CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 1 · Solutions

Posted Wednesday 27 August 2014. Due 5pm Wednesday 03 September 2014.

1. [24 points]

   For each of the following equations, (a) specify whether it is an ODE or a PDE; (b) determine its order; and (c) specify whether it is linear or nonlinear. For those that are linear, specify whether they (d) are homogeneous or inhomogeneous; and (e) have constant or variable coefficients.

$$(1.1)\quad \frac{dv}{dx} + \frac{2}{x}v = 0 \qquad\qquad (1.2)\quad \frac{\partial v}{\partial t} - 3\frac{\partial v}{\partial x} = x - t$$

$$(1.3)\quad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) = 0 \qquad\qquad (1.4)\quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$(1.5)\quad \frac{d^2y}{dx^2} - 7(1-y^2)\frac{dy}{dx} + y = 0 \qquad\qquad (1.6)\quad \frac{d^2}{dx^2}\left(x^2\frac{d^2u}{dx^2}\right) = \sin(x)$$

Solution.

(1.1) [4 points] ODE, first order, linear, homogeneous, variable coefficient
The $2/x$ factor in front of the v is the variable coefficient.

(1.2) [4 points] PDE, first order, linear, inhomogeneous, constant coefficient
The $x - t$ term on the right, which does not involve v, makes the equation inhomogeneous.

(1.3) [4 points] PDE, second order, nonlinear
Using the product rule for partial derivatives, we can write this equation in the equivalent form

$$\frac{\partial u}{\partial t} - 2\left(\frac{\partial u}{\partial x}\right)^2 - 2u\left(\frac{\partial^2 u}{\partial x^2}\right) = 0.$$

The second and third terms on the left hand side make this equation nonlinear.

(1.4) [4 points] PDE, third order, nonlinear
The $u(\partial u/\partial x)$ term makes this equation nonlinear. This a version of the famous Korteweg-de Vries (KdV) equation that describes shallow water waves.

(1.5) [4 points] ODE, second order, nonlinear
The $(1 - y^2)(dy/dx)$ term makes this ODE nonlinear.

(1.6) [4 points] ODE, fourth order, linear, inhomogeneous, variable coefficient
Using the product rule for partial derivatives, we can write this equation in the equivalent form

$$2\frac{d^2u}{dx^2} + 4x\frac{d^3u}{dx^3} + x^2\frac{d^4u}{dx^4} = \sin(x),$$

hence we can see that it is fourth order.

---

2. [21 points]

   (a) Is $v(x) = 1/x^2$ a solution of

$$\frac{dv}{dx} + \frac{2}{x}v = 0?$$

(b) Is $v(x,t) = t(t+x)$ a solution of

$$\frac{\partial v}{\partial t} - 3\frac{\partial v}{\partial x} = x - t?$$

(c) Is $u(x,t) = xe^t$ a solution of

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) = 0?$$

Solution.

(a) [7 points] $v(x) = 1/x^2$ is a solution of (1.1).
To plug $v(x) = 1/x^2$ into the left-hand side of (1.1), we compute $dv/dx = d(x^{-2})/dx = -2x^{-3}$. Substituting this formula, the left-hand side of (1.1) becomes

$$-2x^{-3} + 2x^{-1}x^{-2} = 0.$$

This agrees with the right-hand side of (1.1), so this v is a solution.

(b) [7 points] $v(x,t) = t(t+x)$ is a solution of (1.2).
We compute $\partial v/\partial t = 2t + x$ and $\partial v/\partial x = t$. Thus the left-hand side of (1.2) becomes

$$(2t + x) - 3(t) = x - t.$$

This agrees with the right-hand side of (1.2), so this v is a solution.

(c) [7 points] $u(x,t) = xe^t$ is *not* a solution of (1.3).
We compute $\partial u/\partial t = xe^t$ and $\partial u/\partial x = e^t$. From this it follows that

$$\frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}2xe^{2t} = 2e^{2t}.$$

Thus the left-hand side of (1.3) is

$$xe^t - 2e^{2t},$$

which is nonzero in general, in disagreement with the right-hand side of (1.3).

---

3. [15 points]

A Bernoulli differential equation (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Observe that, if $n = 0$ or $n = 1$, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

Solution. We know that a differential equation is linear if the unknown function and its derivatives appear to the power 1 (products of the unknown function and its derivatives are not allowed) and nonlinear otherwise. Using the definition it can be seen that if $n = 0$ and $n = 1$ Bernoulli differential equation becomes

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and

$$\frac{dy}{dx} + (P(x) - Q(x))y = 0,$$

respectively. We can easily conclude they are linear and For other values of n is nonlinear.

Now,Setting $u = y^{1-n}$,

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx} = \frac{u^n/(1-n)}{1-n}\frac{du}{dx}.$$

the Bernoulli differential equation becomes

$$\frac{u^{n/(1-n)}}{1-n}\frac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$$

or

$$\frac{du}{dx} + (1-n)P(x)u = Q(x)(1-n).$$

---

4. [40 points] Recall the 1D steady-state heat equation with constant diffusivity over the interval $[0, 1]$

$$\begin{aligned} -\tfrac{\partial^2 u}{\partial x^2} &= f \\ u(0) = u(1) &= 0. \end{aligned}$$

Recall from class the finite difference approximation to this problem: given a set of points $x_0, \ldots, x_{N+1}$, solved for the solution $u(x_i)$ at each point by approximating $\frac{\partial^2 u}{\partial x^2}$ with

$$u''(x_i) \approx \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2}, \quad i = 1, \ldots, N$$

(where h is the spacing between points x_{i+1} and x_i) along with the conditions that

$$u(x_0) = u(x_{N+1}) = 0.$$

We will modify this finite difference approximation to accomodate instead the Neumann boundary condition of $u'(1) = 0$ at $x = 1$.

(a) We would like to enforce that $u'(x_{N+1}) = 0$, but if we approximate $u'(x_{N+1})$ with a central difference

$$u'(x_{N+1}) \approx \frac{u(x_{N+\frac{3}{2}}) - u(x_{N+\frac{1}{2}})}{h},$$

we end up with an equation involving $u(x_{N+\frac{3}{2}})$, which does not lie inside the interval $[0, 1]$. Instead, we can define a *backward difference* approximation to the derivative

$$u'(x_{N+1}) \approx \frac{u(x_{N+1}) - u(x_N)}{h} = 0$$

and set this to zero instead. Write out the expression for $u''(x_N)$ in terms of $u(x_i)$ and use the backward difference approximation for $u'(x_{N+1})$ to eliminate $u(x_{N+1})$.

(b) Determine the exact solution to $-u''(x) = 1$ for $u(0) = 0$, $u'(1) = 0$ (hint: the solution is a quadratic function).

(c) Create a MATLAB script that constructs the matrix system $Au = f$ resulting from the finite difference equations when $f = 1$. Plot the computed solution values $u(x_i)$, as well as the error at each point $|u_{\text{exact}}(x_i) - u(x_i)|$, for $i = 0, \ldots, N+1$ for $N = 16, 32, 64, 128$, and label each appropriately.

(d) Suppose we have $u'(0) = u'(1) = 0$. Show that if $u(x)$ is a solution of the steady state heat equation with these boundary conditions, that

$$u + C$$

for any constant C is also a solution to the same steady state heat equation. This shows that there is no *unique* solution to the steady state heat equation under these boundary conditions.

Solution.

(a) We can rewrite the finite difference approximation to $-u''(x) = f(x)$ at point x_N as

$$-\frac{u(x_{N-1}) - 2u(x_N) + u(x_{N+1})}{h^2} = f(x_N).$$

Note that

$$\frac{u(x_{N-1}) - 2u(x_N) + u(x_{N+1})}{h^2} = \frac{u(x_{N-1}) - u(x_N)}{h^2} + \frac{u(x_{N+1}) - u(x_N)}{h^2}$$

Using the backwards difference approximation to $u'(x_{N+1})$, we have

$$\frac{u(x_{N+1}) - u(x_N)}{h^2} = 0$$

which simplifies our finite difference equation at x_N to

$$-\frac{u(x_{N-1}) - u(x_N)}{h^2} = f(x_N).$$

(Note that the boundary condition also implies $x_N = x_{N+1}$).

(b) We can integrate the differential equation twice to get the boundary conditions.

$$\int_0^x -u''(s)ds = \int_0^x 1ds$$

where s is a dummy variable for integration. By the fundamental theorem of calculus, this gives

$$-u'(x) + u'(0) = x.$$

Since we don't know the value of $u'(0)$, we consider it an unknown constant C_1 that we have to determine using our boundary conditions. Repeating the process again gives

$$\int_0^x (-u'(s) + C_1)ds = \int_0^x xds$$

which results in

$$-u(x) + C_1 x + u(0) = \frac{x^2}{2}.$$

We could set $u(0)$ to be a constant C_2 to be determined by the boundary conditions as well; however, since we know $u(0) = 0$ from the boundary conditions, we can go ahead and zero it out. The end result gives

$$u(x) = \frac{x^2}{2} + C_1 x$$

The above form of the equation and the boundary condition $u'(1) = 0$ give the condition that

$$u'(1) = 1 + C_1 = 0$$

implying $C_1 = -1$, and

$$u(x) = \frac{x^2}{2} - x = \frac{1}{2}x(2-x).$$

Alternatively, since the problem specifies the solution is a quadratic, it is possible to simply specify

$$u(x) = ax^2 + bx + c$$

and use the differential equation and boundary conditions to determine the constants.

(c) Since the finite difference equations must be satisfied at each point x_i, they lead to a series of N equations with N unknowns (the values of $u(x_i)$ for $i = 1, \ldots, N$). The matrix system resulting from these equations for homogeneous boundary conditions

$$u(0) = u(1) = 0$$

is

$$\frac{-1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix},$$

where $u_i \approx u(x_i)$. Since we have the boundary condition $u'(1) = 0$ instead, this changes our finite difference equation at point x_N, which corresponds to the final row of our matrix. Thus, our new matrix system for a no-flux boundary condition at $x = 1$ will be

$$\frac{-1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix}.$$

Included is Matlab code that can be used to generate the finite difference solution and the error between it and the exact solution:

```
% HW 1, Problem 4c. CAAM 336, Fall 2014
% solves the steady heat equation u''(x) = 1 with u(0) = 0, u'(1) = 0
clear

uexact = @(x) .5*x.*(2-x);

i = 1;
C = hsv(4); % neat trick: makes a matrix whose values determine colors.
Nlist = [16 32 64 128]; % number of interior points
for N = Nlist
    K = N+1; % number of line segments
    h = 1/K; % spacing between points
    x = linspace(0,1,N+2)'; % need +2 to include x_0 and x_{N+1}

    A = -2*diag(ones(N,1)) + diag(ones(N-1,1),1) + diag(ones(N-1,1),-1);
    A(N,N-1:N) = [1 -1]; % modify last row of matrix for no-flux boundary condition
```

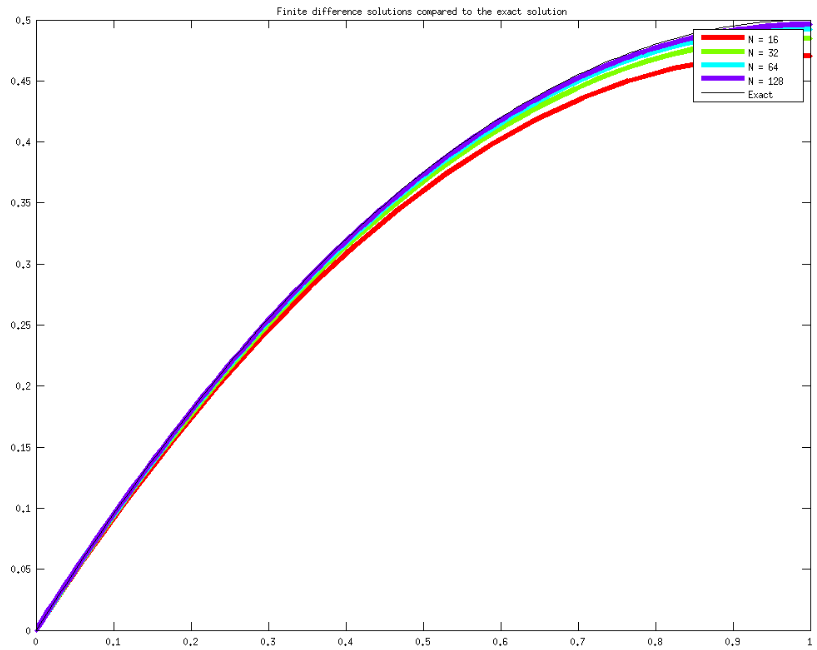

Figure 1: Finite difference solutions for various N

```
    A = -A/h^2;
    b = ones(N,1); % f(x) = 1
    u = A\b;

    figure(1)
    plot(x,[0;u;u(N)],'.-','color',C(i,:),'linewidth',3);
    hold on % append value at x(N+1) = x(N)

    figure(2)
    err = uexact(x)-[0;u;u(end)];
    plot(x,err,'o-','color',C(i,:),'linewidth',3);hold on

    i = i+1;
end
figure(1)
title('Finite difference solutions compared to the exact solution','fontsize',14)
plot(x,uexact(x),'k-')
legend('N = 16','N = 32','N = 64', 'N = 128','Exact')
print(gcf,'-dpng','p4c_sol') % print out graphs to file
figure(2)
title('Error between finite difference and exact solutions','fontsize',14)
legend('N = 16','N = 32','N = 64', 'N = 128')
print(gcf,'-dpng','p4c_error') % print out graphs to file
```

(d) If $u(x)$ is a solution, then $u(x)$ satisfies $u'(0) = u'(1) = 0$ and that

$$-u''(x) = f.$$

Then, notice that $u + C$ also satisfies the same differential equation: $(u + C)'(x) = (u' + C')(x) = u'(x)$, since C is constant. Thus, the boundary conditions are satisfied. Taking two derivatives of $(u + C)''(x) = u''(x)$ for the same reason, which implies that $u + C$ also satisfies the differential equation.

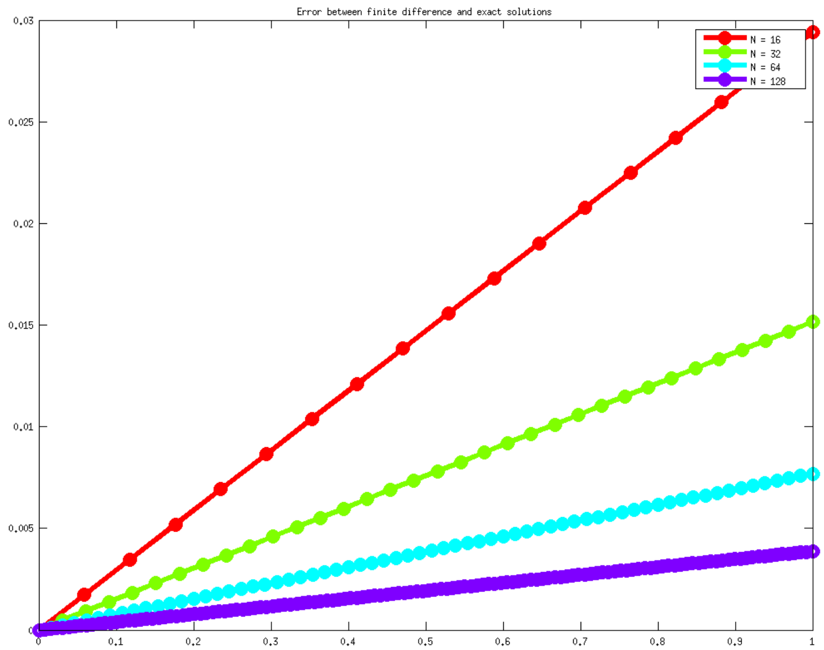

Figure 2: Error between the exact solution and finite difference solution at points x_i.