• The transpose of a matrix and a dot product relation:

∘ The study of eigenvalues and eigenvectors is much easier when the matrix is symmetric. In order to see why this is the case we first discuss the relation between the transpose of a matrix and the dot product.

Definition: Let A be a matrix with entries $(A)_{ij}$. Then the transpose of A is a matrix, denoted by A^T, whose entries are given by $(A^T)_{ij} = (A)_{ji}$. E.g. the ith row of A^T is the i^{th} column of A.

Definition: A matrix A is called symmetric if $A^T = A$. E.g. if $(A)_{ij} = (A)_{ji}$

Ex: If $(\cdot,\cdot)_V$ is a real-valued inner product then the Gram matrix $(G)_{ij} = (w_j, w_i)$ is symmetric since $(G)_{ij} = (w_j, w_i) = (w_i, w_j) = (G)_{ji}$

Lets turn our attention to a relationship satisfied by a matrix, its transpose and the dot product.

Theorem: Let A be an $n\times n$ matrix and u, v be vectors. Then $(Au)\cdot v = u \cdot (A^T v)$

pf: $Au = (u_1 \vec{a}_1 + u_2\vec{a}_2 + \dots + u_n \vec{a}_n)$ where $\vec{a}_i$ is the ith column of A.

$$(Au)\cdot v = u_1\, \vec{a}_1 \cdot v + u_2\, \vec{a}_2\cdot v + \dots + u_n \vec{a}_n \cdot v$$
$$= u_1 \left(\sum_{i=1}^n \vec{a}_{i1} v_i\right) + u_2\left(\sum_{i=1}^n \vec{a}_{i2} v_i\right) + \dots + u_n\left(\sum_{i=1}^n \vec{a}_{in} v_i\right)$$

Now group all of the terms multiplied by v_1 together and do the same for $v_2, v_3, \dots, v_n$.

$$= v_1\left(a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n\right) + v_2\left(a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n\right)$$
$$+ \dots + v_n\left(a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n\right)$$

Now notice that each v_j is multiplied by a quantity of the form: $a_{j1}u_1 + a_{j2}u_2 + \dots + a_{jn}u_n$

which is exactly the dot product of the j^{th} row of matrix A dotted with the vector $\vec{u}$. Recall that the j^{th} row of A is the j^{th} column of the matrix A^T.

This gives: $a_{j1}u_1 + a_{j2}u_2 + \cdots + a_{jn}u_n = \vec{a}_j^T \cdot u$ so that

$$v_1(a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n) + \cdots + v_n(a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n) \text{ is:}$$

$$v_1 \vec{a}_1^T \cdot u + v_2 \vec{a}_2^T \cdot u + \cdots + v_n \vec{a}_n^T \cdot u$$
$$= (v_1\vec{a}_1^T + v_2\vec{a}_2^T + \cdots + v_n\vec{a}_n^T) \cdot u = u \cdot (v_1\vec{a}_1^T + v_2 a_2^T + \cdots + v_n\vec{a}_n^T)$$
$$= u \cdot (A^T v) \quad \blacksquare$$

This seemingly innocuous identity has very nice consequences for symmetric matrices. If A is symmetric then the above becomes:
$(Au)\cdot v = u \cdot (Av)$

<u>Result #1</u>: If A is symmetric and λ is an eigenvalue of A then λ is a <u>real number</u> (e.g. A does not have complex eigenvalues).

- This result comes from the fact that, in general, if $\vec{x}$ and $\vec{y}$ are vectors whose entries are complex numbers then the complex dot product is <u>defined</u> by:
  $(x, y) = x \cdot \bar{y}$ so that $(x,y) = \overline{(y,x)}$.
  We haven't discussed inner products from a vector space V into the complex numbers $\mathbb{C}$ very much in class so refer to pg 72 of your textbook for a proof if interested.

<u>Result #2</u>: Let A be symmetric with eigenvectors $\vec{x}_1$ $\vec{x}_2$ having corresponding eigenvalues λ_1, λ_2. Then if $\lambda_1 \neq \lambda_2$ $(\vec{x}_1, \vec{x}_2) = 0$ e.g. eigenvectors for different eigenvalues are <u>unique.</u>

Recall: λ_1 and λ_2 must be real numbers by result #1.

Pf: $(Ax_1) \cdot x_2 = x_1 \cdot (Ax_2)$

$$\lambda_1 x_1 \cdot x_2 = x_1 \cdot \lambda_2 x_2$$
$$\lambda_1 (x_1, x_2) = \lambda_2 (x_1, x_2)$$
$$\Rightarrow (\lambda_1 - \lambda_2)(x_1, x_2) = 0 \quad \text{but } \lambda_1 - \lambda_2 \neq 0 \text{ so}$$
$(x_1, x_2) = 0$ must be true.

Result #3: (The spectral theorem for matrices)

Suppose that A is a symmetric $n \times n$ matrix. Then there exists an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ for $\mathbb{R}^n$ which are eigenvectors of A.

Note: It may be the case that some of the eigenvectors $\vec{u}_i$ share the same eigenvalue. e.g.
$Au_1 = \lambda u_1$ and $Au_2 = \lambda u_2$

Ex: The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is symmetric. Notice that it is not invertible since $\det(A) = 0$ (Row of zeros).

$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$ so that $\det(A - \lambda I) =$
$(1-\lambda)\lambda^2 - 0 \cdot 0 + 1(0 + \lambda)$
$= \lambda^2 - \lambda^3 + \lambda \rightarrow -\lambda^3 + \lambda^2 + \lambda$

so $\det(A - \lambda I) = p_A(\lambda) = -\lambda(\lambda^2 - \lambda - 1) = 0$ has solutions:

$\lambda_1 = 0 \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \quad \lambda_3 = \frac{1+\sqrt{5}}{2}$

and eigenvectors: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = u_1 \quad \approx \begin{bmatrix} -0.61803 \\ 0 \\ 1 \end{bmatrix} = u_2 \quad \approx \begin{bmatrix} 1.61803 \\ 0 \\ 1 \end{bmatrix} = u_3$

note $(u_1, u_2) = (u_1, u_3) = (u_2, u_3) = 0$.

Note: This example shows that
- The matrix A need not be invertible
- Eigenvalues of zero can still have nonzero eigenvectors.

The spectral method for solving $Ax=b$ for A symmetric:

- Suppose that A is symmetric and we know the n orthonormal eigenvectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ and corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

- Suppose further that none of the eigenvalues λ_i are zero.

Then since the vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ form a basis we can write:

$$\vec{x} = \alpha_1 \vec{u}_1 + \cdots + \alpha_n \vec{u}_n$$
$$\vec{b} = \beta_1 \vec{u}_1 + \cdots + \beta_n \vec{u}_n$$

So that $Ax=b$ becomes:

$$A(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_n \vec{u}_n) = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2 + \cdots + \beta_n \vec{u}_n$$
$$\rightarrow \alpha_1 \lambda_1 \vec{u}_1 + \alpha_2 \lambda_2 \vec{u}_2 + \cdots + \alpha_n \lambda_n u_n = \beta_1 \vec{u}_1 + \cdots + \beta_n \vec{u}_n$$

by the eigenvector property $Au_i = \lambda_i u_i$

Taking the dot product of both sides with the basis vector $\vec{u}_j$ and using $\vec{u}_i \cdot \vec{u}_j = 0$ and $u_j \cdot u_j = 1$ gives:

$$\alpha_j \lambda_j u_j \cdot u_j = \beta_j u_j \cdot u_j \rightarrow \alpha_j = \beta_j / \lambda_j$$

So what is β_j? we have $b \cdot u_j = (\beta_1 \vec{u}_1 + \cdots + \beta_n \vec{u}_n) \cdot u_j$

so that $\beta_j = b \cdot u_j$

thus $\alpha_j = \dfrac{b \cdot u_j}{\lambda_j}$. The coefficients of x thus determined, we now know $\vec{x}$.

So we have talked alot about eigenvectors for an eigenvalue but how do we find them? Well in general this is not a simple task.

If v is an eigenvector of A with eigenvalue λ then
$$Av = \lambda v \rightarrow (A - I\lambda)v = 0 \quad \text{so that} \quad v \in N(A - \lambda I)$$

So we see that the <u>null space</u> of $A - \lambda I$ will contain all of our eigenvectors v for the eigenvalue λ. So we want to analyze that space. If we can find a <u>basis</u> for it then that basis will be a set of linearly independent eigenvectors for the eigenvalue λ.

Lets illustrate with an easy example. The matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ is not symmetric but we can still find its eigenvalues and eigenvectors.

$$\det(A - \lambda I) = p_\lambda(A) = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 8 = -5 - 4\lambda + \lambda^2$$
$$= (\lambda - 5)(\lambda + 1)$$

So the eigenvalues are $\lambda_1 = 5 \quad \lambda_2 = -1$

So for $\lambda_1 = 5$ we want to investigate $N(A - 5I) = N\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right)$

The reduced row echelon form of $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad N\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\right)$$

So $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is in $N\left(\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\right)$ means that

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 - 1/2\, v_2 = 0 \rightarrow v_1 = 1/2\, v_2$$

so v looks like $v = \begin{bmatrix} 1/2\, v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

Now v_1, v_2 were arbitrary so for <u>any</u> value of v_2 the vector $v_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ will be in the null space. Since vectors like this are the only vectors in the nullspace it follows that

$$N\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\right) = \left\{ t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

So not only is $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ an eigenvector for the eigenvalue $\lambda = 5$

it is a basis for the Eigenspace corresponding to $\lambda = 5$.
The eigenspace is defined to be the Null space of $A - \lambda I$.
e.g. all eigenvectors having eigenvalue λ.