CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6 · Solutions

Posted Wednesday 18 March, 2015. Due 5pm Wednesday 25 March, 2015.

Please write your name and instructor on your homework.

1. [30 points: 6 points each]

We have been able to obtain nice formulas for the eigenvalues of the operators that we have considered thus far. This problem illustrates that this is not always the case.

Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx.$$

Let the linear operator $L : V \to C[0,1]$ be defined by

$$Lu = -u''$$

where

$$V = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

Note that if $u \in V$ then u satisfies the homogeneous Robin boundary condition

$$u(0) - u'(0) = 0$$

and the homogeneous Dirichlet boundary condition

$$u(1) = 0.$$

(a) Prove that L is symmetric.

(b) Is zero an eigenvalue of L?

(c) Show that $(Lu, u) \geq 0$ for all $u \in V$. What does this and the answer to part (b) then allow us to say about the eigenvalues of L?

(d) Show that the eigenvalues λ of L must satisfy the equation $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$.

(e) Use MATLAB to plot $g(x) = -\tan(x)$ and $h(x) = x$ on the same figure. Use the command `axis([0 5*pi -5*pi 5*pi])` and make sure that your plot gives an accurate representation of these functions on the region shown on the figure when this command is used. By hand or using MATLAB, mark on your plot the points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Note that $g \notin C[0, 5\pi]$. How many eigenvalues λ does L have which are such that $\sqrt{\lambda} \leq 5\pi$?

Solution.

(a) [8 points] Suppose $u, v \in V$, so that $u(0) - u'(0) = v(0) - v'(0) = u(1) = v(1) = 0$. Integrating by parts twice yields

$$\begin{aligned}
(Lu, v) &= \int_0^1 -u''(x)v(x)\,dx \\
&= \Big[-u'(x)v(x)\Big]_0^1 + \int_0^1 u'(x)v'(x)\,dx, \\
&= -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v'(x)\,dx, \\
&= -u'(1)v(1) + u'(0)v(0) + \Big[u(x)v'(x)\Big]_0^1 - \int_0^1 u(x)v''(x)\,dx \\
&= -u'(1)v(1) + u'(0)v(0) + u(1)v'(1) - u(0)v'(0) + (u, Lv) \\
&= (u, Lv).
\end{aligned}$$

In the last step, two boundary terms are zero because $u(1) = v(1) = 0$. For the other boundary term, note that $v(0) - v'(0) = 0$ implies $v(0) = v'(0)$, so $u'(0)v(0) - u(0)v'(0) = u'(0)v(0) - u(0)v(0) = -(u(0) - u'(0))v(0) = 0$ since $u(0) - u'(0) = 0$. Hence $(Lu, v) = (u, Lv)$ for all $u, v \in V$ and so L is symmetric.

(b) [8 points] Zero is *not* an eigenvalue of L. To see this, we seek a nonzero solution $\psi \in V$ to $L\psi = 0\psi$, i.e., $-\psi''(x) = 0$. The general solution of $-\psi''(x) = 0$ is $\psi(x) = Ax + B$ where A and B are constants. The right boundary condition $\psi(1) = 0$ implies that

$$0 = \psi(1) = A + B,$$

hence $A = -B$. The left boundary condition implies

$$0 = \psi(0) - \psi'(0) = B - A,$$

hence $A = B$. The only solution which satisfies both of these conditions is hence $A = B = 0$, so $\psi(x) = 0$ is the only solution of $L\psi = 0$. Thus zero is not an eigenvalue of L.

(c) [8 points] Suppose $u \in V$, so that $u(0) - u'(0) = u(1) = 0$. Then, integrating by parts gives

$$\begin{aligned}
(Lu, u) &= \int_0^1 -u''(x)u(x)\,dx \\
&= \Big[-u'(x)u(x)\Big]_0^1 + \int_0^1 u'(x)u'(x)\,dx, \\
&= -u'(1)u(1) + u'(0)u(0) + \int_0^1 (u'(x))^2\,dx.
\end{aligned}$$

Now, $(u'(x))^2 \geq 0$ for all $x \in [0, 1]$ and so $\int_0^1 (u'(x))^2\,dx \geq 0$. Moreover, since $u(1) = 0$ we have that $-u'(1)u(1) = 0$ and since $u(0) - u'(0) = 0$ we can say that $u(0) = u'(0)$ from which it follows that $u'(0)u(0) = (u(0))^2 \geq 0$. Therefore, $(Lu, u) \geq 0$ for all $u \in V$.

If λ is an eigenvalue of L then, since L is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $\psi \in V$ which are such that $L\psi = \lambda\psi$ and hence

$$\lambda(\psi, \psi) = (\lambda\psi, \psi) = (L\psi, \psi).$$

The fact that $(Lu, u) \geq 0$ for all $u \in V$ then means that

$$\lambda = \frac{(L\psi, \psi)}{(\psi, \psi)} \geq 0$$

since $(\psi, \psi) > 0$ by the definition of the inner product because ψ is a nonzero function. However, in part (b) we had showed that zero is not an eigenvalue of L and so we can conclude that $\lambda > 0$ for all eigenvalues λ of L.

(d) [8 points] We now know that all eigenvalues λ of L are positive and so the general solution of $L\psi = \lambda\psi$, i.e. $-\psi'' = \lambda\psi$, has the form

$$\psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

where A and B are constants. The left boundary condition gives

$$0 = \psi(0) - \psi'(0) = A\sin(0) + B\cos(0) - A\sqrt{\lambda}\cos(0) + B\sqrt{\lambda}\sin(0) = B - A\sqrt{\lambda},$$

hence $B = A\sqrt{\lambda}$. The right boundary condition gives

$$0 = \psi(1) = A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda}).$$

Substituting the left boundary condition into this last formula, we find

$$0 = A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}).$$

Since we need $A \neq 0$ in order for $\psi \neq 0$, this equation implies that

$$\sqrt{\lambda} = -\frac{\sin(\sqrt{\lambda})}{\cos(\sqrt{\lambda})} = -\tan(\sqrt{\lambda}).$$

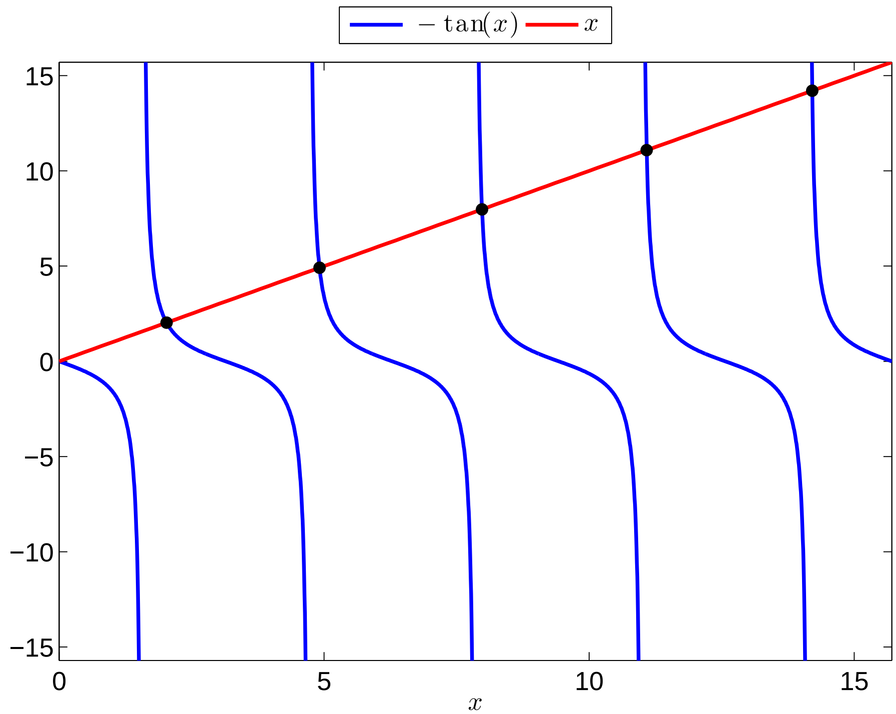
Therefore, the eigenfunctions of L have the form

$$\psi(x) = A(\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\cos(\sqrt{\lambda}x)),\ A \neq 0$$

where the eigenvalues λ are the positive numbers which are such that

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

(e) [8 points] The plot is shown below.



From the plot we can see that there are 5 points where $g(x)$ and $h(x)$ intersect for $x \in (0, 5\pi]$. Hence, since the eigenvalues λ of L are the positive numbers which are such that $g(\sqrt{\lambda}) = h(\sqrt{\lambda})$, L has five eigenvalues λ which are such that $\sqrt{\lambda} \leq 5\pi$.

The code used to produce the plot is below.

```
clear
clc
figure(1)
clf
for j=0:5
    x = linspace((j-1/2)*pi,(j+1/2)*pi,500);
    x = x(2:end-1);
    tanplt = plot(x,-tan(x),'b-','linewidth',2);
    hold on
end
x = linspace(0,5*pi,100);
linplt = plot(x,x,'r-','linewidth',2);
axis([0 5*pi -5*pi 5*pi])
xlabel('$x$','interpreter','latex','fontsize',14)

lgd = legend([tanplt,linplt],'$\ -\tan(x)$', '$x$',...
             'location','northoutside','orientation','horizontal');
set(lgd,'interpreter','latex')
set(gca,'fontsize',14)

guess = [2 5 7.98 11 14.21]';
bracket = [1.6 2.5;
4.8 5;
7.9 8.1;
11 11.2;
14.15 14.3];

ew = zeros(size(guess));
for k=1:length(guess)
    ew(k) = bisect(@(x) x+tan(x),bracket(k,1),bracket(k,2));
    plot(ew(k),ew(k),'k.','markersize',20)
```

```
end
print -depsc2 eigroot
```

The function **bisect** used in the above code is below.

```
function xstar = bisect(f,a,b)

% function xstar = bisect(f,a,b)
% Compute a root of the function f using bisection.
% f:    a function name, e.g., bisect('sin',3,4), or bisect('myfun',0,1)
% a, b: a starting bracket:  f(a)*f(b) < 0.

fa = feval(f,a);
fb = feval(f,b);          % evaluate f at the bracket endpoints
delta = (b-a);            % width of initial bracket
k = 0; fc = inf;          % initialize loop control variables

c = (a+b)/2;
while (delta/(2^k)>1e-18) && abs(fc)>1e-18
    c = (a+b)/2;
    fc = feval(f,c);    % evaluate function at bracket midpoint
    if fa*fc < 0
        b=c;
        fb = fc;      % update new bracket
    else
        a=c;
        fa=fc;
    end
    k = k+1;
%    fprintf(' %3d  %20.14f  %16.8e\n', k, c, fc);
end
xstar = c;
```

2. [30 points: 6 points each]

(a) Consider the boundary value problem

$$\begin{aligned} -\frac{\partial^2 u}{\partial x^2} &= f(x) \\ u'(0) &= 0 \\ u'(1) &= 0. \end{aligned}$$

If we define $Lu = -\frac{\partial^2 u}{\partial x^2}$ and the space $C_N^2[0,1]$

$$C_N^2[0,1] = \{u \in C^2[0,1], u'(0) = u'(1) = 0\},$$

this can be written as an operator equation

$$Lu = f, \quad L : C_N^2[0,1] \to C[0,1].$$

Explain why L is not positive-definite.

(b) Derive (do not just show) that the eigenfunctions $\phi_j(x)$ and corresponding eigenvalues λ_j of the above operator equation are

$$\phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2.$$

Specify the values of j for which these formulas hold. Describe what problems arise with the use of the spectral method for the above problem.

(c) Earlier in the semester, we showed that a solution only exists to the above problem if $\int_0^1 f(x)dx = 0$. Let $u(x)$ be a spectral method solution to $Lu = f$ for some source function $f(x)$. Assume that

$$\int_0^1 L\phi_j(x) = 0$$

for any eigenfunction $\phi_j(x)$, and explain why this implies that $\int_0^1 f(x)dx = 0$.

(d) The above problem is also non-unique: for any solution $u(x)$, $u(x) + C$ is also a solution for constant C. One way to make the solution unique is to add a condition where the average of $u(x)$ is zero:

$$\int_0^1 u(x)dx = 0.$$

To this end, we can redefine our operator equation

$$L_A u = -\frac{\partial^2 u}{\partial x^2}, \quad L_A u = f, \quad L_A : C_A^2[0,1] \to C[0,1].$$

where $C_A^2[0,1]$ contains functions in $C_N^2[0,1]$ with zero average

$$C_A^2[0,1] = \left\{ u \in C^2[0,1], u'(0) = u'(1) = 0, \int_0^1 u(x)dx = 0 \right\}.$$

Show that L_A is positive definite.

(e) Determine eigenfunctions and eigenvalues for the operator L_A, and give an expression for the spectral method solution for the above operator equation. Use these to give the exact spectral method solution to the equation

$$\begin{aligned} -\frac{\partial^2 u}{\partial x^2} &= 1 + \cos(\pi x) \\ u'(0) &= u'(1) = 0. \end{aligned}$$

Solution.

(a) Note that for any constant $C \neq 0$, $\frac{\partial^2 C}{\partial x^2} = 0$. Additionally, $C \in C^2_N([0,1])$. Thus, we can conclude that $LC = 0$. As a result,

$$(LC, C) = (0, C) = 0.$$

This shows L is not positive definite, because otherwise $(Lu, u) = 0$ would imply $u = 0$.

(b) If $L\phi_j = \lambda\phi_j$, then $-\phi_j'' = \lambda\phi_j$, which implies that $\phi_j(x)$ should have the form

$$\phi_j(x) = A\sin(\sqrt{\lambda_j}x) + B\cos(\sqrt{\lambda_j}x)$$

Then,

$$\phi_j'(x) = A\sqrt{\lambda_j}\cos(\sqrt{\lambda_j}x) - B\sqrt{\lambda_j}\sin(\sqrt{\lambda_j}x).$$

The boundary condition $\phi_j'(0) = 0$ then implies that $A = 0$. Likewise, the boundary condition $\phi_j'(1) = 0$ implies that

$$\phi_j'(1) = B\sqrt{\lambda_j}\sin(\sqrt{\lambda_j}x) = 0$$

so that $\sqrt{\lambda_j} = j\pi$, and

$$\phi_j(x) = \cos(j\pi x), \qquad \lambda_j = j\pi^2.$$

The above formula holds for $j = 0, 1, 2, \ldots$, since for $j = 0$, $\phi_j = 1$, which is an eigenfunction of L with eigenvalue $\lambda_0 = 0$. This causes problems with the spectral method — the spectral method gives the solution

$$u(x) = \sum_{j=0}^{\infty} \frac{(f, \phi_j)}{\lambda_j(\phi_j, \phi_j)}\phi_j(x).$$

When $j = 0$, $\lambda_j = 0$, and we end up dividing by zero.

(c) You can show that

$$\int_0^1 L\phi_j(x) = \int_0^1 (j\pi)^2\cos(j\pi x) = (j\pi)^2\left[\frac{\sin(j\pi x)}{j\pi}\right]_0^1 = 0, \quad j \neq 0.$$

If $j = 0$, then $L\phi_0 = 0$, since $\phi_0(x) = 1$. Thus, $\int_0^1 L\phi_j(x) = 0$ for all j. Then, if you assume that there is a solution $u(x)$ to the equation $Lu = f$,

$$\int_0^1 f(x) = \int_0^1 Lu = \int_0^1 \sum_{j=0}^{\infty} \alpha_j L\phi_j(x) = \sum_{j=0}^{\infty} \alpha_j \int_0^1 L\phi_j(x) = 0.$$

(d) If we redefine $L_A : C^2_A[0,1] \to C[0,1]$, then we can show it is positive definite. First note that, by integration by parts, we get

$$(L_A u, u) = \int_0^1 L_A uu = \int_0^1 -u''(x)u(x) = [-u'(x)u(x)]_0^1 + \int_0^1 u'(x)^2 = \int_0^1 u'(x)^2 \geq 0.$$

Thus, we only have to show now that $(L_A u, u) = 0$ implies $u = 0$.

$$(L_A u, u) = 0 \to \int_0^1 u'(x)^2 = 0$$

which implies $u'(x) = 0$, so that $u(x) = C$ for some constant C. However, since we require $u \in C^2_A[0,1]$, $\int_0^1 u = \int_0^1 C = C = 0$, implying that for $(L_A u, u) = 0$, $u = 0$.

(e) The eigenfunctions and eigenvalues of L_A are identical to those derived in part (b); the only difference is that $j = 1, 2, \ldots$ instead of $j = 0, 1, \ldots$.

$$\phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2, \qquad j = 1, 2, \ldots.$$

Using the fact that

$$\int_0^1 (1 + \cos(\pi x)) \cos(j\pi x) = \int_0^1 \cos(j\pi x) + \int_0^1 \cos(\pi x)\cos(j\pi x) = \begin{cases} 1/2, & j = 1 \\ 0, & j \neq 1 \end{cases}$$

we have that representing $f(x)$ using eigenfunctions gives

$$f(x) = \sum_{j=1}^{\infty} \frac{(f, \phi_j)}{(\phi_j, \phi_j)} \phi_j(x) = \cos(\pi x).$$

Notice that using the eigenfunctions of L_A does not *exactly* represent $f(x)$ — we remove 1 from $f(x)$, essentially removing the zero average part of $f(x)$. From part (c), this guarantees the existence of a solution.

The spectral method then gives

$$u(x) = \frac{\cos(\pi x)}{\pi^2}.$$

3. [10 points: 5 points each]

Let $k(x)$ and $p(x)$ be two positive-valued continuous functions on $[0,1]$, and let

$$V = \Big\{ u \in C^2[0,1] : u(0) = \frac{du}{dx}(1) = 0 \Big\}.$$

(a) Derive the weak form of the differential equation

$$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) + p(x)u = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions

$$u(0) = \frac{du}{dx}(1) = 0;$$

that is, transform this differential equation into a problem of the form:

$$\text{Find } u \in V \text{ such that } a(u,v) = (f,v) \text{ for all } v \in V,$$

where $(\cdot,\cdot)$ denotes the usual inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$, and $a(\cdot,\cdot)$ is some bilinear form that you should specify.

(b) Show that the form $a(u,v)$ from part (a) is an inner product for $u, v \in V$.

Solution.

(a) Multiply the differential equation with some function v from the space V and integrate from $x = 0$ to $x = 1$ to obtain

$$\int_0^1 \Big(-\frac{d}{dx}\Big(k(x)\frac{du}{dx}(x)\Big)v(x) + p(x)u(x)v(x) \Big)\,dx = \int_0^1 f(x)v(x)\,dx.$$

Break the integral on the left into pieces to obtain

$$\int_0^1 \Big(-\frac{d}{dx}\Big(k(x)\frac{du}{dx}(x)\Big)\Big)v(x)\,dx + \int_0^1 \Big(p(x)u(x)\Big)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

Integrate the first integral by parts to obtain

$$-\Big[\kappa(x)\frac{du}{dx}(x)v(x)\Big]_0^1 + \int_0^1 k(x)\frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx + \int_0^1 \Big(p(x)u(x)\Big)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

The first term disappears because of the boundary conditions $v(0) = 0$ and $du(1)/dx = 0$. We consolidate the integrals on the left to arrive at the weak problem:

$$\text{Find } u \in V \text{ such that} \quad a(u,v) = (f,v) \quad \text{for all } v \in V,$$

where

$$a(u,v) = \int_0^1 \Big(k(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + p(x)u(x)v(x)\Big)\,dx.$$

(b) To show that the form $a(u,v)$ in part (a) is an inner product, we must verify the three basic properties:

- **Symmetry** is apparent by inspection:

$$\begin{aligned} a(u,v) &= \int_0^1 \Big(k(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + p(x)u(x)v(x)\Big)\,dx \\ &= \int_0^1 \Big(k(x)\frac{dv}{dx}(x)\frac{du}{dx}(x) + p(x)u(x)v(x)\Big)\,dx = a(v,u). \end{aligned}$$

- **Linearity** follows from the linearity of differentiation and integration:

$$
\begin{aligned}
a(\alpha u + \beta v, w) &= \int_0^1 \Big(k(x) \frac{d(\alpha u(x) + \beta v(x))}{dx}(x) \frac{dw}{dx}(x) + p(x)(\alpha u(x) + \beta v(x)) w(x) \Big)\, dx \\
&= \int_0^1 \Big(k(x) \Big(\alpha \frac{du(x)}{dx} + \beta \frac{dv(x)}{dx} \Big) \frac{dw}{dx}(x) + p(x)(\alpha u(x) + \beta v(x)) w(x) \Big)\, dx \\
&= \alpha \int_0^1 \Big(k(x) \frac{du(x)}{dx} \frac{dw}{dx}(x) + p(x) u(x) w(x) \Big)\, dx \\
&\quad + \beta \int_0^1 \Big(k(x) \frac{dv(x)}{dx} \frac{dw}{dx}(x) + p(x) v(x) w(x) \Big)\, dx \\
&= \alpha a(u, w) + \beta a(v, w).
\end{aligned}
$$

- **Positivity** requires that $a(u,u) \geq 0$ and $a(u,u) = 0$ only when $u = 0$. Note that

$$
\begin{aligned}
a(u,u) &= \int_0^1 \Big(k(x) \frac{du}{dx}(x) \frac{du}{dx}(x) + p(x) u(x) u(x) \Big)\, dx \\
&= \int_0^1 \Big(k(x) \Big(\frac{du}{dx}(x) \Big)^2 + p(x) \big(u(x) \big)^2 \Big)\, dx.
\end{aligned}
$$

  Since $k(x)$ and $p(x)$ are both positive for all $x \in [0,1]$, each integrand is non-negative, and hence $a(u,u) \geq 0$. To have $a(u,u) = 0$, we must have $u(x) = 0$ for all $x \in [0,1]$, and $du(x)/dx = 0$ for all $x \in [0,1]$, which is only possible if $u(x) = 0$ for all $x \in [0,1]$, i.e., $u = 0$.

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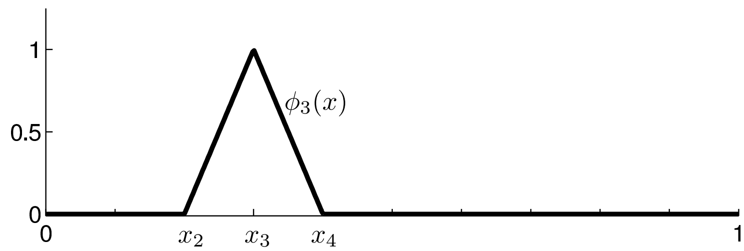
4. [30 points: 6 points each] This problem is meant to introduce you to *hat* functions, which will form the basis (double meaning intented) of the finite element method.

Let $f \in C[0,1]$ be such that $f(x) = \sin(\pi x)$. Suppose that N is a positive integer and define $h = \dfrac{1}{N+1}$ and $x_j = jh$ for $j = 0, 1 \ldots, N+1$. Consider the N hat functions $\phi_k \in C[0,1]$, defined as

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k); \\ \dfrac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1, \ldots, N$. We call these piecewise linear functions *hat functions* because of their shape. As an example, when $N = 9$ and $k = 3$, this function takes the following form.



Let the inner product $(\cdot, \cdot): \ C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_0^1 u(x) v(x)\, dx$$

and let the norm $\|\cdot\|: \ C[0,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

(a) For $j = 1, \ldots, N$, what is $\phi_j(x_k)$ for $k = 0, 1, \ldots, N+1$? Simplify your answer as much as possible.

(b) Show that $\{\phi_1, \ldots, \phi_N\}$ is linearly independent by showing that if $c_k \in \mathbb{R}$ and $\displaystyle\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $c_k = 0$ for $k = 1, \ldots, N$.

(c) By hand, compute (f, ϕ_j) for $j = 1, \ldots, N$.

(d) By hand, compute (ϕ_j, ϕ_k) for $j, k = 1, \ldots, N$. Your final answers should be simplified as much as possible and in your formulas h should be left as h and not be replaced with $1/(N+1)$. You must clearly state which values of j and k each formula you obtain is valid for. An acceptable way to present the final answer would be:
For $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} ? & \text{if } k = j, \\ ? & \text{if } |j - k| = 1, \\ ? & \text{otherwise,} \end{cases}$$

with the question marks replaced with the correct values. Hint: Letting $s = x - x_{j-1}$ yields that

$$\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx = \frac{1}{h^2}\int_{x_{j-1}-x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2\, ds = \frac{1}{h^2}\int_0^h s^2\, ds.$$

(e) Set up a linear system (in MATLAB) and solve it to compute the best approximation f_N to f from $\operatorname{span}\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ for $N = 3$ and $N = 9$. For each of these N, produce a separate plot that superimposes $f_N(x)$ on top of a plot of $f(x)$. The `hat.m` code should help you to produce these plots.

Solution.

(a) [3 points] The definition of ϕ_j yields that $\phi_j(x_k) = 0$ if $k \neq j$. Moreover,

$$\phi_j(x_j) = \frac{x_{j+1} - x_j}{h} = \frac{(j+1)h - jh}{h} = \frac{jh + h - jh}{h} = \frac{h}{h} = 1.$$

Consequently, for $j = 1, \ldots, N$,

$$\phi_j(x_k) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j, \end{cases}$$

for $k = 0, 1, \ldots, N+1$.

(b) [3 points] If $c_k \in \mathbb{R}$ and $\displaystyle\sum_{k=1}^{N} c_k \phi_k(x) = 0$ for all $x \in [0,1]$ then $\displaystyle\sum_{k=1}^{N} c_k \phi_k(x_j) = 0$ for $j = 1, \ldots, N$. The answer to part (a) then allows us to conclude that $c_j = 0$ for $j = 1, \ldots, N$ since $\displaystyle\sum_{k=1}^{N} c_k \phi_k(x_j) = c_j$. Therefore, $c_k = 0$ for $k = 1, \ldots, N$ since $c_j = 0$ for $j = 1, \ldots, N$ is equivalent to $c_k = 0$ for $k = 1, \ldots, N$.

(c) [3 points] For $j = 1, \ldots, N$, integrating by parts yields that

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \sin(\pi x)\, dx &= \left[\frac{x - x_{j-1}}{h}\left(-\frac{\cos(\pi x)}{\pi}\right)\right]_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j} \frac{d}{dx}\left(\frac{x - x_{j-1}}{h}\right)\frac{\cos(\pi x)}{\pi}\, dx \\
&= -\frac{x_j - x_{j-1}}{h}\frac{\cos(\pi x_j)}{\pi} + \int_{x_{j-1}}^{x_j} \frac{1}{h}\frac{\cos(\pi x)}{\pi}\, dx \\
&= -\frac{jh - (j-1)h}{h}\frac{\cos(\pi x_j)}{\pi} + \left[\frac{1}{h}\frac{\sin(\pi x)}{\pi^2}\right]_{x_{j-1}}^{x_j} \\
&= -\frac{\cos(\pi x_j)}{\pi} + \frac{\sin(\pi x_j) - \sin(\pi x_{j-1})}{\pi^2 h}
\end{aligned}$$

and

$$
\begin{aligned}
\int_{x_j}^{x_{j+1}} \frac{x_{j+1}-x}{h} \sin(\pi x)\,dx &= \left[\frac{x_{j+1}-x}{h}\left(-\frac{\cos(\pi x)}{\pi}\right)\right]_{x_j}^{x_{j+1}} + \int_{x_j}^{x_{j+1}} \frac{d}{dx}\left(\frac{x_{j+1}-x}{h}\right)\frac{\cos(\pi x)}{\pi}\,dx \\
&= \frac{x_{j+1}-x_j}{h}\frac{\cos(\pi x_j)}{\pi} - \int_{x_j}^{x_{j+1}} \frac{1}{h}\frac{\cos(\pi x)}{\pi}\,dx \\
&= \frac{(j+1)h-jh}{h}\frac{\cos(\pi x_j)}{\pi} - \left[\frac{1}{h}\frac{\sin(\pi x)}{\pi^2}\right]_{x_j}^{x_{j+1}} \\
&= \frac{\cos(\pi x_j)}{\pi} + \frac{\sin(\pi x_j)-\sin(\pi x_{j+1})}{\pi^2 h}.
\end{aligned}
$$

Hence, for $j = 1, \ldots, N$,

$$
\begin{aligned}
(f, \phi_j) &= \int_{x_{j-1}}^{x_j} \frac{x-x_{j-1}}{h}\sin(\pi x)\,dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1}-x}{h}\sin(\pi x)\,dx \\
&= \frac{2\sin(\pi x_j) - \sin(\pi x_{j-1}) - \sin(\pi x_{j+1})}{\pi^2 h} \\
&= \frac{2\sin(\pi x_j)}{\pi^2 h}(1-\cos(h\pi)).
\end{aligned}
$$

(d) [8 points] For $j = 1, \ldots, N$, letting $s = x - x_{j-1}$ and $t = x - x_{j+1}$ yields that

$$
\begin{aligned}
(\phi_j, \phi_j) &= \int_0^1 (\phi_j(x))^2\,dx \\
&= \int_0^{x_{j-1}} (\phi_j(x))^2\,dx + \int_{x_{j-1}}^{x_j} (\phi_j(x))^2\,dx + \int_{x_j}^{x_{j+1}} (\phi_j(x))^2\,dx + \int_{x_{j+1}}^{1} (\phi_j(x))^2\,dx \\
&= \int_0^{x_{j-1}} 0\,dx + \int_{x_{j-1}}^{x_j} \left(\frac{x-x_{j-1}}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1}-x}{h}\right)^2 dx + \int_{x_{j+1}}^{1} 0\,dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{x-x_{j-1}}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1}-x}{h}\right)^2 dx \\
&= \frac{1}{h^2}\int_{x_{j-1}-x_{j-1}}^{x_j-x_{j-1}} (s + x_{j-1} - x_{j-1})^2\,ds + \frac{1}{h^2}\int_{x_j-x_{j+1}}^{x_{j+1}-x_{j+1}} (x_{j+1} - (t + x_{j+1}))^2\,dt \\
&= \frac{1}{h^2}\int_0^h s^2\,ds + \frac{1}{h^2}\int_{-h}^0 t^2\,dt \\
&= \frac{1}{h^2}\left[\frac{s^3}{3}\right]_0^h + \frac{1}{h^2}\left[\frac{t^3}{3}\right]_{-h}^0 \\
&= \frac{h^3}{3h^2} - \frac{(-h)^3}{3h^2} \\
&= \frac{h}{3} + \frac{h}{3} \\
&= \frac{2h}{3}.
\end{aligned}
$$

Moreover, for $j = 1, \ldots, N-1$,

$$\phi_{j+1}(x) = \begin{cases} \dfrac{x - x_j}{h} & \text{if } x \in [x_j, x_{j+1}); \\ \dfrac{x_{j+2} - x}{h} & \text{if } x \in [x_{j+1}, x_{j+2}); \\ 0 & \text{otherwise;} \end{cases}$$

and so letting $s = x - x_j$ yields that

$$\begin{aligned}
(\phi_{j+1}, \phi_j) &= (\phi_j, \phi_{j+1}) \\
&= \int_0^1 \phi_j(x)\phi_{j+1}(x)\,dx \\
&= \int_0^{x_j} \phi_j(x)\phi_{j+1}(x)\,dx + \int_{x_j}^{x_{j+1}} \phi_j(x)\phi_{j+1}(x)\,dx + \int_{x_{j+1}}^1 \phi_j(x)\phi_{j+1}(x)\,dx \\
&= \int_0^{x_j} 0\,dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h}\frac{x - x_j}{h}\,dx + \int_{x_{j+1}}^1 0\,dx \\
&= \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h}\frac{x - x_j}{h}\,dx \\
&= \frac{1}{h^2}\int_{x_j - x_j}^{x_{j+1} - x_j} \left(x_{j+1} - (s + x_j)\right)\left(s + x_j - x_j\right)\,ds \\
&= \frac{1}{h^2}\int_0^h hs - s^2\,ds \\
&= \frac{1}{h^2}\left[\frac{hs^2}{2} - \frac{s^3}{3}\right]_0^h \\
&= \frac{1}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) \\
&= \frac{3}{6} - \frac{2h}{6} \\
&= \frac{h}{6}.
\end{aligned}$$

Finally, for $j = 1, \ldots, N$,

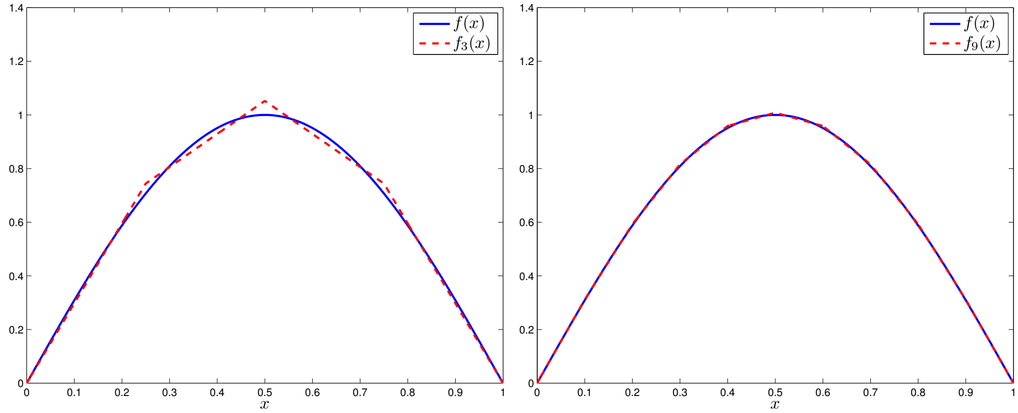
$$(\phi_j, \phi_k) = \int_0^1 \phi_j(x)\phi_k(x)\,dx = \int_0^1 0\,dx = 0$$

if $|j - k| > 1$.

Hence, for $j, k = 1, \ldots, N$,

$$(\phi_j, \phi_k) = \begin{cases} \frac{2h}{3} & \text{if } k = j, \\ \frac{h}{6} & \text{if } |j - k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(e) [8 points] The requested plots are shown below, followed by the MATLAB code that generated them.




```
xx = linspace(0,1,500).';
for N = [3 9]
    h = 1/(N+1);
    x = (0:N+1)*h;
    % set up the matrix from the inner products computed in part (d)
    A = 2*h/3*eye(N) + h/6*diag(ones(N-1,1),1) + h/6*diag(ones(N-1,1),-1);
    % set up the right-hand side vector from the inner products computed in part (c)
    b = 2/(h*pi^2)*(1-cos(h*pi))*sin(h*pi*(1:N).');
    % solve for the coefficients
    c = A\b
    % compute the approximation on fine grid on [0,1]
    fN = zeros(length(xx),1);
    for j=1:N
        fN = fN + c(j)*hat(xx,j,N);
    end
    % plot the function f and the approximation
    figure(2)
    clf
    plot(xx, sin(pi*xx), 'b-','linewidth',2)
    hold on
    plot(xx, fN, 'r--','linewidth',2)
    xlabel('$x$','interpreter','latex','fontsize',16)
    legendStr{1}=['$f(x)$'];
    legendStr{2}=['$f_{' num2str(N) '}(x)$'];
    legend(legendStr,'interpreter','latex','fontsize',16)
    %set(gca,'fontsize',16)
    if (N==3)
        saveas(figure(2),'f_3.eps','epsc')
    elseif (N==9)
        saveas(figure(2),'f_9.eps','epsc')
    end
end
```