CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 9 · Solutions

Posted Wednesday 31 October 2012. Due Wednesday 7 November 2012, 5pm.

1. [40 points: 12 points each for (a) and (b); 6 points for (c); 10 points for (d)]

   (a) Consider the function $u_0(x) = \begin{cases} 1, & x \in [0, 1/3]; \\ 0, & x \in (1/3, 2/3); \\ 1, & x \in [2/3, 1]. \end{cases}$

   Recall that the eigenvalues of the operator $L : C_N^2[0,1] \to C[0,1]$,

   $$Lu = -u''$$

   are $\lambda_n = n^2\pi^2$ for $n = 0, 1, \ldots$ with associated (normalized) eigenfunctions $\psi_0(x) = 1$ and

   $$\psi_n(x) = \sqrt{2}\cos(n\pi x), \qquad n = 1, 2, \ldots.$$

   We wish to write $u_0(x)$ as a series of the form

   $$u_0(x) = \sum_{n=0}^{\infty} a_n(0)\psi_n(x),$$

   where $a_n(0) = (u_0, \psi_n)$.
   Compute these inner products $a_n(0) = (u_0, \psi_n)$ by hand and simplify as much as possible.
   For $m = 0, 2, 4, 80$, plot the partial sums

   $$u_{0,m}(x) = \sum_{n=0}^{m} a_n(0)\psi_n(x).$$

   (You may superimpose these on one single, well-labeled plot if you like.)

   (b) Write down a series solution to the homogeneous heat equation

   $$u_t(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad t \ge 0$$

   with Neumann boundary conditions

   $$u_x(0,t) = u_x(1,t) = 0$$

   and initial condition $u(x,0) = u_0(x)$.
   Create a plot showing the solution at times $t = 0, 0.002, 0.05, 0.1$.
   You will need to truncate your infinite series to show this plot.
   Discuss how the number of terms you use in this infinite series affects the accuracy of your plots.

   (c) Describe the behavior of your solution as $t \to \infty$.
   (To do so, write down a formula for the solution in the limit $t \to \infty$.)

   (d) How would you expect the solution to the inhomogeneous heat equation

   $$u_t(x,t) = u_{xx} + 1, \qquad 0 < x < 1, \quad t \ge 0$$

   with Neumann boundary conditions

   $$u_x(0,t) = u_x(1,t) = 0$$

   to behave as $t \to \infty$?

Solution.

(a) To expand $u_0(x)$ in the form

$$u_0(x) = \sum_{n=0}^{\infty} a_n(0)\psi_n(x),$$

we must compute the coefficients $a_n(0)$. For $n = 0$ we compute

$$a_0(0) = \int_0^1 u_0(x) \cdot 1 \,\mathrm{d}x = \int_0^{1/3} 1 \,\mathrm{d}x + \int_{2/3}^1 1 \,\mathrm{d}x = 2/3.$$

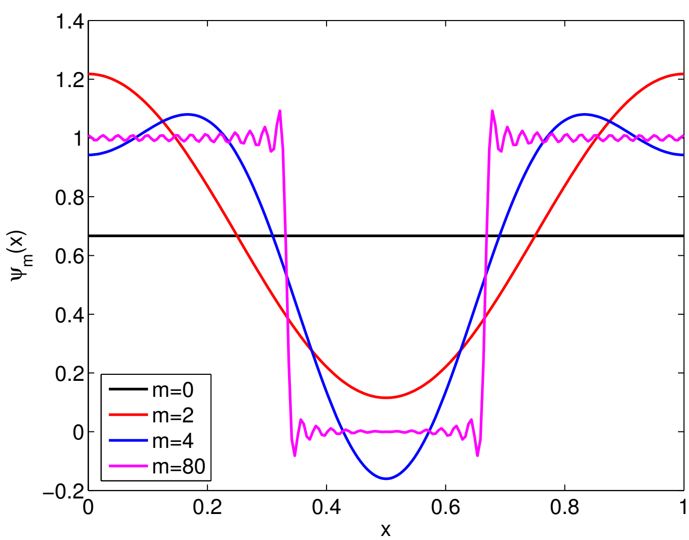
For $n > 0$ we have

$$\begin{aligned} a_n(0) &= \sqrt{2}\int_0^1 u_0(x)\cos(n\pi x)\,dx \\ &= \sqrt{2}\Big(\int_0^{1/3}\cos(n\pi x)\,dx + \int_{2/3}^1 \cos(n\pi x)\,dx\Big) \\ &= \sqrt{2}\Big(\Big[\frac{\sin(n\pi x)}{n\pi}\Big]_0^{1/3} + \Big[\frac{\sin(n\pi x)}{n\pi}\Big]_{2/3}^1 \\ &= \frac{\sqrt{2}(\sin(n\pi/3) - \sin(2n\pi/3))}{n\pi}. \end{aligned}$$

[GRADERS: this last expression is sufficiently simplified to receive full credit.]
Note that $\sin(2n\pi/3) = 2\sin(n\pi/3)\cos(n\pi/3)$, and hence

$$\sin(n\pi/3) - \sin(2n\pi/3) = \sin(n\pi/3)(1 - 2\cos(n\pi/3)).$$

Thus we have $a_n(0) = 0$ in two cases: if n is a multiple of 3, or if $\cos(n\pi/3) = 1/2$. The former occurs when $n = 3, 6, 9, 12, 15, \ldots$, whiles the latter occurs when $n\pi/3 (\mathrm{mod}\ 2\pi) = \pi/3$ or $5\pi/3$, and hence $a_n(0) = 0$ when $n = 1 + 6p$ for integers $p \geq 0$ or $n = -1 + 6p$ for integers $p \geq 1$. Together, this implies that for all odd integers n, $a_n(0) = 0$. We end up with the partial sums shown in the following figure. (MATLAB code follows at the end of this solution.)



(b) We seek a series solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x).$$

Using standard techniques described in class, together with the fact the problem is homogeneous ($f(x,t) = 0$), we find that

$$a_n'(t) + \lambda_n a_n(t) = 0.$$

For $n = 0$ we have

$$a_0'(t) = 0,$$

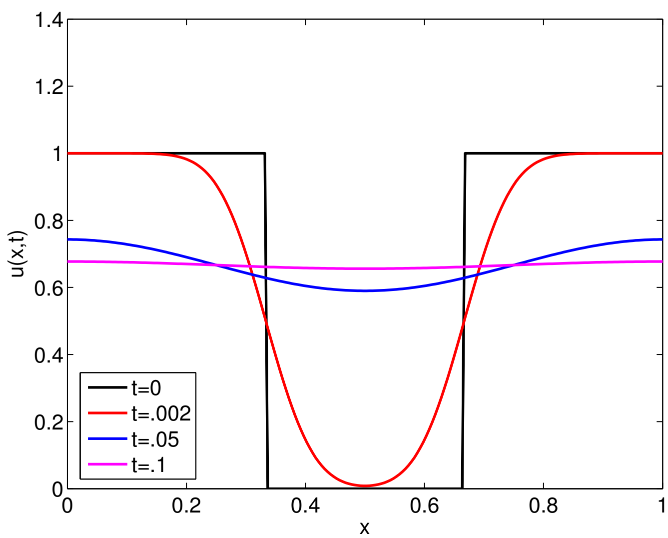
and hence $a_0(t)$ is constant, so we conclude $a_0(t) = a_0(0) = 2/3$. For $n \geq 1$ we have

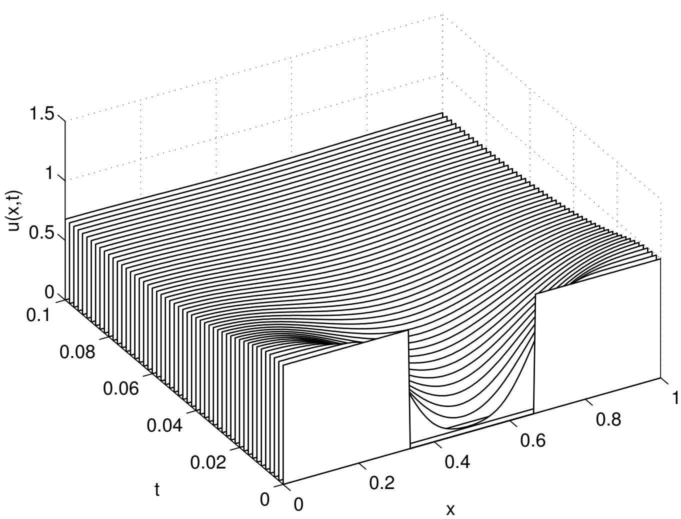
$$a_n(t) = e^{-\lambda_n t} a_n(0),$$

where $\lambda_n = n^2\pi^2$. In sum, we have

$$u(x,t) = 2/3 + \sum_{n=1}^{n} e^{-\lambda_n t} a_n(0)\big(\sqrt{2}\cos(n\pi x)\big).$$

Below we show this plot at the required times, based on taking the sum out to $N = 20$. While the number of terms in the series affects the accuracy of the solution in at early times, the importance of these extra terms decreases as $t \to \infty$.





(c) As is clear from the series formula in part (b) and from the figures, as $t \to \infty$, $u(x,t) \to 2/3$ for all $x \in [0,1]$.

(d) The existence of the limiting solution in part (c) does not contradict the fact that $\lambda_0 = 0$. There is no division by zero, as there is in the analogous steady-state problem $u_{xx} = f(x)$ with homogeneous Neumann conditions. The addition of the source term adds energy to the system, effectively increasing the rate of change of temperature with respect to time (u_t) by one unit. This corresponds to the physical situation of pumping more energy into a bar that is insulated at both ends—and hence energy cannot escape. Thus we expect the heat to grow as $t \to \infty$.

The above paragraph is satisfactory for full credit, but we can actually be quite a bit more precise. The eigenvalue $\lambda_0 = 0$ contributes a constant term to the solution of the PDE $u_t = u_{xx}$, and this constant will be nonzero provided $(u_0, \psi_0) = \int_0^1 u_0(x) \cdot 1\, dx \neq 0$. If u_0 has 'zero mean', i.e., $\int_0^1 u_0(x)\, dx = 0$, then the solution to the homogeneous problem will decay as $t \to \infty$; otherwise, as $t \to \infty$ the solution will approach the nonzero constant (u_0, ψ_0).

To write down the solution to the general inhomogeneous equation $u_t = u_{xx} + f$, we must expand

$$f(x,t) = \sum_{n=0}^{\infty} c_n(t)\psi_n(x).$$

The the coefficients $a_n(t)$ in the expansion of the solution

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x)$$

obey the differential equation

$$a_n'(t) = -\lambda_n a_n(t) + c_n(t).$$

As seen in class, these ODEs have the solutions

$$a_n(t) = e^{-\lambda_n t} a_n(0) + \int_0^t e^{-\lambda_n (t-\tau)} c_n(\tau)\, d\tau.$$

The $a_0(t)$ case is particularly interesting: $a_0(t) = a_0(0) + \int_0^t c_0(\tau)\, d\tau$. Hence we cannot possibly have a steady state solution if $c_0(\tau)$ is bounded away from zero for all $\tau > 0$.

In the case of $f(x,t) = 1$, we have $c_0(t) = 1$ and $c_n(t) = 0$ for $n > 0$, so that

$$a_0(t) = a_0(0) + \int_0^t 1\, d\tau = a_0(0) + t;$$

and for $n > 0$,

$$a_n(t) = e^{-\lambda_n t} a_n(0),$$

thus giving the solution

$$u(x,t) = a_0(0) + t + \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n(0)\psi_n(x).$$

```
% Plot the expansion of the initial data, psi(x)

 x = linspace(0,1,200);
 col = 'krbm';
 figure(1), clf
 fm = zeros(size(x));
 for n=0:2:80
    if n==0, an0 = 2/3;                         % psi(x) = 1 for x in [0,1/3], [2/3,1];
    else, an0 = sqrt(2)*(sin(n*pi/3)-sin(2*n*pi/3))/(n*pi);% psi(x) = 0 otherwise.
    end
```

```
   if n==0, fm = an0*ones(size(fm));
   else, fm = fm + an0*sqrt(2)*cos(n*pi*x);
   end
   if ismember(n,[ 0 2 4 80]),
      plot(x, fm, '-','linewidth',2,'color',col(1)), hold on, col = col(2:end);
   end
end
legend('m=0', 'm=2','m=4','m=80',3)
set(gca,'fontsize',16)
xlabel('x'), ylabel('\psi_m(x)')
print -depsc2 heateqn1

% Compute the solution at at various times.

psi = (x <= 1/3) | (x >= 2/3);    %  initial condition
U = [psi];
col = 'krbmc';
figure(2), clf
plot(x, psi, 'linewidth',2,'color',col(1)), hold on, col = col(2:end);
t = .002:.002:0.1;
tprint = [.002 .05 0.1];
for j=1:length(t)
   for n=0:2:20
      if n==0,
         an0    = 2/3;
         lambda = 0;
         uj     = exp(-lambda*t(j))*an0*ones(size(x));
      else
         an0 = sqrt(2)*(sin(n*pi/3)-sin(2*n*pi/3))/(n*pi);
         lambda = n^2*pi^2;
         uj = uj + exp(-lambda*t(j))*an0*(sqrt(2)*cos(n*pi*x));
      end
   end
   U = [U;uj];
   if ismember(t(j),tprint),
      plot(x, uj, '-','linewidth',2,'color',col(1)), hold on, col = col(2:end);
   end
end
legend('t=0','t=.002','t=.05','t=.1',3)
set(gca,'fontsize',16)
xlabel('x'), ylabel('u(x,t)')
print -depsc2 heateqn2

figure(3), clf
plt = waterfall(x,[0 t],U);
set(plt,'edgecolor','k')       % make the lines black

view(-30,50)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 heateqn3
```

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2. [20 points]
Describe how to solve the heat equation

$$u_t(x,t) = u_{xx}(x,t) + f(x,t), \qquad 0 < x < 1, \quad t \geq 0$$

with *inhomogeneous* Neumann boundary conditions

$$u_x(0,t) = \alpha, \qquad u_x(1,t) = \beta$$

and initial condition $u(x,0) = u_0(x)$.

Solution. We shall attempt to write the solution $u(x)$ in the form

$$u(x,t) = \widehat{u}(x,t) + v(x),$$

where $v(x)$ is some function that we shall construct that satisfies the boundary conditions

$$v'(0) = \alpha, \qquad v'(1) = \beta,$$

and $\widehat{u}(x,t)$ is some function we shall determine by solving the heat equation with homogeneous Neumann boundary conditions:

$$\widehat{u}_x(0,t) = 0, \qquad \widehat{u}_x(1,t) = 0.$$

There are various ways to arrive at the function $v(x)$. Some students noticed that since we know $v'(x)$ at two points, we can define $v'(x)$ to be the line that passes through $(0,\alpha)$ and $(1,\beta)$, i.e.,

$$v'(x) = \alpha + (\beta - \alpha)x.$$

Integrate this polynomial to get

$$v(x) = \alpha x + \tfrac{1}{2}(\beta - \alpha)x^2 + C$$

for any constant C. Taking $C = 0$ gives the cleanest form:

$$v(x) = \alpha x + \tfrac{1}{2}(\beta - \alpha)x^2.$$

Given this formula for $v(x)$, we must determine the constraints on $\widehat{u}(x,t)$. Since $u(x,t) = \widehat{u}(x,t) + v(x)$ is to solve the differential equation $u_t = u_{xx} + f$, we consider:

$$u_t(x,t) = \widehat{u}_t(x,t) + v_t(x) = \widehat{u}_t(x,t),$$

since $v(x)$ is independent of t, and

$$\begin{aligned} u_{xx}(x,t) + f(x,t) &= \widehat{u}_{xx}(x,t) + v_{xx}(x) + f(x,t) \\ &= \widehat{u}_{xx}(x,t) + (\beta - \alpha) + f(x,t) \\ &= \widehat{u}_{xx}(x,t) + \widehat{f}(x,t), \end{aligned}$$

where we have defined

$$\widehat{f}(x,t) = \beta - \alpha + f(x,t).$$

Equating the expressions for u_t and $u_{xx} + f$, we arrive at the partial differential equation

$$\widehat{u}_t(x,t) = \widehat{u}_{xx}(x,t) + \widehat{f}(x,t).$$

To fully specify this PDE, we must determine the boundary conditions and initial condition. At the boundaries, we want

$$\begin{aligned} \alpha &= u_x(0,t) = \widehat{u}_x(0,t) + v_x(0) = \widehat{u}_x(0,t) + \alpha \\ \beta &= u_x(1,t) = \widehat{u}_x(1,t) + v_x(1) = \widehat{u}_x(1,t) + \beta, \end{aligned}$$

so we conclude that

$$\widehat{u}_x(0,t) = \widehat{u}_x(1,t) = 0,$$

as we were intending (since we already know how to solve equations with *homogeneous* Neumann conditions).

What can be said of the initial condition? We want

$$u_0(x) = u(x,0) = \widehat{u}(x,0) + v(x),$$

and so we arrive at the initial condition that $\widehat{u}$ must satisfy:

$$\widehat{u}(x,0) = u_0(x) - v(x).$$

[GRADERS: A complete solutions must specify four essential ingredients: (i) the function v; (ii) the PDE for $\widehat{u}$ with the correct $\widehat{f}$ term; (iii) the boundary conditions for $\widehat{u}$; (iv) the correct shifted initial conditions for $\widehat{u}$. Deduct 5 points each for any of these ingredients that are missing. Note that other corrections are also possible: the correct answer is not unique. ***Please grade carefully!***]

3. [40 points: 8 points for (a); 16 points each for (b) and (c)]
Consider the *fourth order* partial differential equation

$$u_t(x,t) = u_{xx}(x,t) - u_{xxxx}(x,t)$$

with so-called *hinged* boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$$

and initial condition (that should satisfy the boundary conditions)

$$u(x,0) = u_0(x).$$

(This equation is related to a model that arises in the study of thin films.)
To solve this PDE, we introduce the linear operator $L : C_H^4[0,1] \to C[0,1]$, where

$$Lu = -u'' + u''''$$

and

$$C_H^4[0,1] = \{u \in C^4[0,1], u(0) = u''(0) = u(1) = u''(1) = 0\}$$

is the set of C^4 functions that satisfy the hinged boundary conditions.

(a) The operator L has eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

Use this fact to compute a formula for the eigenvalues λ_n, $n = 1, 2, \ldots$.

(b) Suppose the initial condition $u_0(x)$ is expanded in the form

$$u_0(x) = \sum_{n=1}^{\infty} a_n(0)\psi_n(x).$$

Briefly describe how one can write the solution to the PDE $u_t = u_{xx} - u_{xxxx}$ as an infinite sum.

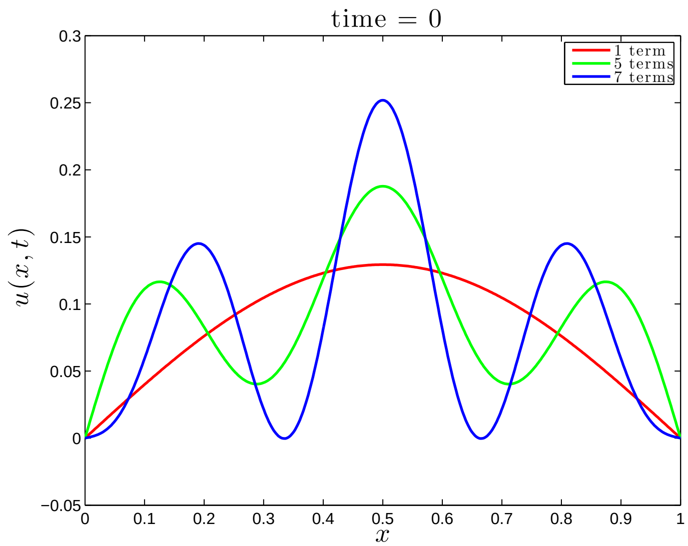
(c) Suppose the initial data is given by

$$u_0(x) = (x - x^2)\sin(3\pi x)^2,$$

with associated coefficients

$$a_n(0) = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3}, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Write a program (you may modify your earlier codes) to compute the solution you describe in part (b) up to seven terms in the infinite sum. At each time $t = 0$; 10^{-5}; 2×10^{-5}; 4×10^{-5}, produce a plot comparing the sum of the first 1, 5, and 7 terms of the series. For example, at time $t = 0$, your plot should appear as shown below. (Alternatively, you can produce attractive 3-dimensional plots over the time interval $t \in [0, 4 \times 10^{-5}]$ using 1, 5, and 7 terms in the series.)



Solution.

(a) Given the eigenfunctions ψ_n, we simply apply L to ψ_n to compute $\lambda_n\psi_n$:

$$\begin{aligned} L\psi_n(x) &= -\psi_n''(x) + \psi_n''''(x) \\ &= -\frac{d^2}{dx^2}(\sqrt{2}\sin(n\pi x)) + \frac{d^4}{dx^4}(\sqrt{2}\sin(n\pi x)) \\ &= n^2\pi^2\sqrt{2}\sin(n\pi x) + n^4\pi^4\sqrt{2}\sin(n\pi x) \\ &= (n^2\pi^2 + n^4\pi^4)(\sqrt{2}\sin(n\pi x)) \\ &= \lambda_n\psi_n(x). \end{aligned}$$

Thus, we identify $\lambda_n = n^2\pi^2 + n^4\pi^4$ for $n = 1, 2, \ldots$.

(b) Following the prodedure outlined in class, we look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

Substituting this equation into the differential equation, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} a_n'(t)\psi_n(x) &= \sum_{n=1}^{\infty} a_n(t)(\psi_n''(x) - \psi_n''''(x)) \\ &= \sum_{n=1}^{\infty} -\lambda_n a_n(t)\psi_n(x). \end{aligned}$$

Taking an inner product of both sides with ψ_k and using the orthonormality of the eigenfunctions, we obtain the scalar differential equations

$$a_k'(t) = -\lambda_k a_k(t),$$

which has the solution

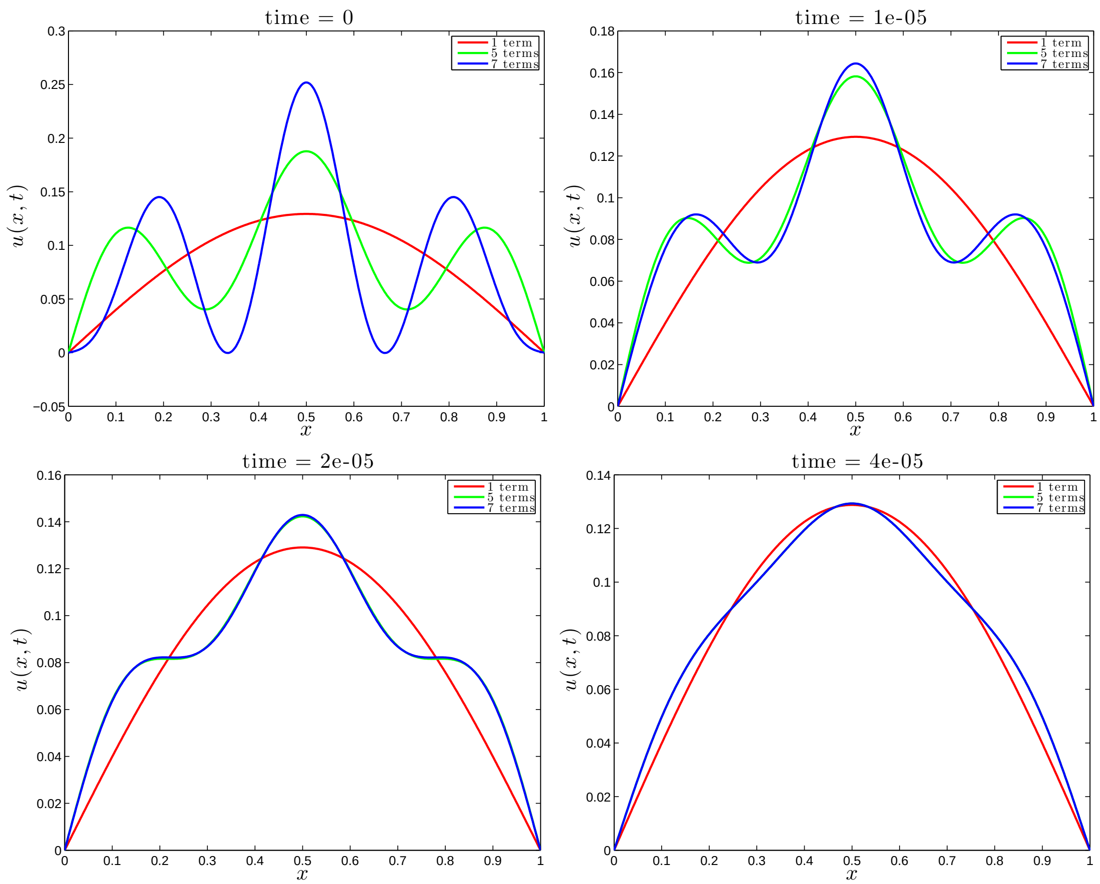
$$a_k(t) = e^{-\lambda_k} a_k(0).$$

Thus, the solution can be written in the series

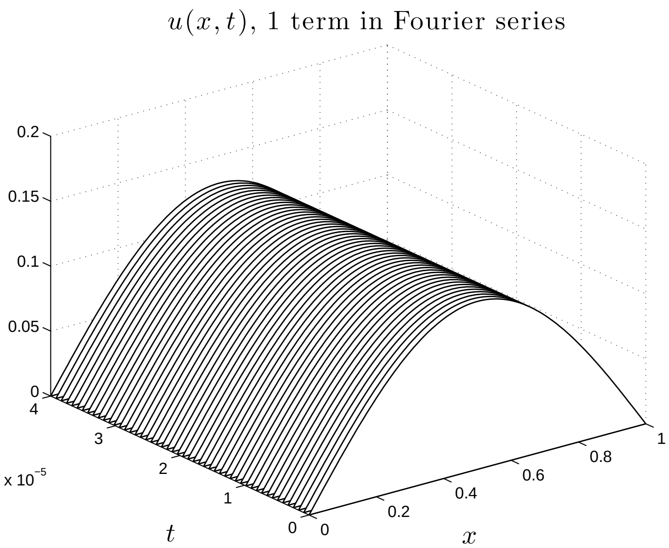
$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n(0)\psi_n(x) \\ &= \sum_{n=1}^{\infty} \sqrt{2} e^{-(n^2\pi^2+n^4\pi^4)t} a_n(0) \sin(n\pi x). \end{aligned}$$

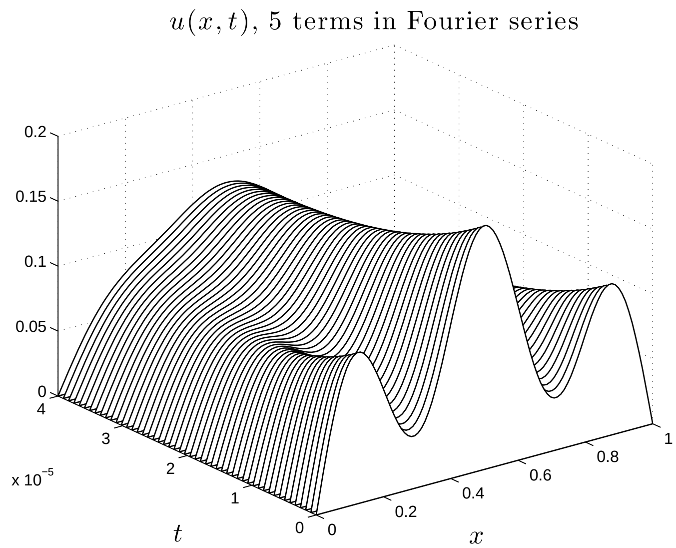
[GRADERS: students need only write down one of these series solutions for $u(x,t)$; they need not include the derivation.]

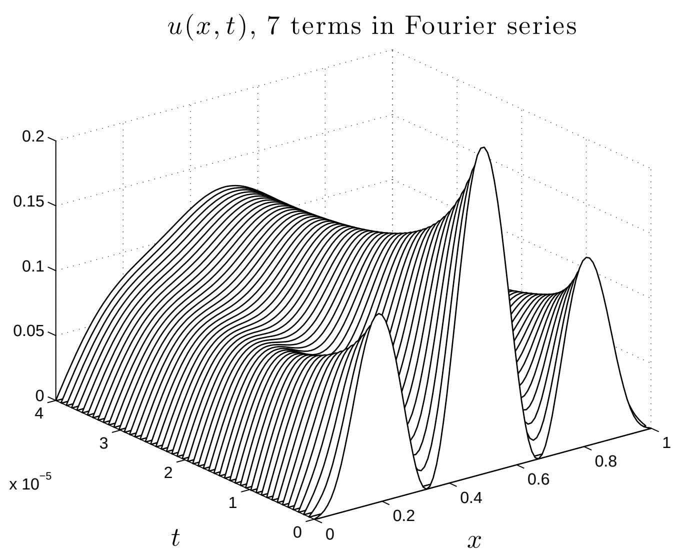
(c) Plots for the four requested times are shown below.



Alternatively, students may produce three-dimensional plots over the same time span for 1, 5, and 7 terms in the Fourier series.







One can produce these plots with the following code.

```
tvec = [0 .00001 .00002 .00004];
x = linspace(0,1,500);
an0 = inline('sqrt(2)*432*(n^4-18*n^2+216)/((36*n-n^3)^3*pi^3)');
lam = inline('n^2*pi^2 + n^4*pi^4');
col = 'rgb';
str = 'abcd';
for j=1:length(tvec)
   figure(1), clf
   t = tvec(j);
   u = zeros(size(x));
   for n=1:2:7
       u = u+exp(-lam(n)*t)*an0(n)*(sqrt(2)*sin(n*pi*x));
       [tf,loc] = ismember(n,[1 5 7]);
       if tf,
          plot(x,u,'-','color',col(loc),'linewidth',2), hold on
       end
   end
   legend('1 term','5 terms', '7 terms')
   xlabel('x','fontsize',20)
   ylabel('u(x,t)','fontsize',20)
   title(sprintf('time = %g',t),'fontsize',20)
```

```
    eval(sprintf('print -depsc2 fourth_%s',str(j)))
    pause(.1)
 end

% surface plot
 tvec = linspace(0, .00004, 50);
 x    = linspace(0, 1, 100);
 U = zeros(length(tvec),length(x),3);
 for j=1:length(tvec)
    t = tvec(j);
    u = zeros(size(x));
    for n=1:2:7
       u = u+exp(-lam(n)*t)*an0(n)*(sqrt(2)*sin(n*pi*x));
       [tf,loc] = ismember(n,[1 5 7]);
       if tf, U(j,:,loc) = u; end
    end
 end
 figure(1), clf
 plt=waterfall(x,tvec,U(:,:,1));
 set(plt,'edgecolor','k')
 axis([0 1 0 .00004 0 .2])
 xlabel('x','fontsize',20), ylabel('t','fontsize',20)
 title('u(x,t), 1 term in Fourier series','fontsize',20)
 print -depsc2 fourth_wf1

 figure(1), clf
 plt=waterfall(x,tvec,U(:,:,2));
 set(plt,'edgecolor','k')
 axis([0 1 0 .00004 0 .2])
 xlabel('x','fontsize',20), ylabel('t','fontsize',20)
 title('u(x,t), 5 terms in Fourier series','fontsize',20)
 print -depsc2 fourth_wf5

 figure(1), clf
 plt=waterfall(x,tvec,U(:,:,3));
 set(plt,'edgecolor','k')
 axis([0 1 0 .00004 0 .2])
 xlabel('x','fontsize',20), ylabel('t','fontsize',20)
 title('u(x,t), 7 terms in Fourier series','fontsize',20)
 print -depsc2 fourth_wf7
```