CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 46 · Solutions

Posted Wednesday 9 April 2014. Due 1pm Friday 25 April 2014.

46. [25 points]

Let the norm $\|\cdot\|: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\|\mathbf{y}\| = \sqrt{\mathbf{y}\cdot\mathbf{y}}.$$

Let the timestep $\Delta t \in \mathbb{R}$ be such that $\Delta t > 0$ and let $t_k = k\Delta t$ for $k = 0, 1, 2, \ldots$. Let

$$\mathbf{A} = \begin{bmatrix} -50 & 49 \\ 49 & -50 \end{bmatrix}$$

and consider the problem of finding $\mathbf{x}(t)$ such that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \geq 0$$

and

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

(a) Compute $\mathbf{x}(t)$.

(b) How does $\|\mathbf{x}(t)\|$ behave as $t \to \infty$?

(c) For $k = 0, 1, 2, \ldots$, let $\mathbf{x}_k$ be the approximation to $\mathbf{x}(t_k)$ obtained using the forward Euler method. What choice of the timestep $\Delta t > 0$ will result in $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$?

(d) For $k = 0, 1, 2, \ldots$, let $\mathbf{x}_k$ be the approximation to $\mathbf{x}(t_k)$ obtained using the backward Euler method. What choice of the timestep $\Delta t > 0$ will result in $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$?

Solution.

(a) [10 points] Since,

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda + 50 & -49 \\ -49 & \lambda + 50 \end{bmatrix}$$

we have that

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= (\lambda + 50)^2 - 49^2 \\ &= \lambda^2 + 100\lambda + 50^2 - 49^2 \\ &= \lambda^2 + 100\lambda + (49 + 1)^2 - 49^2 \\ &= \lambda^2 + 100\lambda + 49^2 + 98 + 1 - 49^2 \\ &= \lambda^2 + 100\lambda + 99 \\ &= (\lambda + 1)(\lambda + 99) \end{aligned}$$

and so

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

when $\lambda = -99$ or $\lambda = -1$. Hence, the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = -99$$

and

$$\lambda_2 = -1.$$

Moreover,

$$(\lambda_1 \mathbf{I} - \mathbf{A}) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -49 & -49 \\ -49 & -49 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -49c_1 - 49c_2 \\ -49c_1 - 49c_2 \end{bmatrix}$$

and so to make this vector zero we need to set $c_2 = -c_1$. Hence, any vector of the form

$$\begin{bmatrix} c_1 \\ -c_1 \end{bmatrix}$$

where c_1 is a nonzero constant is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_1. Let us choose

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Furthermore,

$$(\lambda_2 \mathbf{I} - \mathbf{A}) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 49 & -49 \\ -49 & 49 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 49d_1 - 49d_2 \\ -49d_1 + 49d_2 \end{bmatrix}$$

and so to make this vector zero we need to set $d_2 = d_1$. Hence, any vector of the form

$$\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}$$

where d_1 is a nonzero constant is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_2. Let us choose

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = -99$ and $\lambda_2 = -1$ and eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

which are such that $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_2 = 1$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1 = 0$. Since $\mathbf{A} = \mathbf{A}^T$, if we set

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -99 & 0 \\ 0 & -1 \end{bmatrix}$$

then we have that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

and

$$
\begin{aligned}
e^{t\mathbf{A}} &= \mathbf{V}e^{t\boldsymbol{\Lambda}}\mathbf{V}^T \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{-99t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}}e^{-99t} & \frac{1}{\sqrt{2}}e^{-t} \\ -\frac{1}{\sqrt{2}}e^{-99t} & \frac{1}{\sqrt{2}}e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\left(e^{-t}+e^{-99t}\right) & \frac{1}{2}\left(e^{-t}-e^{-99t}\right) \\ \frac{1}{2}\left(e^{-t}-e^{-99t}\right) & \frac{1}{2}\left(e^{-t}+e^{-99t}\right) \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 = \begin{bmatrix} \frac{1}{2}\left(e^{-t}+e^{-99t}\right) & \frac{1}{2}\left(e^{-t}-e^{-99t}\right) \\ \frac{1}{2}\left(e^{-t}-e^{-99t}\right) & \frac{1}{2}\left(e^{-t}+e^{-99t}\right) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t}+e^{-99t} \\ e^{-t}-e^{-99t} \end{bmatrix}.
$$

(b) [5 points] Since all of the eigenvalues of $\mathbf{A}$ are negative, $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$.

(c) [5 points] Now,

$$\mathbf{x}_k = (\mathbf{I} + \Delta t\mathbf{A})^k \mathbf{x}_0.$$

Moreover, the eigenvalues of $\mathbf{I} + \Delta t\mathbf{A}$ are $1 + \Delta t\lambda_1 = 1 - 99\Delta t$ and $1 + \Delta t\lambda_2 = 1 - \Delta t$ and

$$\mathbf{I} + \Delta t\mathbf{A} = \mathbf{V} \begin{bmatrix} 1 - 99\Delta t & 0 \\ 0 & 1 - \Delta t \end{bmatrix} \mathbf{V}^{-1}.$$

Hence, since all of the eigenvalues of $\mathbf{A}$ are negative, and $-99 < -1$ we can conclude that if

$$\Delta t < \frac{2}{99}$$

then $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$.

(d) [5 points] Now,

$$\mathbf{x}_k = ((\mathbf{I} - \Delta t\mathbf{A})^{-1})^k \mathbf{x}_0.$$

Moreover, the eigenvalues of $(\mathbf{I} - \Delta t\mathbf{A})^{-1}$ are $\frac{1}{1-\Delta t\lambda_1} = \frac{1}{1+99\Delta t}$ and $\frac{1}{1-\Delta t\lambda_2} = \frac{1}{1+\Delta t}$ and

$$(\mathbf{I} - \Delta t\mathbf{A})^{-1} = \mathbf{V} \begin{bmatrix} \frac{1}{1+99\Delta t} & 0 \\ 0 & \frac{1}{1+\Delta t} \end{bmatrix} \mathbf{V}^{-1}.$$

Hence, since all of the eigenvalues of $\mathbf{A}$ are negative, we can conclude that there is no restriction on Δt to obtain $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$.
