CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 49 · Solutions

Posted Wednesday 27 November 2013. Due 1pm Friday 6 December 2013.

49. [25 points] Let $H_D^1(0,1) = \{v \in H^1(0,1) : v(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in H_D^1(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $V_N = \text{span}\{\phi_1, \ldots \phi_{N+1}\}$, let $u_0 \in H_D^1(0,1)$ be such that

$$u_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1/4], \\ 4x - 1 & \text{if } x \in (1/4, 1/2], \\ 3 - 4x & \text{if } x \in (1/2, 3/4], \\ 0 & \text{if } x \in (3/4, 1], \end{cases}$$

and let

$$u_{0,N}(x) = \sum_{j=1}^{N+1} u_0(x_j)\phi_j(x).$$

Note that $u_0 = u_{0,N}$ if and only if $u_0 \in V_N$.

(a) Write a MATLAB function for $u_0(x)$. It should take in as input x. It should return the value $u_0(x)$. It should also be able to take in a vector for $\mathbf{x} = (\hat{x}_1, \ldots, \hat{x}_m)$ and return the vector $u_0(\mathbf{x}) = (u_0(\hat{x}_1), \ldots, u_0(\hat{x}_m))$. Use your function to produce a plot of u_0. For this figure and the ones that you have to produce in part (b), use the command `set(gca,'XTick',[0 0.25 0.5 0.75 1])` to change the location of the tick marks on the x-axis.

(b) Write a MATLAB function for $u_{0,N}(x)$. It should take in as input x and N. It should return the value $u_{0,N}(x)$. It should also be able to take in a vector for $\mathbf{x} = (\hat{x}_1, \ldots, \hat{x}_m)$ and return the vector $u_{0,N}(\mathbf{x}) = (u_{0,N}(\hat{x}_1), \ldots, u_{0,N}(\hat{x}_m))$. On the same figure, plot u_0 as well as $u_{0,N}$ for $N = 3, 4, 5, 6$. On another figure, plot u_0 as well as $u_{0,N}$ for $N = 47, 48, 49, 50$.

(c) For which 2 of the 8 values of N that you plotted for in part (b) is $\max_{x \in [0,1]} |u_0(x) - u_{0,N}(x)|$ the smallest? Use the fact that

$$\begin{aligned} &\text{span}\{\phi_1, \ldots \phi_{N+1}\} \\ = &\ \{v \in C[0,1] : v(0) = 0,\ v(x) = a_j x + b_j,\ \text{where } a_j, b_j \in \mathbb{R},\ \text{if } x \in [x_{j-1}, x_j],\ \text{for } j = 1, \ldots, N+1\}, \end{aligned}$$

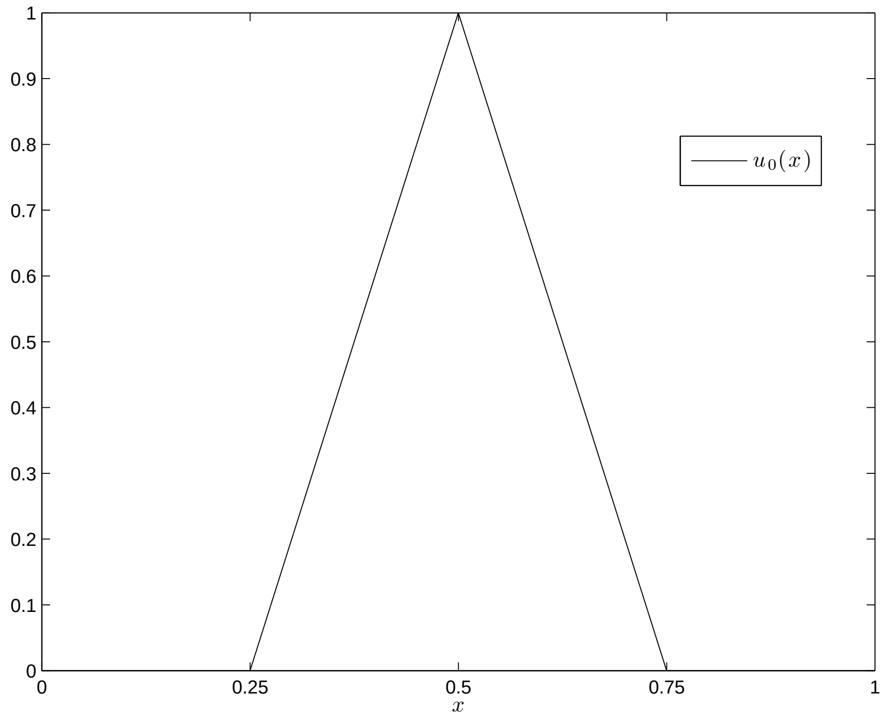
as well as information given previously in the question, to explain your answer.

Solution.

(a) [5 points] One way of coding the function is:

```
function u0=initial(x)

u0=((x>1/4)&(x<=1/2)).*(4*x-1)+((x>1/2)&(x<=3/4)).*(3-4*x);
```

The plot and code used to create it are below.



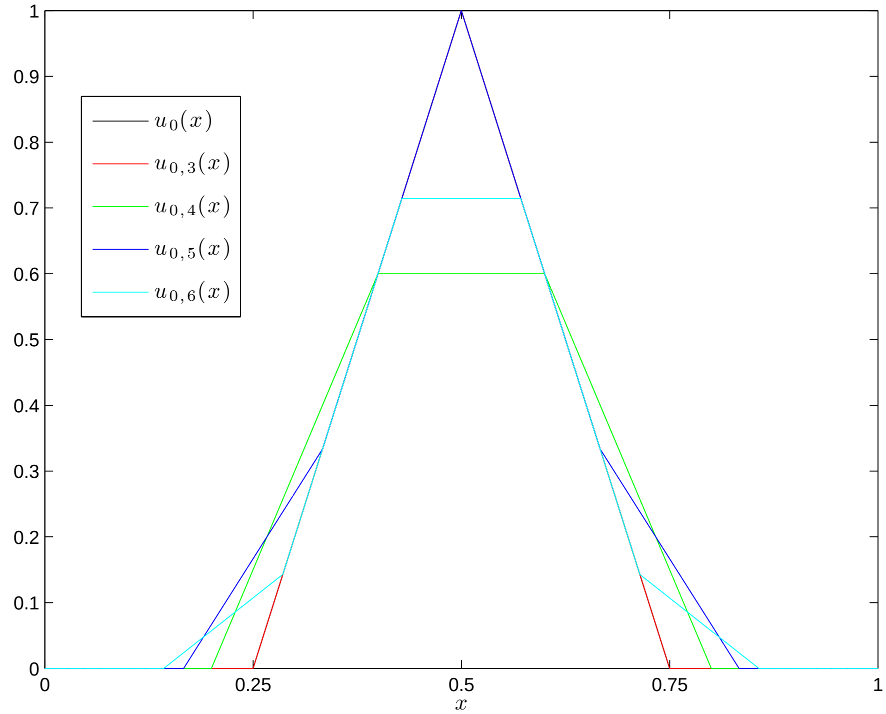

```
clear
clc

x=linspace(0,1,100000);

figure(1)
clf
plot(x,initial(x),'-k')
set(gca,'XTick',[0 0.25 0.5 0.75 1])
xlabel('$x$','interpreter','latex','FontSize',12)
legendstr{1}='$u_0(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(1),'hw49a.eps','epsc')
```

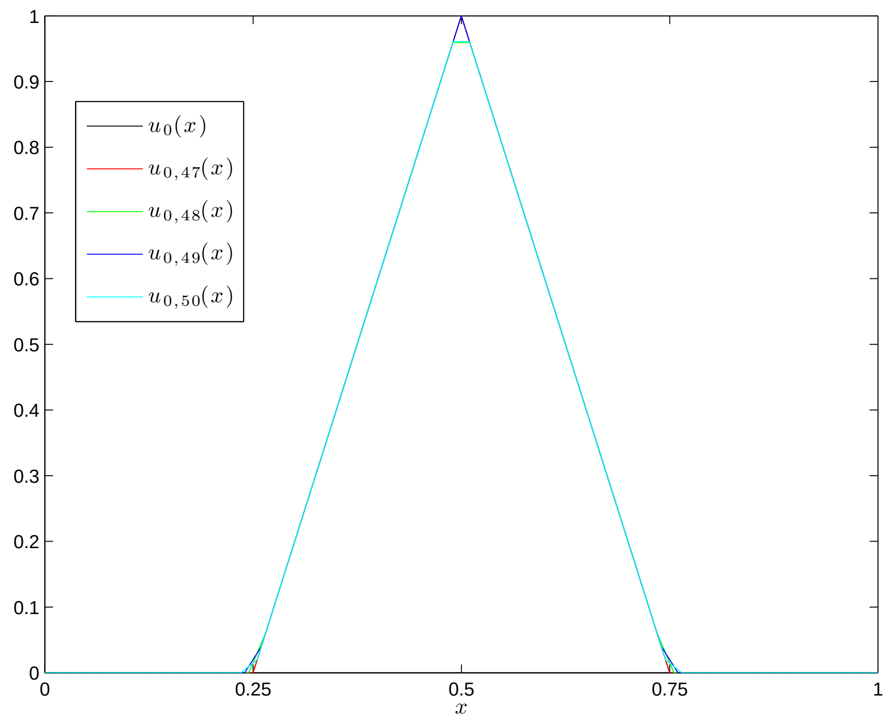
(b) [10 points] One way of coding the function is:

```
function uN0=initialinterpolant(x,N)

h=1/(N+1);
xj=(1:N+1)*h;

u0xj=initial(xj);

uN0=zeros(size(x));
for j=1:N+1
```

```
    uN0=uN0+u0xj(j)*hat(x,j,N);
end
```

The plots and codes used to create them are below.






```
clear
clc
```

```
x=linspace(0,1,100000);

figure(1)
clf
plot(x,initial(x),'-k')
hold on
plot(x,initialinterpolant(x,3),'-r')
plot(x,initialinterpolant(x,4),'-g')
plot(x,initialinterpolant(x,5),'-b')
plot(x,initialinterpolant(x,6),'-c')
set(gca,'XTick',[0 0.25 0.5 0.75 1])
xlabel('$x$','interpreter','latex','FontSize',12)
legendstr{1}='$u_0(x)$';
legendstr{2}='$u_{0,3}\,(x)$';
legendstr{3}='$u_{0,4}\,(x)$';
legendstr{4}='$u_{0,5}\,(x)$';
legendstr{5}='$u_{0,6}\,(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(1),'hw49b1.eps','epsc')

figure(2)
clf
plot(x,initial(x),'-k')
hold on
plot(x,initialinterpolant(x,47),'-r')
plot(x,initialinterpolant(x,48),'-g')
plot(x,initialinterpolant(x,49),'-b')
plot(x,initialinterpolant(x,50),'-c')
set(gca,'XTick',[0 0.25 0.5 0.75 1])
xlabel('$x$','interpreter','latex','FontSize',12)
legendstr{1}='$u_0(x)$';
legendstr{2}='$u_{0,47}\,(x)$';
legendstr{3}='$u_{0,48}\,(x)$';
legendstr{4}='$u_{0,49}\,(x)$';
legendstr{5}='$u_{0,50}\,(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(2),'hw49b2.eps','epsc')
```

(c) [10 points] When $N = 3$ and $N = 47$, $\max_{x\in[0,1]} |u_0(x) - u_{0,N}(x)|$ is smallest since $\max_{x\in[0,1]} |u_0(x) - u_{0,N}(x)| = 0$ because $u_0 = u_{0,N}$. From the definition of u_0 and the information about V_N and $u_{0,N}$ given in the question, it follows that $u_0 = u_{0,N}$ if and only if $\frac{1}{4}, \frac{1}{2}, \frac{3}{4} \in \{x_0, x_1, \ldots x_{N+1}\}$. Now, $jh = \frac{j}{N+1}$, and so $jh = \frac{1}{4}$ when $j = \frac{N+1}{4}$. If $j = \frac{N+1}{4}$ then $0 < j < N+1$ but in order for $jh \in \{x_0, x_1, \ldots x_{N+1}\}$, in addition, j must be an integer. In order for this to be the case $N+1$ must be divisible by 4 and so we can conclude that $\frac{1}{4} \in \{x_0, x_1, \ldots x_{N+1}\}$ if and only if $N = 4m - 1$ where m is a positive integer. Moreover, for such an N, $0 < m < N+1$, $0 < 2m < N+1$ and $0 < 3m < N+1$ and $x_m = \frac{m}{4m} = \frac{1}{4}$, $x_{2m} = \frac{2m}{4m} = \frac{1}{2}$, and $x_{3m} = \frac{3m}{4m} = \frac{3}{4}$. Consequently, $u_0 = u_{0,N}$ if and only if

$$N = 4m - 1$$

where m is a positive integer. For this reason, $u_0 = u_{0,N}$ and hence $\max_{x\in[0,1]} |u_0(x) - u_{0,N}(x)| = 0$ when $N = 3$ and $N = 47$ but $u_0 \neq u_{0,N}$ and hence $\max_{x\in[0,1]} |u_0(x) - u_{0,N}(x)| \neq 0$ when $N = 4, 5, 6, 48, 49, 50$.