Example: The spectral method and the Gram "best approximation" technique

<u>Key idea</u>: The spectral method is similar to the Gram "best approximation" technique.

Suppose that V is a vector space (of dimension n) and A is a symmetric matrix $A: V \to V$. Then the spectral theorem says that we can find an orthonormal basis of eigenvectors of A for the space V. Eg $\{u_1, u_2, \ldots, u_n\}$. Let W be the finite dimensional subspace spanned by the first j basis vectors. Then $W = \text{span}\{u_1, u_2, \ldots, u_j\}$

Let $v \in V$ and suppose we want to find the best approximation $m \in W$ to v. Then we need to solve the Gram problem

$G\vec{x} = \vec{b}$ where b is the vector $b_i = (b, u_i)$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \end{bmatrix}$ is the vector of coefficients for $m = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_j\vec{u}_j$. The matrix G is $G_{ij} = (\vec{u}_j, \vec{u}_i)$ And since the vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ are orthonormal the Gram matrix is the <u>identity</u>. Thus $x_1 = (b, u_1)$, $x_2 = (b, u_2)$, ... $x_j = (b, u_j)$.

<u>Key idea</u>: Symmetric operators give us very nice basis vectors which can be used to solve other problems of interest more easily.

- The same type of thinking holds even when the vector space V is infinite dimensional
- The spectral theorem (e.g. the existence of a basis of orthonormal eigenvectors) generalizes to other types of linear operators.

<u>Ex</u>: Let $V = C_D^2[0,1] = \{ f \mid f$ is twice continuously differentiable and $f(0) = f(1) = 0 \}$.

with inner product $(f,g) = \int_0^1 fg$

Consider the operator $A = \frac{\partial^2}{\partial x^2}$ defined on V. We have already seen that A is linear. We haven't discussed what it means for a differential operator to be symmetric but yet but lets take that for granted for now. We already saw in class that the functions

$f_n = \sin(2\pi n x)$ satisfy $(f_{n_1}, f_{n_2}) = 0$ if $n_1 \neq n_2$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ 1/2$ if $n_1 = n_2$

It follows that the functions $\tilde{f}_n = \sqrt{2}\sin(2\pi n x)$ are orthonormal and linear independent in V. There are infinitely many such functions in V. Notice that each f_n is an eigenvector of the (symmetric) operator $A = \frac{\partial^2}{\partial x^2}$ since

$$A\tilde{f}_n = \frac{\partial^2}{\partial x^2}\tilde{f}_n = \frac{\partial^2}{\partial x^2}\left(\sqrt{2}\sin(2\pi n x)\right) = -\sqrt{2}(2\pi n)^2 \sin(2\pi n x))$$
$$= -\sqrt{2}\, 4\pi^2 n^2 \tilde{f}_n$$

so that $\tilde{f}_n$ is an eigenvector with eigenvalue $\lambda_n = -\sqrt{2}\, 4\pi^2 n^2$.

So we have an infinite set of orthonormal eigenvectors of A given by $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \dots\}$. We dont know if they form a basis for V because V isnt finite dimensional. However we have shown an example that indicates that we might be able to find orthormal eigenfunctions of symmetric (general) linear operators.

So now lets pick $W = \text{Span}\{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_j\}$ and use the Gram procedure to find the best approximation to $g(x) = (x-1)(x)(x-1/2) = x^3 - \frac{3x^2}{2} + \frac{x}{2}$. Since the basis is orthonormal the Gram matrix $G_{ij} = (u_j, u_i)$ is the identity. Thus $x_i = (g, \tilde{f}_i)$.

Lets use the first three component functions
$\tilde{f}_1 = \sqrt{2}\sin(2\pi x)$ $\quad \tilde{f}_2 = \sqrt{2}\sin(4\pi x)$, $\quad \tilde{f}_3 = \sqrt{2}\sin(6\pi x)$

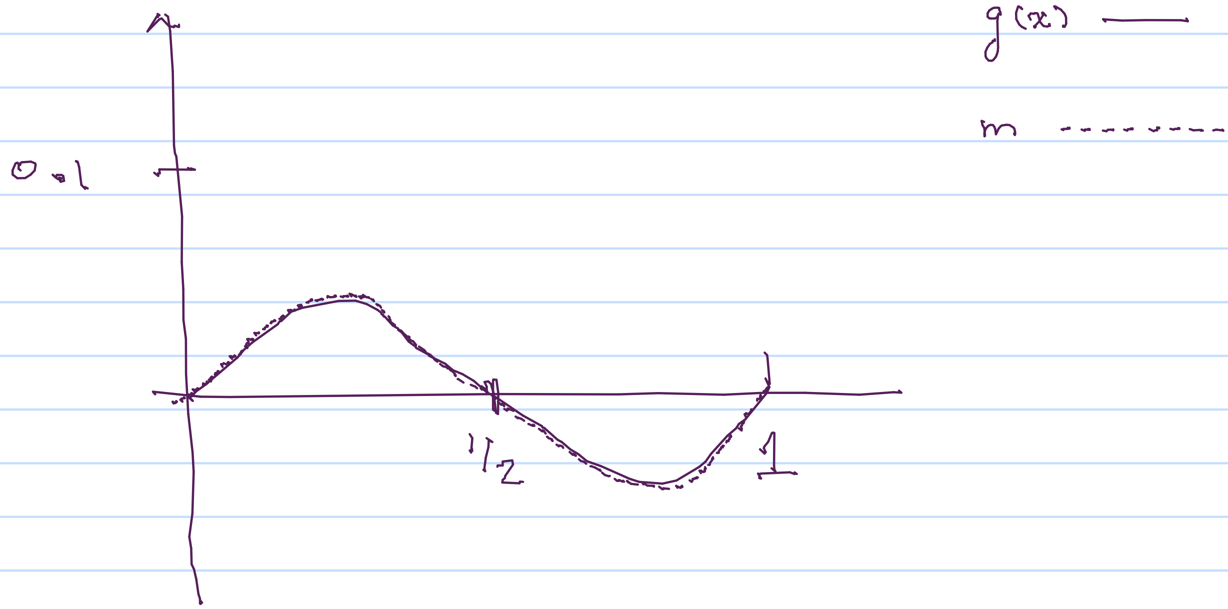
then

$$x_1 = (g, \tilde{f}_1) = \int_0^1 \left(x^3 - \frac{3x^2}{2} + \frac{x}{2}\right)\left(\sqrt{2}\sin(2\pi x)\right) \approx 0.034208$$

$$x_2 = (g, \tilde{f}_2) = \int_0^1 \left(x^3 - \frac{3x^2}{2} + \frac{x}{2}\right)\left(\sqrt{2}\sin(4\pi x)\right) \approx 0.0042760$$

$$x_3 = (g, \tilde{f}_3) = \int_0^1 \left(x^3 - \frac{3x^2}{2} + \frac{x}{2}\right)\left(\sqrt{2}\sin(6\pi x)\right) \approx 0.0012670$$

$\Rightarrow$ $m = x_1\tilde{f}_1(x) + x_2\tilde{f}_2(x) + x_3\tilde{f}_3(x)$ is the minimizer.



0.1
1/2
1
g(x)
m