CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 39

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

39. [25 points]

Let

$$f(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

In this question we will consider the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \quad t \ge 0,$$

and

$$u(1,t) = 1, \quad t \ge 0,$$

and initial condition

$$u(x,0) = x^3, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

The operator L has eigenvalues $\lambda_n = n^2\pi^2$ with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \psi_m(x)\psi_n(x)\,dx = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \ne n. \end{cases}$$

(a) Let $w(x)$ be such that

$$w''(x) = 0,$$

$$w(0) = 0$$

and

$$w(1) = 1.$$

Obtain a formula for $w(x)$.

(b) Let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

$$\hat{u}(0,t) = \hat{u}(1,t) = 0, \qquad t \ge 0,$$

and

$$\hat{u}(x,0) = \hat{u}_0(x), \qquad 0 \le x \le 1,$$

where $\hat{u}_0(x)$ is such that

$$u(x,t) = w(x) + \hat{u}(x,t).$$

Obtain a formula for $\hat{u}_0(x)$.

(c) We can write

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$b_n = \int_0^1 f(x)\psi_n(x)\,dx.$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(d) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(e) Write out a formula for $u(x,t)$.

(f) Plot the approximations to $u(x,t)$ obtained by replacing the upper limit of the summation in your series solution with 20 for $t = 0, 0.1, 0.2, 0.3, 0.5, 1, 2$.