CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 36 · Solutions

Posted Friday 1 November 2013. Due 5pm Wednesday 13 November 2013.

36. [25 points] Let $H^1_D(0,1) = \left\{w \in H^1(0,1) : w(1) = 0\right\}$ and let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Also, let $f \in L^2(0,1)$, let N be a positive integer, and let V_N be a subspace of $H^1_D(0,1)$. Moreover, let $u \in H^1_D(0,1)$ be such that

$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1)$$

and let $u_N \in V_N$ be such that

$$a(u_N,v) = (f,v) \text{ for all } v \in V_N.$$

(a) Show that

$$a(u-u_N, u-u_N) = a(u,u) - a(u_N,u_N).$$

(b) Let $\phi_0, \phi_1, \ldots, \phi_N \in V_N$ and let $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ be the matrix with entries $K_{jk} = a(\phi_{k-1}, \phi_{j-1})$ for $j,k = 1,\ldots,N+1$. Also, let

$$u_N = \sum_{j=0}^{N} \alpha_j \phi_j$$

where, for $k = 1,\ldots,N+1$, $\alpha_{k-1} \in \mathbb{R}$ is the kth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$. Show that

$$\mathbf{c}^T\mathbf{K}\mathbf{c} = a(u_N,u_N).$$

Solution.

(a) [12 points] The properties satisfied by the inner product allow us to say that

$$\begin{aligned} a(u-u_N,u-u_N) &= a(u,u-u_N) - a(u_N,u-u_N) \\ &= a(u,u) - a(u,u_N) - a(u_N,u) + a(u_N,u_N) \\ &= a(u,u) - 2a(u,u_N) + a(u_N,u_N). \end{aligned}$$

Now, $u_N \in V_N$ and so the fact that

$$a(u_N,v) = (f,v) \text{ for all } v \in V_N$$

means that

$$a(u_N,u_N) = (f,u_N).$$

Moreover, $u_N \in H^1_D(0,1)$, since V_N is a subspace of $H^1_D(0,1)$ and $u_N \in V_N$, and so the fact that

$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1)$$

means that

$$a(u,u_N) = (f,u_N).$$

So,

$$a(u,u) - 2a(u,u_N) + a(u_N,u_N) = a(u,u) - 2(f,u_N) + (f,u_N) = a(u,u) - (f,u_N).$$

Therefore,

$$a(u-u_N, u-u_N) = a(u,u) - (f,u_N) = a(u,u) - a(u_N,u_N)$$

because

$$a(u_N,u_N) = (f,u_N).$$

(b) [13 points] We first compute that

$$\mathbf{Kc} = \mathbf{d}$$

where $\mathbf{d} \in \mathbb{R}^N$ is the vector with entries

$$d_j = \sum_{k=1}^{N+1} a(\phi_{k-1},\phi_{j-1})c_k = \sum_{k=1}^{N+1} a(\phi_{k-1},\phi_{j-1})\alpha_{k-1} = \sum_{k=0}^{N} a(\phi_k,\phi_{j-1})\alpha_k$$

for $j = 1,\ldots,N+1$. Moreover, since

$$u_N = \sum_{j=0}^{N} \alpha_j\phi_j = \sum_{k=0}^{N} \alpha_k\phi_k,$$

the properties satisfied by the inner product yield that

$$\sum_{k=0}^{N} a(\phi_k,\phi_{j-1})\alpha_k = a\left(\sum_{k=0}^{N} \alpha_k\phi_k, \phi_{j-1}\right) = a(u_N,\phi_{j-1})$$

and so

$$d_j = a(u_N,\phi_{j-1})$$

for $j = 1,\ldots,N+1$. Therefore,

$$\begin{aligned}
\mathbf{c}^T\mathbf{Kc} &= \mathbf{c}^T\mathbf{d} \\
&= \sum_{j=1}^{N+1} c_j a(u_N,\phi_{j-1}) \\
&= \sum_{j=1}^{N+1} \alpha_{j-1} a(u_N,\phi_{j-1}) \\
&= \sum_{j=0}^{N} \alpha_j a(u_N,\phi_j) \\
&= a\left(u_N, \sum_{j=0}^{N} \alpha_j\phi_j\right) \\
&= a(u_N,u_N)
\end{aligned}$$

by the properties satisfied by the inner product and the fact that

$$u_N = \sum_{j=0}^{N} \alpha_j\phi_j.$$