Recall: We were working towards a spectral method for solving Boundary Value problems that extended the ideas for symmetric matrices.

Lets Briefly review the core concepts from Matrices since we are returning from a long break.

Key Concepts:

1) A a symmetric $n \times n$ matrix
2) Eigenvalues of A exist and are real
3) There is an orthonormal basis of eigenvectors. To solve $Ax=b$ expand x and b in terms of this basis.

Example:

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solve $Ax=b$ using the spectral method.

$$\det(A - \lambda I) = 0 \Rightarrow (4-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$$
$$\Rightarrow \lambda = 3, \ \lambda = 5$$

E_3: Find eigenvectors w/ eigenvalue $\lambda = 3 \Rightarrow$ What is the kernel of $(A - 3I)v$?

$Av = 3v \Rightarrow (A - 3I)v = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 = -v_2$$

Gaussian Elimination, Same kernels.

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$ So that $\text{Eigenspace}\{\lambda = 3\} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$$E_5: \quad \text{Kernel}(A - 5I) = \text{Kernel}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \text{Kernel}\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$
$$= \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

Eigenvectors: $u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note: $u_1 \cdot u_2 = 0$

$w_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ $w_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

So $b = (b, w_1)\vec{w}_1 + (b, w_2)\vec{w}_2$. $(b, w_1) = b \cdot w_1 = -1/\sqrt{2}$

$(b, w_2) = b \cdot w_2 = 5/\sqrt{2}$

$\rightarrow b = -1/\sqrt{2}\,\vec{w}_1 + 5/\sqrt{2}\,\vec{w}_2$

Now we assume $A(\alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2) = -1/\sqrt{2}\,\vec{w}_1 + 5/\sqrt{2}\,\vec{w}_2$

$\rightarrow \alpha_1 3\vec{w}_1 + \alpha_2 5\vec{w}_2 = -1/\sqrt{2}\,\vec{w}_1 + 5/\sqrt{2}\,\vec{w}_2$

taking the dot product of left + right hand sides by $\vec{w}_1$ and $\vec{w}_2$ respectively yields: $\alpha_1 = \frac{-1}{3\sqrt{2}}$ $\alpha_2 = \frac{1}{\sqrt{2}}$ so that $x = \frac{-1}{3\sqrt{2}}\vec{w}_1 + \frac{1}{\sqrt{2}}\vec{w}_2$

- The idea is to remake the spectral methods for solving boundary value problems.

We defined a <u>symmetric differential operator</u> (with respect to the inner product $(\cdot,\cdot)_V$) to be one satisfying $(Lf, g) = (f, Lg)$ for all $f, g \in V$.

We saw that an operator L being symmetric depended upon

-) properties of the operator L
-) properties of the inner product $(\cdot,\cdot)_V$
-) properties of the vector space V.

Remember, too that for boundary value problems the vector space V often carries information about the <u>boundary conditions</u> for the problem.

<u>Ex:</u>

Consider $-\frac{\partial^2}{\partial x^2} u = f$

$u(0) = u(1) = 0$

We showed that $L = -\frac{\partial^2}{\partial x^2}$, $V = C_D^2[0,1] = \{ v \mid v \in C^2[0,1],\ v(0) = v(1) = 0 \}$ equipped with the inner product $(f, g) = \int_0^1 fg\, dx$ <u>Then</u> L is <u>symmetric</u>.

In the "matrix spectral method" we had orthonormal eigenvectors of our matrix operator.

Key idea: when looking for eigenvectors of the differential operator L eg $Lf = \lambda f$, the <u>boundary conditions</u> of the boundary value problem (which show up in the definition of the space V) play a big role.

<u>Ex:</u> The eigenvectors of $L = \frac{-\partial^2}{\partial x^2}$ in $C_0^2[0,\ell] = \{ v \mid v \in C^2[0,\ell],\ v(0) = v(\ell) = 0 \}$ solve the boundary value problem:

$$\begin{cases} -\frac{\partial^2}{\partial x^2} u = \lambda u & \rightarrow \quad \frac{\partial^2}{\partial x^2} u + \lambda u = 0 \\ u(0) = u(\ell) = 0 \end{cases}$$

and are given by: $\psi_n(x) = \sin\left(\frac{n\pi}{\ell} x\right)$ and $\lambda_n = \frac{n^2\pi^2}{\ell^2}$

The orthonormal variant is $\tilde{\psi}_n = \psi_n / \|\psi_n\| = \sqrt{2/\ell}\, \psi_n$

The expansion of $f \in V$ in terms of $\tilde{\psi}_n$ is called the <u>Fourier Series expansion</u> of $f(x)$. Eg: $f = (f, \tilde{\psi}_1)\tilde{\psi}_1 + (f, \tilde{\psi}_2)\tilde{\psi}_2 + (f, \tilde{\psi}_3)\tilde{\psi}_3 + \cdots = \sum c_n \sin\left(\frac{n\pi}{\ell} x\right)$

then $c_n = (f, \tilde{\psi}_n) = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$

<u>Ex:</u> let $\ell = 1$ and $f(x) = 1 - x$ then $c_n = 2\int_0^1 (1-x)\sin(n\pi x)\,dx = \frac{2}{n\pi}$

so the fourier series for $f(x)$ is $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x)$

Solving a boundary value problem with the spectral method:

Suppose we want to solve

$$\begin{cases} -\frac{\partial^2}{\partial x^2} u = f \\ u(0) = u(1) = 0 \end{cases}$$

where $f = 1 - x$

We let $L = \frac{-\partial^2}{\partial x^2}$ and $V = C_D^2[0,1]$. We know L is symmetric and the eigenfunctions of L on V is $\tilde{\psi}_n = \sqrt{2/\ell} \sin(n\pi x)$ with eigenvalues $\lambda_n = n^2\pi^2$

Expanding $f = 1-x$ in terms of these eigenfunctions (e.g. the Fourier Series for f) and the unknown solution $u(x)$ in terms of these eigenfunctions gives:

$$-\frac{\partial^2}{\partial x^2}\left(\sum \alpha_n \sin(n\pi x)\right) = \left(\sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x)\right) \quad (*)$$

where we are using the fact that $f = \sum c_n \sin(n\pi x) = \sum \frac{2}{n\pi} \sin(n\pi x)$ is the Fourier Series for $f(x) = 1-x$. Then $(*)$ is:

$$\sum \alpha_n (n^2\pi^2) \sin(n\pi x) = \sum \frac{2}{n\pi} \sin(n\pi x) \quad (\dagger)$$

or in terms of the basis functions,

$$\sum \alpha_n n^2\pi^2 \tilde{\psi}_n = \sum \frac{2}{n\pi} \tilde{\psi}_n.$$

Integrating both sides of $(\dagger)$ with respect to $\sqrt{2} \sin(j\pi x) = \tilde{\psi}_j$ (e.g. taking the inner product with respect to $\tilde{\psi}_j$) gives:

$$\alpha_j (n^2\pi^2) = \frac{2}{n\pi} \rightarrow \alpha_j = \frac{2}{n^3\pi^3}$$

So that $u(x) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi^3} \sin(n\pi x)$ solves the problem.

Steps:

1. Find the eigenfunctions of L which respect the boundary conditions.
2. Compute the Fourier expansion of the Right-Hand side
3. Expand the unknown function in terms of the eigenfunctions
4. Compare coefficients and solve.