CAAM 336 · DIFFERENTIAL EQUATIONS

Fall 2013 Examination 1

1. [25 points]

Let the operator $L : \; C^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v'' + 9v.$$

Let $u \in C^2[0,1]$ be the solution to the differential equation

$$-u''(x) + 9u(x) = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = \alpha$$

and

$$u(1) = \beta$$

where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Note that

$$(Lu)(x) = -u''(x) + 9u(x)$$

for all $x \in [0,1]$. Let N be an integer which is such that $N \geq 2$ and let $h = \frac{1}{N+1}$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

(a) Determine whether or not L is a linear operator.

(b) By using the approximation

$$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

for $j = 1, \ldots, N$ we can write

$$\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} \approx \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}$$

where $\mathbf{D} \in \mathbb{R}^{N \times (N+2)}$. What are the entries of the matrix $\mathbf{D}$? An acceptable way to present your final answer is

$$D_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

with the question marks replaced with the correct values.

(c) We can use the differential equation and boundary conditions satisfied by u and the approximation from the previous part to write

$$\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ and $\mathbf{b} \in \mathbb{R}^N$. What are the entries of the matrix $\mathbf{A}$ and the vector $\mathbf{b}$? An acceptable way to present your final answer is

$$A_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

and

$$b_j = \begin{cases} ? & \text{if } j = ?; \\ ? & \text{if } j = ?; \\ ? & \text{otherwise;} \end{cases}$$

with the question marks replaced with the correct values.

(d) Let $f(x) = 18$, $\alpha = \beta = 0$ and $N = 2$. Obtain approximations u_1 and u_2 to $u(x_1)$ and $u(x_2)$, respectively, by solving

$$\mathbf{A}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{b}.$$

Solution.

(a) [5 points] If $v \in C^2[0,1]$ and $w \in C^2[0,1]$ then

$$L(v+w) = -(v+w)'' + 9(v+w) = -v'' - w'' + 9v + 9w = -v'' + 9v - w'' + 9w = Lv + Lw$$

and so $L(v+w) = Lv + Lw$ for all $v, w \in C^2[0,1]$. If $v \in C^2[0,1]$ and $\gamma \in \mathbb{R}$ then

$$L(\gamma v) = -(\gamma v)'' + 9(\gamma v) = -\gamma v'' + 9\gamma v = \gamma(-v'' + 9v) = \gamma L v$$

and so $L(\gamma v) = \gamma L v$ for all $v \in C^2[0,1]$ and all $\gamma \in \mathbb{R}$. Consequently, L is a linear operator.

(b) [5 points] For $j = 1, 2, \ldots, N$, using the approximation

$$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

yields that

$$\begin{aligned}(Lu)(x_j) &= -u''(x_j) + 9u(x_j) \approx -\frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + 9u(x_j) \\ &= -\frac{1}{h^2}u(x_{j-1}) + \left(\frac{2}{h^2} + 9\right)u(x_j) - \frac{1}{h^2}u(x_{j+1}).\end{aligned}$$

So,

$$
\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix}
$$

$$
\approx \begin{bmatrix} -\frac{1}{h^2}u(x_0) + \left(\frac{2}{h^2}+9\right)u(x_1) - \frac{1}{h^2}u(x_2) \\ -\frac{1}{h^2}u(x_1) + \left(\frac{2}{h^2}+9\right)u(x_2) - \frac{1}{h^2}u(x_3) \\ \vdots \\ -\frac{1}{h^2}u(x_{N-2}) + \left(\frac{2}{h^2}+9\right)u(x_{N-1}) - \frac{1}{h^2}u(x_N) \\ -\frac{1}{h^2}u(x_{N-1}) + \left(\frac{2}{h^2}+9\right)u(x_N) - \frac{1}{h^2}u(x_{N+1}) \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}
$$

$$
= \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}
$$

where $\mathbf{D} \in \mathbb{R}^{N\times(N+2)}$ is the matrix with entries

$$
D_{jk} = \begin{cases} \frac{2}{h^2}+9 & \text{if } k = j+1; \\ -\frac{1}{h^2} & \text{if } k = j \text{ or } k = j+2; \\ 0 & \text{otherwise.} \end{cases}
$$

(c) [10 points] Since

$$
(Lu)(x) = f(x), \quad 0 < x < 1
$$

we have that

$$
\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix}
$$

for $j = 1, \ldots, N$. Using the approximation obtained in the previous part then yields that

$$\mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix} \approx \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix}$$

Moreover,

$$\begin{aligned} \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix} &= \mathbf{D} \left(\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} + \begin{bmatrix} u(x_0) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ u(x_{N+1}) \end{bmatrix} \right) \\ &= \mathbf{D} \left(\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix} \right) \\ &= \mathbf{D} \begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} + \mathbf{D} \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix} \end{aligned}$$

since $u(x_0) = u(0) = \alpha$ and $u(x_{N+1}) = u(1) = \beta$. Furthermore,

$$
\begin{aligned}
&\mathbf{D}\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix}
-\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2}
\end{bmatrix}
\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix}
\frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\
0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} \\
0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9
\end{bmatrix}
\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix}
\end{aligned}
$$

and so

$$
\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}
$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is the matrix with entries

$$
A_{jk} = \begin{cases} \frac{2}{h^2}+9 & \text{if } k = j; \\ -\frac{1}{h^2} & \text{if } k = j-1 \text{ or } k = j+1; \\ 0 & \text{otherwise.} \end{cases}
$$

and

$$
\mathbf{b} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} - \mathbf{D}\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix}.
$$

Now,

$$
\mathbf{D}\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix}
$$

$$
= \begin{bmatrix}
-\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2}
\end{bmatrix}
\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{\alpha}{h^2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\frac{\beta}{h^2} \end{bmatrix}
$$

and so

$$
\mathbf{b} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} - \begin{bmatrix} -\frac{\alpha}{h^2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\frac{\beta}{h^2} \end{bmatrix} = \begin{bmatrix} f(x_1) + \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) + \frac{\beta}{h^2} \end{bmatrix}.
$$

Hence $\mathbf{b} \in \mathbb{R}^N$ is the vector with entries

$$
b_j = \begin{cases} f(x_1) + \frac{\alpha}{h^2} & \text{if } j = 1; \\ f(x_N) + \frac{\beta}{h^2} & \text{if } j = N; \\ f(x_j) & \text{otherwise.} \end{cases}
$$

(d) [5 points] When $N = 2$, $h = \frac{1}{2+1} = \frac{1}{3}$ and so $h^2 = \frac{1}{3^2} = \frac{1}{9}$ and hence $\frac{1}{h^2} = 9$ and $\frac{2}{h^2} = 18$. Therefore,

$$
\mathbf{A} = \begin{bmatrix} 18+9 & -9 \\ -9 & 18+9 \end{bmatrix} = \begin{bmatrix} 27 & -9 \\ -9 & 27 \end{bmatrix}.
$$

Moreover, when $N = 2$, $f(x) = 18$ and $\alpha = \beta = 0$,

$$
\mathbf{b} = \begin{bmatrix} 18 \\ 18 \end{bmatrix}.
$$

Consequently, we have that

$$\begin{bmatrix} 27 & -9 \\ -9 & 27 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \end{bmatrix}$$

and hence

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{9-1} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 6+2 \\ 2+6 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 8 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

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2. [25 points]

(a) Compute

$$\int_{-1}^{1} x \, dx.$$

(b) Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$\mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to

$$\mathbf{Ac} = \mathbf{g}.$$

(c) Let

$$V_0 = \left\{ w \in C^1[0,1] : \int_0^1 w(x)\, dx = 0 \right\}.$$

Determine whether or not V_0 is a subspace of $C^1[0,1]$.

(d) Let $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} x u(x) v(x)\, dx.$$

Determine whether or not $(\cdot,\cdot)$ is an inner product on $C[-1,1]$.

(e) Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $\phi \in C[a,b]$ be defined by $\phi(x) = 1$ and let the inner product $B(\cdot,\cdot) : C[a,b] \times C[a,b] \to \mathbb{R}$ be defined by

$$B(u,v) = \int_{a}^{b} u(x) v(x)\, dx.$$

Let the linear operator $P_0 : C[a,b] \to C[a,b]$ be defined by

$$P_0 f = \frac{1}{b-a} B(f,\phi).$$

Determine whether or not P_0 is a projection.

Solution.

(a) [1 point] We can compute that

$$\int_{-1}^{1} x\, dx = \left[\frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2}\left(1^2 - (-1)^2\right) = \frac{1}{2}(1-1) = 0.$$

(b) [6 points] Since,

$$\lambda \mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{bmatrix}$$

we have that

$$\det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$

and so

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

when $\lambda = 3$ or $\lambda = 5$. Hence, the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = 3$$

and

$$\lambda_2 = 5.$$

Moreover,

$$(\lambda_1 \mathbf{I} - \mathbf{A}) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ -c_1 - c_2 \end{bmatrix}$$

and so to make this vector zero we need to set $c_2 = -c_1$. Hence, any vector of the form

$$\begin{bmatrix} c_1 \\ -c_1 \end{bmatrix}$$

where c_1 is a nonzero constant is an eigenvector of $\mathbf{A}$ corresoponding to the eigenvalue λ_1. Let us choose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Furthermore,

$$(\lambda_2 \mathbf{I} - \mathbf{A}) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{bmatrix}$$

and so to make this vector zero we need to set $d_2 = d_1$. Hence, any vector of the form

$$\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}$$

where d_1 is a nonzero constant is an eigenvector of $\mathbf{A}$ corresoponding to the eigenvalue λ_2. Let us choose

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\mathbf{A} = \mathbf{A}^T$, $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$ and $\lambda_1 \neq \lambda_2$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Now,

$$\mathbf{g} \cdot \mathbf{v}_1 = 2 - 3 = -1,$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

$$\mathbf{g} \cdot \mathbf{v}_1 = 2 + 3 = 5,$$

and

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 1^2 + 1^2 = 1 + 1 = 2.$$

The spectral method then yields that

$$\begin{aligned}
\mathbf{x} &= \frac{1}{\lambda_1} \frac{\mathbf{g} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{1}{\lambda_2} \frac{\mathbf{g} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\
&= \frac{1}{3} \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5} \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{6} + \frac{3}{6} \\ \frac{1}{6} + \frac{3}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{6} \\ \frac{4}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.
\end{aligned}$$

(c) [10 points] The set V_0 is a subset of $C^1[0,1]$ and the function z defined by $z(x) = 0$ for $x \in [0,1]$ is such that $z \in V_0$ since

$$\int_0^1 z(x)\,dx = \int_0^1 0\,dx = 0.$$

If $f \in V_0$ and $g \in V_0$, then $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = 0$, and so

$$\int_0^1 (f+g)(x)\,dx = \int_0^1 f(x) + g(x)\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = 0 + 0 = 0$$

and hence $f + g \in V_0$ for all $f, g \in V_0$. Also, if $f \in V_0$ and $\alpha \in \mathbb{R}$, then $\int_0^1 f(x)\,dx = 0$, and so

$$\int_0^1 (\alpha f)(x)\,dx = \int_0^1 \alpha f(x)\,dx = \alpha \int_0^1 f(x)\,dx = 0$$

and hence $\alpha f \in V_0$ for all $f \in V_0$ and all $\alpha \in \mathbb{R}$. Consequently, V_0 is a subspace of $C^1[0,1]$.

(d) [3 points] The mapping $(\cdot,\cdot)$ is not an inner product on $C[-1,1]$ since if $w(x) = 1$ for all $x \in [-1,1]$ then $w \in C[-1,1]$ and $w \neq 0$ but

$$(w,w) = \int_{-1}^1 x w(x) w(x)\,dx = \int_{-1}^1 x\,dx = 0$$

and so $(\cdot,\cdot)$ is not positive definite.

(e) [5 points] If $f \in C[a,b]$ then

$$P_0 f = \frac{1}{b-a} B(f,\phi)$$

and so

$$\begin{aligned}
P_0(P_0 f) &= \frac{1}{b-a} B(P_0 f, \phi) \\
&= \frac{1}{b-a} \int_a^b \frac{1}{b-a} B(f,\phi)\,dx \\
&= \frac{1}{b-a} B(f,\phi) \int_a^b \frac{1}{b-a}\,dx \\
&= \frac{1}{b-a} B(f,\phi) \left[\frac{x}{b-a}\right]_a^b \\
&= \frac{1}{b-a} B(f,\phi) \left(\frac{b}{b-a} - \frac{a}{b-a}\right) \\
&= \frac{1}{b-a} B(f,\phi) \frac{b-a}{b-a} \\
&= \frac{1}{b-a} B(f,\phi) \\
&= P_0 f.
\end{aligned}$$

Hence, $P_0(P_0 f) = P_0 f$ for all $f \in C[a,b]$ and so P_0 is a projection.

3. [25 points]

Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L:S\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$S=\{w\in C^2[0,1]: w'(0)=w(1)=0\}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv,w)=(v,Lw)\text{ for all } v,w\in S.$$

Let N be a positive integer and let $f\in C[0,1]$ be defined by

$$f(x)=\begin{cases} 1-2x & \text{if } x\in\left[0,\frac{1}{2}\right];\\ 0 & \text{otherwise.}\end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x)=\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n=1,2,\ldots$. Note that, for $m,n=1,2,\ldots$,

$$(\psi_m,\psi_n)=\begin{cases} 1 & \text{if } m=n;\\ 0 & \text{if } m\neq n.\end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n=1,2,\ldots$.

(b) Compute the best approximation to f from span$\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=f(x),\quad 0<x<1$$

and

$$\tilde{u}'(0)=\tilde{u}(1)=0.$$

(d) What is the best approximation to $\tilde{u}$ from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$?

(e) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

Solution.

(a) [3 points] We can compute that, for $n = 1, 2, \ldots$,

$$\psi_n'(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)\pi \sin\left(\frac{2n-1}{2}\pi x\right).$$

and

$$\psi_n''(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)^2 \pi^2 \cos\left(\frac{2n-1}{2}\pi x\right).$$

and so

$$L\psi_n = -\psi_n'' = \left(\frac{2n-1}{2}\right)^2 \pi^2 \psi_n.$$

Hence,

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \pi^2 = (2n-1)^2 \frac{\pi^2}{4} \text{ for } n = 1, 2, \ldots.$$

(b) [8 points] Since $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$, the best approximation to f from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$f_N = \sum_{n=1}^{N} (f, \psi_n)\psi_n.$$

Now, for $n = 1, 2, \ldots,$

$$\begin{aligned}
& (f, \psi_n) \\
= & \int_0^1 f(x)\psi_n(x)\, dx \\
= & \int_0^{1/2} f(x)\psi_n(x)\, dx + \int_{1/2}^1 f(x)\psi_n(x)\, dx \\
= & \int_0^{1/2} (1-2x)\, \sqrt{2} \cos\left(\frac{2n-1}{2}\pi x\right)\, dx + \int_{1/2}^1 0\, dx \\
= & \sqrt{2} \int_0^{1/2} (1-2x) \cos\left(\frac{2n-1}{2}\pi x\right)\, dx + 0 \\
= & \sqrt{2}\left(\left[(1-2x)\, \frac{2}{(2n-1)\,\pi} \sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} - \int_0^{1/2} (-2)\, \frac{2}{(2n-1)\,\pi} \sin\left(\frac{2n-1}{2}\pi x\right)\, dx\right) \\
= & \sqrt{2}\left(0 - 0 + \frac{4}{(2n-1)\,\pi} \int_0^{1/2} \sin\left(\frac{2n-1}{2}\pi x\right)\, dx\right) \\
= & \sqrt{2}\frac{4}{(2n-1)\,\pi}\left[-\frac{2}{(2n-1)\,\pi} \cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} \\
= & \frac{4\sqrt{2}}{(2n-1)\,\pi}\left(-\frac{2}{(2n-1)\,\pi} \cos\left(\frac{2n-1}{4}\pi\right) - \left(-\frac{2}{(2n-1)\,\pi}\right)\right) \\
= & \frac{8\sqrt{2}}{(2n-1)^2\,\pi^2}\left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
f_N(x) &= \sum_{n=1}^N (f, \psi_n)\psi_n(x) \\
&= \sum_{n=1}^N (f, \psi_n)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right) \\
&= \sum_{n=1}^N \frac{16}{(2n-1)^2\,\pi^2}\left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right).
\end{aligned}$$

(c) [4 points] Now, $\tilde{u}$ is the solution to $L\tilde{u} = f$ and so the spectral method yields the series solution

$$\tilde{u}(x) = \sum_{n=1}^\infty \frac{(f, \psi_n)}{\lambda_n}\psi_n(x) = \sum_{n=1}^\infty \frac{64}{(2n-1)^4\,\pi^4}\left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right).$$

(d) [4 points] The best approximation to $\tilde{u}$ from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$\tilde{u}_N(x) = \sum_{n=1}^N \frac{(f, \psi_n)}{\lambda_n}\psi_n(x) = \sum_{n=1}^N \frac{64}{(2n-1)^4\,\pi^4}\left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right).$$

(e) [6 points] Let $\tilde{u}$ be the solution to $L\tilde{u} = f$ and let $w \in C^2[0,1]$ be such that

$$-w''(x) = 0, \quad 0 < x < 1$$

and

$$w'(0) = w(1) = 1.$$

Then $u(x) = w(x) + \tilde{u}(x)$ will be such that

$$-u''(x) = -w''(x) - \tilde{u}''(x) = 0 + f(x) = f(x);$$

$$u'(0) = w'(0) + \tilde{u}'(0) = 1 + 0 = 1;$$

and

$$u(1) = w(1) + \tilde{u}(1) = 1 + 0 = 1.$$

Now, the general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(0) = 1$ when $A = 1$. Hence, $w(x) = x + B$ and so $w(1) = 1$ when $B = 0$. Consequently,

$$w(x) = x$$

and so

$$u(x) = x + \tilde{u}(x).$$

We can then use the series solution to $L\tilde{u} = f$ that we obtained in part (c) to obtain the series solution

$$u(x) = x + \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right)$$

to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1;$$

$$u'(0) = u(1) = 1.$$

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4. [25 points]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $f_1 \in C[-1,1]$, and $f_2 \in C[-1,1]$ be defined by

$$\phi_1(x) = \frac{1}{\sqrt{2}},$$

$$\phi_2(x) = \frac{\sqrt{3}}{\sqrt{2}} x,$$

$$f_1(x) = \sin(\pi x),$$

and

$$f_2(x) = \cos(\pi x),$$

for all $x \in [-1, 1]$. Note that $\{\phi_1, \phi_2\}$ is linearly independent. Let the inner product $(\cdot, \cdot) : C[-1, 1] \times C[-1, 1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_{-1}^{1} u(x)v(x)\, dx$$

and let the norm $\|\cdot\| : C[-1, 1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

Note that $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$. Also, let $\psi_1 \in C[0, 1]$, $\psi_2 \in C[0, 1]$, $g_1 \in C[0, 1]$, and $g_2 \in C[0, 1]$ be defined by

$$\psi_1(x) = \frac{1}{\sqrt{2}},$$

$$\psi_2(x) = \frac{\sqrt{3}}{\sqrt{2}} x,$$

$$g_1(x) = \sin(\pi x),$$

and

$$g_2(x) = \cos(\pi x),$$

for all $x \in [0, 1]$. Note that $\{\psi_1, \psi_2\}$ is linearly independent. Let the inner product $B(\cdot, \cdot) : C[0, 1] \times C[0, 1] \to \mathbb{R}$ be defined by

$$B(u, v) = \int_{0}^{1} u(x)v(x)\, dx$$

and let the norm $\|\cdot\|_B : C[0, 1] \to \mathbb{R}$ be defined by

$$\|u\|_B = \sqrt{B(u, u)}.$$

Note that

$$(f_1, \phi_2) = \frac{\sqrt{6}}{\pi},$$

$$B(g_1, \psi_1) = \frac{\sqrt{2}}{\pi},$$

$$B(g_1, \psi_2) = \frac{\sqrt{6}}{2\pi},$$

$$B(g_2, \psi_2) = -\frac{\sqrt{6}}{\pi^2},$$

and

$$(f_1, \phi_1) = (f_2, \phi_1) = (f_2, \phi_2) = B(g_2, \psi_1) = 0.$$

(a) Construct the best approximation to f_1 from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(b) Construct the best approximation to f_2 from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) Construct the best approximation to g_1 from span$\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.

(d) Construct the best approximation to g_2 from span$\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.

Solution.

(a) [5 points] Since $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$, the best approximation to f_1 from span $\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$\tilde{f}_1(x) = (f_1, \phi_1)\phi_1(x) + (f_1, \phi_2)\phi_2(x) = 0 + \frac{\sqrt{6}}{\pi}\frac{\sqrt{3}}{\sqrt{2}}x = \frac{\sqrt{2}\sqrt{3}}{\pi}\frac{\sqrt{3}}{\sqrt{2}}x = \frac{\sqrt{9}}{\pi}x = \frac{3}{\pi}x.$$

(b) [4 points] Since $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$, the best approximation to f_2 from span $\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$\tilde{f}_2(x) = (f_2, \phi_1)\phi_1(x) + (f_2, \phi_2)\phi_2(x) = 0 + 0 = 0.$$

(c) [10 points] Now,

$$B(\psi_1, \psi_2) = \int_0^1 \psi_1(x)\psi_2(x)\,dx = \int_0^1 \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{2}}x\,dx = \frac{\sqrt{3}}{2}\int_0^1 x\,dx = \frac{\sqrt{3}}{2}\left[\frac{1}{2}x^2\right]_0^1 = \frac{\sqrt{3}}{2}\frac{1}{2} = \frac{\sqrt{3}}{4}$$

and so $\{\psi_1, \psi_2\}$ is not orthogonal with respect to the inner product $B(\cdot, \cdot)$. Consequently, the best approximation to g_1 from span $\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_1(x) = c_1\psi_1(x) + c_2\psi_2(x)$$

where the coefficients $c_1, c_2 \in \mathbb{R}$ are such that

$$\mathbf{G}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} B(g_1, \psi_1) \\ B(g_1, \psi_2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{6}}{2\pi} \end{bmatrix}$$

where

$$\mathbf{G} = \begin{bmatrix} B(\psi_1, \psi_1) & B(\psi_1, \psi_2) \\ B(\psi_1, \psi_2) & B(\psi_2, \psi_2) \end{bmatrix}.$$

Now,

$$B(\psi_1, \psi_1) = \int_0^1 \psi_1(x)\psi_1(x)\,dx = \int_0^1 \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\,dx = \frac{1}{2}\int_0^1 1\,dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

and

$$B(\psi_2, \psi_2) = \int_0^1 \psi_2(x)\psi_2(x)\,dx = \int_0^1 \frac{\sqrt{3}}{\sqrt{2}}x\frac{\sqrt{3}}{\sqrt{2}}x\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{3}{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{3}{2}\frac{1}{3} = \frac{1}{2}.$$

Hence,

$$\mathbf{G} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}$$

and so

$$\mathbf{G}^{-1} = 16 \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 8 \end{bmatrix}$$

since

$$\frac{1}{2}\frac{1}{2} - \frac{\sqrt{3}}{4}\frac{\sqrt{3}}{4} = \frac{1}{4} - \frac{3}{16} = \frac{4}{16} - \frac{3}{16} = \frac{1}{16}.$$

Therefore

$$\begin{aligned}
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \mathbf{G}^{-1} \begin{bmatrix} \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{6}}{2\pi} \end{bmatrix} \\
&= \begin{bmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 8 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{6}}{2\pi} \end{bmatrix} \\
&= \begin{bmatrix} 8\frac{\sqrt{2}}{\pi} - 4\sqrt{3}\frac{\sqrt{6}}{2\pi} \\ -4\sqrt{3}\frac{\sqrt{2}}{\pi} + 8\frac{\sqrt{6}}{2\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8\sqrt{2}}{\pi} - \frac{2\sqrt{18}}{\pi} \\ \frac{4\sqrt{6}}{\pi} + \frac{4\sqrt{6}}{\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8\sqrt{2}}{\pi} - \frac{2\sqrt{2}\sqrt{9}}{\pi} \\ \frac{4\sqrt{6}}{\pi} + \frac{4\sqrt{6}}{\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8\sqrt{2}}{\pi} - \frac{6\sqrt{2}}{\pi} \\ \frac{4\sqrt{6}}{\pi} + \frac{4\sqrt{6}}{\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\sqrt{2}}{\pi} \\ 0 \end{bmatrix}.
\end{aligned}$$

Consequently, the best approximation to g_1 from span$\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_1(x) = c_1\psi_1(x) + c_2\psi_2(x) = \frac{2\sqrt{2}}{\pi}\frac{1}{\sqrt{2}} + 0 = \frac{2}{\pi}.$$

(d) [6 points] Since $B(\psi_1, \psi_2) \neq 0$, the best approximation to g_2 from span$\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_2(x) = d_1\psi_1(x) + d_2\psi_2(x)$$

where the coefficients $d_1, d_2 \in \mathbb{R}$ are such that

$$\mathbf{G} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} B(g_2, \psi_1) \\ B(g_2, \psi_2) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{\pi^2} \end{bmatrix}$$

where

$$\mathbf{G} = \begin{bmatrix} B(\psi_1, \psi_1) & B(\psi_1, \psi_2) \\ B(\psi_1, \psi_2) & B(\psi_2, \psi_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}.$$

Therefore

$$\begin{aligned} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} &= \mathbf{G}^{-1} \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\ &= \begin{bmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\ &= \begin{bmatrix} 4\sqrt{3}\frac{\sqrt{6}}{\pi^2} \\ -8\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\ &= \begin{bmatrix} 4\frac{\sqrt{18}}{\pi^2} \\ -8\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\ &= \begin{bmatrix} 4\frac{\sqrt{2}\sqrt{9}}{\pi^2} \\ -8\frac{\sqrt{2}\sqrt{3}}{\pi^2} \end{bmatrix} \\ &= \begin{bmatrix} 12\frac{\sqrt{2}}{\pi^2} \\ -8\frac{\sqrt{2}\sqrt{3}}{\pi^2} \end{bmatrix} \end{aligned}$$

Consequently, the best approximation to g_2 from span $\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_2(x) = d_1\psi_1(x) + d_2\psi_2(x) = 12\frac{\sqrt{2}}{\pi^2}\frac{1}{\sqrt{2}} - 8\frac{\sqrt{2}\sqrt{3}}{\pi^2}\frac{\sqrt{3}}{\sqrt{2}}x = \frac{12}{\pi^2} - \frac{24}{\pi^2}x = \frac{12}{\pi^2}(1 - 2x).$$
