CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 15 · Solutions

Posted Wednesday 18 September 2013. Due 5pm Wednesday 25 September 2013.

15. [25 points]
Suppose $\mathcal{V}$ is a vector space with an associated inner product $(\cdot,\cdot): \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ and norm $\|\cdot\|: \mathcal{V}\to\mathbb{R}$ defined by $\|w\| = \sqrt{(w,w)}$. The angle $\angle(u,v)$ between $u\in\mathcal{V}$ and $v\in\mathcal{V}$ is defined via the equation

$$(u,v) = \|u\|\|v\|\cos\angle(u,v).$$

Let $\mathcal{V} = C[0,1]$ and let the inner product $(\cdot,\cdot): C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by $(u,v) = \int_0^1 u(x)v(x)\,dx$. Let m and n be nonnegative integers.

(a) Compute (x^n, x^m).

(b) Compute $\|x^m\|$ and $\|x^n\|$.

(c) Compute $\cos\angle(x^n,x^m)$ between x^n and x^m.

(d) Compute $\cos\angle(x^n,x^{n+1})$ between x^n and x^{n+1}.

(e) What happens to $\angle(x^n,x^{n+1})$ as $n\to\infty$?

Solution.

(a) [5 points] We have

$$(x^n,x^m) = \int_0^1 x^{n+m}\,dx = \frac{1}{n+m+1}.$$

(b) [5 points] We also have

$$\|x^n\| = \sqrt{(x^n,x^n)} = \frac{1}{\sqrt{2n+1}}$$

and

$$\|x^m\| = \sqrt{(x^m,x^m)} = \frac{1}{\sqrt{2m+1}}.$$

(c) [5 points] Altogether, we have the following formula for $\cos\angle(x^n,x^m)$ between x^n and x^m:

$$\cos\angle(x^n,x^m) = \frac{(x^n,x^m)}{\|x^n\|\|x^m\|} = \frac{\sqrt{(2n+1)(2m+1)}}{n+m+1}.$$

(d) [5 points] If $m = n+1$ we have the following formula for $\cos\angle(x^n,x^{n+1})$ between x^n and x^{n+1}:

$$\cos\angle(x^n,x^{n+1}) = \frac{\sqrt{(2n+1)(2n+3)}}{2n+2}.$$

(e) [5 points] As $n\to\infty$, we see that $\cos\angle(x^n,x^{n+1})\to 1$, implying that the angle between x^n and x^{n+1} shrinks to zero as $n\to\infty$.
