

# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 26

Posted Wednesday 9 October 2013. Due 1pm Friday 18 October 2013.

26. [25 points] Let the inner product  $(\cdot, \cdot) : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$(v, w) = \int_0^1 v(x)w(x) dx$$

and let the norm  $\|\cdot\| : C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let  $N$  be a positive integer and let  $\psi_1, \dots, \psi_N \in C[0, 1]$  be such that  $\{\psi_1, \dots, \psi_N\}$  is orthonormal with respect to the inner product  $(\cdot, \cdot)$ . For  $g \in C[0, 1]$ , let

$$g_N = \sum_{n=1}^N \alpha_n \psi_n$$

where  $\alpha_n = (g, \psi_n)$ . Note that  $g_N$  is the best approximation to  $g$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$ . Moreover, let  $u \in C^2[0, 1]$  be such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u(0) = u(1) = 0$$

with  $f(x) = 1$  for all  $x \in [0, 1]$ . Note that  $u(x) = \frac{1}{2}x(1-x)$ .

(a) Show that

$$\|g - g_N\|^2 = \|g\|^2 - \sum_{n=1}^N \alpha_n^2.$$

(b) For the remainder of this question we will just consider the case when

$$\psi_n(x) = \sqrt{2} \sin(n\pi x) \text{ for } n = 1, 2, \dots, N.$$

The best approximation to  $f$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$  is

$$f_N = \sum_{n=1}^N (f, \psi_n) \psi_n$$

where

$$(f, \psi_n) = \frac{\sqrt{2}}{n\pi} (1 - \cos(n\pi)).$$

Produce a **loglog** plot of  $\|f - f_N\|$  for  $N = 1, 2, \dots, 1000000$ .

(c) We can use the spectral method to conclude that the best approximation to  $u$  from  $\text{span}\{\psi_1, \dots, \psi_N\}$  with respect to the norm  $\|\cdot\|$  is

$$u_N = \sum_{n=1}^N (u, \psi_n) \psi_n$$

where

$$(u, \psi_n) = \frac{(f, \psi_n)}{n^2\pi^2} = \frac{\sqrt{2}}{n^3\pi^3} (1 - \cos(n\pi)).$$

Add a **loglog** plot of  $\|u - u_N\|$  for  $N = 1, 2, \dots, 1000000$  to the plot that you produced in part (b).

(Be aware that the norm of the error may appear to flatline or become imaginary around  $10^{-8}$ : this is a consequence of the computer's floating point arithmetic, and so you will not lose points because of this.)