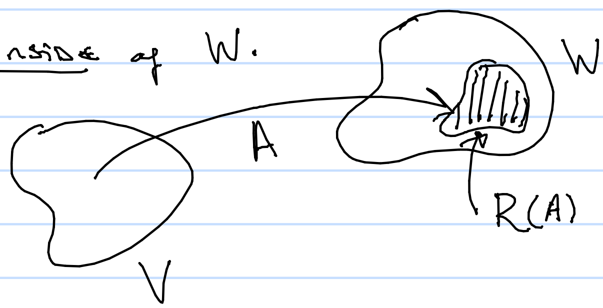
Existence and uniqueness of solutions to the problem $Ax=b$
(Chapter 3.2 of the textbook)

Let $V$ be a vector space and pick your favorite linear operator $A$, mapping elements of $V$ to some other vector space $W$, and fix a choice of vector $b$ in $W$. What does it mean to say that "There exists a solution to $Ax=b$"?

One way to answer this question is to define the <u>Range</u> of the operator $A$. The range is "everything in $W$ that $A$ can 'get to' with inputs from $V$".

<u>Defn</u>: $R(A) = \{ w \in W$ so that we can find an $x$ in $V$ with $Ax = w \}$

So $(A)$ sits <u>inside</u> of $W$.




Note that:

1) The zero vector is in $R(A)$ since $A0=0$ (by $A$ linear)
2) If $y \in R(A)$ and $w \in R(A)$ then $y = Ax$ and $w = Az$ for some $x, z \in V$. Consider $\alpha y + \beta w$ for $\alpha, \beta$ scalars.
Then $\alpha y + \beta w = \alpha A x + \beta A z = A(\alpha x) + A(\beta z) = A(\alpha x + \beta z)$
so that $\alpha y + \beta w$ is in $R(A)$.

$\Rightarrow R(A)$ is a vector subspace of $W$.

<u>IDEA</u>: When $W$ is $n$-dimensional real space ($\mathbb{R}^n$) for some $n$ then the vector subspaces are precisely $\mathbb{R}^d$ for some $0 \leq d \leq n$

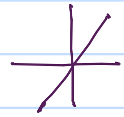
$\Rightarrow$ If $W = \mathbb{R}^n$ and $R(A) = \mathbb{R}^n$ then "$Ax=b$" has a solution for every $b \in W$.

If $R(A) = \mathbb{R}^d$ for $d < n$ then "$Ax=b$" fails to have a solution for infinitely many $b \in W$.

Ex: Consider $A = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ then $A: \mathbb{R}^2 \to \mathbb{R}^2$. What is $R(A)$?

→ if $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$ then $Ax = \begin{bmatrix}x_1 + 2x_2\\ 2x_1 + 4x_2\end{bmatrix} = \begin{bmatrix}1 \cdot (x_1 + 2x_2)\\ 2 \cdot (x_1 + 2x_2)\end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix}1\\ 2\end{bmatrix}$

so that every outcome is a multiple of the vector $\begin{bmatrix}1\\ 2\end{bmatrix}$

Hence $R(A) = \{$ all multiples of $\begin{bmatrix}1\\ 2\end{bmatrix} \}$ =  → $R(A)$ is one-dimensional.

Q: Does $Ax = \begin{bmatrix}-1\\ 0\end{bmatrix}$ have a solution?

Ex: Let $L_N : C^2_N[0,1] \to C[0,1]$ be defined by $L_N[u] = \frac{-\partial^2}{\partial x^2} u$

where, Recall: $C^2_N[0,1] = \{ u \in C[0,1] \mid \frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(1) = 0 \}$

Q: Is $R(A)$ all of $C[0,1]$?

⇒ Suppose that $f$ is in $R(A)$. Then there exists $u$ in $C^2_N[0,1]$ with $-\frac{\partial^2}{\partial x^2} u = f$ so that $\int_0^1 f = -\int_0^1 \frac{\partial^2}{\partial x^2} u = -\frac{\partial u}{\partial x}(0) + \frac{\partial u}{\partial x}(1) = 0$

So we must have $\int_0^1 f \, dx = 0$.

But then $R(A)$ cannot be all of $C[0,1]$ since the function $f(x) = x$ is in $C[0,1]$ but $\int_0^1 f(x) = 1/2 \neq 0$.

## <u>Uniqueness of Solutions to Linear operators</u>

Uniqueness to an expression "$Ax = b$" where $A: V \to W$ is a linear operator between vector spaces $V$ and $W$ means that *if* there is a solution $x$ satisfying $Ax = b$ **then** there is only **ONE** such solution.

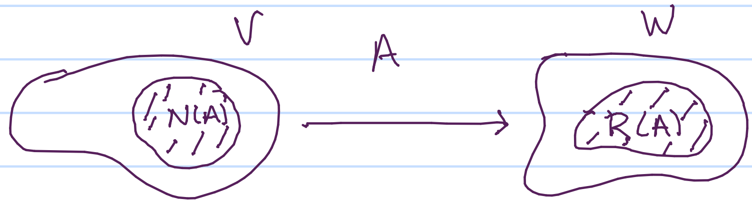
Suppose both $x$ and $z$ solve $Ax = b$, $Az = b$ then $A(x - z) = 0$

• Defn: We define the null space $N(A)$ of an operator $A$ to be $N(A) = \{ z \text{ in } V \mid Az = 0 \}$

• Note that since $A$ is a linear operator then $A0 = 0$

this means that zero is always in $N(A)$. Now suppose that $x$ and $z$ are in $N(A)$ and $\alpha, \beta$ are scalars.

Then $A(\alpha x + \beta z) = A(\alpha x) + A(\beta z) = \alpha A(x) + \beta A(z) = \alpha 0 + \beta 0 = 0$ so that $\alpha x + \beta z$ is in $N(A)$. Hence $N(A)$ is a vector subspace of $V$.




The Null space $N(A)$ of $A$ lives in $V$.

The Range of $A$, $R(A)$, lives $W$

Ex: Consider:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{then } Ax = b \text{ has the form } \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

which by Gaussian elimination becomes:

$$x_1 + x_3 = b_1 \qquad \text{(equation 1)}$$
$$-2x_2 = b_2 \qquad \text{(equation 2)}$$
$$0 = -3b_1 + b_3 \qquad \text{(equation 3)}$$

This means that if $Ax = b$ is to have a solution then we must pick $b$ such that $-3b_1 + b_3 = 0$

Also $x_1 + x_3 = b_1 + b_3 \rightarrow x_1 = b_1 + b_3 - x_3$ so $x_3$ can be anything. (notice there are no restrictions on $x_3$ in the system). Hence $\vec{x}$ has the form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 - x_3 \\ -1/2\, b_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ -1/2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

where $s = x_3$ is a free parameter.

What does this tell us?

- $R(A) = \{ b : \text{there exists } y \text{ with } Ay = b \}$
  - $\rightarrow$ Note that it is required by (equation 3) that $b_3 = 3b_1$. So the choices of $\vec{b}$ we can make look like $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ 3b_1 \end{bmatrix}$ so we have only <u>two</u> free variables, $b_1$ & $b_2$, when

We select the vector $\vec{b}$. This means that $R(A)$ will "look like" two-dimensional real space, $\mathbb{R}^2$.

What about $N(A)$? This is where (equation 1) and (equation 2) are used. These equations say that: $x_1 + x_3 = b_1$ and $-2x_2 = b_2$. So any vector $\vec{x}$ satisfying these requirements will give $A\vec{x} = b$.

- Recall that if $\vec{x}$ and $\vec{z}$ both satisfy $Ax = b = Az$ then the vector $(x - z)$ satisfies $A(x-z) = 0$ so that $(x-z)$ is in the subvector space $N(A)$.

Notice that the equations: $x_1 + x_3 = b_1$ and $-2x_2 = b_2$ say that a vector $x$ will solve $A\vec{x} = b$ if it has the form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 - x_3 \\ -b_2/2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ -b_2/2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So notice that the above says if we <u>pick</u> the vector $y = \begin{bmatrix} b_1 \\ -b_2/2 \\ 0 \end{bmatrix}$ then $y$ satisfies the equation $Ay = b$ for $b \in R(A)$

<u>And</u> it tells us that if we pick any other vector of the form $z = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ that

$A(y+s) = b$ as well. So: $Ay = b$, $A(y+s) = b$ means $A(y - (y+s)) = 0 \Rightarrow As = 0$

So $N(A)$ is described by all possible vectors of the form $s\begin{bmatrix}-1\\0\\1\end{bmatrix}$ where $s$ is some real number.

Writing this thought out in "set notation" gives:

$$N(A) = \left\{ s\begin{bmatrix}-1\\0\\1\end{bmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$