Example for Fourier series Lecture:

* See example 5.5, pg 143.

Suppose that $f(x) = x(1-x)$. Notice that $f(0) = f(1) = 0$.

find the Fourier Sine series for $f(x)$ on the interval $[0,1]$.

- We know the Fourier Sine series for $f(x)$ on $[0,1]$ has the form: $\hat{f}(x) = \sum_{n=1}^{\infty} b_n \sqrt{2} \sin(n\pi x)$ where $b_n = \int_0^1 f(x) \sqrt{2} \sin(n\pi x)$

So we need to compute

$$\int_0^1 x(1-x) \sin(n\pi x) = \int_0^1 x \sin(n\pi x) - \int x^2 \sin(n\pi x)$$

$$\int_0^1 x \sin(n\pi x) = \frac{-x\cos(n\pi x)}{n\pi}\Big|_0^1 - \int_0^1 \left(-\frac{\cos(n\pi x)}{n\pi}\right) dx$$

$vu' = uv - v'u$

$$= -\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\, dx$$

$$= -\frac{(-1)^n}{n\pi} + \frac{1}{(n\pi)^2} \sin(n\pi x)\Big|_0^1$$

$$= \frac{(-1)^{n+1}}{n\pi} + 0.$$

Likewise $\int_0^1 x^2 \sin(n\pi x) = \dfrac{(2-\pi^2 n^2)\cos(\pi n) + 2\sin(\pi n) - 2}{\pi^3 n^3}$

$$= \frac{(2-\pi^2 n^2)(-1)^n + 0 - 2}{\pi^3 n^3}$$

$$= \frac{-2(-1)^{n+1} - 2}{\pi^3 n^3} + \frac{(-1)^{n+1}}{\pi n}$$

$$= \frac{-2\left((-1)^{n+1} - 1\right)}{\pi^3 n^3} + \frac{(-1)^{n+1}}{\pi n}$$

Combining these terms gives

$$\hat{f}(x) \approx \sum_{n=1}^{\infty} \left(\sqrt{2} \int_0^1 x(1-x) \sin(n\pi x)\, dx\right) \sqrt{2} \sin(n\pi x)\, dx$$

$$= \sum_{n=1}^{\infty} \left(2\left\{\int_0^1 x \sin(n\pi x)\, dx - \int_0^1 x^2 \sin(n\pi x)\, dx\right\}\right) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} \left(2\left\{\frac{(-1)^{n+1}}{n\pi} - \left(\frac{-2\left((-1)^{n+1}-1\right)}{\pi^3 n^3} + \frac{(-1)^{n+1}}{\pi n}\right\}\right)\right) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} \left[\frac{4\left((-1)^{n+1}-1\right)}{\pi^3 n^3}\right] \sin(n\pi x)$$

Note: The fourier series is technically:

$$\sum_{n=1}^{\infty} b_n \tilde{\psi}_n = \sum_{n=1}^{\infty} b_n \left(\sqrt{\tfrac{2}{\ell}} \sin(n\pi x)\right) \quad \text{where} \quad b_n = (f, \tilde{\psi}_n).$$

However it is commonplace to take <u>any</u> constant coefficient from $\tilde{\psi}_n$ and group it with b_n. In this case the $\sqrt{\tfrac{2}{\ell}}$ So you will often see

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi}{\ell} x\right) dx$$