

CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 1

Posted Wednesday 22 August 2012. Due Wednesday 29 August 2012, 5pm.

A reminder from the course syllabus: Mathematically rigorous solutions are expected; strive for clarity and elegance. You may collaborate on the problems, but your write-up must be your own independent work. Transcribed solutions and copied MATLAB code are both unacceptable. *You may not consult solutions from previous sections of this class.* Unless it is specified that a particular calculation must be performed ‘by hand,’ you are encouraged to use MATLAB’s Symbolic Math Toolbox (or Mathematica/Wolfram Alpha/Maple) to compute and simplify tedious integrals and derivatives on the problem sets. As always, you must document your calculations clearly.

1. [16 points]

For each of the following equations, specify whether each is (a) an ODE or a PDE; (b) determine its order; (c) specify whether it is linear or nonlinear. For those that are linear, specify whether they are (d) homogeneous or inhomogeneous, and (e) whether they have constant or variable coefficients.

$$(1.1) \quad \frac{dv}{dx} + \frac{2}{x}v = 0$$

$$(1.2) \quad \frac{\partial v}{\partial t} - 3\frac{\partial v}{\partial x} = x - t$$

$$(1.3) \quad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[2u \frac{\partial u}{\partial x} \right] = 0$$

$$(1.4) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$(1.5) \quad \frac{d^2 y}{dx^2} - \mu(1 - y^2) \frac{dy}{dx} + y = 0$$

$$(1.6) \quad \frac{d^2}{dx^2} \left[\rho(x) \frac{d^2 u}{dx^2} \right] = \sin(x)$$

2. [12 points]

Determine whether each of the following functions is a solution of the corresponding differential equation from question 1.

(a) Is $v(x) = 1/x^2$ a solution of (1.1) ?

(b) Is $v(x, t) = t(t + x)$ a solution of (1.2) ?

(c) Is $u(x, t) = xe^t$ a solution of (1.3) ?

3. [34 points]

Consider the temperature function

$$u(x, t) = e^{-\kappa \theta^2 t / (\rho c)} \sin(\theta x)$$

for constant κ , ρ , c , and θ .

(a) Show that this function $u(x, t)$ is a solution of the homogeneous heat equation

$$\rho c \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \ell \text{ and all } t.$$

(b) For which values of θ will u satisfy homogeneous Dirichlet boundary conditions at $x = 0$ and $x = \ell$?

(c) Suppose $\kappa = 2.37$ W/(cm K), $\rho = 2.70$ g/cm³, and $c = 0.897$ J/(g K) (approximate values for aluminum found on Wikipedia), and that the bar has length $\ell = 10$ cm. Let θ be such that $u(x, t)$ satisfies homogeneous Dirichlet boundary conditions as in part (b) and $u(x, t) \geq 0$ for all x and t .

Use MATLAB to plot the solution $u(x, t)$ for $0 \leq x \leq \ell$ and time $0 \leq t \leq 20$ sec.

You may choose to do this in one of the following ways: (1) Plot the solution for $0 \leq x \leq \ell$ at times $t = 0, 4, 8, \dots, 20$ sec., superimposing all six plots on the same axis (helpful commands: `linspace`, `plot`, `hold on`); (2) Create a three-dimensional plot of the data using `surf`, `mesh`, or `waterfall`. In either case, be sure to produce an attractive, well-labeled plot.

4. [20 points]

Consider a bar of metal alloy manufactured such that its thermal conductivity is $\kappa(x) = 1 + \alpha x$ for constant α and $0 \leq x \leq \ell$. You may assume the heat equation for a non-uniform bar:

$$c(x)\rho(x)u_t(x, t) = \frac{\partial}{\partial x} \left(\kappa(x)u_x(x, t) \right).$$

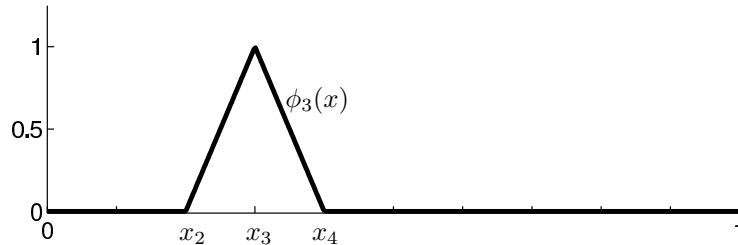
- Determine a general formula for the steady-state temperature distribution of this bar, which should include two free constants. (Assume no additional source term $f(x)$ is present.)
- Find formulas for these free constants in the case that the ends of the bar are submerged in ice baths of γ deg on the left and δ deg on the right.
- Now find formulas for the free constants in the case that the left end has a fixed *heat flux* equal to γ (measured in $\text{J}/(\text{m}^2 \cdot \text{sec})$) and the right end is submerged in an ice bath of δ deg.

5. [18 points]

Suppose $N \geq 1$ is an integer and define $h = 1/(N + 1)$ and $x_k = kh$ for $k = 0, \dots, N + 1$. Consider the $N + 2$ hat functions, defined as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k]; \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

for $x \in [0, 1]$ and $k = 0, \dots, N + 1$. We call these piecewise linear functions *hat functions* because of their shape. They will be important functions later in the course. For example, when $N = 9$ and $k = 3$, this function takes the following form.



- Write a MATLAB function for $\phi_k(x)$. It should take in as input x , k , and N . It should return the value $\phi_k(x)$. It should also be able to take in a vector for $\mathbf{x} = (x_1, \dots, x_m)$ and return the vector $\phi_k(\mathbf{x}) = (\phi_k(x_1), \dots, \phi_k(x_m))$.
- Let $N = 9$. Plot $\phi_0(x), \phi_4(x), \phi_5(x), \phi_6(x), \phi_{10}(x)$ on the same figure. Make sure to:
 - plot each function with a different color;
 - label the axes and provide a title;
 - create an accurate legend for the figure.

Optional but encouraged: Adjust the text sizes (if necessary) to make the labels easily legible, and use the \LaTeX interpreter to make your labels, titles, and legend look stylish. See the MATLAB Primer on the course web site for details.