CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 39 · Solutions

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

39. [25 points]

Let

$$f(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

In this question we will consider the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \quad t \ge 0,$$

and

$$u(1,t) = 1, \quad t \ge 0,$$

and initial condition

$$u(x,0) = x^3, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

The operator L has eigenvalues $\lambda_n = n^2\pi^2$ with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \psi_m(x)\psi_n(x)\,dx = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \ne n. \end{cases}$$

(a) Let $w(x)$ be such that

$$w''(x) = 0,$$

$$w(0) = 0$$

and

$$w(1) = 1.$$

Obtain a formula for $w(x)$.

(b) Let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$
$$\hat{u}(0,t) = \hat{u}(1,t) = 0, \qquad t \ge 0,$$

and

$$\hat{u}(x,0) = \hat{u}_0(x), \qquad 0 \le x \le 1,$$

where $\hat{u}_0(x)$ is such that

$$u(x,t) = w(x) + \hat{u}(x,t).$$

Obtain a formula for $\hat{u}_0(x)$.

(c) We can write

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$b_n = \int_0^1 f(x)\psi_n(x)\, dx.$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(d) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(e) Write out a formula for $u(x,t)$.

(f) Plot the approximations to $u(x,t)$ obtained by replacing the upper limit of the summation in your series solution with 20 for $t = 0, 0.1, 0.2, 0.3, 0.5, 1, 2$.

Solution.

(a) [5 points] The general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w(0) = B$ and so $w(0) = 0$ when $B = 0$. Hence, $w(x) = Ax$ and so $w(1) = A$ and hence $w(1) = 1$ when $A = 1$. Consequently,

$$w(x) = x.$$

(b) [5 points] We can compute that $u(x,t) = w(x) + \hat{u}(x,t)$ will be such that

$$u(x,0) = w(x) + \hat{u}(x,0) = x + \hat{u}_0(x)$$

and so since

$$u(x,0) = x^3$$

we can conclude that

$$\hat{u}_0(x) = x^3 - x.$$

(c) [5 points] Substituting the expressions for $\hat{u}(x,t)$ and $f(x)$ into the partial differential equation yields

$$\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) - \sum_{n=1}^{\infty} a_n(t)\left(-\left(L\psi_n\right)(x)\right) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

and hence

$$\sum_{n=1}^{\infty}\left(a_n'(t)+\lambda_n a_n(t)\right)\psi_n(x) = \sum_{n=1}^{\infty} b_n\psi_n(x).$$

We can then say that

$$\sum_{n=1}^{\infty}\left(a_n'(t)+\lambda_n a_n(t)\right)\int_0^1 \psi_n(x)\psi_m(x)\,dx = \sum_{n=1}^{\infty} b_n \int_0^1 \psi_n(x)\psi_m(x)\,dx$$

for $m = 1,2,\ldots$, from which it follows that

$$a_m'(t)+\lambda_m a_m(t) = b_m$$

for $m = 1,2,\ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m=n, \\ 0 & \text{otherwise,} \end{cases}$$

for $m,n = 1,2,\ldots$. Now, for $n = 1,2,\ldots$,

$$\begin{aligned}
b_n &= \int_0^1 f(x)\psi_n(x)\,dx \\
&= \sqrt{2}\int_0^1 f(x)\sin(n\pi x)\,dx \\
&= \sqrt{2}\left(\int_0^{1/2} f(x)\sin(n\pi x)\,dx + \int_{1/2}^1 f(x)\sin(n\pi x)\,dx\right) \\
&= 2\sqrt{2}\left(\int_0^{1/2} x\sin(n\pi x)\,dx + \int_{1/2}^1 (1-x)\sin(n\pi x)\,dx\right) \\
&= 2\sqrt{2}\left(\left[-\frac{1}{n\pi}x\cos(n\pi x)\right]_0^{1/2} + \frac{1}{n\pi}\int_0^{1/2}\cos(n\pi x)\,dx + \left[-\frac{1}{n\pi}(1-x)\cos(n\pi x)\right]_{1/2}^1 - \frac{1}{n\pi}\int_{1/2}^1 \cos(n\pi x)\,dx\right) \\
&= 2\sqrt{2}\left(-\frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\int_0^{1/2}\cos(n\pi x)\,dx + \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\int_{1/2}^1\cos(n\pi x)\,dx\right) \\
&= \frac{2\sqrt{2}}{n\pi}\left(\int_0^{1/2}\cos(n\pi x)\,dx - \int_{1/2}^1\cos(n\pi x)\,dx\right) \\
&= \frac{2\sqrt{2}}{n\pi}\left(\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^{1/2} - \left[\frac{1}{n\pi}\sin(n\pi x)\right]_{1/2}^1\right) \\
&= \frac{2\sqrt{2}}{n\pi}\left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right) \\
&= \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

Hence, for $n = 1,2,\ldots$,

$$a_n'(t) + n^2\pi^2 a_n(t) = \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

Also,

$$\hat{u}(x,0) = x^3 - x$$

means that

$$\sum_{n=1}^{\infty} a_n(0)\psi_n(x) = x^3 - x$$

and so

$$\sum_{n=1}^{\infty} a_n(0) \int_0^1 \psi_n(x)\psi_m(x)\,dx = \int_0^1 \left(x^3 - x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m(0) = \int_0^1 \left(x^3 - x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

for $m, n = 1, 2, \ldots$. Now, for $n = 1, 2, \ldots$,

$$\begin{aligned}
\int_0^1 \left(x^3 - x\right)\psi_n(x)\,dx &= \sqrt{2}\int_0^1 \left(x^3 - x\right)\sin(n\pi x)\,dx \\
&= \sqrt{2}\left(\left[-\frac{1}{n\pi}\left(x^3 - x\right)\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \left(3x^2 - 1\right)\cos(n\pi x)\,dx\right) \\
&= \frac{\sqrt{2}}{n\pi}\int_0^1 \left(3x^2 - 1\right)\cos(n\pi x)\,dx \\
&= \frac{\sqrt{2}}{n\pi}\left(\left[\frac{1}{n\pi}\left(3x^2 - 1\right)\sin(n\pi x)\right]_0^1 - \frac{6}{n\pi}\int_0^1 x\sin(n\pi x)\,dx\right) \\
&= -\frac{6\sqrt{2}}{n^2\pi^2}\int_0^1 x\sin(n\pi x)\,dx \\
&= -\frac{6\sqrt{2}}{n^2\pi^2}\left(\left[-\frac{1}{n\pi}x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right) \\
&= -\frac{6\sqrt{2}}{n^2\pi^2}\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^1\right) \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi).
\end{aligned}$$

Hence, for $n = 1, 2, \ldots$,

$$a_n(0) = \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi).$$

Therefore, for $n = 1, 2, \ldots$, $a_n(t)$ is the solution to the differential equation

$$a_n'(t) + n^2\pi^2 a_n(t) = \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

with initial condition

$$a_n(0) = \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi).$$

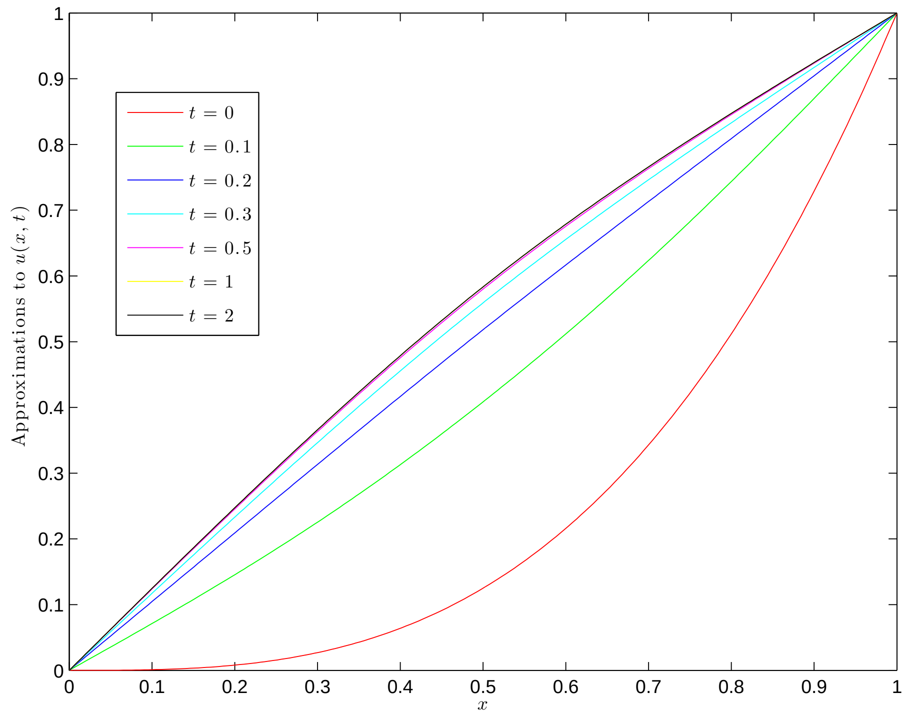
(d) [4 points] For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n(t) &= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi)e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2(s-t)} b_n \, ds \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi)e^{-n^2\pi^2 t} + \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\int_0^t e^{n^2\pi^2(s-t)}\, ds \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi)e^{-n^2\pi^2 t} + \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\left[\frac{1}{n^2\pi^2}e^{n^2\pi^2(s-t)}\right]_{s=0}^{s=t} \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi)e^{-n^2\pi^2 t} + \frac{4\sqrt{2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\left(\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}e^{-n^2\pi^2 t}\right) \\
&= \frac{6\sqrt{2}}{n^3\pi^3}\cos(n\pi)e^{-n^2\pi^2 t} + \frac{4\sqrt{2}}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right) \\
&= \frac{2\sqrt{2}}{n^3\pi^3}\left(3\cos(n\pi)e^{-n^2\pi^2 t} + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right)\right) \\
&= \frac{2\sqrt{2}}{n^3\pi^3}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(3\cos(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{-n^2\pi^2 t}\right).
\end{aligned}$$

(e) [3 points] We can write

$$\begin{aligned}
u(x,t) &= w(x) + \hat{u}(x,t) \\
&= x + \sum_{n=1}^{\infty} a_n(t)\psi_n(x) \\
&= x + \sum_{n=1}^{\infty} \frac{2\sqrt{2}}{n^3\pi^3}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(3\cos(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{-n^2\pi^2 t}\right)\psi_n(x) \\
&= x + \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(3\cos(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{-n^2\pi^2 t}\right)\sin(n\pi x).
\end{aligned}$$

(f) [3 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
col = 'rgbcmyk';
x = linspace(0,1,200);
tvec=[0 0.1 0.2 0.3 0.5 1 2];
figure(1)
clf
for j=1:length(tvec)
    U = x;
    t=tvec(j);
    for n=1:20
        U=U + 4*(2*sin(n*pi/2)/(n*pi)+(3*cos(n*pi)-2*sin(n*pi/2)/(n*pi))*exp(-n^2*pi^2*t
            ))*sin(n*pi*x)/(n^3*pi^3);
    end
    legendStr{j}=['$t=' num2str(t) '$'];
    plot(x,U,col(j))
    hold on
end
legend(legendStr,'interpreter','latex','location','best')
xlabel('$x$','interpreter','latex')
ylabel('Approximations to $u(x,t)$','interpreter','latex')
saveas(figure(1),'hw39f.eps','epsc')
```