CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 11

Posted Friday 21 November 2014. Due Monday 1 December 2014, 5pm.

1. [20 points: 3 points each for (a) and (b); 7 points for (c)-(d)]
   Consider the wave equation posed on the infinite domain $x \in (-\infty, \infty)$:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty, \quad t > 0 \qquad (*)$$

with initial conditions $u(x,0) = \psi(x)$ and $u_t(x,0) = \gamma(x)$.

At a given point $(\widetilde{x}, \widetilde{t})$, with $\widetilde{x} \in (-\infty, \infty)$ and $\widetilde{t} > 0$, the solution $u(\widetilde{x}, \widetilde{t})$ of the wave equation is only affected by some portion of the initial data. In other words, $u(\widetilde{x}, \widetilde{t})$ is only influenced by $\psi(x)$ and $\gamma(x)$ for $x \in [a, b]$, where a and b will depend upon $\widetilde{x}$ and $\widetilde{t}$. This interval $[a, b]$ is called the *domain of dependence* of the solution at $(\widetilde{x}, \widetilde{t})$.

(a) Determine the domain of dependence of the solution to the wave equation $(*)$ at $(\widetilde{x}, \widetilde{t}) = (0, 1)$.

Now consider the heat equation on an unbounded domain:

$$u_t(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty$$

with initial data

$$u(x,0) = \psi(x).$$

Like d'Alembert's solution, there exists a formula for the solution of the heat equation on this domain: for all $t > 0$,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \psi(s)\,\mathrm{d}s.$$

(b) What is the domain of dependence of this solution to the heat equation at $(\widetilde{x}, \widetilde{t}) = (0, 1)$?
Contrast the physical implications of the domains of dependence for the heat and wave equations.

(c) Consider the *wave* equation with discontinuous initial data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0; \end{cases} \qquad \gamma(x) = 0.$$

On one plot, superimpose solutions to this equation at the four times $t = 0, 1/2, 1, 2$.
(Notice how the discontinuity in the initial data is propagated in time.)

(d) Now consider the *heat* equation with the same starting data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

Using the formula for $u(x,t)$ given above, produce solutions to this equation at the four times $t = 0, 0.01, 0.1, 1$. What happens to the discontinuity for $t > 0$?

Important hint: You will need to compute some nasty integrals here that you cannot work out entirely by hand. To produce your plots, use MATLAB's `erfc` command. For example,

$$\frac{2}{\sqrt{\pi}} \int_{\mathtt{z}}^{\infty} e^{-y^2}\,\mathrm{d}y = \mathtt{erfc(z)}.$$

2. [40 points: 8 points each]
Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We will look for solutions to the PDE of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

For simplicity, assume that $d \in (0, \pi)$.

(a) From the differential equation and this form for $u(x,t)$, show that the coefficients $a_k(t)$ must satisfy the ordinary differential equation

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t).$$

(b) Show that the following function satisfies the differential equation in part (a):

$$a_k(t) = C_1 \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) + C_2 \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big)$$

for arbitrary constants C_1 and C_2. (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

(c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{k=1}^{\infty} b_k(0)\psi(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

(d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x\sin(3\pi x)$, for which

$$b_k(0) = \frac{-6k\sqrt{2}(1 + (-1)^k)}{(k^2 - 9)^2\pi^2} \text{ for } k \neq 3, \qquad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x,t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

(e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d?

3. [40 points: 9 points each for (a), (b), (c), 13 points for (e)]
This problem and the next study equations in two dimensions. We begin with the steady-state problem. In place of the one dimensional equation, $-u'' = f$, we now have

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$ for all $0 \le x \le 1$ and $0 \le y \le 1$. The associated operator L is defined as

$$Lu = -(u_{xx} + u_{yy}),$$

acting on the space $C_D^2[0,1]^2$ consisting of twice continuously differentiable functions on $[0,1] \times [0,1]$ with homogeneous boundary conditions. We can solve the differential equation $Lu = f$ using the spectral method just as we have seen in class before. This problem will walk you through the process; you may consult Section 8.2 of the text for hints.

(a) Show that L is symmetric, given the inner product

$$(v,w) = \int_0^1 \int_0^1 v(x,y)w(x,y)\,dx\,dy.$$

(b) Verify that the functions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

are eigenfunctions of L for $j,k = 1,2,\ldots$.
(To do this, you simply need to show that $L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$ for some scalar $\lambda_{j,k}$.)
What is the eigenvalue $\lambda_{j,k}$ associated with $\psi_{j,k}$?

(c) Compute the inner product $(\psi_{j,k}, \psi_{j,k}) = \|\psi_{j,k}\|^2$.

(d) Let $f(x,y) = x(1-y)$. Compute the inner product $(f, \psi_{j,k})$.

(e) The solution to the diffusion equation is given by the spectral method, but now with a double sum to account for all the eigenvalues:

$$u(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

In MATLAB plot the partial sum

$$u_{10}(x,y) = \sum_{j=1}^{10} \sum_{k=1}^{10} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

Hint for 3d plots: To plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$, you could use

```
x = linspace(0,1,40); y = linspace(0,1,40);
[X,Y] = meshgrid(x,y);
Psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,Psi11)
```