CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 28 · Solutions

Posted Wednesday 9 October 2013. Due 1pm Friday 18 October 2013.

28. [25 points] Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\| : C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$S = \{w \in C^2[0,1] : w(0) = w'(1) = 0\}.$$

Note that

$$(Lv,w) = (v,Lw) \text{ for all } v,w \in S.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin\left(\left(n - \frac{1}{2}\right)\pi x\right)$$

for $n = 1, 2, \ldots$. Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = x + \sin(\pi x), \quad 0 < x < 1$$

and

$$\tilde{u}(0) = \tilde{u}'(1) = 0.$$

Note that, for $m, n = 1, 2, \ldots,$

$$(\psi_m, \psi_n) = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

(c) Plot the approximation to $\tilde{u}$ obtained by replacing the upper limit of the summation in your series solution with 20.

(d) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = x + \sin(\pi x), \quad 0 < x < 1$$

and

$$u(0) = u'(1) = 1.$$

(e) Plot the approximation to u obtained by replacing the upper limit of the summation in your series solution with 20.

Solution.

(a) [4 points] We can compute that, for $n = 1, 2, \ldots$,

$$\psi_n'(x) = \sqrt{2}\left(n - \frac{1}{2}\right)\pi \cos\left(\left(n - \frac{1}{2}\right)\pi x\right)$$

and

$$\psi_n''(x) = -\sqrt{2}\left(n - \frac{1}{2}\right)^2 \pi^2 \sin\left(\left(n - \frac{1}{2}\right)\pi x\right)$$

and so

$$L\psi_n = -\psi_n'' = \left(n - \frac{1}{2}\right)^2 \pi^2 \psi_n.$$

Hence,

$$\lambda_n = \left(n - \frac{1}{2}\right)^2 \pi^2 \text{ for } n = 1, 2, \ldots.$$

(b) [8 points] Let $f \in C[0,1]$ be defined by $f(x) = x + \sin(\pi x)$. Then $\tilde{u}$ is the solution to $L\tilde{u} = f$ and so the spectral method yields the series solution

$$\tilde{u}(x) = \sum_{n=1}^{\infty} \frac{(f, \psi_n)}{\lambda_n} \psi_n(x).$$

Now, for $k = 1, 2, \ldots$,

$$\begin{aligned}
\sqrt{2}\int_0^1 x \sin\left(\left(k - \frac{1}{2}\right)\pi x\right)\, dx &= \frac{\sqrt{2}\left(\left(k - \frac{1}{2}\right)\pi \cos\left(\left(k - \frac{1}{2}\right)\pi\right) + \sin\left(\left(k - \frac{1}{2}\right)\pi\right)\right)}{\left(k - \frac{1}{2}\right)^2 \pi^2} \\
&= \frac{4\sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi\right)}{(2k-1)^2 \pi^2}
\end{aligned}$$

and twice integrating by parts shows that

$$\begin{aligned}
&\sqrt{2}\int_0^1 \sin(\pi x) \sin\left(\left(k - \frac{1}{2}\right)\pi x\right)\, dx \\
&= \frac{\sqrt{2}\left(\pi \cos(\pi) \sin\left(\left(k - \frac{1}{2}\right)\pi\right) - \left(k - \frac{1}{2}\right)\pi \sin(\pi) \cos\left(\left(k - \frac{1}{2}\right)\pi\right)\right)}{\left(k - \frac{1}{2}\right)^2 \pi^2 - \pi^2} \\
&= -\frac{\sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi\right)}{\left(\left(k - \frac{1}{2}\right)^2 - 1\right)\pi}.
\end{aligned}$$

We put these pieces together to find that, for $k = 1, 2, \ldots$,

$$\begin{aligned}
(f, \psi_k) &= \frac{4\sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi\right)}{(2k-1)^2 \pi^2} - \frac{\sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi\right)}{\left(\left(k - \frac{1}{2}\right)^2 - 1\right)\pi} \\
&= \sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi\right)\left(\frac{4}{(2k-1)^2 \pi^2} - \frac{1}{\left(\left(k - \frac{1}{2}\right)^2 - 1\right)\pi}\right) \\
&= \sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi\right)\frac{4\left(\left(k - \frac{1}{2}\right)^2 - 1\right) - (2k-1)^2 \pi}{(2k-1)^2\left(\left(k - \frac{1}{2}\right)^2 - 1\right)\pi^2}.
\end{aligned}$$

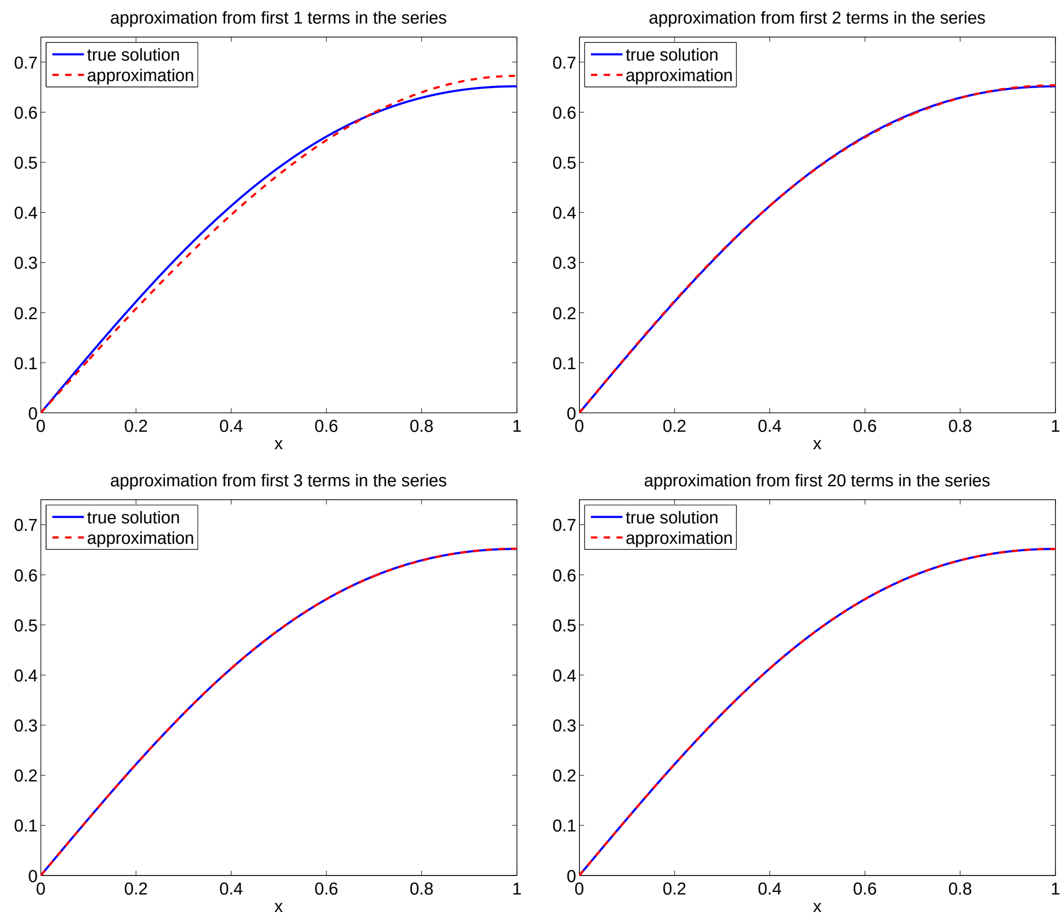
The spectral method thus gives the formula

$$\tilde{u}(x) = \sum_{k=1}^{\infty} 2\sin\left(\left(k - \frac{1}{2}\right)\pi\right)\frac{4\left(\left(k - \frac{1}{2}\right)^2 - 1\right) - (2k-1)^2 \pi}{(2k-1)^4\left(\left(k - \frac{1}{2}\right)^2 - 1\right)\pi^4} \sin\left(\left(k - \frac{1}{2}\right)\pi x\right).$$

(c) [4 points] Though not asked for in the question, the exact solution can be determined to be

$$\tilde{u}(x) = \frac{\sin(\pi x)}{\pi^2} - \frac{x^3}{6} + \frac{x}{2} + \frac{x}{\pi}.$$

The plots below compare the exact solution to the partial sums involving 1, 2, 3, and 20 terms. The code that produced the plots follows.




```
xx = linspace(0,1,1000)';     % fine grid between x=0 and x=1
uN = zeros(size(xx));         % homogeneous boundary conditions
for k=1:20
   figure(1), clf
   plot(xx, sin(pi*xx)/(pi^2) - (xx.^3)/6 + (1/2+1/pi)*xx,'b-','linewidth',2), hold on
   lamk = ((k-1/2)^2)*(pi^2);
   uN = uN + 2*(-1)^k*(pi/(lamk*(lamk-pi^2))-1/(lamk^2))*sin(sqrt(lamk)*xx);
   plot(xx, uN, 'r--','linewidth',2)
   legend('true solution', 'approximation',2)
   set(gca,'fontsize',16)
   xlabel('x')
   title(sprintf('approximation from first %d terms in the series', k))
   axis([0 1 0 0.75])
   if ismember(k,[1 2 3 20]),
       eval(sprintf('print -depsc2 bvps_%d',k))
   end
   pause
end
```

(d) [5 points] Let $\tilde{u}$ be the solution to $L\tilde{u} = f$ and let $w \in C^2[0,1]$ be such that

$$-w''(x) = 0, \quad 0 < x < 1$$

and

$$w(0) = w'(1) = 1.$$

Then $u(x) = w(x) + \tilde{u}(x)$ will be such that

$$-u''(x) = -w''(x) - \tilde{u}''(x) = 0 + f(x) = f(x);$$

$$u(0) = w(0) + \tilde{u}(0) = 1 + 0 = 1;$$

and

$$u'(1) = w'(1) + \tilde{u}'(1) = 1 + 0 = 1.$$

Now, the general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(1) = 1$ when $A = 1$. Hence, $w(x) = x + B$ and so $w(0) = B$ and hence $w(0) = 1$ when $B = 1$. Consequently,

$$w(x) = 1 + x$$

and so

$$u(x) = 1 + x + \tilde{u}(x).$$

We can then use the series solution to $L\tilde{u} = f$ that we obtained in part (c) to obtain the series solution

$$u(x) = 1 + x + \sum_{k=1}^{\infty} 2\sin\left(\left(k - \frac{1}{2}\right)\pi\right) \frac{4\left(\left(k-\frac{1}{2}\right)^2 - 1\right) - (2k-1)^2\pi}{(2k-1)^4\left(\left(k-\frac{1}{2}\right)^2 - 1\right)\pi^4} \sin\left(\left(k - \frac{1}{2}\right)\pi x\right)$$

to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u(0) = u'(1) = 1.$$

(e) [4 points] Though not asked for in the question, the exact solution is

$$u(x) = 1 + x + \frac{\sin(\pi x)}{\pi^2} - \frac{x^3}{6} + \frac{x}{2} + \frac{x}{\pi}.$$

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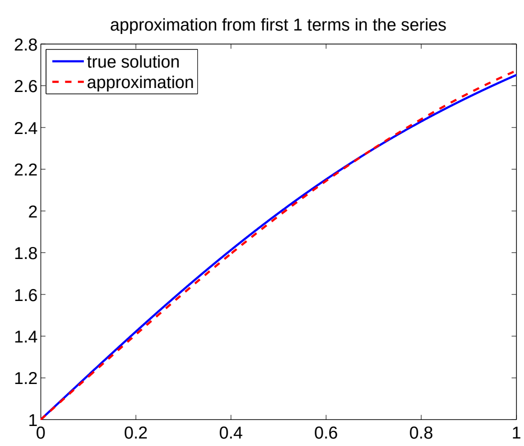

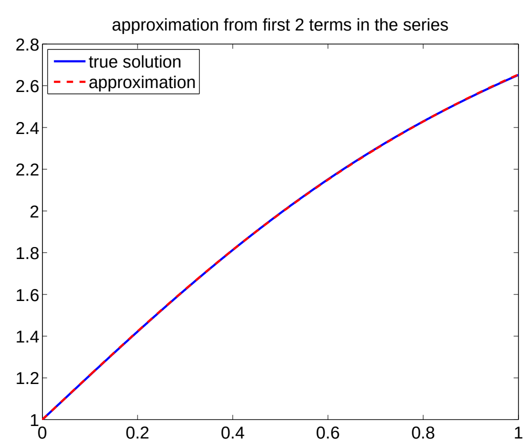

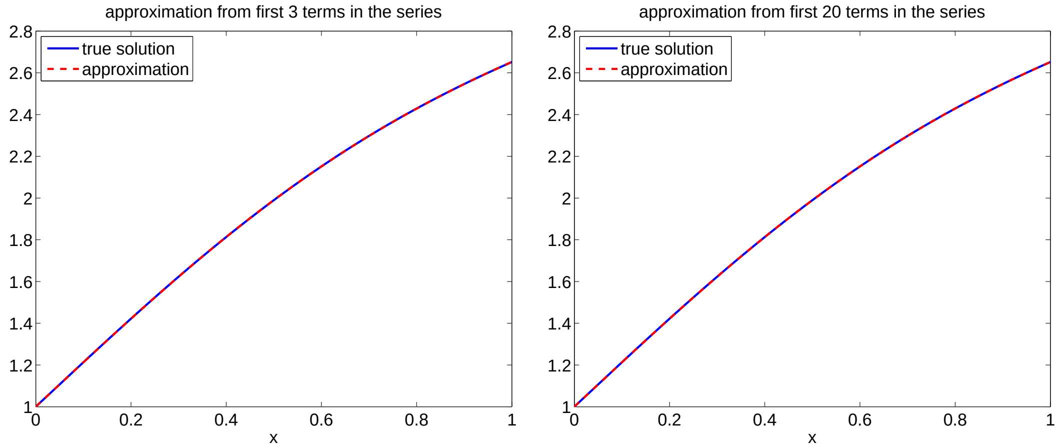


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