CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 7

Posted Wednesday 3 October 2012. Due Wednesday 10 October 2012, 5pm.

Please write the name of your college on your paper.

This problem set counts for 75 points, i.e., three-quarters the value of the earlier problem sets.
Because of the Centennial, late papers are due at 5pm on Monday, 15 October.

General advice: You may compute any integrals you encounter using symbolic mathematics tools such as WolframAlpha, Mathematica, or the Symbolic Math Toolbox in MATLAB.

1. [50 points]
   Use the finite element method to solve the differential equation

$$-(u'(x)\kappa(x))' = 2x, \qquad 0 < x < 1$$

for $\kappa(x) = 1 + x^2$, subject to homogeneous Dirichlet boundary conditions,

$$u(0) = u(1) = 0,$$

with the approximation space V_N given by the piecewise linear *hat functions* that featured on earlier problem sets: For $n \geq 1$, $h = 1/(N+1)$, and $x_k = kh$ for $k = 0, \ldots, N+1$, we have

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write MATLAB code that constructs the stiffness matrix $\mathbf{K}$ for a given value of N, with $\kappa(x) = 1 + x^2$.
[You may edit the `fem_demo1.m` code from the class website. You should compute all necessary integrals (by hand or using a symbolic package) so as to obtain clean formulas that depend on h and the index of the hat functions involved (e.g.,$a(\phi_j, \phi_j)$ can depend on j).]

(b) Write MATLAB code that constructs the load vector $\mathbf{f}$ for a given value of N, with $f(x) = 2x$.

(c) For $N = 7$ and $N = 15$, produce plots comparing your solution u_N to the true solution

$$u(x) = (4/\pi)\tan^{-1}(x) - x.$$

(Note that you can compute $\tan^{-1}(x)$ as `atan(x)` in MATLAB.)

(d) Produce a `loglog` plot showing how the error

$$\max_{x \in [0,1]} |u_N(x) - u(x)|$$

decreases as N increases. (For example, take $N = 8, 16, 32, 64, 128, 256, 512$.) On the same plot, show N^{-2} for the same values of N. If your code from parts (a) and (b) is working, your error curve should have the same slope as the N^{-2} curve. (Consult the `fem_demo1.m` code on the website for a demonstration of the style of plot we intend for part (d); edit this code as you like.)

2. [25 points]
Let $k(x)$ and $p(x)$ be two positive-valued continuous functions on $[0,1]$, and let

$$V = \Big\{u \in C^2[0,1] : u(0) = \frac{du}{dx}(1) = 0\Big\}.$$

(a) Derive the weak form of the differential equation

$$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) + p(x)u = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions

$$u(0) = \frac{du}{dx}(1) = 0;$$

that is, transform this differential equation into a problem of the form:

$$\text{Find } u \in V \text{ such that } a(u,v) = (f,v) \text{ for all } v \in V,$$

where $(\cdot,\cdot)$ denotes the usual inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$, and $a(\cdot,\cdot)$ is some bilinear form that you should specify.

(b) Show that the form $a(u,v)$ from part (a) is an inner product for $u, v \in V$.

3. [25 points]
A classical problem in quantum mechanics models a particle moving in an infinite square well, subject to an infinite potential at a point. The result is a Schrödinger operator posed on $C_D^2[0,1]$ of the form

$$Lu = -u'' + \delta_{1/2}u,$$

where $\delta_{1/2}$ is a "delta function" centered at the location of the infinite potential, $x = 1/2$. A beautiful theory supports these exotic functions (more properly called *distributions*). For this problem, you need only know the following fact: for any function $g \in C[0,1]$,

$$\int_0^1 \delta_{1/2}(x)g(x)\ dx = g(1/2).$$

The equation $Lu = f$ has the equivalent weak form

$$a(u,w) = (f,w) \qquad \text{for all } w \in V = C_D^2[0,1],$$

where

$$a(u,w) = \int_0^1 \Big(u'(x)w'(x) + \delta_{1/2}(x)u(x)w(x)\Big)\,dx.$$

We wish to use the Galerkin method to approximate solutions to $Lu = f$ from the finite dimensional subspace $V_N = \operatorname{span}\{\phi_1, \ldots, \phi_N\}$. Use as basis vectors the eigenfunctions from the problem without the potential at $x = 1/2$:

$$\phi_k(x) = \sqrt{2}\sin(k\pi x).$$

(a) Compute a general formula for $a(\phi_j, \phi_k)$.

(b) Write out (by hand) the stiffness matrix for $N = 5$.

(c) Write down a general formula for the entries in the load vector, (f, ϕ_k), when $f(x) = 1$. (You may use formulas from prior homework.)

(d) Plot your approximate solutions to $-u''(x) + \delta_{1/2}(x)u(x) = 1$ for $N = 5$ and $N = 35$.

4. [25 points]
Consider the *Euler–Bernoulli beam equation*,

$$(\kappa(x)u''(x))'' = f(x), \qquad 0 < x < 1,$$

with boundary conditions describing a beam that is *clamped* at both ends:

$$u(0) = u(1) = 0, \qquad u'(0) = u'(1) = 0.$$

Here $\kappa(x)$ is a positive-valued function that describes the material properties of the beam.

With these boundary conditions, the eigenvalues and eigenvectors of this operator are difficult to compute – even if $\kappa(x) \equiv 1$. We will consider finite element solutions of this problem.

(a) Derive the weak form of the beam equation with the above boundary conditions, i.e., derive the weak problem

$$a(u, v) = (f, v), \quad \text{for all } v \in V = C_D^4[0, 1],$$

where

$$C_D^4[0, 1] = \{u \in C^4[0, 1] : u(0) = u(1) = u'(0) = u'(1) = 0\}.$$

Specify the bilinear form $a(u, v)$, and show that it is an inner product on $C_D^4[0, 1]$.
(Note: for the problem $-(u'\kappa)' = f$, we do not explicitly impose Neumann boundary conditions – they follow 'naturally'. For the beam equation, we must impose all four boundary conditions on the space of test functions, $V = C_D^4[0, 1]$.)

(b) Suppose that $V_n = \text{span}\{\phi_1, \ldots, \phi_n\}$ is an n-dimensional subspace of $C_D^4[0, 1]$.
(Do not assume a particular form for the functions $\phi_1, \ldots, \phi_n$ at this point.)
Show how the Galerkin problem

$$a(u_n, v) = (f, v), \quad \text{for all } v \in V_n$$

leads to the linear system $\mathbf{K}\mathbf{u} = \mathbf{f}$. Be sure to specify the entries of $\mathbf{K}$, $\mathbf{u}$, and $\mathbf{f}$.

(c) Now suppose we take for $\phi_1, \ldots, \phi_n$ the standard piecewise linear 'hat' functions used, for example, in Problem 1. Are these functions suitable for this problem? If so, describe the location of the nonzero entries of the matrix $\mathbf{K}$. If not, *roughly* describe a better choice for the functions $\phi_1, \ldots, \phi_n$ and the explain which entries of $\mathbf{K}$ are nonzero for that choice.