CAAM 336 FALL 2010 MIDTERM

1a) Best approximation: $\dfrac{(v,\phi_1)}{(\phi_1,\phi_1)}\phi_1$

1b)

$$\psi_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \psi_2 = \phi_2 - \frac{(\phi_2,\psi_1)}{(\psi_1,\psi_1)}\psi_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$\psi_3 = \phi_3 - \frac{(\phi_3,\psi_1)}{(\psi_1,\psi_1)}\psi_1 - \frac{(\phi_3,\psi_2)}{(\psi_2,\psi_2)}\psi_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{2}{2}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

1c) Best approximation $= \dfrac{(v,\psi_1)}{(\psi_1,\psi_1)}\psi_1 + \dfrac{(v,\psi_2)}{(\psi_2,\psi_2)}\psi_2 + \dfrac{(v,\psi_3)}{(\psi_3,\psi_3)}$

$$= \frac{0}{1}\psi_1 + \frac{1}{2}\psi_2 + \frac{0}{1}\psi_3 = \begin{bmatrix}0\\1/2\\1/2\\0\end{bmatrix}$$

1d) $\psi_1(x) = 1, \quad \psi_2(x) = \phi_2(x) - \dfrac{(\phi_2,\psi_1)}{(\psi_1,\psi_1)}\psi_1(x) = x - \dfrac{\int_{-1}^1 1\cdot x\,dx}{\int_{-1}^1 1\cdot 1\,dx}1 = x - 0\cdot 1 = x$

$$\psi_3(x) = \phi_3(x) - \frac{(\phi_3,\psi_1)}{(\psi_1,\psi_1)}\psi_1(x) - \frac{(\phi_3,\psi_2)}{(\psi_2,\psi_2)}\psi_2(x)$$

$$= x^2 - \frac{\int_{-1}^1 x^2\,dx}{\int_{-1}^1 1\cdot 1\,dx}1 - \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 x\cdot x\,dx}x = x^2 - \frac{1}{3} - 0$$

$$= x^2 - \frac{1}{3}$$

1e) Best approximation

$$= \frac{(v,\psi_1)}{(\psi_1,\psi_1)}\psi_1 + \frac{(v,\psi_2)}{(\psi_2,\psi_2)}\psi_2 + \frac{(v,\psi_3)}{(\psi_3,\psi_3)}\psi_3$$

$$= 0\cdot 1 + \frac{\int_{-1}^1 x^4\,dx}{\int_{-1}^1 x^2\,dx}x + \underbrace{\frac{\int_{-1}^1 \left(x^5 - \frac{x^3}{3}\right)dx}{\int_{-1}^1 \left(x^2-\frac{1}{3}\right)^2 dx}}_{0} = \frac{3}{5}x$$

2a) L is <u>not</u> symmetric.

$$(Lu,v) = \int_0^1 (u''+cu')v\,dx = \int_0^1 u''v + cu'v\,dx$$
$$= -\int_0^1 u'v'dx + \int_0^1 cu'v\,dx$$

$$(u,Lv) = \int_0^1 u(v''+cv')\,dx = \int_0^1 uv'' + cuv'\,dx$$
$$= -\int_0^1 u'v'dx + \int_0^1 c\,uv'\,dx$$

So $(Lu,v) \neq (u,Lv)$ if $\int_0^1 u'v\,dx \neq \int_0^1 uv'\,dx$.

Show on example <u>with</u> <u>$u,v \in C_0^2[0,1]$</u>: $u(x) = x(1-x)$, $v(x) = x^2(1-x)$

$$\int_0^1 u'v\,dx = -\frac{1}{60}, \quad \int_0^1 uv'\,dx = \frac{1}{60}$$

2b) $L\psi_n = \psi_n'' + c\psi_n'$

$$\psi_n' = e^{-cx/2}\left(-\frac{c}{2}\sin(n\pi x) + (n\pi)\cos(n\pi x)\right)$$

$$\psi_n'' = -\frac{c}{2}e^{-cx/2}\left(-\frac{c}{2}\sin(n\pi x) + (n\pi)\cos(n\pi x)\right)$$
$$+ (n\pi)e^{-cx/2}\left(-\frac{c}{2}\cos(n\pi x) - (n\pi)\sin(n\pi x)\right)$$

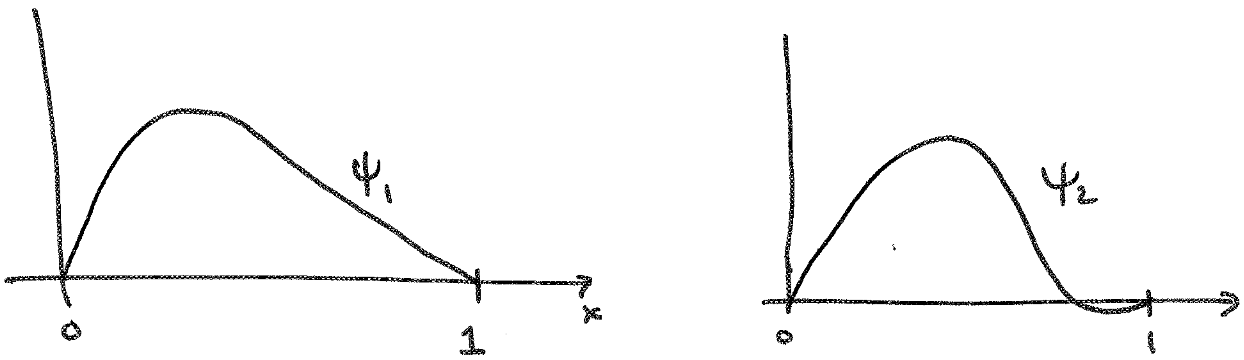
$$L\psi_n = \psi_n'' + c\psi_n' = e^{-cx/2}\Big[\left(-\frac{c^2}{4} - n^2\pi^2\right)\sin(n\pi x)$$
$$+ \left(-\frac{c}{2}(n\pi) - \frac{c}{2}(n\pi) + cn\pi\right)\cos(n\pi x)\Big]$$

$$= \underbrace{\left(-\frac{c^2}{4} - n^2\pi^2\right)}_{\lambda_n}\underbrace{e^{-cx/2}\sin(n\pi x)}_{\psi_n.}$$

Note: $\psi_n(0) = \psi_n(1) = 0$ since $\sin(n\pi x)\Big|_{x=0,1} = 0$.

∎

2c)



2d) Recall that

$$\cos\measuredangle(\psi_1, \psi_2) = \frac{(\psi_1, \psi_2)}{\|\psi_1\| \, \|\psi_2\|}.$$

As $c \to 0$, L approaches a symmetric operator,

so $\measuredangle(\psi_1, \psi_2) \to \pi/2$: eigenvectors become orthogonal in the limit.

As $c \to \infty$, the angle between ψ_1 and ψ_2 shrinks:

so $\measuredangle(\psi_1, \psi_2) \to 0$.

2e) $L\psi_n = \lambda_n \psi_n$ suggests that

$$L\left(\frac{\psi_n}{\lambda_n}\right) = \psi_n$$

Hence, the solution to $Lu = \psi_n$ is

$$u = \frac{\psi_n}{\lambda_n}.$$

3a) $(Lu, w) = \int_0^1 -u''(x)\,w(x)\,dx = -\left[u'(x)\,w(x)\right]_0^1 + \int_0^1 u'(x)\,w'(x)\,dx$

$= -\left[u'(x)\,w(x)\right]_0^1 + \left[u(x)\,w'(x)\right]_0^1 - \int_0^1 u(x)\,w''(x)\,dx$

$= -u'(1)\,w(1) + \underbrace{u'(0)\,w(0)}_{=0 \text{ since } w(0)=0} + u(1)\,w'(1) \underbrace{-u(0)\,w'(0)}_{=0 \text{ since } u(0)=0} - \int_0^1 u(x)\,w''(x)\,dx$

$= \underbrace{-u'(1)\,w(1) + u(1)\,w'(1)}_{} + (u, Lw) = (u, Lw)$

$=0$ since $u(1) - u'(1) = 0 \Rightarrow u(1) = u'(1)$
$w(1) - w'(1) = 0$
$-u'(1)\,w(1) + u(1)\,w'(1) = u(1)\left(w'(1) - w(1)\right) = u(1)\cdot 0 = 0.$

Hence L is symmetric.

3b) $\psi(x) = A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x) \Rightarrow L\psi = \lambda\psi.$

$0 = \psi(0) = A\sin(0) + B\cos(0) = B \Rightarrow \boxed{B=0}$

$\Rightarrow \psi(x) = A\sin(\sqrt{\lambda}\,x)$

$0 = \psi(1) - \psi'(1) = A\left(\sin(\sqrt{\lambda}\cdot 1) - \sqrt{\lambda}\cos(\sqrt{\lambda}\cdot 1)\right)$

$\Rightarrow \boxed{\sqrt{\lambda} = \tan(\sqrt{\lambda}).}$ λ must satisfy this equation to be an eigenvalue.

3c) Is $\lambda = 0$ an eigenvalue?

Note that $\psi(x) = A\sin(0\cdot x) = 0$; $\psi(x) \equiv 0$ is not an eigenfunction, as eigenfunctions cannot be identically zero.

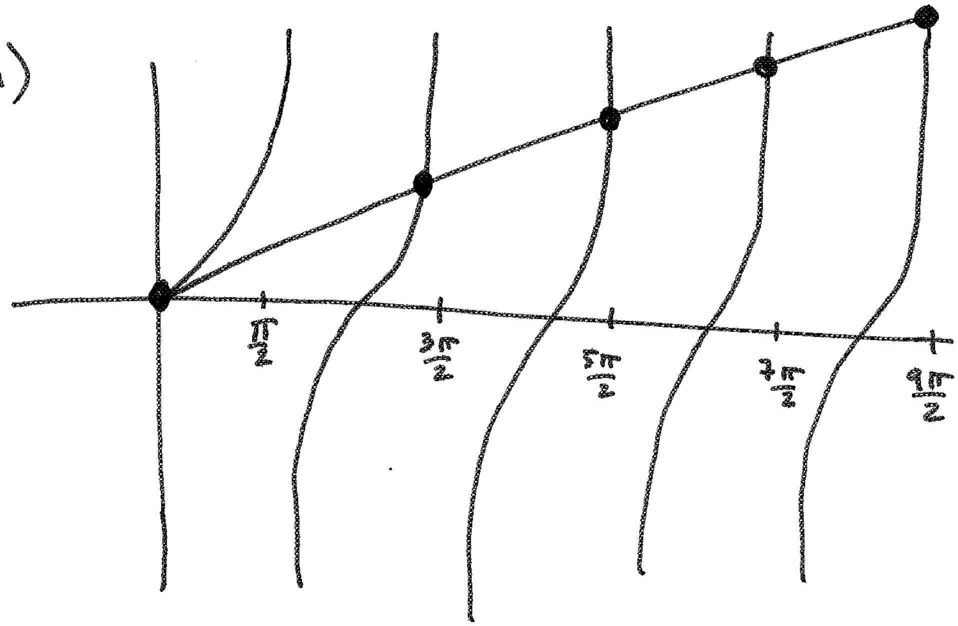
<u>However</u> when $\lambda = 0$, the equation $L\psi = \lambda\psi = 0$ admits solutions of the form $\psi(x) = A + Bx.$

$0 = \psi(0) = A + B\cdot 0 = A \Rightarrow A = 0.$

$0 = \psi(1) - \psi'(1) = B - B = 0 \Rightarrow \psi(x) = x$ <u>is</u> a nonzero eigenfunction associated with $\lambda = 0$.

Hence $\lambda = 0$ <u>is</u> an eigenvalue.

3d)



3e) Since $\tan(x) \to \infty$ as $x \to (n-\frac{1}{2})\pi$, for integers n there will be an intersection of $g(x) = \tan(x)$ with $h(x) = x$ in every interval $[(n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi]$, giving eigenvalues $\lambda_n \in [(n-\frac{1}{2})^2\pi^2, (n+\frac{1}{2})^2\pi^2]$ tending toward $(n+\frac{1}{2})^2\pi^2$ as $n \to \infty$.

3f) If $(f, \psi_0) = (f, x) = 0$, then

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{(f, \psi_n)}{(\psi_n, \psi_n)} \psi_n(x).$$

is one solution, as is anything of the form

$$u(x) = cx + \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{(f, \psi_n)}{(\psi_n, \psi_n)} \psi_n(x)$$

If $(f, \psi_0) = (f, x) \neq 0$, no solution exists.

(Credit was given if the zero eigenvalue was neglected.)

4a) $-u''(x) + 6u(x) = f(x), \quad u(0) = u(1) = 0.$

Multiply by a test function $w \in V$ and integrate:

$$\int_0^1 (-u''(x) + 6u(x))\, w(x)\, dx = \int_0^1 f(x)\, w(x)\, dx = (f, w)$$

$$= -\int_0^1 u''(x) w(x)\, dx + \int_0^1 6u(x) w(x)\, dx$$

$$= -\cancel{\left[u'(x) w(x)\right]_0^1} + \underbrace{\int_0^1 u'(x) w'(x)\, dx + 6\int_0^1 u(x) w(x)\, dx}_{= a(u,w)}$$

since $w(0) = w(1) = 0$

so $a(u,w) = (f, w)$ for all $w \in V$.

4b) Yes: $a(\cdot,\cdot)$ satisfies all properties of an inner product.

It is symmetric $(a(u,w) = a(w,u))$

linear $(a(\alpha u + v, w) = \alpha\, a(u,w) + a(v,w))$

positive $\left(a(u,u) = \int_0^1 u'(x)^2 dx + 6\int_0^1 u(x)^2 dx\right)$

with $a(u,u) = 0 \Leftrightarrow u = 0$.

4c) $a(\phi_j, \phi_k) = \int_0^1 \phi_j'(x)\, \phi_k'(x)\, dx + 6\int_0^1 \phi_j(x)\, \phi_k(x)\, dx$

$h = \frac{1}{N+1} = \frac{1}{3} \Rightarrow$

$$K = \begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & a(\phi_1,\phi_3) \\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & a(\phi_2,\phi_3) \\ a(\phi_3,\phi_1) & a(\phi_3,\phi_2) & a(\phi_3,\phi_3) \end{bmatrix}$$

$$a(\phi_j, \phi_j) = \int_{x_{j-1}}^{x_j} \left(-\frac{1}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{1}{h}\right)^2 dx + 6(\phi_j, \phi_j)$$

$$= \frac{1}{h} + \frac{1}{h} + 6\left(\frac{2h}{3}\right) = \frac{2}{h} + 4h = 8 + \frac{4}{4} = 9$$

4c) continued

$$a(\phi_j, \phi_{j+1}) = \int_{x_j}^{x_{j+1}} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx + 6(\phi_j, \phi_{j+1})$$

$$= -\frac{1}{h} + 6\,\frac{h}{6} = h - \frac{1}{h} = \frac{1}{4} - 4 = -\frac{15}{4}$$

$a(\phi_i, \phi_k) = 0$ otherwise.

$$K = \begin{bmatrix} 9 & -\frac{15}{4} & 0 \\ -\frac{15}{4} & 9 & -\frac{15}{4} \\ 0 & -\frac{15}{4} & 9 \end{bmatrix}$$

4d) $f = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ (f, \phi_3) \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$ for $f(x) = 1$

Since $(f, \phi_j) = \int_0^1 1\, \phi_j(x)\, dx = 1 \cdot \underbrace{h}$

$h =$ area of a hat function.

Bonus For $N=5$, compute the stiffness matrix K.

This requires $a(\phi_j,\phi_k)$ for $j,k=1,..,5$.

$$a(\phi_j,\phi_k) = \int_0^1 (\sqrt{2}\cos(j\pi x))(\sqrt{2}\cos(k\pi x))(j\pi)(k\pi)\,dx \quad (1)$$

$$+ \int_0^1 \delta_{1/2}(x)\sin(j\pi x)\sin(k\pi x)\,dx \quad (2)$$

$$(1) = \begin{cases} j^2\pi^2, & j=k \\ 0, & j\neq k. \end{cases}$$

$$(2) = 2\sin\left(\frac{j\pi}{2}\right)\sin\left(\frac{k\pi}{2}\right)$$

Use
$$\sin\left(\frac{\ell\pi}{2}\right) = \begin{cases} 1, & \ell=1 \\ 0, & \ell=2 \\ -1, & \ell=3 \\ 0, & \ell=4 \\ 1, & \ell=5 \end{cases}$$

To compute

$$K = \begin{bmatrix} \pi^2+2 & 0 & -2 & 0 & 2 \\ 0 & 4\pi^2 & 0 & 0 & 0 \\ -2 & 0 & 9\pi^2+2 & 0 & -2 \\ 0 & 0 & 0 & 16\pi^2 & 0 \\ 2 & 0 & -2 & 0 & 25\pi^2+2 \end{bmatrix}$$