CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 16 · Solutions

Posted Friday 14 February 2014. Due 1pm Friday 21 February 2014.

16. [25 points]
Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $\phi_3 \in C[-1,1]$, and $f \in C[-1,1]$ be defined by

$$\phi_1(x) = 1,$$
$$\phi_2(x) = x,$$
$$\phi_3(x) = 3x^2 - 1,$$

and

$$f(x) = e^x,$$

for all $x \in [-1,1]$. Let the inner product $(\cdot,\cdot): C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\|: C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2, \phi_3\}$ is orthogonal with respect to the inner product $(\cdot,\cdot)$.

(a) By hand, construct the best approximation f_1 to f from $\text{span}\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) By hand, construct the best approximation f_2 to f from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) By hand, construct the best approximation f_3 to f from $\text{span}\{\phi_1, \phi_2, \phi_3\}$ with respect to the norm $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) [4 points] The best approximation to $f(x) = e^x$ from $\text{span}\{\phi_1\}$ with respect to the norm $\|\cdot\|$ is

$$f_1(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x).$$

We compute

$$(\phi_1,\phi_1) = \int_{-1}^{1} 1^2\,dx = [x]_{-1}^{1} = 1 - (-1) = 2$$

and

$$(f,\phi_1) = \int_{-1}^{1} e^x\,dx = [e^x]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

and hence

$$f_1(x) = \frac{1}{2}\left(e - \frac{1}{e}\right).$$

(b) [7 points] Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot,\cdot)$, i.e., $(\phi_1,\phi_2)=0$, the best approximation to $f(x)=e^x$ from span$\{\phi_1,\phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$f_2(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x) + \frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x) = f_1(x) + \frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x).$$

Noting that

$$(\phi_2,\phi_2) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{-1}{3}\right) = \frac{1}{3} - \frac{1}{3} = \frac{2}{3}$$

and

$$(f,\phi_2) = \int_{-1}^{1} xe^x\,dx = [xe^x]_{-1}^{1} - \int_{-1}^{1} e^x\,dx = e^1 - \left(-e^{-1}\right) - (f,\phi_1) = e + \frac{1}{e} - e + \frac{1}{e} = \frac{2}{e}$$

we can compute that

$$f_2(x) = f_1(x) + \frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e}x.$$

(c) [7 points] Since,

$$(\phi_1,\phi_2) = (\phi_1,\phi_3) = (\phi_2,\phi_3) = 0,$$

the best approximation to $f(x)=e^x$ from span$\{\phi_1,\phi_2,\phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$f_3(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x) + \frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x) + \frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x) = f_2(x) + \frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x).$$

Toward this end, compute

$$\begin{aligned}
(\phi_3,\phi_3) &= \int_{-1}^{1} (3x^2-1)^2\,dx \\
&= \int_{-1}^{1} 9x^4 - 6x^2 + 1\,dx \\
&= \int_{-1}^{1} 9x^4\,dx - 6(\phi_2,\phi_2) + (\phi_1,\phi_1) \\
&= \left[\frac{9x^5}{5}\right]_{-1}^{1} - 6\frac{2}{3} + 2 \\
&= \frac{9}{5} - \left(-\frac{9}{5}\right) - \frac{12}{3} + 2 \\
&= \frac{18}{5} - \frac{12}{3} + 2 \\
&= \frac{54}{15} - \frac{60}{15} + \frac{30}{15} \\
&= \frac{24}{15} \\
&= \frac{8}{5}
\end{aligned}$$

and

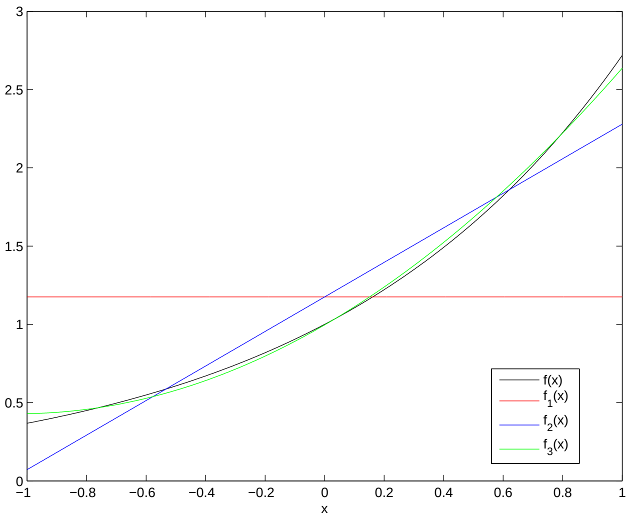
$$(f,\phi_3) = \int_{-1}^{1} (3x^2-1)e^x\,dx$$

$$
\begin{aligned}
&= \int_{-1}^{1} 3x^2 e^x \, dx - (f, \phi_1) \\
&= \left[3x^2 e^x\right]_{-1}^{1} - \int_{-1}^{1} 6x e^x \, dx - \left(e - \frac{1}{e}\right) \\
&= 3e^1 - 3e^{-1} - 6(f, \phi_2) - \left(e - \frac{1}{e}\right) \\
&= 2e - \frac{2}{e} - \frac{12}{e} \\
&= 2e - \frac{14}{e}
\end{aligned}
$$

thus giving

$$
f_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e} x + \frac{5}{4}\left(e - \frac{7}{e}\right)(3x^2 - 1).
$$

(d) [7 points] The following plot compares the best approximations to $f(x)$.



The code use to produce it is below.

```
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=exp(x);
f1=(exp(1)-exp(-1))/2+x-x;
f2=f1+3*exp(-1)*x;
f3=f2+5*(exp(1)-7*exp(-1))*(3*x.^2-1)/4;
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
plot(x,f3,'-g')
xlabel('x')
legend('f(x)','f_1(x)','f_2(x)','f_3(x)','location','best')
saveas(figure(1),'hw16d.eps','epsc')
```