CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 1 · Solutions

Posted Wednesday 22 August 2012. Due Wednesday 29 August 2012, 5pm.

A reminder from the course syllabus: Mathematically rigorous solutions are expected; strive for clarity and elegance. You may collaborate on the problems, but your write-up must be your own independent work. Transcribed solutions and copied MATLAB code are both unacceptable. *You may not consult solutions from previous sections of this class.* Unless it is specified that a particular calculation must be performed 'by hand,' you are encouraged to use MATLAB's Symbolic Math Toolbox (or Mathematica/Wolfram Alpha/Maple) to compute and simplify tedious integrals and derivatives on the problem sets. As always, you must document your calculations clearly.

1. [16 points: 2 points each for (1.1), (1.2); 3 points each for (1.3)–(1.6)]
For each of the following equations, specify whether each is (a) an ODE or a PDE; (b) determine its order; (c) specify whether it is linear or nonlinear. For those that are linear, specify whether they are (d) homogeneous or inhomogeneous, and (e) whether they have constant or variable coefficients.

(1.1) $\displaystyle \frac{dv}{dx} + \frac{2}{x}v = 0$ (1.2) $\displaystyle \frac{\partial v}{\partial t} - 3\frac{\partial v}{\partial x} = x - t$

(1.3) $\displaystyle \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 0$ (1.4) $\displaystyle \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$

(1.5) $\displaystyle \frac{d^2y}{dx^2} - \mu(1-y^2)\frac{dy}{dx} + y = 0$ (1.6) $\displaystyle \frac{d^2}{dx^2}\left[\rho(x)\frac{d^2u}{dx^2}\right] = \sin(x)$

Solution.

(1.1) ODE, first order, linear, homogeneous, variable coefficient
The $2/x$ factor in front of the v is the variable coefficient.

(1.2) PDE, first order, linear, inhomogeneous, constant coefficient
The $x - t$ term on the right, which does not involve v, makes the equation inhomogeneous.

(1.3) PDE, second order, nonlinear
Using the product rule for partial derivatives, we can write this equation in the equivalent form

$$\frac{\partial u}{\partial t} - 2\left(\frac{\partial u}{\partial x}\right)^2 - 2u\left(\frac{\partial^2 u}{\partial x^2}\right) = 0.$$

The second and third terms on the left hand side make this equation nonlinear.

(1.4) PDE, third order, nonlinear
The $u(\partial u/\partial x)$ term makes this equation nonlinear. This a version of the famous Korteweg-de Vries (KdV) equation that describes shallow water waves.

(1.5) ODE, second order, nonlinear
The $(1-y^2)(dy/dt)$ term makes this ODE nonlinear.

(1.6) ODE, fourth order, linear, inhomogeneous, variable coefficient
Using the product rule for partial derivatives, we can write this equation in the equivalent form

$$\frac{d^2\rho}{dx^2}\frac{d^2u}{dx^2} + 2\frac{d\rho}{dx}\frac{d^3u}{dx^3} + \rho(x)\frac{d^4u}{dx^4} = \sin(x),$$

hence we can see that it is fourth order. This equation, attributed to Euler, describes the deflection of a one-dimensional beam with a static load of $\sin(x)$; $\rho(x)$ describes the elasticity of the material that constitutes the beam.

2. [12 points: 4 points each]
Determine whether each of the following functions is a solution of the corresponding differential equation from question 1.

   (a) Is $v(x) = 1/x^2$ a solution of (1.1) ?
   (b) Is $v(x,t) = t(t+x)$ a solution of (1.2) ?
   (c) Is $u(x,t) = xe^t$ a solution of (1.3) ?

Solution.

(a) $v(x) = 1/x^2$ is a solution of (1.1).
To plug $v(x) = 1/x^2$ into the left-hand side of (1.1), we compute $dv/dx = d(x^{-2})/dx = -2x^{-3}$. Substituting this formula, the left-hand side of (1.1) becomes

$$-2x^{-3} + 2x^{-1}x^{-2} = 0.$$

This agrees with the right-hand side of (1.1), so this v is a solution.

(b) $v(x,t) = t(t+x)$ is a solution of (1.2).
We compute $\partial v/\partial t = 2t + x$ and $\partial v/\partial x = t$. Thus the left-hand side of (1.2) becomes

$$(2t + x) - 3(t) = x - t.$$

This agrees with the right-hand side of (1.2), so this v is a solution.

(c) $u(x,t) = xe^t$ is *not* a solution of (1.3).
We compute $\partial u/\partial t = xe^t$ and $\partial u/\partial x = e^t$. From this it follows that

$$\frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}2xe^{2t} = 2e^{2t}.$$

Thus the left-hand side of (1.3) is

$$xe^t - 2e^{2t},$$

which is nonzero in general, in disagreement with the right-hand side of (1.3).

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3. [34 points: 11 points each for (a),(b); 12 points for (c)]
Consider the temperature function

$$u(x,t) = e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

for constant κ, ρ, c, and θ.

   (a) Show that this function $u(x,t)$ is a solution of the homogeneous heat equation

$$\rho c\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \ell \text{ and all } t.$$

   (b) For which values of θ will u satisfy homogeneous Dirichlet boundary conditions at $x = 0$ and $x = \ell$?
   (c) Suppose $\kappa = 2.37$ W/(cm K), $\rho = 2.70$ g/cm^3, and $c = 0.897$ J/(g K) (approximate values for aluminum found on Wikipedia), and that the bar has length $\ell = 10$ cm. Let θ be such that $u(x,t)$ satisfies homogeneous Dirichlet boundary conditions as in part (b) and $u(x,t) \geq 0$ for all x and t.
   Use MATLAB to plot the solution $u(x,t)$ for $0 \leq x \leq \ell$ and time $0 \leq t \leq 20$ sec.
   You may choose to do this in one of the following ways: (1) Plot the solution for $0 \leq x \leq \ell$ at times $t = 0, 4, 8, \ldots, 20$ sec., superimposing all six plots on the same axis (helpful commands: `linspace`, `plot`, `hold on`); (2) Create a three-dimensional plot of the data using `surf`, `mesh`, or `waterfall`. In either case, be sure to produce an attractive, well-labeled plot.

Solution.

(a) We compute

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\kappa\theta^2/(\rho c)e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)\\ \frac{\partial u}{\partial x} &= \theta e^{-\kappa\theta^2 t/(\rho c)}\cos(\theta x)\\ \frac{\partial^2 u}{\partial x^2} &= -\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x).\end{aligned}$$

With these formulas in hand it is easy to verify that

$$\rho c\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}.$$

(b) We wish to find the values of θ that give homogeneous Dirichlet boundary conditions, i.e., $u(0,t) = u(\ell,t) = 0$ for all t. Since $e^{-\kappa\theta^2 t/(\rho c)}$ is positive for all t, we can only get the homogeneous Dirichlet conditions when $\sin(\theta x) = 0$. For any θ, $\sin(\theta\cdot 0) = 0$, so the condition at $x = 0$ is automatically satisfied. To get $\sin(\theta\ell) = 0$, we need $\theta\ell$ to be an integer multiple of π, that is,

$$\theta\ell = \pi n, \qquad n = 0, \pm 1, \pm 2, \dots,$$

or equivalently

$$\theta = \frac{\pi n}{\ell}, \qquad n = 0, \pm 1, \pm 2, \dots.$$

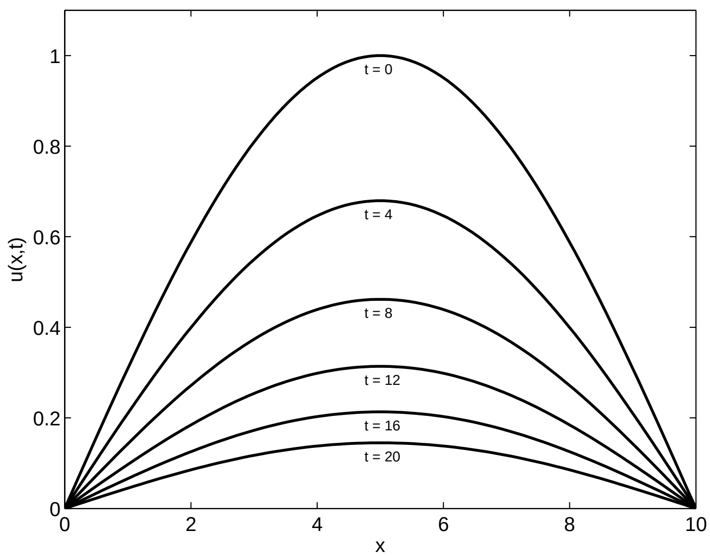
(Notice that if $n = 0$ we have the trivial solution $u(x,t) = 0$ for all x, t. If $n = 1$, we have a solution for which $u(x,t) \geq 0$ for all x, t. For other values of n the solution will be *negative* for some x, t. If our temperature is measured in Kelvin this could be a problem! However, this heat equation takes the same form if we shift to Celsius units, so we needn't be so troubled by the negative values of temperature.)

(c) Since $n = 0$ is trivial, we shall take $n = 1$ ($\theta = \pi/\ell$) to obtain

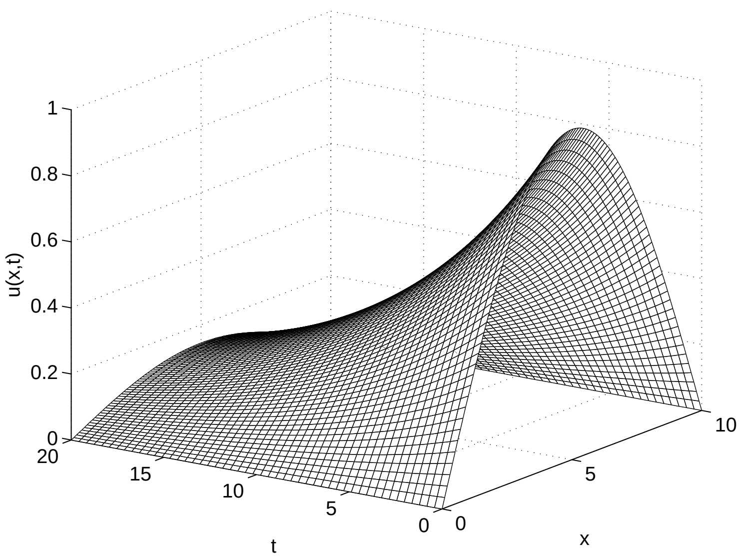
$$\begin{aligned}u(x,t) &= e^{-\kappa\pi^2 t/(\ell^2\rho c)}\sin(\pi x/\ell)\\ &= e^{-2.37\pi^2 t/(100\cdot 2.70\cdot 0.897)}\sin(\pi x/10).\end{aligned}$$

(d) Solutions are shown in the attached plots. Any of these style is acceptable. The MATLAB code that generated these plots follows.

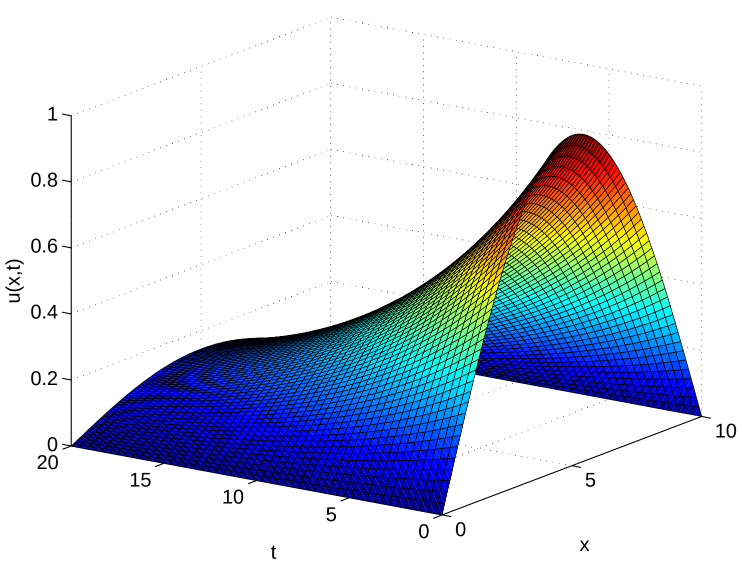
[**GRADERS**: please make a note if students did not include their MATLAB code, but do not take off points for this first time. You should do so in the future, though!]



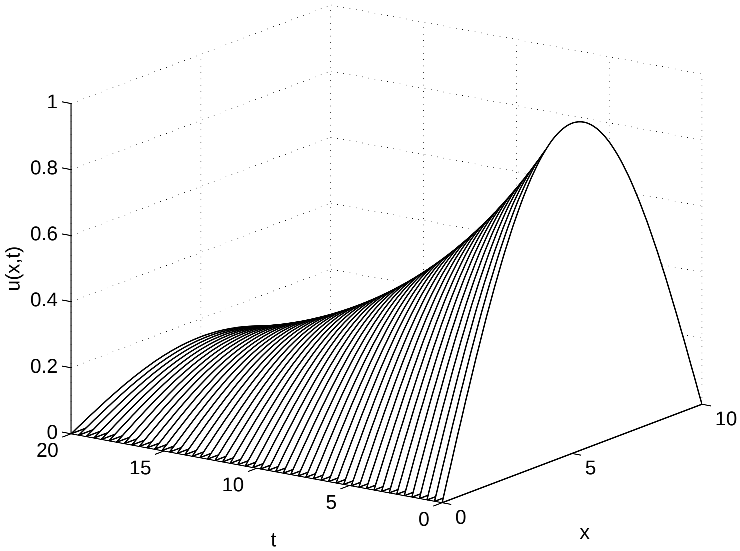
1
0.8
0.6
0.4
0.2
0
u(x,t)
t = 0
t = 4
t = 8
t = 12
t = 16
t = 20
0
2
4
6
8
10
x

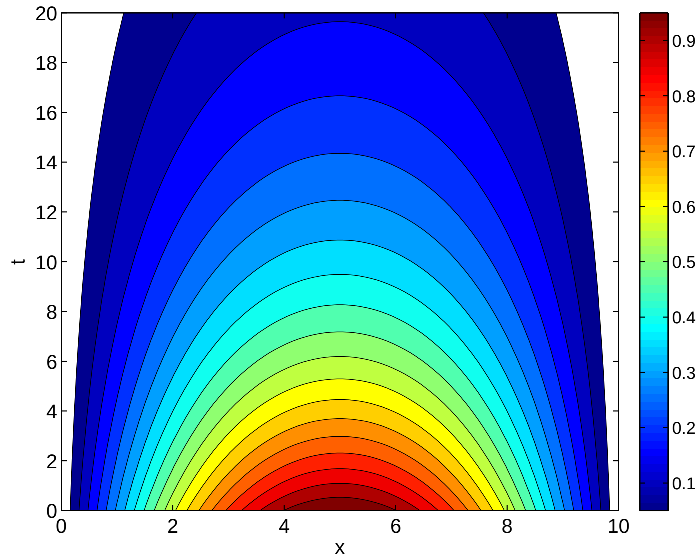


1
0.8
0.6
0.4
0.2
0
u(x,t)
20
15
10
5
0
t
0
5
10
x



1
0.8
0.6
0.4
0.2
0
u(x,t)
20
15
10
5
0
t
0
5
10
x





MATLAB code:

```
 c = .897;
 kappa = 2.37;
 rho = 2.70;
 l = 10;
 theta = pi/l;

% first style: snapshots at t = 0, 4, 8, ..., 20

 t = 0:4:20;
 x = linspace(0,l,100);

 figure(1), clf
 for j=1:length(t)
    u = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x);  % compute u(:,t(j))
    plot(x,u,'k-','linewidth',2), hold on
    text(4.75, max(u)-.03, sprintf('t = %d', t(j)))
 end
 axis([0 10 0 1.1])
 set(gca,'fontsize',14)
 xlabel('x')
```

```
 ylabel('u(x,t)')
 print -depsc2 checksol1

% generate data for 3-d plots

 x = linspace(0,1,100);
 t = linspace(0,20,50);
 U = zeros(length(t), length(x));
 for j=1:length(t)
    U(j,:) = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x);
 end

% mesh plot
 figure(2), clf
 mesh(x,t,U,'edgecolor','k')
 view(-55,20)
 set(gca,'fontsize',14)
 xlabel('x'), ylabel('t'), zlabel('u(x,t)')
 print -depsc2 checksol2

% surf plot
 figure(3), clf
 surf(x,t,U)
 view(-55,20)
 set(gca,'fontsize',14)
 xlabel('x'), ylabel('t'), zlabel('u(x,t)')
 print -depsc2 checksol3

% waterfall plot
 figure(4), clf
 plt = waterfall(x,t,U);
 set(plt,'edgecolor','k')       % make the lines black
 view(-55,20)
 set(gca,'fontsize',14)
 xlabel('x'), ylabel('t'), zlabel('u(x,t)')
 print -depsc2 checksol4

% contour plot
 figure(5), clf
 [cs,h] = contourf(x,t,U,[.05:.05:.95],'k-');
 set(gca,'fontsize',14)
 xlabel('x'), ylabel('t')
 colorbar
 print -depsc2 checksol5
```

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4. [20 points: 8 points for (a); 6 points each for (b), (c)]
Consider a bar of metal alloy manufactured such that its thermal conductivity is $\kappa(x) = 1 + \alpha x$ for constant α and $0 \le x \le \ell$. You may assume the heat equation for a non-uniform bar:

$$c(x)\rho(x)u_t(x,t) = \frac{\partial}{\partial x}\Big(\kappa(x)u_x(x,t)\Big).$$

(a) Determine a general formula for the steady-state temperature distribution of this bar, which should include two free constants. (Assume no additional source term $f(x)$ is present.)

(b) Find formulas for these free constants in the case that the ends of the bar are submerged in ice baths of γ deg on the left and δ deg on the right.

(c) Now find formulas for the free constants in the case that the left end has a fixed *heat flux* equal to γ (measured in $\mathrm{J}/(\mathrm{m}^2 \cdot \mathrm{sec})$) and the right end is submerged in an ice bath of δ deg.

Solution.

(a) Because the bar is at steady state, we have

$$\frac{\partial u}{\partial t}(x) = 0,$$

where we have dropped the t argument from $u(x,t)$, since u is independent of t in this context. The heat equation becomes

$$0 = \frac{d}{dx}\Big(\kappa(x)\frac{d}{dx}u(x)\Big).$$

In other words, the quantity on the right under the first derivative must be a constant. Integrate this equation once to obtain

$$\kappa(x)\frac{d}{dx}u(x) = C_1,$$

where the constant C_1 will be determined later by the boundary conditions. Divide by $\kappa(x)$ to obtain the equation

$$\frac{d}{dx}u(x) = \frac{C_1}{\kappa(x)}.$$

Substituting in the particular equation for this bar, $\kappa(x) = 1 + \alpha x$, we have

$$\frac{d}{dx}u(x) = \frac{C_1}{1+\alpha x},$$

which we can integrate once to obtain the general form of the solution

$$u(x) = \frac{C_1}{\alpha}\log(1+\alpha x) + C_2,$$

where C_1 and C_2 are constants. (As is common in higher mathematics, we use log to denote the natural logarithm rather than ln.)

(b) If the ends are submerged in ice baths of γ and δ degrees, we have the Dirichlet boundary conditions

$$u(0) = \gamma, \qquad u(\ell) = \delta.$$

We must find C_1 and C_2 to satisfy these conditions. This gives two equations in two unknowns:

$$\begin{aligned} \gamma &= \frac{C_1}{\alpha}\log(1+\alpha\cdot 0) + C_2 \\ \delta &= \frac{C_1}{\alpha}\log(1+\alpha\ell) + C_2. \end{aligned}$$

Since $\log(1+\alpha\cdot 0) = \log(1) = 0$, the first equation reduces to

$$\gamma = C_2.$$

Substituting this formula into the second equation and solving for C_1 yields

$$C_1 = \frac{\alpha(\delta-\gamma)}{\log(1+\alpha\ell)}.$$

Thus, the steady state solution with desired Dirichlet boundary conditions is

$$u(x) = (\delta-\gamma)\frac{\log(1+\alpha x)}{\log(1+\alpha\ell)} + \gamma.$$

(c) If the heat flux is fixed at γ J/(m$^2\cdot$sec) on the left end of the bar, we have

$$q(0)=\gamma.$$

Fourier's law of heat conduction gives

$$q(0)=-\kappa(0)\frac{\partial u}{\partial x}(x,t),$$

and hence we have the Neumann boundary condition on the left end:

$$\frac{\partial u}{\partial x}(0)=-\frac{\gamma}{\kappa(0)}=-\frac{\gamma}{1}=-\gamma.$$

[**GRADERS**: If students directly assumed that $\partial u(0)/\partial t=\gamma$ without appeal to Fourier's Law, please deduct 5 points.]

The ice bath on the right hand side gives the Dirichlet condition

$$u(\ell)=\delta.$$

To impose the Neumann condition, we must take a derivative of the general solution

$$u(x)=\frac{C_1}{\alpha}\log(1+\alpha x)+C_2,$$

or just go back to the intermediate step in part (a), to obtain

$$\frac{\partial u}{\partial x}(x)=\frac{C_1}{1+\alpha x},$$

and so

$$\frac{\partial u}{\partial x}(0)=C_1.$$

Thus our boundary conditions again impose two equations in two unknowns:

$$\begin{aligned}-\gamma &= C_1\\ \delta &= \frac{C_1}{\alpha}\log(1+\alpha\ell)+C_2.\end{aligned}$$

Substituting $C_1=-\gamma$ into the second equation and solving for C_2 gives

$$C_2=\delta+\frac{\gamma}{\alpha}\log(1+\alpha\ell).$$

With these constants, the solution becomes

$$\begin{aligned}u(x) &= \frac{\gamma}{\alpha}\big(\log(1+\alpha\ell)-\log(1+\alpha x)\big)+\delta\\ &= \frac{\gamma}{\alpha}\log\Big(\frac{1+\alpha\ell}{1+\alpha x}\Big)+\delta\end{aligned}$$

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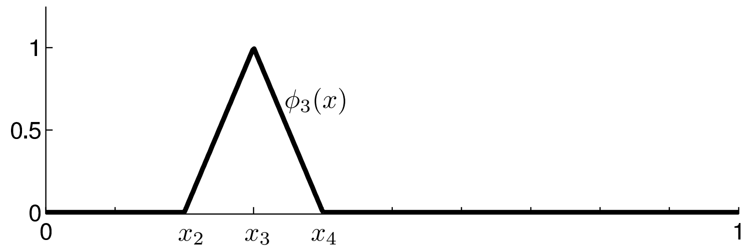
5. [18 points: 10 points for (a); 8 points for (b)]
Suppose $N\ge 1$ is an integer and define $h=1/(N+1)$ and $x_k=kh$ for $k=0,\ldots,N+1$. Consider the $N+2$ hat functions, defined as

$$\phi_k(x)=\begin{cases}(x-x_{k-1})/h, & x\in[x_{k-1},x_k);\\ (x_{k+1}-x)/h, & x\in[x_k,x_{k+1});\\ 0, & \text{otherwise.}\end{cases}$$

for $x\in[0,1]$ and $k=0,\ldots,N+1$. We call these piecewise linear functions *hat functions* because of their shape. They will be important functions later in the course. For example, when $N=9$ and $k=3$, this function takes the following form.



(a) Write a MATLAB function for $\phi_k(x)$. It should take in as input x, k, and N. It should return the value $\phi_k(x)$. It should also be able to take in a vector for $\mathbf{x} = (x_1, \ldots, x_m)$ and return the vector $\phi_k(\mathbf{x}) = (\phi_k(x_1), \ldots, \phi_k(x_m))$.

(b) Let $N = 9$. Plot $\phi_0(x), \phi_4(x), \phi_5(x), \phi_6(x), \phi_{10}(x)$ on the same figure. Make sure to:

- plot each function with a different color;
- label the axes and provide a title;
- create an accurate legend for the figure.

Optional but encouraged: Adjust the text sizes (if necessary) to make the labels easily legible, and use the LaTeX interpreter to make your labels, titles, and legend look stylish. See the MATLAB Primer on the course web site for details.

Solution.

(a) A sample implementation of the hat function follows below. Student solutions should vary widely.

```
 function phi_k = hat(x,k,N)

% function phi_k = hat(x,k,N)
%
% evaluates the hat function phi_k(x), where N denotes the
% size of the mesh, so that phi_k is non-zero on ((k-1)*h,(k+1)*h)
% with h = 1/(N+1).

 h = 1/(N+1);
 xk = [0:N+1]*h;

 if k==0
     phi_k =  ((x>=xk(1))&(x<xk(2))).*((xk(2)-x)/h);
 elseif k==N+1
     phi_k =  ((x>=xk(N+1))&(x<=xk(N+2))).*((x-xk(N+1))/h);
 else
     phi_k =  ((x>=xk(k))&(x<xk(k+1))).*((x-xk(k))/h) ...
                        + ((x>=xk(k+1))&(x<xk(k+2))).*((xk(k+2)-x)/h);
 end
```

(b) The MATLAB script to create the plot is given below. (It is acceptable for students to print their output in black and white.)

```
%% Code for plotting hat functions
N=9;

k=[0 4 5 6 10]; % hat function indices
colors='bgrcmyk';
x=linspace(0,1,1000);
figure; hold on;
ct=0;  % initializing counter for loop
for j=k
```

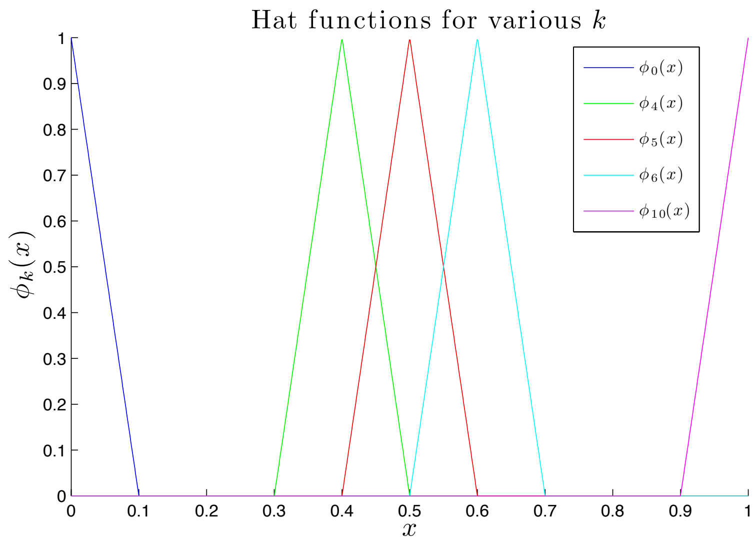


Figure 1:

```
    ct=ct+1;
    plot(x,hat(x,j,N),colors(ct));
    legendStr{ct}=['$\phi_{' num2str(j) '}(x)$'];
end
xlabel('$x$','interpreter','latex','fontsize',16);
ylabel('$\phi_k(x)$','interpreter','latex','fontsize',16);
title('Hat functions for various $k$','interpreter','latex','fontsize',16);
legend(legendStr,'interpreter','latex');
```