CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 5 · Solutions

Posted Wednesday 19 September 2012. Due Wednesday 26 September 2012, 5pm.

All of the problems on this set use the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

1. [30 points: 6 points each for (a) and (b); 9 points each for (c) and (d)]
   Consider the linear operator $L_b : C_b^2[0,1] \to C[0,1]$ defined by

   $$L_b u = -\frac{d^2u}{dx^2},$$

   where

   $$C_b^2[0,1] = \Big\{ u \in C^2[0,1] : \frac{du}{dx}(0) = u(1) = 0 \Big\}.$$

   (a) Is L_b symmetric?

   (b) What is the null space of L_b?
   That is, find all $u \in C_b^2[0,1]$ such that $L_b u(x) = 0$ for all $x \in [0,1]$.

   (c) Show that $(L_b u, u) > 0$ for all nonzero $u \in C_b^2[0,1]$ and explain why this implies that $\lambda > 0$ for all eigenvalues λ.

   (d) Find the eigenvalues and eigenfunctions of L_b.

Solution.

(a) Yes, L_b *is symmetric.*
To prove this, we shall show that for any $u, v \in C_b^2[0,1]$, we have $(L_b u, v) = (u, L_b v)$. He we shall use primes to denote derivation with respect to x. Integrating by parts twice, we have

$$\begin{aligned}(L_b u, v) &= \int_0^1 -u''(x)v(x)\,dx \\ &= -[u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x)\,dx \\ &= -[u'(x)v(x)]_0^1 + [u(x)v'(x)]_0^1 - \int_0^1 u(x)v''(x)\,dx.\end{aligned}$$

Since $u, v \in C_b^2[0,1]$ we have $u'(0) = 0$ and $v(1) = 0$, and hence the first term in square brackets must be zero. Again using $u, v \in C_b^2[0,1]$ we have $v'(0) = 0$ and $u(1) = 0$, and hence the second term in square brackets is also zero. It follows that

$$(L_b u, v) = \int_0^1 u(x)(-v''(x))\,dx = (u, L_b v).$$

(b) The general solution to the differential equation

$$-\frac{d^2u}{dx^2} = 0$$

has the form

$$u(x) = A + Bx$$

for constants A and B that are determined by the boundary conditions. In order for u to be in $C_b^2[0,1]$, we must have $u'(0) = 0$; since $u'(x) = B$, we have $B = 0$. Now $u \in C_b^2[0,1]$ also requires $u(1) = 0$, and since $u(1) = A$, we conclude that $A = 0$ too, meaning that $u(x) = A + Bx = 0$ for all $x \in [0,1]$. Thus, the only element of the null space is the zero function, that is, the null space is trivial, $\mathcal{N}(L_b) = \{0\}$.

(c) [**GRADERS**: please count $(L_b u, u) > 0$ for 5 points, and count $\lambda > 0$ for 4 points.]
We wish to show that $(L_b u, u) > 0$ for all $\in C_b^2[0,1]$. Using the first integration by parts from part (a), we have

$$\begin{aligned}(L_b u, u) &= -[u'(x)u(x)]_0^1 + \int_0^1 u'(x)u'(x)\,dx \\ &= \int_0^1 (u'(x))^2\,dx.\end{aligned}$$

Thus, $(L_b u, u)$ is the integral of a nonnegative function, so it is nonnegative.
This statement implies that all eigenvalues are non-negative, since

$$\lambda(u,u) = (\lambda u, u) = (L_b u, u) \geq 0,$$

and we know that $(u,u) > 0$ for all nonzero u due to the positivity of the inner product.
One can show that $u'(x) \neq 0$ for some $x \in [0,1]$: otherwise u would be a constant, and the only constant that satisfies the boundary conditions is $u(x) = 0$, which is prohibited from consideration in the problem statement. Thus, we can say $(L_b u, u) > 0$ for all nonzero u.

(d) Suppose that $L_b u = \lambda u$. The general solution to the equivalent differential equation

$$-\frac{d^2u}{dx^2}(x) = \lambda u(x)$$

has the form

$$u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

Now we must find values of the constants A, B, and $\sqrt{\lambda}$ that will satisfy the boundary conditions. Since

$$u'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

and thus

$$u'(0) = A\sqrt{\lambda},$$

the condition $u'(0) = 0$ implies that $A = 0$. On the other hand, the condition $u(1) = 0$ implies that

$$u(1) = B\cos(\sqrt{\lambda}) = 0,$$

which can be achieved with nonzero B provided that $\sqrt{\lambda} = (n - 1/2)\pi$ for integers n. We thus have the eigenvalues

$$\lambda_n = (n - 1/2)^2\pi^2$$

with corresponding eigenfunctions

$$u_n(x) = \cos(\sqrt{\lambda_n}x)$$

or any nonzero scaling of the same function, such as the normalized version:

$$u_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x).$$

Since $n = 0, -1, -2, \ldots$ give the same eigenvalues and eigenvectors as $n = 1, 2, 3, \ldots$, we can restrict n to the positive integers.

2. [35 points: 12 points for (a); 7 points for (b); 9 points for (c); 7 points for (d)]
Consider the operator $L_D : C_D^2[0,1] \to C[0,1]$ defined by

$$L_D u = -\frac{d^2 u}{dx^2},$$

with homogeneous Dirichlet boundary conditions imposed by the domain

$$C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}.$$

Recall that the eigenvalues of L_D are $\lambda_n = n^2\pi^2$ with associated normalized eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

("Normalized" means that these eigenfunctions each have norm equal to one, $\|\psi_n\|^2 = (\psi_n, \psi_n) = 1$.)

We wish to study the equation $Lu = f$ for $f(x) = 1$, a problem to be addressed in Lecture 14. First consider the best approximation to f from $\text{span}\{\psi_1, \ldots, \psi_N\}$:

$$f_N = \sum_{n=1}^{N} \frac{(f, \psi_n)}{(\psi_n, \psi_n)} \psi_n = \sum_{n=1}^{N} c_n \psi_n.$$

In class we shall note that $f_N(x)$ *does not converge* to $f(x)$ for all points $x \in [0,1]$: in particular, $f_N(0) = f_N(1) = 0$ for all N, while $f(0) = f(1) = 1$. However, we claimed in class that f_N *does converge to* f *in norm*, that is, $\|f - f_N\| \to 0$ as $N \to \infty$. In this problem, you will justify that statement.

(a) Use the properties of inner products, the orthogonality of the eigenfunctions, the fact that $(\psi_k, \psi_k) = 1$, and $\|f - f_N\|^2 = (f - f_N, f - f_N)$ to derive the general formula

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} c_n^2.$$

(b) For $f(x) = 1$, we computed in class that $c_n = 2\sqrt{2}/(n\pi)$ for odd n, and $c_n = 0$ for even n. Use this expression for c_n and your formula from part (a) to produce a `loglog` plot of the error $\|f - f_N\|$ versus N for all integers $N = 1, \ldots, 10^4$. (Optional: take $N = 1, \ldots, 10^6$. You will need to write efficient MATLAB code for this to run quickly.)

(c) For $f(x) = 1$, the equation $Lu = f$ has the exact solution $u(x) = x(1 - x)/2$. Confirm that the spectral method approximation

$$u_N = \sum_{n=1}^{N} \frac{c_n}{\lambda_n} \psi_n$$

provides the best approximation to u from the subspace $\text{span}\{\psi_1, \ldots, \psi_N\}$. To do this, simply show that the coefficient of ψ_n you would get for the best approximation of u (which requires knowledge of u) matches the coefficient produced by the spectral method (which did not require knowledge of u) for this particular f and u:

$$\frac{(u, \psi_n)}{(\psi_n, \psi_n)} = \frac{c_n}{\lambda_n},$$

where c_n comes from the best approximation to f given in part (b).

(d) Given the result of part (c), the same argument used in part (a) tells us that

$$\|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n^2}.$$

(You do not need to show this explicitly.) Use this formula to produce a `loglog` plot of the error $\|u-u_N\|$ for $N = 1, \ldots, 10^4$ (or $N = 1, \ldots, 10^6$) on the same plot you made in part (b). (Be aware that the error may appear to flatline around 10^{-8}: this is a consequence of the computer's floating point arithmetic, and is not a concern of ours here. To learn more about this phenomenon, take CAAM 353 or CAAM 453!)

Solution.

(a) We compute

$$\begin{aligned}
\|f - f_N\|^2 &= \big(f - f_N, f - f_N\big) \\
&= \Big(f - \sum_{n=1}^{N} c_n\psi_n, f - \sum_{m=1}^{N} c_m\psi_m\Big) \\
&= \Big(f - \sum_{n=1}^{N} c_n\psi_n, f\Big) - \Big(f - \sum_{n=1}^{N} c_n\psi_n, \sum_{m=1}^{N} c_m\psi_m\Big) \\
&= (f,f) - \Big(\sum_{n=1}^{N} c_n\psi_n, f\Big) - \Big(f, \sum_{m=1}^{N} c_m\psi_m\Big) + \Big(\sum_{n=1}^{N} c_n\psi_n, \sum_{m=1}^{N} c_m\psi_m\Big) \\
&= (f,f) - \sum_{n=1}^{N} c_n(\psi_n, f) - \sum_{m=1}^{N} c_m(f, \psi_m) + \sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m(\psi_n, \psi_m) \\
&= (f,f) - \sum_{n=1}^{N} c_n(\psi_n, f) - \sum_{m=1}^{N} c_m(f, \psi_m) + \sum_{n=1}^{N} c_n^2(\psi_n, \psi_n) \\
&= (f,f) - \sum_{n=1}^{N} c_n^2 - \sum_{m=1}^{N} c_m^2 + \sum_{n=1}^{N} c_n^2 \\
&= (f,f) - \sum_{n=1}^{N} c_n^2,
\end{aligned}$$

where at each equal sign we have use: (1) the definition of the norm; (2) the definition of f_N; (3) linearity of the inner product in the second argument; (4) linearity of the inner product in the first component; (5) linear of the inner product; (6) orthogonality of the eigenfunctions: $(\psi_n, \psi_n) = 0$ if $n \neq m$; (7) $(f, \psi_n) = (\psi_n, f) = c_n$ and $(\psi_n, \psi_n) = 1$; (8) simple algebra.

(b) Code and the plot follow at the end of the problem.

(c) You can work out this equality for the specific case of $u(x) = x(1-x)/2$. One computes (e.g., in Mathematica) that

$$(u, \psi_n) = (x(1-x)/2, \psi_n) = \begin{cases} \dfrac{2\sqrt{2}}{n^3\pi^3}, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

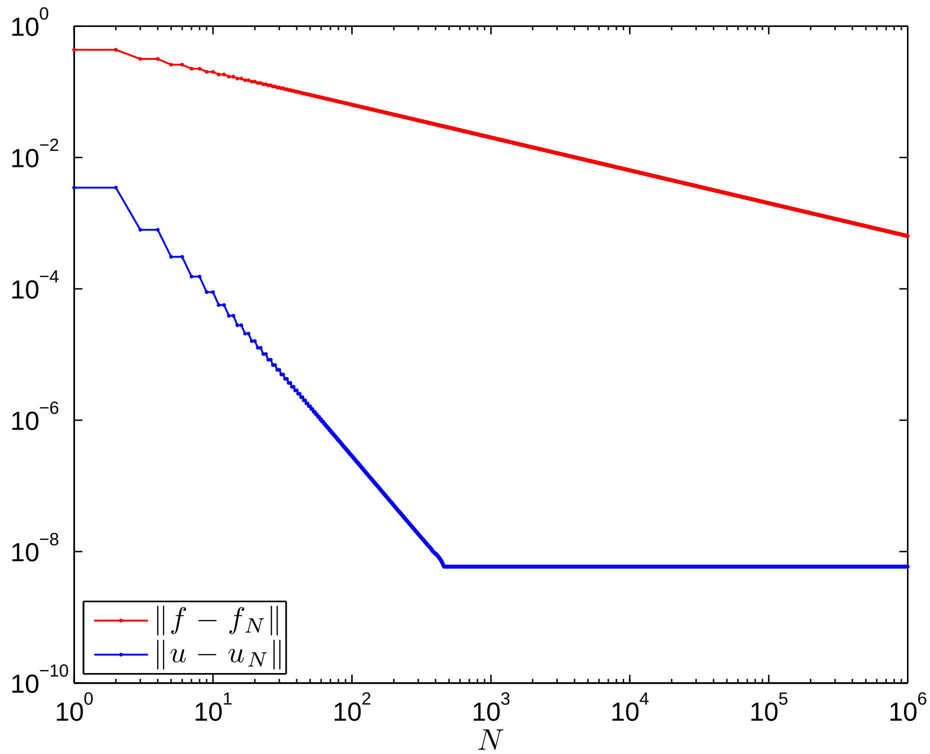
This formula for $(u, \psi_n) = (u, \psi_n)/(\psi_n, \psi_n)$ exactly matches c_n/λ_n. This calculation is all that is needed for full credit.

Alternatively, one can show that this result holds *in general.* Notice that since $Lu = f$ and L is symmetric, we have

$$c_n = \frac{(f,\psi_n)}{(\psi_n,\psi_n)} = \frac{(Lu,\psi_n)}{(\psi_n,\psi_n)} = \frac{(u,L\psi_n)}{(\psi_n,\psi_n)} = \frac{(u,\lambda_n\psi_n)}{(\psi_n,\psi_n)} = \lambda_n \frac{(u,\psi_n)}{(\psi_n,\psi_n)},$$

which is equivalent to the given statement.

(d) The requested plot for (b) and (d) is shown below.



The code that produced the plot above for (b) and (c) is shown below.

```
n = [1:1e6]';
cn = (sqrt(2)/pi)*(1+(-1).^(n+1))./(n);
lamn = pi^2*n.^2;
normf2 = 1;
normu2 = 1/120;
figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')
hold on
loglog(n, sqrt(normu2-cumsum((cn./lamn).^2)),'b.-')

set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$','$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr
```

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3. [35 points: 15 points for (a); 6 points each for (b) and (c); 8 points for (d)]
This problem concerns the same operator from Problem 2, $L : C_D^2[0,1] \to C[0,1]$ defined by

$$L_D u = -\frac{d^2u}{dx^2},$$

with homogeneous Dirichlet boundary conditions imposed via

$$C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}.$$

The eigenvalues and (normalized) eigenfunctions remain as in Problem 2: $\lambda_n = n^2\pi^2$ and $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for $n = 1, 2, \ldots$. Now let $f(x) = x^2(1-x)$.

(a) For this f, compute the coefficients

$$c_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

in the expansion

$$f = \sum_{n=1}^{\infty} c_n \psi_n.$$

You may determine these by hand, by consulting a table of integrals, or by using a symbolic mathematics package like Mathematica or the Symbolic Toolbox in MATLAB.

(b) Produce a plot (or series of plots) comparing $f(x)$ to the partial sums

$$f_N(x) = \sum_{k=1}^{N} c_k \psi_k(x)$$

for $N = 1, \ldots, 10$.

(c) Plot the approximations u_N to the true solution u that you obtain using the spectral method:

$$u_N(x) = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \psi_k(x)$$

for $N = 1, \ldots, 10$.

(d) Now replace the homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ above with the inhomogeneous Dirichlet conditions $u(0) = -1/100$ and $u(1) = 1/100$. Describe how to adjust your solution from part (c) to account for these boundary conditions, and produce a plot of the solution with these inhomogeneous boundary conditions, based on u_{10} from part (c).

Solution.

(a) Expand $f(x) = x^2(1-x) = x^2 - x^3$. We can compute the coefficients (f, ψ_k) as

$$(f, \psi_k) = \int_0^1 x^2\big(\sqrt{2}\sin(k\pi x)\big)\,dx - \int_0^1 x^3\big(\sqrt{2}\sin(k\pi x)\big)\,dx.$$

The first integral on the right can be computed using Mathematica, etc. Alternatively, one can work it out directly:

$$\begin{aligned}
\sqrt{2}\int_0^1 x^2 \sin(k\pi x)\,dx &= \sqrt{2}\bigg(\Big[\frac{-x^2\cos k\pi x}{k\pi}\Big]_0^1 + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\bigg)\\
&= \sqrt{2}\bigg(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\bigg)\\
&= \sqrt{2}\bigg(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^2\pi^2}\Big[x\sin(k\pi x)\Big]_0^1 - \int_0^1 \sin(k\pi x)\,dx\bigg)\\
&= \sqrt{2}\bigg(\frac{(-1)^{k+1}}{k\pi} - \frac{2}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx\bigg)
\end{aligned}$$

$$= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^3\pi^3}\big[\cos(k\pi x)\big]_0^1\Big)$$

$$= \sqrt{2}\Big(\frac{(2-k^2\pi^2)(-1)^k - 2}{k^3\pi^3}\Big).$$

The second integral follows from integrating thrice by parts:

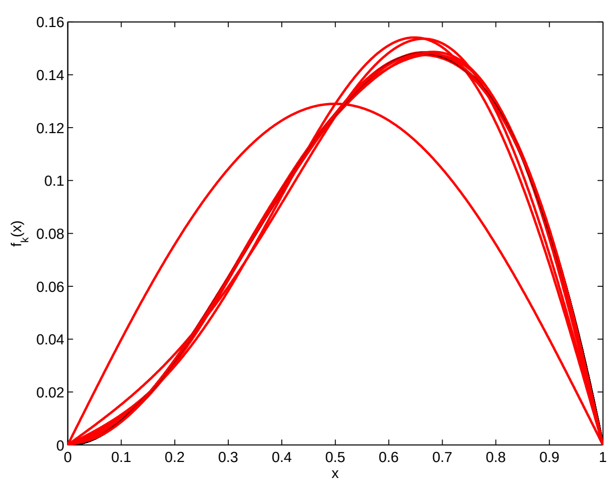
$$\int_0^1 x^3\big(\sqrt{2}\sin(k\pi x)\big)\,dx = \frac{\sqrt{2}(-1)^n(6-k^2\pi^2)}{k^3\pi^3}.$$

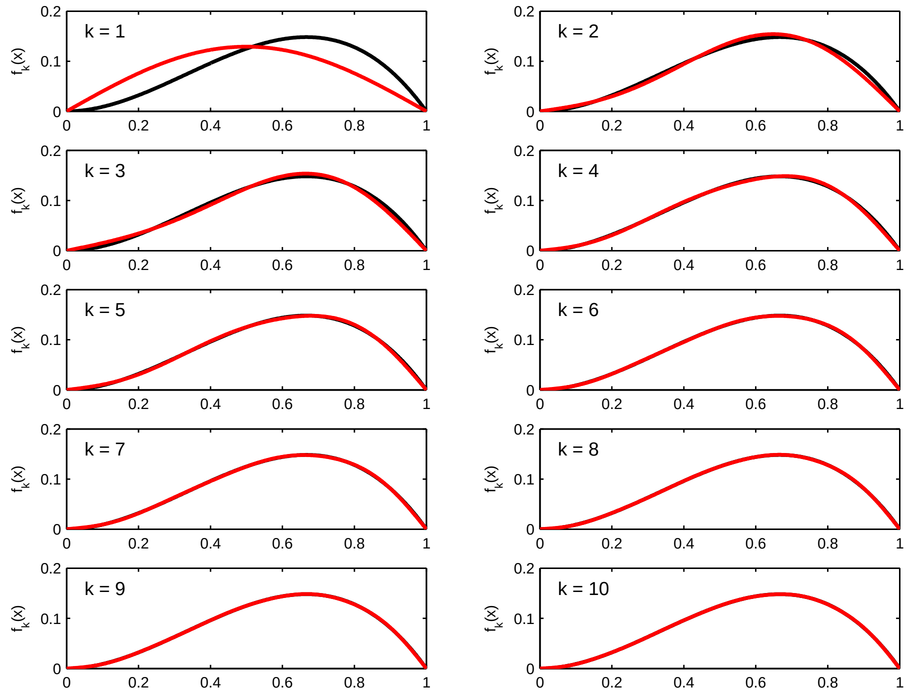
Assembling these results, we simplify to obtain

$$(f,\psi_k) = \frac{\sqrt{2}\big(4(-1)^{k+1}-2\big)}{k^3\pi^3}.$$

$$\begin{aligned}
\sqrt{2}\int_0^1 x^2\sin(k\pi x)\,dx &= \sqrt{2}\Big(\Big[\frac{-x^2\cos k\pi x}{k\pi}\Big]_0^1 + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\Big)\\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\Big)\\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^2\pi^2}\Big[x\sin(k\pi x)\Big]_0^1 - \int_0^1 \sin(k\pi x)\,dx\Big)\\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} - \frac{2}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx\Big)\\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^3\pi^3}\big[\cos(k\pi x)\big]_0^1\Big)\\
&= \sqrt{2}\Big(\frac{(2-k^2\pi^2)(-1)^k - 2}{k^3\pi^3}\Big).
\end{aligned}$$

(b) Partial sums of the series formula for f are shown in the plots below. Code follows at the end of the problem. The function f happens to satisfy homogeneous Dirichlet boundary conditions, and convergence is quite quick.

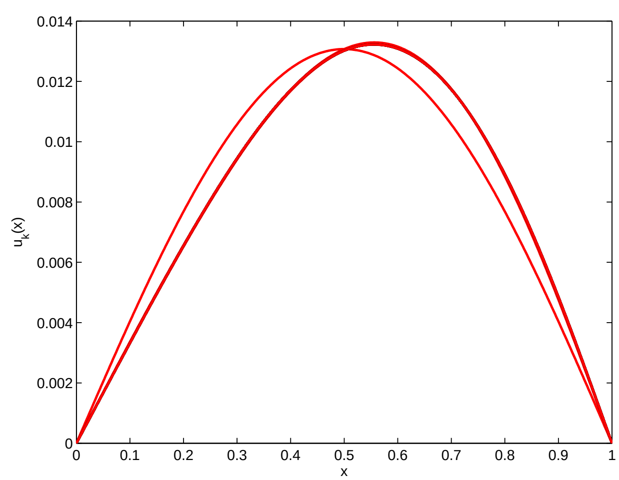


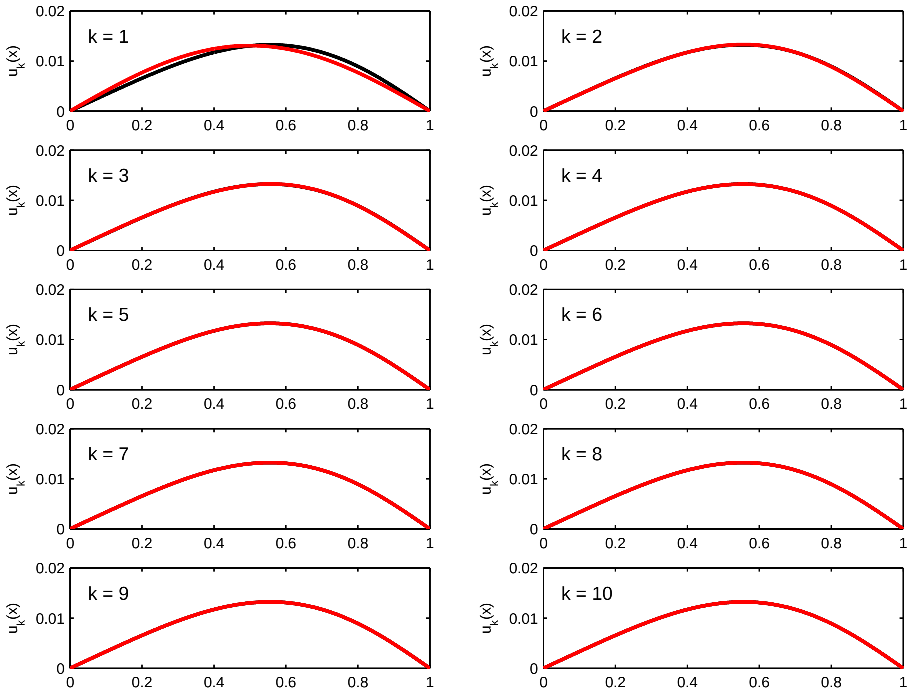


(c) The true solution to this problem (not asked for in the problem statement) can be computed as $u(x) = (2x - 5x^4 + 3x^5)/60$. The spectral method gives $u(x)$ as the series

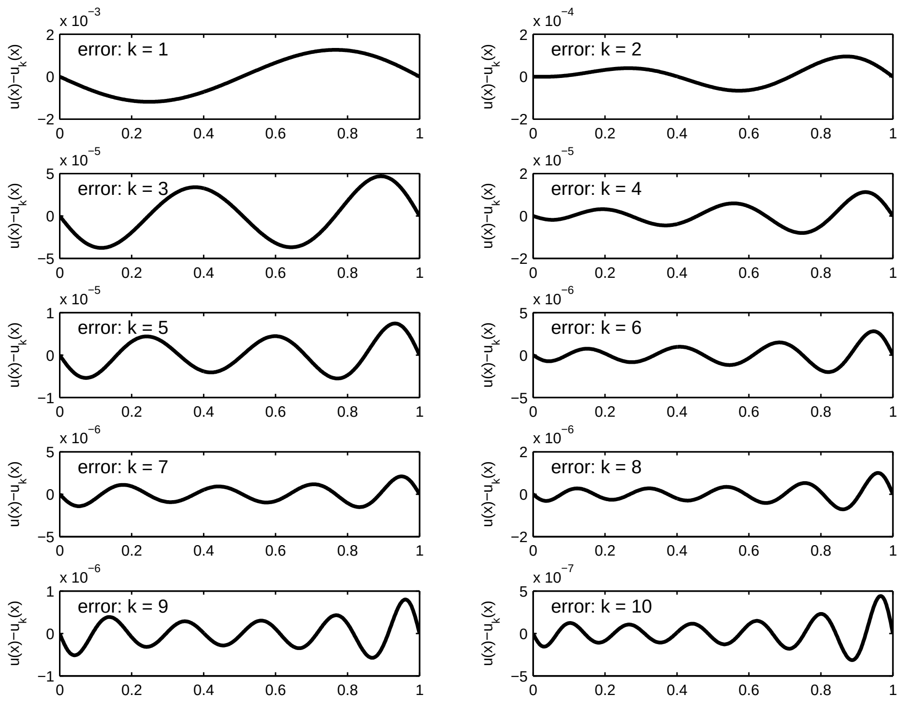
$$u(x) = \sum_{k=1}^{\infty} \sqrt{2}\Big(\frac{4(-1)^{k+1} - 2}{k^5\pi^5}\Big)\big(\sqrt{2}\sin(k\pi x)\big),$$

whose coefficients decay even more rapidly than did the coefficients for f itself, explaining the fantastic convergence rate. The next two plots compare the sum from the spectral method (red lines) to the true solution (black line). The following plot shows the error as a function of x for $k = 1, \ldots, N$.





Plots of the error as k increases:



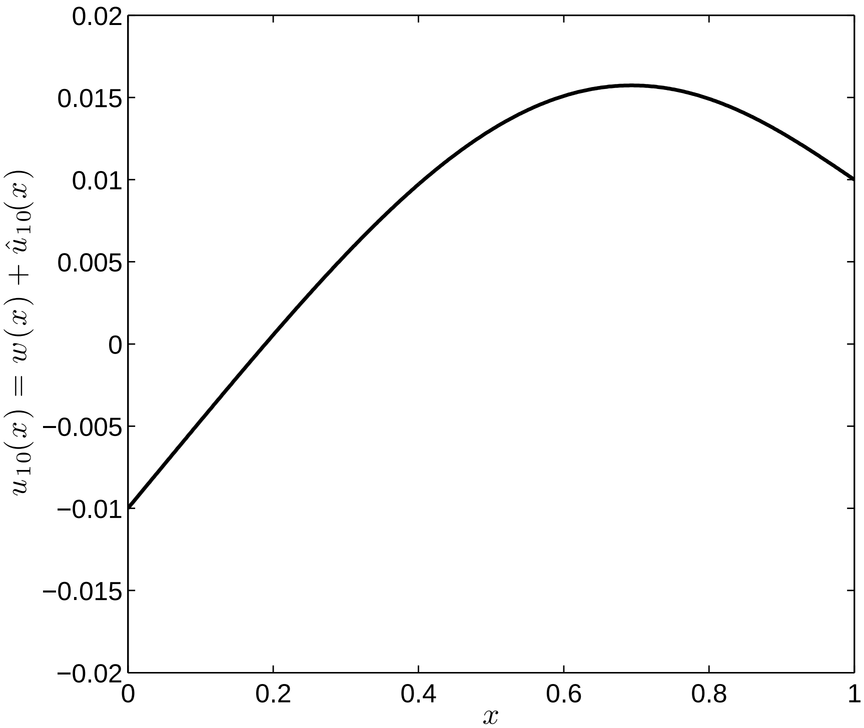
(d) To incorporate inhomogeneous Dirichlet boundary conditions, we will write the solution in the form $u = \widehat{u} + w$, where the correction $w(x) = \alpha + \beta x$ has the property that $-w''(x) = 0$, and $\widehat{u}$ denotes the solution to $L\widehat{u} = f$ with homogeneous Dirichlet boundary conditions; thus $\widehat{u}$ is precisely the solution u worked out in part (c). Notice that $u(0) = w(0) = \widehat{u}(0) = w(0) = \alpha$ and $u(1) = w(1) = \widehat{u}(1) = w(1) = \alpha + \beta$. Since we want $u(0) = -1/100$ and $u(1) = 1/100$, we solve to find:

$$\alpha = -1/100, \qquad \beta = 2/100.$$

Thus, the solution to our equation with these inhomogeneous boundary conditions is:

$$u(x) = -1/100 + (2/100)x + \sum_{k=1}^{\infty} \sqrt{2}\Big(\frac{4(-1)^{k+1}-2}{k^5\pi^5}\Big)\big(\sqrt{2}\sin(k\pi x)\big),$$

The plot of $u_{10}(x) = w(x) + \widehat{u}_{10}(x)$ is shown below.



The plots above were computed using the following MATLAB code.

```
% f(x) = x^2(1-x);

% compute the inner products (f, phi_k) for k=1,...,30
  k = [1:30]';
  ck = sqrt(2)*(4*(-1).^(k+1)-2)./(k.^3*pi^3);

% plot first 10 partial sums fk, all on one plot
 figure(2), clf
 x = linspace(0,1,500)';
 fk = zeros(size(x));
 for k=1:10
    plot(x,(x.^2).*(1-x),'k-','linewidth',2), hold on
    fk = fk + ck(k)*sqrt(2)*sin(k*pi*x);
    plot(x,fk,'r-','linewidth',2)
    xlabel('x'), ylabel('f_k(x)')
 end
 print -depsc2 sineseries2b

% plot first 10 partial sums fk, all on 10 different plot
 figure(3), clf
 x = linspace(0,1,500)';
 fk = zeros(size(x));
 for k=1:10
```

```
   subplot(5,2,k)
   plot(x,(x.^2).*(1-x),'k-','linewidth',2), hold on
   fk = fk + ck(k)*sqrt(2)*sin(k*pi*x);
   plot(x,fk,'r-','linewidth',2)
   axis([0 1 0 0.2])
   set(gca,'fontsize',8)
   text(0.05,0.16,sprintf('k = %d',k))
   ylabel('f_k(x)')
 end
 print -depsc2 sineseries2c


% plot first 10 partial sums uk, all on one plot
 figure(4), clf
 x = linspace(0,1,500)';
 uk = zeros(size(x));
 for k=1:10
   plot(x,(2*x-5*(x.^4)+3*(x.^5))/60,'k-','linewidth',2), hold on
   lamk = k^2*pi^2;
   uk = uk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
   plot(x,uk,'r-','linewidth',2)
   xlabel('x'), ylabel('u_k(x)')
 end
 print -depsc2 sineseries2d

% plot first 10 partial sums uk, all on 10 different plot
 figure(5), clf
 x = linspace(0,1,500)';
 uk = zeros(size(x));
 for k=1:10
   subplot(5,2,k)
   plot(x,(2*x-5*(x.^4)+3*(x.^5))/60,'k-','linewidth',2), hold on
   lamk = k^2*pi^2;
   uk = uk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
   plot(x,uk,'r-','linewidth',2)
   axis([0 1 0 .02])
   set(gca,'fontsize',8)
   text(0.05,0.015,sprintf('k = %d',k))
   ylabel('u_k(x)')
 end
 print -depsc2 sineseries2e

% plot error in first 10 partial sums uk, all on 10 different plot
 figure(6), clf
 x = linspace(0,1,500)';
 uk = zeros(size(x));
 for k=1:10
   subplot(5,2,k)
   lamk = k^2*pi^2;
   uk = uk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
   plot(x,(2*x-5*(x.^4)+3*(x.^5))/60 - uk,'k-','linewidth',2), hold on
   set(gca,'fontsize',8)
   text(0.05,max(ylim)-.175*diff(ylim),sprintf('error: k = %d',k))
   ylabel('u(x)-u_k(x)')
 end
 print -depsc2 sineseries2f

% inhomogeneous boundary condition:
```

```
% plot first 10 partial sums uk, all on 10 different plot
 figure(7), clf
 x = linspace(0,1,500)';
 w = -1/100 + (2/100)*x;
 uhatk = zeros(size(x));
 for k=1:10
    lamk = k^2*pi^2;
    uhatk = uhatk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
    plot(x,w+uhatk,'k-','linewidth',2)
    axis([0 1 -.02 .02])
    set(gca,'fontsize',14)
    ylabel('$u_{10}(x) = w(x) + \hat{u}_{10}(x)$','fontsize',16,'interpreter','latex')
    xlabel('$x$','fontsize',16,'interpreter','latex')
 end
 print -depsc2 sineseries2g
```

Challenge problem [5 bonus points]

Many fluid dynamics problems lead to *advection–diffusion* equations, the simplest example of which is

$$u''(x) + cu'(x) = f(x),$$

for $x \in [0,1]$ with $u(0) = u(1) = 0$. (The u'' term describes diffusion of a fluid; the constant c describes the strength with which the fluid advects across the domain through the cu' term.)

Define the linear operator $L : C^2_D[0,1] \to C[0,1]$ by $Lu = u'' + cu'$.

Determine all the eigenvalues and eigenfunctions of L.
Are the eigenfunctions orthogonal? Explain.

Solution. The eigenvalues are

$$\lambda_n = -\frac{c^2}{4} - n^2\pi^2, \quad n = 1, 2, \ldots$$

with corresponding eigenfunctions

$$\psi_n(x) = e^{-cx/2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

If $c \neq 0$, then these eigenfunctions are not orthogonal (and the operator is not symmetric).

To receive credit, students must show how they derived these eigenvalues and eigenfunctions.

GRADERS: please let me know those students who got this bonus problem correct.
