CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 3 · Solutions

Posted Wednesday 5 September 2012. Due Wednesday 12 September 2012, 5pm.

1. [20 points]
   Determine whether each of the following functions $(\cdot,\cdot)$ determines an inner product on the vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

(a) $\mathcal{V} = C^1[0,1]$, $(u,v) = \int_0^1 u(x)v'(x)\,dx$ (b) $\mathcal{V} = C[0,1]$: $(u,v) = \int_0^1 |u(x)||v(x)|\,dx$

(c) $\mathcal{V} = C[0,1]$: $(u,v) = \int_0^1 u(x)v(x)e^{-x}\,dx$ (d) $\mathcal{V} = C[0,1]$: $(u,v) = \int_0^1 \big(u(x)+v(x)\big)\,dx$

Solution. For each example, we shall check all three properties required for an inner product.

(a) *This function is not an inner product*: it is not symmetric and (u,u) need not be positive.
[**GRADERS**: deduct 2 points if a student only identifies one of the two failed properties.]
The function (u,v) is not symmetric. For example, if $u(x)=1$ and $v(x)=x$, then

$$(u,v) = \int_0^1 u(x)v'(x)\,dx = \int_0^1 1\,dx = 1,$$

yet

$$(v,u) = \int_0^1 v(x)u'(x)\,dx = \int_0^1 0\,dx = 0.$$

The function also fails to satisfy the nonnegativity requirement: For example, if $u(x)=1$, then $(u,u)=0$; if $u(x)=1-x$, then

$$(u,u) = \int_0^1 (1-x)(-1)\,dx = -1/2.$$

[**GRADERS**: students need only to show that $(u,u)=0$ for nonzero u, or $(u,u)<0$ for some u. They need not show both.]
For what it is worth, we note that (u,v) is linear:

$$(\alpha u+\beta v, w) = \alpha\int_0^1 u(x)w'(x)\,dx + \beta\int_0^1 v(x)w'(x)\,dx = \alpha(u,w)+\beta(v,w).$$

(b) *This function is not an inner product*: it is not linear.
The function (u,v) is symmetric, as

$$(u,v) = \int_0^1 |u(x)||v(x)|\,dx = \int_0^1 |v(x)||u(x)|\,dx = (v,u).$$

However, it is not linear:

$$(\alpha u, w) = \int_0^1 |\alpha u(x)||w(x)|\,dx = \int_0^1 |\alpha||u(x)||w(x)|\,dx = |\alpha|(u,w).$$

If $u,w\neq 0$ and $\alpha<0$, then we have $(\alpha u,w)\neq \alpha(u,w)$.

The function (u, u) is nonnegative and positive when $u \neq 0$:

$$(u, u) = \int_0^1 |u(x)|^2 \, dx$$

is the integral of a nonnegative function, and hence is nonnegative; $(u, u) = 0$ if and only if $u(x) = 0$ for all x, i.e., $u = 0$.

(c) *This function is an inner product.*
The function (u, v) is symmetric, as

$$(u, v) = \int_0^1 u(x)v(x)e^{-x} \, dx = \int_0^1 v(x)u(x)e^{-x} \, dx = (v, u).$$

Similarly, the function is also linear:

$$\begin{aligned}(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x} \, dx \\ &= \alpha \int_0^1 u(x)w(x)e^{-x} \, dx + \beta \int_0^1 v(x)w(x)e^{-x} \, dx \\ &= \alpha(u, w) + \beta(v, w).\end{aligned}$$

Lastly, we check

$$(u, u) = \int_0^1 u(x)^2 e^{-x} dx.$$

The function e^{-x} is positive valued on $[0, 1]$, so we have that (u, u) is the integrand of a nonnegative function, and hence is also nonnegative. If $(u, u) = 0$, then $u(x)^2 e^{-x} = 0$ for all $x \in [0, 1]$, which means that $u(x) = 0$ for all such x, i.e., $u = 0$.

(d) *This function is not an inner product*: it is not linear in the first component, and (u, u) need not be positive.
[**GRADERS**: deduct 2 points if a student only identifies one of the two failed properties.]
To test linearity, consider

$$(u+v, w) = \int_0^1 \big(u(x)+v(x)\big)+w(x) \, dx = \int_0^1 u(x)+w(x) \, dx + \int_0^1 v(x) \, dx = (u, w) + \int_0^1 v(x) \, dx.$$

For most choices of v and w (for example, $v(x) = w(x) = 1$), $\int_0^1 v(x) \, dx \neq (v, w)$, so $(\cdot, \cdot)$ is not linear. The function $(\cdot, \cdot)$ also fails to satisfy the nonnegativity requirement. For example, if $u(x) = -1$, then

$$(u, u) = \int_0^1 \big(u(x) + u(x)\big) \, dx = \int_0^1 -2 \, dx = -2 < 0.$$

For what it is worth, note that (u, v) symmetric: $(u, v) = (v, u)$.

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2. [20 points]
Suppose $\mathcal{V}$ is a vector space with an associated inner product. The angle $\angle(u, v)$ between u and $v \in \mathcal{V}$ is defined via the equation

$$(u, v) = \|u\|\|v\| \cos \angle(u, v).$$

Let $\mathcal{V} = C[0, 1]$ and $(u, v) = \int_0^1 u(x)v(x) \, dx$. Compute $\cos \angle(x^n, x^m)$ between $u(x) = x^n$ and $v(x) = x^m$ for nonnegative integers m and n. What happens to $\angle(x^n, x^{n+1})$ as $n \to \infty$?

Solution. This result is a simple calculation. For integers $n, m \geq 0$, we have

$$(x^n, x^m) = \int_0^1 x^{n+m} \, dx = \frac{1}{n + m + 1}.$$

We also have

$$\|x^n\|^2 = (x^n, x^n) = \frac{1}{2n+1}.$$

Altogether, we have the following formula for the angle between x^n and x^m:

$$\cos\angle(x^n, x^m) = \frac{(x^n, x^m)}{\|x^n\|\|x^m\|} = \frac{\sqrt{(2n+1)(2m+1)}}{n+m+1}.$$

As $m = n+1$ and $n \to \infty$, we see that $\cos\angle(x^n, x^{n+1}) \to 1$, implying that the angle between x^n and x^{n+1} shrinks to zero as $n \to \infty$. This observation is all that is required for full credit, but a little manipulation yields even greater insight. We can equivalently write

$$\cos\angle(x^n, x^m) = \sqrt{1 - \frac{(n-m)^2}{(n+m+1)^2}},$$

which for larger m and n gives the approximation

$$\cos\angle(x^n, x^m) \approx 1 - \frac{1}{2}\frac{(n-m)^2}{(n+m+1)^2} - \frac{1}{8}\frac{(n-m)^4}{(n+m+1)^4}.$$

Note that for small θ,

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4.$$

Comparing terms, we arrive at the first-order approximation

$$\angle(x^n, x^m) \approx \frac{|n-m|}{n+m+1}$$

suitable when $n+m$ is large and best when $|n-m|$ is small as well. This approximation confirms the intuition we obtain from comparing graphs of x^n and x^m for large m and n: the graphs look very similar, so we expect the 'angle' between these function in the inner product space to be small. This result of this problem signals that the usual basis for polynomials, $1, x, x^2, x^3, \ldots$ may introduce computational challenges. We prefer to use an orthogonal basis whenever possible.

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3. [25 points]
Consider the polynomials $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = 3x^2 - 1$, which form a basis for the set of all quadratic polynomials. These polynomials are orthogonal in $C[-1,1]$ with the usual inner product

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

(You do not need to prove this.) In the parts below, "best approximation" is defined with respect to this inner product, and the norm it induces.

Let $f(x) = \cos(\pi x)$.

(a) Construct the best approximation $f_1(x) = c_1\phi_1(x)$ to $f(x)$ from $\text{span}\{\phi_1\}$ (i.e., determine c_1 to minimize $\|f - f_1\|$ in $C[-1,1]$).

(b) Construct the best approximation $f_2(x) = c_1\phi_1(x) + c_2\phi_2(x)$ to $f(x)$ from $\text{span}\{\phi_1, \phi_2\}$.

(c) Construct the best approximation $f_3(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ to $f(x)$ from $\text{span}\{\phi_1, \phi_2, \phi_3\}$.

(d) Produce a plot that superimposes your best approximation from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) The best approximation to $f(x) = \cos(\pi x)$ from $\text{span}\{\phi_1\}$ on $[0, 1]$ in the usual inner product is

$$f_1(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x).$$

We compute

$$\begin{aligned}(\phi_1, \phi_1) &= \int_{-1}^{1} 1^2 \, dx = 2 \\ (f, \phi_1) &= \int_{-1}^{1} 1 \cdot \cos(\pi x) \, dx = 0,\end{aligned}$$

and hence the best approximation is trivial:

$$f_1(x) = 0 \cdot \phi_1(x) = 0.$$

(b) To compute the best approximation from $\text{span}\{\phi_1, \phi_2\}$, we can follow two approaches. If we notice that ϕ_1 and ϕ_2 are orthogonal, i.e., $(\phi_1, \phi_2) = 0$, we can directly compute the best approximation as the sum of the best approximation from each of the one dimensional subspaces:

$$f_2(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x).$$

Noting, in addition to the quantities from part (a), that

$$\begin{aligned}(\phi_2, \phi_2) &= \int_{-1}^{1} x^2 \, dx = 2/3 \\ (f, \phi_2) &= \int_{-1}^{1} x \cdot \cos(\pi x) \, dx = 0,\end{aligned}$$

we again find the trivial approximation:

$$f_2(x) = 0 \cdot \phi_1(x) + 0 \cdot \phi_2(x) = 0.$$

Alternatively, we could have arrived at this approximation via the linear system

$$\begin{bmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \end{bmatrix}.$$

Using $(\phi_1, \phi_2) = (\phi_2, \phi_1) = 0$ gives the same trivial result as above.

(c) We can find the coefficients of $f_3(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ by noting that all three basis functions are orthogonal,

$$(\phi_1, \phi_2) = (\phi_1, \phi_3) = (\phi_2, \phi_3) = 0,$$

and thus constructing

$$f_3(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)}\phi_3(x).$$

Toward this end, compute

$$(\phi_3, \phi_3) = \int_{-1}^{1} (3x^2 - 1)^2 \, dx = 8/5$$

$$(\phi_3, f) = \int_{-1}^{1} (3x^2 - 1) \cdot \cos(\pi x) \, dx = -12/\pi^2,$$

thus giving

$$f_3(x) = 0 \cdot \phi_1(x) + 0 \cdot \phi_2(x) + \frac{-12/\pi^2}{8/5} \phi_3(x) = \frac{-15}{2\pi^2}(3x^2 - 1).$$

Alternatively we can solve the linear system,

$$\begin{bmatrix} (\phi_1,\phi_1) & (\phi_1,\phi_2) & (\phi_1,\phi_3) \\ (\phi_2,\phi_1) & (\phi_2,\phi_2) & (\phi_2,\phi_3) \\ (\phi_3,\phi_1) & (\phi_3,\phi_2) & (\phi_3,\phi_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} (\phi_1, f) \\ (\phi_2, f) \\ (\phi_3, f) \end{bmatrix}$$

whose coefficients have already been computed, except for

$$(\phi_3, \phi_1) = (\phi_1, \phi_3) = \int_{-1}^{1} 1(3x^2 - 1) \, dx = 0$$

$$(\phi_3, \phi_2) = (\phi_3, \phi_2) = \int_{-1}^{1} x(3x^2 - 1) \, dx = 0,$$

i.e., the basis vectors ϕ_1, ϕ_2, and ϕ_3 are orthogonal. Since the matrix is diagonal, the coefficients c_1, c_2, and c_3 are simple to find:

$$c_1 = (\phi_1, f)/(\phi_1, \phi_1) = 0$$

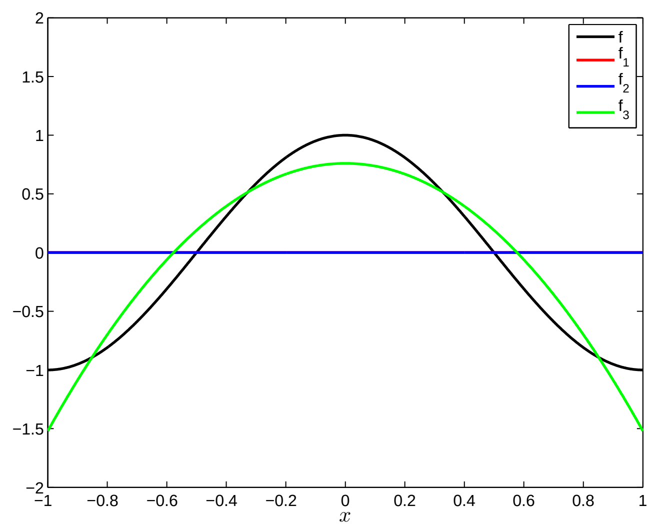
$$c_2 = (\phi_2, f)/(\phi_2, \phi_2) = 0$$

$$c_3 = (\phi_3, f)/(\phi_3, \phi_3) = -15/(2\pi^2).$$

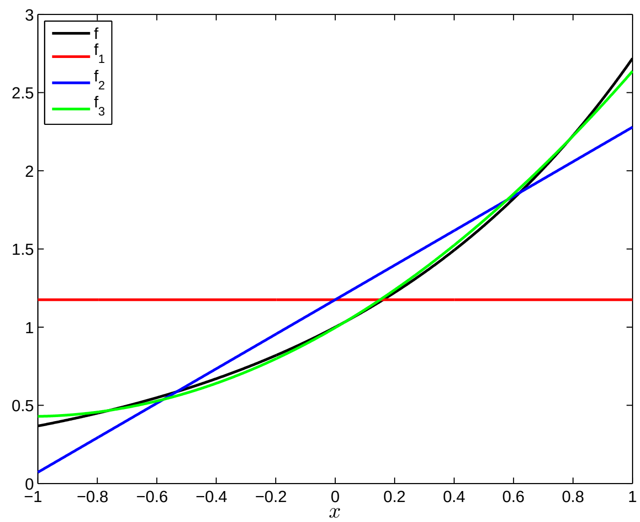
Thus the best approximation to $\cos(\pi x)$ from the quadratic polynomials is

$$f_3(x) = \frac{15}{2\pi^2}(1 - 3x^2).$$

(d) The following plot compares best approximations to $f(x)$. Note that f_2 obscures f_1. [**GRADERS**: the code for making such a plot is so basic that students do not need to include it. If they omitted the code, please do not subtract points here.]



A more interesting series of approximations follows for $f(x) = e^x$, as seen in the following figure.



please turn over

4. [35 points]
Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

We can approximate the differential equation

$$\frac{d^2}{dx^2} u = f(x), \quad x \in (0,1),$$

with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ by the matrix equation

$$\frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix},$$

where $u_j \approx u(x_j)$. (Entries of the matrix that are not specified are zero.)

(a) Suppose that $f(x) = 25\pi^2 \cos(5\pi x)$.
Compute and plot the approximate solutions obtained when $N = 8, 16, 32, 64, 128$.
You may superimpose these on one plot. To solve the linear systems, you may use MATLAB's 'backslash' command: `u = A\f`.
For each value of N compute the maximum error $|u_j - u(x_j)|$, given that the true solution is

$$u(x) = 1 - 2x - \cos(5\pi x).$$

Plot this error using a `loglog` plot with error on the vertical axis and N on the horizontal axis.

(b) Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

$$u(0) = 1, \quad u(1) = 2.$$

Compute and plot solutions for $N = 8, 32, 128$.

(c) Now suppose that we have mixed boundary conditions

$$u(0) = 1, \quad \frac{du}{dx}(1) = -5.$$

The matrix equation will now have the form

$$\frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \ddots & & \\ & & \ddots & \ddots & 1 & 0 \\ & & & 1 & -2 & \star \\ & & & \star & \star & \star \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} f(x_1) - * \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \\ \star \end{bmatrix}.$$

Specify values for the entries marked by $\star$ to impose the approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N-1} - 4u_N + 3u_{N+1}}{2h} = -5.$$

Compute and plot solutions for $N = 8, 32, 128$.

Optional: What happens to the overall accuracy of the approximate solution if, instead of the $O(h^2)$ accurate approximation given above, you only use the $O(h)$ approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N+1} - u_N}{h} \; ?$$

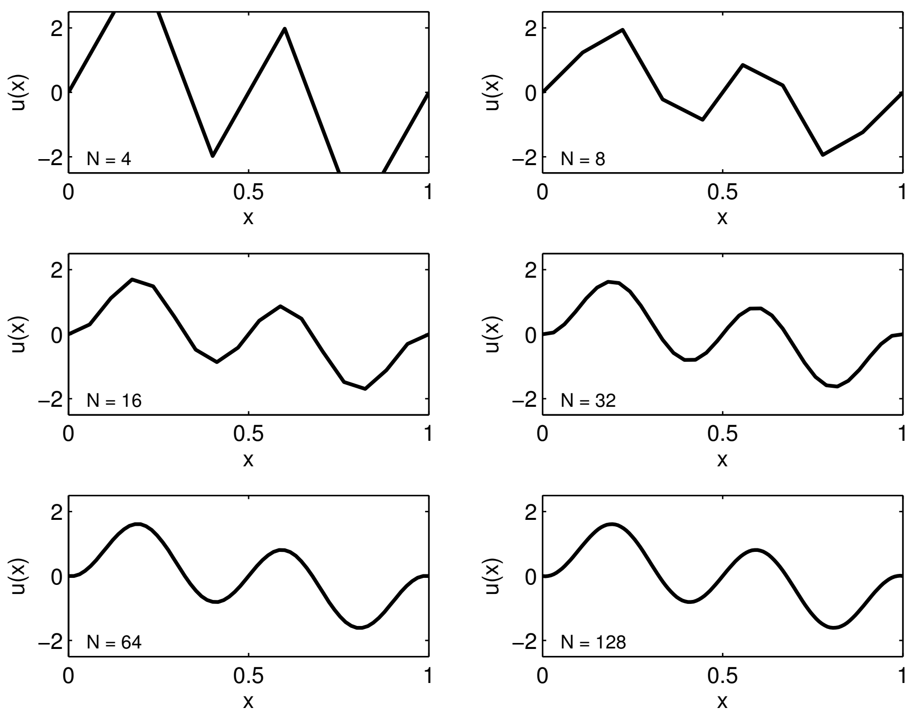
Solution.

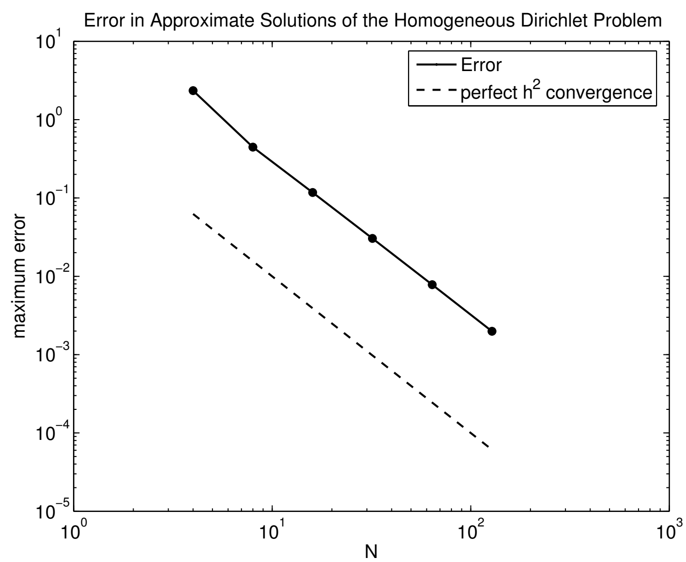
(a) The code listed below produces the desired plots. (We use `subplot` rather than superimposing the plots, and we also include $N = 4$ for the sake of comparison.)

[On the error plot, we include a line showing the rate of convergence if the error was reduced exactly like h^2. We see that the rates (that is, the slope of the true error curve (solid) and this h^2 curve (dashed)) are quite close. The follows from the fact that our approximation to the second derivative makes an error of size h^2; see the note for lectures 5 and 6.]

```
 Nvec = [4 8 16 32 64 128];
 err = zeros(size(Nvec));
 figure(1), clf
 for j=1:length(Nvec)
    N = Nvec(j);
    h = 1/(N+1);
    x = h*[1:N]';
    A = (-2*eye(N)+diag(ones(N-1,1),1)+diag(ones(N-1,1),-1))/(h^2);
    f = 25*pi^2*cos(5*pi*x);
    u = A\f;
% plot the function, adding in the homogeneous values at the boundary;
% this tacks on extra entries for the x and u vectors:
    figure(1), subplot(3,2,j)
    plot([0;x;1],[0;u;0],'k-','linewidth',2), hold on
    axis([0 1 -2.5 2.5])
    xlabel('x')
    ylabel('u(x)')
    text(.05,-2.05,sprintf('N = %d',N))
% compute error
    true_u = 1-2*x-cos(5*pi*x);
    err(j) = max(abs(true_u - u));
 end
 print -depsc2 diffmats_a1.eps

% error plot
 figure(2), clf
 loglog(Nvec, err, 'k.-','linewidth',1.5), hold on
 loglog(Nvec, Nvec.^(-2), 'k--', 'linewidth',1.5)
 legend('Error', 'perfect h^2 convergence',1)
 loglog(Nvec, err, 'k.','markersize',20)
 set(gca,'fontsize',14)
 xlabel('N')
 ylabel('maximum error')
 title('Error in Approximate Solutions of the Homogeneous Dirichlet Problem')
 print -depsc2 diffmats_a2.eps
```





(b) To impose inhomogeneous boundary conditions, we must make a correction to the right hand side vector to account for the fact that $u(0)$ and $u(1)$ are nonzero. For example, when $N = 3$ we have the equation

$$\frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} u(0) \\ u(x_1) \\ u(x_2) \\ u(x_3) \\ u(1) \end{bmatrix} \approx \begin{bmatrix} u''(x_1) \\ u''(x_2) \\ u''(x_3) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}.$$

Suppose we have $u(0) = \alpha$ and $u(1) = \beta$. Then we can write the above approximation in the form

$$\frac{1}{h^2}\begin{bmatrix}1\\0\\0\end{bmatrix}\alpha + \frac{1}{h^2}\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}u(x_1)\\u(x_2)\\u(x_3)\end{bmatrix} + \frac{1}{h^2}\begin{bmatrix}0\\0\\1\end{bmatrix}\beta \approx \begin{bmatrix}f(x_1)\\f(x_2)\\f(x_3)\end{bmatrix}.$$

Moving the two known vectors on the left to the right hand side, we obtain

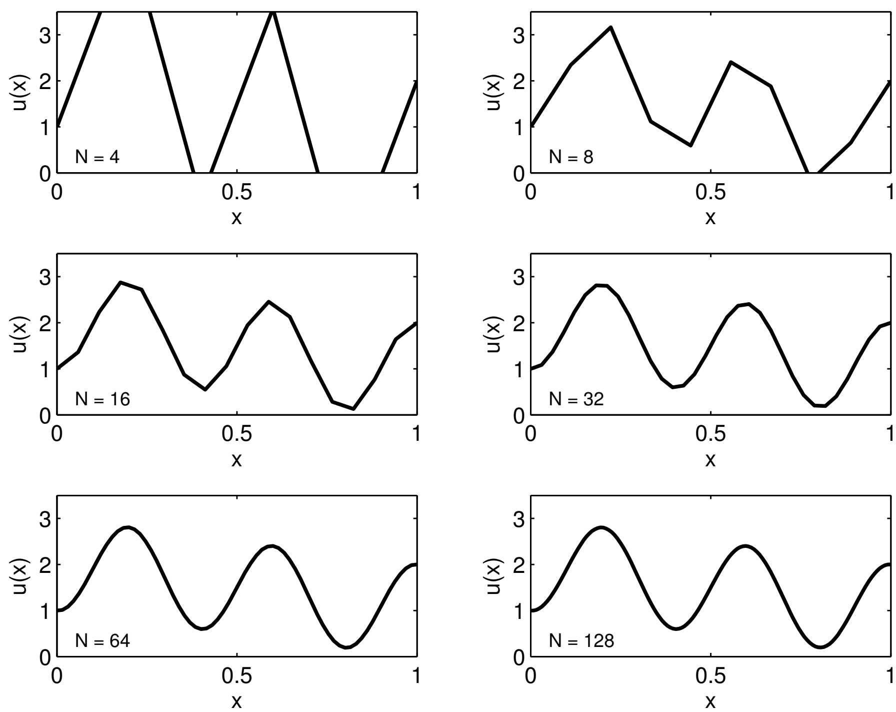
$$\frac{1}{h^2}\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}u(x_1)\\u(x_2)\\u(x_3)\end{bmatrix} \approx \begin{bmatrix}f(x_1)-\alpha/h^2\\f(x_2)\\f(x_3)-\beta/h^2\end{bmatrix}.$$

The formula generalizes to larger values of N: One imposes inhomogeneous Dirichlet boundary conditions by modifying the first and last entries of the right-hand side vector.

[**GRADERS**: It is also possible to impose inhomogeneous Dirichlet boundary conditions for this problem by adding an appropriate line to the solution from part (a). Please give credit for this solution, though I prefer the approach outlined here because it more easily generalizes to a broader class of operators.]

The code below produces the requested plots, plus a few extra.

```
 Nvec = [4 8 16 32 64 128];
 u0 = 1;
 u1 = 2;
 figure(1), clf
 for j=1:length(Nvec)
    N = Nvec(j);
    h = 1/(N+1);
    x = h*[1:N]';
    A = (-2*eye(N)+diag(ones(N-1,1),1)+diag(ones(N-1,1),-1))/(h^2);
    f = 25*pi^2*cos(5*pi*x);
    f(1) = f(1)-u0/(h^2);
    f(N) = f(N)-u1/(h^2);
    u = A\f;
% plot the function, adding in the inhomogeneous values at the boundary;
% this tacks on extra entries for the x and u vectors:
    subplot(3,2,j)
    plot([0;x;1],[u0;u;u1],'k-','linewidth',2), hold on
    axis([0 1 0 3.5])
    xlabel('x')
    ylabel('u(x)')
    set(gca,'ytick',[0:1:3])
    text(.05,.35,sprintf('N = %d',N))
 end
 print -depsc2 diffmats_b
```



(c) The mixed boundary conditions impose a further complication, as now we do not know a value for $u(1) = u(x_{N+1})$. Suppose that $u(0) = \alpha$. Again resorting to the $N = 3$ case for illustrative purposes, we have

$$\frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} \approx \begin{bmatrix} u''(x_1) \\ u''(x_2) \\ u''(x_3) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}.$$

This gives three equations in the four unknowns $u(x_1)$, $u(x_2)$, $u(x_3)$, and $u(x_4)$. We need a further equation, and this gives the opportunity to insert some information about the right boundary condition $u'(1) = \beta$. We can use the difference approximation

$$\beta = u'(1) \approx \frac{u_{N-1} - 4u_N + 3u_{N+1}}{2h}$$

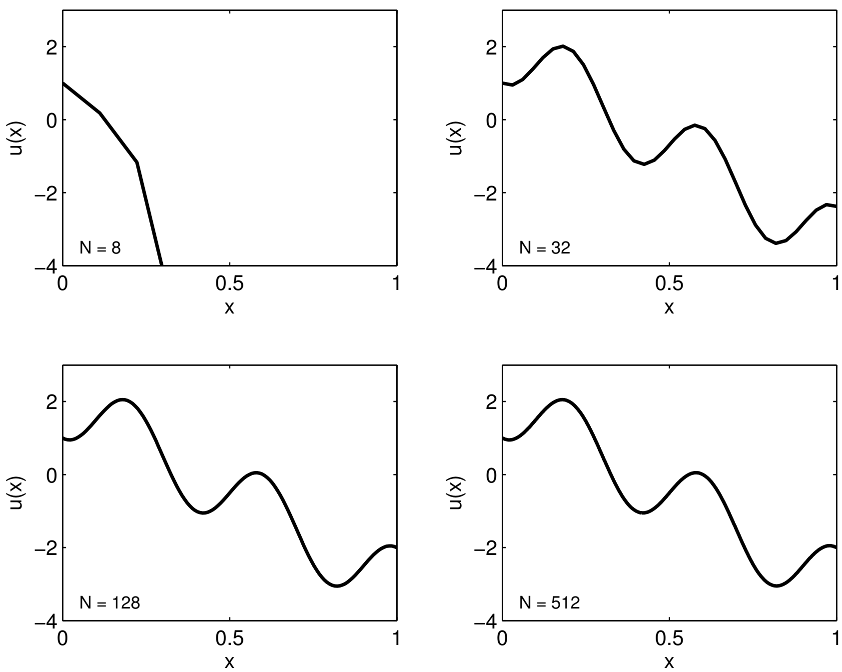
to get an additional equation. We can add this to our previous matrix equation to obtain

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1/2 & -2 & 3/2 \end{bmatrix}\begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} \approx \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_2) \\ f(x_3) \\ \beta/h \end{bmatrix}.$$

This is implemented in the code below.

```
Nvec = [4 8 16 32 64 128];
Nvec = [8 32 128 512];
u0 = 1;
u1prime = -5;
figure(1), clf
for j=1:length(Nvec)
   N = Nvec(j);
   h = 1/(N+1);
   x = h*[1:N]';
```

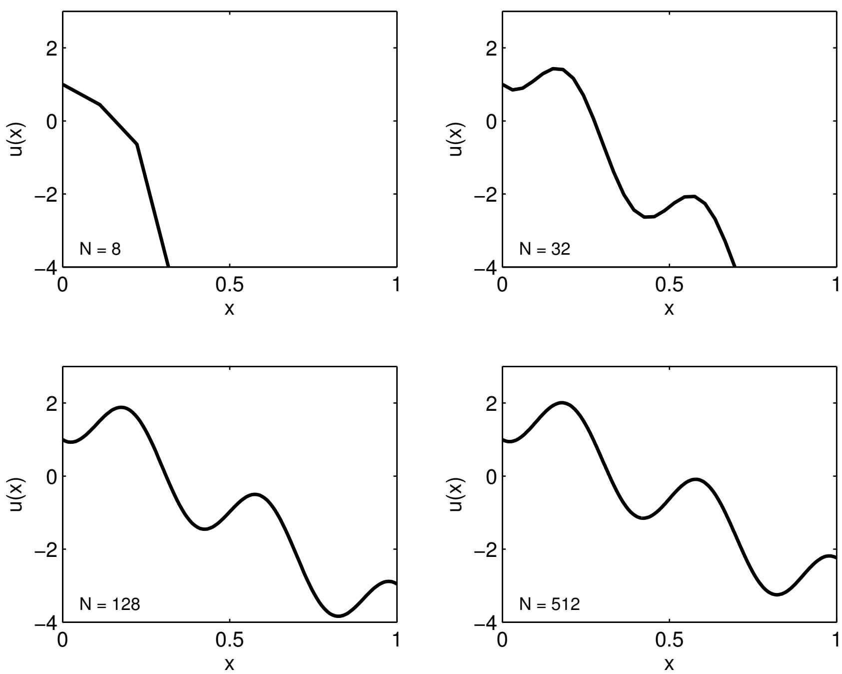
```
    A = (-2*eye(N+1)+diag(ones(N,1),1)+diag(ones(N,1),-1))/(h^2);
    A(N+1,N-1:N+1) = [1/2 -2 3/2]/h^2;
    f = 25*pi^2*cos(5*pi*x);
    f(1) = f(1) - u0/(h^2);
    f = [f;u1prime/h];
    u = A\f;
% plot the function, adding in the inhomogeneous value at the left boundary;
% this tacks extra entries on to the x and u vectors:
    subplot(2,2,j)
    plot([0;x;1],[u0;u],'k-','linewidth',2), hold on
%    axis([0 1 0 3.5])
    axis([0 1 -4 3])
    xlabel('x')
    ylabel('u(x)')
%    set(gca,'ytick',[0:1:3])
    text(.05,-3.5,sprintf('N = %d',N))
 end
```



Why did we use the strange formula to approximate $u'(1)$? Why not use the simpler backward difference approximation

$$\beta = u'(1) \approx \frac{u(x_{N+1}) - u(x_N)}{h}$$

for the last row of the matrix equation? This approximation is only $O(h)$ accurate, whereas the formula used above is $O(h^2)$ accurate. For small h, this makes a big difference. Recall that the other rows use an $O(h^2)$ approximation to the second derivative. By using an $O(h)$ approximation at just one point, $u'(1)$, we destroy the accuracy. This is apparent in the slower convergence shown in the plots below. For the largest value of h ($N = 8$), there is not much difference between this approximation and the one used above. For $n = 32$ and $N = 128$, the difference is quite apparent!


N = 8
N = 32
N = 128
N = 512
u(x)
x
2
0
−2
−4
0
0.5
1