1. [25 points: 5 points each]

Let L be a linear operator on a vector space V and suppose that $L\psi = \lambda\psi$ with $\psi \neq 0$ so that λ is an eigenvalue of L with corresponding eigenfunction ψ.

(a) Show that the operator $\tilde{L}$ defined by

$$\tilde{L}u = Lu + \mu u \qquad u \in W, \qquad \mu \text{ a scalar}$$

is a linear operator on W and that ψ is an eigenfunction for $\tilde{L}$ corresponding to the eigenvalue $\tilde{\lambda} = \lambda + \mu$.

(b) Now consider the operator $\tilde{L}$

$$\tilde{L} : C_M^2[0;1] \to C[0;1]$$

where

$$C_M^2[0;1] = \{u \in C^2[0;1] : u'(0) = u(1) = 0\}.$$

with

$$\tilde{L}u(x) = \frac{-d^2u(x)}{dx^2} + u(x)$$

Show that $\tilde{L}$ is symmetric and positive definite, and calculate the eigenvalues $\tilde{\lambda}_n$ and corresponding eigenfunctions ψ_n of $\tilde{L}$ using part (a). Show that the eigenfunctions are orthogonal; i.e. that

$$(\psi_j, \psi_k) = \int_0^1 \psi_j(x)\psi_k(x)dx = 0, \quad j \neq k.$$

Hint: you may wish to use the trigonometric formula $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$

(c) Let $f(x)$ be a general function and $V_N = \text{span}\{\psi_1, \psi_2, \cdots, \psi_N\}$, where ψ_i were the eigenfunctions above. Explain how to compute the best approximation $f_N \in V_N$ to f. Explicitly give the integral formulas for the coeffcients needed for the best approximation. DO NOT evaluate those integrals, just give the formulas.

(d) Give the spectral method. solution $u(x)$ to the equation

$$\begin{aligned} \frac{-d^2u(x)}{dx^2} + u(x) &= f(x) \\ u'(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

(e) Explain (as specifically as possible) how to modify the solution $u(x)$ given in part (d) so in order to solve the problem with inhomogeneous boundary conditions.

$$\begin{aligned} \frac{-d^2u(x)}{dx^2} + u(x) &= f(x) \\ u'(0) &= 5 \\ u(1) &= 1 \end{aligned}$$

Solution.

(a) Suppose $u, v \in W$. We would like to show following equality is hold for any scalars α and β

$$\tilde{L}(\alpha u + \beta v) = \alpha \tilde{L}u + \beta \tilde{L}v$$

Given L is linear operator

$$\begin{aligned}
\tilde{L}(\alpha u + \beta v) &= L(\alpha u + \beta v) + \mu(\alpha u + \beta v) \\
&= \alpha L u + \beta L v + \mu \alpha u + \mu \beta v \\
&= \alpha L u + \mu \alpha u + \beta L v + \mu \beta v \\
&= \alpha (L u + \mu u) + \beta (L v + \mu v) \\
&= \alpha \tilde{L} u + \beta \tilde{L} v
\end{aligned}$$

as desired.

Now, let λ is an eigenvalue of L with an corresponding eigenfunction $v \in W$. Then $Lv = \lambda v$ with $v \neq 0$. For a scalar μ

$$\tilde{L}v = Lv + \mu v = \lambda v + \mu v = (\lambda + \mu)v \qquad \text{for } v \neq 0$$

That implies $\lambda + \mu$ is an eigenvalue of $\tilde{L}$ with corresponding eigenfunction v.

(b) Symmetry is apparent by integration by parts twice:

$$\begin{aligned}
\left(\tilde{L}u, v\right) &= \int_0^1 (-u'' + u)v \, dx \\
&= \int_0^1 -u''v + uv \, dx \\
&= \left[-u'v\right]_0^1 + \int_0^1 u'v' + uv \, dx && \text{integration by parts} \\
&= \int_0^1 u'v' + uv \, dx && \text{by boundary condition } u'(0) = 0 \text{ and } v(1) = 0 \\
&= \left[-uv'\right]_0^1 + \int_0^1 uv'' + uv \, dx && \text{integration by parts} \\
&= \int_0^1 uv'' + uv \, dx && \text{by boundary condition } u(1) = 0 \text{ and } v'(0) = 0 \\
&= \int_0^1 (v'' + v)u \, dx \\
&= \left(u, \tilde{L}v\right)
\end{aligned}$$

Positive definiteness requires $\left(\tilde{L}u, u\right) \geq 0$ and $\left(\tilde{L}u, u\right) = 0$ if and only if $u = 0$. Then

$$\begin{aligned}
\left(\tilde{L}u, u\right) &= \int_0^1 (-u'' + u)u \, dx \\
&= \int_0^1 -u''u + u^2 \, dx \\
&= \left[-u'u\right]_0^1 + \int_0^1 u'u' + u^2 \, dx \qquad \text{by integration by parts} \\
&= \int_0^1 (u')^2 + u^2 \, dx \qquad \text{by boundary condition } u'(0) = 0 \text{ and } u(1) = 0.
\end{aligned}$$

Since each integrand is non-negative, it follows $\left(\tilde{L}u, u\right) \geq 0$. To have $\left(\tilde{L}u, u\right) = 0$, we must have $u(x) = 0$ for all $x \in [0, 1]$, and $u'(x) = 0$ for all $x \in [0, 1]$, which is only possible if $u(x) = 0$ for all $x \in [0, 1]$, i.e., $u = 0$.

From part (a) we can conclude that if λ is an eigenvalue for $Lu = -u''$ then $\lambda + 1$ is an eigenvalue of $\tilde{L}$. Note that in that case $\mu = 1$. Moreover if v is an eigenfunction for L then it is also for $\tilde{L}$.

Therefore, let first find the eigenvalue of $Lu = -u''$.

$$\begin{aligned}
Lu = \lambda u &\to -u'' = \lambda u \\
&\to u'' + \lambda u = 0
\end{aligned}$$

Characteristic roots of that ODE are $\pm i\sqrt{\lambda}$. Then characteristic solution of the problem is

$$u(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Using the given boundary conditions we conclude that $c_2 = 0$ and $\cos(\sqrt{\lambda}) = 0$. Second equality implies

$$\lambda = \frac{(2n-1)^2}{4}\pi^2 \qquad \text{for} \quad n = 1, 2, \cdots$$

Then eigenvalue and corresponding eigenfunction of $\tilde{L}$ is

$$\begin{aligned}
\tilde{\lambda} &= \lambda + 1 = \frac{(2n-1)^2}{4}\pi^2 + 1 \\
\psi(x) &= \cos(\frac{2n-1}{2}\pi x) \quad \text{for} \quad n = 1, 2, \cdots
\end{aligned}$$

Finally we can show that the eigenfunctions of $\tilde{L}$ form a orthogonal set by showing

$$\begin{aligned}
(\psi_j, \psi_k) &= \int_0^1 \psi_j(x)\psi_k(x)\,dx, \quad j \neq k \\
&= \int_0^1 \cos(\frac{2j-1}{2}\pi x)\ \cos(\frac{2k-1}{2}\pi x)\,dx \\
&= \frac{1}{2}\int_0^1 \cos((j+k-1)\pi x) + \cos((j-k)\pi x)\,dx \\
&= \frac{1}{2}\left[\frac{1}{(j+k-1)\pi}\sin((j+k-1)\pi x) + \frac{1}{(j-k)\pi}\sin((j-k)\pi x)\right]_0^1 \\
&= 0 \qquad \text{for } i, j = 1, 2\cdots
\end{aligned}$$

There is also another elegant way to show eigenfunctions are orthogonal to each other. Let $\tilde{\lambda_1}$ and $\tilde{\lambda_2}$ two distinct eigenvalues of corresponding eigenfunction u and v.

$$\left(\tilde{L}u, v\right) = (\lambda_1 u, v) = \lambda_1 (u, v)$$

Moreover

$$\left(u, \tilde{L}v\right) = (u, \lambda_2 v) = \lambda_2 (u, v)$$

By symmetry we know $\left(\tilde{L}u, v\right) = \left(u, \tilde{L}v\right)$. This is possible if and only if $(u, v) = 0$.

(c) Best approximation f_N to f for a given eigenfunction $\psi_n = \cos(\frac{2n-1}{2}\pi x)$ for $n = 1, 2, ...$ can be computed as follow

$$f_N(x) = \sum_{n=1}^{N} c_n\psi_n = \sum_{n=1}^{\infty} \frac{(f, \psi_n)}{(\psi_n, \psi_n)}\psi_n$$

where

$$(f, \psi_n) = \int_0^1 f(x)\cos(\frac{2n-1}{2}\pi x)\,dx$$

and

$$(\psi_n, \psi_n) = \int_0^1 \cos^2(\frac{2n-1}{2}\pi x)\,dx$$

(d) Let $\tilde{u}$ be the spectral solution of the given BVP then

$$\tilde{u}(x) = \sum_{n=1}^{\infty} \frac{c_n}{\tilde{\lambda_n}}\psi_n$$

where c_n is given in part (c) and $\tilde{\lambda_n}$ in part (b).

(e) A function that satisfies the BC $u'(0) = 5$ and $u(1) = 1$ does not belong to $C^2_M[0; 1]$. To deal with it, we find a function $g(x)$ that does satisfy the BC's. Let $g(x) = 5x - 4$ satisfies this. Let $w(x) = u(x) - p(x)$. We then note that $w(x) = u(x) - p(x)$ belongs to $C^2_M[0; 1]$ and solves the following original differential equation with the right hand side modified:

$$\begin{aligned} \tilde{L}w &= \tilde{L}u - \tilde{L}g = f - \tilde{L}g = f - g \\ w'(0) &= u'(0) - g'(0) = 0 \\ w(1) &= u(1) - g(1) = 0 \end{aligned}$$

or

$$\begin{aligned} \frac{-d^2w(x)}{dx^2} + w(x) &= f(x) - g(x) \\ w'(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Then, we solve for w as above, with $f(x)$ replaced by $f(x) - g(x) = f(x) - (5x - 4)$ Our final solution will be $u(x) = w(x) + g(x)$.

2. [25 points: 5 points each]

   (a) Consider the boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$
$$u'(0) = 0$$
$$u'(1) = 0.$$

   If we define $Lu = -\frac{\partial^2 u}{\partial x^2}$ and the space $C^2_N[0,1]$

$$C^2_N[0,1] = \{u \in C^2[0,1], u'(0) = u'(1) = 0\},$$

   this can be written as an operator equation

$$Lu = f, \quad L : C^2_N[0,1] \to C[0,1].$$

   Explain why L is not positive-definite.

   (b) Derive (do not just show) that the eigenfunctions $\phi_j(x)$ and corresponding eigenvalues λ_j of the above operator equation are

$$\phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2.$$

   Specify the values of j for which these formulas hold. Describe what problems arise with the use of the spectral method for the above problem.

   (c) Earlier in the semester, we showed that a solution only exists to the above problem if $\int_0^1 f(x)dx = 0$. Let $u(x)$ be a spectral method solution to $Lu = f$ for some source function $f(x)$. Assume that

$$\int_0^1 L\phi_j(x) = 0$$

   for any eigenfunction $\phi_j(x)$, and explain why this implies that $\int_0^1 f(x)dx = 0$.

   (d) The above problem is also non-unique: for any solution $u(x)$, $u(x) + C$ is also a solution for constant C. One way to make the solution unique is to add a condition where the average of $u(x)$ is zero:

$$\int_0^1 u(x)dx = 0.$$

   To this end, we can redefine our operator equation

$$L_A u = -\frac{\partial^2 u}{\partial x^2}, \quad L_A u = f, \quad L_A : C^2_A[0,1] \to C[0,1].$$

   where $C^2_A[0,1]$ contains functions in $C^2_N[0,1]$ with zero average

$$C^2_A[0,1] = \left\{u \in C^2[0,1], u'(0) = u'(1) = 0, \int_0^1 u(x)dx = 0\right\}.$$

   Show that L_A is positive definite.

(e) Determine eigenfunctions and eigenvalues for the operator L_A, and give an expression for the spectral method solution for the above operator equation.

Solution.

(a) Note that for any constant $C \neq 0$, $\frac{\partial^2 C}{\partial x^2} = 0$. Additionally, $C \in C_N^2([0,1])$. Thus, we can conclude that $LC = 0$. As a result,

$$(LC, C) = (0, C) = 0.$$

This shows L is not positive definite, because otherwise $(Lu, u) = 0$ would imply $u = 0$.

(b) If $L\phi_j = \lambda\phi_j$, then $-\phi_j'' = \lambda\phi_j$, which implies that $\phi_j(x)$ should have the form

$$\phi_j(x) = A\sin(\sqrt{\lambda_j}x) + B\cos(\sqrt{\lambda_j}x)$$

Then,

$$\phi_j'(x) = A\sqrt{\lambda_j}\cos(\sqrt{\lambda_j}x) - B\sqrt{\lambda_j}\sin(\sqrt{\lambda_j}x).$$

The boundary condition $\phi_j'(0) = 0$ then implies that $A = 0$. Likewise, the boundary condition $\phi_j'(1) = 0$ implies that

$$\phi_j'(1) = B\sqrt{\lambda_j}\sin(\sqrt{\lambda_j}x) = 0$$

so that $\sqrt{\lambda_j} = j\pi$, and

$$\phi_j(x) = \cos(j\pi x), \qquad \lambda_j = j\pi^2.$$

The above formula holds for $j = 0, 1, 2, \ldots$, since for $j = 0$, $\phi_j = 1$, which is an eigenfunction of L with eigenvalue $\lambda_0 = 0$. This causes problems with the spectral method — the spectral method gives the solution

$$u(x) = \sum_{j=0}^{\infty} \frac{(f, \phi_j)}{\lambda_j(\phi_j, \phi_j)}\phi_j(x).$$

When $j = 0$, $\lambda_j = 0$, and we end up dividing by zero.

(c) You can show that

$$\int_0^1 L\phi_j(x) = \int_0^1 (j\pi)^2\cos(j\pi x) = (j\pi)^2\left[\frac{\sin(j\pi x)}{j\pi}\right]_0^1 = 0, \quad j \neq 0.$$

If $j = 0$, then $L\phi_0 = 0$, since $\phi_0(x) = 1$. Thus, $\int_0^1 L\phi_j(x) = 0$ for all j. Then, if you assume that there is a solution $u(x)$ to the equation $Lu = f$,

$$\int_0^1 f(x) = \int_0^1 Lu = \int_0^1 \sum_{j=0}^{\infty}\alpha_j L\phi_j(x) = \sum_{j=0}^{\infty}\alpha_j\int_0^1 L\phi_j(x) = 0.$$

(d) If we redefine $L_A : C_A^2[0,1] \to C[0,1]$, then we can show it is positive definite. First note that, by integration by parts, we get

$$(L_A u, u) = \int_0^1 L_A uu = \int_0^1 -u''(x)u(x) = [-u'(x)u(x)]_0^1 + \int_0^1 u'(x)^2 = \int_0^1 u'(x)^2 \geq 0.$$

Thus, we only have to show now that $(L_A u, u) = 0$ implies $u = 0$.

$$(L_A u, u) = 0 \to \int_0^1 u'(x)^2 = 0$$

which implies $u'(x) = 0$, so that $u(x) = C$ for some constant C. However, since we require $u \in C_A^2[0,1]$, $\int_0^1 u = \int_0^1 C = C = 0$, implying that for $(L_A u, u) = 0$, $u = 0$.

(e) The eigenfunctions and eigenvalues of L_A are identical to those derived in part (b); the only difference is that $j = 1, 2, \ldots$ instead of $j = 0, 1, \ldots$.

$$\phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2, \qquad j = 1, 2, \ldots.$$

3. [25 points: 5 points each]

For this problem, we will consider the behavior of the finite element method with various boundary conditions.

(a) Consider the boundary value problem

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} &= f(x), \quad 0 < x < 1 \\
u'(0) &= \alpha \\
u(1) &= \beta.
\end{aligned}
$$

We set $u(x) = w(x) + g(x)$, where $w(1) = 0$, and $g(1) = \beta$. Let $C_R^2([0,1]) = \{v \in C^2[0,1], v(1) = 0\}$; derive the weak form of the above equation

$$a(w, v) = (f, v) + (???), \quad \forall v \in C_R^2([0,1])$$

where (???) depends explicitly on $v, \alpha, \beta, a(\cdot,\cdot)$ and $g(x)$.

(b) Given some spacing h and points $x_j = jh$, we will use hat functions

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & x_{j-1} \le x < x_j \\ \frac{x_{j+1} - x}{h} & x_j \le x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

and define $V_N = \text{span}\{\phi_0, \dots, \phi_N\}$. Note that

$$\phi_0(x) = \begin{cases} \frac{h-x}{h} & 0 \le 0 < h \\ 0 & \text{otherwise.} \end{cases}$$

The finite element method replaces $C_R^2([0,1])$ with V_N and solves for $w_N \in V_N$ such that

$$a(w_N, \phi_i) = (f, \phi_i) + (???), \quad i = 0, \dots, N.$$

Specify what $g(x)$ you chose, and write out explicitly the $(N+1) \times (N+1)$ matrix system $K\alpha = b$ resulting from the above equations, where

$$K_{ij} = a(\phi_{j-1}, \phi_{i-1}), \quad b_i = (f, \phi_{i-1}) + (???), \qquad 1 \le i, j \le N$$

Notice we have $K_{ij} = a(\phi_{j-1}, \phi_{i-1})$ because $K_{11} = a(\phi_0, \phi_0)$ (we've started counting basis functions $\phi_j(x)$ from $j = 0$ instead of $j = 1$). Specify the values of K_{ij} and b_i explicitly in terms of problem parameters $h, f(x)$ and α, β.

Turn to next page for the rest of the problem.

(c) Consider now the boundary value problem

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} &= f(x), \quad 0 < x < 1 \\
\alpha u(0) - u'(0) &= 0 \\
u'(1) &= 0.
\end{aligned}
$$

Derive the weak form $a(u, v) = (f, v)$ of the above equation by multiplying by a function $v(x) \in C^2[0, 1]$ and integrating by parts.

(d) Let $V_N = \text{span}\{\phi_0, \ldots, \phi_{N+1}\}$, where ϕ_{N+1} is

$$
\phi_{N+1}(x) = \begin{cases} \frac{x-(1-h)}{h} & 1-h \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Write down explicitly the $(N+2) \times (N+2)$ finite element stiffness matrix equation $K\alpha = b$, where

$$
K_{ij} = a(\phi_{j-1}, \phi_{i-1}), \quad b_i = (f, \phi_{i-1}) + (???), \qquad 1 \le i, j \le N+2
$$

Specify the values of K_{ij} and b_i explicitly in terms of $h, f(x), \alpha$.

(e) For what values of α is the matrix K guaranteed to be positive definite? *Hint: use the relationship between the matrix K and the weak form.*

Solution.

(a) Multiply the differential equation with some function $v \in C_R^2([0, 1])$ and integrate from $x = 0$ to $x = 1$ to obtain

$$
\int_0^1 -\frac{\partial^2 u}{\partial x^2} v(x)\, dx = \int_0^1 f(x) v(x)\, dx, \quad 0 < x < 1
$$

Integrate the left hand side by parts to obtain

$$
\left[-\frac{\partial u}{\partial x} v(x)\right]_0^1 + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx = \int_0^1 f(x) v(x)\, dx
$$

$$
\frac{\partial u(0)}{\partial x} v(0) + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx = \int_0^1 f(x) v(x)\, dx \qquad \text{use BC } \tfrac{\partial u(0)}{\partial x} = \alpha
$$

$$
\int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx = \int_0^1 f(x) v(x)\, dx - \alpha v(0)
$$

The first boundary integral at the point 1 disappears because of the boundary conditions $v(1) = 0$. The weak problem:

$$\text{Find } \ u \in C_R^2([0,1]) \ \text{ such that } \ a(u,v) = (f,v) - \alpha v(0) \qquad \text{for all } \ v \in C_R^2([0,1])$$

With the given setting we are looking for a solution $u(x) = w(x) + g(x)$, where $w(1) = 0$, and $g(1) = \beta$. Then problem becomes

$$\text{Find } \ w \in C_R^2([0,1]) \ \text{ such that } \ a(w+g,v) = (f,v) - \alpha v(0) \qquad \text{for all } \ v \in C_R^2([0,1])$$

or equivalently

$$\text{Find } w \in C_R^2([0,1]) \text{ such that } a(w,v) = (f,v) - a(g,v) - \alpha v(0) \qquad \text{for all } v \in C_R^2([0,1])$$

(b) The Galerkin problem is

$$\text{Find } \ w_N \in V_N \ \text{ such that } \ a(w_N,v) = (f,v) - a(g_N,v) - \alpha v(0) \qquad \text{for all } \ v \in V_N$$

We chose g_n satisfy $g_N(1) = \beta$. The simplest such function is $g_N(x) = \beta\phi_{N+1}$. Substituting $w_N(x) = \sum_{i=0}^{N} w_i\phi_i$, choosing $v \in V_N = span\{\phi_0, \phi_1, \cdots \phi_N\}$ we obtain

$$\sum_{j=0}^{N} a(\phi_i,\phi_j)w_i = (f,\phi_i) - a(g_N,\phi_i) - \alpha\phi_i(0) \quad \text{for } \ i = 0,1,2,\cdots N$$

Then right hand side becomes

$$\sum_{j=0}^{N} a(\phi_i,\phi_j)w_i = (f,\phi_i) - \beta a(\phi_{N+1},\phi_i) - \alpha\phi_i(0) \quad \text{for } \ i = 0,1,2,\cdots N$$

where

$$K_{ij} = a(\phi_{j-1},\phi_{i-1}), \quad b_i = (f,\phi_{i-1}) - \beta a(\phi_{N+1},\phi_{i-1}) - \alpha\phi_{i-1}(0), \qquad 1 \le i,j \le N+1$$

More specific

$$b_i = \begin{cases} (f,\phi_{i-1}) & \text{if } \ i = 2,\cdots,N \\ (f,\phi_0) - \alpha\phi_0(0) & \text{if } \ i = 1, \\ (f,\phi_N) - \beta a(\phi_{N+1},\phi_N) & \text{if } \ i = N+1 \end{cases}$$

Now, specify the values of K_{ij} and b_i

When the hat functions ϕ_i and ϕ_j are not neighbors, i.e., $|i-j| > 1$,

$$\int_0^1 \phi_i(x)\phi_j(x)\,dx = 0 \ \text{ and } \ \int_0^1 \frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} = 0$$

Thus we only need to consider the following cases:

$$
\begin{aligned}
K_{ii} = a(\phi_{i-1}, \phi_{i-1}) &= \int_0^1 \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_{i-1}}{\partial x}\, dx \\
&= \int_{x_{i-2}}^{x_i} \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_{i-1}}{\partial x}\, dx \\
&= \int_{x_{i-2}}^{x_{i-1}} \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_{i-1}}{\partial x}\, dx + \int_{x_{i-1}}^{x_i} \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_{i-1}}{\partial x}\, dx \\
&= \int_{x_{i-2}}^{x_{i-1}} (\frac{1}{h})^2\, dx + \int_{x_{i-1}}^{x_i} (\frac{-1}{h})^2\, dx \\
&= \frac{2}{h} \qquad \text{for } \ i = 2, \cdots N+1
\end{aligned}
$$

Since $[x_{i-1}, x_i]$ is the only interval where ϕ_{i-1} and ϕ_i and their derivative are not 0 we have

$$
\begin{aligned}
K_{i(i+1)} = a(\phi_{i-1}, \phi_i) &= \int_0^1 \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_i}{\partial x}\, dx \\
&= \int_{x_{i-1}}^{x_i} \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_i}{\partial x}\, dx \\
&= \int_{x_i}^{x_{i+1}} (-\frac{1}{h})(\frac{1}{h})\, dx \\
&= -\frac{1}{h} \qquad \text{for } \ i = 1, \cdots N+1
\end{aligned}
$$

By symmetry $K_{i(i+1)} = K_{(i+1)i}$ for $i = 1, \cdots N+1$.
Finally,

$$
\begin{aligned}
K_{11} = a(\phi_0, \phi_0) &= \int_0^1 \frac{\partial \phi_0}{\partial x}\frac{\partial \phi_0}{\partial x}\, dx \\
&= \int_{x_0}^{x_1} \frac{\partial \phi_0}{\partial x}\frac{\partial \phi_0}{\partial x}\, dx \\
&= \int_0^h (-\frac{1}{h})(-\frac{1}{h})\, dx \\
&= \frac{1}{h} \qquad \text{for } \ i = 1
\end{aligned}
$$

Therefore $(N+1) \times (N+1)$ matrix K

$$
K = \frac{1}{h}\begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 \\
\vdots & & & \ddots & & \vdots \\
0 & \cdots & & -1 & 2 & -1 \\
0 & & \cdots & & -1 & 2
\end{bmatrix}
$$

So b_i take form

$$b_i = \begin{cases} (f, \phi_{i-1}) = \int_0^1 f(x)\phi_{i-1}(x)\,dx & \text{if } i = 2, \cdots, N \\ \\ (f, \phi_0) - \alpha = \int_0^1 f(x)\phi_0(x)\,dx - \alpha & \text{if } i = 1, \\ \\ (f, \phi_N) - \beta a(\phi_{N+1}, \phi_N) = \int_0^1 f(x)\phi_N(x)\,dx + \frac{1}{h}\beta & \text{if } i = N+1 \end{cases}$$

(c) Multiply the differential equation with some function $v \in C^2([0,1])$ and integrate from $x = 0$ to $x = 1$ to obtain

$$\int_0^1 -\frac{\partial^2 u}{\partial x^2} v(x)\,dx = \int_0^1 f(x)v(x)\,dx, \quad 0 < x < 1$$

Integrate the left hand side by parts to obtain

$$\left[-\frac{\partial u}{\partial x} v(x)\right]_0^1 + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx = \int_0^1 f(x)v(x)\,dx$$

$$-\frac{\partial u(1)}{\partial x} v(1) + \frac{\partial u(0)}{\partial x} v(0) + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx = \int_0^1 f(x)v(x)\,dx$$

Given boundary conditions yields $u'(0) = \alpha u(0)$ and $u'(1) = 0$. Then

$$\begin{aligned} -\frac{\partial u(1)}{\partial x} v(1) + \frac{\partial u(0)}{\partial x} v(0) + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx &= \int_0^1 f(x)v(x)\,dx \\ \alpha u(0)v(0) + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx &= \int_0^1 f(x)v(x)\,dx \quad \text{for all } v \in C^2[0,1] \end{aligned}$$

Then

$$a(u, v) = \alpha u(0)v(0) + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx, \quad (f, v) = \int_0^1 f(x)v(x)\,dx, \quad \text{for all } v \in C^2[0,1]$$

(d) The Galerkin problem is

$$\text{Find } u_N \in V_N \text{ such that } a(u_N, v) = (f, v) \qquad \text{for all } v \in V_N$$

Let u_N be finite element solution of given BVP in part (c). Then $u_N(x) = \sum_{i=0}^{N+1} c_i\phi_i(x)$. Moreover, $v \in V_N = span\{\phi_0, \phi_1, \cdots \phi_{N+1}\}$. Therefore

$$a(u_N, v) = \alpha c_0\phi_0(0)\phi_0(0) + \sum_{j=0}^{N+1} a(\phi_i, \phi_j)c_j, \quad (f, \phi_i) = \int_0^1 f(x)\phi_i(x)\,dx, \quad \text{for} i = 0, 1, \cdots, N+1$$

Note that $\phi_0(0) = 1$. Last equation becomes

$$a(u_N, v) = \alpha c_0 + \sum_{j=0}^{N+1} a(\phi_i, \phi_j) c_j, \quad (f, \phi_i) = \int_0^1 f(x)\phi_i(x)\, dx, \quad \text{for} \quad i = 0, 1, \cdots N+1$$

Where $K_{ij} = a(\phi_i, \phi_j)$. However we should be careful at this point. Because of new boundary conditions bilinear form includes the term αc_0. When we construct our stiffness matrix K because of the coefficient c_0, the matrix entry K_{11} will be

$$K_{11} = \alpha + a(\phi_0, \phi_0) = \alpha + \int_0^1 \frac{\partial \phi_0}{\partial x}\frac{\partial \phi_0}{\partial x}\, dx = \alpha + \frac{1}{h}$$

By part (b) we know that $K_{ii} = \frac{2}{h}$ and $K_{i(i+1)} = K_{(i+1)i} = \frac{-1}{h}$ for $i = 2, \cdots N+1$. Also for $i = N+2$

$$K_{(N+2)(N+2)} = a(\phi_{N+1}, \phi_{N+1}) = \int_0^1 \frac{\partial \phi_{N+1}}{\partial x}\frac{\partial \phi_{N+1}}{\partial x}\, dx = \frac{1}{h}$$

Therefore

$$K = \frac{1}{h}\begin{bmatrix} \alpha h + 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & & \cdots & & -1 & 1 \end{bmatrix}$$

Finally, the right hand side vector b_i

$$b_i = (f, \phi_{i-1}) = \int_0^1 f(x)\phi_{i-1}(x)\, dx \quad \text{if } i = 1, \cdots, N+2$$

(e) Let $0 \neq x \in R^{n+2}$

$$\begin{aligned} x^T K x &= \sum_{i=1}^{n+2}\sum_{j=1}^{n+2} x_i K_{ij} x_j \\ &= x_1(\alpha + a(\phi_0, \phi_0))x_1 + \sum_{i=2}^{n+2}\sum_{j=2}^{n+2} x_i a(\phi_{i-1}, \phi_{j-1}) x_j \\ &= (x_1^2 \alpha + a(x_1\phi_0, x_1\phi_0)) + a(\sum_{i=2}^{n+2} x_i \phi_{i-1}, \sum_{j=2}^{n+2} x_j \phi_{j-1}) \end{aligned}$$

Since $x \neq 0$ and ϕ_i is linearly independent

$$\sum_{i=2}^{n+2} x_i \phi_{i-1} \neq 0$$

Thus

$$a(\sum_{i=2}^{n+2} x_i\phi_{i-1}, \sum_{j=2}^{n+2} x_j\phi_{j-1}) > 0 \text{ and } a(x_1\phi_0, x_1\phi_0) > 0$$

Only term is left $x_1^2\alpha$. If $\alpha > 0$, therefore $x^T K x > 0$ for any $x \neq 0$. Thus K is positive definite.

4. [25 points: (a) = 8, (b) = 9, (c) = 8]

(a) Consider the differential equation with $k(x) > 0$

$$-\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x), \quad 0 < x < 1$$
$$u(0) = 0$$
$$k(1)u'(1) = \beta.$$

Derive the weak form of the above PDE

$$a(u,v) = \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx = \int_0^1 f(x)v(x)dx + (???), \quad \forall v \in C_L^2([0,1]).$$

Give an explicit expression for (???) that depends on β.

(b) An additional advantage of solving the weak form of a differential equation is that it allows for the diffusivity $k(x)$ to be discontinuous (which can happen when two different bars are joined together). This is not technically possible with the strong form of the equation, since $k(x)\frac{\partial u}{\partial x}$ might no longer be differentiable.

Define $k(x)$ to be the discontinuous function

$$k(x) = \begin{cases} k_1 & 0 < x < \frac{1}{2} \\ k_2 & \frac{1}{2} < x < 1, \end{cases}$$

Let us define $h = 1/4$ and define the points

$$x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad x_4 = 1.$$

and span of piecewise linear hat functions $V_N = \text{span}\{\phi_1, \phi_2, \phi_3, \phi_4\}$ where

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & x_{j-1} \le x < x_j \\ \frac{x_{j+1} - x}{h} & x_j \le x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi_4(x) = \begin{cases} \frac{x - x_3}{h} & x_3 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $k_1 = 1$, $k_2 = 2$, construct the 4×4 finite element system $K\alpha = b$, where

$$K_{ij} = a(\phi_j, \phi_i), \quad b_i = (f, \phi_i), \qquad 1 \le i, j \le 4.$$

Write out all expressions in terms of h and β.

Turn to next page for the rest of the problem.

(c) You may ask yourself: "what does it mean for $k(x)$ to be discontinuous?" This part aims to show that it is equivalent to solving two different differential equations that are coupled together by boundary conditions enforcing that both temperature and heat flux are continuous.

Consider the two differential equations with variables $u_1(x)$, $u_2(x)$

$$-k_1 \frac{\partial^2 u_1}{\partial x^2} = f(x), \quad 0 < x < \frac{1}{2}$$
$$-k_2 \frac{\partial^2 u_2}{\partial x^2} = f(x), \quad \frac{1}{2} < x < 1.$$

with boundary conditions

$$u_1(0) = 0$$
$$k_1 u_1'\left(\frac{1}{2}\right) = k_2 u_2'\left(\frac{1}{2}\right).$$

and

$$u_1\left(\frac{1}{2}\right) = u_2\left(\frac{1}{2}\right)$$
$$k_2 u_2'(1) = \beta.$$

We can define two weak forms for the above problems

$$a_1(u_1, v) = \int_0^{1/2} f(x)v(x)$$
$$a_2(u_2, v) = \int_{1/2}^1 f(x)v(x) + \beta v(1).$$

Specify what $a_1(u_1, v)$ and $a_2(u_2, v)$ are, and show that, if we define $u(x)$ piecewise

$$u(x) = \begin{cases} u_1(x), & 0 < x < \frac{1}{2} \\ u_2(x), & \frac{1}{2} \leq x < 1. \end{cases}$$

that $a_1(u_1, v) + a_2(u_2, v) = a(u, v)$.

Solution.

(a) To derive the weak form, we multiply by a test function $v(x)$ such that $v(0) = 0$

$$\int_0^1 -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) v(x) = \int_0^1 f(x)v(x).$$

Integrating the first term by parts gives

$$\int_0^1 -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) v(x) = \left[-k(x)\frac{\partial u}{\partial x}(x)v(x)\right] + \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}.$$

Noting that $v(0) = 0$ and $k(1)\frac{\partial(u}{\partial x(}1) = \beta$, we can simplify the above to

$$-\beta v(1) + \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = \int_0^1 f(x)v(x).$$

Rearranging the unknown solution $u(x)$ on one side gives

$$\int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = \int_0^1 f(x)v(x) + \beta v(1).$$

(b) To compute the finite element system, we note that

$$a(u,v) = \int_0^1 k(x)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}.$$

Assuming that $u, v \in \text{span}\{\phi_1, \ldots, \phi_4\}$, the entries of the stiffness matrix are $K_{ij} = a(\phi_j, \phi_i)$. We note that $K_{ij} = 0$ if $|i-j| > 1$, and that $K_{ij} = K_{ji}$ by symmetry. Using the fact that $k(x) = 1$ for $x < 1/2$ and $k(x) = 2$ for $x \geq 1/2$, we can compute explicitly all entries of K.

For $i = 1$, $K_{ij} \neq 0$ for $j = 1, 2$. Note that $k(x) = 1$ for these integrals.

$$K_{11} = \int_0^1 k(x)\phi_1'(x)\phi_1'(x) = \int_0^{x_2} k_1\frac{1}{h^2} = k_1\frac{1}{h^2}\int_0^{2h} = k_1\frac{2}{h}$$

$$K_{21} = \int_0^1 k(x)\phi_2'(x)\phi_1'(x) = \int_{x_1}^{x_2} k_1\frac{-1}{h}\frac{1}{h} = k_1\frac{-1}{h^2}\int_h^{2h} = -k_1\frac{1}{h}.$$

For $i = 2$, $K_{ij} \neq 0$ for $j = 1, 2, 3$. Here, we have to be a little more careful, since the integral goes from $x_1 = 1/4$ to $x_3 = 3/4$, and $k(x)$ has a jump over this interval. $K_{12} = K_{21}$ by symmetry, so we only need to compute K_{22} and K_{32}

$$K_{22} = \int_0^1 k(x)\phi_2'(x)\phi_2'(x) = \int_{x_1}^{x_3} k(x)\frac{1}{h}\frac{1}{h} = \frac{1}{h^2}\int_{1/4}^{3/4} k(x)$$
$$= \frac{1}{h^2}\left(\int_{1/4}^{1/2} k_1 + \int_{1/2}^{3/4} k_2\right) = (k_1 + k_2)\frac{1}{h}$$

$$K_{32} = \int_0^1 k(x)\phi_3'(x)\phi_2'(x) = \int_{x_2}^{x_3} k(x)\frac{-1}{h}\frac{1}{h} = \frac{-1}{h^2}\int_{1/2}^{3/4} k_2 = -k_2\frac{1}{h}.$$

For $i = 3$, we have roughly the same case as with $i = 1$ since $k(x) = k_2$ for all the integrals in this problem.

$$K_{33} = \int_0^1 k(x)\phi_3'(x)\phi_3'(x) = \int_{x_2}^{x_4} k_2\frac{1}{h^2} = k_2\frac{1}{h^2}\int_0^{2h} = k_2\frac{2}{h}$$

$$K_{34} = \int_0^1 k(x)\phi_4'(x)\phi_3'(x) = \int_{x_3}^{x_4} k_2\frac{-1}{h}\frac{1}{h} = k_2\frac{-1}{h}.$$

Finally, for $i = 4$, we have

$$K_{44} = \int_0^1 k(x)\phi_4'(x)\phi_4'(x) = \int_{x_3}^{x_4} k_2\frac{1}{h^2} = k_2\frac{1}{h}.$$

For $b_i = (f, \phi_i) + \beta\phi_i(1)$, we have (due to the fact that $\phi_i(1) = 0$ unless $i = 4$, in which case $\phi_4(1) = 1$)

$$b_i = \begin{cases} (f, \phi_i), & i < 4 \\ (f, \phi_4) + \beta, & i = 4. \end{cases}$$

(c) If we multiply both equations by a test function $v(x)$ and integrate over their respective domains, we get

$$\int_0^{1/2} -k_1 \frac{\partial^2 u_1}{\partial x^2} v(x) = \int_0^{1/2} f(x)v(x) \int_{1/2}^1 -k_2 \frac{\partial^2 u_2}{\partial x^2} v(x) \qquad = \int_{1/2}^1 f(x)v(x).$$

Integrating by parts gives

$$\begin{aligned} \int_0^{1/2} -k_1 \frac{\partial^2 u_1}{\partial x^2} v(x) &= \left[-k_1 \frac{\partial u_1}{\partial x} v(x) \right]_0^{1/2} + \int_0^{1/2} k_1 \frac{\partial u_1}{\partial x} \frac{\partial v}{\partial x} \\ \int_0^{1/2} -k_2 \frac{\partial^2 u_2}{\partial x^2} v(x) &= \left[-k_2 \frac{\partial u_2}{\partial x} v(x) \right]_{1/2}^{1} + \int_{1/2}^1 k_2 \frac{\partial u_2}{\partial x} \frac{\partial v}{\partial x}. \end{aligned}$$

Using the boundary conditions, we can reduce the above to

$$\begin{aligned} \left[-k_1 \frac{\partial u_1}{\partial x} v(x) \right]_0^{1/2} + \int_0^{1/2} k_1 \frac{\partial u_1}{\partial x} \frac{\partial v}{\partial x} &= -k_1 \frac{\partial u_1}{\partial x}(1/2)v(1/2) + \int_0^{1/2} k_1 \frac{\partial u_1}{\partial x} \frac{\partial v}{\partial x} \\ \left[-k_2 \frac{\partial u_2}{\partial x} v(x) \right]_{1/2}^{1} + \int_{1/2}^1 k_2 \frac{\partial u_2}{\partial x} \frac{\partial v}{\partial x} &= -k_2 \frac{\partial u_2}{\partial x}(1)v(1) + k_2 \frac{\partial u_2}{\partial x}(1/2)v(1/2) + \int_{1/2}^1 k_2 \frac{\partial u_2}{\partial x} \frac{\partial v}{\partial x} \\ &= -\beta v(1) + k_2 \frac{\partial u_2}{\partial x}(1/2)v(1/2) + \int_{1/2}^1 k_2 \frac{\partial u_2}{\partial x} \frac{\partial v}{\partial x}. \end{aligned}$$

We can then adding these two above equations together. Note that the boundary terms at $x = 1/2$ cancel out due to boundary conditions

$$k_2 \frac{\partial u_2}{\partial x}(1/2)v(1/2) - k_1 \frac{\partial u_1}{\partial x}(1/2)v(1/2) = 0.$$

As a result, summing the two bilinear forms and their right hand sides together gives

$$\int_0^{1/2} k_1 \frac{\partial u_1}{\partial x} \frac{\partial v}{\partial x} + \int_{1/2}^1 k_2 \frac{\partial u_2}{\partial x} \frac{\partial v}{\partial x} - \beta v(1) = \int_0^{1/2} f(x)v(x) + \int_{1/2}^1 f(x)v(x).$$