CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 8 · Solutions

Posted Wednesday 22 October, 2014. Due 5pm Wednesday 29 October, 2014.

Please write your name and ***residential college*** *on your homework.*

1. [20 points: 10 points each]

Let $k(x)$ and $p(x)$ be two positive-valued continuous functions on $[0, 1]$, and let

$$V = \Big\{ u \in C^2[0,1] : u(0) = \frac{du}{dx}(1) = 0 \Big\}.$$

(a) Derive the weak form of the differential equation

$$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) + p(x)u = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions

$$u(0) = \frac{du}{dx}(1) = 0;$$

that is, transform this differential equation into a problem of the form:

$$\text{Find } u \in V \text{ such that } a(u, v) = (f, v) \text{ for all } v \in V,$$

where $(\cdot,\cdot)$ denotes the usual inner product $(f, g) = \int_0^1 f(x)g(x)\,dx$, and $a(\cdot,\cdot)$ is some bilinear form that you should specify.

(b) Show that the form $a(u, v)$ from part (a) is an inner product for $u, v \in V$.

Solution.

(a) Multiply the differential equation with some function v from the space V and integrate from $x = 0$ to $x = 1$ to obtain

$$\int_0^1 \Big(-\frac{d}{dx}\Big(k(x)\frac{du}{dx}(x)\Big)v(x) + p(x)u(x)v(x)\Big)\,dx = \int_0^1 f(x)v(x)\,dx.$$

Break the integral on the left into pieces to obtain

$$\int_0^1 \Big(-\frac{d}{dx}\Big(k(x)\frac{du}{dx}(x)\Big)\Big)v(x)\,dx + \int_0^1 \Big(p(x)u(x)\Big)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

Integrate the first integral by parts to obtain

$$-\Big[\kappa(x)\frac{du}{dx}(x)v(x)\Big]_0^1 + \int_0^1 k(x)\frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx + \int_0^1 \Big(p(x)u(x)\Big)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

The first term disappears because of the boundary conditions $v(0) = 0$ and $du(1)/dx = 0$. We consolidate the integrals on the left to arrive at the weak problem:

$$\text{Find } u \in V \text{ such that} \quad a(u, v) = (f, v) \quad \text{for all } v \in V,$$

where

$$a(u, v) = \int_0^1 \Big(k(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + p(x)u(x)v(x)\Big)\,dx.$$

(b) To show that the form $a(u,v)$ in part (a) is an inner product, we must verify the three basic properties:

- **Symmetry** is apparent by inspection:

$$\begin{aligned} a(u,v) &= \int_0^1 \Big(k(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + p(x)u(x)v(x) \Big)\, dx \\ &= \int_0^1 \Big(k(x)\frac{dv}{dx}(x)\frac{du}{dx}(x) + p(x)u(x)v(x) \Big)\, dx \;=\; a(v,u). \end{aligned}$$

- **Linearity** follows from the linearity of differentiation and integration:

$$\begin{aligned} a(\alpha u + \beta v, w) &= \int_0^1 \Big(k(x)\frac{d(\alpha u(x) + \beta v(x))}{dx}(x)\frac{dw}{dx}(x) + p(x)(\alpha u(x) + \beta v(x))w(x) \Big)\, dx \\ &= \int_0^1 \Big(k(x)\Big(\alpha\frac{du(x)}{dx} + \beta\frac{dv(x)}{dx}\Big)\frac{dw}{dx}(x) + p(x)(\alpha u(x) + \beta v(x))w(x) \Big)\, dx \\ &= \alpha \int_0^1 \Big(k(x)\frac{du(x)}{dx}\frac{dw}{dx}(x) + p(x)u(x)w(x) \Big)\, dx \\ &\qquad + \beta \int_0^1 \Big(k(x)\frac{dv(x)}{dx}\frac{dw}{dx}(x) + p(x)v(x)w(x) \Big)\, dx \\ &= \alpha a(u,w) + \beta a(v,w). \end{aligned}$$

- **Positivity** requires that $a(u,u) \geq 0$ and $a(u,u) = 0$ only when $u = 0$. Note that

$$\begin{aligned} a(u,u) &= \int_0^1 \Big(k(x)\frac{du}{dx}(x)\frac{du}{dx}(x) + p(x)u(x)u(x) \Big)\, dx \\ &= \int_0^1 \Big(k(x)\Big(\frac{du}{dx}(x)\Big)^2 + p(x)\big(u(x)\big)^2 \Big)\, dx. \end{aligned}$$

  Since $k(x)$ and $p(x)$ are both positive for all $x \in [0,1]$, each integrand is non-negative, and hence $a(u,u) \geq 0$. To have $a(u,u) = 0$, we must have $u(x) = 0$ for all $x \in [0,1]$, and $du(x)/dx = 0$ for all $x \in [0,1]$, which is only possible if $u(x) = 0$ for all $x \in [0,1]$, i.e., $u = 0$.

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2. [20 points: 10 points each]

Let

$$H^1_D(0,1) = \{w \in H^1(0,1) : \, w(0) = w(1) = 0\}$$

and let the inner product $(\cdot,\cdot)$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and the energy inner product $a(\cdot,\cdot)$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Also, let $f \in L^2(0,1)$, let N be a positive integer, and let V_N be a subspace of $H^1_D(0,1)$. Moreover, let $u \in H^1_D(0,1)$ be such that

$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1)$$

and let $u_N \in V_N$ be such that

$$a(u_N,v) = (f,v) \text{ for all } v \in V_N.$$

(a) Show that

$$a(u-u_N,u-u_N) = a(u,u) - a(u_N,u_N).$$

(b) Let $\phi_1,\ldots,\phi_N \in V_N$ and let $\mathbf{K} \in \mathbb{R}^{N\times N}$ be the matrix with entries $K_{jk} = a(\phi_k,\phi_j)$ for $j,k = 1,\ldots,N$. Also, let

$$u_N = \sum_{j=1}^{N} c_j\phi_j$$

where $c_j \in \mathbb{R}$ is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^N$. Show that

$$\mathbf{c}^T\mathbf{K}\mathbf{c} = a(u_N,u_N).$$

Solution.

(a) [13 points] The properties satisfied by the inner product allow us to say that

$$\begin{aligned} a(u-u_N,u-u_N) &= a(u,u-u_N) - a(u_N,u-u_N) \\ &= a(u,u) - a(u,u_N) - a(u_N,u) + a(u_N,u_N) \\ &= a(u,u) - 2a(u,u_N) + a(u_N,u_N). \end{aligned}$$

Now, $u_N \in V_N$ and so the fact that

$$a(u_N,v) = (f,v) \text{ for all } v \in V_N$$

means that

$$a(u_N,u_N) = (f,u_N).$$

Moreover, $u_N \in H^1_D(0,1)$, since V_N is a subspace of $H^1_D(0,1)$ and $u_N \in V_N$, and so the fact that

$$a(u,v) = (f,v) \text{ for all } v \in H^1_D(0,1)$$

means that

$$a(u, u_N) = (f, u_N).$$

So,

$$a(u,u) - 2a(u,u_N) + a(u_N,u_N) = a(u,u) - 2(f,u_N) + (f,u_N) = a(u,u) - (f,u_N).$$

Therefore,

$$a(u-u_N, u-u_N) = a(u,u) - (f,u_N) = a(u,u) - a(u_N,u_N)$$

because

$$a(u_N, u_N) = (f, u_N).$$

(b) [12 points] We first compute that

$$\mathbf{K}\mathbf{c} = \mathbf{d}$$

where $\mathbf{d} \in \mathbb{R}^N$ is the vector with entries

$$d_j = \sum_{k=1}^{N} a(\phi_k, \phi_j) c_k$$

for $j = 1, \ldots, N$. Moreover, since

$$u_N = \sum_{j=1}^{N} c_j \phi_j = \sum_{k=1}^{N} c_k \phi_k,$$

the properties satisfied by the inner product yield that

$$\sum_{k=1}^{N} a(\phi_k, \phi_j) c_k = a\left(\sum_{k=1}^{N} c_k \phi_k, \phi_j\right) = a(u_N, \phi_j)$$

and so

$$d_j = a(u_N, \phi_j)$$

for $j = 1, \ldots, N$. Therefore,

$$\mathbf{c}^T \mathbf{K}\mathbf{c} = \mathbf{c}^T \mathbf{d} = \sum_{j=1}^{N} c_j a(u_N, \phi_j) = a\left(u_N, \sum_{j=1}^{N} c_j \phi_j\right) = a(u_N, u_N)$$

by the properties satisfied by the inner product and the fact that

$$u_N = \sum_{j=1}^{N} c_j \phi_j.$$

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3. [30 points: 6 points each]

Let $f \in C[0,1]$ be such that $f(x) = \sin(\pi x)$. Suppose that N is a positive integer and define $h = \dfrac{1}{N+1}$ and $x_j = jh$ for $j = 0, 1 \ldots, N+1$. Consider the N hat functions $\phi_k \in C[0,1]$, defined as

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k); \\ \dfrac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise}; \end{cases}$$

for $k = 1, \ldots, N$. Let the inner product $(\cdot, \cdot) : \ C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_0^1 u(x)v(x)\, dx$$

and let the norm $\|\cdot\| : \ C[0,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

(a) For $j = 1, \ldots, N$, what is $\phi_j(x_k)$ for $k = 0, 1, \ldots, N+1$? Simplify your answer as much as possible.

(b) Show that $\{\phi_1, \ldots, \phi_N\}$ is linearly independent by showing that if $c_k \in \mathbb{R}$ and $\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $c_k = 0$ for $k = 1, \ldots, N$.

(c) By hand, compute (f, ϕ_j) for $j = 1, \ldots, N$.

(d) By hand, compute (ϕ_j, ϕ_k) for $j, k = 1, \ldots, N$. Your final answers should be simplified as much as possible and in your formulas h should be left as h and not be replaced with $1/(N+1)$. You must clearly state which values of j and k each formula you obtain is valid for. An acceptable way to present the final answer would be:
For $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} ? & \text{if } k = j, \\ ? & \text{if } |j - k| = 1, \\ ? & \text{otherwise}, \end{cases}$$

with the question marks replaced with the correct values. Hint: Letting $s = x - x_{j-1}$ yields that

$$\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx = \frac{1}{h^2} \int_{x_{j-1}-x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2\, ds = \frac{1}{h^2} \int_0^h s^2\, ds.$$

(e) Set up a linear system (in MATLAB) and solve it to compute the best approximation f_N to f from $\text{span}\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ for $N = 3$ and $N = 9$. For each of these N, produce a separate plot that superimposes $f_N(x)$ on top of a plot of $f(x)$. The `hat.m` code (from Homework 2, either your code or the code from the solutions) should help you to produce these plots.

Solution.

(a) [3 points] The definition of ϕ_j yields that $\phi_j(x_k) = 0$ if $k \neq j$. Moreover,

$$\phi_j(x_j) = \frac{x_{j+1} - x_j}{h} = \frac{(j+1)h - jh}{h} = \frac{jh + h - jh}{h} = \frac{h}{h} = 1.$$

Consequently, for $j = 1, \ldots, N$,

$$\phi_j(x_k) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j, \end{cases}$$

for $k = 0, 1, \ldots, N+1$.

(b) [3 points] If $c_k \in \mathbb{R}$ and $\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $\sum_{k=1}^{N} c_k \phi_k(x_j) = 0$ for $j = 1, \ldots, N$. The answer to part (a) then allows us to conclude that $c_j = 0$ for $j = 1, \ldots, N$ since $\sum_{k=1}^{N} c_k \phi_k(x_j) = c_j$. Therefore, $c_k = 0$ for $k = 1, \ldots, N$ since $c_j = 0$ for $j = 1, \ldots, N$ is equivalent to $c_k = 0$ for $k = 1, \ldots, N$.

(c) [3 points] For $j = 1, \ldots, N$, integrating by parts yields that

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \sin(\pi x)\, dx &= \left[\frac{x - x_{j-1}}{h} \left(-\frac{\cos(\pi x)}{\pi} \right) \right]_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j} \frac{d}{dx} \left(\frac{x - x_{j-1}}{h} \right) \frac{\cos(\pi x)}{\pi}\, dx \\
&= -\frac{x_j - x_{j-1}}{h} \frac{\cos(\pi x_j)}{\pi} + \int_{x_{j-1}}^{x_j} \frac{1}{h} \frac{\cos(\pi x)}{\pi}\, dx \\
&= -\frac{jh - (j-1)h}{h} \frac{\cos(\pi x_j)}{\pi} + \left[\frac{1}{h} \frac{\sin(\pi x)}{\pi^2} \right]_{x_{j-1}}^{x_j} \\
&= -\frac{\cos(\pi x_j)}{\pi} + \frac{\sin(\pi x_j) - \sin(\pi x_{j-1})}{\pi^2 h}
\end{aligned}$$

and

$$\begin{aligned}
\int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \sin(\pi x)\, dx &= \left[\frac{x_{j+1} - x}{h} \left(-\frac{\cos(\pi x)}{\pi} \right) \right]_{x_j}^{x_{j+1}} + \int_{x_j}^{x_{j+1}} \frac{d}{dx} \left(\frac{x_{j+1} - x}{h} \right) \frac{\cos(\pi x)}{\pi}\, dx \\
&= \frac{x_{j+1} - x_j}{h} \frac{\cos(\pi x_j)}{\pi} - \int_{x_j}^{x_{j+1}} \frac{1}{h} \frac{\cos(\pi x)}{\pi}\, dx \\
&= \frac{(j+1)h - jh}{h} \frac{\cos(\pi x_j)}{\pi} - \left[\frac{1}{h} \frac{\sin(\pi x)}{\pi^2} \right]_{x_j}^{x_{j+1}} \\
&= \frac{\cos(\pi x_j)}{\pi} + \frac{\sin(\pi x_j) - \sin(\pi x_{j+1})}{\pi^2 h}.
\end{aligned}$$

Hence, for $j = 1, \ldots, N$,

$$\begin{aligned}
(f, \phi_j) &= \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \sin(\pi x)\, dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \sin(\pi x)\, dx \\
&= \frac{2\sin(\pi x_j) - \sin(\pi x_{j-1}) - \sin(\pi x_{j+1})}{\pi^2 h} \\
&= \frac{2\sin(\pi x_j)}{\pi^2 h} (1 - \cos(h\pi)).
\end{aligned}$$

(d) [8 points] For $j = 1, \ldots, N$, letting $s = x - x_{j-1}$ and $t = x - x_{j+1}$ yields that

$$
\begin{aligned}
(\phi_j, \phi_j) &= \int_0^1 (\phi_j(x))^2 \, dx \\
&= \int_0^{x_{j-1}} (\phi_j(x))^2 \, dx + \int_{x_{j-1}}^{x_j} (\phi_j(x))^2 \, dx + \int_{x_j}^{x_{j+1}} (\phi_j(x))^2 \, dx + \int_{x_{j+1}}^1 (\phi_j(x))^2 \, dx \\
&= \int_0^{x_{j-1}} 0 \, dx + \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h}\right)^2 dx + \int_{x_{j+1}}^1 0 \, dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h}\right)^2 dx \\
&= \frac{1}{h^2} \int_{x_{j-1} - x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2 \, ds + \frac{1}{h^2} \int_{x_j - x_{j+1}}^{x_{j+1} - x_{j+1}} (x_{j+1} - (t + x_{j+1}))^2 \, dt \\
&= \frac{1}{h^2} \int_0^h s^2 \, ds + \frac{1}{h^2} \int_{-h}^0 t^2 \, dt \\
&= \frac{1}{h^2} \left[\frac{s^3}{3}\right]_0^h + \frac{1}{h^2} \left[\frac{t^3}{3}\right]_{-h}^0 \\
&= \frac{h^3}{3h^2} - \frac{(-h)^3}{3h^2} \\
&= \frac{h}{3} + \frac{h}{3} \\
&= \frac{2h}{3}.
\end{aligned}
$$

Moreover, for $j = 1, \ldots, N - 1$,

$$
\phi_{j+1}(x) = \begin{cases} \dfrac{x - x_j}{h} & \text{if } x \in [x_j, x_{j+1}); \\ \dfrac{x_{j+2} - x}{h} & \text{if } x \in [x_{j+1}, x_{j+2}); \\ 0 & \text{otherwise;} \end{cases}
$$

and so letting $s = x - x_j$ yields that

$$
\begin{aligned}
(\phi_{j+1}, \phi_j) &= (\phi_j, \phi_{j+1}) \\
&= \int_0^1 \phi_j(x)\phi_{j+1}(x)\,dx \\
&= \int_0^{x_j} \phi_j(x)\phi_{j+1}(x)\,dx + \int_{x_j}^{x_{j+1}} \phi_j(x)\phi_{j+1}(x)\,dx + \int_{x_{j+1}}^1 \phi_j(x)\phi_{j+1}(x)\,dx \\
&= \int_0^{x_j} 0\,dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1}-x}{h}\frac{x-x_j}{h}\,dx + \int_{x_{j+1}}^1 0\,dx \\
&= \int_{x_j}^{x_{j+1}} \frac{x_{j+1}-x}{h}\frac{x-x_j}{h}\,dx \\
&= \frac{1}{h^2}\int_{x_j-x_j}^{x_{j+1}-x_j} \left(x_{j+1} - (s+x_j)\right)(s + x_j - x_j)\ ds \\
&= \frac{1}{h^2}\int_0^h hs - s^2\,ds \\
&= \frac{1}{h^2}\left[\frac{hs^2}{2} - \frac{s^3}{3}\right]_0^h \\
&= \frac{1}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) \\
&= \frac{3}{6} - \frac{2h}{6} \\
&= \frac{h}{6}.
\end{aligned}
$$

Finally, for $j = 1, \ldots, N$,

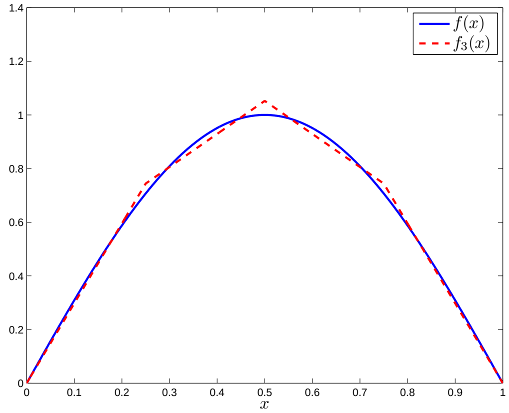
$$
(\phi_j, \phi_k) = \int_0^1 \phi_j(x)\phi_k(x)\,dx = \int_0^1 0\,dx = 0
$$

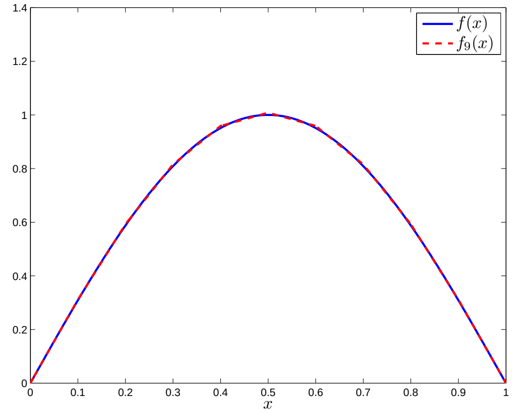
if $|j-k| > 1$.

Hence, for $j, k = 1, \ldots, N$,

$$
(\phi_j, \phi_k) = \begin{cases} \frac{2h}{3} & \text{if } k = j, \\ \frac{h}{6} & \text{if } |j-k| = 1, \\ 0 & \text{otherwise.} \end{cases}
$$

(e) [8 points] The requested plots are shown below, followed by the MATLAB code that generated them.






```
xx = linspace(0,1,500).';
for N = [3 9]
    h = 1/(N+1);
    x = (0:N+1)*h;
    % set up the matrix from the inner products computed in part (d)
    A = 2*h/3*eye(N) + h/6*diag(ones(N-1,1),1) + h/6*diag(ones(N-1,1),-1);
    % set up the right-hand side vector from the inner products computed in part (c)
    b = 2/(h*pi^2)*(1-cos(h*pi))*sin(h*pi*(1:N).');
    % solve for the coefficients
    c = A\b
    % compute the approximation on fine grid on [0,1]
    fN = zeros(length(xx),1);
    for j=1:N
        fN = fN + c(j)*hat(xx,j,N);
    end
    % plot the function f and the approximation
    figure(2)
    clf
    plot(xx, sin(pi*xx), 'b-','linewidth',2)
    hold on
    plot(xx, fN, 'r--','linewidth',2)
    xlabel('$x$','interpreter','latex','fontsize',16)
    legendStr{1}=['$f(x)$'];
    legendStr{2}=['$f_{' num2str(N) '}(x)$'];
    legend(legendStr,'interpreter','latex','fontsize',16)
    %set(gca,'fontsize',16)
    if (N==3)
        saveas(figure(2),'f_3.eps','epsc')
    elseif (N==9)
        saveas(figure(2),'f_9.eps','epsc')
    end
end
```

4. [30 points: 6 points each]
Use the finite element method to solve the differential equation

$$-(u'(x)\kappa(x))' = 2x, \qquad 0 < x < 1$$

for $\kappa(x) = 1 + x^2$, subject to homogeneous Dirichlet boundary conditions,

$$u(0) = u(1) = 0,$$

with the approximation space V_N given by the piecewise linear *hat functions* that featured on earlier problem sets: For $n \geq 1$, $h = 1/(N+1)$, and $x_k = kh$ for $k = 0, \ldots, N+1$, we have

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write MATLAB code that constructs the stiffness matrix $\mathbf{K}$ for a given value of N, with $\kappa(x) = 1 + x^2$.

[You may edit the `fem_demo1.m` code from the class website. You should compute all necessary integrals (by hand or using a symbolic package) so as to obtain clean formulas that depend on h and the index of the hat functions involved (e.g.,$a(\phi_j, \phi_j)$ can depend on j).]

(b) Write MATLAB code that constructs the load vector $\mathbf{f}$ for a given value of N, with $f(x) = 2x$.

(c) For $N = 7$ and $N = 15$, produce plots comparing your solution u_N to the true solution

$$u(x) = (4/\pi)\tan^{-1}(x) - x.$$

(Note that you can compute $\tan^{-1}(x)$ as `atan(x)` in MATLAB.)

(d) Produce a `loglog` plot showing how the error

$$\max_{x \in [0,1]} |u_N(x) - u(x)|$$

decreases as N increases. (For example, take $N = 8, 16, 32, 64, 128, 256, 512$.) On the same plot, show N^{-2} for the same values of N. If your code from parts (a) and (b) is working, your error curve should have the same slope as the N^{-2} curve. (Consult the `fem_demo1.m` code on the website for a demonstration of the style of plot we intend for part (d); edit this code as you like.)

Solution.

(a) First we compute the energy inner product of the basis functions. Note that

$$\frac{d\phi_k}{dx}(x) = \begin{cases} 1/h, & x \in [x_{k-1}, x_k); \\ -1/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have

$$\begin{aligned} a(\phi_j, \phi_j) &= \int_0^1 (1+x^2)\Big(\frac{d\phi_j}{dx}(x)\Big)^2\,dx \\ &= \int_{x_{j-1}}^{x_j} (1+x^2)\Big(\frac{1}{h}\Big)^2\,dx + \int_{x_j}^{x_{j+1}} (1+x^2)\Big(-\frac{1}{h}\Big)^2\,dx \\ &= \frac{1}{h^2}\int_{x_{j-1}}^{x_{j+1}} (1+x^2)\,dx \;=\; \frac{1}{h^2}\Big[x+\frac{x^3}{3}\Big]_{x_{j-1}}^{x_{j+1}} \;=\; \frac{2}{h}+\frac{2h}{3}+2hj^2, \end{aligned}$$

$$\begin{aligned} a(\phi_j, \phi_{j+1}) &= \int_0^1 (1+x^2)\Big(\frac{d\phi_j}{dx}(x)\Big)\Big(\frac{d\phi_{j+1}}{dx}(x)\Big)\,dx \\ &= \int_{x_j}^{x_{j+1}} (1+x^2)\Big(-\frac{1}{h}\Big)\Big(\frac{1}{h}\Big)\,dx \\ &= -\frac{1}{h^2}\int_{x_j}^{x_{j+1}} (1+x^2)\,dx \;=\; -\frac{1}{h^2}\Big[x+\frac{x^3}{3}\Big]_{x_j}^{x_{j+1}} \;=\; -\frac{1}{h} - h\Big(j^2+j+\frac{1}{3}\Big), \end{aligned}$$

and for $|j-k|>1$,

$$a(\phi_j, \phi_k) = 0$$

since $\big(d\phi_j(x)/dx\big)\big(d\phi_k(x)/dx\big) = 0$ for all $x \in [0,1]$ (except at the nodes x_ℓ, where strictly speaking these derivatives are not defined—but these single isolated points do not add anything to the integral). The stiffness matrix is given by

$$\mathbf{K} = \begin{bmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_1,\phi_n) \\ \vdots & \ddots & \vdots \\ a(\phi_n,\phi_1) & \cdots & a(\phi_n,\phi_n) \end{bmatrix}.$$

(b) Next we compute the entries of the load vector:

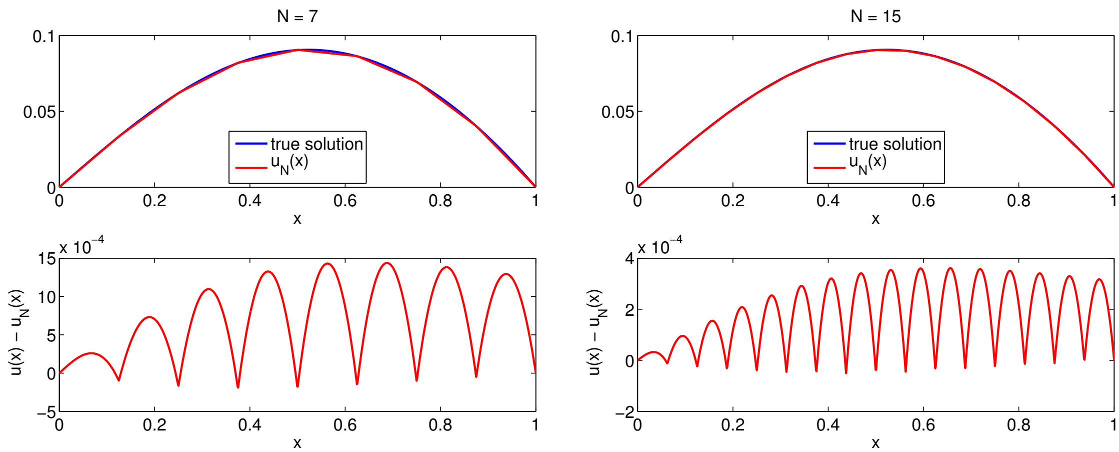
$$\begin{aligned} (f, \phi_j) &= \int_0^1 f(x)\phi_j(x)\,dx \\ &= \int_{x_{j-1}}^{x_j} (2x)\Big(\frac{x-x_{j-1}}{h}\Big)\,dx + \int_{x_j}^{x_{j+1}} (2x)\Big(\frac{x_{j+1}-x}{h}\Big)\,dx \\ &= \frac{1}{h}\Big[\frac{2x^3}{3} - x^2x_{j-1}\Big]_{x_{j-1}}^{x_j} + \frac{1}{h}\Big[x^2x_{j+1} - \frac{2x^3}{3}\Big]_{x_j}^{x_{j+1}} \\ &= 2h^2 j. \end{aligned}$$

The load vector is given by

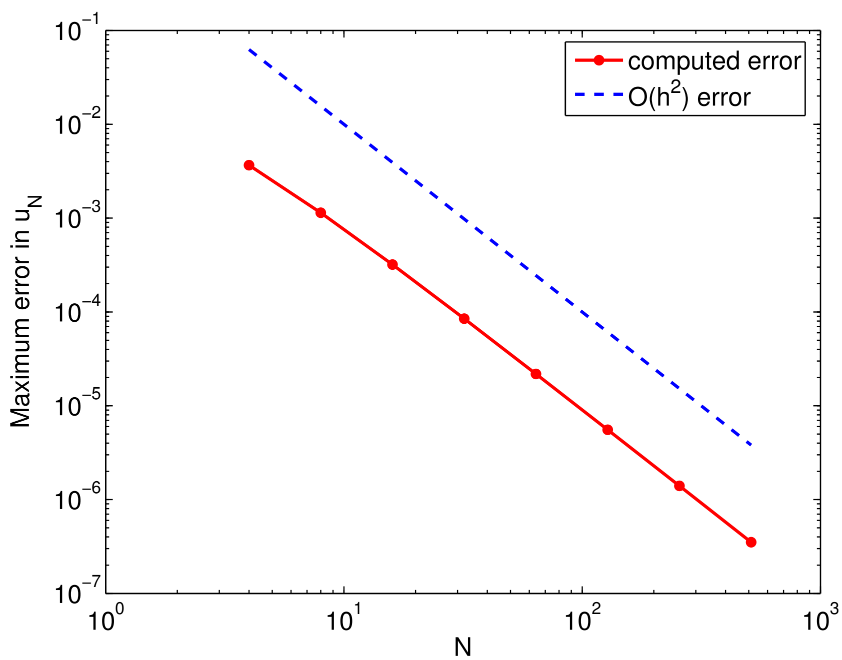
$$\mathbf{f} = [(f,\phi_1) \vdots (f,\phi_n)]^T.$$

The MATLAB code at the end of this problem shows generates these matrices and produces plots similar to those shown in (b) and (c).

(c) The following plots show the solution (and error) at $N = 7$ (left) and $N = 15$ (right).



(d) The following plot shows the decay of the error as a function of N. Notice that the error decays like $1/N^2$.




```
% demo of the finite element method for the problem
%   -d/dx((1+x^2) du/dx)  = 2x, 0 < x < 1, u(0) = u(1) = 0.
% which has exact solution u(x) = (4/pi)*atan(x) - x.

Nvec = [4 8 16 32 64 128 256 512];   % vector of N values we shall use
maxerr = zeros(size(Nvec));          % vector to hold the max errors for each N

% each pass of the following loop handles a new N value...
for j=1:length(Nvec)
   N = Nvec(j);
   h = 1/(N+1);
   x = [1:N]*h;

% construct the stiffness matrix (integrals done by hand)
   maindiag = 2/h + 2*h/3 + 2*h*([1:N].^2);
   offdiag  = -1/h - h*(([1:N-1].^2) + [1:N-1] + 1/3);
```

```
   K = diag(maindiag) + diag(offdiag,1) + diag(offdiag,-1);

% construct the load vector (integrals done by hand)
   f = 2*h^2*[1:N]';

% solve for expansion coefficients of Galerkin approximation
   c = K\f;

% plot the true solution
   xx = linspace(0,1,1000)';     % finely spaced points between 0 and 1.
   u = (4/pi)*atan(xx)-xx;      % true solution

   figure(1), clf
   subplot(2,1,1)
   plot(xx, u, 'b-','linewidth',2)
   hold on

% plot the approximation solution
   uN = zeros(size(xx));
   for k=1:N
      uN = uN + c(k)*hat(xx,k,N);
   end
   plot(xx, uN, 'r-','linewidth',2)
   set(gca,'fontsize',16)
   xlabel('x')
   legend('true solution', 'u_N(x)','location','south')
   title(sprintf('N = %d', N))

% plot the error in the solution for this N
   subplot(2,1,2)
   plot(xx, u-uN, 'r-','linewidth',2)
   set(gca,'fontsize',16)
   xlabel('x')
   ylabel(' u(x) - u_N(x)')

% approximate the maximum error for this value of N
   maxerr(j) = max(abs(u - uN));

   input('hit return to continue')
end

% plot the maximum error
  figure(2), clf
  loglog(Nvec, maxerr, 'r.-', 'linewidth',2, 'markersize',20)
  hold on
  loglog(Nvec, Nvec.^(-2), 'b--', 'linewidth',2)
  legend('computed error', 'O(h^2) error')
  set(gca,'fontsize',16);
  xlabel('N')
  ylabel('Maximum error in u_N')
  print -depsc2 femb.eps
```