CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 3

Posted Friday 5 September 2008. Due Friday 12 September 2008, 5pm.

1. [18 points]
   Recall that a function $f : \mathcal{V} \to \mathcal{W}$ that maps a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a *linear operator* provided (1) $f(u + v) = f(u) + f(v)$ for all u, v in $\mathcal{V}$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbf{R}$ and $v \in \mathcal{V}$.

   Demonstrate whether each of the following functions is a linear operator.
   (Show that both properties hold, or give an example showing that one of the properties must fail.)

   (a) $f : \mathbf{R}^n \to \mathbf{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for a fixed matrix $\mathbf{A} \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$.
   (b) $f : \mathbf{R}^n \to \mathbf{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$ and fixed vector $\mathbf{b} \in \mathbf{R}^{\mathbf{m}}$.
   (c) $f : \mathbf{R}^2 \to \mathbf{R}$, $f(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{x}$.
   (d) $f : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$, $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$.
   (e) $L : C^1[0, 1] \to C[0, 1]$, $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.
   (f) $L : C^2[0, 1] \to C[0, 1]$, $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

2. [20 points]
   (a) Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbf{R}^{\mathbf{2}\times\mathbf{2}}$ such that f is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbf{R}^{\mathbf{2}}$ can be written as $\mathbf{u} = \mathbf{u_1}\mathbf{e_1} + \mathbf{u_2}\mathbf{e_2}$, where

   $$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Since f is linear, we have $f(\mathbf{u}) = \mathbf{u_1}\mathbf{f}(\mathbf{e_1}) + \mathbf{u_2}\mathbf{f}(\mathbf{e_2})$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e_1})$ and $f(\mathbf{e_2})$.
   (b) Now we want to generalize the result in part (a): Show that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^{\mathbf{n}}$.

   (Thus any linear function that maps $\mathbf{R}^n$ to $\mathbf{R}^m$ can be written as a matrix-vector product.)

3. [20 points]
   Determine whether each of the following functions $(\cdot, \cdot)$ determines an inner product on the vector space $\mathcal{V}$. If not, demonstrate which property (or properties) of the inner product are violated.

   (a) $\mathcal{V} = C^1[0, 1]$, $(u, v) = \displaystyle\int_0^1 u'(x)v'(x)\,dx$ (b) $\mathcal{V} = C[0, 1]$: $(u, v) = \displaystyle\int_0^1 |u(x)||v(x)|\,dx$
   (c) $\mathcal{V} = C[0, 1]$: $(u, v) = \displaystyle\int_0^1 u(x)v(x)e^{-x}\,dx$ (d) $\mathcal{V} = C^1[0, 1]$: $(u, v) = \displaystyle\int_0^1 u(x)v'(x)\,dx$

4. [10 points]
   Suppose $\mathcal{V}$ is a vector space with an associated inner product. The angle θ between u and $v \in \mathcal{V}$ is defined via the equation

   $$(u, v) = \|u\|\|v\|\cos\theta.$$

   Let $\mathcal{V} = C[0, 1]$ and $(u, v) = \int_0^1 u(x)v(x)\,dx$. Compute the angle θ between $u(x) = x^n$ and $v(x) = x^m$ for nonnegative integers m and n.

please turn over

5. [32 points + 8 bonus]

Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

We can approximate the differential equation

$$\frac{d^2}{dx^2}u = f(x), \quad x \in (0,1),$$
$$u(0) = u(1) = 0$$

by the matrix equation

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix},$$

where $u_j \approx u(x_j)$. (Entries of the matrix that are not specified are zero.)

(a) Suppose that $f(x) = 25\pi^2 \cos(5\pi x)$.
Compute and plot the approximate solutions obtained when $N = 8, 16, 32, 64, 128$.
You may superimpose these on one plot. To solve the linear systems, you may use MATLAB's 'backslash' command: `u = A\f`.
For each value of N compute the maximum error $|u_j - u(x_j)|$, given that the true solution is

$$u(x) = 1 - 2x - \cos(5\pi x).$$

Plot this error using a `loglog` plot with error on the vertical axis and N on the horizontal axis.

(b) Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

$$u(0) = 1, \quad u(1) = 2.$$

Compute and plot solutions for $N = 8, 32, 128$.

(c) This part of the problem is an optional bonus worth 8 extra points:
Now suppose that we have mixed boundary conditions

$$u(0) = 1, \quad \frac{du}{dx}(1) = -5.$$

The matrix equation will now have the form

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \ddots & & \\ & & \ddots & \ddots & 1 & 0 \\ & & & 1 & -2 & * \\ & & & * & * & * \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} f(x_1) - * \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \\ * \end{bmatrix}.$$

Specify what the entries marked by $*$ should be, keeping in mind the approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N+1} - u_N}{h}.$$

Compute and plot solutions for $N = 8, 32, 128$.