CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 13

Posted Monday 3 February 2014. Due 1pm Friday 14 February 2014.

13. [25 points]

Let the operator $L: \ C^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v'' + 9v.$$

Let $u \in C^2[0,1]$ be the solution to the differential equation

$$-u''(x) + 9u(x) = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = \alpha$$

and

$$u(1) = \beta$$

where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Note that

$$(Lu)(x) = -u''(x) + 9u(x)$$

for all $x \in [0,1]$. Let N be an integer which is such that $N \geq 2$ and let $h = \dfrac{1}{N+1}$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

(a) Determine whether or not L is a linear operator.

(b) By using the approximation

$$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

for $j = 1, \ldots, N$ we can write

$$\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} \approx \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}$$

where $\mathbf{D} \in \mathbb{R}^{N \times (N+2)}$. What are the entries of the matrix $\mathbf{D}$? An acceptable way to present your final answer is

$$D_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

with the question marks replaced with the correct values.

(c) We can use the differential equation and boundary conditions satisfied by u and the approximation from the previous part to write

$$\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ and $\mathbf{b} \in \mathbb{R}^N$. What are the entries of the matrix $\mathbf{A}$ and the vector $\mathbf{b}$? An acceptable way to present your final answer is

$$A_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

and

$$b_j = \begin{cases} ? & \text{if } j = ?; \\ ? & \text{if } j = ?; \\ ? & \text{otherwise;} \end{cases}$$

with the question marks replaced with the correct values.

(d) Let $f(x) = 18$, $\alpha = \beta = 0$ and $N = 2$. Obtain approximations u_1 and u_2 to $u(x_1)$ and $u(x_2)$, respectively, by solving

$$\mathbf{A}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{b}$$

by hand.