CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 18 · Solutions

Posted Wednesday 25 September 2013. Due 5pm Wednesday 2 October 2013.

18. [25 points] Consider the polynomials $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = 3x^2 - 1$, which form a basis for the set of all quadratic polynomials. These polynomials are orthogonal with respect to the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ defined by

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\| : C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Let $f(x) = \cos(\pi x)$.

(a) By hand, construct the best approximation f_1 to f from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) By hand, construct the best approximation f_2 to f from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) By hand, construct the best approximation f_3 to f from span$\{\phi_1, \phi_2, \phi_3\}$ with respect to the norm $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) [4 points] The best approximation to $f(x) = \cos(\pi x)$ from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$ is

$$f_1(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x).$$

In Homework 17 we computed that

$$(\phi_1, \phi_1) = 2.$$

Moreover, we can compute that

$$(f, \phi_1) = \int_{-1}^{1} \cos(\pi x)\,dx = \left[\frac{1}{\pi}\sin(\pi x)\right]_{-1}^{1} = \frac{1}{\pi}\sin(\pi) - \frac{1}{\pi}\sin(-\pi) = 0 - 0 = 0$$

and hence

$$f_1(x) = 0.$$

(b) [7 points] Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot,\cdot)$, i.e., $(\phi_1, \phi_2) = 0$, the best approximation to $f(x) = \cos(\pi x)$ from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$f_2(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x) = f_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x).$$

In Homework 17 we computed that

$$(\phi_2, \phi_2) = \frac{2}{3}.$$

Moreover, we can compute that

$$
\begin{aligned}
(f, \phi_2) &= \int_{-1}^{1} x\cos(\pi x)\,dx \\
&= \left[\frac{1}{\pi}x\sin(\pi x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{\pi}\sin(\pi x)\,dx \\
&= \frac{1}{\pi}\sin(\pi) - \left(-\frac{1}{\pi}\sin(-\pi)\right) - \left[-\frac{1}{\pi^2}\cos(\pi x)\right]_{-1}^{1} \\
&= \frac{1}{\pi^2}\cos(\pi) - \frac{1}{\pi^2}\cos(-\pi) \\
&= -\frac{1}{\pi^2} - \left(-\frac{1}{\pi^2}\right) \\
&= 0
\end{aligned}
$$

and hence

$$f_2(x) = f_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x) = 0.$$

(c) [7 points] Since,

$$(\phi_1, \phi_2) = (\phi_1, \phi_3) = (\phi_2, \phi_3) = 0,$$

the best approximation to $f(x) = \cos(\pi x)$ from $\operatorname{span}\{\phi_1, \phi_2, \phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$f_3(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)}\phi_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)}\phi_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)}\phi_3(x).$$

In Homework 17 we computed that

$$(\phi_3, \phi_3) = \frac{8}{5}.$$

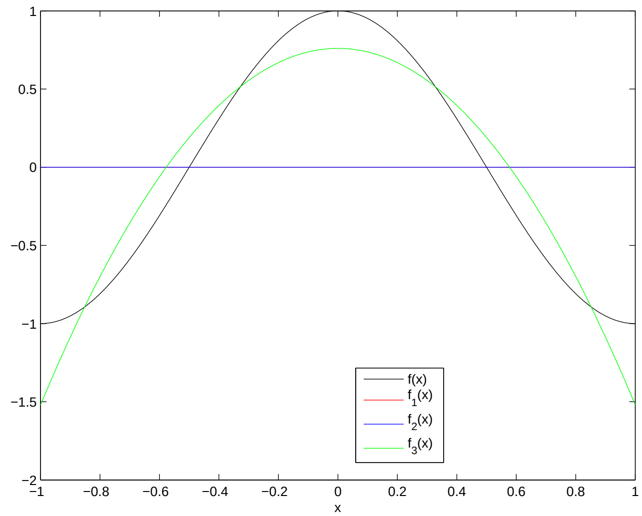
Moreover, we can compute that

$$
\begin{aligned}
(f, \phi_3) &= \int_{-1}^{1}(3x^2 - 1)\cos(\pi x)\,dx \\
&= 3\int_{-1}^{1} x^2\cos(\pi x)\,dx - (f, \phi_1) \\
&= 3\int_{-1}^{1} x^2\cos(\pi x)\,dx \\
&= 3\left(\left[\frac{1}{\pi}x^2\sin(\pi x)\right]_{-1}^{1} - 2\int_{-1}^{1}\frac{1}{\pi}x\sin(\pi x)\,dx\right) \\
&= 3\left(\frac{1}{\pi}\sin(\pi) - \frac{1}{\pi}\sin(-\pi) - \left[-\frac{2}{\pi^2}x\cos(\pi x)\right]_{-1}^{1} - \frac{2}{\pi^2}\int_{-1}^{1}\cos(\pi x)\,dx\right) \\
&= 3\left(-\left(-\frac{2}{\pi^2}\cos(\pi) - \frac{2}{\pi^2}\cos(-\pi)\right) - \frac{2}{\pi^2}(f, \phi_1)\right) \\
&= 3\left(-\frac{2}{\pi^2} - \frac{2}{\pi^2}\right) \\
&= -\frac{12}{\pi^2}
\end{aligned}
$$

thus giving

$$f_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)}\phi_3(x) = -\frac{15}{2\pi^2}(3x^2 - 1).$$

(d) [7 points] The following plot compares the best approximations to $f(x)$. Note that f_2 obscures f_1.



The code use to produce it is below.

```
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=cos(pi*x);
f1=x-x;
f2=f1+0;
f3=f2-15*(3*x.^2-1)/(2*pi^2);
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
plot(x,f3,'-g')
xlabel('x')
legend('f(x)','f_1(x)','f_2(x)','f_3(x)','location','best')
saveas(figure(1),'hw18d.eps','epsc')
```