We discussed that the spectral method for boundary value problems involves ① the differential operator ② the boundary conditions ③ the inner product.

The first few steps of the method are

1) Determine the vector space V where you are solving the problem
2) Make sure the differential operator $\mathcal{L}$ is symmetric on V with respect to the provided inner product
3) Find the eigenvectors and eigenvalues of $\mathcal{L}$ in V.

These three steps are the <u>foundational</u> or <u>setup</u> steps. The next steps are:

4) Expand the right-hand side with respect to the eigenfunctions (e.g. Compute its "Fourier series").
5) Expand the unknown function in terms of the eigenvectors + unknown coefficients
6) Apply the operator $\mathcal{L}$ to the unknown function
7) Compare coefficients

These 4 steps are known as the <u>solution</u> steps. Steps 1-7 constitute the "spectral method".

Today we are going to discuss the <u>foundation steps</u> 1-3 for various boundary conditions.

Type I: homogeneous Boundary Conditions

Model problem
$$-\frac{\partial^2}{\partial x^2} u = f$$
$$u(0) = u(\ell) = 0$$

Differential operator: $\mathcal{L} = -\frac{\partial^2}{\partial x^2}$

Solution Space: $V = C_D^2[0,\ell] = \{ v \in C^2 \mid v(0) = v(\ell) = 0 \}$

We have seen numerous times in class that with respect to the inner product $(f,g)_V = \int fg$ the operator L is symmetric. e.g. that $(Lf,g) = (f,Lg) \quad \forall f,g \in V$.

To find the eigenvectors / eigenvalues in V we want to solve

$$Lu = \lambda u \quad , \quad u \in V$$

this results in the ODE: $-\frac{\partial^2}{\partial x^2} u = \lambda u$
$u(0) = u(1) = 0$

which is equivalent to $\frac{\partial^2}{\partial x^2} u + \lambda u = 0$
$u(0) = u(1) = 0$

Section 4.2 has a discussion on the equation $\frac{\partial^2}{\partial x^2} u + \lambda u = 0$ and the general solution is:

$$C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

applying the boundary conditions gives that $u(0)=0 \to C_1 = 0$
$u(\ell) = 0 \to \sin(\sqrt{\lambda}\,\ell) = 0$ so that $\sqrt{\lambda}\,\ell = n\pi \to \lambda = n^2\pi^2/\ell^2$

So that the eigenvectors are $\psi_n = \sin\left(\frac{n\pi}{\ell} x\right)$ and eigenvalues $\lambda_n = n^2\pi^2/\ell^2$. One can orthonormalize the $\{\psi_n\}$ by defining $\tilde{\psi}_n = \psi_n / \|\psi_n\| = \dfrac{\psi_n}{\sqrt{(\psi_n, \psi_n)}}$

Type II: Mixed Boundary Conditions

Model problem: $-\frac{\partial^2}{\partial x^2} u = f$
$u(0) = 0 \quad , \quad \frac{\partial u}{\partial x}(\ell) = 0$

Differential operator: L
Solution space: $V = C^2_M[0,\ell] = \{ v \in C^2 \mid v(0) = 0, \frac{\partial v}{\partial x}(\ell) = 0 \}$

Q: is V a vector space? this is required by the theory.
Validate that V is a subvector space of C^2.

Step 2 of the foundational steps requires we check that L is symmetric with respect to a selected inner product. What inner product do we select?

Lets try old faithful: $(f,g)_V = \int fg$. This is an inner product on $C^2_M[0,\ell]$ because its an inner product on C^2 and $C^2_M[0,\ell] \subseteq C^2[0,\ell]$ is a subvector space.

Q: is L symmetric with respect to $(\cdot,\cdot)_V$?

$$(Lf,g) = \int \left(-\frac{\partial^2}{\partial x^2} f\right) g = \int \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \underline{\frac{\partial f}{\partial x} g \Big|_0^\ell}$$? why does this equal zero?

$$= -\int f \frac{\partial^2 g}{\partial x^2} + \underline{f \frac{\partial g}{\partial x}\Big|_0^\ell}$$? why does this equal zero?

$$= \int f \left(-\frac{\partial^2}{\partial x^2} g\right) = (f, Lg)$$

So yes. L is symmetric.

Step 3 says to find the eigenvectors and eigenvalues of L in V. So we want to solve:

$$Lu = \lambda u, \quad u \in V$$

$$\longrightarrow \quad -\frac{\partial^2}{\partial x^2} u = \lambda u, \quad u(0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(\ell) = 0$$

this is equivalent to the ODE:

$$\frac{\partial^2}{\partial x^2} u + \lambda u = 0$$

$$u(0) = 0, \quad \frac{\partial u}{\partial x}(\ell) = 0$$

As already discussed Section 4.2 treats the ODE $\frac{\partial^2}{\partial x^2} u + \lambda u = 0$ which has general solution:

$$C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

Applying boundary conditions gives: $C_1 = u(0) = 0$ and $\frac{\partial u}{\partial x}(\ell) = 0$ implies $\sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}\, \ell) = 0$ so that

$$\sqrt{\lambda}\,\ell = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{(2n-1)\pi}{2}$$

Thus $\lambda = \dfrac{(2n-1)^2\pi^2}{4\ell^2}$ $\quad n = 1, 2, 3, \dots$

So that the eigenvectors and eigenvalues are given by:

$$\varphi_n = \sin\left(\frac{(2n-1)\pi}{2l}x\right), \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4l^2} \quad n=1,2,3,\ldots$$

One can orthonormalize by setting $\tilde{\varphi}_n = \frac{\varphi_n}{\sqrt{(\varphi_n,\varphi_n)}}$

Type III: Neumann Boundary conditions

Suppose we want to solve
$$-\frac{\partial^2}{\partial x^2}u = f$$
$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(l) = 0$$

Differential operator: $L = \frac{-\partial^2}{\partial x^2}$

Vector space: $V = C^2_N[0,l] = \left\{ v \in C^2[0,l] \,\middle|\, \frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial x}(l) = 0 \right\}$

Let $(f,g)_V = \int fg$ be the inner product.

- The steps require we show that L is symmetric. The procedure is just integration by parts.

- One small difference here: Notice that L has a nontrivial null space when considered on $C^2_N[0,l]$. Namely the null space of L in $C^2_N[0,l]$ consists of all <u>constant functions</u>. This means that $\lambda_0 = 0$ will be an eigenvalue since there exists nontrivial vectors with $Lv = 0v = 0$. However, for now we will consider the eigenvectors with eigenvalue $\lambda > 0$.

Now what about the eigenvalues + eigenvectors? Here is where

We want to solve: $-\frac{\partial^2}{\partial x^2} u = \lambda u$

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(\ell) = 0$$

As we saw previously the general solution to the above is (for $\lambda > 0$) given by:

$$C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

so that $\frac{\partial u}{\partial x} = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$

then $\frac{\partial u}{\partial x}(0) = 0 \rightarrow C_2 \sqrt{\lambda} = 0 \rightarrow C_2 = 0$

$\frac{\partial u}{\partial x}(\ell) = 0 \rightarrow C_1 \sin(\sqrt{\lambda}\, \ell) = 0 \rightarrow \sqrt{\lambda}\, \ell = \pm n\pi \quad n = 1, 2, 3, \ldots$

so that $\lambda = \frac{n^2 \pi^2}{\ell^2}$

It follows that the eigenvectors of L in $V = C_N^2[0,\ell]$ are $\psi_n = \cos\left(\frac{n\pi}{\ell} x\right)$ with eigenvalues $\lambda_n = \frac{n^2\pi^2}{\ell^2}$ $\quad n = 1, 2, \ldots$

One can orthonormalize (again) by $\tilde{\psi}_n := \frac{\psi_n}{\|\psi_n\|} = \frac{\psi_n}{\sqrt{(\psi_n, \psi_n)}}$