CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 14 · Solutions

Posted Friday 7 February 2014. Due 1pm Friday 14 February 2014.

14. [25 points]

Determine whether or not each of the following mappings is an inner product on the real vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

(a) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 u(x)v'(x)\,dx$ where $\mathcal{V}=C^1[0,1]$.

(b) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 |u(x)||v(x)|\,dx$ where $\mathcal{V}=C[0,1]$.

(c) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 u(x)v(x)e^{-x}\,dx$ where $\mathcal{V}=C[0,1]$.

(d) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_0^1 (u(x)+v(x))\,dx$ where $\mathcal{V}=C[0,1]$.

(e) $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ defined by $(u,v)=\int_{-1}^1 xu(x)v(x)\,dx$ where $\mathcal{V}=C[-1,1]$.

Solution.

(a) [5 points] *This mapping is not an inner product*: it is not symmetric and it is not positive definite. The mapping is not symmetric. For example, if $u(x)=1$ and $v(x)=x$, then

$$(u,v)=\int_0^1 u(x)v'(x)\,dx=\int_0^1 1\,dx=1,$$

yet

$$(v,u)=\int_0^1 v(x)u'(x)\,dx=\int_0^1 0\,dx=0.$$

The mapping is also not positive definite. For example, if $u(x)=1$, then $(u,u)=0$ and if $u(x)=1-x$, then

$$(u,u)=\int_0^1 (1-x)(-1)\,dx=-1/2.$$

For what it is worth, we note that the mapping is linear in the first argument since

$$(\alpha u+\beta v,w)=\alpha\int_0^1 u(x)w'(x)\,dx+\beta\int_0^1 v(x)w'(x)\,dx=\alpha(u,w)+\beta(v,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$. It is also linear in the second argument since

$$(u,\alpha v+\beta w)=\alpha\int_0^1 u(x)v'(x)\,dx+\beta\int_0^1 u(x)w'(x)\,dx=\alpha(u,v)+\beta(u,w)$$

for all $u,v,w\in C^1[0,1]$ and all $\alpha,\beta\in\mathbb{R}$.

(b) [5 points] *This mapping is not an inner product*: it is not linear in the first argument.
If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 |\alpha u(x) + \beta v(x)||w(x)|\, dx$$

and

$$\alpha(u, w) + \beta(v, w) = \alpha \int_0^1 |u(x)||w(x)|\, dx + \beta \int_0^1 |v(x)||w(x)|\, dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = -1$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 |-1||1|\, dx = \int_0^1 1\, dx = 1$$

but

$$\alpha(u, w) + \beta(v, w) = -\int_0^1 |1||1|\, dx = -\int_0^1 1\, dx = -1$$

and so the mapping is not linear in the first argument.
The mapping is symmetric, as

$$(u, v) = \int_0^1 |u(x)||v(x)|\, dx = \int_0^1 |v(x)||u(x)|\, dx = (v, u)$$

for all $u, v \in C[0,1]$.
Moreover, the mapping is positive definite as for all $u \in C[0,1]$

$$(u, u) = \int_0^1 |u(x)|^2\, dx$$

is the integral of a nonnegative function, and hence is nonnegative and $(u, u) = 0$ only if $u = 0$.

(c) [5 points] *This mapping is an inner product.*
The mapping is symmetric, as

$$(u, v) = \int_0^1 u(x)v(x)e^{-x}\, dx = \int_0^1 v(x)u(x)e^{-x}\, dx = (v, u)$$

for all $u, v \in C[0,1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}(\alpha u + \beta v, w) &= \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x}\, dx \\ &= \alpha \int_0^1 u(x)w(x)e^{-x}\, dx + \beta \int_0^1 v(x)w(x)e^{-x}\, dx \\ &= \alpha(u, w) + \beta(v, w)\end{aligned}$$

for all $u, v, w \in C[0,1]$ and all $\alpha, \beta \in \mathbb{R}$.
The function e^{-x} is positive valued for all $x \in [0,1]$, so we have that

$$(u, u) = \int_0^1 (u(x))^2 e^{-x} dx$$

is the integral of a nonnegative function, and hence is also nonnegative. If $(u, u) = 0$ then $(u(x))^2 e^{-x} = 0$ for all $x \in [0,1]$ and, since $e^{-x} > 0$ for all $x \in [0,1]$, this means that $u(x) = 0$ for all $x \in [0,1]$, i.e., $u = 0$. Hence, the mapping is positive definite.

(d) [5 points] *This mapping is not an inner product*: it is not linear in the first argument and it is not positive definite.
If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x) + w(x))\, dx$$

and

$$\alpha(u,w) + \beta(v,w) = \alpha \int_0^1 (u(x) + w(x))\, dx + \beta \int_0^1 (v(x) + w(x))\, dx.$$

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = 2$ and $\beta = 0$ then

$$(\alpha u + \beta v, w) = \int_0^1 (2+1)\, dx = \int_0^1 3\, dx = 3$$

but

$$\alpha(u,w) + \beta(v,w) = 2\int_0^1 (1+1)\, dx = 2\int_0^1 2\, dx = 4$$

and so $(\cdot,\cdot)$ is not linear in the first argument.
The mapping $(\cdot,\cdot)$ is also not positive definite. For example, if $u(x) = -1$, then

$$(u,u) = \int_0^1 (u(x) + u(x))\, dx = \int_0^1 -2\, dx = -2 < 0.$$

The mapping is symmetric, as

$$(u,v) = \int_0^1 (u(x) + v(x))\, dx = \int_0^1 (v(x) + u(x))\, dx = (v,u)$$

for all $u, v \in C[0,1]$.

(e) [5 points] *This mapping is not an inner product*: it is not positive definite.
If $w(x) = 1$ for all $x \in [-1,1]$ then $w \in C[-1,1]$ and $w \neq 0$ but

$$(w,w) = \int_{-1}^1 x w(x) w(x)\, dx = \int_{-1}^1 x\, dx = \left[\frac{1}{2}x^2\right]_{-1}^1 = \frac{1}{2}\left(1^2 - (-1)^2\right) = \frac{1}{2}(1-1) = 0$$

and so $(\cdot,\cdot)$ is not positive definite.
The mapping is symmetric, as

$$(u,v) = \int_{-1}^1 x u(x) v(x)\, dx = \int_{-1}^1 x v(x) u(x)\, dx = (v,u)$$

for all $u, v \in C[-1,1]$.
The mapping is also linear in the first argument since

$$\begin{aligned}
(\alpha u + \beta v, w) &= \int_{-1}^1 x(\alpha u(x) + \beta v(x)) w(x)\, dx \\
&= \alpha \int_{-1}^1 x u(x) w(x)\, dx + \beta \int_{-1}^1 x v(x) w(x)\, dx \\
&= \alpha(u,w) + \beta(v,w)
\end{aligned}$$

for all $u, v, w \in C[-1,1]$ and all $\alpha, \beta \in \mathbb{R}$.