CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 33

Posted Friday 25 October 2013. Due 5pm Wednesday 6 November 2013.

33. [25 points] Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = w(1) = 0\}$ and let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the norm $|||\cdot||| : H^1_D(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{a(v,v)}.$$

Let m be a positive integer and let $f_m \in L^2(0,1)$ be defined by $f_m(x) = \sqrt{2}m^2\pi^2\sin(m\pi x)$. Note that, for $j = 1,\ldots,N$,

$$(f_m,\phi_j) = \frac{2\sqrt{2}\sin(m\pi x_j)}{h}(1-\cos(m\pi h)).$$

Let N be a positive integer, let $h = \dfrac{1}{N+1}$, let $x_j = jh$ for $j = 0,1,\ldots,N+1$, and let $\phi_k \in H^1_D(0,1)$ be defined by

$$\phi_k(x) = \begin{cases} \dfrac{(x - x_{k-1})}{h} & \text{if } x \in [x_{k-1}, x_k); \\ \dfrac{(x_{k+1} - x)}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1,\ldots,N$. Let $V_N = \text{span}\{\phi_1, ..., \phi_N\}$. Let $u_m \in H^1_D(0,1)$ be such that

$$a(u_m, v) = (f_m, v) \text{ for all } v \in H^1_D(0,1).$$

(a) The true solution to the problem of finding $u_m \in H^1_D(0,1)$ such that

$$a(u_m, v) = (f_m, v) \text{ for all } v \in H^1_D(0,1)$$

is $u_m(x) = \sqrt{2}\sin(m\pi x)$. Use integration by parts and the fact that $(u_m, u_m) = 1$ to obtain a formula for $a(u_m, u_m)$.

(b) We can obtain finite element approximations to u_m by finding $u_{m,N} \in V_N$ such that

$$a(u_{m,N}, v) = (f_m, v) \text{ for all } v \in V_N.$$

Write a code which can compute the energy norm of the error

$$|||u_m - u_{m,N}|||.$$

Use your code to produce a `loglog` plot of the energy norm of the error

$$|||u_m - u_{m,N}|||$$

when

$$m = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

for

$$N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191$$

on the same figure with a separate line for each value of m.

(c) Produce a `loglog` plot of the percentage error

$$100\frac{|||u_m - u_{m,N}|||}{|||u_m|||}$$

when

$$m = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

for

$$N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191$$

on the same figure with a separate line for each value of m.

(d) Let $f \in L^2(0,1)$. The problem of finding $u \in H_D^1(0,1)$ such that

$$a(u,v) = (f,v) \text{ for all } v \in H_D^1(0,1)$$

is the weak form of the problem of finding u such that

$$-u''(x) = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0,$$

or equivalently,

$$f(x) + u''(x) = 0, \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

If $\tilde{u}_N \in V_N$ then how does the quantity

$$\sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 \, dx$$

behave as N increases?