CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 38 · Solutions

Posted Wednesday 13 November 2013. Due 5pm Wednesday 20 November 2013.

38. [25 points] Parts (a) and (c) of this question should be done by hand.

Let

$$f(x,t) = \begin{cases} 2x & \text{if } x \in \left[0, \dfrac{1}{2}\right), \\ 2-2x & \text{if } x \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

(a) Use the spectral method to obtain a series solution to the problem of finding the solution $\tilde{u}(x,t)$ to the heat equation

$$\tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0$$

with Dirichlet boundary conditions

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \quad t \geq 0$$

and initial condition

$$\tilde{u}(x,0) = 0, \quad 0 < x < 1.$$

(b) Plot the approximations to $\tilde{u}(x,t)$ obtained by replacing the upper limit of the summation in your series solution with 20 for $t = 0, 0.1, 0.2, 0.3, 0.5, 1, 2$.

(c) By shifting the data and then using the spectral method, obtain a series solution to the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \quad t \geq 0$$

and

$$u(1,t) = 1, \quad t \geq 0$$

and initial condition

$$u(x,0) = x^3, \quad 0 < x < 1.$$

(d) Plot the approximations to $u(x,t)$ obtained by replacing the upper limit of the summation in your series solution with 20 for $t = 0, 0.1, 0.2, 0.3, 0.5, 1, 2$.

Solution.

(a) [8 points] Let

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$. The spectral method yields the series solution

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

where

$$\begin{aligned} a_n(t) &= \int_0^1 0\,dx e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds \\ &= \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds. \end{aligned}$$

Now, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} &\int_0^1 f(x,s)\psi_n(x)\,dx \\ &= \sqrt{2}\left(\int_0^{1/2} f(x,s)\sin(n\pi x)\,dx + \int_{1/2}^1 f(x,s)\sin(n\pi x)\,dx\right) \\ &= 2\sqrt{2}\left(\int_0^{1/2} x\sin(n\pi x)\,dx + \int_{1/2}^1 (1-x)\sin(n\pi x)\,dx\right) \\ &= 2\sqrt{2}\left(\left[-\frac{1}{n\pi}x\cos(n\pi x)\right]_0^{1/2} + \frac{1}{n\pi}\int_0^{1/2}\cos(n\pi x)\,dx + \left[-\frac{1}{n\pi}(1-x)\cos(n\pi x)\right]_{1/2}^1 - \frac{1}{n\pi}\int_{1/2}^1 \cos(n\pi x)\,dx\right) \\ &= 2\sqrt{2}\left(-\frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\int_0^{1/2}\cos(n\pi x)\,dx + \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\int_{1/2}^1 \cos(n\pi x)\,dx\right) \\ &= \frac{2\sqrt{2}}{n\pi}\left(\int_0^{1/2}\cos(n\pi x)\,dx - \int_{1/2}^1 \cos(n\pi x)\,dx\right) \\ &= \frac{2\sqrt{2}}{n\pi}\left(\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^{1/2} - \left[\frac{1}{n\pi}\sin(n\pi x)\right]_{1/2}^1\right) \\ &= \frac{2\sqrt{2}}{n\pi}\left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right) \\ &= \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

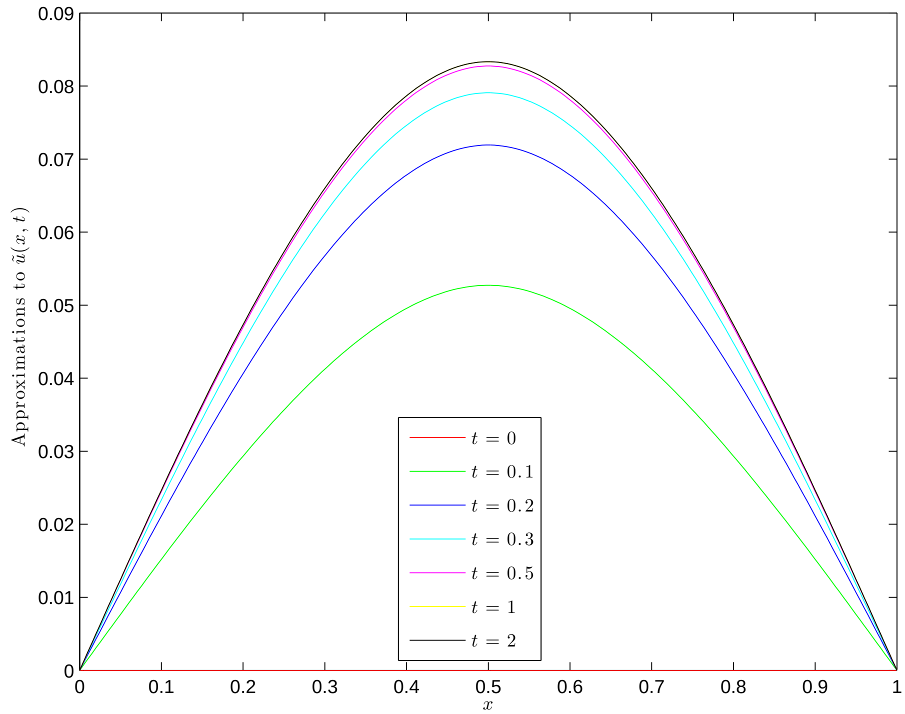
Consequently,

$$\begin{aligned} a_n(t) &= \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds \\ &= \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\int_0^t e^{n^2\pi^2(s-t)}\,ds \\ &= \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\left[\frac{1}{n^2\pi^2}e^{n^2\pi^2(s-t)}\right]_{s=0}^{s=t} \\ &= \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\left(\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}e^{-n^2\pi^2 t}\right) \\ &= \frac{4\sqrt{2}}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right) \end{aligned}$$

and so

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right)\sin(n\pi x).$$

(b) [5 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
col = 'rgbcmyk';
x = linspace(0,1,200);
tvec=[0 0.1 0.2 0.3 0.5 1 2];
figure(1)
clf
for j=1:length(tvec)
    U = zeros(size(x));
    t=tvec(j);
    for n=1:20
        U=U + 8*sin(n*pi/2)*(1-exp(-n^2*pi^2*t))*sin(n*pi*x)/(n^4*pi^4);
    end
    legendStr{j}=['$t=' num2str(t) '$'];
    plot(x,U,col(j))
    hold on
end
legend(legendStr,'interpreter','latex','location','best')
xlabel('$x$','interpreter','latex')
ylabel('Approximations to $\tilde{u}(x,t)$','interpreter','latex')
saveas(figure(1),'hw38b.eps','epsc')
```

(c) [7 points] Let

$$w(x) = x$$

so that

$$w(0) = 0$$

and

$$w(1) = 1.$$

Moreover, let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0;$$

$$\hat{u}(0,t) = \hat{u}(1,t) = 0, \quad t \geq 0;$$

and

$$\hat{u}(x,0) = x^3 - x, \quad 0 < x < 1.$$

Then $u(x,t) = w(x) + \hat{u}(x,t)$ will be such that

$$u_t(x,t) - u_{xx}(x,t) = \hat{u}_t(x,t) - w''(x) - \hat{u}_{xx}(x,t) = \hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0;$$

$$u(0,t) = w(0) + \hat{u}(0,t) = 0 + 0 = 0, \quad t \geq 0;$$
$$u(1,t) = w(1) + \hat{u}(1,t) = 1 + 0 = 1, \quad t \geq 0;$$

and

$$u(x,0) = w(x) + \hat{u}(x,0) = x + x^3 - x = x^3, \quad 0 < x < 1.$$

The spectral method yields that

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} \hat{a}_n(t)\psi_n(x)$$

where

$$\hat{a}_n(t) = \int_0^1 \left(x^3 - x\right)\psi_n(x)\,dx e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds.$$

Now, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned}
\int_0^1 \left(x^3 - x\right)\psi_n(x)\,dx &= \sqrt{2}\int_0^1 \left(x^3 - x\right)\sin(n\pi x)\,dx \\
&= \sqrt{2}\left(\left[-\frac{1}{n\pi}\left(x^3 - x\right)\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \left(3x^2 - 1\right)\cos(n\pi x)\,dx\right) \\
&= \frac{\sqrt{2}}{n\pi}\int_0^1 \left(3x^2 - 1\right)\cos(n\pi x)\,dx \\
&= \frac{\sqrt{2}}{n\pi}\left(\left[\frac{1}{n\pi}\left(3x^2 - 1\right)\sin(n\pi x)\right]_0^1 - \frac{6}{n\pi}\int_0^1 x\sin(n\pi x)\,dx\right) \\
&= -\frac{6\sqrt{2}}{n^2\pi^2}\int_0^1 x\sin(n\pi x)\,dx \\
&= -\frac{6\sqrt{2}}{n^2\pi^2}\left(\left[-\frac{1}{n\pi}x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right) \\
&= -\frac{6\sqrt{2}}{n^2\pi^2}\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^1\right) \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi).
\end{aligned}$$

Moreover, in part (a) we computed that, for $n = 1, 2, 3, \ldots$,

$$\int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds = \frac{4\sqrt{2}}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right).$$

Hence,

$$\begin{aligned}
\hat{a}_n(t) &= \int_0^1 \left(x^3 - x\right)\psi_n(x)\,dx e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi)e^{-n^2\pi^2 t} + \frac{4\sqrt{2}}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right) \\
&= \frac{2\sqrt{2}}{n^3\pi^3}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(3\cos(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{-n^2\pi^2 t}\right)
\end{aligned}$$

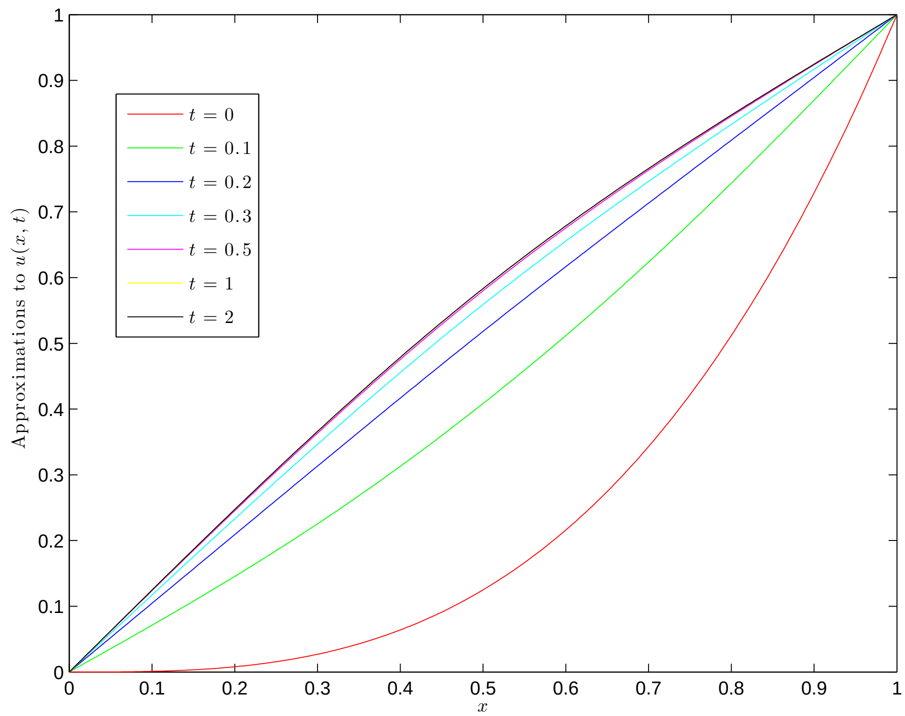
and so

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \left(\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \left(3\cos(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right) e^{-n^2\pi^2 t} \right) \sin(n\pi x).$$

Consequently,

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \left(\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \left(3\cos(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right) e^{-n^2\pi^2 t} \right) \sin(n\pi x).$$

(d) [5 points] The requested plot is below.



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clf
for j=1:length(tvec)
    U = x;
    t=tvec(j);
    for n=1:20
        U=U + 4*(2*sin(n*pi/2)/(n*pi)+(3*cos(n*pi)-2*sin(n*pi/2)/(n*pi))*exp(-n^2*pi^2*t
            ))*sin(n*pi*x)/(n^3*pi^3);
    end
    legendStr{j}=['$t=' num2str(t) '$'];
    plot(x,U,col(j))
    hold on
end
legend(legendStr,'interpreter','latex','location','best')
xlabel('$x$','interpreter','latex')
ylabel('Approximations to $u(x,t)$','interpreter','latex')
saveas(figure(1),'hw38d.eps','epsc')
```