CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 5 · Solutions

Posted Wednesday 24 September, 2014. Due 5pm Wednesday 1 October, 2014.

Please write your name and ***residential college*** *on your homework.*

1. [25 points: 5 points each]

Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

(a) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V}=C^1[0,1]$.

(b) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

(c) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V}=C[0,1]$.

(d) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 (u(x)+v(x))\,dx$ where $\mathcal{V}=C[0,1]$.

(e) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V}=C[-1,1]$.

Solution.

(a) [5 points] *This mapping is not an inner product*: it is not symmetric and it is not positive definite. The mapping is not symmetric. For example, if $u(x)=1$ and $v(x)=x$, then

$$(u,v)=\int_0^1 u(x)v'(x)\,dx=\int_0^1 1\,dx=1,$$

yet

$$(v,u)=\int_0^1 v(x)u'(x)\,dx=\int_0^1 0\,dx=0.$$

The mapping is also not positive definite. For example, if $u(x)=1$, then $(u,u)=0$ and if $u(x)=1-x$, then

$$(u,u)=\int_0^1 (1-x)(-1)\,dx=-1/2.$$

For what it is worth, we note that the mapping is linear in the first argument since

$$(\alpha u+\beta v,w)=\alpha\int_0^1 u(x)w'(x)\,dx+\beta\int_0^1 v(x)w'(x)\,dx=\alpha(u,w)+\beta(v,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$. It is also linear in the second argument since

$$(u,\alpha v+\beta w)=\alpha\int_0^1 u(x)v'(x)\,dx+\beta\int_0^1 u(x)w'(x)\,dx=\alpha(u,v)+\beta(u,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$.

(b) [5 points] *This mapping is not an inner product*: it is not linear in the first argument. If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 |\alpha u(x) + \beta v(x)||w(x)|\, dx$$

and

$$\alpha(u, w) + \beta(v, w) = \alpha \int_0^1 |u(x)||w(x)|\, dx + \beta \int_0^1 |v(x)||w(x)|\, dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = -1$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 |-1||1|\, dx = \int_0^1 1\, dx = 1$$

but

$$\alpha(u, w) + \beta(v, w) = -\int_0^1 |1||1|\, dx = -\int_0^1 1\, dx = -1$$

and so the mapping is not linear in the first argument.
The mapping is symmetric, as

$$(u, v) = \int_0^1 |u(x)||v(x)|\, dx = \int_0^1 |v(x)||u(x)|\, dx = (v, u)$$

for all $u, v \in C[0,1]$.
Moreover, the mapping is positive definite as for all $u \in C[0,1]$

$$(u, u) = \int_0^1 |u(x)|^2\, dx$$

is the integral of a nonnegative function, and hence is nonnegative and $(u, u) = 0$ only if $u = 0$.

(c) [5 points] *This mapping is an inner product.*
The mapping is symmetric, as

$$(u, v) = \int_0^1 u(x)v(x)e^{-x}\, dx = \int_0^1 v(x)u(x)e^{-x}\, dx = (v, u)$$

for all $u, v \in C[0,1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x}\, dx \\ &= \alpha \int_0^1 u(x)w(x)e^{-x}\, dx + \beta \int_0^1 v(x)w(x)e^{-x}\, dx \\ &= \alpha(u, w) + \beta(v, w)\end{aligned}$$

for all $u, v, w \in C[0,1]$ and all $\alpha, \beta \in \mathbb{R}$.
The function e^{-x} is positive valued for all $x \in [0,1]$, so we have that

$$(u, u) = \int_0^1 (u(x))^2 e^{-x} dx$$

is the integral of a nonnegative function, and hence is also nonnegative. If $(u, u) = 0$ then $(u(x))^2 e^{-x} = 0$ for all $x \in [0,1]$ and, since $e^{-x} > 0$ for all $x \in [0,1]$, this means that $u(x) = 0$ for all $x \in [0,1]$, i.e., $u = 0$. Hence, the mapping is positive definite.

(d) [5 points] *This mapping is not an inner product*: it is not linear in the first argument and it is not positive definite.

If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x) + w(x))\, dx$$

and

$$\alpha(u, w) + \beta(v, w) = \alpha \int_0^1 (u(x) + w(x))\, dx + \beta \int_0^1 (v(x) + w(x))\, dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = 2$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 (2+1)\, dx = \int_0^1 3\, dx = 3$$

but

$$\alpha(u, w) + \beta(v, w) = 2\int_0^1 (1+1)\, dx = 2\int_0^1 2\, dx = 4$$

and so $(\cdot, \cdot)$ is not linear in the first argument.

The mapping $(\cdot, \cdot)$ is also not positive definite. For example, if $u(x) = -1$, then

$$(u, u) = \int_0^1 (u(x) + u(x))\, dx = \int_0^1 -2\, dx = -2 < 0.$$

The mapping is symmetric, as

$$(u, v) = \int_0^1 (u(x) + v(x))\, dx = \int_0^1 (v(x) + u(x))\, dx = (v, u)$$

for all $u, v \in C[0,1]$.

(e) [5 points] *This mapping is not an inner product*: it is not positive definite.

If $w(x) = 1$ for all $x \in [-1,1]$ then $w \in C[-1,1]$ and $w \neq 0$ but

$$(w, w) = \int_{-1}^1 x w(x) w(x)\, dx = \int_{-1}^1 x\, dx = \left[\frac{1}{2}x^2\right]_{-1}^1 = \frac{1}{2}\left(1^2 - (-1)^2\right) = \frac{1}{2}(1-1) = 0$$

and so $(\cdot, \cdot)$ is not positive definite.

The mapping is symmetric, as

$$(u, v) = \int_{-1}^1 x u(x) v(x)\, dx = \int_{-1}^1 x v(x) u(x)\, dx = (v, u)$$

for all $u, v \in C[-1,1]$.

The mapping is also linear in the first argument since

$$\begin{aligned}
(\alpha u + \beta v, w) &= \int_{-1}^1 x(\alpha u(x) + \beta v(x)) w(x)\, dx \\
&= \alpha \int_{-1}^1 x u(x) w(x)\, dx + \beta \int_{-1}^1 x v(x) w(x)\, dx \\
&= \alpha(u, w) + \beta(v, w)
\end{aligned}$$

for all $u, v, w \in C[-1,1]$ and all $\alpha, \beta \in \mathbb{R}$.

2. [24 points: 6 points each]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $\phi_3 \in C[-1,1]$, and $f \in C[-1,1]$ be defined by

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = 3x^2 - 1,$$

and

$$f(x) = e^x,$$

for all $x \in [-1,1]$. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\| : C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2, \phi_3\}$ is orthogonal with respect to the inner product $(\cdot,\cdot)$, which is defined on $[-1,1]$.

(a) By hand, construct the best approximation f_1 to f from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) By hand, construct the best approximation f_2 to f from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) By hand, construct the best approximation f_3 to f from span$\{\phi_1, \phi_2, \phi_3\}$ with respect to $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) [4 points] The best approximation to $f(x) = e^x$ from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$ is

$$f_1(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x).$$

We compute

$$(\phi_1,\phi_1) = \int_{-1}^{1} 1^2\,dx = [x]_{-1}^{1} = 1 - (-1) = 2$$

and

$$(f,\phi_1) = \int_{-1}^{1} e^x\,dx = [e^x]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

and hence

$$f_1(x) = \frac{1}{2}\left(e - \frac{1}{e}\right).$$

(b) [7 points] Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot,\cdot)$, i.e., $(\phi_1,\phi_2) = 0$, the best approximation to $f(x) = e^x$ from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$f_2(x) = \frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x) + \frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x) = f_1(x) + \frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x).$$

Noting that

$$(\phi_2,\phi_2)=\int_{-1}^{1} x^2\,dx=\left[\frac{x^3}{3}\right]_{-1}^{1}=\frac{1}{3}-\left(-\frac{-1}{3}\right)=\frac{1}{3}-\frac{1}{3}=\frac{2}{3}$$

and

$$(f,\phi_2)=\int_{-1}^{1} xe^x\,dx=[xe^x]_{-1}^{1}-\int_{-1}^{1} e^x\,dx=e^1-\left(-e^{-1}\right)-(f,\phi_1)=e+\frac{1}{e}-e+\frac{1}{e}=\frac{2}{e}$$

we can compute that

$$f_2(x)=f_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)=\frac{1}{2}\left(e-\frac{1}{e}\right)+\frac{3}{e}x.$$

(c) [7 points] Since,

$$(\phi_1,\phi_2)=(\phi_1,\phi_3)=(\phi_2,\phi_3)=0,$$

the best approximation to $f(x)=e^x$ from span$\{\phi_1,\phi_2,\phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$f_3(x)=\frac{(f,\phi_1)}{(\phi_1,\phi_1)}\phi_1(x)+\frac{(f,\phi_2)}{(\phi_2,\phi_2)}\phi_2(x)+\frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x)=f_2(x)+\frac{(f,\phi_3)}{(\phi_3,\phi_3)}\phi_3(x).$$

Toward this end, compute

$$\begin{aligned}
(\phi_3,\phi_3) &= \int_{-1}^{1}(3x^2-1)^2\,dx\\
&= \int_{-1}^{1} 9x^4-6x^2+1\,dx\\
&= \int_{-1}^{1} 9x^4\,dx-6(\phi_2,\phi_2)+(\phi_1,\phi_1)\\
&= \left[\frac{9x^5}{5}\right]_{-1}^{1}-6\frac{2}{3}+2\\
&= \frac{9}{5}-\left(-\frac{9}{5}\right)-\frac{12}{3}+2\\
&= \frac{18}{5}-\frac{12}{3}+2\\
&= \frac{54}{15}-\frac{60}{15}+\frac{30}{15}\\
&= \frac{24}{15}\\
&= \frac{8}{5}
\end{aligned}$$

and

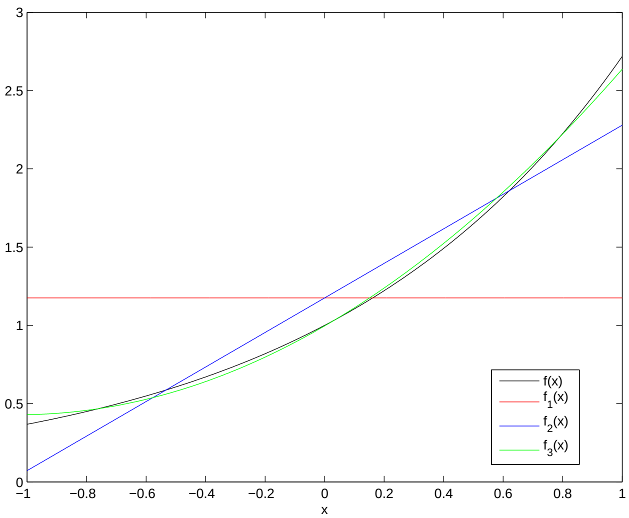
$$\begin{aligned}
(f,\phi_3) &= \int_{-1}^{1}(3x^2-1)e^x\,dx\\
&= \int_{-1}^{1} 3x^2e^x\,dx-(f,\phi_1)\\
&= \left[3x^2e^x\right]_{-1}^{1}-\int_{-1}^{1} 6xe^x\,dx-\left(e-\frac{1}{e}\right)
\end{aligned}$$

$$= 3e^1 - 3e^{-1} - 6(f, \phi_2) - \left(e - \frac{1}{e}\right)$$
$$= 2e - \frac{2}{e} - \frac{12}{e}$$
$$= 2e - \frac{14}{e}$$

thus giving

$$f_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)}\phi_3(x) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e}x + \frac{5}{4}\left(e - \frac{7}{e}\right)(3x^2 - 1).$$

(d) [7 points] The following plot compares the best approximations to $f(x)$.



The code use to produce it is below.

```
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=exp(x);
f1=(exp(1)-exp(-1))/2+x-x;
f2=f1+3*exp(-1)*x;
f3=f2+5*(exp(1)-7*exp(-1))*(3*x.^2-1)/4;
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
plot(x,f3,'-g')
xlabel('x')
legend('f(x)','f_1(x)','f_2(x)','f_3(x)','location','best')
saveas(figure(1),'hw16d.eps','epsc')
```

3. [27 points: 8 points for (a), (b), 11 points for (c)]

Let V be an inner product space (i.e. V a vector space with an inner product). Suppose $\{v_1, v_2, v_3\}$ is a basis for V, and we would like to construct a is possible to construct a new *orthogonal* basis $\{\phi_1, \phi_2, \phi_3\}$ through the following procedure:

$$\begin{aligned}
\phi_1 &= v_1 \\
\phi_2 &= v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1 \\
\phi_3 &= v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2 \\
&\vdots \\
\phi_k &= v_k - \sum_{i=1}^{k-1} \frac{(\phi_i, v_k)}{(\phi_i, \phi_i)}\phi_i
\end{aligned}$$

This is called the *Gram-Schmidt* procedure.

(a) We know that nonzero vectors $u_1, u_2, ..., u_k \in V$ form an orthogonal set if they are orthogonal to each other: i.e. if

$$(u_i, u_j) = 0, \quad i \neq j.$$

Show that ϕ_1, ϕ_2, ϕ_3 form an orthogonal set, i.e. $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$.

(b) Show that if we have an orthogonal set of vectors $\phi_1, \ldots, \phi_k$, then $\phi_1, \ldots, \phi_k$ are linearly independent as well, i.e.

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

is only true if $\alpha_1, \ldots, \alpha_k = 0$.

(c) Since we can define an inner product $(\cdot, \cdot)$ on the function space $C[-1, 1]$ as

$$(u, v) = \int_{-1}^{1} u(x)v(x)\, dx,$$

we can also use the Gram-Schmidt procedure to create orthogonal sets of *functions*. Using the Gram-Schmidt procedure above, compute the orthogonal vectors $\{\phi_1, \phi_2, \phi_3\}$ given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$.

Solution.

(a) We are going to show that $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$. To check that these formulas yield an orthogonal sequence, first compute (ϕ_1, ϕ_2) by substituting the above formula for ϕ_2

$$\begin{aligned}
(\phi_1, \phi_2) &= (\phi_1, v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1) \\
&= (\phi_1, v_2) - (\phi_1, \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1) \\
&= (\phi_1, v_2) - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}(\phi_1, \phi_1) \\
&= (\phi_1, v_2) - (\phi_1, v_2) \\
&= 0.
\end{aligned}$$

Then use the fact that $(\phi_1,\phi_2)=0$, to compute (ϕ_1,ϕ_3). By substituting again the formula for ϕ_3

$$
\begin{aligned}
(\phi_1,\phi_3) &= (\phi_1, v_3 - \frac{(\phi_1,v_3)}{(\phi_1,\phi_1)}\phi_1 - \frac{(\phi_2,v_3)}{(\phi_2,\phi_2)}\phi_2) \\
&= (\phi_1,v_3) - (\phi_1, \frac{(\phi_1,v_3)}{(\phi_1,\phi_1)}\phi_1) - (\phi_1, \frac{(\phi_2,v_3)}{(\phi_2,\phi_2)}\phi_2) \\
&= (\phi_1,v_3) - \frac{(\phi_1,v_3)}{(\phi_1,\phi_1)}(\phi_1,\phi_1) - \frac{(\phi_2,v_3)}{(\phi_2,\phi_2)}\underbrace{(\phi_1,\phi_2)}_{=0} \\
&= (\phi_1,v_3) - (\phi_1,v_3) \\
&= 0
\end{aligned}
$$

Similarly, using the symmetry property of inner product $(\phi_i,\phi_j)=(\phi_j,\phi_i)$ for all i,j. We can show $(\phi_2,\phi_3)=0$.

$$
\begin{aligned}
(\phi_2,\phi_3) &= (\phi_2, v_3 - \frac{(\phi_1,v_3)}{(\phi_1,\phi_1)}\phi_1 - \frac{(\phi_2,v_3)}{(\phi_2,\phi_2)}\phi_2) \\
&= (\phi_2,v_3) - (\phi_2, \frac{(\phi_1,v_3)}{(\phi_1,\phi_1)}\phi_1) - (\phi_2, \frac{(\phi_2,v_3)}{(\phi_2,\phi_2)}\phi_2) \\
&= (\phi_2,v_3) - \frac{(\phi_1,v_3)}{(\phi_1,\phi_1)}\underbrace{(\phi_2,\phi_1)}_{=0} - \frac{(\phi_2,v_3)}{(\phi_2,\phi_2)}(\phi_2,\phi_2) \\
&= (\phi_2,v_3) - (\phi_2,v_3) \\
&= 0.
\end{aligned}
$$

By symmetry we can conclude that $(\phi_2,\phi_3)=(\phi_3,\phi_2)=0$ and $(\phi_1,\phi_3)=(\phi_3,\phi_1)=0$. This completes the proof.

(b) Consider a linear relationship

$$\sum_{i=1}^{k}\alpha_i\phi_i = 0$$

which can be written

$$\alpha_1\phi_1+\alpha_2\phi_2+\cdots+\alpha_k\phi_k = 0.$$

If $1\le i\le k$ then taking the inner product of ϕ_i with both sides of the equation and using the properties of inner product (*Definition 3.32, page 58*),

$$
\begin{aligned}
(\phi_i, \alpha_1\phi_1+\alpha_2\phi_2+\cdots+\alpha_k\phi_k) &= (\phi_i,0) \\
(\phi_i,\alpha_1\phi_1)+(\phi_i,\alpha_2\phi_2)+\cdots+(\phi_i,\alpha_k\phi_k) &= 0 \\
\alpha_1(\phi_i,\phi_1)+\alpha_2(\phi_i,\phi_2)+\cdots+\alpha_k(\phi_i,\phi_k) &= 0 \\
\alpha_i(\phi_i,\phi_i) &= 0
\end{aligned}
$$

shows, since ϕ_i is nonzero, that α_i for $i=1,\cdots,k$ is zero.

(c) We want to construct the new orthogonal bases for V by *Gram-Schmidt* procedure given starting vectors $\{v_1,v_2,v_3\}=\{1,x,x^2\}$. Following the procedure we set

$$\phi_1 = v_1 = 1$$

and

$$\phi_2 \;=\; v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1.$$

We compute

$$(\phi_1, v_2) = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

and

$$(\phi_1, \phi_1) = \int_{-1}^{1} 1\,dx = 2.$$

Now we can compute

$$\phi_2 \;=\; x - \frac{0}{2}(1) = x.$$

Finally for ϕ_3,

$$\phi_3 \;=\; v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2$$

$$(\phi_1, v_3) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

and

$$(\phi_2, v_3) = \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

and

$$(\phi_2, \phi_2) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

Substituting these inner products into the equation for ϕ_3, we get

$$\phi_3 \;=\; x^2 - \frac{(2/3)}{2}(1) - \frac{0}{(2/3)}(x) = x^2 - \frac{1}{3}.$$

This yields $\{\phi_1, \phi_2, \phi_3\} = \{1, x, x^2 - \frac{1}{3}\}$ as desired.

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4. [24 points: 8 each]
One of the most intriguing results in mathematics is the idea that continuous functions (especially those whose derivatives are also continuous) can be very well approximated using combinations of trigonometric functions — i.e. sines and cosines — of different frequencies. This is encapsulated in the idea of *Fourier Series*: a large class of functions $u(x)$ can be represented by the infinite sum

$$u(x) = C + \sum_{j=1}^{\infty} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right).$$

Additionally, it turns out that the finite sum (i.e. a linear combination of sines and cosines)

$$u(x) \approx C + \sum_{j=1}^{n} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right)$$

is often a very good approximation to $u(x)$. We will go more into depth on these ideas later in the semester.

In this problem, unless specified otherwise, we will examine orthogonality properties of sines and cosines using the following inner product on $C^2[0,1]$: for $u, v \in C^2[0,1]$,

$$(u, v) = \int_0^1 u(x)v(x)dx.$$

For all parts, assume j and k are integers.

(a) Show that sines of different frequencies are orthogonal to each other, i.e. that

$$(\sin(j\pi x), \sin(k\pi x)) = \int_0^1 \sin(j\pi x)\sin(k\pi x)dx = 0, \quad j \neq k.$$

(b) Show that cosines of different nonzero frequencies are orthogonal to each other, i.e. that

$$(\cos(j\pi x), \cos(k\pi x)) = \int_0^1 \cos(j\pi x)\cos(k\pi x)dx = 0, \quad j \neq k.$$

(c) Show that sines and cosines of different frequencies are orthogonal to each other *over the interval* $[-1,1]$, i.e. that

$$(\sin(j\pi x), \cos(k\pi x)) = \int_{-1}^1 \sin(j\pi x)\cos(k\pi x)dx = 0, \quad j \neq k.$$

Unlike the previous two parts, cosines and sines of different frequencies are not orthogonal to each other using the inner product on $[0,1]$, and must be shown to be orthogonal using the inner product

$$(u, v) = \int_{-1}^1 u(x)v(x)dx.$$

(d) Show that sines and cosines, in addition to being orthogonal, can easily be made *orthonormal* over $[0,1]$ by scaling by $\sqrt{2}$: i.e.

$$\left\|\sqrt{2}\sin(j\pi x)\right\|^2 = 2\int_0^1 \sin(j\pi x)^2 dx = 1, \qquad \left\|\sqrt{2}\cos(j\pi x)\right\|^2 = 2\int_0^1 \cos(j\pi x)^2 dx = 1.$$

Solution.

(a) Using the sum formula

$$\sin(j\pi x)\sin(k\pi x)dx = \frac{1}{2}\left(\cos((j-k)\pi x) - \cos((j+k)\pi x)\right)$$

we can conclude that $\int_0^1 \sin(j\pi x)\sin(k\pi x)dx$ is

$$\frac{1}{2}\int_0^1 \left(\cos((j-k)\pi x) - \cos((j+k)\pi x)\right) = \frac{1}{2}\left(\frac{\sin((j-k)\pi x)}{(j-k)\pi} - \frac{\sin((j+k)\pi x)}{(j+k)\pi}\right)\Bigg|_0^1.$$

Since j and k are integers, $j-k$ and $j+k$ are also integers. Since $\sin(n\pi x) = 0$ for any integer n, we can conclude that $\sin(j\pi x)\sin(k\pi x)dx = 0$.

(b) Using the sum formula

$$\cos(j\pi x)\cos(k\pi x)dx = \frac{1}{2}\left(\cos((j-k)\pi x) + \cos((j+k)\pi x)\right)$$

we can repeat the steps in (a) to compute $\int_0^1 \cos(j\pi x)\cos(k\pi x)dx$ as

$$\frac{1}{2}\int_0^1 \left(\cos((j-k)\pi x) + \cos((j+k)\pi x)\right) = \frac{1}{2}\left(\frac{\sin((j-k)\pi x)}{(j-k)\pi} + \frac{\sin((j+k)\pi x)}{(j+k)\pi}\right)\Bigg|_0^1 = 0.$$

(c) We may use another sum formula

$$\sin(j\pi x)\cos(k\pi x) = \frac{1}{2}\left(\sin((j+k)\pi x) + \sin((j-k)\pi x)\right)$$

Then, $\int_{-1}^1 \sin(j\pi x)\cos(k\pi x)$ gives

$$\frac{1}{2}\int_{-1}^1 \left(\sin((j+k)\pi x) + \sin((j-k)\pi x)\right) = \frac{-1}{2}\left(\frac{\cos((j+k)\pi x)}{(j+k)\pi} + \frac{\cos((j-k)\pi x)}{(j-k)\pi}\right)\Bigg|_{-1}^1$$

Since $\cos(x)$ is even, evaluating the above at $x = 1$ and $x = -1$ yields the same result, which cancels out to zero.

You may also conclude the above by noting that $\cos(j\pi x)\sin(k\pi x)$ is odd, and thus its integral from $[-1, 1]$ is zero.

(d) By the above sum formulas,

$$\sin(j\pi x)^2 = \frac{1}{2}\left(\cos(0\pi x) - \cos(2j\pi x)\right) = \frac{1}{2}\left(1 - \cos(2j\pi x)\right)$$

so $\|\sin(j\pi x)\|^2 = \int_0^1 \sin(j\pi x)^2$ gives

$$\frac{1}{2} - \int_0^1 \cos(2j\pi x) = \frac{1}{2} - (\cos(2j\pi) - \cos(0)) = \frac{1}{2}$$

since $2j$ is even, and $\cos(2j\pi) = (-1)^(2j) = 1$.

We may conclude similarly that

$$\|\sin(j\pi x)\|^2 = \cos(j\pi x)^2 = \frac{1}{2}\left(\cos(0\pi x) + \cos(2j\pi x)\right) = 2.$$