CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 16

Posted Wednesday 25 September 2013. Due 5pm Wednesday 2 October 2013.

16. [25 points]

Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = jh$ for $j = 0, 1 \ldots, N+1$. Consider the N hat functions $\phi_k \in C[0,1]$, defined as

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k); \\ \dfrac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1, \ldots, N$. Let the inner product $(\cdot, \cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_0^1 u(x)v(x)\,dx$$

and let the norm $\|\cdot\| : C[0,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

(a) For $j = 1, \ldots, N$, what is $\phi_j(x_k)$ for $k = 0, 1, \ldots, N+1$? Simplify your answer as much as possible.

(b) Show that $\{\phi_1, \ldots, \phi_N\}$ is linearly independent by showing that if $c_k \in \mathbb{R}$ and $\displaystyle\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $c_k = 0$ for $k = 1, \ldots, N$.

(c) For $f(x) = \sin(\pi x)$, compute by hand the inner products (f, ϕ_j) for $j = 1, \ldots, N$.

(d) Compute by hand the inner products (ϕ_j, ϕ_k) for $j, k = 1, \ldots, N$. Your final answers should be simplified as much as possible and in your formulas h should be left as h and not be replaced with $1/(N+1)$. You must clearly state which values of j and k each formula you obtain is valid for. An acceptable way to present the final answer would be:
For $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} ? & \text{if } k = j, \\ ? & \text{if } |j - k| = 1, \\ ? & \text{otherwise.} \end{cases}$$

with the question marks replaced with the correct values. Hint:

$$\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx = \frac{1}{h^2} \int_{x_{j-1}-x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2\, ds = \frac{1}{h^2} \int_0^h s^2\, ds$$

where $s = x - x_{j-1}$.

(e) Use your solutions to (c) and (d) to set up a linear system (in MATLAB) and solve it to compute the best approximations $f_N(x)$ to $f(x) = \sin(\pi x)$ from $\operatorname{span}\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ for $N = 3$ and $N = 9$. For each of these N, produce one plot that compares $f_N(x)$ to $f(x)$, and a second plot that shows the error $f(x) - f_N(x)$. The `hat.m` code (from Homework 2, either your code or the code from the solutions) should help you to produce these plots.