CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 43

Posted Wednesday 13 November 2013. Due 5pm Wednesday 27 November 2013.

43. [25 points] Let
$$\Omega = \{(x,y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$
and let $u_0(x,y) = 200xy(1-x)(1-y)(x-\frac{1}{4})(y-\frac{1}{4})$. Note that, for $m,n = 1,2,3,\ldots$,
$$\int_0^1 \int_0^1 2u_0(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy = \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}.$$
In this question we will consider the problem of finding the solution $u(x,y,t)$ to the wave equation
$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$
with homogeneous Dirichlet boundary conditions
$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$
and initial conditions
$$u(x,y,0) = u_0(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$
and
$$u_t(x,y,0) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$
Let
$$C_D^2(\Omega) = \left\{v \in C^2(\Omega) : v(x,0) = v(x,1) = v(0,y) = v(1,y) = 0,\, 0 \le x \le 1,\, 0 \le y \le 1\right\}.$$
Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by
$$(Lv)(x,y) = -\left(v_{xx}(x,y) + v_{yy}(x,y)\right).$$
The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions
$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$
for $j,k = 1,2,\ldots$, which are such that
$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$
for $j,k = 1,2,\ldots$. Recall that in Homework 40 you obtained a formula for $\lambda_{j,k}$ for $j,k = 1,2,\ldots$.

(a) We can write
$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$
What ordinary differential equation and initial conditions does $a_{j,k}(t)$ satisfy for $j,k = 1,2,\ldots$?

(b) Obtain an expression for $a_{j,k}(t)$ for $j,k = 1,2,\ldots$.

(c) Use you answer to part (b) to write out a formula for $u(x,y,t)$.

(d) Plot
$$u_{10}(x,y,t) = \sum_{j=1}^{10}\sum_{k=1}^{10} a_{j,k}(t)\psi_{j,k}(x,y)$$
at times $t = 0, 0.5, 1.0, 1.5, 2.5$. Use the command `zlim([-2 2])` so that the axes on all of your plots are the same.