CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 28

Posted Friday 21 March 2014. Due 1pm Friday 28 March 2014.

28. [25 points]

Let
$$H^1_D(0,1) = \big\{w \in H^1(0,1) :\, w(0) = w(1) = 0\big\}$$
and let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by
$$(v,w) = \int_0^1 v(x)w(x)\,dx$$
and the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by
$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$
Also, let $f \in L^2(0,1)$, let N be a positive integer, and let V_N be a subspace of $H^1_D(0,1)$. Moreover, let $u \in H^1_D(0,1)$ be such that
$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1)$$
and let $u_N \in V_N$ be such that
$$a(u_N,v) = (f,v) \text{ for all } v \in V_N.$$

(a) Show that
$$a(u-u_N, u-u_N) = a(u,u) - a(u_N,u_N).$$

(b) Let $\phi_1,\ldots,\phi_N \in V_N$ and let $\mathbf{K} \in \mathbb{R}^{N\times N}$ be the matrix with entries $K_{jk} = a(\phi_k,\phi_j)$ for $j,k = 1,\ldots,N$. Also, let
$$u_N = \sum_{j=1}^{N} c_j\phi_j$$
where $c_j \in \mathbb{R}$ is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^N$. Show that
$$\mathbf{c}^T\mathbf{K}\mathbf{c} = a(u_N,u_N).$$