CAAM 336 · DIFFERENTIAL EQUATIONS

Examination 1

Posted Wednesday 16 October 2013.

Due no later than 10am Thursday 24 October 2013.

Read and follow all of the below instructions:

1. You may not look at any of the questions on this exam until the time at which you start to take this exam.
2. The maximum amount of time that can be spent taking this exam is 5 consecutive hours.
3. Once you have started taking this exam you may not consult any books, notes, websites, homework questions, homework solutions, or other resources until after you have finished taking this exam.
4. Once you have started taking this exam you may not use any electronic devices until after you have finished taking this exam.
5. You may not discuss this exam with anyone until after you have finished taking this exam and even then you may not discuss this exam with, or in the presence of, anyone in the class who has yet to take this exam.
6. If you do not understand any of the instructions on this page then ask Richard Rankin for clarification. If you were not in class on the eleventh of October then before taking this exam you must email Richard Rankin and let him know that you have read and understood all of the instructions on this page.

You should turn this page in with your exam. This page and all of the pages that make up your exam should be stapled together with this page at the front.

Legibly write your name on the line below:

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Legibly write the dates and times that you started and finished taking this exam below.

Date started: ____________________ Time started: ____________________ am/pm

Date finished: ____________________ Time finished: ____________________ am/pm

Indicate that this is your own individual effort in compliance with the instructions given on the previous page and the honor system by legibly writing out in full and signing the traditional pledge on the lines below.

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1. [25 points]

Let the operator $L: \ C^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v'' + 9v.$$

Let $u \in C^2[0,1]$ be the solution to the differential equation

$$-u''(x) + 9u(x) = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = \alpha$$

and

$$u(1) = \beta$$

where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Note that

$$(Lu)(x) = -u''(x) + 9u(x)$$

for all $x \in [0,1]$. Let N be an integer which is such that $N \geq 2$ and let $h = \frac{1}{N+1}$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

(a) Determine whether or not L is a linear operator.

(b) By using the approximation

$$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

for $j = 1, \ldots, N$ we can write

$$\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} \approx \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}$$

where $\mathbf{D} \in \mathbb{R}^{N \times (N+2)}$. What are the entries of the matrix $\mathbf{D}$? An acceptable way to present your final answer is

$$D_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

with the question marks replaced with the correct values.

(c) We can use the differential equation and boundary conditions satisfied by u and the approximation from the previous part to write

$$\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ and $\mathbf{b} \in \mathbb{R}^N$. What are the entries of the matrix $\mathbf{A}$ and the vector $\mathbf{b}$? An acceptable way to present your final answer is

$$A_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise}; \end{cases}$$

and

$$b_j = \begin{cases} ? & \text{if } j = ?; \\ ? & \text{if } j = ?; \\ ? & \text{otherwise}; \end{cases}$$

with the question marks replaced with the correct values.

(d) Let $f(x) = 18$, $\alpha = \beta = 0$ and $N = 2$. Obtain approximations u_1 and u_2 to $u(x_1)$ and $u(x_2)$, respectively, by solving

$$\mathbf{A}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{b}.$$

2. [25 points]

(a) Compute

$$\int_{-1}^{1} x\,dx.$$

(b) Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$\mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to

$$\mathbf{Ac} = \mathbf{g}.$$

(c) Let

$$V_0 = \left\{ w \in C^1[0,1] : \int_0^1 w(x)\,dx = 0 \right\}.$$

Determine whether or not V_0 is a subspace of $C^1[0,1]$.

(d) Let $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u,v) = \int_{-1}^1 x u(x) v(x)\,dx.$$

Determine whether or not $(\cdot,\cdot)$ is an inner product on $C[-1,1]$.

(e) Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $\phi \in C[a,b]$ be defined by $\phi(x) = 1$ and let the inner product $B(\cdot,\cdot) : C[a,b] \times C[a,b] \to \mathbb{R}$ be defined by

$$B(u,v) = \int_a^b u(x) v(x)\,dx.$$

Let the linear operator $P_0 : C[a,b] \to C[a,b]$ be defined by

$$P_0 f = \frac{1}{b-a} B(f,\phi).$$

Determine whether or not P_0 is a projection.

3. [25 points]

Let the inner product $(\cdot,\cdot) :\ C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x) w(x)\,dx$$

and let the norm $\|\cdot\| :\ C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$S = \{w \in C^2[0,1] : w'(0) = w(1) = 0\}.$$

Note that S is a subspace of $C[0,1]$ and that

$$(Lv, w) = (v, Lw) \text{ for all } v, w \in S.$$

Let N be a positive integer and let $f \in C[0,1]$ be defined by

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. Note that, for $m, n = 1, 2, \ldots$,

$$(\psi_m, \psi_n) = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) Compute the best approximation to f from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1$$

and

$$\tilde{u}'(0) = \tilde{u}(1) = 0.$$

(d) What is the best approximation to $\tilde{u}$ from span $\{\psi_1, \ldots, \psi_N\}$ with respect to the norm $\|\cdot\|$?

(e) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

4. [25 points]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $f_1 \in C[-1,1]$, and $f_2 \in C[-1,1]$ be defined by

$$\phi_1(x) = \frac{1}{\sqrt{2}},$$

$$\phi_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$f_1(x) = \sin(\pi x),$$

and

$$f_2(x) = \cos(\pi x),$$

for all $x \in [-1,1]$. Note that $\{\phi_1, \phi_2\}$ is linearly independent. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx$$

and let the norm $\|\cdot\|: C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$. Also, let $\psi_1 \in C[0,1]$, $\psi_2 \in C[0,1]$, $g_1 \in C[0,1]$, and $g_2 \in C[0,1]$ be defined by

$$\psi_1(x) = \frac{1}{\sqrt{2}},$$

$$\psi_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$g_1(x) = \sin(\pi x),$$

and

$$g_2(x) = \cos(\pi x),$$

for all $x \in [0,1]$. Note that $\{\psi_1, \psi_2\}$ is linearly independent. Let the inner product $B(\cdot,\cdot): C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$B(u,v) = \int_0^1 u(x)v(x)\,dx$$

and let the norm $\|\cdot\|_B: C[0,1] \to \mathbb{R}$ be defined by

$$\|u\|_B = \sqrt{B(u,u)}.$$

Note that

$$(f_1, \phi_2) = \frac{\sqrt{6}}{\pi},$$

$$B(g_1, \psi_1) = \frac{\sqrt{2}}{\pi},$$

$$B(g_1, \psi_2) = \frac{\sqrt{6}}{2\pi},$$

$$B(g_2, \psi_2) = -\frac{\sqrt{6}}{\pi^2},$$

and

$$(f_1, \phi_1) = (f_2, \phi_1) = (f_2, \phi_2) = B(g_2, \psi_1) = 0.$$

(a) Construct the best approximation to f_1 from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(b) Construct the best approximation to f_2 from $\text{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) Construct the best approximation to g_1 from $\text{span}\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.

(d) Construct the best approximation to g_2 from $\text{span}\{\psi_1, \psi_2\}$ with respect to the norm $\|\cdot\|_B$.