Study Highlights: 01/14/2015

Chapter #2: Mathematical Models in ONE Spatial Dimension

- PDE have been used for centuries to model the physical world
- PDE can be very challenging: models are often studied first in one or two dimensions
- Several important PDE that act as Heuristic "building blocks" of more complicated processes:
  - Heat equation
  - WAVE Equation
  - transport equation

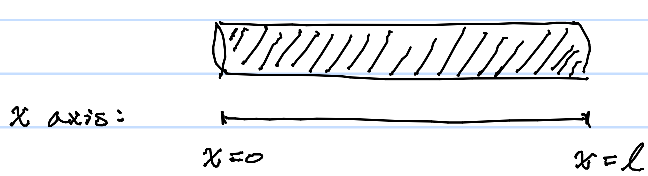
□ This chapter focuses on showing the basics of the <u>Modeling Process</u>. This process typically leans on arguments from <u>Physics / physical principles</u> and sometimes <u>statistics</u>. Typically, to arrive at a model, several simplifying assumptions tend to be made. These assumptions must be kept in mind when considering the solution (either analytic or computed)

▷ <u>Chapter 2.1</u>: Heat flow in a bar, Fourier's law and the heat equation

- Assumptions:
  - Long, thin bar
  - uniform cross sections
  - temperature varies only with x direction

⇒ It will be shown in chapter 11 that these assumptions imply that for all subsequent times the temp. depends only on the x direction

⇒ The model is therefore one dimensional (in space)

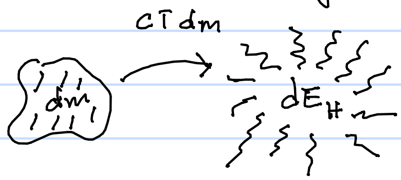


* We want to model the change (or flow) of <u>Heat Energy</u> in the bar

Modeling Assumption :

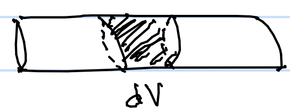
- WE ARE only interested in modeling the heat/energy flow near a reference temperature T_0
- Assume an approximately linear relationship betwee the heat energy in the bar and its temperature T w.r.t a reference temp. T_0. I.E there exists c such that $(E_H) = c(T - T_0) + E_0$

Implication: The heat energy of a small mass, dm, of uniform material at a temperature T is given by $dE_H = c(T - T_0)\,dm + E_0$ where $\underline{\underline{c}}$ the specific heat of the material and T_0 is a reference temperature (so that $T - T_0$ is the temp. w.r.t the reference temp.) and E_0 is the reference heat Energy.



Drawback: Our model will only work for temperatures near a reference temperature but the definition of "NEAR" can vary depending on the specific material in question.

Consider a small volume of the bar then this volume has mass:



$dm = \rho\, dV.$

Letting A be the cross sectional area of the bar g that the volume of the bar is: $dV = A\Delta x$. We assumed the temperature depended only on x: denote the temperature at x and time t as $u(x,t)$.

Then the total energy in dV at time t is:

$$dE_h = \int_x^{x+\Delta x} A\rho\,\underbrace{(u(x,t) - T_0)}_{\text{temp wrt Refence}}\,ds + \underbrace{E_0}_{\text{reference energy}}$$

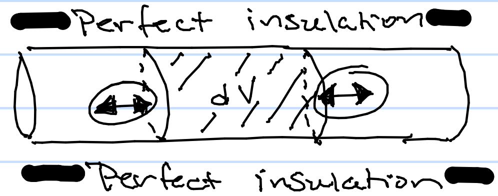
$A\rho\, ds = \rho\,(A\,ds)$
$= \rho\,(\text{infintessimal volume})$
$= \text{infintessimal mass.}$

$$\Rightarrow dE_h = \hat{E}_0 + \int_x^{x+\Delta x} A\rho\, u(x,t)\, ds$$

⇒ Next we consider two different ways to specify the rate of change of heat energy in the small volume dV

1) take time deriv of dE_n

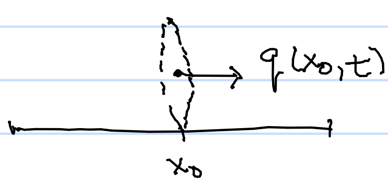
2) Assume the bar is perfect insulated on the boundary. Then the only way for heat energy to change in dV is to flow in or out from the ends or to be added/removed from the <u>inside</u> of dV



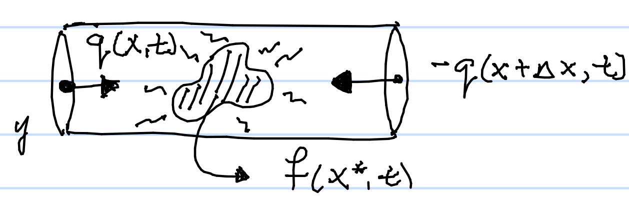
1) $$\frac{d}{dt} E_n = \frac{d}{dt}\left(\hat{E}_0 + \int_x^{x+\Delta x} A\rho\, u(x,t)\, ds \right) = \frac{d}{dt}\int_x^{x+\Delta x} A\rho\, u(x,t)\, ds$$
$$= \int_x^{x+\Delta x} A\rho\, \partial_t u(x,t)\, ds$$

this equality follows from a <u>theorem</u> in mathematics that requires $\partial_t u(x,t)$ be known to be a <u>continuous function</u>. This step adds to the list of modeling assumptions

2) Let $q(x,t)$ denote the rate at which heat energy is flowing through the cross section at x and time t. $q(x,t)$ is positive if heat flows "right" and negative if heat flows "left",



Then the net heat energy in dV is the amount of heat energy flowing <u>in</u> (which can be negative for "flowing out") plus the Energy <u>created</u> or <u>removed</u> (source/sink) in dV. Let "$f(x,t)$" denote the energy creation at x and time t so that the expression $\int_x^{x+\Delta x} f(s,t)\, ds$ is the energy created/removed in $(x, x+\Delta x)$ at time t. f is positive for a <u>source</u> and negative for a <u>sink</u>.



Then the amount of heat energy in dV is: $A(q(x,t) - q(x+\Delta x,t) + \int_x^{x+\Delta x} f(s,t)\,ds$

Which gives: $A\left(q(x,t) - q(x+\Delta x,t)\right) + \int_x^{x+\Delta x} f(s,t)\,ds = \int_x^{x+\Delta x} \left(A - \frac{\partial q}{\partial x}(s,t) + f(s,t)\right) ds$
by the <u>fundamental theorem of calculus</u>.

<u>IDEA</u>: We now have two expressions for the same thing so they must be equal!

$$(1) = (2) \Rightarrow \int_x^{x+\Delta x} Ac\rho\, \partial_t u(s,t)\,ds = -\int_x^{x+\Delta x} \left(A \frac{\partial q}{\partial x}(s,t) + f(s,t)\right) ds$$

$$\Rightarrow \int_x^{x+\Delta x} \left\{Ac\rho\, \partial_t u(s,t) + A\frac{\partial q}{\partial x}(s,t)\right\} ds = \int_x^{x+\Delta x} f(s,t)\,ds$$

Since this holds for any x, Δx and all t it must follow that $A\rho\, \partial_t u(s,t) + A\frac{\partial q}{\partial x}(s,t) = f(s,t)$

- <u>Key IDEA: Fourier's Law:</u> $q \approx \frac{\partial u}{\partial x} \Leftrightarrow$ there exists $K(x)$ with $q(x,t) = K(x)\frac{\partial u}{\partial x}(x,t)$

So that the result is the heat equation:
$$Ac\rho\, \partial_t u(x,t) + A\frac{\partial}{\partial x}\left(-K\frac{\partial u}{\partial x}\right) = f$$
Dividing by A and assuming that c and K could depend on x yields (we just absorb the constant A into the f term):

$$\boxed{c(x)\rho(x)\,\partial_t u(x,t) + \frac{\partial}{\partial x}\left(-K(x)\,\partial_x u(x,t)\right) = f} \qquad (1)$$

▷ <u>Chapter 2.1.1: Boundary Conditions for the Heat Equation:</u>

- Without any mathematics at all we can see that equation (1) cannot be a complete physical model.

- Why? The rod is perfectly insulated on the edges but heat energy flows through cross sections. Therefore, what happens at the ends of the rod?

- The specification of what happens "at the end" are called <u>boundary conditions</u>.

- We will consider two possible boundary conditions:

  1) Perfectly insulated endpoints: corresponds to <u>no heat flux</u> through the endpoints.

     <u>e.g</u> $q(0,t)=0$ and $q(l,t)=0$
     by "Fourier's law" this becomes:
     $-K\frac{\partial u}{\partial x}(0,t)=0$, $-K\frac{\partial u}{\partial x}(l,t)=0$

  2) Ends of the rod are held at a prescribed temperature {without loss of generality assume zero degrees}
     $u(0,t) = u(l,t) = 0$

⇒ Boundary condition terminology:

A) When <u>derivatives</u> are prescribed on the boundary we call this "Neumann Boundary Conditions"

B) When <u>values</u> are prescribed on the boundary we call this "Dirichlet Boundary Conditions"

C) When a derivative is prescribed on one side and a value is prescribed on another this is called a "mixed boundary condition"

- We are still not done? Why?
  <u>A</u>: The heat distribution is changing with time.
  So we need to know the <u>initial heat distribution</u>.
  We need to know the initial value at time $t=0$. IE we need to know $u(x,0)$ for all $0 \leq x \leq l$.

→ Our complete system specifies:
1) The partial differential equation to be solved
2) The boundary conditions
3) The initial value

This type of problem is called an <u>I</u>nitial <u>B</u>oundary <u>V</u>alue <u>P</u>roblem or <u>IBVP</u>.

Chapter 2.1.2: Steady State Heat Flow

- "Steady State" means that nothing changes with time. This means that $\frac{\partial}{\partial t} u = 0$

So the steady state version of equation (1) is given by:

$$\boxed{\frac{\partial}{\partial x}\left(-K(x) \frac{\partial}{\partial x} u(x) \right) = f} \qquad (2)$$

Note: In the steady state case the source term, f, cannot depend on time! If it did, it would cause time variation in the unknown function, u. However we are assuming that $\frac{\partial}{\partial t} u = 0$!

- If we want to solve a steady state problem we dont need to know the initial heat distribution. So the full steady state problem is:
  1) Differential Equation
  2) Boundary Values

Ex:
$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(-K(x) \frac{\partial}{\partial x} u(x) \right) = 0 \\ &u(0) = u(\ell) = 0 \end{aligned}\right]$$
This is a boundary value problem (BVP)

Note that since $\frac{\partial}{\partial t} u = 0$ u depends only on the variable x. Hence this is an ordinary differential equation (ODE).

Ex: Solve the following BVP:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial x^2} u(x) = 0 \\ &u(0) = 4 \quad u(\ell) = 10 \end{aligned}\right]$$
(Steady State) 2nd order homogeneous ODE w/ Dirichlet boundary Conditions

Solution: $\frac{\partial}{\partial x} u(x) = C_1 \rightarrow \quad u(x) = C_1 x + C_2$

Boundary Conditions: $u(0) = C_2 = 4 \qquad u(\ell) = C_1 \ell + 4 = 10$

So that $C_1 = 6/\ell$ and

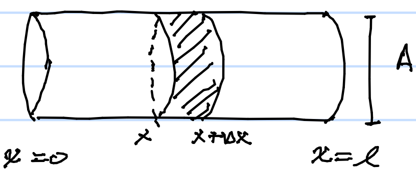
$$u(x) = \frac{6}{\ell} x + 4$$

Chapter 2.1.3 : Diffusion

- Consider a tube filled with liquid (water, say) and suppose the pipe contains a chemical whose concentration varies only with the x direction. Suppose the tube has length ℓ and cross-sectional area A.

- Let $u(x,t)$ be the concentration of chemical, in units of mass per volume, at a point x and time t. Then the mass of chemical from x to $x+\Delta x$ at time t is given by:

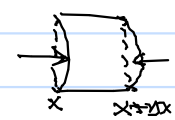
$$\int_x^{x+\Delta x} A\,u(s,t)\,ds$$



The change in mass between x and $x+\Delta x$ is therefore:

$$\frac{\partial}{\partial t}\int_x^{x+\Delta x} A\,u(s,t)\,ds \longrightarrow \int_x^{x+\Delta x} A\,\frac{\partial}{\partial t}u(s,t)\,ds$$

- The chemical mass in $(x, x+\Delta x)$ can change only by
  1) chemical leaving / entering through the endpoints
  2) chemical introduced / removed directly inside



- The amount of chemical mass is therefore $A(q(x,t) - (x+\Delta x,t)) + f$ where $q(x,t)$ is the <u>concentration flux</u> <u>per unit time</u> and f is the source/sink term.

- Assuming that $q(x,t)$ is proportional to $\frac{\partial}{\partial x}u(x,t)$ then gives

$$A\left(q(x,t) - q(x+\Delta x,t)\right) + f = A\left(K\left[\frac{\partial}{\partial x}u(x,t) - \frac{\partial}{\partial x}u(x+\Delta x,t)\right]\right) + f$$
$$= -\int_x^{x+\Delta x} A\,\frac{\partial}{\partial x}\left(K\frac{\partial}{\partial x}u(x,t)\right)ds + f$$

- Equating these expressions gives:

$$\int_x^{x+\Delta x} A\,\frac{\partial}{\partial t}u(s,t)\,ds = -\int_x^{x+\Delta x} A\,\frac{\partial}{\partial x}\left(K\,\frac{\partial}{\partial x}u(s,t)\right)ds + f$$

Since x and $x+\Delta x$ are arbitrary this gives: $\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(K\frac{\partial}{\partial x}u\right) = f$

<u>Note: this is the same as the heat equation!!</u>