CAAM 336 · DIFFERENTIAL EQUATIONS

Fall 2013 Examination 1

1. [25 points]

   Let the operator $L : C^2[0,1] \to C[0,1]$ be defined by

   $$Lv = -v'' + 9v.$$

   Let $u \in C^2[0,1]$ be the solution to the differential equation

   $$-u''(x) + 9u(x) = f(x), \quad 0 < x < 1$$

   with boundary conditions

   $$u(0) = \alpha$$

   and

   $$u(1) = \beta$$

   where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Note that

   $$(Lu)(x) = -u''(x) + 9u(x)$$

   for all $x \in [0,1]$. Let N be an integer which is such that $N \geq 2$ and let $h = \frac{1}{N+1}$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

   (a) Determine whether or not L is a linear operator.

   (b) By using the approximation

   $$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

   for $j = 1, \ldots, N$ we can write

   $$\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} \approx \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}$$

   where $\mathbf{D} \in \mathbb{R}^{N \times (N+2)}$. What are the entries of the matrix $\mathbf{D}$? An acceptable way to present your final answer is

   $$D_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

   with the question marks replaced with the correct values.

(c) We can use the differential equation and boundary conditions satisfied by u and the approximation from the previous part to write

$$\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ and $\mathbf{b} \in \mathbb{R}^N$. What are the entries of the matrix $\mathbf{A}$ and the vector $\mathbf{b}$? An acceptable way to present your final answer is

$$A_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise}; \end{cases}$$

and

$$b_j = \begin{cases} ? & \text{if } j = ?; \\ ? & \text{if } j = ?; \\ ? & \text{otherwise}; \end{cases}$$

with the question marks replaced with the correct values.

(d) Let $f(x) = 18$, $\alpha = \beta = 0$ and $N = 2$. Obtain approximations u_1 and u_2 to $u(x_1)$ and $u(x_2)$, respectively, by solving

$$\mathbf{A}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{b}.$$

2. [25 points]

(a) Compute

$$\int_{-1}^{1} x\,dx.$$

(b) Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$\mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to

$$\mathbf{Ac} = \mathbf{g}.$$

(c) Let

$$V_0 = \left\{ w \in C^1[0,1] : \int_0^1 w(x)\,dx = 0 \right\}.$$

Determine whether or not V_0 is a subspace of $C^1[0,1]$.

(d) Let $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^{1} x u(x) v(x)\,dx.$$

Determine whether or not $(\cdot,\cdot)$ is an inner product on $C[-1,1]$.

(e) Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $\phi \in C[a,b]$ be defined by $\phi(x) = 1$ and let the inner product $B(\cdot,\cdot) : C[a,b] \times C[a,b] \to \mathbb{R}$ be defined by

$$B(u,v) = \int_a^b u(x) v(x)\,dx.$$

Let the linear operator $P_0 : C[a,b] \to C[a,b]$ be defined by

$$P_0 f = \frac{1}{b-a} B(f,\phi).$$

Determine whether or not P_0 is a projection.

3. [25 points]

Let the inner product $(\cdot,\cdot) :\ C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x) w(x)\,dx$$

and let the norm $\|\cdot\| :\ C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$S = \{ w \in C^2[0,1] : w'(0) = w(1) = 0 \}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv, w) = (v, Lw) \text{ for all } v, w \in S.$$

Let N be a positive integer and let $f \in C[0,1]$ be defined by

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$(\psi_m, \psi_n) = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) Compute the best approximation to f from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1$$

and

$$\tilde{u}'(0) = \tilde{u}(1) = 0.$$

(d) What is the best approximation to $\tilde{u}$ from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$?

(e) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

4. [25 points]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $f_1 \in C[-1,1]$, and $f_2 \in C[-1,1]$ be defined by

$$\phi_1(x) = \frac{1}{\sqrt{2}},$$

$$\phi_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$f_1(x) = \sin(\pi x),$$

and

$$f_2(x) = \cos(\pi x),$$

for all $x \in [-1,1]$. Note that $\{\phi_1, \phi_2\}$ is linearly independent. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_{-1}^{1} u(x)v(x)\, dx$$

and let the norm $\|\cdot\|: C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$. Also, let $\psi_1 \in C[0,1]$, $\psi_2 \in C[0,1]$, $g_1 \in C[0,1]$, and $g_2 \in C[0,1]$ be defined by

$$\psi_1(x) = \frac{1}{\sqrt{2}},$$

$$\psi_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$g_1(x) = \sin(\pi x),$$

and

$$g_2(x) = \cos(\pi x),$$

for all $x \in [0,1]$. Note that $\{\psi_1, \psi_2\}$ is linearly independent. Let the inner product $B(\cdot,\cdot): C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$B(u,v) = \int_0^1 u(x)v(x)\,dx$$

and let the norm $\|\cdot\|_B: C[0,1] \to \mathbb{R}$ be defined by

$$\|u\|_B = \sqrt{B(u,u)}.$$

Note that

$$(f_1, \phi_2) = \frac{\sqrt{6}}{\pi},$$

$$B(g_1, \psi_1) = \frac{\sqrt{2}}{\pi},$$

$$B(g_1, \psi_2) = \frac{\sqrt{6}}{2\pi},$$

$$B(g_2, \psi_2) = -\frac{\sqrt{6}}{\pi^2},$$

and

$$(f_1, \phi_1) = (f_2, \phi_1) = (f_2, \phi_2) = B(g_2, \psi_1) = 0.$$

(a) Construct the best approximation to f_1 from $\mathrm{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(b) Construct the best approximation to f_2 from $\mathrm{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) Construct the best approximation to g_1 from $\mathrm{span}\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.

(d) Construct the best approximation to g_2 from $\mathrm{span}\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.