CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 10 · Solutions

Posted Friday 31 January 2014. Due 1pm Friday 7 February 2014.

10. [25 points]

(a) Demonstrate whether or not the set $S_1 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^3\}$ is a subspace of $\mathbb{R}^2$.

(b) Demonstrate whether or not the set $S_2 = \{\mathbf{x} \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

(c) Demonstrate whether or not the set $S_3 = \{f \in C[0,1] : f(x) \geq 0 \text{ for all } x \in [0,1]\}$ is a subspace of $C[0,1]$.

(d) Demonstrate whether or not the set $S_4 = \left\{ f \in C[0,1] : \max_{x\in[0,1]} f(x) \leq 1 \right\}$ is a subspace of $C[0,1]$.

(e) Demonstrate whether or not the set $S_5 = \{f \in C^2[0,1] : f(1) = 1\}$ is a subspace of $C^2[0,1]$.

(f) Demonstrate whether or not the set $S_6 = \{f \in C^2[0,1] : f(1) = 0\}$ is a subspace of $C^2[0,1]$.

Solution.

(a) [4 points] The set S_1 is not a subspace of $\mathbb{R}^2$.
The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set S_1, yet $2\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not, since $2 \neq 2^3 = 8$. Consequently, the set S_1 is not a subspace of $\mathbb{R}^2$.

(b) [4 points] The set S_2 is a subspace of $\mathbb{R}^3$.
The set S_2 is a subset of $\mathbb{R}^3$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a member of the set S_2. Now, suppose $\mathbf{x}$ and $\mathbf{y}$ are members of the set S_2. Then $3x_1 + 2x_2 + x_3 = 0$ and $3y_1 + 2y_2 + y_3 = 0$. Adding these two equations gives

$$3(x_1 + y_1) + 2(x_2 + y_2) + (x_3 + y_3) = 0,$$

and hence $\mathbf{x} + \mathbf{y}$ is also in the set S_2. Multiplying $3x_1 + 2x_2 + x_3 = 0$ by an arbitrary constant $\alpha \in \mathbb{R}$ gives

$$3(\alpha x_1) + 2(\alpha x_2) + \alpha x_3 = 0,$$

and hence $\alpha\mathbf{x}$ is also in the set S_2. Consequently, the set S_2 is a subspace of $\mathbb{R}^3$.

(c) [4 points] The set S_3 is not a subspace of $C[0,1]$.
Let $f(x) = 1$ for $x \in [0,1]$. Then f is in the set S_3, but a scalar multiple, $-1 \cdot f(x) = -1$ for $x \in [0,1]$, takes negative values and thus violates the requirement for membership in the set S_3. Consequently, the set S_3 is not a subspace of $C[0,1]$.

(d) [4 points] The set S_4 is not a subspace of $C[0,1]$.
Let $f(x) = 1$ for $x \in [0,1]$. Then f is in the set S_4, but a scalar multiple, $2 \cdot f(x) = 2$ for $x \in [0,1]$, takes values greater than one and thus violates the requirement for membership in the set S_4. Consequently, the set S_4 is not a subspace of $C[0,1]$.

(e) [4 points] The set S_5 *is not* a subspace of $C^2[0,1]$.
The function z defined by $z(x) = 0$ for $x \in [0,1]$ is not in the set S_5 since $z(1) = 0$ and thus violates the requirement for membership in the set S_5. Consequently, the set S_5 is not a subspace of $C^2[0,1]$.

(f) [5 points] The set S_6 subspace of $C^2[0,1]$.
The set S_6 is a subset of $C^2[0,1]$ and the function z defined by $z(x) = 0$ for $x \in [0,1]$ is in the set S_6. If f and g are in the set S_6, then $f(1) = g(1) = 0$, so

$$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and hence $f + g$ is in the set S_6. Also, if f is in the set S_6 and $\alpha \in \mathbb{R}$, then

$$(\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0$$

and hence αf is in the set S_6. Consequently, the set S_6 is a subspace of $C^2[0,1]$.