

# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 27 · Solutions

Posted Wednesday 9 October 2013. Due 1pm Friday 18 October 2013.

27. [25 points] Let the inner product  $(\cdot, \cdot) : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$(v, w) = \int_0^1 v(x)w(x) dx$$

and let the norm  $\|\cdot\| : C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let the linear operator  $L : C_D^2[0, 1] \rightarrow C[0, 1]$  be defined by

$$Lv = -v''$$

where

$$C_D^2[0, 1] = \{w \in C^2[0, 1] : w(0) = w(1) = 0\}.$$

Recall that the operator  $L$  has eigenvalues

$$\lambda_n = n^2\pi^2$$

with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2} \sin(n\pi x)$$

for  $n = 1, 2, \dots$ . Let  $N$  be a positive integer, let  $f \in C[0, 1]$  be defined by  $f(x) = x^2(1 - x)$  and let  $u$  be the solution to

$$Lu = f.$$

- (a) Compute the best approximation  $f_N$  to  $f$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$ .
- (b) Use the spectral method to compute the best approximation  $u_N$  to  $u$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$ .
- (c) Produce a plot comparing  $f$  to  $f_N$  for  $N = 1, 2, 3, 4, 5, 6$ .
- (d) Plot the approximations  $u_N$  to  $u$  that you obtained using the spectral method for  $N = 1, 2, 3, 4, 5, 6$ .
- (e) Write down the series solution to

$$Lu = f$$

that is obtained using the spectral method.

- (f) By shifting the data and then using a series solution that you have obtained previously in this question, obtain a series solution to the problem of finding  $\tilde{u} \in C^2[0, 1]$  such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{100}$$

and

$$\tilde{u}(1) = \frac{1}{100}.$$

- (g) Let  $\tilde{u}_N$  be the series solution that you obtained in part (f) but with  $\infty$  replaced by  $N$ . Plot  $\tilde{u}_N$  for  $N = 1, 2, 3, 4, 5, 6$ .

**Solution.**

- (a) [5 points] Since  $f(x) = x^2(1-x) = x^2 - x^3$ , we have that, for  $k = 1, 2, \dots$ ,

$$\begin{aligned}
 (f, \psi_k) &= \sqrt{2} \int_0^1 (x^2 - x^3) \sin(k\pi x) dx \\
 &= \sqrt{2} \left( \left[ -\frac{1}{k\pi} (x^2 - x^3) \cos(k\pi x) \right]_0^1 + \frac{1}{k\pi} \int_0^1 (2x - 3x^2) \cos(k\pi x) dx \right) \\
 &= \frac{\sqrt{2}}{k\pi} \int_0^1 (2x - 3x^2) \cos(k\pi x) dx \\
 &= \frac{\sqrt{2}}{k\pi} \left( \left[ \frac{1}{k\pi} (2x - 3x^2) \sin(k\pi x) \right]_0^1 - \frac{1}{k\pi} \int_0^1 (2 - 6x) \sin(k\pi x) dx \right) \\
 &= -\frac{\sqrt{2}}{k^2\pi^2} \int_0^1 (2 - 6x) \sin(k\pi x) dx \\
 &= -\frac{\sqrt{2}}{k^2\pi^2} \left( \left[ -\frac{1}{k\pi} (2 - 6x) \cos(k\pi x) \right]_0^1 - \frac{6}{k\pi} \int_0^1 \cos(k\pi x) dx \right) \\
 &= -\frac{\sqrt{2}}{k^2\pi^2} \left( \frac{4}{k\pi} \cos(k\pi) + \frac{2}{k\pi} - \frac{6}{k\pi} \left[ \frac{1}{k\pi} \sin(k\pi x) \right]_0^1 \right) \\
 &= \frac{-2\sqrt{2}}{k^3\pi^3} (1 + 2 \cos(k\pi)).
 \end{aligned}$$

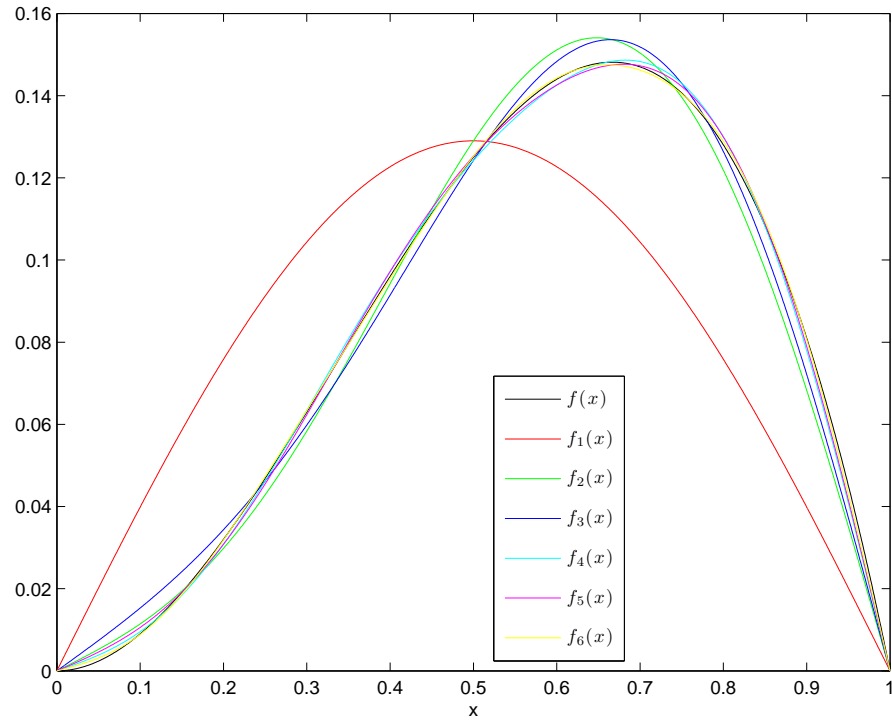
Hence, the best approximation to  $f$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$  is

$$\begin{aligned}
 f_N(x) &= \sum_{j=1}^N (f, \psi_j) \psi_j(x) \\
 &= \sum_{j=1}^N \frac{-2\sqrt{2}}{j^3\pi^3} (1 + 2 \cos(j\pi)) \sqrt{2} \sin(j\pi x) \\
 &= \sum_{j=1}^N \frac{-4}{j^3\pi^3} (1 + 2 \cos(j\pi)) \sin(j\pi x).
 \end{aligned}$$

- (b) [6 points] The best approximation to  $u$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$  is

$$u_N(x) = \sum_{j=1}^N \frac{(f, \psi_j)}{\lambda_j} \psi_j(x) = \sum_{j=1}^N \frac{-4}{j^5\pi^5} (1 + 2 \cos(j\pi)) \sin(j\pi x).$$

- (c) [2 points] The requested plot is below. Note that the function  $f$  happens to satisfy homogeneous Dirichlet boundary conditions, and convergence is quite quick.

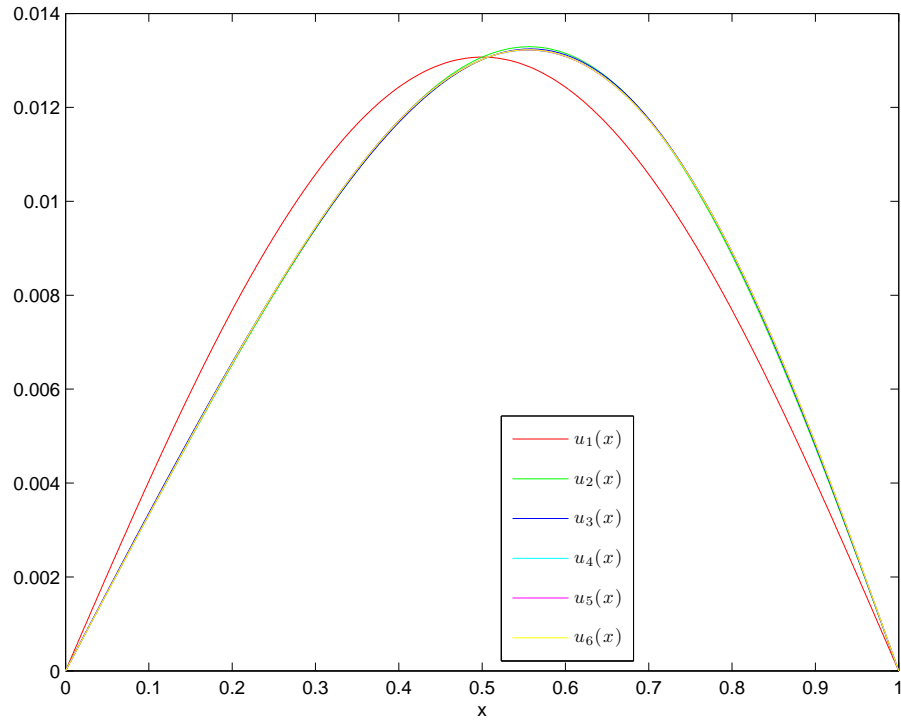


The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(1)
clf
legendStr{1}=['$f(x)$'];
plot(x,(x.^2).*(1-x),'k-')
hold on
fk = zeros(size(x));
for k=1:6
    fk = fk + 2*(4*(-1).^(k+1)-2)./(k.^3*pi^3)*sin(k*pi*x);
    plot(x,fk,colors(k))
    legendStr{k+1}=['$f_{'$ num2str(k) '$}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(gcf,'hw27c','eps')
```

(d) [2 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(2)
clf
uk = zeros(size(x));
for k=1:6
    uk = uk + 2*(4*(-1).^(k+1)-2)./(k.^5*pi^5)*sin(k*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=[ '$u_{' num2str(k) ' }(x)$' ];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(2),'hw27d','eps')
```

(e) [2 points] The series solution that we obtain using the spectral method is

$$u(x) = \sum_{j=1}^{\infty} \frac{-4}{j^5 \pi^5} (1 + 2 \cos(j\pi)) \sin(j\pi x).$$

(f) [6 points] Let  $u$  be the solution to  $Lu = f$  and let  $w \in C^2[0, 1]$  be such that

$$-w''(x) = 0, \quad 0 < x < 1;$$

$$w(0) = -\frac{1}{100}$$

and

$$w(1) = \frac{1}{100}.$$

Then  $\tilde{u}(x) = w(x) + u(x)$  will be such that

$$-\tilde{u}''(x) = -w''(x) - u''(x) = 0 + f(x) = f(x);$$

$$\tilde{u}(0) = w(0) + u(0) = -\frac{1}{100} + 0 = -\frac{1}{100};$$

and

$$\tilde{u}(1) = w(1) + u(1) = \frac{1}{100} + 0 = \frac{1}{100}.$$

Now, the general solution to

$$-w''(x) = 0$$

is  $w(x) = Ax + B$  where  $A$  and  $B$  are constants. Moreover,  $w(0) = B$  and so  $w(0) = -\frac{1}{100}$  when  $B = -\frac{1}{100}$ . Hence,  $w(x) = Ax - \frac{1}{100}$  and so  $w(1) = A - \frac{1}{100}$  and hence  $w(1) = \frac{1}{100}$  when  $A = \frac{1}{100} + \frac{1}{100} = \frac{2}{100} = \frac{1}{50}$ . Consequently,

$$w(x) = \frac{1}{50}x - \frac{1}{100}$$

and so

$$\tilde{u}(x) = \frac{1}{50}x - \frac{1}{100} + u(x).$$

We can then use the series solution to  $Lu = f$  that we obtained in part (e) to obtain the series solution

$$\tilde{u}(x) = \frac{1}{50}x - \frac{1}{100} + \sum_{j=1}^{\infty} \frac{-4}{j^5\pi^5} (1 + 2\cos(j\pi)) \sin(j\pi x)$$

to the problem of finding  $\tilde{u} \in C^2[0, 1]$  such that

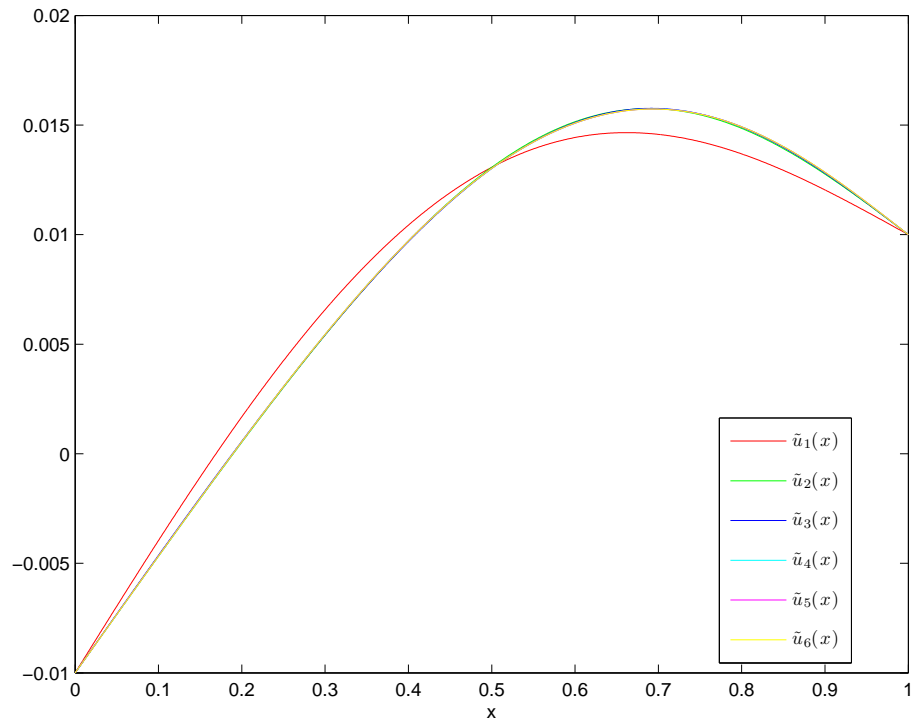
$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{100}$$

and

$$\tilde{u}(1) = \frac{1}{100}.$$

(g) [2 points] The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(3)
clf
uk = -1/100+2*x/100;
for k=1:6
    uk = uk + 2*(4*(-1).^(k+1)-2)./(k.^5*pi^5)*sin(k*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$\tilde{u}_{'$ num2str(k) '}'(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(3),'hw27g','eps')
```

---