CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 13 · Solutions

Posted Monday 16 September 2013. Due 5pm Wednesday 25 September 2013.

11. [25 points]

   (a) Are there any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ which are such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and the set

   $$\mathcal{P}_{\mathbf{u},\mathbf{v}} = \big\{\mathbf{w} \in \mathbb{R}^2 : \ \mathbf{w} = a\mathbf{u} + b\mathbf{v} \text{ where } a, b \in \mathbb{R} \text{ are such that } a \geq 0 \text{ and } b \geq 0\big\}$$

   is a subspace of $\mathbb{R}^2$?

   (b) Are there any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ which are such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and the set

   $$\mathcal{Q}_{\mathbf{u},\mathbf{v}} = \big\{\mathbf{w} \in \mathbb{R}^2 : \ \mathbf{w} = a\mathbf{u} + b\mathbf{v} \text{ where } a, b \in \mathbb{R} \text{ are such that } ab \geq 0\big\}$$

   is a subspace of $\mathbb{R}^2$?

Solution.

(a) [12 points] There are no vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ which are such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and the set $\mathcal{P}_{\mathbf{u},\mathbf{v}}$ is a subspace of $\mathbb{R}^2$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. The set $\mathcal{P}_{\mathbf{u},\mathbf{v}}$ is not a subspace of $\mathbb{R}^2$, because even though it is closed under addition, it is not closed under scalar multiplication. If $\mathbf{x} \in \mathcal{P}_{\mathbf{u},\mathbf{v}}$, then $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$, where $a, b \in \mathbb{R}$ are such that $a \geq 0$ and $b \geq 0$. The choice

$$a = 1, b = 1$$

satisfies these conditions, but then

$$-\mathbf{x} = -a\mathbf{u} - b\mathbf{v} = -\mathbf{u} - \mathbf{v} \notin \mathcal{P}_{\mathbf{u},\mathbf{v}}$$

since the fact that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent means that the only $\tilde{a}, \tilde{b} \in \mathbb{R}$ for which

$$-\mathbf{u} - \mathbf{v} = \tilde{a}\mathbf{u} + \tilde{b}\mathbf{v}$$

are $\tilde{a} = \tilde{b} = -1$ which are such that $\tilde{\alpha} < 0$ and $\tilde{\beta} < 0$.

(b) [13 points] There are no vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ which are such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and the set $\mathcal{Q}_{\mathbf{u},\mathbf{v}}$ is a subspace of $\mathbb{R}^2$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. The set $\mathcal{Q}_{\mathbf{u},\mathbf{v}}$ is not a subspace of $\mathbb{R}^2$, because even though it is closed under scalar multiplication, it is not closed under addition. If $\mathbf{x}, \mathbf{y} \in \mathcal{Q}_{\mathbf{u},\mathbf{v}}$, then

$$\mathbf{x} = a_1\mathbf{u} + b_1\mathbf{v}, \ \mathbf{y} = a_2\mathbf{u} + b_2\mathbf{v},$$

where $a_1, b_1, a_2, b_2 \in \mathbb{R}$ are such that $a_1 b_1 \geq 0$, and $a_2 b_2 \geq 0$. The choice

$$a_1 = 2, b_1 = 1, a_2 = -1, b_2 = -2$$

satisfies these conditions, but then

$$\mathbf{x} + \mathbf{y} = (a_1 + a_2)\mathbf{u} + (b_1 + b_2)\mathbf{v} = \mathbf{u} - \mathbf{v} \notin \mathcal{Q}_{\mathbf{u},\mathbf{v}}$$

since the fact that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent means that the only $\tilde{a}, \tilde{b} \in \mathbb{R}$ for which

$$\mathbf{u} - \mathbf{v} = \tilde{a}\mathbf{u} + \tilde{b}\mathbf{v}$$

are $\tilde{a} = 1$ and $\tilde{b} = -1$ which are such that $\tilde{a}\tilde{b} < 0$.
