CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 21 · Solutions

Posted Wednesday 19 February 2014. Due 1pm Friday 28 February 2014.

21. [25 points]

Let the inner product $(\cdot,\cdot):\ C[-1,1]\times C[-1,1]\to\mathbb{R}$ be defined by

$$(u,v)=\int_{-1}^{1} u(x)v(x)\,dx.$$

Let the linear operator $P_e:\ C[-1,1]\to C[-1,1]$ be defined by

$$(P_e f)(x)=\frac{1}{2}\left(f(x)+f(-x)\right)$$

and let the linear operator $P_o:\ C[-1,1]\to C[-1,1]$ be defined by

$$(P_o f)(x)=\frac{1}{2}\left(f(x)-f(-x)\right).$$

Note that P_e and P_o project functions onto their even and odd parts, respectively.

(a) Verify that P_e and P_o are projections.

(b) For all $f\in C[-1,1]$, verify that $P_e f$ and $P_o f$ are orthogonal with respect to the inner product $(\cdot,\cdot)$.

(c) Is P_e+P_o a projection? Note that $P_e+P_o:\ C[-1,1]\to C[-1,1]$ is defined by

$$(P_e+P_o)f=P_e f+P_o f.$$

(d) Let $a,b\in\mathbb{R}$ be such that $a<b$. Let $\phi\in C[a,b]$ be defined by $\phi(x)=1$ and let the inner product $B(\cdot,\cdot):C[a,b]\times C[a,b]\to\mathbb{R}$ be defined by

$$B(u,v)=\int_a^b u(x)v(x)\,dx.$$

Let the linear operator $P:C[a,b]\to C[a,b]$ be defined by

$$Pf=\frac{1}{b-a}B(f,\phi)\phi.$$

Determine whether or not P is a projection.

Solution.

(a) [6 points] If $f\in C[-1,1]$ then

$$(P_e f)(x)=\frac{1}{2}\left(f(x)+f(-x)\right)$$

and so

$$\begin{aligned}
(P_e(P_e f))(x) &= \frac{1}{2}\left((P_e f)(x) + (P_e f)(-x)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(-x) + f(-(-x))\right)\right) \\
&= = \frac{1}{2}\left(\frac{1}{2}f(x) + \frac{1}{2}f(-x) + \frac{1}{2}f(-x) + \frac{1}{2}f(x)\right) \\
&= \frac{1}{2}\left(f(x) + f(-x)\right) \\
&= (P_e f)(x).
\end{aligned}$$

Thus we conclude that $P_e(P_e f) = P_e f$ for all $f \in C[-1,1]$ which means that P_e is a projection. In the same way, if $f \in C[-1,1]$ then

$$(P_o f)(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

and so

$$\begin{aligned}
(P_o(P_o f))(x) &= \frac{1}{2}\left((P_o f)(x) - (P_o f)(-x)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(f(x) - f(-x)\right) - \frac{1}{2}\left(f(-x) - f(-(-x))\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}f(x) - \frac{1}{2}f(-x) - \frac{1}{2}f(-x) + \frac{1}{2}f(x)\right) \\
&= \frac{1}{2}\left(f(x) - f(-x)\right) \\
&= (P_o f)(x).
\end{aligned}$$

Thus we conclude that $P_o(P_o f) = P_o f$ for all $f \in C[-1,1]$, which means that P_o is a also projection.

(b) [6 points] If $f \in C[-1,1]$ then

$$\begin{aligned}
(P_e f, P_o f) &= \int_{-1}^{1} (P_e f)(x)(P_o f)(x)\,dx \\
&= \int_{-1}^{1} \frac{1}{4}\left(f(x) + f(-x)\right)\left(f(x) - f(-x)\right)\,dx \\
&= \frac{1}{4}\int_{-1}^{1} \left((f(x))^2 - f(x)f(-x) + f(x)f(-x) - (f(-x))^2\right)\,dx \\
&= \frac{1}{4}\int_{-1}^{1} \left((f(x))^2 - (f(-x))^2\right)\,dx \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\,dx - \int_{-1}^{1} (f(-x))^2\,dx\right) \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\,dx + \int_{-(-1)}^{-1} (f(y))^2\,dy\right) \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\,dx + \int_{1}^{-1} (f(y))^2\,dy\right) \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\,dx - \int_{-1}^{1} (f(y))^2\,dy\right) \\
&= 0
\end{aligned}$$

where we let $y = -x$.

(c) [6 points] If $f \in C[-1,1]$ then

$$((P_e + P_o)f)(x) = (P_e f)(x) + (P_o f)(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) = f(x),$$

and so

$$((P_e + P_o)((P_e + P_o)f))(x) = ((P_e + P_o)f)(x).$$

Hence, $(P_e + P_o)((P_e + P_o)f) = (P_e + P_o)f$ for all $f \in C[-1,1]$ and so $P_e + P_o$ is a projection.

(d) [7 points] If $f \in C[a,b]$ then

$$Pf = \frac{1}{b-a} B(f,\phi)\phi$$

and so

$$P(Pf) \;=\; \frac{1}{b-a} B(Pf,\phi)\phi.$$

Now,

$$\begin{aligned}
B(Pf,\phi) &= \int_a^b \frac{1}{b-a} B(f,\phi)\phi(x)\phi(x)\,dx \\
&= \int_a^b \frac{1}{b-a} B(f,\phi)\,dx \\
&= B(f,\phi)\int_a^b \frac{1}{b-a}\,dx \\
&= B(f,\phi)\left[\frac{x}{b-a}\right]_a^b \\
&= B(f,\phi)\left(\frac{b}{b-a} - \frac{a}{b-a}\right) \\
&= B(f,\phi)\frac{b-a}{b-a} \\
&= B(f,\phi).
\end{aligned}$$

Consequently,

$$\begin{aligned}
P(Pf) &= \frac{1}{b-a} B(Pf,\phi)\phi \\
&= \frac{1}{b-a} B(f,\phi)\phi \\
&= Pf.
\end{aligned}$$

Hence, $P(Pf) = Pf$ for all $f \in C[a,b]$ and so P is a projection.