CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 7 · Solutions

Posted Wednesday 4 September 2013. Due 5pm Wednesday 11 September 2013.

7. [25 points]
Consider the temperature function

$$u(x,t) = e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

for constant κ, ρ, c, and θ.

(a) Show that this function $u(x,t)$ is a solution of the homogeneous heat equation

$$\rho c\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \ell \text{ and all } t.$$

(b) For which values of θ will u satisfy homogeneous Dirichlet boundary conditions at $x = 0$ and $x = \ell$?

(c) Suppose $\kappa = 2.37$ W/(cm K), $\rho = 2.70$ g/cm^3, and $c = 0.897$ J/(g K) (approximate values for aluminum found on Wikipedia), and that the bar has length $\ell = 10$ cm. Let θ be such that $u(x,t)$ satisfies homogeneous Dirichlet boundary conditions as in part (b) and $u(x,t) \geq 0$ for $0 \leq x \leq \ell$ and all t.

Use MATLAB to plot the solution $u(x,t)$ for $0 \leq x \leq \ell$ and time $0 \leq t \leq 20$ sec.
You may choose to do this in one of the following ways: (1) Plot the solution for $0 \leq x \leq \ell$ at times $t = 0, 4, 8, \ldots, 20$ sec., superimposing all six plots on the same axis (helpful commands: `linspace`, `plot`, `hold on`); (2) Create a three-dimensional plot of the data using `surf`, `mesh`, or `waterfall`. In either case, be sure to produce an attractive, well-labeled plot.

Solution.

(a) [8 points] We compute

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\kappa\theta^2}{\rho c}e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)\\
\frac{\partial u}{\partial x} &= \theta e^{-\kappa\theta^2 t/(\rho c)}\cos(\theta x)\\
\frac{\partial^2 u}{\partial x^2} &= -\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x).
\end{aligned}$$

Hence,

$$\rho c\frac{\partial u}{\partial t} = -\kappa\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

and

$$\kappa\frac{\partial^2 u}{\partial x^2} = -\kappa\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

from which it can be seen that

$$\rho c\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}.$$

(b) [8 points] We wish to find the values of θ that give homogeneous Dirichlet boundary conditions, i.e., $u(0,t) = u(\ell,t) = 0$ for all t. Since $e^{-\kappa\theta^2 t/(\rho c)}$ is positive for all t, we can only get the homogeneous Dirichlet conditions when $\sin(\theta x) = 0$. For any θ, $\sin(\theta \cdot 0) = 0$, so the condition at $x = 0$ is automatically satisfied. To get $\sin(\theta\ell) = 0$, we need $\theta\ell$ to be an integer multiple of π, that is,

$$\theta\ell = \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

or equivalently

$$\theta = \frac{\pi n}{\ell}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

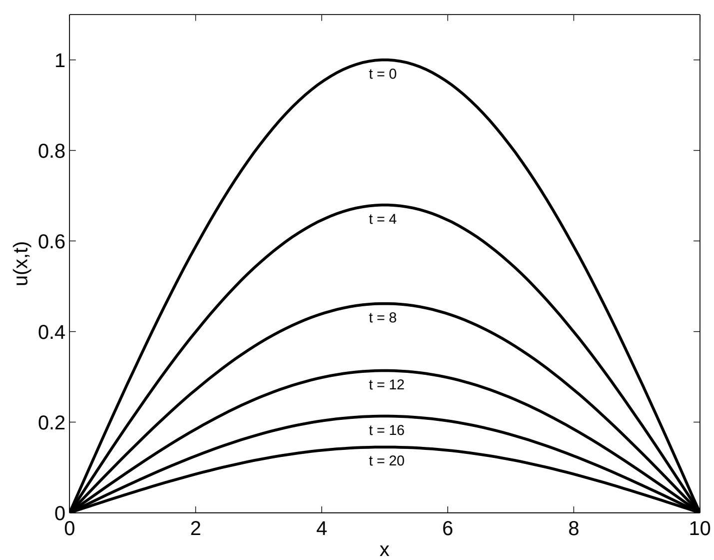
(c) [9 points]

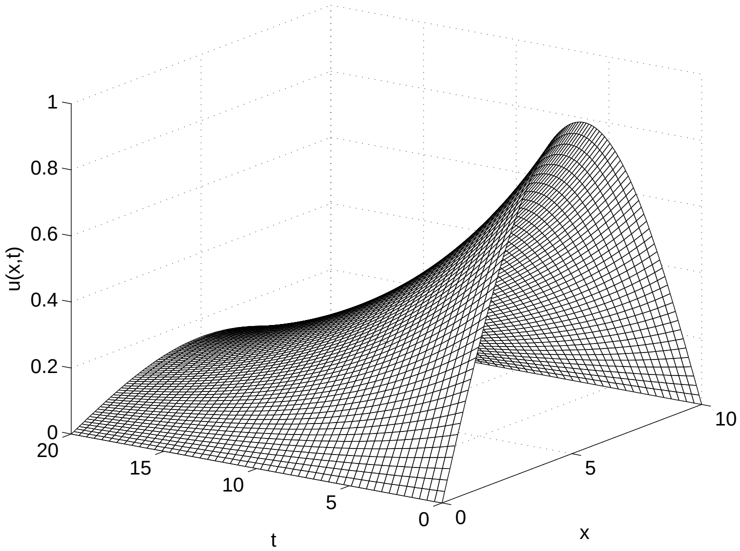
Notice that if $n = 0$ we have the trivial solution $u(x,t) = 0$ for $0 \leq x \leq \ell$ and all t. If $n = 1$, we have a solution for which $u(x,t) \geq 0$ for $0 \leq x \leq \ell$ and all t. For other values of n the solution will be *negative* for some $x \in [0, \ell]$. If our temperature is measured in Kelvin this could be a problem! However, this heat equation takes the same form if we shift to Celsius units, so we needn't be so troubled by the negative values of temperature.

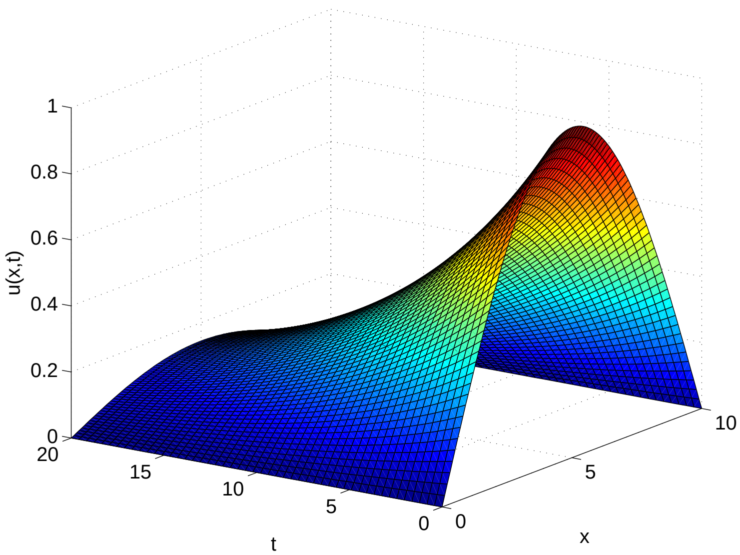
Since $n = 0$ is trivial, we shall take $n = 1$ ($\theta = \pi/\ell$) to obtain

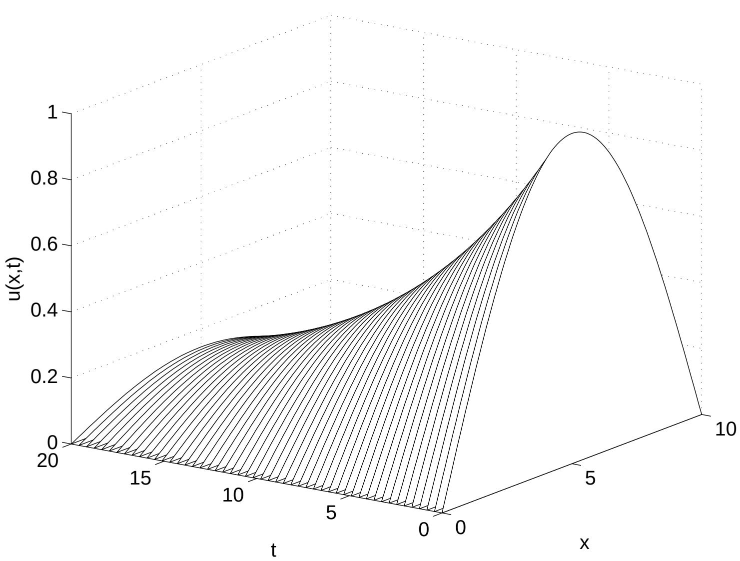
$$\begin{aligned} u(x,t) &= e^{-\kappa\pi^2 t/(\ell^2 \rho c)} \sin(\pi x/\ell) \\ &= e^{-2.37\pi^2 t/(100 \cdot 2.70 \cdot 0.897)} \sin(\pi x/10). \end{aligned}$$

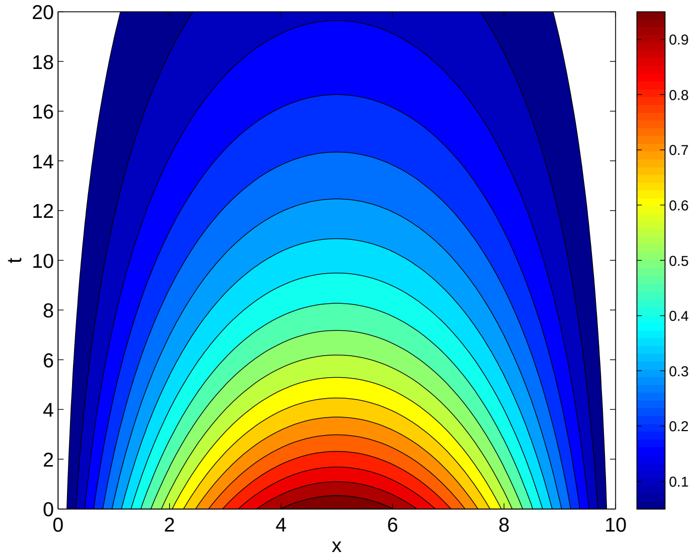
Solutions are shown in the attached plots. Any of these style is acceptable. The MATLAB code that generated these plots follows.




u(x,t)
1
0.8
0.6
0.4
0.2
0
20
15
10
5
0
t
0
5
10
x


u(x,t)
1
0.8
0.6
0.4
0.2
0
20
15
10
5
0
t
0
5
10
x


u(x,t)
1
0.8
0.6
0.4
0.2
0
20
15
10
5
0
t
0
5
10
x



MATLAB code:

```
c = .897;
kappa = 2.37;
rho = 2.70;
l = 10;
theta = pi/l;
% first style: snapshots at t = 0, 4, 8, ..., 20
t = 0:4:20;
x = linspace(0,l,100);
figure(1), clf
for j=1:length(t)
    u = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x); % compute u(:,t(j))
    plot(x,u,'k-','linewidth',2), hold on
    text(4.75, max(u)-.03, sprintf('t = %d', t(j)))
end
axis([0 10 0 1.1])
set(gca,'fontsize',14)
xlabel('x')
ylabel('u(x,t)')
print -depsc2 checksol1
% generate data for 3-d plots
x = linspace(0,l,100);
t = linspace(0,20,50);
U = zeros(length(t), length(x));
for j=1:length(t)
    U(j,:) = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x);
end
% mesh plot
figure(2), clf
mesh(x,t,U,'edgecolor','k')
view(-55,20)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 checksol2
% surf plot
figure(3), clf
surf(x,t,U)
view(-55,20)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 checksol3
% waterfall plot
```

```
figure(4), clf
plt = waterfall(x,t,U);
set(plt,'edgecolor','k') % make the lines black
view(-55,20)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 checksol4
% contour plot
figure(5), clf
[cs,h] = contourf(x,t,U,[.05:.05:.95],'k-');
set(gca,'fontsize',14)
xlabel('x'), ylabel('t')
colorbar
print -depsc2 checksol5
```