CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 11 · Solutions

Posted Monday 3 February 2014. Due 1pm Friday 14 February 2014.

11. [25 points]

Demonstrate whether or not each of the following is a linear operator.

(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

(c) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

(d) $L : C^1[0, 1] \to C[0, 1]$ defined by $(Lu)(x) = u(x)u'(x)$.

(e) $L : C^2[0, 1] \to C[0, 1]$ defined by $(Lu)(x) = u''(x) - \sin(x)u'(x) + \cos(x)u(x)$.

Solution.

(a) [5 points] Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + \mathbf{b}$$

but

$$f(\mathbf{u}) + f(\mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{b} + \mathbf{A}\mathbf{v} + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + 2\mathbf{b}$$

and so $f(\mathbf{u} + \mathbf{v})$ does not equal $f(\mathbf{u}) + f(\mathbf{v})$ when $\mathbf{b} \neq \mathbf{0}$. Hence, f *is not* a linear operator.

(b) [5 points] Suppose $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

$$f(\alpha\mathbf{x}) = (\alpha\mathbf{x})^T(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^T\mathbf{x}$$

and

$$\alpha f(\mathbf{x}) = \alpha\mathbf{x}^T\mathbf{x}.$$

However, if $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\alpha = 2$ then $\mathbf{x}^T\mathbf{x} = 1$ and so

$$f(\alpha\mathbf{x}) = 2^2 = 4$$

but

$$\alpha f(\mathbf{x}) = 2.$$

Hence, f *is not* a linear operator.

(c) [5 points] Suppose $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n\times n}$. Then

$$f(\mathbf{X} + \mathbf{Y}) = \mathbf{A}(\mathbf{X} + \mathbf{Y}) + (\mathbf{X} + \mathbf{Y})\mathbf{B} = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B} + \mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{B} = f(\mathbf{X}) + f(\mathbf{Y}),$$

and if $\alpha \in \mathbb{R}$, then

$$f(\alpha\mathbf{X}) = \mathbf{A}(\alpha\mathbf{X}) + (\alpha\mathbf{X})\mathbf{B} = \alpha(\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}) = \alpha f(\mathbf{X}).$$

Hence, f *is* a linear operator.

(d) [5 points] Suppose that $u \in C^1[0,1]$ and $\alpha \in \mathbb{R}$. Then

$$\alpha(Lu)(x) = \alpha u(x)u'(x)$$

and

$$(L(\alpha u))(x) = (\alpha u)(x)(\alpha u)'(x) = \alpha^2 u(x)u'(x).$$

However, if $u(x) = x$ and $\alpha = 2$ then

$$\alpha(Lu)(x) = 2x$$

but

$$(L(\alpha u))(x) = 2^2 x = 4x.$$

Hence, L *is not* a linear operator.

(e) [5 points] Suppose that $u, v \in C^2[0,1]$. Then

$$\begin{aligned}(L(u+v))(x) &= (u+v)''(x) - \sin(x)(u+v)'(x) + \cos(x)(u+v)(x)\\ &= u''(x) - \sin(x)u'(x) + \cos(x)u(x) + v''(x) - \sin(x)v'(x) + \cos(x)v(x)\\ &= (Lu)(x) + (Lv)(x),\end{aligned}$$

and for all $\alpha \in \mathbb{R}$,

$$\begin{aligned}(L(\alpha u))(x) &= (\alpha u)''(x) - \sin(x)(\alpha u)'(x) + \cos(x)(\alpha u)(x)\\ &= \alpha\left(u''(x) - \sin(x)u'(x) + \cos(x)u(x)\right)\\ &= \alpha(Lu)(x).\end{aligned}$$

Hence, L *is* a linear operator.