CAAM 336 · DIFFERENTIAL EQUATIONS

Fall 2013 Examination 2

1. [5 points]
   Let $\alpha \in \mathbb{R}$, let $\beta \in \mathbb{R}$, let $\gamma \in \mathbb{R}$ be such that $\gamma \neq 0$ and let $\mu \in \mathbb{R}$ be such that $\mu > 0$.

   (a) Verify that
   $$p(t) = \alpha \cos\left(\sqrt{\mu} t\right) + \frac{\beta}{\sqrt{\mu}} \sin\left(\sqrt{\mu} t\right)$$
   satisfies
   $$-p''(t) = \mu p(t),$$
   $$p(0) = \alpha$$
   and
   $$p'(0) = \beta.$$

   (b) Verify that
   $$q(t) = \alpha e^{\gamma t} + \frac{\beta}{\gamma}\left(e^{\gamma t} - 1\right)$$
   satisfies
   $$q'(t) = \gamma q(t) + \beta$$
   and
   $$q(0) = \alpha.$$

Solution.

(a) [3 points] We can compute that
$$p'(t) = -\sqrt{\mu}\alpha \sin\left(\sqrt{\mu} t\right) + \beta \cos\left(\sqrt{\mu} t\right)$$
and that
$$p''(t) = -\mu\alpha \cos\left(\sqrt{\mu} t\right) - \sqrt{\mu}\beta \sin\left(\sqrt{\mu} t\right).$$
We can then conclude that
$$-p''(t) = \mu\alpha \cos\left(\sqrt{\mu} t\right) + \sqrt{\mu}\beta \sin\left(\sqrt{\mu} t\right) = \mu\left(\alpha \cos\left(\sqrt{\mu} t\right) + \frac{\beta}{\sqrt{\mu}} \sin\left(\sqrt{\mu} t\right)\right) = \mu p(t)$$
thus verifying that
$$-p''(t) = \mu p(t).$$

Moreover,

$$p(0) = \alpha \cos(0) + \frac{\beta}{\sqrt{\mu}} \sin(0) = \alpha.$$

Furthermore,

$$p'(0) = -\sqrt{\mu}\alpha \sin(0) + \beta \cos(0) = \beta.$$

(b) [2 points] We can compute that

$$q'(t) = \gamma\alpha e^{\gamma t} + \beta e^{\gamma t}$$

and that

$$\gamma q(t) + \beta = \gamma\alpha e^{\gamma t} + \beta\left(e^{\gamma t} - 1\right) + \beta = \gamma\alpha e^{\gamma t} + \beta e^{\gamma t} - \beta + \beta = \gamma\alpha e^{\gamma t} + \beta e^{\gamma t}$$

thus verifying that

$$q'(t) = \gamma q(t) + \beta.$$

Moreover,

$$q(0) = \alpha e^0 + \frac{\beta}{\gamma}\left(e^0 - 1\right) = \alpha + \frac{\beta}{\gamma}(1-1) = \alpha.$$

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2. [5 points]

Let $f \in C[0,1]$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Let u be such that

$$-4u''(x) + 9u(x) = f(x), \quad 0 < x < 1;$$

$$-4u'(0) = \alpha$$

and

$$4u'(1) = \rho.$$

(a) It can be shown that

$$\int_0^1 \left(4u'(x)v'(x) + 9u(x)v(x)\right)\,dx = g(f,\alpha,\rho,v) \text{ for all } v \in C^2[0,1].$$

Obtain a formula for $g(f,\alpha,\rho,v)$.

Solution.

(a) [5 points] If $v \in C^2[0,1]$, then

$$\int_0^1 \left(-4u''(x) + 9u(x)\right)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

since

$$-4u''(x) + 9u(x) = f(x), \quad 0 < x < 1.$$

Integration by parts then yields that

$$\begin{aligned}
&\int_0^1 \big(-4u''(x) + 9u(x)\big)\, v(x)\, dx \\
&= -4\int_0^1 u''(x)v(x)\, dx + 9\int_0^1 u(x)v(x)\, dx \\
&= -4\left[u'(x)v(x)\right]_0^1 + 4\int_0^1 u'(x)v'(x)\, dx + 9\int_0^1 u(x)v(x)\, dx \\
&= -4u'(1)v(1) - \big(-4u'(0)v(0)\big) + \int_0^1 \big(4u'(x)v'(x) + 9u(x)v(x)\big)\; dx
\end{aligned}$$

from which we can conclude that

$$-\rho v(1) - \alpha v(0) + \int_0^1 \big(4u'(x)v'(x) + 9u(x)v(x)\big)\; dx = \int_0^1 f(x)v(x)\, dx$$

since $-4u'(0) = \alpha$ and $4u'(1) = \rho$. Therefore,

$$\int_0^1 \big(4u'(x)v'(x) + 9u(x)v(x)\big)\; dx = g(f,\alpha,\rho,v) \text{ for all } v \in C^2[0,1]$$

where

$$g(f,\alpha,\rho,v) = \int_0^1 f(x)v(x)\, dx + \alpha v(0) + \rho v(1).$$

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3. [20 points]

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\, dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\, dx.$$

Let $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v,w) = a(v,w) + (v,w).$$

Let the norm $|||\cdot||| : H^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{B(v,v)}.$$

Let $f \in L^2(0,1)$, let $\rho \in \mathbb{R}$, let $H^1_D(0,1) = \big\{w \in H^1(0,1) : w(0) = 0\big\}$ and let $u \in H^1_D(0,1)$ be such that

$$B(u,v) = (f,v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Moreover, let N be a positive integer, let V_N be a subspace of $H^1_D(0,1)$ and let $u_N \in V_N$ be such that

$$B(u_N,v) = (f,v) + \rho v(1) \text{ for all } v \in V_N.$$

(a) Use the fact that $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$ and the fact that $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$ to show that $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$. Recall that $H^1(0,1) = \{v \in L^2(0,1) : v' \in L^2(0,1)\}$.

(b) Show that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N.$$

(c) Show that

$$|||u - u_N|||^2 = |||u|||^2 - |||u_N|||^2 .$$

(d) Show that

$$|||u_N|||^2 \leq |||u|||^2 .$$

Solution.

(a) [9 points] Since $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$,

$$(\alpha w_1 + \beta w_2, w_3) = \alpha(w_1, w_3) + \beta(w_2, w_3) \text{ for all } w_1, w_2, w_3 \in H^1(0,1) \text{ and all } \alpha, \beta \in \mathbb{R}$$

because if $w_1, w_2 \in H^1(0,1)$ then $w_1, w_2 \in L^2(0,1)$. Also, since $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$,

$$a(\alpha w_1 + \beta w_2, w_3) = \alpha a(w_1, w_3) + \beta a(w_2, w_3) \text{ for all } w_1, w_2, w_3 \in H^1(0,1) \text{ and all } \alpha, \beta \in \mathbb{R}.$$

Hence, for all $w_1, w_2, w_3 \in H^1(0,1)$ and all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} B(\alpha w_1 + \beta w_2, w_3) &= a(\alpha w_1 + \beta w_2, w_3) + (\alpha w_1 + \beta w_2, w_3) \\ &= \alpha a(w_1, w_3) + \beta a(w_2, w_3) + \alpha(w_1, w_3) + \beta(w_2, w_3) \\ &= \alpha\left(a(w_1, w_3) + (w_1, w_3)\right) + \beta\left(a(w_2, w_3) + (w_2, w_3)\right) \\ &= \alpha B(w_1, w_3) + \beta B(w_2, w_3). \end{aligned}$$

Therefore, $B(\cdot,\cdot)$ is linear in the first argument.
Moreover, since $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$,

$$(w_1, w_2) = (w_2, w_1) \text{ for all } w_1, w_2 \in H^1(0,1)$$

because if $w_1, w_2 \in H^1(0,1)$ then $w_1, w_2 \in L^2(0,1)$. Furthermore, since $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$,

$$a(w_1, w_2) = a(w_2, w_1) \text{ for all } w_1, w_2 \in H^1(0,1).$$

Hence, for all $w_1, w_2 \in H^1(0,1)$,

$$\begin{aligned} B(w_1, w_2) &= a(w_1, w_2) + (w_1, w_2) \\ &= a(w_2, w_1) + (w_2, w_1) \\ &= B(w_2, w_1). \end{aligned}$$

Therefore, $B(\cdot,\cdot)$ is symmetric.
It then follows that, for all $w_1, w_2, w_3 \in H^1(0,1)$ and all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} B(w_1, \alpha w_2 + \beta w_3) &= B(\alpha w_2 + \beta w_3, w_1) \\ &= \alpha B(w_2, w_1) + \beta B(w_3, w_1) \\ &= \alpha B(w_1, w_2) + \beta B(w_1, w_3). \end{aligned}$$

Therefore, $B(\cdot,\cdot)$ is linear in the second argument.
Consequently, $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$.

(b) [3 points] Since V_N is a subspace of $H_D^1(0,1)$, the fact that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H_D^1(0,1)$$

means that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in V_N.$$

Moreover,

$$a(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_N.$$

Therefore the properties satisfied by a symmetric bilinear form allow us to say that, for all $v \in V_N$,

$$\begin{aligned} B(u - u_N, v) &= B(u, v) - B(u_N, v) \\ &= (f, v) + \rho v(1) - ((f, v) + \rho v(1)) \\ &= 0. \end{aligned}$$

Consequently,

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N.$$

(c) [5 points] The properties satisfied by a symmetric bilinear form allow us to say that

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u, u - u_N) - B(u_N, u - u_N) \\ &= B(u, u) - B(u, u_N) - B(u_N, u) + B(u_N, u_N) \\ &= B(u, u) - 2B(u, u_N) + B(u_N, u_N). \end{aligned}$$

Now, $u_N \in V_N$ and so the fact that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N$$

means that

$$B(u - u_N, u_N) = 0$$

and hence

$$B(u, u_N) = B(u_N, u_N)$$

since the properties satisfied by a symmetric bilinear form mean that

$$B(u - u_N, u_N) = B(u, u_N) - B(u_N, u_N).$$

Therefore

$$\begin{aligned} B(u-u_N, u-u_N) &= B(u,u) - 2B(u_N,u_N) + B(u_N,u_N) \\ &= B(u,u) - B(u_N,u_N). \end{aligned}$$

The definition of the norm $|||\cdot|||$ then allows us to conclude that

$$|||u-u_N|||^2 = |||u|||^2 - |||u_N|||^2 .$$

(d) [3 points] Since $|||u-u_N||| \in \mathbb{R}$, we can say that

$$|||u-u_N|||^2 \geq 0$$

and so since

$$|||u-u_N|||^2 = |||u|||^2 - |||u_N|||^2$$

we can conclude that

$$|||u|||^2 - |||u_N|||^2 \geq 0.$$

Hence,

$$|||u_N|||^2 \leq |||u|||^2 .$$

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4. [25 points]

Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let $\phi_0 \in H^1(0,1)$ be defined by

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

let $\phi_j \in H^1_D(0,1)$ be defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let $\phi_{N+1} \in H^1_D(0,1)$ be defined by

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x) w(x)\, dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the symmetric bilinear form $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v,w) = a(v,w) + (v,w).$$

Also, let $f \in L^2(0,1)$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Moreover, let $u \in H^1(0,1)$ be such that $u(0) = \alpha$ and

$$B(u,v) = (f,v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Let $V_N = \operatorname{span}\{\phi_0, \phi_1, \ldots, \phi_{N+1}\}$ and let $V_{N,D} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_{N+1}\}$. Let $u_N \in V_N$ be such that $u_N(0) = \alpha$ and

$$B(u_N, v) = (f,v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

(a) We can write

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j$$

where, for $j = 1, 2, \ldots, N+1$, c_j is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$ which is the solution to

$$\mathbf{K}\mathbf{c} = \mathbf{b}.$$

What are the entries of the matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ and the vector $\mathbf{b} \in \mathbb{R}^{N+1}$?

(b) Show that

$$B(u - u_N, u - u_N) = B(u,u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0).$$

(c) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = 0$, $\rho = 0$ and $N = 1$. Note that, when $N = 1$,

$$\int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$\int_0^{1/2} \phi_0(x)\phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\phi_2(x)\,dx = \frac{1}{6};$$

and

$$\int_0^{1/2} \phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\,dx = \frac{1}{4}.$$

(d) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = -1$, $\rho = 1$ and $N = 1$.

Solution.

(a) [5 points] The function

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j$$

is such that $u_N(0) = \alpha$ and will be such that

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}$$

when, for $j = 1, 2, \ldots, N+1$, the c_j are such that

$$B\left(\alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j, \phi_k\right) = (f, \phi_k) + \rho\phi_k(1) \text{ for } k = 1, 2, \ldots, N+1,$$

or equivalently,

$$\sum_{j=1}^{N+1} c_j B(\phi_j, \phi_k) = (f, \phi_k) + \rho\phi_k(1) - \alpha B(\phi_0, \phi_k) \text{ for } k = 1, 2, \ldots, N+1.$$

We can write this system of equations in the form

$$\mathbf{Kc} = \mathbf{b}$$

where $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ is the matrix with entries

$$K_{jk} = B(\phi_k, \phi_j)$$

for $j, k = 1, 2, \ldots, N+1$ and $\mathbf{b} \in \mathbb{R}^{N+1}$ is the vector with entries

$$b_j = (f, \phi_j) + \rho\phi_j(1) - \alpha B(\phi_0, \phi_j) = \begin{cases} (f, \phi_1) - \alpha B(\phi_0, \phi_1) & \text{if } j = 1, \\ (f, \phi_{N+1}) + \rho & \text{if } j = N+1, \\ (f, \phi_j) & \text{otherwise.} \end{cases}$$

for $j = 1, 2, \ldots, N+1$.

(b) [6 points] Since $V_{N,D}$ is a subspace of $H^1_D(0, 1)$, the fact that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0, 1)$$

means that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

Moreover,

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

Therefore the properties satisfied by a symmetric bilinear form allow us to say that, for all $v \in V_{N,D}$,

$$\begin{aligned} B(u - u_N, v) &= B(u, v) - B(u_N, v) \\ &= (f, v) + \rho v(1) - ((f, v) + \rho v(1)) \\ &= 0. \end{aligned}$$

Consequently,

$$B(u - u_N, v) = 0 \text{ for all } v \in V_{N,D}.$$

The properties satisfied by a symmetric bilinear form allow us to say that

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u, u - u_N) - B(u_N, u - u_N) \\ &= B(u,u) - B(u, u_N) - B(u_N, u) + B(u_N, u_N) \\ &= B(u,u) - 2B(u, u_N) + B(u_N, u_N). \end{aligned}$$

Now, $u_N - \alpha\phi_0 \in V_{N,D}$ and so the fact that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_{N,D}$$

means that

$$B(u - u_N, u_N - \alpha\phi_0) = 0$$

and hence

$$B(u, u_N) = B(u_N, u_N) + \alpha B(u - u_N, \phi_0)$$

since the properties satisfied by a symmetric bilinear form mean that

$$\begin{aligned} B(u - u_N, u_N - \alpha\phi_0) &= B(u - u_N, u_N) - \alpha B(u - u_N, \phi_0) \\ &= B(u, u_N) - B(u_N, u_N) - \alpha B(u - u_N, \phi_0). \end{aligned}$$

Therefore,

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u,u) - 2\left(B(u_N, u_N) + \alpha B(u - u_N, \phi_0)\right) + B(u_N, u_N) \\ &= B(u,u) - 2B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0) + B(u_N, u_N) \\ &= B(u,u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0). \end{aligned}$$

(c) [9 points] When $N = 1$, $f(x) = 2$, $\alpha = 0$ and $\rho = 0$,

$$\mathbf{K} = \begin{bmatrix} B(\phi_1, \phi_1) & B(\phi_2, \phi_1) \\ B(\phi_1, \phi_2) & B(\phi_2, \phi_2) \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \end{bmatrix}$$

where

$$\phi_1(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2x - 1 & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

and hence

$$\phi_1'(x) = \begin{cases} 2 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ -2 & \text{if } x \in \left(\frac{1}{2}, 1\right); \end{cases}$$

and

$$\phi_2'(x) = \begin{cases} 0 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ 2 & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Now,

$$(\phi_1, \phi_1) = \int_0^1 \phi_1(x)\phi_1(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx + \int_{1/2}^1 \phi_1(x)\phi_1(x)\,dx = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3};$$

$$(\phi_1, \phi_2) = \int_0^1 \phi_1(x)\phi_2(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$(\phi_2, \phi_1) = (\phi_1, \phi_2) = \frac{1}{12};$$

and

$$(\phi_2, \phi_2) = \int_0^1 \phi_2(x)\phi_2(x)\,dx = \int_{1/2}^1 \phi_2(x)\phi_2(x)\,dx = \frac{1}{6}.$$

Moreover,

$$\begin{aligned} a(\phi_1, \phi_1) &= \int_0^1 \phi_1'(x)\phi_1'(x)\,dx \\ &= \int_0^{1/2} \phi_1'(x)\phi_1'(x)\,dx + \int_{1/2}^1 \phi_1'(x)\phi_1'(x)\,dx \\ &= \int_0^{1/2} 4\,dx + \int_{1/2}^1 4\,dx \\ &= \frac{4}{2} + \frac{4}{2} \\ &= 4; \end{aligned}$$

$$a(\phi_1, \phi_2) = \int_0^1 \phi_1'(x)\phi_2'(x)\,dx = \int_{1/2}^1 \phi_1'(x)\phi_2'(x)\,dx = \int_{1/2}^1 -4\,dx = -\frac{4}{2} = -2;$$

$$a(\phi_2, \phi_1) = a(\phi_1, \phi_2) = -2;$$

and

$$a(\phi_2, \phi_2) = \int_0^1 \phi_2'(x)\phi_2'(x)\,dx = \int_{1/2}^1 \phi_2'(x)\phi_2'(x)\,dx = \int_{1/2}^1 4\,dx = \frac{4}{2} = 2.$$

Consequently,

$$B(\phi_1, \phi_1) = a(\phi_1, \phi_1) + (\phi_1, \phi_1) = 4 + \frac{1}{3} = \frac{12}{3} + \frac{1}{3} = \frac{13}{3};$$

$$B(\phi_1, \phi_2) = a(\phi_1, \phi_2) + (\phi_1, \phi_2) = -2 + \frac{1}{12} = -\frac{24}{12} + \frac{1}{12} = -\frac{23}{12};$$

$$B(\phi_2, \phi_1) = B(\phi_1, \phi_2) = -\frac{23}{12};$$

and

$$B(\phi_2, \phi_2) = a(\phi_2, \phi_2) + (\phi_2, \phi_2) = 2 + \frac{1}{6} = \frac{12}{6} + \frac{1}{6} = \frac{13}{6}.$$

Furthermore,

$$(f,\phi_1) = 2\int_0^1 \phi_1(x)\,dx = 2\left(\int_0^{1/2} \phi_1(x)\,dx + \int_{1/2}^1 \phi_1(x)\,dx\right) = 2\left(\frac{1}{4}+\frac{1}{4}\right) = \frac{4}{4} = 1;$$

and

$$(f,\phi_2) = 2\int_0^1 \phi_2(x)\,dx = 2\left(\int_0^{1/2} \phi_2(x)\,dx + \int_{1/2}^1 \phi_2(x)\,dx\right) = 2\left(0+\frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}.$$

Hence,

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix} B(\phi_1,\phi_1) & B(\phi_2,\phi_1) \\ B(\phi_1,\phi_2) & B(\phi_2,\phi_2) \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{3} & -\frac{23}{12} \\ -\frac{23}{12} & \frac{13}{6} \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{b} &= \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.\end{aligned}$$

(d) [5 points] When $N = 1$, $f(x) = 2$, $\alpha = -1$ and $\rho = 1$,

$$\mathbf{K} = \begin{bmatrix} B(\phi_1,\phi_1) & B(\phi_2,\phi_1) \\ B(\phi_1,\phi_2) & B(\phi_2,\phi_2) \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} (f,\phi_1) + B(\phi_0,\phi_1) \\ (f,\phi_2) + 1 \end{bmatrix}$$

where

$$\phi_0(x) = \begin{cases} 1-2x & \text{if } x \in \left[0,\frac{1}{2}\right); \\ 0 & \text{if } x \in \left[\frac{1}{2},1\right]; \end{cases}$$

$$\phi_1(x) = \begin{cases} 2x & \text{if } x \in \left[0,\frac{1}{2}\right); \\ 2-2x & \text{if } x \in \left[\frac{1}{2},1\right]; \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 0 & \text{if } x \in \left[0,\frac{1}{2}\right); \\ 2x-1 & \text{if } x \in \left[\frac{1}{2},1\right]; \end{cases}$$

and hence

$$\phi_0'(x) = \begin{cases} -2 & \text{if } x \in \left(0,\frac{1}{2}\right); \\ 0 & \text{if } x \in \left(\frac{1}{2},1\right); \end{cases}$$

$$\phi_1'(x) = \begin{cases} 2 & \text{if } x \in \left(0,\frac{1}{2}\right); \\ -2 & \text{if } x \in \left(\frac{1}{2},1\right); \end{cases}$$

and

$$\phi_2'(x) = \begin{cases} 0 & \text{if } x \in \left(0, \frac{1}{2}\right); \\ 2 & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Now,

$$(\phi_0, \phi_1) = \int_0^1 \phi_0(x)\phi_1(x)\,dx = \int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \frac{1}{12};$$

and

$$a(\phi_0, \phi_1) = \int_0^1 \phi_0'(x)\phi_1'(x)\,dx = \int_0^{1/2} -4\,dx = -\frac{4}{2} = -2.$$

Consequently,

$$B(\phi_0, \phi_1) = a(\phi_0, \phi_1) + (\phi_0, \phi_1) = -2 + \frac{1}{12} = -\frac{24}{12} + \frac{1}{12} = -\frac{23}{12}.$$

Moreover, in part (c) we computed that

$$B(\phi_1, \phi_1) = \frac{13}{3};$$

$$B(\phi_1, \phi_2) = B(\phi_2, \phi_1) = -\frac{23}{12};$$

$$B(\phi_2, \phi_2) = \frac{13}{6};$$

$$(f, \phi_1) = 1;$$

and

$$(f, \phi_2) = \frac{1}{2}.$$

Hence,

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} B(\phi_1, \phi_1) & B(\phi_2, \phi_1) \\ B(\phi_1, \phi_2) & B(\phi_2, \phi_2) \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{3} & -\frac{23}{12} \\ -\frac{23}{12} & \frac{13}{6} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{b} &= \begin{bmatrix} (f, \phi_1) + B(\phi_0, \phi_1) \\ (f, \phi_2) + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{23}{12} \\ \frac{1}{2} + 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{12}{12} - \frac{23}{12} \\ \frac{1}{2} + \frac{2}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11}{12} \\ \frac{3}{2} \end{bmatrix}. \end{aligned}$$

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5. [35 points]

Let

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

In this question we will consider the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = 1, \quad t \ge 0,$$

and

$$u_x(1,t) = 2, \quad t \ge 0,$$

and initial condition

$$u(x,0) = x^2 + 1, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w'(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin\left(\frac{2n-1}{2} \pi x \right)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \psi_m(x) \psi_n(x)\, dx = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \ne n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) For $n = 1, 2, \ldots$, compute

$$\int_0^1 f(x) \psi_n(x)\, dx.$$

(c) Let $w(x)$ be such that

$$w''(x) = 0,$$

$$w(0) = 1$$

and

$$w'(1) = 2.$$

Obtain a formula for $w(x)$.

(d) Let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$
$$\hat{u}(0,t) = \hat{u}_x(1,t) = 0, \quad t \ge 0,$$

and

$$\hat{u}(x,0) = \hat{u}_0(x), \qquad 0 \le x \le 1,$$

where $\hat{u}_0(x)$ is such that

$$u(x,t) = w(x) + \hat{u}(x,t).$$

Obtain a formula for $\hat{u}_0(x)$.

(e) For $n = 1, 2, \ldots$, compute

$$\int_0^1 \hat{u}_0(x)\psi_n(x)\,dx.$$

(f) We can write

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$b_n = \int_0^1 f(x)\psi_n(x)\,dx.$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(g) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(h) Write out a formula for $u(x,t)$.

Solution.

(a) [4 points] We can compute that, for $n = 1, 2, \ldots$,

$$\psi_n'(x) = \sqrt{2}\left(\frac{2n-1}{2}\right)\pi\cos\left(\frac{2n-1}{2}\pi x\right).$$

and

$$\psi_n''(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)^2\pi^2\sin\left(\frac{2n-1}{2}\pi x\right).$$

and so

$$L\psi_n = -\psi_n'' = \left(\frac{2n-1}{2}\right)^2\pi^2\psi_n.$$

Hence,

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \pi^2 = (2n-1)^2\,\frac{\pi^2}{4} \text{ for } n = 1, 2, \ldots.$$

(b) [4 points] For $n = 1, 2, \ldots,$

$$\begin{aligned}
&\int_0^1 f(x)\psi_n(x)\,dx \\
&= \int_0^{1/2} f(x)\psi_n(x)\,dx + \int_{1/2}^1 f(x)\psi_n(x)\,dx \\
&= \int_0^{1/2} (1-2x)\,\sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx + \int_{1/2}^1 0\,dx \\
&= \sqrt{2}\int_0^{1/2} (1-2x)\sin\left(\frac{2n-1}{2}\pi x\right)\,dx + 0 \\
&= \sqrt{2}\left(\left[-(1-2x)\,\frac{2}{(2n-1)\,\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} - \int_0^{1/2} \frac{4}{(2n-1)\,\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \sqrt{2}\left(0 + \frac{2}{(2n-1)\,\pi} - \frac{4}{(2n-1)\,\pi}\int_0^{1/2} \cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - 2\int_0^{1/2} \cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - 2\left[\frac{2}{(2n-1)\,\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2}\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - \frac{4}{(2n-1)\,\pi}\sin\left(\frac{2n-1}{4}\pi\right) - 0\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - \frac{4}{(2n-1)\,\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}$$

(c) [5 points] The general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(1) = 2$ when $A = 2$. Hence, $w(x) = 2x + B$ and so $w(0) = B$ and hence $w(0) = 1$ when $B = 1$. Consequently,

$$w(x) = 1 + 2x.$$

(d) [5 points] We can compute that $u(x,t) = w(x) + \hat{u}(x,t)$ will be such that

$$u(x,0) = w(x) + \hat{u}(x,0) = 1 + 2x + \hat{u}_0(x)$$

and so since

$$u(x,0) = x^2 + 1$$

we can conclude that

$$\hat{u}_0(x) = x^2 + 1 - (1 + 2x) = x^2 - 2x.$$

(e) [4 points] For $n = 1, 2, \ldots,$

$$\begin{aligned}
&\int_0^1 \hat{u}_0(x)\psi_n(x)\,dx \\
&= \int_0^1 \left(x^2 - 2x\right)\psi_n(x)\,dx \\
&= \sqrt{2}\int_0^1 \left(x^2 - 2x\right)\sin\left(\frac{2n-1}{2}\pi x\right)\,dx \\
&= \sqrt{2}\left(\left[-\left(x^2 - 2x\right)\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^1 + \int_0^1 (2x-2)\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\pi}\left(0 - 0 + \int_0^1 (2x-2)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\pi}\left(\left[(2x-2)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^1 - \int_0^1 \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(0 - 0 - \int_0^1 \sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(-\left[-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^1\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(0 - \frac{2}{(2n-1)\pi}\right) \\
&= -\frac{16\sqrt{2}}{(2n-1)^3\pi^3}.
\end{aligned}$$

(f) [5 points] Substituting the expressions for $\hat{u}(x,t)$ and $f(x)$ into the partial differential equation yields

$$\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) - \sum_{n=1}^{\infty} a_n(t)\left(-\left(L\psi_n\right)(x)\right) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

and hence

$$\sum_{n=1}^{\infty}\left(a_n'(t) + \lambda_n a_n(t)\right)\psi_n(x) = \sum_{n=1}^{\infty} b_n\psi_n(x).$$

We can then say that

$$\sum_{n=1}^{\infty} \left(a_n'(t) + \lambda_n a_n(t)\right) \int_0^1 \psi_n(x)\psi_m(x)\,dx = \sum_{n=1}^{\infty} b_n \int_0^1 \psi_n(x)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m'(t) + \lambda_m a_m(t) = b_m$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

for $m, n = 1, 2, \ldots$. Hence, for $n = 1, 2, \ldots$,

$$a_n'(t) + (2n-1)^2 \frac{\pi^2}{4} a_n(t) = \frac{2\sqrt{2}}{(2n-1)\,\pi} \left(1 - \frac{4}{(2n-1)\,\pi} \sin\left(\frac{2n-1}{4}\pi\right)\right).$$

Also,

$$\hat{u}(x,0) = x^2 - 2x$$

means that

$$\sum_{n=1}^{\infty} a_n(0)\psi_n(x) = x^2 - 2x$$

and so

$$\sum_{n=1}^{\infty} a_n(0) \int_0^1 \psi_n(x)\psi_m(x)\,dx = \int_0^1 \left(x^2 - 2x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m(0) = \int_0^1 \left(x^2 - 2x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

for $m, n = 1, 2, \ldots$. Hence, for $n = 1, 2, \ldots$,

$$a_n(0) = -\frac{16\sqrt{2}}{(2n-1)^3\,\pi^3}.$$

Therefore, for $n = 1, 2, \ldots$, $a_n(t)$ is the solution to the differential equation

$$a_n'(t) = -(2n-1)^2 \frac{\pi^2}{4} a_n(t) + \frac{2\sqrt{2}}{(2n-1)\,\pi} \left(1 - \frac{4}{(2n-1)\,\pi} \sin\left(\frac{2n-1}{4}\pi\right)\right)$$

with initial condition

$$a_n(0) = -\frac{16\sqrt{2}}{(2n-1)^3\,\pi^3}.$$

(g) [5 points] From question 1(b) we have that, for $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n(t) &= -\frac{16\sqrt{2}}{(2n-1)^3\pi^3} e^{-(2n-1)^2\pi^2 t/4} \\
&\quad -\frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(1-\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\left(e^{-(2n-1)^2\pi^2 t/4}-1\right) \\
&= -\frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(2e^{-(2n-1)^2\pi^2 t/4}+\left(1-\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\left(e^{-(2n-1)^2\pi^2 t/4}-1\right)\right) \\
&= -\frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(\left(3-\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)-1\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(\left(\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)-3\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +1-\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}$$

(h) [3 points] We can write

$$\begin{aligned}
u(x,t) &= w(x)+\hat{u}(x,t) \\
&= 1+2x+\sum_{n=1}^{\infty} a_n(t)\psi_n(x) \\
&= 1+2x+\sum_{n=1}^{\infty}\frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(\left(\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)-3\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +1-\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\psi_n(x) \\
&= 1+2x+\sum_{n=1}^{\infty}\frac{16}{(2n-1)^3\pi^3}\left(\left(\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)-3\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +1-\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\sin\left(\frac{2n-1}{2}\pi x\right).
\end{aligned}$$

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6. [10 points]

In this question we will consider the problem of finding the solution $u(x,t)$ to the wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = u_x(1,t) = 0, \quad t \ge 0,$$

and initial conditions

$$u(x,0) = 0, \qquad 0 \le x \le 1,$$

and

$$u_t(x,0) = x^2 - 2x, \qquad 0 \le x \le 1.$$

Let

$$S = \{w \in C^2[0,1] : w(0) = w'(1) = 0\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. Recall that you obtained a formula for the eigenvalues of L in question 5.

(a) We can write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

What ordinary differential equation and initial conditions does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(b) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(c) Write out a formula for $u(x,t)$.

Solution.

(a) [6 points] Substituting the expression for $u(x,t)$ into the partial differential equation yields

$$\sum_{n=1}^{\infty} a_n''(t)\psi_n(x) = \sum_{n=1}^{\infty} a_n(t)\left(-\left(L\psi_n\right)(x)\right)$$

and hence

$$\sum_{n=1}^{\infty} a_n''(t)\psi_n(x) = \sum_{n=1}^{\infty} \left(-\lambda_n\right) a_n(t)\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$\lambda_n = (2n-1)^2\,\frac{\pi^2}{4}.$$

We can then say that

$$\sum_{n=1}^{\infty} a_n''(t)\int_0^1 \psi_n(x)\psi_m(x)\,dx = \sum_{n=1}^{\infty} \left(-\lambda_n\right) a_n(t)\int_0^1 \psi_n(x)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m''(t) = -\lambda_m a_m(t)$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

for $m, n = 1, 2, \ldots$.
Also,

$$u(x,0) = 0$$

means that

$$\sum_{n=1}^{\infty} a_n(0)\psi_n(x) = 0$$

and so

$$\sum_{n=1}^{\infty} a_n(0) \int_0^1 \psi_n(x)\psi_m(x)\,dx = \int_0^1 0\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m(0) = 0$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

for $m, n = 1, 2, \ldots$.
Moreover,

$$u_t(x,t) = \sum_{n=1}^{\infty} a_n'(t)\psi_n(x).$$

Hence,

$$u_t(x,0) = x^2 - 2x$$

means that

$$\sum_{n=1}^{\infty} a_n'(0)\psi_n(x) = x^2 - 2x$$

and so

$$\sum_{n=1}^{\infty} a_n'(0) \int_0^1 \psi_n(x)\psi_m(x)\,dx = \int_0^1 \left(x^2 - 2x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m'(0) = \int_0^1 \left(x^2 - 2x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

for $m, n = 1, 2, \ldots$.

Hence, for $n = 1, 2, \ldots$, $a_n(t)$ is the solution to the differential equation

$$a_n''(t) = -\left(2n-1\right)^2 \frac{\pi^2}{4} a_n(t),$$

or equivalently,

$$-a_n''(t) = \left(2n-1\right)^2 \frac{\pi^2}{4} a_n(t)$$

with initial conditions

$$a_n(0) = 0$$

and

$$a_n'(0) = -\frac{16\sqrt{2}}{\left(2n-1\right)^3 \pi^3}$$

since in question 5(e) we computed that

$$\int_0^1 \left(x^2 - 2x\right) \psi_n(x)\, dx = -\frac{16\sqrt{2}}{\left(2n-1\right)^3 \pi^3}.$$

(b) [2 points] From question 1(a) we have that, for $n = 1, 2, \ldots$,

$$a_n(t) = \frac{-32\sqrt{2}}{\left(2n-1\right)^4 \pi^4} \sin\left(\frac{2n-1}{2}\pi t\right).$$

(c) [2 points] We can write

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\psi_n(x) \\ &= \sum_{n=1}^{\infty} \frac{-32\sqrt{2}}{\left(2n-1\right)^4 \pi^4} \sin\left(\frac{2n-1}{2}\pi t\right) \psi_n(x) \\ &= \sum_{n=1}^{\infty} \frac{-64}{\left(2n-1\right)^4 \pi^4} \sin\left(\frac{2n-1}{2}\pi t\right) \sin\left(\frac{2n-1}{2}\pi x\right). \end{aligned}$$
