

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 27

Posted Wednesday 9 October 2013. Due 1pm Friday 18 October 2013.

27. [25 points] Let the inner product $(\cdot, \cdot) : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x) dx$$

and let the norm $\|\cdot\| : C[0, 1] \rightarrow \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let the linear operator $L : C_D^2[0, 1] \rightarrow C[0, 1]$ be defined by

$$Lv = -v''$$

where

$$C_D^2[0, 1] = \{w \in C^2[0, 1] : w(0) = w(1) = 0\}.$$

Recall that the operator L has eigenvalues

$$\lambda_n = n^2\pi^2$$

with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin(n\pi x)$$

for $n = 1, 2, \dots$. Let N be a positive integer, let $f \in C[0, 1]$ be defined by $f(x) = x^2(1 - x)$ and let u be the solution to

$$Lu = f.$$

- (a) Compute the best approximation f_N to f from $\text{span}\{\psi_1, \dots, \psi_N\}$ with respect to the norm $\|\cdot\|$.
- (b) Use the spectral method to compute the best approximation u_N to u from $\text{span}\{\psi_1, \dots, \psi_N\}$ with respect to the norm $\|\cdot\|$.
- (c) Produce a plot comparing f to f_N for $N = 1, 2, 3, 4, 5, 6$.
- (d) Plot the approximations u_N to u that you obtained using the spectral method for $N = 1, 2, 3, 4, 5, 6$.
- (e) Write down the series solution to

$$Lu = f$$

that is obtained using the spectral method.

- (f) By shifting the data and then using a series solution that you have obtained previously in this question, obtain a series solution to the problem of finding $\tilde{u} \in C^2[0, 1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{100}$$

and

$$\tilde{u}(1) = \frac{1}{100}.$$

- (g) Let \tilde{u}_N be the series solution that you obtained in part (f) but with ∞ replaced by N . Plot \tilde{u}_N for $N = 1, 2, 3, 4, 5, 6$.