CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 8

Posted Wednesday 22 October, 2014. Due 5pm Wednesday 29 October, 2014.

*Please write your name and **residential college** on your homework.*

1. [20 points: 10 points each]

   Let $k(x)$ and $p(x)$ be two positive-valued continuous functions on $[0,1]$, and let

   $$V = \Big\{u \in C^2[0,1] : u(0) = \frac{du}{dx}(1) = 0\Big\}.$$

   (a) Derive the weak form of the differential equation

   $$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) + p(x)u = f(x), \quad 0 < x < 1,$$

   subject to the boundary conditions

   $$u(0) = \frac{du}{dx}(1) = 0;$$

   that is, transform this differential equation into a problem of the form:

   $$\text{Find } u \in V \text{ such that } a(u,v) = (f,v) \text{ for all } v \in V,$$

   where $(\cdot,\cdot)$ denotes the usual inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$, and $a(\cdot,\cdot)$ is some bilinear form that you should specify.

   (b) Show that the form $a(u,v)$ from part (a) is an inner product for $u, v \in V$.

2. [20 points: 10 points each]

Let
$$H_D^1(0,1) = \{w \in H^1(0,1) : w(0) = w(1) = 0\}$$
and let the inner product $(\cdot,\cdot)$ be defined by
$$(v,w) = \int_0^1 v(x)w(x)\,dx$$
and the energy inner product $a(\cdot,\cdot)$ be defined by
$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$
Also, let $f \in L^2(0,1)$, let N be a positive integer, and let V_N be a subspace of $H_D^1(0,1)$. Moreover, let $u \in H_D^1(0,1)$ be such that
$$a(u,v) = (f,v) \text{ for all } v \in H_D^1(0,1)$$
and let $u_N \in V_N$ be such that
$$a(u_N,v) = (f,v) \text{ for all } v \in V_N.$$

(a) Show that
$$a(u-u_N, u-u_N) = a(u,u) - a(u_N,u_N).$$

(b) Let $\phi_1,\ldots,\phi_N \in V_N$ and let $\mathbf{K} \in \mathbb{R}^{N\times N}$ be the matrix with entries $K_{jk} = a(\phi_k,\phi_j)$ for $j,k = 1,\ldots,N$. Also, let
$$u_N = \sum_{j=1}^{N} c_j\phi_j$$
where $c_j \in \mathbb{R}$ is the jth entry of the vector $\mathbf{c} \in \mathbb{R}^N$. Show that
$$\mathbf{c}^T\mathbf{K}\mathbf{c} = a(u_N,u_N).$$

3. [30 points: 6 points each]

Let $f \in C[0,1]$ be such that $f(x) = \sin(\pi x)$. Suppose that N is a positive integer and define $h = \dfrac{1}{N+1}$ and $x_j = jh$ for $j = 0, 1 \ldots, N+1$. Consider the N hat functions $\phi_k \in C[0,1]$, defined as

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k); \\ \dfrac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1, \ldots, N$. Let the inner product $(\cdot, \cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_0^1 u(x)v(x)\, dx$$

and let the norm $\|\cdot\| : C[0,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

(a) For $j = 1, \ldots, N$, what is $\phi_j(x_k)$ for $k = 0, 1, \ldots, N+1$? Simplify your answer as much as possible.

(b) Show that $\{\phi_1, \ldots, \phi_N\}$ is linearly independent by showing that if $c_k \in \mathbb{R}$ and $\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $c_k = 0$ for $k = 1, \ldots, N$.

(c) By hand, compute (f, ϕ_j) for $j = 1, \ldots, N$.

(d) By hand, compute (ϕ_j, ϕ_k) for $j, k = 1, \ldots, N$. Your final answers should be simplified as much as possible and in your formulas h should be left as h and not be replaced with $1/(N+1)$. You must clearly state which values of j and k each formula you obtain is valid for. An acceptable way to present the final answer would be:
For $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} ? & \text{if } k = j, \\ ? & \text{if } |j - k| = 1, \\ ? & \text{otherwise,} \end{cases}$$

with the question marks replaced with the correct values. Hint: Letting $s = x - x_{j-1}$ yields that

$$\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h} \right)^2 dx = \frac{1}{h^2} \int_{x_{j-1} - x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2\, ds = \frac{1}{h^2} \int_0^h s^2\, ds.$$

(e) Set up a linear system (in MATLAB) and solve it to compute the best approximation f_N to f from span$\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ for $N = 3$ and $N = 9$. For each of these N, produce a separate plot that superimposes $f_N(x)$ on top of a plot of $f(x)$. The `hat.m` code (from Homework 2, either your code or the code from the solutions) should help you to produce these plots.

4. [30 points: 6 points each]
Use the finite element method to solve the differential equation

$$-(u'(x)\kappa(x))' = 2x, \qquad 0 < x < 1$$

for $\kappa(x) = 1 + x^2$, subject to homogeneous Dirichlet boundary conditions,

$$u(0) = u(1) = 0,$$

with the approximation space V_N given by the piecewise linear *hat functions* that featured on earlier problem sets: For $n \geq 1$, $h = 1/(N+1)$, and $x_k = kh$ for $k = 0, \ldots, N+1$, we have

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write MATLAB code that constructs the stiffness matrix $\mathbf{K}$ for a given value of N, with $\kappa(x) = 1 + x^2$.

[You may edit the `fem_demo1.m` code from the class website. You should compute all necessary integrals (by hand or using a symbolic package) so as to obtain clean formulas that depend on h and the index of the hat functions involved (e.g.,$a(\phi_j, \phi_j)$ can depend on j).]

(b) Write MATLAB code that constructs the load vector $\mathbf{f}$ for a given value of N, with $f(x) = 2x$.

(c) For $N = 7$ and $N = 15$, produce plots comparing your solution u_N to the true solution

$$u(x) = (4/\pi)\tan^{-1}(x) - x.$$

(Note that you can compute $\tan^{-1}(x)$ as `atan(x)` in MATLAB.)

(d) Produce a `loglog` plot showing how the error

$$\max_{x \in [0,1]} |u_N(x) - u(x)|$$

decreases as N increases. (For example, take $N = 8, 16, 32, 64, 128, 256, 512$.) On the same plot, show N^{-2} for the same values of N. If your code from parts (a) and (b) is working, your error curve should have the same slope as the N^{-2} curve. (Consult the `fem_demo1.m` code on the website for a demonstration of the style of plot we intend for part (d); edit this code as you like.)