# The spectral method for solving Boundary Value Problems

Recall: we saw that many of the nice features for the theory of algebraic linear operators were also relevant for the study of linear differential operators.

We are going to explore extending the spectral method for linear algebraic systems to solve the linear differential operator

$$-K\frac{\partial^2 u}{\partial x^2} = f \quad 0 < x < \ell$$
$$u(0) = 0$$
$$u(\ell) = 0$$

This system can be viewed as $L_D u = f$ where $L_D : C_D^2[0,1] \to C[0,1]$ is the differential operator $L_D = -K\frac{\partial^2}{\partial x^2}$

The spectral method for $Ax = b$ relied on the fact that if $A$ was symmetric we could find a basis of orthogonal eigenvectors of $A$ for $\mathbb{R}^n$. The really important part to notice is that $A: \mathbb{R}^n \to \mathbb{R}^n$ so we were able to find a basis of orthogonal eigenvectors for the domain and the range of $A$.

We saw that $L_D$ is symmetric. Consequently its eigenvalues are real and its eigenvectors are orthogonal. We also saw that $L_D$ had the added property that its eigenvalues are positive. So in order to bridge the gap to the spectral method we need to know: Can every function in $C_D^2[0,1]$ and $C[0,1]$ be represented as sums of eigenvectors of $L_D$?

- What are the eigenvectors + eigenvalues of $L_D$?

  To find eigenvalues and eigenvectors we need to solve the problem: $L_D u = \lambda u$ in the vector space $C_D^2[0,1]$. Ie we want to solve the problem:

$$-K\frac{\partial^2 u}{\partial x^2} = \lambda u$$
$$u(0) = u(\ell) = 0$$

Since $\lambda > 0$ we can write $\lambda = \theta^2$ for some $\theta$ and consider the ODE $K\frac{\partial^2 u}{\partial x^2} + u = 0$. From section 4.2 the general form of the solution to this problem is:

$$u(x) = C_1\cos(\theta x) + C_2\sin(\theta x)$$

we are looking for a solution in $C_D^2[0,l]$ so we see that $u(0)=0 \Rightarrow c_1=0$ must follow.

Thus $u(l)=0$ gives $c_2 \sin(\theta l)=0$. This is true for $c_2=0$ which gives $u(x)=0$ but we are interested in nonzero solutions so that $\theta l = \pm n\pi$, $n=1,2,3,\ldots$ must follow. Recalling that $\lambda=\theta^2$ gives: $\lambda = \frac{n^2\pi^2}{l^2}$ for $n=1,2,3,\ldots$

So $L_D$ has <u>infinitely many</u> eigenvalues $\lambda_n = \frac{n^2\pi^2}{l^2}$ with corresponding eigenvectors $\psi_n(x) = \sin\left(\frac{n\pi}{l}x\right)$ $n=1,2,3,\ldots$

<u>Note</u>: we know that $(\psi_n,\psi_m)=0$ for $n\neq m$ because $L_D$ is symmetric!

<u>Note</u>: $\|\psi_n\|_{L^2} = (\psi_n,\psi_n) = \int_0^l \sin^2\left(\frac{n\pi}{l}x\right)dx = \frac{l}{2}$

It follows that $\tilde{\psi}_n = \sqrt{\frac{2}{l}}\,\psi_n(x)$ has unit length and $\{\tilde{\psi}_m\}_{m=1}^{\infty}$ are <u>orthonormal</u>.

- The Fourier Series:

Recall that the spectral method used the fact that every vector of $\mathbb{R}^n$, the domain and Range of the symmetric matrix $A$, could be written as a sum of the eigenvectors of $A$. In order to have a "spectral method" for linear differential operators we need the same sort of fact.

The analogous question becomes:
"Can every 'vector' in $C_D^2[0,l]$ and $C[0,l]$, the domain and range of $L_D$, be written as a sum of the functions $\tilde{\psi}_n = \sqrt{\frac{2}{l}}\sin\left(\frac{n\pi}{l}x\right)$?"

The answer is "Yes and No". If we restrict ourselves to finitely many of the eigenfunctions $\tilde{\psi}_n$ then, No. To see this just consider the function $g(x)=x$ in $C[0,l]$. No <u>finite</u> sum of eigenfunctions $\tilde{\psi}_n$ can give us $g(x)$ since $g(l)=l\neq 0$ but $\tilde{\psi}_n(l)=0$ for every value of $n$.

However, all is not lost! A result from mathematics tells us that if we fix an $\varepsilon > 0$ we <u>can</u> find a finite sum of the eigenfunctions $\tilde{\psi}_n(x)$ such that

$$\| \sum_{j=1}^{M} c_j \tilde{\psi}_j(x) - g(x) \|_{L^2} < \varepsilon$$

And in fact we can do this for any function $g(x)$ in $C[0,1]$.

How can we find this sum: $\sum_{j=1}^{M} c_j \tilde{\psi}_j(x)$ approximating $g(x) \in C[0,1]$?

Let $V^K = \text{span}\{\tilde{\psi}_1, \tilde{\psi}_2, \dots, \tilde{\psi}_K\}$ be the span of the first $K$ eigenfunctions of $L_D$. Then each $V^K$ is a finite dimensional subspace of $C[0,1]$ and $V^1 \subseteq V^2 \subseteq V^3 \subseteq \dots \subseteq C[0,1]$.

We know how to find the best approximation to $g(x) \in C[0,1]$ in a finite dimensional subspace. We can solve the Gram problem: $Gx = b$!

Suppose we want to solve this problem in the subspace $V^k$ then $G$ is the matrix $G_{ij} = (\tilde{\psi}_j, \tilde{\psi}_i)$ and $b_i = (g, \tilde{\psi}_i)$. Since the vectors $\{\tilde{\psi}_m\}$ are <u>orthonormal</u> the matrix $G$ is the <u>identity matrix</u>.

Therefore $Gx = b$ gives, directly, $x_i = (g, \tilde{\psi}_i) = \sqrt{\frac{2}{\ell}} \int_0^\ell g(x) \sin\left(\frac{n\pi}{\ell} x\right)$ so the best approximation to $g(x)$ in $V^k$ is:

$$m^K(x) = x_1 \tilde{\psi}_1 + x_2 \tilde{\psi}_2 + \dots + x_k \tilde{\psi}_k = \sum_{i=1}^{k} (g, \tilde{\psi}_i) \tilde{\psi}_i(x)$$

the best approximation to $g(x)$ from $V^k$

$$= \sum_{i=1}^{k} \left( \sqrt{\frac{\ell}{2}} \int_0^\ell g(x) \sin\left(\frac{n\pi}{\ell} x\right) \right) \sqrt{\frac{\ell}{2}} \sin\left(\frac{n\pi}{\ell} x\right)$$

$$(*) \quad \left[ = \sum_{i=1}^{k} \left( \frac{2}{\ell} \int_0^\ell g(x) \sin\left(\frac{n\pi}{\ell} x\right) dx \right) \sin\left(\frac{n\pi}{\ell} x\right) \right]$$

Now as $k \mapsto \infty$ we know (from a result in mathematics) that the best approximation to $g(x)$ from $V^k$ gets better and better.

That is, $m^K(x) \longmapsto g(x)$ and we say "$m^k(x)$ <u>converges</u> to $g(x)$ in the $L^2$ sense" (meaning $\| m^k(x) - g(x) \|_{L^2} \to 0$.)

It is in this sense that we write:

$$g(x) = \sum_{i=1}^{\infty} (g, \tilde{\psi}_i)\, \tilde{\psi}_i(x) = \sum_{i=1}^{\infty} \underbrace{\left( \frac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\frac{n\pi}{\ell} x\right) dx \right)}_{(\blacksquare)} \sin\left(\frac{n\pi}{\ell} x\right) \qquad (**)$$

The right-hand side of $(**)$ is called the "fourier sine series" of $g(x)$ and the term $(\blacksquare)$ is called the "$i^{th}$ fourier coefficient" of $g(x)$ and denoted as "$C_n$"

- Important note: the eigenfunctions of $L_D$ depended on both the differential operator and the domain of definition $C_0^2[0,1]$.

Ex: what are the fourier coefficients of the function $f(x) = 1 - x$ on the interval $[0,1]$?

Solution $C_n = \frac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\frac{n\pi}{\ell} x\right) dx \rightarrow 2 \int_0^1 (1-x) \sin(n\pi x)\, dx$
$= \frac{2}{n\pi}$

So the fourier series for $g(x) = 1 - x$ is $\sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x)$