CAAM 336 · DIFFERENTIAL EQUATIONS

Examination 1

Posted Tuesday, 16 October 2012.

Due no later than 5pm on Monday, 22 October 2012.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Time started: __________________ Time completed: __________________
7. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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8. Staple this page to the front of your exam.

1. [20 points: 5 points per part]

   Consider the problem $-u''(x) = f(x)$ for $x \in [0,1]$ with Dirichlet boundary conditions, where

$$f(x) = \begin{cases} 0, & x < 1/2; \\ 1, & x \geq 1/2. \end{cases}$$

   Let $V = C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$ with the usual inner product

$$(u,v) = \int_0^1 u(x)v(x)\,dx.$$

   Define $L : V \to C[0,1]$ by $Lu = -u''$.

   (a) Write down the eigenvalues $\lambda_1, \lambda_2, \ldots$ and eigenfunctions $\psi_1, \psi_2, \ldots$ of L. (You need not compute these from scratch if you remember them.)

   (b) Compute $(f, \psi_j)/(\psi_j, \psi_j)$ for $j = 1, 2, \ldots$.

   (c) Write the spectral method solution to $-u''(x) = f(x)$ with $u(0) = u(1) = 0$.

   (d) Explain how to modify your solution in part (c) to accommodate the inhomogeneous boundary conditions $u(0) = -1$, $u(1) = 1$.

2. [28 points: 5 points each for (a)–(e); 3 points for (f)]

   In class we considered inner products for vectors in $\mathbb{R}^n$ and functions in $C[0,1]$, and used these inner products to generate best approximations. This problem will introduce you to an inner product of *matrices* in $\mathbb{R}^{n\times n}$. (Related math arises in many modern applications; e.g., at the roots of how Netflix recommends movies.)

   (a) Let V be a vector space endowed with an inner product, and let $V_n = \text{span}\{\phi_1, \ldots, \phi_n\}$ denote an n-dimensional subspace of V. We wish to approximate some vector $f \in V$.

   - Explain how to compute $f_n = c_1\phi_1 + \cdots + c_n\phi_n$, the best approximation to f from V_n. (That is, explain how to determine the unknown coefficients $c_1, \ldots, c_n$.)
   - How does the procedure simplify when the basis vectors $\phi_1, \ldots, \phi_n$ are orthogonal?

   For the rest of the problem, consider this specific case: Let $V = \mathbb{R}^{2\times 2}$, the set of all 2×2 matrices, and define the inner product

$$(\mathbf{A}, \mathbf{B}) = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}b_{jk},$$

   where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

   With this inner product we associate the norm $\|\mathbf{A}\| = \sqrt{(\mathbf{A}, \mathbf{A})}$.

   (b) Verify that $(\cdot,\cdot)$ is an inner product on $V = \mathbb{R}^{2\times 2}$.

   (c) Consider the subspace of V consisting of all symmetric matrices:

$$V_3 = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

   Compute the best approximation $\mathbf{M}_3$ to the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

   from the subspace V_3.

   (d) Now let $\mathbf{M} \in \mathbb{R}^{2\times 2}$ be any 2×2 matrix, and let $\mathbf{M}_3$ be its best approximation from V_3. Explain why the error $\mathbf{M} - \mathbf{M}_3$ must *always* have zero diagonal entries.

   (e) Now consider the subspace

$$\widehat{V}_3 = \text{span}\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}.$$

   What is the best approximation $\widehat{\mathbf{M}}_3$ from $\widehat{V}_3$ to

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}?$$

   (f) Write down a basis for the set of symmetric matrices in $\mathbb{R}^{3\times 3}$ (i.e., 3×3 matrices). In general, what is the dimension of the set of symmetric matrices in $\mathbb{R}^{n\times n}$?

3. [28 points: 4 points per part]

This question addresses the problem $-u''(x) = f(x)$ for $x \in [0,1]$ with Neumann boundary conditions: $u'(0) = u'(1) = 0$. Let

$$V = C_N^2[0,1] = \{u \in C^2[0,1] : u'(0) = u'(1) = 0\},$$

with the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

Define the operator $L : V \to C[0,1]$ by

$$Lu = -u'', \qquad u'(0) = u'(1) = 0.$$

(a) Is L symmetric?

(b) What is the null space of L? That is, describe all vectors $v \in V$ such that $Lv = 0$.

(c) Compute all the eigenvalues and corresponding eigenfunctions of L.

(d) Now suppose we wish to solve $Lu = f$ for the unknown u. State the conditions on f for which this problem has:

- no solutions;
- infinitely many solutions;
- a unique solution.

(e) Suppose f is some function such that $Lu = f$ has infinitely many solutions. Using the spectral method (and other terms as appropriate), write down a general formula for all of them.

(f) Now pose the problem $-u'' = f$ with Neumman boundary conditions $u'(0) = u'(1) = 0$ as before, but additionally require solutions to satisfy $\int_0^1 u(x)\,dx = 0$ (that is, to have an average value of zero). We call the resulting operator $\widehat{L}$, which operates on the space

$$\widehat{V} = \{u \in C^2[0,1] : u'(0) = u'(1) = 0 \text{ and } \int_0^1 u(x)\,dx = 0\}.$$

How do the eigenvalues and eigenfunctions of $\widehat{L}$ differ from those of L that you found in problem (b)? Be precise.

(g) Suppose $\int_0^1 f(x)\,dx = 0$. Write down the spectral method solution u to $\widehat{L}u = f$.

4. [24 points: 4 points per part]

This problem addresses the finite element solution of the problem $-u''(x) = f(x)$ for $x \in [0,1]$ with Neumann boundary conditions $u'(0) = u'(1) = 0$ considered in the last problem.

Let

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx, \qquad (u,v) = \int_0^1 u(x)v(x)\,dx.$$

(a) Suppose $-u''(x) = f(x)$ with $u'(0) = u'(1) = 0$. Show that $a(u,v) = (f,v)$ for all $v \in V = C^2[0,1]$ (i.e., do not assume that v satisfies any particular boundary conditions).

Let N be a positive integer, and define $h = 1/(N+1)$. Define the hat functions, for $k = 1,\ldots,N$,

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise;} \end{cases}$$

with the "half-hat" functions supported on the far ends of the domain:

$$\phi_0(x) = \begin{cases} (x_1 - x)/h, & x \in [x_0, x_1); \\ 0, & \text{otherwise;} \end{cases} \qquad \phi_{N+1}(x) = \begin{cases} (x - x_N)/h, & x \in [x_N, x_{N+1}]; \\ 0, & \text{otherwise.} \end{cases}$$

(b) We seek the approximate solution $u_N = \sum_{j=0}^{N+1} c_j\phi_j$ such that $a(u_N, \phi_k) = (f, \phi_k)$ for $k = 0,\ldots,N+1$. Write down the general form of the stiffness matrix $\mathbf{K}$ and load vector $\mathbf{f}$, such that the coefficients $\mathbf{c}$ should satisfy the linear system $\mathbf{Kc} = \mathbf{f}$.
(That is, specify the entires of $\mathbf{K}$ and $\mathbf{f}$.)

(c) Why did we include ϕ_0 and ϕ_{N+1} in part (b)?
How would our solution differ had we only used $\phi_1,\ldots,\phi_N$?

(d) For $N = 1$, the stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

Zero is an eigenvalue of $\mathbf{K}$. Find a corresponding eigenvector $\mathbf{v}$ such that $\mathbf{Kv} = \mathbf{0}$.

(e) Write your answer $\mathbf{v}$ to part (d) in the form

$$\mathbf{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

Draw a plot of the corresponding function $v(x) = \alpha\phi_0(x) + \beta\phi_1(x) + \gamma\phi_2(x)$ for $x \in [0,1]$.
How does this relate to question 3(b)?

(f) For which of the following load vectors will $\mathbf{Kc} = \mathbf{f}$ have a solution? Explain.

$$\mathbf{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$