CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 33 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

33. [25 points]

Let $f \in C[0,1]$, let $\alpha \in \mathbb{R}$ and let $\beta \in \mathbb{R}$. Let $p \in C^2[0,1]$ be such that

$$-p''(x) = f(x), \quad 0 < x < 1;$$

$$-p'(0) = \alpha$$

and

$$p(1) = \beta.$$

Let $u \in C^2[0,1]$ be such that

$$-4u''(x) + 9u(x) = f(x), \quad 0 < x < 1;$$

$$-4u'(0) = \alpha$$

and

$$4u'(1) = \beta.$$

Let $C_D^2[0,1] = \{w \in C^2[0,1] : w(1) = 0\}$.

(a) It can be shown that

$$\int_0^1 p'(x)v'(x)\,dx = q(f,\alpha,v) \text{ for all } v \in C_D^2[0,1].$$

Obtain a formula for $q(f,\alpha,v)$.

(b) It can be shown that

$$\int_0^1 \left(4u'(x)v'(x) + 9u(x)v(x)\right)\,dx = g(f,\alpha,\beta,v) \text{ for all } v \in C^2[0,1].$$

Obtain a formula for $g(f,\alpha,\beta,v)$.

Solution.

(a) [12 points] If $v \in C_D^2[0,1]$, then

$$-\int_0^1 p''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

Integration by parts then yields that

$$-[p'(x)v(x)]_0^1 + \int_0^1 p'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

from which we can conclude that

$$p'(0)v(0) + \int_0^1 p'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

since $v(1) = 0$. Hence,

$$-\alpha v(0) + \int_0^1 p'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

since $-p'(0) = \alpha$. Therefore,

$$\int_0^1 p'(x)v'(x)\,dx = q(f, \alpha, v) \text{ for all } v \in C_D^2[0,1]$$

where

$$q(f, \alpha, v) = \int_0^1 f(x)v(x)\,dx + \alpha v(0).$$

(b) [13 points] If $v \in C^2[0,1]$, then

$$\int_0^1 \left(-4u''(x) + 9u(x)\right)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

since

$$-4u''(x) + 9u(x) = f(x), \quad 0 < x < 1.$$

Integration by parts then yields that

$$\begin{aligned}
&\int_0^1 \left(-4u''(x) + 9u(x)\right)v(x)\,dx \\
&= -4\int_0^1 u''(x)v(x)\,dx + 9\int_0^1 u(x)v(x)\,dx \\
&= -4\left[u'(x)v(x)\right]_0^1 + 4\int_0^1 u'(x)v'(x)\,dx + 9\int_0^1 u(x)v(x)\,dx \\
&= -4u'(1)v(1) - \left(-4u'(0)v(0)\right) + \int_0^1 \left(4u'(x)v'(x) + 9u(x)v(x)\right)\,dx
\end{aligned}$$

from which we can conclude that

$$-\beta v(1) - \alpha v(0) + \int_0^1 \left(4u'(x)v'(x) + 9u(x)v(x)\right)\,dx = \int_0^1 f(x)v(x)\,dx$$

since $-4u'(0) = \alpha$ and $4u'(1) = \beta$. Therefore,

$$\int_0^1 \left(4u'(x)v'(x) + 9u(x)v(x)\right)\,dx = g(f, \alpha, \beta, v) \text{ for all } v \in C^2[0,1]$$

where

$$g(f, \alpha, \rho, v) = \int_0^1 f(x)v(x)\,dx + \alpha v(0) + \beta v(1).$$