CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 2

Posted Wednesday 3, September 2014. Due 5pm Wednesday 10, September 2014.

*Please write your name and **residential college** on your homework.*

1. [28 points: 8 each (a),(b), 12 for (c)] In this problem, we will derive a finite difference discretization for the equation

$$\alpha u(x) - \frac{\partial^2 u(x)}{\partial x^2} = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u'(0) = 0, \quad u'(1) = 0.$$

Since we cannot solve the equation exactly, we will wish to satisfy it at a finite number of points x_i, such that

$$\alpha u(x_i) - \frac{\partial^2 u(x_i)}{\partial x^2} = f(x_i), \quad 0 < x_i < 1.$$

To do so, we will replace $\frac{\partial^2 u(x_i)}{\partial x^2}$ with a finite difference approximation using $u(x_i)$ at the 5 points

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad x_4 = 1.$$

(a) Using the finite difference approximations for the boundary conditions

$$u'(0) = u'(x_0) \approx \frac{u(x_1) - u(x_0)}{h} = 0$$

and

$$u'(1) = u'(x_4) \approx \frac{u(x_4) - u(x_3)}{h} = 0,$$

write down the finite difference approximation to the differential equation at $0 < x_1, x_2, x_3 < 1$, and construct explicitly the matrix system $\mathbf{Au} = \mathbf{b}$ resulting from the finite difference approximation of $\alpha u(x) - u''(x) = f(x)$, where

$$\mathbf{u} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \end{bmatrix}.$$

(b) Solve the above system for $\alpha = 1$, $f(x) = e^x$, and report the solution at the interior points $u(x_1), u(x_2), u(x_3)$.

(c) Consider the case where $\alpha = 0$, or

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1 \\ u'(0) &= 0 \\ u'(1) &= 0. \end{aligned}$$

In the previous homework, we showed that this equation does not have a unique solution; as a result, neither does the finite difference system for this system. Verify that $\mathbf{e}$ is in the null space of $\mathbf{A}$, where $\mathbf{e} = (1, 1, 1)^T$ is the vector of all ones. Suppose that, instead of 5 points, we have $N + 2$ points. Would the vector of all ones be in the null space of $\mathbf{A}$ for arbitrary N as well?

2. [27 points: 9 each]

Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = ih$ for $i = 0, \ldots, N+1$. We can approximate the differential equation

$$-u''(x) = f(x), \quad 0 < x < 1,$$

with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ by the matrix equation

$$\frac{-1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix},$$

where $u_i \approx u(x_i)$. (Entries of the matrix that are not specified are zero.)

(a) Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

$$u(0) = 1, \quad u(1) = 2.$$

(b) Suppose that we have

$$\begin{aligned} -u''(x) &= (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1, \\ u(0) &= 1 \\ u(1) &= 2. \end{aligned}$$

Since this differential equation is linear, we can split up the solution into

$$u(x) = u_1(x) + u_2(x),$$

where $u_1(x)$ satisfies

$$\begin{aligned} -u_1''(x) &= 0, \quad 0 < x < 1, \\ u_1(0) &= 1 \\ u_1(1) &= 2 \end{aligned}$$

and $u_2(x)$ satisfies the equation

$$\begin{aligned} -u_2''(x) &= (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1, \\ u_2(0) &= 0 \\ u_2(1) &= 0. \end{aligned}$$

Determine $u_1(x), u_2(x)$ and the exact solution $u(x)$.

(c) Compute and plot the approximate solutions for $N = 8, 16, 32, 64$, and compare it to the exact solution $u(x)$.

3. [23 points] Using Taylor series expansions

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 + \frac{u'''(x)}{3!}\Delta x^3 + \dots$$

and

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 - \frac{u'''(x)}{3!}\Delta x^3 + \dots$$

show that the second order finite difference approximation

$$u''(x) \approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}$$

has accuracy $O(\Delta x^2)$. In other words, if $u''(x)$ is the exact second derivative, show that

$$\left| u''(x) - \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2} \right| = O(\Delta x^2).$$

4. [22 points] Consider the time-dependent homogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$\begin{aligned} u(0,t) &= 0, \qquad t > 0, \\ u(1,t) &= 0, \qquad t > 0 \end{aligned}$$

and initial condition $u(x,0) = \psi(x)$. We wish to approximate the solution $u(x,t)$ at spatial points $x_0, x_1, \dots, x_N, x_{N+1}$, and at times $t_0, t_1, t_2, \dots$. To do so, we will approximate $\frac{\partial u}{\partial t}$ at points x_i and time t_j with a forward difference in time

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{dt}$$

where $dt = t_{j+1} - t_j$. Similarly, we will approximate $\frac{\partial^2 u}{\partial x^2}$ with a 2nd order central difference at time t_j

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}$$

where $h = x_{i+1} - x_i$.

If we denote $u(x_i, t_j) = u_i^j$, this finite difference scheme in x and t can be written as

$$\frac{u_i^{j+1} - u_i^j}{dt} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}.$$

(a) Write a MATLAB code that, given $u(x_i, t_j) = u_i^j$, uses the above finite difference scheme to compute $u(x_i, t_{j+1}) = u_i^{j+1}$ at the next timestep. For the initial condition $u_i^0 = u(x_i, t_0) = u(x_i, 0) = \psi(x_i)$, use the discontinuous function

$$\psi(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$$

Take $N = 32$ and $dt = 1/10000$. Plot the solution at $t = 0$, and after 10, 50, and 100 timesteps. What happens (qualitatively) to the solution as t increases? Specifically, what happens to the parts of $\psi(x)$ that have sharp corners as t increases?