CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 47

Posted Wednesday 20 November 2013. Due 5pm Wednesday 4 December 2013.

47. [25 points] Let the norm $\|\cdot\|:\ \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\|\mathbf{y}\| = \sqrt{\mathbf{y}\cdot\mathbf{y}}.$$

Let the timestep $\Delta t \in \mathbb{R}$ be such that $\Delta t > 0$ and let $t_k = k\Delta t$ for $k = 0, 1, 2, \ldots$. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and consider the problem of finding $\mathbf{x}(t)$ such that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \geq 0$$

and

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Compute $\mathbf{x}(t)$. Note that for real numbers t,

$$e^{it} = \cos(t) + i\sin(t)$$

and

$$e^{-it} = \cos(t) - i\sin(t).$$

(b) How does $\|\mathbf{x}(t)\|$ behave as t increases?

(c) For $k = 0, 1, 2, \ldots$, let $\mathbf{x}_k$ be the approximation to $\mathbf{x}(t_k)$ obtained using the forward Euler method. For all choices of the timestep $\Delta t > 0$, how will $\|\mathbf{x}_k\|$ behave as $k \to \infty$?

(d) For $k = 0, 1, 2, \ldots$, let $\mathbf{x}_k$ be the approximation to $\mathbf{x}(t_k)$ obtained using the backward Euler method. For all choices of the timestep $\Delta t > 0$, how will $\|\mathbf{x}_k\|$ behave as $k \to \infty$?