CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 23 · Solutions

Posted Monday 24 February 2014. Due 1pm Friday 14 March 2014.

23. [25 points]

Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx.$$

Consider the linear operator $L : C_m^2[0,1] \to C[0,1]$ defined by

$$Lu = -u''$$

where

$$C_m^2[0,1] = \left\{u \in C^2[0,1] : u'(0) = u(1) = 0\right\}.$$

(a) Is L symmetric?

(b) What is the null space of L?

(c) Show that $(Lu, u) \geq 0$ for all $u \in C_m^2[0,1]$ and explain why this and the answer to part (b) mean that $\lambda > 0$ for all eigenvalues λ of L.

(d) Find the eigenvalues and eigenfunctions of L.

Solution.

(a) [5 points] Yes, L is symmetric.
Let $u, v \in C_m^2[0,1]$. Integrating by parts twice, we have

$$\begin{aligned}
(Lu, v) &= \int_0^1 -u''(x)v(x)\,dx \\
&= -[u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x)\,dx \\
&= -[u'(x)v(x)]_0^1 + [u(x)v'(x)]_0^1 - \int_0^1 u(x)v''(x)\,dx.
\end{aligned}$$

Since $u, v \in C_m^2[0,1]$ we have $u'(0) = 0$ and $v(1) = 0$, and hence the first term in square brackets must be zero. Again using the fact that $u, v \in C_m^2[0,1]$ we have $v'(0) = 0$ and $u(1) = 0$, and hence the second term in square brackets is also zero. It follows that

$$(Lu, v) = \int_0^1 u(x)(-v''(x))\,dx = (u, Lv)$$

for all $u, v \in C_m^2[0,1]$.

(b) [5 points] The general solution to the differential equation

$$-u''(x) = 0$$

has the form

$$u(x) = A + Bx$$

for constants A and B. In order for u to be in $C_m^2[0,1]$, we must have $u'(0) = 0$ and so since $u'(x) = B$, we must have $B = 0$. Now $u \in C_m^2[0,1]$ also requires $u(1) = 0$, and since $u(1) = A$, we conclude that $A = 0$ too, meaning that $u(x) = A + Bx = 0$ for all $x \in [0,1]$. Thus, the only element of the null space is the zero function, that is, $\mathcal{N}(L) = \{0\}$.

(c) [7 points] Let $u \in C_m^2[0,1]$. Using the first integration by parts from part (a), we have

$$\begin{aligned}(Lu, u) &= -[u'(x)u(x)]_0^1 + \int_0^1 u'(x)u'(x)\,dx \\ &= \int_0^1 (u'(x))^2\,dx.\end{aligned}$$

Thus, (Lu, u) is the integral of a nonnegative function, so it is nonnegative. Consequently, $(Lu, u) \geq 0$ for all $u \in C_m^2[0,1]$.

This statement implies that all eigenvalues of L are non-negative, since if λ is an eigenfunction of L then, since L is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $u \in C_m^2[0,1]$ which are such that $Lu = \lambda u$ and hence

$$\lambda(u, u) = (\lambda u, u) = (Lu, u) \geq 0,$$

and so, since we know that $(u, u) > 0$ for all nonzero $u \in C_m^2[0,1]$ due to the positivite-definiteness of the inner product, we have that

$$\lambda = \frac{(Lu, u)}{(u, u)} \geq 0.$$

If zero was an eigenvalue of L, then there would exist nonzero $u \in C_m^2[0,1]$ which were such that $Lu = 0$. However, we showed in part (b) that there were no nonzero $u \in C_m^2[0,1]$ which satisfied this and so zero cannot be an eigenvalue of L and hence we can say that $\lambda > 0$ for all eigenvalues λ of L.

(d) [8 points] The eigenvalues of L are the real numbers $\lambda > 0$ for which there exist nonzero $u \in C_m^2[0,1]$ which are such that $Lu = \lambda u$. When $\lambda > 0$, the general solution to the equivalent differential equation

$$-u''(x) = \lambda u(x)$$

has the form

$$u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

where A and B are constants. Since

$$u'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

and thus

$$u'(0) = A\sqrt{\lambda},$$

the boundary condition $u'(0) = 0$ implies that $A = 0$. On the other hand, the boundary condition $u(1) = 0$ implies that

$$u(1) = B\cos(\sqrt{\lambda}) = 0,$$

which can be achieved with nonzero B provided that $\sqrt{\lambda} = (n - 1/2)\pi$ for positive integers n. We thus have that L has eigenvalues

$$\lambda_n = (n - 1/2)^2\pi^2$$

with corresponding eigenfunctions

$$u_n(x) = B_n \cos(\sqrt{\lambda_n}x) = B_n \cos((n - 1/2)\pi x)$$

for nonzero constants B_n, for $n = 1, 2, 3, \ldots$.
