CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 2 · Solutions

Posted Wednesday 3 September 2014. Due 5pm Wednesday 10 September 2014.

1. [24 points] In this problem, we will derive a finite difference discretization for the equation

$$\alpha u(x) - \frac{\partial u(x)}{\partial x}2 = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u'(0) = 0, \quad u'(1) = 0.$$

Since we cannot solve the equation exactly, we will wish to satisfy it at a finite number of points x_i, such that

$$\alpha u(x_i) - \frac{\partial u(x_i)}{\partial x}2 = f(x_i), \quad 0 < x_i < 1.$$

To do so, we will replace $\frac{\partial u(x_i)}{\partial x}2$ with a finite difference approximation using $u(x_i)$ at the 5 points

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad x_4 = 1.$$

(a) Using the finite difference approximations for the boundary conditions

$$u'(0) = u'(x_0) \approx \frac{u(x_1) - u(x_0)}{h} = 0$$

and

$$u'(1) = u'(x_4) \approx \frac{u(x_4) - u(x_3)}{h} = 0,$$

write down the finite difference approximation to the differential equation at $0 < x_1, x_2, x_3 < 1$, and construct explicitly the matrix system $\mathbf{Au} = \mathbf{b}$ resulting from the finite difference approximation of $\alpha u(x) - u''(x) = f(x)$, where

$$\mathbf{u} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \end{bmatrix}.$$

(b) Solve the above system for $\alpha = 1$, $f(x) = e^x$, and report the solution at the interior points $u(x_1), u(x_2), u(x_3)$.

(c) Consider the case where $\alpha = 0$, or

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1 \\ u'(0) &= 0 \\ u'(1) &= 0. \end{aligned}$$

In the previous homework, we showed that this equation does not have a unique solution; as a result, neither does the finite difference system for this system. Verify that $\mathbf{e}$ is in the null space of $\mathbf{A}$, where $\mathbf{e} = (1, 1, 1)^T$ is the vector of all ones. Suppose that, instead of 5 points, we have $N + 2$ points. Would the vector of all ones be in the null space of $\mathbf{A}$ for arbitrary N as well?

Solution.

(a) The finite difference approximation for second derivative $u''(x)$ around the point x_i can be written by central difference difference scheme as follows

$$\frac{\partial^2 u}{\partial x^2}(x_i) = u''(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}.$$

where $i = 1, 2, 3$. Note that from boundary conditions we have

$$u'(0) = u'(x_0) \approx \frac{u(x_1) - u(x_0)}{h} = 0$$

which can be written $u(x_1) = u(x_0)$, and

$$u'(1) = u'(x_4) \approx \frac{u(x_4) - u(x_3)}{h} = 0,$$

similarly, we can write $u(x_4) = u(x_3)$

Let $u(x_i) = u_i$ for simplicity. Then, the finite difference approximations for equation $\alpha u(x) - u''(x) = f(x)$ at the point x_i are given by:

$$\alpha u_i - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i.$$

If we reorganize above scheme, our discretised PDE becomes;

$$(\alpha + \frac{2}{h^2})u_i - \frac{1}{h^2}(u_{i+1} + u_{i-1}) = f_i.$$

Then

$$\begin{aligned}
(\alpha + \frac{2}{h^2})u_1 - \frac{1}{h^2}(u_2 + u_0) &= f_1 \qquad for\ \ i = 1 \\
(\alpha + \frac{2}{h^2})u_2 - \frac{1}{h^2}(u_3 + u_1) &= f_2 \qquad for\ \ i = 2 \\
(\alpha + \frac{2}{h^2})u_3 - \frac{1}{h^2}(u_4 + u_2) &= f_3 \qquad for\ \ i = 1
\end{aligned}$$

Using boundary conditions which are given as $u_0 = u_1$ and $u_4 = u_3$ we get

$$\begin{aligned}
(\alpha + \frac{1}{h^2})u_1 - \frac{1}{h^2}u_2 &= f_1 \qquad for\ \ i = 1 \\
(\alpha + \frac{2}{h^2})u_2 - \frac{1}{h^2}(u_3 + u_1) &= f_2 \qquad for\ \ i = 2 \\
(\alpha + \frac{1}{h^2})u_3 - \frac{1}{h^2}u_4 &= f_3 \qquad for\ \ i = 1
\end{aligned}$$

We can rewrite in matrix form $\mathbf{Au} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} \alpha + \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ -\frac{1}{h^2} & \alpha + \frac{2}{h^2} & -\frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \alpha + \frac{1}{h^2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\mathbf{b}},$$

since $h = 1/4$ we end up with the following linear system,

$$\underbrace{\begin{bmatrix} \alpha + \frac{1}{(1/4)^2} & -\frac{1}{(1/4)^2} & 0 \\ -\frac{1}{(1/4)^2} & \alpha + \frac{2}{(1/4)^2} & -\frac{1}{(1/4)^2} \\ 0 & -\frac{1}{(1/4)^2} & \alpha + \frac{1}{(1/4)^2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\mathbf{b}},$$

(b) if $\alpha = 1$ and $f(x) = e^x$ and our system of ODE becomes

$$\begin{bmatrix} 17 & -16 & 0 \\ -16 & 33 & -16 \\ 0 & -16 & 17 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} e^{1/4} \\ e^{2/4} \\ e^{3/4} \end{bmatrix}$$

By solving the given system such that $u = A^{-1}b$ we get

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1.6591 \\ 1.6825 \\ 1.7081 \end{bmatrix}$$

(c) First let us remember definition of null space. The null space of an mxn matrix $\mathbf{A}$, denoted $Null\ \mathbf{A}$, is the set of all solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$. Written in set notation, we have

$$Null\ \mathbf{A} = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \text{ and } \mathbf{Ax} = \mathbf{0}\}$$

To prove that e is in the null space of A, we must show that $\mathbf{Ae} = \mathbf{0}$. By $1(b)$ we can say that if $\alpha = 0$

$$\mathbf{A} = \begin{bmatrix} 16 & -16 & 0 \\ -16 & 32 & -16 \\ 0 & -16 & 16 \end{bmatrix}$$

Then it is easy to show that

$$\mathbf{Ae} = \begin{bmatrix} 16 & -16 & 0 \\ -16 & 32 & -16 \\ 0 & -16 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

Assume that u is the solution of the given problem $\mathbf{Au} = \mathbf{f}$ for NxN matrix $\mathbf{A}$. In the previous homework, we showed that this equation does not have a unique solution. Then, let assume v is another solution of this problem. Then by linearity we can say that $u + Cv$ is also solution for this problem for arbitrary constant C. Since u is one solution then it satisfies the following linear system

$$\mathbf{Au} = \mathbf{f}$$

and $u + Cv$ also another solution and satisfy this system too

$$\mathbf{A(u + Cv) = f} \Rightarrow \underbrace{\mathbf{Au}}_{=\mathbf{f}} + \mathbf{CAv = f} \Rightarrow \mathbf{CAv = 0}$$

Then for $N \times N$ matrix $\mathbf{A}$ we show that v is the solution of the homogeneous equation $\mathbf{Av} = \mathbf{0}$. This implies v is in the null space of $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$.

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2. [21 points] Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = ih$ for $i = 0, \dots, N+1$. We can approximate the differential equation

$$-u''(x) = f(x), \quad 0 < x < 1,$$

with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ by the matrix equation

$$\frac{-1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix},$$

where $u_i \approx u(x_i)$. (Entries of the matrix that are not specified are zero.)

(a) Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

$$u(0) = 1, \quad u(1) = 2.$$

(b) Suppose that we have

$$\begin{aligned} -u''(x) &= (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1, \\ u(0) &= 1 \\ u(1) &= 2. \end{aligned}$$

Since this differential equation is linear, we can split up the solution into

$$u(x) = u_1(x) + u_2(x),$$

where $u_1(x)$ satisfies

$$\begin{aligned} -u_1''(x) &= 0, \quad 0 < x < 1, \\ u_1(0) &= 1 \\ u_1(1) &= 2 \end{aligned}$$

and $u_2(x)$ satisfies the equation

$$\begin{aligned} -u_2''(x) &= (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1, \\ u_2(0) &= 0 \\ u_2(1) &= 0. \end{aligned}$$

Determine $u_1(x), u_2(x)$ and the exact solution $u(x)$.

(c) Compute and plot the approximate solutions for $N = 8, 16, 32, 64$, and compare it to the exact solution $u(x)$.

Solution.

(a) Since boundary conditions are applied at $u(x_0) = u_0$ and $u(x_{N+1}) = u_{N+1}$, they only show up in the finite difference equations for x_1 and x_N. The finite difference equation at x_1 approximates $-u''(x_1) = f(x_1)$ via

$$-\frac{u_2 - 2u_1 + u_0}{h^2} = f_1.$$

Since $u_0 = u(x_0) = 1$ is known, we can modify the above equation to be

$$-\frac{u_2 - 2u_1}{h^2} = f_1 + \frac{1}{h^2}.$$

Similarly, at $u_N = u(x_N)$, we approximate $-u''(x_N) = f(x_N)$ via

$$-\frac{u_{N+1} - 2u_N + u_{N-1}}{h^2} = f_1.$$

Since $u_{N+1} = u(x_{N+1}) = 2$ is known, we can modify the above equation to be

$$-\frac{-2u_N + u_{N-1}}{h^2} = f_1 + \frac{2}{h^2}.$$

This leads to the system of equations

$$\frac{-1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 + \frac{1}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N + \frac{2}{h^2} \end{bmatrix},$$

(b) Since $-u_1''(x) = 0$, we know that u_1 should be a linear polynomial, or that

$$u_1(x) = ax + b.$$

Boundary conditions then give

$$u(0) = 1 = b, \qquad u(1) = 2 = a + b$$

or that $a = 1$, $b = 1$, and $u_1 = x + 1$.
To solve $-u_2''(x) = (2\pi)^2 \sin(2\pi x)$ with zero boundary conditions, we can note that $\sin(2\pi x)$ satisfies zero boundary conditions, and then observe that taking the negative of two derivatives of $\sin(2\pi x)$ gives back

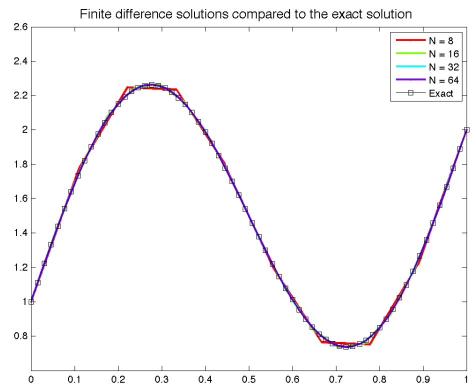
$$-\frac{\partial \sin(2\pi x)}{\partial x}2 = (2\pi^2)\sin(2\pi x).$$

This implies that $u(x) = \sin(2\pi x)$ satisfies both the boundary conditions and the differential equation with inhomogenous source term.

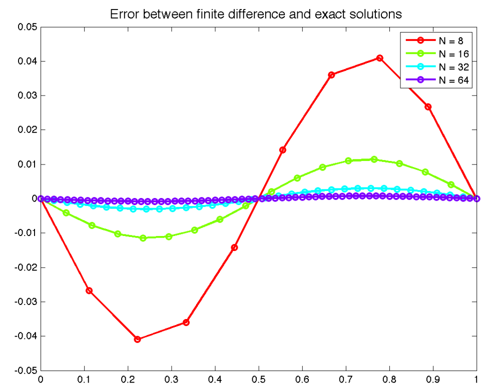
(c) Included is Matlab code that can be used to generate the finite difference solution, exact solution, and the error between it and the exact solution.

Graders: please do not take off if the students did not plot the error — we only asked for a comparison of the exact solution to the computed solutions.

```
% HW 2, Problem 2c. CAAM 336, Fall 2014
% solves the steady heat equation u''(x) = (2 pi)^2 sin(2 pi x)
% with u(0) = 1, u(1) = 2
clear

uexact = @(x) sin(2*pi*x) + x + 1;

i = 1;
C = hsv(4); % neat trick: makes a matrix whose values determine colors.
Nlist = [8 16 32 64]; % number of interior points
for N = Nlist
    K = N+1; % number of line segments
    h = 1/K; % spacing between points
    x = (0:N+1)/(N+1);
    x = x(:); % makes x a column vector.

    A = -2*diag(ones(N,1)) + diag(ones(N-1,1),1) + diag(ones(N-1,1),-1);
    A = -A/h^2;
    b = (2*pi)^2*sin(2*pi*x(2:end-1));
    b(1) = b(1) + 1/h^2; % modify b for inhomogeneous BCs
    b(N) = b(N) + 2/h^2; % modify b for inhomogeneous BCs
    u = A\b;

    figure(1)
    plot(x,[1;u;2],'.-','color',C(i,:),'linewidth',2);
    hold on

    figure(2)
```



(a) Finite difference solutions for various N



(b) Error between the exact solution and finite difference solution at points x_i.

```
    err = uexact(x)-[1;u;2];
    plot(x,err,'o-','color',C(i,:),'linewidth',2);hold on

    i = i+1;
end
figure(1)
title('Finite difference solutions compared to the exact solution','fontsize',14)
plot(x,uexact(x),'ks-')
legend('N = 8','N = 16','N = 32','N = 64','Exact')
print(gcf,'-dpng','p2c_sol') % print out graphs to file
figure(2)
title('Error between finite difference and exact solutions','fontsize',14)
legend('N = 8','N = 16','N = 32','N = 64','Exact')
print(gcf,'-dpng','p2c_error') % print out graphs to file
```

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3. [15 points] Using Taylor series expansions

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 + \frac{u'''(x)}{3!}\Delta x^3 + \dots$$

and

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 - \frac{u'''(x)}{3!}\Delta x^3 + \dots$$

show that the second order finite difference approximation

$$u''(x) \approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}$$

has accuracy $O(\Delta x^2)$. In other words, if $u''(x)$ is the exact second derivative, show that

$$\left| u''(x) - \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2} \right| = O(\Delta x^2).$$

Solution.

Adding together the Taylor expansions

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 + \frac{u'''(x)}{3!}\Delta x^3 + \dots$$
$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{u''(x)}{2}\Delta x^2 - \frac{u'''(x)}{3!}\Delta x^3 + \dots$$

gives

$$u(x + \Delta x) + u(x - \Delta x) = 2u(x) + u''(x)\Delta x^2 + 2\frac{u''''(x)}{4!}\Delta x^4 + \dots .$$

Subtracting $2u(x)$ from both sides and dividing by Δx^2 gives

$$\frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{\Delta x^2} = u''(x) + 2\frac{u''''(x)}{4!}\Delta x^2 + \dots ,$$

implying that the truncation error decreases at the same rate that Δx^2 decreases (if Δx is small enough). This implies the 2nd order central finite difference formula is $O(\Delta x^2)$ accurate.

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4. [40 points] Consider the time-dependent homogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0, t) = 0, \qquad t > 0,$$
$$u(1, t) = 0, \qquad t > 0$$

and initial condition $u(x, 0) = \psi(x)$. We wish to approximate the solution $u(x, t)$ at spatial points $x_0, x_1, \dots, x_N, x_{N+1}$, and at times $t_0, t_1, t_2, \dots$. To do so, we will approximate $\frac{\partial u}{\partial t}$ at points x_i and time t_j with a forward difference in time

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{dt}$$

where $dt = t_{j+1} - t_j$. Similarly, we will approximate $\frac{\partial u}{\partial x}2$ with a 2nd order central difference at time t_j

$$\frac{\partial u}{\partial x}2(x_i, t_j) \approx \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}$$

where $h = x_{i+1} - x_i$.

If we denote $u(x_i, t_j) = u_i^j$, this finite difference scheme in x and t can be written as

$$\frac{u_i^{j+1} - u_i^j}{dt} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}.$$

(a) Write a MATLAB code that, given $u(x_i, t_j) = u_i^j$, uses the above finite difference scheme to compute $u(x_i, t_{j+1}) = u_i^{j+1}$ at the next timestep. For the initial condition $u_i^0 = u(x_i, t_0) = u(x_i, 0) = \psi(x_i)$, use the discontinuous function

$$\psi(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1 - x) & 1/2 \le x \le 1 \end{cases}$$

Take $N = 32$ and $dt = 1/10000$. Plot the solution at $t = 0$, and after 10, 50, and 100 timesteps. What happens (qualitatively) to the solution as t increases? Specifically, what happens to the parts of $\psi(x)$ that have sharp corners as t increases?

Solution.

(a) The given finite difference scheme in x and t can be written as

$$\frac{u_i^{j+1} - u_i^j}{dt} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}.$$

or alternatively,

$$u_i^{j+1} = (1-2r)u_i^j + r(u_{i+1}^j + u_{i-1}^j).$$

for $i = 1, 2 \cdot N$ where $r = \frac{dt}{h^2}$. The matrix system resulting from these equations for homogeneous boundary conditions is

$$\underbrace{\begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_{N_1}^{j+1} \\ u_N^{j+1} \end{bmatrix}}_{U^{j+1}} = \underbrace{\begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & r & 1-2r \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1^j \\ u_2^j \\ \vdots \\ u_{N-1}^j \\ u_N^k \end{bmatrix}}_{U^j},$$

Note that for U^0 we know the solution by initial conditions. To find the solution next time step we simply need to solve $AU^0 = U^1$. We will find U^1. In Matlab solving this linear system successively we are able to find solution at $j+1$ time step,i.e.; $AU^j = U^{j+1}$, we can get solutions for the next time steps.

Included is Matlab code that can be used to generate the finite difference solution for successive time steps:

```
%%HW 2, Problem 4a. CAAM 336, Fall 2014
%% Heat equation u_t=u_xx - explicit finite difference scheme on the interval 0<x<1 with
    boundary
%% conditions u(0,t)=0, u(1,t)=0 and given initial condition u(x,0) =1-2*(x-1/2).*sign(x
    -1/2)



% number of interior unknowns
M=32;
% time step and spatial step size
dt = 0.0001;
dx =(1-0)/(M+1);

% number of time iterations
N =100;

% final time of the computation
Tf = N*dt;

% the mesh ratio
r = dt/dx^2;

tvals = 0:dt:Tf;
xvals = 0:dx:1;

N=length(tvals);
J=length(xvals);

u0 = @(x) 1-2*(x-1/2).*sign(x-1/2);

u = zeros(J,N);
u(:,1) = u0(xvals);


for n = 1:N-1
    u(2:J-1,n+1) = r*u(3:J,n) + (1-2*r)*u(2:J-1,n)+r*u(1:J-2,n);
    u(1,n+1) = 0;
    u(J,n+1) = 0;
end
```

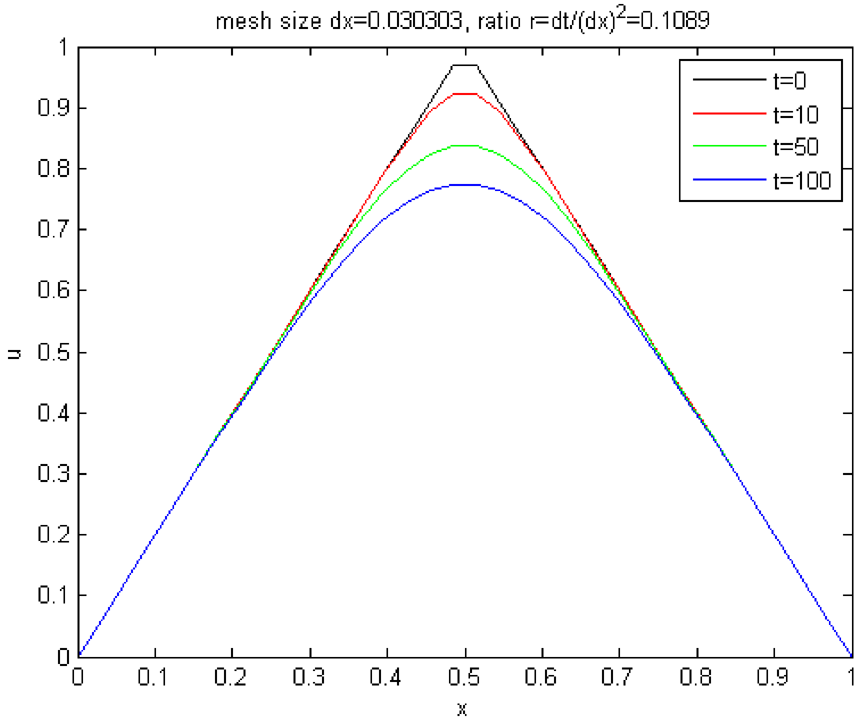

Figure 1: Finite difference solutions for various t

```
figure(1)
plot(xvals,u(:,1),'k');hold on %soluton at t=0
xlabel x; ylabel u;
title(strcat('mesh size dx= ',num2str(dx),...
    ', ratio r=dt/(dx)^2= ',num2str(r)))
plot(xvals,u(:,11),'r');hold on
plot(xvals,u(:,51),'g');hold on
plot(xvals,u(:,100),'b');hold on
legend('t=0', 't=10','t=50','t=100')
hold off

figure(2)
surf(xvals, tvals, u')
xlabel x; ylabel t; zlabel u
title(strcat('mesh size dx= ',num2str(dx),...
    ', ratio r=dt/(dx)^2= ',num2str(r)))
```

Notice that when we advance in time we get more smooth solutions in other word sharp corner of initial condition turn out parabolic shape. Also there is another fact that we might notice solution is decreasing when $t \to \infty$. One of the interesting properties of the heat equation is the maximum principle that says If u satisfies the heat equation for $0 < x < 1$ and $0 < t < T$ then the maximum value of u occurs at $t = 0$ (at the initial condition) or for $x = 0$ or $x = 1$ (at the boundary point).

Note: Students does not have to add both 2D plot or 3D surface. One of the plot should be enough.

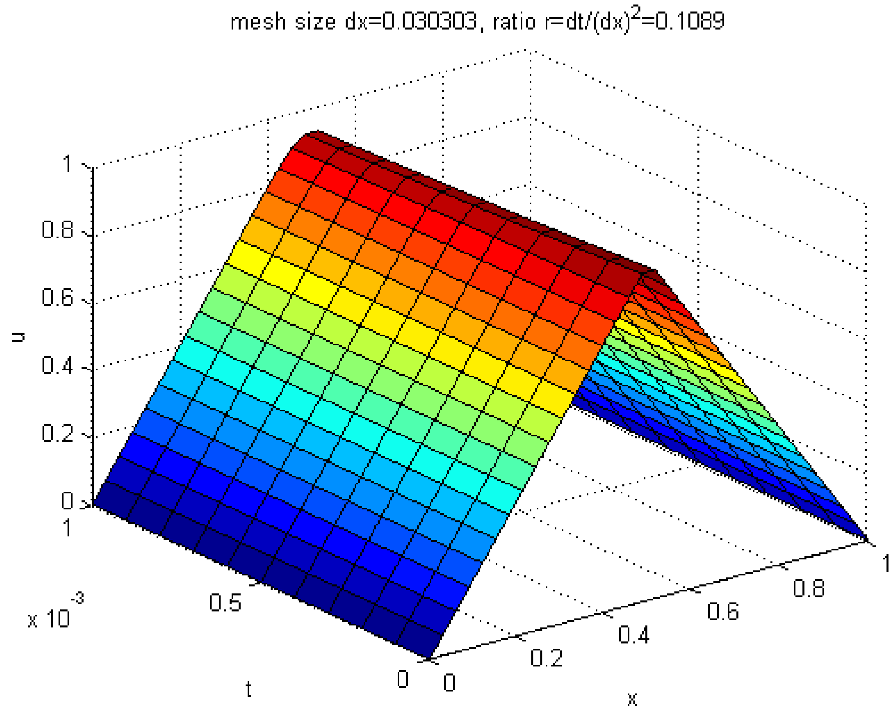

Figure 2: Surface plot for finite difference solution $t = 10$.

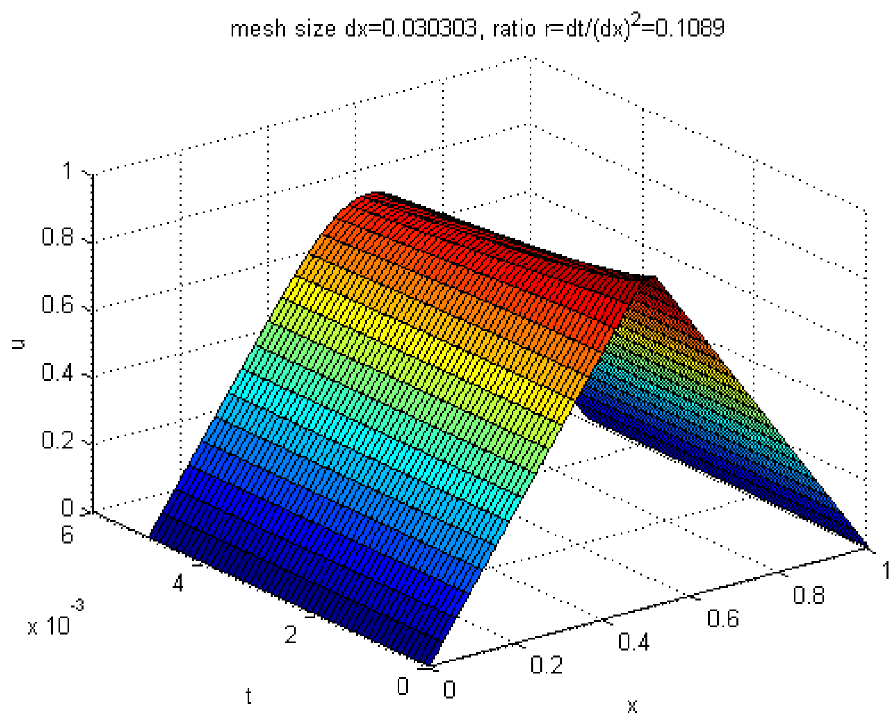

Figure 3: Surface plot for finite difference solution $t = 50$.

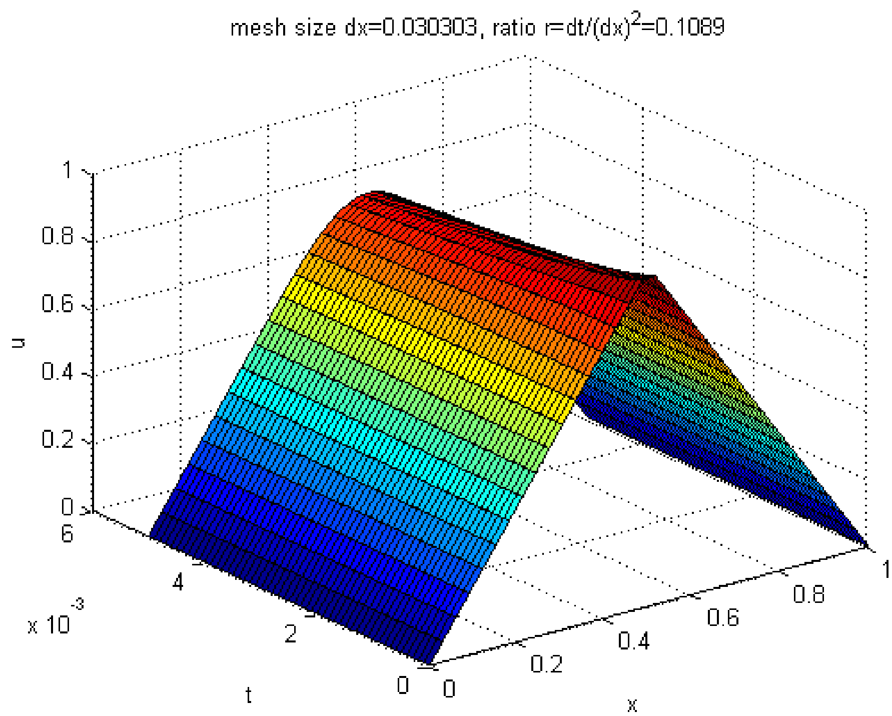

Figure 4: Surface plot for finite difference solution $t = 100$.