Lets recap the discussion from last time and do some examples.

Recall that if A is symmetric we know one very big fact:

"Let A be a symmetric $n \times n$ matrix defined on a vector space V of dimension n. Then there exists ortonormal eigenvectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ of A which form a basis for V."

We saw last time that A doesn't even need to be invertible! All we need is ① A a non-zero matrix and ② A symmetric

We also discussed the notion of an <u>eigenspace</u> for an eigenvalue λ as $E_\lambda = N(A - \lambda I)$. Recall that the null space of <u>any</u> matrix (in this case $A - \lambda I$) is a subvector space and therefore has a basis. Another way to think about E_λ is as the <u>subvector space of all eigenvectors with eigenvalue λ.</u>

Finally we discussed the spectral method for solving $Ax=b$. This method requires that we already have:
1) The eigenvalues for a matrix A
2) An <u>ortonormal</u> set of eigenvectors for A.

So to use the spectral method to solve $Ax=b$ we first need to <u>find</u> its eigenvalues and an ortonormal set of eigenvectors. This can be a very challenging task for large matrices but the approach is more applicable to more general linear operators such as those we see in the context of partial differential equations.

So first off suppose that we have found two eigenvectors V_1, V_2 for eigenvalues λ_1, λ_2 of a symmetric matrix A.

The first thing we notice is that if V is an eigenvector of A with eigenvalue λ then so is αV where $\alpha \neq 0$ is any number. This follows from $A(\alpha V) = \alpha A V = \alpha(\lambda V) = \lambda(\alpha V)$.

We already know that if $\lambda_1 \neq \lambda_2$ then $(V_1, V_2) = 0$ so all that is left to do is give them unit length. Hence the vectors $u_1 = v_1/\|v_1\|$ and $u_2 = v_2/\|u_2\|$ are <u>orthonormal</u>.

But what if $\lambda_1 = \lambda_2$? This is exactly what happens when the Eigenspace E_{λ_1} has dimension higher than <u>one</u>. In this case we <u>aren't</u> guaranteed that $\langle v_1, v_2 \rangle = 0$. However we can make them orthonormal by using the <u>Gram-Schmidt process</u>.

That is we define $\tilde{v}_1 = v_1$ and $\tilde{v}_2 = v_2 - \mathrm{proj}_{\tilde{v}_1}(v_2)$. Then $\langle \tilde{v}_1, \tilde{v}_2 \rangle = 0$ so we can define $u_1 = \tilde{v}_1/\|\tilde{v}_1\|$ and $u_2 = \tilde{v}_2/\|\tilde{v}_2\|$. Then $\{u_1, u_2\}$ are eigenvectors of A and are <u>orthonormal</u>.

* <u>Note</u>: the Gram-Schmidt process preserves the eigenvector status of eigenvectors with the same eigenvalue. Can you explain why this is true?

Lets do a <u>fun</u> example of the spectral method:

<u>Q</u>: Solve $Ax = b$ using the spectral method where
$$A = \begin{bmatrix} 164 & -48 \\ -48 & 136 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 116 \\ 88 \end{bmatrix}$$

<u>Note</u>: the inner product is, implicitly, the dot product.

<u>Recall</u>: to solve this problem I need an orthonormal set of eigenvectors + their eigenvalues.

<u>Step 1</u>: Eigenvalues
$$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 164-\lambda & -48 \\ -48 & 136-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (164-\lambda)(136-\lambda) - 48^2 = 0$$
$$\Rightarrow \lambda^2 - 300\lambda + 22304 - 2304 = 0$$
$$\Rightarrow \lambda^2 - 300\lambda + 20000 = 0$$
$$\Rightarrow (x-100)(x-200)$$

So the eigenvalues are $\lambda_1 = 100$ $\lambda_2 = 200$.

Key idea: We have two distinct eigenvalues. We therefore know that if V_1 is an eigenvector corresponding to λ_1 and V_2 an eigenvector corresponding to λ_2 that $V_1 \cdot V_2 = 0$.

Question: How many eigenvectors can have eigenvalue λ_1?
Answer: ONLY ONE! How do we know? A is a linear map from $A: \mathbb{R}^2 \to \mathbb{R}^2$. We know that E_{λ_1} and E_{λ_2} are subspaces of $\mathbb{R}^2$ and that they are not empty. Furthermore we know that nothing in E_{λ_1} can be in E_{λ_2} and vice versa (if v were a vector in both it would be an eigenvector with two different eigenvalues - this can't happen). So we know that $\dim(\mathbb{R}^2) \geq \dim(E_1) + \dim(E_2)$. But this means that E_{λ_1} can't have more than one linearly independent vector and neither can E_{λ_2}? If they did then we would have at least three linearly independent vectors in $\mathbb{R}^2$ and so the dimension of $\mathbb{R}^2$ would be at least three! We know that $\dim(\mathbb{R}^2) = 2$. So we know that E_{λ_1} contains one eigenvector and E_{λ_2} contains one eigenvector. Since $\lambda_1 \neq \lambda_2$ we also know these eigenvectors will be orthogonal.

Lets find them:

$$E_{\lambda_1} = E_{100} = N(A - 100\lambda) = N\left(\begin{bmatrix} 64 & -48 \\ -48 & 36 \end{bmatrix}\right)$$

The reduced row echelon form of $\begin{bmatrix} 64 & -48 \\ -48 & 36 \end{bmatrix}$ is $\begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$ so that the null space is all vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $\begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

which means $x_1 = 3/4\, x_2 \Rightarrow \vec{x}$ looks like $\vec{x} = \alpha \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ for any value α.

Thus: $E_{100} = \text{Span}\left\{ \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \right\}$ and $V_1 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 100$.

Likewise $E_{200} = N(A - 200\lambda) = N\left(\begin{bmatrix} -36 & -48 \\ -48 & -64 \end{bmatrix}\right)$

The reduced row echelon form of $\begin{bmatrix} -36 & -48 \\ -48 & -64 \end{bmatrix}$ is $\begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}$

So that $\vec{x}$ is in the nullspace $N\left(\begin{bmatrix} -36 & -48 \\ -48 & -64 \end{bmatrix}\right)$ if and only if

$\begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = -4/3\, x_2$ so that all vectors of the nullspace consists of all vectors of the form $\alpha \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ so that

$E_{200} = N(A - 200I) = \text{Span}\left\{\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}\right\}$ and $V_2 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ is

an eigenvector for the eigenvalue $\lambda_2 = 200$

Check: One readily sees that $V_1 \cdot V_2 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} = 0$

So $\{V_1, V_2\}$ are orthogonal and hence form a basis for $\mathbb{R}^2$ (since there are two of them, they are linearly independent and $\dim(\mathbb{R}^2) = 2$)

Let make them into an orthonormal basis:

Key idea: they are already orthogonal so all we have to do is give them unit length

$$u_1 = V_1/\|V_1\| = \left(\frac{1}{\sqrt{(3/4)^2 + 1^2}}\right) V_1 = \frac{4}{5} V_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$u_2 = V_2/\|V_2\| = \left(\frac{1}{\sqrt{(4/3)^2 + 1^2}}\right) V_2 = \frac{3}{5} V_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

Then $\{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A.

Now we can solve $Ax = b$ using the spectral method.

We have:

$$b = \beta_1 u_1 + \beta_2 u_2 \quad \text{where} \quad \beta_1 = (b, u_1) \quad \beta_2 = (b, u_2)$$

Here $b = \begin{bmatrix} 116 \\ 88 \end{bmatrix} \rightarrow \beta_1 = (b, u_1) = \begin{bmatrix} 116 \\ 88 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = 140$

$\beta_2 = (b, u_2) = \begin{bmatrix} 116 \\ 88 \end{bmatrix} \cdot \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} = -40$

So $b = 140\, \vec{u}_1 - 40\, \vec{u}_2$

(or $\begin{bmatrix} 140 \\ -40 \end{bmatrix}$ with respect to the $\{\vec{u}_1, \vec{u}_2\}$ basis)

The spectral method finds the coefficients α_i of the unknown vector $\vec{x} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$ by using the fact that $A u_1 = \lambda_1 u_1$ and $A u_2 = \lambda_2 u_2$

Then $Ax = b \Rightarrow A(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2$

$$\Rightarrow \alpha_1 A\vec{u}_1 + \alpha_2 A \vec{u}_2 = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2$$

$$\Rightarrow \alpha_1 \lambda_1 \vec{u}_1 + \alpha_2 \lambda \vec{u}_2 = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2$$

So that $\alpha_1 = \dfrac{\beta_1}{\lambda_1} = \dfrac{(b, u_1)}{\lambda_1}$ and $\alpha_2 = \dfrac{\beta_2}{\lambda_2} = \dfrac{(b, u_2)}{\lambda_2}$

Thus for us $\alpha_1 = \dfrac{\beta_1}{100} = \dfrac{140}{100} = 1.4$, $\alpha_2 = \dfrac{\beta_2}{200} = \dfrac{-40}{200} = -0.2$

So that $x = 1.4 \vec{u}_1 - 0.2 \vec{u}_2 \sim \begin{bmatrix} 1.31667 \\ 1.2 \end{bmatrix}$