• Announcement: Exam #1: 02/23/2015 and Exam #2: 03/20/2015 •

Inner Products and orthogonal bases for vector spaces (Chapter 3.4 Continued)

In the last lecture we introduced the concept of an inner product on a vector space.

Example Consider the space of functions $C[0,1]$ then $(f,g)=\int_0^1 fg$ is an inner product on $C[0,1]$.

Lets show this is true by verifying the requirements of an inner product:

1) $(f,g)=\int_0^1 fg = \int_0^1 gf = (g,f)$ for all $f,g \in C[0,1]$

2) $(af+bg,w)= \int (af+bg)w = a\int fw + b\int gw$
$= a(f,w)+b(g,w)$

3) $(f,f) = \int f^2 \geq 0$ and $\int f^2 = 0 \Rightarrow f=0$.

Recall: last time we discussed bases of vector spaces and mentioned that some bases were "better than others" as they could greatly simplify solving certain problems. We also discussed the idea of two orthogonal vectors in n-dimensional real space $\mathbb{R}^n$.

Definition: Let V a vector space with an inner product $(\cdot,\cdot)_V$ then two vectors f,g in V are called orthogonal with respect to $(\cdot,\cdot)_V$ if $(f,g)=0$.

• A higly desireable property for a basis of a vector space is that the basis $B=\{v_1,v_2,\ldots,v_n\}$ of V consist of mutually orthogonal vectors.

Definition: A basis $B=\{v_1,v_2,\ldots,v_n\}$ of V is called an orthogonal basis if $(v_i,v_j)=0$ whenever $i\neq j$.

An orthogonal basis is very nice. For example if we know that $\{v_1, v_2, \dots, v_n\}$ are an orthogonal basis for V then if $\vec{w}$ is any vector we can <u>easily</u> determine how to break $\vec{w}$ down into its basis components.

How? we know that $\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \vec{\alpha_n} v_n$ since the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are a basis of V. Since the basis is orthonormal we can find the coefficients α_i by taking the inner product of $\vec{w}$ with $\vec{v}_i$.

$$(\vec{w}, \vec{v}_i) = (\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n, \vec{v}_i)$$
$$= \alpha_1 (\vec{v}_1, \vec{v}_i) + \alpha_2 (\vec{v}_2, \vec{v}_i) + \dots + \alpha_n (\vec{v}_n, \vec{v}_i)$$

Recall that $(\vec{v}_i, \vec{v}_j) = 0$ if $i \neq j$ so the above is:

$$(\vec{w}, \vec{v}_i) = \alpha_i (\vec{v}_i, \vec{v}_i)$$

Since $(\cdot, \cdot)$ is an inner product (and $\vec{v}_i \neq 0$) we know that $(\vec{v}_i, \vec{v}_i) > 0$ so we can solve for α_i to get:

$$\alpha_i = (\vec{w}, \vec{v}_i) / (\vec{v}_i, \vec{v}_i)$$

- A more desirable property than orthogonality is <u>orthonormality</u>

<u>Definition</u>: A basis $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of the vector space V is called <u>orthonormal</u> with respect to the inner product $(\cdot, \cdot)_V$ if it is orthogonal and, further, $(v_i, v_i)_V = 1$ for $i = 1, 2, \dots, n$

Some nice applications of orthonormal bases:

1) $B = \{v_1, v_2, \dots, v_n\}$ orthonormal basis of V and $\vec{w} \in V$ then $\vec{w} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$ where $\alpha_i = (\vec{w}, \vec{v}_i)$

2) Suppose that a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ with the property that the columns are <u>orthonormal</u> then the inverse of A is $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. That is $AA^T = A^T A = Id.$

- Building an orthonormal basis

We have seen that:

1) vector spaces have bases
2) A single vector space can have multiple bases
3) The selection of a particular base for a vector space can make some problems easier
4) Bases that are <u>orthogonal</u> are nice. Bases that are orthonormal are <u>nicer</u>!

Q: Can we construct an orthonormal basis for a (finite dimensional) vector space, V?

A: Yes! The process is called "the Gram-Schmidt process" and it goes like this:

- Suppose that V is a finite dimensional vector space and let $B = \{v_1, v_2, \dots, v_n\}$ be any basis
- Suppose that $(\cdot,\cdot)$ is an inner product defined on V
- Define a new basis $\tilde{B} = \{w_1, w_2, \dots, w_n\}$ which is orthonormal with respect to the inner product $(\cdot,\cdot)$ as

1) Define $\vec{p}_1 = v_1 \longrightarrow \vec{w}_1 = \vec{p}_1 / (p_1, p_1)^{1/2}$

2) Define $\vec{p}_2 = \vec{v}_2 - \dfrac{(v_2, p_1)}{(p_1, p_1)} p_1 \longrightarrow \vec{w}_2 = \vec{p}_2 / (p_2, p_2)^{1/2}$

3) Define $\vec{p}_3 = \vec{v}_3 - \dfrac{(v_3, p_2)}{(p_2, p_2)} \vec{p}_2 - \dfrac{(v_3, p_1)}{(p_1, p_1)} \vec{p}_1 \longrightarrow \vec{w}_3 = \vec{p}_3 / (p_3, p_3)^{1/2}$

$\vdots$

n) Define $\vec{p}_n = \vec{v}_n - \sum_{i=1}^{n-1} \dfrac{(v_n, p_i)}{(p_i, p_i)} p_i \longrightarrow \vec{w}_n = \vec{p}_n / (\vec{p}_n, \vec{p}_n)^{1/2}$

Notice that $(p_i, p_j) = 0$ if $i \neq j$ (hence $(w_i, w_j) = 0$ for $i \neq j$) and that $(w_i, w_i)^{1/2} = 1$

Example

Consider $V = \mathbb{R}^3$ with the inner product $(x,y) = \vec{x}\cdot\vec{y}$.

Consider the basis:

$$V_1 = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 \\ 11 \\ 13 \end{bmatrix}$$

$$p_1 = V_1 \longrightarrow w_1 = p_1 / (p_1,p_1)^{1/2} = 1/\sqrt{58} \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$$

note: $(w_1, w_1) = \left(1/58\,(3^2 + 7^2)\right)^{1/2} = 1^{1/2} = 1.$

$$p_2 = V_2 - \frac{(V_2, p_1)}{(p_1,p_1)} p_1$$

Now $(V_2, p_1) = -6$, $(p_1,p_1) = 58$ so $\dfrac{(V_2,p_1)}{(p_1,p_1)} p_1 = \dfrac{-3}{29}\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$

so that $V_2 - \dfrac{(V_2,p_1)}{(p_1,p_1)}p_1$ is $\begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} + \dfrac{3}{29}\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -49/29 \\ 5 \\ 21/29 \end{bmatrix}$

note: $(p_2,p_1) = (V_2,p_1) - \dfrac{(V_2,p_1)}{(p_1,p_1)}(p_1,p_1) = 0.$

and $(p_2,p_1) = (-49/29)\,3 + (5)\,0 + (21/29)\cdot 7$
$= -147/29 + 0 + 147/29 = 0$

so that $\vec{w}_2 = \vec{p}_2 / (p_2,p_2)^{1/2}$:

now $(p_2,p_2) = 823/29$ so

$$w_2 = \sqrt{\frac{29}{823}} \begin{bmatrix} -49/29 \\ 5 \\ 21/29 \end{bmatrix}$$

Q: Can you compute $\vec{p}_3$ and $\vec{w}_3$?

once found the vectors $\tilde{B} = \{w_1, w_2, w_3\}$ form an orthonormal basis for $\mathbb{R}^3$.