CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 35 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

35. [25 points] Let

$$H^1_D(0,1) = \{w \in H^1(0,1) : \ w(0) = w(1) = 0\}$$

and let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the norm $|||\cdot||| : \ H^1_D(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{a(v,v)}.$$

Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0,1,\ldots,N+1$. Let the continuous piecewise linear hat functions $\widehat{\phi}_j \in H^1_D(0,1)$ be defined by

$$\widehat{\phi}_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1,\ldots,N$. Also, let the continuous piecewise quadratic functions $\phi_j \in H^1_D(0,1)$ be defined by

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1,\ldots,N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in H^1_D(0,1)$ be defined by

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1,\ldots,N+1$. Let $\widehat{V}_N = \text{span}\{\widehat{\phi}_1,\ldots,\widehat{\phi}_N\}$ and let $V_N = \text{span}\{\phi_1,\ldots,\phi_N,\psi_1,\ldots,\psi_{N+1}\}$. Also, let $f \in L^2(0,1)$ be defined by

$$f(x) = \frac{12\sqrt{35}}{\sqrt{17}}x(1-x)$$

and let $u \in H^1_D(0,1)$ be such that

$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1).$$

Note that $a(u,u) = 1$ and that

$$(f, \widehat{\phi}_j) = -\frac{2\sqrt{35}}{\sqrt{17}} h(h^2 + 6x_j^2 - 6x_j)$$

for $j = 1, \ldots, N$;

$$(f, \phi_j) = \frac{2\sqrt{35}}{5\sqrt{17}} h(h^2 - 10x_j^2 + 10x_j)$$

for $j = 1, \ldots, N$; and

$$(f, \psi_j) = -\frac{4\sqrt{35}}{5\sqrt{17}} h(3h^2 - 10hx_j + 5h + 10x_j^2 - 10x_j)$$

for $j = 1, \ldots, N+1$.

We can obtain a finite element approximation to u by finding $\widehat{u}_N \in \widehat{V}_N$ such that

$$a(\widehat{u}_N, v) = (f, v) \text{ for all } v \in \widehat{V}_N.$$

However, we can obtain a better finite element approximation to u by finding $u_N \in V_N$ such that

$$a(u_N, v) = (f, v) \text{ for all } v \in V_N.$$

The stiffness matrix associated with finding u_N is

$$\mathbf{K} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{R} \end{bmatrix}$$

where $\mathbf{P} \in \mathbb{R}^{N \times N}$ is the matrix with entries

$$P_{jk} = a(\phi_j, \phi_k);$$

$\mathbf{Q} \in \mathbb{R}^{N \times N+1}$ is the matrix with entries

$$Q_{jk} = a(\phi_j, \psi_k);$$

and $\mathbf{R} \in \mathbb{R}^{N+1 \times N+1}$ is the matrix with entries

$$R_{jk} = a(\psi_j, \psi_k);$$

and the load vector associated with finding u_N is

$$\mathbf{b} = \begin{bmatrix} \mathbf{d} \\ \mathbf{g} \end{bmatrix}$$

where $\mathbf{d} \in \mathbb{R}^N$ is the vector with entries

$$d_j = (f, \phi_j);$$

and $\mathbf{g} \in \mathbb{R}^{N+1}$ is the vector with entries

$$g_j = (f, \psi_j).$$

(a) Write a code which can compute the energy norm of the error

$$|||u - u_N|||\,.$$

Use your code to produce a `loglog` plot of the energy norm of the error

$$|||u - u_N|||$$

for $N = 1, 3, 7, 15, 31, 63, 127$. On the same figure plot

$$|||u - \widehat{u}_N|||\,;$$

$$|||u - \widetilde{u}_N|||\,;$$

and

$$|||u - u_N^*|||\,;$$

for the same values of N, where $\widetilde{u}_N \in \operatorname{span}\{\phi_1, \ldots, \phi_N\}$ is such that

$$a(\widetilde{u}_N, v) = (f, v) \text{ for all } v \in \operatorname{span}\{\phi_1, \ldots, \phi_N\}$$

and $u_N^* \in \operatorname{span}\{\psi_1, \ldots, \psi_{N+1}\}$ is such that

$$a(u_N^*, v) = (f, v) \text{ for all } v \in \operatorname{span}\{\psi_1, \ldots, \psi_{N+1}\}.$$

Note that even though using the Galerkin method means that our approximations will be the best approximations, from the spaces that we are using, with respect to the energy norm $|||\cdot|||$, this does not mean that approximations obtained in this way will actually be any good.

(b) Since obtaining u_N involves solving a larger system of equations than that which has to be solved in order to obtain $\widehat{u}_N$, a fairer comparison of the accuracy of $\widehat{u}_N$ and u_N would be to plot $|||u - \widehat{u}_N|||$ and $|||u - u_N|||$ against the dimension of the spaces $\widehat{V}_N$ and V_N, respectively, instead of N. Produce a `loglog` plot showing this.

(c) Fill in the blanks in the below table where we use $\dim(\mathrm{W})$ to denote the dimension of a space W. If done correctly the table should show the factor that $|||u - \widehat{u}_N|||$ goes down by between each consecutive pair of values of N, and of the dimension of $\widehat{V}_N$, for which we computed $|||u - \widehat{u}_N|||$. If you wish you can reproduce the table yourself and so do not necessarily have to print out this page and fill it in.

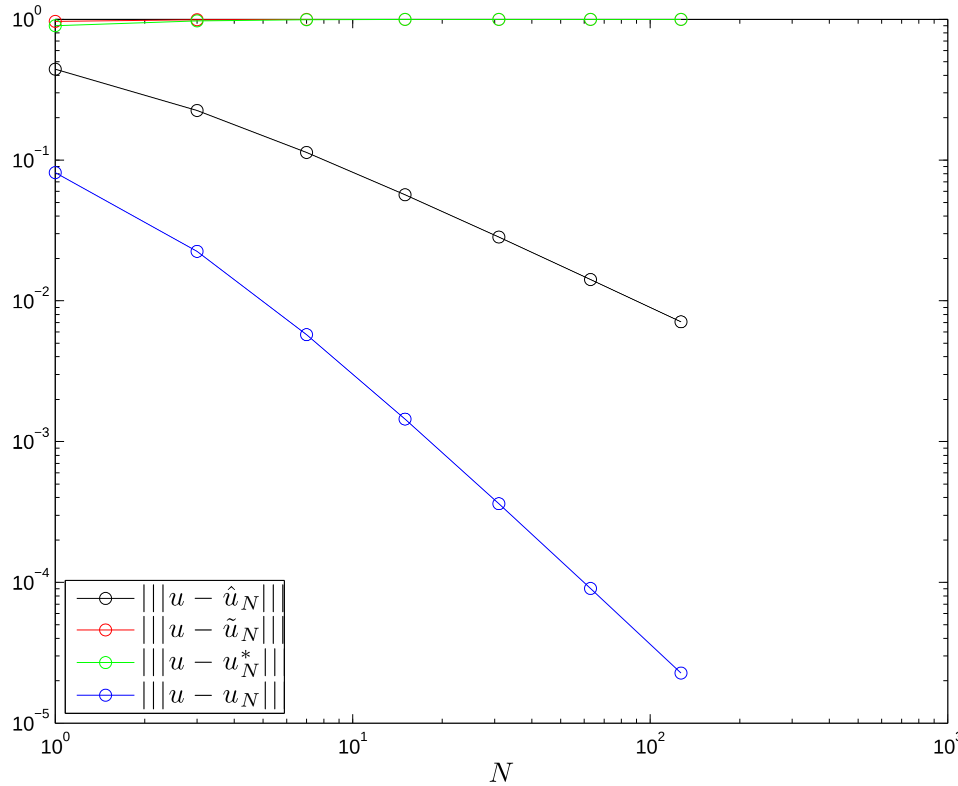
N_1	N_2	$\dim\left(\widehat{V}_{N_1}\right)$	$\dim\left(\widehat{V}_{N_2}\right)$	$\dfrac{\lvert\lvert\lvert u - \widehat{u}_{N_1}\rvert\rvert\rvert}{\lvert\lvert\lvert u - \widehat{u}_{N_2}\rvert\rvert\rvert}$
1	3			1.9688
3	7			
7	15			
15	31			
31	63			
63	127			

(d) Fill in the blanks in the below table where we use $\dim(\mathrm{W})$ to denote the dimension of a space W. If done correctly the table should show the factor that $|||u - u_N|||$ goes down by between each consecutive pair of values of N, and of the dimension of V_N, for which we computed $|||u - u_N|||$. If you wish you can reproduce the table yourself and so do not necessarily have to print out this page and fill it in.

N_1	N_2	$\dim(V_{N_1})$	$\dim(V_{N_2})$	$\dfrac{\lvert\lvert\lvert u - u_{N_1}\rvert\rvert\rvert}{\lvert\lvert\lvert u - u_{N_2}\rvert\rvert\rvert}$
1	3			3.6181
3	7			
7	15			
15	31			
31	63			
63	127			

Solution.

(a) [10 points] The requested plot is below.

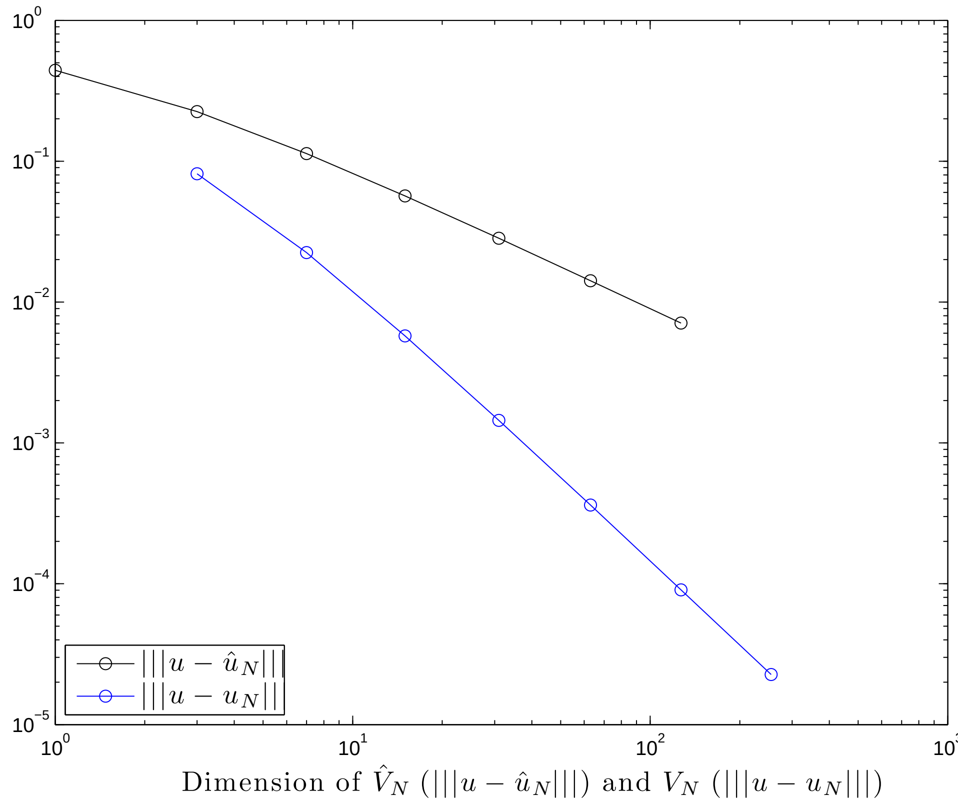


The code used to produce the results shown in this part and in parts (b), (c) and (d) is:

```
clear
clc

Nvec=2.^(1:1:7)-1;
energyerrhat=zeros(size(Nvec));
energyerr1=zeros(size(Nvec));
energyerr2=zeros(size(Nvec));
energyerr=zeros(size(Nvec));
for j=1:length(Nvec)
    N=Nvec(j);
    h=1/(N+1);

    Khat=sparse(N,N);
    Khat=Khat+sparse(1:N-1,2:N,-1/h,N,N);
    Khat=Khat+Khat.';
    Khat=Khat+sparse(1:N,1:N,2/h,N,N);

    P=sparse(N,N);
    P=P+sparse(1:N-1,2:N,1/(3*h),N,N);
    P=P+P.';
    P=P+sparse(1:N,1:N,14/(3*h),N,N);

    Q=sparse(N,N+1);
    Q=Q+sparse(1:N,1:N,-8/(3*h),N,N+1);
    Q=Q+sparse(1:N,2:N+1,-8/(3*h),N,N+1);
```

```
        R=sparse(N+1,N+1);
        R=R+sparse(1:N+1,1:N+1,16/(3*h),N+1,N+1);

        K=[P Q; Q.' R];

        fhat=(-2*sqrt(35/17)*h*(h^2+6*((1:N)*h).^2-6*(1:N)*h)).';

        d=((2/5)*sqrt(35/17)*h*(h^2-10*((1:N)*h).^2+10*((1:N)*h))).';

        g=(-(4/5)*sqrt(35/17)*h*(3*h^2-10*h*((1:N+1)*h)+5*h+10*((1:N+1)*h).^2-10*((1:N+1)*h)
            )).';

        f=[d; g];

        chat=Khat\fhat;
        c1=P\d;
        c2=R\g;
        c=K\f;

        energyerrhat(j)=sqrt(1-(chat.')*Khat*chat);
        energyerr1(j)=sqrt(1-(c1.')*P*c1);
        energyerr2(j)=sqrt(1-(c2.')*R*c2);
        energyerr(j)=sqrt(1-(c.')*K*c);
    end

    figure(1)
    clf
    loglog(Nvec,energyerrhat,'-ok')
    hold on
    loglog(Nvec,energyerr1,'-or')
    loglog(Nvec,energyerr2,'-og')
    loglog(Nvec,energyerr,'-ob')
    legendstr{1}='$|||u-\hat{u}_N|||$';
    legendstr{2}='$|||u-\tilde{u}_N|||$';
    legendstr{3}='$|||u-u_N^*|||$';
    legendstr{4}='$|||u-u_N|||$';
    legend(legendstr,'interpreter','latex','FontSize',14,'Location','Southwest')
    xlabel('$N$','interpreter','latex','FontSize',14)
    saveas(figure(1),'hw35a.eps','epsc')

    figure(2)
    clf
    loglog(Nvec,energyerrhat,'-ok')
    hold on
    loglog(2*Nvec+1,energyerr,'-ob')
    legendstr2{1}='$|||u-\hat{u}_N|||$';
    legendstr2{2}='$|||u-u_N|||$';
    legend(legendstr2,'interpreter','latex','FontSize',14,'Location','Southwest')
    xlabel('Dimension of $\hat{V}_N$ ($|||u-\hat{u}_N|||$) and $V_N$ ($|||u-u_N|||$)','
        interpreter','latex','FontSize',14)
    saveas(figure(2),'hw35b.eps','epsc')

    reductionhat=zeros(6,1);
    reduction=zeros(6,1);
    for j=1:length(Nvec)-1
        reductionhat(j)=energyerrhat(j)/energyerrhat(j+1);
        reduction(j)=energyerr(j)/energyerr(j+1);
    end
    display(reductionhat)
    display(reduction)
```

(b) [5 points] The requested plot is below.



(c) [5 points] The completed table is shown below.

N_1	N_2	$\dim\left(\widehat{\mathrm{V}}_{\mathrm{N}_1}\right)$	$\dim\left(\widehat{\mathrm{V}}_{\mathrm{N}_2}\right)$	$\dfrac{\lvert\lvert\lvert u - \widehat{u}_{N_1}\rvert\rvert\rvert}{\lvert\lvert\lvert u - \widehat{u}_{N_2}\rvert\rvert\rvert}$
1	3	1	3	1.9688
3	7	3	7	1.9863
7	15	7	15	1.9962
15	31	15	31	1.9990
31	63	31	63	1.9998
63	127	63	127	1.9999

(d) [5 points] The completed table is shown below.

N_1	N_2	$\dim\left(\mathrm{V}_{\mathrm{N}_1}\right)$	$\dim\left(\mathrm{V}_{\mathrm{N}_2}\right)$	$\dfrac{\lvert\lvert\lvert u - u_{N_1}\rvert\rvert\rvert}{\lvert\lvert\lvert u - u_{N_2}\rvert\rvert\rvert}$
1	3	3	7	3.6181
3	7	7	15	3.9121
7	15	15	31	3.9784
15	31	31	63	3.9946
31	63	63	127	3.9985
63	127	127	255	3.9915