CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 7 · Solutions

Posted Wednesday 22 January 2014. Due 1pm Friday 31 January 2014.

7. [25 points]

Consider the temperature function

$$u(x,t) = e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

for constant κ, ρ, c, and θ.

(a) Show that this function $u(x,t)$ is a solution of the homogeneous heat equation

$$\rho c\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \ell \text{ and all } t.$$

(b) For which values of θ will u satisfy homogeneous Dirichlet boundary conditions at $x = 0$ and $x = \ell$?

(c) Suppose $\kappa = 2.37$ W/(cm K), $\rho = 2.70$ g/cm^3, and $c = 0.897$ J/(g K) (approximate values for aluminum found on Wikipedia), and that the bar has length $\ell = 10$ cm. Let θ be such that $u(x,t)$ satisfies homogeneous Dirichlet boundary conditions as in part (b) and $u(x,t) > 0$ for $0 < x < \ell$ and all t. Use MATLAB to plot the solution $u(x,t)$ for $0 \le x \le \ell$ at times $t = 0, 4, 8, 12, 16, 20$ seconds, superimposing all six plots on the same axis.

Solution.

(a) [8 points] We can compute that

$$\frac{\partial u}{\partial t} = -\frac{\kappa\theta^2}{\rho c}e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x),$$

$$\frac{\partial u}{\partial x} = \theta e^{-\kappa\theta^2 t/(\rho c)}\cos(\theta x),$$

and

$$\frac{\partial^2 u}{\partial x^2} = -\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x).$$

Hence,

$$\rho c\frac{\partial u}{\partial t} = -\kappa\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

and

$$\kappa\frac{\partial^2 u}{\partial x^2} = -\kappa\theta^2 e^{-\kappa\theta^2 t/(\rho c)}\sin(\theta x)$$

from which it can be seen that

$$\rho c\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}.$$

(b) [8 points] We wish to find the values of θ that give homogeneous Dirichlet boundary conditions, i.e., $u(0,t) = u(\ell,t) = 0$ for all t. Since $e^{-\kappa\theta^2 t/(\rho c)}$ is positive for all t, we can only get the homogeneous Dirichlet conditions when $\sin(\theta x) = 0$. For any θ, $\sin(\theta \cdot 0) = 0$, so the condition at $x = 0$ is automatically satisfied. To get $\sin(\theta\ell) = 0$, we need $\theta\ell$ to be an integer multiple of π, that is,

$$\theta\ell = \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

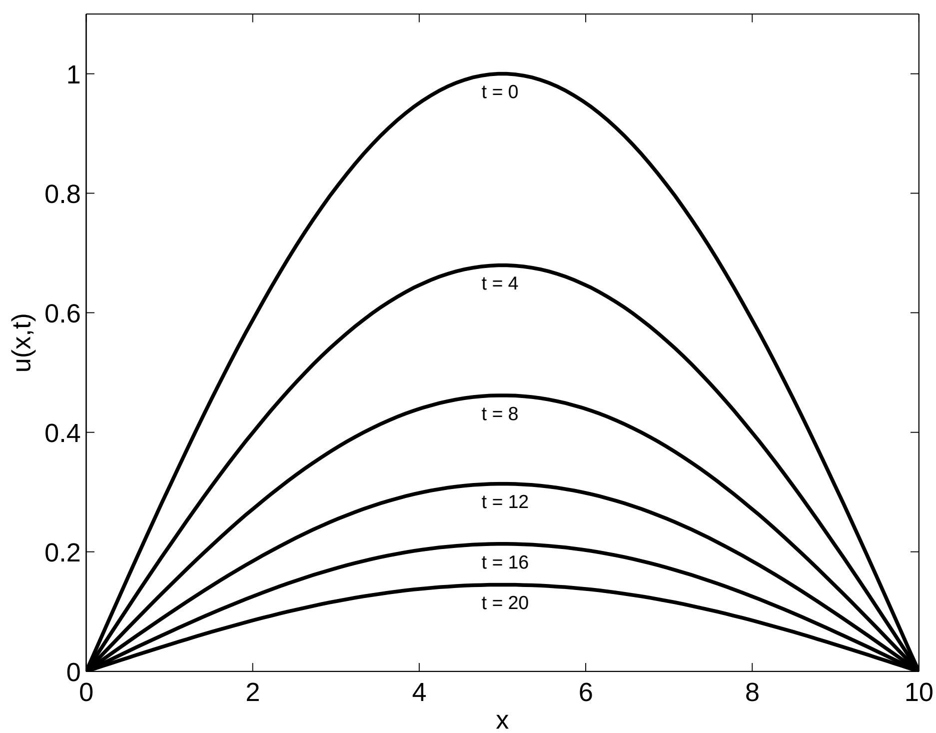
or equivalently,

$$\theta = \frac{\pi n}{\ell}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

(c) [9 points] Notice that if $n = 0$ we have the trivial solution $u(x,t) = 0$ for $0 < x < \ell$ and all t. If $n = 1$, we have a solution for which $u(x,t) > 0$ for $0 < x < \ell$ and all t. For other values of n the solution will be *negative* for some $x \in (0,\ell)$. If our temperature is measured in Kelvin this could be a problem! However, this heat equation takes the same form if we shift to Celsius units, so we needn't be so troubled by the negative values of temperature. Consequently, we shall take $n = 1$ ($\theta = \pi/\ell$) to obtain

$$\begin{aligned} u(x,t) &= e^{-\kappa\pi^2 t/(\ell^2 \rho c)} \sin(\pi x/\ell) \\ &= e^{-2.37\pi^2 t/(100 \cdot 2.70 \cdot 0.897)} \sin(\pi x/10). \end{aligned}$$

The requested plot is below.



The MATLAB code that generated the plot is below.

```
c = .897;
kappa = 2.37;
rho = 2.70;
l = 10;
theta = pi/l;
```

```
t = 0:4:20;
x = linspace(0,l,100);
figure(1), clf
for j=1:length(t)
    u = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x); % compute u(:,t(j))
    plot(x,u,'k-','linewidth',2), hold on
    text(4.75, max(u)-.03, sprintf('t = %d', t(j)))
end
axis([0 10 0 1.1])
set(gca,'fontsize',14)
xlabel('x')
ylabel('u(x,t)')
print -depsc2 checksol1
```
