CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 8 · Solutions

Posted Wednesday 29 January 2014. Due 1pm Friday 7 February 2014.

8. [25 points]

(a) Consider the forward difference approximation

$$u'(x) \approx \frac{u(x+h) - u(x)}{h}.$$

For $u(x) = \exp(2x)$, compute (in MATLAB) the error

$$\left| u'(1/2) - \frac{u(1/2+h) - u(1/2)}{h} \right|,$$

for $N = 2, 4, 8, 16, 32, 64, 128, 256, 512$ (powers of 2) with $h = 1/(N+1)$. When h is small enough, this error should be proportional to h. Present your results in a table like the one below but with the missing entries filled in.

N	error
2	2.2920610
4	
8	
16	
32	
64	
128	
256	
512	

(b) Consider the centered difference approximation

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}.$$

For $u(x) = \exp(2x)$, compute (in MATLAB) the error

$$\left| u'(1/2) - \frac{u(1/2+h) - u(1/2-h)}{2h} \right|$$

for $N = 2, 4, 8, 16, 32, 64, 128, 256, 512$ with $h = 1/(N+1)$. When h is small enough, this error should be proportional to h^2. Present your results in a table like the one below but with the missing entries filled in.

N	error
2	0.4117528
4	
8	
16	
32	
64	
128	
256	
512	

(c) Use MATLAB's `loglog` command to produce a plot of the error for the approximations considered in part (a) and part (b) for $N = 2, 4, 8, 16, 32, 64, 128, 256, 512$. Use the `hold on` command so that the plot showing the errors from part (a) is on the same figure as the plot showing the errors from part (b).

(d) By inspecting the plot you have created, estimate the value of N that you need to approximate $u'(1/2)$ to an error of 10^{-2} using the approximations in part (a) and part (b).

Solution.

(a) [8 points] The error is shown in the table below.

N	error
2	2.2920610
4	1.2480270
8	0.6514086
16	0.3327167
32	0.1681236
64	0.0845039
128	0.0423625
256	0.0212089
512	0.0106114

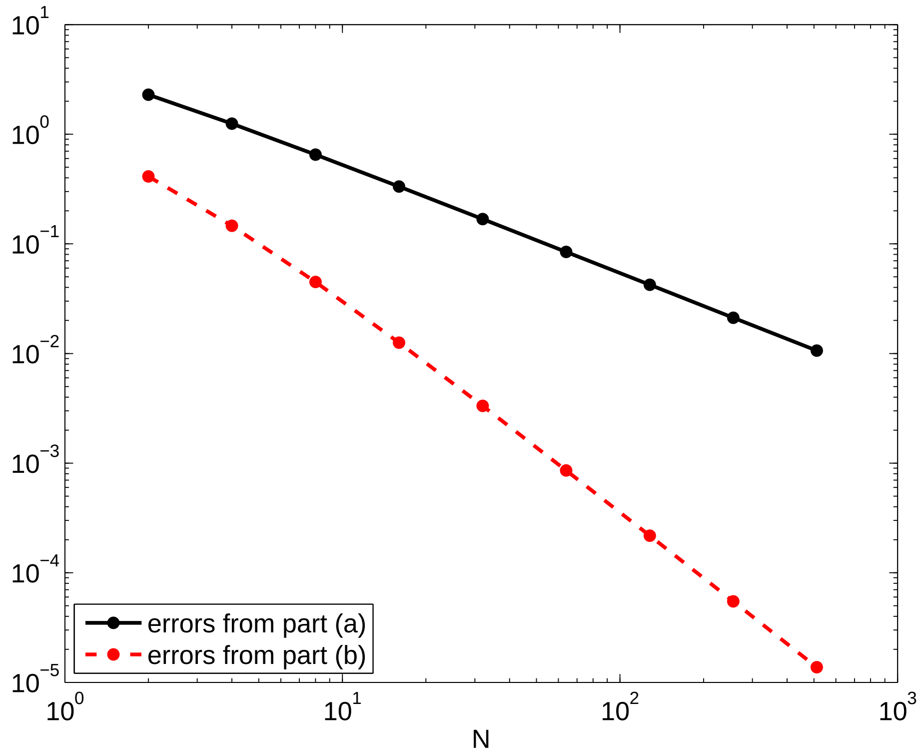
The code that generated the results shown in this part, part (b) and part (c) is below.

```
u = inline('exp(2*x)');
uprime = inline('2*exp(2*x)');
Nvec = 2.^(1:9).';
err = zeros(size(Nvec));
x = 1/2;
fprintf('\n part (a)\n')
for k=1:length(Nvec)
    N = Nvec(k);
    h = 1/(N+1);
    deriv = (u(x+h)-u(x))/h;
    err(k) = abs(uprime(x)-deriv);
    fprintf(' %3d %10.7f\n', N, err(k));
end
loglog(Nvec,err,'k.-','linewidth',2,'markersize',20)
fprintf('\n part (b)\n')
for k=1:length(Nvec)
    N = Nvec(k);
    h = 1/(N+1);
    deriv = (u(x+h)-u(x-h))/(2*h);
    err(k) = abs(uprime(x)-deriv);
    fprintf(' %3d %10.7f\n', N, err(k));
end
hold on
loglog(Nvec,err,'r--','linewidth',2,'marker','.','markersize',20)
set(gca,'fontsize',14)
xlabel('N', 'fontsize',14)
legend('errors from part (a)','errors from part (b)',3)
print -depsc2 findiff.eps
```

(b) [8 points] The error for the $O(h^2)$ centered difference approximation is shown in the table below.

N	error
2	0.4117528
4	0.1461393
8	0.0448560
16	0.0125498
32	0.0033288
64	0.0008579
128	0.0002178
256	0.0000549
512	0.0000138

(c) [5 points] The errors for the approximation in part (b) decay much more rapidly than the errors for the approximation in part (a). This is made clear by the plot below.



(d) [4 points] Roughly speaking, we can estimate that we need to take $N = a$, where $512 < a \leq 600$, to approximate $u'(1/2)$ to an error of 10^{-2} using the approximation in part (a) and $N = b$, where $16 < b \leq 20$, to approximate $u'(1/2)$ to an error of 10^{-2} using the approximation in part (b). When used in the context of solving differential equations, the improved accuracy of the centered difference formula allows one to work with smaller matrices than required for the forward difference formula, potentially delivering a great speed-up in run-time.