CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 27 · Solutions

Posted Friday 28 February 2014. Due 1pm Friday 14 March 2014.

27. [25 points]

All parts of this question should be done by hand.

Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L:S\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$S=\{w\in C^2[0,1]: w'(0)=w(1)=0\}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv,w)=(v,Lw)\text{ for all } v,w\in S.$$

Let N be a positive integer and let $f\in C[0,1]$ be defined by

$$f(x)=\begin{cases}1-2x & \text{if } x\in\left[0,\frac{1}{2}\right];\\ 0 & \text{otherwise.}\end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x)=\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n=1,2,\ldots$. Note that, for $m,n=1,2,\ldots$,

$$(\psi_m,\psi_n)=\begin{cases}1 & \text{if } m=n;\\ 0 & \text{if } m\neq n.\end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n=1,2,\ldots$.

(b) Compute the best approximation to f from span $\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=f(x),\quad 0<x<1$$

and

$$\tilde{u}'(0)=\tilde{u}(1)=0.$$

(d) What is the best approximation to $\tilde{u}$ from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$?

(e) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

Solution.

(a) [3 points] We can compute that, for $n = 1, 2, \ldots$,

$$\psi_n'(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)\pi \sin\left(\frac{2n-1}{2}\pi x\right).$$

and

$$\psi_n''(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)^2 \pi^2 \cos\left(\frac{2n-1}{2}\pi x\right).$$

and so

$$L\psi_n = -\psi_n'' = \left(\frac{2n-1}{2}\right)^2 \pi^2 \psi_n.$$

Hence,

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \pi^2 = (2n-1)^2 \frac{\pi^2}{4} \text{ for } n = 1, 2, \ldots.$$

(b) [8 points] Since $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$, the best approximation to f from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$f_N = \sum_{n=1}^{N} (f, \psi_n)\psi_n.$$

Now, for $n = 1, 2, \ldots$,

$$
\begin{aligned}
&(f, \psi_n) \\
&= \int_0^1 f(x)\psi_n(x)\,dx \\
&= \int_0^{1/2} f(x)\psi_n(x)\,dx + \int_{1/2}^1 f(x)\psi_n(x)\,dx \\
&= \int_0^{1/2} (1-2x)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)\,dx + \int_{1/2}^1 0\,dx \\
&= \sqrt{2}\int_0^{1/2} (1-2x)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx + 0 \\
&= \sqrt{2}\left(\left[(1-2x)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} - \int_0^{1/2}(-2)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right) \\
&= \sqrt{2}\left(0 - 0 + \frac{4}{(2n-1)\pi}\int_0^{1/2}\sin\left(\frac{2n-1}{2}\pi x\right)\,dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{2}\frac{4}{(2n-1)\,\pi}\left[-\frac{2}{(2n-1)\,\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} \\
&= \frac{4\sqrt{2}}{(2n-1)\,\pi}\left(-\frac{2}{(2n-1)\,\pi}\cos\left(\frac{2n-1}{4}\pi\right)-\left(-\frac{2}{(2n-1)\,\pi}\right)\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\,\pi^2}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_N(x) &= \sum_{n=1}^{N}(f,\psi_n)\psi_n(x) \\
&= \sum_{n=1}^{N}(f,\psi_n)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right) \\
&= \sum_{n=1}^{N}\frac{16}{(2n-1)^2\,\pi^2}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right)\cos\left(\frac{2n-1}{2}\pi x\right).
\end{aligned}
$$

(c) [4 points] Now, $\tilde{u}$ is the solution to $L\tilde{u} = f$ and so the spectral method yields the series solution

$$
\tilde{u}(x) = \sum_{n=1}^{\infty}\frac{(f,\psi_n)}{\lambda_n}\psi_n(x) = \sum_{n=1}^{\infty}\frac{64}{(2n-1)^4\,\pi^4}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right)\cos\left(\frac{2n-1}{2}\pi x\right).
$$

(d) [4 points] The best approximation to $\tilde{u}$ from span $\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$
\tilde{u}_N(x) = \sum_{n=1}^{N}\frac{(f,\psi_n)}{\lambda_n}\psi_n(x) = \sum_{n=1}^{N}\frac{64}{(2n-1)^4\,\pi^4}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right)\cos\left(\frac{2n-1}{2}\pi x\right).
$$

(e) [6 points] Let $\tilde{u}$ be the solution to $L\tilde{u} = f$ and let $w \in C^2[0,1]$ be such that

$$
-w''(x) = 0, \quad 0 < x < 1
$$

and

$$
w'(0) = w(1) = 1.
$$

Then $u(x) = w(x) + \tilde{u}(x)$ will be such that

$$
-u''(x) = -w''(x) - \tilde{u}''(x) = 0 + f(x) = f(x);
$$

$$
u'(0) = w'(0) + \tilde{u}'(0) = 1 + 0 = 1;
$$

and

$$
u(1) = w(1) + \tilde{u}(1) = 1 + 0 = 1.
$$

Now, the general solution to

$$
-w''(x) = 0
$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(0) = 1$ when $A = 1$. Hence, $w(x) = x + B$ and so $w(1) = 1$ when $B = 0$. Consequently,

$$
w(x) = x
$$

and so

$$u(x) = x + \tilde{u}(x).$$

We can then use the series solution to $L\tilde{u} = f$ that we obtained in part (c) to obtain the series solution

$$u(x) = x + \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right)$$

to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1;$$

$$u'(0) = u(1) = 1.$$
