CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 10 · Solutions

Posted Wednesday 22 April 2015, due Friday 24 April 2015, 5pm. Accepted without late penalty until Wednesday 29 April 2015, 5pm. *This homework is extra credit.*

1. [20 points: 10 points each]
Consider the wave equation

$$u_{tt}(x,t) = u_{xx}(x,t)$$

for $0 \le x \le 1$ and $t \ge 0$ subject to the *mixed* boundary conditions

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0$$

for all $t \ge 0$ and initial conditions

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} a_n(0)\psi_n(x), \qquad u_t(x,0) = v_0(x) = \sum_{n=1}^{\infty} b_n\psi_n(x),$$

where the functions ψ_n are the eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x)$$

of the operator

$$Lu = -u_{xx}$$

with initial conditions $u(0) = u_x(1) = 0$ and eigenvalues $\lambda_n = (n-1/2)^2\pi^2$ for $n = 1, 2, \ldots$.
(Recall that you computed these eigenvalues and eigenfunctions on Problem Set 5.)

(a) We wish to write the solution to this wave equation in the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

Show that the coefficients $a_k(t)$ obey the ordinary differential equation

$$a_k''(t) = -\lambda_j a_k(t)$$

subject to the initial values $a_k(0)$ and $a_k'(0) = b_k(0)$ obtained from u_0 and v_0, and write down the solution to the differential equation in part (a).

(b) Use your solution to part (b) to write out a formula for the solution $u(x,t)$. Write a MATLAB program to compute solutions to this differential equation with initial conditions

$$u_0(x) = 0, \qquad v_0(x) = x + \sin(\pi x).$$

Submit your code, along with a surface plot showing the solution over the spatial interval $x \in [0,1]$ and the time interval $t \in [0,10]$.

Solution.

(a) If we write the solution in the form

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\phi_j(x)$$

for any $t \geq 0$, and substitute this series into the differential equation, we obtain

$$\sum_{j=1}^{\infty} \frac{d^2 a_j}{dt^2}(t)\phi_j(x) = c^2 \sum_{j=1}^{\infty} a_j(t) \frac{d^2}{dx^2}\phi_j(x).$$

Since ϕ_j is an eigenfunction of the operator L, we simply have

$$\sum_{j=1}^{\infty} \frac{d^2 a_j}{dt^2}(t)\phi_j(x) = -c^2 \sum_{j=1}^{\infty} a_j(t)\lambda_j\phi_j(x).$$

Take the inner product of both sides with ϕ_k and use the orthogonality of the eigenfunctions to obtain

$$\frac{d^2 a_j}{dt^2}(t) = -c^2\lambda_j a_j(t).$$

At time $t = 0$, we have

$$u(x,0) = \psi(x) = \sum_{n=1}^{\infty} b_n\phi_n(x), \qquad \frac{\partial u}{\partial t}(x,0) = \gamma(x) = \sum_{n=1}^{\infty} d_n\phi_n(x),$$

which we can compare with the expansion

$$u(x,0) = \sum_{j=1}^{\infty} a_j(0)\phi_j(x), \qquad \frac{\partial u}{\partial t}(x,0) = \sum_{j=1}^{\infty} \frac{da_j}{dt}(0)\phi_j(x)$$

to identify the initial conditions

$$a_j(0) = b_j, \qquad \frac{\partial u}{\partial t}(x,0) = d_j.$$

The differential equation derived in part (a) has a general solution of the form

$$a_j(t) = A\sin(c\sqrt{\lambda_j}t) + B\cos(c\sqrt{\lambda_j}t).$$

The constants A and B are determined by the boundary conditions. At $t = 0$, the above formula takes the value

$$a_j(0) = A\sin(0) + B\cos(0) = B.$$

Hence we conclude that

$$B = b_j.$$

The derivative of our formula for a_j gives

$$\frac{da_j}{dt}(0) = c\sqrt{\lambda_j}A\cos(0) + c\sqrt{\lambda_j}B\sin(0) = c\sqrt{\lambda_j}A,$$

and since we require $(da_j/dt)(0) = d_j$, we have

$$A = \frac{d_j}{c\sqrt{\lambda_j}} = \frac{d_j}{c(j-1/2)\pi}.$$

Thus we have

$$a_j(t) = \left(\frac{d_j}{c(j-1/2)\pi}\right)\sin(c\sqrt{\lambda_j}t) + b_j\cos(c\sqrt{\lambda_j}t).$$

(b) With this formula for $a_j(t)$, we can write down the full solution to the differential equation

$$u(x,t) = \sum_{j=1}^{\infty} \left[\left(\frac{d_j}{c(j-1/2)\pi} \right) \sin(c\sqrt{\lambda_j}t) + b_j \cos(c\sqrt{\lambda_j}t) \right] \left(\sqrt{2} \sin(\sqrt{\lambda_j}x) \right).$$

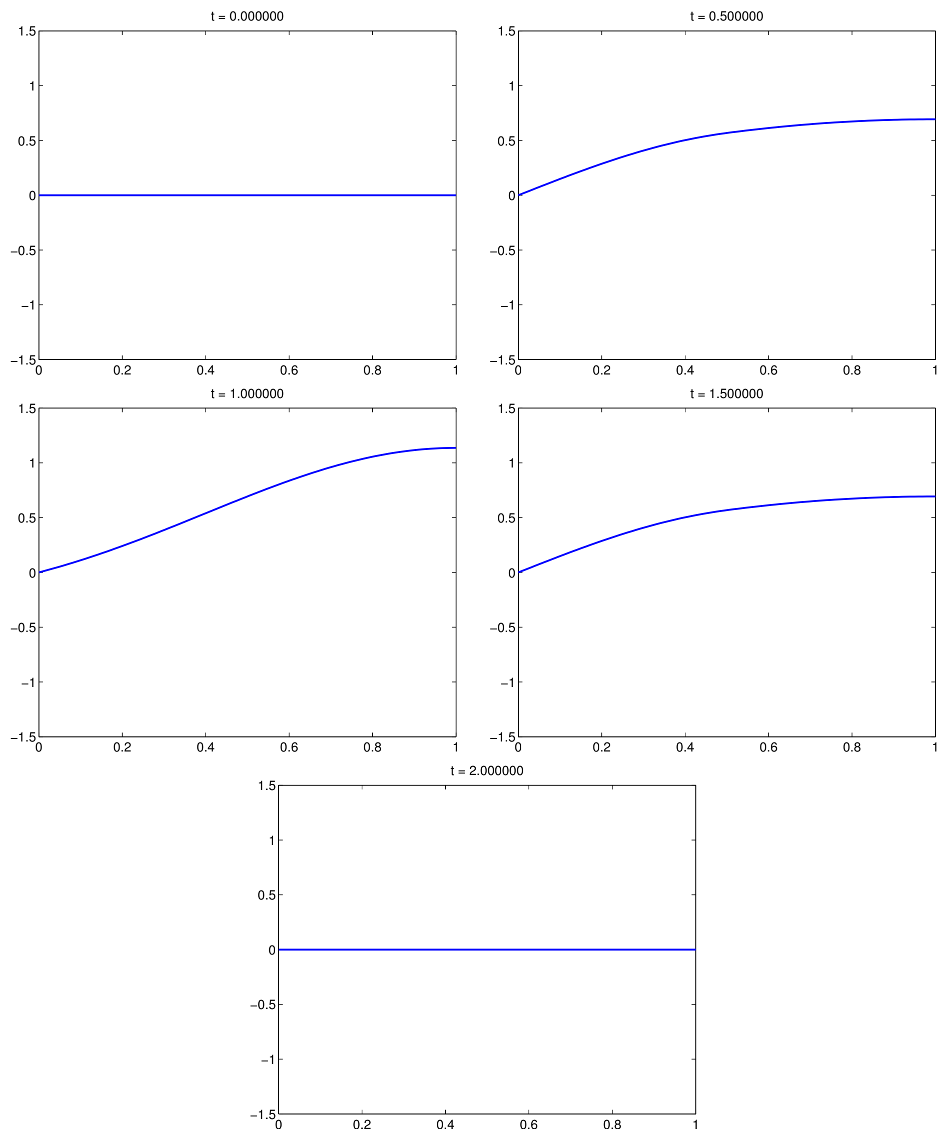
MATLAB code to solve this problem with $c = 1$ is included below. Since $u(0,t) = 0$ we have $b_j = 0$. We can use Mathematica or MATLAB to compute the d_j that give

$$\frac{\partial u}{\partial t}(0,t) = x + \sin(\pi x) = \sum_{j=1}^{\infty} d_j \phi_j(x).$$

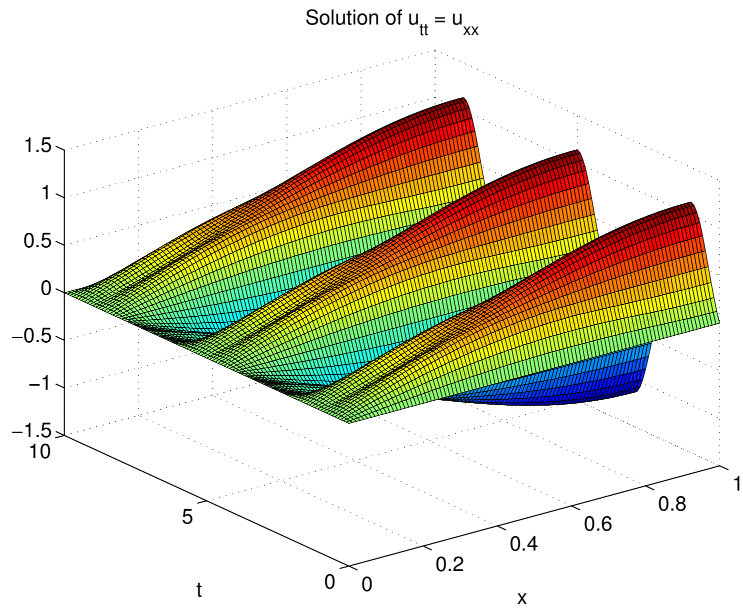
In particular, since the eigenfunctions are normalized, we have

$$\begin{aligned} d_j = (x + \sin(\pi x), \phi_j) &= \frac{4\sqrt{2}(-1)^j}{\pi^2} \left(\frac{\pi}{4j^2 - 4j - 3} - \frac{1}{(1-2j)^2} \right) \\ &= \sqrt{2}(-1)^{j+1} \left(\frac{1}{\lambda_j} + \frac{\pi}{\pi^2 - \lambda_j} \right). \end{aligned}$$

The solutions at times $t = 0, 0.5, 1.0, 1.5, 2.0$ are shown in the plots below.



These plots are rather boring, so we also show the action in three dimensions.




```
% solution of the homogeneous wave equation with mixed boundary conditions
 xx = linspace(0,1,100)';
 nmax = 100;

 t = linspace(0,10,100);
 t = 0:.1:10;
 U = zeros(length(t), length(xx));
 figure(1), clf
 for k=1:length(t)
    u = zeros(size(xx));
    for n=1:nmax
       lam = (n-.5)^2*pi^2;
       d_n = sqrt(2)*((-1).^(n+1))*(1/lam+pi/(pi^2-lam));
       phi_n = sqrt(2)*sin(sqrt(lam)*xx);
       u = u + d_n/sqrt(lam)*sin(sqrt(lam)*t(k))*phi_n;
    end
    plot(xx,u, 'b-','linewidth',2)
    U(k,:) = u;
    set(gca,'fontsize',14)
    title(sprintf('t = %f', t(k)))
    axis([0 1 -1.5 1.5])
    drawnow
    if min(abs(t(k)-[0 .5 1 1.5 2])) < 1e-8,
         eval(sprintf('print -depsc2 mixed_%d',t(k)*2));
    end
 end

 figure(2), clf
 surf(xx,t,U)
 set(gca,'fontsize',14)
 xlabel('x'), ylabel('t')
 title(sprintf('Solution of u_{tt} = u_{xx}'))
 print -depsc2 mixed_3d
```

2. [30 points: 6 points each]
Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We will look for solutions to the PDE of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

For simplicity, assume that $d \in (0,\pi)$.

(a) From the differential equation and this form for $u(x,t)$, show that the coefficients $a_k(t)$ must satisfy the ordinary differential equation

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t).$$

(b) Show that the following function satisfies the differential equation in part (a):

$$a_k(t) = C_1 \exp\Big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\Big) + C_2 \exp\Big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\Big)$$

for arbitrary constants C_1 and C_2. (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

(c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{k=1}^{\infty} b_k(0)\psi(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

(d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x\sin(3\pi x)$, for which

$$b_k(0) = \frac{-6k\sqrt{2}(1 + (-1)^k)}{(k^2-9)^2\pi^2} \quad \text{for } k \neq 3, \qquad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x,t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

(e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d?

Solution.

(a) Follow the usual methodology: Substitute the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

into the differential equation $u_{tt} = u_{xx} - 2du_t$ to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} a_n''(t)\psi_n(x) &= \sum_{n=1}^{\infty} a_n(t)\psi_n''(x) - 2d\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) \\ &= -\sum_{n=1}^{\infty} \lambda_n a_n(t)\psi_n(x) - 2d\sum_{n=1}^{\infty} a_n'(t)\psi_n(x).\end{aligned}$$

Take the inner product with the eigenfunction ψ_k and use orthogonality of the eigenfunctions to obtain

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t)$$

as required.

(b) We first compute two derivatives of the proposed formula for a_k:

$$\begin{aligned}a_k'(t) &= C_1\big(-d + \sqrt{d^2 - k^2\pi^2}\big)\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\big(-d - \sqrt{d^2 - k^2\pi^2}\big)\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big)\end{aligned}$$

$$\begin{aligned}a_k''(t) &= C_1\big(-d - \sqrt{d^2 - k^2\pi^2}\big)^2\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\big(-d + \sqrt{d^2 - k^2\pi^2}\big)^2\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big).\end{aligned}$$

We wish to verify that $a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t)$, where $\lambda_k = k^2\pi^2$. We can see that

$$\begin{aligned}-\lambda_k a_k(t) - 2da_k'(t) &= C_1\Big(-\lambda_k - 2d\big(-d + \sqrt{d^2 - k^2\pi^2}\big)\Big)\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\Big(-\lambda_k - 2d\big(-d - \sqrt{d^2 - k^2\pi^2}\big)\Big)\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &= C_1\big(-k^2\pi^2 + 2d^2 - 2d\sqrt{d^2 - k^2\pi^2}\big)\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\big(-k^2\pi^2 + 2d^2 + 2d\sqrt{d^2 - k^2\pi^2}\big)\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &= C_1\big(-d + \sqrt{d^2 - k^2\pi^2}\big)^2\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\big(-d - \sqrt{d^2 - k^2\pi^2}\big)^2\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big).\end{aligned}$$

This final formula agrees with the formula for $a_k''(t)$ we computed earlier, and thus we have confirmed that this is a general solution for our differential equation.

(c) We need to now compute C_1 and C_2 so that $a_k(0) = 0$ and $a_k'(0) = b_k(0)$. At $t = 0$, the general solution becomes

$$a_k(0) = C_1\exp(0) + C_2\varepsilon(0) = C_1 + C_2,$$

so $a_k(0) = 0$ requires that

$$C_1 = -C_2.$$

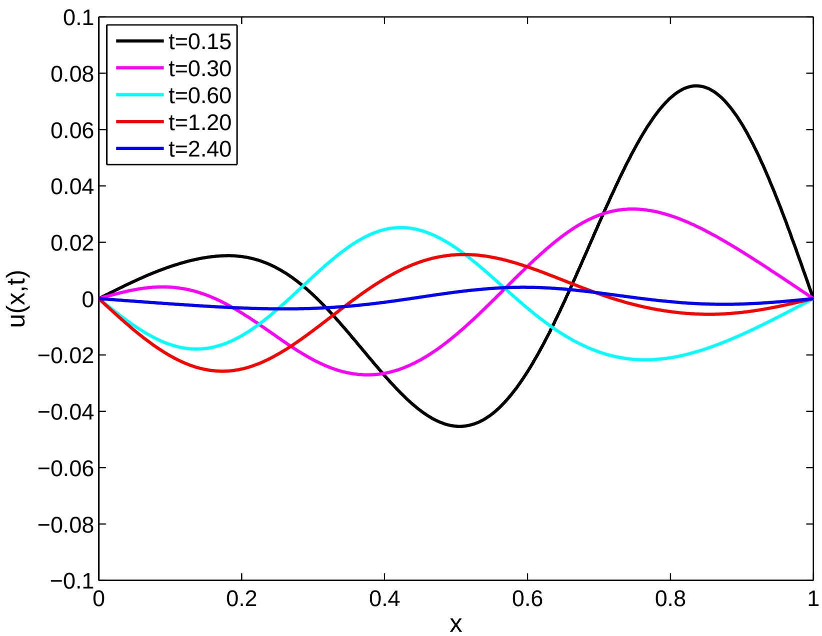
Taking the formula for $a_k'(t)$ in part (b) and evaluating at $t = 0$ gives

$$a_k'(0) = C_1\big(-d + \sqrt{d^2 - k^2\pi^2}\big) + C_2\big(-d - \sqrt{d^2 - k^2\pi^2}\big).$$

So with $C_1 = -C2$ and $a_k'(0) = b_k(0)$, we arrive at

$$C_1 = -C_2 = \frac{b_k(0)}{2\sqrt{d^2 - k^2\pi^2}}.$$

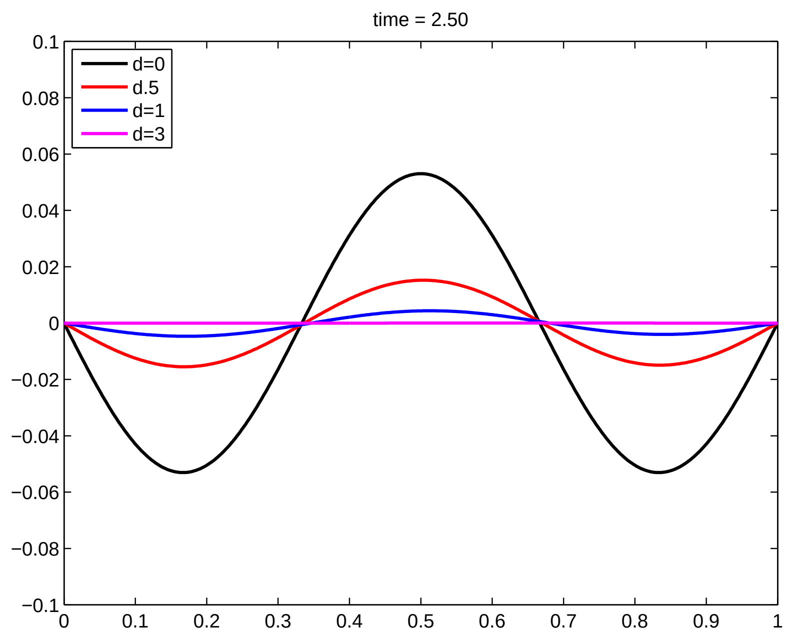
(d) The requested solutions, varying in t with fixed d, are collected in the plot below.




```
tvec = [.15 .30 .60 1.20 2.40];
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

d = 1;

col = 'kmcrb';
figure(1), clf
for m=1:length(tvec)
   t = tvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
```

```
   plot(xx,u,'-','linewidth',2,'color',col(m)), hold on
   ylim([-.1 .1])
   pause
end
legend('t=0.15','t=0.30','t=0.60','t=1.20','t=2.40',2)
set(gca,'fontsize',14)
xlabel('x','fontsize',16)
ylabel('u(x,t)','fontsize',16)
print -depsc2 damp1.eps
```

(e) The requested solutions, now varying in d with fixed t, are collected in the plot below. As the damping parameter increases on $(0, \pi)$, the solution gets increasingly smaller in amplitude at this time.




```
t = 2.5;
dvec = [0 .5 1 3];
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

figure(1), clf
cvec = 'krbm';
for m=1:length(dvec)
   d = dvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
   plot(xx,u,'-','linewidth',2,'color',cvec(m)), hold on
```

```
   ylim([-.1 .1])
   title(sprintf('time = %3.2f', t))
end
legend('d=0','d=.5','d=1','d=3',2)
print -depsc2 damp2.eps
```