CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 11 · Solutions

Posted Friday 13 September 2013. Due 5pm Wednesday 25 September 2013.

11. [25 points]
Demonstrate whether or not each of the following is a linear operator.

(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

(c) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

(d) $L : C^1[0, 1] \to C[0, 1]$ defined by $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

(e) $L : C^2[0, 1] \to C[0, 1]$ defined by $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

Solution.

(a) [5 points] This function *is not* a linear operator.
Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + \mathbf{b}$$

but

$$f(\mathbf{u}) + f(\mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{b} + \mathbf{A}\mathbf{v} + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + 2\mathbf{b}$$

and so $f(\mathbf{u} + \mathbf{v})$ does not equal $f(\mathbf{u}) + f(\mathbf{v})$ when $\mathbf{b} \neq \mathbf{0}$.

(b) [5 points] This function *is not* a linear operator.
Suppose $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

$$f(\alpha\mathbf{x}) = (\alpha\mathbf{x})^T(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^T\mathbf{x} = \alpha^2 f(\mathbf{x}),$$

and thus if $\alpha \neq \pm 1$, we have $f(\alpha\mathbf{x}) \neq \alpha f(\mathbf{x})$.

(c) [5 points] This function *is* a linear operator.
Suppose $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n\times n}$. Then

$$f(\mathbf{X} + \mathbf{Y}) = \mathbf{A}(\mathbf{X} + \mathbf{Y}) + (\mathbf{X} + \mathbf{Y})\mathbf{B} = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B} + \mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{B} = f(\mathbf{X}) + f(\mathbf{Y}),$$

and if $\alpha \in \mathbb{R}$, then

$$f(\alpha\mathbf{X}) = \mathbf{A}(\alpha\mathbf{X}) + (\alpha\mathbf{X})\mathbf{B} = \alpha(\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}) = \alpha f(\mathbf{X}).$$

(d) [5 points] This function *is not* a linear operator.
Suppose that $u(x) = x$. Then

$$Lu = u\frac{du}{dx} = x \cdot 1 = x,$$

yet for any $\alpha \in \mathbb{R}$ we have

$$L(\alpha u) = (\alpha u)\frac{d(\alpha u)}{dx} = (\alpha x) \cdot \alpha = \alpha^2 x,$$

so if $\alpha \neq \pm 1$, we have $L(\alpha u) \neq \alpha Lu$.

(e) [5 points] This function *is* a linear operator.
Suppose that $u, v \in C^2[0,1]$. Then

$$\begin{aligned} L(u+v) &= \frac{d^2(u+v)}{dx^2} - \sin(x)\frac{d(u+v)}{dx} + \cos(x)(u+v) \\ &= \frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u + \frac{d^2v}{dx^2} - \sin(x)\frac{dv}{dx} + \cos(x)v \\ &= Lu + Lv, \end{aligned}$$

and for any $\alpha \in \mathbb{R}$,

$$L(\alpha u) = \frac{d^2(\alpha u)}{dx^2} - \sin(x)\frac{d(\alpha u)}{dx} + \cos(x)(\alpha u) = \alpha\Big(\frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u\Big) = \alpha L(u).$$