CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 4

Posted Wednesday 12 September 2012. Due Wednesday 19 September 2012, 5pm.

1. [20 points]
   The equation $x_1 + x_2 + x_3 = 0$ defines a plane in $\mathbb{R}^3$ that passes through the origin.

   (a) Find two linearly independent vectors in $\mathbb{R}^3$ whose span is this plane.

   (b) Find the point in this plane closest (in the standard Euclidean norm, $\|\mathbf{z}\| = \sqrt{\mathbf{z}^T\mathbf{z}}$) to the vector

   $$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   by formulating this as a best approximation problem. (You may use MATLAB to invert a matrix.)

2. [25 points]
   Recall that a linear operator P is a projection from the vector space V to the vector space V provided $P^2 = P$, that is, $P(Pf) = Pf$ for all $f \in V$. Consider $V = C[-1,1]$ with the usual inner product

   $$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx,$$

   and the two linear operators P_e and P_o the project a function onto their even and odd parts. That is,

   $$(P_e f)(x) = \frac{f(x) + f(-x)}{2}, \qquad (P_o f)(x) = \frac{f(x) - f(-x)}{2}.$$

   (a) Show that P_e and P_o are projections.

   (b) Verify that $P_e f$ and $P_o f$ are orthogonal for any $f \in C[-1,1]$.

   (c) Is $P_e + P_o$ a projection? Explain.

3. [25 points]
   Suppose that

   $$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (a) Compute *by hand* the eigenvalues and eigenvectors of this matrix.

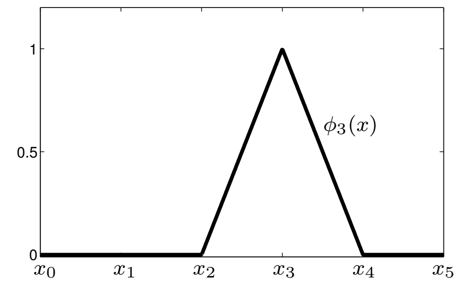
   (b) Verify *by hand* that these eigenvectors are orthogonal.

   (c) Solve the linear system $\mathbf{Ax} = \mathbf{b}$ using the spectral method, where

   $$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

4. [30 points]
Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_k = kh$ for $k = 0, \ldots, N+1$. Consider the N *hat functions*, defined as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$



The plot to the right shows $\phi_3(x)$ for $N = 4$.

Consider the standard inner product on $C[0,1]$,

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

(a) Compute the inner products (ϕ_j, ϕ_k) for $k = 1, \ldots, N$,
obtaining answers that depend on N (or h) only. Consider the following cases individually:

- (ϕ_j, ϕ_j) for $j = 1, \ldots, N$;
- (ϕ_j, ϕ_{j+1}) for $j = 1, \ldots, N-1$;
- (ϕ_j, ϕ_k) for $|j - k| > 1$.

(b) For $f(x) = \sin(\pi x)$, compute the inner products (ϕ_k, f) for $k = 1, \ldots, N$.

(c) Use your solutions to (a) and (b) to set up a linear system (in MATLAB) and solve it to compute the best approximations $f_N(x)$ from $\text{span}\{\phi_1, \ldots, \phi_N\}$ to $f(x) = \sin(\pi x)$ for $N = 3$ and $N = 9$ over the interval $[0,1]$ with the standard inner product.

For each of these N, use the `hat.m` code (from Problem Set 1, either your code or from the solutions) to plot your best approximations. For each N, produce one plot that compares $f_N(x)$ to $f(x)$, and a second plot that shows the error $f(x) - f_N(x)$.

[Be careful: Are the basis functions used for the best approximation orthogonal?]