CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 5 · Solutions

Posted Wednesday 24 September, 2014. Due 5pm Wednesday 1 October, 2014.

Please write your name and ***residential college*** *on your homework.*

1. [25 points: 5 points each]

Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

(a) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V}=C^1[0,1]$.

(b) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

(c) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V}=C[0,1]$.

(d) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 (u(x)+v(x))\,dx$ where $\mathcal{V}=C[0,1]$.

(e) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V}=C[-1,1]$.

Solution.

(a) [5 points] *This mapping is not an inner product*: it is not symmetric and it is not positive definite. The mapping is not symmetric. For example, if $u(x)=1$ and $v(x)=x$, then

$$(u,v)=\int_0^1 u(x)v'(x)\,dx=\int_0^1 1\,dx=1,$$

yet

$$(v,u)=\int_0^1 v(x)u'(x)\,dx=\int_0^1 0\,dx=0.$$

The mapping is also not positive definite. For example, if $u(x)=1$, then $(u,u)=0$ and if $u(x)=1-x$, then

$$(u,u)=\int_0^1 (1-x)(-1)\,dx=-1/2.$$

For what it is worth, we note that the mapping is linear in the first argument since

$$(\alpha u+\beta v,w)=\alpha\int_0^1 u(x)w'(x)\,dx+\beta\int_0^1 v(x)w'(x)\,dx=\alpha(u,w)+\beta(v,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$. It is also linear in the second argument since

$$(u,\alpha v+\beta w)=\alpha\int_0^1 u(x)v'(x)\,dx+\beta\int_0^1 u(x)w'(x)\,dx=\alpha(u,v)+\beta(u,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$.

(b) [5 points] *This mapping is not an inner product*: it is not linear in the first argument.
If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 |\alpha u(x) + \beta v(x)||w(x)|\, dx$$

and

$$\alpha(u,w) + \beta(v,w) = \alpha \int_0^1 |u(x)||w(x)|\, dx + \beta \int_0^1 |v(x)||w(x)|\, dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = -1$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 |-1||1|\, dx = \int_0^1 1\, dx = 1$$

but

$$\alpha(u,w) + \beta(v,w) = -\int_0^1 |1||1|\, dx = -\int_0^1 1\, dx = -1$$

and so the mapping is not linear in the first argument.
The mapping is symmetric, as

$$(u,v) = \int_0^1 |u(x)||v(x)|\, dx = \int_0^1 |v(x)||u(x)|\, dx = (v,u)$$

for all $u, v \in C[0,1]$.
Moreover, the mapping is positive definite as for all $u \in C[0,1]$

$$(u,u) = \int_0^1 |u(x)|^2\, dx$$

is the integral of a nonnegative function, and hence is nonnegative and $(u,u) = 0$ only if $u = 0$.

(c) [5 points] *This mapping is an inner product.*
The mapping is symmetric, as

$$(u,v) = \int_0^1 u(x)v(x)e^{-x}\, dx = \int_0^1 v(x)u(x)e^{-x}\, dx = (v,u)$$

for all $u, v \in C[0,1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x}\, dx \\ &= \alpha \int_0^1 u(x)w(x)e^{-x}\, dx + \beta \int_0^1 v(x)w(x)e^{-x}\, dx \\ &= \alpha(u,w) + \beta(v,w)\end{aligned}$$

for all $u, v, w \in C[0,1]$ and all $\alpha, \beta \in \mathbb{R}$.
The function e^{-x} is positive valued for all $x \in [0,1]$, so we have that

$$(u,u) = \int_0^1 (u(x))^2 e^{-x} dx$$

is the integral of a nonnegative function, and hence is also nonnegative. If $(u,u) = 0$ then $(u(x))^2 e^{-x} = 0$ for all $x \in [0,1]$ and, since $e^{-x} > 0$ for all $x \in [0,1]$, this means that $u(x) = 0$ for all $x \in [0,1]$, i.e., $u = 0$. Hence, the mapping is positive definite.

(d) [5 points] *This mapping is not an inner product*: it is not linear in the first argument and it is not positive definite.

If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x) + w(x))\, dx$$

and

$$\alpha(u, w) + \beta(v, w) = \alpha \int_0^1 (u(x) + w(x))\, dx + \beta \int_0^1 (v(x) + w(x))\, dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = 2$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 (2+1)\, dx = \int_0^1 3\, dx = 3$$

but

$$\alpha(u, w) + \beta(v, w) = 2\int_0^1 (1+1)\, dx = 2\int_0^1 2\, dx = 4$$

and so $(\cdot, \cdot)$ is not linear in the first argument.

The mapping $(\cdot, \cdot)$ is also not positive definite. For example, if $u(x) = -1$, then

$$(u, u) = \int_0^1 (u(x) + u(x))\, dx = \int_0^1 -2\, dx = -2 < 0.$$

The mapping is symmetric, as

$$(u, v) = \int_0^1 (u(x) + v(x))\, dx = \int_0^1 (v(x) + u(x))\, dx = (v, u)$$

for all $u, v \in C[0,1]$.

(e) [5 points] *This mapping is not an inner product*: it is not positive definite.

If $w(x) = 1$ for all $x \in [-1, 1]$ then $w \in C[-1,1]$ and $w \neq 0$ but

$$(w, w) = \int_{-1}^1 x w(x) w(x)\, dx = \int_{-1}^1 x\, dx = \left[\frac{1}{2}x^2\right]_{-1}^1 = \frac{1}{2}\left(1^2 - (-1)^2\right) = \frac{1}{2}(1-1) = 0$$

and so $(\cdot, \cdot)$ is not positive definite.

The mapping is symmetric, as

$$(u, v) = \int_{-1}^1 x u(x) v(x)\, dx = \int_{-1}^1 x v(x) u(x)\, dx = (v, u)$$

for all $u, v \in C[-1,1]$.

The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_{-1}^1 x(\alpha u(x) + \beta v(x)) w(x)\, dx \\ &= \alpha \int_{-1}^1 x u(x) w(x)\, dx + \beta \int_{-1}^1 x v(x) w(x)\, dx \\ &= \alpha(u, w) + \beta(v, w)\end{aligned}$$

for all $u, v, w \in C[-1,1]$ and all $\alpha, \beta \in \mathbb{R}$.

2. [24 points: 6 points each]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $\phi_3 \in C[-1,1]$, and $f \in C[-1,1]$ be defined by

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = 3x^2 - 1,$$

and

$$f(x) = e^x,$$

for all $x \in [-1,1]$. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\| : C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2, \phi_3\}$ is orthogonal with respect to the inner product $(\cdot,\cdot)$, which is defined on $[-1,1]$.

(a) By hand, construct the best approximation f_1 to f from $\operatorname{span}\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) By hand, construct the best approximation f_2 to f from $\operatorname{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) By hand, construct the best approximation f_3 to f from $\operatorname{span}\{\phi_1, \phi_2, \phi_3\}$ with respect to $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) [4 points] The best approximation to $f(x) = e^x$ from $\operatorname{span}\{\phi_1\}$ with respect to the norm $\|\cdot\|$ is

$$f_1(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x).$$

We compute

$$(\phi_1,\phi_1) = \int_{-1}^{1} 1^2\,dx = [x]_{-1}^{1} = 1 - (-1) = 2$$

and

$$(f,\phi_1) = \int_{-1}^{1} e^x\,dx = [e^x]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

and hence

$$f_1(x) = \frac{1}{2}\left(e - \frac{1}{e}\right).$$

(b) [7 points] Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot,\cdot)$, i.e., $(\phi_1,\phi_2)=0$, the best approximation to $f(x)=e^x$ from $\operatorname{span}\{\phi_1,\phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$f_2(x)=\frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)=f_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x).$$

Noting that

$$(\phi_2,\phi_2)=\int_{-1}^{1}x^2\,dx=\left[\frac{x^3}{3}\right]_{-1}^{1}=\frac{1}{3}-\left(-\frac{-1}{3}\right)=\frac{1}{3}-\frac{1}{3}=\frac{2}{3}$$

and

$$(f,\phi_2)=\int_{-1}^{1}xe^x\,dx=[xe^x]_{-1}^{1}-\int_{-1}^{1}e^x\,dx=e^1-\left(-e^{-1}\right)-(f,\phi_1)=e+\frac{1}{e}-e+\frac{1}{e}=\frac{2}{e}$$

we can compute that

$$f_2(x)=f_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)=\frac{1}{2}\left(e-\frac{1}{e}\right)+\frac{3}{e}x.$$

(c) [7 points] Since,

$$(\phi_1,\phi_2)=(\phi_1,\phi_3)=(\phi_2,\phi_3)=0,$$

the best approximation to $f(x)=e^x$ from $\operatorname{span}\{\phi_1,\phi_2,\phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$f_3(x)=\frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)+\frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x)=f_2(x)+\frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x).$$

Toward this end, compute

$$\begin{aligned}
(\phi_3,\phi_3) &= \int_{-1}^{1}(3x^2-1)^2\,dx\\
&= \int_{-1}^{1}9x^4-6x^2+1\,dx\\
&= \int_{-1}^{1}9x^4\,dx-6(\phi_2,\phi_2)+(\phi_1,\phi_1)\\
&= \left[\frac{9x^5}{5}\right]_{-1}^{1}-6\frac{2}{3}+2\\
&= \frac{9}{5}-\left(-\frac{9}{5}\right)-\frac{12}{3}+2\\
&= \frac{18}{5}-\frac{12}{3}+2\\
&= \frac{54}{15}-\frac{60}{15}+\frac{30}{15}\\
&= \frac{24}{15}\\
&= \frac{8}{5}
\end{aligned}$$

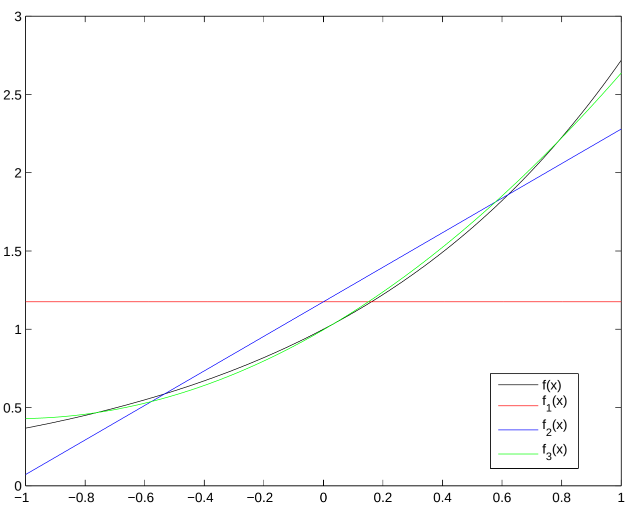
and

$$\begin{aligned}
(f,\phi_3) &= \int_{-1}^{1}(3x^2-1)e^x\,dx \\
&= \int_{-1}^{1} 3x^2e^x\,dx - (f,\phi_1) \\
&= \left[3x^2e^x\right]_{-1}^{1} - \int_{-1}^{1} 6xe^x\,dx - \left(e-\frac{1}{e}\right) \\
&= 3e^1 - 3e^{-1} - 6(f,\phi_2) - \left(e-\frac{1}{e}\right) \\
&= 2e - \frac{2}{e} - \frac{12}{e} \\
&= 2e - \frac{14}{e}
\end{aligned}$$

thus giving

$$f_3(x) = f_2(x) + \frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x) = \frac{1}{2}\left(e-\frac{1}{e}\right) + \frac{3}{e}x + \frac{5}{4}\left(e-\frac{7}{e}\right)(3x^2-1).$$

(d) [7 points] The following plot compares the best approximations to $f(x)$.



The code use to produce it is below.

```
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=exp(x);
f1=(exp(1)-exp(-1))/2+x-x;
f2=f1+3*exp(-1)*x;
f3=f2+5*(exp(1)-7*exp(-1))*(3*x.^2-1)/4;
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
```

```
plot(x,f3,'-g')
xlabel('x')
legend('f(x)','f_1(x)','f_2(x)','f_3(x)','location','best')
saveas(figure(1),'hw16d.eps','epsc')
```

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3. [27 points: 8 points for (a), (b), 11 points for (c)]

Let V be an inner product space (i.e. V a vector space with an inner product). Suppose $\{v_1, v_2, v_3\}$ is a basis for V, and we would like to construct a is possible to construct a new *orthogonal* basis $\{\phi_1, \phi_2, \phi_3\}$ through the following procedure:

$$\begin{aligned}
\phi_1 &= v_1 \\
\phi_2 &= v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1 \\
\phi_3 &= v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2 \\
&\vdots \\
\phi_k &= v_k - \sum_{i=1}^{k-1} \frac{(\phi_i, v_k)}{(\phi_i, \phi_i)}\phi_i
\end{aligned}$$

This is called the *Gram-Schmidt* procedure.

(a) We know that nonzero vectors $u_1, u_2, ..., u_k \in V$ form an orthogonal set if they are orthogonal to each other: i.e. if

$$(u_i, u_j) = 0, \quad i \neq j.$$

Show that ϕ_1, ϕ_2, ϕ_3 form an orthogonal set, i.e. $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$.

(b) Show that if we have an orthogonal set of vectors $\phi_1, \ldots, \phi_k$, then $\phi_1, \ldots, \phi_k$ are linearly independent as well, i.e.

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

is only true if $\alpha_1, \ldots, \alpha_k = 0$.

(c) Since we can define an inner product $(\cdot, \cdot)$ on the function space $C[-1, 1]$ as

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx,$$

we can also use the Gram-Schmidt procedure to create orthogonal sets of *functions*. Using the Gram-Schmidt procedure above, compute the orthogonal vectors $\{\phi_1, \phi_2, \phi_3\}$ given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$.

Solution.

(a) We are going to show that $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$. To check that these formulas yield an orthogonal sequence, first compute (ϕ_1, ϕ_2) by substituting the above formula for ϕ_2

$$\begin{aligned}
(\phi_1, \phi_2) &= (\phi_1, v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1) \\
&= (\phi_1, v_2) - (\phi_1, \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1) \\
&= (\phi_1, v_2) - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}(\phi_1, \phi_1) \\
&= (\phi_1, v_2) - (\phi_1, v_2) \\
&= 0.
\end{aligned}$$

Then use the fact that $(\phi_1, \phi_2) = 0$, to compute (ϕ_1, ϕ_3). By substituting again the formula for ϕ_3

$$\begin{aligned}
(\phi_1, \phi_3) &= (\phi_1, v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_1, v_3) - (\phi_1, \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1) - (\phi_1, \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_1, v_3) - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}(\phi_1, \phi_1) - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\underbrace{(\phi_1, \phi_2)}_{=0} \\
&= (\phi_1, v_3) - (\phi_1, v_3) \\
&= 0
\end{aligned}$$

Similarly, using the symmetry property of inner product $(\phi_i, \phi_j) = (\phi_j, \phi_i)$ for all i, j. We can show $(\phi_2, \phi_3) = 0$.

$$\begin{aligned}
(\phi_2, \phi_3) &= (\phi_2, v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_2, v_3) - (\phi_2, \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1) - (\phi_2, \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_2, v_3) - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\underbrace{(\phi_2, \phi_1)}_{=0} - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}(\phi_2, \phi_2) \\
&= (\phi_2, v_3) - (\phi_2, v_3) \\
&= 0.
\end{aligned}$$

By symmetry we can conclude that $(\phi_2, \phi_3) = (\phi_3, \phi_2) = 0$ and $(\phi_1, \phi_3) = (\phi_3, \phi_1) = 0$. This completes the proof.

(b) Consider a linear relationship

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

which can be written

$$\alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_k\phi_k = 0.$$

If $1 \leq i \leq k$ then taking the inner product of ϕ_i with both sides of the equation and using the properties of inner product (*Definition 3.32, page 58*),

$$\begin{aligned}
(\phi_i, \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_k\phi_k) &= (\phi_i, 0) \\
(\phi_i, \alpha_1\phi_1) + (\phi_i, \alpha_2\phi_2) + \cdots + (\phi_i, \alpha_k\phi_k) &= 0 \\
\alpha_1(\phi_i, \phi_1) + \alpha_2(\phi_i, \phi_2) + \cdots + \alpha_k(\phi_i, \phi_k) &= 0 \\
\alpha_i(\phi_i, \phi_i) &= 0
\end{aligned}$$

shows, since ϕ_i is nonzero, that α_i for $i = 1, \cdots, k$ is zero.

(c) We want to construct the new orthogonal bases for V by *Gram-Schmidt* procedure given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$. Following the procedure we set

$$\phi_1 \;=\; v_1 = 1$$

and

$$\phi_2 \;=\; v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1.$$

We compute

$$(\phi_1, v_2) = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

and

$$(\phi_1, \phi_1) = \int_{-1}^{1} 1\,dx = 2.$$

Now we can compute

$$\phi_2 \;=\; x - \frac{0}{2}(1) = x.$$

Finally for ϕ_3,

$$\phi_3 \;=\; v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2$$

$$(\phi_1, v_3) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

and

$$(\phi_2, v_3) = \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

and

$$(\phi_2, \phi_2) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

Substituting these inner products into the equation for ϕ_3, we get

$$\phi_3 \;=\; x^2 - \frac{(2/3)}{2}(1) - \frac{0}{(2/3)}(x) = x^2 - \frac{1}{3}.$$

This yields $\{\phi_1, \phi_2, \phi_3\} = \{1, x, x^2 - \frac{1}{3}\}$ as desired.
