CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 43 · Solutions

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

43. [25 points]

Let
$$\Omega = \{(x,y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$
and let $f(x,y,t) = (x-\frac{1}{2})^3(y-\frac{1}{2})e^{-t}$. Note that, for $m,n = 1,2,\ldots$,
$$\int_0^1\int_0^1 2f(x,y,t)\sin(m\pi x)\sin(n\pi y)\,dx\,dy = \frac{(1+(-1)^m)(1+(-1)^n)(m^2\pi^2-24)}{8m^3n\pi^4}e^{-t}.$$
In this question we will consider the problem of finding the solution $u(x,y,t)$ to the heat equation
$$u_t(x,y,t) - (u_{xx}(x,y,t)+u_{yy}(x,y,t)) = f(x,y,t), \qquad 0\le x\le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$
with homogeneous Dirichlet boundary conditions
$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \qquad 0\le x\le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$
and initial condition
$$u(x,y,0) = 0, \qquad 0\le x\le 1, \quad 0 \le y \le 1.$$
Let
$$C^2_D(\Omega) = \left\{v \in C^2(\Omega) : v(x,0) = v(x,1) = v(0,y) = v(1,y) = 0,\, 0\le x\le 1,\, 0\le y\le 1\right\}.$$
Let the linear operator $L : C^2_D(\Omega) \to C(\Omega)$ be defined by
$$(Lv)(x,y) = -(v_{xx}(x,y)+v_{yy}(x,y)).$$
The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions
$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$
for $j,k = 1,2,\ldots$, which are such that
$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$
for $j,k = 1,2,\ldots$. Recall that in Homework 42 you obtained a formula for $\lambda_{j,k}$ for $j,k = 1,2,\ldots$.

(a) We can write
$$u(x,y,t) = \sum_{j=1}^\infty\sum_{k=1}^\infty a_{j,k}(t)\psi_{j,k}(x,y)$$
and
$$f(x,y,t) = \sum_{j=1}^\infty\sum_{k=1}^\infty c_{j,k}(t)\psi_{j,k}(x,y)$$
where
$$c_{j,k}(t) = \int_0^1\int_0^1 f(x,y,t)\psi_{j,k}(x,y)\,dx\,dy.$$
What ordinary differential equation and initial condition does $a_{j,k}(t)$ satisfy for $j,k = 1,2,\ldots$?

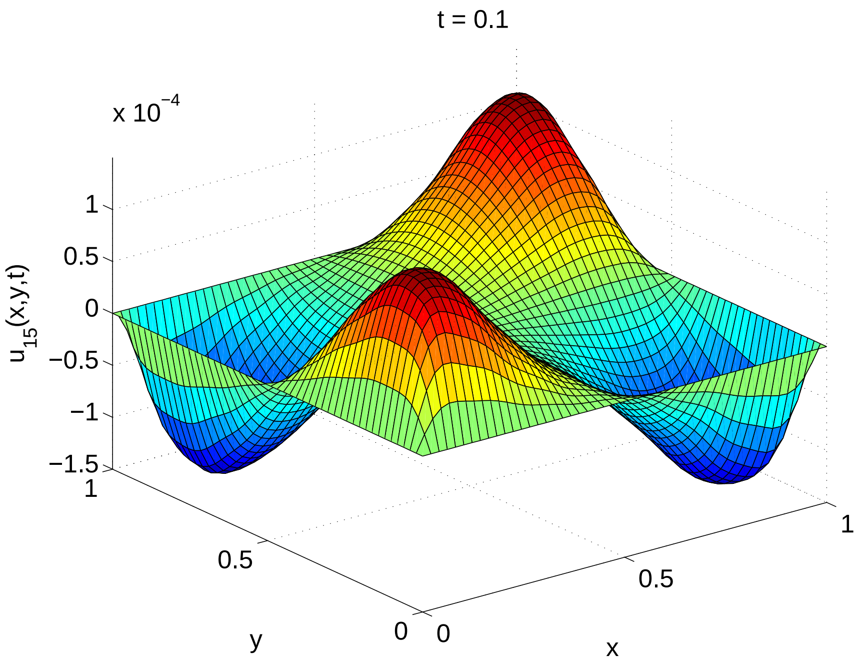
(b) Obtain an expression for $a_{j,k}(t)$ for $j, k = 1, 2, \ldots$.

(c) Use you answer to part (b) to write out a formula for $u(x, y, t)$.

(d) Plot

$$u_{15}(x, y, t) = \sum_{j=1}^{15} \sum_{k=1}^{15} a_{j,k}(t)\psi_{j,k}(x, y)$$

at the six times $t = 0, 0.001, 0.01, 0.1, 1, 2$. Use the command `zlim([-.00015 .00015])` so that the axes on all of your plots are the same. Your plot for $t = 0.1$ should resemble the plot below.



Solution.

(a) [7 points] Substituting the expressions for $u(x, y, t)$ and $f(x, y, t)$ into the partial differential equation yields

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a'_{j,k}(t)\psi_{j,k}(x, y) - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\left(-\left(L\psi_{j,k}\right)(x, y)\right) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x, y)$$

and hence

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(a'_{j,k}(t) + \lambda_{j,k}a_{j,k}(t)\right)\psi_{j,k}(x, y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x, y).$$

We can then say that

$$\begin{aligned}&\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(a'_{j,k}(t) + \lambda_{j,k}a_{j,k}(t)\right)\int_0^1\int_0^1 \psi_{j,k}(x, y)\psi_{m,n}(x, y)\,dx\,dy \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\int_0^1\int_0^1 \psi_{j,k}(x, y)\psi_{m,n}(x, y)\,dx\,dy,\end{aligned}$$

for $m, n = 1, 2, \ldots$, from which it follows that

$$a'_{m,n}(t) + \lambda_{m,n} a_{m,n}(t) = c_{m,n}(t),$$

for $m, n = 1, 2, \ldots$, since

$$\int_0^1 \int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{if } j \neq m \text{ or } k \neq n \end{cases}$$

for $j, k, m, n = 1, 2, \ldots$.

Also,

$$u(x, y, 0) = 0$$

means that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y) = 0$$

and so

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0) \int_0^1 \int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \int_0^1 \int_0^1 0\,dx\,dy,$$

for $m, n = 1, 2, \ldots$, from which it follows that

$$a_{m,n}(0) = 0,$$

for $m, n = 1, 2, \ldots$, since

$$\int_0^1 \int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{if } j \neq m \text{ or } k \neq n \end{cases}$$

for $j, k, m, n = 1, 2, \ldots$, and

$$\int_0^1 \int_0^1 0\,dx\,dy = 0.$$

Hence, for $j, k = 1, 2, \ldots$, $a_{j,k}(t)$ is the solution to the differential equation

$$a'_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t) + c_{j,k}(t)$$

with initial condition

$$a_{j,k}(0) = 0.$$

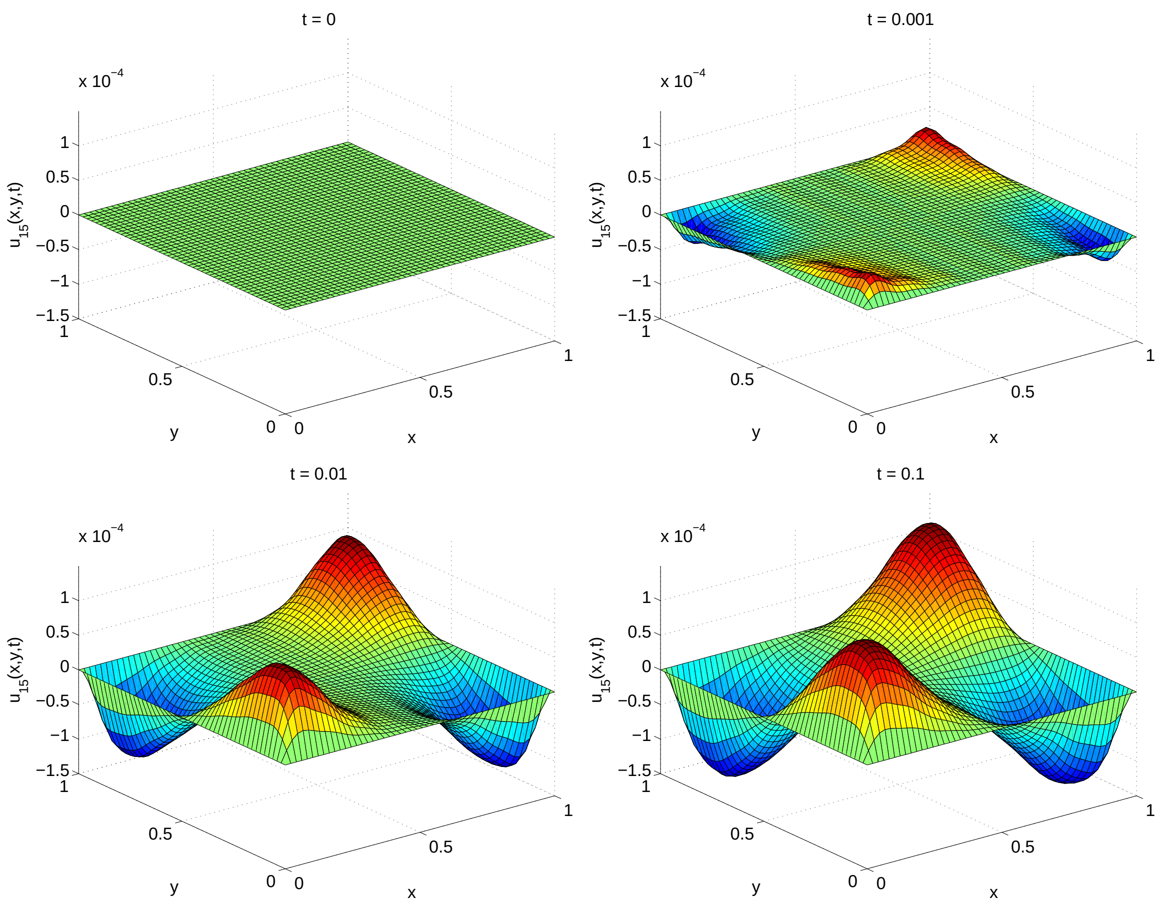
(b) [4 points] For $j, k = 1, 2, \ldots$,

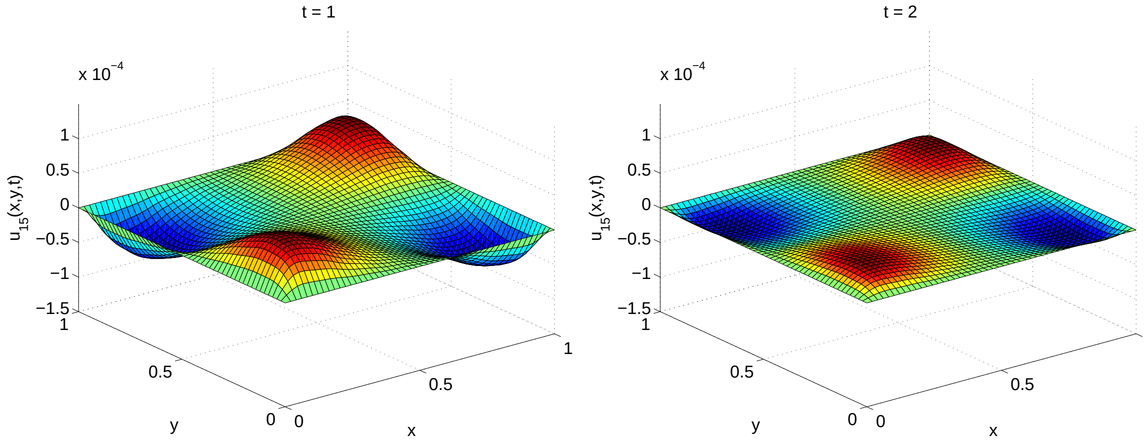
$$\begin{aligned}
a_{j,k}(t) &= \int_0^t e^{\lambda_{j,k}(s-t)} c_{j,k}(s)\,ds \\
&= \int_0^t e^{\lambda_{j,k}(s-t)} \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2 - 24)}{8j^3k\pi^4} e^{-s}\,ds \\
&= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2 - 24)}{8j^3k\pi^4} \int_0^t e^{(\lambda_{j,k}-1)s - \lambda_{j,k}t}\,ds \\
&= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2 - 24)}{8j^3k\pi^4} \left[\frac{1}{\lambda_{j,k}-1} e^{(\lambda_{j,k}-1)s - \lambda_{j,k}t}\right]_{s=0}^{s=t} \\
&= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2 - 24)}{8j^3k\pi^4} \left(\frac{1}{\lambda_{j,k}-1} e^{-t} - \frac{1}{\lambda_{j,k}-1} e^{-\lambda_{j,k}t}\right) \\
&= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2 - 24)}{8j^3k\pi^4\left(\pi^2\left(j^2+k^2\right)-1\right)} \left(e^{-t} - e^{-\pi^2\left(j^2+k^2\right)t}\right).
\end{aligned}$$

(c) [4 points] We can write

$$
\begin{aligned}
u(x,y,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4\left(\pi^2\left(j^2+k^2\right)-1\right)}\left(e^{-t}-e^{-\pi^2(j^2+k^2)t}\right)\psi_{j,k}(x,y) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{4j^3k\pi^4\left(\pi^2\left(j^2+k^2\right)-1\right)}\left(e^{-t}-e^{-\pi^2(j^2+k^2)t}\right)\sin(j\pi x)\sin(k\pi y).
\end{aligned}
$$

(d) [10 points] Plots at the requested times are shown below, followed by the code that generated them.






```
clear
clc
npts = 50;
x = linspace(0,1,npts);
y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
tvec = [0 .001 0.01 0.1 1 2];
for m=1:length(tvec)
    t = tvec(m);
    figure(1)
    clf
    U = zeros(npts,npts);
    n=15;
    for j=1:n
        for k=1:n
            cjk = (1+(-1)^j)*(1+(-1)^k)*(j^2*pi^2-24)/(8*j^3*k*pi^4); %  (x-1/2)^3 (y-
                1/2)
            lamjk = pi^2*(j^2+k^2);
            psijk = 2*sin(j*pi*X).*sin(k*pi*Y);
            U = U + (exp(-t)-exp(-lamjk*t))/(lamjk-1)*cjk*psijk;
        end
    end
    surf(X,Y,U)
    set(gca,'fontsize',16)
    xlabel('x')
    ylabel('y')
    zlabel('u_{15}(x,y,t)')
    title(sprintf(' t = %g', t))
    zlim([-.00015 .00015])
    eval(sprintf('print -depsc2 heat2d%d',m))
 end
```