CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 31 · Solutions

Posted Monday 21 October 2013. Due 5pm Wednesday 30 October 2013.

31. [25 points] Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = w(1) = 0\}$ and let the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\widehat{\phi}_j \in H^1_D(0,1)$ be such that

$$\widehat{\phi}_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$. Also, let the continuous piecewise quadratic functions $\phi_j \in H^1_D(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in H^1_D(0,1)$ be such that

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N+1$.

(a) By hand, obtain formulas for

i. $a(\phi_j, \phi_k)$ for $j, k = 1, \ldots, N$;

ii. $a(\psi_j, \psi_k)$ for $j, k = 1, \ldots, N+1$;

iii. $a(\phi_j, \psi_k)$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$.

Your final answers should be in terms of h and you must clearly state which values of j and k each formula you obtain is valid for. For example, if you were asked to obtain formulas for $a(\widehat{\phi}_j, \widehat{\phi}_k)$ for $j, k = 1, \ldots, N$ then an acceptable way to present the answer would be:
For $j, k = 1, \ldots, N$,

$$a(\widehat{\phi}_j, \widehat{\phi}_k) = \begin{cases} \dfrac{2}{h} & \text{if } k = j, \\ -\dfrac{1}{h} & \text{if } |j-k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution.

(a) [25 points]

i. For $j = 1, \ldots, N$,

$$\phi_j'(x) = \begin{cases} \dfrac{(2x - x_{j-1} - x_j) + 2(x - x_{j-1})}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ \dfrac{-2(x_{j+1} - x) - (x_j + x_{j+1} - 2x)}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \dfrac{4x - 3x_{j-1} - x_j}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ \dfrac{4x - 3x_{j+1} - x_j}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

and so we can compute that, for $j = 1, \ldots, N$,

$$\begin{aligned}
& a(\phi_j, \phi_j) \\
&= \int_0^1 \left(\phi_j'(x)\right)^2 \, dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{4x - 3x_{j-1} - x_j}{h^2}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{4x - 3x_{j+1} - x_j}{h^2}\right)^2 dx \\
&= \int_{x_{j-1}-x_{j-1}}^{x_j - x_{j-1}} \left(\frac{4\left(s + x_{j-1}\right) - 3x_{j-1} - x_j}{h^2}\right)^2 ds + \int_{x_j - x_{j+1}}^{x_{j+1}-x_{j+1}} \left(\frac{4\left(t + x_{j+1}\right) - 3x_{j+1} - x_j}{h^2}\right)^2 dt \\
&= \int_0^h \left(\frac{4s + x_{j-1} - x_j}{h^2}\right)^2 ds + \int_{-h}^0 \left(\frac{4t + x_{j+1} - x_j}{h^2}\right)^2 dt \\
&= \int_0^h \left(\frac{4s - h}{h^2}\right)^2 ds + \int_{-h}^0 \left(\frac{4t + h}{h^2}\right)^2 dt \\
&= \left[\frac{1}{3}\frac{h^2}{4}\left(\frac{4s - h}{h^2}\right)^3\right]_0^h + \left[\frac{1}{3}\frac{h^2}{4}\left(\frac{4t + h}{h^2}\right)^3\right]_{-h}^0 \\
&= \frac{h^2}{12}\left(\frac{3h}{h^2}\right)^3 - \frac{h^2}{12}\left(\frac{-h}{h^2}\right)^3 + \frac{h^2}{12}\left(\frac{h}{h^2}\right)^3 - \frac{h^2}{12}\left(\frac{-3h}{h^2}\right)^3 \\
&= \frac{h^2}{12}\left(\frac{27h^3}{h^6} + \frac{h^3}{h^6} + \frac{h^3}{h^6} + \frac{27h^3}{h^6}\right) \\
&= \frac{h^2}{12}\frac{56}{h^3} \\
&= \frac{14}{3h}
\end{aligned}$$

where $s = x - x_{j-1}$ and $t = x - x_{j+1}$.
Moreover, for $j = 1, \ldots, N-1$,

$$\phi_{j+1}'(x) = \begin{cases} \dfrac{4x - 3x_j - x_{j+1}}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ \dfrac{4x - 3x_{j+2} - x_{j+1}}{h^2} & \text{if } x \in (x_{j+1}, x_{j+2}), \\ 0 & \text{otherwise,} \end{cases}$$

and so we can compute that, for $j = 1, \ldots, N-1$,

$$a(\phi_{j+1}, \phi_j) = a(\phi_j, \phi_{j+1})$$

$$
\begin{aligned}
&= \int_0^1 \phi_j'(x)\phi_{j+1}'(x)\,dx \\
&= \int_{x_j}^{x_{j+1}} \frac{4x-3x_{j+1}-x_j}{h^2}\frac{4x-3x_j-x_{j+1}}{h^2}\,dx \\
&= \int_{x_j-x_j}^{x_{j+1}-x_j} \frac{4\,(s+x_j)-3x_{j+1}-x_j}{h^2}\frac{4\,(s+x_j)-3x_j-x_{j+1}}{h^2}\,ds \\
&= \int_0^h \frac{4s-3h}{h^2}\frac{4s-h}{h^2}\,ds \\
&= \frac{1}{h^4}\int_0^h 16s^2-16hs+3h^2\,ds \\
&= \frac{1}{h^4}\left[\frac{16}{3}s^3-8hs^2+3h^2s\right]_0^h \\
&= \frac{1}{h^4}\left(\frac{16}{3}h^3-8h^3+3h^3\right) \\
&= \frac{1}{h^4}\frac{1}{3}h^3 \\
&= \frac{1}{3h}
\end{aligned}
$$

where $s = x - x_j$.
Finally, we note that, for $j, k = 1, \ldots, N$,

$$a(\phi_j, \phi_k) = 0 \text{ if } |j-k| > 1.$$

For $j, k = 1, \ldots, N$,

$$a(\phi_j, \phi_k) = \begin{cases} \dfrac{14}{3h} & \text{if } k = j, \\ \dfrac{1}{3h} & \text{if } |j-k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

ii. For $j = 1, \ldots, N+1$,

$$
\begin{aligned}
\psi_j'(x) &= \begin{cases} \dfrac{4(x_j-x)-4(x-x_{j-1})}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{4(x_j+x_{j-1}-2x)}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

and so we can compute that, for $j = 1, \ldots, N+1$,

$$
\begin{aligned}
a(\psi_j, \psi_j) &= \int_0^1 \left(\psi_j'(x)\right)^2\,dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{4\,(x_j+x_{j-1}-2x)}{h^2}\right)^2 dx \\
&= \int_{x_{j-1}-x_{j-1}}^{x_j-x_{j-1}} \left(\frac{4\,(x_j+x_{j-1}-2\,(s+x_{j-1}))}{h^2}\right)^2 ds \\
&= \int_0^h \left(\frac{4\,(x_j-x_{j-1}-2s)}{h^2}\right)^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{16}{h^4}\int_0^h (h-2s)^2\ ds \\
&= \frac{16}{h^4}\left[-\frac{1}{6}(h-2s)^3\right]_0^h \\
&= \frac{16}{h^4}\left(-\frac{1}{6}(-h)^3+\frac{1}{6}h^3\right) \\
&= \frac{16}{h^4}\left(\frac{1}{6}h^3+\frac{1}{6}h^3\right) \\
&= \frac{16}{h^4}\frac{h^3}{3} \\
&= \frac{16}{3h}
\end{aligned}
$$

where $s = x - x_{j-1}$.

Finally, we note that, for $j, k = 1, \ldots, N+1$,

$$a(\psi_j, \psi_k) = 0 \text{ if } j \neq k.$$

Therefore, for $j, k = 1, \ldots, N+1$,

$$a(\psi_j, \psi_k) = \begin{cases} \dfrac{16}{3h} & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

iii. We can compute that, for $j = 1, \ldots, N$,

$$
\begin{aligned}
a(\phi_j, \psi_j) &= \int_0^1 \phi_j'(x)\psi_j'(x)\,dx \\
&= \int_{x_{j-1}}^{x_j} \frac{4x-3x_{j-1}-x_j}{h^2}\,\frac{4(x_j+x_{j-1}-2x)}{h^2}\,dx \\
&= \frac{4}{h^4}\int_{x_{j-1}-x_{j-1}}^{x_j-x_{j-1}} (4s-h)(h-2s)\,ds \\
&= \frac{4}{h^4}\int_0^h 6hs-8s^2-h^2\,ds \\
&= \frac{4}{h^4}\left[3hs^2-\frac{8}{3}s^3-h^2s\right]_0^h \\
&= \frac{4}{h^4}\left(3h^3-\frac{8}{3}h^3-h^3\right) \\
&= \frac{4}{h^4}\left(-\frac{2}{3}h^3\right) \\
&= -\frac{8}{3h}
\end{aligned}
$$

where $s = x - x_{j-1}$.

Moreover, for $j = 1, \ldots, N$,

$$\psi_{j+1}'(x) = \begin{cases} \dfrac{4(x_{j+1}+x_j-2x)}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

and so we can compute that, for $j = 1, \ldots, N$,

$$a(\phi_j, \psi_{j+1}) = \int_0^1 \phi_j'(x)\psi_{j+1}'(x)\,dx$$

$$
\begin{aligned}
&= \int_{x_j}^{x_{j+1}} \frac{4x - 3x_{j+1} - x_j}{h^2} \frac{4(x_{j+1} + x_j - 2x)}{h^2}\, dx \\
&= \int_{x_j - x_j}^{x_{j+1} - x_j} \frac{4(s + x_j) - 3x_{j+1} - x_j}{h^2} \frac{4(x_{j+1} + x_j - 2(s + x_j))}{h^2}\, ds \\
&= \frac{4}{h^4} \int_0^h (4s - 3h)(h - 2s)\, ds \\
&= \frac{4}{h^4} \int_0^h 10hs - 8s^2 - 3h^2\, ds \\
&= \frac{4}{h^4} \left[5hs^2 - \frac{8}{3}s^3 - 3h^2 s \right]_0^h \\
&= \frac{4}{h^4} \left(5h^3 - \frac{8}{3}h^3 - 3h^3 \right) \\
&= \frac{4}{h^4} \left(-\frac{2}{3}h^3 \right) \\
&= -\frac{8}{3h}
\end{aligned}
$$

where $s = x - x_j$.
Finally, we note that, for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$,

$$a(\phi_j, \psi_k) = 0 \text{ if } k - j > 1 \text{ or } k - j < 0.$$

Therefore, for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$,

$$a(\phi_j, \psi_k) = \begin{cases} -\dfrac{8}{3h} & \text{if } k = j \text{ or } k = j+1, \\ 0 & \text{otherwise.} \end{cases}$$