CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Examination 3

Instructions:

1. Time limit: **3 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You are allowed one cheat sheet to refer to during the exam.
   You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions throughly (but succintly!) and justify all your answers.
   Show your work for partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points: 5 points for (a), 10 points for (b)-(c)] When studying new numerical techniques for modeling fluid phenomena periodic boundary conditions are often employed as they simulate an 'infinite' space in a finite interval $[-1, 1]$. As we saw in exam two the operator $L = -\frac{\partial^2}{\partial x^2}$ with periodic boundary conditions in space $u(\cdot, -1) = u(\cdot, 1)$, $\frac{\partial}{\partial_x} u(\cdot, -1) = \frac{\partial}{\partial_x} u(\cdot, 1)$. Has eigenvalues $\lambda_0 = 0$ with eigenvector $\Psi_0 = 1$ and $\lambda_n = (n\pi)^2$ corresponding to the pair of eigenvectors $\Psi_{1,n} = sin(n\pi x)$ and $\Psi_{2,n} = cos(n\pi x)$. All of the eigenvectors are mutually orthogonal with respect to the inner product

$$(u, v) = \int_{-1}^{1} uv \, dx$$

In this problem we will consider a modified wave equation with periodic boundary conditions on the interval $[-1, 1]$. The value $p > 0$ is a fixed constant.

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + pu = 0 \\ \begin{array}{cc} u(0, x) = G(x) & \frac{\partial u}{\partial t}(0, x) = H(x) \\ u(t, -1) = u(t, 1) & \frac{\partial u}{\partial x}(t, -1) = \frac{\partial u}{\partial x}(t, 1) \end{array} \end{gathered} \tag{1}$$

(a) Show that the modified spatial operator $-\frac{\partial^2 u}{\partial x^2} + pu$ with periodic boundary conditions has eigenvalue $\tilde{\lambda}_0 = (\lambda_0 + p)$ with eigenvector $\tilde{\Psi}_0 = \Psi_0$ and that $\tilde{\Psi}_{1,n} = \Psi_{1,n}$, $\tilde{\Psi}_{2,n} = \Psi_{2,n}$ are eigenvectors which both have eigenvalue $\tilde{\lambda}_j = (\lambda_j + p)$.

(b) As a result of part (a) we write the solution to the system (1) as in equation (2) and expand the initial conditions as given in equations (3) and (4).

$$u(t, x) = \gamma_0(t) + \Sigma_{k=1}^{\infty} \left(\alpha_k(t) \tilde{\Psi}_{1,n}(x) + \beta_k(t) \tilde{\Psi}_{2,n}(x) \right) \tag{2}$$

$$G(x) = g_0 + \Sigma_{k=1}^{\infty} \left(a_k \Psi_{1,n}(x) + b_k \tilde{\Psi}_{2,n}(x) \right) \tag{3}$$

$$H(x) = h_0 + \Sigma_{k=1}^{\infty} \left(c_k \Psi_{1,n}(x) + d_k \tilde{\Psi}_{2,n}(x) \right) \tag{4}$$

Use these expansions in the modified damped wave equation (1) and write down the system of Initial boundary value problems (ordinary differential equations) determining the unknown coefficients $\gamma_0(t)$ and $\alpha_k(t)$, $\beta_k(t)$ for $k = 1, 2, \ldots$.

(c) Solve the modified damped wave equation (1) with $p = \pi^2$, $G(x) = 1 + sin(\pi x)$ and $H(x) = 1 + cos(\pi x)$. You will need the fact that the solution to the Initial value problem (5) is given by the equation $y(t) = Acos(\theta t) + \frac{B}{\theta} sin(\theta t)$

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} + \theta^2 y &= 0 \\ y(0) &= A \\ \frac{\partial y}{\partial t} &= B \end{aligned} \tag{5}$$

2. [20 points: 10 points for (a), 5 points for each of (b)-(c)] In the last part of the semester we have seen the finite element method applied to the time-dependent heat equation; the most fundamental being the case of homogeneous Dirichlet boundary conditions as in (6).

$$\begin{gathered}\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f(t,x) \\ \\ u(0,x) = \psi(x) \\ u(t,0) = 0 \quad u(t,l) = 0\end{gathered} \tag{6}$$

We discretized equation using the finite element method and the space of linear hat functions $S_0^N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$ where ϕ_i is the hat function corresponding to the internal mesh point x_i on a uniformly spaced mesh of the interval $[0,l]$. The discrete solution for (6) can be represented as $u_h(t,x) = \Sigma_{i=1}^N \alpha_i(t)\phi_i(x)$ and the initial condition can be approximated as $\psi_h(x) = \sum_{i=1}^N \psi(x_i)\phi_i(x)$. Let $\vec{\alpha}$, $\vec{\psi}$, and $\vec{f}$ denote coefficient vectors with $\vec{\alpha}_i = \alpha_i(t)$, $\vec{\psi}_i = \psi(x_i)$.

(a) The finite element method applied to (6) results in a first-order system of ordinary differential equations for $\vec{\alpha}(t)$. Write down this system. Specify the entries of the right-hand side vector $\vec{f}$ in addition to each matrix in the formulation. That is, clearly indicate formulas for the components $\vec{f}_i$ and B_{ij} for every matrix B involved in your answer.

(b) Write down the numerical scheme for the Backward Euler method applied to the linear system of ordinary differential equations $\frac{\partial \vec{y}}{\partial t} = A\vec{y}(t) + \vec{g}(t)$ where A is a matrix. Your answer should be a formula for $\vec{y}(t_{n+1})$ in terms of $\vec{y}(t_n)$

(c) Explain how you would deal with *inhomogeneous* Dirichlet boundary conditions $u(t,0) = a$, $u(t,l) = b$. If any additions are made to the function space S_0^N draw a picture of the additional shape functions. (Hint: Recall that the problem can be split into two parts; what are these parts and how do you deal with them?)

3. [20 points: 10 points for (a), 5 points for each of (b)-(c)] When studying new numerical techniques for modeling fluid phenomena periodic boundary conditions are often employed as they simulate an 'infinite' space in a finite interval $[-1,1]$. As we saw in exam two the operator $L = -\frac{\partial^2}{\partial x^2}$ with periodic boundary conditions in space $u(\cdot,-1) = u(\cdot,1)$, $\frac{\partial}{\partial_x}u(\cdot,-1) = \frac{\partial}{\partial_x}u(\cdot,1)$. Has eigenvalues $\lambda_0 = 0$ with eigenvector $\Psi_0 = 1$ and $\lambda_n = (n\pi)^2$ corresponding to the pair of eigenvectors $\Psi_{1,n} = sin(n\pi x)$ and $\Psi_{2,n} = cos(n\pi x)$. All of the eigenvectors are mutually orthogonal with respect to the inner product

$$(u,v) = \int_{-1}^{1} uv\,dx$$

In this problem we will consider a modified wave equation with periodic boundary conditions on the interval $[-1,1]$. The value $p > 0$ is a fixed constant.

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + pu = 0\\ \begin{array}{ll} u(0,x) = G(x) & \frac{\partial u}{\partial t}(0,x) = H(x)\\ u(t,-1) = u(t,1) & \frac{\partial u}{\partial x}(t,-1) = \frac{\partial u}{\partial x}(t,1)\end{array}\end{gathered} \tag{7}$$

(a) Show that the modified spatial operator $-\frac{\partial^2 u}{\partial x^2} + pu$ with periodic boundary conditions has eigenvalue $\tilde{\lambda}_0 = (\lambda_0 + p)$ with eigenvector $\tilde{\Psi}_0 = \Psi_0$ and that $\tilde{\Psi}_{1,n} = \Psi_{1,n}$, $\tilde{\Psi}_{2,n} = \Psi_{2,n}$ are eigenvectors which both have eigenvalue $\tilde{\lambda}_j = (\lambda_j + p)$.

(b) As a result of part (a) we write the solution to the system (1) as in equation (2) and expand the initial conditions as given in equations (3) and (4).

$$u(t,x) = \gamma_0(t) + \Sigma_{k=1}^{\infty}\left(\alpha_k(t)\tilde{\Psi}_{1,n}(x) + \beta_k(t)\tilde{\Psi}_{2,n}(x)\right) \tag{8}$$

$$G(x) = g_0 + \Sigma_{k=1}^{\infty}\left(a_k\Psi_{1,n}(x) + b_k\tilde{\Psi}_{2,n}(x)\right) \tag{9}$$

$$H(x) = h_0 + \Sigma_{k=1}^{\infty}\left(c_k\Psi_{1,n}(x) + d_k\tilde{\Psi}_{2,n}(x)\right) \tag{10}$$

Use these expansions in the modified damped wave equation (1) and write down the system of Initial boundary value problems (ordinary differential equations) determining the unknown coefficients $\gamma_0(t)$ and $\alpha_k(t)$, $\beta_k(t)$ for $k = 1,2,\ldots$.

(c) Solve the modified damped wave equation (1) with $p = \pi^2$, $G(x) = 1 + sin(\pi x)$ and $H(x) = 1 + cos(\pi x)$. You will need the fact that the solution to the Initial value problem (5) is given by the equation $y(t) = Acos(\theta t) + \frac{B}{\theta}sin(\theta t)$

$$\begin{gathered}\frac{\partial^2 y}{\partial t^2} + \theta^2 y = 0\\ y(0) = A\\ \frac{\partial y}{\partial t} = B\end{gathered} \tag{11}$$