CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 8

Posted Monday 22 October 2012. Due Monday 29 October 2012, 5pm. Corrected 25 October 2012.

1. [50 points]
   Consider the following three matrices:

$$\text{(i)}\quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(ii)}\quad \mathbf{A} = \begin{bmatrix} -50 & 49 \\ 49 & -50 \end{bmatrix} \qquad \text{(iii)}\quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (a) For each of the matrices (i)–(iii), compute the matrix exponential $e^{t\mathbf{A}}$.
   You may use `eig` for the eigenvalues and eigenvectors, but please construct the matrix exponential "by hand" (not with `expm`). For diagonalizable $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, recall the formula $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{\Lambda}}\mathbf{V}^{-1}$. If you encounter a complex eigenvalue $\lambda = \alpha + i\beta$, you may use the formula

$$e^{\lambda} = e^{\alpha+i\beta} = e^{\alpha}(\cos(\beta) + i\sin(\beta)).$$

   (b) Use your answers in part (a) to explain the behavior of solutions $\mathbf{x}(t)$ to the differential equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ as $t \to \infty$, given that $\mathbf{x}(0) = [2,\ 0]^T$ (e.g., specify and explain exponential growth, exponential decay, or neither) for each of the three matrices (i)–(iii).

   (c) For the matrix (ii), describe how large one can choose the time step Δt so that the forward Euler method applied to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$,

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t\mathbf{A}\mathbf{x}_k,$$

   will produce a solution that qualitatively matches the behavior of the true solution (i.e., the approximations $\mathbf{x}_k$ should grow, decay, or remain of the same size as the true solution does).
   Answer the same question for the backward Euler method

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t\mathbf{A}\mathbf{x}_{k+1}.$$

   (d) For the matrix in (iii), describe how the forward Euler method behaves *for all* Δt as $k \to \infty$ for $\mathbf{x}(0) = [1, 1]^T$. Now describe how the backward Euler method must behave as $k \to \infty$ for the same matrix and initial condition.

2. [50 points]
   There exist a host of alternatives to the forward and backward Euler methods for approximating the solution of the differential equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. For example, the *trapezoid method* has the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \tfrac{1}{2}\Delta t\mathbf{A}(\mathbf{x}_k + \mathbf{x}_{k+1}),$$

   where $\Delta t > 0$ is the time-step, and $\mathbf{x}_k \approx \mathbf{x}(t_k)$ for $t_k = k\Delta t$.

   (a) Like backward Euler, the trapezoid method is an *implicit* technique: $\mathbf{x}_{k+1}$ appears on both the right and left hand side of the formula above that defines it. Describe how to find $\mathbf{x}_{k+1}$ given $\mathbf{x}_k$. In particular, what linear system of algebraic equations needs to be solved at each step? (For comparison, the backward Euler method requires the solution of the system $(\mathbf{I} - \Delta t\mathbf{A})\mathbf{x}_{k+1} = \mathbf{x}_k$ for the unknown $\mathbf{x}_{k+1}$ at each step.)

   (b) Consider the matrix and initial condition

$$\mathbf{A} = \begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   Approximate the solution to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ on the interval $t \in [0, 5]$ for time step $\Delta t = .05$. Produce a `semilogy` plot showing $t_k = k\Delta t$ versus $\|\mathbf{x}_k\|$ for $k = 0, \ldots, 100$. (Use the `norm` command in MATLAB to compute $\|\mathbf{x}_k\|$.)

(c) We wish to understand how the error in our approximation at time $t = 1$ improves as we run the simulation with smaller and smaller Δt values. Produce a `loglog` plot showing Δt versus the error in the trapezoid rule and backward Euler approximations for the matrix and initial condition in part (b) at time $t = 1$. To compute the error, first find the exact solution $\mathbf{x}(1) = e^{\mathbf{A}}\mathbf{x}(0)$ using the `expm` command, then compute the norms $\|\widehat{\mathbf{x}} - \mathbf{x}(1)\|$, where $\widehat{\mathbf{x}}$ denotes your approximation to $\mathbf{x}(1)$ from the trapezoid or backward Euler methods. Start your plot with $\Delta t = 1/2$ and use sufficiently many smaller values of Δt to make the trend in your plot clear. For which method does the error decay more rapidly as $\Delta t \to 0$?

(d) Forward Euler iterates can be written as $\mathbf{x}_k = (\mathbf{I} + \Delta t\mathbf{A})^k \mathbf{x}_0$, while backward Euler iterates can be written as $\mathbf{x}_k = (\mathbf{I} - \Delta t\mathbf{A})^{-k}\mathbf{x}_0$.

Work out a similar formula for the iterates $\mathbf{x}_k$ generated by the trapezoid method.

Suppose $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ is symmetric, and all of its eigenvalues λ_j, $j = 1, \ldots, n$, are negative. How must you choose the time step Δt to ensure that the iterates $\mathbf{x}_k$ generated by the trapezoid method converge to zero, $\|\mathbf{x}_k\| \to 0$, as $k \to \infty$?