CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 33 · Solutions

Posted Friday 25 October 2013. Due 5pm Wednesday 6 November 2013.

33. [25 points] Let $H_D^1(0,1) = \{w \in H^1(0,1) : w(0) = w(1) = 0\}$ and let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and the inner product $a(\cdot,\cdot) : H_D^1(0,1) \times H_D^1(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let the norm $|||\cdot||| : H_D^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{a(v,v)}.$$

Let m be a positive integer and let $f_m \in L^2(0,1)$ be defined by $f_m(x) = \sqrt{2}m^2\pi^2\sin(m\pi x)$. Note that, for $j = 1,\ldots,N$,

$$(f_m,\phi_j) = \frac{2\sqrt{2}\sin(m\pi x_j)}{h}(1-\cos(m\pi h)).$$

Let N be a positive integer, let $h = \dfrac{1}{N+1}$, let $x_j = jh$ for $j = 0,1,\ldots,N+1$, and let $\phi_k \in H_D^1(0,1)$ be defined by

$$\phi_k(x) = \begin{cases} \dfrac{(x-x_{k-1})}{h} & \text{if } x \in [x_{k-1},x_k); \\ \dfrac{(x_{k+1}-x)}{h} & \text{if } x \in [x_k,x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1,\ldots,N$. Let $V_N = \text{span}\{\phi_1,\ldots,\phi_N\}$. Let $u_m \in H_D^1(0,1)$ be such that

$$a(u_m,v) = (f_m,v) \text{ for all } v \in H_D^1(0,1).$$

(a) The true solution to the problem of finding $u_m \in H_D^1(0,1)$ such that

$$a(u_m,v) = (f_m,v) \text{ for all } v \in H_D^1(0,1)$$

is $u_m(x) = \sqrt{2}\sin(m\pi x)$. Use integration by parts and the fact that $(u_m,u_m) = 1$ to obtain a formula for $a(u_m,u_m)$.

(b) We can obtain finite element approximations to u_m by finding $u_{m,N} \in V_N$ such that

$$a(u_{m,N},v) = (f_m,v) \text{ for all } v \in V_N.$$

Write a code which can compute the energy norm of the error

$$|||u_m - u_{m,N}|||.$$

Use your code to produce a `loglog` plot of the energy norm of the error

$$|||u_m - u_{m,N}|||$$

when

$$m = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

for

$$N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191$$

on the same figure with a separate line for each value of m.

(c) Produce a `loglog` plot of the percentage error

$$100 \frac{|||u_m - u_{m,N}|||}{|||u_m|||}$$

when

$$m = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

for

$$N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191$$

on the same figure with a separate line for each value of m.

(d) Let $f \in L^2(0,1)$. The problem of finding $u \in H_D^1(0,1)$ such that

$$a(u,v) = (f,v) \text{ for all } v \in H_D^1(0,1)$$

is the weak form of the problem of finding u such that

$$-u''(x) = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0,$$

or equivalently,

$$f(x) + u''(x) = 0, \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

If $\tilde{u}_N \in V_N$ then how does the quantity

$$\sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} \left(f(x) + \tilde{u}_N''(x)\right)^2 dx$$

behave as N increases?

Solution.

(a) [5 points] Integration by parts yields that

$$\begin{aligned}
& a(u_m, u_m) \\
&= \int_0^1 \frac{d}{dx}\left(\sqrt{2}\sin(m\pi x)\right) \frac{d}{dx}\left(\sqrt{2}\sin(m\pi x)\right) dx \\
&= \left[\frac{d}{dx}\left(\sqrt{2}\sin(m\pi x)\right)\sqrt{2}\sin(m\pi x)\right]_0^1 - \int_0^1 \frac{d^2}{dx^2}\left(\sqrt{2}\sin(m\pi x)\right)\sqrt{2}\sin(m\pi x)\, dx \\
&= \left[2m\pi\cos(m\pi x)\sin(m\pi x)\right]_0^1 + \int_0^1 \sqrt{2}m^2\pi^2\sin(m\pi x)\sqrt{2}\sin(m\pi x)\, dx \\
&= 2m\pi\cos(m\pi)\sin(m\pi) - 2m\pi\cos(0)\sin(0) + m^2\pi^2(u_m, u_m) \\
&= m^2\pi^2(u_m, u_m)
\end{aligned}$$

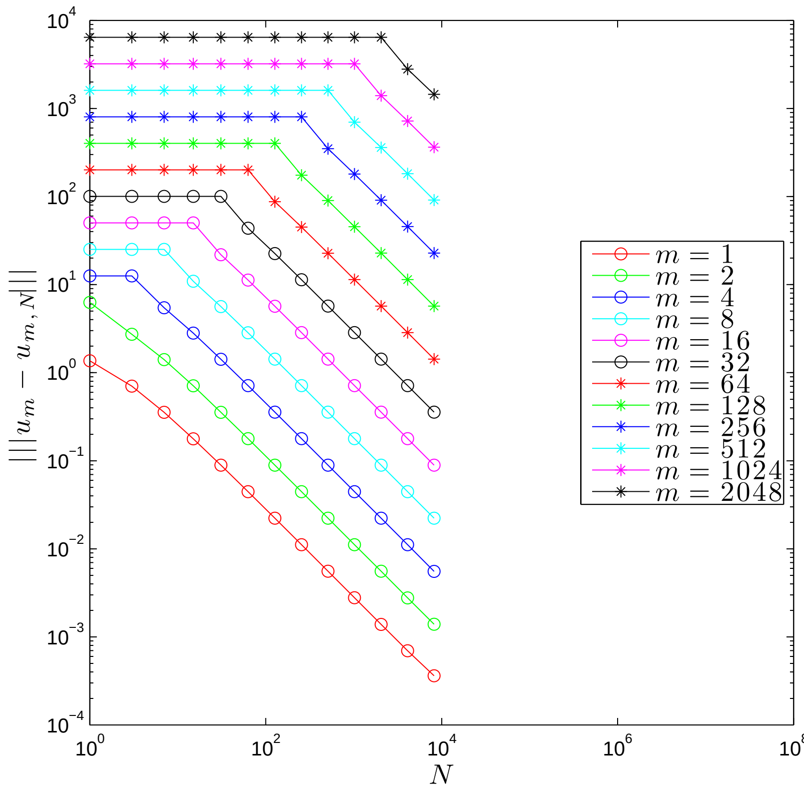
since $\sin(0) = 0$ and $\sin(m\pi) = 0$ when m is a positive integer. Therefore, the fact that $(u_m, u_m) = 1$ allows us to conclude that

$$a(u_m, u_m) = m^2\pi^2.$$

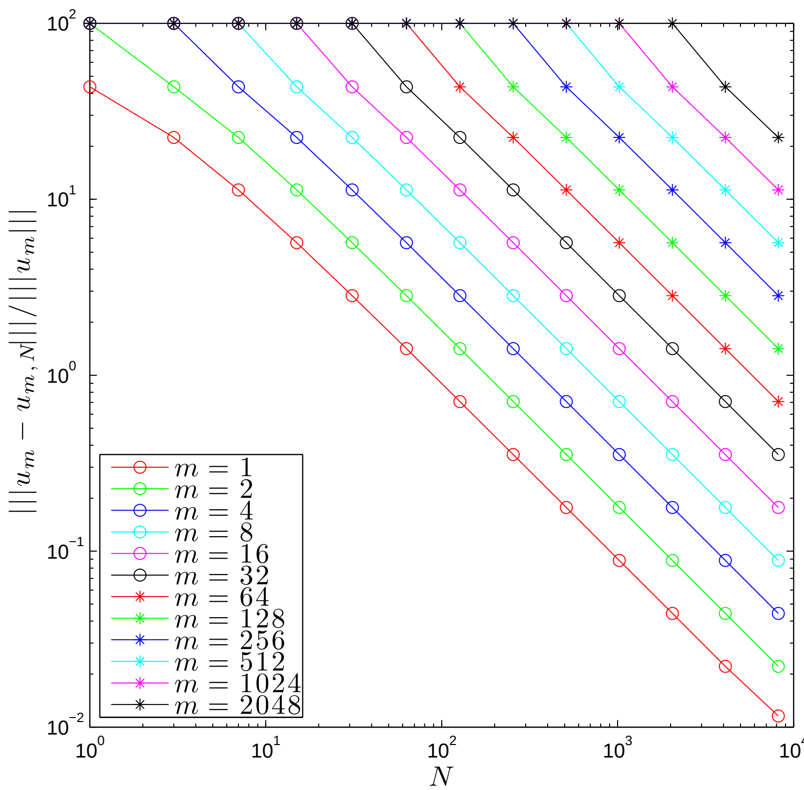
(b) [10 points] The code for this part and part (c) is:

```
clear
clc
figure(1)
clf
figure(2)
clf

markercolor='-or-og-ob-oc-om-ok-*r-*g-*b-*c-*m-*k';

mvec=2.^(0:1:11);
Nvec=2.^(0:1:13)-1;

for i=1:length(mvec)
    m=mvec(i);
    energyerr=zeros(size(Nvec));
    for j=1:length(Nvec)
        N=Nvec(j);
        h=1/(N+1);

        % construct the stiffness matrix
        K=sparse(N,N);
        K=K+sparse(1:N-1,2:N,-1/h,N,N);
        K=K+K.';
        K=K+sparse(1:N,1:N,2/h,N,N);

        % construct the load vector
        f=(2*sqrt(2)/h)*(1-cos(h*m*pi))*sin((1:N).'*h*m*pi);

        % solve for the coefficients
        c=K\f;

        energyerr(j)=sqrt(m^2*pi^2-(c.')*K*c);
    end

    % plot the energy norm of the error
    figure(1)
    loglog(Nvec,energyerr,markercolor(3*i-2:3*i))
    hold on

    % plot the percentage error
    figure(2)
    loglog(Nvec,100*energyerr/(m*pi),markercolor(3*i-2:3*i))
    hold on

    legendStr{i}=['$m=' num2str(m) '$'];
end

  figure(1)
  axis([1 10^8 10^(-4) 10^4])
  axis square
  legend(legendStr,'interpreter','latex','FontSize',14,'Location','East')
  xlabel('$N$','interpreter','latex','FontSize',14)
  ylabel('$|||u_m-u_{m,N}|||$','interpreter','latex','FontSize',14)
  saveas(figure(1),'hw33b.eps','epsc')

  figure(2)
  axis square
  legend(legendStr,'interpreter','latex','FontSize',14,'Location','Southwest')
  xlabel('$N$','interpreter','latex','FontSize',14)
  ylabel('$|||u_m-u_{m,N}|||/|||u_m|||$','interpreter','latex','FontSize',14)
  saveas(figure(2),'hw33c.eps','epsc')
```

The requested plot is below.



(c) [5 points] The requested plot is below.



(d) [5 points] For $j = 0, \ldots, N$, when $x \in (x_j, x_{j+1})$, $\phi_k''(x) = 0$ for $k = 1, \ldots, N$. Hence, if $\tilde{u}_N \in V_N$ then for $j = 0, \ldots, N$, when $x \in (x_j, x_{j+1})$, $\tilde{u}_N''(x) = 0$. Therefore,

$$\sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 \, dx = \sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (f(x))^2 \, dx = \int_0^1 (f(x))^2 \, dx$$

for all $\tilde{u}_N \in V_N$. So, if $\tilde{u}_N \in V_N$, then the value of

$$\sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 \, dx$$

does not change as N increases as for all positive integers N

$$\sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 \, dx = \int_0^1 (f(x))^2 \, dx.$$
