# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 30 · Solutions

Posted Monday 21 October 2013. Due 5pm Wednesday 30 October 2013.

30. [25 points] Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\widehat{\phi}_j \in C[0,1]$ be such that

$$\widehat{\phi}_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$. Also, let the continuous piecewise quadratic functions $\phi_j \in C[0,1]$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in C[0,1]$ be such that

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N+1$.

(a) What is

i. $\phi_j\left(x_k\right)$ for $j = 1, \ldots, N$ and $k = 0, \ldots, N+1$;

ii. $\phi_j\left(\dfrac{x_{k-1} + x_k}{2}\right)$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$;

iii. $\psi_j\left(x_k\right)$ for $j = 1, \ldots, N+1$ and $k = 0, \ldots, N+1$;

iv. $\psi_j\left(\dfrac{x_{k-1} + x_k}{2}\right)$ for $j, k = 1, \ldots, N+1$.

(b) Show that $\{\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_{N+1}\}$ is linearly independent by showing that if $\alpha_j, \beta_j \in \mathbb{R}$ and $\displaystyle\sum_{j=1}^{N} \alpha_j \phi_j(x) + \sum_{j=1}^{N+1} \beta_j \psi_j(x) = 0$ for all $x \in [0,1]$ then $\alpha_k = 0$ for $k = 1, \ldots, N$ and $\beta_k = 0$ for $k = 1, \ldots, N+1$.

(c) Obtain an expression for

$$\phi_j + \frac{1}{2}\left(\psi_j + \psi_{j+1}\right)$$

for $j = 1, \ldots, N$.

(d) For $j = 1, \ldots, N$, is $\widehat{\phi}_j \in \operatorname{span}\{\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_{N+1}\}$?

(e) Is $\{\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_{N+1}, \widehat{\phi}_1, \ldots, \widehat{\phi}_N\}$ linearly independent?

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Solution.

(a) [5 points]

i. For $j = 1, \ldots, N$ and $k = 0, 1, \ldots, N+1$, the definition of $\phi_j$ yields that $\phi_j(x_k) = 0$ if $k \neq j$. Moreover, for $j = 1, \ldots, N$,

$$\phi_j(x_j) = \frac{(x_j + x_{j+1} - 2x_j)(x_{j+1} - x_j)}{h^2} = \frac{(x_{j+1} - x_j)(x_{j+1} - x_j)}{h^2} = \frac{h^2}{h^2} = 1.$$

Consequently, for $j = 1, \ldots, N$,

$$\phi_j(x_k) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j, \end{cases}$$

for $k = 0, 1, \ldots, N+1$.

ii. For $j = 1, \ldots, N$, the definition of $\phi_j$ yields that $\phi_j\left(\frac{x_{k-1} + x_k}{2}\right) = 0$ for $k = 1, \ldots, N+1$.

iii. For $j = 1, \ldots, N+1$, the definition of $\psi_j$ yields that $\psi_j(x_k) = 0$ for $k = 0, 1, \ldots, N+1$.

iv. For $j, k = 1, \ldots, N+1$, the definition of $\psi_j$ yields that $\psi_j\left(\frac{x_{k-1} + x_k}{2}\right) = 0$ if $k \neq j$. Moreover, for $j = 1, \ldots, N+1$,

$$\psi_j(x) = \frac{4(x - x_{j-1})(x_j - x)}{h^2} = \frac{(2x - 2x_{j-1})(2x_j - 2x)}{h^2}$$

and so

$$\psi_j\left(\frac{x_{j-1} + x_j}{2}\right) = \frac{(x_{j-1} + x_j - 2x_{j-1})(2x_j - (x_{j-1} + x_j))}{h^2} = \frac{(x_j - x_{j-1})(x_j - x_{j-1})}{h^2} = \frac{h^2}{h^2} = 1.$$

Consequently, for $j = 1, \ldots, N+1$,

$$\psi_j\left(\frac{x_{k-1} + x_k}{2}\right) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j, \end{cases}$$

for $k = 1, \ldots, N+1$.

(b) [5 points] If $\alpha_j, \beta_j \in \mathbb{R}$ and $\sum_{j=1}^{N} \alpha_j \phi_j(x) + \sum_{j=1}^{N+1} \beta_j \phi_j(x) = 0$ for all $x \in [0,1]$ then $\sum_{j=1}^{N} \alpha_j \phi_j(x_k) + \sum_{j=1}^{N+1} \beta_j \phi_j(x_k) = 0$ for $k = 1, \ldots, N$. The answer to parts (a)i. and (a)iii. then allows us to conclude that $\alpha_k = 0$ for $k = 1, \ldots, N$ since $\sum_{j=1}^{N} \alpha_j \phi_j(x_k) + \sum_{j=1}^{N+1} \beta_j \phi_j(x_k) = \alpha_k$. Moreover, if $\alpha_j, \beta_j \in \mathbb{R}$ and $\sum_{j=1}^{N} \alpha_j \phi_j(x) + \sum_{j=1}^{N+1} \beta_j \phi_j(x) = 0$ for all $x \in [0,1]$ then $\sum_{j=1}^{N} \alpha_j \phi_j\left(\frac{x_{k-1} + x_k}{2}\right) + \sum_{j=1}^{N+1} \beta_j \phi_j\left(\frac{x_{k-1} + x_k}{2}\right) = 0$ for $k = 1, \ldots, N+1$. The answer to parts (a)ii. and (a)iv. then allows us to conclude that $\beta_k = 0$ for $k = 1, \ldots, N+1$ since $\sum_{j=1}^{N} \alpha_j \phi_j\left(\frac{x_{k-1} + x_k}{2}\right) + \sum_{j=1}^{N+1} \beta_j \phi_j\left(\frac{x_{k-1} + x_k}{2}\right) = \beta_k$. Therefore, if $\alpha_j, \beta_j \in \mathbb{R}$ and $\sum_{j=1}^{N} \alpha_j \phi_j(x) + \sum_{j=1}^{N+1} \beta_j \phi_j(x) = 0$ for all $x \in [0,1]$ then $\alpha_j = 0$ for $j = 1, \ldots, N$ and $\beta_j = 0$ for $j = 1, \ldots, N+1$.

(c) [5 points] Since

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$,

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N+1$, and

$$\psi_{j+1}(x) = \begin{cases} \dfrac{4(x - x_j)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 0, \ldots, N$, we have that, for $j = 1, \ldots, N$,

$$\begin{aligned}
&\phi_j(x) + \frac{1}{2}\left(\psi_j(x) + \psi_{j+1}(x)\right) \\
&= \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j) + 2(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x) + 2(x - x_j)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{(x - x_{j-1})((2x - x_{j-1} - x_j) + 2(x_j - x))}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)((x_j + x_{j+1} - 2x) + 2(x - x_j))}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{(x - x_{j-1})(x_j - x_{j-1})}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)(x_{j+1} - x_j)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, for $j = 1, \ldots, N$,

$$\phi_j + \frac{1}{2}\left(\psi_j + \psi_{j+1}\right) = \widehat{\phi}_j.$$

(d) [5 points] Yes, since from part (c) we have that

$$\widehat{\phi}_j = \phi_j + \frac{1}{2}\psi_j + \frac{1}{2}\psi_{j+1}$$

and so $\widehat{\phi}_j \in \mathrm{span}\{\phi_1, \ldots, \phi_{\mathrm{N}}, \psi_1, \ldots, \psi_{\mathrm{N}+1}\}$.

(e) [5 points] No, as for any $j = 1, \ldots, N$,

$$\phi_j + \frac{1}{2}\psi_j + \frac{1}{2}\psi_{j+1} - \widehat{\phi}_j = 0.$$