CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 21 · Solutions

Posted Wednesday 25 September 2013. Due 5pm Wednesday 9 October 2013.

21. [25 points] Let the inner product $(\cdot,\cdot):\ C[-1,1]\times C[-1,1]\to\mathbb{R}$ be defined by

$$(u,v)=\int_{-1}^{1}u(x)v(x)\,dx.$$

Let the linear operator $P_e:\ C[-1,1]\to C[-1,1]$ be defined by

$$(P_ef)(x)=\frac{f(x)+f(-x)}{2}$$

and let the linear operator $P_o:\ C[-1,1]\to C[-1,1]$ be defined by

$$(P_of)(x)=\frac{f(x)-f(-x)}{2}.$$

Note that P_e and P_o project functions onto their even and odd parts, respectively.

(a) Verify that P_e and P_o are projections.

(b) For all $f\in C[-1,1]$, verify that P_ef and P_of are orthogonal with respect to the inner product $(\cdot,\cdot)$.

(c) Is P_e+P_o a projection? Note that $P_e+P_o:\ C[-1,1]\to C[-1,1]$ is defined by $(P_e+P_o)f=P_ef+P_of$.

Solution.

(a) [8 points] If $f\in C[-1,1]$ then

$$(P_ef)(x)=\frac{f(x)+f(-x)}{2},$$

and so

$$(P_e(P_ef))(x)=\frac{\left(\dfrac{f(x)+f(-x)}{2}\right)+\left(\dfrac{f(-x)+f(x)}{2}\right)}{2}=\frac{f(x)+f(-x)}{2}=(P_e)f(x).$$

Thus we conclude that $P_e(P_ef)=P_ef$ for all $f\in C[-1,1]$, which means that P_e is a projection. In the same way, if $f\in C[-1,1]$ then

$$(P_of)(x)=\frac{f(x)-f(-x)}{2},$$

and so

$$(P_o(P_of))(x)=\frac{\left(\dfrac{f(x)-f(-x)}{2}\right)-\left(\dfrac{f(-x)-f(x)}{2}\right)}{2}=\frac{f(x)-f(-x)}{2}=(P_of)(x).$$

Thus we conclude that $P_o(P_of)=P_of$ for all $f\in C[-1,1]$, which means that P_o is a also projection.

(b) [9 points] If $f \in C[-1,1]$ then

$$
\begin{aligned}
(P_e f, P_o f) &= \int_{-1}^{1} (P_e f)(x)(P_o f)(x)\,dx \\
&= \int_{-1}^{1} \frac{(f(x)+f(-x))\,(f(x)-f(-x))}{4}\,dx \\
&= \frac{1}{4}\int_{-1}^{1} (f(x))^2 - f(x)f(-x) + f(x)f(-x) - (f(-x))^2\;dx \\
&= \frac{1}{4}\int_{-1}^{1} (f(x))^2 - (f(-x))^2\;dx \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\;dx - \int_{-1}^{1} (f(-x))^2\;dx\right) \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\;dx + \int_{-(-1)}^{-1} (f(y))^2\;dy\right) \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\;dx + \int_{1}^{-1} (f(y))^2\;dy\right) \\
&= \frac{1}{4}\left(\int_{-1}^{1} (f(x))^2\;dx - \int_{-1}^{1} (f(y))^2\;dy\right) \\
&= 0
\end{aligned}
$$

where $y = -x$.

(c) [8 points] If $f \in C[-1,1]$ then

$$((P_e + P_o)f)(x) = (P_e f)(x) + (P_o f)(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = f(x),$$

and so

$$((P_e + P_o)((P_e + P_o)f))(x) = ((P_e + P_o)f)(x).$$

Hence, $(P_e + P_o)((P_e + P_o)f) = (P_e + P_o)f$ for all $f \in C[-1,1]$ and so $P_e + P_o$ is a projection.