CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Examination 1

Instructions:

1. Time limit: **3 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You are allowed one cheat sheet to refer to during the exam.
   You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions throughly (but succintly!) and justify all your answers.
   Show your work for partial credit.
5. Print your name on the line below:

   ____
6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points: (a) = 5, (b),(c) = 10] In this problem, we investigate the spectral method for the steady heat equation

$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

with *periodic* boundary conditions

$$u(0) = u(1)$$
$$\frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$

(a) Show that for $u(x)$ defined as

$$u(x) = \sum_{j=1}^{\infty} \left(c_j \sin(j\pi x) + d_j \cos(j\pi x)\right)$$

that $u(x)$ satisfies periodic boundary conditions

$$u(0) = u(1)$$
$$\frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$

(b) Let

(c)

(d)

2. [25 points:] In this problem, we consider the finite element method for the equation

$$-\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x).$$

This models the steady state distribution of temperature in a bar, where $k(x)$ is the diffusivity of the bar at the point x. Diffusivity must be positive for the equation to be physically realistic; however, if $k(x)$ is not positive, the finite element method may run into issues as well.

It may be helpful to use the fact that the determinant of a 2×2 matrix is

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

(a) Show that, if $k(x) > 0$ for $0 < x < 1$ that $a(u, v)$

$$a(u, v) = \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

defines an inner product. Then, explain why, if $k(x) = 0$ over some interval $[a, b] \subset [0, 1]$, $a(u, v)$ may not be positive definite (and hence not an inner product).

(b) Compute the finite element matrix $K_{ij} = a(\phi_j, \phi_i)$ for $N = 2$ and $k(x)$

$$k(x) = x.$$

(c) Let $k(x)$ be the function

$$k(x) = \begin{cases} 0 & x \leq 2/3 \\ 1 & x > 2/3. \end{cases}$$

Verify that the finite element stiffness matrix for $N = 2$ is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compute the eigenvalues of this matrix. Explain what complications arise when attempting to solve this system using the spectral method.

3. [25 points:]

4. [25 points:]