CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6 · Solutions

Posted Wednesday 18 March, 2015. Due 5pm Wednesday 25 March, 2015.

Please write your name and instructor on your homework.

1. [10 points: 5 points each]

All parts of this question should be done by hand.

(a) Let

$$\mathbf{D} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{c} \in \mathbb{R}^2$ to

$$\mathbf{Dc} = \mathbf{g}.$$

(b) Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Use the spectral method to obtain the solution $\mathbf{x} \in \mathbb{R}^3$ to

$$\mathbf{Ax} = \mathbf{b}.$$

Solution.

(a) [14 points] Since,

$$\lambda\mathbf{I} - \mathbf{D} = \begin{bmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{bmatrix}$$

we have that

$$\det(\lambda\mathbf{I} - \mathbf{D}) = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$

and so

$$\det(\lambda\mathbf{I} - \mathbf{D}) = 0$$

when $\lambda = 3$ or $\lambda = 5$. Hence, the eigenvalues of $\mathbf{D}$ are

$$\lambda_1 = 3$$

and

$$\lambda_2 = 5.$$

Moreover,

$$(\lambda_1\mathbf{I} - \mathbf{D}) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -f_1 - f_2 \\ -f_1 - f_2 \end{bmatrix}$$

and so to make this vector zero we need to set $f_2 = -f_1$. Hence, any vector of the form

$$\begin{bmatrix} f_1 \\ -f_1 \end{bmatrix}$$

where f_1 is a nonzero constant is an eigenvector of $\mathbf{D}$ corresponding to the eigenvalue λ_1. Let us choose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Furthermore,

$$(\lambda_2 \mathbf{I} - \mathbf{D}) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{bmatrix}$$

and so to make this vector zero we need to set $d_2 = d_1$. Hence, any vector of the form

$$\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}$$

where d_1 is a nonzero constant is an eigenvector of $\mathbf{D}$ corresponding to the eigenvalue λ_2. Let us choose

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\mathbf{D} = \mathbf{D}^T$, $\mathbf{D}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\mathbf{D}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$ and $\lambda_1 \neq \lambda_2$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Now,

$$\mathbf{g} \cdot \mathbf{v}_1 = 2 - 3 = -1,$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

$$\mathbf{g} \cdot \mathbf{v}_2 = 2 + 3 = 5,$$

and

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 1^2 + 1^2 = 1 + 1 = 2.$$

The spectral method then yields that

$$\begin{aligned}
\mathbf{c} &= \frac{1}{\lambda_1} \frac{\mathbf{g} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{1}{\lambda_2} \frac{\mathbf{g} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\
&= \frac{1}{3} \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5} \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{6} + \frac{3}{6} \\ \frac{1}{6} + \frac{3}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{6} \\ \frac{4}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.
\end{aligned}$$

(b) [14 points] For this matrix $\mathbf{A}$ we have

$$\lambda \mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix},$$

and hence the characteristic polynomial is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - 3)(\lambda^2 - 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial, which we label

$$\lambda_1 = -1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3.$$

To compute the eigenvectors associated with the eigenvalue $\lambda_1 = -1$, we seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -u_2 + u_3 \\ u_2 - u_3 \end{bmatrix}.$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = u_2$. Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_1 = -1$.
To compute the eigenvectors associated with the eigenvalue $\lambda_2 = 1$ we now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_2 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ u_2 + u_3 \\ u_3 + u_2 \end{bmatrix}.$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = -u_2$. Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ -u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_2 = 1$.
To compute the eigenvectors associated with the eigenvalue $\lambda_3 = 3$ we now seek $\mathbf{u} = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$(\lambda_3 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2 + u_3 \\ u_2 + 3u_3 \end{bmatrix}.$$

To make the second component zero we need $u_2 = -u_3/3$, while to make the third component zero we need $u_3 = -u_2/3$. The only way to accomplish both is to set $u_2 = u_3 = 0$. Thus any vector of the form

$$\begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, \quad u_1 \neq 0$$

is an eigenvector associated with the eigenvalue $\lambda_3 = 3$.
We choose the eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We can compute that

$$\mathbf{u}_1^T \mathbf{u}_2 = \mathbf{u}_2^T \mathbf{u}_1 = 0 \cdot 0 + (1/\sqrt{2}) \cdot (1/\sqrt{2}) + (1/\sqrt{2}) \cdot (-1/\sqrt{2}) = 0,$$

$$\mathbf{u}_1^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_1 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (1/\sqrt{2}) \cdot 0 = 0,$$

and

$$\mathbf{u}_2^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_2 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (-1/\sqrt{2}) \cdot 0 = 0.$$

Now, for $j = 1, 2, 3$, $\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j$ and $\mathbf{u}_j^T \mathbf{u}_j = 1$. Since $\mathbf{A} = \mathbf{A}^T$, the spectral method then yields that

$$\mathbf{x} = \sum_{j=1}^{3} \frac{1}{\lambda_j} \frac{\mathbf{u}_j^T \mathbf{b}}{\mathbf{u}_j^T \mathbf{u}_j} \mathbf{u}_j = \sum_{j=1}^{3} \frac{\mathbf{u}_j^T \mathbf{b}}{\lambda_j} \mathbf{u}_j.$$

We can compute that

$$\mathbf{u}_1^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (1/\sqrt{2}) \cdot 3 = \sqrt{2},$$

$$\mathbf{u}_2^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (-1/\sqrt{2}) \cdot 3 = -2\sqrt{2},$$

and

$$\mathbf{u}_3^T \mathbf{b} = 1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 3 = 2,$$

and hence

$$\mathbf{x} = \frac{\sqrt{2}}{-1} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{-2\sqrt{2}}{1} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -3 \\ 1 \end{bmatrix}.$$

We can multiply $\mathbf{A}\mathbf{x}$ out to verify that the desired $\mathbf{b}$ is obtained.

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2. [20 points: 5 points each]

Let the inner product $(\cdot, \cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x) w(x)\, dx.$$

Consider the linear operator $L : C_m^2[0,1] \to C[0,1]$ defined by

$$Lu = -u''$$

where

$$C_m^2[0,1] = \left\{ u \in C^2[0,1] : u'(0) = u(1) = 0 \right\}.$$

(a) Is L symmetric?

(b) What is the null space of L?

(c) Show that $(Lu, u) \geq 0$ for all $u \in C_m^2[0,1]$ and explain why this and the answer to part (b) mean that $\lambda > 0$ for all eigenvalues λ of L.

(d) Find the eigenvalues and eigenfunctions of L.

Solution.

(a) [5 points] Yes, L is symmetric.
Let $u, v \in C_m^2[0,1]$. Integrating by parts twice, we have

$$\begin{aligned}(Lu, v) &= \int_0^1 -u''(x)v(x)\,dx \\ &= -[u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x)\,dx \\ &= -[u'(x)v(x)]_0^1 + [u(x)v'(x)]_0^1 - \int_0^1 u(x)v''(x)\,dx.\end{aligned}$$

Since $u, v \in C_m^2[0,1]$ we have $u'(0) = 0$ and $v(1) = 0$, and hence the first term in square brackets must be zero. Again using the fact that $u, v \in C_m^2[0,1]$ we have $v'(0) = 0$ and $u(1) = 0$, and hence the second term in square brackets is also zero. It follows that

$$(Lu, v) = \int_0^1 u(x)(-v''(x))\,dx = (u, Lv)$$

for all $u, v \in C_m^2[0,1]$.

(b) [5 points] The general solution to the differential equation

$$-u''(x) = 0$$

has the form

$$u(x) = A + Bx$$

for constants A and B. In order for u to be in $C_m^2[0,1]$, we must have $u'(0) = 0$ and so since $u'(x) = B$, we must have $B = 0$. Now $u \in C_m^2[0,1]$ also requires $u(1) = 0$, and since $u(1) = A$, we conclude that $A = 0$ too, meaning that $u(x) = A + Bx = 0$ for all $x \in [0,1]$. Thus, the only element of the null space is the zero function, that is, $\mathcal{N}(L) = \{0\}$.

(c) [7 points] Let $u \in C_m^2[0,1]$. Using the first integration by parts from part (a), we have

$$\begin{aligned}(Lu, u) &= -[u'(x)u(x)]_0^1 + \int_0^1 u'(x)u'(x)\,dx \\ &= \int_0^1 (u'(x))^2\,dx.\end{aligned}$$

Thus, (Lu, u) is the integral of a nonnegative function, so it is nonnegative. Consequently, $(Lu, u) \geq 0$ for all $u \in C_m^2[0,1]$.
This statement implies that all eigenvalues of L are non-negative, since if λ is an eigenfunction of L then, since L is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $u \in C_m^2[0,1]$ which are such that $Lu = \lambda u$ and hence

$$\lambda(u, u) = (\lambda u, u) = (Lu, u) \geq 0,$$

and so, since we know that $(u, u) > 0$ for all nonzero $u \in C_m^2[0,1]$ due to the positivite-definiteness of the inner product, we have that

$$\lambda = \frac{(Lu, u)}{(u, u)} \geq 0.$$

If zero was an eigenvalue of L, then there would exist nonzero $u \in C_m^2[0,1]$ which were such that $Lu = 0$. However, we showed in part (b) that there were no nonzero $u \in C_m^2[0,1]$ which satisfied this and so zero cannot be an eigenvalue of L and hence we can say that $\lambda > 0$ for all eigenvalues λ of L.

(d) [8 points] The eigenvalues of L are the real numbers $\lambda > 0$ for which there exist nonzero $u \in C_m^2[0,1]$ which are such that $Lu = \lambda u$. When $\lambda > 0$, the general solution to the equivalent differential equation

$$-u''(x) = \lambda u(x)$$

has the form

$$u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

where A and B are constants. Since

$$u'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

and thus

$$u'(0) = A\sqrt{\lambda},$$

the boundary condition $u'(0) = 0$ implies that $A = 0$. On the other hand, the boundary condition $u(1) = 0$ implies that

$$u(1) = B\cos(\sqrt{\lambda}) = 0,$$

which can be achieved with nonzero B provided that $\sqrt{\lambda} = (n - 1/2)\pi$ for positive integers n. We thus have that L has eigenvalues

$$\lambda_n = (n - 1/2)^2\pi^2$$

with corresponding eigenfunctions

$$u_n(x) = B_n\cos(\sqrt{\lambda_n}x) = B_n\cos((n - 1/2)\pi x)$$

for nonzero constants B_n, for $n = 1, 2, 3, \ldots$.

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3. [20 points: 4 points (b), 7 points (a),(c),(d)]
Define the inner product (u, v) to be

$$(u, v) = \int_0^1 u(x)v(x)\, dx$$

and let the norm $\|v(x)\|$ be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let N be a positive integer and let $\phi_1, \ldots, \phi_N \in C[0, 1]$ be such that $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$. We wish to approximate a continuous function $f(x)$ with $f_N(x)$

$$f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

where

$$\phi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \ldots.$$

and where $\alpha_n = (f, \phi_n)$. (Note that f_N is the best approximation to g from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\| \cdot \|$.)

(a) Assume that $f_N \to f$ as $N \to \infty$. Show that, since $\phi_1, \ldots, \phi_N$ are orthonormal,

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2.$$

(b) The best approximation to $f(x) = x(1 - x)$ has coefficients α_n which satisfy

$$\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3}(1 - (-1)^n).$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 5$. On a separate figure, plot the norm of the error $\|f - f_N\|$ using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using `loglog` in MATLAB.

(c) For $f(x) = 1$ (which does not satisfy the same boundary conditions as $\phi_n(x)$!), we computed in class that

$$c_n = 2\sqrt{2}/(n\pi)$$

for odd n, and $c_n = 0$ for even n. Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 100$. On a separate figure, plot the norm of the error $\|f - f_N\|$ using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using `loglog` in MATLAB.

You may have noticed that the rate at which the coefficients $\alpha_n \to 0$ determines how fast the error decreases — this is not coincidental!

(d) For $f(x) = 1$, the equation $Lu = f$ has the exact solution $u(x) = x(1 - x)/2$. Given the result of part (c), the same argument used in part (a) tells us that

$$\|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n^2}.$$

(You do not need to show this explicitly.) Use this formula to produce a `loglog` plot of the error $\|u - u_N\|$ for $N = 1, \ldots, 100$ on the same plot you made in part (b). (Be aware that the error may appear to flatline around 10^{-8}: this is a consequence of the computer's floating point arithmetic, and is not a concern of ours here. To learn more about this phenomenon, take CAAM 453!)

Solution.

(a) [10 points] We have that

$$
\begin{aligned}
\|f - f_N\|^2 &= (f - f_N, f - f_N) \\
&= \left(f - \sum_{n=1}^{N} \alpha_n \psi_n, f - \sum_{m=1}^{N} \alpha_m \psi_m\right) \\
&= \left(f - \sum_{n=1}^{N} \alpha_n \psi_n, f\right) - \sum_{m=1}^{N} \alpha_m \left(f - \sum_{n=1}^{N} \alpha_n \psi_n, \psi_m\right) \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{m=1}^{N} \alpha_m \sum_{n=1}^{N} \alpha_n (\psi_n, \psi_m) \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n^2 (\psi_n, \psi_n) \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n^2 \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n^2 \\
&= (f, f) - \sum_{n=1}^{N} \alpha_n^2 \\
&= \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2,
\end{aligned}
$$

where at each equal sign we have used: (1) the definition of the norm $\|\cdot\|$; (2) the definition of g_N; (3) linearity of the inner product in the second argument; (4) linearity of the inner product in the first argument; (5) the fact that $(\psi_n, \psi_m) = 0$ if $n \neq m$, for $m, n = 1, 2, \ldots, N$, since $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$; (6) the fact that $(\psi_n, \psi_n) = 1$, for $n = 1, 2, \ldots, N$, since $\{\psi_1, \ldots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$; (7) the fact that $(f, \psi_n) = (\psi_n, f) = \alpha_n$; (8) algebra; (9) the definition of the norm $\|\cdot\|$.

(b) [10 points] First calculate the norm of f

$$
\|f\|^2 = \int_0^1 (f(x))^2 \, dx = \int_0^1 x^2 (1 - x)^2 \, dx = \frac{1}{30}.
$$

Then

$$
\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2,
$$

where

$$
\alpha_n = \frac{2\sqrt{2}}{n^3 \pi^3} \left(1 - (-1)^n\right).
$$

and so

$$
\|f - f_N\|^2 = \frac{1}{30} - \sum_{n=1}^{N} \alpha_n^2.
$$

The requested plots are shown below.

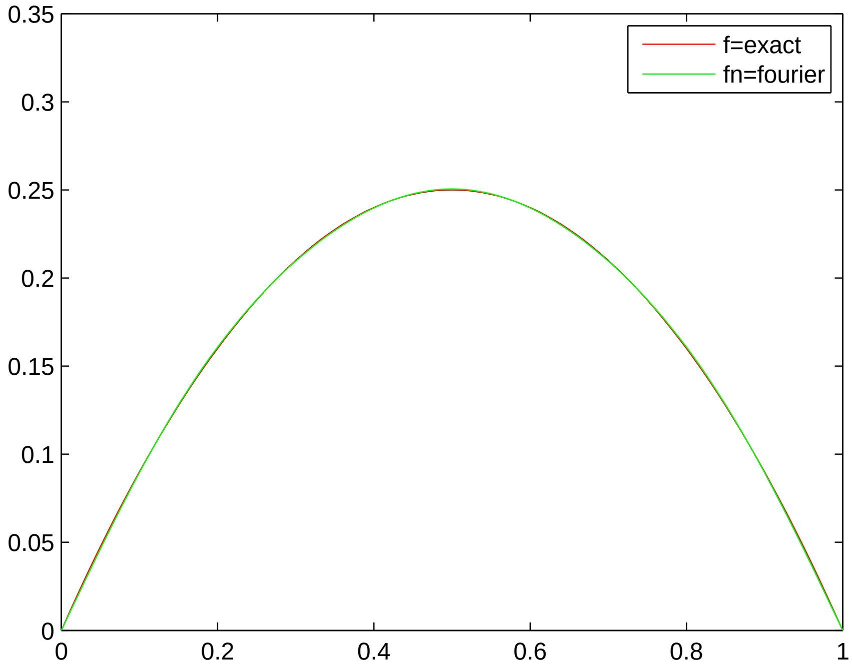


Figure 1: Comparison of the true function $f(x)$ and $f_N(x)$ for $N = 5$

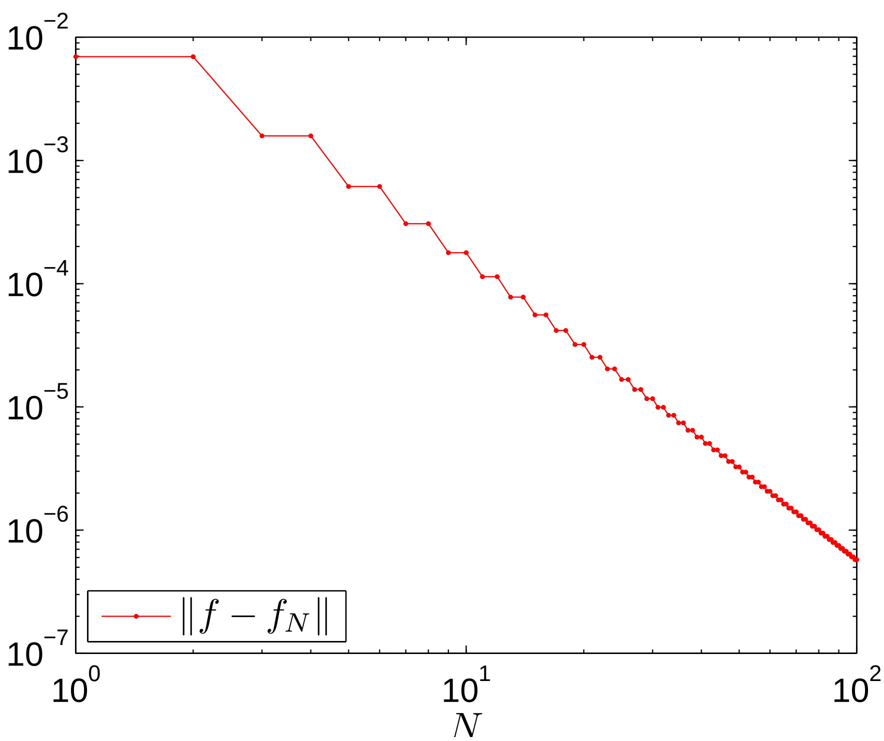


Figure 2: Norm of the error for $N = 1, 2 \cdots, 100$ on a log-log scale

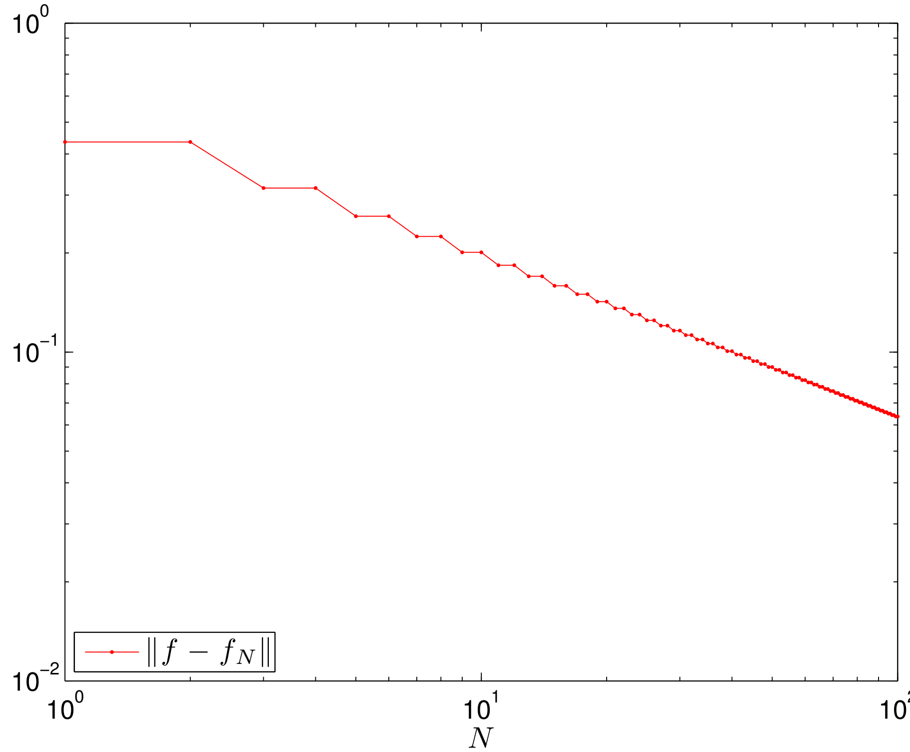


Figure 3: Plot for (c).

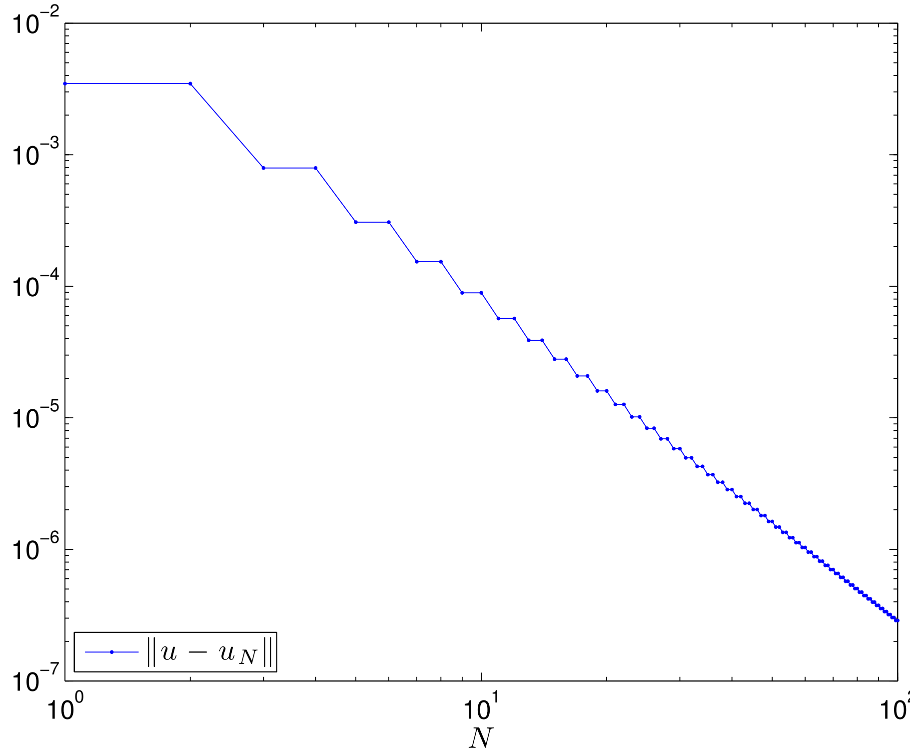


Figure 4: Plot for (d).

(c) The requested plot for (c) and (d) is shown below.

The code that produced the plot above for (c) and (d) is shown below.

```
n = [1:100]';
cn = (sqrt(2)/pi)*(1+(-1).^(n+1))./(n);
lamn = pi^2*n.^2;
normf2 = 1;
normu2 = 1/120;

figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')
set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$','$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr1

figure(2)
loglog(n, sqrt(normu2-cumsum((cn./lamn).^2)),'b.-')
set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr2
```

4. [25 points: 4 points (a), 7 points (b)-(d)]
This problem concerns the same operator from class and previous problems, $L : C_D^2[0,1] \to C[0,1]$ defined by

$$L_D u = -\frac{d^2u}{dx^2},$$

with homogeneous Dirichlet boundary conditions imposed via

$$C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}.$$

The eigenvalues and (normalized) eigenfunctions remain as they were before: $\lambda_n = n^2\pi^2$ and $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for $n = 1, 2, \ldots$. Now let $f(x) = x^2(1-x)$.

(a) For this f, compute the coefficients

$$c_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

in the expansion

$$f = \sum_{n=1}^{\infty} c_n \psi_n.$$

You may determine these by hand, by consulting a table of integrals, or by using a symbolic mathematics package like Mathematica or the Symbolic Toolbox in MATLAB.

(b) Produce a plot (or series of plots) comparing $f(x)$ to the partial sums

$$f_N(x) = \sum_{k=1}^{N} c_k \psi_k(x)$$

for $N = 1, \ldots, 10$.

(c) Plot the approximations u_N to the true solution u that you obtain using the spectral method:

$$u_N(x) = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \psi_k(x)$$

for $N = 1, \ldots, 10$.

(d) Now replace the homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ above with the inhomogeneous Dirichlet conditions $u(0) = -1/100$ and $u(1) = 1/100$. Describe how to adjust your solution from part (c) to account for these boundary conditions, and produce a plot of the solution with these inhomogeneous boundary conditions, based on u_{10} from part (c).

Solution.

(a) Expand $f(x) = x^2(1-x) = x^2 - x^3$. We can compute the coefficients (f, ψ_k) as

$$(f, \psi_k) = \int_0^1 x^2\big(\sqrt{2}\sin(k\pi x)\big)\,dx - \int_0^1 x^3\big(\sqrt{2}\sin(k\pi x)\big)\,dx.$$

The first integral on the right can be computed using Mathematica, etc. Alternatively, one can work it out directly:

$$\begin{aligned}\sqrt{2}\int_0^1 x^2 \sin(k\pi x)\,dx &= \sqrt{2}\bigg(\Big[\frac{-x^2\cos k\pi x}{k\pi}\Big]_0^1 + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\bigg)\\ &= \sqrt{2}\bigg(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\bigg)\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^2\pi^2}\Big[x\sin(k\pi x)\Big]_0^1 - \int_0^1 \sin(k\pi x)\,dx\Big) \\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} - \frac{2}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx\Big) \\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^3\pi^3}\big[\cos(k\pi x)\big]_0^1\Big) \\
&= \sqrt{2}\Big(\frac{(2-k^2\pi^2)(-1)^k - 2}{k^3\pi^3}\Big).
\end{aligned}
$$

The second integral follows from integrating thrice by parts:

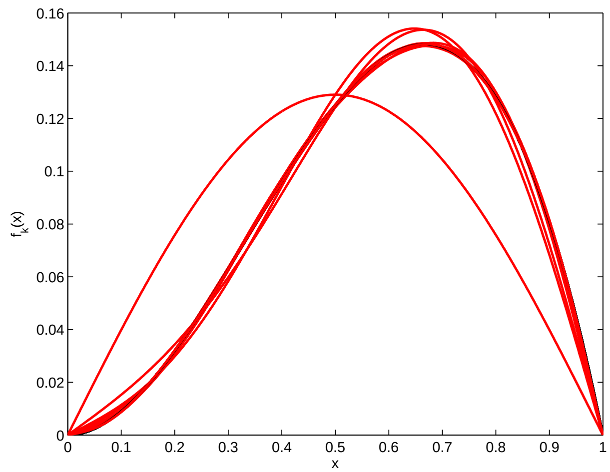
$$\int_0^1 x^3\big(\sqrt{2}\sin(k\pi x)\big)\,dx = \frac{\sqrt{2}(-1)^n(6-k^2\pi^2)}{k^3\pi^3}.$$

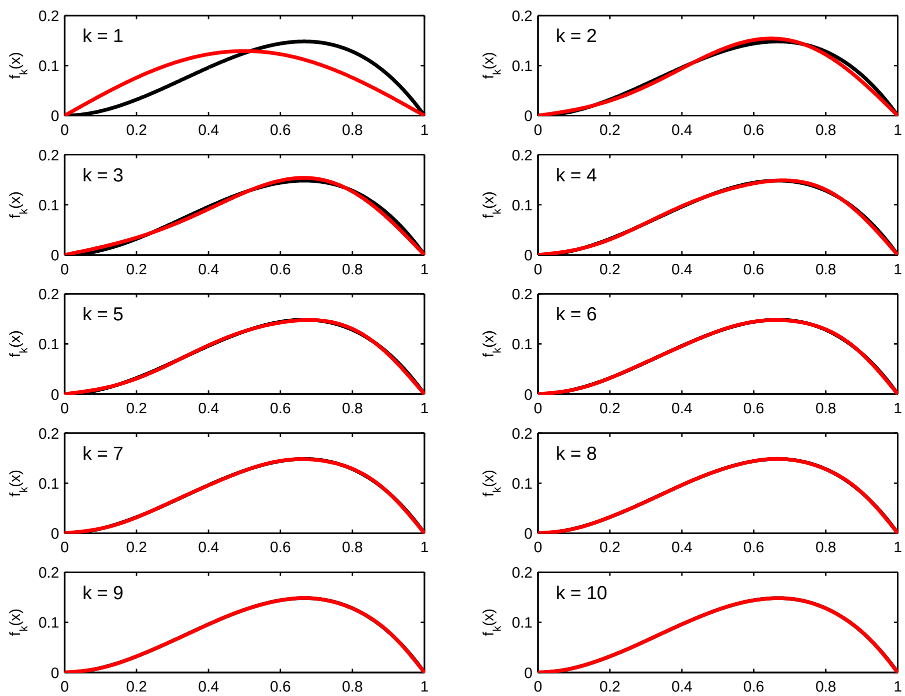
Assembling these results, we simplify to obtain

$$(f,\psi_k) = \frac{\sqrt{2}\big(4(-1)^{k+1}-2\big)}{k^3\pi^3}.$$

$$
\begin{aligned}
\sqrt{2}\int_0^1 x^2\sin(k\pi x)\,dx &= \sqrt{2}\Big(\Big[\frac{-x^2\cos k\pi x}{k\pi}\Big]_0^1 + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\Big) \\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k\pi}\int_0^1 x\cos(k\pi x)\,dx\Big) \\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^2\pi^2}\Big[x\sin(k\pi x)\Big]_0^1 - \int_0^1 \sin(k\pi x)\,dx\Big) \\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} - \frac{2}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx\Big) \\
&= \sqrt{2}\Big(\frac{(-1)^{k+1}}{k\pi} + \frac{2}{k^3\pi^3}\big[\cos(k\pi x)\big]_0^1\Big) \\
&= \sqrt{2}\Big(\frac{(2-k^2\pi^2)(-1)^k - 2}{k^3\pi^3}\Big).
\end{aligned}
$$

(b) Partial sums of the series formula for f are shown in the plots below. Code follows at the end of the problem. The function f happens to satisfy homogeneous Dirichlet boundary conditions, and convergence is quite quick.

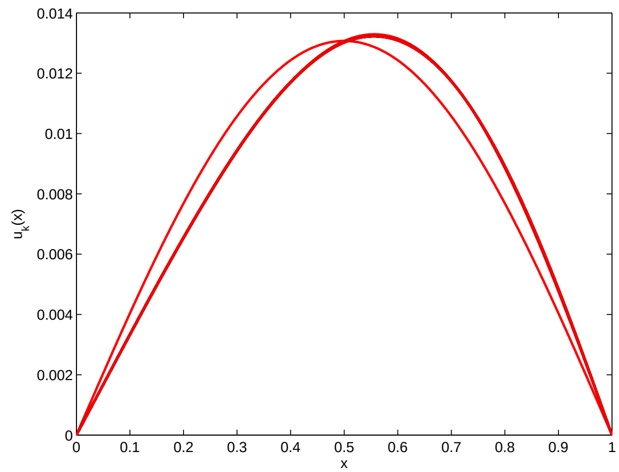


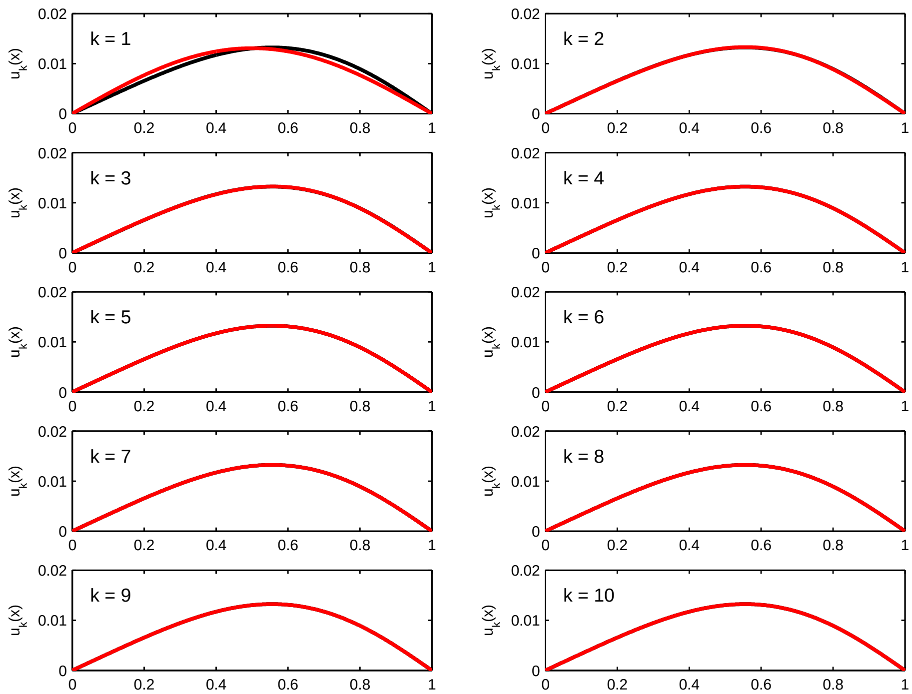


(c) The true solution to this problem (not asked for in the problem statement) can be computed as $u(x) = (2x - 5x^4 + 3x^5)/60$. The spectral method gives $u(x)$ as the series

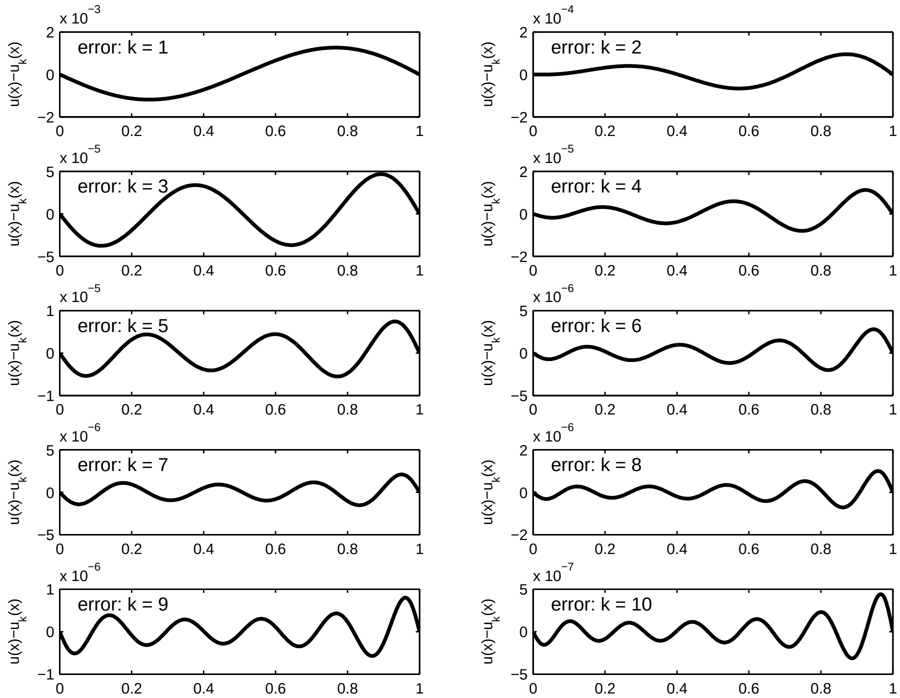
$$u(x) = \sum_{k=1}^{\infty} \sqrt{2}\Big(\frac{4(-1)^{k+1} - 2}{k^5\pi^5}\Big)\big(\sqrt{2}\sin(k\pi x)\big),$$

whose coefficients decay even more rapidly than did the coefficients for f itself, explaining the fantastic convergence rate. The next two plots compare the sum from the spectral method (red lines) to the true solution (black line). The following plot shows the error as a function of x for $k = 1, \ldots, N$.





Plots of the error as k increases:



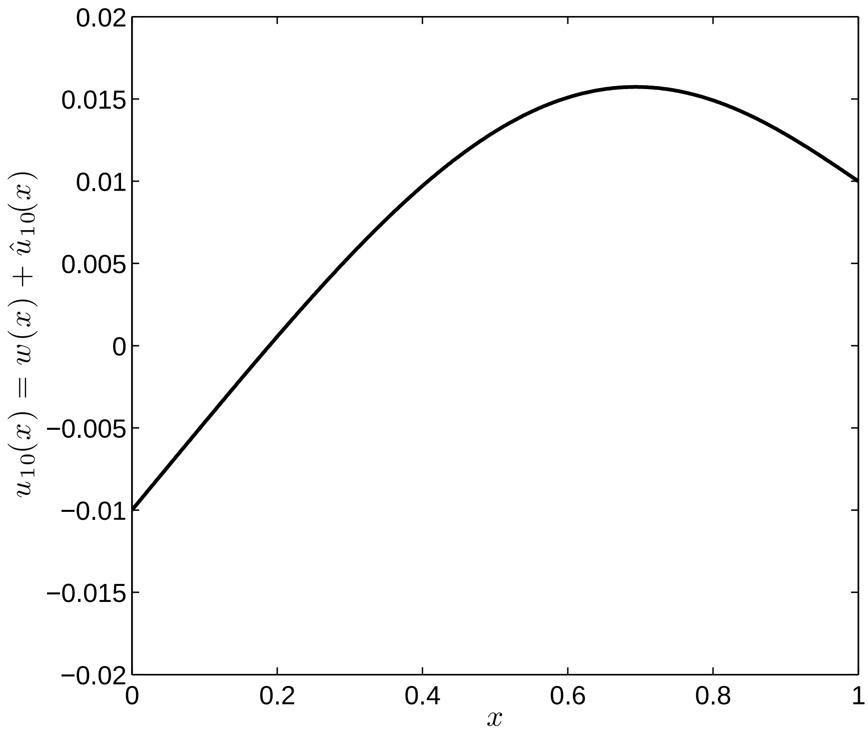
(d) To incorporate inhomogeneous Dirichlet boundary conditions, we will write the solution in the form $u = \widehat{u} + w$, where the correction $w(x) = \alpha + \beta x$ has the property that $-w''(x) = 0$, and $\widehat{u}$ denotes the solution to $L\widehat{u} = f$ with homogeneous Dirichlet boundary conditions; thus $\widehat{u}$ is precisely the solution u worked out in part (c). Notice that $u(0) = w(0) = \widehat{u}(0) = w(0) = \alpha$ and $u(1) = w(1) = \widehat{u}(1) = w(1) = \alpha + \beta$. Since we want $u(0) = -1/100$ and $u(1) = 1/100$, we solve to find:

$$\alpha = -1/100, \qquad \beta = 2/100.$$

Thus, the solution to our equation with these inhomogeneous boundary conditions is:

$$u(x) = -1/100 + (2/100)x + \sum_{k=1}^{\infty} \sqrt{2}\Big(\frac{4(-1)^{k+1} - 2}{k^5\pi^5}\Big)\big(\sqrt{2}\sin(k\pi x)\big),$$

The plot of $u_{10}(x) = w(x) + \widehat{u}_{10}(x)$ is shown below.



The plots above were computed using the following MATLAB code.

```
% f(x) = x^2(1-x);

% compute the inner products (f, phi_k) for k=1,...,30
  k = [1:30]';
  ck = sqrt(2)*(4*(-1).^(k+1)-2)./(k.^3*pi^3);

% plot first 10 partial sums fk, all on one plot
 figure(2), clf
 x = linspace(0,1,500)';
 fk = zeros(size(x));
 for k=1:10
    plot(x,(x.^2).*(1-x),'k-','linewidth',2), hold on
    fk = fk + ck(k)*sqrt(2)*sin(k*pi*x);
    plot(x,fk,'r-','linewidth',2)
    xlabel('x'), ylabel('f_k(x)')
 end
 print -depsc2 sineseries2b

% plot first 10 partial sums fk, all on 10 different plot
 figure(3), clf
 x = linspace(0,1,500)';
 fk = zeros(size(x));
 for k=1:10
    subplot(5,2,k)
    plot(x,(x.^2).*(1-x),'k-','linewidth',2), hold on
    fk = fk + ck(k)*sqrt(2)*sin(k*pi*x);
    plot(x,fk,'r-','linewidth',2)
    axis([0 1 0 0.2])
    set(gca,'fontsize',8)
    text(0.05,0.16,sprintf('k = %d',k))
```

```
   ylabel('f_k(x)')
 end
 print -depsc2 sineseries2c


% plot first 10 partial sums uk, all on one plot
 figure(4), clf
 x = linspace(0,1,500)';
 uk = zeros(size(x));
 for k=1:10
   plot(x,(2*x-5*(x.^4)+3*(x.^5))/60,'k-','linewidth',2), hold on
   lamk = k^2*pi^2;
   uk = uk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
   plot(x,uk,'r-','linewidth',2)
   xlabel('x'), ylabel('u_k(x)')
 end
 print -depsc2 sineseries2d

% plot first 10 partial sums uk, all on 10 different plot
 figure(5), clf
 x = linspace(0,1,500)';
 uk = zeros(size(x));
 for k=1:10
   subplot(5,2,k)
   plot(x,(2*x-5*(x.^4)+3*(x.^5))/60,'k-','linewidth',2), hold on
   lamk = k^2*pi^2;
   uk = uk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
   plot(x,uk,'r-','linewidth',2)
   axis([0 1 0 .02])
   set(gca,'fontsize',8)
   text(0.05,0.015,sprintf('k = %d',k))
   ylabel('u_k(x)')
 end
 print -depsc2 sineseries2e

% plot error in first 10 partial sums uk, all on 10 different plot
 figure(6), clf
 x = linspace(0,1,500)';
 uk = zeros(size(x));
 for k=1:10
   subplot(5,2,k)
   lamk = k^2*pi^2;
   uk = uk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
   plot(x,(2*x-5*(x.^4)+3*(x.^5))/60 - uk,'k-','linewidth',2), hold on
   set(gca,'fontsize',8)
   text(0.05,max(ylim)-.175*diff(ylim),sprintf('error: k = %d',k))
   ylabel('u(x)-u_k(x)')
 end
 print -depsc2 sineseries2f

% inhomogeneous boundary condition:
% plot first 10 partial sums uk, all on 10 different plot
 figure(7), clf
 x = linspace(0,1,500)';
 w = -1/100 + (2/100)*x;
 uhatk = zeros(size(x));
 for k=1:10
   lamk = k^2*pi^2;
```

```
    uhatk = uhatk + ck(k)/lamk*sqrt(2)*sin(k*pi*x);
    plot(x,w+uhatk,'k-','linewidth',2)
    axis([0 1 -.02 .02])
    set(gca,'fontsize',14)
    ylabel('$u_{10}(x) = w(x) + \hat{u}_{10}(x)$','fontsize',16,'interpreter','latex')
    xlabel('$x$','fontsize',16,'interpreter','latex')
end
print -depsc2 sineseries2g
```

5. [25 points: 5 points each]

Let the symmetric linear operator $L : C_M^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$C_M^2[0,1] = \{w \in C^2[0,1] : w'(0) = w(1) = 0\}.$$

Let N be a positive integer and let $f \in C[0,1]$ be defined by

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 0 & \text{otherwise.} \end{cases}$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\phi_n(x) = \sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)$$

for $n = 1, 2, \ldots$. We have that, for $m, n = 1, 2, \ldots$,

$$(\phi_m, \phi_n) = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) Compute f_N, the best approximation to f from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$. Plot f_N for $N = 1, 2, 3, 4, 5, 6$.

(c) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = f(x), \quad 0 < x < 1$$

and

$$\tilde{u}'(0) = \tilde{u}(1) = 0.$$

(d) By shifting the data, obtain an infinite series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = f(x), \quad 0 < x < 1$$

and

$$u'(0) = u(1) = 1.$$

(e) Let u_N be the series solution that you obtained in part (d) but with ∞ replaced by N, i.e.

$$u_N(x) = p(x) + \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

where $p(x)$ is the linear function added to match the nonzero boundary conditions in (d). Plot u_N for $N = 1, 2, 3, 4, 5, 6$.

Solution.

(a) [3 points] We can compute that, for $n = 1, 2, \ldots$,

$$\phi_n'(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)\pi \sin\left(\frac{2n-1}{2}\pi x\right).$$

and

$$\phi_n''(x) = -\sqrt{2}\left(\frac{2n-1}{2}\right)^2 \pi^2 \cos\left(\frac{2n-1}{2}\pi x\right).$$

and so

$$L\phi_n = -\phi_n'' = \left(\frac{2n-1}{2}\right)^2 \pi^2 \phi_n.$$

Hence,

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \pi^2 = (2n-1)^2 \frac{\pi^2}{4} \text{ for } n = 1, 2, \ldots.$$

(b) [8 points] Since $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$, the best approximation to f from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$ is

$$f_N = \sum_{n=1}^{N} (f, \phi_n)\phi_n.$$

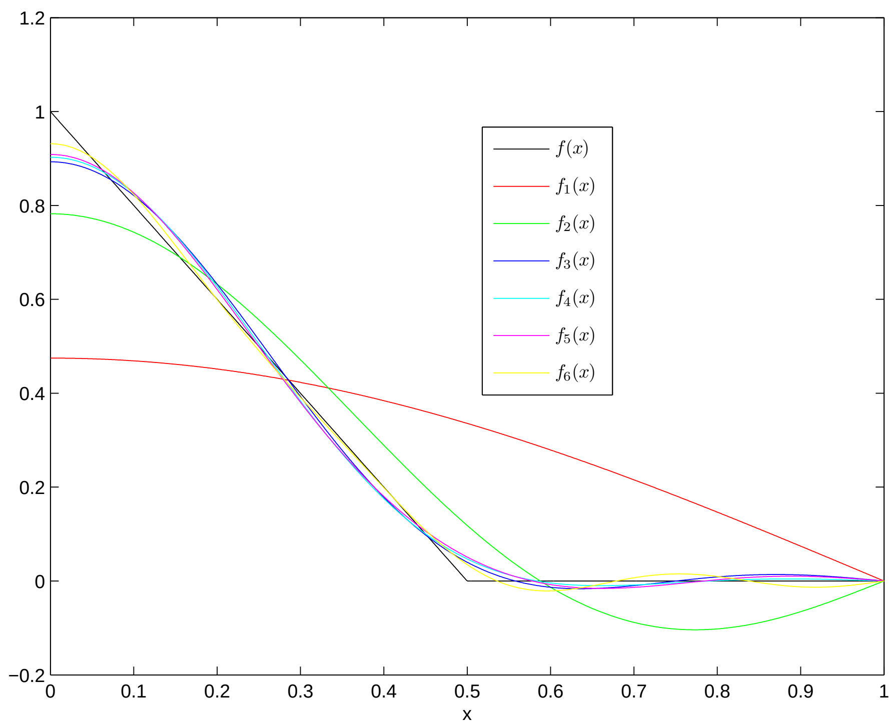
Now, for $n = 1, 2, \ldots$,

$$\begin{aligned}
&(f, \phi_n) \\
&= \int_0^1 f(x)\phi_n(x)\,dx \\
&= \int_0^{1/2} f(x)\phi_n(x)\,dx + \int_{1/2}^1 f(x)\phi_n(x)\,dx \\
&= \int_0^{1/2} (1-2x)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right)dx + \int_{1/2}^1 0\,dx \\
&= \sqrt{2}\int_0^{1/2} (1-2x)\cos\left(\frac{2n-1}{2}\pi x\right)dx + 0 \\
&= \sqrt{2}\left(\left[(1-2x)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} - \int_0^{1/2}(-2)\frac{2}{(2n-1)\pi}\sin\left(\frac{2n-1}{2}\pi x\right)dx\right) \\
&= \sqrt{2}\left(0 - 0 + \frac{4}{(2n-1)\pi}\int_0^{1/2}\sin\left(\frac{2n-1}{2}\pi x\right)dx\right) \\
&= \sqrt{2}\frac{4}{(2n-1)\pi}\left[-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\pi x\right)\right]_0^{1/2} \\
&= \frac{4\sqrt{2}}{(2n-1)\pi}\left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{4}\pi\right) - \left(-\frac{2}{(2n-1)\pi}\right)\right) \\
&= \frac{8\sqrt{2}}{(2n-1)^2\pi^2}\left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned} f_N(x) &= \sum_{n=1}^{N} (f,\phi_n)\phi_n(x) \\ &= \sum_{n=1}^{N} (f,\phi_n)\sqrt{2}\cos\left(\frac{2n-1}{2}\pi x\right) \\ &= \sum_{n=1}^{N} \frac{16}{(2n-1)^2\pi^2}\left(1-\cos\left(\frac{2n-1}{4}\pi\right)\right)\cos\left(\frac{2n-1}{2}\pi x\right). \end{aligned}$$

The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(1)
clf
legendStr{1}=['$f(x)$'];
plot(x,-(x-1/2)+(x-1/2).*sign(x-1/2),'k-')
hold on
fk = zeros(size(x));
for k=1:6
    fk = fk + 16*(1-cos(((2*k-1)/4)*pi))./((2*k-1).^2*pi^2)*cos(((2*k-1)/2)*pi*x);
    plot(x,fk,colors(k))
    legendStr{k+1}=['$f_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(1),'hw72c','epsc')
```

(c) [4 points] Now, $\tilde{u}$ is the solution to $L\tilde{u} = f$ and so the spectral method yields the series solution

$$\tilde{u}(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n} \phi_n(x) = \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right).$$

(d) [6 points] Let $\tilde{u}$ be the solution to $L\tilde{u} = f$ and let $w \in C^2[0,1]$ be such that

$$-w''(x) = 0, \quad 0 < x < 1$$

and

$$w'(0) = w(1) = 1.$$

Then $u(x) = w(x) + \tilde{u}(x)$ will be such that

$$-u''(x) = -w''(x) - \tilde{u}''(x) = 0 + f(x) = f(x);$$

$$u'(0) = w'(0) + \tilde{u}'(0) = 1 + 0 = 1;$$

and

$$u(1) = w(1) + \tilde{u}(1) = 1 + 0 = 1.$$

Now, the general solution to

$$-w''(x) = 0$$

is $w(x) = Ax + B$ where A and B are constants. Moreover, $w'(x) = A$ and so $w'(0) = 1$ when $A = 1$. Hence, $w(x) = x + B$ and so $w(1) = 1$ when $B = 0$. Consequently,

$$w(x) = x$$

and so

$$u(x) = x + \tilde{u}(x).$$

We can then use the series solution to $L\tilde{u} = f$ that we obtained in part (c) to obtain the series solution

$$u(x) = x + \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right)$$

to the problem of finding $u \in C^2[0,1]$ such that

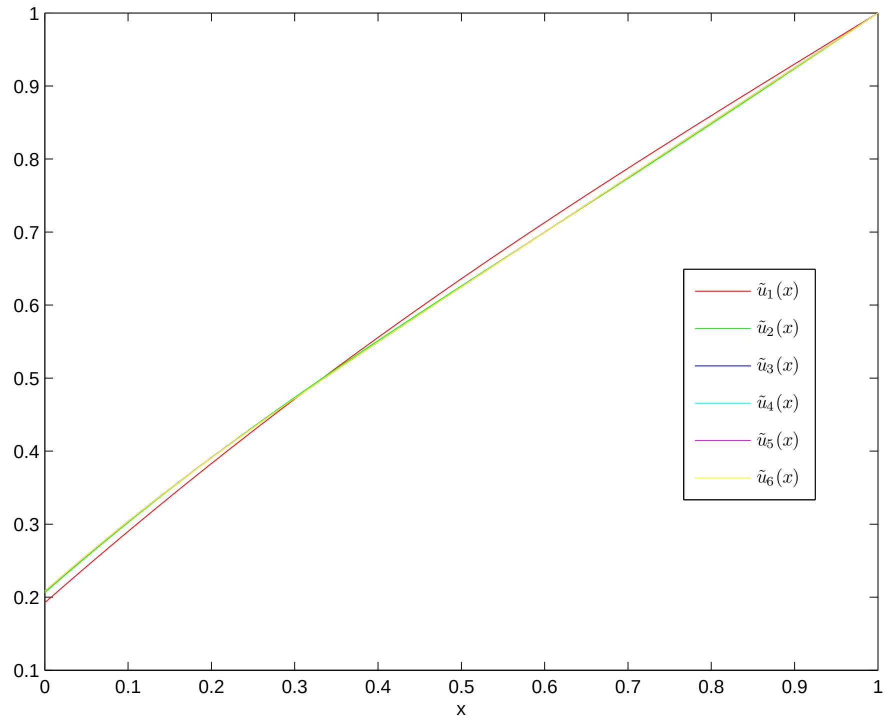
$$-u''(x) = f(x), \quad 0 < x < 1;$$

$$u'(0) = u(1) = 1.$$

(e) [4 points] The best approximation $\tilde{u_N}$ to $\tilde{u}$ from part (d) is

$$\tilde{u}_N(x) = x + \sum_{n=1}^{N} \frac{(f, \phi_n)}{\lambda_n} \phi_n(x) = \sum_{n=1}^{N} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos\left(\frac{2n-1}{4}\pi\right)\right) \cos\left(\frac{2n-1}{2}\pi x\right).$$

The requested plot is below.



The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);

figure(3)
clf
uk = x;
for k=1:6
    uk = uk + 64*(1-cos(((2*k-1)/4)*pi))./((2*k-1).^4*pi^4)*cos(((2*k-1)/2)*pi*x);
    plot(x,uk,colors(k))
    hold on
    legendStr{k}=['$\tilde{u}_{' num2str(k) '}(x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(3),'hw72e','epsc')
```