CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 38 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

38. [25 points]

All parts of this question should be done by hand.

Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let $\phi_0 \in H^1(0,1)$ be defined by

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

let $\phi_j \in H^1_D(0,1)$ be defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let $\phi_{N+1} \in H^1_D(0,1)$ be defined by

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the symmetric bilinear form $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v, w) = a(v, w) + (v, w).$$

Also, let $f \in L^2(0,1)$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Moreover, let $u \in H^1(0,1)$ be such that $u(0) = \alpha$ and

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Let $V_N = \operatorname{span}\{\phi_0, \phi_1, \ldots, \phi_{N+1}\}$ and let $V_{N,D} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_{N+1}\}$. Let $u_N \in V_N$ be such that $u_N(0) = \alpha$ and

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

(a) We can write

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j$$

where, for $j = 1, 2, \ldots, N+1$, c_j is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$ which is the solution to

$$\mathbf{K}\mathbf{c} = \mathbf{b}.$$

What are the entries of the matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ and the vector $\mathbf{b} \in \mathbb{R}^{N+1}$?

(b) Show that

$$B(u - u_N, u - u_N) = B(u, u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0).$$

(c) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = 0$, $\rho = 0$ and $N = 1$. Note that, when $N = 1$,

$$\int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$\int_0^{1/2} \phi_0(x)\phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\phi_2(x)\,dx = \frac{1}{6};$$

and

$$\int_0^{1/2} \phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\,dx = \frac{1}{4}.$$

(d) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = -1$, $\rho = 1$ and $N = 1$.

Solution.

(a) [5 points] The function

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j$$

is such that $u_N(0) = \alpha$ and will be such that

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}$$

when, for $j = 1, 2, \ldots, N+1$, the c_j are such that

$$B\left(\alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j, \phi_k\right) = (f, \phi_k) + \rho\phi_k(1) \text{ for } k = 1, 2, \ldots, N+1,$$

or equivalently,

$$\sum_{j=1}^{N+1} c_j B(\phi_j, \phi_k) = (f, \phi_k) + \rho\phi_k(1) - \alpha B(\phi_0, \phi_k) \text{ for } k = 1, 2, \ldots, N+1.$$

We can write this system of equations in the form

$$\mathbf{K}\mathbf{c} = \mathbf{b}$$

where $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ is the matrix with entries

$$K_{jk} = B(\phi_k, \phi_j)$$

for $j, k = 1, 2, \ldots, N+1$ and $\mathbf{b} \in \mathbb{R}^{N+1}$ is the vector with entries

$$b_j = (f, \phi_j) + \rho\phi_j(1) - \alpha B(\phi_0, \phi_j) = \begin{cases} (f, \phi_1) - \alpha B(\phi_0, \phi_1) & \text{if } j = 1, \\ (f, \phi_{N+1}) + \rho & \text{if } j = N+1, \\ (f, \phi_j) & \text{otherwise.} \end{cases}$$

for $j = 1, 2, \ldots, N+1$.

(b) [6 points] Since $V_{N,D}$ is a subspace of $H^1_D(0,1)$, the fact that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1)$$

means that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

Moreover,

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

Therefore the properties satisfied by a symmetric bilinear form allow us to say that, for all $v \in V_{N,D}$,

$$\begin{aligned} B(u - u_N, v) &= B(u, v) - B(u_N, v) \\ &= (f, v) + \rho v(1) - ((f, v) + \rho v(1)) \\ &= 0. \end{aligned}$$

Consequently,

$$B(u - u_N, v) = 0 \text{ for all } v \in V_{N,D}.$$

The properties satisfied by a symmetric bilinear form allow us to say that

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u, u - u_N) - B(u_N, u - u_N) \\ &= B(u, u) - B(u, u_N) - B(u_N, u) + B(u_N, u_N) \\ &= B(u, u) - 2B(u, u_N) + B(u_N, u_N). \end{aligned}$$

Now, $u_N - \alpha\phi_0 \in V_{N,D}$ and so the fact that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_{N,D}$$

means that

$$B(u - u_N, u_N - \alpha\phi_0) = 0$$

and hence

$$B(u, u_N) = B(u_N, u_N) + \alpha B(u - u_N, \phi_0)$$

since the properties satisfied by a symmetric bilinear form mean that

$$\begin{aligned} B(u - u_N, u_N - \alpha\phi_0) &= B(u - u_N, u_N) - \alpha B(u - u_N, \phi_0) \\ &= B(u, u_N) - B(u_N, u_N) - \alpha B(u - u_N, \phi_0). \end{aligned}$$

Therefore,

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u, u) - 2\left(B(u_N, u_N) + \alpha B(u - u_N, \phi_0)\right) + B(u_N, u_N) \\ &= B(u, u) - 2B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0) + B(u_N, u_N) \\ &= B(u, u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0). \end{aligned}$$

(c) [9 points] When $N = 1$, $f(x) = 2$, $\alpha = 0$ and $\rho = 0$,

$$\mathbf{K} = \left[\begin{array}{cc} B(\phi_1, \phi_1) & B(\phi_2, \phi_1) \\ B(\phi_1, \phi_2) & B(\phi_2, \phi_2) \end{array} \right]$$

and

$$\mathbf{b} = \left[\begin{array}{c} (f, \phi_1) \\ (f, \phi_2) \end{array} \right]$$

where

$$\phi_1(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2x - 1 & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

and hence

$$\phi_1'(x) = \begin{cases} 2 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ -2 & \text{if } x \in \left(\frac{1}{2}, 1\right); \end{cases}$$

and

$$\phi_2'(x) = \begin{cases} 0 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ 2 & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Now,

$$(\phi_1, \phi_1) = \int_0^1 \phi_1(x)\phi_1(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx + \int_{1/2}^1 \phi_1(x)\phi_1(x)\,dx = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3};$$

$$(\phi_1, \phi_2) = \int_0^1 \phi_1(x)\phi_2(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$(\phi_2, \phi_1) = (\phi_1, \phi_2) = \frac{1}{12};$$

and

$$(\phi_2, \phi_2) = \int_0^1 \phi_2(x)\phi_2(x)\,dx = \int_{1/2}^1 \phi_2(x)\phi_2(x)\,dx = \frac{1}{6}.$$

Moreover,

$$\begin{aligned} a(\phi_1, \phi_1) &= \int_0^1 \phi_1'(x)\phi_1'(x)\,dx \\ &= \int_0^{1/2} \phi_1'(x)\phi_1'(x)\,dx + \int_{1/2}^1 \phi_1'(x)\phi_1'(x)\,dx \\ &= \int_0^{1/2} 4\,dx + \int_{1/2}^1 4\,dx \\ &= \frac{4}{2} + \frac{4}{2} \\ &= 4; \end{aligned}$$

$$a(\phi_1, \phi_2) = \int_0^1 \phi_1'(x)\phi_2'(x)\,dx = \int_{1/2}^1 \phi_1'(x)\phi_2'(x)\,dx = \int_{1/2}^1 -4\,dx = -\frac{4}{2} = -2;$$

$$a(\phi_2, \phi_1) = a(\phi_1, \phi_2) = -2;$$

and

$$a(\phi_2, \phi_2) = \int_0^1 \phi_2'(x)\phi_2'(x)\,dx = \int_{1/2}^1 \phi_2'(x)\phi_2'(x)\,dx = \int_{1/2}^1 4\,dx = \frac{4}{2} = 2.$$

Consequently,

$$B(\phi_1,\phi_1) = a(\phi_1,\phi_1) + (\phi_1,\phi_1) = 4 + \frac{1}{3} = \frac{12}{3} + \frac{1}{3} = \frac{13}{3};$$

$$B(\phi_1,\phi_2) = a(\phi_1,\phi_2) + (\phi_1,\phi_2) = -2 + \frac{1}{12} = -\frac{24}{12} + \frac{1}{12} = -\frac{23}{12};$$

$$B(\phi_2,\phi_1) = B(\phi_1,\phi_2) = -\frac{23}{12};$$

and

$$B(\phi_2,\phi_2) = a(\phi_2,\phi_2) + (\phi_2,\phi_2) = 2 + \frac{1}{6} = \frac{12}{6} + \frac{1}{6} = \frac{13}{6}.$$

Furthermore,

$$(f,\phi_1) = 2\int_0^1 \phi_1(x)\,dx = 2\left(\int_0^{1/2} \phi_1(x)\,dx + \int_{1/2}^1 \phi_1(x)\,dx\right) = 2\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{4}{4} = 1;$$

and

$$(f,\phi_2) = 2\int_0^1 \phi_2(x)\,dx = 2\left(\int_0^{1/2} \phi_2(x)\,dx + \int_{1/2}^1 \phi_2(x)\,dx\right) = 2\left(0 + \frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}.$$

Hence,

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix} B(\phi_1,\phi_1) & B(\phi_2,\phi_1) \\ B(\phi_1,\phi_2) & B(\phi_2,\phi_2) \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{3} & -\frac{23}{12} \\ -\frac{23}{12} & \frac{13}{6} \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{b} &= \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.\end{aligned}$$

(d) [5 points] When $N = 1$, $f(x) = 2$, $\alpha = -1$ and $\rho = 1$,

$$\mathbf{K} = \begin{bmatrix} B(\phi_1,\phi_1) & B(\phi_2,\phi_1) \\ B(\phi_1,\phi_2) & B(\phi_2,\phi_2) \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} (f,\phi_1) + B(\phi_0,\phi_1) \\ (f,\phi_2) + 1 \end{bmatrix}$$

where

$$\phi_0(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

$$\phi_1(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2x - 1 & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

and hence

$$\phi_0'(x) = \begin{cases} -2 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ 0 & \text{if } x \in \left(\frac{1}{2}, 1\right); \end{cases}$$

$$\phi_1'(x) = \begin{cases} 2 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ -2 & \text{if } x \in \left(\frac{1}{2}, 1\right); \end{cases}$$

and

$$\phi_2'(x) = \begin{cases} 0 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ 2 & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Now,

$$(\phi_0, \phi_1) = \int_0^1 \phi_0(x)\phi_1(x)\,dx = \int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \frac{1}{12};$$

and

$$a(\phi_0, \phi_1) = \int_0^1 \phi_0'(x)\phi_1'(x)\,dx = \int_0^{1/2} -4\,dx = -\frac{4}{2} = -2.$$

Consequently,

$$B(\phi_0, \phi_1) = a(\phi_0, \phi_1) + (\phi_0, \phi_1) = -2 + \frac{1}{12} = -\frac{24}{12} + \frac{1}{12} = -\frac{23}{12}.$$

Moreover, in part (c) we computed that

$$B(\phi_1, \phi_1) = \frac{13}{3};$$

$$B(\phi_1, \phi_2) = B(\phi_2, \phi_1) = -\frac{23}{12};$$

$$B(\phi_2, \phi_2) = \frac{13}{6};$$

$$(f, \phi_1) = 1;$$

and

$$(f, \phi_2) = \frac{1}{2}.$$

Hence,

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} B(\phi_1, \phi_1) & B(\phi_2, \phi_1) \\ B(\phi_1, \phi_2) & B(\phi_2, \phi_2) \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{3} & -\frac{23}{12} \\ -\frac{23}{12} & \frac{13}{6} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{b} &= \begin{bmatrix} (f, \phi_1) + B(\phi_0, \phi_1) \\ (f, \phi_2) + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{23}{12} \\ \frac{1}{2} + 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{12}{12} - \frac{23}{12} \\ \frac{1}{2} + \frac{2}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11}{12} \\ \frac{3}{2} \end{bmatrix}. \end{aligned}$$