CAAM 336 · DIFFERENTIAL EQUATIONS

Midterm 1

Tuesday 7 October, 2014.

Please write your name and ***residential college*** *on your homework.*

1. [25 points: 5 points each]

   (a) Let $\mathcal{V}$ be the space of all continuously differentiable real valued functions on $[a, b]$ and $(\cdot,\cdot) : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ defined by

   $$(f, g) = \int_a^b f(x)g(x)\,dx + \int_a^b f'(x)g'(x)\,dx$$

   where $\mathcal{V} = C^1[a, b]$.
   Prove that $(\cdot,\cdot)$ is an inner product on $\mathcal{V}$.

   (b) Let $\mathcal{V} = C^1[a, b]$. Define $\|f\| = \displaystyle\int_a^b |f(x)|\,dx + \int_a^b |f'(x)|\,dx$. Prove that this defines a norm on $\mathcal{V}$ bu using the fact: *if a continuous function is positive anywhere, its integral is also positive.*

   (c) Let V be a finite dimensional vector space with an inner product and let u be a vector in V. Let $Lu : V \to V$ be the linear operator defined by

   $$Lu(v) = v - 2(v, u)u$$

   for all $v \in V$. Prove that Lu is unitary, i.e., $(Lu(v), Lu(w)) = (v, w)$ for all $v, w \in V$.

   (d) Let W be inner product space defined by $(f, g) = \displaystyle\int_0^1 f(x)g(x)\,dx$ where $W = C[0, 1]$. Using the orthonormal bases

   $$\{\phi_1, \phi_2, \phi_3\} = \{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\}$$

   i Find the linear polynomial $p(x)$ that best approximates $g(x) = \sin(\pi x)$.

   ii Find the quadratic polynomial $k(x)$ that best approximates $g(x) = \sin(\pi x)$.

   Hint:

   $$\int x sin(\pi x)\,dx = \frac{\sin(\pi x) - \pi x\cos(\pi x)}{\pi^2} + constant$$

   and

   $$\int x^2 sin(\pi x)\,dx = \frac{2\pi x\sin(\pi x) + (2 - \pi^2 x^2)\cos(\pi x)}{\pi^3} + constant$$

2. [25 points: 5 points each]
Using Backward Euler method we can approximate second derivative as follows

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &\approx \frac{\partial}{\partial x}\left(\frac{u(x)-u(x-\Delta x)}{\Delta x}\right)\\ &\approx \frac{\frac{u(x)-u(x-\Delta x)}{\Delta x} - \frac{u(x-\Delta x)-u(x-2\Delta x)}{\Delta x}}{\Delta x}\\ &= \frac{u(x)-2u(x-\Delta x)+u(x-2\Delta x)}{\Delta x^2}\end{aligned}$$

Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x)-2u(x-\Delta x)+u(x-2\Delta x)}{\Delta x^2}$$

where the term $\Delta x = x_{i+1} - x_i = x_i - x_{i-1}$ represents a constant spatial interval.

(a) Show that the second order backward finite difference approximation

$$u''(x) \approx \frac{u(x)-2u(x-\Delta x)+u(x-2\Delta x)}{\Delta x^2}$$

has accuracy $O(\Delta x)$. In other words, if $u''(x)$ is the exact second derivative, show that

$$\left| u''(x) - \frac{u(x+\Delta x)-2u(x)+u(x-\Delta x)}{\Delta x^2} \right| = O(\Delta x).$$

(b) Four point backward finite difference formula for the second derivative given as follows

$$u''(x) \approx \frac{2u(x)-5u(x-\Delta x)+4u(x-2\Delta x)-u(x-3\Delta x)}{\Delta x^2}$$

Show that this approximation has accuracy $O(\Delta x)^2$.

3. [25 points: 5 points each]
The periodic heat equation, namely

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} && x \in [-1,1], 0 \le t \le \infty \\
u(-1,t) &= u(1,t) && 0 \le t \le \infty \\
\frac{\partial u}{\partial x}(-1,t) &= \frac{\partial u}{\partial x}(1,t) && 0 \le t \le \infty \\
u(x,0) &= \cos(\pi x) && x \in [-1,1], t = 0
\end{aligned}
$$

Since we cannot solve the equation exactly, we want to approximate exact solution to a finite number of points (x_i, t_j), such that

$$\frac{\partial u(x_i,t_j)}{\partial t} = \frac{\partial^2 u(x_i,t_j)}{\partial x^2}, \quad -1 \le x_i \le 1, \qquad 0 \le t_j < \infty$$

To do so, we will approximate time and space derivative using finite difference(FD) method for $u(x_i, t_j)$ at the 5 points

$$x_0 = -1, \quad x_1 = \frac{-1}{2}, \quad x_2 = 0, \quad x_3 = \frac{1}{2}, \quad x_4 = 1 \quad \text{and} \quad t_0 = 0$$

(a) Using forward difference in time and central difference in space around the point (x_i, t_j) write down the FD formula for the heat equation at $-1 \le x_0, x_1, x_2, x_3, x_4 \le 1$.

(b) Using the fact

$$
\begin{aligned}
u(-1,t) &= u(1,t) && 0 \le t < \infty \\
\frac{\partial u}{\partial x}(-1,t) &= \frac{\partial u}{\partial x}(1,t) && 0 \le t < \infty
\end{aligned}
$$

Show that $u(x_{-1}, t_j) = u(x_3, t_j)$ and $u(x_5, t_j) = u(x_1, t_j)$. Note that x_{-1} and x_5 is not in our region. Hoverer using the given information we can find the values of those points. This technique is called *ghost point* or *fictitious point* idea.

(c) Using the information from (a) and (b) construct explicitly the matrix system $\mathbf{U^{j+1}} = \mathbf{AU^j}$ resulting from the finite difference approximation of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, where

$$\mathbf{U^j} = \begin{bmatrix} u(x_0,t_j) \\ u(x_1,t_j) \\ u(x_2,t_j) \\ u(x_3,t_j) \\ u(x_4,t_j) \end{bmatrix}.$$

Write down the matrix $\mathbf{A}$ and $\mathbf{U^0}$ explicitly.