# CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 6

Posted Wednesday 26 September 2012. Due Wednesday 3 October 2012, 5pm.

***Please write the name of your college on your paper.***

General advice: You may compute any integrals you encounter using symbolic mathematics tools such as WolframAlpha, Mathematica, or the Symbolic Math Toolbox in MATLAB.

1. [60 points]
   Solve the following boundary value problems using the spectral method.
   For each problem, (i) write down the expansions of the right hand side functions as linear combinations of the eigenfunctions; (ii) write down the sum for the solution $u$ obtained from the spectral method; and (iii) produce a plot showing the sum of the first twenty terms in the series for $u$.

   For parts (c)–(e) you may use the eigenvalues and eigenfunctions computed in Problem 1(d) of Problem Set 5, and the results of Section 5.2.3 of the text.

   (a) $-\dfrac{d^2u}{dx^2}(x) = e^x, \quad u(0) = 1,\ u(1) = 0.$

   (b) $-\dfrac{d^2u}{dx^2}(x) - 10u(x) = e^x, \quad u(0) = 0,\ u(1) = 0.$

   (c) $-\dfrac{d^2u}{dx^2}(x) = x + \sin(\pi x), \quad u(0) = \dfrac{du}{dx}(1) = 0.$

   (d) $-\dfrac{d^2u}{dx^2}(x) = x + \sin(\pi x), \quad u(0) = \dfrac{du}{dx}(1) = 1.$

   (e) $-\dfrac{d^2u}{dx^2}(x) = f(x), \quad \dfrac{du}{dx}(0) = u(1) = 0$, where $f(x) = \begin{cases} 1, & 0 < x < 1/2; \\ 0, & 1/2 < x < 1. \end{cases}$
   (This $f$ is not continuous; follow the usual procedure and see if you obtain a sensible answer.)

2. [40 points]
   For the problems we have considered thus far, the eigenvalues have always satisfied nice formulas that are fairly easy to compute. This problem illustrates that this is not always the case.
   Consider the equation
   $$-u''(x) = f(x), \quad x \in [0, 1]$$
   with a homogeneous *Robin condition* on the left,
   $$u(0) - u'(0) = 0$$
   and a homogeneous Dirichlet boundary condition on the right,
   $$u(1) = 0.$$
   Define the linear operator $L : V \to C[0, 1]$ via $Lu = -u''$ with
   $$V = \{u \in C^2[0, 1] : u(0) - u'(0) = u(1) = 0\}.$$

(a) Prove that $L$ is symmetric.

(b) Is zero an eigenvalue of $L$? That is, does there exist a nontrivial solution to $-u''(x) = 0$ with these boundary conditions?

(c) Compute the eigenfunctions of $L$ associated with nonzero eigenvalues, and show that these eigenvalues $\lambda$ must satisfy the equation

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

(d) In MATLAB, create a plot of $g(x) = -\tan(x)$ for $x \in [0, 5\pi]$ and superimpose (`hold on`) a plot of $h(x) = x$. By hand, mark the points where these two functions intersect on your plot.

(e) Use your plot in (d) to argue that $L$ has infinitely many eigenvalues, with $(n - \frac{1}{2})^2\pi^2 < \lambda_n < (n + \frac{1}{2})^2\pi^2$. What value does $\lambda_n$ tend to as $n$ becomes large?

(f) Estimate the first four eigenvalues to at least six digits of accuracy.
You will need to find the points of intersection you marked in part (d).
Please *don't* just try to 'eyeball' these by zooming in on your plot!
Instead, either use MATLAB's `fzero` function, or write your own implementation of a root-finding algorithm (Newton's method, bisection, etc.).

(g) [optional: 5 bonus points]
Recall the finite difference matrices you worked with in Problem Set 3. Figure out how to construct a similar discretization of $L\psi = \lambda\psi$ for the linear operator in this problem, paying particular attention to the boundary condition at $x = 1$. You should arrive at a matrix equation of the form $\mathbf{Dv} = \lambda\mathbf{v}$. Compute the eigenvalues of $\mathbf{D}$ using MATLAB's `eig` command. How do these approximations compare to the values you computed in part (f), as you take finer and finer discretizations $(N \to \infty)$?
Hint: The first row of your matrix equation $\mathbf{Dv} = \lambda\mathbf{v}$ should encode the equation

$$\frac{-u_0 + 2u_1 - u_2}{h^2} = \lambda u_1.$$

Obtain a formula for $u_0$ in terms of $u_1$ and $u_2$ via the boundary condition $u(0) - u'(0) = 0$ and the approximation

$$u'(0) = \frac{-3u_0 + 4u_1 - u_2}{2h} + O(h^2),$$

and use these values to update the $u_1$ and $u_2$ entries in the first row of $\mathbf{D}$....