

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 1 · Solutions

Posted Monday 26 August 2013. Due 5pm Wednesday 4 September 2013.

1. [25 points]

Let $f(x) = 1$ for $x \in [0, 1]$. Also, let

$$\phi_n(x) = \sin(n\pi x)$$

and let

$$a_n = \frac{\int_0^1 f(x)\phi_n(x)dx}{\int_0^1 (\phi_n(x))^2 dx}$$

for $n = 1, 2, 3, \dots$. We can approximate f as a linear combination of the periodic functions ϕ_n for $n = 1, 2, 3, \dots, N$. We call the approximation f_N which is defined as

$$f_N(x) = \sum_{n=1}^N a_n \phi_n(x)$$

for $x \in [0, 1]$. In this course, you will learn in what sense a_n are the correct coefficients to best approximate f as a linear combination of the ϕ_n for $n = 1, 2, 3, \dots, N$.

- (a) Compute the coefficients a_n symbolically using the MATLAB functions **syms**, **int**, and **simplify** as needed.
- (b) Use the fact that n is an integer to simplify the formula that you obtained for a_n .
- (c) Simplify your formula for a_n further by considering the cases when n is odd and n is even separately.
- (d) On the same figure plot $f_N(x)$ for $N = 4, 16, 64, 256$. Make sure to:
 - Use a different color for each value of N ;
 - label the axes and provide a title;
 - create an accurate legend for the figure;
 - adjust the text sizes if necessary to make everything easily legible;
 - use the LATEX interpreter to make your labels, titles, and legend look stylish.

Solution.

- (a) [5 points] We can use the MATLAB code:

```
syms x n;  
phin=sin(n*pi*x);  
a_n=simplify((int(phin,0,1)/int(phin^2,0,1)))
```

to compute that

$$a_n = \frac{-8 \sin^2\left(\frac{n\pi}{2}\right)}{\sin(2n\pi) - 2n\pi}.$$

(b) [5 points] If n is an integer then $\sin(2n\pi) = 0$ and therefore

$$a_n = \frac{-8 \sin^2\left(\frac{n\pi}{2}\right)}{-2n\pi} = \frac{4 \sin^2\left(\frac{n\pi}{2}\right)}{\pi n}.$$

(c) [5 points] Now,

$$\sin^2\left(\frac{n\pi}{2}\right) = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Consequently,

$$a_n = \begin{cases} \frac{4}{n\pi}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

(d) [10 points] We can use the MATLAB code:

```
clear
clc
p=10000;
x=linspace(0,1,p);
colors='rgbk';
N=[4 16 64 256];
figure(1)
clf
for i=1:size(N,2);
    fn=zeros(1,p);
    for n=1:2:N(i)
        fn=fn+4/(n*pi)*sin(n*pi*x);
    end
    plot(x,fn,colors(i));
    hold on
    legendStr{i}=[ '$N=' num2str(N(i)) '$' ];
end
xlabel('$x$', 'interpreter', 'latex', 'fontsize', 16);
ylabel('$f_N(x)$', 'interpreter', 'latex', 'fontsize', 16);
title('Approximations $f_N(x)$ to $f(x)=1$', 'interpreter', 'latex', 'fontsize', 16);
legend(legendStr, 'interpreter', 'latex', 'Location', 'best');
saveas(figure(1), 'hw1d.eps', 'eps')
```

to produce the plot:

