CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 7 · Solutions

Posted Wednesday 15 October, 2014. Due 5pm Wednesday 22 October, 2014.

*Please write your name and **residential college** on your homework.*

1. [50 points: 10 points each]

Let the inner product $(\cdot,\cdot): C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|: C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let the linear operator $L: C_D^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$C_D^2[0,1] = \{w \in C^2[0,1] : w(0) = w(1) = 0\}.$$

Recall that the operator L has eigenvalues

$$\lambda_n = n^2\pi^2$$

with corresponding eigenfunctions

$$\phi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$. Let N be a positive integer, let $f \in C[0,1]$ be defined by $f(x) = 8x^2(1-x)$ and let u be the solution to

$$Lu = f.$$

(a) Compute the best approximation f_N to f from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$.

(b) Write down the infinite series solution to

$$Lu = f$$

that is obtained using the spectral method, i.e.

$$u(x) = \sum_{j=1}^{\infty} \alpha_j \phi_j(x)$$

where α_j are coefficients to be specified. Given this above series, determine the best approximation u_N to u from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Plot the approximations u_N to u that you obtained using the spectral method for $N = 1, 2, 3, 4, 5, 6$.

(d) By shifting the data and then using an infinite series solution that you have obtained previously in this question, obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{4}$$

and

$$\tilde{u}(1) = \frac{1}{4}.$$

(e) Let $\tilde{u}_N$ be the series solution that you obtained in part (d) but with ∞ replaced by N, i.e.

$$\tilde{u}_N(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

Plot $\tilde{u}_N$ for $N = 1, 2, 3, 4, 5, 6$.

Solution.

(a) [5 points] Since $f(x) = 8x^2(1-x) = 8\left(x^2 - x^3\right)$, we have that, for $k = 1, 2, \ldots$,

$$\begin{aligned}
(f, \psi_k) &= 8\sqrt{2}\int_0^1 \left(x^2 - x^3\right) \sin\left(k\pi x\right)\, dx \\
&= 8\sqrt{2}\left(\left[-\frac{1}{k\pi}\left(x^2 - x^3\right)\cos\left(k\pi x\right)\right]_0^1 + \frac{1}{k\pi}\int_0^1 \left(2x - 3x^2\right)\cos(k\pi x)\, dx\right) \\
&= \frac{8\sqrt{2}}{k\pi}\int_0^1 \left(2x - 3x^2\right)\cos(k\pi x)\, dx \\
&= \frac{8\sqrt{2}}{k\pi}\left(\left[\frac{1}{k\pi}\left(2x - 3x^2\right)\sin\left(k\pi x\right)\right]_0^1 - \frac{1}{k\pi}\int_0^1 (2 - 6x)\sin(k\pi x)\, dx\right) \\
&= -\frac{8\sqrt{2}}{k^2\pi^2}\int_0^1 (2 - 6x)\sin(k\pi x)\, dx \\
&= -\frac{8\sqrt{2}}{k^2\pi^2}\left(\left[-\frac{1}{k\pi}(2 - 6x)\cos\left(k\pi x\right)\right]_0^1 - \frac{6}{k\pi}\int_0^1 \cos(k\pi x)\, dx\right) \\
&= -\frac{8\sqrt{2}}{k^2\pi^2}\left(\frac{4}{k\pi}\cos\left(k\pi\right) + \frac{2}{k\pi} - \frac{6}{k\pi}\left[\frac{1}{k\pi}\sin(k\pi x)\right]_0^1\right) \\
&= \frac{-16\sqrt{2}}{k^3\pi^3}\left(1 + 2\cos\left(k\pi\right)\right) \\
&= \frac{-16\sqrt{2}}{k^3\pi^3}\left(1 + 2\left(-1\right)^k\right).
\end{aligned}$$

Hence, the best approximation to f from $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$\begin{aligned}
f_N(x) &= \sum_{j=1}^{N} (f, \psi_j)\psi_j(x) \\
&= \sum_{j=1}^{N} \frac{-16\sqrt{2}}{j^3\pi^3}\left(1 + 2\left(-1\right)^j\right)\sqrt{2}\sin(j\pi x) \\
&= \sum_{j=1}^{N} \frac{-32}{j^3\pi^3}\left(1 + 2\left(-1\right)^j\right)\sin(j\pi x).
\end{aligned}$$

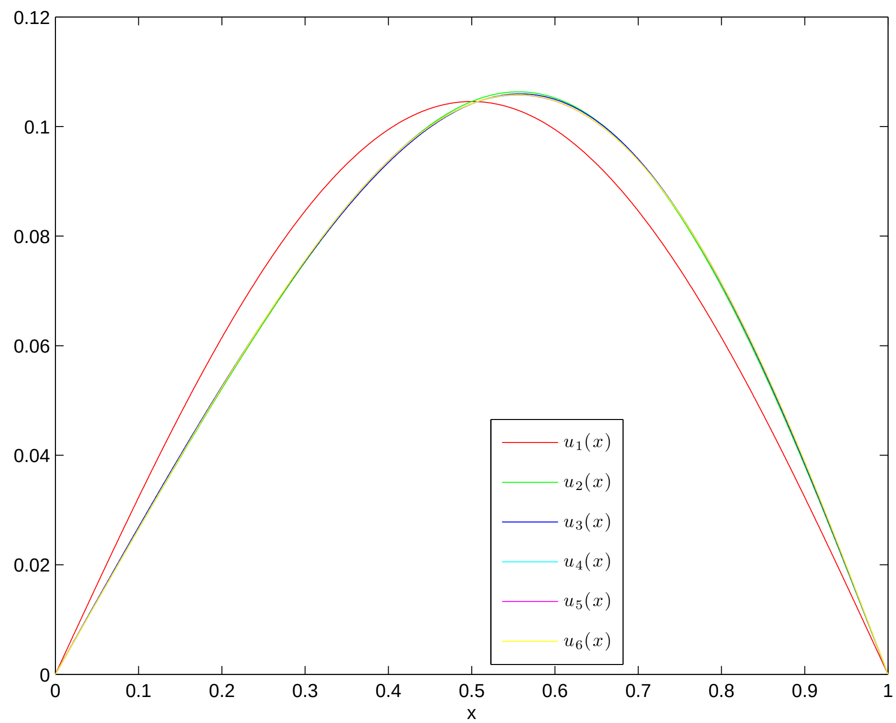
(b) [6 points] The series solution that we obtain using the spectral method is

$$u(x) = \sum_{j=1}^{\infty} \frac{(f, \psi_j)}{\lambda_j} \psi_j(x) = \sum_{j=1}^{\infty} \frac{-32}{j^5 \pi^5} \left(1 + 2(-1)^j\right) \sin(j\pi x).$$

The best approximation to u from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$u_N(x) = \sum_{j=1}^{N} \frac{(f, \psi_j)}{\lambda_j} \psi_j(x) = \sum_{j=1}^{N} \frac{-32}{j^5 \pi^5} \left(1 + 2(-1)^j\right) \sin(j\pi x).$$

(c) [2 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(2)
clf
uk = zeros(size(x));
for k=1:6
    uk = uk + 32*(2*(-1).^(k+1)-1)./(k.^5*pi^5)*sin(k*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$u_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(2),'hw26d','epsc')
```

(d) [6 points] Let u be the solution to $Lu = f$ and let $w \in C^2[0,1]$ be such that

$$-w''(x) = 0, \quad 0 < x < 1;$$

$$w(0) = -\frac{1}{4}$$

and

$$w(1) = \frac{1}{4}.$$

Then $\tilde{u}(x) = w(x) + u(x)$ will be such that

$$-\tilde{u}''(x) = -w''(x) - u''(x) = 0 + f(x) = f(x);$$

$$\tilde{u}(0) = w(0) + u(0) = -\frac{1}{4} + 0 = -\frac{1}{4};$$

and

$$\tilde{u}(1) = w(1) + u(1) = \frac{1}{4} + 0 = \frac{1}{4}.$$

Now, the general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w(0) = B$ and so $w(0) = -\frac{1}{4}$ when $B = -\frac{1}{4}$. Hence, $w(x) = Ax - \frac{1}{4}$ and so $w(1) = A - \frac{1}{4}$ and hence $w(1) = \frac{1}{4}$ when $A = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$. Consequently,

$$w(x) = \frac{1}{2}x - \frac{1}{4}$$

and so

$$\tilde{u}(x) = \frac{1}{2}x - \frac{1}{4} + u(x).$$

We can then use the series solution to $Lu = f$ that we obtained in part (e) to obtain the series solution

$$\tilde{u}(x) = \frac{1}{2}x - \frac{1}{4} + \sum_{j=1}^{\infty} \frac{-32}{j^5\pi^5}\left(1 + 2(-1)^j\right)\sin(j\pi x)$$

to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

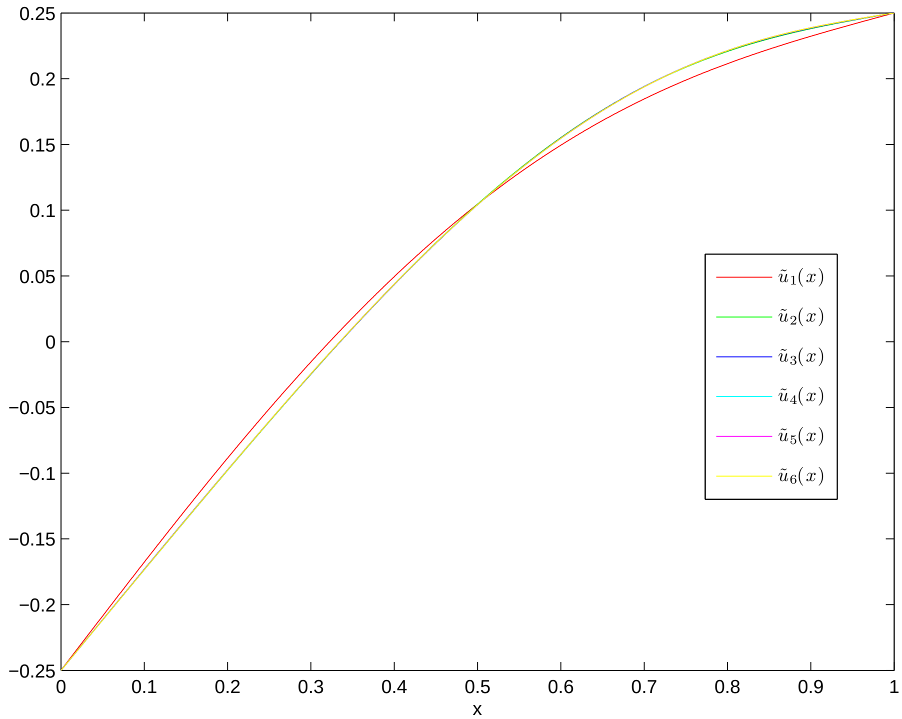
$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{4}$$

and

$$\tilde{u}(1) = \frac{1}{4}.$$

(e) [2 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(3)
clf
uk = x/2-1/4;
for k=1:6
    uk = uk + 32*(2*(-1).^(k+1)-1)./(k.^5*pi^5)*sin(k*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$\tilde{u}_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(3),'hw26g','epsc')
```

2. [50 points: 10 points each]

All parts of this question should be done by hand.

Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L:S\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$S=\{w\in C^2[0,1]:w'(0)=w(1)=0\}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv,w)=(v,Lw)\text{ for all }v,w\in S.$$

Let N be a positive integer and let $f\in C[0,1]$ be defined by

$$f(x)=\begin{cases}1-2x & \text{if } x\in\left[0,\frac{1}{2}\right];\\ 0 & \text{otherwise.}\end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\phi_n(x)=\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n=1,2,\ldots$. Note that, for $m,n=1,2,\ldots$,

$$(\phi_m,\phi_n)=\begin{cases}1 & \text{if } m=n;\\ 0 & \text{if } m\neq n.\end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n=1,2,\ldots$.

(b) Compute f_N, the best approximation to f from span$\{\phi_1,\ldots,\phi_N\}$ with respect to the norm $\|\cdot\|$. Plot f_N for $N=1,2,3,4,5,6$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=f(x),\quad 0<x<1$$

and

$$\tilde{u}'(0)=\tilde{u}(1)=0.$$

(d) By shifting the data, obtain an infinite series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

(e) Let $\tilde{u}_N$ be the series solution that you obtained in part (d) but with ∞ replaced by N, i.e.

$$\tilde{u}_N(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

Plot $\tilde{u}_N$ for $N = 1, 2, 3, 4, 5, 6$.

Solution.

(a) [3 points] We can compute that, for $n = 1, 2, \ldots$,

$$\phi_n'(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)\pi \sin\left(\frac{2n-1}{2}\pi x\right).$$

and

$$\phi_n''(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)^2 \pi^2 \cos\left(\frac{2n-1}{2}\pi x\right).$$

and so

$$L\phi_n = -\phi_n'' = \left(\frac{2n-1}{2}\right)^2 \pi^2 \phi_n.$$

Hence,

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \pi^2 = (2n-1)^2 \frac{\pi^2}{4} \text{ for } n = 1, 2, \ldots.$$

(b) [8 points] Since $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$, the best approximation to f from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ is

$$f_N = \sum_{n=1}^{N} (f, \phi_n)\phi_n.$$

Now, for $n = 1, 2, \ldots$,

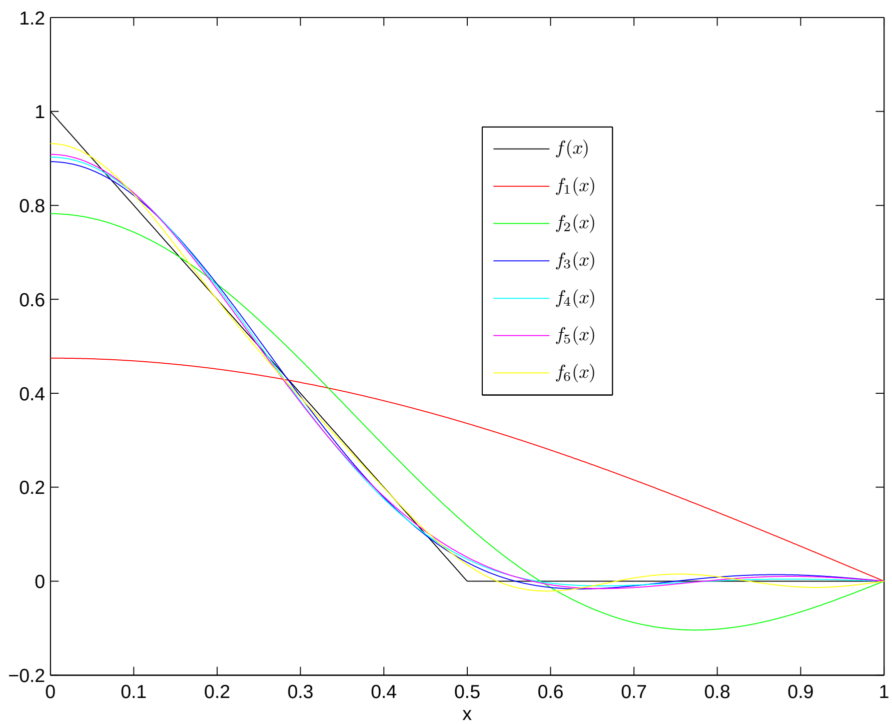
$$\begin{aligned}
&(f, \phi_n) \\
&= \int_0^1 f(x)\phi_n(x)\,dx \\
&= \int_0^{1/2} f(x)\phi_n(x)\,dx + \int_{1/2}^1 f(x)\phi_n(x)\,dx \\
&= \int_0^{1/2} (1-2x)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx + \int_{1/2}^1 0\,dx \\
&= \sqrt{2}\int_0^{1/2} (1-2x)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx + 0
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2}\left(\left[(1-2x)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} - \int_0^{1/2}(-2)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \sqrt{2}\left(0-0+\frac{4}{(2n-1)\pi}\int_0^{1/2}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \sqrt{2}\frac{4}{(2n-1)\pi}\left[-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} \\
&= \frac{4\sqrt{2}}{(2n-1)\pi}\left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{4}\pi\right) - \left(-\frac{2}{(2n-1)\pi}\right)\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_N(x) &= \sum_{n=1}^{N}(f,\phi_n)\phi_n(x) \\
&= \sum_{n=1}^{N}(f,\phi_n)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right) \\
&= \sum_{n=1}^{N}\frac{16}{(2n-1)^2\pi^2}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right)\cos\left(\frac{2n-1}{2}\pi x\right).
\end{aligned}
$$

The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
```

```
colors='rgbcmy';
x = linspace(0,1,1000);

figure(1)
clf
legendStr{1}=['$f(x)$'];
plot(x,-(x-1/2)+(x-1/2).*sign(x-1/2),'k-')
hold on
fk = zeros(size(x));
for k=1:6
    fk = fk + 16*(1-cos(((2*k-1)/4)*pi))./((2*k-1).^2*pi^2)*cos(((2*k-1)/2)*pi*x);
    plot(x,fk,colors(k))
    legendStr{k+1}=['$f_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(1),'hw72c','epsc')
```

(c) [4 points] Now, $\tilde{u}$ is the solution to $L\tilde{u} = f$ and so the spectral method yields the series solution

$$\tilde{u}(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n} \phi_n(x) = \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right).$$

(d) [6 points] Let $\tilde{u}$ be the solution to $L\tilde{u} = f$ and let $w \in C^2[0,1]$ be such that

$$-w''(x) = 0, \quad 0 < x < 1$$

and

$$w'(0) = w(1) = 1.$$

Then $u(x) = w(x) + \tilde{u}(x)$ will be such that

$$-u''(x) = -w''(x) - \tilde{u}''(x) = 0 + f(x) = f(x);$$

$$u'(0) = w'(0) + \tilde{u}'(0) = 1 + 0 = 1;$$

and

$$u(1) = w(1) + \tilde{u}(1) = 1 + 0 = 1.$$

Now, the general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(0) = 1$ when $A = 1$. Hence, $w(x) = x + B$ and so $w(1) = 1$ when $B = 0$. Consequently,

$$w(x) = x$$

and so

$$u(x) = x + \tilde{u}(x).$$

We can then use the series solution to $L\tilde{u} = f$ that we obtained in part (c) to obtain the series solution

$$u(x) = x + \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right)$$

to the problem of finding $u \in C^2[0,1]$ such that

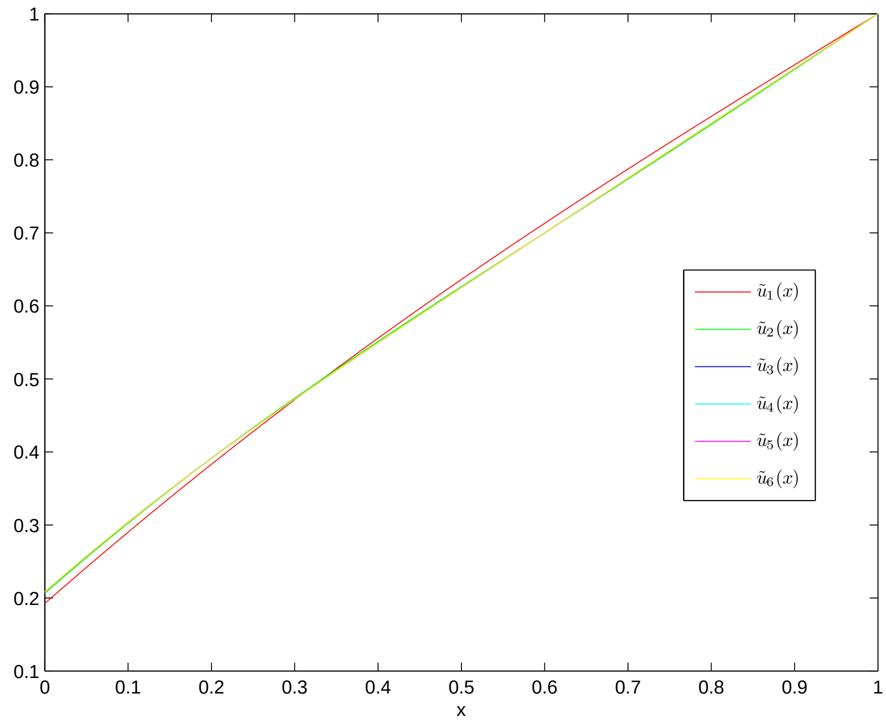
$$-u''(x) = f(x), \quad 0 < x < 1;$$

$$u'(0) = u(1) = 1.$$

(e) [4 points] The best approximation $\tilde{u_N}$ to $\tilde{u}$ from part (d) is

$$\tilde{u}_N(x) = x + \sum_{n=1}^{N} \frac{(f,\phi_n)}{\lambda_n}\phi_n(x) = \sum_{n=1}^{N} \frac{64}{(2n-1)^4 \pi^4}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right)\cos\left(\frac{2n-1}{2}\pi x\right).$$

The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(3)
clf
uk = x;
for k=1:6
    uk = uk + 64*(1-cos(((2*k-1)/4)*pi))./((2*k-1).^4*pi^4)*cos(((2*k-1)/2)*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$\tilde{u}_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(3),'hw72e','epsc')
```