# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 3

Posted Wednesday 28, January 2015. Due 5pm Wednesday 4, February 2015.

*Please write your name and instructor on your homework.*

1. [20 points: 10 each]

   (a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that $f$ is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

   $$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Since $f$ is linear, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

   (b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

   (Thus any linear function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix-vector product.)

2. [30 points: 5 each]

   Recall that a function $f : \mathcal{V} \to \mathcal{W}$ that maps a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a *linear operator* provided (1) $f(u + v) = f(u) + f(v)$ for all $u, v$ in $\mathcal{V}$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$.

   Demonstrate whether each of the following functions is a linear operator.
   (Show that both properties hold, or give an example showing that one of the properties must fail.)

   (a) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$.

   (b) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

   (d) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

   (e) $L : C^1[0, 1] \to C[0, 1]$, $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

   (f) $L : C^2[0, 1] \to C[0, 1]$, $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

3. [25 points: 5 points for (a), 10 points each for (b)-(c)]

In this problem we'll consider a linear operator mapping to and from a very specific vector space, and use it to explore what an operator inverse can look like.

Consider the $V$ defined as

$$V = \left\{ u(x) = \sum_{j=1}^{N} c_j \sin(j\pi x), \quad c_j \in \mathbb{R} \right\}.$$

In other words, $V$ is the set of all functions that are linear combinations of a finite number of different sine functions. This means that, for each $u \in V$, there is a set of coefficients $c_1, \ldots, c_N$ that is also associated with $u$.

(a) Show that $V$ is a subspace of the vector space $C_D^2[0,1]$, where

$$C_D^2[0,1] = \{u(x) \in C^2[0,1], \quad u(0) = u(1) = 0\}.$$

(b) Let the operator $L$ be defined as

$$Lu = -\frac{\partial^2 u}{\partial x^2}$$

Show that, for $u \in V$, $Lu \in V$. This shows that $L$ can be viewed as

$$L : V \to V,$$

a map from $V$ to $V$.

(c) We can define the operator $\tilde{L} : V \to V$ as

$$\tilde{L}u = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x).$$

Show that both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$.

Since both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$, we can refer to $\tilde{L}$ as the inverse $L^{-1}$ of $L : V \to V$.

4. [25 points: 5 each]

Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

(a) $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $(u,v) = \int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V} = C^1[0,1]$.

(b) $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $(u,v) = \int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V} = C[0,1]$.

(c) $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $(u,v) = \int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V} = C[0,1]$.

(d) $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $(u,v) = \int_0^1 (u(x)+v(x))\,dx$ where $\mathcal{V} = C[0,1]$.

(e) $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $(u,v) = \int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V} = C[-1,1]$.