CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 42

Posted Wednesday 13 November 2013. Due 5pm Wednesday 27 November 2013.

42. [25 points] Let

$$\Omega = \{(x, y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$

and let $f(x, y, t) = (x - \frac{1}{2})^3 (y - \frac{1}{2}) e^{-t}$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \int_0^1 2 f(x, y, t) \sin(m\pi x) \sin(n\pi y)\, dx\, dy = \frac{(1 + (-1)^m)(1 + (-1)^n)(m^2\pi^2 - 24)}{8 m^3 n \pi^4} e^{-t}.$$

In this question we will consider the problem of finding the solution $u(x, y, t)$ to the heat equation

$$u_t(x, y, t) - (u_{xx}(x, y, t) + u_{yy}(x, y, t)) = f(x, y, t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

with homogeneous Dirichlet boundary conditions

$$u(x, 0, t) = u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

and initial condition

$$u(x, y, 0) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Let

$$C_D^2(\Omega) = \left\{ v \in C^2(\Omega) : v(x, 0) = v(x, 1) = v(0, y) = v(1, y) = 0,\, 0 \le x \le 1,\, 0 \le y \le 1 \right\}.$$

Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by

$$(Lv)(x, y) = -\left(v_{xx}(x, y) + v_{yy}(x, y)\right).$$

The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions

$$\psi_{j,k}(x, y) = 2 \sin(j\pi x) \sin(k\pi y)$$

for $j, k = 1, 2, \ldots$, which are such that

$$L\psi_{j,k} = \lambda_{j,k} \psi_{j,k}$$

for $j, k = 1, 2, \ldots$. Recall that in Homework 40 you obtained a formula for $\lambda_{j,k}$ for $j, k = 1, 2, \ldots$.

(a) We can write

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}(t) \psi_{j,k}(x, y)$$

and

$$f(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(t) \psi_{j,k}(x, y)$$

where

$$c_{j,k}(t) = \int_0^1 \int_0^1 f(x, y, t) \psi_{j,k}(x, y)\, dx\, dy.$$

What ordinary differential equation and initial condition does $a_{j,k}(t)$ satisfy for $j, k = 1, 2, \ldots$?

(b) Obtain an expression for $a_{j,k}(t)$ for $j, k = 1, 2, \ldots$.

(c) Use you answer to part (b) to write out a formula for $u(x, y, t)$.

(d) Plot

$$u_{15}(x, y, t) = \sum_{j=1}^{15} \sum_{k=1}^{15} a_{j.k}(t) \psi_{j,k}(x, y)$$

at the four times $t = 0, 0.005, 0.1, 2$. Use the command `zlim([-.00016 .00016])` so that the axes on all of your plots are the same. Your plot for $t = 0.1$ should resemble the plot below.