CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 5

Posted Wednesday 24 September, 2014. Due 5pm Wednesday 1 October, 2014.

*Please write your name and **residential college** on your homework.*

1. [25 points: 5 points each]

   Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

   (a) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v) = \int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V} = C^1[0,1]$.

   (b) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v) = \int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V} = C[0,1]$.

   (c) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v) = \int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V} = C[0,1]$.

   (d) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v) = \int_0^1 (u(x)+v(x))\,dx$ where $\mathcal{V} = C[0,1]$.

   (e) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v) = \int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V} = C[-1,1]$.

2. [24 points: 6 points each]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $\phi_3 \in C[-1,1]$, and $f \in C[-1,1]$ be defined by

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = 3x^2 - 1,$$

and

$$f(x) = e^x,$$

for all $x \in [-1,1]$. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\| : C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2, \phi_3\}$ is orthogonal with respect to the inner product $(\cdot,\cdot)$, which is defined on $[-1,1]$.

(a) By hand, construct the best approximation f_1 to f from $\text{span}\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) By hand, construct the best approximation f_2 to f from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) By hand, construct the best approximation f_3 to f from $\text{span}\{\phi_1, \phi_2, \phi_3\}$ with respect to $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

3. [27 points: 8 points for (a), (b), 11 points for (c)]

Let V be an inner product space (i.e. V a vector space with an inner product). Suppose $\{v_1, v_2, v_3\}$ is a basis for V, and we would like to construct a is possible to construct a new *orthogonal* basis $\{\phi_1, \phi_2, \phi_3\}$ through the following procedure:

$$\begin{aligned}
\phi_1 &= v_1 \\
\phi_2 &= v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1 \\
\phi_3 &= v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2 \\
&\vdots \\
\phi_k &= v_k - \sum_{i=1}^{k-1} \frac{(\phi_i, v_k)}{(\phi_i, \phi_i)}\phi_i
\end{aligned}$$

This is called the *Gram-Schmidt* procedure.

(a) We know that nonzero vectors $v_1, v_2, ..., v_k \in V$ form an orthogonal set if they are orthogonal to each other: i.e. if

$$(v_i, v_j) = 0, \quad i \neq j.$$

Show that ϕ_1, ϕ_2, ϕ_3 form an orthogonal set, i.e. $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$.

(b) Show that if we have an orthogonal set of vectors $\phi_1, \ldots, \phi_k$, then $\phi_i, \ldots, \phi_k$ are linearly independent as well, i.e.

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

is only true if $\alpha_1, \ldots, \alpha_k = 0$.

(c) Since we can define an inner product $(\cdot, \cdot)$ on the function space $C[-1, 1]$ as

$$(u, v) = \int_{-1}^{1} u(x)v(x)\, dx,$$

we can also use the Gram-Schmidt procedure to create orthogonal sets of *functions*. Using the Gram-Schmidt procedure above, compute the orthogonal vectors $\{\phi_1, \phi_2, \phi_3\}$ given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$.

4. [24 points: 8 each]
One of the most intriguing results in mathematics is the idea that continuous functions (especially those whose derivatives are also continuous) can be very well approximated using combinations of trigonometric functions — i.e. sines and cosines — of different frequencies. This is encapsulated in the idea of *Fourier Series*: a large class of functions $u(x)$ can be represented by the infinite sum

$$u(x) = C + \sum_{j=1}^{\infty} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right).$$

Additionally, it turns out that the finite sum (i.e. a linear combination of sines and cosines)

$$u(x) \approx C + \sum_{j=1}^{n} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right)$$

is often a very good approximation to $u(x)$. We will go more into depth on these ideas later in the semester.

In this problem, unless specified otherwise, we will examine orthogonality properties of sines and cosines using the following inner product on $C^2[0,1]$: for $u, v \in C^2[0,1]$,

$$(u, v) = \int_0^1 u(x)v(x)dx.$$

For all parts, assume j and k are integers.

(a) Show that sines of different frequencies are orthogonal to each other, i.e. that

$$(\sin(j\pi x), \sin(k\pi x)) = \int_0^1 \sin(j\pi x)\sin(k\pi x)dx = 0, \quad j \neq k.$$

(b) Show that cosines of different nonzero frequencies are orthogonal to each other, i.e. that

$$(\cos(j\pi x), \cos(k\pi x)) = \int_0^1 \cos(j\pi x)\cos(k\pi x)dx = 0, \quad j \neq k.$$

(c) Show that sines and cosines of different frequencies are orthogonal to each other *over the interval* $[-1, 1]$, i.e. that

$$(\sin(j\pi x), \cos(k\pi x)) = \int_{-1}^1 \sin(j\pi x)\cos(k\pi x)dx = 0, \quad j \neq k.$$

Unlike the previous two parts, cosines and sines of different frequencies are not orthogonal to each other using the inner product on $[0, 1]$, and must be shown to be orthogonal using the inner product

$$(u, v) = \int_{-1}^1 u(x)v(x)dx.$$

(d) Show that sines and cosines, in addition to being orthogonal, can easily be made *orthonormal* over $[0, 1]$ by scaling by $\sqrt{2}$: i.e.

$$\left\|\sqrt{2}\sin(j\pi x)\right\|^2 = 2\int_0^1 \sin(j\pi x)^2 dx = 1, \qquad \left\|\sqrt{2}\cos(j\pi x)\right\|^2 = 2\int_0^1 \cos(j\pi x)^2 dx = 1.$$