CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 18

Posted Friday 14 February 2014. Due 1pm Friday 21 February 2014.

18. [25 points]

All parts of this question should be done by hand.

Let the inner product $(\cdot,\cdot) : \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined by

$$(\mathbf{A},\mathbf{B}) = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}b_{jk},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

With this inner product we associate the norm $\|\cdot\| : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\|\mathbf{A}\| = \sqrt{(\mathbf{A},\mathbf{A})}$. In the parts below best approximation is defined to be best approximation with respect to this norm.

(a) Verify that $(\cdot,\cdot)$ is an inner product on $\mathbb{R}^{2\times 2}$.

(b) Consider the subspace of $\mathbb{R}^{2\times 2}$ consisting of symmetric matrices:

$$V_3 = \{\mathbf{A} \in \mathbb{R}^{2\times 2} : \mathbf{A} = \mathbf{A}^T\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Compute the best approximation $\mathbf{M}_3$ to the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

from the subspace V_3.

(c) Now let $\mathbf{M} \in \mathbb{R}^{2\times 2}$ be any 2×2 matrix, and let $\mathbf{M}_3$ be its best approximation from V_3. Explain why the error $\mathbf{M} - \mathbf{M}_3$ must *always* have zero diagonal entries.

(d) Carefully consider the subspace

$$\widehat{V}_3 = \{\mathbf{A} \in \mathbb{R}^{2\times 2} : \mathbf{A} = \mathbf{A}^T\} = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}.$$

What is the best approximation $\widehat{\mathbf{M}}_3$ to the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

from the subspace $\widehat{V}_3$?