CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 40 · Solutions

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

40. [25 points]

All parts of this question should be done by hand.

Let

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

In this question we will consider the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = 1, \quad t \ge 0,$$

and

$$u_x(1,t) = 2, \quad t \ge 0,$$

and initial condition

$$u(x,0) = x^2 + 1, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w'(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \psi_m(x)\psi_n(x)\,dx = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \ne n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) For $n = 1, 2, \ldots$, compute

$$\int_0^1 f(x)\psi_n(x)\,dx.$$

(c) Let $w(x)$ be such that

$$w''(x) = 0,$$
$$w(0) = 1$$

and

$$w'(1) = 2.$$

Obtain a formula for $w(x)$.

(d) Let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

$$\hat{u}(0,t) = \hat{u}_x(1,t) = 0, \qquad t \ge 0,$$

and

$$\hat{u}(x,0) = \hat{u}_0(x), \qquad 0 \le x \le 1,$$

where $\hat{u}_0(x)$ is such that

$$u(x,t) = w(x) + \hat{u}(x,t).$$

Obtain a formula for $\hat{u}_0(x)$.

(e) For $n = 1, 2, \ldots$, compute

$$\int_0^1 \hat{u}_0(x)\psi_n(x)\,dx.$$

(f) We can write

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$b_n = \int_0^1 f(x)\psi_n(x)\,dx.$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(g) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(h) Write out a formula for $u(x,t)$.

Solution.

(a) [2 points] We can compute that, for $n = 1, 2, \ldots$,

$$\psi_n'(x) = \sqrt{2}\left(\frac{2n-1}{2}\right)\pi\cos\left(\frac{2n-1}{2}\pi x\right).$$

and

$$\psi_n''(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)^2\pi^2\sin\left(\frac{2n-1}{2}\pi x\right).$$

and so

$$L\psi_n = -\psi_n'' = \left(\frac{2n-1}{2}\right)^2\pi^2\psi_n.$$

Hence,

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2\pi^2 = (2n-1)^2\frac{\pi^2}{4} \text{ for } n = 1, 2, \ldots.$$

(b) [3 points] For $n = 1, 2, \ldots,$

$$
\begin{aligned}
&\int_0^1 f(x)\psi_n(x)\,dx \\
&= \int_0^{1/2} f(x)\psi_n(x)\,dx + \int_{1/2}^1 f(x)\psi_n(x)\,dx \\
&= \int_0^{1/2} (1-2x)\,\sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx + \int_{1/2}^1 0\,dx \\
&= \sqrt{2}\int_0^{1/2} (1-2x)\sin\left(\frac{2n-1}{2}\pi x\right)\,dx + 0 \\
&= \sqrt{2}\left(\left[-(1-2x)\,\frac{2}{(2n-1)\,\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} - \int_0^{1/2}\frac{4}{(2n-1)\,\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \sqrt{2}\left(0 + \frac{2}{(2n-1)\,\pi} - \frac{4}{(2n-1)\,\pi}\int_0^{1/2}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - 2\int_0^{1/2}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - 2\left[\frac{2}{(2n-1)\,\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2}\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - \frac{4}{(2n-1)\,\pi}\sin\left(\frac{2n-1}{4}\pi\right) - 0\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\,\pi}\left(1 - \frac{4}{(2n-1)\,\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}
$$

(c) [3 points] The general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(1) = 2$ when $A = 2$. Hence, $w(x) = 2x + B$ and so $w(0) = B$ and hence $w(0) = 1$ when $B = 1$. Consequently,

$$w(x) = 1 + 2x.$$

(d) [4 points] We can compute that $u(x,t) = w(x) + \hat{u}(x,t)$ will be such that

$$u(x,0) = w(x) + \hat{u}(x,0) = 1 + 2x + \hat{u}_0(x)$$

and so since

$$u(x,0) = x^2 + 1$$

we can conclude that

$$\hat{u}_0(x) = x^2 + 1 - (1 + 2x) = x^2 - 2x.$$

(e) [3 points] For $n = 1, 2, \ldots$,

$$
\begin{aligned}
&\int_0^1 \hat{u}_0(x)\psi_n(x)\,dx \\
&= \int_0^1 \left(x^2 - 2x\right)\psi_n(x)\,dx \\
&= \sqrt{2}\int_0^1 \left(x^2 - 2x\right)\sin\left(\frac{2n-1}{2}\pi x\right)\,dx \\
&= \sqrt{2}\left(\left[-\left(x^2 - 2x\right)\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^1 + \int_0^1 (2x-2)\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\pi}\left(0 - 0 + \int_0^1 (2x-2)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{2\sqrt{2}}{(2n-1)\pi}\left(\left[(2x-2)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^1 - \int_0^1 \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(0 - 0 - \int_0^1 \sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(-\left[-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^1\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(0 - \frac{2}{(2n-1)\pi}\right) \\
&= -\frac{16\sqrt{2}}{(2n-1)^3\pi^3}.
\end{aligned}
$$

(f) [3 points] Substituting the expressions for $\hat{u}(x,t)$ and $f(x)$ into the partial differential equation yields

$$\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) - \sum_{n=1}^{\infty} a_n(t)\left(-\left(L\psi_n\right)(x)\right) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

and hence

$$\sum_{n=1}^{\infty}\left(a_n'(t) + \lambda_n a_n(t)\right)\psi_n(x) = \sum_{n=1}^{\infty} b_n\psi_n(x).$$

We can then say that

$$\sum_{n=1}^{\infty}\left(a_n'(t) + \lambda_n a_n(t)\right)\int_0^1 \psi_n(x)\psi_m(x)\,dx = \sum_{n=1}^{\infty} b_n\int_0^1 \psi_n(x)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m'(t) + \lambda_m a_m(t) = b_m$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

for $m, n = 1, 2, \ldots$. Hence, for $n = 1, 2, \ldots$,

$$a_n'(t) + (2n-1)^2\frac{\pi^2}{4}a_n(t) = \frac{2\sqrt{2}}{(2n-1)\pi}\left(1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right).$$

Also,

$$\hat{u}(x,0) = x^2 - 2x$$

means that

$$\sum_{n=1}^{\infty} a_n(0)\psi_n(x) = x^2 - 2x$$

and so

$$\sum_{n=1}^{\infty} a_n(0)\int_0^1 \psi_n(x)\psi_m(x)\,dx = \int_0^1 \left(x^2-2x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m(0) = \int_0^1 \left(x^2-2x\right)\psi_m(x)\,dx$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

for $m, n = 1, 2, \ldots$. Hence, for $n = 1, 2, \ldots$,

$$a_n(0) = -\frac{16\sqrt{2}}{(2n-1)^3\pi^3}.$$

Therefore, for $n = 1, 2, \ldots$, $a_n(t)$ is the solution to the differential equation

$$a_n'(t) = -(2n-1)^2\frac{\pi^2}{4}a_n(t) + \frac{2\sqrt{2}}{(2n-1)\pi}\left(1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)$$

with initial condition

$$a_n(0) = -\frac{16\sqrt{2}}{(2n-1)^3\pi^3}.$$

(g) [4 points] For $n = 1, 2, \ldots$,

$$\begin{aligned}
a_n(t) &= \int_0^1 \left(x^2-2x\right)\psi_n(x)\,dx e^{-(2n-1)^2\pi^2 t/4} + \int_0^t e^{(2n-1)^2\pi^2(s-t)/4} b_n\,ds \\
&= \int_0^1 \left(x^2-2x\right)\psi_n(x)\,dx e^{-(2n-1)^2\pi^2 t/4} + b_n\int_0^t e^{(2n-1)^2\pi^2(s-t)/4}\,ds \\
&= \int_0^1 \left(x^2-2x\right)\psi_n(x)\,dx e^{-(2n-1)^2\pi^2 t/4} + b_n\left[\frac{4}{(2n-1)^2\pi^2}e^{(2n-1)^2\pi^2(s-t)/4}\right]_{s=0}^{s=t} \\
&= \int_0^1 \left(x^2-2x\right)\psi_n(x)\,dx e^{-(2n-1)^2\pi^2 t/4} + b_n\left(\frac{4}{(2n-1)^2\pi^2} - \frac{4}{(2n-1)^2\pi^2}e^{-(2n-1)^2\pi^2 t/4}\right) \\
&= \int_0^1 \left(x^2-2x\right)\psi_n(x)\,dx e^{-(2n-1)^2\pi^2 t/4} + b_n\frac{4}{(2n-1)^2\pi^2}\left(1 - e^{-(2n-1)^2\pi^2 t/4}\right) \\
&= -\frac{16\sqrt{2}}{(2n-1)^3\pi^3}e^{-(2n-1)^2\pi^2 t/4} \\
&\quad + \frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\left(1 - e^{-(2n-1)^2\pi^2 t/4}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(2e^{-(2n-1)^2\pi^2 t/4} + \left(1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\left(e^{-(2n-1)^2\pi^2 t/4} - 1\right)\right) \\
&= -\frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(\left(3 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. + \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right) - 1\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(\left(\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right) - 3\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}
$$

(h) [3 points] We can write

$$
\begin{aligned}
u(x,t) &= w(x) + \hat{u}(x,t) \\
&= 1 + 2x + \sum_{n=1}^{\infty} a_n(t)\psi_n(x) \\
&= 1 + 2x + \sum_{n=1}^{\infty} \frac{8\sqrt{2}}{(2n-1)^3\pi^3}\left(\left(\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right) - 3\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\psi_n(x) \\
&= 1 + 2x + \sum_{n=1}^{\infty} \frac{16}{(2n-1)^3\pi^3}\left(\left(\frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right) - 3\right)e^{-(2n-1)^2\pi^2 t/4}\right. \\
&\quad \left. +1 - \frac{4}{(2n-1)\pi}\sin\left(\frac{2n-1}{4}\pi\right)\right)\sin\left(\frac{2n-1}{2}\pi x\right).
\end{aligned}
$$
