CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 39 · Solutions

Posted Wednesday 13 November 2013. Due 5pm Wednesday 20 November 2013.

39. [25 points] Parts (a) and (c) of this question should be done by hand.

    Let

$$f(x,t) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right), \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

(a) Use the spectral method to obtain a series solution to the problem of finding the solution $\tilde{u}(x,t)$ to the heat equation

$$\tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0$$

with Neumann boundary conditions

$$\tilde{u}_x(0,t) = \tilde{u}_x(1,t) = 0, \quad t \geq 0$$

and initial condition

$$\tilde{u}(x,0) = 0, \quad 0 < x < 1.$$

(b) Plot the approximations to $\tilde{u}(x,t)$ obtained by replacing the upper limit of the summation in your series solution with 20 for $t = 0, 0.1, 0.2, 0.3, 0.5, 1, 2$.

(c) By shifting the data and then using the spectral method, obtain a series solution to the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0$$

with Neumann boundary conditions

$$u_x(0,t) = 0, \quad t \geq 0$$

and

$$u_x(1,t) = 2, \quad t \geq 0$$

and initial condition

$$u(x,0) = x^2, \quad 0 < x < 1.$$

(d) Plot the approximations to $u(x,t)$ obtained by replacing the upper limit of the summation in your series solution with 20 for $t = 0, 0.1, 0.2, 0.3, 0.5, 1, 2$.

Solution.

(a) [7 points] Let

$$\psi_0(x) = 1$$

and let

$$\psi_n(x) = \sqrt{2}\cos(n\pi x)$$

for $n = 1, 2, \ldots$. The spectral method yields the series solution

$$\tilde{u}(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x)$$

where

$$
\begin{aligned}
a_0(t) &= \int_0^1 0\,dx + \int_0^t \int_0^1 f(x,s)\psi_0(x)\,dx\,ds \\
&= \int_0^t \int_0^1 f(x,s)\psi_0(x)\,dx\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
a_n(t) &= \int_0^1 0\,dx e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds \\
&= \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds
\end{aligned}
$$

for $n = 1, 2, \ldots$.

Now,

$$
\begin{aligned}
&\int_0^1 f(x,s)\psi_0(x)\,dx \\
&= \left(\int_0^{1/2} f(x,s)\,dx + \int_{1/2}^1 f(x,s)\,dx\right) \\
&= 2\left(\int_0^{1/2} x\,dx + \int_{1/2}^1 1 - x\,dx\right) \\
&= 2\left(\left[\frac{1}{2}x^2\right]_0^{1/2} + \left[x - \frac{1}{2}x^2\right]_{1/2}^1\right) \\
&= 2\left(\frac{1}{8} - 0 + 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{8}\right) \\
&= 2\left(\frac{1}{8} + \frac{8}{8} - \frac{4}{8} - \frac{4}{8} + \frac{1}{8}\right) \\
&= \frac{4}{8} \\
&= \frac{1}{2}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
a_0(t) &= \int_0^t \int_0^1 f(x,s)\psi_0(x)\,dx\,ds \\
&= \int_0^t \frac{1}{2}\,ds \\
&= \left[\frac{1}{2}s\right]_{s=0}^{s=t} \\
&= \frac{1}{2}t.
\end{aligned}
$$

Also, for $n = 1, 2, 3, \ldots$,

$$
\begin{aligned}
&\int_0^1 f(x,s)\psi_n(x)\,dx \\
&= \sqrt{2}\left(\int_0^{1/2} f(x,s)\cos(n\pi x)\,dx + \int_{1/2}^1 f(x,s)\cos(n\pi x)\,dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sqrt{2}\left(\int_0^{1/2} x\cos(n\pi x)\,dx + \int_{1/2}^1 (1-x)\cos(n\pi x)\,dx\right) \\
&= 2\sqrt{2}\left(\left[\frac{1}{n\pi}x\sin(n\pi x)\right]_0^{1/2} - \frac{1}{n\pi}\int_0^{1/2}\sin(n\pi x)\,dx + \left[\frac{1}{n\pi}(1-x)\sin(n\pi x)\right]_{1/2}^1 + \frac{1}{n\pi}\int_{1/2}^1 \sin(n\pi x)\,dx\right) \\
&= 2\sqrt{2}\left(\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}x\right) - \frac{1}{n\pi}\int_0^{1/2}\sin(n\pi x)\,dx - \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}x\right) + \frac{1}{n\pi}\int_{1/2}^1 \sin(n\pi x)\,dx\right) \\
&= \frac{2\sqrt{2}}{n\pi}\left(\int_{1/2}^1 \sin(n\pi x)\,dx - \int_0^{1/2}\sin(n\pi x)\,dx\right) \\
&= \frac{2\sqrt{2}}{n\pi}\left(\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_{1/2}^1 - \left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^{1/2}\right) \\
&= \frac{2\sqrt{2}}{n\pi}\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\right) \\
&= \frac{2\sqrt{2}}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right).
\end{aligned}
$$

Consequently,

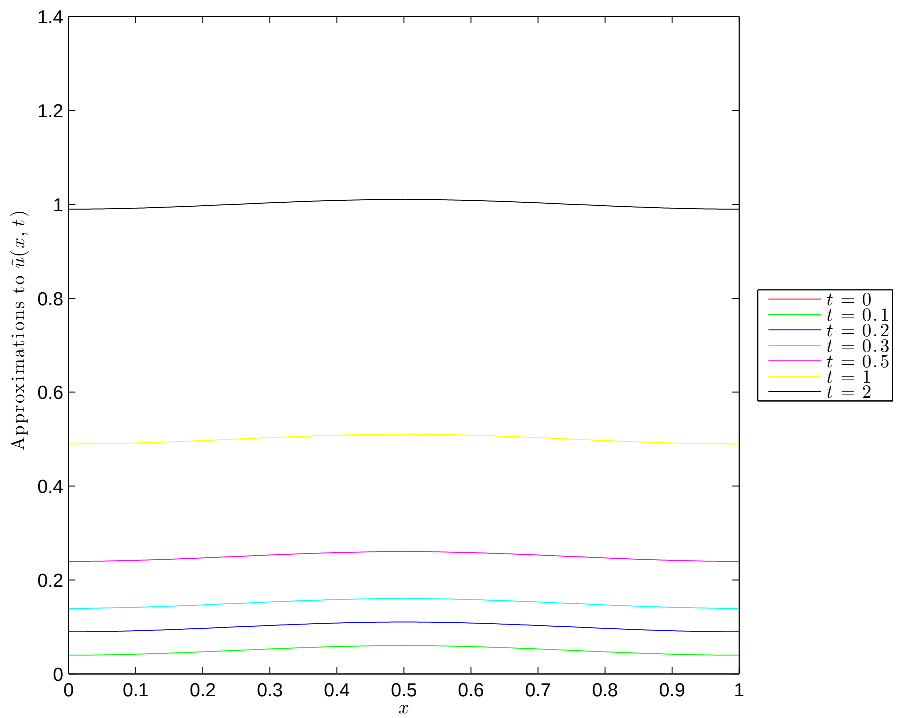
$$
\begin{aligned}
a_n(t) &= \int_0^t e^{n^2\pi^2(s-t)}\int_0^1 f(x,s)\psi_n(x)\,dx\,ds \\
&= \frac{2\sqrt{2}}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\int_0^t e^{n^2\pi^2(s-t)}\,ds \\
&= \frac{2\sqrt{2}}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\left[\frac{1}{n^2\pi^2}e^{n^2\pi^2(s-t)}\right]_{s=0}^{s=t} \\
&= \frac{2\sqrt{2}}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\left(\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}e^{-n^2\pi^2 t}\right) \\
&= \frac{2\sqrt{2}}{n^4\pi^4}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\left(1 - e^{-n^2\pi^2 t}\right)
\end{aligned}
$$

for $n = 1, 2, \ldots$.

Hence,

$$
\tilde{u}(x,t) = \frac{1}{2}t + \sum_{n=1}^{\infty}\frac{4}{n^4\pi^4}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\left(1 - e^{-n^2\pi^2 t}\right)\cos(n\pi x).
$$

(b) [5 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
col = 'rgbcmyk';
x = linspace(0,1,200);
tvec=[0 0.1 0.2 0.3 0.5 1 2];
figure(1)
clf
for j=1:length(tvec)
    U = zeros(size(x));
    t=tvec(j);
    U=U+t/2;
    for n=1:20
        U=U + 4*(2*cos(n*pi/2)-cos(n*pi)-1)*(1-exp(-n^2*pi^2*t))*cos(n*pi*x)/(n^4*pi^4);
    end
    legendStr{j}=['$t=' num2str(t) '$'];
    plot(x,U,col(j))
    hold on
end
legend(legendStr,'interpreter','latex','location','eastoutside')
xlabel('$x$','interpreter','latex')
ylabel('Approximations to $\tilde{u}(x,t)$','interpreter','latex')
saveas(figure(1),'hw39b.eps','epsc')
```

(c) [8 points] Let

$$w(x) = x^2$$

so that

$$w'(0) = 0$$

and

$$w'(1) = 2.$$

Moreover, let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x,t) + w''(x) = f(x,t) + 2, \quad 0 < x < 1, \quad t > 0;$$

$$\hat{u}(0,t) = \hat{u}(1,t) = 0, \quad t \geq 0;$$

and

$$\hat{u}(x,0) = x^2 - x^2 = 0, \quad 0 < x < 1.$$

Then $u(x,t) = w(x) + \hat{u}(x,t)$ will be such that

$$u_t(x,t) - u_{xx}(x,t) = \hat{u}_t(x,t) - w''(x) - \hat{u}_{xx}(x,t) = f(x,t) + 2 - 2 = f(x,t), \quad 0 < x < 1, \quad t > 0;$$

$$u_x(0,t) = w'(0) + \hat{u}_x(0,t) = 0 + 0 = 0, \quad t \geq 0;$$
$$u_x(1,t) = w'(1) + \hat{u}_x(1,t) = 2 + 0 = 2, \quad t \geq 0;$$

and

$$u(x,0) = w(x) + \hat{u}(x,0) = x^2 + 0 = x^2, \quad 0 < x < 1.$$

The spectral method yields that

$$\hat{u}(x,t) = \sum_{n=0}^{\infty} \hat{a}_n(t)\psi_n(x)$$

where

$$\begin{aligned}\hat{a}_0(t) &= \int_0^1 0\,dx + \int_0^t \int_0^1 \left(f(x,s) + 2\right)\psi_0(x)\,dx\,ds \\ &= \int_0^t \int_0^1 \left(f(x,s) + 2\right)\psi_0(x)\,dx\,ds\end{aligned}$$

and

$$\begin{aligned}\hat{a}_n(t) &= \int_0^1 0\,dx e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 \left(f(x,s) + 2\right)\psi_n(x)\,dx\,ds \\ &= \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 \left(f(x,s) + 2\right)\psi_n(x)\,dx\,ds\end{aligned}$$

for $n = 1, 2, 3, \ldots$.
Now,

$$\int_0^1 2\psi_0(x)\,dx = 2\int_0^1 \psi_0(x)\psi_0(x)\,dx = 2$$

and in part (a) we computed that

$$\int_0^t \int_0^1 f(x,s)\psi_0(x)\,dx\,ds = \frac{1}{2}t$$

and so, for $n = 1, 2, 3, \ldots$,

$$\int_0^t \int_0^1 \left(f(x,s) + 2\right)\psi_0(x)\,dx\,ds = \frac{1}{2}t + \int_0^t 2\,ds = \frac{1}{2}t + [2s]_{s=0}^{s=t} = \frac{1}{2}t + 2t = \frac{1}{2}t + \frac{4}{2}t = \frac{5}{2}t.$$

Consequently,

$$\begin{aligned}\hat{a}_0(t) &= \int_0^t \int_0^1 \left(f(x,s) + 2\right)\psi_0(x)\,dx\,ds \\ &= \frac{5}{2}t.\end{aligned}$$

Also, for $n = 1, 2, 3, \ldots$,

$$\int_0^1 2\psi_n(x)\,dx = 2\int_0^1 \psi_0(x)\psi_n(x)\,dx = 0$$

and in part (a) we computed that, for $n = 1, 2, 3, \ldots$,

$$\int_0^t e^{n^2\pi^2(s-t)} \int_0^1 f(x,s)\psi_n(x)\,dx\,ds = \frac{4\sqrt{2}}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right)$$

and so, for $n = 1, 2, 3, \ldots$,

$$\int_0^t e^{n^2\pi^2(s-t)} \int_0^1 \left(f(x,s) + 2\right)\psi_n(x)\,dx\,ds = \frac{4\sqrt{2}}{n^4\pi^4}\sin\left(\frac{n\pi}{2}\right)\left(1 - e^{-n^2\pi^2 t}\right).$$

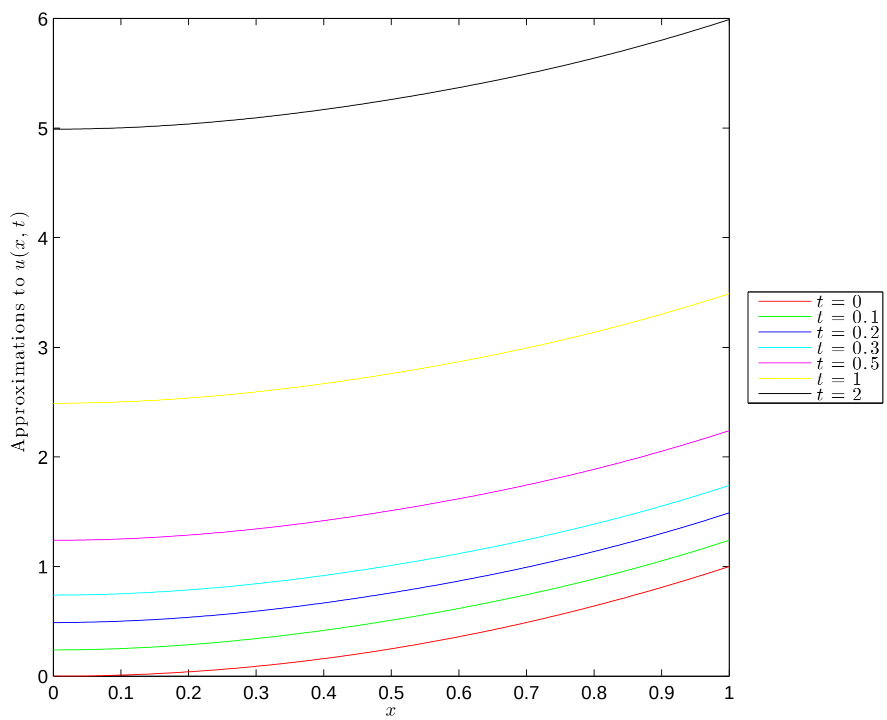
Consequently,

$$\begin{aligned}\hat{a}_n(t) &= \int_0^t e^{n^2\pi^2(s-t)} \int_0^1 \left(f(x,s) + 2\right)\psi_n(x)\,dx\,ds \\ &= \frac{2\sqrt{2}}{n^4\pi^4}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\left(1 - e^{-n^2\pi^2 t}\right)\end{aligned}$$

for $n = 1, 2, 3, \ldots$.
Hence,

$$u(x,t) = x^2 + \frac{5}{2}t + \sum_{n=1}^{\infty} \frac{4}{n^4\pi^4}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)\left(1 - e^{-n^2\pi^2 t}\right)\cos(n\pi x).$$

(d) [5 points] The requested plot is below.



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    end
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end
legend(legendStr,'interpreter','latex','location','eastoutside')
xlabel('$x$','interpreter','latex')
ylabel('Approximations to $u(x,t)$','interpreter','latex')
saveas(figure(1),'hw39d.eps','epsc')
```
