CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 25

Posted Monday 7 October 2013. Due 1pm Friday 18 October 2013.

25. [25 points] We have been able to obtain nice formulas for the eigenvalues of the operators that we have considered thus far. This problem illustrates that this is not always the case.

    Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

    $$(v, w) = \int_0^1 v(x)w(x)\,dx.$$

    Let the linear operator $L : V \to C[0,1]$ be defined by

    $$Lu = -u''$$

    where

    $$V = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

    Note that if $u \in V$ then u satisfies the homogeneous Robin boundary condition

    $$u(0) - u'(0) = 0$$

    and the homogeneous Dirichlet boundary condition

    $$u(1) = 0.$$

    (a) Prove that L is symmetric.

    (b) Is zero an eigenvalue of L?

    (c) Show that $(Lu, u) \geq 0$ for all $u \in V$. What does this and the answer to part (b) then allow us to say about the eigenvalues of L?

    (d) Show that the eigenvalues λ of L must satisfy the equation $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$.

    (e) Use MATLAB to plot $g(x) = -\tan(x)$ and $h(x) = x$ on the same figure. Use the command `axis([0 5*pi -5*pi 5*pi])` and make sure that your plot gives an accurate representation of these functions on the region shown on the figure when this command is used. By hand or using MATLAB, mark on your plot the points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Note that $g \notin C[0, 5\pi]$. How many eigenvalues λ does L have which are such that $\sqrt{\lambda} \leq 5\pi$?