CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 37

Posted Friday 1 November 2013. Due 5pm Wednesday 13 November 2013.

37. [25 points] Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in C[0,1]$ be such that

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h} & \text{if } x \in [x_0, x_1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $H_D^1(0,1) = \{v \in H^1(0,1) : v(1) = 0\}$. Let the inner product $(\cdot,\cdot) : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x)\,dx$$

and let the symmetric bilinear form $a(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x)\,dx.$$

Also, let the norm $|||\cdot||| : H_D^1(0,1) \to \mathbb{R}$ be defined by

$$|||v||| = \sqrt{a(v,v)}.$$

Let $u \in C^2[0,1]$ be the solution of

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1; \\ u'(0) &= \alpha; \\ u(1) &= \beta; \end{aligned}$$

where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$.

(a) It can be shown that

$$a(u, v) = g(f, \alpha, v) \text{ for all } v \in \{w \in C^2[0,1] : w(1) = 0\}$$

where $g(f, \alpha, v)$ is a function of f, α and v. Obtain a formula for $g(f, \alpha, v)$.

(b) For the remainder of this question we shall just consider the case when

$$f(x) = 12x^2 - 24x + 4.$$

Note that in this case you obtained a formula for u in a previous homework. For this part we will just consider the case when $\alpha = \beta = 0$. In this case, we can obtain finite element approximations u_N to u by finding $u_N \in \text{span}\{\phi_0, \dots, \phi_{\text{N}}\}$ such that

$$a(u_N, v) = g(f, 0, v) \text{ for all } v \in \text{span}\{\phi_0, \dots, \phi_{\text{N}}\}.$$

Write a code which can obtain u_N and u_N^* where $u_N^* \in \text{span}\{\phi_1, \dots, \phi_{\text{N}}\}$ is such that

$$a(u_N^*, v) = g(f, 0, v) \text{ for all } v \in \text{span}\{\phi_1, \dots, \phi_{\text{N}}\}.$$

On the same figure, plot u as well as u_N and u_N^* for $N = 3$ and $N = 7$.

(c) For the case when $\alpha = \beta = 0$, plot

$$|||u - u_N|||$$

and

$$|||u - u_N^*|||$$

for $N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767$.

(d) Modify your code so that it can obtain finite element approximations u_N to u by finding $u_N \in \text{span}\{\phi_0, \dots, \phi_{\text{N+1}}\}$ such that $u_N(1) = \beta$ and

$$a(u_N, v) = g(f, \alpha, v) \text{ for all } v \in \text{span}\{\phi_0, \dots, \phi_{\text{N}}\}$$

for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. For the particular case of $\alpha = -1$ and $\beta = 1$, use your code to obtain u_N for $N = 3, 7, 15, 31$ and on the same figure plot u and u_N for $N = 3, 7, 15, 31$.