CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 16 · Solutions

Posted Wednesday 25 September 2013. Due 5pm Wednesday 2 October 2013.

16. [25 points]

Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = jh$ for $j = 0, 1 \ldots, N+1$. Consider the N hat functions $\phi_k \in C[0,1]$, defined as

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k); \\ \dfrac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1, \ldots, N$. Let the inner product $(\cdot, \cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_0^1 u(x)v(x)\, dx$$

and let the norm $\|\cdot\| : C[0,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

(a) For $j = 1, \ldots, N$, what is $\phi_j(x_k)$ for $k = 0, 1, \ldots, N+1$? Simplify your answer as much as possible.

(b) Show that $\{\phi_1, \ldots, \phi_N\}$ is linearly independent by showing that if $c_k \in \mathbb{R}$ and $\displaystyle\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $c_k = 0$ for $k = 1, \ldots, N$.

(c) For $f(x) = \sin(\pi x)$, compute by hand the inner products (f, ϕ_j) for $j = 1, \ldots, N$.

(d) Compute by hand the inner products (ϕ_j, ϕ_k) for $j, k = 1, \ldots, N$. Your final answers should be simplified as much as possible and in your formulas h should be left as h and not be replaced with $1/(N+1)$. You must clearly state which values of j and k each formula you obtain is valid for. An acceptable way to present the final answer would be:
For $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} ? & \text{if } k = j, \\ ? & \text{if } |j - k| = 1, \\ ? & \text{otherwise.} \end{cases}$$

with the question marks replaced with the correct values. Hint:

$$\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h} \right)^2 dx = \frac{1}{h^2} \int_{x_{j-1}-x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2\, ds = \frac{1}{h^2} \int_0^h s^2\, ds$$

where $s = x - x_{j-1}$.

(e) Use your solutions to (c) and (d) to set up a linear system (in MATLAB) and solve it to compute the best approximations $f_N(x)$ to $f(x) = \sin(\pi x)$ from $\text{span}\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ for $N = 3$ and $N = 9$. For each of these N, produce one plot that compares $f_N(x)$ to $f(x)$, and a second plot that shows the error $f(x) - f_N(x)$. The `hat.m` code (from Homework 2, either your code or the code from the solutions) should help you to produce these plots.

Solution.

(a) [3 points] The definition of ϕ_j yields that $\phi_j(x_k) = 0$ if $k \neq j$. Moreover,

$$\phi_j(x_j) = \frac{x_{j+1} - x_j}{h} = \frac{(j+1)h - jh}{h} = \frac{jh + h - jh}{h} = \frac{h}{h} = 1.$$

Consequently, for $j = 1, \dots, N$,

$$\phi_j(x_k) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j, \end{cases}$$

for $k = 0, 1, \dots, N+1$.

(b) [3 points] If $c_k \in \mathbb{R}$ and $\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $\sum_{k=1}^{N} c_k \phi_k(x_j) = 0$ for $j = 1, \dots, N$. The answer to part (a) then allows us to conclude that $c_j = 0$ for $j = 1, \dots, N$ since $\sum_{k=1}^{N} c_k \phi_k(x_j) = c_j$. Therefore, $c_k = 0$ for $k = 1, \dots, N$ since $c_j = 0$ for $j = 1, \dots, N$ is equivalent to $c_k = 0$ for $k = 1, \dots, N$.

(c) [3 points] Integrating by parts yields that

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \sin(\pi x)\, dx &= \left[\frac{x - x_{j-1}}{h} \left(-\frac{\cos(\pi x)}{\pi} \right) \right]_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j} \frac{d}{dx} \left(\frac{x - x_{j-1}}{h} \right) \frac{\cos(\pi x)}{\pi}\, dx \\
&= -\frac{x_j - x_{j-1}}{h} \frac{\cos(\pi x_j)}{\pi} + \int_{x_{j-1}}^{x_j} \frac{1}{h} \frac{\cos(\pi x)}{\pi}\, dx \\
&= -\frac{jh - (j-1)h}{h} \frac{\cos(\pi x_j)}{\pi} + \left[\frac{1}{h} \frac{\sin(\pi x)}{\pi^2} \right]_{x_{j-1}}^{x_j} \\
&= -\frac{\cos(\pi x_j)}{\pi} + \frac{\sin(\pi x_j) - \sin(\pi x_{j-1})}{\pi^2 h}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \sin(\pi x)\, dx &= \left[\frac{x_{j+1} - x}{h} \left(-\frac{\cos(\pi x)}{\pi} \right) \right]_{x_j}^{x_{j+1}} + \int_{x_j}^{x_{j+1}} \frac{d}{dx} \left(\frac{x_{j+1} - x}{h} \right) \frac{\cos(\pi x)}{\pi}\, dx \\
&= \frac{x_{j+1} - x_j}{h} \frac{\cos(\pi x_j)}{\pi} - \int_{x_j}^{x_{j+1}} \frac{1}{h} \frac{\cos(\pi x)}{\pi}\, dx \\
&= \frac{(j+1)h - jh}{h} \frac{\cos(\pi x_j)}{\pi} - \left[\frac{1}{h} \frac{\sin(\pi x)}{\pi^2} \right]_{x_j}^{x_{j+1}} \\
&= \frac{\cos(\pi x_j)}{\pi} + \frac{\sin(\pi x_j) - \sin(\pi x_{j+1})}{\pi^2 h}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\phi_j, f) &= \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \sin(\pi x)\, dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \sin(\pi x)\, dx \\
&= \frac{2\sin(\pi x_j) - \sin(\pi x_{j-1}) - \sin(\pi x_{j+1})}{\pi^2 h} \\
&= \frac{2\sin(\pi x_j)}{\pi^2 h} (1 - \cos(h\pi)).
\end{aligned}$$

(d) [8 points] For $j = 1, \ldots, N$,

$$
\begin{aligned}
(\phi_j, \phi_j) &= \int_0^1 (\phi_j(x))^2 \, dx \\
&= \int_0^{x_{j-1}} (\phi_j(x))^2 \, dx + \int_{x_{j-1}}^{x_j} (\phi_j(x))^2 \, dx + \int_{x_j}^{x_{j+1}} (\phi_j(x))^2 \, dx + \int_{x_{j+1}}^1 (\phi_j(x))^2 \, dx \\
&= \int_0^{x_{j-1}} 0 \, dx + \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h} \right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h} \right)^2 dx + \int_{x_{j+1}}^1 0 \, dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h} \right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h} \right)^2 dx \\
&= \frac{1}{h^2} \int_{x_{j-1} - x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2 \, ds + \frac{1}{h^2} \int_{x_j - x_{j+1}}^{x_{j+1} - x_{j+1}} (x_{j+1} - (t + x_{j+1}))^2 \, dt \\
&= \frac{1}{h^2} \int_0^h s^2 \, ds + \frac{1}{h^2} \int_{-h}^0 t^2 \, dt \\
&= \frac{1}{h^2} \left[\frac{s^3}{3} \right]_0^h + \frac{1}{h^2} \left[\frac{t^3}{3} \right]_{-h}^0 \\
&= \frac{h^3}{3h^2} - \frac{(-h)^3}{3h^2} \\
&= \frac{h}{3} + \frac{h}{3} \\
&= \frac{2h}{3}
\end{aligned}
$$

where $s = x - x_{j-1}$ and $t = x - x_{j+1}$.
Moreover, for $j = 1, \ldots, N-1$,

$$
\phi_{j+1}(x) = \begin{cases} \dfrac{x - x_j}{h} & \text{if } x \in [x_j, x_{j+1}); \\ \dfrac{x_{j+2} - x}{h} & \text{if } x \in [x_{j+1}, x_{j+2}); \\ 0 & \text{otherwise;} \end{cases}
$$

and

$$
\begin{aligned}
(\phi_{j+1}, \phi_j) &= (\phi_j, \phi_{j+1}) \\
&= \int_0^1 \phi_j(x) \phi_{j+1}(x) \, dx \\
&= \int_0^{x_j} \phi_j(x) \phi_{j+1}(x) \, dx + \int_{x_j}^{x_{j+1}} \phi_j(x) \phi_{j+1}(x) \, dx + \int_{x_{j+1}}^1 \phi_j(x) \phi_{j+1}(x) \, dx \\
&= \int_0^{x_j} 0 \, dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \frac{x - x_j}{h} \, dx + \int_{x_{j+1}}^1 0 \, dx \\
&= \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \frac{x - x_j}{h} \, dx \\
&= \frac{1}{h^2} \int_{x_j - x_j}^{x_{j+1} - x_j} (x_{j+1} - (s + x_j)) (s + x_j - x_j) \, ds \\
&= \frac{1}{h^2} \int_0^h hs - s^2 \, ds \\
&= \frac{1}{h^2} \left[\frac{hs^2}{2} - \frac{s^3}{3} \right]_0^h
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) \\
&= \frac{3}{6} - \frac{2h}{6} \\
&= \frac{h}{6}
\end{aligned}$$

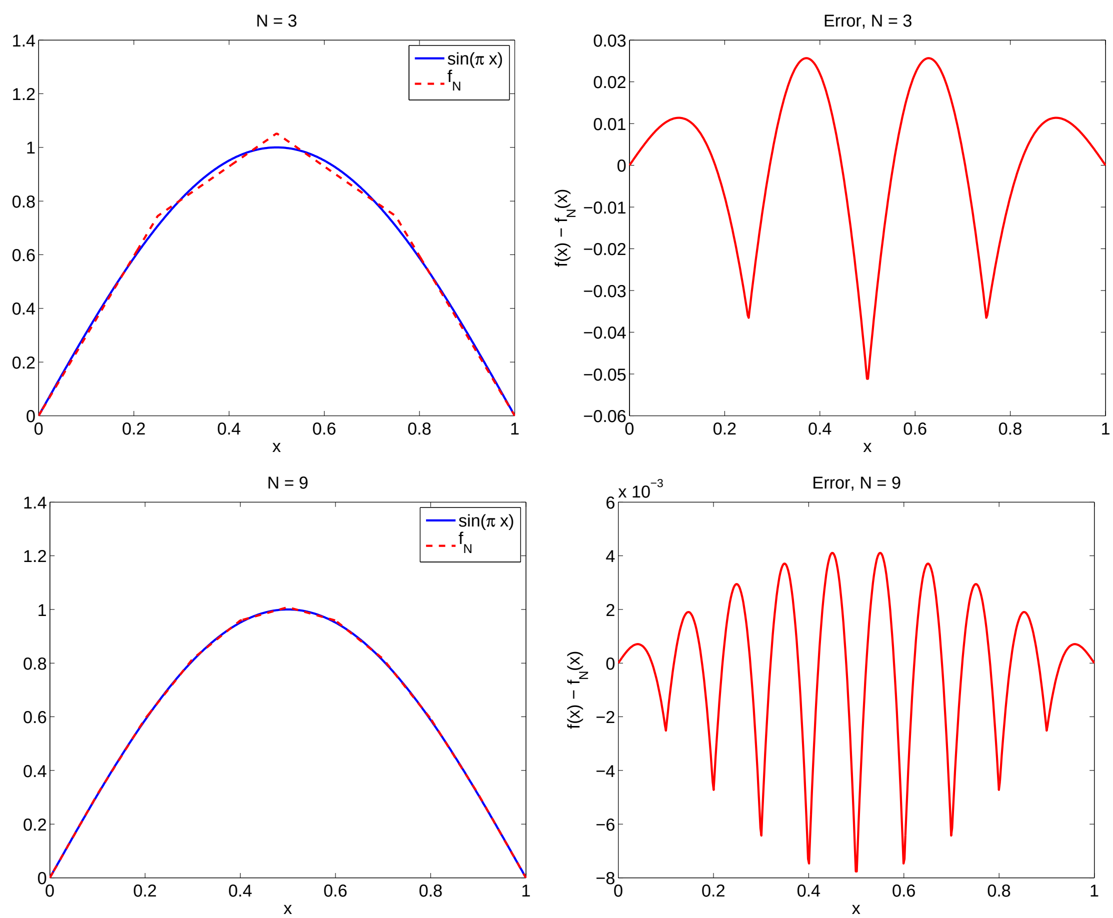
where $s = x - x_j$.
Finally, for $j = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \int_0^1 \phi_j(x)\phi_k(x)\,dx = \int_0^1 0\,dx = 0$$

if $|j - k| > 1$.
Hence, for $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} \frac{2h}{3} & \text{if } k = j, \\ \frac{h}{6} & \text{if } |j-k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(e) [8 points] The requested plots are shown below, followed by the MATLAB code that generated them.




```
xx = linspace(0,1,500).';
```

```
for N = [3 9]
    h = 1/(N+1);
    x = (0:N+1)*h;
    % set up the matrix from the inner products computed in part (d)
    A = 2*h/3*eye(N) + h/6*diag(ones(N-1,1),1) + h/6*diag(ones(N-1,1),-1);
    % set up the right-hand side vector from the inner products in part (c)
    b = 2/(h*pi^2)*(1-cos(h*pi))*sin(h*pi*(1:N).');
    % solve for the coefficients
    c = A\b;
    % compute the approximation on fine grid on [0,1]
    fN = zeros(length(xx),1);
    for j=1:N
        fN = fN + c(j)*hat(xx,j,N);
    end
    % plot the function f and the approximation
    figure(2), clf
    plot(xx, sin(pi*xx), 'b-','linewidth',2), hold on
    plot(xx, fN, 'r--','linewidth',2)
    legend('sin(\pi x)', 'f_N')
    set(gca,'fontsize',16)
    xlabel('x'), title(sprintf('N = %d', N))
    % plot the error
    figure(3), clf
    plot(xx, sin(pi*xx)-fN, 'r-','linewidth',2)
    set(gca,'fontsize',16)
    xlabel('x'), title(sprintf('Error, N = %d', N))
    ylabel('f(x) - f_N(x)')
    if (N==3)
        saveas(figure(2),'f_3a.eps','epsc')
        saveas(figure(3),'f_3b.eps','epsc')
    elseif (N==9)
        saveas(figure(2),'f_9a.eps','epsc')
        saveas(figure(3),'f_9b.eps','epsc')
    end
end
```