CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 14 · Solutions

Posted Wednesday 18 September 2013. Due 5pm Wednesday 25 September 2013.

14. [25 points]
Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

   (a) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V}=C^1[0,1]$.

   (b) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

   (c) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V}=C[0,1]$.

   (d) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\displaystyle\int_0^1 u(x)+v(x)\,dx$ where $\mathcal{V}=C[0,1]$.

Solution.

(a) [6 points] *This mapping is not an inner product*: it is not symmetric and it is not positive definite.

The mapping is not symmetric. For example, if $u(x)=1$ and $v(x)=x$, then

$$(u,v)=\int_0^1 u(x)v'(x)\,dx=\int_0^1 1\,dx=1,$$

yet

$$(v,u)=\int_0^1 v(x)u'(x)\,dx=\int_0^1 0\,dx=0.$$

The mapping is also not positive definite. For example, if $u(x)=1$, then $(u,u)=0$ and if $u(x)=1-x$, then

$$(u,u)=\int_0^1 (1-x)(-1)\,dx=-1/2.$$

For what it is worth, we note that the mapping is linear in the first argument since

$$(\alpha u+\beta v,w)=\alpha\int_0^1 u(x)w'(x)\,dx+\beta\int_0^1 v(x)w'(x)\,dx=\alpha(u,w)+\beta(v,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$. It is also linear in the second argument since

$$(u,\alpha v+\beta w)=\alpha\int_0^1 u(x)v'(x)\,dx+\beta\int_0^1 u(x)w'(x)\,dx=\alpha(u,v)+\beta(u,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$.

(b) [6 points] *This mapping is not an inner product*: it is not linear in the first argument.

If $u,w\in C[0,1]$ and $\alpha\in\mathbb{R}$ then

$$(\alpha u,w)=\int_0^1 |\alpha u(x)||w(x)|\,dx=\int_0^1 |\alpha||u(x)||w(x)|\,dx=|\alpha|(u,w).$$

Hence, if $u \neq 0$, $w \neq 0$ and $\alpha < 0$, then $(\alpha u, w) \neq \alpha(u, w)$ and so the mapping is not linear in the first argument.
The mapping is symmetric, as

$$(u, v) = \int_0^1 |u(x)||v(x)|\, dx = \int_0^1 |v(x)||u(x)|\, dx = (v, u)$$

for all $u, v \in C[0, 1]$.
Moreover, the mapping is positive definite as for $u \in C[0, 1]$

$$(u, u) = \int_0^1 |u(x)|^2\, dx$$

is the integral of a nonnegative function, and hence is nonnegative and $(u, u) = 0$ only if $u = 0$.

(c) [7 points] *This mapping is an inner product.*

The mapping is symmetric, as

$$(u, v) = \int_0^1 u(x)v(x)e^{-x}\, dx = \int_0^1 v(x)u(x)e^{-x}\, dx = (v, u)$$

for all $u, v \in C[0, 1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x}\, dx \\ &= \alpha \int_0^1 u(x)w(x)e^{-x}\, dx + \beta \int_0^1 v(x)w(x)e^{-x}\, dx \\ &= \alpha(u, w) + \beta(v, w).\end{aligned}$$

for all $u, v, w \in C[0, 1]$ and all $\alpha, \beta \in \mathbb{R}$.
Lastly, we check

$$(u, u) = \int_0^1 u(x)^2 e^{-x} dx.$$

The function e^{-x} is positive valued for all $x \in [0, 1]$, so we have that (u, u) is the integrand of a nonnegative function, and hence is also nonnegative. If $(u, u) = 0$, then $u(x)^2e^{-x} = 0$ for all $x \in [0, 1]$, which means that $u(x) = 0$ for all $x \in [0, 1]$, i.e., $u = 0$. Hence, the mapping is positive definite.

(d) [6 points] *This mapping is not an inner product*: it is not linear in the first argument and it is not positive definite.

If $u, v, w \in C[0, 1]$ then

$$(u + v, w) = \int_0^1 u(x) + v(x) + w(x)\, dx = \int_0^1 u(x) + w(x)\, dx + \int_0^1 v(x)\, dx = (u, w) + \int_0^1 v(x)\, dx.$$

For most choices of v and w (for example, $v(x) = w(x) = 1$), $\int_0^1 v(x)\, dx \neq (v, w)$, so $(\cdot, \cdot)$ is not linear in the first argument.
The mapping $(\cdot, \cdot)$ is also not positive definite. For example, if $u(x) = -1$, then

$$(u, u) = \int_0^1 u(x) + u(x)\, dx = \int_0^1 -2\, dx = -2 < 0.$$

The mapping is symmetric, as

$$(u, v) = \int_0^1 u(x) + v(x)\,dx = \int_0^1 v(x) + u(x)\,dx = (v, u)$$

for all $u, v \in C[0, 1]$.