CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 26 · Solutions

Posted Friday 28 February 2014. Due 1pm Friday 14 March 2014.

26. [25 points]

Let the inner product $(\cdot,\cdot): \ C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|: \ C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let the linear operator $L : C_D^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$C_D^2[0,1] = \{w \in C^2[0,1] : w(0) = w(1) = 0\}.$$

Recall that the operator L has eigenvalues

$$\lambda_n = n^2\pi^2$$

with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$. Let N be a positive integer, let $f \in C[0,1]$ be defined by $f(x) = 8x^2(1-x)$ and let u be the solution to

$$Lu = f.$$

(a) Compute the best approximation f_N to f from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$.

(b) Use the spectral method to compute the best approximation u_N to u from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Produce a plot comparing f to f_N for $N = 1, 2, 3, 4, 5, 6$.

(d) Plot the approximations u_N to u that you obtained using the spectral method for $N = 1, 2, 3, 4, 5, 6$.

(e) Write down the series solution to

$$Lu = f$$

that is obtained using the spectral method.

(f) By shifting the data and then using a series solution that you have obtained previously in this question, obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{4}$$

and

$$\tilde{u}(1) = \frac{1}{4}.$$

(g) Let $\tilde{u}_N$ be the series solution that you obtained in part (f) but with ∞ replaced by N. Plot $\tilde{u}_N$ for $N = 1, 2, 3, 4, 5, 6$.

Solution.

(a) [5 points] Since $f(x) = 8x^2(1-x) = 8\left(x^2 - x^3\right)$, we have that, for $k = 1, 2, \ldots$,

$$\begin{aligned}
(f, \psi_k) &= 8\sqrt{2}\int_0^1 \left(x^2 - x^3\right)\sin\left(k\pi x\right)\,dx \\
&= 8\sqrt{2}\left(\left[-\frac{1}{k\pi}\left(x^2 - x^3\right)\cos\left(k\pi x\right)\right]_0^1 + \frac{1}{k\pi}\int_0^1 \left(2x - 3x^2\right)\cos(k\pi x)\,dx\right) \\
&= \frac{8\sqrt{2}}{k\pi}\int_0^1 \left(2x - 3x^2\right)\cos(k\pi x)\,dx \\
&= \frac{8\sqrt{2}}{k\pi}\left(\left[\frac{1}{k\pi}\left(2x - 3x^2\right)\sin\left(k\pi x\right)\right]_0^1 - \frac{1}{k\pi}\int_0^1 (2-6x)\sin(k\pi x)\,dx\right) \\
&= -\frac{8\sqrt{2}}{k^2\pi^2}\int_0^1 (2-6x)\sin(k\pi x)\,dx \\
&= -\frac{8\sqrt{2}}{k^2\pi^2}\left(\left[-\frac{1}{k\pi}(2-6x)\cos\left(k\pi x\right)\right]_0^1 - \frac{6}{k\pi}\int_0^1 \cos(k\pi x)\,dx\right) \\
&= -\frac{8\sqrt{2}}{k^2\pi^2}\left(\frac{4}{k\pi}\cos\left(k\pi\right) + \frac{2}{k\pi} - \frac{6}{k\pi}\left[\frac{1}{k\pi}\sin(k\pi x)\right]_0^1\right) \\
&= \frac{-16\sqrt{2}}{k^3\pi^3}\left(1 + 2\cos\left(k\pi\right)\right) \\
&= \frac{-16\sqrt{2}}{k^3\pi^3}\left(1 + 2\left(-1\right)^k\right).
\end{aligned}$$

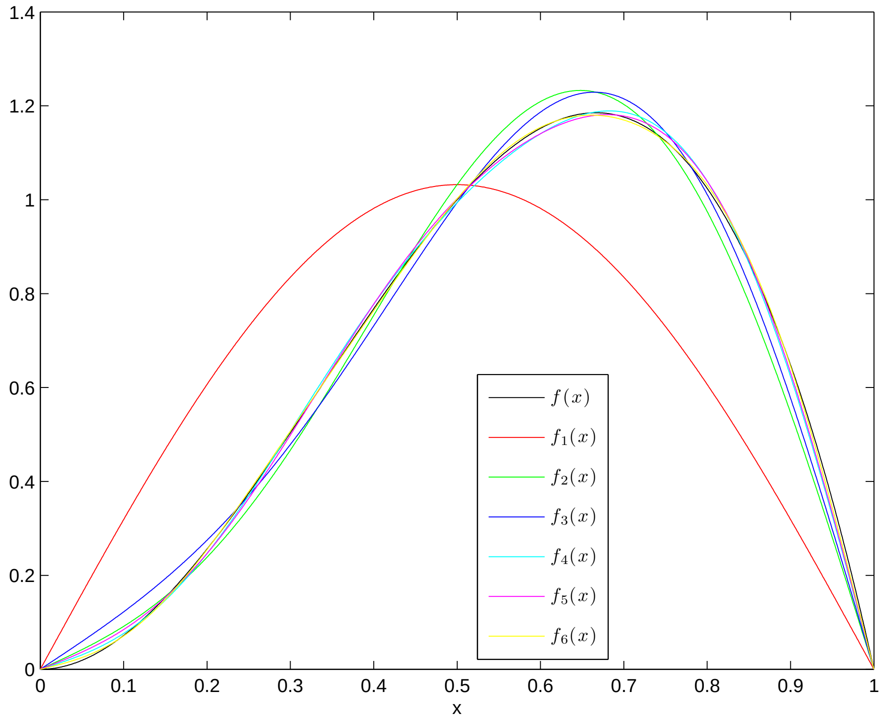
Hence, the best approximation to f from $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$\begin{aligned}
f_N(x) &= \sum_{j=1}^{N}(f, \psi_j)\psi_j(x) \\
&= \sum_{j=1}^{N}\frac{-16\sqrt{2}}{j^3\pi^3}\left(1 + 2\left(-1\right)^j\right)\sqrt{2}\sin(j\pi x) \\
&= \sum_{j=1}^{N}\frac{-32}{j^3\pi^3}\left(1 + 2\left(-1\right)^j\right)\sin(j\pi x).
\end{aligned}$$

(b) [6 points] The best approximation to u from $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$u_N(x) = \sum_{j=1}^{N}\frac{(f, \psi_j)}{\lambda_j}\psi_j(x) = \sum_{j=1}^{N}\frac{-32}{j^5\pi^5}\left(1 + 2\left(-1\right)^j\right)\sin(j\pi x).$$

(c) [2 points] The requested plot is below. Note that the function f happens to satisfy homogeneous Dirichlet boundary conditions, and convergence is quite quick.

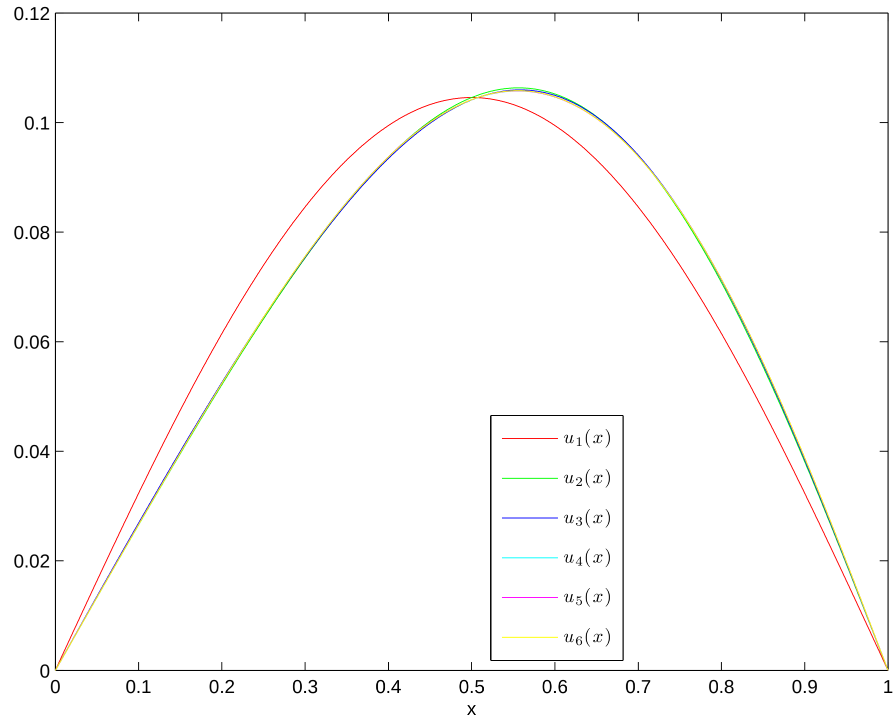


The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(1)
clf
legendStr{1}=['$f(x)$'];
plot(x,8*(x.^2).*(1-x),'k-')
hold on
fk = zeros(size(x));
for k=1:6
    fk = fk + 32*(2*(-1).^(k+1)-1)./(k.^3*pi^3)*sin(k*pi*x);
    plot(x,fk,colors(k))
    legendStr{k+1}=['$f_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(1),'hw26c','epsc')
```

(d) [2 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(2)
clf
uk = zeros(size(x));
for k=1:6
    uk = uk + 32*(2*(-1).^(k+1)-1)./(k.^5*pi^5)*sin(k*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$u_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(2),'hw26d','epsc')
```

(e) [2 points] The series solution that we obtain using the spectral method is

$$u(x) = \sum_{j=1}^{\infty} \frac{-32}{j^5\pi^5} \left(1 + 2\,(-1)^j\right) \sin(j\pi x).$$

(f) [6 points] Let u be the solution to $Lu = f$ and let $w \in C^2[0,1]$ be such that

$$-w''(x) = 0, \quad 0 < x < 1;$$

$$w(0) = -\frac{1}{4}$$

and

$$w(1) = \frac{1}{4}.$$

Then $\tilde{u}(x) = w(x) + u(x)$ will be such that

$$-\tilde{u}''(x) = -w''(x) - u''(x) = 0 + f(x) = f(x);$$

$$\tilde{u}(0) = w(0) + u(0) = -\frac{1}{4} + 0 = -\frac{1}{4};$$

and

$$\tilde{u}(1) = w(1) + u(1) = \frac{1}{4} + 0 = \frac{1}{4}.$$

Now, the general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w(0) = B$ and so $w(0) = -\frac{1}{4}$ when $B = -\frac{1}{4}$. Hence, $w(x) = Ax - \frac{1}{4}$ and so $w(1) = A - \frac{1}{4}$ and hence $w(1) = \frac{1}{4}$ when $A = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$. Consequently,

$$w(x) = \frac{1}{2}x - \frac{1}{4}$$

and so

$$\tilde{u}(x) = \frac{1}{2}x - \frac{1}{4} + u(x).$$

We can then use the series solution to $Lu = f$ that we obtained in part (e) to obtain the series solution

$$\tilde{u}(x) = \frac{1}{2}x - \frac{1}{4} + \sum_{j=1}^{\infty} \frac{-32}{j^5\pi^5}\left(1 + 2\,(-1)^j\right)\sin(j\pi x)$$

to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

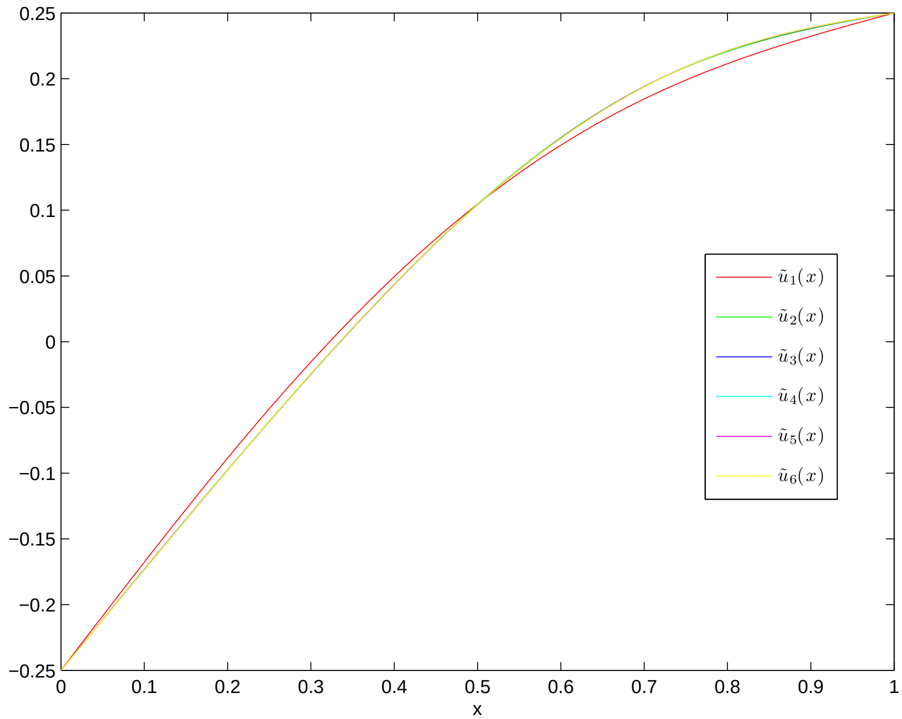
$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{4}$$

and

$$\tilde{u}(1) = \frac{1}{4}.$$

(g) [2 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(3)
clf
uk = x/2-1/4;
for k=1:6
    uk = uk + 32*(2*(-1).^(k+1)-1)./(k.^5*pi^5)*sin(k*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$\tilde{u}_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(3),'hw26g','epsc')
```