CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 37

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

37. [25 points]

Let the symmetric bilinear form $(\cdot,\cdot) : L^2(0,1)\times L^2(0,1)\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1)\times H^1(0,1)\to\mathbb{R}$ be defined by

$$a(v,w)=\int_0^1 v'(x)w'(x)\,dx.$$

Let $B(\cdot,\cdot) : H^1(0,1)\times H^1(0,1)\to\mathbb{R}$ be defined by

$$B(v,w)=a(v,w)+(v,w).$$

Let the norm $|||\cdot||| : H^1(0,1)\to\mathbb{R}$ be defined by

$$|||v|||=\sqrt{B(v,v)}.$$

Let $f\in L^2(0,1)$, let $\rho\in\mathbb{R}$, let $H^1_D(0,1)=\{w\in H^1(0,1) : w(0)=0\}$ and let $u\in H^1_D(0,1)$ be such that

$$B(u,v)=(f,v)+\rho v(1) \text{ for all } v\in H^1_D(0,1).$$

Moreover, let N be a positive integer, let V_N be a subspace of $H^1_D(0,1)$ and let $u_N\in V_N$ be such that

$$B(u_N,v)=(f,v)+\rho v(1) \text{ for all } v\in V_N.$$

(a) Use the fact that $(\cdot,\cdot)$ is a symmetric bilinear form on $L^2(0,1)$ and the fact that $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$ to show that $B(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$. Recall that $H^1(0,1)=\{v\in L^2(0,1) : v'\in L^2(0,1)\}$.

(b) Show that

$$B(u-u_N,v)=0 \text{ for all } v\in V_N.$$

(c) Show that

$$|||u-u_N|||^2=|||u|||^2-|||u_N|||^2.$$

(d) Show that

$$|||u_N|||^2\le |||u|||^2.$$