CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 1

Posted Wednesday 27, August 2014. Due 5pm Wednesday 3, September 2014.

Please write your name and ***residential college*** *on your homework.*

1. [24 points: 4 each]

   For each of the following equations, (a) specify whether it is an ODE or a PDE; (b) determine its order; and (c) specify whether it is linear or nonlinear. For those that are linear, specify whether they (d) are homogeneous or inhomogeneous; and (e) have constant or variable coefficients.

   (1.1) $\dfrac{dv}{dx} + \dfrac{2}{x}v = 0$ (1.2) $\dfrac{\partial v}{\partial t} - 3\dfrac{\partial v}{\partial x} = x - t$

   (1.3) $\dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(2u\dfrac{\partial u}{\partial x}\right) = 0$ (1.4) $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \dfrac{\partial^3 u}{\partial x^3} = 0$

   (1.5) $\dfrac{d^2 y}{dx^2} - 7(1-y^2)\dfrac{dy}{dx} + y = 0$ (1.6) $\dfrac{d^2}{dx^2}\left(x^2\dfrac{d^2 u}{dx^2}\right) = \sin(x)$

2. [21 points: 7 each]

   (a) Is $v(x) = 1/x^2$ a solution of
   $$\frac{dv}{dx} + \frac{2}{x}v = 0?$$

   (b) Is $v(x,t) = t(t+x)$ a solution of
   $$\frac{\partial v}{\partial t} - 3\frac{\partial v}{\partial x} = x - t?$$

   (c) Is $u(x,t) = xe^t$ a solution of
   $$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) = 0?$$

3. [15 points]

   A Bernoulli differential equation (named after James Bernoulli) is of the form
   $$\frac{dy}{dx} + P(x)y = Q(x)y^n$$
   Observe that, if $n = 0$ or $n = 1$, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation
   $$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

4. [40 points: 14 each for (a),(c), 7 for (b), 5 for (d)]

Recall the 1D steady-state heat equation with constant diffusivity over the interval $[0, 1]$

$$-\frac{\partial^2 u}{\partial x^2} = f$$
$$u(0) = u(1) = 0.$$

Recall from class the finite difference approximation to this problem: given a set of points $x_0, \ldots, x_{N+1}$, solved for the solution $u(x_i)$ at each point by approximating $\frac{\partial^2 u}{\partial x^2}$ with

$$u''(x_i) \approx \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2}, \quad i = 1, \ldots, N$$

(where h is the spacing between points x_{i+1} and x_i) along with the conditions that

$$u(x_0) = u(x_{N+1}) = 0.$$

We will modify this finite difference approximation to accomodate instead the Neumann boundary condition of $u'(1) = 0$ at $x = 1$.

(a) We would like to enforce that $u'(x_{N+1}) = 0$, but if we approximate $u'(x_{N+1})$ with a central difference

$$u'(x_{N+1}) \approx \frac{u(x_{N+\frac{3}{2}}) - u(x_{N+\frac{1}{2}})}{h},$$

we end up with an equation involving $u(x_{N+\frac{3}{2}})$, which does not lie inside the interval $[0, 1]$. Instead, we can define a *backward difference* approximation to the derivative

$$u'(x_{N+1}) \approx \frac{u(x_{N+1}) - u(x_N)}{h} = 0$$

and set this to zero instead. Write out the expression for $u''(x_N)$ in terms of $u(x_i)$ and use the backward difference approximation for $u'(x_{N+1})$ to eliminate $u(x_{N+1})$.

(b) Determine the exact solution to $-u''(x) = 1$ for $u(0) = 0$, $u'(1) = 0$ (hint: the solution is a quadratic function).

(c) Create a MATLAB script that constructs the matrix system $Au = f$ resulting from the finite difference equations when $f = 1$. Plot the computed solution values $u(x_i)$, as well as the error at each point $|u_{\text{exact}}(x_i) - u(x_i)|$, for $i = 0, \ldots, N + 1$ for $N = 16, 32, 64, 128$, and label each appropriately.

(d) Suppose we have $u'(0) = u'(1) = 0$. Show that if $u(x)$ is a solution of the steady state heat equation with these boundary conditions, that

$$u + C$$

for any constant C is also a solution to the same steady state heat equation. This shows that there is no *unique* solution to the steady state heat equation under these boundary conditions.