CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 36 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

36. [25 points]
Let
$$H^1_D(0,1) = \{v \in H^1(0,1) : v(1) = 0\}.$$
Let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by
$$(v,w) = \int_0^1 v(x)w(x)\,dx$$
and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by
$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$
Also, let the norm $|||\cdot||| : H^1_D(0,1) \to \mathbb{R}$ be defined by
$$|||v||| = \sqrt{a(v,v)}.$$
Let $\alpha, \beta \in \mathbb{R}$ and let $f \in L^2(0,1)$ be defined by
$$f(x) = 12x^2 - 24x + 4.$$
Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let $\phi_0 \in H^1(0,1)$ be such that
$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$
let $\phi_j \in H^1(0,1)$ be such that
$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$
for $j = 1, \ldots, N$ and let $\phi_{N+1} \in H^1(0,1)$ be such that
$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$
Let $u \in H^1(0,1)$ be such that $u(1) = \beta$ and
$$a(u,v) = (f,v) + \alpha v(0) \text{ for all } v \in H^1_D(0,1).$$
Note that
$$u(x) = -x^4 + 4x^3 - 2x^2 - \alpha x + \alpha + \beta - 1.$$

(a) For this part and the next, we will just consider the case when $\alpha = \beta = 0$. In this case, we can obtain finite element approximations u_N to u by finding $u_N \in \text{span}\{\phi_0, \dots, \phi_{\text{N}}\}$ such that

$$a(u_N, v) = (f, v) + \alpha v(0) \text{ for all } v \in \text{span}\{\phi_0, \dots, \phi_{\text{N}}\}.$$

Write a code which can obtain u_N and u_N^* where $u_N^* \in \text{span}\{\phi_1, \dots, \phi_{\text{N}}\}$ is such that

$$a(u_N^*, v) = (f, v) + \alpha v(0) \text{ for all } v \in \text{span}\{\phi_1, \dots, \phi_{\text{N}}\}.$$

On the same figure, plot u as well as u_N and u_N^* for $N = 3$ and $N = 7$.

(b) For the case when $\alpha = \beta = 0$, plot

$$|||u - u_N||| = \sqrt{a(u, u) - a(u_N, u_N)}$$

and

$$|||u - u_N^*||| = \sqrt{a(u, u) - a(u_N^*, u_N^*)}$$

for $N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767$.

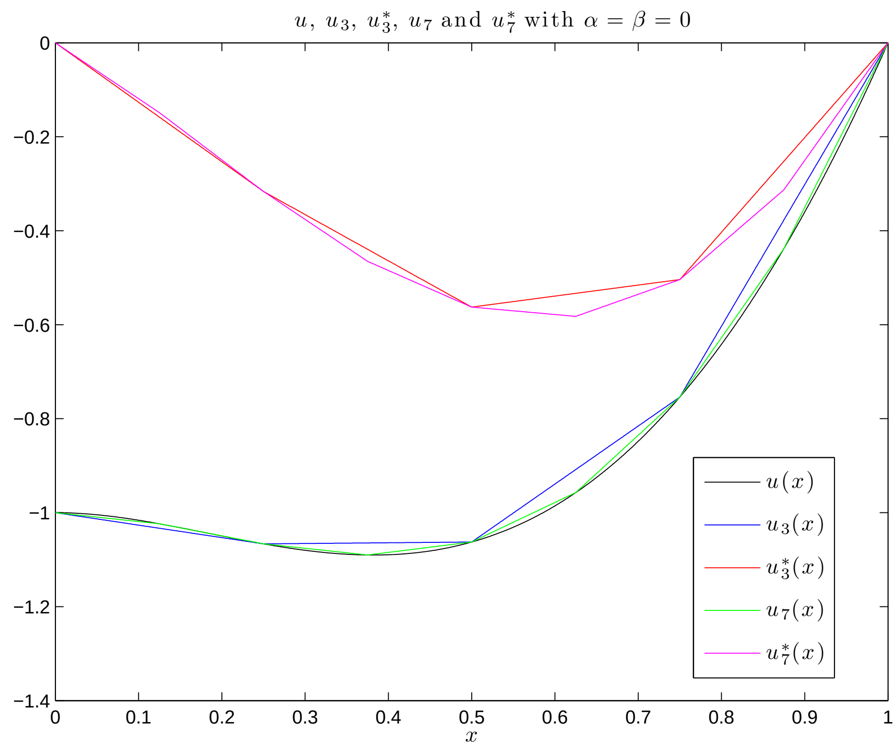
(c) Modify your code so that it can obtain finite element approximations u_N to u by finding $u_N \in \text{span}\{\phi_0, \dots, \phi_{\text{N+1}}\}$ such that $u_N(1) = \beta$ and

$$a(u_N, v) = (f, v) + \alpha v(0) \text{ for all } v \in \text{span}\{\phi_0, \dots, \phi_{\text{N}}\}$$

for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. For the particular case of $\alpha = -1$ and $\beta = 1$, use your code to obtain u_N for $N = 3, 7, 15, 31$ and on the same figure plot u and u_N for $N = 3, 7, 15, 31$.

Solution.

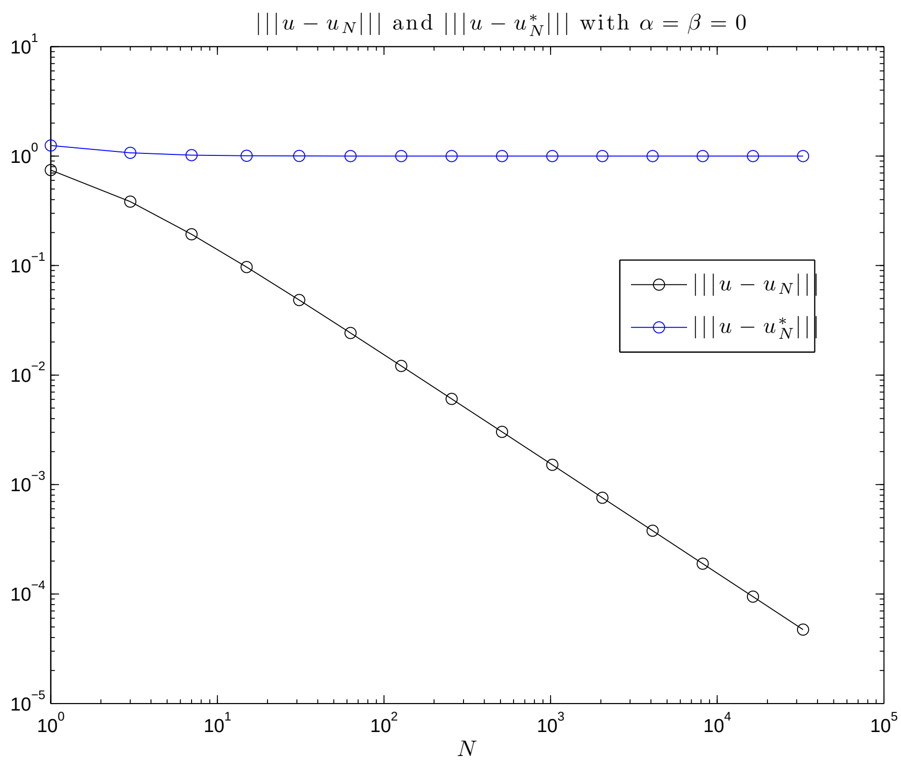
(a) [8 points] The plot and code used to create it, and the plot shown in part (b), are below. Note that the below code uses the MATLAB function which you had to write in Homework 2.



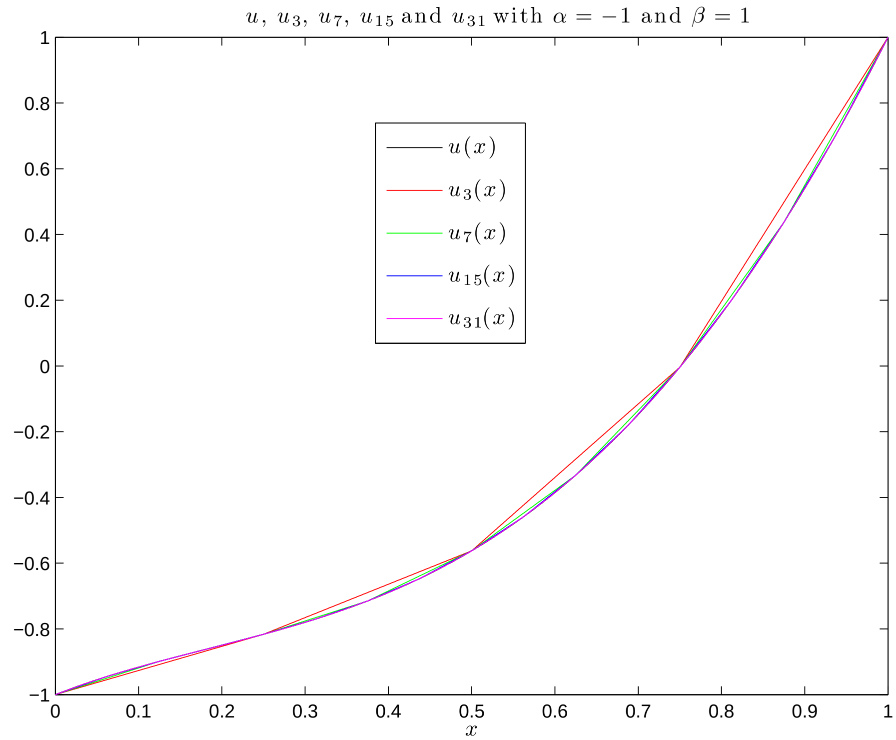
```
clear
clc

figure(1)
clf
x=linspace(0,1,1000).';
u=-x.^4+4*x.^3-2*x.^2-1;
plot(x, u, '-k')
hold on

Nvec=2.^(1:15)-1;
energyerr=zeros(size(Nvec));
energyerrhD=zeros(size(Nvec));
for j=1:length(Nvec)
    N=Nvec(j);
    h=1/(N+1);

    K=sparse(N+1,N+1);
    K=K+sparse(1:N,2:N+1,-1/h,N+1,N+1);
    K=K+K.';
    K=K+sparse(1,1,1/h,N+1,N+1);
    K=K+sparse(2:N+1,2:N+1,2/h,N+1,N+1);

    KhD=K(2:N+1,2:N+1);

    f=zeros(N+1,1);
    f(1)=h*(h^2-4*h+2);
    f(2:N+1)=2*h*(h^2+6*((1:N)*h).^2-12*((1:N)*h)+2);

    fhD=f(2:N+1);

    c=K\f;

    chD=KhD\fhD;

    energyerr(j)=sqrt(296/105-(c.')*K*c);
    energyerrhD(j)=sqrt(296/105-(chD.')*KhD*chD);

    if N==3
       uN = zeros(size(x));
       for k=1:N+1
           uN = uN + c(k)*hat(x,k-1,N);
       end
       plot(x, uN, '-b')
       uNhD = zeros(size(x));
       for k=1:N
           uNhD = uNhD + chD(k)*hat(x,k,N);
       end
       plot(x, uNhD, '-r')
    end
    if N==7
        uN = zeros(size(x));
        for k=1:N+1
            uN = uN + c(k)*hat(x,k-1,N);
        end
        plot(x, uN, '-g')
        uNhD = zeros(size(x));
        for k=1:N
            uNhD = uNhD + chD(k)*hat(x,k,N);
        end
        plot(x, uNhD, '-m')
    end
end

title('$u$, $u_3$, $u_3^*$, $u_7$ and $u_7^*$ with $\alpha=\beta=0$','interpreter','
    latex','FontSize',12)
xlabel('$x$','interpreter','latex','FontSize',12)
```

```
legendstr{1}='$u(x)$';
legendstr{2}='$u_3(x)$';
legendstr{3}='$u_3^*(x)$';
legendstr{4}='$u_7(x)$';
legendstr{5}='$u_7^*(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(1),'hw36a.eps','epsc')

figure(2)
clf
loglog(Nvec,energyerr,'-ok')
hold on
loglog(Nvec,energyerrhD,'-ob')
title('$|||u-u_N|||$ and $|||u-u_N^*|||$ with $\alpha=\beta=0$','interpreter','latex','
    FontSize',12)
xlabel('$N$','interpreter','latex','FontSize',12)
legendstr2{1}='$|||u-u_N|||$';
legendstr2{2}='$|||u-u_N^*|||$';
legend(legendstr2,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(2),'hw36b.eps','epsc')
```

(b) [7 points] The plot is below.



(c) [10 points] The plot and code used to create it are below. Note that the below code uses the MATLAB function which you had to write in Homework 2.




```
clear
clc

alpha=-1;
beta=1;

figure(1)
clf
x=linspace(0,1,1000).';
u=-x.^4+4*x.^3-2*x.^2-alpha*x+alpha+beta-1;
plot(x, u, '-k')
hold on

colors='rgbm';
Nvec=2.^(2:5)-1;
for j=1:length(Nvec)
    N=Nvec(j);
    h=1/(N+1);

    K=sparse(N+1,N+1);
    K=K+sparse(1:N,2:N+1,-1/h,N+1,N+1);
    K=K+K.';
    K=K+sparse(1,1,1/h,N+1,N+1);
    K=K+sparse(2:N+1,2:N+1,2/h,N+1,N+1);

    f=zeros(N+1,1);
    f(1)=h*(h^2-4*h+2)+alpha;
    f(2:N+1)=2*h*(h^2+6*((1:N)*h).^2-12*((1:N)*h)+2);
    f(N+1)=f(N+1)+beta/h;

    c=K\f;

    uN = zeros(size(x));
    for k=1:N+1
        uN = uN + c(k)*hat(x,k-1,N);
```

```
    end
    uN = uN + beta*hat(x,N+1,N);
    plot(x, uN, colors(j))
end

title('$u$, $u_3$, $u_7$, $u_{15}$ and $u_{31}$ with $\alpha=-1$ and $\beta=1$','
    interpreter','latex','FontSize',12)
xlabel('$x$','interpreter','latex','FontSize',12)
legendstr{1}='$u(x)$';
legendstr{2}='$u_3(x)$';
legendstr{3}='$u_7(x)$';
legendstr{4}='$u_{15}(x)$';
legendstr{5}='$u_{31}(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(1),'hw36c.eps','epsc')
```
