CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 6 · Solutions

Posted Wednesday 26 September 2012. Due Wednesday 3 October 2012, 5pm.

General advice: You may compute any integrals you encounter using symbolic mathematics tools such as WolframAlpha, Mathematica, or the Symbolic Math Toolbox in MATLAB.

1. [60 points: 13 points for (a), (b), (c), and (e); 8 points for (d)]
   Solve the following boundary value problems using the spectral method.
   For each problem, (i) write down the expansions of the right hand side functions as linear combinations of the eigenfunctions; (ii) write down the sum for the solution u obtained from the spectral method; and (iii) produce a plot showing the sum of the first twenty terms in the series for u.

   For parts (c)–(e) you may use the eigenvalues and eigenfunctions computed in Problem 1(d) of Problem Set 5, and the results of Section 5.2.3 of the text.

   (a) $-\dfrac{d^2u}{dx^2}(x) = e^x, \quad u(0) = 1,\ u(1) = 0.$

   (b) $-\dfrac{d^2u}{dx^2}(x) - 10u(x) = e^x, \quad u(0) = 0,\ u(1) = 0.$

   (c) $-\dfrac{d^2u}{dx^2}(x) = x + \sin(\pi x), \quad u(0) = \dfrac{du}{dx}(1) = 0.$

   (d) $-\dfrac{d^2u}{dx^2}(x) = x + \sin(\pi x), \quad u(0) = \dfrac{du}{dx}(1) = 1.$

   (e) $-\dfrac{d^2u}{dx^2}(x) = f(x), \quad \dfrac{du}{dx}(0) = u(1) = 0$, where $f(x) = \begin{cases} 1, & 0 < x < 1/2; \\ 0, & 1/2 < x < 1. \end{cases}$
   (This f is not continuous; follow the usual procedure and see if you obtain a sensible answer.)

Solution.

(a) First we solve the problem with homogeneous Dirichlet boundary conditions using the spectral method, then we will add a correction to satisfy the inhomogeneous boundary conditions.
For the operator with homogeneous Dirichlet conditions, we have eigenvalues $\lambda_k = k^2\pi^2$ with associated (normalized) eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We shall call the solution to the problem with homogeneous Dirichlet conditions $\widehat{u}$, which is given by the spectral method

$$\widehat{u} = \sum_{k=1}^{\infty} \frac{(f, \psi_k)}{\lambda_k}\psi_k.$$

To compute (e^x, ψ_k), integrate twice by parts to obtain

$$\begin{aligned}
\int_0^1 e^x\sqrt{2}\sin(k\pi x)\,dx &= \sqrt{2}\big[e^x\sin(k\pi x)\big]_0^1 - \frac{\sqrt{2}}{k\pi}\int_0^1 e^x\cos(k\pi x)\,dx \\
&= -\frac{\sqrt{2}}{k\pi}\int_0^1 e^x\cos(k\pi x)\,dx \\
&= -\frac{\sqrt{2}}{k\pi}\Big(\big[e^x\cos(k\pi x)\big]_0^1 + \frac{1}{k\pi}\int_0^1 e^x\sin(k\pi x)\,dx\Big)
\end{aligned}$$

$$= \frac{\sqrt{2}}{k\pi}\big(1 - e(-1)^k\big) - \frac{\sqrt{2}}{k^2\pi^2}\int_0^1 e^x \sin(k\pi x)\,dx,$$

from which we discern that

$$\sqrt{2}\Big(1 + \frac{1}{k^2\pi^2}\Big)\int_0^1 e^x \sin(k\pi x)\,dx = \frac{\sqrt{2}}{k\pi}(1 - (-1)^k e).$$

We conclude that

$$(f, \psi_k) = \int_0^1 e^x \sqrt{2}\sin(k\pi x)\,dx = \frac{\sqrt{2}k\pi}{1 + k^2\pi^2}(1 - (-1)^k e).$$

The spectral method thus gives

$$\widehat{u} = \sum_{k=1}^{\infty} \frac{(f, \psi_k)}{\lambda_k}\psi_k = \sum_{k=1}^{\infty} \frac{2(1 - (-1)^k e)}{k\pi(1 + k^2\pi^2)}\sin(k\pi x).$$

Now we need to add some function w to $\widehat{u}$ that will produce a $u = \widehat{u} + w$ that satisfies both the differential equation and the inhomogeneous boundary conditions. We want

$$-\frac{d^2u}{dx^2} = -\frac{d^2\widehat{u}}{dx^2} - \frac{d^2w}{dx^2} = e^x,$$

but since we already have

$$-\frac{d^2\widehat{u}}{dx^2} = e^x,$$

we need

$$-\frac{d^2w}{dx^2} = 0.$$

In other words, w must be for the form $w(x) = \alpha + \beta x$ for some constants α and β. Now note that

$$u(0) = \widehat{u}(0) + w(0) = 0 + w(0), \qquad u(1) = \widehat{u}(1) + w(1) = 0 + w(1),$$

so to satisfy the inhomogeneous conditions we need

$$w(0) = 1, \qquad w(1) = 0.$$

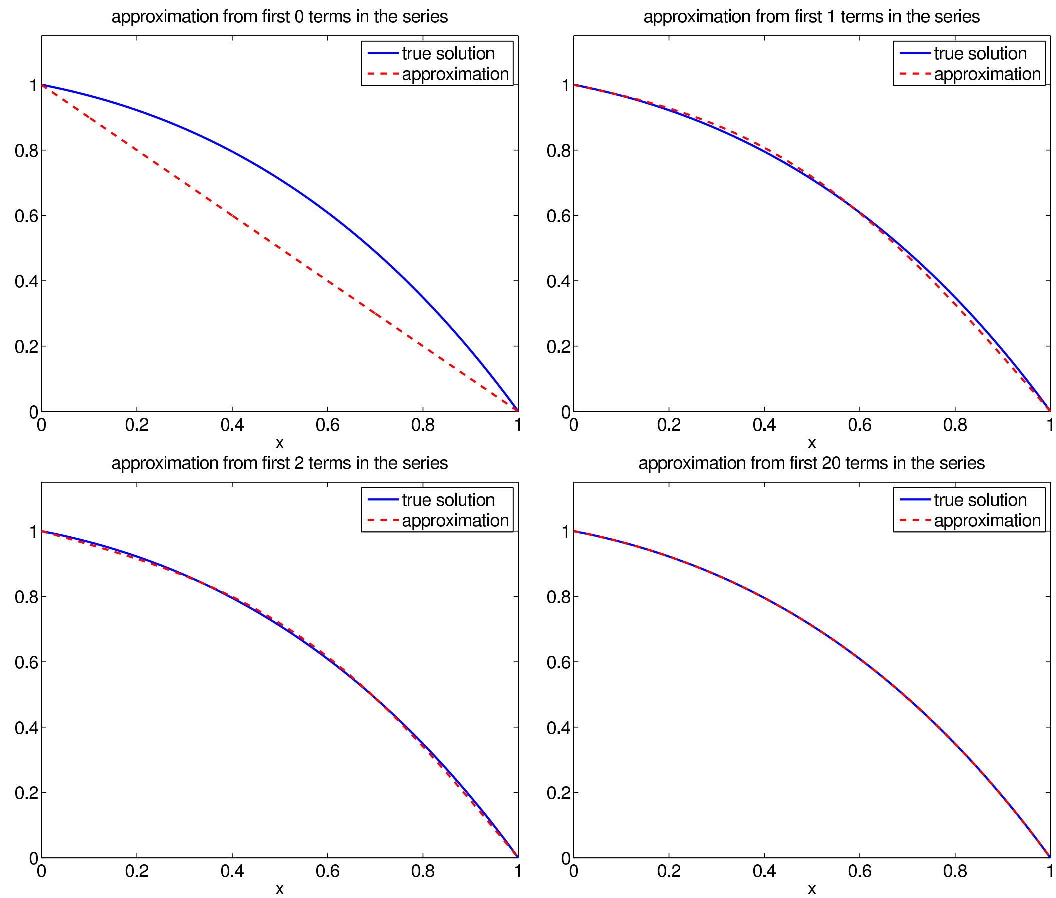
This is only satisfied if we take $\alpha = 1$, $\beta = -1$. The final series formula for u is thus

$$u(x) = 1 - x + \sum_{k=1}^{\infty} \frac{2(1 - (-1)^k e)}{k\pi(1 + k^2\pi^2)}\sin(k\pi x).$$

Though it is not asked for in the problem, the exact solution is

$$u(x) = -e^x + (e - 2)x + 2.$$

The plots below compare the exact solution to the partial sums involving 0, 1, 2, and 20 terms. The code that produced the plots follows.




```
xx = linspace(0,1,1000)';     % fine grid between x=0 and x=1
uN = ones(size(xx));          % u0 = 1 for the boundary condition
for k=0:20
   figure(1), clf
   plot(xx, -exp(xx)+(exp(1)-1)*xx+2,'b-','linewidth',2), hold on
   if k>0
      uN = uN + 2*(1-exp(1)*(-1)^k)/(k*pi*(1+k^2*pi^2))*sin(k*pi*xx);
   end
   plot(xx, uN, 'r--','linewidth',2)
   legend('true solution', 'approximation')
   set(gca,'fontsize',16)
   xlabel('x')
   title(sprintf('approximation from first %d terms in the series', k))
   axis([0 1 0.9 1.35])
   if ismember(k,[0 1 2 20]),
       eval(sprintf('print -depsc2 bvps1_%d',k))
   end
   pause
end
```

(b) To solve this problem, identify the linear operator as

$$Lu = -u'' - 10u,$$

acting the space $C^2_D[0,1]$. To find the eigenvalues λ_k and eigenfunctions ψ_k, notice that $L\psi_k = \lambda_k\psi_k$ is equivalent to

$$-\psi_k'' - 10\psi_k = \lambda_k\psi_k,$$

which can be rewritten as

$$-\psi_k'' = (\lambda_k + 10)\psi_k.$$

Thus, the eigenfunctions ψ_k will be the same as our usual eigenfunctions for $-u''$ on $C_D^2[0,1]$:

$$\psi_k(x) = \sqrt{2}\sin(k\pi x).$$

The eigenvalues must satisfy $\lambda_k + 10 = k^2\pi^2$, in other words,

$$\lambda_k = k^2\pi^2 - 10.$$

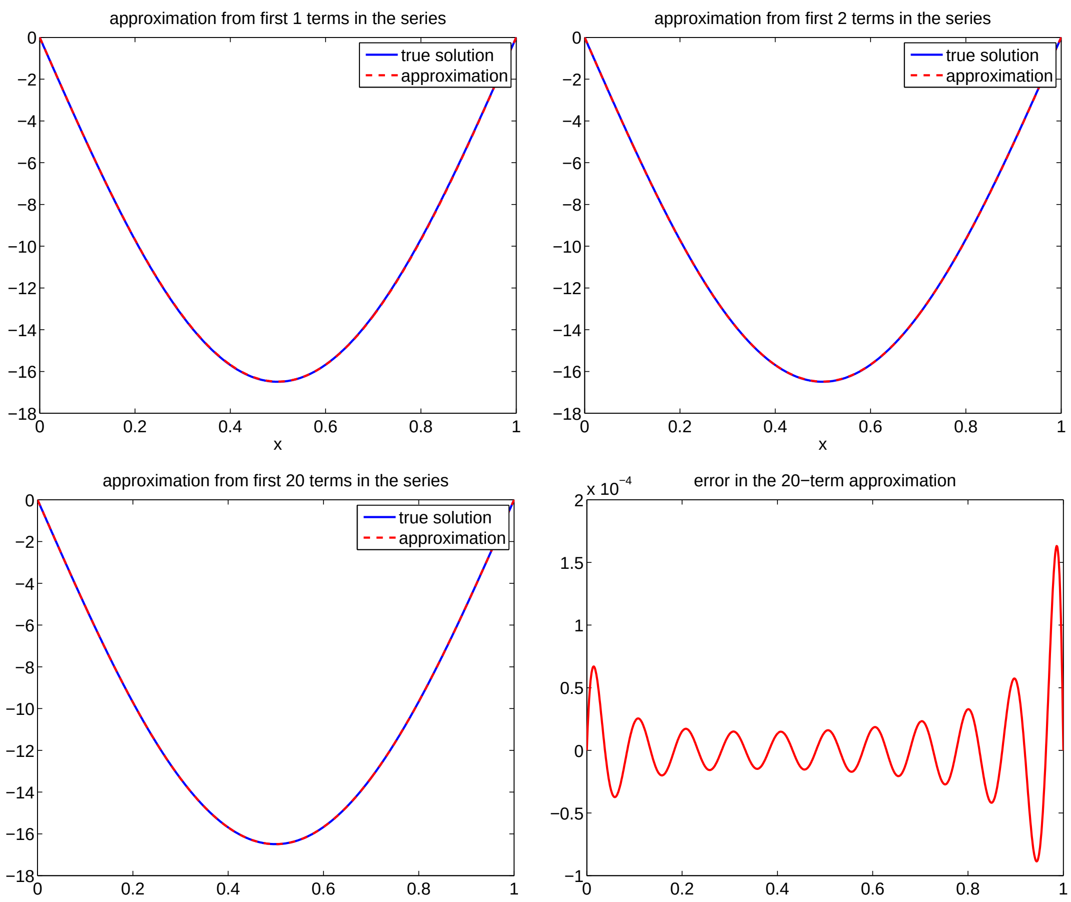
Now we are prepared to solve the problem using the spectral method. Since the right hand side $f(x) = e^x$ is the same as in part (a), and the eigenfunctions are the same, we can simply use the expression we have already derived:

$$f(x) = \sum_{k=1}^{\infty} \frac{(f,\psi_k)}{(\psi_k,\psi_k)}\psi_k = \sum_{k=1}^{\infty} \frac{2(1-(-1)^k e)k\pi}{(1+k^2\pi^2)}\sin(k\pi x).$$

To get the solution $u(x)$, we merely divide each term in the series for f by the eigenvalues:

$$u(x) = \sum_{k=1}^{\infty} \frac{(f,\psi_k)}{(\psi_k,\psi_k)}\frac{1}{\lambda_k}\psi_k = \sum_{k=1}^{\infty} \frac{2(1-(-1)^k e)k\pi}{(1+k^2\pi^2)(k^2\pi^2-10)}\sin(k\pi x).$$

The plots below compare the exact solution to the partial sums involving 1, 2, and 20 terms, along with the error in the 20-term approximation. Code to produce these plots follows. The first term in the series gives an excellent approximation! Why? Because when $k = 1$, the first eigenvalue $\lambda_1 = \pi^2 - 10 \approx -0.130$ is quite a bit smaller than $\lambda_2 = 4\pi^2 - 10 \approx 29.478$, so the first term is quite a bit larger than the next one.



```
xx = linspace(0,1,1000)';     % fine grid between x=0 and x=1
uN = ones(size(xx));          % u0 = 1 for the boundary condition
for k=0:20
   figure(1), clf
   plot(xx, -exp(xx)+(exp(1)-1)*xx+2,'b-','linewidth',2), hold on
   if k>0
      uN = uN + 2*(1-exp(1)*(-1)^k)/(k*pi*(1+k^2*pi^2))*sin(k*pi*xx);
   end
   plot(xx, uN, 'r--','linewidth',2)
   legend('true solution', 'approximation')
   set(gca,'fontsize',16)
   xlabel('x')
   title(sprintf('approximation from first %d terms in the series', k))
   axis([0 1 0.9 1.35])
   if ismember(k,[0 1 2 20]),
       eval(sprintf('print -depsc2 bvps1_%d',k))
   end
   pause
end
```

(c) One can readily compute (or look up in the textbook) the eigenvalues and (orthonormal) eigenfunctions for these boundary conditions:

$$\begin{aligned}\lambda_k &= (k-1/2)^2\pi^2, \quad k = 1, 2, \ldots \\ \psi_k &= \sqrt{2}\sin(\sqrt{\lambda_k}x).\end{aligned}$$

We now need to compute the inner products $(x + \sin(\pi x), \psi_k)$, which we can do in pieces:

$$\begin{aligned}(x, \psi_k) &= \sqrt{2}\int_0^1 x\sin(\sqrt{\lambda_k}x)\,dx \\ &= \frac{\sqrt{2}(\sqrt{\lambda_k}\cos(\sqrt{\lambda_k}) + \sin(\sqrt{\lambda_k})}{\lambda_k} = \frac{\sqrt{2}(-1)^{k+1}}{\lambda_k}.\end{aligned}$$

twice integrating by parts shows that

$$\int_0^1 \sin(\alpha x)\sin(\beta)x\,dx = \frac{\alpha\cos(\alpha)\sin(\beta) - \beta\sin(\alpha)\cos(\beta)}{\beta^2-\alpha^2},$$

and hence

$$(\sin(\pi x), \psi_k) = \frac{\sqrt{2}(-1)^k}{((k-1/2)^2-1)\pi} = \frac{\sqrt{2}(-1)^k\pi}{(\lambda_k-\pi^2)}.$$

We put these pieces together to find that

$$(x + \sin(\pi x), \psi_k) = \frac{\sqrt{2}(-1)^{k+1}}{\lambda_k} + \frac{\sqrt{2}(-1)^k\pi}{(\lambda_k-\pi^2)}.$$

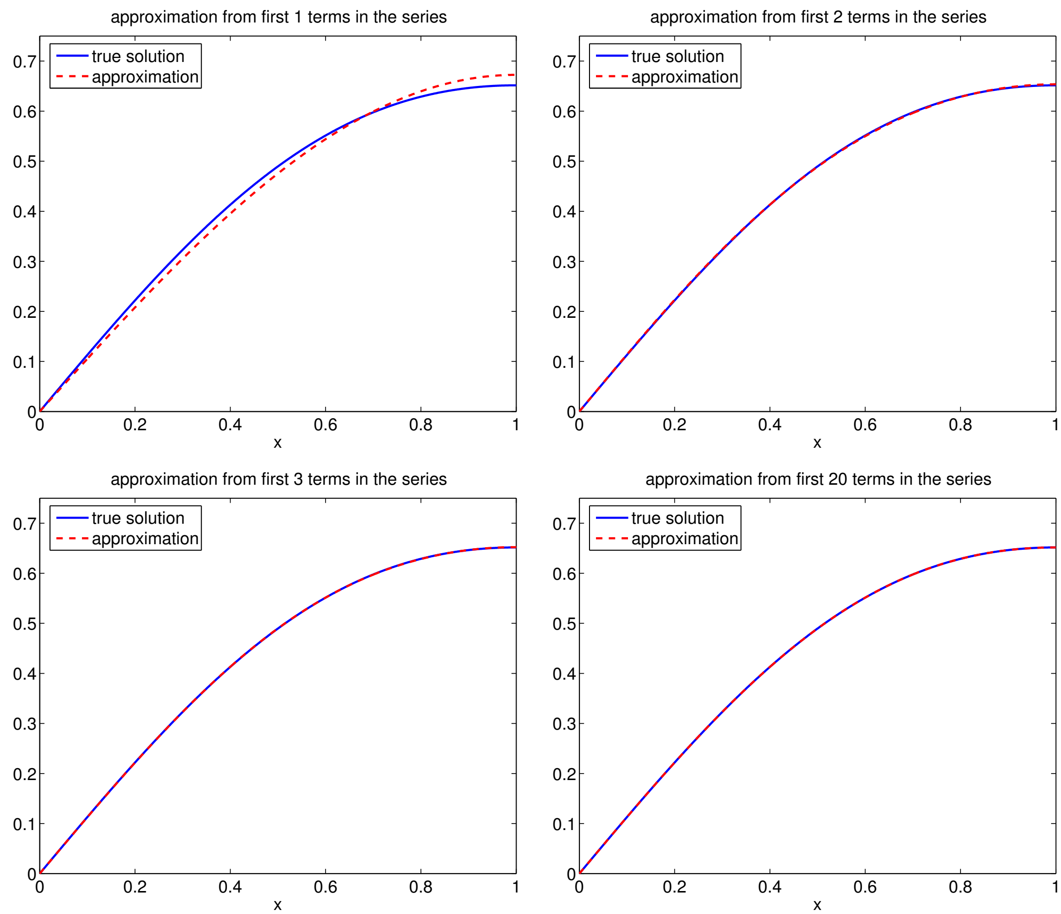
The spectral method thus gives the formula

$$\begin{aligned}u(x) &= \sum_{k=1}^{\infty} 2(-1)^k\Big(\frac{\pi}{\lambda_k(\lambda_k-\pi^2)} - \frac{1}{\lambda_k^2}\Big)\sin(\sqrt{\lambda_k}x) \\ &= \sum_{k=1}^{\infty}\frac{2(-1)^k\big((\pi-1)\lambda_k-\pi^2\big)}{\lambda_k^2(\lambda_k-\pi^2)}\sin(\sqrt{\lambda_k}x).\end{aligned}$$

The true solution can be determined to be

$$u(x) = \frac{\sin(\pi x)}{\pi^2} - \frac{x^3}{6} + \frac{x}{2} + \frac{x}{\pi}.$$

The plots below compare the exact solution to the partial sums involving 1, 2, 3, and 20 terms. The code that produced the plots follows.




```
xx = linspace(0,1,1000)';     % fine grid between x=0 and x=1
uN = zeros(size(xx));         % homogeneous boundary conditions
for k=1:20
   figure(1), clf
   plot(xx, sin(pi*xx)/(pi^2) - (xx.^3)/6 + (1/2+1/pi)*xx,'b-','linewidth',2), hold on
   lamk = ((k-1/2)^2)*(pi^2);
   uN = uN + 2*(-1)^k*(pi/(lamk*(lamk-pi^2))-1/(lamk^2))*sin(sqrt(lamk)*xx);
   plot(xx, uN, 'r--','linewidth',2)
   legend('true solution', 'approximation',2)
   set(gca,'fontsize',16)
   xlabel('x')
   title(sprintf('approximation from first %d terms in the series', k))
   axis([0 1 0 0.75])
   if ismember(k,[1 2 3 20]),
       eval(sprintf('print -depsc2 bvps2_%d',k))
   end
   pause
end
```

(d) Notice that this problem is identical to part (c), except now with the inhomogeneous boundary conditions $u(0) = u'(1) = 1$. As in part (a), we seek a solution $u = \widehat{u} + w$, where $\widehat{u}$ satisfies the equation with homogeneous boundary conditions (from part (c)),

$$\begin{aligned}\widehat{u}(x) &= \sum_{k=1}^{\infty} \frac{2(-1)^k\big((\pi-1)\lambda_k - \pi^2\big)}{\lambda_k^2(\lambda_k-\pi^2)} \sin(\sqrt{\lambda_k}x)\\ &= \frac{\sin(\pi x)}{\pi^2} - \frac{x^3}{6} + \frac{x}{2} + \frac{x}{\pi},\end{aligned}$$

and w is a function with $-w''(x) = 0$, i.e., $w(x) = \alpha + \beta x$. In this case we require

$$u(0) = \widehat{u}(0) + w(0) = \alpha = 1$$

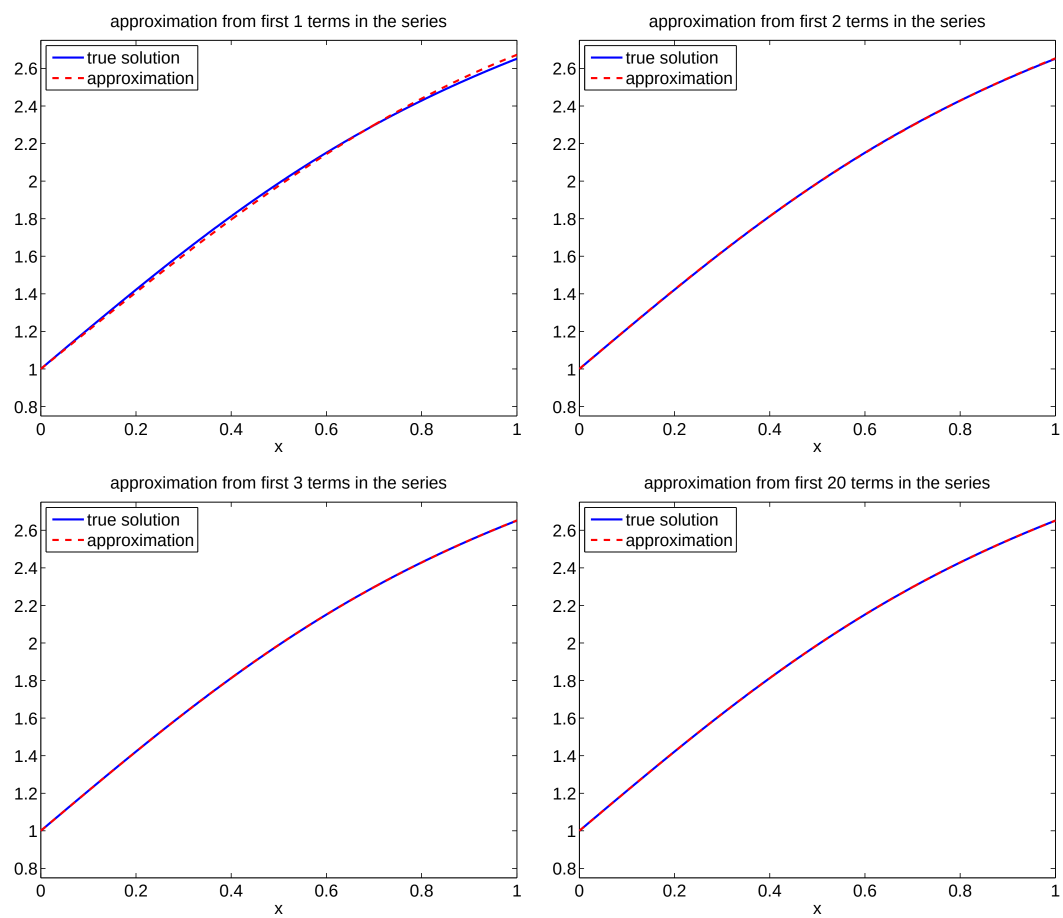
and

$$u'(1) = \widehat{u}'(1) + w'(0) = beta = 1.$$

Hence, the solution is given by

$$\begin{aligned} u(x) &= 1 + x + \sum_{k=1}^{\infty} \frac{2(-1)^k \big((\pi - 1)\lambda_k - \pi^2\big)}{\lambda_k^2(\lambda_k - \pi^2)} \sin(\sqrt{\lambda_k} x) \\ &= 1 + x + \frac{\sin(\pi x)}{\pi^2} - \frac{x^3}{6} + \frac{x}{2} + \frac{x}{\pi}. \end{aligned}$$

The plots below compare the exact solution to the partial sums involving 1, 2, 3, and 20 terms. The code that produced the plots follows.




```
xx = linspace(0,1,1000)';     % fine grid between x=0 and x=1
uN = zeros(size(xx));         % homogeneous boundary conditions
for k=1:20
   figure(1), clf
   plot(xx, sin(pi*xx)/(pi^2) - (xx.^3)/6 + (1/2+1/pi)*xx,'b-','linewidth',2), hold on
   lamk = ((k-1/2)^2)*(pi^2);
   uN = uN + 2*(-1)^k*(pi/(lamk*(lamk-pi^2))-1/(lamk^2))*sin(sqrt(lamk)*xx);
   plot(xx, uN, 'r--','linewidth',2)
   legend('true solution', 'approximation',2)
   set(gca,'fontsize',16)
   xlabel('x')
```

```
    title(sprintf('approximation from first %d terms in the series', k))
    axis([0 1 0 0.75])
    if ismember(k,[1 2 3 20]),
        eval(sprintf('print -depsc2 bvps2_%d',k))
    end
    pause
end
```

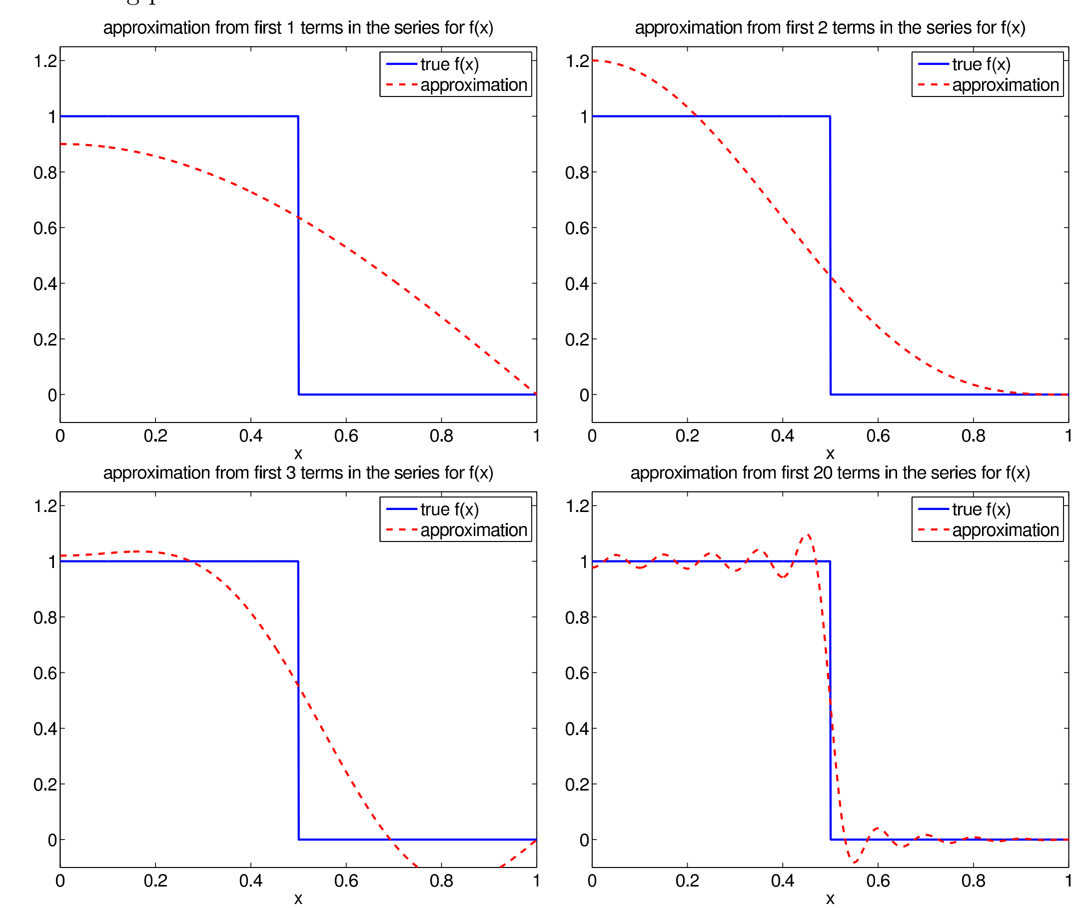
(e) On Problem Set 5, the eigenvalues and eigenfunctions for this example were determined to be

$$\begin{aligned}\lambda_k &= (k-1/2)^2\pi^2, \quad k=1,2,\ldots \\ \psi_k &= \sqrt{2}\cos(\lambda_k x).\end{aligned}$$

Though the function f is discontinuous, we proceed as usual, computing

$$\begin{aligned}(f,\psi_k) = \int_0^1 f(x)\psi_k(x)\,dx &= \int_0^{1/2} \psi_k(x)\,dx \\ &= \int_0^{1/2} \sqrt{2}\cos(\sqrt{\lambda_k}x)\,dx = \sqrt{2}\left[\frac{\sin(\sqrt{\lambda_k}x)}{\sqrt{\lambda_k}}\right]_0^{1/2} = \frac{\sqrt{2}\sin(\sqrt{\lambda_k}/2)}{\sqrt{\lambda_k}}.\end{aligned}$$

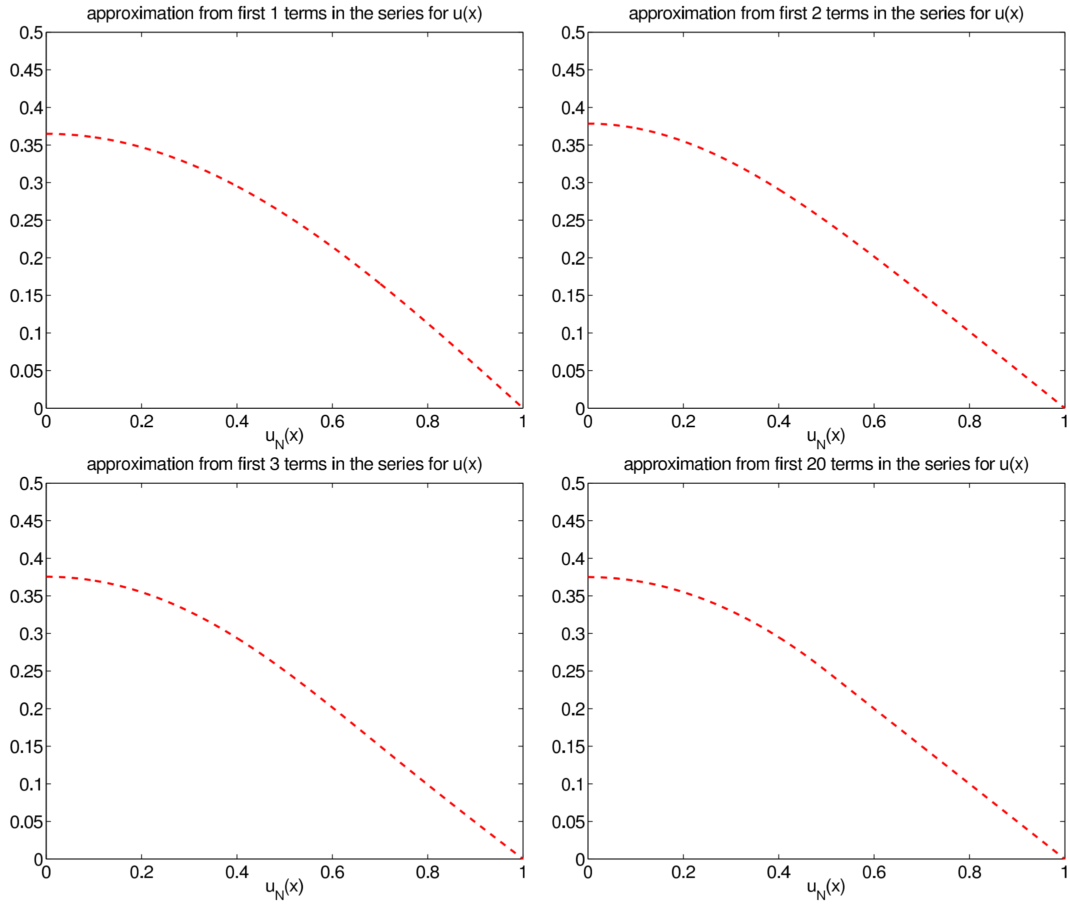
Note the slow decay of these coefficients, even though f satisfies the same boundary conditions as the eigenfunctions! This is due to the discontinuity on the interior of the domain, as seen in the following plots.



The spectral method thus gives the solution as the formula

$$u(x) = \sum_{k=0}^{\infty} = \frac{2\sin(\sqrt{\lambda_k}/2)}{\lambda_k^{3/2}}\cos(\sqrt{\lambda_k}x).$$

The plots below show the partial sums involving 1, 2, 3, and 20 terms. The code that produced the plots follows.




```
xx = linspace(0,1,1000)';     % fine grid between x=0 and x=1
f  = (xx<.5);                 % define the right hand side function
fN = zeros(size(xx));
uN = zeros(size(xx));
for k=1:20
   ck =  (sqrt(2)*sin((2*k-1)*pi/4))/((k-.5)*pi);
   lamk = ((k-1/2)^2)*(pi^2);
   fN = fN + ck*(sqrt(2)*cos(sqrt(lamk)*xx));       % construct the approximation to the RHS
   uN = uN + ck/lamk*(sqrt(2)*cos(sqrt(lamk)*xx)); % spectral method for u

   figure(1), clf
     plot(xx, f, 'b-','linewidth',2), hold on
     plot(xx, fN, 'r--','linewidth',2)
     legend('true f(x)', 'approximation',1)
     set(gca,'fontsize',16)
     xlabel('x')
     title(sprintf('approximation from first %d terms in the series for f(x)', k))
     axis([0 1 -.1 1.25])
     if ismember(k,[1 2 3 20]),
         eval(sprintf('print -depsc2 bvps3_%db',k))
     end
   figure(2), clf
     plot(xx, uN, 'r--','linewidth',2)
     set(gca,'fontsize',16)
     xlabel('x')
     xlabel('u_N(x)')
     title(sprintf('approximation from first %d terms in the series for u(x)', k))
     axis([0 1 0 0.5])
```

```
        if ismember(k,[1 2 3 20]),
            eval(sprintf('print -depsc2 bvps3_%da',k))
        end
      pause
   end
```

---

2. [40 points: (a)=7 points; (b)=5 points; (c)=6 points; (d)=5 points; (e)=7 points; (f)=5 points]
For the problems we have considered thus far, the eigenvalues have always satisfied nice formulas that are fairly easy to compute. This problem illustrates that this is not always the case.

   Consider the equation

$$-u''(x) = f(x), \quad x \in [0,1]$$

   with a homogeneous *Robin condition* on the left,

$$u(0) - u'(0) = 0$$

   and a homogeneous Dirichlet boundary condition on the right,

$$u(1) = 0.$$

   Define the linear operator $L : V \to C[0,1]$ via $Lu = -u''$ with

$$V = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

   (a) Prove that L is symmetric.

   (b) Is zero an eigenvalue of L? That is, does there exist a nontrivial solution to $-u''(x) = 0$ with these boundary conditions?

   (c) Compute the eigenfunctions of L associated with nonzero eigenvalues, and show that these eigenvalues λ must satisfy the equation

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

   (d) In MATLAB, create a plot of $g(x) = -\tan(x)$ for $x \in [0, 5\pi]$ and superimpose (`hold on`) a plot of $h(x) = x$. By hand, mark the points where these two functions intersect on your plot.

   (e) Use your plot in (d) to argue that L has infinitely many eigenvalues,
   with $(n - \frac{1}{2})^2\pi^2 < \lambda_n < (n + \frac{1}{2})^2\pi^2$. What value does λ_n tend to as n becomes large?

   (f) Estimate the first four eigenvalues to at least six digits of accuracy.
   You will need to find the points of intersection you marked in part (d).
   Please *don't* just try to 'eyeball' these by zooming in on your plot!
   Instead, either use MATLAB's `fzero` function, or write your own implementation of a root-finding algorithm (Newton's method, bisection, etc.).

   (g) [optional: 5 bonus points]
   Recall the finite difference matrices you worked with in Problem Set 3. Figure out how to construct a similar discretization of $L\psi = \lambda\psi$ for the linear operator in this problem, paying particular attention to the boundary condition at $x = 1$. You should arrive at a matrix equation of the form $\mathbf{Dv} = \lambda\mathbf{v}$. Compute the eigenvalues of $\mathbf{D}$ using MATLAB's `eig` command. How do these approximations compare to the values you computed in part (f), as you take finer and finer discretizations ($N \to \infty$)?
   Hint: The first row of your matrix equation $\mathbf{Dv} = \lambda\mathbf{v}$ should encode the equation

$$\frac{-u_0 + 2u_1 - u_2}{h^2} = \lambda u_1.$$

Obtain a formula for u_0 in terms of u_1 and u_2 via the boundary condition $u(0) - u'(0) = 0$ and the approximation

$$u'(0) = \frac{-3u_0 + 4u_1 - u_2}{2h} + O(h^2),$$

and use these values to update the u_1 and u_2 entries in the first row of $\mathbf{D}$....

Solution.

(a) Suppose $u, v \in V$, so that $u(0) - u'(0) = v(0) - v'(0) = u(1) = v(1) = 0$. Then integrating by parts twice,

$$\begin{aligned}
(Lu, v) &= \int_0^1 -u''(x)v(x)\,dx \\
&= \Big[-u'(x)v(x)\Big]_0^1 + \int_0^1 \int_0^1 -u'(x)v'(x)\,dx, \\
&= u'(1)v(1) - u'(0)v(0) + \int_0^1 \int_0^1 -u'(x)v'(x)\,dx, \\
&= u'(1)v(1) - u'(0)v(0) + \Big[u(x)v'(x)\Big]_0^1 - \int_0^1 -u(x)v''(x)\,dx \\
&= u'(1)v(1) - u'(0)v(0) + u(0)v'(0) - u(1)v'(1) - (u, Lv) \\
&= -u'(0)v(0) + u(0)v'(0) - (u, Lv).
\end{aligned}$$

In the last step, two boundary terms are zero because $u(1) = v(1) = 0$. For the other boundary term, note that $v(0) - v'(0) = 0$ implies $v(0) = v'(0)$, so $-u'(0)v(0) + u(0)v'(0) = u(0)v(0) - u'(0)v(0) = (u(0) - u'(0))v(0) = 0$ since $u(0) - u'(0) = 0$. Hence $(Lu, v) = (u, Lv)$, so L is symmetric.

[**GRADERS**: Please deduct 3 points if students do not clearly explain why the boundary term at $x = 0$ is zero.]

(b) Zero is *not* an eigenvalue of L. To see this, we seek a nonzero solution $\psi \in V$ to $L\psi = 0\psi$, i.e., $-\psi''(x) = 0$. The general solution of $-\psi''(x) = 0$ is $\psi(x) = Ax + B$. The right boundary condition $\psi(1) = 0$ implies that

$$0 = \psi(1) = A + B,$$

hence $A = -B$. The left boundary condition implies

$$0 = \psi(0) - \psi'(0) = B - A,$$

hnece $A = B$. The only solution to both boundary conditions is hence $A = B = 0$, so the trivial solution $\psi(x) = 0$ is the only solution of $L\psi = 0$ in V. Thus zero is not an eigenvalue of L.

(c) We now know that all eigenvalues λ are nonzero. For $\lambda \neq 0$, the general solution of $L\psi = \lambda\psi$, i.e. $-\psi'' = \lambda\psi$, has the form

$$\psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

The left boundary condition gives

$$0 = \psi(0) - \psi(1) = A\sin(\sqrt{\lambda}0) + B\cos(\sqrt{\lambda}0) - A\sqrt{\lambda}\cos(\sqrt{\lambda}0) + B\sqrt{\lambda}\sin(\sqrt{\lambda}0) = B - A\sqrt{\lambda},$$

hence $B = A\sqrt{\lambda}$. The right boundary condition gives

$$0 = \psi(1) = A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda}).$$

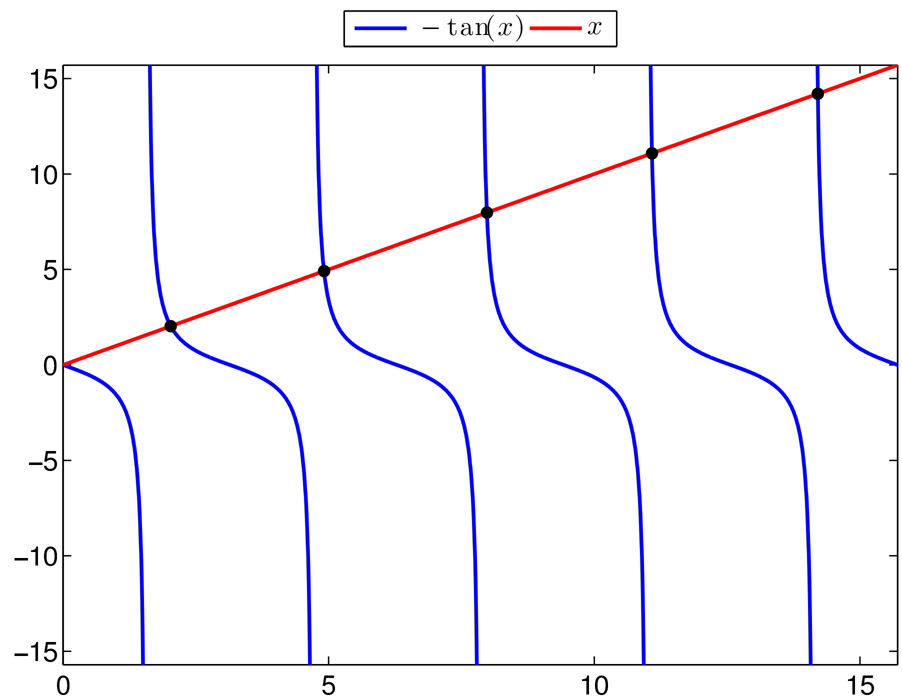
Substituting the left boundary condition into this last formula, we find

$$0 = A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}).$$

Since we need $A \neq 0$ to obtain nontrivial solutions, this equation implies

$$\sqrt{\lambda} = -\frac{\sin(\sqrt{\lambda})}{\cos(\sqrt{\lambda})} = -\tan(\sqrt{\lambda}).$$

(d) The plot is shown below. The code that produced it follows at the end of this solution.



(e) On every interval $((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$, the function $-\tan(x)$ takes on every value on $(-\infty, \infty)$ exactly once, so there will be precisely one intersection between $-\tan(x)$ and x on each of these intervals. Hence, exactly one eigenvalue falls in every interval $((n-\frac{1}{2})^2\pi^2, (n+\frac{1}{2})^2\pi^2)$. As x gets larger and larger, this intersection occurs closer and closer to the left end of the interval, so as $n \to \infty$, we expect

$$\lambda_n \approx (n-\tfrac{1}{2})^2\pi^2.$$

(f) Since the plot shows 5 points of intersection for $\lambda > 0$, we compute the first 5 eigenvalues, even though the problem only asks for the first four. The table below also compares the eigenvalue to the $(n-\frac{1}{2})^2\pi^2$ as discussed in the last subproblem (not required).

n	$\sqrt{\lambda_n}$	λ_n	$(n-1/2)^2\pi^2$
1	2.0287578381	4.1158583657	2.4674011003
2	4.9131804394	24.1393420304	22.2066099025
3	7.9786657124	63.6591065504	61.6850275068
4	11.0855384065	122.8891617619	120.9026539133
5	14.2074367252	201.8512583003	199.8594891221

MATLAB's `fzero` works for some of these roots, but the rapid rate of growth of $-\tan(x)$ causes this algorithm to perform poorly for some roots. Hence we turned to the bisection algorithm, which includes the root between two points, and successively cuts that interval in half as it zeros in on the root. This algorithm is not as fast as `fzero`, but is more reliable.

[**GRADERS**: Please deduct 2 points if students got some incorrect roots using `fzero` and didn't note that these roots are wrong. If they got some roots correct but noted that there was a problem with `fzero` for some of the roots, you can give full credit. Please deduct one point if students only report $\sqrt{\lambda_n}$ and not λ_n as requested in the problem.]

```
 figure(1), clf
 for j=0:5
    x = linspace((j-1/2)*pi,(j+1/2)*pi,500);
    x = x(2:end-1);
    tanplt = plot(x,-tan(x),'b-','linewidth',2);
    hold on
 end
 x = linspace(0,5*pi,100);
 linplt = plot(x,x,'r-','linewidth',2);
 ylim([-5*pi 5*pi])
 xlim([0 5*pi])

 lgd = legend([tanplt,linplt],'$\ -\tan(x)$', '$x$',...
        'location','northoutside','orientation','horizontal');
 set(lgd,'interpreter','latex')
 set(gca,'fontsize',14)

 guess = [2 5 7.98 11 14.21]';
 bracket = [1.6 2.5;
            4.8 5;
            7.9 8.1;
            11 11.2;
            14.15 14.3];

 ew = zeros(size(guess));
 fprintf('   sqrt(lambda)          lambda     (n-1/2)^2 pi^2 \n', ew(k), ew(k)^2)
 for k=1:length(guess)
%     ew(k) = fzero(@(x) x+tan(x),guess(k));
     ew(k) = bisect(@(x) x+tan(x),bracket(k,1),bracket(k,2));
     plot(ew(k),ew(k),'k.','markersize',20)
     fprintf(' %15.10f   %15.10f  %15.10f\n', ew(k), ew(k)^2, (k-1/2)^2*pi^2)
 end
 print -depsc2 eigroot

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 function xstar = bisect(f,a,b)

% function xstar = bisect(f,a,b)
% Compute a root of the function f using bisection.
% f:    a function name, e.g., bisect('sin',3,4), or bisect('myfun',0,1)
% a, b: a starting bracket:  f(a)*f(b) < 0.

 fa = feval(f,a);
 fb = feval(f,b);          % evaluate f at the bracket endpoints
 delta = (b-a);            % width of initial bracket
 k = 0; fc = inf;          % initialize loop control variables

 c = (a+b)/2;
 while (delta/(2^k)>1e-18) & abs(fc)>1e-18
    c = (a+b)/2; fc = feval(f,c);   % evaluate function at bracket midpoint
    if fa*fc < 0, b=c; fb = fc;     % update new bracket
    else a=c; fa=fc; end
    k = k+1;
%    fprintf(' %3d  %20.14f  %16.8e\n', k, c, fc);
 end
 xstar = c;
```

(g) [bonus question] The eigenvalues of $\mathbf{D}$ for various values of N are plotted below. Indeed, we see increasingly accurate approximations to the eigenvalues computed via a completely different method in part (f).

N	λ_1	λ_2	λ_3	λ_4	λ_5
8	4.0841721152	23.7107767862	61.0866729525	112.5994697441	171.7282531156
16	4.1058135169	23.9757051976	62.7295637315	119.5758989350	192.8815393298
32	4.1130303588	24.0887676846	63.3662538535	121.8628790245	199.1474454253
64	4.1151083041	24.1253264546	63.5766828167	122.5995099114	201.0898102312
128	4.1156652470	24.1356573886	63.6372636884	122.8121852697	201.6487031669
256	4.1158093721	24.1383977521	63.6534869375	122.8693278028	201.7990311254
512	4.1158460272	24.1391030403	63.6576815597	122.8841286367	201.8380002820