CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 17 · Solutions

Posted Friday 14 February 2014. Due 1pm Friday 21 February 2014.

17. [25 points]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $\phi_3 \in C[-1,1]$, and $f \in C[-1,1]$ be defined by

$$\phi_1(x) = 1,$$

$$\phi_2(x) = x,$$

$$\phi_3(x) = 3x^2 - 1,$$

and

$$f(x) = \cos(\pi x),$$

for all $x \in [-1,1]$. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\| : C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2, \phi_3\}$ is orthogonal with respect to the inner product $(\cdot,\cdot)$.

(a) By hand, construct the best approximation f_1 to f from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) By hand, construct the best approximation f_2 to f from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) By hand, construct the best approximation f_3 to f from span$\{\phi_1, \phi_2, \phi_3\}$ with respect to the norm $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) [4 points] The best approximation to $f(x) = \cos(\pi x)$ from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$ is

$$f_1(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x).$$

In Homework 16 we computed that

$$(\phi_1,\phi_1) = 2.$$

Moreover, we can compute that

$$(f,\phi_1) = \int_{-1}^{1} \cos(\pi x)\,dx = \left[\frac{1}{\pi}\sin(\pi x)\right]_{-1}^{1} = \frac{1}{\pi}\sin(\pi) - \frac{1}{\pi}\sin(-\pi) = 0 - 0 = 0$$

and hence

$$f_1(x) = 0.$$

(b) [7 points] Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot,\cdot)$, i.e., $(\phi_1,\phi_2)=0$, the best approximation to $f(x)=\cos(\pi x)$ from span$\{\phi_1,\phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$f_2(x)=\frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)=f_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x).$$

In Homework 16 we computed that

$$(\phi_2,\phi_2)=\frac{2}{3}.$$

Moreover, we can compute that

$$\begin{aligned}
(f,\phi_2) &= \int_{-1}^{1} x\cos(\pi x)\,dx \\
&= \left[\frac{1}{\pi}x\sin(\pi x)\right]_{-1}^{1}-\int_{-1}^{1}\frac{1}{\pi}\sin(\pi x)\,dx \\
&= \frac{1}{\pi}\sin(\pi)-\left(-\frac{1}{\pi}\sin(-\pi)\right)-\left[-\frac{1}{\pi^2}\cos(\pi x)\right]_{-1}^{1} \\
&= \frac{1}{\pi^2}\cos(\pi)-\frac{1}{\pi^2}\cos(-\pi) \\
&= -\frac{1}{\pi^2}-\left(-\frac{1}{\pi^2}\right) \\
&= 0
\end{aligned}$$

and hence

$$f_2(x)=f_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)=0.$$

(c) [7 points] Since,

$$(\phi_1,\phi_2)=(\phi_1,\phi_3)=(\phi_2,\phi_3)=0,$$

the best approximation to $f(x)=\cos(\pi x)$ from span$\{\phi_1,\phi_2,\phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$f_3(x)=\frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)+\frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x)=f_2(x)+\frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x).$$

In Homework 16 we computed that

$$(\phi_3,\phi_3)=\frac{8}{5}.$$

Moreover, we can compute that

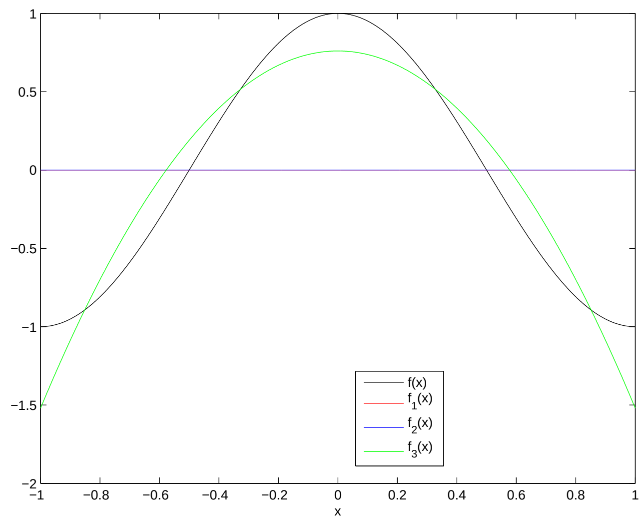
$$\begin{aligned}
(f,\phi_3) &= \int_{-1}^{1}(3x^2-1)\cos(\pi x)\,dx \\
&= 3\int_{-1}^{1}x^2\cos(\pi x)\,dx-(f,\phi_1) \\
&= 3\int_{-1}^{1}x^2\cos(\pi x)\,dx \\
&= 3\left(\left[\frac{1}{\pi}x^2\sin(\pi x)\right]_{-1}^{1}-2\int_{-1}^{1}\frac{1}{\pi}x\sin(\pi x)\,dx\right) \\
&= 3\left(\frac{1}{\pi}\sin(\pi)-\frac{1}{\pi}\sin(-\pi)-\left[-\frac{2}{\pi^2}x\cos(\pi x)\right]_{-1}^{1}-\frac{2}{\pi^2}\int_{-1}^{1}\cos(\pi x)\,dx\right)
\end{aligned}$$

$$
\begin{aligned}
&= 3\left(-\left(-\frac{2}{\pi^2}\cos(\pi) - \frac{2}{\pi^2}\cos(-\pi)\right) - \frac{2}{\pi^2}(f,\phi_1)\right)\\
&= 3\left(-\frac{2}{\pi^2} - \frac{2}{\pi^2}\right)\\
&= -\frac{12}{\pi^2}
\end{aligned}
$$

thus giving

$$f_3(x) = f_2(x) + \frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x) = -\frac{15}{2\pi^2}(3x^2-1).$$

(d) [7 points] The following plot compares the best approximations to $f(x)$. Note that f_2 obscures f_1.



The code use to produce it is below.

```
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=cos(pi*x);
f1=x-x;
f2=f1+0;
f3=f2-15*(3*x.^2-1)/(2*pi^2);
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
plot(x,f3,'-g')
xlabel('x')
legend('f(x)','f_1(x)','f_2(x)','f_3(x)','location','best')
saveas(figure(1),'hw17d.eps','epsc')
```