CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 48

Posted Wednesday 27 November 2013. Due 5pm Wednesday 4 December 2013.

48. [25 points] Let $H_D^1(0,1) = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in H_D^1(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$. Let $V_N = \text{span}\{\phi_1, ..., \phi_N\}$. Let $\rho \in C[0,1]$ be such that $\rho(x) > 0$ for all $x \in [0,1]$, let $c \in C[0,1]$ be such that $c(x) > 0$ for all $x \in [0,1]$ and let $\kappa \in C[0,1]$ be such that $\kappa(x) > 0$ for all $x \in [0,1]$. Let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(u,v) = \int_0^1 \rho(x)c(x)u(x)v(x)\,dx$$

and let the inner product $a(\cdot,\cdot) : H_D^1(0,1) \times H_D^1(0,1) \to \mathbb{R}$ be defined by

$$a(u,v) = \int_0^1 \kappa(x)u'(x)v'(x)\,dx.$$

Let $\mathbf{M} \in \mathbb{R}^{N\times N}$ be the matrix with entries

$$M_{jk} = (\phi_k, \phi_j)$$

and let $\mathbf{K} \in \mathbb{R}^{N\times N}$ be the matrix with entries

$$K_{jk} = a(\phi_k, \phi_j).$$

For $\mathbf{w} \in \mathbb{R}^N$, let

$$\hat{w}_N = \sum_{j=1}^{N} w_j \phi_j$$

where $w_j \in \mathbb{R}$ is the jth entry of the vector $\mathbf{w}$.

In class we had stated that the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real. In class we had also stated that the eigenvalues of $-\mathbf{M}^{-1}\mathbf{K}$ are negative since the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are positive. This question will walk you through the process of showing that the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are positive given that we know that the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real.

(a) For $\mathbf{w} \in \mathbb{R}^N$, show that

$$\mathbf{w}^T\mathbf{M}\mathbf{w} = (\hat{w}_N, \hat{w}_N).$$

(b) Show that if $\mathbf{M}^{-1}\mathbf{K}\mathbf{w} = \lambda\mathbf{w}$, for $\lambda \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^N$, then

$$a(\hat{w}_N, \hat{w}_N) = \lambda(\hat{w}_N, \hat{w}_N).$$

In addition to the information given previously in the question you may use the fact that

$$\mathbf{w}^T\mathbf{K}\mathbf{w} = a(\hat{w}_N, \hat{w}_N).$$

(c) Use the properties satisfied by inner products to show that if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$ then $\lambda > 0$.