# CAAM 336 · DIFFERENTIAL EQUATIONS
## Homework 31

Posted Monday 21 October 2013. Due 5pm Wednesday 30 October 2013.

31. [25 points] Let $H_D^1(0,1) = \{w \in H^1(0,1) : w(0) = w(1) = 0\}$ and let the inner product $a(\cdot,\cdot) : H_D^1(0,1) \times H_D^1(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\widehat{\phi}_j \in H_D^1(0,1)$ be such that

$$\widehat{\phi}_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\[2mm] \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\[2mm] 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$. Also, let the continuous piecewise quadratic functions $\phi_j \in H_D^1(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\[2mm] \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\[2mm] 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in H_D^1(0,1)$ be such that

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\[2mm] 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N+1$.

(a) By hand, obtain formulas for
   i. $a(\phi_j, \phi_k)$ for $j, k = 1, \ldots, N$;
   ii. $a(\psi_j, \psi_k)$ for $j, k = 1, \ldots, N+1$;
   iii. $a(\phi_j, \psi_k)$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$.

Your final answers should be in terms of $h$ and you must clearly state which values of $j$ and $k$ each formula you obtain is valid for. For example, if you were asked to obtain formulas for $a(\widehat{\phi}_j, \widehat{\phi}_k)$ for $j, k = 1, \ldots, N$ then an acceptable way to present the answer would be:
For $j, k = 1, \ldots, N$,

$$a(\widehat{\phi}_j, \widehat{\phi}_k) = \begin{cases} \dfrac{2}{h} & \text{if } k = j, \\[2mm] -\dfrac{1}{h} & \text{if } |j - k| = 1, \\[2mm] 0 & \text{otherwise.} \end{cases}$$