CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 22 · Solutions

Posted Wednesday 19 February 2014. Due 1pm Friday 28 February 2014.

22. [25 points]

All parts of this question should be done by hand.

(a) Let

$$\mathbf{D} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$\mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to

$$\mathbf{Dc} = \mathbf{g}.$$

(b) Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{x} \in \mathbb{R}^3$ to

$$\mathbf{Ax} = \mathbf{b}.$$

Solution.

(a) [10 points] Since,

$$\lambda\mathbf{I} - \mathbf{D} = \begin{bmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{bmatrix}$$

we have that

$$\det(\lambda\mathbf{I} - \mathbf{D}) = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$

and so

$$\det(\lambda\mathbf{I} - \mathbf{D}) = 0$$

when $\lambda = 3$ or $\lambda = 5$. Hence, the eigenvalues of $\mathbf{D}$ are

$$\lambda_1 = 3$$

and

$$\lambda_2 = 5.$$

Moreover,

$$(\lambda_1\mathbf{I} - \mathbf{D})\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -f_1 - f_2 \\ -f_1 - f_2 \end{bmatrix}$$

and so to make this vector zero we need to set $f_2 = -f_1$. Hence, any vector of the form

$$\begin{bmatrix} f_1 \\ -f_1 \end{bmatrix}$$

where f_1 is a nonzero constant is an eigenvector of $\mathbf{D}$ corresponding to the eigenvalue λ_1. Let us choose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Furthermore,

$$(\lambda_2 \mathbf{I} - \mathbf{D}) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{bmatrix}$$

and so to make this vector zero we need to set $d_2 = d_1$. Hence, any vector of the form

$$\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}$$

where d_1 is a nonzero constant is an eigenvector of $\mathbf{D}$ corresponding to the eigenvalue λ_2. Let us choose

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\mathbf{D} = \mathbf{D}^T$, $\mathbf{D}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\mathbf{D}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$ and $\lambda_1 \neq \lambda_2$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Now,

$$\mathbf{g} \cdot \mathbf{v}_1 = 2 - 3 = -1,$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

$$\mathbf{g} \cdot \mathbf{v}_2 = 2 + 3 = 5,$$

and

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 1^2 + 1^2 = 1 + 1 = 2.$$

The spectral method then yields that

$$\begin{aligned} \mathbf{c} &= \frac{1}{\lambda_1} \frac{\mathbf{g} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{1}{\lambda_2} \frac{\mathbf{g} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\ &= \frac{1}{3} \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5} \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{6} + \frac{3}{6} \\ \frac{1}{6} + \frac{3}{6} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{6} \\ \frac{4}{6} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}. \end{aligned}$$

(b) [15 points] For this matrix $\mathbf{A}$ we have

$$\lambda \mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix},$$

and hence the characteristic polynomial is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - 3)(\lambda^2 - 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial, which we label

$$\lambda_1 = -1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3.$$

To compute the eigenvectors associated with the eigenvalue $\lambda_1 = -1$, we seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -u_2 + u_3 \\ u_2 - u_3 \end{bmatrix}.$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = u_2$. Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_1 = -1$.
To compute the eigenvectors associated with the eigenvalue $\lambda_2 = 1$ we now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_2 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ u_2 + u_3 \\ u_3 + u_2 \end{bmatrix}.$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = -u_2$. Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ -u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_2 = 1$.
To compute the eigenvectors associated with the eigenvalue $\lambda_3 = 3$ we now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_3 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2 + u_3 \\ u_2 + 3u_3 \end{bmatrix}.$$

To make the second component zero we need $u_2 = -u_3/3$, while to make the third component zero we need $u_3 = -u_2/3$. The only way to accomplish both is to set $u_2 = u_3 = 0$. Thus any vector of the form

$$\begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, \quad u_1 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_3 = 3$.
We choose the eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We can compute that

$$\mathbf{u}_1^T\mathbf{u}_2 = \mathbf{u}_2^T\mathbf{u}_1 = 0\cdot 0 + (1/\sqrt{2})\cdot(1/\sqrt{2}) + (1/\sqrt{2})\cdot(-1/\sqrt{2}) = 0,$$

$$\mathbf{u}_1^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_1 = 0\cdot 1 + (1/\sqrt{2})\cdot 0 + (1/\sqrt{2})\cdot 0 = 0,$$

and

$$\mathbf{u}_2^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_2 = 0\cdot 1 + (1/\sqrt{2})\cdot 0 + (-1/\sqrt{2})\cdot 0 = 0.$$

Now, for $j = 1, 2, 3$, $\mathbf{A}\mathbf{u}_j = \lambda_j\mathbf{u}_j$ and $\mathbf{u}_j^T\mathbf{u}_j = 1$. Since $\mathbf{A} = \mathbf{A}^T$, the spectral method then yields that

$$\mathbf{x} = \sum_{j=1}^{3} \frac{1}{\lambda_j}\frac{\mathbf{u}_j^T\mathbf{b}}{\mathbf{u}_j^T\mathbf{u}_j}\mathbf{u}_j = \sum_{j=1}^{3}\frac{\mathbf{u}_j^T\mathbf{b}}{\lambda_j}\mathbf{u}_j.$$

We can compute that

$$\mathbf{u}_1^T\mathbf{b} = 0\cdot 2 + (1/\sqrt{2})\cdot(-1) + (1/\sqrt{2})\cdot 3 = \sqrt{2},$$

$$\mathbf{u}_2^T\mathbf{b} = 0\cdot 2 + (1/\sqrt{2})\cdot(-1) + (-1/\sqrt{2})\cdot 3 = -2\sqrt{2},$$

and

$$\mathbf{u}_3^T\mathbf{b} = 1\cdot 2 + 0\cdot(-1) + 0\cdot 3 = 2,$$

and hence

$$\mathbf{x} = \frac{\sqrt{2}}{-1}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{-2\sqrt{2}}{1}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -3 \\ 1 \end{bmatrix}.$$

We can multiply $\mathbf{A}\mathbf{x}$ out to verify that the desired $\mathbf{b}$ is obtained.