CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 42

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

42. [25 points]

Let

$$\Omega = \{(x,y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$

and let $f \in C(\Omega)$ be defined by $f(x,y) = x(1-y)$. In this question we will consider the problem of finding the solution $u(x,y)$ to the steady-state heat equation

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Let

$$C_D^2(\Omega) = \left\{ v \in C^2(\Omega) : v(x,0) = v(x,1) = v(0,y) = v(1,y) = 0,\, 0 \le x \le 1,\, 0 \le y \le 1 \right\}.$$

Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by

$$(Lv)(x,y) = -\left(v_{xx}(x,y) + v_{yy}(x,y)\right).$$

Let the inner product $(\cdot,\cdot) : C(\Omega) \times C(\Omega) \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 \int_0^1 v(x,y) w(x,y)\, dx\, dy.$$

(a) Show that L is symmetric by showing that

$$(Lv, w) = (v, Lw) \text{ for all } v, w \in C_D^2(\Omega).$$

(b) The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

for $j,k = 1,2,\ldots$, which are such that

$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$

for $j,k = 1,2,\ldots$. Obtain a formula for $\lambda_{j,k}$ for $j,k = 1,2,\ldots$.

(c) Let

$$C_D^2[0,1] = \left\{ v \in C^2[0,1] : v(0) = v(1) = 0 \right\}$$

and let the linear operator $L_1 : C_D^2[0,1] \to C[0,1]$ be defined by

$$L_1 w = -w''.$$

Use what you know about the eigenfunctions of L_1 to compute $(\psi_{j,k}, \psi_{m,n})$ for $j,k,m,n = 1,2,\ldots$.

(d) The solution to $Lu = f$ that we obtain using the spectral method is

$$u(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

Plot

$$u_N(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y)$$

for $N = 1,2,3,4,5,10$. Note that, for $j,k = 1,2,\ldots,$

$$(f,\psi_{j,k}) = 2\frac{(-1)^{j+1}}{jk\pi^2}.$$

Also note that to plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$ you could use

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x = linspace(0,1,50);
y = linspace(0,1,50);
[X,Y] = meshgrid(x,y);
psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,psi11)
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