CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 9 · Solutions

Posted Wednesday 15 April, 2015. Due 5pm Wednesday 22 April, 2015.

Please write your name and instructor on your homework. This homework is out of only 75 points instead of the usual 100.

1. [30 points: 15 points each]
We wish to approximate the solution to the heat equation

$$u_t(x,t) = u_{xx}(x,t) + 100tx, \qquad 0 \le x \le 1,\ t \ge 0$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = 0$$

using the finite element method (method of lines). Let $N \ge 1$, $h = 1/(N+1)$, and $x_k = kh$ for $k = 0, \ldots, N+1$. We shall construct approximations using the hat functions

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

The approximate solution shall have the form

$$u_N(x,t) = \sum_{k=1}^{N} a_k(t)\phi_k(x).$$

(a) Write down the system of ordinary differential equations that determines the coefficients $a_k(t)$, $k = 1, \ldots, N$. Specify the entries in the mass and stiffness matrices and the load vector. (You may use results from previous homeworks and class as convenient.)

(b) Write a MATLAB code that uses the backward Euler method to solve for the coefficients $a_k(t)$. Plot your approximate solution $u_N(x,t)$ at time $t = 1$.
Choose N and Δt so that your solution appears to be accurate.
Verify this accuracy by superimposing on your plot the computed solution at $t = 1$ obtained by using space and time steps that are ten times smaller.

Solution.

(a) As derived in class, the system of ordinary differential equations that governs the behavior of the coefficients $a_k(t)$ is given by

$$\mathbf{M}\frac{d\mathbf{a}}{dt}(t) + \mathbf{K}\mathbf{a}(t) = \mathbf{f}(t),$$

where $\mathbf{M}$ and $\mathbf{K}$ are the *mass* and *stiffness* matrices whose (j,k) entries are given by (ϕ_j, ϕ_k) and $a(\phi_j, \phi_k)$. We also have When the basis functions $\{\phi_j\}_{j=1}^{N}$ are hat functions, we have

$$\mathbf{M}_{j,k} = (\phi_j, \phi_k) = \begin{cases} 2h/3 & \text{if } j = k; \\ h/6 & \text{if } |j-k| = 1; \\ 0 & \text{otherwise.} \end{cases},$$

as computed in Problem 3 of Problem Set 6, and

$$\mathbf{K}_{j,k} = a(\phi_j, \phi_k) = \begin{cases} 2/h & \text{if } j = k; \\ -1/h & \text{if } |j-k| = 1; \\ 0 & \text{otherwise.} \end{cases},$$

as computed in class.

(b) The backward Euler method for this differential equation takes the form

$$\mathbf{a}_{k+1} = (\mathbf{I} + \Delta t \mathbf{M}^{-1}\mathbf{K})^{-1}(\mathbf{a}_k + \Delta t \mathbf{M}^{-1}\mathbf{f}(t_{k+1})),$$

which can be rearranged in the more computationally appealing form

$$\mathbf{a}_{k+1} = (\mathbf{M} + \Delta t \mathbf{K})^{-1}(\mathbf{M}\mathbf{a}_k + \Delta t \mathbf{f}(t_{k+1})).$$

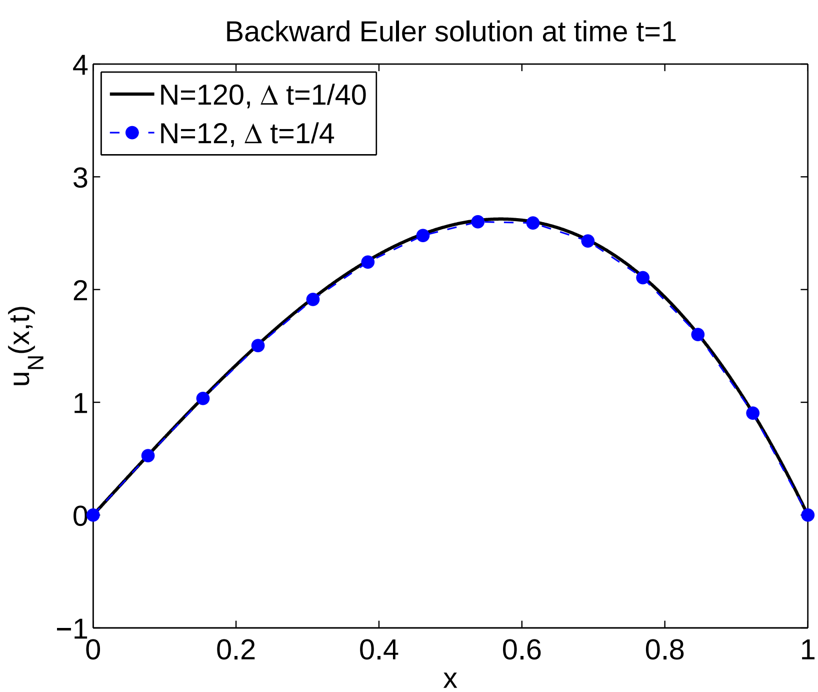
For this $f(x,t) = 100tx$, we have for fixed t that

$$\mathbf{f}_j(t) = (f(\cdot, t), \phi_j) = 100t \int_0^1 x\phi_j(x)\, dx,$$

and (as can be readily computed with Mathematica)

$$\int_0^1 x\phi_j(x)\, dx = \int_{x_{j-1}}^{x_j} \frac{x(x - x_{j-1})}{h}\, dx + \int_{x_j}^{x_{j+1}} \frac{x(x_{j+1} - x)}{h}\, dx = \Big(\frac{hx_j}{2} - \frac{h^2}{6}\Big) + \Big(\frac{hx_j}{2} + \frac{h^2}{6}\Big) = x_j h = jh^2.$$

The backward Euler method produces remarkably resilient solutions to this problem. The plot below compares the solution obtained the $N = 8$ and $\Delta t = 1/2$ (dots) to that computed with $N = 80$ and $\Delta t = 1/20$.



MATLAB code to solve the problem for a specified N and Δt is given below.

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N = input('enter N:  ');
dt = input('enter dt: ');
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 h = 1/(N+1);
 x = h*[1:N];

 M = (h/6)*(4*eye(N)+diag(ones(N-1,1),1)+diag(ones(N-1,1),-1));
 K = (1/h)*(2*eye(N)-diag(ones(N-1,1),1)-diag(ones(N-1,1),-1));

 tfinal = 1;
 xx = linspace(0,1,1000)';     % finely spaced points between 0 and 1.

 k = [1:N]';
 f = 100*h^2*k;       % (f,phi_k) = 100*exp(-t)*h^2*k:
                      % compute it once, but add the "exp(-t)" part at each step
 a0 = zeros(N,1);     % initial condition

 akp1 = a0;           % a_{k+1} will be computed in this variable
 L = chol(M)';
 for k=0:tfinal/dt-1
    t = (k+1)*dt;
    akp1 = (M+dt*K)\(M*akp1+dt*(exp(-t)*f));  % compute solution at t_{k+1}
 end

% plot the solution on a fine grid.
 uN = zeros(size(xx));
 for j=1:N
   uN = uN + akp1(j)*hat(xx,j,N);
 end
 figure(1), clf
 plot(xx,uN,'b-','linewidth',2)
 axis([0 1 -1.25 7])
 set(gca,'fontsize',18)
 xlabel('x')
 ylabel('u_N(x,t)')
 title(sprintf('time = %f', t))
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2. [45 points: (a)-(c) 10 points each, (d) 15 points]

(a) Let B be defined as the matrix

$$B = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

Using trigonometric identities, verify that the eigenvalues λ_i and eigenvectors v_i of B are

$$\lambda_i = 2\cos\left(\frac{i\pi}{N+1}\right), \qquad v_i = \begin{bmatrix} \sin\left(\frac{i\pi}{N+1}\right) \\ \sin\left(\frac{2i\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{(N-1)i\pi}{N+1}\right) \\ \sin\left(\frac{Ni\pi}{N+1}\right) \end{bmatrix}, \qquad i = 1, \dots N.$$

(Note: some of you may remember this problem from CAAM 335, Spring 2014. This is intentional, and meant to give additional practice to those who did not enjoy the luxury of a semester-long CAAM excursion into matrix theory.)

(b) The finite difference matrix A for the steady heat equation

$$-\kappa \frac{\partial^2 u}{\partial x^2} = f(x)$$

(with zero Dirichlet boundary conditions) can be written as

$$A = \frac{\kappa}{h^2}(2I - B),$$

use part (a) to determine the eigenvalues of A in terms of κ, h, and i, N. *Hint: you may use that, if B has eigenvalues λ_i and eigenvectors v_i, then $\alpha I + B$ has eigenvalues $\alpha + \lambda_i$, αB has eigenvalues $\alpha\lambda_i$, and both have eigenvectors v_i.*

(c) We may represent any vector $u \in \mathbb{R}^n$ as a linear combination of the eigenvectors

$$u = \sum_{j=1}^{N} \alpha_j v_j.$$

Use this representation to show that

$$A^n u = \sum_{j=1}^{N} \alpha_j \lambda_j^n v_j.$$

(d) If we use finite differences, the forward Euler update for the unsteady heat equation may be written as

$$u_{k+1} = (I - dtA)u_k.$$

Show that

$$u_{k+1} = (I - dtA)^{k+1} u_0.$$

and explain, using the result of part (b) and (c), why $dt > \frac{h^2}{2\kappa}$ will cause the solution to blow up as $i \to \infty$. *Hint: if $|a| > 1$, then $|a|^k \to \infty$ as k increases. What are the eigenvalues of $(I - dtA)$?*

Solution.

(a) We need to show that $Av_j = \lambda_j v_j$ for each $j = 1\ldots, N$. We will do so by showing that each entry of the vector Av_j matches the corresponding entry of $\lambda_j v_j$. There are three cases to study: the first entry; the kth entry, $2 \le k \le N-1$; the last entry.

- For the first entry, we want to show that $(Av_j)_1 = (\lambda_j v_j)_1$. Substituting in the formulas for v_j and λ_j, we see that

$$(Av_j)_1 = \sin\left(\frac{2j\pi}{N+1}\right), \qquad (\lambda_j v_j) = 2\cos\left(\frac{j\pi}{N+1}\right)\sin\left(\frac{j\pi}{N+1}\right).$$

Using the double-angle identity $2\cos(\theta)\sin(\theta) = \sin(2\theta)$, we see that

$$2\cos\left(\frac{j\pi}{N+1}\right)\sin\left(\frac{j\pi}{N+1}\right) = \sin\left(\frac{2j\pi}{N+1}\right),$$

and so $(Av_j)_1 = (\lambda_j v_j)_1$.

- For the interior entries, we need to show that $(Av_j)_k = (\lambda_j v_j)_k$ for $k = 2, \ldots, N-1$, where

$$(Av_j)_k = \sin\left(\frac{j(k-1)\pi}{N+1}\right) + \sin\left(\frac{j(k+1)\pi}{N+1}\right), \qquad (\lambda_j v_j)_k = 2\cos\left(\frac{j\pi}{N+1}\right)\sin\left(\frac{kj\pi}{N+1}\right).$$

Recall the "product-to-sum" formula $2\cos(\phi)\sin(\theta) = \sin(\theta+\phi) + \sin(\theta-\phi)$. With $\phi = j\pi/(N+1)$ and $\theta = kj\pi/(N+1)$, we have

$$\begin{aligned}(\lambda_j v_j)_k &= 2\cos\left(\frac{j\pi}{N+1}\right)\sin\left(\frac{jk\pi}{N+1}\right)\\ &= \sin\left(\frac{(k+1)j\pi}{N+1}\right) + \sin\left(\frac{(k-1)j\pi}{N+1}\right)\\ &= (Av_j)_k,\end{aligned}$$

as required.

- To show that $(Av_j)_N = (\lambda_j v_j)_N$, we consider

$$(Av_j)_N = \sin\left(\frac{(N-1)j\pi}{N+1}\right), \qquad (\lambda_j v_j)_N = 2\cos\left(\frac{j\pi}{N+1}\right)\sin\left(\frac{Nj\pi}{N+1}\right).$$

As we use the identity $2\cos(\phi)\sin(\theta) = \sin(\theta+\phi) + \sin(\theta-\phi)$. With $\phi = j\pi/(N+1)$ and $\theta = Nj\pi/(N+1)$, we have

$$\begin{aligned}(\lambda_j v_j)_N &= 2\cos\left(\frac{j\pi}{N+1}\right)\sin\left(\frac{jk\pi}{N+1}\right)\\ &= \sin\left(\frac{(N+1)j\pi}{N+1}\right) + \sin\left(\frac{N-1)j\pi}{N+1}\right)\\ &= \sin\left(j\pi\right) + \sin\left(\frac{N-1)j\pi}{N+1}\right)\\ &= \sin\left(\frac{N-1)j\pi}{N+1}\right),\end{aligned}$$

where the last step used the fact that j is an integer. Notice that this last quantity is precisely $(Av_j)_N$, so we have shown that $(Av_j)_N = (\lambda_j v_j)_N$.

(b) This is simply algebraic manipulation: since the eigenvalues of B are

$$\mu_i = 2\cos\left(\frac{i\pi}{N+1}\right).$$

The eigenvalues of $2I - B$ are then $2 - \mu_i$. Similarly, scaling by a constant scales the eigenvalues by that constant. The eigenvalues of $A = \frac{\kappa}{h^2}(2I - B)$ are

$$\lambda_i = \frac{\kappa}{h^2}(2 - \mu_i) = \frac{\kappa}{h^2}\left(2 - 2\cos\left(\frac{i\pi}{N+1}\right)\right), \quad i = 1, \dots, N.$$

(c) If we take

$$u = \sum_{j=1}^{N} \alpha_j v_j$$

and multiply by A, we have

$$Au = A(\sum_{j=1}^{N} \alpha_j v_j) = \sum_{j=1}^{N} \alpha_j A v_j = \sum_{j=1}^{N} \alpha_j \lambda_j v_j.$$

Applying A twice, we have

$$A^2 u = A(Au) = A(\sum_{j=1}^{N} \alpha_j \lambda_j v_j) = \sum_{j=1}^{N} \alpha_j \lambda_j A v_j = \sum_{j=1}^{N} \alpha_j \lambda_j^2 v_j.$$

Continuing this pattern shows that

$$A^n u = \sum_{j=1}^{N} \alpha_j \lambda_j^n v_j.$$

(d) Since $u_1 = (I - dtA)u_0$, the formula for u_2 is

$$u_2 = (I - dtA)u_1 = (I - dtA)^2 u_0.$$

Continuing this shows that

$$u_{k+1} = (I - dtA)u_k = (I - dtA)^2 u_{k-1} = \dots = (I - dtA)^{k+1} u_0.$$

By part (c), if $u_0 = \sum \alpha_j v_j$, then u_{k+1} is

$$(I - dtA)^{k+1} u = \sum_{j=1}^{N} \alpha_j \mu_j^{k+1} v_j,$$

where μ_j are the eigenvalues of $(I - dtA)$. By (b), these eigenvalues depend on dt such that

$$\mu_j = 1 - dt\left(\frac{\kappa}{h^2}(2 - 2\cos\left(\frac{i\pi}{N+1}\right)\right).$$

If $|\mu_j| > 1$, then u_{k+1} will blow up as k increases. We note that

$$-1 < \cos\left(\frac{i\pi}{N+1}\right) < 1$$

so we can conclude that

$$0 < 2 - 2\cos\left(\frac{i\pi}{N+1}\right) < 4.$$

Then, our largest eigenvalue μ_j is found when $2 - 2\cos\left(\frac{i\pi}{N+1}\right)$ is closest to 4.

$$|\mu_j| < \left|1 - dt4\frac{\kappa}{h^2}\right|.$$

If we choose $dt = h^2/(2\kappa)$, this reduces to

$$|\mu_j| < |1 - 2| = 1.$$

Choosing $dt < h^2/(2\kappa)$ guarantees the solution will not blow up. Choosing dt larger means the solution may tend to infinity as k increases.

Graders: please grade generously on this problem. Full mathematical rigor is not required, only a correct reasoning.
