CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 25 · Solutions

Posted Monday 7 October 2013. Due 5pm Friday 18 October 2013.

25. [25 points] We have been able to obtain nice formulas for the eigenvalues of the operators that we have considered thus far. This problem illustrates that this is not always the case.

    Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

    $$(v,w) = \int_0^1 v(x)w(x)\,dx.$$

    Let the linear operator $L : V \to C[0,1]$ be defined by

    $$Lu = -u''$$

    where

    $$V = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

    Note that if $u \in V$ then u satisfies the homogeneous Robin boundary condition

    $$u(0) - u'(0) = 0$$

    and the homogeneous Dirichlet boundary condition

    $$u(1) = 0.$$

    (a) Prove that L is symmetric.

    (b) Is zero an eigenvalue of L?

    (c) Show that $(Lu,u) \geq 0$ for all $u \in V$. What does this and the answer to part (b) then allow us to say about the eigenvalues of L?

    (d) Show that the eigenvalues λ of L must satisfy the equation $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$.

    (e) Use MATLAB to plot $g(x) = -\tan(x)$ and $h(x) = x$ on the same figure. Use the command `axis([0 5*pi -5*pi 5*pi])` and make sure that your plot gives an accurate representation of these functions on the region shown on the figure when this command is used. By hand or using MATLAB, mark on your plot the points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Note that $g \notin C[0,5\pi]$. How many eigenvalues λ does L have which are such that $\sqrt{\lambda} \leq 5\pi$?

Solution.

(a) [5 points] Suppose $u, v \in V$, so that $u(0) - u'(0) = v(0) - v'(0) = u(1) = v(1) = 0$. Integrating by parts twice yields

$$\begin{aligned}
(Lu, v) &= \int_0^1 -u''(x)v(x)\,dx \\
&= \Big[-u'(x)v(x)\Big]_0^1 + \int_0^1 u'(x)v'(x)\,dx, \\
&= -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v'(x)\,dx,
\end{aligned}$$

$$
\begin{aligned}
&= -u'(1)v(1) + u'(0)v(0) + \Big[u(x)v'(x)\Big]_0^1 - \int_0^1 u(x)v''(x)\,dx \\
&= -u'(1)v(1) + u'(0)v(0) + u(1)v'(1) - u(0)v'(0) + (u, Lv) \\
&= (u, Lv).
\end{aligned}
$$

In the last step, two boundary terms are zero because $u(1) = v(1) = 0$. For the other boundary term, note that $v(0) - v'(0) = 0$ implies $v(0) = v'(0)$, so $u'(0)v(0) - u(0)v'(0) = u'(0)v(0) - u(0)v(0) = -(u(0) - u'(0))v(0) = 0$ since $u(0) - u'(0) = 0$. Hence $(Lu, v) = (u, Lv)$ for all $u, v \in V$ and so L is symmetric.

(b) [3 points] Zero is *not* an eigenvalue of L. To see this, we seek a nonzero solution $\psi \in V$ to $L\psi = 0\psi$, i.e., $-\psi''(x) = 0$. The general solution of $-\psi''(x) = 0$ is $\psi(x) = Ax + B$ where A and B are constants. The right boundary condition $\psi(1) = 0$ implies that

$$0 = \psi(1) = A + B,$$

hence $A = -B$. The left boundary condition implies

$$0 = \psi(0) - \psi'(0) = B - A,$$

hence $A = B$. The only solution which satisfies both of these conditions is hence $A = B = 0$, so $\psi(x) = 0$ is the only solution of $L\psi = 0$. Thus zero is not an eigenvalue of L.

(c) [7 points] Suppose $u \in V$, so that $u(0) - u'(0) = u(1) = 0$. Then, integrating by parts gives

$$
\begin{aligned}
(Lu, u) &= \int_0^1 -u''(x)u(x)\,dx \\
&= \Big[-u'(x)u(x)\Big]_0^1 + \int_0^1 u'(x)u'(x)\,dx, \\
&= -u'(1)u(1) + u'(0)u(0) + \int_0^1 (u'(x))^2\,dx.
\end{aligned}
$$

Now, $(u'(x))^2 \geq 0$ for all $x \in [0,1]$ and so $\int_0^1 (u'(x))^2\,dx \geq 0$. Moreover, since $u(1) = 0$ we have that $-u'(1)u(1) = 0$ and since $u(0) - u'(0) = 0$ we can say that $u(0) = u'(0)$ from which it follows that $u'(0)u(0) = (u(0))^2 \geq 0$. Therefore, $(Lu, u) \geq 0$ for all $u \in V$.

If λ is an eigenvalue of L then, since L is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $\psi \in V$ which are such that $L\psi = \lambda\psi$ and hence

$$\lambda(\psi, \psi) = (\lambda\psi, \psi) = (L\psi, \psi).$$

The fact that $(Lu, u) \geq 0$ for all $u \in V$ then means that

$$\lambda = \frac{(L\psi, \psi)}{(\psi, \psi)} \geq 0$$

since $(\psi, \psi) > 0$ by the definition of the inner product because ψ is a nonzero function. However, in part (b) we had showed that zero is not an eigenvalue of L and so we can conclude that $\lambda > 0$ for all eigenvalues λ of L.

(d) [5 points] We now know that all eigenvalues λ of L are positive and so the general solution of $L\psi = \lambda\psi$, i.e. $-\psi'' = \lambda\psi$, has the form

$$\psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

where A and B are constants. The left boundary condition gives

$$0 = \psi(0) - \psi'(0) = A\sin(0) + B\cos(0) - A\sqrt{\lambda}\cos(0) + B\sqrt{\lambda}\sin(0) = B - A\sqrt{\lambda},$$

hence $B = A\sqrt{\lambda}$. The right boundary condition gives

$$0 = \psi(1) = A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda}).$$

Substituting the left boundary condition into this last formula, we find

$$0 = A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}).$$

Since we need $A \neq 0$ in order for $\psi \neq 0$, this equation implies that

$$\sqrt{\lambda} = -\frac{\sin(\sqrt{\lambda})}{\cos(\sqrt{\lambda})} = -\tan(\sqrt{\lambda}).$$

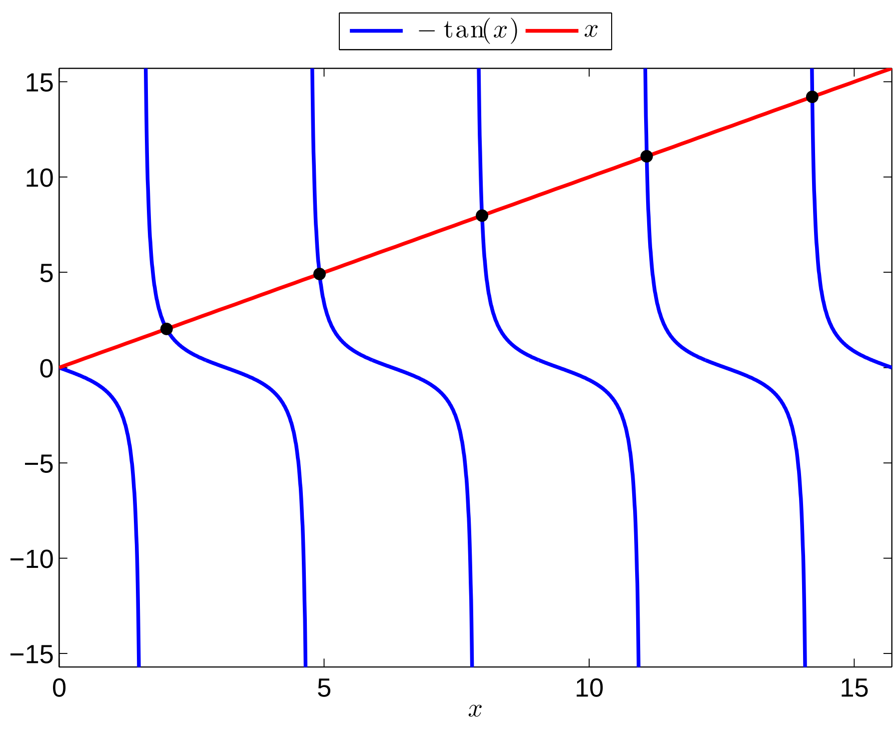
Therefore, the eigenfunctions of L have the form

$$\psi(x) = A(\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\cos(\sqrt{\lambda}x)),\ A \neq 0$$

where the eigenvalues λ are the positive numbers which are such that

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

(e) [5 points] The plot is shown below.



From the plot we can see that there are 5 points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Hence, since the eigenvalues λ of L are the positive numbers which are such that $g(\sqrt{\lambda}) = h(\sqrt{\lambda})$, L has five eigenvalues λ which are such that $\sqrt{\lambda} \leq 5\pi$.

The code used to produce the plot is below.

```matlab
clear
clc
figure(1)
clf
for j=0:5
    x = linspace((j-1/2)*pi,(j+1/2)*pi,500);
    x = x(2:end-1);
    tanplt = plot(x,-tan(x),'b-','linewidth',2);
    hold on
end
x = linspace(0,5*pi,100);
linplt = plot(x,x,'r-','linewidth',2);
axis([0 5*pi -5*pi 5*pi])
xlabel('$x$','interpreter','latex','fontsize',14)

lgd = legend([tanplt,linplt],'$\ -\tan(x)$', '$x$',...
             'location','northoutside','orientation','horizontal');
set(lgd,'interpreter','latex')
set(gca,'fontsize',14)

guess = [2 5 7.98 11 14.21]';
bracket = [1.6 2.5;
4.8 5;
7.9 8.1;
11 11.2;
14.15 14.3];

ew = zeros(size(guess));
for k=1:length(guess)
    ew(k) = bisect(@(x) x+tan(x),bracket(k,1),bracket(k,2));
    plot(ew(k),ew(k),'k.','markersize',20)
end
print -depsc2 eigroot
```

The function `bisect` used in the above code is below.

```matlab
function xstar = bisect(f,a,b)

% function xstar = bisect(f,a,b)
% Compute a root of the function f using bisection.
% f:    a function name, e.g., bisect('sin',3,4), or bisect('myfun',0,1)
% a, b: a starting bracket:  f(a)*f(b) < 0.

fa = feval(f,a);
fb = feval(f,b);         % evaluate f at the bracket endpoints
delta = (b-a);           % width of initial bracket
k = 0; fc = inf;         % initialize loop control variables

c = (a+b)/2;
while (delta/(2^k)>1e-18) && abs(fc)>1e-18
    c = (a+b)/2;
    fc = feval(f,c);   % evaluate function at bracket midpoint
    if fa*fc < 0
        b=c;
        fb = fc;     % update new bracket
    else
        a=c;
        fa=fc;
    end
    k = k+1;
%    fprintf(' %3d  %20.14f  %16.8e\n', k, c, fc);
end
xstar = c;
```