We are going to work with the <u>Solution Steps</u>, steps 4–7, from the last lecture in order to solve a selection of Boundary Value Problems. Recall that steps 4–7 were:

4) Expand the right-hand side (e.g. f) in terms of the eigenfunctions ψ_n.
5) Expand the unknown function $u = \sum \alpha_n \tilde{\psi}_n$ where the α_n are <u>unknown</u>
6) Apply the operator, L, to this expansion of u (using the fact that the $\tilde{\psi}_n$ are eigenvectors of L with eigenvalue λ_n)
7) Compare coefficients (take inner product of both sides with ψ_k) to determine the unknown coefficients α_k

<u>BVP #1:</u>

$$\begin{cases} -\frac{\partial^2}{\partial x^2} u = x(1-x) \\ u(0) = u(1) = 0 \end{cases}$$

We see that this is a problem of Type I (from last lecture) with $l = 1$. So we know the eigenfunctions are $\psi_n = \sin(n\pi x)$ with eigenvalues $\lambda_n = n^2\pi^2$

Note: We could find eigenvectors $\tilde{\psi}_n$ with unit length by computing $(\psi_n, \psi_n) = \int_0^1 \sin^2(n\pi x)\,dx = 1/2$ so that $\tilde{\psi}_n = \psi_n / \sqrt{(\psi_n, \psi_n)} = \sqrt{2}\sin(n\pi x)$ has the desired unit length.

However, the book does not do this. The impact of this choice is that the expansion for the function f as well as the solution u will have their coefficients in the direction of the given eigenvectors <u>rescaled.</u> For example consider the basis $\tilde{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\tilde{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ for $\mathbb{R}^2$. Then the vector $v = \begin{bmatrix}2\\1\end{bmatrix} = 2\tilde{e}_1 + \tilde{e}_2$. However if I use the basis $\hat{e}_1 = \begin{bmatrix}2\\0\end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix}0\\2\end{bmatrix}$, which doesn't have unit length, $v = \hat{e}_1 + \frac{1}{2}\hat{e}_2$

The first step of the solution process is:

4) expand f in terms of the basis functions ψ_n

So we want to write $f = \sum b_n \psi_n$ for coefficients b_n. To find the k^{th} coefficient b_k we take the inner product of both sides with respect to ψ_k and use <u>orthogonality</u> to get:

$$(f, \psi_k) = b_k(\psi_k, \psi_k) \rightarrow b_k = \frac{(f, \psi_k)}{(\psi_k, \psi_k)}$$

For our problem, $f(x) = x(1-x)$ and $\varphi_n = \sin(n\pi x)$. Hence

$$(\varphi_n, \varphi_n) = \int_0^1 \sin^2(n\pi x) = 1/2.$$

Therefore $f = \sum b_n \varphi_n$ where $b_n = \dfrac{(f, \varphi_n)}{(\varphi_n, \varphi_n)} = 2\int_0^1 x(1-x)\sin(n\pi x)$

One way to evaluate this integral is to break it into two pieces:

$2\int_0^1 x\sin(n\pi x)$ and $-2\int_0^1 x^2 \sin(n\pi x)$. Each integral can then be solved using applications of integration by parts.

$\underline{\int_0^1 x\sin(n\pi x)}$: Let $u = x$ and $dv = \sin(n\pi x)$ then $uv - v\,du = u\,dv$ gives:

$$-\frac{x\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x) = -\frac{\cos(n\pi)}{n\pi} + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_0^1$$

$$= \frac{-1}{n\pi}(-1)^n + \frac{1}{(n\pi)^2}(0) = \frac{(-1)^{n+1}}{n\pi}$$

$\int_0^1 x^2\sin(n\pi x)$: Let $u = x^2$ and $dv = \sin(n\pi x)$. Then $uv - v\,du = u\,dv$ gives:

$$-\frac{x^2\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x).$$

We know from the first integral (above) that $\dfrac{-x^2}{n\pi}\cos(n\pi x)\Big|_0^1 = \dfrac{(-1)^{n+1}}{n\pi}$

So we just need to evaluate $\frac{2}{n\pi}\int_0^1 x\cos(n\pi x)$. Letting $u = x$ and $dv = \cos(n\pi x)$ another application of integration by parts gives:

$$\int_0^1 x\cos(n\pi x) = \frac{x\sin(n\pi x)}{n\pi}\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x) = 0 + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1$$

$$= \frac{1}{(n\pi)^2}(-1)^n - \frac{1}{(n\pi)^2}.$$

Therefore: $$\int_0^1 x^2\sin(n\pi x) = \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left(\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right)$$

Using these results gives that:

$$b_k = 2\int_0^1 x(1-x)\sin(n\pi x) = 2\int_0^1 x\sin(n\pi x) - 2\int_0^1 x^2\sin(n\pi x)$$

$$= 2\left(\frac{(-1)^{n+1}}{n\pi} - \left\{\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\left(\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right)\right\}\right)$$

$$= \frac{-4\left((-1)^n - 1\right)}{(n\pi)^3} = \frac{4\left(1 + (-1)^{n+1}\right)}{(n\pi)^3}$$

Therefore: $$f = \sum_{n=1}^{\infty} \frac{4\left(1+(-1)^{n+1}\right)}{(n\pi)^3}\sin(n\pi x)$$

<u>Notice</u> that if n is <u>even</u> the coefficient $b_n = 0$.

Now we have completed the <u>first step</u> of the solution phase. As you can see it can be alot of work! The good news is, in fact, it is the majority of the computational work.

<u>Step 5:</u> Expand the unknown function $u(x)$ in terms of the eigenvectors ψ_n: $u(x) = \sum \alpha_n \psi_n(x) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x)$

* we can't directly compute the α_n as we did for $f = x(1-x)$ because we dont know what $u(x)$ is!

<u>Step 6:</u> Apply the operator L to the expansion of $u(x)$ and use the fact that ψ_n is an eigenvector of L.

we have $Lu = L\left(\sum_{n=1}^{\infty} \alpha_n \psi_n\right) = \sum_{n=1}^{\infty} \alpha_n L\psi_n = \sum_{n=1}^{\infty} \alpha_n \lambda_n \psi_n$

for us $L = -\partial^2/\partial x^2$ and we already determined that for type I problems (homogeneous Dirichlet boundary conditions) $\lambda_n = n^2\pi^2/\ell^2$

Since $\ell = 1$ in this problem $\lambda_n = n^2\pi^2$.

Putting this all together we have:

$$Lu = \sum_{n=1}^{\infty} \alpha_n \lambda_n \psi_n \longleftrightarrow -\partial^2/\partial x^2\, u = \sum_{n=1}^{\infty} \alpha_n (n\pi)^2 \sin(n\pi x)$$

<u>Step 7:</u> The entire system is now:

$$\sum_{n=1}^{\infty} \alpha_n (n\pi)^2 \sin(n\pi x) = \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{(n\pi)^3} \sin(n\pi x)$$

the abstract form of the above is: $\sum_{n=1}^{\infty} \alpha_n \lambda_n \psi_n = \sum_{n=1}^{\infty} \frac{(f,\psi_n)}{(\psi_n,\psi_n)} \psi_n$

Step 7 is to isolate the unknown coefficient α_k by taking the inner product of both sides with ψ_k and using that the eigenvectors are orthogonal to get: $\alpha_k \lambda_k (\psi_k,\psi_k) = \frac{(f,\psi_k)}{\cancel{(\psi_k,\psi_k)}} \cancel{(\psi_k,\psi_k)} = (f,\psi_k)$

this gives $\alpha_k \lambda_k = \frac{(f,\psi_k)}{(\psi_k,\psi_k)} = b_k$ (from step 4)

For us this means: $\alpha_k = \frac{4(1+(-1)^{n+1})}{(n\pi)^5}$ and the solution to the boundary value problem #1 is therefore:

$$u(x) = \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{(n\pi)^5} \sin(n\pi x)$$

BVP #2 Consider the problem:

$$-\frac{\partial^2}{\partial x^2} u = 1$$
$$u(0) = 0$$
$$\frac{\partial u}{\partial x}(1) = 0$$

First off we see that $l = 1$ and $f(x) = 1$. We are also in the case of type II (Mixed) boundary conditions. Last time we saw that the eigenvectors and eigenvalues for this type of problem are:

$$\psi_n(x) = \sin\left((2n-1)\pi x\right), \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4}$$

where we have used the fact that $l=1$.

Step 4: Compute the coefficients of $f = \sum_{n=1}^{\infty} b_k \psi_n(x)$.

$$b_k = \frac{(f, \psi_k)}{(\psi_k, \psi_k)}$$

we have: $(\psi_k, \psi_k) = \int_0^1 \sin^2((2k-1)\pi x) = \frac{1}{2} + \frac{\sin(4\pi k)}{4\pi - 8\pi k} = 1/2$

where we have used the fact that $k = 1, 2, 3, \ldots$

(Note: the steps for computing integrals are skipped for this problem. These steps entail most of the "real work")

$$(f, \psi_k) = (1, \psi_k) = \int_0^1 \sin((2k-1)\pi x) = \frac{-2}{\pi(1-2k)} \cos(\pi k) = \frac{2}{\pi(2k-1)} \cos(\pi k)$$

Now since $k = 1, 2, \ldots$ $\cos(\pi k) = (-1)^k$ so the above is:

$$(f, \psi_k) = \frac{2(-1)^k}{\pi(2k-1)} \quad k = 1, 2, \ldots$$

Hence: $b_k = \frac{(f, \psi_k)}{(\psi_k, \psi_k)} \Rightarrow b_k = \frac{4(-1)^k}{\pi(2k-1)} \Rightarrow 1 = f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(2n-1)} \sin((2n-1)\pi x)$

Step 5: Write $u(x) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x)$

Step 6: Applying L to $u(x)$ and setting equal to the right-hand side:

$$Lu = f \Rightarrow L\left(\sum \alpha_n \psi_n\right) = \sum b_n \psi_n \Rightarrow \sum \alpha_n \lambda_n \psi_n = \sum b_n \psi_n$$

where we have used that ψ_n are eigenvectors of L w/ eigenvalue λ_n

Step 7: take the inner product of both sides with respect to ψ_k and use the orthogonality of the eigenvectors to get: $\alpha_k = b_k / \lambda_k$

so $\alpha_k = \frac{4(-1)^k}{\pi(2k-1)} \frac{4}{(2k-1)^2\pi^2} = \frac{(-1)^k 16}{(2k-1)^3\pi^3}$ so that

$$u(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 16}{(2n-1)^3\pi^3} \sin((2n-1)\pi x)$$

is the solution to the boundary value problem.

An introduction to "Shifting the data" for solving Inhomogeneous problems.

First assume we want to solve an Inhomogeneous problem such as:

$$-\frac{\partial^2}{\partial x^2} u = f \qquad\qquad -\frac{\partial^2}{\partial x^2} u = f$$
$$u(0) = a \qquad \text{OR} \qquad u(0) = a$$
$$u(\ell) = b \qquad\qquad \frac{du}{dx}(\ell) = b.$$

Notice that the theory has an Issue here. If we try to define our "vector spaces" for the solutions as

$$V_1 = \{ v \in C^2 \mid v(0) = a,\ v(\ell) = b \} \quad \text{or} \quad V_2 = \{ v \in C^2 \mid v(0) = a\ \frac{dv}{dx}(\ell) = b \}$$

then V_1 and V_2 are not vector spaces! So the general approach won't work. A common method in mathematics is to try and take a problem that you don't have an approach for and transform it into one you do have an approach for.

So suppose I CAN find the solution to a different problem:

$$Lw = 0 \quad \text{OR, respectively} \quad L = 0$$
$$w(0) = a \qquad\qquad w(0) = a$$
$$w(\ell) = b \qquad\qquad \frac{\partial w}{\partial x}(\ell) = b$$

How do we solve the augmented problem? It will just depend on L.

but lets be optimistic and assume we can find a function w that solves it. Then define: $v(x) = u(x) - w(x)$. What problem does $v(x)$ solve?

$$Lv = L(u-w) = Lu - Lw = f - 0 = f$$
$$v(0) = u(0) - w(0) = a - a = 0$$
$$v(\ell) = u(\ell) - w(\ell) = b - b = 0$$
$$\left(\text{or } \frac{\partial v}{\partial x}(\ell) = \frac{\partial u}{\partial x}(\ell) - \frac{\partial w}{\partial x}(\ell) = b - b = 0\right)$$

$$Lv = f \qquad \text{-OR-} \qquad Lv = f$$
$$v(0) = v(\ell) = 0 \qquad\qquad v(0) = 0$$
$$\qquad\qquad\qquad \frac{\partial v}{\partial x}(\ell) = 0$$

The function $V(x)$ will solve the case of homogeneous boundary conditions! We know how to find $V(x)$... so if we can find $W(x)$ then: $u(x) = V(x) + W(x)$ will solve the inhomogeneous problem we started with!

Note: The form of the problem:

$$\begin{array}{lcl} Lw = 0 & & Lw = 0 \\ w(0) = a & \text{-OR-} & w(0) = a \\ w(l) = b & & \frac{\partial w}{\partial x}(0) = b \end{array}$$

Says: find a function $w \in C^2[0,l]$ which is in the nullspace of L that satisfies the boundary conditions

So for $L = \frac{-\partial^2}{\partial x^2}$ $Lw = 0$ and $w \in C^2[0,l]$ means that $w(x) = c + dx$ for c, d unknown. To find c, d use whatever boundary conditions correspond to your problem.

e.g. if $w(0) = a$, $w(l) = b$ then $c = a$ and $d = \frac{b-a}{l}$.

if $w(0) = a$, $\frac{\partial w}{\partial x}(l) = b$ then $c = a$ and $d = b$.

The general solution to the inhomogeneous problem is then:

$u(x) = V(x) + (c + dx)$ where c, d are determined as above

and $V(x)$ solves:

$$\begin{array}{lcl} Lv = f & & Lv = f \\ v(0) = 0 & \text{-OR-} & v(0) = 0 \\ v(l) = 0 & & \frac{\partial v}{\partial x}(l) = 0 \end{array}$$

and can be found using the spectral method given by Steps 1-7!