CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 40 · Solutions

Posted Wednesday 13 November 2013. Due 5pm Wednesday 20 November 2013.

40. [25 points] Let

$$\Omega = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$

and let $f \in C(\Omega)$ be defined by $f(x, y) = x(1 - y)$. In this question we will consider the problem of finding the solution $u(x, y)$ to the steady-state heat equation

$$-(u_{xx}(x, y) + u_{yy}(x, y)) = f(x, y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Let

$$C_D^2(\Omega) = \left\{ v \in C^2(\Omega) : v(x, 0) = v(x, 1) = v(0, y) = v(1, y) = 0,\ 0 \le x \le 1,\ 0 \le y \le 1 \right\}.$$

Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by

$$(Lv)(x, y) = -\left(v_{xx}(x, y) + v_{yy}(x, y)\right).$$

Let the inner product $(\cdot, \cdot) : C(\Omega) \times C(\Omega) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 \int_0^1 v(x, y) w(x, y)\, dx\, dy.$$

(a) Show that L is symmetric by showing that

$$(Lv, w) = (v, Lw) \text{ for all } v, w \in C_D^2(\Omega).$$

(b) The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions

$$\psi_{j,k}(x, y) = 2 \sin(j\pi x) \sin(k\pi y)$$

for $j, k = 1, 2, \ldots$, which are such that

$$L\psi_{j,k} = \lambda_{j,k} \psi_{j,k}$$

for $j, k = 1, 2, \ldots$. Obtain a formula for $\lambda_{j,k}$ for $j, k = 1, 2, \ldots$.

(c) Let

$$C_D^2[0, 1] = \left\{ v \in C^2[0, 1] : v(0) = v(1) = 0 \right\}$$

and let the linear operator $L_1 : C_D^2[0, 1] \to C[0, 1]$ be defined by

$$L_1 w = -w''.$$

Use what you know about the eigenfunctions of L_1 to compute $(\psi_{j,k}, \psi_{m,n})$ for $j, k, m, n = 1, 2, \ldots$.

(d) The solution to $Lu = f$ that we obtain using the spectral method is

$$u(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})}\, \psi_{j,k}(x, y).$$

Plot

$$u_N(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y)$$

for $N = 1,2,3,10$. Note that, for $j,k = 1,2,\ldots,$

$$(f,\psi_{j,k}) = 2\frac{(-1)^{j+1}}{jk\pi^2}.$$

Also note that to plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$ you could use

```
x = linspace(0,1,50);
y = linspace(0,1,50);
[X,Y] = meshgrid(x,y);
psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,psi11)
```

Solution.

(a) [6 points] If $v \in C_D^2(\Omega)$ and $w \in C_D^2(\Omega)$ then

$$\begin{aligned}
(Lv,w) &= -\int_0^1\int_0^1 (v_{xx}(x,y)+v_{yy}(x,y))\,w(x,y)\,dx\,dy\\
&= -\int_0^1\int_0^1 v_{xx}(x,y)w(x,y)\,dx\,dy - \int_0^1\int_0^1 v_{yy}(x,y)w(x,y)\,dx\,dy\\
&= -\int_0^1\int_0^1 v_{xx}(x,y)w(x,y)\,dx\,dy - \int_0^1\int_0^1 v_{yy}(x,y)w(x,y)\,dy\,dx\\
&= -\int_0^1 [v_x(x,y)w(x,y)]_{x=0}^{x=1} - \int_0^1 v_x(x,y)w_x(x,y)\,dx\,dy\\
&\quad -\int_0^1 [v_y(x,y)w(x,y)]_{y=0}^{y=1} - \int_0^1 v_y(x,y)w_y(x,y)\,dy\,dx\\
&= -\int_0^1 [v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1} + \int_0^1 v(x,y)w_{xx}(x,y)\,dx\,dy\\
&\quad -\int_0^1 [v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1} + \int_0^1 v(x,y)w_{yy}(x,y)\,dy\,dx\\
&= -\int_0^1 [v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1}\;dy - \int_0^1\int_0^1 v(x,y)w_{xx}(x,y)\,dx\,dy\\
&\quad -\int_0^1 [v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1}\;dx - \int_0^1\int_0^1 v(x,y)w_{yy}(x,y)\,dy\,dx\\
&= -\int_0^1 [v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1}\;dy - \int_0^1\int_0^1 v(x,y)w_{xx}(x,y)\,dx\,dy\\
&\quad -\int_0^1 [v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1}\;dx - \int_0^1\int_0^1 v(x,y)w_{yy}(x,y)\,dx\,dy\\
&= -\int_0^1 [v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1}\;dy\\
&\quad -\int_0^1 [v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1}\;dx\\
&\quad -\int_0^1\int_0^1 v(x,y)\,(w_{xx}(x,y)+w_{yy}(x,y))\;dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^1 v_x(1,y)w(1,y) - v_x(0,y)w(0,y) - v(1,y)w_x(1,y) + v(0,y)w_x(0,y)\,dy \\
&\quad -\int_0^1 v_y(x,1)w(x,1) - v_y(x,0)w(x,0) - v(x,1)w_y(x,1) + v(x,0)w_y(x,0)\,dx \\
&\quad +(v, Lw) \\
&= (v, Lw)
\end{aligned}
$$

since $w(1,y) = w(0,y) = v(1,y) = v(0,y) = w(x,1) = w(x,0) = v(x,1) = v(x,0) = 0$ because $v, w \in C_D^2(\Omega)$. Consequently, $(Lv, w) = (v, Lw)$ for all $v, w \in C_D^2(\Omega)$.

(b) [4 points] We can compute that, for $j, k = 1, 2, \ldots$,

$$
\begin{aligned}
(L\psi_{j,k})(x,y) &= -\frac{\partial^2}{\partial x^2}(2\sin(j\pi x)\sin(k\pi y)) - \frac{\partial^2}{\partial y^2}(2\sin(j\pi x)\sin(k\pi y)) \\
&= 2j^2\pi^2\sin(j\pi x)\sin(k\pi y) + 2k^2\pi^2\sin(j\pi x)\sin(k\pi y) \\
&= 2\left(j^2+k^2\right)\pi^2\sin(j\pi x)\sin(k\pi y) \\
&= \left(j^2+k^2\right)\pi^2\psi_{j,k}(x,y).
\end{aligned}
$$

Hence,

$$\lambda_{j,k} = (j^2+k^2)\pi^2 \text{ for } j, k = 1, 2, \ldots.$$

(c) [7 points] We have that

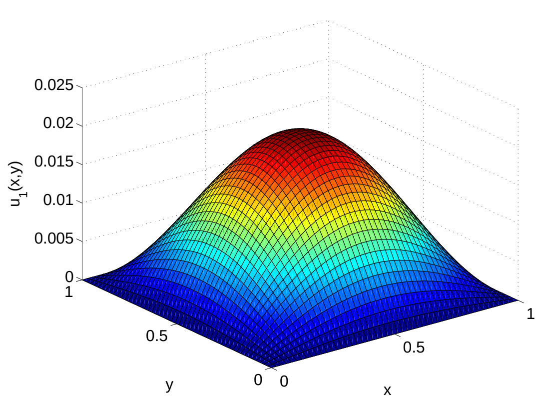
$$
\begin{aligned}
(\psi_{j,k}, \psi_{m,n}) &= \int_0^1\int_0^1 2\sin(j\pi x)\sin(k\pi y)2\sin(m\pi x)\sin(n\pi y)\,dx\,dy \\
&= \int_0^1 2\sin(k\pi y)\sin(n\pi y)\int_0^1 2\sin(j\pi x)\sin(m\pi x)\,dx\,dy \\
&= \int_0^1 2\sin(j\pi x)\sin(m\pi x)\,dx\int_0^1 2\sin(k\pi y)\sin(n\pi y)\,dy \\
&= \int_0^1 \sqrt{2}\sin(j\pi x)\sqrt{2}\sin(m\pi x)\,dx\int_0^1 \sqrt{2}\sin(k\pi y)\sqrt{2}\sin(n\pi y)\,dy \\
&= \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{if } j \neq m \text{ or } k \neq n \end{cases}
\end{aligned}
$$

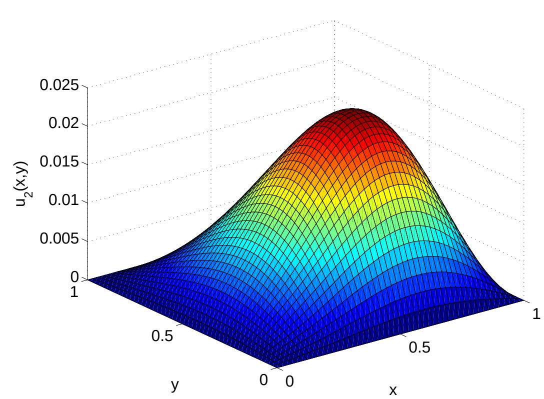
for $j, k, m, n = 1, 2, 3, \ldots$ since

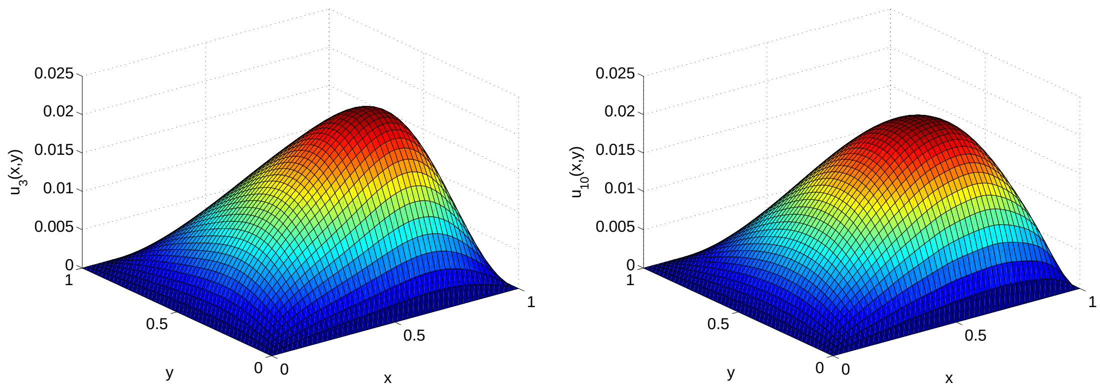
$$\int_0^1 \sqrt{2}\sin(m\pi s)\sqrt{2}\sin(n\pi s)\,ds = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

for $m, n = 1, 2, 3, \ldots$.

(d) [8 points] The requested plots and the code used to produce them are below.








```
clear
clc
npts = 50;
x = linspace(0,1,npts);
y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
for n=1:10
    figure(1)
    clf
    U = zeros(npts,npts);
    for j=1:n
        for k=1:n
            U = U + 4*(-1)^(j+1)/(j*k*pi^2)*sin(j*pi*X).*sin(k*pi*Y)/(j^2+k^2)/(pi^2);
        end
    end
    surf(X,Y,U), drawnow
    set(gca,'fontsize',16)
    xlabel('x')
    ylabel('y')
    zlabel(['u_{' num2str(n) '}(x,y)'])
    if ismember(n,[1 2 3 10]),
        eval(sprintf('print -depsc2 twoD%d', n))
    end
    pause
end
```