CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 4

Posted Wednesday 17 September 2014. Due 5pm Wednesday 24 September 2014.

*Please write your name and **residential college** on your homework.*

1. [30 points - 5 points each]
   For each part, if the set is not a vector space, please show what properties of a vector space are violated. Otherwise, show that all properties of a vector space are satisfied.

   (a) Demonstrate whether or not the set $S_1 = \{\mathbf{x} \in \mathbb{R}^2 : \ x_2 = x_1^3\}$ is a subspace of $\mathbb{R}^2$.

   (b) Demonstrate whether or not the set $S_2 = \{\mathbf{x} \in \mathbb{R}^3 : \ 3x_1 + 2x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

   (c) Demonstrate whether or not the set $S_3 = \{f \in C[0,1] : \ f(x) \geq 0 \text{ for all } x \in [0,1]\}$ is a subspace of $C[0,1]$.

   (d) Demonstrate whether or not the set $S_4 = \left\{f \in C[0,1] : \ \max_{x\in[0,1]} f(x) \leq 1\right\}$ is a subspace of $C[0,1]$.

   (e) Demonstrate whether or not the set $S_5 = \{f \in C^2[0,1] : \ f(1) = 1\}$ is a subspace of $C^2[0,1]$.

   (f) Demonstrate whether or not the set $S_6 = \{f \in C^2[0,1] : \ f(1) = 0\}$ is a subspace of $C^2[0,1]$.

2. [25 points - 5 points each]

   Demonstrate whether or not each of the following is a linear operator.

   (a) $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

   (c) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

   (d) $L : C^1[0,1] \to C[0,1]$ defined by $(Lu)(x) = u(x)u'(x)$.

   (e) $L : C^2[0,1] \to C[0,1]$ defined by $(Lu)(x) = u''(x) - \sin(x)u'(x) + \cos(x)u(x)$.

3. [24 points - 10 points for (a), 14 points for (b)]

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since f is a linear operator, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

(b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

4. [21 points - 7 points each]
In this problem we'll consider a linear operator mapping to and from a very specific vector space, and use it to explore what an operator inverse can look like.

Consider the V defined as

$$V = \left\{ u(x) = \sum_{j=1}^{N} c_j \sin(j\pi x), \quad c_j \in \mathbb{R} \right\}.$$

In other words, V is the set of all functions that are linear combinations of a finite number of different sine functions. This means that, for each $u \in V$, there is a set of coefficients $c_1, \dots, c_N$ that is also associated with u.

(a) Show that V is a subspace of the vector space C_D^2, where

$$C_D^2 = \{u(x) \in C^2[0,1], \quad u(0) = u(1) = 0\}.$$

(b) Let the operator L be defined as

$$Lu = -\frac{\partial^2 u}{\partial x^2}.$$

Show that, for $u \in V$, $Lu \in V$. This shows that L can be viewed as

$$L : V \to V,$$

a map from V to V.

(c) We can define the operator $\tilde{L} : V \to V$ as

$$\tilde{L}u = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x).$$

Show that both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$.

Since both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$, we can refer to $\tilde{L}$ as the inverse L^{-1} of $L : V \to V$.