CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 45 · Solutions

Posted Wednesday 2 April 2014. Due 1pm Friday 25 April 2014.

45. [25 points]

Let the norm $\|\cdot\|: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\|\mathbf{y}\| = \sqrt{\mathbf{y}\cdot\mathbf{y}}.$$

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and consider the problem of finding $\mathbf{x}(t)$ such that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t > 0$$

and

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

(a) Compute $\mathbf{x}(t)$.

(b) How does $\|\mathbf{x}(t)\|$ behave as $t \to \infty$?

Solution.

(a) [20 points] Since,

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}$$

we have that

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

and so

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

when $\lambda = -1$ or $\lambda = 1$. Hence, the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = -1$$

and

$$\lambda_2 = 1.$$

Moreover,

$$(\lambda_1\mathbf{I} - \mathbf{A})\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ -c_1 - c_2 \end{bmatrix}$$

and so to make this vector zero we need to set $c_2 = -c_1$. Hence, any vector of the form

$$\begin{bmatrix} c_1 \\ -c_1 \end{bmatrix}$$

where c_1 is a nonzero constant is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_1. Let us choose

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Furthermore,

$$(\lambda_2 \mathbf{I} - \mathbf{A}) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{bmatrix}$$

and so to make this vector zero we need to set $d_2 = d_1$. Hence, any vector of the form

$$\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}$$

where d_1 is a nonzero constant is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_2. Let us choose

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ and eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

which are such that $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_2 = 1$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1 = 0$. Since $\mathbf{A} = \mathbf{A}^T$, if we set

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

then we have that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

and

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V} e^{t\mathbf{\Lambda}} \mathbf{V}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} e^{-t} & \frac{1}{\sqrt{2}} e^t \\ -\frac{1}{\sqrt{2}} e^{-t} & \frac{1}{\sqrt{2}} e^t \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}. \end{aligned}$$

Hence,

$$\mathbf{x}(t) = e^{t\mathbf{A}} \mathbf{x}_0 = \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix}.$$

(b) [5 points] Since $\lambda_2 > 0$ and $\mathbf{v}_2 \cdot \mathbf{x}_0 = \frac{2}{\sqrt{2}} \neq 0$, $\|\mathbf{x}(t)\| \to \infty$ as $t \to \infty$.
