CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 27

Posted Friday 28 February 2014. Due 1pm Friday 14 March 2014.

27. [25 points]

All parts of this question should be done by hand.

Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L:S\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$S=\{w\in C^2[0,1]: w'(0)=w(1)=0\}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv,w)=(v,Lw)\text{ for all } v,w\in S.$$

Let N be a positive integer and let $f\in C[0,1]$ be defined by

$$f(x)=\begin{cases} 1-2x & \text{if } x\in\left[0,\frac{1}{2}\right]; \\ 0 & \text{otherwise.}\end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x)=\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n=1,2,\ldots$. Note that, for $m,n=1,2,\ldots$,

$$(\psi_m,\psi_n)=\begin{cases} 1 & \text{if } m=n; \\ 0 & \text{if } m\neq n.\end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n=1,2,\ldots$.

(b) Compute the best approximation to f from span$\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=f(x),\quad 0<x<1$$

and

$$\tilde{u}'(0)=\tilde{u}(1)=0.$$

(d) What is the best approximation to $\tilde{u}$ from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$?

(e) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$