CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 48 · Solutions

Posted Wednesday 16 April 2014. Due 1pm Friday 25 April 2014.

48. [25 points]

Let $H^1_D(0,1) = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in H^1_D(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N$. Let $V_N = \text{span}\{\phi_1, ..., \phi_N\}$. Let $\rho \in C[0,1]$ be such that $\rho(x) > 0$ for all $x \in [0,1]$, let $c \in C[0,1]$ be such that $c(x) > 0$ for all $x \in [0,1]$ and let $\kappa \in C[0,1]$ be such that $\kappa(x) > 0$ for all $x \in [0,1]$. Let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(u, v) = \int_0^1 \rho(x)c(x)u(x)v(x)\,dx$$

and let the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by

$$a(u, v) = \int_0^1 \kappa(x)u'(x)v'(x)\,dx.$$

Let $\mathbf{M} \in \mathbb{R}^{N \times N}$ be the matrix with entries

$$M_{jk} = (\phi_k, \phi_j)$$

and let $\mathbf{K} \in \mathbb{R}^{N \times N}$ be the matrix with entries

$$K_{jk} = a(\phi_k, \phi_j).$$

For $\mathbf{w} \in \mathbb{R}^N$, let

$$\hat{w}_N = \sum_{j=1}^{N} w_j \phi_j$$

where $w_j \in \mathbb{R}$ is the jth entry of the vector $\mathbf{w}$.

In class we had stated that the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real. In class we had also stated that the eigenvalues of $-\mathbf{M}^{-1}\mathbf{K}$ are negative since the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are positive. This question will walk you through the process of showing that the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are positive given that we know that the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real.

(a) For $\mathbf{w} \in \mathbb{R}^N$, show that

$$\mathbf{w}^T\mathbf{M}\mathbf{w} = (\hat{w}_N, \hat{w}_N).$$

(b) Show that if $\mathbf{M}^{-1}\mathbf{K}\mathbf{w} = \lambda\mathbf{w}$, for $\lambda \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^N$, then

$$a(\hat{w}_N, \hat{w}_N) = \lambda(\hat{w}_N, \hat{w}_N).$$

In addition to the information given previously in the question you may use the fact that

$$\mathbf{w}^T\mathbf{K}\mathbf{w} = a(\hat{w}_N, \hat{w}_N).$$

(c) Use the properties satisfied by inner products to show that if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$ then $\lambda > 0$.

Solution.

(a) [7 points] We first compute that

$$\mathbf{M}\mathbf{w} = \mathbf{g}$$

where $\mathbf{g} \in \mathbb{R}^N$ is the vector with entries

$$g_j = \sum_{k=1}^{N} (\phi_k, \phi_j) w_k$$

for $j = 1, \ldots, N$. Moreover, since

$$\hat{w}_N = \sum_{j=1}^{N} w_j \phi_j = \sum_{k=1}^{N} w_k \phi_k,$$

the properties satisfied by the inner product yield that

$$\sum_{k=1}^{N} (\phi_k, \phi_j) w_k = \left(\sum_{k=1}^{N} w_k \phi_k, \phi_j \right) = (\hat{w}_N, \phi_j)$$

and so

$$g_j = (\hat{w}_N, \phi_j)$$

for $j = 1, \ldots, N$. Therefore,

$$\mathbf{w}^T\mathbf{M}\mathbf{w} = \mathbf{w}^T\mathbf{g} = \sum_{j=1}^{N} w_j (\hat{w}_N, \phi_j) = \left(\hat{w}_N, \sum_{j=1}^{N} w_j \phi_j \right) = (\hat{w}_N, \hat{w}_N)$$

by the properties satisfied by the inner product and the fact that

$$\hat{w}_N = \sum_{j=1}^{N} w_j \phi_j.$$

(b) [8 points] If $\mathbf{M}^{-1}\mathbf{K}\mathbf{w} = \lambda\mathbf{w}$, for $\lambda \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^N$, then

$$\mathbf{K}\mathbf{w} = \lambda\mathbf{M}\mathbf{w}$$

and hence

$$\mathbf{w}^T\mathbf{K}\mathbf{w} = \lambda\mathbf{w}^T\mathbf{M}\mathbf{w}$$

from which it follows that

$$a(\hat{w}_N, \hat{w}_N) = \lambda(\hat{w}_N, \hat{w}_N)$$

using the information given in parts (a) and (b).

(c) [10 points] Let $\lambda \in \mathbb{R}$ be an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$. Then there exist nonzero $\mathbf{v} \in \mathbb{R}^N$ which are such that

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{v} = \lambda\mathbf{v}.$$

Let $\mathbf{w} \in \mathbb{R}^N$ be such that

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{w} = \lambda\mathbf{w}$$

and $\mathbf{w} \neq \mathbf{0}$. From part (b) we then have that

$$a(\hat{w}_N, \hat{w}_N) = \lambda(\hat{w}_N, \hat{w}_N).$$

Moreover, since $\mathbf{w} \neq \mathbf{0}$ then $\hat{w}_N \neq 0$ because $\{\phi_1, ..., \phi_N\}$ is linearly independent. Hence, since $\hat{w}_N \neq 0$, the properties satisfied by an inner product mean that $a(\hat{w}_N, \hat{w}_N) > 0$ and so

$$\lambda(\hat{w}_N, \hat{w}_N) > 0.$$

Consequently,

$$\lambda > 0$$

since, because $\hat{w}_N \neq 0$, the properties satisfied by an inner product mean that $(\hat{w}_N, \hat{w}_N) > 0$.
