Class announcement: Exam dates: Exam #1 : 02/20 Exam 2: 03/30
Exam 3 : During Finals week. Date will be set by registrar. (non-comprehensive)

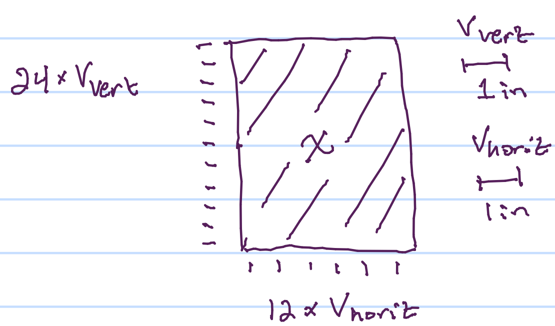
Basis and dimension, Continued

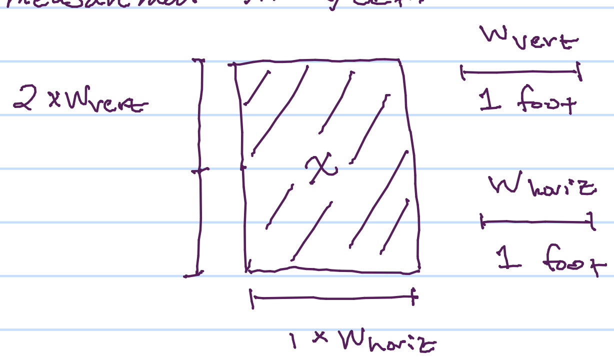
One final important concept related to basis and dimension of a vector space is representing a vector with respect to different bases.

- We already saw that a vector space can have more than one basis. Eg $B_1 = \{v_1, v_2, \dots, v_n\}$, $B_2 = \{w_1, w_2, \dots, w_n\}$

- A basis is just a way to view the vectors in a vector space. They are like a ruler. For example if $\vec{x}$ were an object to be measured and we expressed the measurement of $\vec{x}$ as $\vec{x} = \begin{bmatrix} \text{horizontal length} \\ \text{vertical length} \end{bmatrix}$

Let $\{V_{vert}, V_{horiz}\}$ be unit measurements in inches and $\{W_{vert}, W_{horiz}\}$ be unit measurement in feet.

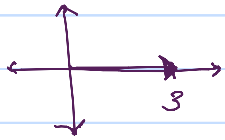




$\Rightarrow \quad \vec{x} = \begin{bmatrix} 24 \\ 12 \end{bmatrix} \qquad\qquad \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Then $\vec{x}$ can be measured in two ways: in inches (by the basis $\{V_{vert}, V_{horiz}\}$) or in feet (by the basis $\{W_{vert}, W_{horiz}\}$)
Notice that $\vec{x}$ is the same in both cases – it is just measured differently by the two bases.

Ex: $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
are two different bases of $\mathbb{R}^2$ and the measure vectors $x \in \mathbb{R}^2$ differently.

Eg if $\vec{x} =$ 

then $\vec{x} = 3\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix}$

So that $\vec{x}$ has the representation given by $\vec{x} = \begin{bmatrix}3\\0\end{bmatrix}$ when "measured" according to the basis $B_1 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$. Likewise $\vec{x} = \frac{3}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}$ so that if $\vec{x}$ is "measured" by the basis $B_2 = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ it looks like $\begin{bmatrix}3/2\\3/2\end{bmatrix}$

Key IDEA: We saw how a choice of basis can impact "what vectors look like". The same is true for matrices! For example consider the bases $V = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $W = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$. If we pick a matrix $A_V = \begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}$ written in terms of the basis V then solving the matrix equation $A_V x = b$ is easy! However if A is "viewed" with the basis W then it looks like $A_W = \begin{bmatrix}3/2 & -5/2\\3/2 & 5/2\end{bmatrix}$ so solving "$A_W x = \tilde{b}$" is a little bit more difficult!

Remark: figuring out what A_W looks like given A_V is done by a process called "change of base". We havent discussed how that process is done (See any youtube video on the subject if interested) however the important idea is that Viewing the "$Ax=b$" problem with a "smart" choice of basis can really simplify your life!

In general the choice of basis for a vector space doesn't change any quantitative attribute for the space. However, when solving problems such as "$Ax=b$" the choice of basis can have a big impact on the ease of computing the solution!

In general a basis that has the property of "orthogonality" will greatly simplify the process of solving such problems. Before we can discuss "orthogonality", and how to construct bases that have the property, we need to introduce the concept of inner products and inner product spaces.

Inner Products and Inner Product Spaces (chapter 3.4)

Everyone remembers the Dot product from calculus:
for $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\vec{x} \cdot \vec{y} = \sum_{i=0}^{n} x_i y_i$

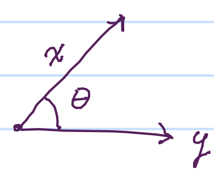
Ex: In $\mathbb{R}^2$ $\vec{a} = (a_1, a_2)$ $\vec{b} = (b_1, b_2)$ then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

What happens if we look at $\vec{x} \cdot \vec{x}$? $\vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + \dots + x_n^2$

Recall: the definition of the length of a vector:
$\vec{x} \in \mathbb{R}^n$, $\vec{x} = (x_1, x_2, \dots, x_n)$ then $\|x\| = \sqrt{(x_1^2 + x_2^2 + \dots + x_n^2)}$
So this means that the length of a vector can be expressed in terms of the dot product:

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$

In fact a very famous mathematical fact is that if two vectors $\vec{x}$ and $\vec{y}$ are separated by an angle of θ then:
$\vec{x} \cdot \vec{y} = \|x\| \|y\| \cos(\theta)$



So what happens if two vectors in $\mathbb{R}^2$ are perpendicular? that is if $\theta = 90°$? Then $\vec{x} \cdot \vec{y} = \|x\| \|y\| \cos(90°) = 0$.
Conversely if $\vec{x} \cdot \vec{y} = 0$ (and neither of $\vec{x}$ or $\vec{y}$ is the zero vector, $\vec{0}$) Then it must be the case that $\theta = 90°$ or $\theta = 180°$.
So we see that the dot product makes determining perpendicular vectors very easy!

Definition: Two vectors $\vec{x}$ and $\vec{y}$ are called orthogonal if $\vec{x} \cdot \vec{y} = 0$

IDEA: The concept of "orthogonality" is a generalized version of the idea of "perpendicular" except we no longer need to appeal to specific angles. This is very helpful in high dimensions where you have many angles that can separate two vectors.

Lets notice a few properties of the dot product. Lets work in $\mathbb{R}^2$ so things are easy to see. All of these results are easily shown to be true in higher dimensions.

1) $\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 = y_1x_1 + y_2x_2 = \vec{y}\cdot\vec{x}$ for any $\vec{x},\vec{y} \in \mathbb{R}^2$

2)
$$\begin{aligned}(a\vec{x}+b\vec{y})\cdot\vec{w} &= (ax_1+by_1)w_1 + (ax_2+by_2)w_2 \\ &= ax_1w_1 + by_1w_1 + ax_2w_2 + by_2w_2 \\ &= a(x_1w_1+x_2w_2) + b(y_1w_1+y_2w_2) \\ &= a\,\vec{x}\cdot\vec{w} + b\,\vec{y}\cdot\vec{w}\end{aligned}$$

3) $\vec{x}\cdot\vec{x} = x_1^2 + x_2^2$ which is ≥ 0 for every $\vec{x}\in\mathbb{R}^2$. Furthermore $\vec{x}\cdot\vec{x}=0$ means that $\vec{x}=0$ (i.e. $\vec{x}$ is a zero length vector and the zero vector is our only zero length vector. Hence $\vec{x}=0$ must follow).

These three properties of the dot product end up being very important for many applications. So important, in fact, that they are isolated and given their own elevated status.

<u>Definition</u>: Let V be a vector space and consider a function taking <u>two</u> vectors from V as input and producing a real number as output. Instead of denoting this function as $f(\vec{v},\vec{w})$ we will drop the "f" and just use the symbol $(\vec{v},\vec{w})$. We say that $(\vec{v},\vec{w})$ is an <u>inner product</u> on the vector space $\vec{V}$ if it has the following qualities:

1) $(\vec{v},\vec{w}) = (\vec{w},\vec{v})$ for every $\vec{v},\vec{w}$ in V

2) $(a\vec{v}+b\vec{w},\vec{x}) = a(\vec{v},\vec{x}) + b(\vec{w},\vec{x})$ for all $\vec{v},\vec{w},\vec{x}$ in V.

3) $(\vec{v},\vec{v}) \geq 0$ for every $\vec{v}\in V$ and $(\vec{v},\vec{v})=0$ if and only if $\vec{v}=\vec{0}$.

Definition: If V is a vector space and there exists an inner product $(\cdot,\cdot)$ defined on V then V together with the inner product $(\cdot,\cdot)$ is called an <u>inner product space</u>.

this notation shows you that the inner product function takes <u>two</u> inputs from V.

Remark: Just like a vector space can have several bases there are many different inner products that can be defined on vector spaces. They are not unique.

We can now talk about the dot product in terms of our new concepts:

The dot product $\vec{x}\cdot\vec{y}$ defines an <u>inner product</u> on n-dimensional real space $\mathbb{R}^n$ where $(\vec{x},\vec{y}) = \vec{x}\cdot\vec{y} = x_1^2 + x_2^2 + \cdots + x_n^2$ and $\mathbb{R}^n$ together with the dot product $(\vec{x},\vec{y}) = \vec{x}\cdot\vec{y}$ is an <u>inner product space</u>