Alternate types of Fourier Series: Derivation and examples

Recall: the Fourier Sine Series was derived from

1) Looking for eigenvalues and eigenvectors of the operator
$$L = -K\frac{\partial^2}{\partial x^2}. \quad \text{ie solving } L u = \lambda u$$

2) Applying the boundary conditions $u(0) = u(\ell) = 0$ to the general solution $C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$ of the eigenvalue problem $Lu = \lambda u$ to get that
$u(x) = \sin(\frac{n\pi}{\ell}x)$ are the eigenvectors of L corresponding to eigenvalue $\frac{n^2\pi^2}{\ell^2}$

3) Then the functions $\tilde{u}_n(x)$ defined by $\tilde{u}_n(x) = \sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x)$ are orthonormal eigenvectors of the operator L.

4) Then we considered the subvector spaces V^K of $C_D^2[0,1]$ Defined by $V^K = \text{span}\{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_K\}$

5) Given any function $g(x)$ in $C[0,1]$ we found the best approximation to $g(x)$ in V^K was given by:
$$m_K(x) = \sum_{i=1}^{K} (g, \tilde{u}_i)\, \tilde{u}_i(x)$$
$$= \sum_{i=1}^{K} \left(\frac{2}{\ell}\int_0^\ell g(x)\sin(\frac{n\pi}{\ell}x)\,dx\right) \sin(\frac{n\pi}{\ell}x)$$

6) Then we commented that a result from mathematics tells us that the higher the value of K we select the closer $m_K(x)$ is to $g(x)$ in the sense that the error $\|g(x) - m_K(x)\|_{L^2}$ measured by the L^2 inner product norm goes to zero.

7) Since the approximation continues to get better then the function $m_\infty(x) = \lim_{K\to\infty} m_K(x)$ is going to give us the best possible result. This function is
$$m_\infty(x) = \sum_{i=1}^{\infty} (g, \tilde{u}_i)\, \tilde{u}_i(x)$$
$$= \sum_{i=1}^{\infty} \left(\frac{2}{\ell}\int_0^\ell g(x)\sin(\frac{n\pi}{\ell}x)\,dx\right) \sin(\frac{n\pi}{\ell}x)$$
and is called the

Fourier Sine Series of $g(x)$.

Now we investigate the effects of changing the boundary conditions.

Consider the Boundary value problem:

$$\begin{cases} -k\dfrac{\partial^2 u}{\partial x^2} = f(x) , & 0<x<\ell \\ u(0)=0 \\ \dfrac{\partial u}{\partial x}(\ell)=0 \end{cases}$$

Our differential operator is $L = -k\frac{\partial^2}{\partial x^2}$ but now we are considering L on the space

$$C^2_M[0,\ell] = \left\{ f \mid f(0)=0,\ \tfrac{\partial f}{\partial x}(\ell)=0 \right\}$$

Recall that the <u>domain</u> of L plays a big role in whether or not L is <u>symmetric</u> and whether or not its eigenvalues are positive.

Lets check symmetry:

Let $f, g \in C^D_m[0,\ell]$. We need to verify that $(Lf,g) = (f,Lg)$ we have:

$$(Lf,g) = \int_0^\ell \left(-k\frac{\partial^2}{\partial x^2} f\right) g = \int_0^\ell k\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + -k\frac{\partial f}{\partial x} g\Big|_0^\ell$$

$$= \int_0^\ell k\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + -k\cancel{\frac{\partial f}{\partial x}(\ell)}^{0} g(\ell) + k\frac{\partial f}{\partial x}(0)\cancel{g(0)}^{0}$$

$$= \int_0^\ell f\left(-k\frac{\partial^2}{\partial x^2} g\right) + k f\frac{\partial g}{\partial x}\Big|_0^\ell$$

$$= (f, Lg) + k f(\ell)\cancel{\frac{\partial g}{\partial x}(\ell)}^{0} - k\cancel{f(0)}^{0}\frac{\partial g}{\partial x}(0)$$

$$= (f, Lg)$$

So L is still symmetric on $C^D_m[0,\ell]$ but the <u>way</u> that the boundary term became zero was different for $C^2_m[0,\ell]$ than it was for $C^2_D[0,\ell]$. Since L is symmetric we know that eigenvectors for distinct eigenvalues are orthogonal and all eigenvalues are <u>real</u>.

The fact that eigenvalues are still <u>positive</u> follows from the fact that if $\tilde{u}$ is a unit eigenvector of L with eigenvalue λ then:

$$\lambda = \lambda\cdot 1 = \lambda(\tilde{u},\tilde{u}) = (\lambda\tilde{u},\tilde{u}) = (L\tilde{u},\tilde{u})$$

$$= \int_0^\ell \left(-k\frac{\partial^2}{\partial x^2}\tilde{u}\right)\tilde{u} = k\int_0^\ell \frac{\partial}{\partial x}\tilde{u}\,\frac{\partial}{\partial x}\tilde{u} + \cancel{k\frac{\partial}{\partial x}u\, u\Big|_0^\ell}^{0} = k\int\left(\frac{\partial}{\partial x}u\right)^2 > 0$$

So each eigenvalue λ satisfies $\lambda = \theta^2 > 0$ for some θ. Finding the eigenvalues for L on $C_m^2[0,l]$ then amounts to solving:

$$\frac{\partial^2}{\partial x^2} u + \theta^2 u = 0$$
$$u(0) = 0$$
$$\frac{\partial u}{\partial x}(l) = 0$$

We mentioned that the general solution to "$\frac{\partial^2}{\partial x^2} u + \theta^2 u = 0$" is given by: $u(x) = C_1 \cos(\theta x) + C_2 \sin(\theta x)$

The boundary conditions give: $u(0) = C_1 = 0$
and $\frac{\partial}{\partial x} u(l) = 0 \Rightarrow C_2 \theta \cos(\theta l) = 0$

which means that $\theta l = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}$ for $n = 1, 2, \ldots$
so that:

$$\theta = \frac{(2n-1)\pi}{2l}, \quad n = 1, 2, \ldots$$

Therefore the eigenvalues are $\lambda_n = \theta_n^2 = \dfrac{(2n-1)^2 \pi^2}{4l^2}$ $n = 1, 2, \ldots$

and the corresponding eigenvectors are $\psi_n = \sin\left(\dfrac{(2n-1)\pi}{2l} x\right)$

a computation shows that $(\psi_n, \psi_n) = l/2$ so that $\tilde{\psi}_n = \sqrt{\frac{2}{l}}\, \psi_n(x)$ defines an orthonormal set of eigenvectors $\{\tilde{\psi}_n\}_{n=1}^{\infty}$ to L defined on $C_m^2[0,l]$.

Now if we let $f(x)$ be any continuous function then the best approximation to $f(x)$ from the vector space $V^k = \operatorname{Span}\{\tilde{\psi}_1, \ldots, \tilde{\psi}_k\}$ is:

$$m_k(x) = \sum_{n=1}^{k} b_n \sqrt{\frac{2}{l}} \sin\left(\frac{(2n-1)\pi}{2l} x\right)$$

$$\text{where } b_n = (f, \tilde{\psi}_n) = \sqrt{\frac{2}{l}} \int_0^l f(x) \sin\left(\frac{(2n-1)\pi}{2l} x\right) dx$$

so that

$$m_k(x) = \sum_{n=1}^{k} \frac{2}{l} \left(\int_0^l f(x) \sin\left(\frac{(2n-1)\pi}{2l} x\right) dx \right) \sin\left(\frac{(2n-1)\pi}{2l} x\right)$$

Like we saw in the previous case where L was considered as defined on $C_0^2[0,l]$ as $k \mapsto \infty$ $m_k(x)$ is a better approximation for $f(x)$. Hence

$$m_\infty(x) = \lim_{k\to\infty} m_k(x) = \sum_{n=1}^{\infty} \frac{2}{\ell}\left(\int_0^\ell f(x)\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)dx\right)\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

is the "best" approximation for $f(x)$ in $C_m^0[0,\ell]$ in terms of the eigenvectors of L. $m_\infty(x)$ is called the

Fourier quarter-wave sine series of $f(x)$.

If time permits:

* Example 5.7 pg 146
* Excercise 3 pg 147