CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 47 · Solutions

Posted Wednesday 20 November 2013. Due 5pm Wednesday 4 December 2013.

47. [25 points] Let the norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\|\mathbf{y}\| = \sqrt{\mathbf{y}\cdot\mathbf{y}}.$$

Let the timestep $\Delta t \in \mathbb{R}$ be such that $\Delta t > 0$ and let $t_k = k\Delta t$ for $k = 0, 1, 2, \ldots$. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and consider the problem of finding $\mathbf{x}(t)$ such that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \geq 0$$

and

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Compute $\mathbf{x}(t)$. Note that for real numbers t,

$$e^{it} = \cos(t) + i\sin(t)$$

and

$$e^{-it} = \cos(t) - i\sin(t).$$

(b) How does $\|\mathbf{x}(t)\|$ behave as t increases?

(c) For $k = 0, 1, 2, \ldots$, let $\mathbf{x}_k$ be the approximation to $\mathbf{x}(t_k)$ obtained using the forward Euler method. For all choices of the timestep $\Delta t > 0$, how will $\|\mathbf{x}_k\|$ behave as $k \to \infty$?

(d) For $k = 0, 1, 2, \ldots$, let $\mathbf{x}_k$ be the approximation to $\mathbf{x}(t_k)$ obtained using the backward Euler method. For all choices of the timestep $\Delta t > 0$, how will $\|\mathbf{x}_k\|$ behave as $k \to \infty$?

Solution.

(a) [10 points] The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = -i$ and $\lambda_2 = i$ and eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

which are such that $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. If we set

$$\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

then we have that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

and

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{\Lambda}}\mathbf{V}^{-1} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left(e^{it} + e^{-it}\right) & \frac{i}{2}\left(e^{-it} - e^{it}\right) \\ \frac{i}{2}\left(e^{it} - e^{-it}\right) & \frac{1}{2}\left(e^{it} + e^{-it}\right) \end{bmatrix}. \end{aligned}$$

Now,

$$e^{it} + e^{-it} = \cos(t) + i\sin(t) + \cos(t) - i\sin(t) = 2\cos(t),$$

$$\begin{aligned} i\left(e^{it} - e^{-it}\right) &= i\left(\cos(t) + i\sin(t) - (\cos(t) - i\sin(t))\right) \\ &= i\left(\cos(t) + i\sin(t) - \cos(t) + i\sin(t)\right) \\ &= 2i^2\sin(t) \\ &= -2\sin(t) \end{aligned}$$

and

$$i\left(e^{-it} - e^{it}\right) = -i\left(e^{it} - e^{-it}\right) = 2\sin(t).$$

Therefore,

$$e^{t\mathbf{A}} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Hence,

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) - \sin(t) \end{bmatrix}.$$

(b) [5 points] We can compute that, for each $t \in \mathbb{R}$,

$$\begin{aligned} \|\mathbf{x}(t)\|^2 &= (\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2 \\ &= (\cos(t))^2 + 2\cos(t)\sin(t) + (\sin(t))^2 + (\cos(t))^2 - 2\cos(t)\sin(t) + (\sin(t))^2 \\ &= 2\left((\cos(t))^2 + (\sin(t))^2\right) \\ &= 2. \end{aligned}$$

Hence, for all $t \geq 0$,

$$\|\mathbf{x}(t)\| = \sqrt{2}$$

and so $\|\mathbf{x}(t)\|$ does not change as t increases.

(c) [5 points] Now,

$$\mathbf{x}_k = (\mathbf{I} + \Delta t\mathbf{A})^k\mathbf{x}_0.$$

Moreover, the eigenvalues of $\mathbf{I} + \Delta t\mathbf{A}$ are $1 + \Delta t\lambda_1 = 1 - \Delta t i$ and $1 + \Delta t\lambda_2 = 1 + \Delta t i$ and

$$\mathbf{I} + \Delta t\mathbf{A} = \mathbf{V}\begin{bmatrix} 1 - \Delta t i & 0 \\ 0 & 1 + \Delta t i \end{bmatrix}\mathbf{V}^{-1}.$$

Furthermore, for all choices of the timestep $\Delta t > 0$,

$$|1 - \Delta t i| = \sqrt{1 + (\Delta t)^2} > 1$$

and

$$|1+\Delta t i| = \sqrt{1+(\Delta t)^2} > 1.$$

Hence, for all choices of the timestep $\Delta t > 0$, $\|\mathbf{x}_k\| \to \infty$ as $k \to \infty$.

(d) [5 points] Now,

$$\mathbf{x}_k = ((\mathbf{I} - \Delta t\mathbf{A})^{-1})^k \mathbf{x}_0.$$

Moreover, the eigenvalues of $(\mathbf{I} - \Delta t\mathbf{A})^{-1}$ are $\frac{1}{1-\Delta t\lambda_1} = \frac{1}{1+\Delta t i}$ and $\frac{1}{1-\Delta t\lambda_2} = \frac{1}{1-\Delta t i}$ and

$$(\mathbf{I} - \Delta t\mathbf{A})^{-1} = \mathbf{V}\begin{bmatrix} \frac{1}{1+\Delta t i} & 0 \\ 0 & \frac{1}{1-\Delta t i} \end{bmatrix}\mathbf{V}^{-1}.$$

Furthermore, for all choices of the timestep $\Delta t > 0$,

$$\left|\frac{1}{1+\Delta t i}\right| = \frac{|1|}{|1+\Delta t i|} = \frac{1}{\sqrt{1+(\Delta t)^2}} < 1$$

and

$$\left|\frac{1}{1-\Delta t i}\right| = \frac{|1|}{|1-\Delta t i|} = \frac{1}{\sqrt{1+(\Delta t)^2}} < 1$$

since

$$\sqrt{1+(\Delta t)^2} > 1.$$

Hence, for all choices of the timestep $\Delta t > 0$, $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$.