CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 28 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 28 March 2014.

28. [25 points]
Let
$$H^1_D(0,1) = \big\{ w \in H^1(0,1) : w(0) = w(1) = 0 \big\}$$
and let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by
$$(v,w) = \int_0^1 v(x)w(x)\,dx$$
and the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by
$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$
Also, let $f \in L^2(0,1)$, let N be a positive integer, and let V_N be a subspace of $H^1_D(0,1)$. Moreover, let $u \in H^1_D(0,1)$ be such that
$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1)$$
and let $u_N \in V_N$ be such that
$$a(u_N,v) = (f,v) \text{ for all } v \in V_N.$$

(a) Show that
$$a(u-u_N, u-u_N) = a(u,u) - a(u_N,u_N).$$

(b) Let $\phi_1,\ldots,\phi_N \in V_N$ and let $\mathbf{K} \in \mathbb{R}^{N\times N}$ be the matrix with entries $K_{jk} = a(\phi_k,\phi_j)$ for $j,k = 1,\ldots,N$. Also, let
$$u_N = \sum_{j=1}^N c_j\phi_j$$
where $c_j \in \mathbb{R}$ is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^N$. Show that
$$\mathbf{c}^T\mathbf{K}\mathbf{c} = a(u_N,u_N).$$

Solution.

(a) [13 points] The properties satisfied by the inner product allow us to say that
$$\begin{aligned} a(u-u_N, u-u_N) &= a(u, u-u_N) - a(u_N, u-u_N) \\ &= a(u,u) - a(u,u_N) - a(u_N,u) + a(u_N,u_N) \\ &= a(u,u) - 2a(u,u_N) + a(u_N,u_N). \end{aligned}$$

Now, $u_N \in V_N$ and so the fact that
$$a(u_N,v) = (f,v) \text{ for all } v \in V_N$$

means that

$$a(u_N, u_N) = (f, u_N).$$

Moreover, $u_N \in H^1_D(0,1)$, since V_N is a subspace of $H^1_D(0,1)$ and $u_N \in V_N$, and so the fact that

$$a(u, v) = (f, v) \text{ for all } v \in H^1_D(0,1)$$

means that

$$a(u, u_N) = (f, u_N).$$

So,

$$a(u,u) - 2a(u, u_N) + a(u_N, u_N) = a(u,u) - 2(f, u_N) + (f, u_N) = a(u,u) - (f, u_N).$$

Therefore,

$$a(u - u_N, u - u_N) = a(u,u) - (f, u_N) = a(u,u) - a(u_N, u_N)$$

because

$$a(u_N, u_N) = (f, u_N).$$

(b) [12 points] We first compute that

$$\mathbf{Kc} = \mathbf{d}$$

where $\mathbf{d} \in \mathbb{R}^N$ is the vector with entries

$$d_j = \sum_{k=1}^{N} a(\phi_k, \phi_j) c_k$$

for $j = 1, \ldots, N$. Moreover, since

$$u_N = \sum_{j=1}^{N} c_j \phi_j = \sum_{k=1}^{N} c_k \phi_k,$$

the properties satisfied by the inner product yield that

$$\sum_{k=1}^{N} a(\phi_k, \phi_j) c_k = a\left(\sum_{k=1}^{N} c_k \phi_k, \phi_j\right) = a(u_N, \phi_j)$$

and so

$$d_j = a(u_N, \phi_j)$$

for $j = 1, \ldots, N$. Therefore,

$$\mathbf{c}^T \mathbf{Kc} = \mathbf{c}^T \mathbf{d} = \sum_{j=1}^{N} c_j a(u_N, \phi_j) = a\left(u_N, \sum_{j=1}^{N} c_j \phi_j\right) = a(u_N, u_N)$$

by the properties satisfied by the inner product and the fact that

$$u_N = \sum_{j=1}^{N} c_j \phi_j.$$