# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 44 · Solutions

Posted Friday 15 November 2013. Due 5pm Wednesday 27 November 2013.

44. [25 points] Let the norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\|\mathbf{y}\| = \sqrt{\mathbf{y} \cdot \mathbf{y}}.$$

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and consider the problem of finding $\mathbf{x}(t)$ such that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t > 0$$

and

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

(a) Compute $\mathbf{x}(t)$.

(b) How does $\|\mathbf{x}(t)\|$ behave as $t \to \infty$?

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Solution.

(a) [20 points] The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ and eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

which are such that $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_2 = 1$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1 = 0$. Since $\mathbf{A} = \mathbf{A}^T$, if we set

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

then we have that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

and

$$e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{\Lambda}}\mathbf{V}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^t + e^{-t}\right) & \frac{1}{2}\left(e^t - e^{-t}\right) \\ \frac{1}{2}\left(e^t - e^{-t}\right) & \frac{1}{2}\left(e^t + e^{-t}\right) \end{bmatrix}.$$

Hence,

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 = \begin{bmatrix} \frac{1}{2}\left(e^t + e^{-t}\right) & \frac{1}{2}\left(e^t - e^{-t}\right) \\ \frac{1}{2}\left(e^t - e^{-t}\right) & \frac{1}{2}\left(e^t + e^{-t}\right) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix}.$$

(b) [5 points] Since $\lambda_2 > 0$ and $\mathbf{v}_2 \cdot \mathbf{x}_0 = \frac{2}{\sqrt{2}} \neq 0$, $\|\mathbf{x}(t)\| \to \infty$ as $t \to \infty$.