CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 12 · Solutions

Posted Friday 26 November 2010. Due Friday 3 December 2010, 5pm.

This problem set counts for 100 points, plus a 20 point bonus.

1. [50 points: 15 points for (a); 10 points for (b); 5 points for (c); 20 points for (d)]
On Problem Set 10, you solved the heat equation on a two-dimensional square domain. Now we will investigate the wave equation on the same domain, a model of a vibrating membrane stretched over a square frame—that is, a square drum:

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t),$$

with $0 \le x \le 1$, and $0 \le y \le 1$, and $t \ge 0$. Take homogeneous Dirichlet boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$$

for all x and y such that $0 \le x \le 1$ and $0 \le y \le 1$ and all $t \ge 0$, and consider the initial conditions

$$u(x,y,0) = u_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y), \qquad u_t(x,y,0) = v_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}(0)\psi_{j,k}(x,y).$$

Here $\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$, for $j,k \ge 1$, are the eigenfunctions of the operator

$$Lu = -(u_{xx} + u_{yy}),$$

with homogeneous Dirichlet boundary conditions, as in Problem Set 10. You may use without proof that these eigenfunctions are orthogonal, and use the eigenvalues $\lambda_{j,k} = (j^2 + k^2)\pi^2$ computed for Problem Set 10.

   (a) We wish to write the solution to the wave equation in the form

   $$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$

   Show that the coefficients $a_{j,k}(t)$ obey the ordinary differential equation

   $$a''_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t)$$

   with initial conditions

   $$a_{j,k}(0), \qquad a'_{j,k}(0) = b_{j,k}(0)$$

   derived from the initial conditions u_0 and v_0.

   (b) Write down the solution to the differential equation in part (a).

   (c) Use your solution to part (b) to write out a formula for the solution $u(x,y,t)$.

   (d) Suppose the drum begins with zero velocity, $v_0(x,y) = 0$, and displacement

   $$u_0(x,y) = 200xy(1-x)(1-y)(x-1/4)(y-1/4) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\psi_{j,k}(x,y).$$

   Submit surface (or contour) plots of the solution at times $t = 0, 0.5, 1.0, 1.5, 2.5$, using $j = 1,\ldots,10$ and $k = 1,\ldots,10$ in the series.

Solution. This question follows the same pattern as the first problem on this problem set.

(a) Substitute the formula

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$

into the two dimensional wave equation to obtain

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{d^2 a_{j,k}}{dt^2}(t)\psi_{j,k}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\psi_{j,k}(x,y),$$

which implies

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{d^2 a_{j,k}}{dt^2}(t)\psi_{j,k}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} -a_{j,k}(t)\lambda_{j,k}\psi_{j,k}(x,y).$$

Take the inner product of both sides with $\psi_{m,n}$ and use the orthogonality of the eigenfunctions to obtain

$$a''_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t).$$

The initial value $a_{j,k}(0)$ is simply the (j,k) coefficient in the expansion of the initial condition,

$$u(x,y,0) = u_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y).$$

As the differential equation describing $a_{j,k}$ is of second order, we require a second initial condition to determine a unique solution. Since

$$u_t(x,y,0) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a'_{j,k}(0)\psi_{j,k}(x,y),$$

and

$$v_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}\psi_{j,k}(x,y),$$

the second initial condition we require is simply the (j,k) coefficient of v_0:

$$a'_{j,k}(0) = b_{j,k}.$$

(b) As we have seen often in this class, the equation $a''_{j,k}(t) = -\lambda_{j,k}a_{j,k}(t)$ has solutions of the form

$$a_{j,k}(t) = A\sin(\sqrt{\lambda_{j,k}}t) + B\cos(\sqrt{\lambda_{j,k}}t),$$

with A and B determined by the initial conditions. In particular, evaluating the general solution at $t=0$ immediately gives

$$B = a_{j,k}(0).$$

Computing the derivative

$$a'_{j,k}(t) = A\sqrt{\lambda_{j,k}}\cos(\sqrt{\lambda_{j,k}}t) - B\sqrt{\lambda_{j,k}}\sin(\sqrt{\lambda_{j,k}}t)$$

and evaluating it at $t=0$ then gives

$$A = \frac{b_{j,k}(0)}{\sqrt{\lambda_{j,k}}}.$$

(c) From this formula for $a_{j,k}(t)$ we obtain the solution

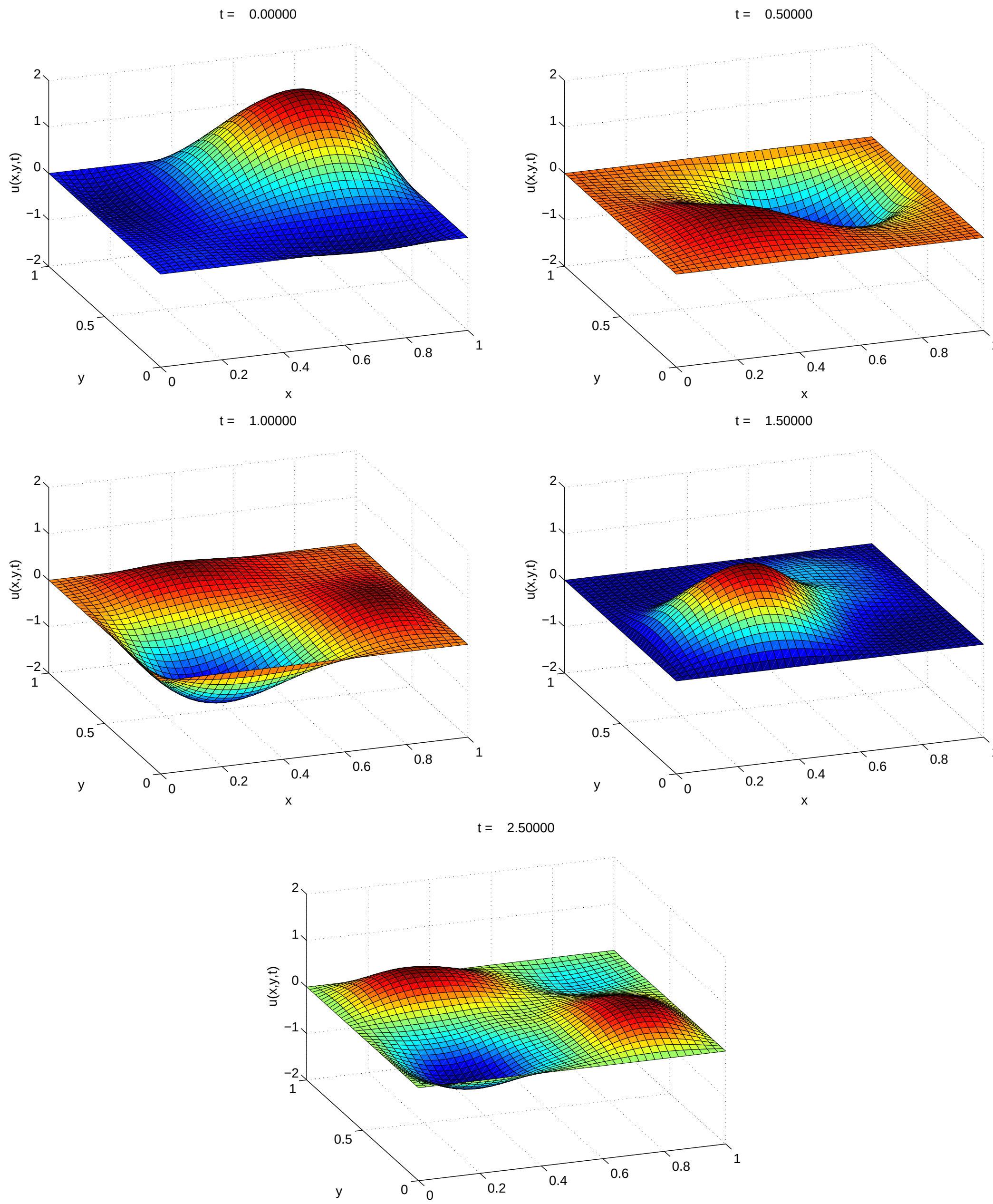
$$u(x,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big[\Big(\frac{b_{j,k}(0)}{\sqrt{\lambda_{j,k}}}\Big)\sin(\sqrt{\lambda_{j,k}}t) + a_{j,k}(0)\cos(\sqrt{\lambda_{j,k}}t)\Big]\big(2\sin(\sqrt{\lambda_{j,k}}x)\sin(\sqrt{\lambda_{j,k}}y)\big).$$

(d) Zero initial velocity implies that $b_{j,k}(0) = 0$ for all (j,k) pairs. The solution thus simplifies to

$$u(x,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\cos(\sqrt{\lambda_{j,k}}t)\big(2\sin(\sqrt{\lambda_{j,k}}x)\sin(\sqrt{\lambda_{j,k}}y)\big).$$

with

$$a_{j,k}(0) = \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}.$$



The code that produced these plots follows.

```
N = 40;
x = linspace(0,1,N); y = linspace(0,1,N);
[X,Y] = meshgrid(x,y);
nmax = 10;

tvec = 0:.05:4;
tvec = [0 0.5 1.0 1.5 2.5];
for m=1:length(tvec)
   t = tvec(m)
   figure(1), clf
   U = zeros(N,N);
   for j=1:nmax
      for k=1:nmax
         ajk = 100*(5+7*(-1)^j)*(5+7*(-1)^k)/(j^3*k^3*pi^6);
         lamjk = (j^2+k^2)*(pi^2);
         psijk = 2*sin(j*pi*X).*sin(k*pi*Y);
         U = U + ajk*cos(sqrt(lamjk)*t)*psijk;
      end
   end
   surf(X,Y,U)
   axis([0 1 0 1 -2 2]) %, caxis([-8 8])
   set(gca,'fontsize',18)
   xlabel('x'), ylabel('y'), zlabel('u(x,y,t)')
   title(sprintf('t = %10.5f\n', t))
   view(-20, 30)
   eval(sprintf('print -depsc2 wave2d_%d',m))
end
```

---

2. [50 points: 10 points per part]
Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We will look for solutions to the PDE of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

For simplicity, assume that $d \in (0,\pi)$.

(a) From the differential equation and this form for $u(x,t)$, show that the coefficients $a_k(t)$ must satisfy the ordinary differential equation

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t).$$

(b) Show that the following function satisfies the differential equation in part (a):

$$a_k(t) = C_1 \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) + C_2 \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big)$$

for arbitrary constants C_1 and C_2. (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

(c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{k=1}^{\infty} b_k(0)\psi(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

(d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x\sin(3\pi x)$, for which

$$b_k(0) = \frac{-6k\sqrt{2}(1 + (-1)^k)}{(k^2 - 9)^2\pi^2} \quad \text{for } k \neq 3, \qquad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x, t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

(e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d?

Solution.

(a) Follow the usual methodology: Substitute the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

into the differential equation $u_{tt} = u_{xx} - 2du_t$ to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} a_n''(t)\psi_n(x) &= \sum_{n=1}^{\infty} a_n(t)\psi_n''(x) - 2d\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) \\ &= -\sum_{n=1}^{\infty} \lambda_n a_n(t)\psi_n(x) - 2d\sum_{n=1}^{\infty} a_n'(t)\psi_n(x).\end{aligned}$$

Take the inner product with the eigenfunction ψ_k and use orthogonality of the eigenfunctions to obtain

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t)$$

as required.

(b) We first compute two derivatives of the proposed formula for a_k:

$$\begin{aligned}a_k'(t) &= C_1\big(-d + \sqrt{d^2 - k^2\pi^2}\big)\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\big(-d - \sqrt{d^2 - k^2\pi^2}\big)\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big)\end{aligned}$$

$$\begin{aligned}a_k''(t) &= C_1\big(-d - \sqrt{d^2 - k^2\pi^2}\big)^2\exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\ &\quad + C_2\big(-d + \sqrt{d^2 - k^2\pi^2}\big)^2\exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big).\end{aligned}$$

We wish to verify that $a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t)$, where $\lambda_k = k^2\pi^2$. We can see that

$$\begin{aligned}
-\lambda_k a_k(t) - 2da_k'(t) &= C_1\Big(- \lambda_k - 2d\big(- d + \sqrt{d^2 - k^2\pi^2}\big)\Big) \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\
&\quad + C_2\Big(- \lambda_k - 2d\big(- d - \sqrt{d^2 - k^2\pi^2}\big)\Big) \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big) \\
&= C_1\big(- k^2\pi^2 + 2d^2 - 2d\sqrt{d^2 - k^2\pi^2}\big) \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\
&\quad + C_2\big(- k^2\pi^2 + 2d^2 + 2d\sqrt{d^2 - k^2\pi^2}\big) \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big) \\
&= C_1\big(- d + \sqrt{d^2 - k^2\pi^2}\big)^2 \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) \\
&\quad + C_2\big(- d - \sqrt{d^2 - k^2\pi^2}\big)^2 \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big).
\end{aligned}$$

This final formula agrees with the formula for $a_k''(t)$ we computed earlier, and thus we have confirmed that this is a general solution for our differential equation.

(c) We need to now compute C_1 and C_2 so that $a_k(0) = 0$ and $a_k'(0) = b_k(0)$. At $t = 0$, the general solution becomes

$$a_k(0) = C_1 \exp(0) + C_2\varepsilon(0) = C_1 + C_2,$$

so $a_k(0) = 0$ requires that

$$C_1 = -C_2.$$

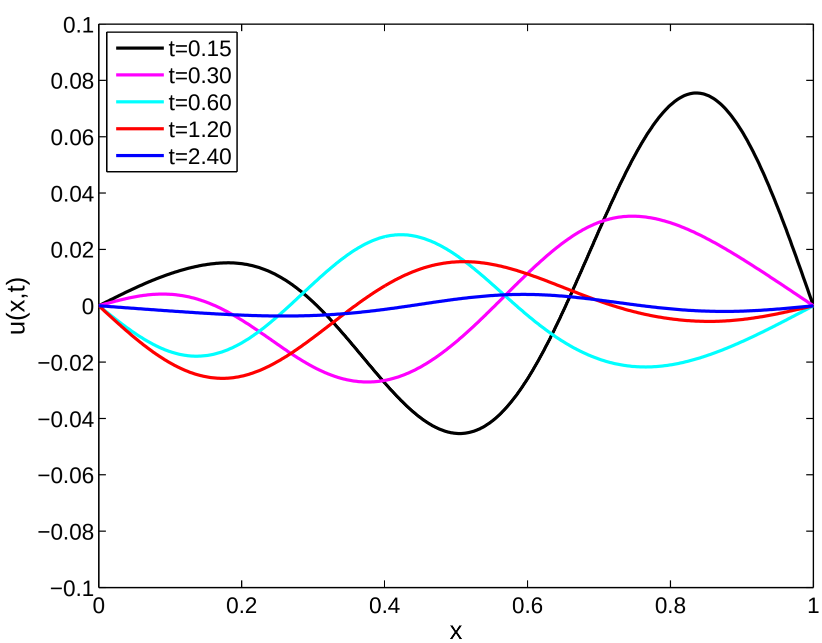
Taking the formula for $a_k'(t)$ in part (b) and evaluating at $t = 0$ gives

$$a_k'(0) = C_1\big(- d + \sqrt{d^2 - k^2\pi^2}\big) + C_2\big(- d - \sqrt{d^2 - k^2\pi^2}\big).$$

So with $C_1 = -C2$ and $a_k'(0) = b_k(0)$, we arrive at

$$C_1 = -C_2 = \frac{b_k(0)}{2\sqrt{d^2 - k^2\pi^2}}.$$

(d) The requested solutions, varying in t with fixed d, are collected in the plot below.



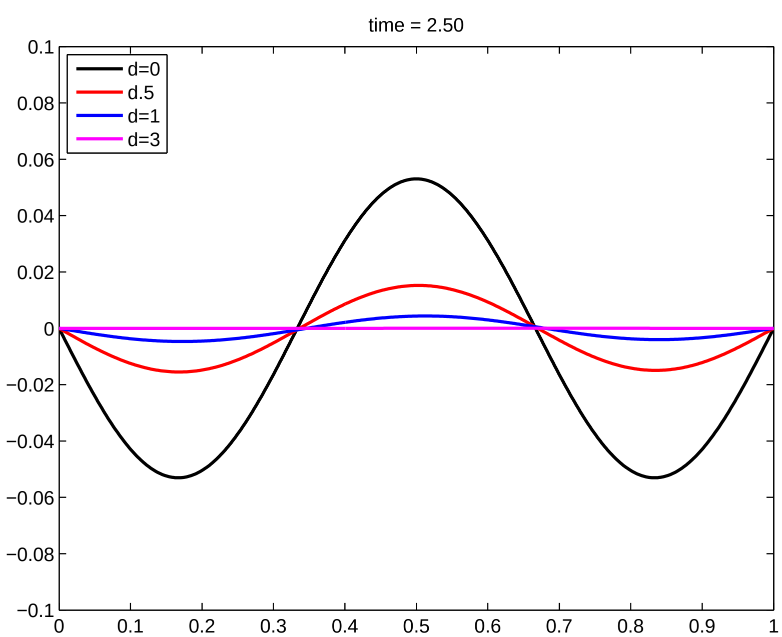
```
tvec = [.15 .30 .60 1.20 2.40];
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

d = 1;

col = 'kmcrb';
figure(1), clf
for m=1:length(tvec)
   t = tvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
   plot(xx,u,'-','linewidth',2,'color',col(m)), hold on
   ylim([-.1 .1])
   pause
end
legend('t=0.15','t=0.30','t=0.60','t=1.20','t=2.40',2)
set(gca,'fontsize',14)
xlabel('x','fontsize',16)
ylabel('u(x,t)','fontsize',16)
print -depsc2 damp1.eps
```

(e) The requested solutions, now varying in d with fixed t, are collected in the plot below. As the damping parameter increases on $(0, \pi)$, the solution gets increasingly smaller in amplitude at this time.




```
t = 2.5;
```

```
dvec = [0 .5 1 3];
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

figure(1), clf
cvec = 'krbm';
for m=1:length(dvec)
   d = dvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
   plot(xx,u,'-','linewidth',2,'color',cvec(m)), hold on
   ylim([-.1 .1])
   title(sprintf('time = %3.2f', t))
end
legend('d=0','d=.5','d=1','d=3',2)
print -depsc2 damp2.eps
```

bonus [20 points: 4 points per part]

In class we will see that the wave equation $u_{tt} = u_{xx}$ can be written as a first order system. We will introduce the velocity variable

$$v(x,t) = u_t(x,t),$$

then notice that the wave equation is equivalent to the first order equation

$$\frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & 0 \end{bmatrix}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}$$

where I is the identity operator (i.e., $Iv = v$ for any function v). The first row of this system gives $u_t = v$, while the second row gives $v_t = u_{xx}$, which is just another way of writing $u_{tt} = u_{xx}$.

We also have the usual initial conditions,

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix}.$$

This system can be analyzed in terms of its eigenvalues and eigenfunctions. In this problem, we will use the same approach to study the damped wave equation from Problem 2.

(a) Verify that the damped wave equation $u_{tt} = u_{xx} - 2du_t$ is equivalent to the system

$$\frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & -2dI \end{bmatrix}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}.$$

(b) For any value of d, the operator

$$A(d) = \begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & -2dI \end{bmatrix}$$

has all its eigenvectors of the same form:

$$\Psi_{\pm k}(x) = \begin{bmatrix} \sin(k\pi x) \\ \lambda_{\pm k}\sin(k\pi x) \end{bmatrix}.$$

Use these eigenvectors to determine the eigenvalues $\lambda_{\pm k}$.

(c) Plot the eigenvalues from part (b) in the complex plane for $d = 0, \pi/2, \pi$, and $3\pi/2$.

(d) What value of $d \geq 0$ optimizes the *asymptotic decay* of the system? That is, what value of d moves *all* the eigenvalues as far to the left in the complex plane as possible? Said yet another way, which d minimizes the real part of the rightmost eigenvalue? Does larger damping always imply faster asymptotic decay?

(e) Confirm this result by plotting the solution at $t = 1.5$ for $d = 2.5, 3.2, 20$ on the same plot with vertical axis $[-.002, .002]$, using the same values for u_0 and v_0 as in part (d) of Problem 2. (You should simply adapt the code you developed for Problem 2 to produce these plots.)

Solution.

(a) The first row of this operator matrix differential equation gives

$$u_t(x,t) = v(x,t),$$

which is simply the definition of v. The second row gives

$$v_t(x,t) = u_{xx}(x,t) - 2dv(x,t).$$

Substituting in our definition $v = u_t$ (and hence $v_t = u_{tt}$), we obtain

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

which is the damped wave equation.

(b) Multiply $A(d)$ against $\Psi_{\pm k}$ to obtain

$$\begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & -2dI \end{bmatrix} \begin{bmatrix} \sin(k\pi x) \\ \lambda_{\pm k}\sin(k\pi x) \end{bmatrix} = \begin{bmatrix} \lambda_{\pm k}\sin(k\pi x) \\ (d^2/dx^2)(\sin(k\pi x)) - 2d\lambda_{\pm k}\sin(k\pi x) \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_{\pm k}\sin(k\pi x) \\ -k^2\pi^2\sin(k\pi x)) - 2d\lambda_{\pm k}\sin(k\pi x) \end{bmatrix}.$$

We want this vector to be of the form

$$\lambda_{\pm k}\begin{bmatrix} \sin(k\pi x) \\ \lambda_{\pm k}\sin(k\pi x) \end{bmatrix};$$

the first component is already in the correct form, but the second component requires that

$$-k^2\pi^2 - 2d\lambda_{\pm k} = \lambda_{\pm k}^2.$$

That is, we must solve the quadratic equation

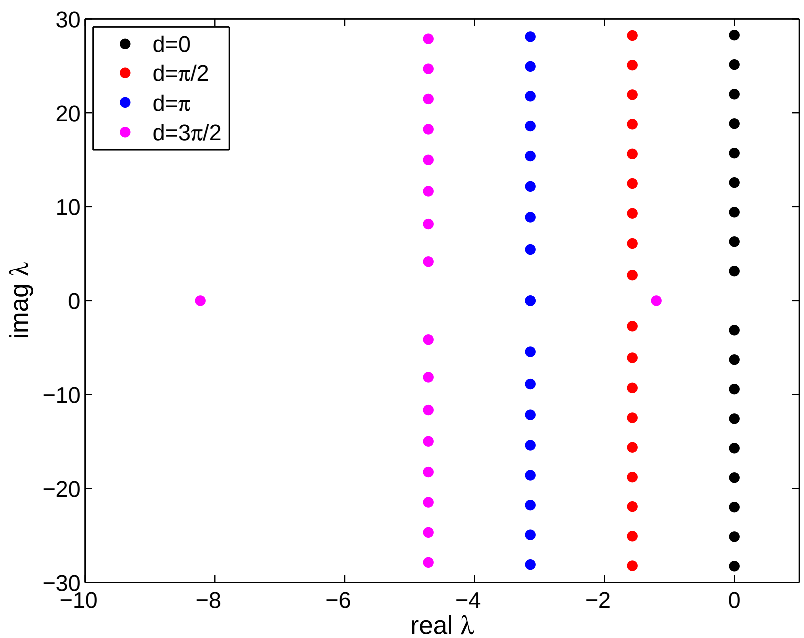
$$\lambda_{\pm k}^2 + 2d\lambda_{\pm k} + k^2\pi^2 = 0,$$

which, via the quadratic formula, gives

$$\lambda_{\pm k} = -d \pm \sqrt{d^2 - k^2\pi^2}.$$

(It is no coincidence that these are the values that appear in the exponential in the solution $a_k(t)$ to the ODE in the previous problem!)

(c) The eigenvalues for $d = 0, \pi/2, \pi, 3\pi/2$ are plotted below.




```
d = [0 pi/2 pi 3*pi/2];
col = 'krbm'
k = 1:10;
figure(1), clf
for j=1:length(d)
    ew = [-d(j)+sqrt(d(j)^2-k.^2*pi^2) -d(j)-sqrt(d(j)^2-k.^2*pi^2)];
```

```
    plot(real(ew),imag(ew),'.','markersize',20,'color',col(j)), hold on
end
axis([-10 1 -30 30])
set(gca,'fontsize',14)
xlabel('real \lambda','fontsize',16)
ylabel('imag \lambda','fontsize',16)
legend('d=0','d=\pi/2','d=\pi','d=3\pi/2',2)
print -depsc2 optdamp1
```

(d) For $d \in [0, \pi]$, the eigenvalues $\lambda_{\pm k}$ will all fall on a vertical line in the complex plane with real part $-d$. When $d = \pi$, we have a double eigenvalue, $\lambda_{\pm 1} = -\pi$, and for $d > \pi$, we will always have a real rightmost eigenvalue of

$$\lambda_{+1} = -d + \sqrt{d^2 - \pi^2}.$$

Taking a derivative with respect to d, we have for $d > \pi$,

$$\lambda'_{+1} = -1 + \frac{d}{\sqrt{d^2 - \pi^2}} = -1 + \frac{1}{\sqrt{1 - \pi^2/d^2}} > 0,$$

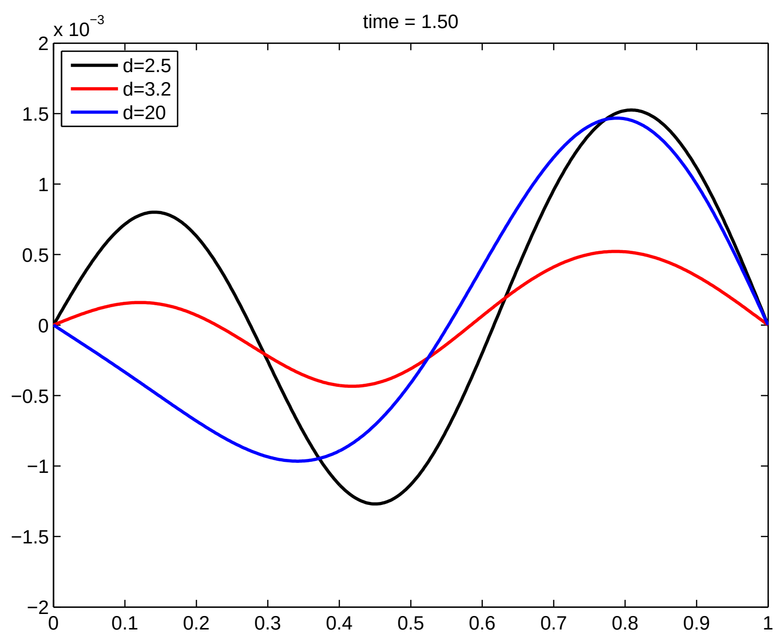
and thus we conclude that

$$d = \pi$$

maximizes the real part of the rightmost eigenvalue, and hence the energy decay rate of the system. (It takes some careful mathematics to show that the rightmost eigenvalue determines the energy decay rate in this infinite dimensional system; this was accomplished in a 1994 paper by S. J. Cox and E. Zuazua.)

[GRADERS: Students can receive full credit for solutions that are less rigorous than the one presented here.]

(e) The solutions for $d = 2.5$, 3.2, and 20 at time $t = 1.5$ are shown in the plot below. Note that despite the large damping term at $d = 20$, the solution remains large. We say that the system is *overdamped*. Near optimal damping, $d \approx \pi$, we can smaller solutions.




```
t = 1.5;
dvec = [2.5 3.2 20];
```

```
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

figure(1), clf
cvec = 'krb';
for m=1:length(dvec)
   d = dvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
   plot(xx,u,'-','linewidth',2,'color',cvec(m)), hold on
   ylim([-.002 .002])
   title(sprintf('time = %3.2f', t))
end
legend('d=2.5','d=3.2','d=20',2)
print -depsc2 optdamp2
```