CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6

Posted Wednesday 8 October, 2014. Due 5pm Wednesday 15 October, 2014.

Please write your name and ***residential college*** *on your homework.*

1. [28 points: 14 points each]

   All parts of this question should be done by hand.

   (a) Let
   $$\mathbf{D} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$
   Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to
   $$\mathbf{Dc} = \mathbf{g}.$$

   (b) Let
   $$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$
   Use the spectral method to obtain the solution $\mathbf{x} \in \mathbb{R}^3$ to
   $$\mathbf{Ax} = \mathbf{b}.$$

2. [24 points: 6 points each]

   Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by
   $$(v,w) = \int_0^1 v(x)w(x)\,dx.$$
   Consider the linear operator $L : C_m^2[0,1] \to C[0,1]$ defined by
   $$Lu = -u''$$
   where
   $$C_m^2[0,1] = \left\{ u \in C^2[0,1] : u'(0) = u(1) = 0 \right\}.$$

   (a) Is L symmetric?

   (b) What is the null space of L?

   (c) Show that $(Lu,u) \geq 0$ for all $u \in C_m^2[0,1]$ and explain why this and the answer to part (b) mean that $\lambda > 0$ for all eigenvalues λ of L.

   (d) Find the eigenvalues and eigenfunctions of L.

3. [32 points: 10 points for (a) and (b), 12 points for (c)]
Define the inner product (u,v) to be

$$(u,v) = \int_0^1 u(x)v(x)\,dx$$

and let the norm $\|v(x)\|$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let N be a positive integer and let $\phi_1,\ldots,\phi_N \in C[0,1]$ be such that $\{\phi_1,\ldots,\phi_N\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$. We wish to approximate a continuous function $f(x)$ with $f_N(x)$

$$f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

where

$$\phi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1,2,\ldots.$$

and where $\alpha_n = (f,\phi_n)$. (Note that f_N is the best approximation to g from span$\{\phi_1,\ldots,\phi_N\}$ with respect to the norm $\|\cdot\|$.)

(a) Assume that $f_N \to f$ as $N \to \infty$. Show that, since $\phi_1,\ldots,\phi_N$ are orthonormal,

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2.$$

(b) The best approximation to $f(x) = x(1-x)$ has coefficients α_n which satisfy

$$\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3}(1-(-1)^n).$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 5$. On a separate figure, plot the error using the above formula for $N = 1,2,\ldots,100$ on a log-log scale by using `loglog` in MATLAB.

(c) Verify that the best approximation to the function $f(x) = 1-x$ (which does not satisfy the same boundary conditions as $\phi_n(x)$!) has coefficients

$$\alpha_n = \frac{\sqrt{2}}{\pi n}.$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 100$. On a separate figure, plot the error using the above formula for $N = 1,2,\ldots,100$ on a log-log scale by using `loglog` in MATLAB.

You may have noticed that the rate at which the coefficients $\alpha_n \to 0$ determines how fast the error decreases — this is not coincidental!

4. [40 points: 8 points each]
This problem concerns the same operator from class and previous problems, $L : C^2_D[0,1] \to C[0,1]$ defined by

$$L_D u = -\frac{d^2u}{dx^2},$$

with homogeneous Dirichlet boundary conditions imposed via

$$C^2_D[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}.$$

The eigenvalues and (normalized) eigenfunctions remain as they were before: $\lambda_n = n^2\pi^2$ and $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for $n = 1, 2, \ldots$. Now let $f(x) = x^2(1-x)$.

(a) For this f, compute the coefficients

$$c_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

in the expansion

$$f = \sum_{n=1}^{\infty} c_n \psi_n.$$

You may determine these by hand, by consulting a table of integrals, or by using a symbolic mathematics package like Mathematica or the Symbolic Toolbox in MATLAB.

(b) Produce a plot (or series of plots) comparing $f(x)$ to the partial sums

$$f_N(x) = \sum_{k=1}^{N} c_k \psi_k(x)$$

for $N = 1, \ldots, 10$.

(c) Plot the approximations u_N to the true solution u that you obtain using the spectral method:

$$u_N(x) = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \psi_k(x)$$

for $N = 1, \ldots, 10$.

(d) Now replace the homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ above with the inhomogeneous Dirichlet conditions $u(0) = -1/100$ and $u(1) = 1/100$. Describe how to adjust your solution from part (c) to account for these boundary conditions, and produce a plot of the solution with these inhomogeneous boundary conditions, based on u_{10} from part (c).