CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 20 · Solutions

Posted Friday 14 February 2014. Due 1pm Friday 28 February 2014.

20. [25 points]

All parts of this question should be done by hand.

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $f_1 \in C[-1,1]$, and $f_2 \in C[-1,1]$ be defined by

$$\phi_1(x) = \frac{1}{\sqrt{2}},$$

$$\phi_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$f_1(x) = \sin(\pi x),$$

and

$$f_2(x) = \cos(\pi x),$$

for all $x \in [-1,1]$. Note that $\{\phi_1, \phi_2\}$ is linearly independent. Let the inner product $(\cdot,\cdot):\ C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx$$

and let the norm $\|\cdot\|:\ C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$. Also, let $\psi_1 \in C[0,1]$, $\psi_2 \in C[0,1]$, $g_1 \in C[0,1]$, and $g_2 \in C[0,1]$ be defined by

$$\psi_1(x) = \frac{1}{\sqrt{2}},$$

$$\psi_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$g_1(x) = \sin(\pi x),$$

and

$$g_2(x) = \cos(\pi x),$$

for all $x \in [0,1]$. Note that $\{\psi_1, \psi_2\}$ is linearly independent. Let the inner product $B(\cdot,\cdot):\ C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$B(u,v) = \int_{0}^{1} u(x)v(x)\,dx$$

and let the norm $\|\cdot\|_B:\ C[0,1] \to \mathbb{R}$ be defined by

$$\|u\|_B = \sqrt{B(u,u)}.$$

Note that

$$(f_1, \phi_2) = \frac{\sqrt{6}}{\pi},$$

$$B(g_1, \psi_1) = \frac{\sqrt{2}}{\pi},$$

$$B(g_1, \psi_2) = \frac{\sqrt{6}}{2\pi},$$

$$B(g_2, \psi_2) = -\frac{\sqrt{6}}{\pi^2},$$

and

$$(f_1, \phi_1) = (f_2, \phi_1) = (f_2, \phi_2) = B(g_2, \psi_1) = 0.$$

(a) Construct the best approximation to f_1 from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(b) Construct the best approximation to f_2 from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) Construct the best approximation to g_1 from $\text{span}\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.

(d) Construct the best approximation to g_2 from $\text{span}\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.

Solution.

(a) [5 points] Since $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$, the best approximation to f_1 from span $\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$\tilde{f}_1(x) = (f_1, \phi_1)\phi_1(x) + (f_1, \phi_2)\phi_2(x) = 0 + \frac{\sqrt{6}}{\pi}\frac{\sqrt{3}}{\sqrt{2}}x = \frac{\sqrt{2}\sqrt{3}}{\pi}\frac{\sqrt{3}}{\sqrt{2}}x = \frac{\sqrt{9}}{\pi}x = \frac{3}{\pi}x.$$

(b) [4 points] Since $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$, the best approximation to f_2 from span $\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$\tilde{f}_2(x) = (f_2, \phi_1)\phi_1(x) + (f_2, \phi_2)\phi_2(x) = 0 + 0 = 0.$$

(c) [10 points] Now,

$$B(\psi_1, \psi_2) = \int_0^1 \psi_1(x)\psi_2(x)\,dx = \int_0^1 \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{2}}x\,dx = \frac{\sqrt{3}}{2}\int_0^1 x\,dx = \frac{\sqrt{3}}{2}\left[\frac{1}{2}x^2\right]_0^1 = \frac{\sqrt{3}}{2}\frac{1}{2} = \frac{\sqrt{3}}{4}$$

and so $\{\psi_1, \psi_2\}$ is not orthogonal with respect to the inner product $B(\cdot,\cdot)$. Consequently, the best approximation to g_1 from span $\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_1(x) = c_1\psi_1(x) + c_2\psi_2(x)$$

where the coefficients $c_1, c_2 \in \mathbb{R}$ are such that

$$\mathbf{G}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} B(g_1, \psi_1) \\ B(g_1, \psi_2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{6}}{2\pi} \end{bmatrix}$$

where

$$\mathbf{G} = \begin{bmatrix} B(\psi_1, \psi_1) & B(\psi_1, \psi_2) \\ B(\psi_1, \psi_2) & B(\psi_2, \psi_2) \end{bmatrix}.$$

Now,

$$B(\psi_1, \psi_1) = \int_0^1 \psi_1(x)\psi_1(x)\,dx = \int_0^1 \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\,dx = \frac{1}{2}\int_0^1 1\,dx = \frac{1}{2}\left[x\right]_0^1 = \frac{1}{2}$$

and

$$B(\psi_2, \psi_2) = \int_0^1 \psi_2(x)\psi_2(x)\,dx = \int_0^1 \frac{\sqrt{3}}{\sqrt{2}}x\frac{\sqrt{3}}{\sqrt{2}}x\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{3}{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{3}{2}\frac{1}{3} = \frac{1}{2}.$$

Hence,

$$\mathbf{G} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}$$

and so

$$\mathbf{G}^{-1} = 16\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 8 \end{bmatrix}$$

since

$$\frac{1}{2}\frac{1}{2} - \frac{\sqrt{3}}{4}\frac{\sqrt{3}}{4} = \frac{1}{4} - \frac{3}{16} = \frac{4}{16} - \frac{3}{16} = \frac{1}{16}.$$

Therefore

$$\begin{aligned}
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \mathbf{G}^{-1}\begin{bmatrix} \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{6}}{2\pi} \end{bmatrix} \\
&= \begin{bmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 8 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{6}}{2\pi} \end{bmatrix} \\
&= \begin{bmatrix} 8\frac{\sqrt{2}}{\pi} - 4\sqrt{3}\frac{\sqrt{6}}{2\pi} \\ -4\sqrt{3}\frac{\sqrt{2}}{\pi} + 8\frac{\sqrt{6}}{2\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8\sqrt{2}}{\pi} - \frac{2\sqrt{18}}{\pi} \\ \frac{4\sqrt{6}}{\pi} + \frac{4\sqrt{6}}{\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8\sqrt{2}}{\pi} - \frac{2\sqrt{2}\sqrt{9}}{\pi} \\ \frac{4\sqrt{6}}{\pi} + \frac{4\sqrt{6}}{\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8\sqrt{2}}{\pi} - \frac{6\sqrt{2}}{\pi} \\ \frac{4\sqrt{6}}{\pi} + \frac{4\sqrt{6}}{\pi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\sqrt{2}}{\pi} \\ 0 \end{bmatrix}.
\end{aligned}$$

Consequently, the best approximation to g_1 from span $\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_1(x) = c_1\psi_1(x) + c_2\psi_2(x) = \frac{2\sqrt{2}}{\pi}\frac{1}{\sqrt{2}} + 0 = \frac{2}{\pi}.$$

(d) [6 points] Since $B(\psi_1, \psi_2) \neq 0$, the best approximation to g_2 from span $\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_2(x) = d_1\psi_1(x) + d_2\psi_2(x)$$

where the coefficients $d_1, d_2 \in \mathbb{R}$ are such that

$$\mathbf{G}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} B(g_2, \psi_1) \\ B(g_2, \psi_2) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{\pi^2} \end{bmatrix}$$

where

$$\mathbf{G} = \begin{bmatrix} B(\psi_1,\psi_1) & B(\psi_1,\psi_2) \\ B(\psi_1,\psi_2) & B(\psi_2,\psi_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}.$$

Therefore

$$\begin{aligned}
\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} &= \mathbf{G}^{-1} \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\
&= \begin{bmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\
&= \begin{bmatrix} 4\sqrt{3}\frac{\sqrt{6}}{\pi^2} \\ -8\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\
&= \begin{bmatrix} 4\frac{\sqrt{18}}{\pi^2} \\ -8\frac{\sqrt{6}}{\pi^2} \end{bmatrix} \\
&= \begin{bmatrix} 4\frac{\sqrt{2}\sqrt{9}}{\pi^2} \\ -8\frac{\sqrt{2}\sqrt{3}}{\pi^2} \end{bmatrix} \\
&= \begin{bmatrix} 12\frac{\sqrt{2}}{\pi^2} \\ -8\frac{\sqrt{2}\sqrt{3}}{\pi^2} \end{bmatrix}
\end{aligned}$$

Consequently, the best approximation to g_2 from span $\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$ is

$$\tilde{g}_2(x) = d_1\psi_1(x) + d_2\psi_2(x) = 12\frac{\sqrt{2}}{\pi^2}\frac{1}{\sqrt{2}} - 8\frac{\sqrt{2}\sqrt{3}}{\pi^2}\frac{\sqrt{3}}{\sqrt{2}}x = \frac{12}{\pi^2} - \frac{24}{\pi^2}x = \frac{12}{\pi^2}\left(1-2x\right).$$
