a) Show that L is symmetric, given the inner product

$$(v,w) = \int_0^1 \int_0^1 v(x,y)w(x,y)\,dx\,dy.$$

   (b) Verify that the functions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

   are eigenfunctions of L for $j,k = 1,2,\ldots$.
   (To do this, you simply need to show that $L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$ for some scalar $\lambda_{j,k}$.)

   (c) What is the eigenvalue $\lambda_{j,k}$ associated with $\psi_{j,k}$?

   (d) Compute the inner product $(\psi_{j,k}, \psi_{j,k}) = \|\psi_{j,k}\|^2$.

   (e) Let $f(x,y) = x(1-y)$. Compute the inner product $(f, \psi_{j,k})$.

   (f) The solution to the diffusion equation is given by the spectral method, but now with a double sum to account for all the eigenvalues:

$$u(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

   In MATLAB plot the partial sum

$$u_{10}(x,y) = \sum_{j=1}^{10}\sum_{k=1}^{10} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

   Hint for 3d plots: To plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$, you could use

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x = linspace(0,1,40); y = linspace(0,1,40);
[X,Y] = meshgrid(x,y);
Psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,Psi11)
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please see the next page...

2. [50 points: 20 points for (a); 10 points for (b); 20 points for (c)]
We now consider the time-dependent heat equation in two dimensions,

$$u_t(x,y,t) = (u_{xx}(x,y,t) + u_{yy}(x,y,t)) + f(x,y,t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions $u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$ for all $0 \le x \le 1$, $0 \le y \le 1$, and $t \ge 0$, and initial condition $u(x,y,0) = u_0(x,y)$. We can consider this problem in the abstract setting of $u_t = -Lu + f$, where, as in the previous problem,

$$Lu = -(u_{xx} + u_{yy}),$$

acting on the space $C_D^2[0,1]^2$. Recall that the eigenvalues $\lambda_{j,k}$ and associated eigenfunctions $\psi_{j,k}$ of this operator were studied in the previous problem.

(a) The solution to the two-dimensional heat equation takes the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big(e^{-\lambda_{j,k}t}a_{j,k}(0) + \int_0^t e^{-\lambda_{j,k}(t-\tau)}c_{j,k}(\tau)\,\mathrm{d}\tau\Big)\psi_{j,k}(x,y).$$

Give a brief derivation of this equation, explaining what the values $a_{j,k}(0)$ and $c_{j,k}(\tau)$ denote, and what ordinary differential equation needs to be solved for each (j,k) pair. (You do not need to derive the solution to that equation from scratch; it should take a familiar form, and you can just quote the solution for equations of this form.)

(b) Suppose $u_0(x,y) = 0$ and $f(x,y,t) = (x-1/2)^3(y-1/2)\mathrm{e}^{-t}$. Simplify the formula in part (a) as much as possible. That is, write out $a_{j,k}(0)$, $c_{j,k}(t)$, and compute a formula for

$$\int_0^t e^{-\lambda_{j,k}(t-\tau)}c_{j,k}(\tau)\,\mathrm{d}\tau.$$

(c) Plot the partial Fourier series solution

$$u_{15}(x,y,t) = \sum_{j=1}^{15}\sum_{k=1}^{15}\Big(e^{-\lambda_{j,k}t}a_{j,k}(0) + \int_0^t e^{-\lambda_{j,k}(t-\tau)}c_{j,k}(\tau)\,\mathrm{d}\tau\Big)\psi_{j,k}(x,y)$$

at the four times $t = 0, 0.005, 0.1, 2$ for the values of u_0 and f given in part (b). Your solution for $t = 0.1$ should resemble the plot below.