CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 9

Posted Wednesday 29 October, 2014. Due 5pm Wednesday 5 November, 2014.

*Please write your name and **residential college** on your homework.*

1. [30 points: 10 points each]
   Consider the following BVP with inhomogeneous boundary conditions conditions:

$$\begin{aligned} -((1+x^2)u')' &= x, \ 0 < x < 1, \\ u(0) &= 1, \\ u(1) &= 2. \end{aligned}$$

   (a) Let $x_0 = 0, x_1, \ldots, x_N, x_{N+1} = 1$ be a grid of points where $x_i = ih$. Compute the finite element solution of this BVP using piecewise linear basis functions

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i); \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}); \\ 0 & \text{otherwise;} \end{cases}$$

   Plot the Galerkin solutions with $N = 4, 8, 16, 32$ superimposed on each other. *You may wish to start with the codes from HW 8.*

   (b) In general, inhomogeneous boundary conditions are treated by decomposing $u(x)$ into

$$u(x) = w(x) + g(x)$$

   where $w(0) = w(1) = 0$ and $g(x)$ is any function satisfying inhomogeneous boundary conditions (this is referred to as the *lift*). We should make sure that the finite element solution does not depend on what lift we choose.

   Let $g(x) = 1 + x$; compute what modifications must be made to the load vector in order to compute the solution in this case.

   (c) Using the above modifications for $g(x) = 1 + x$, plot in MATLAB the solution $u_N(x)$ for $N = 4, 8, 16, 32$. Verify that these solutions should look identical to the solutions from (a).

2. [40 points: 10 points each]
Consider the linear differential equation,

$$\begin{aligned} -u'' + u &= f \qquad 0 < x < 1 \\ u(0) &= 0 \\ u'(1) &= 0 \end{aligned}$$

Let $f(x) = x(1-x)$. Suppose that N is a positive integer and define $h = \dfrac{1}{N+1}$ and $x_i = ih$ for $i = 0, 1 \dots, N+1$. Consider the hat functions $\phi_i \in C[0,1]$, defined as

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i); \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}); \\ 0 & \text{otherwise}; \end{cases}$$

for $i = 1, \dots, N+1$.
Let the stiffness matrix K be defined as

$$K_{ij} = \int_0^1 \phi_j'(x)\phi_i'(x)dx$$

Likewise, let the mass matrix M be defined as

$$M_{ij} = \int_0^1 \phi_j(x)\phi_i(x)dx$$

(a) Show that the finite element matrix A for the weak form of the equation

$$-u'' + u = f$$

can be defined as $A = K + M$. Specify the entries M_{ij} and K_{ij} (you may use the results of previous homework).

(b) Show that A is positive definite. *Hint: Use the weak form.*

(c) Write MATLAB code to solve the finite element system

$$A\alpha = b$$

for the approximate solution $u_N(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$ of the differential equation using the finite element method. Produce a plot that compares the approximate solution u_N for $N = 4$ and $N = 8$ with the true solution

$$u(x) = -x^2 + x - 2 + \frac{e(2e-1)}{1+e^2}e^{-x} + \frac{e+2}{1+e^2}e^x$$

Hint: If you'd like, you can use the Matlab code called posted on the course webpage. You may also use the Matlab function quad for numerical integration.

(d) Describe what modifications to the load vector b are necessary to compute the solution to the problem with inhomogeneous Neumann boundary condition

$$\begin{aligned} -u'' + u &= f \qquad 0 < x < 1 \\ u(0) &= 0 \\ u'(1) &= 1. \end{aligned}$$

Modify your Matlab code to accommodate these changes, and produce a plot of the solution for $N = 4, 8, 16$. *You do not need to compare against the exact solution for this problem.*

3. [30 points: 10 points each]
Consider the *Euler Bernoulli beam equation*,

$$(k(x)u''(x))'' = f(x), \qquad 0 < x < 1,$$

with boundary conditions describing a beam that is *clamped* at both ends:

$$u(0) = u(1) = 0; \qquad u'(0) = u'(1) = 0$$

Here $k(x)$ is a positive-valued function that describes the material properties of the beam. With these boundary conditions, the eigenvalues and eigenvectors of this operator are diffcult to compute even if $k(x) = 1$. We will consider finite element solutions of this problem.

(a) Derive the weak form of the beam equation with the above boundary conditions, i.e., derive the weak problem

$$a(u, v) = (f, v); \qquad \text{for all } v \in V = C_D^4[0, 1],$$

where

$$C_D^4[0, 1] = \{u \in C^4[0, 1] : u(0) = u(1) = u'(0) = u'(1) = 0\}.$$

Specify the bilinear form $a(u, v)$, and show that it is an inner product on $C_D^4[0, 1]$
Note: for the problem $-(ku')' = f$, we do not explicitly impose Neumann boundary conditions, they follow 'naturally'. For the beam equation, we must impose all four boundary conditions on the space of test functions, $V = C_D^4[0, 1]$.

(b) Suppose that $V_n = span\{\phi_1, \cdots, \phi_n\}$ is an n-dimensional subspace of $C_D^4[0, 1]$. (Do not assume a particular form for the functions $\phi_1, \cdots, \phi_n$ at this point.)
Show how the Galerkin problem

$$a(u_n, v) = (f, v), \qquad \text{for all } v \in V_n$$

leads to the linear system $Ku = f$. Be sure to specify the entries of K, u, and f .

(c) Now suppose we take for $\phi_1, \cdots, \phi_n$ the standard piecewise linear 'hat' functions used, for example, in Problem 2. Are these functions suitable for this problem? If so, describe the location of the nonzero entries of the matrix K. If not, roughly describe a better choice for the functions $\phi_1, \cdots, \phi_n$ and the explain which entries of K are nonzero for that choice.