CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 4 · Solutions

Posted Wednesday 4, February 2015. Due 5pm Wednesday 11, February 2015.

Please write your name and instructor on your homework.

1. [18 points: 9 points each]
   The equation $x_1 + x_2 + x_3 = 0$ defines a plane in $\mathbb{R}^3$ that passes through the origin.

   (a) Find two linearly independent vectors in $\mathbb{R}^3$ whose span is this plane.

   (b) Find the point in this plane closest (in the standard Euclidean norm, $\|\mathbf{z}\| = \sqrt{\mathbf{z}^T\mathbf{z}}$) to the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   by formulating this as a best approximation problem. (You may use MATLAB to invert a matrix.)

Solution.

(a) Since two linearly independent vectors determine a plane, we simply need to find two linearly independent vectors that satisfy $x_1 + x_2 + x_3 = 0$. One can do this by inspection, for example, and find

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

However, it would be nice to have an orthogonal basis for this space. To do that, pick one vector, say the first vector given above; set the second vector to be $(\alpha, \beta, \gamma)^T$. We would like the this vector to be in the plane:

$$\alpha + \beta + \gamma = 0$$

and to be orthogonal to the first vector:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha - \beta + 0 = 0.$$

This gives two equations in three unknowns, which will be satisfied if $\beta = \alpha$ and $\gamma = -2\alpha$ for any α, i.e., we have the vector

$$\begin{bmatrix} \alpha \\ \alpha \\ -2\alpha \end{bmatrix}.$$

With $\alpha = 1$, we have two orthogonal vectors whose span is the desired plane:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

(b) The closest point in the plane to the vector $\mathbf{v}$ is found solving the usual best-approximation problem matrix equation:

$$\begin{bmatrix} \mathbf{x}^T\mathbf{x} & \mathbf{x}^T\mathbf{y} \\ \mathbf{y}^T\mathbf{x} & \mathbf{y}^T\mathbf{y} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T\mathbf{v} \\ \mathbf{y}^T\mathbf{v} \end{bmatrix},$$

that is,

$$\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The orthogonality of the vectors $\mathbf{x}$ and $\mathbf{y}$ make this an easy problem to solve:

$$c_1 = 1/2, \qquad c_2 = -1/6.$$

Thus, the best approximation to $\mathbf{v} = (1, 0, 1)^T$ is the vector

$$\widehat{\mathbf{v}} = c_1\mathbf{x} + c_2\mathbf{y} = \begin{bmatrix} 1/2 - 1/6 \\ -1/2 - 1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

We can verify this answer by checking (1) that $\widehat{\mathbf{v}}$ is in the desired plane: $1/3 - 2/3 + 1/3 = 0$, and (2) verifying that the error

$$\mathbf{v} - \widehat{\mathbf{v}} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

is orthogonal to the two basis vectors $\mathbf{x}$ and $\mathbf{y}$ for the plane, $(\mathbf{v} - \widehat{\mathbf{v}})^T\mathbf{x} = (\mathbf{v} - \widehat{\mathbf{v}})^T\mathbf{y} = 0$.

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2. [24 points: 6 points each]

Let $\phi_1 \in C[-1,1]$, $\phi_2 \in C[-1,1]$, $\phi_3 \in C[-1,1]$, and $f \in C[-1,1]$ be defined by

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = 3x^2 - 1,$$

and

$$f(x) = e^x,$$

for all $x \in [-1,1]$. Let the inner product $(\cdot,\cdot) : C[-1,1] \times C[-1,1] \to \mathbb{R}$ be defined by

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

Let the norm $\|\cdot\| : C[-1,1] \to \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u,u)}.$$

Note that $\{\phi_1, \phi_2, \phi_3\}$ is **orthogonal** with respect to the inner product $(\cdot,\cdot)$, which is defined on $[-1,1]$.

(a) Construct the best approximation f_1 to f from $\mathrm{span}\{\phi_1\}$ with respect to the norm $\|\cdot\|$.

(b) Construct the best approximation f_2 to f from $\mathrm{span}\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$.

(c) Construct the best approximation f_3 to f from $\mathrm{span}\{\phi_1, \phi_2, \phi_3\}$ with respect to $\|\cdot\|$.

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of $f(x)$.

Solution.

(a) [4 points] The best approximation to $f(x) = e^x$ from span$\{\phi_1\}$ with respect to the norm $\|\cdot\|$ is

$$f_1(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} \phi_1(x).$$

We compute

$$(\phi_1, \phi_1) = \int_{-1}^{1} 1^2 \, dx = [x]_{-1}^{1} = 1 - (-1) = 2$$

and

$$(f, \phi_1) = \int_{-1}^{1} e^x \, dx = [e^x]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

and hence

$$f_1(x) = \frac{1}{2}\left(e - \frac{1}{e}\right).$$

(b) [7 points] Since ϕ_1 and ϕ_2 are orthogonal with respect to the inner product $(\cdot, \cdot)$, i.e., $(\phi_1, \phi_2) = 0$, the best approximation to $f(x) = e^x$ from span$\{\phi_1, \phi_2\}$ with respect to the norm $\|\cdot\|$ is

$$f_2(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} \phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x) = f_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x).$$

Noting that

$$(\phi_2, \phi_2) = \int_{-1}^{1} x^2 \, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{-1}{3}\right) = \frac{1}{3} - \frac{1}{3} = \frac{2}{3}$$

and

$$(f, \phi_2) = \int_{-1}^{1} x e^x \, dx = [x e^x]_{-1}^{1} - \int_{-1}^{1} e^x \, dx = e^1 - \left(-e^{-1}\right) - (f, \phi_1) = e + \frac{1}{e} - e + \frac{1}{e} = \frac{2}{e}$$

we can compute that

$$f_2(x) = f_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e} x.$$

(c) [7 points] Since,

$$(\phi_1, \phi_2) = (\phi_1, \phi_3) = (\phi_2, \phi_3) = 0,$$

the best approximation to $f(x) = e^x$ from span$\{\phi_1, \phi_2, \phi_3\}$ with respect to the norm $\|\cdot\|$ is

$$f_3(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} \phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x).$$

Toward this end, compute

$$
\begin{aligned}
(\phi_3, \phi_3) &= \int_{-1}^{1} (3x^2 - 1)^2 \, dx \\
&= \int_{-1}^{1} 9x^4 - 6x^2 + 1 \, dx \\
&= \int_{-1}^{1} 9x^4 \, dx - 6(\phi_2, \phi_2) + (\phi_1, \phi_1) \\
&= \left[\frac{9x^5}{5} \right]_{-1}^{1} - 6\frac{2}{3} + 2 \\
&= \frac{9}{5} - \left(-\frac{9}{5} \right) - \frac{12}{3} + 2 \\
&= \frac{18}{5} - \frac{12}{3} + 2 \\
&= \frac{54}{15} - \frac{60}{15} + \frac{30}{15} \\
&= \frac{24}{15} \\
&= \frac{8}{5}
\end{aligned}
$$

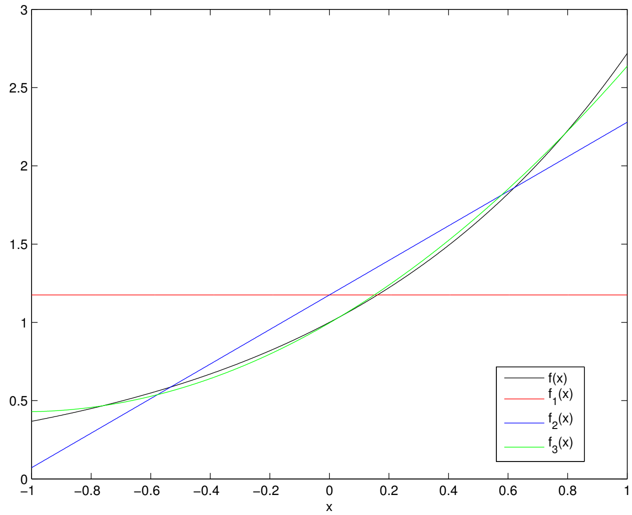
and

$$
\begin{aligned}
(f, \phi_3) &= \int_{-1}^{1} (3x^2 - 1)e^x \, dx \\
&= \int_{-1}^{1} 3x^2 e^x \, dx - (f, \phi_1) \\
&= \left[3x^2 e^x \right]_{-1}^{1} - \int_{-1}^{1} 6xe^x \, dx - \left(e - \frac{1}{e} \right) \\
&= 3e^1 - 3e^{-1} - 6(f, \phi_2) - \left(e - \frac{1}{e} \right) \\
&= 2e - \frac{2}{e} - \frac{12}{e} \\
&= 2e - \frac{14}{e}
\end{aligned}
$$

thus giving

$$
f_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x) = \frac{1}{2} \left(e - \frac{1}{e} \right) + \frac{3}{e} x + \frac{5}{4} \left(e - \frac{7}{e} \right) (3x^2 - 1).
$$

(d) [7 points] The following plot compares the best approximations to $f(x)$.



The code use to produce it is below.

```
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=exp(x);
f1=(exp(1)-exp(-1))/2+x-x;
f2=f1+3*exp(-1)*x;
f3=f2+5*(exp(1)-7*exp(-1))*(3*x.^2-1)/4;
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
plot(x,f3,'-g')
xlabel('x')
legend('f(x)','f_1(x)','f_2(x)','f_3(x)','location','best')
saveas(figure(1),'hw16d.eps','epsc')
```

3. [30 points: 10 points each]

(a) Show that if we have an orthogonal set of vectors $\phi_1, \ldots, \phi_k$, then $\phi_1, \ldots, \phi_k$ are linearly independent as well, i.e.

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

is only true if $\alpha_1, \ldots, \alpha_k = 0$.

(b) Let V be an inner product space (i.e. V a vector space with an inner product). Suppose $\{v_1, v_2, v_3\}$ is a basis for V, and we would like to construct a is possible to construct a new *orthogonal* basis $\{\phi_1, \phi_2, \phi_3\}$ through the following procedure:

$$\begin{aligned}
\phi_1 &= v_1 \\
\phi_2 &= v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1 \\
\phi_3 &= v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2 \\
&\vdots \\
\phi_k &= v_k - \sum_{i=1}^{k-1} \frac{(\phi_i, v_k)}{(\phi_i, \phi_i)}\phi_i
\end{aligned}$$

This is called the *Gram-Schmidt* procedure.

We refer to nonzero vectors $u_1, u_2, ..., u_k \in V$ an orthogonal set if they are orthogonal to each other: i.e. if

$$(u_i, u_j) = 0, \quad i \neq j.$$

Assuming we have v_1, v_2, v_3 and we define ϕ_1, ϕ_2, ϕ_3 under the above process, show that ϕ_1, ϕ_2, ϕ_3 form an orthogonal set, i.e.

$$(\phi_i, \phi_j) = 0, \qquad \text{if } 1 \leq i \neq j \leq 3.$$

(c) Since we can define an inner product $(\cdot, \cdot)$ on the function space $C[-1, 1]$ as

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx,$$

we can also use the Gram-Schmidt procedure to create orthogonal sets of *functions*. Using the Gram-Schmidt procedure above, compute the orthogonal vectors $\{\phi_1, \phi_2, \phi_3\}$ given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$.

Solution.

(a) We are going to show that $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$. To check that these formulas yield an orthogonal sequence, first compute (ϕ_1, ϕ_2) by substituting the above formula for ϕ_2

$$\begin{aligned}
(\phi_1, \phi_2) &= (\phi_1, v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1) \\
&= (\phi_1, v_2) - (\phi_1, \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1) \\
&= (\phi_1, v_2) - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}(\phi_1, \phi_1) \\
&= (\phi_1, v_2) - (\phi_1, v_2) \\
&= 0.
\end{aligned}$$

Then use the fact that $(\phi_1, \phi_2) = 0$, to compute (ϕ_1, ϕ_3). By substituting again the formula for ϕ_3

$$\begin{aligned}
(\phi_1, \phi_3) &= (\phi_1, v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_1, v_3) - (\phi_1, \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1) - (\phi_1, \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_1, v_3) - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}(\phi_1, \phi_1) - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\underbrace{(\phi_1, \phi_2)}_{=0} \\
&= (\phi_1, v_3) - (\phi_1, v_3) \\
&= 0
\end{aligned}$$

Similarly, using the symmetry property of inner product $(\phi_i, \phi_j) = (\phi_j, \phi_i)$ for all i, j. We can show $(\phi_2, \phi_3) = 0$.

$$\begin{aligned}
(\phi_2, \phi_3) &= (\phi_2, v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_2, v_3) - (\phi_2, \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1) - (\phi_2, \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2) \\
&= (\phi_2, v_3) - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\underbrace{(\phi_2, \phi_1)}_{=0} - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}(\phi_2, \phi_2) \\
&= (\phi_2, v_3) - (\phi_2, v_3) \\
&= 0.
\end{aligned}$$

By symmetry we can conclude that $(\phi_2, \phi_3) = (\phi_3, \phi_2) = 0$ and $(\phi_1, \phi_3) = (\phi_3, \phi_1) = 0$. This completes the proof.

(b) Consider a linear relationship

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

which can be written

$$\alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_k\phi_k = 0.$$

If $1 \le i \le k$ then taking the inner product of ϕ_i with both sides of the equation and using the properties of inner product (*Definition 3.32, page 58*),

$$\begin{aligned}
(\phi_i, \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_k\phi_k) &= (\phi_i, 0) \\
(\phi_i, \alpha_1\phi_1) + (\phi_i, \alpha_2\phi_2) + \cdots + (\phi_i, \alpha_k\phi_k) &= 0 \\
\alpha_1(\phi_i, \phi_1) + \alpha_2(\phi_i, \phi_2) + \cdots + \alpha_k(\phi_i, \phi_k) &= 0 \\
\alpha_i(\phi_i, \phi_i) &= 0
\end{aligned}$$

shows, since ϕ_i is nonzero, that α_i for $i = 1, \cdots, k$ is zero.

(c) We want to construct the new orthogonal bases for V by *Gram-Schmidt* procedure given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$. Following the procedure we set

$$\phi_1 = v_1 = 1$$

and

$$\phi_2 \;=\; v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)}\phi_1.$$

We compute

$$(\phi_1, v_2) = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

and

$$(\phi_1, \phi_1) = \int_{-1}^{1} 1\,dx = 2.$$

Now we can compute

$$\phi_2 \;=\; x - \frac{0}{2}(1) = x.$$

Finally for ϕ_3,

$$\phi_3 \;=\; v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)}\phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)}\phi_2$$

$$(\phi_1, v_3) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

and

$$(\phi_2, v_3) = \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

and

$$(\phi_2, \phi_2) = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

Substituting these inner products into the equation for ϕ_3, we get

$$\phi_3 \;=\; x^2 - \frac{(2/3)}{2}(1) - \frac{0}{(2/3)}(x) = x^2 - \frac{1}{3}.$$

This yields $\{\phi_1, \phi_2, \phi_3\} = \{1, x, x^2 - \frac{1}{3}\}$ as desired.

4. [28 points: 7 points each]
One of the most intriguing results in mathematics is the idea that continuous functions (especially those whose derivatives are also continuous) can be very well approximated using combinations of trigonometric functions — i.e. sines and cosines — of different frequencies. This is encapsulated in the idea of *Fourier Series*: a large class of functions $u(x)$ can be represented by the infinite sum

$$u(x) = C + \sum_{j=1}^{\infty} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right).$$

Additionally, it turns out that the finite sum (i.e. a linear combination of sines and cosines)

$$u(x) \approx C + \sum_{j=1}^{n} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right)$$

is often a very good approximation to $u(x)$. We will go more into depth on these ideas later in the semester.

In this problem, unless specified otherwise, we will examine orthogonality properties of sines and cosines using the following inner product on $C^2[0,1]$: for $u, v \in C^2[0,1]$,

$$(u, v) = \int_0^1 u(x)v(x)dx.$$

For all parts, assume j and k are integers.

(a) Show, by hand, that sines of different frequencies are orthogonal to each other, i.e. that

$$(\sin(j\pi x), \sin(k\pi x)) = \int_0^1 \sin(j\pi x)\sin(k\pi x)dx = 0, \quad j \neq k.$$

(b) Show, by hand, that cosines of different nonzero frequencies are orthogonal to each other, i.e. that

$$(\cos(j\pi x), \cos(k\pi x)) = \int_0^1 \cos(j\pi x)\cos(k\pi x)dx = 0, \quad j \neq k.$$

(c) Show, by hand, that sines and cosines of different frequencies are orthogonal to each other *over the interval* $[-1, 1]$, i.e. that

$$(\sin(j\pi x), \cos(k\pi x)) = \int_{-1}^1 \sin(j\pi x)\cos(k\pi x)dx = 0, \quad j \neq k.$$

Unlike the previous two parts, cosines and sines of different frequencies are not orthogonal to each other using the inner product on $[0, 1]$, and must be shown to be orthogonal using the inner product

$$(u, v) = \int_{-1}^1 u(x)v(x)dx.$$

(d) Show, by hand, that sines and cosines, in addition to being orthogonal, can easily be made *orthonormal* over $[0, 1]$ by scaling by $\sqrt{2}$: i.e. that

$$\left\|\sqrt{2}\sin(j\pi x)\right\|^2 = 2\int_0^1 \sin(j\pi x)^2 dx = 1, \qquad \left\|\sqrt{2}\cos(j\pi x)\right\|^2 = 2\int_0^1 \cos(j\pi x)^2 dx = 1.$$

Solution.

(a) Using the sum formula

$$\sin(j\pi x)\sin(k\pi x)dx = \frac{1}{2}\left(\cos((j-k)\pi x) - \cos((j+k)\pi x)\right)$$

we can conclude that $\int_0^1 \sin(j\pi x)\sin(k\pi x)dx$ is

$$\frac{1}{2}\int_0^1 \left(\cos((j-k)\pi x) - \cos((j+k)\pi x)\right) = \frac{1}{2}\left(\frac{\sin((j-k)\pi x)}{(j-k)\pi} - \frac{\sin((j+k)\pi x)}{(j+k)\pi}\right)\Bigg|_0^1.$$

Since j and k are integers, $j-k$ and $j+k$ are also integers. Since $\sin(n\pi x) = 0$ for any integer n, we can conclude that $\sin(j\pi x)\sin(k\pi x)dx = 0$.

(b) Using the sum formula

$$\cos(j\pi x)\cos(k\pi x)dx = \frac{1}{2}\left(\cos((j-k)\pi x) + \cos((j+k)\pi x)\right)$$

we can repeat the steps in (a) to compute $\int_0^1 \cos(j\pi x)\cos(k\pi x)dx$ as

$$\frac{1}{2}\int_0^1 \left(\cos((j-k)\pi x) + \cos((j+k)\pi x)\right) = \frac{1}{2}\left(\frac{\sin((j-k)\pi x)}{(j-k)\pi} + \frac{\sin((j+k)\pi x)}{(j+k)\pi}\right)\Bigg|_0^1 = 0.$$

(c) We may use another sum formula

$$\sin(j\pi x)\cos(k\pi x) = \frac{1}{2}\left(\sin((j+k)\pi x) + \sin((j-k)\pi x)\right)$$

Then, $\int_{-1}^1 \sin(j\pi x)\cos(k\pi x)$ gives

$$\frac{1}{2}\int_{-1}^1 \left(\sin((j+k)\pi x) + \sin((j-k)\pi x)\right) = \frac{-1}{2}\left(\frac{\cos((j+k)\pi x)}{(j+k)\pi} + \frac{\cos((j-k)\pi x)}{(j-k)\pi}\right)\Bigg|_{-1}^1$$

Since $\cos(x)$ is even, evaluating the above at $x=1$ and $x=-1$ yields the same result, which cancels out to zero.

You may also conclude the above by noting that $\cos(j\pi x)\sin(k\pi x)$ is odd, and thus its integral from $[-1,1]$ is zero.

(d) By the above sum formulas,

$$\sin(j\pi x)^2 = \frac{1}{2}\left(\cos(0\pi x) - \cos(2j\pi x)\right) = \frac{1}{2}\left(1 - \cos(2j\pi x)\right)$$

so $\|\sin(j\pi x)\|^2 = \int_0^1 \sin(j\pi x)^2$ gives

$$\frac{1}{2} - \int_0^1 \cos(2j\pi x) = \frac{1}{2} - (\cos(2j\pi) - \cos(0)) = \frac{1}{2}$$

since $2j$ is even, and $\cos(2j\pi) = (-1)^(2j) = 1$.

We may conclude similarly that

$$\|\sin(j\pi x)\|^2 = \cos(j\pi x)^2 = \frac{1}{2}\left(\cos(0\pi x) + \cos(2j\pi x)\right) = 2.$$
