Recall: New exam Dates:

02/23 · Exam #1

03/20 · Exam #2

Announce workshop on 02/27.

Recall: we discussed inner products last time:

- $(u,v) = (v,u)$
- $(au+bv, w) = a(u,w) + b(v,w)$
- $(u,u) \geq 0$ and $(u,u)=0$ only if $u=0$

Goals:

- L^2 inner Product
- L^2 orthogonal functions (example)
- Orthonormality of a set of vectors
- Inner products and geometry:
  - Dot product length, angle
  - general inner product length, angle
  - Dot product projection (example in 2D)
  - general inner product projection
- Define the general projection operator

L^2 inner Product:

One can show that if we consider the vector space $C[a,b]$ of continuous functions then:

- $(f,g) = \int_a^b fg\,dx$ is an inner product on $C[a,b]$

  this is called the L^2 inner product.

- The associated norm is called the L^2 norm and is given by: $\|f\|_{L^2} = (f,f)^{1/2} = \sqrt{\int_a^b f^2\,dx}$

Ex: $f = x(1-x)$ in $C[0,1]$ then $\|f\|_{L^2} = \sqrt{\int_0^1 x^2(1-x)^2\,dx} \approx 0.1826$

in $\mathbb{R}^d$ we say two vectors are close if their euclidean distance is small. e.g. if $|\vec{x}-\vec{y}|$ is small. In general we say two vectors in an inner product space are <u>Close</u> if $\|f-g\| = (f-g, f-g)^{1/2}$ is small.

Ex: with f as above and $g(x) = \frac{8}{\pi^3}\sin(\pi x)$ we have:

$$\|f-g\|_{L^2} = \sqrt{\int_0^1 \left(x(1-x) - \frac{8}{\pi^3}\sin(\pi x)\right)^2} \approx 0.006940$$

<u>Note</u>: that $\|f-g\|_{L^2}$ is also called "the mean square error" of f and g.

<u>Recall</u>: we defined the general notion of orthogonality of two vectors in an inner product space to be $(f,g)=0$.

Consider the functions $f = \sin((2\pi)x)$ and $g = \cos((2\pi)x)$ on $[0,1]$ then one can show that

$$(f,g) = \int_0^1 \sin((2\pi)x)\cos((2\pi)x)\,dx = -\frac{\cos(4\pi x)}{8\pi}\Big|_0^1 = 0.$$

In general we can also show that for $f_n = \sin(n(2\pi)x)$ and $g_m = \cos(m(2\pi)x)$ the following is true:

$$(f_{n_1}, f_{n_2}) = \int_0^1 \sin(n_1(2\pi)x)\sin(n_2(2\pi)x) \qquad (n_1 \neq n_2)$$

$$= \frac{1}{4\pi}\left[\frac{\sin(2\pi(n_1-n_2))}{n_1-n_2} - \frac{\sin(2\pi(n_1+n_2))}{n_1+n_2}\right]_0^1$$

so that when $n_1 \neq n_2$ $(f_{n_1}, f_{n_2}) = 0$

if $n_1 = n_2$ then: $(f_{n_1}, f_{n_1}) = 1/2$.

Likewise one can show that
$(g_{m_1}, g_{m_2}) = 0$ for $m_1 \neq m_2$ and that $(g_{m_1}, g_{m_1}) = 1/2$.

We also have:
$$(f_n, g_m) = \int_0^1 \sin(n x) \cos(m x)\, dx = 0 \quad \text{for any } n, m.$$

If we define $\tilde{f}_n = \frac{1}{\sqrt{2}} f_n$ and $\tilde{g}_m = \frac{1}{\sqrt{2}} g_m$ then $\{ \tilde{f}_n, \tilde{g}_m \mid n=1,2,\ldots \ m=1,2,\ldots \}$ are an <u>orthonormal set of functions</u>.

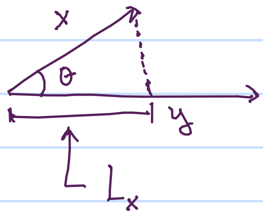
Hence if $u = \sum_{i=1}^{\infty} \alpha_i \tilde{f}_i + \sum_{j=1}^{\infty} \beta_j \tilde{g}_j$ then we can find α_k (or β_k) by computing
$$(u, \tilde{f}_k) = \Big(\sum_{i=1}^{\infty} \alpha_i \tilde{f}_i + \sum_{j=1}^{\infty} \beta_j \tilde{g}_j, \tilde{f}_k \Big)$$
$$= \sum_{i=1}^{\infty} \alpha_i (\tilde{f}_i, \tilde{f}_k) + \beta_j \sum_{j=1}^{\infty} (\tilde{g}_j, \tilde{f}_k)$$
$$= \alpha_k.$$

<u>Note</u>: This is true for any orthonormal set of vectors $\{v_1, v_2, \ldots, v_j\}$.
If $x = \sum \lambda_m v_m$ then $\lambda_m = (x, v_m)$ by orthonormality.

Projections:

Recall in $\mathbb{R}^2$:

The projection of $\vec{x}$ onto $\vec{y}$ is intuitively the "shadow" $\vec{x}$ "casts" on $\vec{y}$.



- Dropping a horizontal down to y then

$$\cos(\theta) = \left| \frac{L_x}{\|x\|} \right| = \frac{\text{adjacent}}{\text{hypotenuse}}$$

so that $L_x = \cos(\theta)\|x\|$

Recall that $x \cdot y = \|x\| \|y\| \cos(\theta)$ so that

- $$L_x = \cos(\theta)\|x\| = \frac{x \cdot y}{\|y\|}$$

- So the vector with length L_x in the direction of y is

$$\underbrace{\left(\frac{x \cdot y}{\|y\|}\right)}_{\text{length}} \cdot \underbrace{\frac{y}{\|y\|}}_{\text{unit vector in direction of } y.}$$

- which equals $\left(\frac{x \cdot y}{\|y\|^2}\right) y$

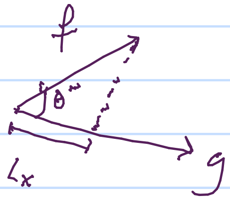
Now Recall that $x \cdot y$ is an example of an inner product

then we can generalize this idea of projections to inner products.

▷ Let $(\cdot , \cdot)_V$ be an inner product on V and define the "generalized angle" to be

$$\cos(\text{"}\theta\text{"}) = \frac{(f, g)}{\|f\| \|g\|}$$

Then



by analogy we can define

Lx to be: $\dfrac{(f,g)}{\|g\|} = \dfrac{(f,g)}{(g,g)^{1/2}}$ ← analogous to $\cos(\text{"}\theta\text{"})\,\|f\|$

So that the projection of f onto g can be *defined* as:

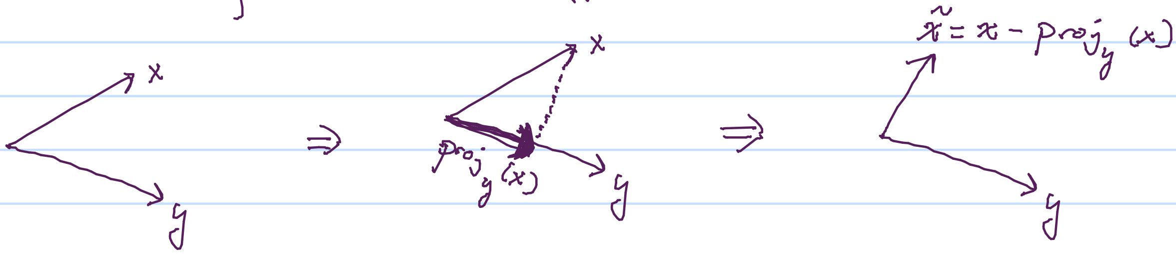
$$\mathrm{proj}_g(f) = \underbrace{\left(\frac{(f,g)}{(g,g)^{1/2}}\right)}_{\text{The generalized "Shadow" of } f \text{ on } g.} \underbrace{\frac{g}{(g,g)^{1/2}}}_{\text{unit vector in the direction of } g.} = \frac{(f,g)}{(g,g)}\, g.$$

Suppose we do this. Notice that the vectors f and g may not be orthogonal to begin with. That is $(f,g) \neq 0$.

However if we define: $\tilde{f} = f - \mathrm{proj}_g(f) = f - \dfrac{(f,g)}{(g,g)}\, g$

then $(\tilde{f}, g) = \left(f - \dfrac{(f,g)}{(g,g)}\, g,\ g\right) = (f,g) - \dfrac{(f,g)}{(g,g)}(g,g) = 0.$

This is exactly like what happens in $\mathbb{R}^2$!



Notice that in $\mathbb{R}^2$ the vector $\mathrm{proj}_y(x)$ is the "closest" vector to x that lies on y.

Another way to say this is that if we let $\vec{w}$ be any vector lying on $\vec{y}$ then $\|\vec{x} - \overrightarrow{\mathrm{proj}}_y(x)\| \leq \|\vec{x} - \vec{w}\|$ for *all* such $\vec{w}$.

This idea is captured generally by the *projection theorem*.