Eigenvalues + Eigenvectors of a symmetric Matrix: Chap 3.5

We previously discussed in class how solving problems like $Ax=b$ can be made simpler by choosing a "good" basis. We mentioned that Orthogonality of a basis was a very desireable trait.

We will see that when a linear operator has a property called "Symmetry" this leads naturally to a set of orthogonal vectors, called eigenvectors, that can make solving problems like $Ax=b$ (or $Lu=f$) much easier.

First, we discuss eigenvalues and eigenvectors of a Symmetric matrix in order to set the stage for more general linear operators.

Defn: Let A be a matrix then a number λ is called an eigenvalue of A if there exists $x \neq 0$ such that $Ax = \lambda x$

Note: λ can be a complex number even if the matrix A is real.

Suppose $x \neq 0$ and $Ax = \lambda x$. Then $(A-\lambda I)x = 0$ and $x \neq 0$ means that $x \in N(A-\lambda I)$. This means that the matrix $A-\lambda I$ has a non-trivial null space and is therefore not invertible. Recall that a matrix B is not invertible iff $\det(B) = 0$.

Hence this means that $\det(A-\lambda I) = 0$.

⇒ Do you remember how to compute a determinant? Review: What is the determinant of a 2×2 matrix, a 3×3 matrix and 4×4 matrix? Familiarize yourself with the method of cofactor expansion for finding the determinant (of a square matrix)

Note: One can show that "$\det(A-\lambda I)$" is a *polynomial* in terms of λ. If A is an $n\times n$ matrix then $p_A(\lambda) = \det(A-\lambda I)$ is a polynomial of degree n.

Therefore "$\det(A-\lambda I)=0$" is "$p_A(\lambda)=0$". That is, we are looking for the (n) roots of the polynomial $p_A(\lambda)$.

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Then $A-\lambda I$ is $\begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

so $\det(A-\lambda I)$ is $p_A(\lambda) = (1-\lambda)(2-\lambda) - 1 = 2 - 3\lambda + \lambda^2 - 1$
$= \lambda^2 - 3\lambda + 1$

so $\det(A-\lambda I) = p_A(\lambda) = 0$ means that λ solves: $\lambda^2 - 3\lambda + 1 = 0$

so that $\lambda = \dfrac{3 \pm \sqrt{5}}{2}$ are the eigenvalues.

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

So that $\det(A - \lambda I) = p_A(\lambda) = (1-\lambda)^3$ so $p_A(\lambda) = 0$ has only one solution, $\lambda = 1$. There is only one eigenvalue, $\lambda = 1$.

Definition:
A vector $x \neq 0$ solving $Ax = \lambda x$ where λ is an eigenvalue of A is called an eigenvector. Eigenvectors are associated with their eigenvalues.

Ex:
$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda = \dfrac{3 \pm \sqrt{5}}{2}$

$\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ has eigenvector $x_1 = \begin{bmatrix} \frac{1}{2}(-1+\sqrt{5}) \\ 1 \end{bmatrix}$

$\lambda_2 = \dfrac{3-\sqrt{5}}{2}$ has eigenvector $x_2 = \begin{bmatrix} \frac{1}{2}(-1-\sqrt{5}) \\ 1 \end{bmatrix}$

Ex: The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has only one eigenvalue λ and only one linearly independent eigenvector given by: $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$