CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 7

Posted Wednesday 15 October, 2014. Due 5pm Wednesday 22 October, 2014.

Please write your name and ***residential college*** *on your homework.*

1. [50 points: 10 points each]

   Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

   $$(v,w)=\int_0^1 v(x)w(x)\,dx$$

   and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

   $$\|v\|=\sqrt{(v,v)}.$$

   Let the linear operator $L: C_D^2[0,1]\to C[0,1]$ be defined by

   $$Lv=-v''$$

   where

   $$C_D^2[0,1]=\{w\in C^2[0,1]: w(0)=w(1)=0\}.$$

   Recall that the operator L has eigenvalues

   $$\lambda_n=n^2\pi^2$$

   with corresponding eigenfunctions

   $$\phi_n(x)=\sqrt{2}\sin(n\pi x)$$

   for $n=1,2,\ldots$. Let N be a positive integer, let $f\in C[0,1]$ be defined by $f(x)=8x^2(1-x)$ and let u be the solution to

   $$Lu=f.$$

   (a) Compute the best approximation f_N to f from span $\{\phi_1,\ldots,\phi_N\}$ with respect to the norm $\|\cdot\|$.

   (b) Write down the infinite series solution to

   $$Lu=f$$

   that is obtained using the spectral method, i.e.

   $$u(x)=\sum_{j=1}^{\infty}\alpha_j\phi_j(x)$$

   where α_j are coefficients to be specified. Given this above series, determine the best approximation u_N to u from span $\{\phi_1,\ldots,\phi_N\}$ with respect to the norm $\|\cdot\|$.

   (c) Plot the approximations u_N to u that you obtained using the spectral method for $N=1,2,3,4,5,6$.

(d) By shifting the data and then using an infinite series solution that you have obtained previously in this question, obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1;$$

$$\tilde{u}(0) = -\frac{1}{4}$$

and

$$\tilde{u}(1) = \frac{1}{4}.$$

(e) Let $\tilde{u}_N$ be the series solution that you obtained in part (d) but with ∞ replaced by N, i.e.

$$\tilde{u}_N(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

Plot $\tilde{u}_N$ for $N = 1, 2, 3, 4, 5, 6$.

2. [50 points: 10 points each]

All parts of this question should be done by hand.

Let the inner product $(\cdot,\cdot): C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|: C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L: S\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$S=\{w\in C^2[0,1]: w'(0)=w(1)=0\}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv,w)=(v,Lw) \text{ for all } v,w\in S.$$

Let N be a positive integer and let $f\in C[0,1]$ be defined by

$$f(x)=\begin{cases} 1-2x & \text{if } x\in\left[0,\frac{1}{2}\right]; \\ 0 & \text{otherwise.}\end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\phi_n(x)=\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n=1,2,\ldots$. Note that, for $m,n=1,2,\ldots$,

$$(\phi_m,\phi_n)=\begin{cases} 1 & \text{if } m=n; \\ 0 & \text{if } m\neq n.\end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n=1,2,\ldots$.

(b) Compute f_N, the best approximation to f from span $\{\phi_1,\ldots,\phi_N\}$ with respect to the norm $\|\cdot\|$. Plot f_N for $N=1,2,3,4,5,6$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=f(x),\quad 0<x<1$$

and

$$\tilde{u}'(0)=\tilde{u}(1)=0.$$

(d) By shifting the data, obtain an infinite series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

(e) Let $\tilde{u}_N$ be the series solution that you obtained in part (d) but with ∞ replaced by N, i.e.

$$\tilde{u}_N(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

Plot $\tilde{u}_N$ for $N = 1, 2, 3, 4, 5, 6$.