CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6 · Solutions

Posted Wednesday 8 October, 2014. Due 5pm Wednesday 15 October, 2014.

*Please write your name and **residential college** on your homework.*

1. [28 points: 14 points each]

   All parts of this question should be done by hand.

   (a) Let
   $$\mathbf{D} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$
   and
   $$\mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$
   Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to
   $$\mathbf{D}\mathbf{c} = \mathbf{g}.$$

   (b) Let
   $$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$
   and
   $$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$
   Use the spectral method to obtain the solution $\mathbf{x} \in \mathbb{R}^3$ to
   $$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Solution.

(a) [14 points] Since,
$$\lambda\mathbf{I} - \mathbf{D} = \begin{bmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{bmatrix}$$
we have that
$$\det(\lambda\mathbf{I} - \mathbf{D}) = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$
and so
$$\det(\lambda\mathbf{I} - \mathbf{D}) = 0$$
when $\lambda = 3$ or $\lambda = 5$. Hence, the eigenvalues of $\mathbf{D}$ are
$$\lambda_1 = 3$$
and
$$\lambda_2 = 5.$$

Moreover,

$$(\lambda_1\mathbf{I}-\mathbf{D})\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -f_1-f_2 \\ -f_1-f_2 \end{bmatrix}$$

and so to make this vector zero we need to set $f_2 = -f_1$. Hence, any vector of the form

$$\begin{bmatrix} f_1 \\ -f_1 \end{bmatrix}$$

where f_1 is a nonzero constant is an eigenvector of $\mathbf{D}$ corresponding to the eigenvalue λ_1. Let us choose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Furthermore,

$$(\lambda_2\mathbf{I}-\mathbf{D})\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1-d_2 \\ -d_1+d_2 \end{bmatrix}$$

and so to make this vector zero we need to set $d_2 = d_1$. Hence, any vector of the form

$$\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}$$

where d_1 is a nonzero constant is an eigenvector of $\mathbf{D}$ corresponding to the eigenvalue λ_2. Let us choose

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\mathbf{D} = \mathbf{D}^T$, $\mathbf{D}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $\mathbf{D}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ and $\lambda_1 \neq \lambda_2$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Now,

$$\mathbf{g}\cdot\mathbf{v}_1 = 2-3 = -1,$$

$$\mathbf{v}_1\cdot\mathbf{v}_1 = 1^2+(-1)^2 = 1+1 = 2,$$

$$\mathbf{g}\cdot\mathbf{v}_2 = 2+3 = 5,$$

and

$$\mathbf{v}_2\cdot\mathbf{v}_2 = 1^2+1^2 = 1+1 = 2.$$

The spectral method then yields that

$$\begin{aligned}
\mathbf{c} &= \frac{1}{\lambda_1}\frac{\mathbf{g}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{1}{\lambda_2}\frac{\mathbf{g}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 \\
&= \frac{1}{3}\frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}\frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{6}+\frac{3}{6} \\ \frac{1}{6}+\frac{3}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{6} \\ \frac{4}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.
\end{aligned}$$

(b) [14 points] For this matrix $\mathbf{A}$ we have

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix},$$

and hence the characteristic polynomial is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - 3)(\lambda^2 - 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial, which we label

$$\lambda_1 = -1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3.$$

To compute the eigenvectors associated with the eigenvalue $\lambda_1 = -1$, we seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -u_2 + u_3 \\ u_2 - u_3 \end{bmatrix}.$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = u_2$. Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_1 = -1$.
To compute the eigenvectors associated with the eigenvalue $\lambda_2 = 1$ we now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_2\mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ u_2 + u_3 \\ u_3 + u_2 \end{bmatrix}.$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = -u_2$. Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ -u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_2 = 1$.
To compute the eigenvectors associated with the eigenvalue $\lambda_3 = 3$ we now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_3\mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2 + u_3 \\ u_2 + 3u_3 \end{bmatrix}.$$

To make the second component zero we need $u_2 = -u_3/3$, while to make the third component zero we need $u_3 = -u_2/3$. The only way to accomplish both is to set $u_2 = u_3 = 0$. Thus any vector of the form

$$\begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, \quad u_1 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_3 = 3$.

We choose the eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We can compute that

$$\mathbf{u}_1^T \mathbf{u}_2 = \mathbf{u}_2^T \mathbf{u}_1 = 0 \cdot 0 + (1/\sqrt{2}) \cdot (1/\sqrt{2}) + (1/\sqrt{2}) \cdot (-1/\sqrt{2}) = 0,$$

$$\mathbf{u}_1^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_1 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (1/\sqrt{2}) \cdot 0 = 0,$$

and

$$\mathbf{u}_2^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_2 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (-1/\sqrt{2}) \cdot 0 = 0.$$

Now, for $j = 1, 2, 3$, $\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j$ and $\mathbf{u}_j^T \mathbf{u}_j = 1$. Since $\mathbf{A} = \mathbf{A}^T$, the spectral method then yields that

$$\mathbf{x} = \sum_{j=1}^{3} \frac{1}{\lambda_j} \frac{\mathbf{u}_j^T \mathbf{b}}{\mathbf{u}_j^T \mathbf{u}_j} \mathbf{u}_j = \sum_{j=1}^{3} \frac{\mathbf{u}_j^T \mathbf{b}}{\lambda_j} \mathbf{u}_j.$$

We can compute that

$$\mathbf{u}_1^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (1/\sqrt{2}) \cdot 3 = \sqrt{2},$$

$$\mathbf{u}_2^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (-1/\sqrt{2}) \cdot 3 = -2\sqrt{2},$$

and

$$\mathbf{u}_3^T \mathbf{b} = 1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 3 = 2,$$

and hence

$$\mathbf{x} = \frac{\sqrt{2}}{-1} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{-2\sqrt{2}}{1} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -3 \\ 1 \end{bmatrix}.$$

We can multiply $\mathbf{A}\mathbf{x}$ out to verify that the desired $\mathbf{b}$ is obtained.

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2. [40 points: 8 points each]

We have been able to obtain nice formulas for the eigenvalues of the operators that we have considered thus far. This problem illustrates that this is not always the case.
Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\, dx.$$

Let the linear operator $L : V \to C[0,1]$ be defined by

$$Lu = -u''$$

where

$$V = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

Note that if $u \in V$ then u satisfies the homogeneous Robin boundary condition

$$u(0) - u'(0) = 0$$

and the homogeneous Dirichlet boundary condition

$$u(1) = 0.$$

(a) Prove that L is symmetric.

(b) Is zero an eigenvalue of L?

(c) Show that $(Lu, u) \geq 0$ for all $u \in V$. What does this and the answer to part (b) then allow us to say about the eigenvalues of L?

(d) Show that the eigenvalues λ of L must satisfy the equation $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$.

(e) Use MATLAB to plot $g(x) = -\tan(x)$ and $h(x) = x$ on the same figure. Use the command `axis([0 5*pi -5*pi 5*pi])` and make sure that your plot gives an accurate representation of these functions on the region shown on the figure when this command is used. By hand or using MATLAB, mark on your plot the points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Note that $g \notin C[0, 5\pi]$. How many eigenvalues λ does L have which are such that $\sqrt{\lambda} \leq 5\pi$?

Solution.

(a) [8 points] Suppose $u, v \in V$, so that $u(0) - u'(0) = v(0) - v'(0) = u(1) = v(1) = 0$. Integrating by parts twice yields

$$\begin{aligned}(Lu, v) &= \int_0^1 -u''(x)v(x)\, dx \\ &= \Big[-u'(x)v(x)\Big]_0^1 + \int_0^1 u'(x)v'(x)\, dx, \\ &= -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v'(x)\, dx,\end{aligned}$$

$$
\begin{aligned}
&= -u'(1)v(1) + u'(0)v(0) + \Big[u(x)v'(x)\Big]_0^1 - \int_0^1 u(x)v''(x)\,dx \\
&= -u'(1)v(1) + u'(0)v(0) + u(1)v'(1) - u(0)v'(0) + (u, Lv) \\
&= (u, Lv).
\end{aligned}
$$

In the last step, two boundary terms are zero because $u(1) = v(1) = 0$. For the other boundary term, note that $v(0) - v'(0) = 0$ implies $v(0) = v'(0)$, so $u'(0)v(0) - u(0)v'(0) = u'(0)v(0) - u(0)v(0) = -(u(0) - u'(0))v(0) = 0$ since $u(0) - u'(0) = 0$. Hence $(Lu, v) = (u, Lv)$ for all $u, v \in V$ and so L is symmetric.

(b) [8 points] Zero is *not* an eigenvalue of L. To see this, we seek a nonzero solution $\psi \in V$ to $L\psi = 0\psi$, i.e., $-\psi''(x) = 0$. The general solution of $-\psi''(x) = 0$ is $\psi(x) = Ax + B$ where A and B are constants. The right boundary condition $\psi(1) = 0$ implies that

$$0 = \psi(1) = A + B,$$

hence $A = -B$. The left boundary condition implies

$$0 = \psi(0) - \psi'(0) = B - A,$$

hence $A = B$. The only solution which satisfies both of these conditions is hence $A = B = 0$, so $\psi(x) = 0$ is the only solution of $L\psi = 0$. Thus zero is not an eigenvalue of L.

(c) [8 points] Suppose $u \in V$, so that $u(0) - u'(0) = u(1) = 0$. Then, integrating by parts gives

$$
\begin{aligned}
(Lu, u) &= \int_0^1 -u''(x)u(x)\,dx \\
&= \Big[-u'(x)u(x)\Big]_0^1 + \int_0^1 u'(x)u'(x)\,dx, \\
&= -u'(1)u(1) + u'(0)u(0) + \int_0^1 (u'(x))^2\,dx.
\end{aligned}
$$

Now, $(u'(x))^2 \geq 0$ for all $x \in [0, 1]$ and so $\int_0^1 (u'(x))^2\,dx \geq 0$. Moreover, since $u(1) = 0$ we have that $-u'(1)u(1) = 0$ and since $u(0) - u'(0) = 0$ we can say that $u(0) = u'(0)$ from which it follows that $u'(0)u(0) = (u(0))^2 \geq 0$. Therefore, $(Lu, u) \geq 0$ for all $u \in V$.

If λ is an eigenvalue of L then, since L is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $\psi \in V$ which are such that $L\psi = \lambda\psi$ and hence

$$\lambda(\psi, \psi) = (\lambda\psi, \psi) = (L\psi, \psi).$$

The fact that $(Lu, u) \geq 0$ for all $u \in V$ then means that

$$\lambda = \frac{(L\psi, \psi)}{(\psi, \psi)} \geq 0$$

since $(\psi, \psi) > 0$ by the definition of the inner product because ψ is a nonzero function. However, in part (b) we had showed that zero is not an eigenvalue of L and so we can conclude that $\lambda > 0$ for all eigenvalues λ of L.

(d) [8 points] We now know that all eigenvalues λ of L are positive and so the general solution of $L\psi = \lambda\psi$, i.e. $-\psi'' = \lambda\psi$, has the form

$$\psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

where A and B are constants. The left boundary condition gives

$$0 = \psi(0) - \psi'(0) = A\sin(0) + B\cos(0) - A\sqrt{\lambda}\cos(0) + B\sqrt{\lambda}\sin(0) = B - A\sqrt{\lambda},$$

hence $B = A\sqrt{\lambda}$. The right boundary condition gives

$$0 = \psi(1) = A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda}).$$

Substituting the left boundary condition into this last formula, we find

$$0 = A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}).$$

Since we need $A \neq 0$ in order for $\psi \neq 0$, this equation implies that

$$\sqrt{\lambda} = -\frac{\sin(\sqrt{\lambda})}{\cos(\sqrt{\lambda})} = -\tan(\sqrt{\lambda}).$$

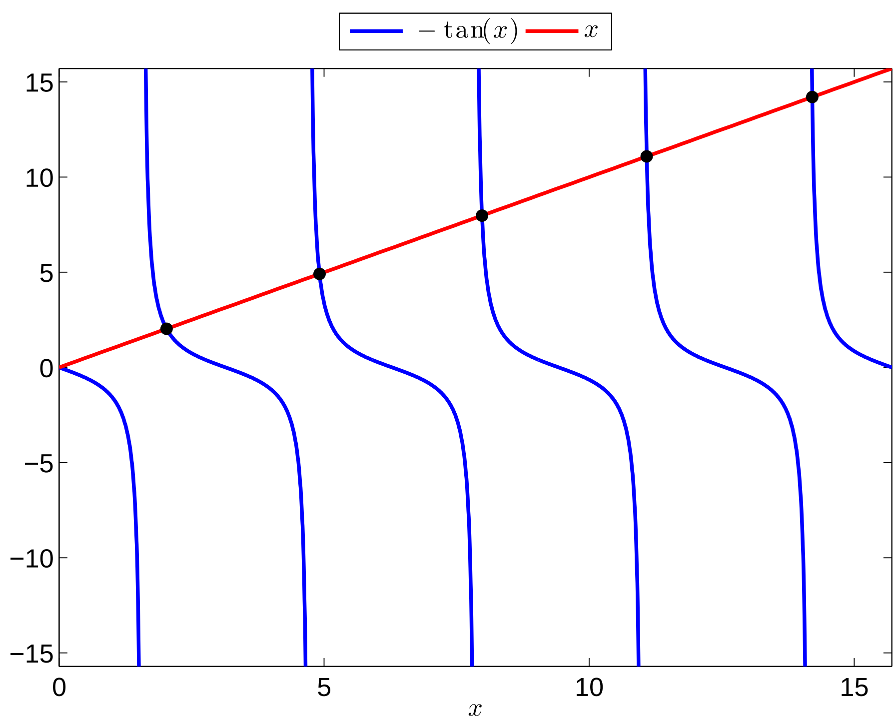
Therefore, the eigenfunctions of L have the form

$$\psi(x) = A(\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\cos(\sqrt{\lambda}x)),\ A \neq 0$$

where the eigenvalues λ are the positive numbers which are such that

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

(e) [8 points] The plot is shown below.



From the plot we can see that there are 5 points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Hence, since the eigenvalues λ of L are the positive numbers which are such that $g(\sqrt{\lambda}) = h(\sqrt{\lambda})$, L has five eigenvalues λ which are such that $\sqrt{\lambda} \leq 5\pi$.

The code used to produce the plot is below.

```
clear
clc
figure(1)
clf
for j=0:5
    x = linspace((j-1/2)*pi,(j+1/2)*pi,500);
    x = x(2:end-1);
    tanplt = plot(x,-tan(x),'b-','linewidth',2);
    hold on
end
x = linspace(0,5*pi,100);
linplt = plot(x,x,'r-','linewidth',2);
axis([0 5*pi -5*pi 5*pi])
xlabel('$x$','interpreter','latex','fontsize',14)

lgd = legend([tanplt,linplt],'$\ -\tan(x)$', '$x$',...
             'location','northoutside','orientation','horizontal');
set(lgd,'interpreter','latex')
set(gca,'fontsize',14)

guess = [2 5 7.98 11 14.21]';
bracket = [1.6 2.5;
4.8 5;
7.9 8.1;
11 11.2;
14.15 14.3];

ew = zeros(size(guess));
for k=1:length(guess)
    ew(k) = bisect(@(x) x+tan(x),bracket(k,1),bracket(k,2));
    plot(ew(k),ew(k),'k.','markersize',20)
end
print -depsc2 eigroot
```

The function `bisect` used in the above code is below.

```
function xstar = bisect(f,a,b)

% function xstar = bisect(f,a,b)
% Compute a root of the function f using bisection.
% f:    a function name, e.g., bisect('sin',3,4), or bisect('myfun',0,1)
% a, b: a starting bracket:  f(a)*f(b) < 0.

fa = feval(f,a);
fb = feval(f,b);           % evaluate f at the bracket endpoints
delta = (b-a);             % width of initial bracket
k = 0; fc = inf;           % initialize loop control variables

c = (a+b)/2;
while (delta/(2^k)>1e-18) && abs(fc)>1e-18
    c = (a+b)/2;
    fc = feval(f,c);    % evaluate function at bracket midpoint
    if fa*fc < 0
        b=c;
        fb = fc;      % update new bracket
    else
        a=c;
        fa=fc;
    end
    k = k+1;
```

```
%    fprintf(' %3d  %20.14f  %16.8e\n', k, c, fc);
end
xstar = c;
```

3. [32 points: 10 points for (a) and (b), 12 points for (c)]
Define the inner product (u, v) to be

$$(u, v) = \int_0^1 u(x)v(x)\,dx$$

and let the norm $\|v(x)\|$ be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let N be a positive integer and let $\phi_1, \ldots, \phi_N \in C[0, 1]$ be such that $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$. We wish to approximate a continuous function $f(x)$ with $f_N(x)$

$$f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

where

$$\phi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \ldots.$$

and where $\alpha_n = (f, \phi_n)$. (Note that f_N is the best approximation to g from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$.)

(a) Assume that $f_N \to f$ as $N \to \infty$. Show that, since $\phi_1, \ldots, \phi_N$ are orthonormal,

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2.$$

(b) The best approximation to $f(x) = x(1 - x)$ has coefficients α_n which satisfy

$$\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3}(1 - (-1)^n).$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 5$. On a separate figure, plot the error using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using `loglog` in MATLAB.

(c) Verify that the best approximation to the function $f(x) = 1 - x$ (which does not satisfy the same boundary conditions as $\phi_n(x)$!) has coefficients

$$\alpha_n = \frac{\sqrt{2}}{\pi n}.$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 100$. On a separate figure, plot the error using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using `loglog` in MATLAB.

You may have noticed that the rate at which the coefficients $\alpha_n \to 0$ determines how fast the error decreases — this is not coincidental!

Solution.

(a) [10 points] We have that

$$\begin{aligned}
\|f - f_N\|^2 &= (f - f_N, f - f_N) \\
&= \left(f - \sum_{n=1}^{N} \alpha_n \psi_n, f - \sum_{m=1}^{N} \alpha_m \psi_m\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(f - \sum_{n=1}^{N} \alpha_n \psi_n, f\right) - \sum_{m=1}^{N} \alpha_m \left(f - \sum_{n=1}^{N} \alpha_n \psi_n, \psi_m\right) \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{m=1}^{N} \alpha_m \sum_{n=1}^{N} \alpha_n (\psi_n, \psi_m) \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n^2 (\psi_n, \psi_n) \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n^2 \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n^2 \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n^2 \\
&= \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2,
\end{aligned}
$$

where at each equal sign we have used: (1) the definition of the norm $\|\cdot\|$; (2) the definition of g_N; (3) linearity of the inner product in the second argument; (4) linearity of the inner product in the first argument; (5) the fact that $(\psi_n, \psi_m) = 0$ if $n \neq m$, for $m, n = 1, 2, \ldots, N$, since $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$; (6) the fact that $(\psi_n, \psi_n) = 1$, for $n = 1, 2, \ldots, N$, since $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$; (7) the fact that $(f, \psi_n) = (\psi_n, f) = \alpha_n$; (8) algebra; (9) the definition of the norm $\|\cdot\|$.

(b) [10 points] First calculate the norm of f

$$\|f\|^2 = \int_0^1 (f(x))^2 \, dx = \int_0^1 x^2(1-x)^2 \, dx = \frac{1}{30}.$$

Then

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2,$$

where

$$\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3} \left(1 - (-1)^n\right).$$

and so

$$\|f - f_N\|^2 = \frac{1}{30} - \sum_{n=1}^{N} \alpha_n^2.$$

The requested plots are shown below.

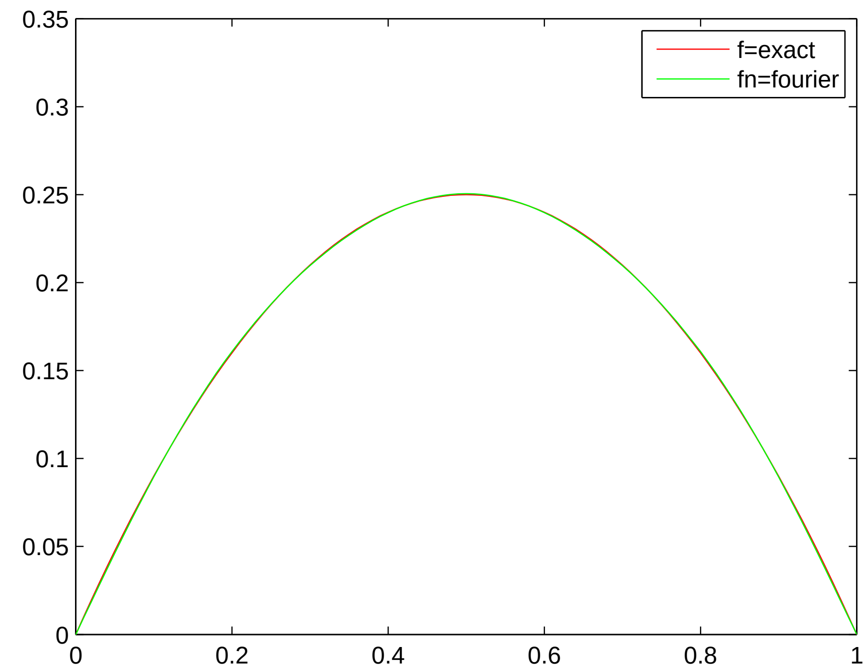


Figure 1: Comparison of the true function $f(x)$ and $f_N(x)$ for $N = 5$

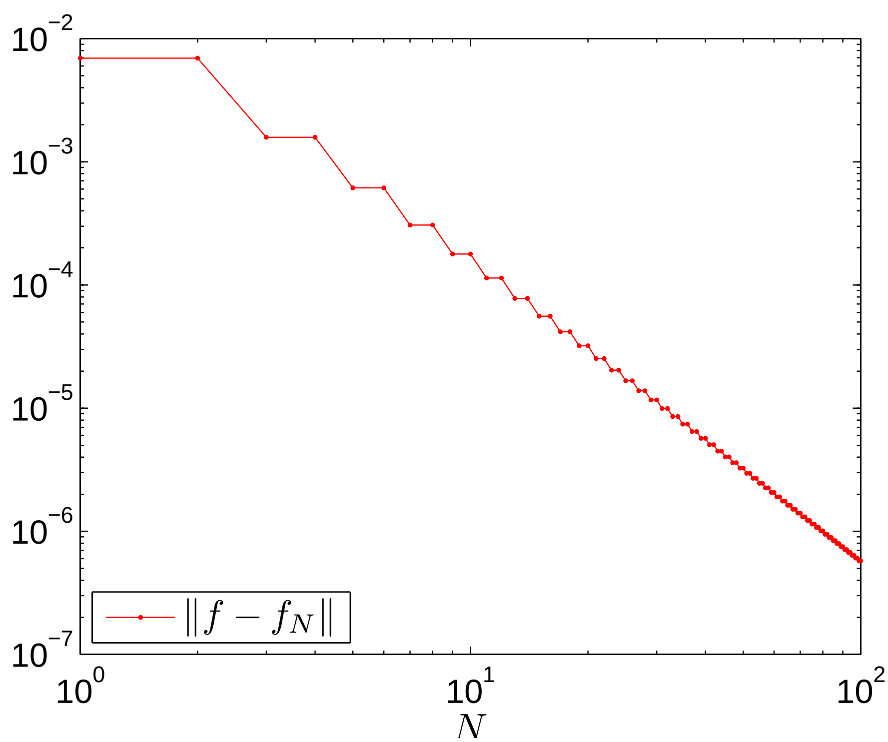


Figure 2: Norm of the error for $N = 1, 2 \cdots, 100$ on a log-log scale

The codes that produced the plots are shown below, respectively.

```
% Sine series for f(x)=x*(1-x) on [0,1]

clear c;
n=5;
for j=1:n
    c(j)=(2*sqrt(2)/(pi^3))*(1-(-1)^(j))/(j^3);
end;
xgrid=0:0.01:1;
hold off
fn=zeros(size(xgrid));
for j=1:n
    fn=fn+ c(j)*sqrt(2)*sin(j*pi*xgrid);
end

figure(1)
plot(xgrid, xgrid.*(1-xgrid), 'r', xgrid, fn, 'g');
legend('f=exact', 'fn=fourier')


%The norm of the error
n = [1:1e2]';
cn = (2*sqrt(2)/pi.^3)*(1-(-1).^(n))./(n.^3);
normf2 = 1/30;
figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')

set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr
```

(c) [12 points] The best approximation to the function

$$f(x) = 1 - x$$

using orthonormal eigenfunction $\phi_n(x)$ can be written as follows

$$f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

where

$$\alpha_n = (f, \phi_n) = \int_0^1 (1-x)\sqrt{2}\sin(n\pi x)\,dx = \sqrt{2}\left[\frac{n\pi(x-1)\cos(n\pi x) - \sin(n\pi x)}{n^2\pi^2}\right]_0^1 = \frac{\sqrt{2}}{n\pi}$$

Now, calculate the norm of f

$$\|f\|^2 = \int_0^1 (f(x))^2\ dx = \int_0^1 (1-x)^2\,dx = \frac{1}{3}.$$

Then

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2,$$

and so

$$\|f - f_N\|^2 = \frac{1}{3} - \sum_{n=1}^{N} \alpha_n^2.$$

The requested plots are shown below.

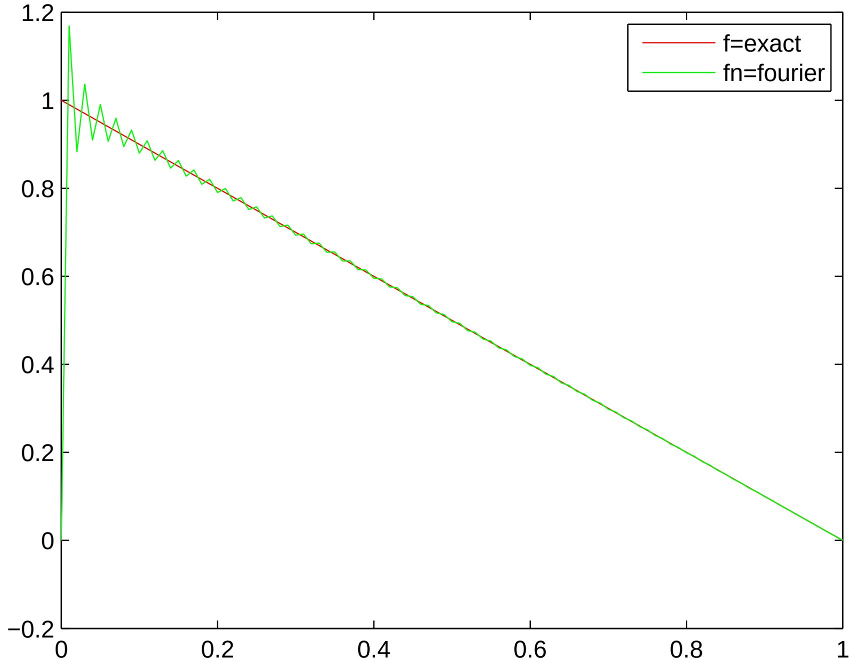


Figure 3: Comparison of the true function $f(x)$ and $f_N(x)$ for $N = 100$

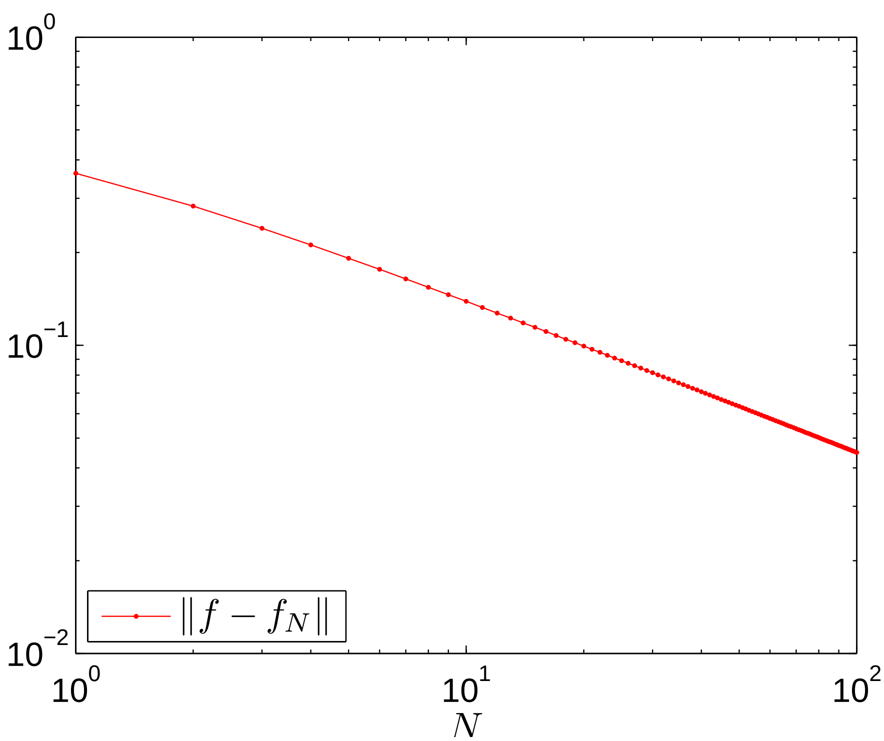


Figure 4: Norm of the error for $N = 1, 2 \cdots, 100$ on a log-log scale

The codes that produced the plots are shown below, respectively.

```
% Sine series for f(x)=(1-x) on [0,1]

clear c;
n=100;
for j=1:n
    c(j)=(sqrt(2))/(pi*j);
end;
xgrid=0:0.01:1;
hold off
fn=zeros(size(xgrid));
for j=1:n
    fn=fn+ c(j)*sqrt(2)*sin(j*pi*xgrid);
end

figure(1)
plot(xgrid, (1-xgrid), 'r', xgrid, fn, 'g');
legend('f=exact', 'fn=fourier')



%the norm of the error
n = [1:1e2]';
cn = (sqrt(2))./(pi*n);
normf2 = 1/3;
figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')

set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr
```