CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 23 · Solutions

Posted Wednesday 2 October 2013. Due 5pm Wednesday 9 October 2013.

23. [25 points]

Let $N \geq 1$ be an integer and define $h = 1/(N+1)$ and $x_k = kh$ for $k = 0, \ldots, N+1$. Consider the $N+2$ hat functions, defined for $x \in [0,1]$ as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise}; \end{cases}$$

for $k = 1, \ldots, N$, with

$$\phi_0(x) = \begin{cases} (x_1 - x)/h, & x \in [x_0, x_1); \\ 0, & \text{otherwise}; \end{cases}$$

and

$$\phi_{N+1}(x) = \begin{cases} (x - x_N)/h, & x \in [x_N, x_{N+1}]; \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\phi_k \in C[0,1]$ for $k = 0, \ldots, N+1$. Let $V_N = \text{span}\{\phi_0, \phi_1, \ldots, \phi_{N+1}\}$ and let $(\cdot,\cdot) : V_N \times V_N \to \mathbb{R}$ be defined by

$$(u, v) = \sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} u'(x) v'(x)\, dx.$$

(a) What value is taken by $w(x) = \displaystyle\sum_{j=0}^{N+1} \phi_j(x)$ for all $x \in [0,1]$?

(b) Why is $(\cdot,\cdot)$ not an inner product on V_N?

(c) For the remainder of this question we shall just consider the case when $N = 1$. In this case the symmetric matrix

$$\mathbf{K} = \begin{bmatrix} (\phi_0,\phi_0) & (\phi_0,\phi_1) & (\phi_0,\phi_2) \\ (\phi_0,\phi_1) & (\phi_1,\phi_1) & (\phi_1,\phi_2) \\ (\phi_0,\phi_2) & (\phi_1,\phi_2) & (\phi_2,\phi_2) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

What are the eigenvalues corresponding to the eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ of $\mathbf{K}$?

(d) For both $\mathbf{f} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, use an eigenvector of $\mathbf{K}$ to determine whether or not there exist solutions $\mathbf{c} \in \mathbb{R}^3$ to $\mathbf{Kc} = \mathbf{f}$ and if solutions exist, use the spectral method to obtain the solutions $\mathbf{c} \in \mathbb{R}^3$ to $\mathbf{Kc} = \mathbf{f}$.

(e) Why do there exist solutions $\mathbf{c} \in \mathbb{R}^3$ to

$$\mathbf{Kc} = \begin{bmatrix} \displaystyle\int_0^1 g(x)\phi_0(x)dx \\ \displaystyle\int_0^1 g(x)\phi_1(x)dx \\ \displaystyle\int_0^1 g(x)\phi_2(x)dx \end{bmatrix}$$

for all $g \in C[0,1]$ which are such that $\displaystyle\int_0^1 g(x)dx = 0$?

Solution.

(a) [5 points] For $k = 0, \ldots, N$, when $x \in [x_k, x_{k+1}]$,

$$\sum_{j=0}^{N+1} \phi_j(x) = \phi_k(x) + \phi_{k+1}(x) = \frac{x_{k+1} - x}{h} + \frac{x - x_k}{h} = \frac{x_{k+1} - x + x - x_k}{h} = \frac{x_{k+1} - x_k}{h} = \frac{h}{h} = 1.$$

Therefore $w(x) = 1$ for all $x \in [0, 1]$.

(b) [5 points] The function w from part (a) is such that $w \in V_N$ and $w(x) = 1$ for all $x \in [0, 1]$. However, $(w, w) = 0$. Hence, the inner product is not positive-definite as there are nonzero $u \in V_N$ for which $(u, u) = 0$.

(c) [3 points] We have that

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

So, 0 is the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, 2 is the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and 6 is the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(d) [7 points] Since a 3×3 matrix can only have three eigenvalues, the eigenvalues of K are 0, 2 and 6. Since only one of these is zero and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0, then, since $\mathbf{K} = \mathbf{K}^T$, there will only exist solutions to $\mathbf{Kc} = \mathbf{f}$ if

$$\mathbf{f} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$$

Now,

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 + 2 + 2 = 6 \neq 0$$

and so there exist no solutions to $\mathbf{Kc} = \mathbf{f}$ when $\mathbf{f} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. However,

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 - 3 + 1 = 0$$

and so there are infinitely many solutions to $\mathbf{Kc} = \mathbf{f}$ when $\mathbf{f} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and since $\mathbf{K} = \mathbf{K}^T$ we can use the spectral method to obtain these solutions.

Henceforth, let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. Then, for any $\alpha \in \mathbb{R}$,

$$\mathbf{c} = \frac{1}{2}\frac{\mathbf{f}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \frac{1}{6}\frac{\mathbf{f}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3 + \alpha\mathbf{v}_1$$

is a solution to $\mathbf{Kc} = \mathbf{f}$. Now, $\mathbf{f}\cdot\mathbf{v}_2 = 1$ and $\mathbf{f}\cdot\mathbf{v}_3 = 9$. Also, $\mathbf{v}_2\cdot\mathbf{v}_2 = 2$ and $\mathbf{v}_3\cdot\mathbf{v}_3 = 6$. So,

$$\frac{1}{2}\frac{\mathbf{f}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \frac{1}{6}\frac{\mathbf{f}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3 = \frac{1}{2}\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{6}\frac{9}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 0 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ -1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}.$$

Therefore, the solutions $\mathbf{c} \in \mathbb{R}^3$ to $\mathbf{Kc} = \mathbf{f}$ are

$$\mathbf{c} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} + \alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for any $\alpha \in \mathbb{R}$.

(e) [5 points] As in part (d), there will exist solutions to

$$\mathbf{Kc} = \begin{bmatrix} \int_0^1 g(x)\phi_0(x)dx \\ \int_0^1 g(x)\phi_1(x)dx \\ \int_0^1 g(x)\phi_2(x)dx \end{bmatrix}$$

if and only if

$$\begin{bmatrix} \int_0^1 g(x)\phi_0(x)dx \\ \int_0^1 g(x)\phi_1(x)dx \\ \int_0^1 g(x)\phi_2(x)dx \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$$

Now,

$$\begin{aligned}
&\begin{bmatrix} \int_0^1 g(x)\phi_0(x)dx \\ \int_0^1 g(x)\phi_1(x)dx \\ \int_0^1 g(x)\phi_2(x)dx \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \int_0^1 g(x)\phi_0(x)dx + \int_0^1 g(x)\phi_1(x)dx + \int_0^1 g(x)\phi_2(x)dx \\
&= \int_0^1 g(x)\left(\phi_0(x) + \phi_1(x) + \phi_2(x)\right)dx \\
&= \int_0^1 g(x)dx
\end{aligned}$$

by part (a). Hence, if $g \in C[0,1]$ is such that $\displaystyle\int_0^1 g(x)dx = 0$ then

$$\begin{bmatrix} \displaystyle\int_0^1 g(x)\phi_0(x)dx \\ \displaystyle\int_0^1 g(x)\phi_1(x)dx \\ \displaystyle\int_0^1 g(x)\phi_2(x)dx \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

and there will exist solutions to

$$\mathbf{Kc} = \begin{bmatrix} \displaystyle\int_0^1 g(x)\phi_0(x)dx \\ \displaystyle\int_0^1 g(x)\phi_1(x)dx \\ \displaystyle\int_0^1 g(x)\phi_2(x)dx \end{bmatrix}.$$
