CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 31

Posted Friday 21 March 2014. Due 1pm Friday 28 March 2014.

31. [25 points]

Let

$$H^1_D(0,1) = \{w \in H^1(0,1) : w(0) = w(1) = 0\}$$

and let the inner product $a(\cdot,\cdot) : H^1_D(0,1) \times H^1_D(0,1) \to \mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx.$$

Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\widehat{\phi}_j \in H^1_D(0,1)$ be such that

$$\widehat{\phi}_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N$. Also, let the continuous piecewise quadratic functions $\phi_j \in H^1_D(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in H^1_D(0,1)$ be such that

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N+1$.

(a) By hand, obtain formulas for

i. $a(\phi_j, \phi_k)$ for $j, k = 1, \ldots, N$;

ii. $a(\psi_j, \psi_k)$ for $j, k = 1, \ldots, N+1$;

iii. $a(\phi_j, \psi_k)$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$.

Your final answers should be in terms of h and you must clearly state which values of j and k each formula you obtain is valid for. For example, if you were asked to obtain formulas for $a\left(\widehat{\phi}_j, \widehat{\phi}_k\right)$

for $j, k = 1, \ldots, N$ then an acceptable way to present the answer would be:
For $j, k = 1, \ldots, N$,

$$a\left(\widehat{\phi}_j, \widehat{\phi}_k\right) = \begin{cases} \dfrac{2}{h} & \text{if } k = j, \\ -\dfrac{1}{h} & \text{if } |j - k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$