CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 38

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

38. [25 points]

All parts of this question should be done by hand.

Let $H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let $\phi_0 \in H^1(0,1)$ be defined by

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

let $\phi_j \in H^1_D(0,1)$ be defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let $\phi_{N+1} \in H^1_D(0,1)$ be defined by

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the symmetric bilinear form $B(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$B(v, w) = a(v, w) + (v, w).$$

Also, let $f \in L^2(0,1)$, let $\alpha \in \mathbb{R}$ and let $\rho \in \mathbb{R}$. Moreover, let $u \in H^1(0,1)$ be such that $u(0) = \alpha$ and

$$B(u, v) = (f, v) + \rho v(1) \text{ for all } v \in H^1_D(0,1).$$

Let $V_N = \operatorname{span}\{\phi_0, \phi_1, \ldots, \phi_{N+1}\}$ and let $V_{N,D} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_{N+1}\}$. Let $u_N \in V_N$ be such that $u_N(0) = \alpha$ and

$$B(u_N, v) = (f, v) + \rho v(1) \text{ for all } v \in V_{N,D}.$$

(a) We can write

$$u_N = \alpha\phi_0 + \sum_{j=1}^{N+1} c_j \phi_j$$

where, for $j = 1, 2, \ldots, N+1$, c_j is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$ which is the solution to

$$\mathbf{K}\mathbf{c} = \mathbf{b}.$$

What are the entries of the matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ and the vector $\mathbf{b} \in \mathbb{R}^{N+1}$?

(b) Show that

$$B(u - u_N, u - u_N) = B(u, u) - B(u_N, u_N) - 2\alpha B(u - u_N, \phi_0).$$

(c) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = 0$, $\rho = 0$ and $N = 1$. Note that, when $N = 1$,

$$\int_0^{1/2} \phi_0(x)\phi_1(x)\,dx = \int_{1/2}^{1} \phi_1(x)\phi_2(x)\,dx = \frac{1}{12};$$

$$\int_0^{1/2} \phi_0(x)\phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^{1} \phi_1(x)\phi_1(x)\,dx = \int_{1/2}^{1} \phi_2(x)\phi_2(x)\,dx = \frac{1}{6};$$

and

$$\int_0^{1/2} \phi_0(x)\,dx = \int_0^{1/2} \phi_1(x)\,dx = \int_{1/2}^{1} \phi_1(x)\,dx = \int_{1/2}^{1} \phi_2(x)\,dx = \frac{1}{4}.$$

(d) Construct $\mathbf{K}$ and $\mathbf{b}$ in the case when $f(x) = 2$, $\alpha = -1$, $\rho = 1$ and $N = 1$.