CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Examination 1

Instructions:

1. Time limit: **3 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You are allowed one cheat sheet to refer to during the exam.
   You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions throughly (but succintly!) and justify all your answers.
   Show your work for partial credit.
5. Print your name on the line below:

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6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points: (a) = 5, (b),(c) = 10] Consider the steady heat equation with *periodic* boundary conditions
$$-\frac{\partial^2 u}{\partial x^2} = f(x), \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$
Define $C_P^2[0,1]$ as the space of C^2 continuous functions with periodic boundary conditions
$$C_P^2[0,1] = \left\{ u \in C^2[0,1], \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}. \right\}$$
and define the operator $L : C_P^2[0,1] \to C[0,1]$ such that $Lu = -\frac{\partial^2 u}{\partial x^2}$. Note that this operator has a zero eigenvalue with zero eigenfunction $\phi_0(x) = 1$.

   (a) Define ϕ_j, ψ_j
$$\phi_j(x) = \sin(j\pi x), \quad \psi_j(x) = \cos(j\pi x), \quad j = 1, 2, \ldots.$$
Verify that ϕ_j, ψ_j, are eigenfunctions with eigenvalues $\lambda_j = (j\pi)^2$, and that linear combinations $f(x)$ of ϕ_j, ψ_j, and ψ_0
$$f(x) = d_0 + \sum_{j=1}^{\infty} \left(c_j \sin(j\pi x) + d_j \cos(j\pi x)\right)$$
satisfy periodic boundary conditions.

   (b) Show for the operator $A : C_P^2[0,1] \to C[0,1]$, defined as
$$Au = u + Lu$$
that ϕ_j, ψ_j are eigenfunctions of A with eigenvalues $\mu_j = 1 + (j\pi)^2$, and that ϕ_0 is an eigenfunction with eigenvalue $\mu_0 = 1$. Explain briefly why we may solve
$$u(x) - \frac{\partial^2 u(x)}{\partial x^2} = f(x)$$
using the spectral method, but not the steady state heat equation $-\frac{\partial^2 u}{\partial x^2} = f(x)$.

   (c) Assume that $f(x)$ and $u(x)$ may be represented as
$$f(x) = d_0 + \sum_{j=1}^{\infty} \left(c_j \sin(j\pi x) + d_j \cos(j\pi x)\right),$$
$$u(x) = \beta_0 + \sum_{j=1}^{\infty} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right).$$
Write down the spectral method solution for the equation
$$u(x) - \frac{\partial^2 u(x)}{\partial x^2} = f(x), \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$
In other words, give expressions for the coefficients $\alpha_j, \beta_j, \beta_0$ that depend on c_j, d_j, d_0, and λ_j or μ_j. *You may use without proof that ϕ_j, ψ_j, and ψ_0 are linearly independent, or that they are orthogonal under the inner product $(f,g) = \int_0^1 f(x)g(x)dx$.*

   (d) Use the spectral method to solve for $u(x)$ such that
$$u(x) - \frac{\partial^2 u(x)}{\partial x^2} = 1 + \cos(\pi x), \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$

2. [25 points:] In this problem, we consider the finite element method for the equation

$$-\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x).$$

This models the steady state distribution of temperature in a bar, where $k(x)$ is the diffusivity of the bar at the point x. Diffusivity must be positive for the equation to be physically realistic; however, if $k(x)$ is not positive, the finite element method may run into issues as well.

It may be helpful to use the fact that the determinant of a 2×2 matrix is

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

(a) If $k(x) > 0$ for $0 < x < 1$, the weak form

$$a(u,v) = \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

defines an inner product for $C_D^2[0,1]$. Explain why, if $k(x) = 0$ over some interval $[a,b] \subset [0,1]$, $a(u,v)$ may not be positive definite (and hence not an inner product).

(b) Assume that $N = 2$, such that there are two hat functions $\phi_1(x), \phi_2(x)$ centered around points

$$x_1 = 1/3, \quad x_2 = 2/3.$$

Give a formula for the entries of the 2×2 finite element matrix $K_{ij} = a(\phi_j, \phi_i)$ for $k(x)$

$$k(x) = x.$$

where ϕ_j are the hat functions

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & x_{j-1} < x \leq x_j \\ \frac{x_{j+1} - x}{h} & x_j < x \leq x_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $k(x)$ be the function

$$k(x) = \begin{cases} 0 & x \leq 2/3 \\ 1 & x > 2/3. \end{cases}$$

Compute the finite element stiffness matrix for $N = 2$. Verify that $A_{11} = A_{21} = 0$. Explain what complication arises if one attempts to solve this system.

3. [25 points:] In the homeworks, and in class, we used the spectral method to solve several problems of the form $Lu = f$ with boundary conditions. The spectral method has conditions that must be satisfied before we can apply it. In this problem we will explore the impact of boundary conditions on several theoretical facets of the spectral method.

   Consider the boundary value problem

$$\begin{cases} \frac{-\partial^2}{\partial_x^2} u = f \\ u(1) = a_{11} u(0) + a_{12} \frac{\partial u}{\partial_x}(0) \\ \frac{\partial u}{\partial x}(1) = a_{21} u(0) + a_{22} \frac{\partial u}{\partial_x}(0) \end{cases} \tag{1}$$

   Define a matrix A containing the boundary value coefficients by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

   Recall: the determinant of a 2×2 matrix is $det(A) = a_{11}a_{22} - a_{12}a_{21}$.

   (a) Define a space of functions V_A to be those functions in $C^2[0, 1]$ satisfying the boundary conditions of the boundary value problem of equation (1). Show that V_A is a vector space by showing that $0 \in V_A$ and $\alpha f + g \in V_A$ whenever α is a real number and f, g are in V_A.

   (b) Consider the usual inner product $(f, g) = \int_0^1 f g \, dx$ on V and define $L = -\frac{\partial^2}{\partial_x^2}$. Whether or not L is symmetric can be shown to depend on the value of the determinant of the matrix A. What should $det(A)$ be in order for L to be symmetric on V_A? (your answer should be a number, show all work).

   (c) Consider the case where $a_{11} = a_{22} = 1$ and $a_{12} = a_{22} = 0$. Is zero an eigenvalue of L in V_A?

   (d) Consider the case where $a_{11} = 1$, $a_{22} = -1$, $a_{12} = 1$ and $a_{21} = -2$. What is the null space of L in V_A?

4. [25 points:]

In the homeworks and in class we used the hat function basis $\{\phi_i(x)\}_{i=1}^N$ defined on an evenly spaced mesh $\{x_0, x_1, \dots x_N, x_{N+1}\}$ of the interval $[0,1]$ where $x_0 = 0$, $x_{N+1} = 1$ and $h = x_n - x_{n-1}$ is a constant value to boundary value problems like equation (2) using the finite element method.

$$\begin{cases} -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f \\ u(0) = 0, u(1) = 0 \end{cases} \tag{2}$$

Suppose that the number of internal mesh points, N, on the interval $[0,1]$ is odd. Since the mesh is assumed to be evenly spaced the middle point $x_{(N+1)/2} = 1/2$. Define a basis $\{\psi_i\}_{i=1}^N$ by $\psi_i = \phi_i$ for $i \neq \frac{N+1}{2}$ For $i = \frac{N+1}{2}$ we define ψ_i to be the 'super-hat' function of formula (3). Define $V_h = \text{span}\{\psi_0, \dots, \psi_N\}$

$$\psi_{(N+1)/2} = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ -2x + 2 & x \in [\frac{1}{2}, 1] \end{cases} \tag{3}$$

(a) Show that the set of functions $\{\psi_0, \dots, \psi_N\}$ are linearly independent so that they define a basis of V_h

(b) Write the discrete weak problem, also called the variational formulation, for the finite element method corresponding to solving the problem (2) in V_h.

(c) If the solution $u_h(x) \in V_h$ is written $u_h = \Sigma_{i=1}^N \alpha_i \psi_i(x)$ then the discrete weak problem is a matrix problem $A\alpha = \mathbf{f}$ where $A_{ij} = a(\psi_j, \psi_i)$ and $\mathbf{f}_i = \int_0^1 f\psi_i$. Supposing that $k(x) = f(x) = 1$ determine the general form for the entries for the matrix A and vector $\mathbf{f}$.

(d) Explain in one sentence the effect of the large support set of the super-hat function on the matrix A. (A 'support set' of a function $f(x)$ can be thought of as the set of points where f(x) is nonzero)

5. [25 points:] Consider the *Euler Bernoulli beam equation*,

$$(k(x)u''(x))'' = f(x), \qquad 0 < x < 1,$$

Here $k(x)$ is a positive-valued function that describes the material properties of the beam.

(a) Verify that, for $k(x) = 1$, the functions

$$\sin(\mu_j x), \quad \cos(\mu_j x), \quad \sinh(\mu_j x), \quad \cosh(\mu_j x)$$

are all eigenfunctions the operator $Lu = u''''$ with eigenvalues λ_j, where $\mu_j = \lambda_j^{1/4}$ and where hyperbolic sine/cosine are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Hint: the derivatives of sinh *and* cosh *may be written in terms of* sinh *and* cosh.

(b) Consider boundary conditions describing a beam that is *clamped* at both ends:

$$u(0) = u(1) = 0; \qquad u'(0) = u'(1) = 0.$$

With these boundary conditions, the eigenvalues and eigenvectors of this operator are diffcult to compute, even if $k(x) = 1$. As a result, we will consider the finite element approximation of this problem.

Derive the weak form of the beam equation with the above boundary conditions, i.e., derive the weak problem

$$a(u, v) = (f, v); \qquad \text{for all } v \in V = C_D^4[0, 1],$$

where

$$C_D^4[0, 1] = \{u \in C^4[0, 1] : u(0) = u(1) = u'(0) = u'(1) = 0\}.$$

Specify the bilinear form $a(u, v)$, and show that it is an inner product on $C_D^4[0, 1]$

Note: for the problem $-(ku')' = f$, we do not explicitly impose Neumann boundary conditions, they follow 'naturally'. For the beam equation, we must impose all four boundary conditions on the space of test functions, $V = C_D^4[0, 1]$.

(c) Suppose that $V_n = \mathit{span}\{\phi_1, \cdots, \phi_n\}$ is an n-dimensional subspace of $C_D^4[0, 1]$. (Do not assume a particular form for the functions $\phi_1, \cdots, \phi_n$ at this point.)

Show how the Galerkin problem

$$a(u_n, v) = (f, v), \qquad \text{for all } v \in V_n$$

leads to the linear system $Ku = f$. Be sure to specify the entries of K, u, and f .

(d) Now suppose we take for $\phi_1, \cdots, \phi_n$ the standard piecewise linear 'hat' functions used, for example, in Problem 2. Are these functions suitable for this problem? If so, describe the location of the nonzero entries of the matrix K. If not, roughly describe a better choice for the functions $\phi_1, \cdots, \phi_n$ and the explain which entries of K are nonzero for that choice.