CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 5

Posted Wednesday 19 September 2012. Due Wednesday 26 September 2012, 5pm.

Please write the name of your college on your paper.

All of the problems on this set use the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

1. [30 points]
   Consider the linear operator $L_b : C_b^2[0,1] \to C[0,1]$ defined by

   $$L_b u = -\frac{d^2u}{dx^2},$$

   where

   $$C_b^2[0,1] = \Big\{u \in C^2[0,1] : \frac{du}{dx}(0) = u(1) = 0\Big\}.$$

   (a) Is L_b symmetric?

   (b) What is the null space of L_b?
   That is, find all $u \in C_b^2[0,1]$ such that $L_b u(x) = 0$ for all $x \in [0,1]$.

   (c) Show that $(L_b u, u) > 0$ for all nonzero $u \in C_b^2[0,1]$ and explain why this implies that $\lambda > 0$ for all eigenvalues λ.

   (d) Find the eigenvalues and eigenfunctions of L_b.

2. [35 points]
   Consider the operator $L_D : C_D^2[0,1] \to C[0,1]$ defined by

   $$L_D u = -\frac{d^2u}{dx^2},$$

   with homogeneous Dirichlet boundary conditions imposed by the domain

   $$C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}.$$

   Recall that the eigenvalues of L_D are $\lambda_n = n^2\pi^2$ with associated normalized eigenfunctions

   $$\psi_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

   ("Normalized" means that these eigenfunctions each have norm equal to one, $\|\psi_n\|^2 = (\psi_n, \psi_n) = 1$.)

   We wish to study the equation $Lu = f$ for $f(x) = 1$, a problem to be addressed in Lecture 14. First consider the best approximation to f from $\mathrm{span}\{\psi_1, \ldots, \psi_N\}$:

   $$f_N = \sum_{n=1}^{N} \frac{(f, \psi_n)}{(\psi_n, \psi_n)}\psi_n = \sum_{n=1}^{N} c_n \psi_n.$$

   In class we shall note that $f_N(x)$ *does not converge* to $f(x)$ for all points $x \in [0,1]$: in particular, $f_N(0) = f_N(1) = 0$ for all N, while $f(0) = f(1) = 1$. However, we claimed in class that f_N *does converge to* f *in norm*, that is, $\|f - f_N\| \to 0$ as $N \to \infty$. In this problem, you will justify that statement.

(a) Use the properties of inner products, the orthogonality of the eigenfunctions, the fact that $(\psi_k, \psi_k) = 1$, and $\|f - f_N\|^2 = (f - f_N, f - f_N)$ to derive the general formula

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} c_n^2.$$

(b) For $f(x) = 1$, we computed in class that $c_n = 2\sqrt{2}/(n\pi)$ for odd n, and $c_n = 0$ for even n. Use this expression for c_n and your formula from part (a) to produce a `loglog` plot of the error $\|f - f_N\|$ versus N for all integers $N = 1, \ldots, 10^4$. (Optional: take $N = 1, \ldots, 10^6$. You will need to write efficient MATLAB code for this to run quickly.)

(c) For $f(x) = 1$, the equation $Lu = f$ has the exact solution $u(x) = x(1 - x)/2$. Confirm that the spectral method approximation

$$u_N = \sum_{n=1}^{N} \frac{c_n}{\lambda_n} \psi_n$$

provides the best approximation to u from the subspace $\text{span}\{\psi_1, \ldots, \psi_N\}$. To do this, simply show that the coefficient of ψ_n you would get for the best approximation of u (which requires knowledge of u) matches the coefficient produced by the spectral method (which did not require knowledge of u) for this particular f and u:

$$\frac{(u, \psi_n)}{(\psi_n, \psi_n)} = \frac{c_n}{\lambda_n},$$

where c_n comes from the best approximation to f given in part (b).

(d) Given the result of part (c), the same argument used in part (a) tells us that

$$\|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n^2}.$$

(You do not need to show this explicitly.) Use this formula to produce a `loglog` plot of the error $\|u - u_N\|$ for $N = 1, \ldots, 10^4$ (or $N = 1, \ldots, 10^6$) on the same plot you made in part (b). (Be aware that the error may appear to flatline around 10^{-8}: this is a consequence of the computer's floating point arithmetic, and is not a concern of ours here. To learn more about this phenomenon, take CAAM 353 or CAAM 453!)

3. [35 points]
This problem concerns the same operator from Problem 2, $L : C_D^2[0, 1] \to C[0, 1]$ defined by

$$L_D u = -\frac{d^2 u}{dx^2},$$

with homogeneous Dirichlet boundary conditions imposed via

$$C_D^2[0, 1] = \{u \in C^2[0, 1] : u(0) = u(1) = 0\}.$$

The eigenvalues and (normalized) eigenfunctions remain as in Problem 2: $\lambda_n = n^2\pi^2$ and $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for $n = 1, 2, \ldots$. Now let $f(x) = x^2(1 - x)$.

(a) For this f, compute the coefficients

$$c_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

in the expansion

$$f = \sum_{n=1}^{\infty} c_n \psi_n.$$

You may determine these by hand, by consulting a table of integrals, or by using a symbolic mathematics package like Mathematica or the Symbolic Toolbox in MATLAB.

(b) Produce a plot (or series of plots) comparing $f(x)$ to the partial sums

$$f_N(x) = \sum_{k=1}^{N} c_k \psi_k(x)$$

for $N = 1, \ldots, 10$.

(c) Plot the approximations u_N to the true solution u that you obtain using the spectral method:

$$u_N(x) = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \psi_k(x)$$

for $N = 1, \ldots, 10$.

(d) Now replace the homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ above with the inhomogeneous Dirichlet conditions $u(0) = -1/100$ and $u(1) = 1/100$. Describe how to adjust your solution from part (c) to account for these boundary conditions, and produce a plot of the solution with these inhomogeneous boundary conditions, based on u_{10} from part (c).

Challenge problem [5 bonus points]

Many fluid dynamics problems lead to *advection–diffusion* equations, the simplest example of which is

$$u''(x) + cu'(x) = f(x),$$

for $x \in [0, 1]$ with $u(0) = u(1) = 0$. (The u'' term describes diffusion of a fluid; the constant c describes the strength with which the fluid advects across the domain through the cu' term.)

Define the linear operator $L : C_D^2[0, 1] \to C[0, 1]$ by $Lu = u'' + cu'$.

Determine all the eigenvalues and eigenfunctions of L.
Are the eigenfunctions orthogonal? Explain.