CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 28

Posted Wednesday 9 October 2013. Due 1pm Friday 18 October 2013.

28. [25 points] Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L:S\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$S=\{w\in C^2[0,1]:w(0)=w'(1)=0\}.$$

Note that

$$(Lv,w)=(v,Lw)\text{ for all }v,w\in S.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x)=\sqrt{2}\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)$$

for $n=1,2,\ldots$. Obtain a formula for the eigenvalues λ_n for $n=1,2,\ldots$.

(b) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=x+\sin(\pi x),\quad 0<x<1$$

and

$$\tilde{u}(0)=\tilde{u}'(1)=0.$$

Note that, for $m,n=1,2,\ldots$,

$$(\psi_m,\psi_n)=\begin{cases}1 & \text{if } m=n;\\ 0 & \text{if } m\neq n.\end{cases}$$

(c) Plot the approximation to $\tilde{u}$ obtained by replacing the upper limit of the summation in your series solution with 20.

(d) By shifting the data, obtain a series solution to the problem of finding $u\in C^2[0,1]$ such that

$$-u''(x)=x+\sin(\pi x),\quad 0<x<1$$

and

$$u(0)=u'(1)=1.$$

(e) Plot the approximation to u obtained by replacing the upper limit of the summation in your series solution with 20.