CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 23

Posted Wednesday 2 October 2013. Due 5pm Wednesday 9 October 2013.

23. [25 points]

Let $N \geq 1$ be an integer and define $h = 1/(N+1)$ and $x_k = kh$ for $k = 0, \ldots, N+1$. Consider the $N+2$ hat functions, defined for $x \in [0,1]$ as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise}; \end{cases}$$

for $k = 1, \ldots, N$, with

$$\phi_0(x) = \begin{cases} (x_1 - x)/h, & x \in [x_0, x_1); \\ 0, & \text{otherwise}; \end{cases}$$

and

$$\phi_{N+1}(x) = \begin{cases} (x - x_N)/h, & x \in [x_N, x_{N+1}]; \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\phi_k \in C[0,1]$ for $k = 0, \ldots, N+1$. Let $V_N = \text{span}\{\phi_0, \phi_1, \ldots, \phi_{N+1}\}$ and let $(\cdot,\cdot) : V_N \times V_N \to \mathbb{R}$ be defined by

$$(u, v) = \sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} u'(x)v'(x)\,dx.$$

(a) What value is taken by $w(x) = \displaystyle\sum_{j=0}^{N+1} \phi_j(x)$ for all $x \in [0,1]$?

(b) Why is $(\cdot,\cdot)$ not an inner product on V_N?

(c) For the remainder of this question we shall just consider the case when $N = 1$. In this case the symmetric matrix

$$\mathbf{K} = \begin{bmatrix} (\phi_0,\phi_0) & (\phi_0,\phi_1) & (\phi_0,\phi_2) \\ (\phi_0,\phi_1) & (\phi_1,\phi_1) & (\phi_1,\phi_2) \\ (\phi_0,\phi_2) & (\phi_1,\phi_2) & (\phi_2,\phi_2) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

What are the eigenvalues corresponding to the eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ of $\mathbf{K}$?

(d) For both $\mathbf{f} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, use an eigenvector of $\mathbf{K}$ to determine whether or not there exist solutions $\mathbf{c} \in \mathbb{R}^3$ to $\mathbf{Kc} = \mathbf{f}$ and if solutions exist, use the spectral method to obtain the solutions $\mathbf{c} \in \mathbb{R}^3$ to $\mathbf{Kc} = \mathbf{f}$.

(e) Why do there exist solutions $\mathbf{c} \in \mathbb{R}^3$ to

$$\mathbf{Kc} = \begin{bmatrix} \displaystyle\int_0^1 g(x)\phi_0(x)dx \\ \displaystyle\int_0^1 g(x)\phi_1(x)dx \\ \displaystyle\int_0^1 g(x)\phi_2(x)dx \end{bmatrix}$$

for all $g \in C[0,1]$ which are such that $\displaystyle\int_0^1 g(x)dx = 0$?