CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 20 · Solutions

Posted Wednesday 25 September 2013. Due 5pm Wednesday 9 October 2013.

20. [25 points] All parts of this question should be done by hand.

Let $v_1(x) = \dfrac{\sqrt{3}}{\sqrt{2}}x$, $v_2(x) = \dfrac{\sqrt{3}}{\sqrt{2}}(3x^2 - x - 1)$ and $f(x) = \dfrac{\sqrt{2}}{\sqrt{3}}\cos(\pi x)$ and let

$$C_z^1[-1,1] = \left\{ v \in C^1[-1,1] : \int_{-1}^{1} v(x)\,dx = 0 \right\}.$$

Note that $v_1 \in C_z^1[-1,1]$, $v_2 \in C_z^1[-1,1]$ and $f \in C_z^1[-1,1]$. Let the inner product $(\cdot,\cdot) : C_z^1[-1,1] \times C_z^1[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx$$

and let the norm $\|\cdot\| : C_z^1[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Also, let the inner product $a(\cdot,\cdot) : C_z^1[-1,1] \times C_z^1[-1,1] \to \mathbb{R}$ be defined by

$$a(u,v) = \int_{-1}^{1} (2+x)u'(x)v'(x)\,dx$$

and let the norm $\|\cdot\|_a : C_z^1[-1,1] \to \mathbb{R}$ be defined by

$$\|u\|_a = \sqrt{a(u,u)}.$$

Moreover, let the inner product $B(\cdot,\cdot) : C_z^1[-1,1] \times C_z^1[-1,1] \to \mathbb{R}$ be defined by

$$B(u,v) = a(u,v) + (u,v)$$

and the norm $\|\cdot\|_B : C_z^1[-1,1] \to \mathbb{R}$ be defined by

$$\|u\|_B = \sqrt{B(u,u)}.$$

Note that $(v_1,v_1) = 1$; $(v_2,v_2) = \dfrac{17}{5}$; $(f,v_1) = 0$; $(f,v_2) = -\dfrac{12}{\pi^2}$; $a(v_1,v_1) = 6$; $a(v_2,v_2) = 66$; $a(f,v_1) = -2$ and $a(f,v_2) = -22$.

(a) Use the fact that $(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are inner products on $C_z^1[-1,1]$ to verify that $B(\cdot,\cdot)$ is an inner product on $C_z^1[-1,1]$.

(b) What is the best approximation to f from $\operatorname{span}\{v_1\}$ with respect to the norm $\|\cdot\|$?

(c) What is the best approximation to f from $\operatorname{span}\{v_1\}$ with respect to the norm $\|\cdot\|_a$?

(d) What is the best approximation to f from $\operatorname{span}\{v_1\}$ with respect to the norm $\|\cdot\|_B$?

(e) What is the best approximation to f from $\operatorname{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|_a$?

(f) What is the best approximation to f from $\operatorname{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|$?

Solution.

(a) [6 points] If $u \in C_z^1[-1,1]$ and $v \in C_z^1[-1,1]$ then

$$B(u,v) = a(u,v) + (u,v) = a(v,u) + (v,u) = B(u,v)$$

since $a(u,v) = a(v,u)$ and $(u,v) = (v,u)$ because $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are inner products on $C_z^1[-1,1]$. So

$$B(u,v) = B(u,v) \text{ for all } u, v \in C_z^1[-1,1].$$

If $u, v, w \in C_z^1[-1,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned} & B(\alpha u + \beta v, w) \\ = \; & a(\alpha u + \beta v, w) + (\alpha u + \beta v, w) \\ = \; & \alpha a(u,w) + \beta a(v,w) + \alpha(u,w) + \beta(v,w) \\ = \; & \alpha\left(a(u,w) + (u,w)\right) + \beta\left(a(v,w) + (v,w)\right) \\ = \; & \alpha B(u,w) + \beta B(v,w) \end{aligned}$$

since $a(\alpha u + \beta v, w) = \alpha a(u,w) + \beta a(v,w)$ and $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w)$ because $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are inner products on $C_z^1[-1,1]$. So

$$B(\alpha u + \beta v, w) = \alpha B(u,w) + \beta B(v,w) \text{ for all } u, v, w \in C_z^1[-1,1] \text{ and all } \alpha, \beta \in \mathbb{R}.$$

If $u \in C_z^1[-1,1]$ then $B(u,u) \geq 0$ since $a(u,u) \geq 0$ and $(u,u) \geq 0$ because $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are inner products on $C_z^1[-1,1]$. Moreover, if $B(u,u) = 0$ then

$$a(u,u) + (u,u) = 0,$$

or equivalently,

$$a(u,u) = -(u,u)$$

and, since $a(u,u) \geq 0$ and $(u,u) \geq 0$, this can only hold when $a(u,u) = (u,u) = 0$ which then implies that $u = 0$ because of either the fact that $a(u,u) = 0$ only if $u = 0$ since $a(\cdot,\cdot)$ is an inner product on $C_z^1[-1,1]$ or the fact that $(u,u) = 0$ only if $u = 0$ since $(\cdot,\cdot)$ is an inner product on $C_z^1[-1,1]$. So

$$B(u,u) \geq 0 \text{ for all } u \in C_z^1[-1,1]$$

with

$$B(u,u) = 0 \text{ only if } u = 0.$$

Consequently, $B(\cdot,\cdot)$ is an inner product on $C_z^1[-1,1]$.

(b) [2 points] The best approximation to f from $\operatorname{span}\{v_1\}$ with respect to the norm $\|\cdot\|$ is

$$b_1(x) = \frac{(f, v_1)}{(v_1, v_1)} v_1(x) = 0$$

since $(f, v_1) = 0$.

(c) [2 points] The best approximation to f from $\operatorname{span}\{v_1\}$ with respect to the norm $\|\cdot\|_a$ is

$$b_2(x) = \frac{a(f, v_1)}{a(v_1, v_1)} v_1(x) = \frac{-2}{6} v_1(x) = \frac{-1}{3} \frac{\sqrt{3}}{\sqrt{2}} x = \frac{-1}{\sqrt{6}} x.$$

(d) [3 points] The definition of $B(\cdot,\cdot)$ means that

$$B(v_1, v_1) = a(v_1, v_1) + (v_1, v_1) = 6 + 1 = 7$$

and

$$B(f, v_1) = a(f, v_1) + (f, v_1) = -2 + 0 = -2.$$

Therefore, the best approximation to f from span$\{v_1\}$ with respect to the norm $\|\cdot\|_B$ is

$$b_3(x) = \frac{B(f, v_1)}{B(v_1, v_1)} v_1(x) = \frac{-2}{7}\frac{\sqrt{3}}{\sqrt{2}} x = -\frac{\sqrt{6}}{7} x.$$

(e) [5 points] We first compute that

$$v_1'(x) = \frac{\sqrt{3}}{\sqrt{2}}$$

and

$$v_2'(x) = \frac{\sqrt{3}}{\sqrt{2}}(6x - 1)$$

and then compute that

$$\begin{aligned}
a(v_1, v_2) = a(v_2, v_1) &= \int_{-1}^{1} (2 + x) v_1'(x) v_2'(x)\, dx \\
&= \int_{-1}^{1} (2 + x) \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{3}}{\sqrt{2}} (6x - 1)\, dx \\
&= \frac{3}{2} \int_{-1}^{1} 6x^2 + 11x - 2)\, dx \\
&= \frac{3}{2} \left[2x^3 + \frac{1}{2}x^2 - 2x \right]_{-1}^{1} \\
&= \frac{3}{2} \left(2 + \frac{1}{2} - 2 - \left(-2 + \frac{1}{2} + 2 \right) \right) \\
&= 0.
\end{aligned}$$

Therefore, v_1 and v_2 is orthogonal with respect to the inner product $a(\cdot,\cdot)$ and hence the best approximation to f from span$\{v_1, v_2\}$ with respect to the norm $\|\cdot\|_a$ is

$$\begin{aligned}
b_4(x) &= \frac{a(f, v_1)}{a(v_1, v_1)} v_1(x) + \frac{a(f, v_2)}{a(v_2, v_2)} v_2(x) \\
&= \frac{-2}{6} v_1(x) + \frac{-22}{66} v_2(x) \\
&= -\frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}} x - \frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}}(3x^2 - x - 1) \\
&= \frac{1}{\sqrt{6}}(1 - 3x^2).
\end{aligned}$$

(f) [7 points] We first compute that

$$\begin{aligned}
(v_1, v_2) = (v_2, v_1) &= \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} x \frac{\sqrt{3}}{\sqrt{2}} (3x^2 - x - 1)\, dx \\
&= \frac{3}{2} \int_{-1}^{1} (3x^3 - x^2 - x)\, dx \\
&= \frac{3}{2} \left[\frac{3}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{-1}^{1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{2}\left(\frac{3}{4} - \frac{1}{3} - \frac{1}{2} - \left(\frac{3}{4} + \frac{1}{3} - \frac{1}{2}\right)\right) \\
&= \frac{3}{2}\left(-\frac{2}{3}\right) \\
&= -1.
\end{aligned}
$$

Therefore, v_1 and v_2 are not orthogonal with respect to the inner product $(\cdot,\cdot)$ and hence the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|$ is

$$b_5(x) = c_1 v_1(x) + c_2 v_2(x)$$

where the coefficients $c_1, c_2 \in \mathbb{R}$ are such that

$$\begin{bmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_1, v_2) & (v_2, v_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (f, v_1) \\ (f, v_2) \end{bmatrix}$$

and hence are such that

$$\begin{bmatrix} 1 & -1 \\ -1 & \frac{17}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{12}{\pi^2} \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} c_1 - c_2 \\ -c_1 + \frac{17}{5} c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{12}{\pi^2} \end{bmatrix}.$$

Now, the first rows implies that $c_1 = c_2$ which mean that the second row yields that $\left(\frac{17}{5} - 1\right) c_2 = -\frac{12}{\pi^2}$ which implies that $\frac{12}{5} c_2 = -\frac{12}{\pi^2}$ and hence $c_2 = -\frac{5}{\pi^2}$. Therefore, the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|$ is

$$b_5(x) = c_1 v_1(x) + c_2 v_2(x) = c_2(v_1(x) + v_2(x)) = -\frac{5}{\pi^2}\frac{\sqrt{3}}{\sqrt{2}}(x + 3x^2 - x - 1) = \frac{5}{\pi^2}\frac{\sqrt{3}}{\sqrt{2}}(1 - 3x^2).$$