CAAM 336 · DIFFERENTIAL EQUATIONS

Examination 2

Posted Monday 9 December 2013.

Due no later than 12 noon Tuesday 17 December 2013.

Read and follow all of the below instructions:

1. You may not look at any of the questions on this exam until the time at which you start to take this exam.

2. The maximum amount of time that can be spent taking this exam is 5 consecutive hours.

3. Once you have started taking this exam you may not consult any books, notes, websites, homework questions, homework solutions, or other resources until after you have finished taking this exam.

4. Once you have started taking this exam you may not use any electronic devices until after you have finished taking this exam.

5. You may not discuss this exam with anyone until after you have finished taking this exam and even then you may not discuss this exam with, or in the presence of, anyone in the class who has yet to take this exam.

6. If you do not understand any of the instructions on this page then ask Richard Rankin for clarification.

7. Before taking this exam you must email Richard Rankin and let him know that you have read and understood all of the instructions on this page.

You should turn this page in with your exam. This page and all of the pages that make up your exam should be stapled together with this page at the front.

Legibly write your name on the line below:

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Legibly write the dates and times that you started and finished taking this exam below.

Date started: ____________________ Time started: ____________________ am/pm

Date finished: ____________________ Time finished: ____________________ am/pm

Indicate that this is your own individual effort in compliance with the instructions given on the previous page and the honor system by legibly writing out in full and signing the traditional pledge on the lines below.

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1. [5 points]

   Let $\alpha \in \mathbb{R}$, let $\beta \in \mathbb{R}$, let $\gamma \in \mathbb{R}$ and let $\mu \in \mathbb{R}$ be such that $\mu > 0$.

   (a) Verify that
   $$p(t) = \alpha \cos\left(\sqrt{\mu} t\right) + \frac{\beta}{\sqrt{\mu}} \sin\left(\sqrt{\mu} t\right)$$
   satisfies
   $$-p''(t) = \mu p(t),$$
   $$p(0) = \alpha$$
   and
   $$p'(0) = \beta.$$

   (b) Verify that
   $$q(t) = \alpha e^{\gamma t} + \frac{\beta}{\gamma}\left(e^{\gamma t} - 1\right)$$
   satisfies
   $$q'(t) = \gamma q(t) + \beta$$
   and
   $$q(0) = \alpha.$$

2. [5 points]

   Let $f \in C[0,1]$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Let u be such that
   $$-4u''(x) + 9u(x) = f(x), \quad 0 < x < 1;$$
   $$-4u'(0) = \alpha$$
   and
   $$4u'(1) = \rho.$$

   (a) It can be shown that
   $$\int_0^1 4u'(x)v'(x) + 9u(x)v(x)\,dx = g(f, \alpha, \rho, v) \text{ for all } v \in C^2[0,1].$$
   Obtain a formula for $g(f, \alpha, \rho, v)$.

3. [20 points]

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v,w) = a(v,w) + (v,w).$$

Let the norm $|||\cdot||| : H^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{B(v,v)}.$$

Let $f \in L^2(0,1)$, let $\rho \in \mathbb{R}$, let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$ and let $u \in H^1_D(0,1)$ be such that

$$B(u,v) = (f,v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Moreover, let N be a positive integer, let V_N be a subspace of $H^1_D(0,1)$ and let $u_N \in V_N$ be such that

$$B(u_N,v) = (f,v) + \rho v(1) \text{ for all } v \in V_N.$$

(a) Use the fact that $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$ and the fact that $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$ to show that $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$. Recall that $H^1(0,1) = \{v \in L^2(0,1) : v' \in L^2(0,1)\}$.

(b) Show that

$$B(u - u_N, v) = 0 \text{ for all } v \in V_N.$$

(c) Show that

$$|||u - u_N|||^2 = |||u|||^2 - |||u_N|||^2.$$

(d) Show that

$$|||u_N|||^2 \leq |||u|||^2.$$

4. [25 points]

Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let $\phi_0 \in H^1(0,1)$ be defined by

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

let $\phi_j \in H^1_D(0,1)$ be defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \dots, N$ and let $\phi_{N+1} \in H^1_D(0,1)$ be defined by

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the symmetric bilinear form $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v, w) = a(v, w) + (v, w).$$

Also, let $f \in L^2(0,1)$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Moreover, let $u \in H^1(0,1)$ be such that $u(0) = \alpha$ and

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Let $V_N = \operatorname{span}\{\phi_0, \phi_1, \dots, \phi_{N+1}\}$ and let $V_{N,D} = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{N+1}\}$. Let $u_N \in V_N$ be such that $u_N(0) = \alpha$ and

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

(a) We can write

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j\phi_j$$

where, for $j = 1, 2, \dots, N+1$, c_j is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$ which is the solution to

$$\mathbf{Kc} = \mathbf{b}.$$

What are the entries of the matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ and the vector $\mathbf{b} \in \mathbb{R}^{N+1}$?

(b) Show that

$$B(u - u_N, u - u_N) = B(u, u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0).$$

(c) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = 0$, $\rho = 0$ and $N = 1$. Note that, when $N = 1$,

$$\int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$\int_0^{1/2} \phi_0(x)\phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\phi_2(x)\,dx = \frac{1}{6};$$

and

$$\int_0^{1/2} \phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\,dx = \int_{1/2}^1 \phi_1(x)\,dx = \int_{1/2}^1 \phi_2(x)\,dx = \frac{1}{4}.$$

(d) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = -1$, $\rho = 1$ and $N = 1$.

5. [35 points]

Let

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

In this question we will consider the problem of finding the solution $u(x,t)$ to the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = 1, \quad t \ge 0,$$

and

$$u_x(1,t) = 2, \quad t \ge 0,$$

and initial condition

$$u(x,0) = x^2 + 1, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w'(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \psi_m(x)\psi_n(x)\,dx = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \ne n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) For $n = 1, 2, \ldots$, compute

$$\int_0^1 f(x)\psi_n(x)\,dx.$$

(c) Let $w(x)$ be such that

$$w''(x) = 0,$$

$$w(0) = 1$$

and

$$w'(1) = 2.$$

Obtain a formula for $w(x)$.

(d) Let $\hat{u}(x,t)$ be such that

$$\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

$$\hat{u}(0,t) = \hat{u}_x(1,t) = 0, \quad t \ge 0,$$

and

$$\hat{u}(x,0) = \hat{u}_0(x), \qquad 0 \le x \le 1,$$

where $\hat{u}_0(x)$ is such that

$$u(x,t) = w(x) + \hat{u}(x,t).$$

Obtain a formula for $\hat{u}_0(x)$.

(e) For $n = 1, 2, \ldots$, compute

$$\int_0^1 \hat{u}_0(x)\psi_n(x)\,dx.$$

(f) We can write

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n\psi_n(x)$$

where, for $n = 1, 2, \ldots$,

$$b_n = \int_0^1 f(x)\psi_n(x)\,dx.$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(g) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(h) Write out a formula for $u(x,t)$.

6. [10 points]

In this question we will consider the problem of finding the solution $u(x,t)$ to the wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = u_x(1,t) = 0, \quad t \ge 0,$$

and initial conditions

$$u(x,0) = 0, \qquad 0 \le x \le 1,$$

and

$$u_t(x,0) = x^2 - 2x, \qquad 0 \le x \le 1.$$

Let

$$S = \left\{ w \in C^2[0,1] : w(0) = w'(1) = 0 \right\}$$

and let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''.$$

The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin\left(\frac{2n-1}{2} \pi x \right)$$

for $n = 1, 2, \ldots$. Recall that you obtained a formula for the eigenvalues of L in question 5.

(a) We can write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \psi_n(x).$$

What ordinary differential equation and initial conditions does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

(b) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(c) Write out a formula for $u(x,t)$.