CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 25

Posted Friday 28 February 2014. Due 1pm Friday 14 March 2014.

25. [25 points]

Let the inner product $(\cdot,\cdot): C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|: C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let N be a posivitive integer and let $\psi_1, \ldots, \psi_N \in C[0,1]$ be such that $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$. For $g \in C[0,1]$, let

$$g_N = \sum_{n=1}^{N} \alpha_n \psi_n$$

where $\alpha_n = (g, \psi_n)$. Note that g_N is the best approximation to g from span$\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$. Moreover, let $u \in C^2[0,1]$ be such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u(0) = u(1) = 0$$

with $f \in C[0,1]$ being defined by $f(x) = 1$ for all $x \in [0,1]$. Note that $u(x) = \frac{1}{2}x(1-x)$.

(a) Show that

$$\|g - g_N\|^2 = \|g\|^2 - \sum_{n=1}^{N} \alpha_n^2.$$

(b) For the remainder of this question we will just consider the case when

$$\psi_n(x) = \sqrt{2}\sin(n\pi x) \text{ for } n = 1, 2, \ldots.$$

The best approximation to f from span$\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$f_N = \sum_{n=1}^{N} (f, \psi_n)\psi_n.$$

Produce a `loglog` plot of $\|f - f_N\|$ for $N = 1, 2, \ldots, 1000000$. Note that, for $n = 1, 2, \ldots$,

$$(f, \psi_n) = \frac{\sqrt{2}}{n\pi}\left(1 - (-1)^n\right).$$

(c) We can use the spectral method to conclude that the best approximation to u from $\mathrm{span}\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$u_N = \sum_{n=1}^{N} (u, \psi_n)\psi_n$$

where

$$(u, \psi_n) = \frac{(f, \psi_n)}{n^2\pi^2} = \frac{\sqrt{2}}{n^3\pi^3}\left(1 - (-1)^n\right).$$

Add a `loglog` plot of $\|u - u_N\|$ for $N = 1, 2, \ldots, 1000000$ to the plot that you produced in part (b).

(Be aware that the norm of the error may appear to flatline or become imaginary around 10^{-8}: this is a consequence of the computer's floating point arithmetic, and so you will not lose points because of this.)