CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 11 · Solutions

Posted Friday 21 November 2014. Due Monday 1 December 2014, 5pm.

1. [20 points: 3 points each for (a) and (b); 7 points for (c)-(d)]
Consider the wave equation posed on the infinite domain $x \in (-\infty, \infty)$:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty, \quad t > 0 \tag{$*$}$$

with initial conditions $u(x,0) = \psi(x)$ and $u_t(x,0) = \gamma(x)$.
At a given point $(\widetilde{x}, \widetilde{t})$, with $\widetilde{x} \in (-\infty, \infty)$ and $\widetilde{t} > 0$, the solution $u(\widetilde{x}, \widetilde{t})$ of the wave equation is only affected by some portion of the initial data. In other words, $u(\widetilde{x}, \widetilde{t})$ is only influenced by $\psi(x)$ and $\gamma(x)$ for $x \in [a, b]$, where a and b will depend upon $\widetilde{x}$ and $\widetilde{t}$. This interval $[a, b]$ is called the *domain of dependence* of the solution at $(\widetilde{x}, \widetilde{t})$.

(a) Determine the domain of dependence of the solution to the wave equation $(*)$ at $(\widetilde{x}, \widetilde{t}) = (0, 1)$.

Now consider the heat equation on an unbounded domain:

$$u_t(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty$$

with initial data

$$u(x,0) = \psi(x).$$

Like d'Alembert's solution, there exists a formula for the solution of the heat equation on this domain: for all $t > 0$,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \psi(s)\, \mathrm{d}s.$$

(b) What is the domain of dependence of this solution to the heat equation at $(\widetilde{x}, \widetilde{t}) = (0, 1)$?
Contrast the physical implications of the domains of dependence for the heat and wave equations.

(c) Consider the *wave* equation with discontinuous initial data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0; \end{cases} \qquad \gamma(x) = 0.$$

On one plot, superimpose solutions to this equation at the four times $t = 0, 1/2, 1, 2$.
(Notice how the discontinuity in the initial data is propagated in time.)

(d) Now consider the *heat* equation with the same starting data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

Using the formula for $u(x,t)$ given above, produce solutions to this equation at the four times $t = 0, 0.01, 0.1, 1$. What happens to the discontinuity for $t > 0$?

Important hint: You will need to compute some nasty integrals here that you cannot work out entirely by hand. To produce your plots, use MATLAB's `erfc` command. For example,

$$\frac{2}{\sqrt{\pi}} \int_{\mathtt{z}}^{\infty} e^{-y^2}\, \mathrm{d}y = \mathtt{erfc(z)}.$$

Solution.

(a) Recall that the general solution of the wave equation on an unbounded domain takes the form

$$u(x,t) = \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) + \tfrac{1}{2}\int_{x-t}^{x+t} \gamma(s)\, ds.$$

From this it follows that

$$u(0,1) = \tfrac{1}{2}\big(\psi(-1) + \psi(1)\big) + \tfrac{1}{2}\int_{-1}^{1} \gamma(s)\, ds.$$

Since the solution at $x = 0$ and $t = 1$ depends on $\gamma(s)$ for all $s \in [-1,1]$ and ϕ at $x = \pm 1 \in [-1,1,$ we note that the domain of dependence is the interval $[-1,1]$.

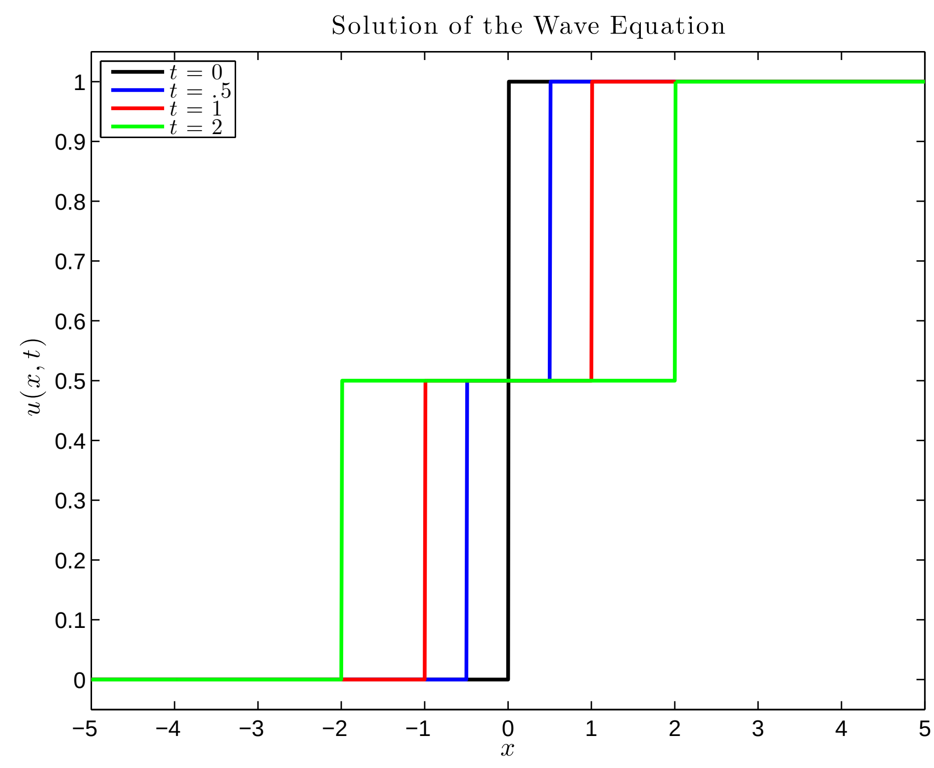
(b) Since the solution to the heat equation in an unbounded domain depends on $\phi(s)$ for all values of $s \in (-\infty, \infty)$, the domain of dependence at $x = 0$ and $t = 1$ consists of the entire real line, $(-\infty, \infty)$.

[Graders: please accept a variety of reasonable solutions to this problem.] For the wave equation, the initial condition takes a finite time to propagate: for example, it takes one full time unit for the value of $\psi(\pm 1)$ to affect the solution u at the point $x = 0$. In contrast, in the heat equation, with its domain of dependence of $(-\infty, \infty)$, the value of the initial condition at any single point *instantaneously* affects the solution at all other points. So, the initial distribution of heat instantly affects the heat at all other points.

(c) Since $\gamma(x) = 0$ for all x, the solution to the wave equation is simply

$$u(x,t) = \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) = \begin{cases} 0, & x < -t; \\ 1/2 & -t \le x < t; \\ 1, & x > t. \end{cases}$$

The requested plot of the solution is shown below; code follows at the end of the problem.



(d) For the specified initial condition, the solution takes the form

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \psi(s)\,\mathrm{d}s = \frac{1}{2\sqrt{\pi t}} \int_{0}^{\infty} e^{\frac{-(s-x)^2}{4t}}\,\mathrm{d}s.$$

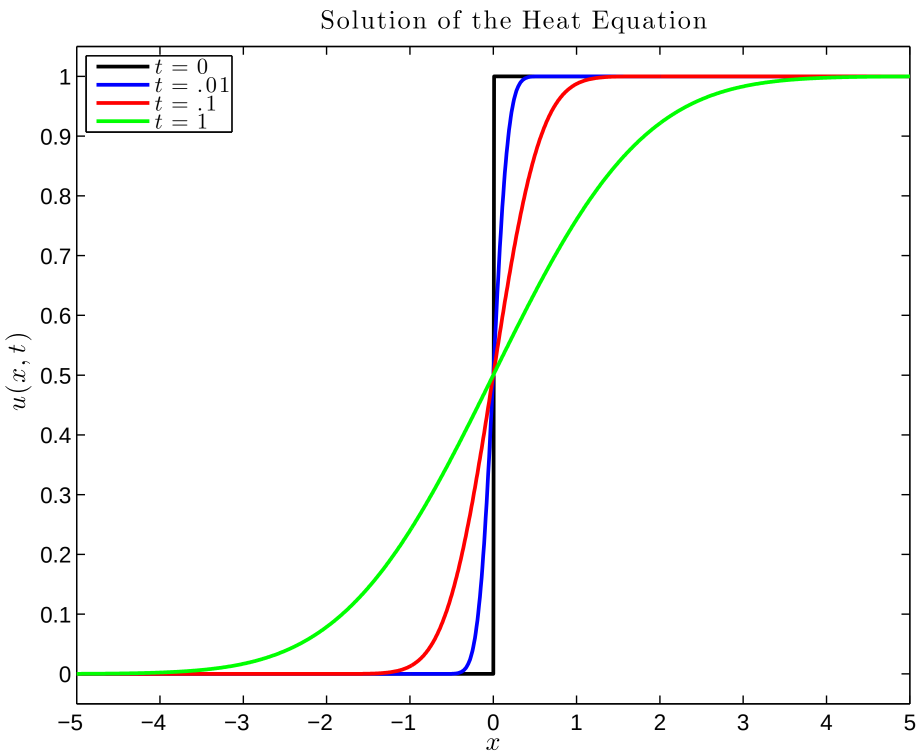
To compute the integral, use the substitution $y = (s-x)/(2\sqrt{t})$, so that $\mathrm{d}y = 1/(2\sqrt{t})\,\mathrm{d}s$ to compute

$$\int_0^{\infty} e^{\frac{-(s-x)^2}{4t}}\,\mathrm{d}s = 2\sqrt{t}\int_{-x/(2\sqrt{t})}^{\infty} e^{-y^2}\,\mathrm{d}y.$$

Hence

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2\sqrt{t})}^{\infty} e^{-y^2}\mathrm{d}y = \frac{1}{2}\texttt{erfc}(-x/(2\sqrt{t})).$$

The requested plot of the solution is shown below; code follows at the end of the problem.




```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initial condition
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 x = linspace(-5,5,1001);
 psi = x>0;

 col = 'kbrg';

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% wave equation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 tvec = [0 .5 1 2];
 figure(1), clf
 for k=1:length(tvec)
    t = tvec(k);
    if t==0,
```

```
      plot(x,psi,'k-','linewidth',2); hold on
   else
      uxt = .5*(x-t>0)+.5*(x+t>0);
      plot(x,uxt,'-','color',col(k),'linewidth',2); hold on
   end
end
ylim([-.05 1.05])
legend('$t=0$','$t=.5$','$t=1$','$t=2$',2)
set(legend,'interpreter','latex')
xlabel('$x$','fontsize',14','interpreter','latex')
ylabel('$u(x,t)$','fontsize',14','interpreter','latex')
title('Solution of the Wave Equation','fontsize',14','interpreter','latex')
print -depsc2 heat_v_wave1


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% heat equation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tvec = [0 .01 .1 1];
figure(2), clf
for k=1:length(tvec)
   t = tvec(k);
   if t==0,
      plot(x,psi,'k-','linewidth',2); hold on
   else
      uxt = (1/2)*erfc(-x/(2*sqrt(t)));
      plot(x,uxt,'-','color',col(k),'linewidth',2); hold on
   end
end
legend('$t=0$','$t=.01$','$t=.1$','$t=1$',2)
set(legend,'interpreter','latex')
ylim([-.05 1.05])
xlabel('$x$','fontsize',14','interpreter','latex')
ylabel('$u(x,t)$','fontsize',14','interpreter','latex')
title('Solution of the Heat Equation','fontsize',14','interpreter','latex')
print -depsc2 heat_v_wave2
```

2. [40 points: 8 points each]
Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We will look for solutions to the PDE of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

For simplicity, assume that $d \in (0, \pi)$.

(a) From the differential equation and this form for $u(x,t)$, show that the coefficients $a_k(t)$ must satisfy the ordinary differential equation

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t).$$

(b) Show that the following function satisfies the differential equation in part (a):

$$a_k(t) = C_1 \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) + C_2 \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big)$$

for arbitrary constants C_1 and C_2. (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

(c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{k=1}^{\infty} b_k(0)\psi(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

(d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x\sin(3\pi x)$, for which

$$b_k(0) = \frac{-6k\sqrt{2}(1 + (-1)^k)}{(k^2 - 9)^2\pi^2} \text{ for } k \neq 3, \qquad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x,t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

(e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d?

Solution.

(a) Follow the usual methodology: Substitute the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x)$$

into the differential equation $u_{tt} = u_{xx} - 2du_t$ to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} a_n''(t)\psi_n(x) &= \sum_{n=1}^{\infty} a_n(t)\psi_n''(x) - 2d\sum_{n=1}^{\infty} a_n'(t)\psi_n(x)\\ &= -\sum_{n=1}^{\infty}\lambda_n a_n(t)\psi_n(x) - 2d\sum_{n=1}^{\infty} a_n'(t)\psi_n(x).\end{aligned}$$

Take the inner product with the eigenfunction ψ_k and use orthogonality of the eigenfunctions to obtain

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t)$$

as required.

(b) We first compute two derivatives of the proposed formula for a_k:

$$\begin{aligned}a_k'(t) &= C_1\big(-d+\sqrt{d^2-k^2\pi^2}\big)\exp\big((-d+\sqrt{d^2-k^2\pi^2})t\big)\\ &\quad + C_2\big(-d-\sqrt{d^2-k^2\pi^2}\big)\exp\big((-d-\sqrt{d^2-k^2\pi^2})t\big)\end{aligned}$$

$$\begin{aligned}a_k''(t) &= C_1\big(-d-\sqrt{d^2-k^2\pi^2}\big)^2\exp\big((-d+\sqrt{d^2-k^2\pi^2})t\big)\\ &\quad + C_2\big(-d+\sqrt{d^2-k^2\pi^2}\big)^2\exp\big((-d-\sqrt{d^2-k^2\pi^2})t\big).\end{aligned}$$

We wish to verify that $a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t)$, where $\lambda_k = k^2\pi^2$. We can see that

$$\begin{aligned}-\lambda_k a_k(t) - 2da_k'(t) &= C_1\Big(-\lambda_k - 2d\big(-d+\sqrt{d^2-k^2\pi^2}\big)\Big)\exp\big((-d+\sqrt{d^2-k^2\pi^2})t\big)\\ &\quad + C_2\Big(-\lambda_k - 2d\big(-d-\sqrt{d^2-k^2\pi^2}\big)\Big)\exp\big((-d-\sqrt{d^2-k^2\pi^2})t\big)\\[1ex] &= C_1\big(-k^2\pi^2 + 2d^2 - 2d\sqrt{d^2-k^2\pi^2}\big)\exp\big((-d+\sqrt{d^2-k^2\pi^2})t\big)\\ &\quad + C_2\big(-k^2\pi^2 + 2d^2 + 2d\sqrt{d^2-k^2\pi^2}\big)\exp\big((-d-\sqrt{d^2-k^2\pi^2})t\big)\\[1ex] &= C_1\big(-d+\sqrt{d^2-k^2\pi^2}\big)^2\exp\big((-d+\sqrt{d^2-k^2\pi^2})t\big)\\ &\quad + C_2\big(-d-\sqrt{d^2-k^2\pi^2}\big)^2\exp\big((-d-\sqrt{d^2-k^2\pi^2})t\big).\end{aligned}$$

This final formula agrees with the formula for $a_k''(t)$ we computed earlier, and thus we have confirmed that this is a general solution for our differential equation.

(c) We need to now compute C_1 and C_2 so that $a_k(0) = 0$ and $a_k'(0) = b_k(0)$. At $t = 0$, the general solution becomes

$$a_k(0) = C_1\exp(0) + C_2\varepsilon(0) = C_1 + C_2,$$

so $a_k(0) = 0$ requires that

$$C_1 = -C_2.$$

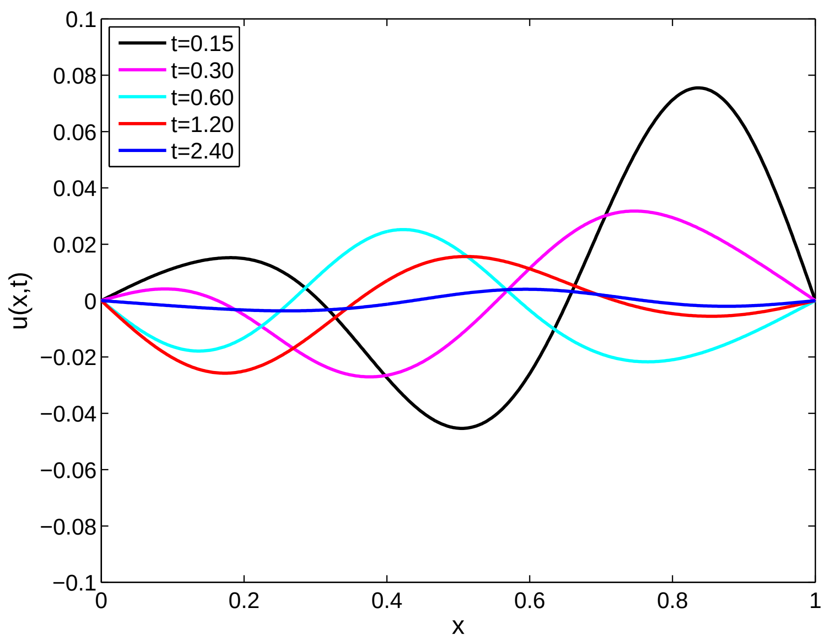
Taking the formula for $a_k'(t)$ in part (b) and evaluating at $t = 0$ gives

$$a_k'(0) = C_1\big(-d + \sqrt{d^2 - k^2\pi^2}\big) + C_2\big(-d - \sqrt{d^2 - k^2\pi^2}\big).$$

So with $C_1 = -C2$ and $a_k'(0) = b_k(0)$, we arrive at

$$C_1 = -C_2 = \frac{b_k(0)}{2\sqrt{d^2 - k^2\pi^2}}.$$

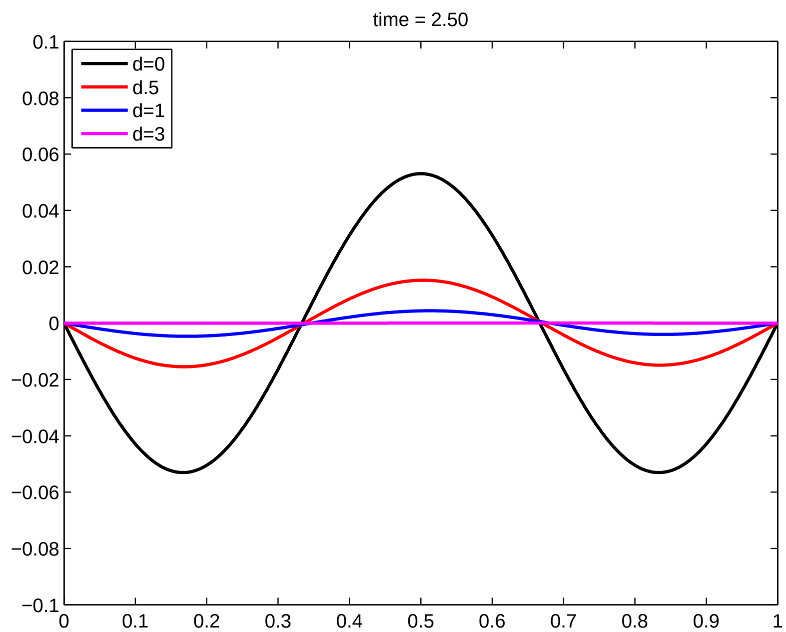
(d) The requested solutions, varying in t with fixed d, are collected in the plot below.




```
tvec = [.15 .30 .60 1.20 2.40];
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

d = 1;

col = 'kmcrb';
figure(1), clf
for m=1:length(tvec)
   t = tvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
```

```
   plot(xx,u,'-','linewidth',2,'color',col(m)), hold on
   ylim([-.1 .1])
   pause
end
legend('t=0.15','t=0.30','t=0.60','t=1.20','t=2.40',2)
set(gca,'fontsize',14)
xlabel('x','fontsize',16)
ylabel('u(x,t)','fontsize',16)
print -depsc2 damp1.eps
```

(e) The requested solutions, now varying in d with fixed t, are collected in the plot below. As the damping parameter increases on $(0, \pi)$, the solution gets increasingly smaller in amplitude at this time.




```
t = 2.5;
dvec = [0 .5 1 3];
xx = linspace(0,1,500)';

ak0 = zeros(10,1);
bk0 = zeros(10,1);
k = [1:20]';
bk0 = -6*sqrt(2)*(1+(-1).^k).*k./((k.^2-9).^2*pi^2);
bk0(3) = sqrt(2)/4;

figure(1), clf
cvec = 'krbm';
for m=1:length(dvec)
   d = dvec(m)
   u = zeros(size(xx));
   for k=1:length(bk0)
       psik = sqrt(2)*sin(k*pi*xx);
       dis = sqrt(d^2-k^2*pi^2);
       ak = bk0(k)*(exp((-d+dis)*t)-exp((-d-dis)*t))/(2*dis);
       u = u+ak*psik;
   end
   plot(xx,u,'-','linewidth',2,'color',cvec(m)), hold on
```

```
    ylim([-.1 .1])
    title(sprintf('time = %3.2f', t))
end
legend('d=0','d=.5','d=1','d=3',2)
print -depsc2 damp2.eps
```

3. [40 points: 9 points each for (a), (b), (c), 13 points for (e)]
This problem and the next study equations in two dimensions. We begin with the steady-state problem. In place of the one dimensional equation, $-u'' = f$, we now have

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$ for all $0 \le x \le 1$ and $0 \le y \le 1$. The associated operator L is defined as

$$Lu = -(u_{xx} + u_{yy}),$$

acting on the space $C_D^2[0,1]^2$ consisting of twice continuously differentiable functions on $[0,1] \times [0,1]$ with homogeneous boundary conditions. We can solve the differential equation $Lu = f$ using the spectral method just as we have seen in class before. This problem will walk you through the process; you may consult Section 8.2 of the text for hints.

(a) Show that L is symmetric, given the inner product

$$(v,w) = \int_0^1 \int_0^1 v(x,y)w(x,y)\,dx\,dy.$$

(b) Verify that the functions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

are eigenfunctions of L for $j,k = 1,2,\ldots$.
(To do this, you simply need to show that $L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$ for some scalar $\lambda_{j,k}$.)
What is the eigenvalue $\lambda_{j,k}$ associated with $\psi_{j,k}$?

(c) Compute the inner product $(\psi_{j,k}, \psi_{j,k}) = \|\psi_{j,k}\|^2$.

(d) Let $f(x,y) = x(1-y)$. Compute the inner product $(f, \psi_{j,k})$.

(e) The solution to the diffusion equation is given by the spectral method, but now with a double sum to account for all the eigenvalues:

$$u(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

In MATLAB plot the partial sum

$$u_{10}(x,y) = \sum_{j=1}^{10}\sum_{k=1}^{10} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

Hint for 3d plots: To plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$, you could use

```
x = linspace(0,1,40); y = linspace(0,1,40);
[X,Y] = meshgrid(x,y);
Psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,Psi11)
```

Solution.

(a) To show that L is symmetric, we must show that $(Lu, v) = (u, Lv)$ for all $u, v \in C_D^2[0,1]^2$. We can establish this result by integrating by parts twice in each spatial dimension:

$$\begin{aligned}
(Lu, v) &= -\int_0^1 \int_0^1 \big(u_{xx}(x,y) + u_{yy}(x,y)\big)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= -\int_0^1 \Big(\int_0^1 u_{xx}(x,y)v(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y - \int_0^1 \Big(\int_0^1 u_{yy}(x,y)v(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= \int_0^1 \Big(-\Big[u_x(x,y)v(x,y)\Big]_{x=0}^1 + \Big[u(x,y)v_x(x,y)\Big]_{x=0}^1 - \int_0^1 u(x,y)v_{xx}(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y \\
&\quad + \int_0^1 \Big(-\Big[u_y(x,y)v(x,y)\Big]_{y=0}^1 + \Big[u(x,y)v_y(x,y)\Big]_{y=0}^1 - \int_0^1 u(x,y)v_{yy}(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= -\int_0^1 \Big(\int_0^1 u(x,y)v_{xx}(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y - \int_0^1 \Big(\int_0^1 u(x,y)v_{yy}(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= -\int_0^1 \int_0^1 u(x,y)\big(v_{xx}(x,y) + v_{yy}(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= (u, Lv).
\end{aligned}$$

More generally, you can appeal to Green's Second Identity, which amounts to integration by parts in higher dimensions. Let $\Omega := [0,1] \times [0,1] \subset \mathbb{R}^2$ denote the domain $0 \le x \le 1$ and $0 \le y \le 1$, and $\partial\Omega$ its boundary. We write $-Lu = -u_{xx} - u_{yy} = -\Delta u$. Green's Second Identity gives

$$\int_\Omega \big((\Delta u)v - u(\Delta v)\big)\,\mathrm{d}V = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,\mathrm{d}S,$$

where $\partial u/\partial n$ and $\partial v/\partial n$ denote derivatives with respect to the outward-pointing normal direction. Thus

$$(Lu, v) = -\int_\Omega (\Delta u)v\,dV = -\int_\Omega u(\Delta v)\,dV + \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,\mathrm{d}S = -\int_\Omega u(\Delta v)\,dV = (u, Lv),$$

since u and v are zero on the boundary $\partial\Omega$.

(b) We simply compute

$$\begin{aligned}
L\psi_{j,k} = -\Big(\frac{\partial^2\psi_{j,k}}{\partial x^2} + \frac{\partial^2\psi_{j,k}}{\partial y^2}\Big) &= -\frac{\partial}{\partial x^2}(\sin(j\pi x)\sin(k\pi y) - \frac{\partial}{\partial y^2}(\sin(j\pi x)\sin(k\pi y) \\
&= j^2\pi^2\sin(j\pi x)\sin(k\pi y) + k^2\pi^2\sin(j\pi x)\sin(k\pi y) \\
&= (j^2 + k^2)\pi^2\sin(j\pi x)\sin(k\pi y) \\
&= \lambda_{j,k}\psi_{j,k}(x,y).
\end{aligned}$$

One can also notice that $\psi_{j,k}(x,y)$ satisfies the necessary boundary conditions

$$\psi_{j,k}(0,y) = \psi_{j,k}(1,y) = \psi_{j,k}(x,0) = \psi_{j,k}(x,1) = 0.$$

[GRADERS: please deduct 2 points if the student forgot to check the boundary conditions.]
Thus $\psi_{j,k}(x,y) = \sin(j\pi x)\sin(k\pi x)$ is an eigenfunction for the operator L.

(c) The computation in part (b) reveals the eigenvalue to be $\lambda_{j,k} = (j^2 + k^2)\pi^2$.

(d) The inner product computation reduces to the product of single integrals:

$$\begin{aligned}(\psi_{j,k}, \psi_{j,k}) &= 4\int_0^1\int_0^1 \sin(j\pi x)^2 \sin(k\pi y)^2\,dx\,dy\\ &= 4\Big(\int_0^1 \sin(j\pi x)^2\,dx\Big)\Big(\int_0^1 \sin(k\pi y)^2\,dy\Big)\\ &= 1.\end{aligned}$$

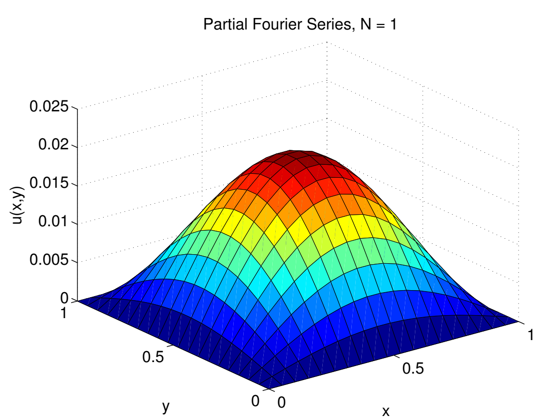
(e) This inner product also breaks into the products of two integrals that are straightforward to compute:

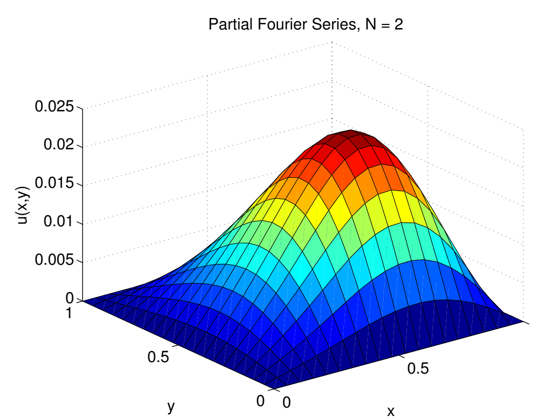
$$\begin{aligned}(f, \psi_{j,k}) &= 2\int_0^1\int_0^1 x(1-y)\sin(j\pi x)\sin(k\pi y)\,dx\,dy\\ &= 2\Big(\int_0^1 x\sin(j\pi x)\,dx\Big)\Big(\int_0^1 (1-y)\sin(k\pi y)^2\,dy\Big)\\ &= 2\Big(\frac{(-1)^{j+1}}{j\pi}\Big)\Big(\frac{1}{k\pi}\Big)\\ &= 2\frac{(-1)^{j+1}}{jk\pi^2}.\end{aligned}$$

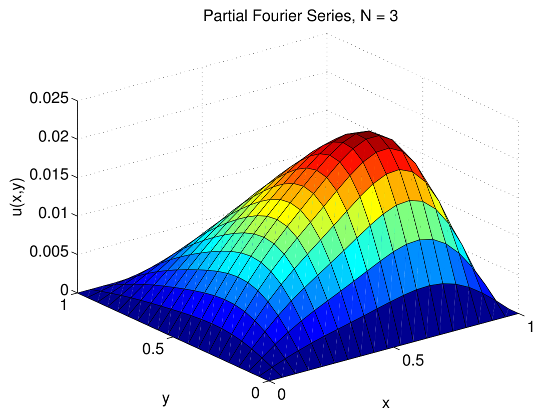
(f) The code below plots the partial sum

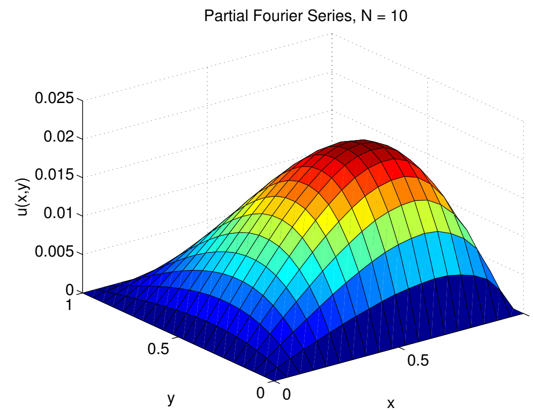
$$u_N(x,y) = \sum_{j=1}^N\sum_{k=1}^N \frac{(f,\psi_{j,k})}{\lambda_{j,k}(\psi_{j,k},\psi_{j,k})}\psi_{j,k}(x,y)$$

for various values of N, as shown in the plots below.










```
npts = 20;
x = linspace(0,1,npts); y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
for n=1:10
   figure(1), clf
   U = zeros(N,N);
   for j=1:n
      for k=1:n
         U = U + 4*(-1)^(j+1)/(j*k*pi^2)*sin(j*pi*X).*sin(k*pi*Y)/(j^2+k^2)/(pi^2);
      end
   end
   surf(X,Y,U), drawnow
   set(gca,'fontsize',16)
   xlabel('x')
   ylabel('y')
   zlabel('u(x,y)')
   title(sprintf(' Partial Fourier Series, N = %d', n))
   if ismember(n,[1 2 3 10]),
      eval(sprintf('print -depsc2 twoD%d', n))
   end
   pause
end
```