CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 37 · Solutions

Posted Friday 1 November 2013. Due 5pm Wednesday 13 November 2013.

37. [25 points] Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in C[0,1]$ be such that

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $H_D^1(0,1) = \{v \in H^1(0,1) : v(1) = 0\}$. Let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Also, let the norm $|||\cdot||| : H_D^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{a(v,v)}.$$

Let $u \in C^2[0,1]$ be the solution of

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1; \\ u'(0) &= \alpha; \\ u(1) &= \beta; \end{aligned}$$

where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$.

(a) It can be shown that

$$a(u, v) = g(f, \alpha, v) \text{ for all } v \in \{w \in C^2[0,1] : w(1) = 0\}$$

where $g(f, \alpha, v)$ is a function of f, α and v. Obtain a formula for $g(f, \alpha, v)$.

(b) For the remainder of this question we shall just consider the case when

$$f(x) = 12x^2 - 24x + 4.$$

Note that in this case you obtained a formula for u in a previous homework. For this part we will just consider the case when $\alpha = \beta = 0$. In this case, we can obtain finite element approximations u_N to u by finding $u_N \in \text{span}\{\phi_0, \ldots, \phi_\text{N}\}$ such that

$$a(u_N, v) = g(f, 0, v) \text{ for all } v \in \text{span}\{\phi_0, \ldots, \phi_\text{N}\}.$$

Write a code which can obtain u_N and u_N^* where $u_N^* \in \text{span}\{\phi_1, \ldots, \phi_\text{N}\}$ is such that

$$a(u_N^*, v) = g(f, 0, v) \text{ for all } v \in \text{span}\{\phi_1, \ldots, \phi_\text{N}\}.$$

On the same figure, plot u as well as u_N and u_N^* for $N = 3$ and $N = 7$.

(c) For the case when $\alpha = \beta = 0$, plot

$$|||u - u_N|||$$

and

$$|||u - u_N^*|||$$

for $N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767$.

(d) Modify your code so that it can obtain finite element approximations u_N to u by finding $u_N \in \text{span}\{\phi_0, \ldots, \phi_\text{N+1}\}$ such that $u_N(1) = \beta$ and

$$a(u_N, v) = g(f, \alpha, v) \text{ for all } v \in \text{span}\{\phi_0, \ldots, \phi_\text{N}\}$$

for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. For the particular case of $\alpha = -1$ and $\beta = 1$, use your code to obtain u_N for $N = 3, 7, 15, 31$ and on the same figure plot u and u_N for $N = 3, 7, 15, 31$.

Solution.

(a) [4 points] If $v \in \{w \in C^2[0,1] : w(1) = 0\}$, then

$$-\int_0^1 u''(x)v(x)\,dx = (f, v).$$

Integration by parts then yields that

$$-[u'(x)v(x)]_0^1 + a(u, v) = (f, v)$$

from which we can conclude that

$$\alpha v(0) + a(u, v) = (f, v)$$

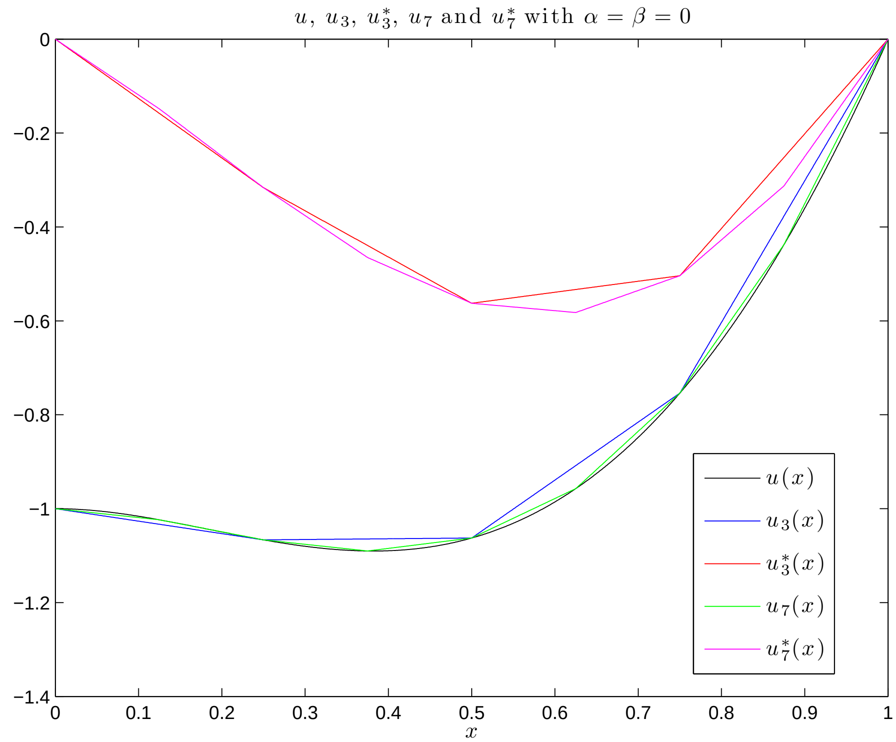
since $v(1) = 0$ and $u'(0) = \alpha$. Therefore,

$$a(u, v) = g(f, \alpha, v) \text{ for all } v \in \{w \in C^2[0,1] : w(1) = 0\}$$

where

$$g(f, \alpha, v) = (f, v) - \alpha v(0).$$

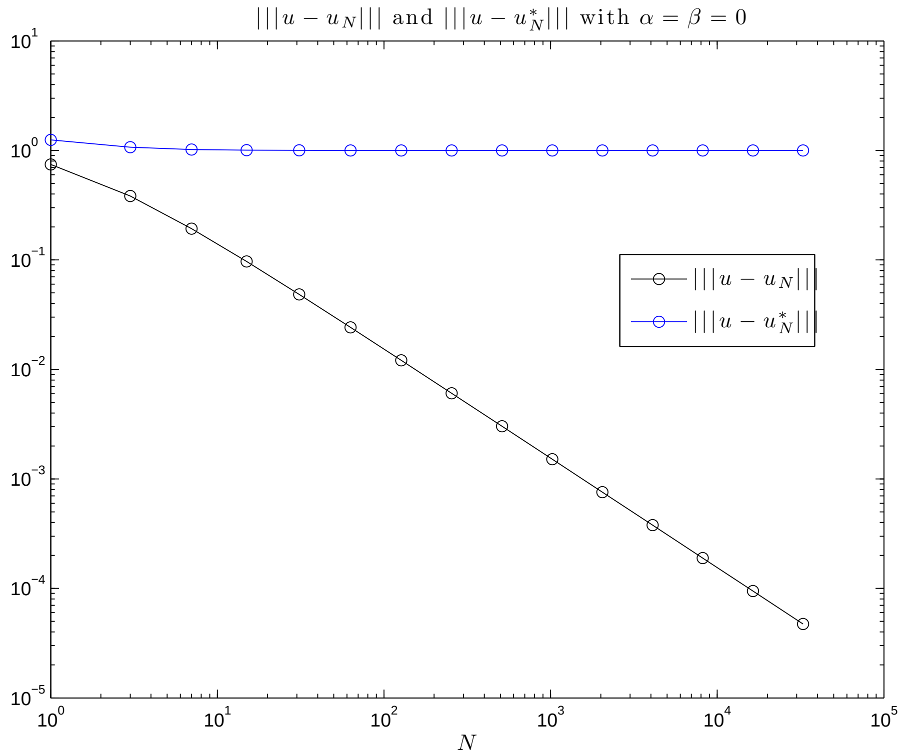
(b) [8 points] The plot and code used to create it, and the plot shown in part (c), are below. Note that the below code uses the MATLAB function which you had to write in Homework 2.



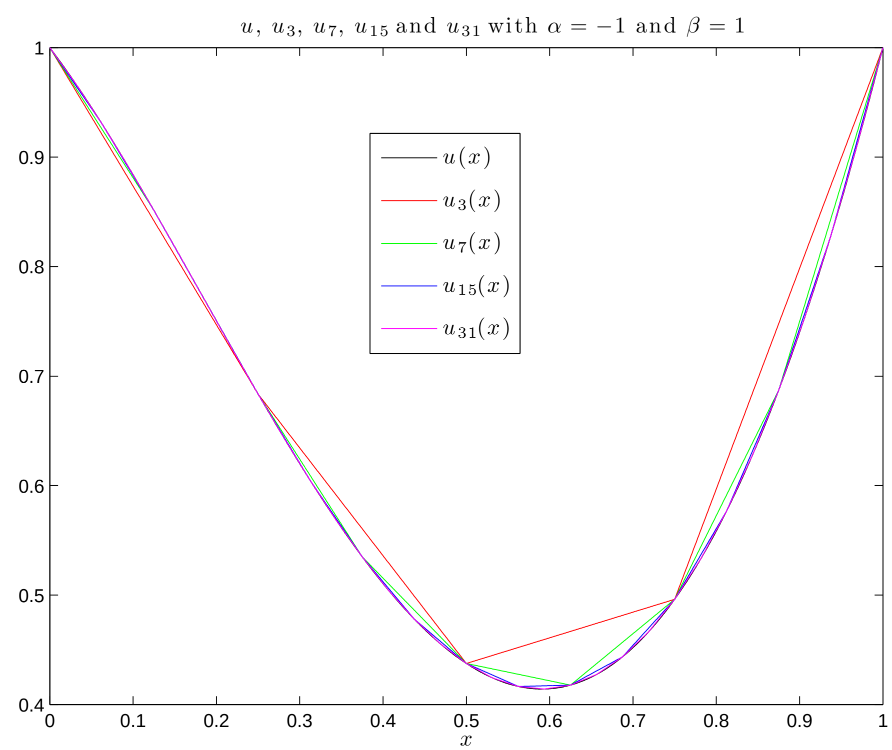

```
clear
clc

figure(1)
clf
x=linspace(0,1,1000).';
u=-x.^4+4*x.^3-2*x.^2-1;
plot(x, u, '-k')
hold on

Nvec=2.^(1:15)-1;
energyerr=zeros(size(Nvec));
energyerrhD=zeros(size(Nvec));
for j=1:length(Nvec)
    N=Nvec(j);
    h=1/(N+1);

    K=sparse(N+1,N+1);
    K=K+sparse(1:N,2:N+1,-1/h,N+1,N+1);
    K=K+K.';
    K=K+sparse(1,1,1/h,N+1,N+1);
    K=K+sparse(2:N+1,2:N+1,2/h,N+1,N+1);

    KhD=K(2:N+1,2:N+1);

    f=zeros(N+1,1);
    f(1)=h*(h^2-4*h+2);
    f(2:N+1)=2*h*(h^2+6*((1:N)*h).^2-12*((1:N)*h)+2);

    fhD=f(2:N+1);

    c=K\f;

    chD=KhD\fhD;
```

```
    energyerr(j)=sqrt(296/105-(c.')*K*c);
    energyerrhD(j)=sqrt(296/105-(chD.')*KhD*chD);

    if N==3
       uN = zeros(size(x));
       for k=1:N+1
           uN = uN + c(k)*hat(x,k-1,N);
       end
       plot(x, uN, '-b')
       uNhD = zeros(size(x));
       for k=1:N
           uNhD = uNhD + chD(k)*hat(x,k,N);
       end
       plot(x, uNhD, '-r')
    end
    if N==7
        uN = zeros(size(x));
        for k=1:N+1
            uN = uN + c(k)*hat(x,k-1,N);
        end
        plot(x, uN, '-g')
        uNhD = zeros(size(x));
        for k=1:N
            uNhD = uNhD + chD(k)*hat(x,k,N);
        end
        plot(x, uNhD, '-m')
    end
end

title('$u$, $u_3$, $u_3^*$, $u_7$ and $u_7^*$ with $\alpha=\beta=0$','interpreter','
    latex','FontSize',12)
xlabel('$x$','interpreter','latex','FontSize',12)
legendstr{1}='$u(x)$';
legendstr{2}='$u_3(x)$';
legendstr{3}='$u_3^*(x)$';
legendstr{4}='$u_7(x)$';
legendstr{5}='$u_7^*(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(1),'hw37c.eps','epsc')

figure(2)
clf
loglog(Nvec,energyerr,'-ok')
hold on
loglog(Nvec,energyerrhD,'-ob')
title('$|||u-u_N|||$ and $|||u-u_N^*|||$ with $\alpha=\beta=0$','interpreter','latex','
    FontSize',12)
xlabel('$N$','interpreter','latex','FontSize',12)
legendstr2{1}='$|||u-u_N|||$';
legendstr2{2}='$|||u-u_N^*|||$';
legend(legendstr2,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(2),'hw37d.eps','epsc')
```

(c) [6 points] The plot is below.



(d) [7 points] The plot and code used to create it are below. Note that the below code uses the MATLAB function which you had to write in Homework 2.



```
clear
clc

alpha=-1;
beta=1;

figure(1)
clf
x=linspace(0,1,1000).';
u=-x.^4+4*x.^3-2*x.^2+alpha*x+beta-alpha-1;
plot(x, u, '-k')
hold on

colors='rgbm';
Nvec=2.^(2:5)-1;
for j=1:length(Nvec)
    N=Nvec(j);
    h=1/(N+1);

    K=sparse(N+1,N+1);
    K=K+sparse(1:N,2:N+1,-1/h,N+1,N+1);
    K=K+K.';
    K=K+sparse(1,1,1/h,N+1,N+1);
    K=K+sparse(2:N+1,2:N+1,2/h,N+1,N+1);

    f=zeros(N+1,1);
    f(1)=h*(h^2-4*h+2)-alpha;
    f(2:N+1)=2*h*(h^2+6*((1:N)*h).^2-12*((1:N)*h)+2);
    f(N+1)=f(N+1)+beta/h;

    c=K\f;

    uN = zeros(size(x));
    for k=1:N+1
        uN = uN + c(k)*hat(x,k-1,N);
    end
    uN = uN + beta*hat(x,N+1,N);
    plot(x, uN, colors(j))
end

title('$u$, $u_3$, $u_7$, $u_{15}$ and $u_{31}$ with $\alpha=-1$ and $\beta=1$','
    interpreter','latex','FontSize',12)
xlabel('$x$','interpreter','latex','FontSize',12)
legendstr{1}='$u(x)$';
legendstr{2}='$u_3(x)$';
legendstr{3}='$u_7(x)$';
legendstr{4}='$u_{15}(x)$';
legendstr{5}='$u_{31}(x)$';
legend(legendstr,'interpreter','latex','FontSize',12,'location','best')
saveas(figure(1),'hw37e.eps','epsc')
```