CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 30

Posted Monday 21 October 2013. Due 5pm Wednesday 30 October 2013.

30. [25 points] Let N be a positive integer, let $h = \frac{1}{N+1}$ and let $x_k = kh$ for k = 0, 1, ..., N+1. Let the continuous piecewise linear hat functions $\hat{\phi}_j \in C[0,1]$ be such that

$$\widehat{\phi}_j(x) = \begin{cases} \frac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \frac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j=1,\ldots,N$. Also, let the continuous piecewise quadratic functions $\phi_j\in C[0,1]$ be such that

$$\phi_j(x) = \begin{cases} \frac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \frac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j=1,\ldots,N$ and let the continuous piecewise quadratic bubble functions $\psi_j\in C[0,1]$ be such that

$$\psi_j(x) = \begin{cases} \frac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases}$$

for j = 1, ..., N + 1.

(a) What is

$$\begin{array}{l} \text{i. } \phi_j\left(x_k\right) \text{ for } j=1,\ldots,N \text{ and } k=0,\ldots,N+1;\\ \text{ii. } \phi_j\left(\frac{x_{k-1}+x_k}{2}\right) \text{ for } j=1,\ldots,N \text{ and } k=1,\ldots,N+1;\\ \text{iii. } \psi_j\left(x_k\right) \text{ for } j=1,\ldots,N+1 \text{ and } k=0,\ldots,N+1;\\ \text{iv. } \psi_j\left(\frac{x_{k-1}+x_k}{2}\right) \text{ for } j,k=1,\ldots,N+1. \end{array}$$

- (b) Show that $\{\phi_1,\ldots,\phi_N,\psi_1,\ldots,\psi_{N+1}\}$ is linearly independent by showing that if $\alpha_j,\beta_j\in\mathbb{R}$ and $\sum_{j=1}^N\alpha_j\phi_j(x)+\sum_{j=1}^{N+1}\beta_j\psi_j(x)=0$ for all $x\in[0,1]$ then $\alpha_k=0$ for $k=1,\ldots,N$ and $\beta_k=0$ for $k=1,\ldots,N+1$.
- (c) Obtain an expression for

$$\phi_j + \frac{1}{2} \left(\psi_j + \psi_{j+1} \right)$$

for j = 1, ..., N.

- (d) For j = 1, ..., N, is $\hat{\phi}_j \in \text{span}\{\phi_1, ..., \phi_N, \psi_1, ..., \psi_{N+1}\}$?
- (e) Is $\{\phi_1, \dots, \phi_N, \psi_1, \dots, \psi_{N+1}, \widehat{\phi}_1, \dots, \widehat{\phi}_N\}$ linearly independent?