# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 10 · Solutions

Posted Wednesday 11 September 2013. Due 5pm Wednesday 18 September 2013.

10. [25 points]
    Demonstrate whether or not each of the following sets is a real vector space. You do not need to show that $\mathbb{R}^2$, $\mathbb{R}^3$, $C[0,1]$, or $C^2[0,1]$ are real vector spaces.

    (a) $\{\mathbf{x} \in \mathbb{R}^2 : \ x_2 = x_1^3\}$

    (b) $\{\mathbf{x} \in \mathbb{R}^3 : \ 3x_1 + 2x_2 + x_3 = 0\}$

    (c) $\{f \in C[0,1] : \ f(x) \geq 0 \text{ for all } x \in [0,1]\}$

    (d) $\left\{f \in C[0,1] : \ \max\limits_{x\in[0,1]} f(x) \leq 1\right\}$

    (e) $\{f \in C^2[0,1] : \ f(1) = 1\}$

    (f) $\{f \in C^2[0,1] : \ f(1) = 0\}$

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Solution.

(a) [4 points] This set *is not* a real vector space.
The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set, yet $2\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not, since $2 \neq 2^3 = 8$.

(b) [4 points] This set *is* a real vector space.
The set is a subset of $\mathbb{R}^3$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a member of this set. Now, suppose $\mathbf{x}$ and $\mathbf{y}$ are members of this set. Then $3x_1 + 2x_2 + x_3 = 0$ and $3y_1 + 2y_2 + y_3 = 0$. Adding these two equations gives

$$3(x_1 + y_1) + 2(x_2 + y_2) + (x_3 + y_3) = 0,$$

and hence $\mathbf{x} + \mathbf{y}$ is also in the set. Multiplying $3x_1 + 2x_2 + x_3 = 0$ by an arbitrary constant $\alpha \in \mathbb{R}$ gives

$$3(\alpha x_1) + 2(\alpha x_2) + \alpha x_3 = 0,$$

and hence $\alpha\mathbf{x}$ is also in the set. Consequently, the set is a subspace of $\mathbb{R}^3$ and is hence a real vector space.

(c) [4 points] This set *is not* a real vector space.
Let $f(x) = 1$ for $x \in [0,1]$. Then $f$ is in the set, but a scalar multiple, $-1 \cdot f(x) = -1$ for $x \in [0,1]$, takes negative values and thus violates the requirement for membership in the set.

(d) [4 points] This set *is not* a real vector space.
Let $f(x) = 1$ for $x \in [0,1]$. Then $f$ is in the set, but a scalar multiple, $2 \cdot f(x) = 2$ for $x \in [0,1]$, takes values greater than one and thus violates the requirement for membership in the set.

(e) [4 points] This set *is not* a real vector space.
The function $z$ defined by $z(x) = 0$ for $x \in [0,1]$ is not in the set since $z(1) = 0$ and thus violates the requirement for membership in the set.

(f) [5 points] This set *is* a real vector space.
The set is a subset of $C^2[0,1]$ and the function $z$ defined by $z(x) = 0$ for $x \in [0,1]$ is in the set. If $f$ and $g$ are in the set, then $f(1) = g(1) = 0$, so

$$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and hence $f+g$ is in the set. Also, if $f$ is in the set and $\alpha \in \mathbb{R}$, then

$$(\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0$$

and hence $\alpha f$ is in the set. Consequently, the set is a subspace of $C^2[0,1]$ and is hence a real vector space.

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