

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 34 · Solutions

Posted Friday 21 March 2014. Due 1pm Friday 11 April 2014.

34. [25 points]

Let

$$H_D^1(0, 1) = \{w \in H^1(0, 1) : w(0) = w(1) = 0\}$$

and let the inner product $(\cdot, \cdot) : L^2(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 v(x)w(x) dx$$

and the inner product $a(\cdot, \cdot) : H_D^1(0, 1) \times H_D^1(0, 1) \rightarrow \mathbb{R}$ be defined by

$$a(v, w) = \int_0^1 v'(x)w'(x) dx.$$

Let the norm $|||\cdot||| : H_D^1(0, 1) \rightarrow \mathbb{R}$ be defined by

$$|||v||| = \sqrt{a(v, v)}.$$

Let m be a positive integer and let $f_m \in L^2(0, 1)$ be defined by $f_m(x) = \sqrt{2}m^2\pi^2 \sin(m\pi x)$. Note that, for $j = 1, \dots, N$,

$$(f_m, \phi_j) = \frac{2\sqrt{2}\sin(m\pi x_j)}{h}(1 - \cos(m\pi h)).$$

Let N be a positive integer, let $h = \frac{1}{N+1}$, let $x_j = jh$ for $j = 0, 1, \dots, N+1$, and let $\phi_k \in H_D^1(0, 1)$ be defined by

$$\phi_k(x) = \begin{cases} \frac{(x - x_{k-1})}{h} & \text{if } x \in [x_{k-1}, x_k]; \\ \frac{(x_{k+1} - x)}{h} & \text{if } x \in [x_k, x_{k+1}]; \\ 0 & \text{otherwise;} \end{cases}$$

for $k = 1, \dots, N$. Let $V_N = \text{span}\{\phi_1, \dots, \phi_N\}$. Let $u_m \in H_D^1(0, 1)$ be such that

$$a(u_m, v) = (f_m, v) \text{ for all } v \in H_D^1(0, 1).$$

(a) The true solution to the problem of finding $u_m \in H_D^1(0, 1)$ such that

$$a(u_m, v) = (f_m, v) \text{ for all } v \in H_D^1(0, 1)$$

is $u_m(x) = \sqrt{2}\sin(m\pi x)$. Use integration by parts and the fact that $(u_m, u_m) = 1$ to obtain a formula for $a(u_m, u_m)$.

(b) We can obtain finite element approximations to u_m by finding $u_{m,N} \in V_N$ such that

$$a(u_{m,N}, v) = (f_m, v) \text{ for all } v \in V_N.$$

Write a code which can compute the energy norm of the error

$$|||u_m - u_{m,N}|||.$$

Use your code to produce a **loglog** plot of the energy norm of the error

$$|||u_m - u_{m,N}|||$$

when

$$m = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

for

$$N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191$$

on the same figure with a separate line for each value of m .

(c) Produce a **loglog** plot of the percentage error

$$100 \frac{|||u_m - u_{m,N}|||}{|||u_m|||}$$

when

$$m = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

for

$$N = 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191$$

on the same figure with a separate line for each value of m .

(d) Let $f \in L^2(0, 1)$. The problem of finding $u \in H_D^1(0, 1)$ such that

$$a(u, v) = (f, v) \text{ for all } v \in H_D^1(0, 1)$$

is the weak form of the problem of finding u such that

$$-u''(x) = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0,$$

or equivalently,

$$f(x) + u''(x) = 0, \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

If $\tilde{u}_N \in V_N$ then how does the quantity

$$\sum_{j=0}^N \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 dx$$

behave as N increases?

Solution.

(a) [5 points] Integration by parts yields that

$$\begin{aligned} & a(u_m, u_m) \\ &= \int_0^1 \frac{d}{dx} \left(\sqrt{2} \sin(m\pi x) \right) \frac{d}{dx} \left(\sqrt{2} \sin(m\pi x) \right) dx \\ &= \left[\frac{d}{dx} \left(\sqrt{2} \sin(m\pi x) \right) \sqrt{2} \sin(m\pi x) \right]_0^1 - \int_0^1 \frac{d^2}{dx^2} \left(\sqrt{2} \sin(m\pi x) \right) \sqrt{2} \sin(m\pi x) dx \\ &= [2m\pi \cos(m\pi x) \sin(m\pi x)]_0^1 + \int_0^1 \sqrt{2} m^2 \pi^2 \sin(m\pi x) \sqrt{2} \sin(m\pi x) dx \\ &= 2m\pi \cos(m\pi) \sin(m\pi) - 2m\pi \cos(0) \sin(0) + m^2 \pi^2 (u_m, u_m) \\ &= m^2 \pi^2 (u_m, u_m) \end{aligned}$$

since $\sin(0) = 0$ and $\sin(m\pi) = 0$ when m is a positive integer. Therefore, the fact that $(u_m, u_m) = 1$ allows us to conclude that

$$a(u_m, u_m) = m^2 \pi^2.$$

(b) [10 points] The code for this part and part (c) is:

```
clear
clc
figure(1)
clf
figure(2)
clf

markercolor='-or-og-ob-oc-om-ok-*r-*g-*b-*c-*m-*k';

mvec=2.^(0:1:11);
Nvec=2.^(0:1:13)-1;

for i=1:length(mvec)
    m=mvec(i);
    energyerr=zeros(size(Nvec));
    for j=1:length(Nvec)
        N=Nvec(j);
        h=1/(N+1);

        % construct the stiffness matrix
        K=sparse(N,N);
        K=K+sparse(1:N-1,2:N,-1/h,N,N);
        K=K+K.';
        K=K+sparse(1:N,1:N,2/h,N,N);

        % construct the load vector
        f=(2*sqrt(2)/h)*(1-cos(h*m*pi))*sin((1:N).'*h*m*pi);

        % solve for the coefficients
        c=K\f;

        energyerr(j)=sqrt(m^2*pi^2-(c.)*K*c);
    end

    % plot the energy norm of the error
    figure(1)
    loglog(Nvec,energyerr,markercolor(3*i-2:3*i))
    hold on

    % plot the percentage error
    figure(2)
    loglog(Nvec,100*energyerr/(m*pi),markercolor(3*i-2:3*i))
    hold on

    legendStr{i}=['$m=' num2str(m) '$'];
end

figure(1)
axis([1 10^8 10^(-4) 10^4])
axis square
legend(legendStr,'interpreter','latex','FontSize',14,'Location','East')
xlabel('$N$', 'interpreter','latex','FontSize',14)
ylabel('$||u_m-u_{m,N}||$', 'interpreter','latex','FontSize',14)
saveas(figure(1),'hw34b.eps','eps')

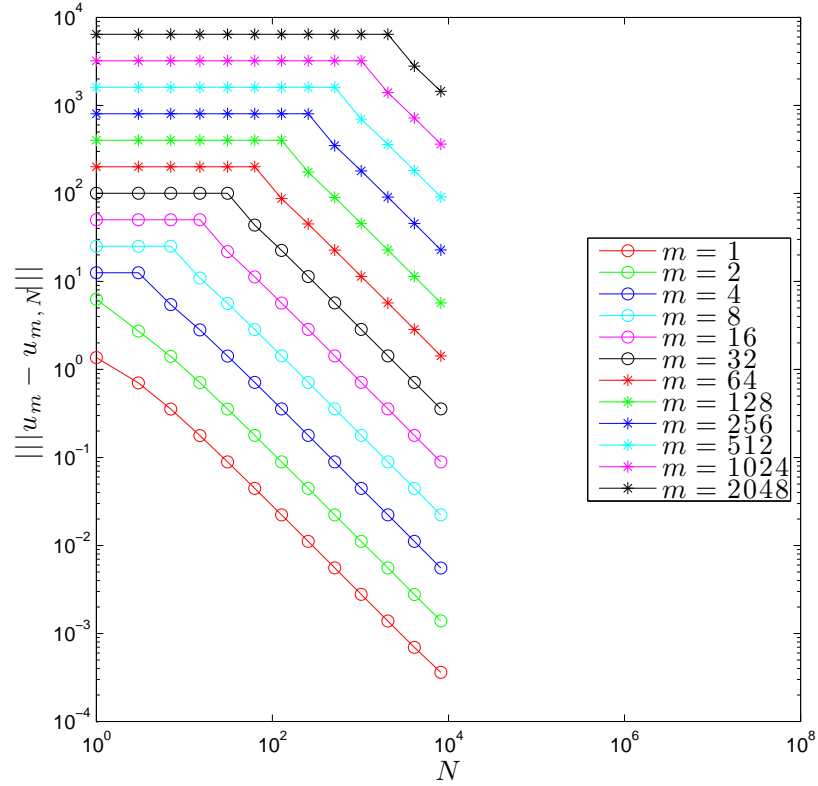
figure(2)
axis square
```

```

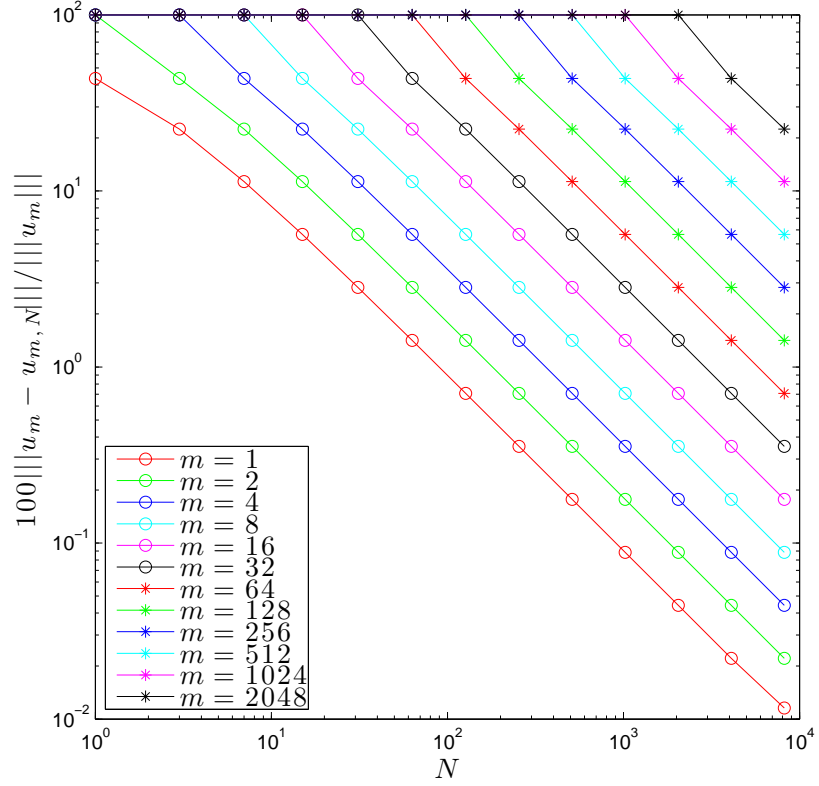
legend(legendStr,'interpreter','latex','FontSize',14,'Location','Southwest')
xlabel('$N$', 'interpreter','latex','FontSize',14)
ylabel('$100 ||u_m - u_{m,N}|| / ||u_m||$', 'interpreter','latex','FontSize',14)
saveas(figure(2), 'hw34c.eps', 'eps')

```

The requested plot is below.



(c) [5 points] The requested plot is below.



- (d) [5 points] For $j = 0, \dots, N$, when $x \in (x_j, x_{j+1})$, $\phi_k''(x) = 0$ for $k = 1, \dots, N$. Hence, if $\tilde{u}_N \in V_N$ then for $j = 0, \dots, N$, when $x \in (x_j, x_{j+1})$, $\tilde{u}_N''(x) = 0$. Therefore,

$$\sum_{j=0}^N \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 dx = \sum_{j=0}^N \int_{x_j}^{x_{j+1}} (f(x))^2 dx = \int_0^1 (f(x))^2 dx$$

for all $\tilde{u}_N \in V_N$. So, if $\tilde{u}_N \in V_N$, then the value of

$$\sum_{j=0}^N \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 dx$$

does not change as N increases as for all positive integers N

$$\sum_{j=0}^N \int_{x_j}^{x_{j+1}} (f(x) + \tilde{u}_N''(x))^2 dx = \int_0^1 (f(x))^2 dx.$$