CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 1 · Solutions

Posted Monday 13 January 2014. Due 1pm Monday 27 January 2014.

1. [25 points]

   Let $f(x) = 1$ for $x \in [0, 1]$. Also, let
   $$\phi_n(x) = \sin(n\pi x)$$
   and let
   $$a_n = \frac{\displaystyle\int_0^1 f(x)\phi_n(x)dx}{\displaystyle\int_0^1 (\phi_n(x))^2 dx}$$
   for $n = 1, 2, 3, \ldots$. We can approximate f as a linear combination of the periodic functions ϕ_n for $n = 1, 2, 3, \ldots, N$. We call the approximation f_N which is defined as
   $$f_N(x) = \sum_{n=1}^{N} a_n \phi_n(x)$$
   for $x \in [0, 1]$. In this course, you will learn in what sense a_n are the correct coefficients to best approximate f as a linear combination of the ϕ_n for $n = 1, 2, 3, \ldots, N$.

   (a) Compute the coefficients a_n symbolically using the MATLAB functions **syms**, **int**, and **simplify** as needed.

   (b) Use the fact that n is an integer to simplify the formula that you obtained for a_n.

   (c) Simplify your formula for a_n further by considering the cases when n is odd and n is even separately.

   (d) On the same figure plot $f_N(x)$ for $N = 4, 16, 64, 256$. Make sure to:
   - Use a different color for each value of N;
   - label the axes and provide a title;
   - create an accurate legend for the figure;
   - adjust the text sizes if necessary to make everything easily legible;
   - use the LATEX interpreter to make your labels, titles, and legend look stylish.

Solution.

(a) [5 points] We can use the MATLAB code:

```
syms x n;
phin=sin(n*pi*x);
a_n=simplify((int(phin,0,1)/int(phin^2,0,1)))
```

to compute that
$$a_n = \frac{-8\sin^2\left(\frac{n\pi}{2}\right)}{\sin(2n\pi) - 2n\pi}.$$

(b) [5 points] If n is an integer then $\sin(2n\pi) = 0$ and therefore

$$a_n = \frac{-8\sin^2\left(\frac{n\pi}{2}\right)}{-2n\pi} = \frac{4\sin^2\left(\frac{n\pi}{2}\right)}{\pi n}.$$

(c) [5 points] Now,

$$\sin^2\left(\frac{n\pi}{2}\right) = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

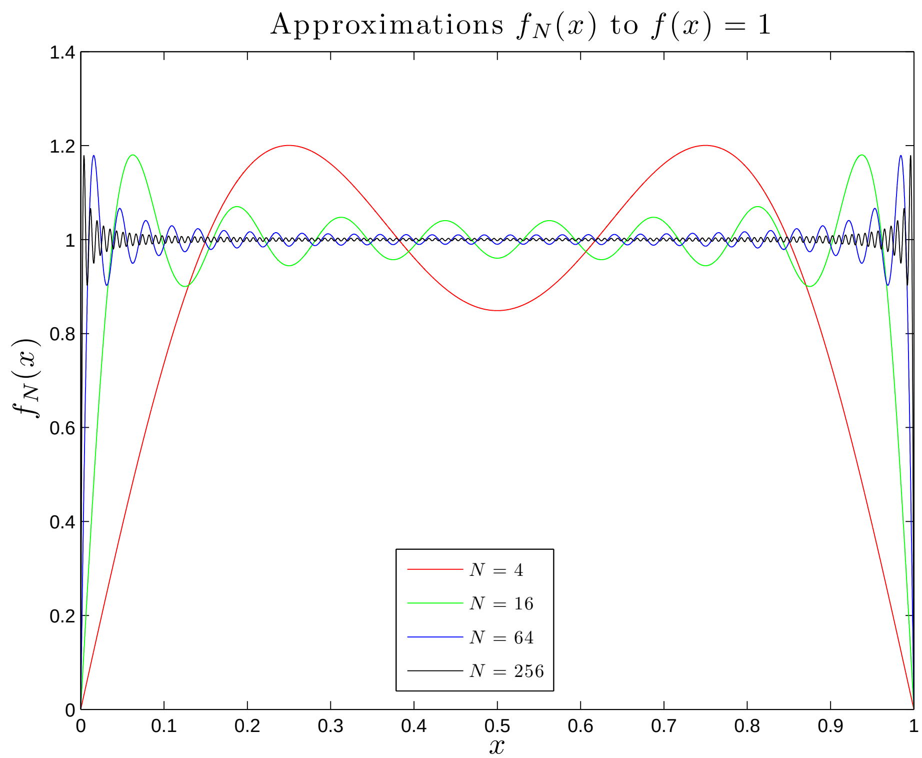
Consequently,

$$a_n = \begin{cases} \dfrac{4}{n\pi}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

(d) [10 points] We can use the MATLAB code:

```
clear
clc
p=10000;
x=linspace(0,1,p);
colors='rgbk';
N=[4 16 64 256];
figure(1)
clf
for i=1:size(N,2);
    fn=zeros(1,p);
    for n=1:2:N(i)
        fn=fn+4/(n*pi)*sin(n*pi*x);
    end
    plot(x,fn,colors(i));
    hold on
    legendStr{i}=['$N=' num2str(N(i)) '$'];
end
xlabel('$x$','interpreter','latex','fontsize',16);
ylabel('$f_N(x)$','interpreter','latex','fontsize',16);
title('Approximations $f_N(x)$ to $f(x)=1$','interpreter','latex','fontsize',16);
legend(legendStr,'interpreter','latex','Location','best');
saveas(figure(1),'hw1d.eps','epsc')
```

to produce the plot:



Approximations $f_N(x)$ to $f(x) = 1$
$f_N(x)$
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$N = 4$
$N = 16$
$N = 64$
$N = 256$