CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 12

Posted Monday December 1, 2014. Due Friday 5 December 2014, 5pm. Accepted without penalty until Monday, December 8, 5pm.

This problem set counts for 50 points, plus a bonus problem.

1. [50 points]
On Problem Set 10, you solved the heat equation on a two-dimensional square domain. Now we will investigate the wave equation on the same domain, a model of a vibrating membrane stretched over a square frame—that is, a square drum:

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t),$$

with $0 \le x \le 1$, and $0 \le y \le 1$, and $t \ge 0$. Take homogeneous Dirichlet boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$$

for all x and y such that $0 \le x \le 1$ and $0 \le y \le 1$ and all $t \ge 0$, and consider the initial conditions

$$u(x,y,0) = u_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y), \qquad u_t(x,y,0) = v_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}(0)\psi_{j,k}(x,y).$$

Here $\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$, for $j,k \ge 1$, are the eigenfunctions of the operator

$$Lu = -(u_{xx} + u_{yy}),$$

with homogeneous Dirichlet boundary conditions, as in Problem Set 10. You may use without proof that these eigenfunctions are orthogonal, and use the eigenvalues $\lambda_{j,k} = (j^2 + k^2)\pi^2$ computed for Problem Set 10.

(a) We wish to write the solution to the wave equation in the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$

Show that the coefficients $a_{j,k}(t)$ obey the ordinary differential equation

$$a''_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t)$$

with initial conditions

$$a_{j,k}(0), \qquad a'_{j,k}(0) = b_{j,k}(0)$$

derived from the initial conditions u_0 and v_0.

(b) Write down the solution to the differential equation in part (a).

(c) Use your solution to part (b) to write out a formula for the solution $u(x,y,t)$.

(d) Suppose the drum begins with zero velocity, $v_0(x,y) = 0$, and displacement

$$u_0(x,y) = 200xy(1-x)(1-y)(x-1/4)(y-1/4) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\psi_{j,k}(x,y).$$

Submit surface (or contour) plots of the solution at times $t = 0, 0.5, 1.0, 1.5, 2.5$, using $j = 1,\dots,10$ and $k = 1,\dots,10$ in the series.

2. [Bonus] Consider the heterogeneous wave equation with

$$
\begin{aligned}
\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= f(x,t) \\
u(x,0) &= \psi(x) \\
\frac{\partial u}{\partial t}(x,0) &= 0 \\
u(0,t) &= 0 \\
\frac{\partial u}{\partial x}(1,t) &= 0
\end{aligned}
$$

In seismic imaging problems (i.e. sonar, radar, finding oil, etc), the wave equation can be used to simulate a sound wave propagating in the x direction through a medium. Here, we assume $\rho(x)$ is the density of the medium, and that it only changes in the direction of propagation.

(a) Formulate a weak form for the above equation.

(b) Let $f = 0$ and $\rho(x) = 1$, which reduces to the standard wave equation with $c = 1$. For a pulse initial condition

$$\psi(x) = xe^{-100x^2}$$

compute the finite element solution (using the matrix exponential) with $N = 64$ and $dt = .015$. Create a 3D plot of the solution by using `surf` to plot the solution at equally spaced times from 0 to the final time $T = 2$. Note: If you compute your solution at points x_j and times t_i,

$$x_j = 0, h, \ldots, 1-h, 1, \qquad t_i = 0, dt, \ldots, 2$$

and form a matrix

$$U_{ij} = u(x_j, t_i)$$

then you may use `surf(X,T,U)` to compute a 3D plot of the solution, where `X` and `T` are vectors of the points x_i and times t_j. You may also wish to use the command `shading interp` to remove mesh lines from the 3D solution plot.

(c) Let $\rho(x)$ now be a discontinuous function

$$\rho(x) = \begin{cases} k_1, & x < .5 \\ k_2, & x \geq .5. \end{cases}$$

Give a formula depending on j and/or x_j for the entries of the mass matrix.

(d) Take $k_1 = .25$ and $k_2 = 1$. Compute the finite element solution using the same initial condition and N, dt as in (b). What effect does the discontinuity have on the behavior of the solution over x and t?