CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 3

Posted Wednesday 10, September 2014. Due 5pm Wednesday 17, September 2014.

*Please write your name and **residential college** on your homework.*

1. [28 points: 7 points each]

   (a) Let B be defined as the matrix

$$B = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

   Using trigonometric identities, verify that the eigenvalues λ_i and eigenvectors v_i of B are

$$\lambda_i = 2\cos\left(\frac{i\pi}{N+1}\right), \qquad v_i = \begin{bmatrix} \sin\left(\frac{i\pi}{N+1}\right) \\ \sin\left(\frac{2i\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{(N-1)i\pi}{N+1}\right) \\ \sin\left(\frac{Ni\pi}{N+1}\right) \end{bmatrix}, \qquad i = 1, \dots N.$$

   (Note: some of you may remember this problem from CAAM 335, Spring 2014. This is intentional, and meant to give additional practice to those who did not enjoy the luxury of a semester-long CAAM excursion into matrix theory.)

   (b) For A defined as

$$A = \frac{\kappa}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix},$$

   show that A is positive-definite by showing $x^T Ax > 0$ for any nonzero vector x (hint: $x^T Ax = x^T(Ax)$, and terms should cancel).

   (c) Since A can be defined as

$$A = \frac{\kappa}{h^2}(2I - B),$$

   use part (a) to determine the eigenvalues of A in terms of κ, h, and N.

   (d) Show that, since A has an orthonormal eigenvector expansion

$$A = V\Lambda V^T$$

   where $V^T V = I$, that $x^T Ax > 0$ for any x implies that the eigenvalues $\lambda_i > 0$. Hint: choose x very specifically to show a single eigenvalue is positive.

2. [22 points: 11 points each] Consider the time-dependent heat equation with no source

$$\begin{aligned} \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= 0, \quad x \in (0,1) \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= \psi(x). \end{aligned}$$

Before we even try to solve this equation over time, it would be good to verify that this equation is *stable* in time — in other words, that $u(x,t)$ doesn't blow up as $t \to \infty$. This can be done by deriving an "energy estimate".

(a) Consider, as with the previous problem, discretizing using finite differences in space, but not in time. In other words, by specifying grid points x_i, and substituting in a finite difference approximation of $\frac{\partial^2 u(x_i,t)}{\partial x^2}$ in the heat equation gives, for $\vec{u}_i(t) = u_i(t)$,

$$\frac{d\vec{u}(t)}{dt} + A\vec{u}(t) = 0.$$

Multiply the entire equation on the left by $\vec{u}(t)^T$ to derive the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}(t)\|^2 + \vec{u}(t)^T A\vec{u}(t) = 0.$$

Use the fact that A is symmetric positive-definite (from Problem 1) to conclude that $\frac{d}{dt}\|\vec{u}(t)\|^2 < 0$ for all times t, and explain why this implies $\vec{u}(t)$ will not approach ∞ as $t \to \infty$.

Hint: to simplify $\vec{u(t)}^T \frac{d\vec{u(t)}}{dt} = \frac{d}{dt} u(t)^T u(t)$, write out the dot product in terms of

$$\vec{u(t)}^T \frac{d\vec{u(t)}}{dt} = u_1(t)\frac{du_1(t)}{dt} + u_2(t)\frac{du_2(t)}{dt} + \ldots + u_N(t)\frac{du_N(t)}{dt}$$

and use the fact that for a function $f(t)$,

$$\frac{df}{dt} f = \frac{1}{2}\frac{d(f^2)}{dt}.$$

(b) There is also an energy estimate that we can derive for the exact differential equation. Multiply the time-dependent heat equation by the solution $u(x,t)$ and integrate over x in the domain $(0,1)$ to get

$$\int_0^1 \left(\frac{\partial u}{\partial t} u - \kappa \frac{\partial^2 u}{\partial x^2} u\right) dx = 0$$

Using again the fact that for a function $f(t)$,

$$\frac{\partial f}{\partial t} f = \frac{1}{2}\frac{\partial (f^2)}{\partial t}$$

as well as integration by parts, derive the energy estimate

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 u^2 dx + \kappa \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx = 0.$$

If $\kappa > 0$, explain qualitatively why this statement implies that $u(t)$ will not approach ∞ as $t \to \infty$.

(Hint: the quantity

$$\int_0^1 u^2 dx$$

can be thought of as measuring the *size* of u - if u is really large in magnitude, then $\int_0^1 u^2 dx$ will also be large, since u^2 will be positive and the integral gives the measure of area under a curve.)

3. [50 points: 8 points for (a), 12 points for (c), 10 points for (b), (d), (e)] The 1D heat equation with $\kappa = 1$ over the interval $[0, 1]$ is given by

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

with boundary conditions and initial condition

$$\begin{aligned} u(0,t) = u(1,t) &= 0 \qquad t > 0, \\ u(x,0) &= \sin(\pi x). \end{aligned}$$

As we've seen in class, *centered* finite difference approximations are more accurate than both forwards/backwards difference approximations. To this end, we would like to find a way to leverage central differences for our approximation of the time derivative $\frac{\partial u}{\partial t}$.

The trick to doing so is to write down the finite difference equations in space and time at the point $(x_i, t_{j+1/2})$

$$\frac{\partial u}{\partial t}(x_i, t_{j+1/2}) = \frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1/2}).$$

We can then proceed in two steps:

- Central differences *in time* then gives us

$$\frac{\partial u}{\partial t}(x_i, t_{j+1/2}) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{dt}$$

  as an approximation for $\frac{\partial u(x_i, t_{j+1/2})}{\partial t}$, where $dt = t_{j+1} - t_j$ is time step.

- To approximate the term $\frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1/2})$ we can average our finite difference approximations for $\frac{\partial^2 u}{\partial x^2}(x_i, t_j + 1)$ and $\frac{\partial^2 u}{\partial x^2}(x_i, t_j)$: defining $u(x_i, t_j) = u_i^n$, we can set

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1/2}) \approx \frac{1}{2}\left[\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{h^2}\right].$$

  where $h = x_{i+1} - x_i$ is the grid spacing/mesh size in x.

Notice now that, if we combine the above two approximations, we no longer have any terms involving $t_{j+1/2}$! We have just defined the *Crank-Nicolson* scheme for u_i^j

$$\frac{u_i^{j+1} - u_i^j}{dt} = \frac{1}{2}\left[\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{h^2}\right]$$

Turn to the next page for the rest of Problem 3.

(a) We know that $\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}$ is an $O(\Delta t^2)$ approximation to $\frac{\partial u(x,t+\Delta t/2)}{\partial t}$. Show that

$$\frac{1}{2}\left[\frac{u(x+\Delta x,t+\Delta t)-2u(x,t+\Delta t)+u(x-\Delta x,t+\Delta t)}{\Delta x^2}+\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2}\right]$$

is an $O(\Delta x^2+\Delta t^2)$ approximation to $\frac{\partial^2 u(x,t+\frac{\Delta}{2})}{\partial x^2}$. With this, we can conclude Crank-Nicolson is a second order accurate approximation to the PDE in both space and time, or that

$$\left|\frac{\partial u(x,t+\Delta t/2)}{\partial t}-\frac{\partial^2 u(x,t+\Delta t/2)}{\partial x^2}-\text{Crank-Nicolson formula}\right|=O(\Delta t^2+\Delta x^2).$$

(b) Write the Crank-Nicolson scheme as an update step

$$\mathbf{u}^{j+1}=(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{B})\mathbf{u}^j,$$

specifying exactly what the matrices $\mathbf{A}$ and $\mathbf{B}$ are.

(c) As with any timestepping method, we can rewrite the Crank-Nicolson scheme as

$$\mathbf{u}^{j+1}=((\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{B}))^{j+1}\mathbf{u}^0.$$

Show that Crank-Nicolson scheme is unconditionally stable by showing that, for eigenvalues λ_i of $(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{B})$,

$$\lambda_i^j<\infty,\quad \text{for any } j>0.$$

(Hint: $\mathbf{I}+\mathbf{A}$ and $\mathbf{I}-\mathbf{B}$ should have the same eigenvectors.)

(d) Create a Matlab script that implements the Crank-Nicolson method. Compute the numerical solution at points x_i and times t_j and plot the computed solution values u_i^j for $i=0,...,N+1$ and $j=0,10,50$ where $N=8,16,32$.

(e) Given that $u(x,t)=e^{-\pi^2 t}\sin(\pi x)$ is the exact solution for the above problem, plot the error at each point $|u_{\text{exact}}(x_i,t_j)-u_i^j)|$, for $i=0,...,N+1$ and $j=0,10,50$ for $N=8,16,32$ for 3 successive time steps (use $dt=h$).