1. [25 points: (a)-(c) = 6, (d)=7]

Consider the wave equation with initial conditions

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} &= f(x,t), \quad 0 < x < 1, \quad t \geq 0 \\
u(x,0) &= \psi(x) \\
\frac{\partial u}{\partial t}(x,0) &= 0
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,t) &= 0 \\
u(1,t) &= 0.
\end{aligned}$$

The operator $Lu = -u_{xx}$ with the above boundary conditions has normalized eigenfunctions $\phi_j(x)$ and eigenvalues λ_j

$$\lambda_j = \left(\frac{2j-1}{2}\pi\right)^2, \quad \phi_j(x) = \sqrt{2}\cos\left(\frac{2j-1}{2}\pi x\right).$$

Note that $(\phi_j, \phi_k) = 0$ and $(\phi_j, \phi_j) = 1$, where $(u, v) = \int_0^1 u(x)v(x)dx$.

We will use the spectral method for this problem, such that

$$u(x,t) = \sum_{j=1}^{\infty} \alpha_j(t)\phi_j(x).$$

(a) Show that

$$\alpha_j(0) = \int_0^1 \psi(x)\phi_j(x)dx.$$

(b) Let

$$f(x,t) = \sum_{j=1}^{\infty} \beta_j(t)\phi_j(x).$$

Explain why, if $u(x,t)$ satisfies the wave equation, the coefficients $\alpha_j(t)$ obey

$$\frac{\partial^2 \alpha_j}{\partial t^2} + c^2\lambda_j\alpha_j(t) = \beta_j(t).$$

with the given initial condition $\alpha_j(0)$.

(c) Assume $f(x,t) = 0$. Derive an exact formula for $\alpha_j(t)$ in terms of c, j and $\alpha_j(0)$.

(d) Suppose $f(x,t) = 0$, and let $c = 2$ and $\psi(x) = \cos(\pi x/2)$. Give an explicit expression for $u(x,t)$ in terms of j. What is the smallest time t at which $u(x,t) = 0$?

Solution.

(a) Initial condition for heat equation given as $\psi(x)$ by problem. Then

$$u(x,0) = \sum_{j=1}^{\infty} \alpha_j(0)\phi_j(x) = \psi(x).$$

Take the inner product with ϕ_k (orthonormal eigenfunctions), then

$$\alpha_j(0) = (\psi, \phi_j) = \int_0^1 \psi(x)\phi_j(x)dx \qquad j = 1, 2\cdots$$

(b) By plug in $u(x,t) = \sum_{j=1}^{\infty} \alpha_j(t)\phi_j(x)$ into the PDE, we can express the left hand side as a cosine series:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \sum_{j=1}^{\infty}\left(\frac{\partial^2 \alpha_j}{\partial t^2} + c^2\lambda_j\alpha_j(t)\right)\phi_j(x)$$

Setting this equal to the series for $f(x,t)$, we obtain

$$\sum_{j=1}^{\infty}\left(\frac{\partial^2 \alpha_j}{\partial t^2} + c^2\lambda_j\alpha_j(t)\right)\phi_j(x) = \sum_{j=1}^{\infty}\beta_j(t)\phi_j(x)$$

Taking the inner product with ϕ_k (orthonormal eigenfunctions), then we get following ODEs

$$\frac{\partial^2 \alpha_j}{\partial t^2} + c^2\lambda_j\alpha_j(t) = \beta_j(t) \qquad j = 1, 2\cdots$$

with the given initial conditions in part (a)

(c) If $f(x,t) = 0$, then we obtain the following homogeneous ODE for $j = 1, 2\cdots$

$$\begin{aligned}
\frac{\partial^2 \alpha_j}{\partial t^2} + c^2\lambda_j\alpha_j(t) = 0& \\
\alpha_j(0) &= \int_0^1 \psi(x)\phi_j dx \\
\alpha_j'(0) &= 0 \text{ (because of initial velocity is zero)}
\end{aligned}$$

Then, solution to that system is

$$\alpha_j(t) = \alpha_j(0)\cos(\sqrt{\lambda_j}t) \qquad j = 1, 2\cdots$$

(d) Since $u(x,t) = \sum_{j=1}^{\infty} \alpha_j(t)\phi_j(x)$,

$$u(x,0) = \psi(x) = \cos(\pi x/2) = \sum_{j=1}^{\infty}\alpha_j(0)\phi_j(x).$$

However, since $\phi_1(x) = \sqrt{2}\cos(\pi x/2)$, we know that $\alpha_1(0) = 1/\sqrt{2}$ and $\alpha_j(0) = 0$ for $j > 0$. The exact solution can then be written

$$u(x,t) = \cos(\pi t/2)\cos(\pi x/2).$$

This will be zero when $\cos(\pi t) = 0$, or when $t = 1/2$.

2. [28 points: (a)-(d) = 7]

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad x \in (0,1),\ \ t \geq 0$$

with boundary conditions and initial conditions:

$$\begin{aligned} u(0,t) &= \frac{\partial u}{\partial x}(1,t) = 0, \\ u(x,0) &= u_0(x), \\ u_t(x,0) &= v_0(x). \end{aligned}$$

(a) Derive the weak form of the equation with test function $v(x)$. We discretize/apply the finite element method by assuming that $v(x) = \phi_i(x)$ and

$$u(x,t) = \sum_{j=1}^{N+1} \alpha_j(t)\phi_j(x), \qquad \phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h} & x_{j-1} \leq x < x_j \\ \frac{x_{j+1}-x}{h} & x_j \leq x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}.$$

Show how this process leads to a system of ordinary differential equations of the form

$$\begin{aligned} \mathbf{M}\alpha''(t) + \mathbf{K}\alpha(t) &= \mathbf{F}(t), \\ \alpha(0) &= \alpha_0, \\ \alpha'(0) &= b_0 \end{aligned}$$

where $\alpha(t)$ is a vector of $\alpha_j(t)$. Specify the entries of $\mathbf{M}$, $\mathbf{K}$, $\mathbf{F}(t)$, α_0 and b_0. You may leave these expressions in either inner product/integral form, or in terms of h.

(b) Write the above matrix equation as a system of first order equations (with only single derivatives in time) with vector variables α and z. For this first order system of equations, use forward Euler approximations

$$\begin{aligned} \frac{\partial \alpha}{\partial t} &\approx \frac{\alpha^{i+1} - \alpha^i}{dt}, \quad \alpha(t) \approx \alpha^i \\ \frac{\partial z}{\partial t} &\approx \frac{z^{i+1} - z^i}{dt}, \quad z(t) \approx z^i \end{aligned}$$

and write an update equation for z^{i+1} and α^{i+1}, given z^i, α^i.

(c) Consider now the damped wave equation with constant density ρ and damping ϵ.

$$\rho\frac{\partial^2 u}{\partial t^2} + \epsilon\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in [0,1],\ \ t \geq 0$$

Give the weak form of this equation. Specify how the matrix equation changes from the matrix equation in part (a).

(d) Write the second order matrix equation for the above damped wave equation as a system of first order equations. Derive the Forward Euler update formulas for α^{i+1} and z^{i+1} for these new equations.

Solution.

(a) Multiply both side with the test function $v \in V = C^2_M[0,1] = \{u \in C^2[0,1] : u(0) = 0\}$ and integrate from 0 to 1

$$\int_0^1 \frac{\partial^2 u}{\partial t^2} v\, dx - \int_0^1 \frac{\partial^2 u}{\partial x^2} v\, dx = \int_0^1 f(x,t) v\, dx, \qquad \forall v \in V$$

Now, integrate by parts we get following weak form

$$\int_0^1 \frac{\partial^2 u}{\partial t^2} v\, dx + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx = \int_0^1 f(x,t) v dx, \qquad \forall v \in V$$

Note that, boundary integral is zero because of the zero boundary conditions.
Now we would like to write Galerkin problem with the finite element space $V_N = span\{\phi_1, \phi_2, \cdots \phi_N\}$. We plug the finite element solution $u(x,t) = \sum_{j=1}^N \alpha_j(t)\phi_j(x)$ into the weak form and choose $v \in V_N$

$$\int_0^1 \frac{\partial^2 \left(\sum_{j=1}^N \alpha_j(t)\phi_j(x)\right)}{\partial t^2}\phi_i(x)dx + \int_0^1 \frac{\partial \left(\sum_{j=1}^N \alpha_j(t)\phi_j(x)\right)}{\partial x}\frac{\partial \phi_i(x)}{\partial x}dx = \int_0^1 f(x,t)\phi_i(x)dx$$
$$\sum_{j=1}^N \left(\frac{\partial^2 \alpha_j(t)}{\partial t^2}\int_0^1 \phi_j(x)\phi_i(x)dx + \alpha_j(t)\int_0^1 \frac{\partial \phi_j(x)}{\partial x}\frac{\partial \phi_i(x)}{\partial x}dx\right) = \int_0^1 f(x,t)\phi_i(x)dx$$

This is the N equations N unknown 2nd order linear system with

$$\begin{aligned} \mathbf{M}\alpha''(t) + \mathbf{K}\alpha(t) &= \mathbf{F}(t), \\ \alpha(0) &= \alpha_0 \\ \alpha'(0) &= b_0 \end{aligned}$$

where $\mathbf{M}$ *mass* matrix with the entries whose (j,k) entries

$$\mathbf{M}_{i,j} = (\phi_i, \phi_j) = \int_0^1 \phi_j(x)\phi_i(x)dx = \begin{cases} 2h/3 & \text{if } i = j; \\ h/6 & \text{if } |i-j| = 1; \\ h/3 & \text{if } i = j = N; \\ 0 & \text{otherwise.} \end{cases},$$

and $\mathbf{K}$ is *stiffness* matrix with the entries

$$\mathbf{K}_{i,j} = a(\phi_j, \phi_k) = \int_0^1 \frac{\partial \phi_j(x)}{\partial x}\frac{\partial \phi_i(x)}{\partial x}dx = \begin{cases} 2/h & \text{if } i = j; \\ -1/h & \text{if } |i-j| = 1; \\ 1/h & \text{if } i = j = N; \\ 0 & \text{otherwise.} \end{cases},$$

as computed in class. Moreover

$$\mathbf{F}(t) = (f, \phi_i) = \int_0^1 f(x,t)\phi_i(x)dx$$

We have

$$u(x,0) = u_0(x)$$

and $u_0(x)$ can be approximated by its linear interpolant

$$u_0(x) = \sum_{j=1}^{N} u_0(x_j)\phi_j(x)$$

We therefore require that

$$\alpha_j(0) = u_0(x_j).$$

Similarly we require

$$\alpha_j'(0) = v_0(x_j)$$

These lead to initial conditions

$$\alpha(0) = \alpha_0 = \begin{bmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_N) \end{bmatrix}, \ \alpha'(0) = b_0 = \begin{bmatrix} v_0(x_1) \\ v_0(x_2) \\ \vdots \\ v_0(x_N) \end{bmatrix}.$$

(b) To apply one of the numerical method for ODEs , we must first convert to a 1st order system. We will define

$$\begin{aligned} \alpha(t) &= \alpha(t), \\ z(t) &= \frac{d\alpha(t)}{dt} \end{aligned}$$

where $\alpha, z \in R^N$. Then

$$\begin{aligned} \frac{d\alpha(t)}{dt} &= z, \\ \frac{dz(t)}{dt} &= \alpha''(t) = -\mathbf{M}^{-1}\mathbf{K}\alpha + \mathbf{M}^{-1}\mathbf{F} \end{aligned}$$

we have also have initial conditions

$$\begin{aligned} \alpha(0) &= \alpha_0, \\ z(0) &= \alpha'(0) = b_0. \end{aligned}$$

We have thus obtained the $2N \times 2N$ system of IVPs

$$\begin{aligned} \frac{d\alpha(t)}{dt} &= z, \qquad \alpha(0) = \alpha_0 \\ \frac{dz(t)}{dt} &= -\mathbf{M}^{-1}\mathbf{K}\alpha + \mathbf{M}^{-1}\mathbf{F}, \qquad z(0) = b_0. \end{aligned}$$

Alternatively

$$\underbrace{\begin{bmatrix} \alpha'(t) \\ z'(t) \end{bmatrix}}_{\mathbf{U}'} = \underbrace{\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha \\ z \end{bmatrix}}_{\mathbf{U}} + \underbrace{\begin{bmatrix} 0 \\ \mathbf{M}^{-1}\mathbf{F} \end{bmatrix}}_{\mathbf{G}}$$

So system can be written as

$$\mathbf{U}' = \mathbf{A}\mathbf{U} + \mathbf{G}.$$

The forward Euler method for this problem is

$$\frac{\mathbf{U}^{i+1} - \mathbf{U}^i}{dt} = \mathbf{A}\mathbf{U}^i + \mathbf{G}^i$$

or

$$\mathbf{U}^{i+1} = (I + dt\mathbf{A})\mathbf{U}^i + dt\mathbf{G}^i$$

where $\mathbf{U}^0 = [\alpha_0,\ b_0]^T$.

(c) Similar to part (a) the weak from of damped wave equation with constant density ρ and damping ϵ is

$$\int_0^1 \rho \frac{\partial^2 u}{\partial t^2} v \, dx + \int_0^1 \epsilon \frac{\partial u}{\partial t} v \, dx + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \, dx = 0, \qquad \forall v \in V$$

Similarly the Galerkin problem turns out

$$\sum_{j=1}^{N} \left(\frac{\partial^2 \alpha_j(t)}{\partial t^2} \int_0^1 \rho \phi_j(x)\phi_i(x) \, dx + \frac{\partial \alpha_j(t)}{\partial t} \int_0^1 \epsilon \phi_j(x)\phi_i(x) \, dx + \alpha_j(t) \int_0^1 \frac{\partial \phi_j(x)}{\partial x}\frac{\partial \phi_i(x)}{\partial x} \, dx \right) = 0$$

This lead to 2nd order ODE

$$\begin{aligned} \rho\mathbf{M}\alpha''(t) + \epsilon\mathbf{M}\alpha'(t) + \mathbf{K}\alpha(t) &= 0, \\ \alpha(0) &= \alpha_0 \\ \alpha'(0) &= b_0 \end{aligned}$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The entires are the same as before.

(d) Similar to part (b), let

$$\begin{aligned} \alpha(t) &= \alpha(t), \\ z(t) &= \frac{d\alpha(t)}{dt} \end{aligned}$$

where $\alpha, z \in R^N$. Then

$$\begin{aligned} \frac{d\alpha(t)}{dt} &= z, \\ \frac{dz(t)}{dt} &= \alpha''(t) = -\frac{1}{\rho}\mathbf{M}^{-1}\mathbf{K}\alpha - \frac{\epsilon}{\rho}\mathbf{M}^{-1}\mathbf{M}\alpha' = -\mathbf{M}^{-1}\mathbf{K}\alpha - \frac{\epsilon}{\rho}z \end{aligned}$$

Together with initial conditions, we have thus obtained the $2N \times 2N$ system of IVPs

$$\underbrace{\begin{bmatrix} \alpha'(t) \\ z'(t) \end{bmatrix}}_{\mathbf{U}'} = \underbrace{\begin{bmatrix} 0 & \mathbf{I} \\ -\frac{1}{\rho}\mathbf{M}^{-1}\mathbf{K} & -\frac{\epsilon}{\rho}\mathbf{I} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha \\ z \end{bmatrix}}_{\mathbf{U}}$$

So system can be written as

$$\mathbf{U}' = \mathbf{A}\mathbf{U}.$$

The forward Euler method for this problem is

$$\frac{\mathbf{U}^{i+1} - \mathbf{U}^{i}}{dt} = \mathbf{A}\mathbf{U}^{i}$$

or

$$\mathbf{U}^{i+1} = (I + dt\mathbf{A})\mathbf{U}^{i}$$

where $\mathbf{U}^0 = [\alpha_0,\ b_0]^T$.

3. [19 points: (a)-(b) = 8, (c) = 3]

Consider the following wave equation:

$$u_{tt} - u_{xx} = 0, \qquad -\infty < x < \infty,\ t \geq 0.$$

(a) Let $u_t(x,0) = \gamma(x) = 0$. What is the domain of dependence of the solution at $x = t = 1$?

(b) Let the wave equation have initial conditions

$$u(x,0) = \psi(x) = \begin{cases} x+1 & \text{if } -1 \leq x < 0, \\ 1-x & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise} \end{cases}$$
$$u_t(x,0) = \gamma(x) = 0, \qquad -\infty < x < \infty,$$

Find the exact solution to this problem in terms of $\psi(x)$, and write the solution at $t = 1/2$ and $t = 10$.

(c) What qualitative differences would you expect between the solution to the heat equation on $-\infty < x < \infty$ with initial condition given above and the solution to the wave equation?

Solution.

(a) Recall that the general solution of the wave equation on an unbounded domain takes the form

$$u(x,t) = \tfrac{1}{2}\big(\psi(x-t) + \psi(x+t)\big) + \tfrac{1}{2}\int_{x-t}^{x+t} \gamma(s)\,ds.$$

From this it follows that

$$u(1,1) = \tfrac{1}{2}\big(\psi(0) + \psi(0)\big) + \tfrac{1}{2}\int_0^0 \gamma(s)\,ds = \psi(0)$$

Since the solution at $x = 1$ and $t = 1$ depends on ψ at $x = 0$, we note that the domain of dependence is a single point which is $x = 0$.

(b) D'Alemberts general solution to the wave equation on $-\infty < x < \infty$ gives (with $c = 1$):

$$u(x,t) = \frac{1}{2}\left[\psi(x-t) + \psi(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \gamma(s)\,ds = \frac{1}{2}\left[\psi(x-t) + \psi(x+t)\right]$$

Therefore,

$$u(x,1/2) = \frac{1}{2}\left[\psi(x-1/2) + \psi(x+1/2)\right] = \begin{cases} \frac{1}{2}(x+\frac{3}{2}) & \text{if } \frac{-3}{2} \leq x < \frac{-1}{2}, \\ \frac{1}{2} & \text{if } \frac{-1}{2} \leq x < \frac{1}{2}, \\ -\frac{1}{2}(x-\frac{3}{2}) & \text{if } \frac{1}{2} \leq x < \frac{3}{2}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$u(x,10) = \frac{1}{2}\left[\psi(x-10) + \psi(x+10)\right] = \begin{cases} \frac{1}{2}(x+11) & \text{if } -11 \le x < -10, \\ -\frac{1}{2}(x+10) & \text{if } -10 \le x < -9, \\ \frac{1}{2}(x-9) & \text{if } 9 \le x < 10, \\ -\frac{1}{2}(x+11) & \text{if } 10 \le x < 11, \\ 0 & \text{otherwise} \end{cases}$$

(c) As we saw in our solutions to the heat equation on a finite interval, heat equation solutions are smooth; the initial profile immediately becomes smooth (C^∞), widens, and flattens out in time (in this case approaches 0) as the heat spreads throughout the bar. The fact that the temperature profile becomes (C^∞) immediately (i.e., for any $t > 0$) implies that the heat has an infinite speed of propagation (which is, of course, unphysical in that it violates the theory of relativity). On the other hand, the profile itself will not propagate left or right, but will remain centered at $x = 0$, with its maximum value also at $x = 0$. In contrast, as seen in part (a), the wave equation does not yield a smooth solution, but rather preserves any sharp corners or discontinuities in the initial data, and the solution is composed of left- and right-going waves that propagate at a finite speed $c = 1$. The consequences these differences have on solutions methods relate primarily to the difference in smoothness of the solutions. Since the wave equation preserves the singularities (discontinuities, corners, etc.) in the initial data, the effectiveness of the solution method is limited by its ability to accurately represent such singularities. As we discussed in class, however, Fourier series converge more slowly for singular functions than for smooth functions and piecewise-linear basis functions provide a relatively poor approximation to singular functions. Therefore, either more terms in the Fourier series representation or more basis functions in the finite element discretization are required to achieve a desired accuracy in solving the wave equation than in solving the heat equation.

4. [28 points: (a) = 6, (b)-(c) = 7, (d) = 8]

Consider the heat equation with initial condition $\psi(x)$ and *periodic* boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(x,t), \quad -1 < x < 1, \quad t \geq 0 \\ u(x,0) &= \psi(x) \\ u(-1,t) &= u(1,t) \\ \frac{\partial u}{\partial x}(-1,t) &= \frac{\partial u}{\partial x}(1,t). \end{aligned}$$

We will use the spectral method: assume the solution $u(x,t)$ is of the form

$$u(x,t) = \frac{1}{\sqrt{2}}\alpha_0(t) + \sum_{j=1}^{\infty} \alpha_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty} \beta_j(t)\cos(j\pi x).$$

Note that $1/\sqrt{2}, \sin(j\pi x), \cos(j\pi x)$ are *orthonormal* with respect to the inner product on $[-1,1]$

$$(u,v) = \int_{-1}^{1} u(x)v(x)dx,$$

This means specifically that, for $j \neq k$

$$\int_{-1}^{1} \cos(j\pi x)\cos(k\pi x)dx = \int_{-1}^{1} \cos(j\pi x)\sin(k\pi x)dx = \int_{-1}^{1} \sin(j\pi x)\sin(k\pi x)dx = 0,$$

$$\int_{-1}^{1} \sin(j\pi x)dx = \int_{-1}^{1} \cos(j\pi x)dx = 0,$$

$$\int_{-1}^{1} (\sin(j\pi x))^2\, dx = \int_{-1}^{1} (\cos(j\pi x))^2\, dx = 1.$$

(a) Show that $\alpha_0(t), \alpha_j(t)$, and $\beta_j(t)$ satisfy initial conditions

$$\alpha_0(0) = \frac{1}{\sqrt{2}}\int_{-1}^{1} \psi(x)dx, \quad \alpha_j(0) = \int_{-1}^{1} \psi(x)\sin(j\pi x)dx, \quad \beta_j(0) = \int_{-1}^{1} \psi(x)\cos(j\pi x)dx.$$

(b) Suppose that we can represent $f(x,t)$ as

$$f(x,t) = \frac{1}{\sqrt{2}}a_0(t) + \sum_{j=1}^{\infty} a_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty} b_j(t)\cos(j\pi x).$$

Using the formula for $u(x,t)$, show that the coefficients $\alpha_0(t), \alpha_j(t), \beta_j(t)$ satisfy

$$\begin{aligned} \frac{\partial \alpha_0}{\partial t} &= a_0(t) \\ \frac{\partial \alpha_j}{\partial t} + (j\pi)^2 \alpha_j(t) &= a_j(t) \\ \frac{\partial \beta_j}{\partial t} + (j\pi)^2 \beta_j(t) &= b_j(t). \end{aligned}$$

(c) Assume that $f = 0$, such that $a_0 = a_j = b_j = 0$ for all $j > 0$. Give explicit expressions for $\alpha_0(t), \alpha_j(t)$, and $\beta_j(t)$ that depend only on j and the initial conditions $\alpha_0(0), \alpha_j(0), \beta_j(0)$.

(d) Assume $f = 0$. What function will $u(x,t)$ approach as $t \to \infty$? Be as specific as possible.

Solution.

(a) The spectral solution of problem is given

$$u(x,t) = \frac{1}{\sqrt{2}}\alpha_0(t) + \sum_{j=1}^{\infty} \alpha_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty} \beta_j(t)\cos(j\pi x).$$

then initial condition $(t = 0)$

$$u(x,0) = \frac{1}{\sqrt{2}}\alpha_0(0) + \sum_{j=1}^{\infty} \alpha_j(0)\sin(j\pi x) + \sum_{j=1}^{\infty} \beta_j(0)\cos(j\pi x) = \psi(x)$$

Take inner product of both side with $\frac{1}{\sqrt{2}}$

$$\left(\frac{1}{\sqrt{2}}\alpha_0(0) + \sum_{j=1}^{\infty} \alpha_j(0)\sin(j\pi x) + \sum_{j=1}^{\infty} \beta_j(0)\cos(j\pi x), \frac{1}{\sqrt{2}}\right) = \left(\psi(x), \frac{1}{\sqrt{2}}\right)$$

which is equal

$$\left(\frac{1}{\sqrt{2}}\alpha_0(0), \frac{1}{\sqrt{2}}\right) = \left(\psi(x), \frac{1}{\sqrt{2}}\right), \text{(because of the orthonormal eigenfunctions.)}$$

where

$$\left(\frac{1}{\sqrt{2}}\alpha_0(0), \frac{1}{\sqrt{2}}\right) = \frac{1}{2}\int_{-1}^{1} \alpha_0(0)dx = \alpha_0(0).$$

Therefore

$$\alpha_0(0) = \frac{1}{\sqrt{2}}(\psi(x), 1) = \frac{1}{\sqrt{2}}\int_{-1}^{1} \psi(x)dx.$$

Now, we will repeat same steps for the eigenfunction $\sin(k\pi x)$. Take inner product of both sides with $\sin(k\pi x)$

$$\left(\frac{1}{\sqrt{2}}\alpha_0(0) + \sum_{j=1}^{\infty} \alpha_j(0)\sin(j\pi x) + \sum_{j=1}^{\infty} \beta_j(0)\cos(j\pi x), \sin(k\pi x)\right) = (\psi(x), \sin(k\pi x))$$

which is equal

$$\left(\sum_{j=1}^{\infty}\alpha_j(0)\sin(j\pi x), \sin(k\pi x)\right) = (\psi(x), \sin(k\pi x)), \text{(because of the orthonormal eigenfunctions.)}$$

Moreover when $j \neq k$, $(\sin(j\pi x), \sin(k\pi x)) = 0$. Therefore, we get

$$(\alpha_j(0)\sin(j\pi x), \sin(j\pi x)) = (\psi(x), \sin(j\pi x)),$$

where

$$(\alpha_j(0)\sin(j\pi x), \sin(j\pi x)) = \alpha_j(0)\int_{-1}^{1}(\sin(j\pi x))^2 dx = \alpha_j(0).$$

Hence

$$\alpha_j(0) = (\psi(x), \sin(j\pi x)) = \int_{-1}^{1}\psi(x)\sin(j\pi x)dx.$$

Similarly, we repeat the same procedure with $\cos(j\pi x)$ and we get,

$$(\beta_j(0)\cos(j\pi x), \cos(j\pi x)) = (\psi(x), \cos(j\pi x)),$$

which implies

$$\beta_j(0) = (\psi(x), \cos(j\pi x)) = \int_{-1}^{1}\psi(x)\cos(j\pi x)dx.$$

(b) We will plug the spectral solution of $u(x,t)$ into the given PDE

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \frac{\partial\left(\frac{1}{\sqrt{2}}\alpha_0(t) + \sum_{j=1}^{\infty}\alpha_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty}\beta_j(t)\cos(j\pi x)\right)}{\partial t} \\
&\quad - \frac{\partial^2\left(\frac{1}{\sqrt{2}}\alpha_0(t) + \sum_{j=1}^{\infty}\alpha_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty}\beta_j(t)\cos(j\pi x)\right)}{\partial x^2} \\
&= \frac{1}{\sqrt{2}}\alpha_0'(t) + \sum_{j=1}^{\infty}\alpha_j'(t)\sin(j\pi x) + \sum_{j=1}^{\infty}\beta_j'(t)\cos(j\pi x) \\
&\quad + \sum_{j=1}^{\infty}\alpha_j(t)(j\pi)^2\sin(j\pi x) + \sum_{j=1}^{\infty}\beta_j(t)(j\pi)^2\cos(j\pi x) \\
&= \frac{1}{\sqrt{2}}\alpha_0'(t) + \sum_{j=1}^{\infty}\left[\alpha_j'(t) + (j\pi)^2\alpha_j(t)\right]\sin(j\pi x) + \sum_{j=1}^{\infty}\left[\beta_j'(t) + (j\pi)^2\beta_j(t)\right]\cos(j\pi x) \\
&= f(x,t) \\
&= \frac{1}{\sqrt{2}}a_0(t) + \sum_{j=1}^{\infty}a_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty}b_j(t)\cos(j\pi x)
\end{aligned}$$

Therefore, we got

$$\frac{1}{\sqrt{2}}\alpha_0'(t) + \sum_{j=1}^{\infty}\left[\alpha_j'(t) + (j\pi)^2\alpha_j(t)\right]\sin(j\pi x) + \sum_{j=1}^{\infty}\left[\beta_j'(t) + (j\pi)^2\beta_j(t)\right]\cos(j\pi x)$$
$$= \frac{1}{\sqrt{2}}a_0(t) + \sum_{j=1}^{\infty} a_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty} b_j(t)\cos(j\pi x)$$

Similar to part (a), taking inner product with $1/\sqrt{2}, \sin(j\pi x)$ and $\cos(j\pi x)$, respectively we get desired result

$$\begin{aligned}\alpha_0'(t) &= a_0(t)\\ \alpha_j'(t) + (j\pi)^2\alpha_j(t) &= a_j(t)\\ \beta_j'(t) + (j\pi)^2\beta_j(t) &= b_j(t).\end{aligned}$$

(c) When $f = 0$, we have following equations with given initial conditions

$$\begin{aligned}\alpha_0'(t) &= 0, & \alpha_0(0) &= \frac{1}{\sqrt{2}}\int_{-1}^{1}\psi(x)dx\\ \alpha_j'(t) + (j\pi)^2\alpha_j(t) &= 0 & \alpha_j(0) &= \int_{-1}^{1}\psi(x)\sin(j\pi x)dx\\ \beta_j'(t) + (j\pi)^2\beta_j(t) &= 0 & \beta_j(0) &= \int_{-1}^{1}\psi(x)\cos(j\pi x)dx.\end{aligned}$$

Solutions of the above system are given respectively

$$\begin{aligned}\alpha_0(t) &= \alpha_0(0),\\ \alpha_j(t) &= \alpha_j(0)e^{-(j\pi)^2 t}\\ \beta_j(t) &= \beta_j(0)e^{-(j\pi)^2 t}\end{aligned}$$

(d) if $f = 0$ by part (c), we have the following spectral solution

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2}}\alpha_0(t) + \sum_{j=1}^{\infty}\alpha_j(t)\sin(j\pi x) + \sum_{j=1}^{\infty}\beta_j(t)\cos(j\pi x)\\ &= \frac{1}{\sqrt{2}}\alpha_0(0) + \sum_{j=1}^{\infty}\alpha_j(0)e^{-(j\pi)^2 t}\sin(j\pi x) + \sum_{j=1}^{\infty}\beta_j(0)e^{-(j\pi)^2 t}\cos(j\pi x)\end{aligned}$$

Note that when $t \to \infty$, $e^{-(j\pi)^2 t} \to 0$. This implies our solution

$$u(x,t) \to \frac{1}{\sqrt{2}}\alpha_0(0)$$

5. [Bonus: 3 points: 1 point each]

This problem deals with the advection-diffusion equation. We wish to solve the time-dependent version of this important equation in physics and engineering:

$$u_{tt}(x,t) = u_{xx}(x,t) - cu_x(x,t)$$

for $x \in [0,1]$ with $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = u_0(x)$. (The $-u_{xx}$ term describes diffusion of a fluid; the constant c describes the strength with which the fluid moves across the domain through the cu_x term).

Define the linear operator $L : C_D^2[0,1] \to C[0,1]$ by $Lu = -u'' + cu'$. The eigenvalues of L are

$$\lambda_n = n^2\pi^2 + \frac{c^2}{4}, \qquad n = 1, 2, \cdots$$

with corresponding eigenfunctions

$$\psi_n(x) = e^{cx/2}\sin(n\pi x), \qquad n = 1, 2, \cdots$$

(a) This L is not symmetric (you do not need to show this). To construct solutions for this case, we must introduce the notion of the *adjoint*, which generalizes the transpose of a matrix. An operator

$$L^* : C_D^2[0,1] \to C[0,1]$$

is the adjoint of

$$L : C_D^2[0,1] \to C[0,1]$$

with $C_D^2[0,1] \subset C[0,1]$ provided $(Lu, v) = (u, L^*v)$ for all $u, v \in C_D^2[0,1]$, where $(u,v) = \int_0^1 u(x)v(x)dx$. Show that the adjoint L^* of L is given by

$$L^*u = -u'' - cu'$$

That is, show that this definition of L^* gives $(Lu, v) = (u, L^*v)$ for all $u, v \in C_D^2[0,1]$.

(b) The adjoint L^* has the same eigenvalues as L, but its eigenfunctions are instead

$$\tilde{\psi}_n(x) = e^{-cx/2}\sin(n\pi x), \qquad n = 1, 2, \cdots.$$

It turns out the eigenfunctions of L and L^* are biorthogonal:

$$(\psi_n, \tilde{\psi}_m) = 0, \qquad n \neq m.$$

Show that, when expanding a function f in the eigenfunctions ψ_n, the coefficients will involve eigenfunctions of both L and L^* :

$$f(x) = \sum_{n=1}^{\infty} \frac{\left(f, \tilde{\psi}_n\right)}{\left(\psi_n, \tilde{\psi}_n\right)} \psi_n(x).$$

(c) Adapt the spectral method to the advection-diffusion equation

$$u_t(x,t) = u_{xx}(x,t) - cu_x(x,t), \qquad u(0,t) = u(1,t) = 0$$

with the initial condition $u(x,0) = u_0(x)$. Specifically, write down a series solution for $u(x,t)$ in terms the eigenfunctions $\psi_n(x)$, which should involve $u_0(x), c, n, \{\psi_n\}$ and $\{\tilde{\psi}_n\}$.

Solution.

(a) We compute the adjoint as follows: for $u, v \in C^2_D[0,1]$

$$\begin{aligned}
(Lu, v) &= \int_0^1 (-u''(x,t) + cu'(x,t))v(x)dx \\
&= \int_0^1 -u''(x,t)v(x)dx + \int_0^1 cu'(x,t)v(x)dx \\
&= \big[u'(x,t)v(x)\big]_0^1 + \int_0^1 u'(x,t)v'(x)dx + [cu(x,t))v(x)]_0^1 - \int_0^1 cu(x,t)v'(x)dx \\
&= \int_0^1 u'(x,t)v'(x)dx - \int_0^1 cu(x,t)v'(x)dx \\
&= \big[u(x,t)v'(x)\big]_0^1 - \int_0^1 u(x,t)v''(x)dx - \int_0^1 cu(x,t)v'(x)dx \\
&= -\int_0^1 u(x,t)v''(x)dx - \int_0^1 cu(x,t)v'(x)dx \\
&= \int_0^1 u(x,t)(-v''(x) - cv'(x))dx \\
&= (u, L^*v)
\end{aligned}$$

Hence the adjoint is given by $L^*u = -u'' - cu'$ for $u \in C^2_D[0,1]$.

(b) Let $f_N = \sum_{n=1}^N c_n\psi_n$ be a best approximation to f. Imposing the conditions $(f - f_N, w) = 0$ for $w = \tilde{\psi}_n$ with $n = 1, 2, \cdots, N$ gives

$$\begin{aligned}
0 &= (f - f_N, \tilde{\psi}_n) \\
&= (f, \tilde{\psi}_n) - (f_N, \tilde{\psi}_n) \\
&= (f, \tilde{\psi}_n) - (\sum_{m=1}^N c_m\psi_m, \tilde{\psi}_n) \\
&= (f, \tilde{\psi}_n) - \sum_{m=1}^N c_m(\psi_m, \tilde{\psi}_n) \\
&= (f, \tilde{\psi}_n) - c_n(\psi_n, \tilde{\psi}_n)
\end{aligned}$$

Rearrange this equation to give an expression for c_n,

$$c_n = \frac{(f, \tilde{\psi}_n)}{(\psi_n, \tilde{\psi}_n)}$$

Hence suggesting a nonsymmetric version of the spectral method:

$$f(x) = \sum_{n=1}^{\infty} \frac{\left(f, \tilde{\psi}_n\right)}{\left(\psi_n, \tilde{\psi}_n\right)} \psi_n(x).$$

(c) We seek the solution $u(x,t)$ in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

Note that

$$u_t(x,t) = \sum_{n=1}^{\infty} a_n'(t)\psi_n(x).$$

and

$$Lu(x,t) = \sum_{n=1}^{\infty} a_n(t)\lambda_n\psi_n(x).$$

Equate these two, and take inner products with ψ_k to obtain

$$a_k'(t) = -\lambda_k a_k(t)$$

i.e.;

$$a_k(t) = e^{-\lambda_k t} a_k(0).$$

Now, expand the initial condition

$$u_0(x) = \sum_{n=1}^{\infty} \frac{\left(u_0, \tilde{\psi}_n\right)}{\left(\psi_n, \tilde{\psi}_n\right)} \psi_n(x).$$

to see that

$$a_k(0) = \frac{\left(u_0, \tilde{\psi}_n\right)}{\left(\psi_n, \tilde{\psi}_n\right)}$$

Hence the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\left(u_0, \tilde{\psi}_n\right)}{\left(\psi_n, \tilde{\psi}_n\right)} \psi_n(x).$$