CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Examination 1

Instructions:

1. Time limit: **3 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You are allowed one cheat sheet to refer to during the exam.
   You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions throughly (but succintly!) and justify all your answers.
   Show your work for partial credit.
5. Print your name on the line below:

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6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points: 5 points for (a), 10 points for (b)-(c)] In the homeworks, and in class, we used the spectral method to solve several problems of the form $Lu = f$ with boundary conditions. The spectral method has conditions that must be satisfied before we can apply it. In this problem we will explore the impact of boundary conditions on several theoretical facets of the spectral method.

   We'll study the steady state heat equation

$$-\frac{\partial^2 u(x)}{\partial x^2} = f(x).$$

   We'll also use the usual inner product

$$(f, g) = \int_0^1 f(x)g(x)\,dx$$

   (a) Derive the spectral method eigenfunctions and eigenvalues for the steady heat equation under the boundary conditions

$$\frac{\partial u(0)}{\partial x} = u(1) = 0.$$

   (b) Consider now boundary conditions

$$u(1) = \frac{\partial u(1)}{\partial x} = 0.$$

   Define the vector space $C_R^2[0,1]$ and the operator $L : C_R^2[0,1] \to C[0,1]$ such that

$$C_R^2[0,1] = \left\{ u \in C^2[0,1], \quad u(1) = \frac{\partial u(1)}{\partial x} = 0. \right\}, \quad Lu = -\frac{\partial^2 u}{\partial x^2}.$$

   Show that under these boundary conditions, L is not symmetric, i.e.

$$(Lu, v) \neq (u, Lv).$$

   (c) Consider the more general set of boundary conditions

$$u(1) = a_{11}u(0) + a_{12}\frac{\partial u(0)}{\partial x} \tag{1}$$

$$\frac{\partial u(1)}{\partial x} = a_{21}u(0) + a_{22}\frac{\partial u(0)}{\partial x} \tag{2}$$

   We can define a vector space of functions V_A to be those functions in $C^2[0,1]$ satisfying the boundary conditions of the boundary value problem of equation (1) and (2). Define $L : V_A \to C[0,1]$ as

$$Lu = -\frac{\partial^2 u}{\partial x^2}.$$

   We also define a matrix A containing the boundary condition coefficients

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

   Whether or not L is symmetric turns out to depend on the determinant of the matrix A. What should $\det(A)$ be in order for L to be symmetric on V_A? *Hint: the determinant of a* 2×2 *matrix is* $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. *Your answer should be a number.*

2. [20 points: 10 each] In this problem, we consider the finite element method for the equation

$$-\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x).$$

This models the steady state distribution of temperature in a bar, where $k(x)$ is the diffusivity of the bar at the point x. Diffusivity must be positive for the equation to be physically realistic; however, if $k(x)$ is not positive, the finite element method may run into issues as well.

It may be helpful to use the fact that the determinant of a 2×2 matrix is

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

(a) If $k(x) > 0$ for $0 < x < 1$, the weak form

$$a(u,v) = \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

defines an inner product for $C_D^2[0,1]$. Explain why, if $k(x) = 0$ over some interval $[a,b] \subset [0,1]$, $a(u,v)$ may not be positive definite (and hence not an inner product).

(b) Assume that $N = 2$, such that there are two hat functions $\phi_1(x), \phi_2(x)$ centered around points

$$x_1 = 1/3, \quad x_2 = 2/3.$$

Let $k(x)$ be the function

$$k(x) = \begin{cases} 0 & x \leq 2/3 \\ 1 & x > 2/3. \end{cases}$$

Compute the finite element stiffness matrix for $N = 2$. Verify that $A_{11} = A_{21} = 0$. Explain what complication arises if one attempts to solve this system/invert this matrix.

3. [28 points: 7 points each] Consider the steady heat equation with *periodic* boundary conditions

$$-\frac{\partial^2 u}{\partial x^2} = f(x), \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$

Define $C^2_P[0,1]$ as the space of C^2 continuous functions with periodic boundary conditions

$$C^2_P[0,1] = \left\{ u \in C^2[0,1], \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}. \right\}$$

and define the operator $L : C^2_P[0,1] \to C[0,1]$ such that $Lu = -\frac{\partial^2 u}{\partial x^2}$. Note that this operator has a zero eigenvalue with zero eigenfunction $\psi_0(x) = 1$.

(a) Define ϕ_j, ψ_j

$$\phi_j(x) = \sin(j\pi x), \quad \psi_j(x) = \cos(j\pi x), \quad j = 1, 2, \ldots.$$

Verify that ϕ_j, ψ_j, are also eigenfunctions of L with eigenvalues $\lambda_j = (j\pi)^2$, and that linear combinations $f(x)$ of ϕ_j, ψ_j, and ψ_0

$$f(x) = d_0 + \sum_{j=1}^{\infty} \left(c_j \sin(j\pi x) + d_j \cos(j\pi x)\right)$$

satisfy periodic boundary conditions.

(b) Show for the operator $A : C^2_P[0,1] \to C[0,1]$, defined as

$$Au = u + Lu = u - \frac{\partial^2 u}{\partial x^2}$$

that ϕ_j, ψ_j are eigenfunctions of A with eigenvalues $\mu_j = 1 + (j\pi)^2$, and that ψ_0 is an eigenfunction with eigenvalue $\mu_0 = 1$. Explain briefly why we may solve

$$u(x) - \frac{\partial^2 u(x)}{\partial x^2} = f(x)$$

using the spectral method, but not the steady state heat equation $-\frac{\partial^2 u}{\partial x^2} = f(x)$.

(c) Assume that $f(x)$ and $u(x)$ may be represented as

$$f(x) = d_0 + \sum_{j=1}^{\infty} \left(c_j \sin(j\pi x) + d_j \cos(j\pi x)\right),$$

$$u(x) = \beta_0 + \sum_{j=1}^{\infty} \left(\alpha_j \sin(j\pi x) + \beta_j \cos(j\pi x)\right).$$

Derive that the spectral method solution for the equation

$$u(x) - \frac{\partial^2 u(x)}{\partial x^2} = f(x), \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$

is given by the above expression for $u(x)$ where

$$\beta_0 = d_0, \quad \alpha_j = \frac{c_j}{1 + (j\pi)^2}, \quad \beta_j = \frac{d_j}{1 + (j\pi)^2}.$$

(d) Solve for $u(x)$ such that

$$u(x) - \frac{\partial^2 u(x)}{\partial x^2} = 1 + \cos(\pi x), \quad u(0) = u(1), \quad \frac{\partial u(0)}{\partial x} = \frac{\partial u(1)}{\partial x}.$$

4. [27 points: 8 points (a)-(c), 3 points (d)] Consider the *Euler Bernoulli beam equation*,

$$(k(x)u''(x))'' = f(x), \qquad 0 < x < 1,$$

Here $k(x)$ is a positive-valued function that describes the material properties of the beam.

(a) Define the operator $L : C^4[0,1] \to C[0,1]$

$$Lu = (k(x)u'')''$$

Verify that, for $k(x) = 1$, the functions

$$\sin(\mu_j x), \quad \cos(\mu_j x), \quad \sinh(\mu_j x), \quad \cosh(\mu_j x)$$

are all eigenfunctions of L with eigenvalues λ_j, where $\mu_j = \lambda_j^{1/4}$ and where hyperbolic sine/cosine are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Hint: the derivatives of sinh *and* cosh *may be written in terms of* sinh *and* cosh.

(b) Consider boundary conditions describing a beam that is *clamped* at both ends:

$$u(0) = u(1) = 0; \qquad u'(0) = u'(1) = 0.$$

With these boundary conditions, the eigenvalues and eigenvectors of this operator are diffcult to compute, even if $k(x) = 1$. As a result, we will consider the finite element approximation of this problem.

Derive the weak form of the beam equation with the above boundary conditions, i.e., derive the weak problem

$$a(u, v) = (f, v); \qquad \text{for all } v \in V = C_D^4[0,1],$$

where

$$C_D^4[0,1] = \{u \in C^4[0,1] : u(0) = u(1) = u'(0) = u'(1) = 0\}.$$

Specify the bilinear form $a(u, v)$, and show that it is an inner product on $C_D^4[0,1]$

(c) Suppose that $V_n = span\{\phi_1, \cdots, \phi_n\}$ is an n-dimensional subspace of $C_D^4[0,1]$. (Do not assume a particular form for the functions $\phi_1, \cdots, \phi_n$ at this point.)

Show how the Galerkin problem

$$a(u_n, v) = (f, v), \qquad \text{for all } v \in V_n$$

leads to the linear system $Ku = f$. Be sure to specify the entries of K, u, and f .

(d) Now suppose we take for $\phi_1, \cdots, \phi_n$ the standard piecewise linear 'hat' functions used, for example, in Problem 2. Are these functions suitable for this problem? If so, describe the location of the nonzero entries of the matrix K. If not, roughly describe a better choice for the functions $\phi_1, \cdots, \phi_n$ and the explain which entries of K are nonzero for that choice.