Chapter 5.1: The analogy between Boundary value problems and linear algebraic systems.

Consider the time-independent model boundary value problem in one dimension given by:

$$-K \frac{\partial^2}{\partial x^2} u = f$$

$$u(0) = 0 \quad u(\ell) = 0$$

Define the differential operator $L = -K \frac{\partial^2}{\partial x^2}$. Recall from class that we approximated L using central finite differences and retrieved a linear system of the form $A\vec{u} = \vec{f}$. We also discussed what it could "possibly indicate" for the continuous problem if the matrix A was singular.

As it turns out there are many analogies between the theory of matrices and the theory of linear differential operators. This is due to the fact that they can both be seen as "linear operators on a vector space".

We asked the following questions about a matrix A:

- given a vector b can we find an x with $Ax = b$?
- if so, is x uniquely determined?
- How can we go about finding such an $\vec{x}$ if it exists?

We can ask the same sorts of questions about the differential operator L. Just as $A^{m \times n}: \mathbb{R}^n \to \mathbb{R}^m$ the differential operator is defined in terms of the vector spaces $L: C^2[0,1] \to C[0,1]$ and is linear we can therefore assume it would be reasonable to ask:

- given f in $C[0,1]$ can we find $u \in C^2[0,1]$ with $Lu = f$
- is such a u uniquely determined?
- How can we find u?

Things are not exactly the same. For example $Ax = b$ represents m equations in n unknowns (for A an $m \times n$ matrix) whereas $Lu = f$ must hold at every point $x \in [0,1]$ and so represents an infinite number of equations and infinitely many unknowns.

However, certain concepts of matrix theory still play a key role in the theory for linear differential operators. For example the solution to $Ax=b$ is unique only if the nullspace of A, $N(A)$, contains only the zero vector, $N(A)=\{0\}$.

The same thing is true for L. If $g \in N(L)$ and u solves $Lu=f$ then $u+g$ does as well.

Notice that if $g \in C[0,1]$ then the function defined by $\tilde{g}(x) = \int_0^x \int_0^s g(z)\,dz\,ds$ satisfies $L\tilde{g}=g$ so that the Range of L is $R(L)=C[0,1]$. But notice that every function of the form $a+bx$ is contained in $N(L)$. In fact it is not hard to see that $N(L) = \{a+bx \mid a,b \in \mathbb{R}\}$.
This means that given $g \in C[0,1]$ $\tilde{g}+a+bx$ satisfies $L(\tilde{g}+a+bx)=g$.

So in a sense there are <u>too many</u> functions in $C^2[0,1]$ to guarantee a unique solution to the problem "$Lu=f$". So we get rid of these "extra" functions by imposing <u>boundary conditions</u>. Enforcing "$u(0)=u(1)=0$" means considering the operator L as mapping $L: C^2_D[0,1] \to C[0,1]$.

Notice though: if we look at $L: C^2_D[0,1] \to C[0,1]$ what is $N(L)$? A function "$a+bx$" is in $C^2_D[0,1]$ if and only if $a=b=0$! So $N(L)=\{0\}$ when L is considered from $C^2_D[0,1] \to C[0,1]$ and solutions to $Lu=f$ are unique when they exist. In fact, $L: C^2_D[0,1] \to C[0,1]$ <u>will</u> have a solution for each $f \in C[0,1]$.

So there is a natural duality between the linear algebra concepts of nullspace, range, existence and uniqueness and the theory of linear differential operators.

$$\begin{array}{lccc}
 & \text{uniqueness} & & \text{existence} \\
\text{Linear Algebra} & [\,N(A)\,] & \text{---} & [\,R(A)\,] \\
 & \updownarrow & & \updownarrow \\
\text{Linear diff. operator} & [\,N(L)\,] & \text{---} & [\,R(L)\,] \\
 & \text{uniqueness} & & \text{existence}
\end{array}$$

Another linear algebra concept with a natural duality is the idea of a <u>symmetric</u> operator.

Recall: The matrix A and its transpose A^T satisfied the relation

$$(Ax)\cdot y = x\cdot(A^T y) \quad \text{for vectors } \vec{x}, \vec{y}.$$

Written as an inner product the above is: $(Ax, y) = (x, A^T y)$

And we said that a matrix $A^{n\times n}$ is symmetric if $A = A^T$. This is <u>equivalent</u> to saying that: for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ $(Ax, y) = (x, Ay)$

This is exactly the observation we will use to define symmetric linear operators.

<u>Definition</u>: Let S be a vector subspace of $C^k[a,b]$ and let L be a linear operator $L: S \to C[0,1]$ then we say L is symmetric iff for every $u, v \in S$ $(Lu, v) = (u, Lv)$ where the inner product is the $L^2[a,b]$ inner product $(f,g) = \int_a^b fg\,dx$

<u>Theorem</u>: The operator $L = -k\frac{\partial^2 u}{\partial x^2}$ is symmetric when considered as $L: C_D^2[a,b] \to C[0,1]$. (Recall $C_D^2[a,b] = \{ f \in C^2[a,b] \mid f(a) = f(b) = 0 \}$).

<u>Proof</u>: We need to show $(Lf, g) = (f, Lg)$ $\forall f, g \in C_D^2[a,b]$.

We have:

$$(Lf, g) = \int_a^b \left(-k\frac{\partial^2}{\partial x^2} f\right) g = \int_a^b k\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + -k\frac{\partial f}{\partial x} g \Big|_a^b$$

(using integration by parts)

$$= \int_a^b k\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + 0 \leftarrow \text{using } g(a) = g(b) = 0$$

using integration by parts again $\rightarrow$

$$= \int_a^b f\left(-k\frac{\partial^2 g}{\partial x^2}\right) + kf\frac{\partial g}{\partial x}\Big|_a^b = \int_a^b f\left(-k\frac{\partial^2 g}{\partial x^2}\right) + 0$$

using f in $C_D^2[a,b]$

$$= (f, Lg) \text{ by the definition of } L.$$

So $(Lf, g) = (f, Lg)$ for every f, g in $C_D^2[a,b]$ so that L is symmetric.

Now recall that a symmetric, A, had several nice properties:

① All eigenvalues of A were real ② Eigenvectors for different eigenvalues were orthogonal ③ We could find a <u>basis</u> of orthonormal eigenvectors.

It can be proven directly (see pg 136) that if $\mathcal{L}$ is symmetric

① eigenvalues of $\mathcal{L}$ must be real numbers

② eigenvectors (sometimes called eigenfunctions) for different eigenvalues are orthogonal.

The symmetric operator $\mathcal{L} = -k\frac{\partial^2}{\partial x^2}$ has one other nice property that is unique to it. All of its eigenvalues are <u>positive</u>.

to see this let u be any eigenvector of $\mathcal{L}$ with eigenvalue λ. Then $\tilde{u} = u/\|u\|$ is a unit length eigenvector of $\mathcal{L}$ with eigenvalue λ. It follows that

$$\begin{aligned}
\lambda &= \lambda(\tilde{u}, \tilde{u}) \\
&= (\lambda \tilde{u}, \tilde{u}) \\
&= (\mathcal{L}\tilde{u}, \tilde{u}) = \int_a^b \left(-k\frac{\partial^2}{\partial x^2}\tilde{u}\right)\tilde{u}
\end{aligned}$$

Integration by parts

$$\begin{cases}
= \int_a^b k \frac{\partial}{\partial x}\tilde{u}\frac{\partial}{\partial x}\tilde{u} + k\left(\frac{\partial}{\partial x}\tilde{u}\right)\tilde{u}\Big|_a^b \\
= \int_a^b k\left(\frac{\partial}{\partial x}\tilde{u}\right)^2 + 0 \quad > 0
\end{cases}$$

(here we have assumed $k>0$ of course)

So we have the following for symmetric operators:

① Eigenvalues are real ② Eigenvalues are orthogonal

<u>Key</u> <u>IDEA</u>: If we can find "enough" eigenvectors of $\mathcal{L}$ so that we can write any function f in the domain of $\mathcal{L}$ in terms of the eigenvalues then we will have the <u>spectral method</u> for solving $\mathcal{L}u = f$ just like we did for solving the matrix equation $Ax = b$! This is exactly what we will see in Section <u>5.2</u>.