CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 37 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

37. [25 points]

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v,w) = a(v,w) + (v,w).$$

Let the norm $|||\cdot||| : H^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{B(v,v)}.$$

Let $f \in L^2(0,1)$, let $\rho \in \mathbb{R}$, let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$ and let $u \in H^1_D(0,1)$ be such that

$$B(u,v) = (f,v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Moreover, let N be a positive integer, let V_N be a subspace of $H^1_D(0,1)$ and let $u_N \in V_N$ be such that

$$B(u_N,v) = (f,v) + \rho v(1) \text{ for all } v \in V_N.$$

(a) Use the fact that $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$ and the fact that $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$ to show that $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$. Recall that $H^1(0,1) = \{v \in L^2(0,1) : v' \in L^2(0,1)\}$.

(b) Show that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N.$$

(c) Show that

$$|||u - u_N|||^2 = |||u|||^2 - |||u_N|||^2.$$

(d) Show that

$$|||u_N|||^2 \le |||u|||^2.$$

Solution.

(a) [10 points] Since $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$,

$$(\alpha w_1 + \beta w_2, w_3) = \alpha(w_1, w_3) + \beta(w_2, w_3) \text{ for all } w_1, w_2, w_3 \in H^1(0,1) \text{ and all } \alpha, \beta \in \mathbb{R}$$

because if $w_1, w_2 \in H^1(0,1)$ then $w_1, w_2 \in L^2(0,1)$. Also, since $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$,

$$a(\alpha w_1 + \beta w_2, w_3) = \alpha a(w_1, w_3) + \beta a(w_2, w_3) \text{ for all } w_1, w_2, w_3 \in H^1(0,1) \text{ and all } \alpha, \beta \in \mathbb{R}.$$

Hence, for all $w_1, w_2, w_3 \in H^1(0,1)$ and all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} B(\alpha w_1 + \beta w_2, w_3) &= a(\alpha w_1 + \beta w_2, w_3) + (\alpha w_1 + \beta w_2, w_3) \\ &= \alpha a(w_1, w_3) + \beta a(w_2, w_3) + \alpha(w_1, w_3) + \beta(w_2, w_3) \\ &= \alpha\left(a(w_1, w_3) + (w_1, w_3)\right) + \beta\left(a(w_2, w_3) + (w_2, w_3)\right) \\ &= \alpha B(w_1, w_3) + \beta B(w_2, w_3). \end{aligned}$$

Therefore, $B(\cdot,\cdot)$ is linear in the first argument.
Moreover, since $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$,

$$(w_1, w_2) = (w_2, w_1) \text{ for all } w_1, w_2 \in H^1(0,1)$$

because if $w_1, w_2 \in H^1(0,1)$ then $w_1, w_2 \in L^2(0,1)$. Furthermore, since $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$,

$$a(w_1, w_2) = a(w_2, w_1) \text{ for all } w_1, w_2 \in H^1(0,1).$$

Hence, for all $w_1, w_2 \in H^1(0,1)$,

$$\begin{aligned} B(w_1, w_2) &= a(w_1, w_2) + (w_1, w_2) \\ &= a(w_2, w_1) + (w_2, w_1) \\ &= B(w_2, w_1). \end{aligned}$$

Therefore, $B(\cdot,\cdot)$ is symmetric.
It then follows that, for all $w_1, w_2, w_3 \in H^1(0,1)$ and all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} B(w_1, \alpha w_2 + \beta w_3) &= B(\alpha w_2 + \beta w_3, w_1) \\ &= \alpha B(w_2, w_1) + \beta B(w_3, w_1) \\ &= \alpha B(w_1, w_2) + \beta B(w_1, w_3). \end{aligned}$$

Therefore, $B(\cdot,\cdot)$ is linear in the second argument.
Consequently, $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$.

(b) [5 points] Since V_N is a subspace of $H^1_D(0,1)$, the fact that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1)$$

means that

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in V_N.$$

Moreover,

$$a(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_N.$$

Therefore the properties satisfied by a symmetric bilinear form allow us to say that, for all $v \in V_N$,

$$\begin{aligned} B(u - u_N, v) &= B(u, v) - B(u_N, v) \\ &= (f, v) + \rho v(1) - ((f, v) + \rho v(1)) \\ &= 0. \end{aligned}$$

Consequently,

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N.$$

(c) [5 points] The properties satisfied by a symmetric bilinear form allow us to say that

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u, u - u_N) - B(u_N, u - u_N) \\ &= B(u, u) - B(u, u_N) - B(u_N, u) + B(u_N, u_N) \\ &= B(u, u) - 2B(u, u_N) + B(u_N, u_N). \end{aligned}$$

Now, $u_N \in V_N$ and so the fact that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N$$

means that

$$B(u - u_N, u_N) = 0$$

and hence

$$B(u, u_N) = B(u_N, u_N)$$

since the properties satisfied by a symmetric bilinear form mean that

$$B(u - u_N, u_N) = B(u, u_N) - B(u_N, u_N).$$

Therefore

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u, u) - 2B(u_N, u_N) + B(u_N, u_N) \\ &= B(u, u) - B(u_N, u_N). \end{aligned}$$

The definition of the norm $|||\cdot|||$ then allows us to conclude that

$$|||u - u_N|||^2 = |||u|||^2 - |||u_N|||^2 .$$

(d) [5 points] Since $|||u - u_N||| \in \mathbb{R}$, we can say that

$$|||u - u_N|||^2 \geq 0$$

and so since

$$|||u - u_N|||^2 = |||u|||^2 - |||u_N|||^2$$

we can conclude that

$$|||u|||^2 - |||u_N|||^2 \geq 0.$$

Hence,

$$|||u_N|||^2 \leq |||u|||^2 .$$