CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 32 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 28 March 2014.

32. [25 points]

Determine whether or not each of the following mappings is a bilinear form on the real vector space $\mathcal{V}$.

(a) $B(\cdot,\cdot):\ \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $B(u,v)=\displaystyle\int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V}=C^1[0,1]$.

(b) $B(\cdot,\cdot):\ \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $B(u,v)=\displaystyle\int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

(c) $B(\cdot,\cdot):\ \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $B(u,v)=\displaystyle\int_0^1 u(x)|v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

(d) $B(\cdot,\cdot):\ \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $B(u,v)=\displaystyle\int_0^1 u(x)+v(x)\,dx$ where $\mathcal{V}=C[0,1]$.

(e) $B(\cdot,\cdot):\ \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $B(u,v)=\displaystyle\int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V}=C[-1,1]$.

Solution.

(a) [5 points] *This mapping is a bilinear form*:

The mapping is linear in the first argument since

$$\begin{aligned}
B(\alpha u+\beta v,w) &= \int_0^1 (\alpha u(x)+\beta v(x))\,w'(x)\,dx\\
&= \int_0^1 \alpha u(x)w'(x)\,dx+\int_0^1 \beta v(x)w'(x)\,dx\\
&= \alpha\int_0^1 u(x)w'(x)\,dx+\beta\int_0^1 v(x)w'(x)\,dx\\
&= \alpha B(u,w)+\beta B(v,w)
\end{aligned}$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$. It is also linear in the second argument since

$$\begin{aligned}
B(u,\alpha v+\beta w) &= \int_0^1 u(x)\,(\alpha v+\beta w)'(x)\,dx\\
&= \int_0^1 u(x)\,(\alpha v'(x)+\beta w'(x))\,dx\\
&= \int_0^1 \alpha u(x)v'(x)\,dx+\int_0^1 \beta u(x)w'(x)\,dx\\
&= \alpha\int_0^1 u(x)v'(x)\,dx+\beta\int_0^1 u(x)w'(x)\,dx\\
&= \alpha B(u,v)+\beta B(u,w)
\end{aligned}$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$.

(b) [5 points] *This mapping is not a bilinear form*:

Let $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Then

$$B(\alpha u + \beta v, w) = \int_0^1 |\alpha u(x) + \beta v(x)||w(x)|\,dx$$

and

$$\alpha B(u, w) + \beta B(v, w) = \alpha \int_0^1 |u(x)||w(x)|\,dx + \beta \int_0^1 |v(x)||w(x)|\,dx.$$

However, if $u(x) = v(x) = w(x) = 1$, $\alpha = -1$ and $\beta = 0$, then

$$B(\alpha u + \beta v, w) = \int_0^1 |-1||1|\,dx = \int_0^1 1\,dx = 1$$

but

$$\alpha B(u, w) + \beta B(v, w) = -\int_0^1 |1||1|\,dx - \int_0^1 1\,dx = -1.$$

Hence, the mapping is not linear in the first argument.

Alternatively, we could have shown that the mapping is not linear in the second argument.

(c) [5 points] *This mapping is not a bilinear form*:

Let $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Then

$$B(u, \alpha v + \beta w) = \int_0^1 u(x)|\alpha v(x) + \beta w(x)|\,dx$$

and

$$\alpha B(u, v) + \beta B(u, w) = \alpha \int_0^1 u(x)|v(x)|\,dx + \beta \int_0^1 u(x)|w(x)|\,dx.$$

However, if $u(x) = v(x) = w(x) = 1$, $\alpha = -1$ and $\beta = 0$, then

$$B(u, \alpha v + \beta w) = \int_0^1 |-1|\,dx = \int_0^1 1\,dx = 1$$

but

$$\alpha B(u, v) + \beta B(u, w) = -\int_0^1 |1|\,dx = -\int_0^1 1\,dx = -1.$$

Hence, the mapping is not linear in the second argument.

(d) [5 points] *This mapping is not a bilinear form*:

Let $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Then

$$B(\alpha u + \beta v, w) = \int_0^1 \alpha u(x) + \beta v(x) + w(x)\,dx$$

and

$$\alpha B(u, w) + \beta B(v, w) = \alpha \int_0^1 u(x) + w(x)\,dx + \beta \int_0^1 v(x) + w(x)\,dx.$$

However, if $u(x) = v(x) = w(x) = 1$ and $\alpha = \beta = 1$, then

$$B(\alpha u + \beta v, w) = \int_0^1 1 + 1 + 1\,dx = \int_0^1 3\,dx = 3$$

but

$$\alpha B(u,w) + \beta B(v,w) = \int_0^1 1+1\,dx + \int_0^1 1+1\,dx = \int_0^1 2\,dx + \int_0^1 2\,dx = 2+2 = 4.$$

Hence, the mapping is not linear in the first argument.
Alternatively, we could have shown that the mapping is not linear in the second argument.

(e) [5 points] *This mapping is a bilinear form*:
The mapping is linear in the first argument since

$$\begin{aligned} B(\alpha u + \beta v, w) &= \int_{-1}^1 x(\alpha u(x) + \beta v(x))w(x)\,dx \\ &= \int_{-1}^1 (\alpha x u(x) w(x) + \beta x v(x) w(x))\,dx \\ &= \alpha \int_{-1}^1 x u(x) w(x)\,dx + \beta \int_{-1}^1 x v(x) w(x)\,dx \\ &= \alpha B(u,w) + \beta B(v,w) \end{aligned}$$

for all $u, v, w \in C[-1,1]$ and all $\alpha, \beta \in \mathbb{R}$.
The mapping is also linear in the second argument since

$$\begin{aligned} B(u, \alpha v + \beta w) &= \int_{-1}^1 x u(x)(\alpha v(x) + \beta w(x))\,dx \\ &= \int_{-1}^1 (\alpha x u(x) v(x) + \beta x u(x) w(x))\,dx \\ &= \alpha \int_{-1}^1 x u(x) v(x)\,dx + \beta \int_{-1}^1 x u(x) w(x)\,dx \\ &= \alpha B(u,v) + \beta B(u,w) \end{aligned}$$

for all $u, v, w \in C[-1,1]$ and all $\alpha, \beta \in \mathbb{R}$.
