CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 50

Posted Wednesday 16 April 2014. Due 1pm Friday 25 April 2014.

50. [25 points]

Let $H_D^1(0,1) = \{v \in H^1(0,1) : v(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in H_D^1(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let the timestep $\Delta t \in \mathbb{R}$ be such that $\Delta t > 0$ and let $t_k = k\Delta t$ for $k = 0, 1, 2, \ldots$. Let $V_N = \text{span}\{\phi_1, \ldots \phi_{N+1}\}$, let $u_0 \in H_D^1(0,1)$ be such that

$$u_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1/4], \\ 4x - 1 & \text{if } x \in (1/4, 1/2], \\ 3 - 4x & \text{if } x \in (1/2, 3/4], \\ 0 & \text{if } x \in (3/4, 1], \end{cases}$$

and let

$$u_{0,N}(x) = \sum_{j=1}^{N+1} u_0(x_j)\phi_j(x).$$

Let $u(x,t)$ be the solution of

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\left((\sin(\pi x) + 1)\frac{\partial u}{\partial x}(x,t)\right) &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, \quad t \geq 0, \\ \frac{\partial u}{\partial x}(1,t) &= 0, \quad t \geq 0, \\ u(x,0) &= u_0(x), \quad 0 < x < 1. \end{aligned}$$

(a) We can obtain finite element approximations to u by finding u_N such that

$$u_N(x,t) = \sum_{j=1}^{N+1} \alpha_j(t)\phi_j(x)$$

where the $\alpha_j(t)$ are such that

$$\int_0^1 \frac{\partial u_N}{\partial t}(x,t)v(x)\,dx + \int_0^1 (\sin(\pi x) + 1)\frac{\partial u_N}{\partial x}(x,t)v'(x)\,dx = 0 \text{ for all } v \in V_N$$

and

$$u_N(x,0) = u_{0,N}(x), \quad 0 < x < 1.$$

Let

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_{N+1}(t) \end{bmatrix}.$$

What system of ordinary differential equations and initial condition does $\boldsymbol{\alpha}(t)$ satisfy?

(b) For the remainder of the question, your answers and codes should not feature the inverse of any matrices. What system of equations has to be solved in order to use the forward Euler method to compute an approximation $\boldsymbol{\alpha}^{(k)}$ to $\boldsymbol{\alpha}(t_k)$ for $k = 1, 2, 3, \ldots$?

(c) Use the forward Euler method with a timestep of $\Delta t = 10^{-5}$ to obtain approximations to $u_N(x,t)$ (and hence $u(x,t)$) at $t = 0.001, 0.01, 0.1, 0.2$ for $N = 15, 31$. Produce a plot showing these approximations but use a different figure for each value of N. Also plot $u_N(x,0)$ on both of your figures.

(d) Repeat part (c) but take the timestep $\Delta t = 10^{-4}$.

(e) What system of equations has to be solved in order to use the backward Euler method to compute an approximation $\boldsymbol{\alpha}^{(k)}$ to $\boldsymbol{\alpha}(t_k)$ for $k = 1, 2, 3, \ldots$?

(f) Use the backward Euler method with a timestep of $\Delta t = 10^{-3}$ to obtain approximations to $u_N(x,t)$ (and hence $u(x,t)$) at $t = 0.001, 0.01, 0.1, 0.2$ for $N = 15, 31$. Produce a plot showing these approximations but use a different figure for each value of N. Also plot $u_N(x,0)$ on both of your figures.