CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 3 · Solutions

Posted Wednesday 28, January 2015. Due 5pm Wednesday 4, February 2015.

Please write your name and instructor on your homework.

1. [20 points: 10 each]

   (a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{Au}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

   $$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Since f is linear, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

   (b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$.

   (Thus any linear function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix-vector product.)

Solution.

(a) Write $\mathbf{u} \in \mathbb{R}^2$ in the form

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}.$$

Any matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ are the columns of $\mathbf{A}$. Now the matrix-vector product $\mathbf{Au}$ is a linear combination of the columns of $\mathbf{A}$:

$$\mathbf{Au} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2. \tag{$*$}$$

We are trying to find a formula for $\mathbf{A}$ (or, equivalently, for $\mathbf{a}_1$ and $\mathbf{a}_2$) such that $f(\mathbf{u}) = \mathbf{Au}$. Using the hint, we have

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2). \tag{$**$}$$

Comparing $(*)$ and $(**)$ and equating like terms, we see that

$$\mathbf{a}_1 = f(\mathbf{e}_1), \qquad \mathbf{a}_2 = f(\mathbf{e}_2),$$

and hence

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) \end{bmatrix}.$$

(b) We follow the same idea as in part (a). Write $\mathbf{u} \in \mathbb{R}^n$ as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{A} \in \mathbb{R}^{m\times n}$ by column,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. Equating like terms in

$$\mathbf{Au} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots u_n\mathbf{a}_n$$

and

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2) + \cdots u_n f(\mathbf{e}_n),$$

we arrive at

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \cdots & f(\mathbf{e}_n) \end{bmatrix}.$$

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2. [30 points: 5 each]

   Recall that a function $f : \mathcal{V} \to \mathcal{W}$ that maps a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a *linear operator* provided (1) $f(u+v) = f(u) + f(v)$ for all u, v in $\mathcal{V}$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$.

   Demonstrate whether each of the following functions is a linear operator.
   (Show that both properties hold, or give an example showing that one of the properties must fail.)

   (a) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{Au}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$.

   (b) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{Au} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

   (d) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $f(\mathbf{X}) = \mathbf{AX} + \mathbf{XB}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

   (e) $L : C^1[0,1] \to C[0,1]$, $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

   (f) $L : C^2[0,1] \to C[0,1]$, $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

Solution.

(a) This function *is* a linear operator.
Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{Au} + \mathbf{Av} = f(\mathbf{u}) + f(\mathbf{v}),$$

and

$$f(\alpha\mathbf{v}) = \mathbf{A}(\alpha\mathbf{v}) = \alpha\mathbf{Av} = \alpha f(\mathbf{v}).$$

(b) This function *is not* a linear operator (provided $\mathbf{b} \neq \mathbf{0}$).
Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}+\mathbf{v}) + \mathbf{b} = \mathbf{Au} + \mathbf{Avb} = f(\mathbf{u}) + \mathbf{Av},$$

which does not equal $f(\mathbf{u}) + f(\mathbf{v})$ when $\mathbf{b} \neq \mathbf{0}$.
[**GRADERS**: please *do not* take off points if the solutions fail to note the special case of $\mathbf{b} = \mathbf{0}$.]

(c) This function *is not* a linear operator.
Suppose $\mathbf{x} \in \mathbb{R}^n$. Then

$$f(\alpha\mathbf{x}) = (\alpha\mathbf{x})^T(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^T\mathbf{x} = \alpha^2 f(\mathbf{x}),$$

and thus if $\alpha \neq \pm 1$, we have $f(\alpha\mathbf{x}) \neq \alpha f(\mathbf{x})$.

(d) This function *is* a linear operator.
Suppose $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n\times n}$. Then

$$f(\mathbf{X}+\mathbf{Y}) = \mathbf{A}(\mathbf{X}+\mathbf{Y}) + (\mathbf{X}+\mathbf{Y})\mathbf{B} = \mathbf{AX}+\mathbf{XB}+\mathbf{AY}+\mathbf{YB} = f(\mathbf{X}) + f(\mathbf{Y}),$$

and if $\alpha \in \mathbb{R}$, then

$$f(\alpha\mathbf{X}) = \mathbf{A}(\alpha\mathbf{X}) + (\alpha\mathbf{X})\mathbf{B} = \alpha(\mathbf{AX}+\mathbf{XB}) = \alpha f(\mathbf{X}).$$

(e) This function *is not* a linear operator.
Suppose that $u(x) = x$. Then

$$Lu = u\frac{du}{dx} = x \cdot 1 = x,$$

yet for any $\alpha \in \mathbb{R}$ we have

$$L(\alpha u) = (\alpha u)\frac{d(\alpha u)}{dx} = (\alpha x)\cdot\alpha = \alpha^2 x,$$

so if $\alpha \neq \pm 1$, we have $L(\alpha u) \neq \alpha L u$.

(f) This function *is* a linear operator.
Suppose that $u, v \in C^2[0,1]$. Then

$$\begin{aligned} L(u+v) &= \frac{d^2(u+v)}{dx^2} - \sin(x)\frac{d(u+v)}{dx} + \cos(x)(u+v) \\ &= \frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u + \frac{d^2v}{dx^2} - \sin(x)\frac{dv}{dx} + \cos(x)v \\ &= Lu + Lv, \end{aligned}$$

and for any $\alpha \in \mathbb{R}$,

$$L(\alpha u) = \frac{d^2(\alpha u)}{dx^2} - \sin(x)\frac{d(\alpha u)}{dx} + \cos(x)(\alpha u) = \alpha\Big(\frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u\Big) = \alpha L(u).$$

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3. [25 points: 5 points for (a), 10 points each for (b)-(c)]

In this problem we'll consider a linear operator mapping to and from a very specific vector space, and use it to explore what an operator inverse can look like.

Consider the V defined as

$$V = \left\{ u(x) = \sum_{j=1}^{N} c_j \sin(j\pi x), \quad c_j \in \mathbb{R} \right\}.$$

In other words, V is the set of all functions that are linear combinations of a finite number of different sine functions. This means that, for each $u \in V$, there is a set of coefficients $c_1, \ldots, c_N$ that is also associated with u.

(a) Show that V is a subspace of the vector space $C^2_D[0,1]$, where

$$C^2_D[0,1] = \{u(x) \in C^2[0,1], \quad u(0) = u(1) = 0\}.$$

(b) Let the operator L be defined as

$$Lu = -\frac{\partial^2 u}{\partial x^2}$$

Show that, for $u \in V$, $Lu \in V$. This shows that L can be viewed as

$$L : V \to V,$$

a map from V to V.

(c) We can define the operator $\tilde{L} : V \to V$ as

$$\tilde{L}u = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x).$$

Show that both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$.

Since both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$, we can refer to $\tilde{L}$ as the inverse L^{-1} of $L : V \to V$.

Solution.

Let V be defined as

$$V = \left\{ u \in C^2_D[0,1] : u(x) = \sum_{j=1}^{N} c_j \sin(j\pi x), \quad c_j \in \mathbb{R} \right\}.$$

(a) Suppose $u(x) \in V$. Then,

$$u(0) = \sum_{j=1}^{N} c_j \sin(0), \qquad u(1) = \sum_{j=1}^{N} c_j \sin(j\pi)$$

Since $\sin(0) = 0$, $u(0) = 0$. Similarly, since j is an integer in the above expression, $\sin(j\pi 1) = \sin(j\pi) = 0$, since sine evaluated at any multiple of π is zero, and $u(1) = 0$ as well. Since the sum of sines is a continuous function, any $u(x) \in V$ is also contained in $C^2_D[0,1]$, and V is contained in $C^2_D[0,1]$.

Next, we just need to show that V itself is a vector space. If $u(x)$ and $v(x)$ are in the set of V, then we have the representations

$$u(x) = \sum_{j=1}^{N} d_j \sin(j\pi x), \qquad v(x) = \sum_{j=1}^{N} e_j \sin(j\pi x)$$

for $d_j, e_j \in \mathbb{R}$. We wish to show that, for any $\alpha, \beta \in \mathbb{R}$, the sum $w(x) = \alpha u(x) + \beta v(x)$ is contained in V as well.

$$\begin{aligned} w(x) = \alpha u(x) + \beta v(x) &= \alpha \sum_{j=1}^{N} d_j \sin(j\pi x) + \beta \sum_{j=1}^{N} e_j \sin(j\pi x) \\ &= \sum_{j=1}^{N} \underbrace{\alpha d_j}_{\tilde{d}_j} \sin(j\pi x) + \sum_{j=1}^{N} \underbrace{\beta e_j}_{\tilde{e}_j} \sin(j\pi x) \\ &= \sum_{j=1}^{N} \underbrace{(\tilde{d}_j + \tilde{e}_j)}_{c_j} \sin(j\pi x) \\ &= \sum_{j=1}^{N} c_j \sin(j\pi x) \end{aligned}$$

Therefore $w \in V$, and since V is contained in $C_D^2[0,1]$ as well, V is a subspace of $C_D^2[0,1]$.

(b) The operator L be defined as

$$Lu = -\frac{\partial^2 u}{\partial x^2}.$$

We are going to show that for all $u \in V$, $Lu \in V$ as well. Let

$$u = \sum_{j=1}^{N} c_j \sin(j\pi x) \in V$$

for $c_j \in \mathbb{R}$. Then

$$\begin{aligned} Lu = -\frac{\partial^2 u}{\partial x^2} &= -\frac{\partial^2}{\partial x^2} \left(\sum_{j=1}^{N} c_j \sin(j\pi x) \right) \\ &= -\frac{\partial}{\partial x} \left[\sum_{j=1}^{N} c_j \frac{\partial(\sin(j\pi x))}{\partial x} \right] \\ &= -\frac{\partial}{\partial x} \left[\sum_{j=1}^{N} c_j (j\pi) \cos(j\pi x) \right] \\ &= -\sum_{j=1}^{N} c_j (j\pi) \frac{\partial(\cos(j\pi x))}{\partial x} \\ &= \sum_{j=1}^{N} c_j (j\pi)^2 \sin(j\pi x). \end{aligned}$$

If we define $\tilde{c}_j = c_j (j\pi)^2 \in \mathbb{R}$ then

$$Lu = \sum_{j=1}^{N} \tilde{c}_j \sin(j\pi x)$$

which is also in V for all $\tilde{c}_j \in \mathbb{R}$. This shows that L can be viewed as

$$L : V \to V,$$

a map from V to V.

(c) We define the operator $\tilde{L} : V \to V$ as

$$\tilde{L}u = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x).$$

We will show that $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$. First, let us show $L\tilde{L}u = u$. From part (b) we can deduce that if $u = \sum_{j=1}^{N} c_j \sin(j\pi x)$

$$Lu = L\left(\sum_{j=1}^{N} c_j \sin(j\pi x)\right) = \sum_{j=1}^{N} c_j (j\pi)^2 \sin(j\pi x)$$

Then

$$\begin{aligned} L\tilde{L}u = L(\tilde{L}u) \;&=\; L\left(\sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x)\right) \\ &= \sum_{j=1}^{N} \frac{c_j (j\pi)^2}{(j\pi)^2} \sin(j\pi x) \\ &= \sum_{j=1}^{N} c_j \sin(j\pi x) \\ &= u \end{aligned}$$

Now, let us show $\tilde{L}Lu = u$ as well. Remember that from problem if $u = \sum_{j=1}^{N} c_j \sin(j\pi x)$ then $\tilde{L}u$ is given as follows

$$\tilde{L}u = \tilde{L}\left(\sum_{j=1}^{N} c_j \sin(j\pi x)\right) = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x)$$

Then, similarly

$$\begin{aligned} \tilde{L}Lu = \tilde{L}(Lu) \;&=\; \tilde{L}\left(\sum_{j=1}^{N} c_j (j\pi)^2 \sin(j\pi x)\right) \\ &= \sum_{j=1}^{N} \frac{c_j (j\pi)^2}{(j\pi)^2} \sin(j\pi x) \\ &= \sum_{j=1}^{N} c_j \sin(j\pi x) \\ &= u \end{aligned}$$

Since both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$, we can refer to $\tilde{L}$ as the inverse L^{-1} of $L : V \to V$.

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4. [25 points: 5 each]

Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

(a) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V}=C^1[0,1]$.

(b) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

(c) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V}=C[0,1]$.

(d) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 (u(x)+v(x))\,dx$ where $\mathcal{V}=C[0,1]$.

(e) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V}=C[-1,1]$.

Solution.

(a) [5 points] *This mapping is not an inner product*: it is not symmetric and it is not positive definite. The mapping is not symmetric. For example, if $u(x)=1$ and $v(x)=x$, then

$$(u,v)=\int_0^1 u(x)v'(x)\,dx=\int_0^1 1\,dx=1,$$

yet

$$(v,u)=\int_0^1 v(x)u'(x)\,dx=\int_0^1 0\,dx=0.$$

The mapping is also not positive definite. For example, if $u(x)=1$, then $(u,u)=0$ and if $u(x)=1-x$, then

$$(u,u)=\int_0^1 (1-x)(-1)\,dx=-1/2.$$

For what it is worth, we note that the mapping is linear in the first argument since

$$(\alpha u+\beta v,w)=\alpha\int_0^1 u(x)w'(x)\,dx+\beta\int_0^1 v(x)w'(x)\,dx=\alpha(u,w)+\beta(v,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$. It is also linear in the second argument since

$$(u,\alpha v+\beta w)=\alpha\int_0^1 u(x)v'(x)\,dx+\beta\int_0^1 u(x)w'(x)\,dx=\alpha(u,v)+\beta(u,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$.

(b) [5 points] *This mapping is not an inner product*: it is not linear in the first argument. If $u,v,w\in C[0,1]$ and $\alpha,\beta\in\mathbb{R}$ then

$$(\alpha u+\beta v,w)=\int_0^1 |\alpha u(x)+\beta v(x)||w(x)|\,dx$$

and

$$\alpha(u,w)+\beta(v,w)=\alpha\int_0^1 |u(x)||w(x)|\,dx+\beta\int_0^1 |v(x)||w(x)|\,dx.$$

However, if $u(x)=1$, $v(x)=0$, $w(x)=1$, $\alpha=-1$ and $\beta=0$ then

$$(\alpha u+\beta v,w)=\int_0^1 |-1||1|\,dx=\int_0^1 1\,dx=1$$

but

$$\alpha(u,w)+\beta(v,w) = -\int_0^1 |1||1|\,dx = -\int_0^1 1\,dx = -1$$

and so the mapping is not linear in the first argument.
The mapping is symmetric, as

$$(u,v) = \int_0^1 |u(x)||v(x)|\,dx = \int_0^1 |v(x)||u(x)|\,dx = (v,u)$$

for all $u, v \in C[0,1]$.
Moreover, the mapping is positive definite as for all $u \in C[0,1]$

$$(u,u) = \int_0^1 |u(x)|^2\,dx$$

is the integral of a nonnegative function, and hence is nonnegative and $(u,u) = 0$ only if $u = 0$.

(c) [5 points] *This mapping is an inner product.*
The mapping is symmetric, as

$$(u,v) = \int_0^1 u(x)v(x)e^{-x}\,dx = \int_0^1 v(x)u(x)e^{-x}\,dx = (v,u)$$

for all $u, v \in C[0,1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x}\,dx \\ &= \alpha \int_0^1 u(x)w(x)e^{-x}\,dx + \beta \int_0^1 v(x)w(x)e^{-x}\,dx \\ &= \alpha(u,w) + \beta(v,w)\end{aligned}$$

for all $u, v, w \in C[0,1]$ and all $\alpha, \beta \in \mathbb{R}$.
The function e^{-x} is positive valued for all $x \in [0,1]$, so we have that

$$(u,u) = \int_0^1 (u(x))^2 e^{-x} dx$$

is the integral of a nonnegative function, and hence is also nonnegative. If $(u,u) = 0$ then $(u(x))^2 e^{-x} = 0$ for all $x \in [0,1]$ and, since $e^{-x} > 0$ for all $x \in [0,1]$, this means that $u(x) = 0$ for all $x \in [0,1]$, i.e., $u = 0$. Hence, the mapping is positive definite.

(d) [5 points] *This mapping is not an inner product*: it is not linear in the first argument and it is not positive definite.
If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x) + w(x))\,dx$$

and

$$\alpha(u,w) + \beta(v,w) = \alpha \int_0^1 (u(x) + w(x))\,dx + \beta \int_0^1 (v(x) + w(x))\,dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = 2$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 (2+1)\,dx = \int_0^1 3\,dx = 3$$

but

$$\alpha(u,w) + \beta(v,w) = 2\int_0^1 (1+1)\,dx = 2\int_0^1 2\,dx = 4$$

and so $(\cdot,\cdot)$ is not linear in the first argument.
The mapping $(\cdot,\cdot)$ is also not positive definite. For example, if $u(x) = -1$, then

$$(u,u) = \int_0^1 (u(x) + u(x))\,dx = \int_0^1 -2\,dx = -2 < 0.$$

The mapping is symmetric, as

$$(u,v) = \int_0^1 (u(x) + v(x))\,dx = \int_0^1 (v(x) + u(x))\,dx = (v,u)$$

for all $u, v \in C[0,1]$.

(e) [5 points] *This mapping is not an inner product*: it is not positive definite.
If $w(x) = 1$ for all $x \in [-1,1]$ then $w \in C[-1,1]$ and $w \neq 0$ but

$$(w,w) = \int_{-1}^1 xw(x)w(x)\,dx = \int_{-1}^1 x\,dx = \left[\frac{1}{2}x^2\right]_{-1}^1 = \frac{1}{2}\left(1^2 - (-1)^2\right) = \frac{1}{2}(1-1) = 0$$

and so $(\cdot,\cdot)$ is not positive definite.
The mapping is symmetric, as

$$(u,v) = \int_{-1}^1 xu(x)v(x)\,dx = \int_{-1}^1 xv(x)u(x)\,dx = (v,u)$$

for all $u, v \in C[-1,1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_{-1}^1 x(\alpha u(x) + \beta v(x))w(x)\,dx \\ &= \alpha\int_{-1}^1 xu(x)w(x)\,dx + \beta\int_{-1}^1 xv(x)w(x)\,dx \\ &= \alpha(u,w) + \beta(v,w)\end{aligned}$$

for all $u, v, w \in C[-1,1]$ and all $\alpha, \beta \in \mathbb{R}$.