CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6 · Solutions

Posted Wednesday 22 January 2014. Due 1pm Friday 31 January 2014.

6. [25 points]

Consider a bar of metal alloy manufactured such that its thermal conductivity is $\kappa(x) = e^{x/\ell}$ for $0 \le x \le \ell$.

(a) Use the homogeneous steady-state heat equation

$$\frac{d}{dx}\left(\kappa(x)u'(x)\right) = 0, \quad 0 \le x \le \ell$$

to determine a general formula for the steady-state temperature distribution $u(x)$ of this bar. The solution should include two arbitrary constants.

(b) Determine the steady-state temperature distribution $u(x)$ satisfying the boundary conditions

$$-\kappa\,(0)\,u'\,(0) = -K\,(u\,(0) - T)$$

and

$$u(\ell) = \gamma.$$

The condition at the left end $x = 0$ is called a Robin boundary condition. In the context of this problem it arises when the bar is in contact with air moving at constant speed. The moving air carries heat away from the bar, and this is known as convection. The temperature of the air is T and K is the convection coefficient.

Solution.

(a) [13 points] Integrate this equation once to obtain

$$\kappa(x)u'(x) = C_1,$$

where the constant C_1 will be determined later by the boundary conditions. Divide by $\kappa(x)$ to obtain the equation

$$u'(x) = \frac{C_1}{\kappa(x)}.$$

Substituting $\kappa(x) = e^{x/\ell}$, we have

$$u'(x) = C_1 e^{-x/\ell},$$

which we can integrate once to obtain the general form of the solution

$$u(x) = -C_1 \ell e^{-x/\ell} + C_2, \tag{1}$$

where C_1 and C_2 are constants.

(b) [12 points] We begin with the general formula (1). Since $\kappa(0) = 1$, $u(0) = C_2 - C_1\ell$, plug them into the boundary condition at $\ell = 0$ to obtain

$$u'(0) = K(C_2 - C_1\ell - T).$$

Since
$$u'(x) = C_1 e^{-x/\ell},$$
we have
$$u'(0) = C_1.$$
So
$$C_1 = K(C_2 - C_1\ell - T). \tag{2}$$
The boundary condition at the right end gives
$$u(\ell) = -C_1\ell e^{-1} + C_2 = \gamma. \tag{3}$$
The solution of the system of equations (2)-(3) is
$$\begin{aligned} C_1 &= \frac{K(\gamma - T)}{1 + K\ell(1 - e^{-1})} \\ C_2 &= \gamma + \frac{\ell e^{-1} K(\gamma - T)}{1 + K\ell(1 - e^{-1})}. \end{aligned}$$
Thus
$$u(x) = \frac{\ell K(T - \gamma)}{1 + K\ell(1 - e^{-1})}(e^{-x/\ell} - e^{-1}) + \gamma.$$
