CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 11 · Solutions

Posted Thursday 18 November 2010. Due Tuesday 23 November 2010, 5pm.

This problem set counts for 75 points.
Late problem sets are due by 5pm on Wednesday 24 November 2010.

1. [30 points: 7 points each for (a), (b), (c); 9 points for (d)]
This question concerns the homogeneous wave equation on an unbounded spatial domain:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty, \quad t > 0.$$

Find the solution $u(x,t)$ to this equation with the following initial conditions:

(a) $u(x,0) = 2\sin(x)e^{-x^2}, \quad u_t(x,0) = 0;$

(b) $u(x,0) = 0, \qquad u_t(x,0) = -\dfrac{2x}{(1+x^2)^2};$

(c) $u(x,0) = 2\sin(x)e^{-x^2}, \quad u_t(x,0) = -\dfrac{2x}{(1+x^2)^2}.$

(d) Produce a plot (or plots) showing your solution to part (c) over $-10 \le x \le 10$ at times $t = 0, 1, 2, 3, 4, 5$.

Solution.

(a) D'Alembert's solution takes the form

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) \\ &= \sin(x-t)e^{-(x-t)^2} + \sin(x+t)e^{-(x+t)^2}. \end{aligned}$$

(b) D'Alembert's solution is now

$$u(x,t) = \tfrac{1}{2}\int_{x-t}^{x+t} \gamma(s)\,ds.$$

We can compute

$$\int -\frac{2x}{(1+x^2)^2}\,dx = \frac{1}{1+x^2} + \text{constant},$$

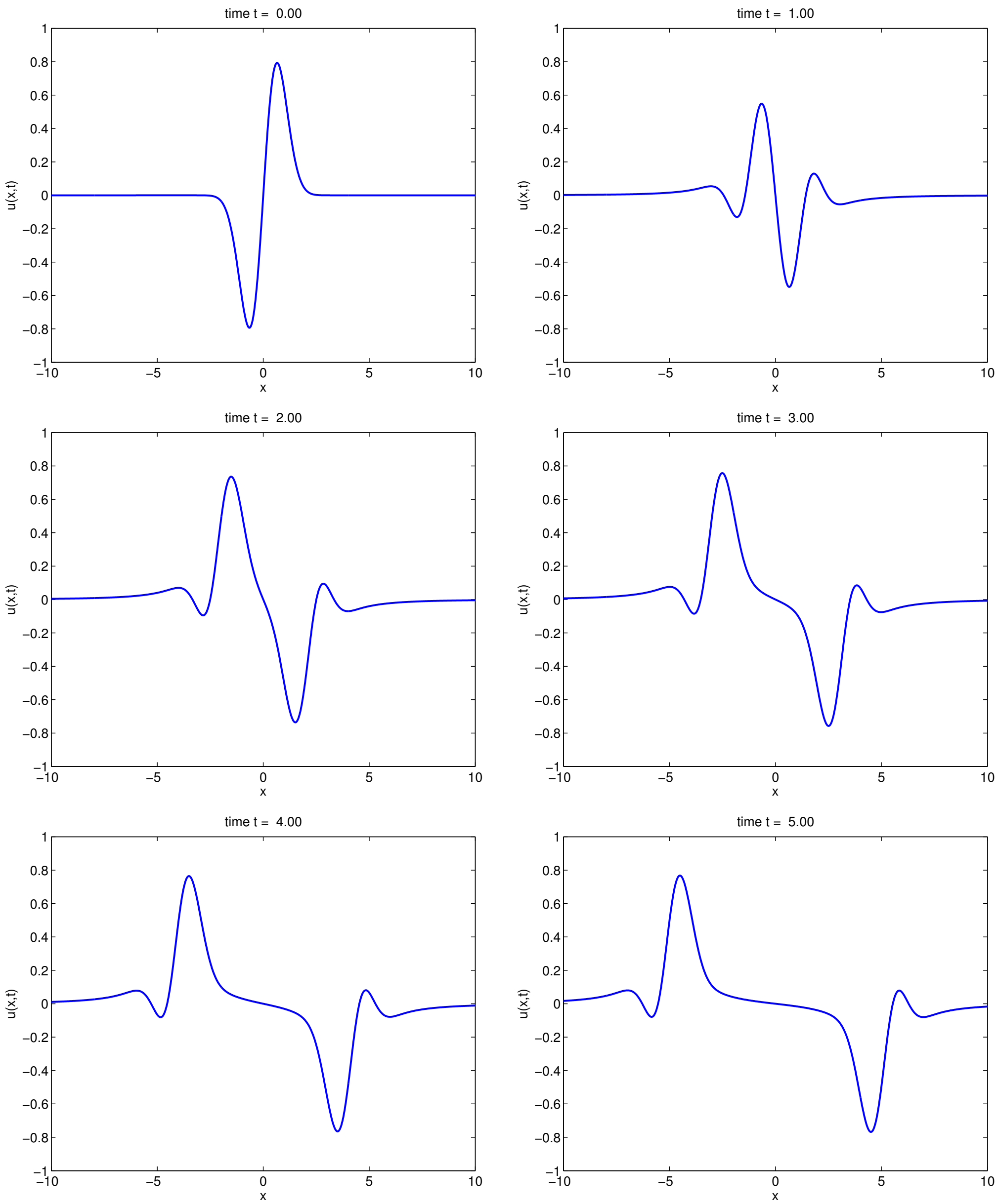
and so

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) \\ &= \tfrac{1}{2}\Big(\frac{1}{1+(x+t)^2} - \frac{1}{1+(x-t)^2}\Big). \end{aligned}$$

(c) When the nonzero boundary conditions from (a) and (b) are combined, we simply sum the solutions to the two previous parts:

$$u(x,t) = \sin(x-t)e^{-(x-t)^2} + \sin(x+t)e^{-(x+t)^2} + \tfrac{1}{2}\Big(\frac{1}{1+(x+t)^2} - \frac{1}{1+(x-t)^2}\Big).$$

(d) The following plots show the solution in part (c) at times $t = 0, 1, 2, 3, 4, 5$, with the code that produced the plots following.




```
x = linspace(-10,10,1000);
t = 0:.05:5;
figure(1), clf
for k=1:length(t)
   plot(x, sin(x-t(k)).*exp(-(x-t(k)).^2)...
                +sin(x+t(k)).*exp(-(x+t(k)).^2) ...
          +.5*(1./(1+(x+t(k)).^2) - 1./(1+(x-t(k)).^2)), 'b-','linewidth',2)
   axis([-10 10 -1 1])
   set(gca,'fontsize',14)
   xlabel('x'), ylabel('u(x,t)')
   title(sprintf('time t = %5.2f', t(k)))
   if min(abs(t(k)-[0:1:5])) < 1e-10
      eval(sprintf('print -depsc2 dalembert%d', round(t(k))))
   end
   pause
end
```

2. [10 points]
Consider the wave equation on an infinite spatial domain, $x \in (-\infty, \infty)$, but now with a forcing term:

$$u_{tt}(x,t) = u_{xx}(x,t) + f(t), \qquad x \in (-\infty, \infty)$$

subject again to the initial data

$$u(x,0) = \psi(x), \qquad u_t(x,0) = \gamma(x).$$

Verify that the formula

$$u(x,t) = \widehat{u}(x,t) + t\int_0^t f(\tau)\,\mathrm{d}\tau - \int_0^t \tau f(\tau)\,\mathrm{d}\tau$$

is a solution to this inhomogeneous wave equation, where $\widehat{u}(x,t)$ denotes the solution of the homogeneous problem with the same initial data.

Solution. To verify the formula, we begin by taking temporal derivatives of the proposed solution:

$$\begin{aligned} u_t(x,t) &= \widehat{u}_t(x,t) + \Big(\frac{\mathrm{d}}{\mathrm{d}t}t\Big)\int_0^t f(\tau)\,\mathrm{d}\tau + t\Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)\,\mathrm{d}\tau\Big) - \Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \tau f(\tau)\,\mathrm{d}\tau\Big) \\ &= \widehat{u}_t(x,t) + \int_0^t f(\tau)\,\mathrm{d}\tau + tf(t) - tf(t) \\ &= \widehat{u}_t(x,t) + \int_0^t f(\tau)\,\mathrm{d}\tau \end{aligned}$$

$$\begin{aligned} u_{tt}(x,t) &= \widehat{u}_{tt}(x,t) + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)\,\mathrm{d}\tau\Big) \\ &= \widehat{u}_{tt}(x,t) + f(t). \end{aligned}$$

Now since $u_{xx}(x,t) = \widehat{u}_{xx}(x,t)$, so we can confirm that

$$u_{tt}(x,t) = \widehat{u}_{tt}(x,t) + f(t) = u_{xx}(x,t) + f(t),$$

as required.

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3. [35 points: 5 points each for (a) and (b); 10 points for (c); 15 points for (d)]
Consider the wave equation posed on the infinite domain $x \in (-\infty, \infty)$:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty, \quad t > 0 \tag{$*$}$$

with initial conditions $u(x,0) = \psi(x)$ and $u_t(x,0) = \gamma(x)$.

At a given point $(\widetilde{x}, \widetilde{t})$, with $\widetilde{x} \in (-\infty, \infty)$ and $\widetilde{t} > 0$, the solution $u(\widetilde{x}, \widetilde{t})$ of the wave equation is only affected by some portion of the initial data. In other words, $u(\widetilde{x}, \widetilde{t})$ is only influenced by $\psi(x)$ and $\gamma(x)$ for $x \in [a,b]$, where a and b will depend upon $\widetilde{x}$ and $\widetilde{t}$. This interval $[a,b]$ is called the *domain of dependence* of the solution at $(\widetilde{x}, \widetilde{t})$.

(a) Determine the domain of dependence of the solution to the wave equation $(*)$ at $(\widetilde{x}, \widetilde{t}) = (0,1)$.

Now consider the heat equation on an unbounded domain:

$$u_t(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty$$

with initial data

$$u(x,0) = \psi(x).$$

Like d'Alembert's solution, there exists a formula for the solution of the heat equation on this domain: for all $t > 0$,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \psi(s)\, \mathrm{d}s.$$

(b) What is the domain of dependence of this solution to the heat equation at $(\widetilde{x}, \widetilde{t}) = (0,1)$? Contrast the physical implications of the domains of dependence for the heat and wave equations.

(c) Consider the *wave* equation with discontinuous initial data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0; \end{cases} \qquad \gamma(x) = 0.$$

On one plot, superimpose solutions to this equation at the four times $t = 0, 1/2, 1, 2$. (Notice how the discontinuity in the initial data is propagated in time.)

(d) Now consider the *heat* equation with the same starting data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

Using the formula for $u(x,t)$ given above, produce solutions to this equation at the four times $t = 0, 0.01, 0.1, 1$. What happens to the discontinuity for $t > 0$?

Important hint: You will need to compute some nasty integrals here that you cannot work out entirely by hand. To produce your plots, use MATLAB's `erfc` command. For example,

$$\frac{2}{\sqrt{\pi}} \int_{\mathtt{z}}^{\infty} e^{-y^2}\, \mathrm{d}y = \mathtt{erfc(z)}.$$

Solution.

(a) Recall that the general solution of the wave equation on an unbounded domain takes the form

$$u(x,t) = \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) + \tfrac{1}{2}\int_{x-t}^{x+t} \gamma(s)\, ds.$$

From this it follows that

$$u(0,1) = \tfrac{1}{2}\big(\psi(-1) + \psi(1)\big) + \tfrac{1}{2}\int_{-1}^{1} \gamma(s)\, ds.$$

Since the solution at $x = 0$ and $t = 1$ depends on $\gamma(s)$ for all $s \in [-1,1]$ and ϕ at $x = \pm 1 \in [-1, 1,$ we note that the domain of dependence is the interval $[-1,1]$.

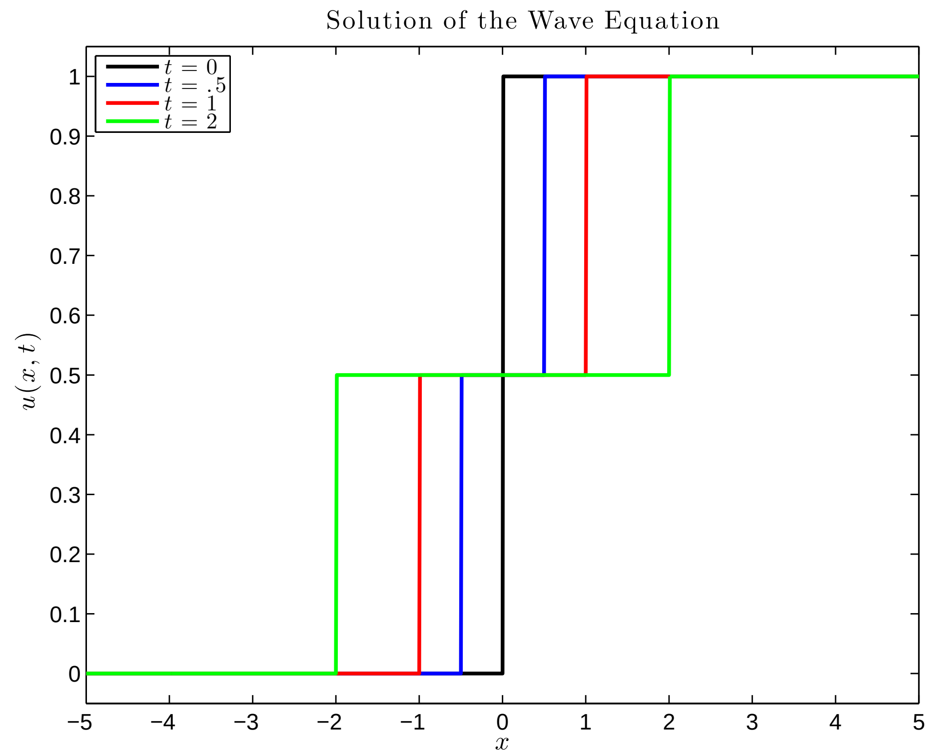
(b) Since the solution to the heat equation in an unbounded domain depends on $\phi(s)$ for all values of $s \in (-\infty, \infty)$, the domain of dependence at $x = 0$ and $t = 1$ consists of the entire real line, $(-\infty, \infty)$.

[Graders: please accept a variety of reasonable solutions to this problem.] For the wave equation, the initial condition takes a finite time to propagate: for example, it takes one full time unit for the value of $\psi(\pm 1)$ to affect the solution u at the point $x = 0$. In contrast, in the heat equation, with its domain of dependence of $(-\infty, \infty)$, the value of the initial condition at any single point *instantaneously* affects the solution at all other points. So, the initial distribution of heat instantly affects the heat at all other points.

(c) Since $\gamma(x) = 0$ for all x, the solution to the wave equation is simply

$$u(x,t) = \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) = \begin{cases} 0, & x < -t; \\ 1/2 & -t \le x < t; \\ 1, & x > t. \end{cases}$$

The requested plot of the solution is shown below; code follows at the end of the problem.



(d) For the specified initial condition, the solution takes the form

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \psi(s)\,\mathrm{d}s = \frac{1}{2\sqrt{\pi t}} \int_{0}^{\infty} e^{\frac{-(s-x)^2}{4t}}\,\mathrm{d}s.$$

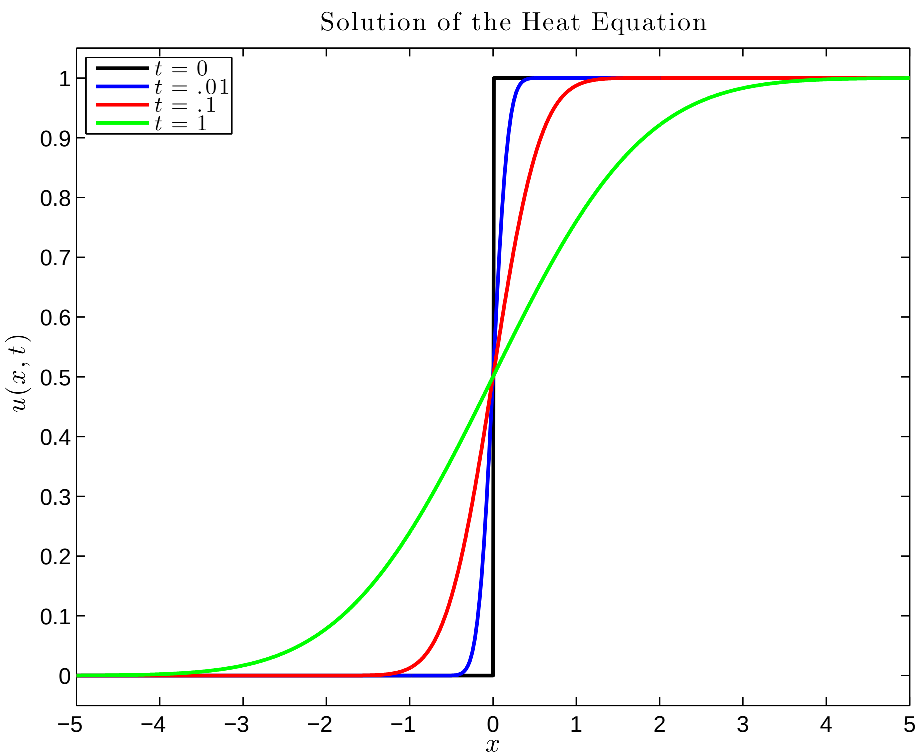
To compute the integral, use the substitution $y = (s-x)/(2\sqrt{t})$, so that $\mathrm{d}y = 1/(2\sqrt{t})\,\mathrm{d}s$ to compute

$$\int_0^{\infty} e^{\frac{-(s-x)^2}{4t}}\,\mathrm{d}s = 2\sqrt{t}\int_{-x/(2\sqrt{t})}^{\infty} e^{-y^2}\,\mathrm{d}y.$$

Hence

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2\sqrt{t})}^{\infty} e^{-y^2}\mathrm{d}y = \frac{1}{2}\texttt{erfc}(-x/(2\sqrt{t})).$$

The requested plot of the solution is shown below; code follows at the end of the problem.




```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initial condition
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 x = linspace(-5,5,1001);
 psi = x>0;

 col = 'kbrg';

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% wave equation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 tvec = [0 .5 1 2];
 figure(1), clf
 for k=1:length(tvec)
    t = tvec(k);
    if t==0,
       plot(x,psi,'k-','linewidth',2); hold on
    else
       uxt = .5*(x-t>0)+.5*(x+t>0);
       plot(x,uxt,'-','color',col(k),'linewidth',2); hold on
    end
 end
 ylim([-.05 1.05])
 legend('$t=0$','$t=.5$','$t=1$','$t=2$',2)
 set(legend,'interpreter','latex')
 xlabel('$x$','fontsize',14','interpreter','latex')
 ylabel('$u(x,t)$','fontsize',14','interpreter','latex')
 title('Solution of the Wave Equation','fontsize',14','interpreter','latex')
 print -depsc2 heat_v_wave1
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% heat equation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 tvec = [0 .01 .1 1];
 figure(2), clf
 for k=1:length(tvec)
    t = tvec(k);
    if t==0,
       plot(x,psi,'k-','linewidth',2); hold on
    else
       uxt = (1/2)*erfc(-x/(2*sqrt(t)));
       plot(x,uxt,'-','color',col(k),'linewidth',2); hold on
    end
 end
 legend('$t=0$','$t=.01$','$t=.1$','$t=1$',2)
 set(legend,'interpreter','latex')
 ylim([-.05 1.05])
 xlabel('$x$','fontsize',14','interpreter','latex')
 ylabel('$u(x,t)$','fontsize',14','interpreter','latex')
 title('Solution of the Heat Equation','fontsize',14','interpreter','latex')
 print -depsc2 heat_v_wave2
```