CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 2

Posted Wednesday 29 August 2012. Due Wednesday 5 September 2012, 5pm.

1. [24 points]
   Consider the following sets of functions. Demonstrate whether or not each is a vector space (with addition and scalar multiplication defined in the obvious way).

   (a) $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^3\}$
   (b) $\{\mathbf{x} \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
   (c) $\{f \in C[0,1] : f(x) \geq 0 \text{ for all } x \in [0,1]\}$
   (d) $\{f \in C[0,1] : \max_{x\in[0,1]} f(x) \leq 1\}$
   (e) $\{f \in C^1[0,1] : f'(0) = 0\}$
   (f) $\{f \in C^2[0,1] : f''(x) = 0 \text{ for all } x \in [0,1]\}$

2. [14 points]

   (a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

   $$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Since f is linear, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

   (b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

   (Thus any linear function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix-vector product.)

3. [24 points]
   Recall that a function $f : \mathcal{V} \to \mathcal{W}$ that maps a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a *linear operator* provided (1) $f(u + v) = f(u) + f(v)$ for all u, v in $\mathcal{V}$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$.

   Demonstrate whether each of the following functions is a linear operator.
   (Show that both properties hold, or give an example showing that one of the properties must fail.)

   (a) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$.
   (b) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.
   (c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.
   (d) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.
   (e) $L : C^1[0,1] \to C[0,1]$, $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.
   (f) $L : C^2[0,1] \to C[0,1]$, $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

4. [12 points]
Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and suppose $f : \mathcal{V} \to \mathcal{W}$ is a linear operator.

The *range* of f is the set of all vectors in $\mathcal{W}$ that can be written in the form $f(v)$ for some $v \in \mathcal{V}$:

$$\mathcal{R}(f) = \{f(v) : v \in \mathcal{V}\}.$$

Show that $\mathcal{R}(f)$ is a subspace of $\mathcal{W}$.

(The *range* generalizes the notion of *column space* from matrix theory.)

5. [26 points]

(a) In class we considered the 'forward difference' approximation

$$u'(x) \approx \frac{u(x+h) - u(x)}{h}.$$

Let $u(x) = \exp(2x)$. For each value $N = 2, 4, 8, 16, \ldots, 512$ (powers of 2), compute (in MATLAB) the error

$$\left| u'(1/2) - \frac{u(1/2+h) - u(1/2)}{h} \right|,$$

where $h = 1/(N+1)$. Print out these errors, and use MATLAB's `loglog` command to produce a plot of N versus the corresponding error. (In class, we showed that this error should be proportional to h as $h \to 0$.)

(b) Consider the 'centered difference' approximation

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}.$$

Repeat part (a) with this approximation: That is, for $u(x) = \exp(2x)$, compute the error

$$\left| u'(1/2) - \frac{u(1/2+h) - u(1/2-h)}{2h} \right|$$

for $N = 2, 4, 8, 16, 32, \ldots, 512$ (powers of 2) with $h = 1/(N+1)$. Print out these errors, and use MATLAB's `loglog` command to produce a plot of N versus the corresponding error. (In class, we showed that this error should be proportional to h^2 as $h \to 0$.)

Use the `hold on` *command to superimpose the plot for (b) on your plot for part (a): you should only turn in one plot for this problem.*

(c) By inspecting the plot you have created, estimate the value of N that you need to approximate $u'(1/2)$ to an error of 10^{-2} using the methods in part (a) and part (b).

Challenge problem (4 bonus points): Given an integer $N \geq 1$, define $h = 1/(N+1)$ and consider the grid of points $x_j = jh$ for $j = 0, \ldots, N+1$. It is often desirable to construct an approximation to $u'(x_0)$ whose accuracy is proportional to h^2 as $h \to 0$. The centered difference in part (b) above is unsuitable, as it would require a value $u(x_{-1}) = u(-h)$, and $-h$ is outside the domain $[0, 1]$. Show an alternative way to approximate $u'(x_0)$ using only the values $u(x_0)$, $u(x_1)$, and $u(x_2)$, i.e., find coefficients α, β, and γ such that

$$u'(x_0) = \alpha u(x_0) + \beta u(x_1) + \gamma u(x_2) + O(h^2).$$