CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 43

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

43. [25 points]

Let

$$\Omega = \{(x,y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$

and let $f(x,y,t) = (x-\frac{1}{2})^3(y-\frac{1}{2})e^{-t}$. Note that, for $m,n = 1,2,\ldots$,

$$\int_0^1\int_0^1 2f(x,y,t)\sin(m\pi x)\sin(n\pi y)\,dx\,dy = \frac{(1+(-1)^m)(1+(-1)^n)(m^2\pi^2-24)}{8m^3n\pi^4}e^{-t}.$$

In this question we will consider the problem of finding the solution $u(x,y,t)$ to the heat equation

$$u_t(x,y,t) - (u_{xx}(x,y,t) + u_{yy}(x,y,t)) = f(x,y,t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

with homogeneous Dirichlet boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

and initial condition

$$u(x,y,0) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Let

$$C_D^2(\Omega) = \left\{ v \in C^2(\Omega) : v(x,0) = v(x,1) = v(0,y) = v(1,y) = 0,\, 0 \le x \le 1,\, 0 \le y \le 1 \right\}.$$

Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by

$$(Lv)(x,y) = -\left(v_{xx}(x,y) + v_{yy}(x,y)\right).$$

The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

for $j,k = 1,2,\ldots$, which are such that

$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$

for $j,k = 1,2,\ldots$. Recall that in Homework 42 you obtained a formula for $\lambda_{j,k}$ for $j,k = 1,2,\ldots$.

(a) We can write

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y)$$

and

$$f(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x,y)$$

where

$$c_{j,k}(t) = \int_0^1\int_0^1 f(x,y,t)\psi_{j,k}(x,y)\,dx\,dy.$$

What ordinary differential equation and initial condition does $a_{j,k}(t)$ satisfy for $j,k = 1,2,\ldots$?

(b) Obtain an expression for $a_{j,k}(t)$ for $j, k = 1, 2, \ldots$.

(c) Use you answer to part (b) to write out a formula for $u(x, y, t)$.

(d) Plot

$$u_{15}(x, y, t) = \sum_{j=1}^{15} \sum_{k=1}^{15} a_{j,k}(t)\psi_{j,k}(x, y)$$

at the six times $t = 0, 0.001, 0.01, 0.1, 1, 2$. Use the command `zlim([-.00015 .00015])` so that the axes on all of your plots are the same. Your plot for $t = 0.1$ should resemble the plot below.