CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 2 · Solutions

Posted Wednesday 29 August 2012. Due Wednesday 5 September 2012, 5pm.

1. [24 points: 4 points per part]
   Consider the following sets of functions. Demonstrate whether or not each is a vector space (with addition and scalar multiplication defined in the obvious way).

   (a) $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^3\}$

   (b) $\{\mathbf{x} \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

   (c) $\{f \in C[0,1] : f(x) \geq 0 \text{ for all } x \in [0,1]\}$

   (d) $\{f \in C[0,1] : \max_{x\in[0,1]} f(x) \leq 1\}$

   (e) $\{f \in C^1[0,1] : f'(0) = 0\}$

   (f) $\{f \in C^2[0,1] : f''(x) = 0 \text{ for all } x \in [0,1]\}$

Solution.

(a) This set *is not* a vector space.
The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set, yet $2\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not, since $2 \neq 2^3 = 8$.

(b) This set *is* a vector space.
Suppose $\mathbf{x}$ and $\mathbf{y}$ are members of this set. Then $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$. Adding these two equations gives

$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = 0,$$

and hence $\mathbf{x} + \mathbf{y}$ is also in the set. Multiplying $x_1 + 2x_2 + 3x_3 = 0$ by an arbitrary constant $\alpha \in \mathbb{R}$ gives

$$(\alpha x_1) + 2(\alpha x_2) + 3(\alpha x_3) = 0,$$

and hence $\alpha\mathbf{x}$ is also in the set.

(c) This set *is not* a vector space.
The function $f(x) = 1$ for all x is in the set, but a scalar multiple, $-1 \cdot f(x) = -1$ for all x, takes negative values and thus violates the requirement for membership in the set.

(d) This set *is not* a vector space.
The function $f(x) = 1$ for all x is in the set, but a scalar multiple, $2 \cdot f(x) = 2$ for all x, takes values greater than 1 and thus violates the requirement for membership in the set.

(e) This set *is* vector space.
If f and g are in the set, then $f'(0) = g'(0) = 0$, so

$$\frac{d(f+g)}{dx}(0) = f'(0) + g'(0) = 0 + 0 = 0.$$

Similarly, for any scalar α,

$$\frac{d(\alpha f)}{dx}(0) = \alpha f'(0) = \alpha \cdot 0 = 0.$$

(f) This set *is* a vector space.
Suppose f and g are both members of this set. Then

$$\frac{d^2(f+g)}{dx^2} = \frac{d^2 f}{dx^2} + \frac{d^2 g}{dx^2} = 0 + 0 = 0,$$

and thus $f+g$ is in the set. For any $\alpha \in \mathbb{R}$ note that

$$\frac{d^2(\alpha f)}{dx^2} = \alpha \frac{d^2 f}{dx^2} = \alpha \cdot 0 = 0,$$

and hence αf is also in the set.
Said another way: any member of the set must be a function of the form $f(x) = \beta + \gamma x$, i.e., a line. The addition of two lines is still a line, and a scalar multiple of a line is also a line.

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2. [14 points: 8 points for (a); 6 points for (b)]

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since f is linear, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

(b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

(Thus any linear function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix-vector product.)

Solution.

(a) Write $\mathbf{u} \in \mathbb{R}^2$ in the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Any matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ are the columns of $\mathbf{A}$. Now the matrix-vector product $\mathbf{A}\mathbf{u}$ is a linear combination of the columns of $\mathbf{A}$:

$$\mathbf{A}\mathbf{u} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2. \tag{$*$}$$

We are trying to find a formula for $\mathbf{A}$ (or, equivalently, for $\mathbf{a}_1$ and $\mathbf{a}_2$) such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Using the hint, we have

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2). \tag{$**$}$$

Comparing $(*)$ and $(**)$ and equating like terms, we see that

$$\mathbf{a}_1 = f(\mathbf{e}_1), \qquad \mathbf{a}_2 = f(\mathbf{e}_2),$$

and hence

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) \end{bmatrix}.$$

(b) Follow the same tack as in part (a). Write $\mathbf{u} \in \mathbb{R}^n$ as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{A} \in \mathbb{R}^{m\times n}$ by column,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. Equating like terms in

$$\mathbf{A}\mathbf{u} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots u_n\mathbf{a}_n$$

and

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2) + \cdots u_n f(\mathbf{e}_n),$$

we arrive at

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \cdots & f(\mathbf{e}_n) \end{bmatrix}.$$

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3. [24 points: 4 points per part]
Recall that a function $f : \mathcal{V} \to \mathcal{W}$ that maps a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a *linear operator* provided (1) $f(u + v) = f(u) + f(v)$ for all u, v in $\mathcal{V}$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$.

Demonstrate whether each of the following functions is a linear operator.
(Show that both properties hold, or give an example showing that one of the properties must fail.)

(a) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$.
(b) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.
(c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.
(d) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.
(e) $L : C^1[0,1] \to C[0,1]$, $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.
(f) $L : C^2[0,1] \to C[0,1]$, $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

Solution.

(a) This function *is* a linear operator.
Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} = f(\mathbf{u}) + f(\mathbf{v}),$$

and

$$f(\alpha\mathbf{v}) = \mathbf{A}(\alpha\mathbf{v}) = \alpha\mathbf{A}\mathbf{v} = \alpha f(\mathbf{v}).$$

(b) This function *is not* a linear operator (provided $\mathbf{b} \neq \mathbf{0}$).
Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}\mathbf{b} = f(\mathbf{u}) + \mathbf{A}\mathbf{v},$$

which does not equal $f(\mathbf{u}) + f(\mathbf{v})$ when $\mathbf{b} \neq \mathbf{0}$.
[**GRADERS**: please *do not* take off points if the solutions fail to note the special case of $\mathbf{b} = \mathbf{0}$.]

(c) This function *is not* a linear operator.
Suppose $\mathbf{x} \in \mathbb{R}^n$. Then
$$f(\alpha\mathbf{x}) = (\alpha\mathbf{x})^T(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^T\mathbf{x} = \alpha^2 f(\mathbf{x}),$$
and thus if $\alpha \neq \pm 1$, we have $f(\alpha\mathbf{x}) \neq \alpha f(\mathbf{x})$.

(d) This function *is* a linear operator.
Suppose $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n\times n}$. Then
$$f(\mathbf{X}+\mathbf{Y}) = \mathbf{A}(\mathbf{X}+\mathbf{Y}) + (\mathbf{X}+\mathbf{Y})\mathbf{B} = \mathbf{AX}+\mathbf{XB}+\mathbf{AY}+\mathbf{YB} = f(\mathbf{X})+f(\mathbf{Y}),$$
and if $\alpha \in \mathbb{R}$, then
$$f(\alpha\mathbf{X}) = \mathbf{A}(\alpha\mathbf{X}) + (\alpha\mathbf{X})\mathbf{B} = \alpha(\mathbf{AX}+\mathbf{XB}) = \alpha f(\mathbf{X}).$$

(e) This function *is not* a linear operator.
Suppose that $u(x) = x$. Then
$$Lu = u\frac{du}{dx} = x\cdot 1 = x,$$
yet for any $\alpha \in \mathbb{R}$ we have
$$L(\alpha u) = (\alpha u)\frac{d(\alpha u)}{dx} = (\alpha x)\cdot\alpha = \alpha^2 x,$$
so if $\alpha \neq \pm 1$, we have $L(\alpha u) \neq \alpha Lu$.

(f) This function *is* a linear operator.
Suppose that $u, v \in C^2[0,1]$. Then
$$\begin{aligned} L(u+v) &= \frac{d^2(u+v)}{dx^2} - \sin(x)\frac{d(u+v)}{dx} + \cos(x)(u+v) \\ &= \frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u + \frac{d^2v}{dx^2} - \sin(x)\frac{dv}{dx} + \cos(x)v \\ &= Lu + Lv, \end{aligned}$$
and for any $\alpha \in \mathbb{R}$,
$$L(\alpha u) = \frac{d^2(\alpha u)}{dx^2} - \sin(x)\frac{d(\alpha u)}{dx} + \cos(x)(\alpha u) = \alpha\Big(\frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u\Big) = \alpha L(u).$$

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4. [12 points: give partial credit for the proof]
Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and suppose $f : \mathcal{V} \to \mathcal{W}$ is a linear operator.

The *range* of f is the set of all vectors in $\mathcal{W}$ that can be written in the form $f(v)$ for some $v \in \mathcal{V}$:
$$\mathcal{R}(f) = \{f(v) : v \in \mathcal{V}\}.$$
Show that $\mathcal{R}(f)$ is a subspace of $\mathcal{W}$.

(The *range* generalizes the notion of *column space* from matrix theory.)

Solution. To show that $\mathcal{R}(f)$ is a subspace of $\mathcal{W}$, we must check that (1) If $w_1, w_2 \in \mathcal{R}(f)$, then $w_1 + w_2 \in \mathcal{R}(f)$ and (2) If $w \in \mathcal{R}(f)$ and $\alpha \in \mathbb{R}$, then $\alpha w \in \mathcal{R}(f)$.

(a) Suppose $w_1, w_2 \in \mathcal{R}(f)$. To be in $\mathcal{R}(f)$, we must find som $v \in V$ such that $f(v) = w_1 + w_2$. Since w_1 and w_2 are in $\mathcal{R}(f)$, there exist $v_1, v_2 \in V$ such that $f(v_1) = w_1$ and $f(v_2) = w_2$. Define $v = v_1 + v_2$. Since f is a linear operator, $f(v) = f(v_1 + v_2) = f(v_1) + f(v_2) = w_1 + w_2$. Hence, $w_1 + w_2 \in \mathcal{R}(f)$.

(b) Suppose $w_1 \in \mathcal{R}(f)$ and $\alpha \in \mathbb{R}$. Since $w_1 \in \mathcal{R}(f)$, there exists some $v_1 \in V$ such that $f(v_1) = w_1$. Define $v = \alpha v_1$. Since f is a linear operator, we have $f(\alpha v_1) = \alpha f(v_1) = \alpha w_1$, so $\alpha w_1 \in \mathcal{R}(f)$.

[**GRADERS:** The book also stipulates that a subspace must contain the zero vector. Some students will verify that linearity of f implies that $f(0) = 0$, and hence $0 \in \mathcal{R}(f)$. This point was not emphasized in the lectures, so please do mark off points if students omitted it.]

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5. [26 points: 10 points each for (a) and (b); 6 points for (c); 4 bonus points]

(a) In class we considered the 'forward difference' approximation

$$u'(x) \approx \frac{u(x+h) - u(x)}{h}.$$

Let $u(x) = \exp(2x)$. For each value $N = 2, 4, 8, 16, \ldots, 512$ (powers of 2), compute (in MATLAB) the error

$$\left| u'(1/2) - \frac{u(1/2+h) - u(1/2)}{h} \right|,$$

where $h = 1/(N+1)$. Print out these errors, and use MATLAB's `loglog` command to produce a plot of N versus the corresponding error. (In class, we showed that this error should be proportional to h as $h \to 0$.)

(b) Consider the 'centered difference' approximation

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}.$$

Repeat part (a) with this approximation: That is, for $u(x) = \exp(2x)$, compute the error

$$\left| u'(1/2) - \frac{u(1/2+h) - u(1/2-h)}{2h} \right|$$

for $N = 2, 4, 8, 16, 32, \ldots, 512$ (powers of 2) with $h = 1/(N+1)$. Print out these errors, and use MATLAB's `loglog` command to produce a plot of N versus the corresponding error. (In class, we showed that this error should be proportional to h^2 as $h \to 0$.)

Use the `hold on` *command to superimpose the plot for (b) on your plot for part (a): you should only turn in one plot for this problem.*

(c) By inspecting the plot you have created, estimate the value of N that you need to approximate $u'(1/2)$ to an error of 10^{-2} using the methods in part (a) and part (b).

Challenge problem (4 bonus points): Given an integer $N \geq 1$, define $h = 1/(N+1)$ and consider the grid of points $x_j = jh$ for $j = 0, \ldots, N+1$. It is often desirable to construct an approximation to $u'(x_0)$ whose accuracy is proportional to h^2 as $h \to 0$. The centered difference in part (b) above is unsuitable, as it would require a value $u(x_{-1}) = u(-h)$, and $-h$ is outside the domain $[0, 1]$. Show an alternative way to approximate $u'(x_0)$ using only the values $u(x_0)$, $u(x_1)$, and $u(x_2)$, i.e., find coefficients α, β, and γ such that

$$u'(x_0) = \alpha u(x_0) + \beta u(x_1) + \gamma u(x_2) + O(h^2).$$

Solution.

(a) The error is shown in the table below. The plot is shown with the solution to part (c). The code that generated this plot follows at the end of the problem.

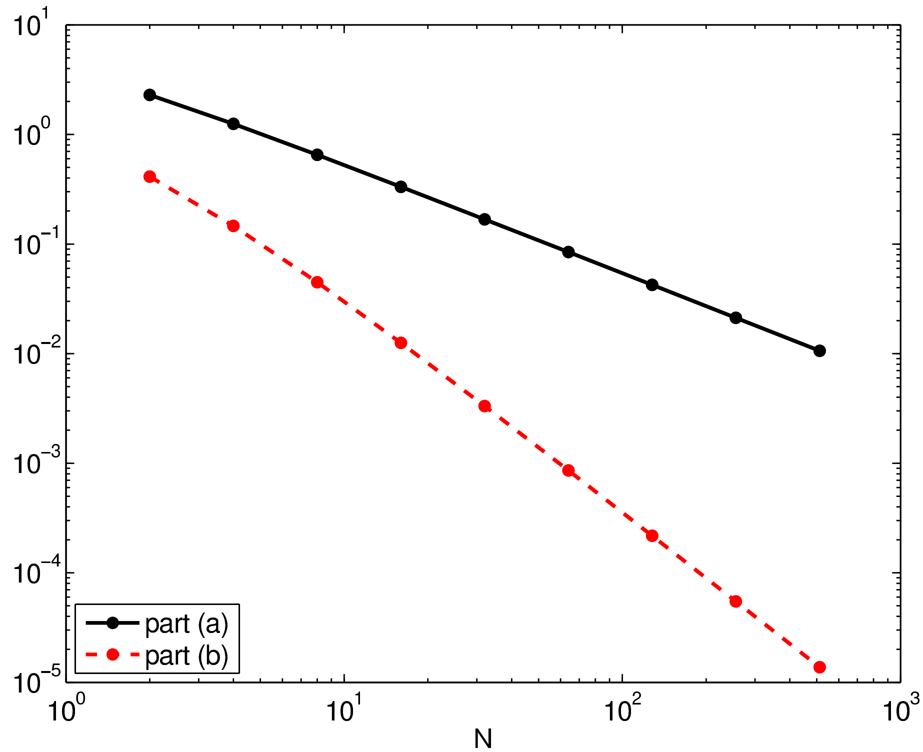
N	error
2	2.2920610
4	1.2480270
8	0.6514086
16	0.3327167
32	0.1681236
64	0.0845039
128	0.0423625
256	0.0212089
512	0.0106114

(b) The $O(h^2)$ centered difference approximation gives

N	error
2	0.4117528
4	0.1461393
8	0.0448560
16	0.0125498
32	0.0033288
64	0.0008579
128	0.0002178
256	0.0000549
512	0.0000138

These errors decay much more rapidly than for the analogous expansion in part (a). This is made clear by the plot below, generated by the following code.

(c) Roughly speaking, the forward difference requires $N \approx 512$ before it is accurate to two digits, while the centered difference only requires $N \approx 16$. (When used in the context of solving differential equations, the improved accuracy of the centered difference formula allows one to work with smaller matrices than required for the forward difference formula, potentially delivering a great speed-up in run-time.)




```
u = inline('exp(2*x)');
uprime = inline('2*exp(2*x)');

Nvec = 2.^[1:9]';
err = zeros(size(Nvec));
x = 1/2;
fprintf('\n part (a)\n')
for k=1:length(Nvec)
   N = Nvec(k);
   h = 1/(N+1);
   deriv = (u(x+h)-u(x))/h;
   err(k) = abs(uprime(x)-deriv);
   fprintf(' %3d   %10.7f\n', N, err(k));
end
loglog(Nvec,err,'k.-','linewidth',2,'markersize',20)

fprintf('\n part (b)\n')
for k=1:length(Nvec)
   N = Nvec(k);
   h = 1/(N+1);
   deriv = (u(x+h)-u(x-h))/(2*h);
   err(k) = abs(uprime(x)-deriv);
   fprintf(' %3d   %10.7f\n', N, err(k));
end
hold on
loglog(Nvec,err,'r--','linewidth',2,'marker','.','markersize',20)
set(gca,'fontsize',14)
xlabel('N', 'fontsize',14)
legend('part (a)','part (b)',3)
print -depsc2 findiff.eps
```

Bonus problem.

[**GRADERS**: please only award bonus credit for students who get this problem entirely correct: no partial credit on the bonus.]

By manipulation of Taylor series, one arrives at the expansion

$$u'(x_0) = \frac{-3u(x_0) + 4u(x_1) - u(x_2)}{2h} + O(h^2),$$

i.e., $\alpha = -3/(2h)$, $\beta = 2/h$, and $\gamma = -1/(2h)$.

On the other end of the domain, one has

$$u'(x_{N+1}) = \frac{u(x_{N-1}) - 4u(x_N) + 3u(x_{N+1})}{2h} + O(h^2).$$
