CAAM 336 · DIFFERENTIAL EQUATIONS

Examination 1

Instructions:

1. Time limit: **3 uninterrupted hours**.
2. There are four questions worth a total of 100 points, plus a 5 point bonus.
   Please do not look at the questions until you begin the exam.
3. You are allowed one cheat sheet to refer to during the exam.
   You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points: (a) = 7, (b), (c) = 9]

   In this problem, we will investigate the order of accuracy for an alternative finite difference approximation for the second derivative. Let $u(x)$ be a smooth function with the Taylor series expansion around the point x

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 + \frac{u'''(x)}{3!}\Delta x^3 + \dots$$

   (a) Derive the finite difference approximation to $u''(x)$

$$u''(x) \approx \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{\Delta x^2}$$

   with backwards difference formulas. In other words, approximate $u''(x) = (u'(x))'$ using a backwards difference formula involving the terms

$$u'(x), \quad u'(x-\Delta x),$$

   and then approximate $u'(x)$ and $u'(x-\Delta x)$ using backwards difference formulas involving

$$u(x), \quad u(x-\Delta x), \quad u(x-2\Delta x).$$

   (b) Show that the second order backward finite difference approximation

$$u''(x) \approx \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{\Delta x^2}$$

   has accuracy $O(\Delta x)$. In other words, if $u''(x)$ is the exact second derivative, show that

$$\left| u''(x) - \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{\Delta x^2} \right| = O(\Delta x).$$

   (c) Show that the 3-point forward difference formula for $u'(x)$

$$u'(x) \approx \frac{-3u(x) + 4u(x+\Delta x) - u(x+2\Delta x)}{2\Delta x}$$

   has accuracy $O(\Delta x^2)$. (In general, the more points a finite difference formula involves, the higher the order of accuracy, which is why the 3 point forward difference is more accurate than the $O(\Delta x)$ two-point forward difference formula.)

Solution.

(a) Using Backward Euler method we can approximate second derivative as follows

$$\begin{aligned} u''(x) = (u'(x))' &\approx \frac{u'(x) - u'(x-\Delta x)}{\Delta x} \\ &\approx \frac{\frac{u(x)-u(x-\Delta x)}{\Delta x} - \frac{u(x-\Delta x)-u(x-2\Delta x)}{\Delta x}}{\Delta x} \\ &= \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{\Delta x^2} \end{aligned}$$

Then

$$u''(x) \approx \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{\Delta x^2}$$

where the term $\Delta x = x_{i+1} - x_i = x_i - x_{i-1}$ represents a constant spatial interval.

(b) Using Taylor series expansion

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{u''(x)}{2!}\Delta x^2 - \frac{u'''(x)}{3!}\Delta x^3 + \frac{u''''(x)}{4!}\Delta x^4 \dots$$
$$u(x-2\Delta x) = u(x) - u'(x)2\Delta x + \frac{u''(x)}{2!}(2\Delta x)^2 - \frac{u'''(x)}{3!}(2\Delta x)^3 + \frac{u''''(x)}{4!}(2\Delta x)^4 \dots$$

Multiplying $u(x-\Delta x)$ with -2 and adding with $u(x-2\Delta x)$ gives

$$-2u(x-\Delta x) + u(x-2\Delta x) = -u(x) + u''(x)\Delta x^2 - u'''(x)\Delta x^3 + \dots$$

Adding $u(x)$ from both sides and dividing by Δx^2 gives

$$\frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{\Delta x^2} = u''(x) - u'''(x)\Delta x + \dots,$$

implying that the truncation error decreases at the same rate that Δx decreases (if Δx is small enough). This implies the 2nd order backward finite difference formula is $O(\Delta x)$ accurate.

(c) Taylor series expansion for $u(x+\Delta x)$ and $u(x+2\Delta x)$ can be written as follows, respectively

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{u''(x)}{2!}\Delta x^2 + \frac{u'''(x)}{3!}\Delta x^3 + \frac{u''''(x)}{4!}\Delta x^4 \dots$$
$$u(x+2\Delta x) = u(x) + u'(x)2\Delta x + \frac{u''(x)}{2!}(2\Delta x)^2 + \frac{u'''(x)}{3!}(2\Delta x)^3 + \frac{u''''(x)}{4!}(2\Delta x)^4 \dots$$

Multiplying $u(x+\Delta x)$ with 4, $u(x+2\Delta x)$ with -1 and adding them together gives

$$4u(x+\Delta x) - u(x+2\Delta x) = 3u(x) + 2u'(x)\Delta x - \frac{2}{3}u'''(x)\Delta x^3 + \dots$$

Subtracting $3u(x)$ from both sides and dividing by $2\Delta x$ gives

$$\frac{-3u(x) + 4u(x+\Delta x) - u(x+2\Delta x)}{2\Delta x} = u'(x) - \frac{1}{3}u'''(x)\Delta x^2 \dots$$

implying that the first order, three point forward difference formula is $O(\Delta x^2)$ accurate.

2. [32 points: (a), (b) = 8, (c) = 10, (d) = 6]

Consider the steady state heat equation

$$-\frac{\partial^2 u(x)}{\partial x^2} = f(x), \quad 0 < x < 1.$$

We wish to discretize the above system using finite differences at four points

$$x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1.$$

such that $h = 1/3$. We wish to solve for the values of the solution $u(x_i) = u_i$ for $0 < x_i < 1$.

(a) Write out the finite difference equations for the above differential equation at all points $0 < x_i < 1$, using the second order finite difference approximation

$$u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

Consider the inhomogeneous boundary conditions

$$u(0) = 1, \qquad u(1) = 2.$$

Write down the resulting finite difference equations for these boundary conditions.

(b) Consider the inhomogeneous boundary conditions

$$u(0) = 1, \qquad -u'(1) = 2.$$

Use the backwards difference approximation

$$u'(1) \approx \frac{u_3 - u_2}{h}$$

to impose the above boundary conditions. Write down the resulting finite difference equations for these boundary conditions.

(c) Consider now the case of a Robin or Cauchy boundary condition on the left

$$u(0) = 1, \qquad -u'(1) + \alpha u(1) = 2 + 2\alpha$$

Use the backward difference for $u'(1)$ in (c) to modify the finite difference equations to accommodate a Robin boundary condition. *Hint: eliminate u_3 from the system using the above boundary condition. You can check that if $\alpha = 0$, you recover the answer to (b), and if $\alpha \to \infty$, you recover the answer to (a).*

(d) Solve, for the above Robin boundary condition, the 2×2 finite difference system for the values of u_1 and u_2, and recover the value of u_3 from the given boundary condition. You may use the fact that the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Give solution values u_1, u_2 in terms of α and $f(x_i)$, and explain how to compute u_3 given u_2.

If you are unable to do part (c), you may solve the system from part (b) for half credit.

Solution.

(a) Since we wish to satisfy the finite difference version of the equation

$$-u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i)$$

at both points x_1, x_2, we have

$$\begin{aligned} -u''(x_1) &\approx -\tfrac{1}{h^2}(u_2 - 2u_1 + u_0) = f(x_1) \\ -u''(x_2) &\approx -\tfrac{1}{h^2}(u_3 - 2u_2 + u_1) = f(x_2). \end{aligned}$$

Since $u_0 = u(x_0) = u(0) = 1$, we may replace it with the value of the boundary condition, and likewise with $u_3 = u(x_3) = u(1) = 2$. Then, the above equations become

$$\begin{aligned} \frac{1}{h^2}(-u_2 + 2u_1) &= f(x_1) + \frac{1}{h^2} \\ \frac{1}{h^2}(-u_3 + 2u_2) &= f(x_2) + \frac{2}{h^2}. \end{aligned}$$

(b) Since the first equation

$$-u''(x_1) \approx \frac{1}{h^2}(-u_2 + 2u_1) = f(x_1) + \frac{1}{h^2}$$

does not involve u_3, it does not change. The second equation

$$-u''(x_2) \approx -\frac{1}{h^2}(u_3 - 2u_2 + u_1) = f(x_2)$$

will be modified to accommodate the new boundary condition. Approximating $u'(1) \approx \frac{u_3 - u_2}{h}$, we have

$$-u'(1) \approx -\frac{u_3 - u_2}{h} = 2.$$

Then, we may rewrite the second finite difference equation as

$$\frac{1}{h}\left(-\frac{u_3 - u_2}{h} - \frac{1}{h}(-u_2 + u_1)\right) = \frac{1}{h}\left(2 + \frac{1}{h}(u_2 - u_1)\right) = f(x_2)$$

Rearranging unknowns to the left hand side gives

$$\frac{1}{h^2}\left(u_2 - u_1\right) = f(x_2) - \frac{2}{h}.$$

(c) Once again, the first equation does not change, since the left boundary condition is the same as before. The Cauchy/Robin boundary condition can be approximated as

$$-u'(1) + \alpha u(1) \approx -\frac{u_3 - u_2}{h} + \alpha u_3 = 2 + 2\alpha.$$

Rearranging, we can use the above to define u_3 in terms of u_2

$$u_3 = \frac{2 + 2\alpha - u_2/h}{\alpha - 1/h}.$$

We can substitute this directly into the second equation to get

$$-\frac{1}{h^2}\left(\frac{2+2\alpha-u_2/h}{\alpha-1/h}-2u_2+u_1\right)=f(x_2)$$

Rearranging gives us the second finite difference equation

$$\frac{1}{h^2}\left(\left(2-\frac{1}{h(\alpha-1/h)}\right)u_2-u_1\right)=f(x_2)+\frac{2+2\alpha}{h^2(\alpha-1/h)}.$$

A quick check shows that if $\alpha=0$, we recover part (b), while if $\alpha\to\infty$, we recover part (a).

(d) We can put the above equations into matrix form:

$$\frac{1}{h^2}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} f \\ g \end{bmatrix}$$

where

$$a=2,\quad b=-1$$

correspond to the first equation, and

$$c=2+\frac{1}{h(\alpha-1/h)},\quad d=-1$$

and

$$f=f(x_1)+\frac{1}{h^2},\quad g=f(x_2)+\frac{2+2\alpha}{h^2(\alpha-1/h)}.$$

3. [15 points: (a) = 9, (b)= 6]

Consider, for $\alpha, \kappa > 0$, the partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + \alpha u(x,t) - \kappa \frac{\partial^2 u(x,t)}{\partial x^2} = 0.$$

with initial/boundary conditions

$$\begin{aligned} u(x,0) &= \psi(x) \\ u(0,t) &= 0 \\ u(1,t) &= 0. \end{aligned}$$

Define finite difference points $x_0, x_1, \ldots, x_N, x_{N+1}$; we wish to simulate the solution $u(x,t)$ at points x_i and times t_j. Further, define vector of solution values u^j and finite difference matrix A such that

$$u^j = \begin{bmatrix} u(x_1,t_j) \\ u(x_2,t_j) \\ \vdots \\ u(x_N,t_j) \end{bmatrix}, \qquad A = \frac{\kappa}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

You may use that Au^j gives back the vector of finite difference approximations of $-\kappa \frac{\partial^2 u(x_i,t_j)}{\partial x^2}$ at all points x_i.

(a) Using a forward difference approximation of the derivative $\frac{\partial u}{\partial t}$, derive that the resulting finite difference equations for the above equation satisfy the update formula

$$u^{j+1} = [(1 - dt\alpha)I - dtA]\, u^j$$

and that this formula can be rewritten as

$$u^{j+1} = [(1 - dt\alpha)I - dtA]^{j+1}\, u^0.$$

(b) Derive an expression for λ_i, the eigenvalues of

$$(1 - dt\alpha)I - dtA$$

in terms of μ_i, the eigenvalues of A. Assuming that

$$0 < \mu_i \le 4\frac{\kappa}{h^2}, \quad i = 1, \ldots, N,$$

derive a bound on dt

$$dt \le (???)$$

in terms of κ, h^2, and α which guarantees $|\lambda_i| \le 1$.

Solution.

(a) If we choose to satisfy our differential equation at points x_i and times t_j, it becomes

$$\frac{\partial u_i^j}{\partial t} + \alpha u_i^j - \kappa \frac{\partial^2 u_i^j}{\partial x^2} = 0$$

where $u_i^j = u(x_i, t_j)$. If we replace the second derivative term by

$$-\kappa \frac{\partial^2 u_i^j}{\partial x^2} = \frac{-\kappa}{h^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right)$$

we get

$$\frac{\partial u_i^j}{\partial t} + \alpha u_i^j + \frac{\kappa}{h^2}\left(-u_{i+1}^j + 2u_i^j - u_{i-1}^j\right) = 0.$$

At this point, we may express this in matrix form, to get

$$\frac{\partial u^j}{\partial t} + \alpha u^j + Au^j = \frac{\partial u^j}{\partial t} + (\alpha I + A)u^j = 0$$

Applying a forward difference approximation in time gives

$$\frac{\partial u^j}{\partial t} \approx \frac{u^{j+1} - u^j}{dt}$$

which, substituted in, gives

$$\frac{u^{j+1} - u^j}{dt} + (\alpha I + A)u^j = 0.$$

Now, since we wish to use the previous time solution u^j to determine u^{j+1}, we multiply the equation on both sides by dt and then rearrange the equation to read

$$u^{j+1} = u^j - dt(\alpha I + A)u^j = (I - dt(\alpha I + A))u^j.$$

Since

$$u^1 = (I - dt(\alpha I + A))u^0, \qquad u^2 = (I - dt(\alpha I + A))u^1 = (I - dt(\alpha I + A))^2 u^0, \ldots$$

we can see that

$$u^{j+1} = (I - dt(\alpha I + A))^{j+1} u^0.$$

(b) Assuming that A has eigenvalues μ_i, $\alpha I + A$ has eigenvalues $\alpha + \mu_i$. Similarly, if a matrix is scaled by a constant, the eigenvalues are also scaled by that same constant.

Thus, if we add $\alpha I + A$, the eigenvalues are $\alpha + \mu_i$. Scaling this by dt gives that the matrix $(I - dt(\alpha I + A))$ has eigenvalues

$$\lambda_i = 1 - dt(\alpha + \mu_i).$$

Since

$$0 < \mu_i < 4\kappa/h^2,$$

$dt(\alpha + \mu_i)$ is always positive, and $|\lambda_i| = |1 - dt(\alpha + \mu_i)|$ is largest when $dt(\alpha + \mu_i)$ is most positive. Since μ_i has a maximum value, $|\lambda_i|$ is smaller than

$$|\lambda_i| \leq |1 - dt(\alpha + 4\kappa/h^2)|.$$

If

$$dt(\alpha + 4\kappa/h^2) > 2,$$

then $|\lambda_i| > 1$, and we violate the stability condition. Thus, we require

$$dt(\alpha + 4\kappa/h^2) \leq 2$$

which, after rearrangement, gives

$$dt \leq \frac{2}{(\alpha + 4\kappa/h^2)}$$

4. [30 points: (a) = 8, (b) = 7, (c) = 15 each, 5 each part]

(a) Let $\mathcal{V} = C^1[0,1]$ be the space of all continuously differentiable real valued functions on the interval $[0,1]$, and let $(\cdot,\cdot): \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by

$$(f,g) = \int_0^1 f(x)g(x)\,dx + \int_0^1 f'(x)g'(x)\,dx$$

Prove that $(\cdot,\cdot)$ is an inner product on $\mathcal{V}$.

(b) Let V be a vector space with an inner product and let u be a vector in V such that

$$\|u\|^2 = (u,u) = 1.$$

Let $L_u : V \to V$ be the operator defined by

$$L_u(v) = v - 2\,(v,u)\,u$$

for all $v \in V$. Show that L_u is linear, and that L_u satisfies

$$(L_u(v), L_u(w)) = (v,w)$$

for all $v, w \in V$ (this implies that L_u is *unitary*, and that $\|L_u(v)\| = \|v\|$).

(c) Let $W \subset C[-1,1]$ now be a vector space with inner product

$$(f,g) = \int_{-1}^1 f(x)g(x)\,dx$$

We will use the orthonormal basis

$$\{\phi_1, \phi_2, \phi_3\} = \left\{ \frac{1}{\sqrt{2}}, \quad \sqrt{\frac{3}{2}}x, \quad \sqrt{\frac{5}{8}}(3x^2 - 1) \right\}.$$

i Verify that ϕ_1, ϕ_2, ϕ_3 have unit norm (i.e. $\|\phi_i\| = 1$ for $i = 1,2,3$).

i Find the linear polynomial $p(x)$ that best approximates $g(x) = \sin(\pi x)$ from $\mathrm{span}\{\phi_1, \phi_2\}$.

ii Find the quadratic polynomial $q(x)$ that best approximates $g(x) = \sin(\pi x)$ from $\mathrm{span}\{\phi_1, \phi_2, \phi_3\}$.

You may use the following without proof:

$$\int_{-1}^1 x\sin(\pi x)\,dx = \frac{2}{\pi}$$

$$\int_{-1}^1 x^2\sin(\pi x)\,dx = 0.$$

Solution.

(a) We want to show $(\cdot,\cdot)$ is an inner product on $\mathcal{V}$.

- The mapping is symmetric, as
$$(f,g) = \int_0^1 f(x)g(x)\,dx + \int_0^1 f'(x)g'(x)\,dx = \int_0^1 g(x)f(x)\,dx + \int_0^1 g'(x)f'(x)\,dx = (g,f)$$
for all $f, g \in C^1[0,1]$.
- The mapping is also linear in the first argument since
$$\begin{aligned}(\alpha f + \beta g, h) &= \int_0^1 (\alpha f(x) + \beta g(x))h(x)\,dx + \int_0^1 (\alpha f'(x) + \beta g'(x))h'(x)\,dx \\ &= \alpha \int_0^1 f(x)h(x)\,dx + \beta \int_0^1 g(x)h(x)\,dx + \alpha \int_0^1 f'(x)h'(x)\,dx + \beta \int_0^1 g'(x)h'(x)\,dx \\ &= \alpha \left(\int_0^1 f(x)h(x)\,dx + \int_0^1 f'(x)h'(x)\,dx \right) + \beta \left(\int_0^1 g(x)h(x)\,dx + \beta \int_0^1 g'(x)h'(x)\,dx \right) \\ &= \alpha\,(f,h) + \beta\,(g,h)\end{aligned}$$
for all $f, g, h \in C^1[0,1]$ and all $\alpha, \beta \in \mathbb{R}$. Together with symmetry it implies inner product linear also second argument.
- The mapping is also positive definite for all $x \in [0,1]$
$$(f,f) = \int_0^1 (f(x))^2\,dx + \int_0^1 (f'(x))^2\,dx$$
is the integral of a nonnegative function, and hence is also nonnegative. If $(f,f) = 0$ then $(f(x))^2 = 0$ for all $x \in [0,1]$ and, this means that $f(x) = 0$ for all $x \in [0,1]$, i.e., $f = 0$. Hence, the mapping is positive definite.

(b) Suppose that $v, w \in V$. Then
$$\begin{aligned}L_u(\alpha v + \beta w) &= \alpha v + \beta w - 2\,(\alpha v + \beta w, u)\,u \\ &= \alpha v + \beta w - 2\,(\alpha v, u)\,u - 2\,(\beta w, u)\,u \\ &= \alpha v - 2\alpha\,(v,u)\,u + \beta w - 2\beta\,(w,u)\,u \\ &= \alpha\,\underbrace{(v - 2\,(v,u)\,u)}_{L_u(v)} + \beta\,\underbrace{(w - 2\,(w,u)\,u)}_{L_u(w)} \\ &= \alpha L_u(v) + \beta L_u(w)\end{aligned}$$
for all $\alpha, \beta \in \mathbb{R}$. This shows that L_u is linear.
Now, we want to show that $(L_u(v), L_u(w)) = (v,w)$ for all $v, w \in V$.

$$\begin{aligned}(L_u(v), L_u(w)) &= (v - 2\,(v,u)\,u, w - 2\,(w,u)\,u) \\ &= (v,w) - 2\,(v, (w,u)\,u) - 2\,((v,u)\,u, w) + 4\,((v,u)\,u, (w,u)\,u) \\ &= (v,w) - 2\,(w,u)\,(v,u) - 2\,(v,u)\,(u,w) + 4\,(v,u)\,(w,u)\,\underbrace{(u,u)}_{=1} \\ &= (v,w) - 2\,(w,u)\,(v,u) - 2\,(v,u)\,(u,w) + 4\,(v,u)\,(w,u) \\ &= (v,w) - 2\,(w,u)\,(v,u) - 2\,(v,u)\,(u,w) + 4\,(v,u)\,(u,w) \\ &= (v,w) - 2\,(v,u)\,(w,u) + 2\,(v,u)\,(u,w) \\ &= (v,w) - 2\,(v,u)\,(u,w) + 2\,(v,u)\,(u,w) \\ &= (v,w)\,.\end{aligned}$$

Shows L_u is *unitary*.

(c) Using the given orthogonal basis

$$\{\phi_1, \phi_2, \phi_3\} = \left\{ \frac{1}{\sqrt{2}}, \quad \sqrt{\frac{3}{2}}x, \quad \sqrt{\frac{5}{8}}(3x^2 - 1) \right\}.$$

i. We will show that $\|\phi_i\|^2 = (\phi_i, \phi_i) = 1$ for $i = 1, 2, 3$ with respect to defined norm.

$$(\phi_1, \phi_1) = \int_{-1}^{1} \frac{1}{2}\, dx = \left[\frac{x}{2}\right]_{-1}^{1} = 1$$

$$(\phi_2, \phi_2) = \int_{-1}^{1} \frac{3}{2}x^2\, dx = \left[\frac{1}{2}x^3\right]_{-1}^{1} = 1$$

$$(\phi_3, \phi_3) = \int_{-1}^{1} \frac{5}{8}(3x^2 - 1)^2\, dx = \left[\frac{5}{8}(\frac{9x^5}{5} - 2x^3 + x)\right]_{-1}^{1} = 1$$

ii. Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot, \cdot)$, i.e., $(\phi_1, \phi_2) = 0$, the best approximation to $g(x) = \sin(\pi x)$ from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$p(x) = \frac{(g, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) + \frac{(g, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x) = (g, \phi_1)\,\phi_1(x) + (g, \phi_2)\,\phi_2(x)$$

We can compute

$$(g, \phi_1) = \int_{-1}^{1} \frac{1}{\sqrt{2}} \sin(\pi x)\, dx = \frac{1}{\sqrt{2}}\left[\frac{-\cos(\pi x)}{\pi}\right]_{-1}^{1} = 0$$

and

$$(g, \phi_2) = \int_{-1}^{1} \sqrt{\frac{3}{2}}x \sin(\pi x)\, dx = \sqrt{\frac{3}{2}} \int_{-1}^{1} x \sin(\pi x)\, dx = \sqrt{\frac{3}{2}}\frac{2}{\pi} = \frac{\sqrt{6}}{\pi}.$$

Then

$$p(x) = (g, \phi_1)\,\phi_1(x) + (g, \phi_2)\,\phi_2(x) = \frac{\sqrt{6}}{\pi}\sqrt{\frac{3}{2}}x = \frac{3}{\pi}x$$

iii. Similar to part (ii) the best approximaton to $g(x) = \sin(\pi x)$ from $\text{span}\{\phi_1, \phi_2, \phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$\begin{aligned} q(x) &= \frac{(g, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) + \frac{(g, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x) + \frac{(g, \phi_3)}{(\phi_3, \phi_3)}\phi_3(x) \\ &= (g, \phi_1)\,\phi_1(x) + (g, \phi_2)\,\phi_2(x) + (g, \phi_3)\,\phi_3(x) \end{aligned}$$

Now let us compute

$$(g, \phi_3) = \int_{-1}^{1} \sqrt{\frac{5}{8}}(3x^2 - 1)\sin(\pi x)\, dx = \sqrt{\frac{5}{8}} \int_{-1}^{1} (3x^2 \sin(\pi x) - \sin(\pi x))\, dx = 0$$

From part (ii) we have $(g, \phi_1) = 0$ and $(g, \phi_2) = \dfrac{\sqrt{6}}{\pi}$. Then the best approximation

$$q(x) = (g, \phi_1)\, \phi_1(x) + (g, \phi_2)\, \phi_2(x) + (g, \phi_3)\, \phi_3(x) = \frac{3}{\pi} x$$

Figure 1: A circular bar.

5. [Bonus: (a) = 4, (b) = 1]

   Recall that the heat equation models heat flow in an insulated bar of length 1, such that $0 < x < 1$ denotes a coordinate system from end of the bar to the other. Suppose that one end of the bar is now attached to the other end, to form a circular bar. In this case, we may still model using the heat equation

$$\frac{\partial u(x,t)}{\partial t} - \kappa \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad 0 < x < 1$$

   with *periodic* boundary conditions

$$u(1,t) = u(0,t)$$

   and

$$\frac{\partial u(1,t)}{\partial x} = \frac{\partial u(0,t)}{\partial x}$$

   for all $t \geq 0$.

   (a) Suppose we are given points on a line

$$0 = x_0, \quad x_1, \quad x_2, \quad \ldots, \quad x_N, \quad x_{N+1} = 1.$$

   Construct the finite difference equations for the steady state heat equation

$$\kappa \frac{\partial^2 u(x,t)}{\partial x^2} = f(x), \quad 0 < x < 1$$

   with periodic boundary conditions

$$u(1) = u(0), \qquad \frac{\partial u(1)}{\partial x} = \frac{\partial u(0)}{\partial x}.$$

   (You will need to approximate $\frac{\partial u(1)}{\partial x}$ and $\frac{\partial u(0)}{\partial x}$ using finite differences).
   In particular, specify the finite difference equations at the point x_1, the point x_i where $1 < i < N$, and the last point x_N, and write out the matrix system that results from the above discretization.

(b) Does the steady state heat equation with periodic boundary conditions yield a unique solution? Why or why not?

Solution.

(a) We will construct the finite difference equations for the steady state heat equation

$$\kappa\frac{\partial^2 u(x,t)}{\partial x^2} = f(x), \quad 0 < x < 1$$

with periodic boundary conditions

$$u(1) = u(0), \qquad \frac{\partial u(1)}{\partial x} = \frac{\partial u(0)}{\partial x}.$$

For the second derivative $\frac{\partial^2 u(x,t)}{\partial x^2}$ we will use second order central difference

$$\frac{\partial^2 u(x_i,t)}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

where $h = x_{i+1} - x_i$. Then finite difference approximation to $\kappa\frac{\partial^2 u(x,t)}{\partial x^2} = f(x)$ at the point x_i can be written

$$\kappa\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i \qquad i = 1, 2 \cdots, n$$

from boundary conditions it can be written

$$u_0 = u_{N+1}$$

and

$$\frac{u_{N+1} - u_N}{h} = \frac{u_1 - u_0}{h} \Rightarrow u_0 - u_N = u_1 - u_0 \Rightarrow u_0 = \frac{u_1 + u_N}{2}$$

Now, we would like specify finite difference solution at x_1, x_i and x_N. For $i = 1$

$$\kappa\frac{u_2 - 2u_1 + u_0}{h^2} = f_1$$

from the given boundary conditions we have $u_0 = \frac{u_1+u_N}{2}$. Then

$$\kappa\frac{u_2 - 2u_1 + \frac{u_1+u_N}{2}}{h^2} = f_1 \Rightarrow \kappa(u_2 - \frac{3}{2}u_1 + \frac{u_N}{2}) = h^2 f_1$$

At the point x_i

$$\kappa\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i \Rightarrow \kappa(u_{i+1} - 2u_i + u_{i-1}) = h^2 f_i$$

At the point x_N

$$\kappa\frac{u_{N+1} - 2u_N + u_{N-1}}{h^2} = f_N$$

from the given boundary conditions we have $u_0 = u_{N+1} = \frac{u_1+u_N}{2}$. Then

$$\kappa\frac{\frac{u_1+u_N}{2} - 2u_N + u_{N-1}}{h^2} = f_N \Rightarrow \kappa(\frac{u_1}{2} - \frac{3}{2}u_N + u_{N-1}) = h^2 f_N$$

As a result we have following linear system of equations for $i = 1, 2, \cdots, N$

$$\begin{aligned}
\kappa\frac{u_2 - \frac{3}{2}u_1 + \frac{u_N}{2}}{h^2} &= f_1\\
\kappa\frac{u_3 - 2u_2 + u_1}{h^2} &= f_2\\
&\vdots\\
\kappa\frac{u_N - 2u_{N-1} + u_{N-2}}{h^2} &= f_{N-1}\\
\kappa\frac{\frac{u_1}{2} - \frac{3}{2}u_N + u_{N-1}}{h^2} &= f_N
\end{aligned}$$

The matrix system is

$$\underbrace{\frac{-\kappa}{h^2}\begin{bmatrix} \frac{3}{2} & -1 & \cdots & & & \frac{-1}{2}\\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & \ddots & \\ & & \ddots & & \ddots\ 2 & -1\\ \frac{-1}{2} & \cdots & & & -1 & \frac{3}{2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_1\\ u_2\\ \vdots\\ u_{N-1}\\ u_N \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} f_1\\ f_2\\ \vdots\\ f_{N-1}\\ f_N \end{bmatrix}}_{\mathbf{f}}$$

As a result we get the matrix system $\mathbf{Ax} = \mathbf{f}$

(b) Define the operator $L_P u : C_P^2[0,1] \to C[0,1]$ by

$$L_P u = \kappa\frac{\partial^2 u(x,t)}{\partial x^2}$$

where $C_P^2[0,1] = \{u \in C^2[0,1] : u(1) = u(0), \frac{\partial u(1)}{\partial x} = \frac{\partial u(0)}{\partial x}\}$ If $L_P u = 0$ then u satisfy the BVP

$$\begin{aligned}
\kappa\frac{\partial^2 u(x,t)}{\partial x^2} &= 0 \qquad 0 < x < 1\\
u(1) &= u(0)\\
\frac{\partial u(1)}{\partial x} &= \frac{\partial u(0)}{\partial x}.
\end{aligned}$$

The differential equation implies that $u(x) = C_1 x + C_2$ for some constant C_1 and C_2. The two boundary conditions lead to conclusion that $C_1 = 0$ however, the constant C_2 can have any value and both the differential equation and two boundary conditions will be satisfied. This shows that every constant function $u(x) = C_2$ satisfies $L_P u = 0$. That is , the null space of L_P is the space of constant function.

$$Null(L_P) = \{C_2 \quad \text{for all} \quad C_2 \in \mathbb{R}\}$$

Since the null space of L_P is not trivial, we know that the operator equation $L_P u = f$ cannot have unique solution. If there is one solution, there must be infinitely many, i.e., Since the null pace is the set of constant functions, all solutions of $L_P u = f$ differ by constant.