

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 41

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

41. [25 points]

Let $u_0(x) = (x - x^2) (\sin(3\pi x))^2$. Note that, for $n = 1, 2, \dots$,

$$\int_0^1 \sqrt{2} u_0(x) \sin(n\pi x) dx = \begin{cases} \frac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3 \pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Consider the problem of finding the solution $u(x, t)$ to the the fourth order partial differential equation

$$u_t(x, t) = u_{xx}(x, t) - u_{xxxx}(x, t), \quad 0 < x < 1, \quad t > 0$$

with so-called hinged boundary conditions

$$u(0, t) = u_{xx}(0, t) = u(1, t) = u_{xx}(1, t) = 0, \quad t \geq 0$$

and initial condition

$$u(x, 0) = u_0(x), \quad 0 < x < 1.$$

This equation is related to a model that arises in the study of thin films. Let

$$C_H^4[0, 1] = \{v \in C^4[0, 1] : v(0) = v''(0) = v(1) = v''(1) = 0\}.$$

Let the linear operator $L : C_H^4[0, 1] \rightarrow C[0, 1]$ be defined by

$$Lv = -v'' + v''''.$$

(a) The operator L has eigenvalues $\lambda_n \in \mathbb{R}$ and eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin(n\pi x)$$

for $n = 1, 2, \dots$, which are such that

$$L\psi_n = \lambda_n \psi_n$$

for $n = 1, 2, \dots$. Obtain a formula for λ_n for $n = 1, 2, \dots$.

(b) We can write

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \psi_n(x).$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \dots$?

(c) Obtain an expression for $a_n(t)$ for $n = 1, 2, \dots$.

(d) Use you answer to part (c) to write out a formula for $u(x, t)$.

(e) Let

$$u_N(x, t) = \sum_{n=1}^N a_n(t) \psi_n(x).$$

For each time $t = 0, 10^{-5}, 2 \times 10^{-5}, 4 \times 10^{-5}$, produce a plot comparing $u_1(x, t)$, $u_3(x, t)$, $u_5(x, t)$, $u_7(x, t)$ and $u_9(x, t)$. For example, at time $t = 0$, your plot should appear as shown below.

