CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 8 · Solutions

Posted Wednesday 9 April, 2015. Due 5pm Wednesday 16 April, 2015.

Please write your name and instructor on your homework.

1. [40 points: 10 points each]
   Consider the following BVP with inhomogeneous boundary conditions conditions:

$$\begin{aligned} -((1+x^2)u')' &= x,\ 0<x<1, \\ u(0) &= 1, \\ u(1) &= 2. \end{aligned}$$

(a) Let $x_0 = 0, x_1, \ldots, x_N, x_{N+1} = 1$ be a grid of points where $x_i = ih$. Compute the finite element solution of this BVP using piecewise linear basis functions

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i); \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}); \\ 0 & \text{otherwise;} \end{cases}$$

Plot the Galerkin solutions with $N = 4, 8, 16, 32$ superimposed on each other. *You may wish to start with the codes from HW 8.*

(b) In general, inhomogeneous boundary conditions are treated by decomposing $u(x)$ into

$$u(x) = w(x) + g(x)$$

where $w(0) = w(1) = 0$ and $g(x)$ is any function satisfying inhomogeneous boundary conditions (this is referred to as the *lift*). We should make sure that the finite element solution does not depend on what lift we choose.

Let $g(x) = 1 + x$; compute what modifications must be made to the load vector in order to compute the solution in this case.

(c) Using the above modifications for $g(x) = 1 + x$, plot in MATLAB the solution $u_N(x)$ for $N = 4, 8, 16, 32$. Verify that these solutions should look identical to the solutions from (a).

Solution.

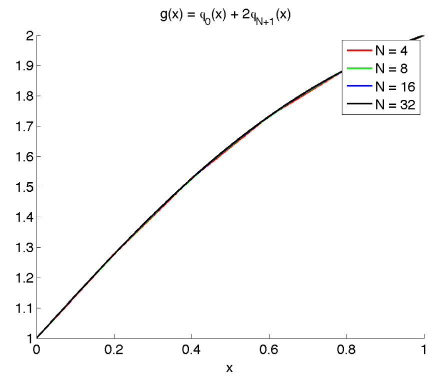
(a) In class, we showed that the resulting finite element system $K\alpha = b$ satisfied

$$K_{ij} = a(\phi_j, \phi_i), \quad b_i = \begin{cases} (f, \phi_1) - u(0)a(\phi_0, \phi_1) & i = 1 \\ (f, \phi_i) & 1 < i < N \\ (f, \phi_N) - u(1)a(\phi_{N+1}, \phi_N) & i = N. \end{cases}$$

In Homework 8, we computed K_{ij}, so here we will focus only on computing b_i.

Since $a(u, v) = \int_0^1 (1 + x^2)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$, and ϕ_0 is piecewise linear and nonzero only on the interval $[x_0, x_1] = [0, h]$,

$$a(\phi_0, \phi_1) = \int_0^h (1 + x^2)\frac{-1}{h^2} = \frac{1}{h^2}\left[x + \frac{x^3}{3}\right]_0^h = \frac{1}{h} + \frac{h}{3}.$$



Similarly, ϕ_{N+1} is nonzero only on $[x_N, x_{N+1}] = [1-h, h]$, which gives

$$a(\phi_{N+1}, \phi_N) = \int_{1-h}^{1} (1+x^2)\frac{-1}{h^2} = \frac{1}{h^2}\left[x + \frac{x^3}{3}\right]_{1-h}^{1} = \frac{2h}{3} + \frac{4}{h} - 2.$$

Then, since $a(\phi_0, \phi_1)$ and $a(\phi_{N+1}, \phi_N)$ are now given, we can compute b_i using the above formula. The graph produced by the finite element solution for $N = 4, 8, 16, 32$ is shown below

(b) By changing $g(x)$ to $1 + x$ and having $u(x) = w(x) + g(x)$, the more general form of the finite element equation needs to hold:

$$a(w, \phi_i) = (f, \phi_i) - a(g, \phi_i), \quad i = 1, \ldots, N.$$

Then, since $\frac{\partial g}{\partial x} = 1$, we can write

$$a(g, \phi_i) = \int_0^1 (1+x^2)\frac{\partial \phi_i}{\partial x} = \frac{1}{h}\left(\int_{x_{i-1}}^{x_i} (1+x^2) - \int_{x_i}^{x_{i+1}} (1+x^2)\right)$$

Plugging in $x_i = ih$ reduces the above to $a(g, \phi_i) = -2ih^2$.
Another alternative way to solve this problem is to notice that you can represent $g(x)$ exactly using a linear combination of $\phi_0, \ldots, \phi_{N+1}$

$$g(x) = \sum_{j=0}^{N+1} g(x_j)\phi_j(x).$$

You can then use the fact that

$$\begin{aligned} a(g, \phi_i) &= \sum_{j=1}^{N} g(x_j)a(\phi_j, \phi_i) + g(0)a(\phi_0, \phi_i) + g(1)a(\phi_{N+1}, \phi_N) \\ &= \sum_{j=1}^{N} g(x_j)K_{ij} + a(\phi_0, \phi_i) + 2a(\phi_{N+1}, \phi_N), \end{aligned}$$

which reduces down to using K_{ij} (from Hw 8) and the solution from part (a) for $a(\phi_0, \phi_i)$ and $a(\phi_{N+1}, \phi_i)$, which are zero unless $i = 1$ or $i = N$.

(c) The figure produced using $g(x) = 1 + x$ is as follow:

The code to compute both part (a) and (c) is below as well.

```
% demo of the finite element method for the problem
%   -d/dx((1+x^2) du/dx)  = x, 0 < x < 1, u(0) = u(1) = 0.
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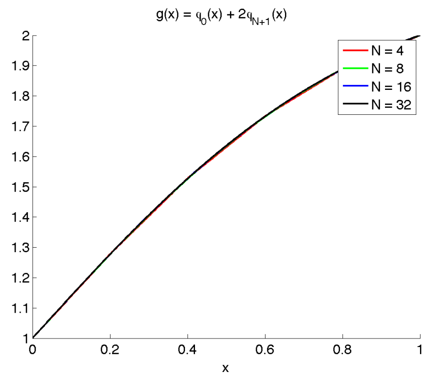
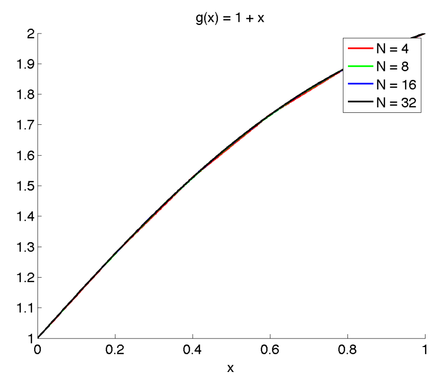


(a) $g(x) = \phi_0(x) + 2\phi_{N+1}(x)$



(b) $g(x) = 1 + x$

```
Nvec = [4 8 16 32];   % vector of N values we shall use

color ='rgbk';

% each pass of the following loop handles a new N value...
for j=1:length(Nvec)
    N = Nvec(j);
    h = 1/(N+1);
    x = [1:N]*h;

    % construct the stiffness matrix (integrals done by hand)
    maindiag = 2/h + 2*h/3 + 2*h*([1:N].^2);
    offdiag  = -1/h - h*(([1:N-1].^2) + [1:N-1] + 1/3);
    K = diag(maindiag) + diag(offdiag,1) + diag(offdiag,-1);

    % construct the load vector (integrals done by hand)
    f = h^2*[1:N]';
    f(1)=f(1) + 1/h + (h/3);
    f(N)=f(N) + 2*h/3 + 4/h - 2;

    % solve for expansion coefficients of Galerkin approximation
    c = K\f;

    % plot the true solution
    xx = linspace(0,1,1000)';     % finely spaced points between 0 and 1.

    % plot the approximation solution
    uN = zeros(size(xx));
    for k=1:N
        uN = uN + c(k)*hat(xx,k,N);
    end
    uN = uN + (xx < h).*(h-xx)/h;
    uN = uN + 2*(xx > (1-h)).*(xx-(1-h))/h;

    figure(1);hold on
    plot(xx, uN, color(j),'linewidth',2)
    set(gca,'fontsize',16)
    xlabel('x')
    axis([0 1 1 2])

    % =================================================================

    % use now g(x) = 1+x, recompute f and add a(g,\phi_i) terms
    f = h^2*[1:N]';
    f = f + 2*[1:N]'*h^2;

    % solve for expansion coefficients of Galerkin approximation
```

```
    c = K\f;

    % plot the approximation solution
    uN = zeros(size(xx));
    for k=1:N
        uN = uN + c(k)*hat(xx,k,N);
    end
    g = @(x) 1+x;
    uN = uN + g(xx);

    figure(2);hold on
    plot(xx, uN, color(j),'linewidth',2)
    set(gca,'fontsize',16)
    xlabel('x')
    tag{j} = sprintf('N = %d', N);
    axis([0 1 1 2])
end
figure(1)
legend(tag)
title('g(x) = \phi_0(x) + 2\phi_{N+1}(x)')
print('-dpng',gcf,'p1a')
figure(2)
title('g(x) = 1 + x')
legend(tag)
print('-dpng',gcf,'p1c')
```

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2. [40 points: 10 points each]

(a) Consider the function $u_0(x) = \begin{cases} 1, & x \in [0, 1/3]; \\ 0, & x \in (1/3, 2/3); \\ 1, & x \in [2/3, 1]. \end{cases}$

Recall that the eigenvalues of the operator $L : C_N^2[0,1] \to C[0,1]$,

$$Lu = -u''$$

are $\lambda_n = n^2\pi^2$ for $n = 0, 1, \ldots$ with associated (normalized) eigenfunctions $\psi_0(x) = 1$ and

$$\psi_n(x) = \sqrt{2}\cos(n\pi x), \qquad n = 1, 2, \ldots.$$

We wish to write $u_0(x)$ as a series of the form

$$u_0(x) = \sum_{n=0}^{\infty} a_n(0)\psi_n(x),$$

where $a_n(0) = (u_0, \psi_n)$.

Compute these inner products $a_n(0) = (u_0, \psi_n)$ by hand and simplify as much as possible.

For $m = 0, 2, 4, 80$, plot the partial sums

$$u_{0,m}(x) = \sum_{n=0}^{m} a_n(0)\psi_n(x).$$

(You may superimpose these on one single, well-labeled plot if you like.)

(b) Write down a series solution to the homogeneous heat equation

$$u_t(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad t \geq 0$$

with Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

and initial condition $u(x,0) = u_0(x)$.

Create a plot showing the solution at times $t = 0, 0.002, 0.05, 0.1$.

You will need to truncate your infinite series to show this plot.

Discuss how the number of terms you use in this infinite series affects the accuracy of your plots.

(c) Describe the behavior of your solution as $t \to \infty$.

(To do so, write down a formula for the solution in the limit $t \to \infty$.)

(d) How would you expect the solution to the inhomogeneous heat equation

$$u_t(x,t) = u_{xx} + 1, \qquad 0 < x < 1, \quad t \geq 0$$

with Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

to behave as $t \to \infty$?

Solution.

(a) To expand $u_0(x)$ in the form

$$u_0(x) = \sum_{n=0}^{\infty} a_n(0)\psi_n(x),$$

we must compute the coefficients $a_n(0)$. For $n = 0$ we compute

$$a_0(0) = \int_0^1 u_0(x) \cdot 1\,\mathrm{d}x = \int_0^{1/3} 1\,\mathrm{d}x + \int_{2/3}^1 1\,\mathrm{d}x = 2/3.$$

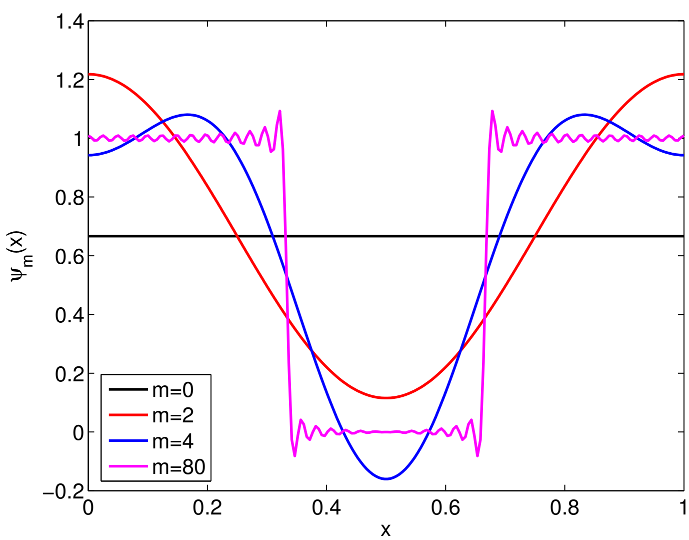
For $n > 0$ we have

$$\begin{aligned} a_n(0) &= \sqrt{2}\int_0^1 u_0(x)\cos(n\pi x)\,dx \\ &= \sqrt{2}\Big(\int_0^{1/3} \cos(n\pi x)\,dx + \int_{2/3}^1 \cos(n\pi x)\,dx\Big) \\ &= \sqrt{2}\Big(\Big[\frac{\sin(n\pi x)}{n\pi}\Big]_0^{1/3} + \Big[\frac{\sin(n\pi x)}{n\pi}\Big]_{2/3}^1 \\ &= \frac{\sqrt{2}(\sin(n\pi/3) - \sin(2n\pi/3))}{n\pi}. \end{aligned}$$

[GRADERS: this last expression is sufficiently simplified to receive full credit.]
Note that $\sin(2n\pi/3) = 2\sin(n\pi/3)\cos(n\pi/3)$, and hence

$$\sin(n\pi/3) - \sin(2n\pi/3) = \sin(n\pi/3)(1 - 2\cos(n\pi/3)).$$

Thus we have $a_n(0) = 0$ in two cases: if n is a multiple of 3, or if $\cos(n\pi/3) = 1/2$. The former occurs when $n = 3, 6, 9, 12, 15, \ldots$, whiles the latter occurs when $n\pi/3 (\mathrm{mod}\ 2\pi) = \pi/3$ or $5\pi/3$, and hence $a_n(0) = 0$ when $n = 1 + 6p$ for integers $p \geq 0$ or $n = -1 + 6p$ for integers $p \geq 1$. Together, this implies that for all odd integers n, $a_n(0) = 0$. We end up with the partial sums shown in the following figure. (MATLAB code follows at the end of this solution.)



(b) We seek a series solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x).$$

Using standard techniques described in class, together with the fact the problem is homogeneous ($f(x,t) = 0$), we find that

$$a_n'(t) + \lambda_n a_n(t) = 0.$$

For $n = 0$ we have

$$a_0'(t) = 0,$$

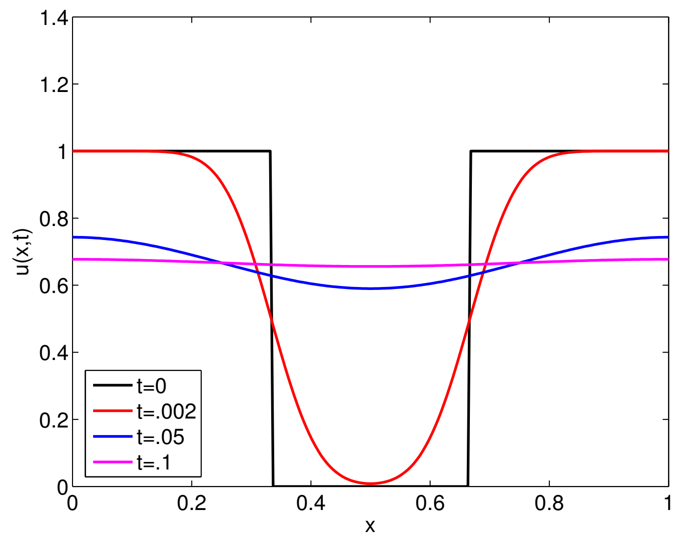
and hence $a_0(t)$ is constant, so we conclude $a_0(t) = a_0(0) = 2/3$. For $n \geq 1$ we have

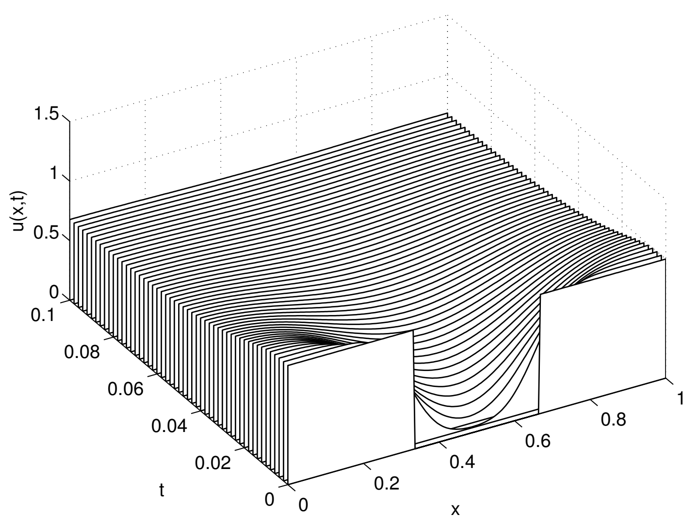
$$a_n(t) = e^{-\lambda_n t} a_n(0),$$

where $\lambda_n = n^2\pi^2$. In sum, we have

$$u(x,t) = 2/3 + \sum_{n=1}^{n} e^{-\lambda_n t} a_n(0) \big(\sqrt{2}\cos(n\pi x)\big).$$

Below we show this plot at the required times, based on taking the sum out to $N = 20$. While the number of terms in the series affects the accuracy of the solution in at early times, the importance of these extra terms decreases as $t \to \infty$.





(c) As is clear from the series formula in part (b) and from the figures, as $t \to \infty$, $u(x,t) \to 2/3$ for all $x \in [0,1]$.

(d) The existence of the limiting solution in part (c) does not contradict the fact that $\lambda_0 = 0$. There is no division by zero, as there is in the analogous steady-state problem $u_{xx} = f(x)$ with homogeneous Neumann conditions. The addition of the source term adds energy to the system, effectively increasing the rate of change of temperature with respect to time (u_t) by one unit. This corresponds to the physical situation of pumping more energy into a bar that is insulated at both ends—and hence energy cannot escape. Thus we expect the heat to grow as $t \to \infty$.

The above paragraph is satisfactory for full credit, but we can actually be quite a bit more precise. The eigenvalue $\lambda_0 = 0$ contributes a constant term to the solution of the PDE $u_t = u_{xx}$, and this constant will be nonzero provided $(u_0, \psi_0) = \int_0^1 u_0(x) \cdot 1\, dx \neq 0$. If u_0 has 'zero mean', i.e., $\int_0^1 u_0(x)\, dx = 0$, then the solution to the homogeneous problem will decay as $t \to \infty$; otherwise, as $t \to \infty$ the solution will approach the nonzero constant (u_0, ψ_0).

To write down the solution to the general inhomogeneous equation $u_t = u_{xx} + f$, we must expand

$$f(x,t) = \sum_{n=0}^{\infty} c_n(t)\psi_n(x).$$

The the coefficients $a_n(t)$ in the expansion of the solution

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x)$$

obey the differential equation

$$a_n'(t) = -\lambda_n a_n(t) + c_n(t).$$

As seen in class, these ODEs have the solutions

$$a_n(t) = e^{-\lambda_n t} a_n(0) + \int_0^t e^{-\lambda_n(t-\tau)} c_n(\tau)\, d\tau.$$

The $a_0(t)$ case is particularly interesting: $a_0(t) = a_0(0) + \int_0^t c_0(\tau)\, d\tau$. Hence we cannot possibly have a steady state solution if $c_0(\tau)$ is bounded away from zero for all $\tau > 0$.

In the case of $f(x,t) = 1$, we have $c_0(t) = 1$ and $c_n(t) = 0$ for $n > 0$, so that

$$a_0(t) = a_0(0) + \int_0^t 1\, d\tau = a_0(0) + t;$$

and for $n > 0$,

$$a_n(t) = e^{-\lambda_n t} a_n(0),$$

thus giving the solution

$$u(x,t) = a_0(0) + t + \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n(0)\psi_n(x).$$

```
% Plot the expansion of the initial data, psi(x)

 x = linspace(0,1,200);
 col = 'krbm';
 figure(1), clf
 fm = zeros(size(x));
 for n=0:2:80
    if n==0, an0 = 2/3;                          % psi(x) = 1 for x in [0,1/3], [2/3,1];
    else, an0 = sqrt(2)*(sin(n*pi/3)-sin(2*n*pi/3))/(n*pi);% psi(x) = 0 otherwise.
    end
    if n==0, fm = an0*ones(size(fm));
    else, fm = fm + an0*sqrt(2)*cos(n*pi*x);
    end
```

```
   if ismember(n,[ 0 2 4 80]),
      plot(x, fm, '-','linewidth',2,'color',col(1)), hold on, col = col(2:end);
   end
end
legend('m=0', 'm=2','m=4','m=80',3)
set(gca,'fontsize',16)
xlabel('x'), ylabel('\psi_m(x)')
print -depsc2 heateqn1

% Compute the solution at at various times.

psi = (x <= 1/3) | (x >= 2/3);    %  initial condition
U = [psi];
col = 'krbmc';
figure(2), clf
plot(x, psi, 'linewidth',2,'color',col(1)), hold on, col = col(2:end);
t = .002:.002:0.1;
tprint = [.002 .05 0.1];
for j=1:length(t)
   for n=0:2:20
      if n==0,
         an0    = 2/3;
         lambda = 0;
         uj     = exp(-lambda*t(j))*an0*ones(size(x));
      else
         an0 = sqrt(2)*(sin(n*pi/3)-sin(2*n*pi/3))/(n*pi);
         lambda = n^2*pi^2;
         uj = uj + exp(-lambda*t(j))*an0*(sqrt(2)*cos(n*pi*x));
      end
   end
   U = [U;uj];
   if ismember(t(j),tprint),
      plot(x, uj, '-','linewidth',2,'color',col(1)), hold on, col = col(2:end);
   end
end
legend('t=0','t=.002','t=.05','t=.1',3)
set(gca,'fontsize',16)
xlabel('x'), ylabel('u(x,t)')
print -depsc2 heateqn2

figure(3), clf
plt = waterfall(x,[0 t],U);
set(plt,'edgecolor','k')       % make the lines black

view(-30,50)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 heateqn3
```

3. [30 points: 10 points each]
Consider the *fourth order* partial differential equation with so-called *hinged* boundary conditions

$$u_t(x,t) = u_{xx}(x,t) - u_{xxxx}(x,t)$$
$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$$

and initial condition $u(x,0) = u_0(x)$ (that should satisfy the boundary conditions) (This equation is related to a model that arises in the study of thin films.)

To solve this PDE, we introduce the linear operator $L : C^4_H[0,1] \to C[0,1]$, where

$$Lu = -u'' + u''''$$

and $C^4_H[0,1] = \{u \in C^4[0,1], u(0) = u''(0) = u(1) = u''(1) = 0\}$ is the set of C^4 functions that satisfy the hinged boundary conditions.

(a) The operator L has eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

Use this fact to compute a formula for the eigenvalues λ_n, $n = 1, 2, \ldots$.

(b) Suppose the initial condition $u_0(x)$ is expanded in the form

$$u_0(x) = \sum_{n=1}^{\infty} a_n(0)\psi_n(x).$$

Briefly describe how one can write the solution to the PDE $u_t = u_{xx} - u_{xxxx}$ as an infinite sum.

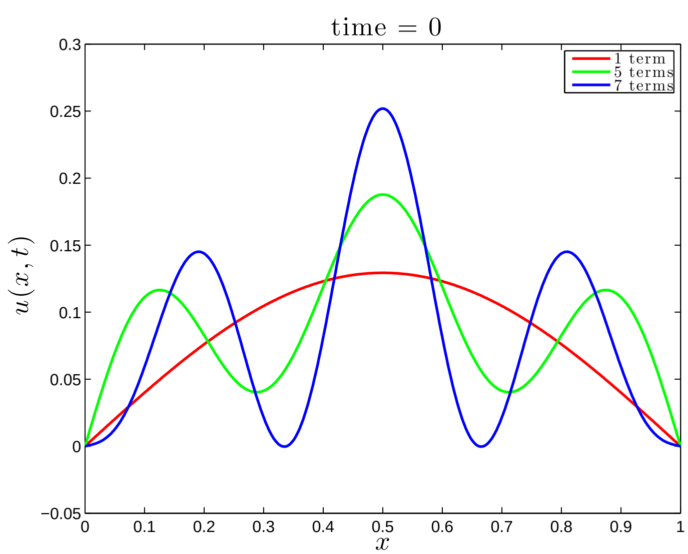
(c) Suppose the initial data is given by

$$u_0(x) = (x - x^2)\sin(3\pi x)^2,$$

with associated coefficients

$$a_n(0) = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3}, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Write a program (you may modify your earlier codes) to compute the solution you describe in part (b) up to seven terms in the infinite sum. At each time $t = 0$; 10^{-5}; 2×10^{-5}; 4×10^{-5}, produce a plot comparing the sum of the first 1, 5, and 7 terms of the series. For example, at time $t = 0$, your plot should appear as shown below. (Alternatively, you can produce attractive 3-dimensional plots over the time interval $t \in [0, 4 \times 10^{-5}]$ using 1, 5, and 7 terms in the series.)



Solution.

(a) Given the eigenfunctions ψ_n, we simply apply L to ψ_n to compute $\lambda_n\psi_n$:

$$\begin{aligned} L\psi_n(x) &= -\psi_n''(x) + \psi_n''''(x) \\ &= -\frac{d^2}{dx^2}(\sqrt{2}\sin(n\pi x)) + \frac{d^4}{dx^4}(\sqrt{2}\sin(n\pi x)) \\ &= n^2\pi^2\sqrt{2}\sin(n\pi x) + n^4\pi^4\sqrt{2}\sin(n\pi x) \\ &= (n^2\pi^2 + n^4\pi^4)(\sqrt{2}\sin(n\pi x) \\ &= \lambda_n\psi_n(x). \end{aligned}$$

Thus, we identify $\lambda_n = n^2\pi^2 + n^4\pi^4$ for $n = 1, 2, \ldots$.

(b) Following the prodedure outlined in class, we look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

Substituting this equation into the differential equation, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} a_n'(t)\psi_n(x) &= \sum_{n=1}^{\infty} a_n(t)(\psi_n''(x) - \psi_n''''(x)) \\ &= \sum_{n=1}^{\infty} -\lambda_n a_n(t)\psi_n(x). \end{aligned}$$

Taking an inner product of both sides with ψ_k and using the orthonormality of the eigenfunctions, we obtain the scalar differential equations

$$a_k'(t) = -\lambda_k a_k(t),$$

which has the solution

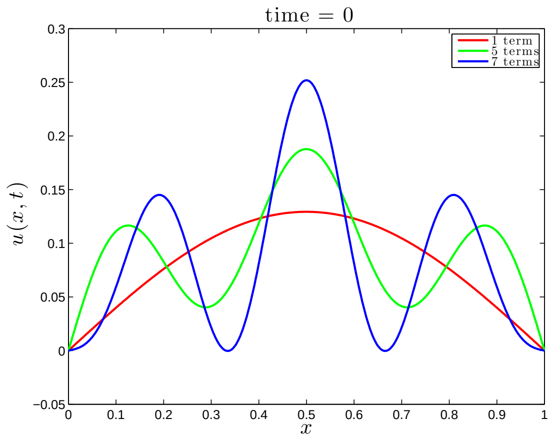
$$a_k(t) = e^{-\lambda_k} a_k(0).$$

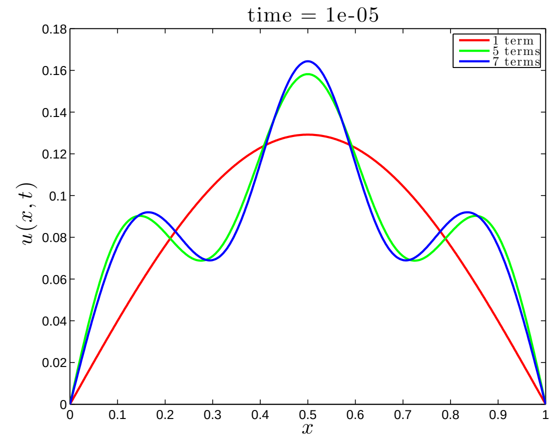
Thus, the solution can be written in the series

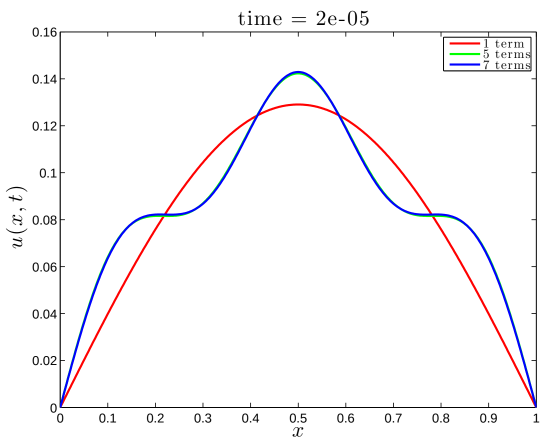
$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n(0)\psi_n(x) \\ &= \sum_{n=1}^{\infty} \sqrt{2}e^{-(n^2\pi^2+n^4\pi^4)t} a_n(0)\sin(n\pi x). \end{aligned}$$

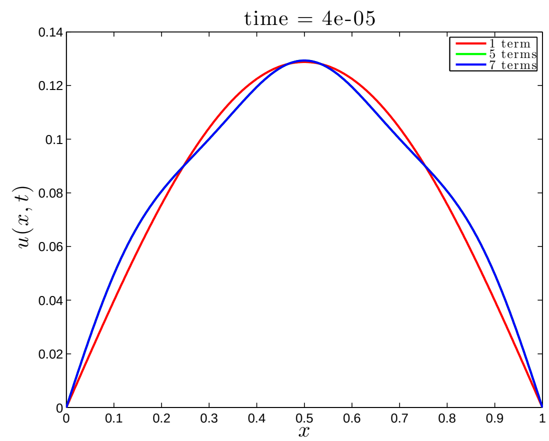
[GRADERS: students need only write down one of these series solutions for $u(x,t)$; they need not include the derivation.]

(c) Plots for the four requested times are shown below.

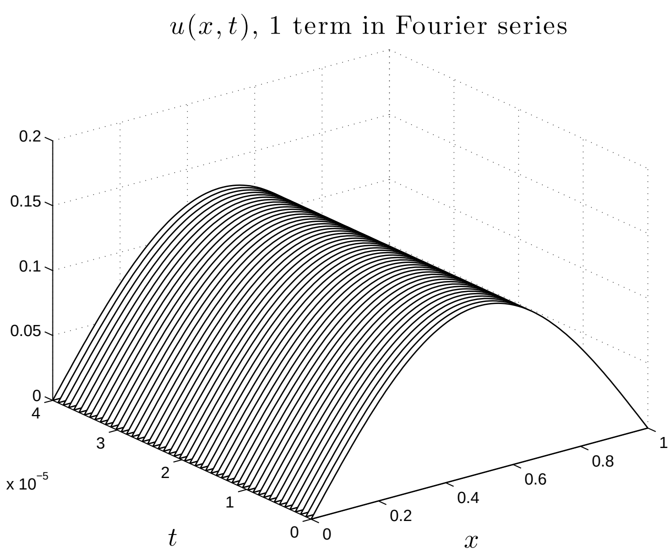


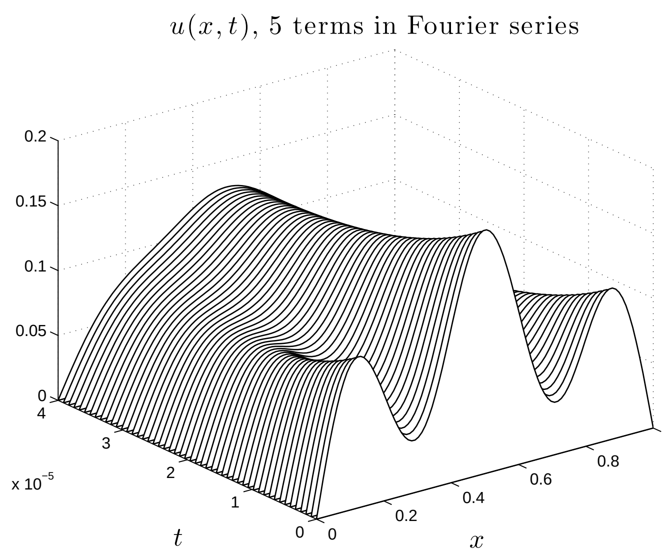


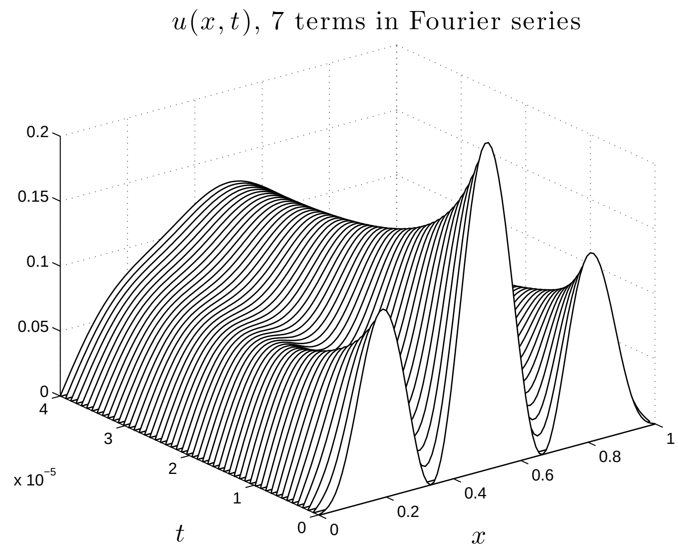




Alternatively, students may produce three-dimensional plots over the same time span for 1, 5, and 7 terms in the Fourier series.







One can produce these plots with the following code.

```
 tvec = [0 .00001 .00002 .00004];
 x = linspace(0,1,500);
 an0 = inline('sqrt(2)*432*(n^4-18*n^2+216)/((36*n-n^3)^3*pi^3)');
 lam = inline('n^2*pi^2 + n^4*pi^4');
 col = 'rgb';
 str = 'abcd';
 for j=1:length(tvec)
    figure(1), clf
    t = tvec(j);
    u = zeros(size(x));
    for n=1:2:7
        u = u+exp(-lam(n)*t)*an0(n)*(sqrt(2)*sin(n*pi*x));
        [tf,loc] = ismember(n,[1 5 7]);
        if tf,
           plot(x,u,'-','color',col(loc),'linewidth',2), hold on
        end
    end
    legend('1 term','5 terms', '7 terms')
    xlabel('x','fontsize',20)
    ylabel('u(x,t)','fontsize',20)
    title(sprintf('time = %g',t),'fontsize',20)
    eval(sprintf('print -depsc2 fourth_%s',str(j)))
    pause(.1)
 end

% surface plot
 tvec = linspace(0, .00004, 50);
 x    = linspace(0, 1, 100);
 U = zeros(length(tvec),length(x),3);
 for j=1:length(tvec)
    t = tvec(j);
    u = zeros(size(x));
    for n=1:2:7
       u = u+exp(-lam(n)*t)*an0(n)*(sqrt(2)*sin(n*pi*x));
       [tf,loc] = ismember(n,[1 5 7]);
       if tf, U(j,:,loc) = u; end
    end
 end
 figure(1), clf
 plt=waterfall(x,tvec,U(:,:,1));
 set(plt,'edgecolor','k')
 axis([0 1 0 .00004 0 .2])
 xlabel('x','fontsize',20), ylabel('t','fontsize',20)
```

```
title('u(x,t), 1 term in Fourier series','fontsize',20)
print -depsc2 fourth_wf1

figure(1), clf
plt=waterfall(x,tvec,U(:,:,2));
set(plt,'edgecolor','k')
axis([0 1 0 .00004 0 .2])
xlabel('x','fontsize',20), ylabel('t','fontsize',20)
title('u(x,t), 5 terms in Fourier series','fontsize',20)
print -depsc2 fourth_wf5

figure(1), clf
plt=waterfall(x,tvec,U(:,:,3));
set(plt,'edgecolor','k')
axis([0 1 0 .00004 0 .2])
xlabel('x','fontsize',20), ylabel('t','fontsize',20)
title('u(x,t), 7 terms in Fourier series','fontsize',20)
print -depsc2 fourth_wf7
```