CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 41 · Solutions

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

41. [25 points]

Let $u_0(x) = (x - x^2)\,(\sin(3\pi x))^2$. Note that, for $n = 1, 2, \ldots$,

$$\int_0^1 \sqrt{2}u_0(x)\sin(n\pi x)\,dx = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Consider the problem of finding the solution $u(x,t)$ to the the fourth order partial differential equation

$$u_t(x,t) = u_{xx}(x,t) - u_{xxxx}(x,t), \quad 0 < x < 1, \quad t > 0$$

with so-called hinged boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0, \quad t \ge 0$$

and initial condition

$$u(x,0) = u_0(x), \quad 0 < x < 1.$$

This equation is related to a model that arises in the study of thin films. Let

$$C_H^4[0,1] = \{v \in C^4[0,1] : v(0) = v''(0) = v(1) = v''(1) = 0\}.$$

Let the linear operator $L : C_H^4[0,1] \to C[0,1]$ be defined by

$$Lv = -v'' + v''''.$$

(a) The operator L has eigenvalues $\lambda_n \in \mathbb{R}$ and eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$, which are such that

$$L\psi_n = \lambda_n \psi_n$$

for $n = 1, 2, \ldots$. Obtain a formula for λ_n for $n = 1, 2, \ldots$.

(b) We can write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

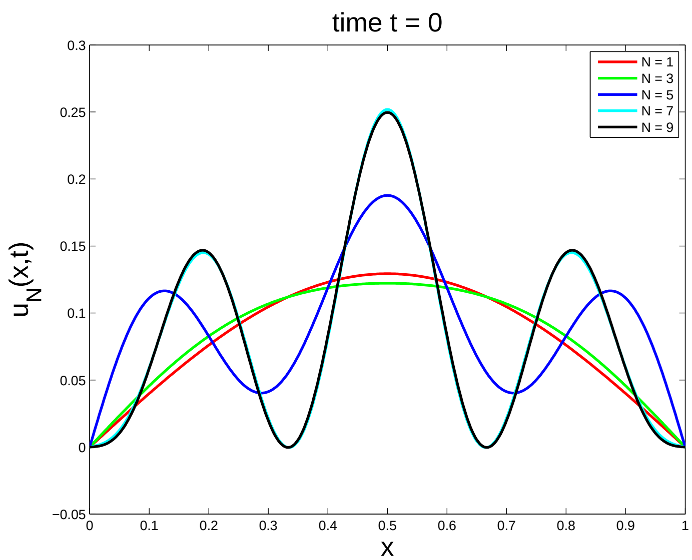
(c) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(d) Use you answer to part (c) to write out a formula for $u(x,t)$.

(e) Let

$$u_N(x,t) = \sum_{n=1}^{N} a_n(t)\psi_n(x).$$

For each time $t = 0, 10^{-5}, 2 \times 10^{-5}, 4 \times 10^{-5}$, produce a plot comparing $u_1(x,t)$, $u_3(x,t)$, $u_5(x,t)$, $u_7(x,t)$ and $u_9(x,t)$. For example, at time $t = 0$, your plot should appear as shown below.



Solution.

(a) [4 points] We can compute that, for $n = 1, 2, \ldots$,

$$\begin{aligned}
(L\psi_n)(x) &= -\psi_n''(x) + \psi_n''''(x) \\
&= -\frac{d^2}{dx^2}(\sqrt{2}\sin(n\pi x)) + \frac{d^4}{dx^4}(\sqrt{2}\sin(n\pi x)) \\
&= n^2\pi^2\sqrt{2}\sin(n\pi x) + n^4\pi^4\sqrt{2}\sin(n\pi x) \\
&= (n^2\pi^2 + n^4\pi^4)\psi_n(x).
\end{aligned}$$

Hence,

$$\lambda_n = n^2\pi^2 + n^4\pi^4 \text{ for } n = 1, 2, \ldots.$$

(b) [7 points] Substituting the expression for $u(x,t)$ into the partial differential equation yields

$$\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) = \sum_{n=1}^{\infty} a_n(t)(\psi_n''(x) - \psi_n''''(x))$$

and hence

$$\sum_{n=1}^{\infty} a_n'(t)\psi_n(x) = \sum_{n=1}^{\infty}(-\lambda_n)a_n(t)\psi_n(x).$$

We can then say that

$$\sum_{n=1}^{\infty} a_n'(t)\int_0^1 \psi_n(x)\psi_m(x)\,dx = \sum_{n=1}^{\infty}(-\lambda_n)a_n(t)\int_0^1 \psi_n(x)\psi_m(x)\,dx,$$

for $m = 1, 2, \ldots$, from which it follows that

$$a'_m(t) = -\lambda_m a_m(t),$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

for $m, n = 1, 2, \ldots$.

Also,

$$u(x, 0) = u_0(x)$$

means that

$$\sum_{n=1}^{\infty} a_n(0)\psi_n(x) = u_0(x)$$

and so

$$\sum_{n=1}^{\infty} a_n(0) \int_0^1 \psi_n(x)\psi_m(x)\,dx = \int_0^1 u_0(x)\psi_m(x)\,dx,$$

for $m = 1, 2, \ldots$, from which it follows that

$$a_m(0) = \int_0^1 u_0(x)\psi_m(x)\,dx,$$

for $m = 1, 2, \ldots$, since

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

for $m, n = 1, 2, \ldots$.

Hence, for $n = 1, 2, \ldots$, $a_n(t)$ is the solution to the differential equation

$$a'_n(t) = -\lambda_n a_n(t)$$

with initial condition

$$a_n(0) = \int_0^1 u_0(x)\psi_n(x)\,dx.$$

(c) [3 points] For $n = 1, 2, \ldots$,

$$\begin{aligned} a_n(t) &= \int_0^1 u_0(x)\psi_n(x)\,dx e^{-\lambda_n t} \\ &= \int_0^1 \sqrt{2}u_0(x)\sin(n\pi x)\,dx e^{-(n^2\pi^2+n^4\pi^4)t} \\ &= b_n e^{-(n^2\pi^2+n^4\pi^4)t} \end{aligned}$$

where

$$b_n = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

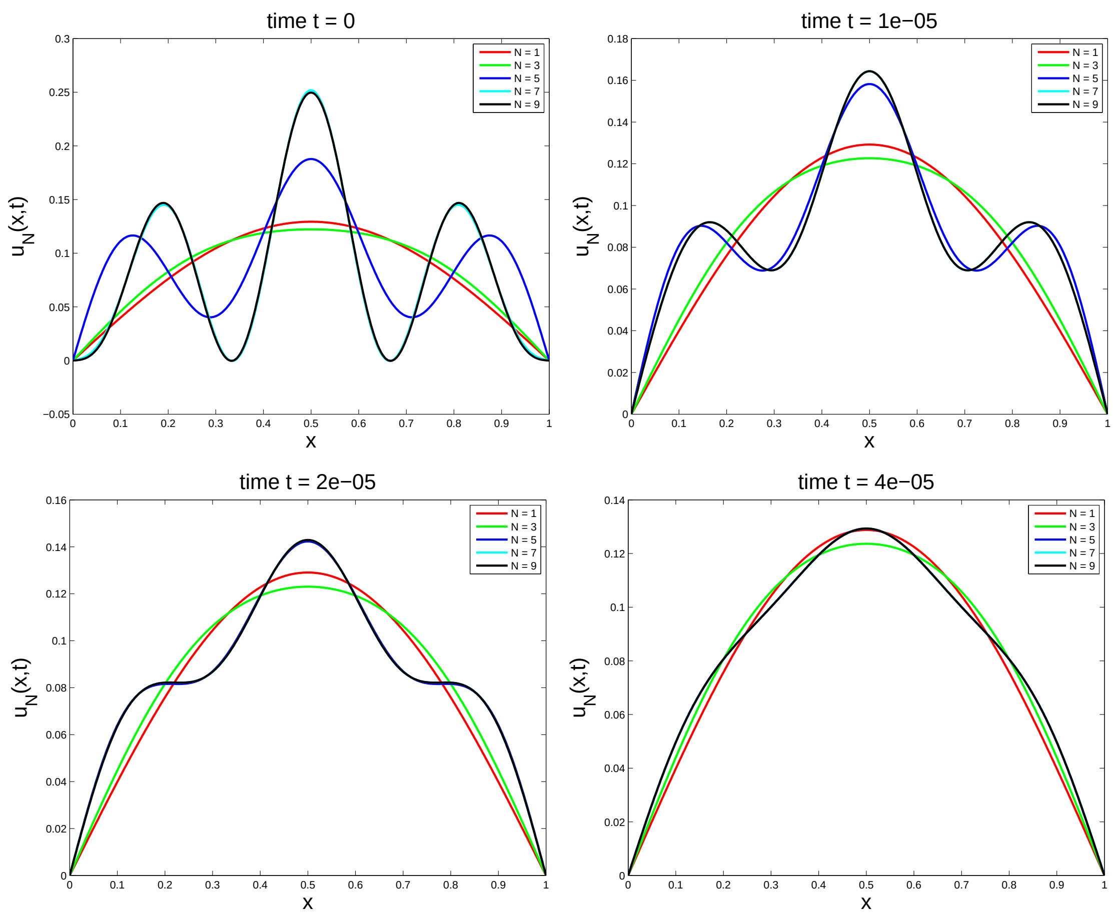
(d) [3 points] We can write

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\psi_n(x) \\ &= \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2+n^4\pi^4)t}\psi_n(x) \\ &= \sum_{n=1}^{\infty} c_n e^{-(n^2\pi^2+n^4\pi^4)t}\sin(n\pi x) \end{aligned}$$

where

$$c_n = \begin{cases} \dfrac{864(n^4-18n^2+216)}{(36n-n^3)^3\pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

(e) [8 points] Plots for the four requested times are shown below.



One can produce these plots with the following code.

```
clear
clc
tvec = [0 .00001 .00002 .00004];
x = linspace(0,1,500);
```

```
an0 = inline('sqrt(2)*432*(n^4-18*n^2+216)/((36*n-n^3)^3*pi^3)');
lam = inline('n^2*pi^2 + n^4*pi^4');
col = 'rgbck';
str = 'abcd';
for j=1:length(tvec)
    figure(1)
    clf
    t = tvec(j);
    u = zeros(size(x));
    for n=1:2:9
        u = u+exp(-lam(n)*t)*an0(n)*(sqrt(2)*sin(n*pi*x));
        [tf,loc] = ismember(n,[1 3 5 7 9]);
        if tf
            plot(x,u,'-','color',col(loc),'linewidth',2)
            hold on
        end
    end
    legend('N = 1','N = 3','N = 5','N = 7','N = 9')
    xlabel('x','fontsize',20)
    ylabel('u_N(x,t)','fontsize',20)
    title(sprintf('time t = %g',t),'fontsize',20)
    eval(sprintf('print -depsc2 fourth_%s',str(j)))
    pause(.1)
end
```