

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 31 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 28 March 2014.

31. [25 points]

Let

$$H_D^1(0, 1) = \{w \in H^1(0, 1) : w(0) = w(1) = 0\}$$

and let the inner product $a(\cdot, \cdot) : H_D^1(0, 1) \times H_D^1(0, 1) \rightarrow \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x) dx.$$

Let N be a positive integer, let $h = \frac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \dots, N+1$. Let the continuous piecewise linear hat functions $\hat{\phi}_j \in H_D^1(0, 1)$ be such that

$$\hat{\phi}_j(x) = \begin{cases} \frac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j], \\ \frac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}], \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \dots, N$. Also, let the continuous piecewise quadratic functions $\phi_j \in H_D^1(0, 1)$ be such that

$$\phi_j(x) = \begin{cases} \frac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j], \\ \frac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}], \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \dots, N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in H_D^1(0, 1)$ be such that

$$\psi_j(x) = \begin{cases} \frac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j], \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \dots, N+1$.

(a) By hand, obtain formulas for

- i. $a(\phi_j, \phi_k)$ for $j, k = 1, \dots, N$;
- ii. $a(\psi_j, \psi_k)$ for $j, k = 1, \dots, N+1$;
- iii. $a(\phi_j, \psi_k)$ for $j = 1, \dots, N$ and $k = 1, \dots, N+1$.

Your final answers should be in terms of h and you must clearly state which values of j and k each formula you obtain is valid for. For example, if you were asked to obtain formulas for $a(\hat{\phi}_j, \hat{\phi}_k)$

for $j, k = 1, \dots, N$ then an acceptable way to present the answer would be:
 For $j, k = 1, \dots, N$,

$$a(\widehat{\phi}_j, \widehat{\phi}_k) = \begin{cases} \frac{2}{h} & \text{if } k = j, \\ -\frac{1}{h} & \text{if } |j - k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution.

(a) [25 points]

i. For $j = 1, \dots, N$,

$$\begin{aligned} \phi'_j(x) &= \begin{cases} \frac{(2x - x_{j-1} - x_j) + 2(x - x_{j-1})}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ \frac{-2(x_{j+1} - x) - (x_j + x_{j+1} - 2x)}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{4x - 3x_{j-1} - x_j}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ \frac{4x - 3x_{j+1} - x_j}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and so, for $j = 1, \dots, N$, letting $s = x - x_{j-1}$ and $t = x - x_{j+1}$ yields that

$$\begin{aligned} &a(\phi_j, \phi_j) \\ &= \int_0^1 (\phi'_j(x))^2 dx \\ &= \int_{x_{j-1}}^{x_j} \left(\frac{4x - 3x_{j-1} - x_j}{h^2} \right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{4x - 3x_{j+1} - x_j}{h^2} \right)^2 dx \\ &= \int_{x_{j-1}-x_{j-1}}^{x_j-x_{j-1}} \left(\frac{4(s + x_{j-1}) - 3x_{j-1} - x_j}{h^2} \right)^2 ds + \int_{x_j-x_{j+1}}^{x_{j+1}-x_{j+1}} \left(\frac{4(t + x_{j+1}) - 3x_{j+1} - x_j}{h^2} \right)^2 dt \\ &= \int_0^h \left(\frac{4s + x_{j-1} - x_j}{h^2} \right)^2 ds + \int_{-h}^0 \left(\frac{4t + x_{j+1} - x_j}{h^2} \right)^2 dt \\ &= \int_0^h \left(\frac{4s - h}{h^2} \right)^2 ds + \int_{-h}^0 \left(\frac{4t + h}{h^2} \right)^2 dt \\ &= \left[\frac{1}{3} \frac{h^2}{4} \left(\frac{4s - h}{h^2} \right)^3 \right]_0^h + \left[\frac{1}{3} \frac{h^2}{4} \left(\frac{4t + h}{h^2} \right)^3 \right]_{-h}^0 \\ &= \frac{h^2}{12} \left(\frac{3h}{h^2} \right)^3 - \frac{h^2}{12} \left(\frac{-h}{h^2} \right)^3 + \frac{h^2}{12} \left(\frac{h}{h^2} \right)^3 - \frac{h^2}{12} \left(\frac{-3h}{h^2} \right)^3 \\ &= \frac{h^2}{12} \left(\frac{27h^3}{h^6} + \frac{h^3}{h^6} + \frac{h^3}{h^6} + \frac{27h^3}{h^6} \right) \\ &= \frac{h^2}{12} \frac{56}{h^3} \\ &= \frac{14}{3h}. \end{aligned}$$

Moreover, for $j = 1, \dots, N-1$,

$$\phi'_{j+1}(x) = \begin{cases} \frac{4x - 3x_j - x_{j+1}}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ \frac{4x - 3x_{j+2} - x_{j+1}}{h^2} & \text{if } x \in (x_{j+1}, x_{j+2}), \\ 0 & \text{otherwise,} \end{cases}$$

and so, for $j = 1, \dots, N-1$, letting $s = x - x_j$ yields that

$$\begin{aligned} a(\phi_{j+1}, \phi_j) &= a(\phi_j, \phi_{j+1}) \\ &= \int_0^1 \phi'_j(x) \phi'_{j+1}(x) dx \\ &= \int_{x_j}^{x_{j+1}} \frac{4x - 3x_{j+1} - x_j}{h^2} \frac{4x - 3x_j - x_{j+1}}{h^2} dx \\ &= \int_{x_j - x_j}^{x_{j+1} - x_j} \frac{4(s + x_j) - 3x_{j+1} - x_j}{h^2} \frac{4(s + x_j) - 3x_j - x_{j+1}}{h^2} ds \\ &= \int_0^h \frac{4s - 3h}{h^2} \frac{4s - h}{h^2} ds \\ &= \frac{1}{h^4} \int_0^h 16s^2 - 16hs + 3h^2 ds \\ &= \frac{1}{h^4} \left[\frac{16}{3} s^3 - 8hs^2 + 3h^2 s \right]_0^h \\ &= \frac{1}{h^4} \left(\frac{16}{3} h^3 - 8h^3 + 3h^3 \right) \\ &= \frac{1}{h^4} \frac{1}{3} h^3 \\ &= \frac{1}{3h}. \end{aligned}$$

Finally, we note that, for $j, k = 1, \dots, N$,

$$a(\phi_j, \phi_k) = 0 \text{ if } |j - k| > 1.$$

Therefore, for $j, k = 1, \dots, N$,

$$a(\phi_j, \phi_k) = \begin{cases} \frac{14}{3h} & \text{if } k = j, \\ \frac{1}{3h} & \text{if } |j - k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

ii. For $j = 1, \dots, N+1$,

$$\begin{aligned} \psi'_j(x) &= \begin{cases} \frac{4(x_j - x) - 4(x - x_{j-1})}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{4(x_j + x_{j-1} - 2x)}{h^2} & \text{if } x \in (x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and so, for $j = 1, \dots, N + 1$, letting $s = x - x_{j-1}$ yields that

$$\begin{aligned}
a(\psi_j, \psi_j) &= \int_0^1 (\psi'_j(x))^2 dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{4(x_j + x_{j-1} - 2x)}{h^2} \right)^2 dx \\
&= \int_{x_{j-1}-x_{j-1}}^{x_j-x_{j-1}} \left(\frac{4(x_j + x_{j-1} - 2(s + x_{j-1}))}{h^2} \right)^2 ds \\
&= \int_0^h \left(\frac{4(x_j - x_{j-1} - 2s)}{h^2} \right)^2 ds \\
&= \frac{16}{h^4} \int_0^h (h - 2s)^2 ds \\
&= \frac{16}{h^4} \left[-\frac{1}{6} (h - 2s)^3 \right]_0^h \\
&= \frac{16}{h^4} \left(-\frac{1}{6} (-h)^3 + \frac{1}{6} h^3 \right) \\
&= \frac{16}{h^4} \left(\frac{1}{6} h^3 + \frac{1}{6} h^3 \right) \\
&= \frac{16}{h^4} \frac{h^3}{3} \\
&= \frac{16}{3h}.
\end{aligned}$$

Finally, we note that, for $j, k = 1, \dots, N + 1$,

$$a(\psi_j, \psi_k) = 0 \text{ if } j \neq k.$$

Therefore, for $j, k = 1, \dots, N + 1$,

$$a(\psi_j, \psi_k) = \begin{cases} \frac{16}{3h} & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

iii. For $j = 1, \dots, N$, letting $s = x - x_{j-1}$ yields that

$$\begin{aligned}
a(\phi_j, \psi_j) &= \int_0^1 \phi'_j(x) \psi'_j(x) dx \\
&= \int_{x_{j-1}}^{x_j} \frac{4x - 3x_{j-1} - x_j}{h^2} \frac{4(x_j + x_{j-1} - 2x)}{h^2} dx \\
&= \frac{4}{h^4} \int_{x_{j-1}-x_{j-1}}^{x_j-x_{j-1}} (4s - h)(h - 2s) ds \\
&= \frac{4}{h^4} \int_0^h 6hs - 8s^2 - h^2 ds \\
&= \frac{4}{h^4} \left[3hs^2 - \frac{8}{3}s^3 - h^2s \right]_0^h \\
&= \frac{4}{h^4} \left(3h^3 - \frac{8}{3}h^3 - h^3 \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{4}{h^4} \left(-\frac{2}{3} h^3 \right) \\
&= -\frac{8}{3h}.
\end{aligned}$$

Moreover, for $j = 1, \dots, N$,

$$\psi'_{j+1}(x) = \begin{cases} \frac{4(x_{j+1} + x_j - 2x)}{h^2} & \text{if } x \in (x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

and so, for $j = 1, \dots, N$, letting $s = x - x_j$ yields that

$$\begin{aligned}
a(\phi_j, \psi_{j+1}) &= \int_0^1 \phi'_j(x) \psi'_{j+1}(x) dx \\
&= \int_{x_j}^{x_{j+1}} \frac{4x - 3x_{j+1} - x_j}{h^2} \frac{4(x_{j+1} + x_j - 2x)}{h^2} dx \\
&= \int_{x_j - x_j}^{x_{j+1} - x_j} \frac{4(s + x_j) - 3x_{j+1} - x_j}{h^2} \frac{4(x_{j+1} + x_j - 2(s + x_j))}{h^2} ds \\
&= \frac{4}{h^4} \int_0^h (4s - 3h)(h - 2s) ds \\
&= \frac{4}{h^4} \int_0^h 10hs - 8s^2 - 3h^2 ds \\
&= \frac{4}{h^4} \left[5hs^2 - \frac{8}{3}s^3 - 3h^2s \right]_0^h \\
&= \frac{4}{h^4} \left(5h^3 - \frac{8}{3}h^3 - 3h^3 \right) \\
&= \frac{4}{h^4} \left(-\frac{2}{3}h^3 \right) \\
&= -\frac{8}{3h}.
\end{aligned}$$

Finally, we note that, for $j = 1, \dots, N$ and $k = 1, \dots, N + 1$,

$$a(\phi_j, \psi_k) = 0 \text{ if } k - j > 1 \text{ or } k - j < 0.$$

Therefore, for $j = 1, \dots, N$ and $k = 1, \dots, N + 1$,

$$a(\phi_j, \psi_k) = \begin{cases} -\frac{8}{3h} & \text{if } k = j \text{ or } k = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$
