CAAM 336 · DIFFERENTIAL EQUATIONS

Fall 2013 Examination 2

1. [5 points]
   Let $\alpha \in \mathbb{R}$, let $\beta \in \mathbb{R}$, let $\gamma \in \mathbb{R}$ be such that $\gamma \neq 0$ and let $\mu \in \mathbb{R}$ be such that $\mu > 0$.

   (a) Verify that
   $$p(t) = \alpha \cos\left(\sqrt{\mu} t\right) + \frac{\beta}{\sqrt{\mu}} \sin\left(\sqrt{\mu} t\right)$$
   satisfies
   $$-p''(t) = \mu p(t),$$
   $$p(0) = \alpha$$
   and
   $$p'(0) = \beta.$$

   (b) Verify that
   $$q(t) = \alpha e^{\gamma t} + \frac{\beta}{\gamma}\left(e^{\gamma t} - 1\right)$$
   satisfies
   $$q'(t) = \gamma q(t) + \beta$$
   and
   $$q(0) = \alpha.$$

2. [5 points]
   Let $f \in C[0,1]$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Let u be such that
   $$-4u''(x) + 9u(x) = f(x), \quad 0 < x < 1;$$
   $$-4u'(0) = \alpha$$
   and
   $$4u'(1) = \rho.$$

   (a) It can be shown that
   $$\int_0^1 \big(4u'(x)v'(x) + 9u(x)v(x)\big)\, dx = g(f, \alpha, \rho, v) \text{ for all } v \in C^2[0,1].$$
   Obtain a formula for $g(f, \alpha, \rho, v)$.

3. [20 points]

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v, w) = a(v, w) + (v, w).$$

Let the norm $|||\cdot||| : H^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{B(v, v)}.$$

Let $f \in L^2(0,1)$, let $\rho \in \mathbb{R}$, let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$ and let $u \in H^1_D(0,1)$ be such that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Moreover, let N be a positive integer, let V_N be a subspace of $H^1_D(0,1)$ and let $u_N \in V_N$ be such that

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_N.$$

(a) Use the fact that $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$ and the fact that $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$ to show that $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$. Recall that $H^1(0,1) = \{v \in L^2(0,1) : v' \in L^2(0,1)\}$.

(b) Show that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N.$$

(c) Show that

$$|||u - u_N|||^2 = |||u|||^2 - |||u_N|||^2.$$

(d) Show that

$$|||u_N|||^2 \le |||u|||^2.$$

4. [25 points]

Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let $\phi_0 \in H^1(0,1)$ be defined by

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

let $\phi_j \in H^1_D(0,1)$ be defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let $\phi_{N+1} \in H^1_D(0,1)$ be defined by

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the symmetric bilinear form $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v, w) = a(v, w) + (v, w).$$

Also, let $f \in L^2(0,1)$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Moreover, let $u \in H^1(0,1)$ be such that $u(0) = \alpha$ and

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Let $V_N = \operatorname{span}\{\phi_0, \phi_1, \ldots, \phi_{N+1}\}$ and let $V_{N,D} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_{N+1}\}$. Let $u_N \in V_N$ be such that $u_N(0) = \alpha$ and

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

(a) We can write

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j$$

where, for $j = 1, 2, \ldots, N+1$, c_j is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$ which is the solution to

$$\mathbf{Kc} = \mathbf{b}.$$

What are the entries of the matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ and the vector $\mathbf{b} \in \mathbb{R}^{N+1}$?

(b) Show that

$$B(u - u_N, u - u_N) = B(u, u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0).$$

(c) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = 0$, $\rho = 0$ and $N = 1$. Note that, when $N = 1$,

$$\int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$\int_0^{1/2} \phi_0(x)\phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\phi_2(x)\,dx = \frac{1}{6};$$

and

$$\int_0^{1/2} \phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\,dx = \frac{1}{4}.$$

(d) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = -1$, $\rho = 1$ and $N = 1$.

5. [35 points]

Let

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

In this question we will consider the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = 1, \quad t \ge 0,$$

and

$$u_x(1,t) = 2, \quad t \ge 0,$$

and initial condition

$$u(x,0) = x^2 + 1, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{w \in C^2[0,1] : w(0) = w'(1) = 0\right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \psi_m(x)\psi_n(x)\,dx = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \ne n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) For $n = 1, 2, \ldots$, compute

$$\int_0^1 f(x)\psi_n(x)\, dx.$$

(c) Let $w(x)$ be such that

$$w''(x) = 0,$$

$$w(0) = 1$$

and

$$w'(1) = 2.$$

Obtain a formula for $w(x)$.

(d) Let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

$$\hat{u}(0,t) = \hat{u}_x(1,t) = 0, \quad t \ge 0,$$

and

$$\hat{u}(x,0) = \hat{u}_0(x), \qquad 0 \le x \le 1,$$

where $\hat{u}_0(x)$ is such that

$$u(x,t) = w(x) + \hat{u}(x,t).$$

Obtain a formula for $\hat{u}_0(x)$.

(e) For $n = 1, 2, \ldots$, compute

$$\int_0^1 \hat{u}_0(x)\psi_n(x)\, dx.$$

(f) We can write

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$b_n = \int_0^1 f(x)\psi_n(x)\, dx.$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(g) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(h) Write out a formula for $u(x,t)$.

6. [10 points]

In this question we will consider the problem of finding the solution $u(x,t)$ to the wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 \leq x \leq 1, \quad t \geq 0,$$

with boundary conditions

$$u(0,t) = u_x(1,t) = 0, \quad t \geq 0,$$

and initial conditions

$$u(x,0) = 0, \qquad 0 \leq x \leq 1,$$

and

$$u_t(x,0) = x^2 - 2x, \qquad 0 \leq x \leq 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w'(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin\left(\frac{2n-1}{2} \pi x \right)$$

for $n = 1, 2, \ldots$. Recall that you obtained a formula for the eigenvalues of L in question 5.

(a) We can write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \psi_n(x).$$

What ordinary differential equation and initial conditions does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(b) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(c) Write out a formula for $u(x,t)$.