1. Let us consider the problem of finding $\boldsymbol{x}(t)$ such that

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

and

$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

where the vector $\boldsymbol{x}_0 \in \mathbb{R}^N$ and the symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$. Let $\Delta t > 0$ and let $t_k = k\Delta t$ for non-negative integers k. We can obtain approximations $\boldsymbol{x}_k$ to $\boldsymbol{x}(t_k)$ by using the formula

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t(1-\theta)\boldsymbol{A}\boldsymbol{x}_k + \Delta t\theta\boldsymbol{A}\boldsymbol{x}_{k+1}$$

for some choice of $\theta \in [0,1]$.

(a) What linear system of equations has to be solved to compute $\boldsymbol{x}_{k+1}$? Your answer should not feature the inverse of any matrices.

(b) In order to analyze the behavior of this method it will be convenient to first write

$$\boldsymbol{x}_k = \boldsymbol{B}^k\boldsymbol{x}_0$$

where $\boldsymbol{B} \in \mathbb{R}^{N\times N}$. What matrix is $\boldsymbol{B}$?

(c) Let the matrix $\boldsymbol{A}$ have eigenvalues $\lambda_1, \ldots, \lambda_N$ with corresponding orthonormal eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$. The matrix $\boldsymbol{B}$ has the same eigenvectors as $\boldsymbol{A}$. What are the eigenvalues of $\boldsymbol{B}$?

(d) We can write

$$\boldsymbol{x}_0 = \sum_{j=1}^{N}(\boldsymbol{x}_0, \boldsymbol{v}_j)\boldsymbol{v}_j$$

where $(\boldsymbol{x}_0, \boldsymbol{v}_j) = \boldsymbol{v}_j^T\boldsymbol{x}_0$. We can also write

$$\boldsymbol{x}_k = \sum_{j=1}^{N} c_j\boldsymbol{v}_j$$

where the coefficients $c_j \in \mathbb{R}$. Use the fact that $\boldsymbol{x}_k = \boldsymbol{B}^k\boldsymbol{x}_0$ to obtain a formula for the coefficients c_j.

(e) If all of the eigenvalues of $\boldsymbol{A}$ are positive and $\theta > 0$, why should the choice of $\Delta t = \frac{1}{\theta\lambda_j}$ be avoided for all $j = 1, \ldots, N$?

(f) If all of the eigenvalues of $\boldsymbol{A}$ are negative and $0 \le \theta < \frac{1}{2}$, what restriction should be placed on Δt so that $\|\boldsymbol{x}_k\| \to 0$ as $k \to \infty$ for all $\boldsymbol{x}_0$?

(g) If all of the eigenvalues of $\boldsymbol{A}$ are negative and $\frac{1}{2} < \theta < 1$, how will $\|\boldsymbol{x}_k\|$ behave as $k \to \infty$?

(h) For certain choices of θ the method in this problem is actually a method that we have looked at previously in this course. Which methods that we have looked at previously do the choices of $\theta = 0$, $\theta = \frac{1}{2}$ and $\theta = 1$ correspond to?

2. Let N be a positive integer and let $h = \frac{1}{N+1}$ and $x_k = kh$. Let

$$V_N = \operatorname{span}\{\phi_0, \phi_1, ..., \phi_{N+1}\}$$

where the ϕ_j are the usual piecewise linear hat functions which are such that

$$\phi_j(x_i) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \ne j \end{cases}$$

for $i, j = 0, 1, \ldots, N+1$.

Consider the problem of finding $u(x,t)$ such that

$$u_t(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0$$

with Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0,\ t > 0$$

and initial condition

$$u(x,0) = u_0(x)\ \ 0 \le x \le 1.$$

We can obtain a finite element approximation

$$u_N(x,t) = \sum_{j=0}^{N+1} \alpha_j(t)\,\phi_j(x)$$

to u by finding $\boldsymbol{\alpha}(t)$ such that

$$\boldsymbol{M}\boldsymbol{\alpha}'(t) = -\boldsymbol{K}\boldsymbol{\alpha}(t)$$

for matrices $\boldsymbol{M}$ and $\boldsymbol{K}$.

(a) Write down the general forms of $\boldsymbol{M}$ and $\boldsymbol{K}$.

(b) What should we take $\boldsymbol{\alpha}(0)$ to be in order for

$$u_N(x_i, 0) = u_0(x_i)$$

for $i = 0, 1, \ldots, N+1$.

(c) When $N = 1$, the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}$ is such that

$$\boldsymbol{M}^{-1}\boldsymbol{K}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},\ \boldsymbol{M}^{-1}\boldsymbol{K}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-12\\0\\12\end{bmatrix},\ \text{and } \boldsymbol{M}^{-1}\boldsymbol{K}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}48\\-48\\48\end{bmatrix}.$$

We can write

$$-\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{W} = \boldsymbol{W}\boldsymbol{D}$$

where the matrix

$$\boldsymbol{W} = \begin{bmatrix}1 & -1 & 1\\ -1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}.$$

Write down the diagonal matrix $\boldsymbol{D}$.

(d) The eigenvectors of $-\boldsymbol{M}^{-1}\boldsymbol{K}$ are not orthonormal and so $\boldsymbol{W}^{-1} \neq \boldsymbol{W}^T$. However, since the matrix $\boldsymbol{W}$ is symmetric there exist matrices $\boldsymbol{V}$ and $\boldsymbol{\Lambda}$ which are such that $\boldsymbol{W} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ and $\boldsymbol{V}^{-1} = \boldsymbol{V}^T$. The matrix $\boldsymbol{W}$ has eigenvalues $-\sqrt{2}, \sqrt{2}, 2$ with corresponding eigenvectors $\begin{bmatrix}-1\\-\sqrt{2}\\1\end{bmatrix}, \begin{bmatrix}-1\\\sqrt{2}\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$. Construct the matrices $\boldsymbol{V}$ and $\boldsymbol{\Lambda}$.

(e) Compute $\boldsymbol{W}^{-1}$. You may use the fact that $\left(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\right)^{-1} = \boldsymbol{V}\boldsymbol{\Lambda}^{-1}\boldsymbol{V}^{-1}$.

(f) By writing $-\boldsymbol{M}^{-1}\boldsymbol{K}$ in an appropriate form, compute the exact solution to

$$\boldsymbol{M}\boldsymbol{\alpha}'(t) = -\boldsymbol{K}\boldsymbol{\alpha}(t)$$

with initial condition

$$\boldsymbol{\alpha}(0) = \begin{bmatrix} \alpha_0(0) \\ \alpha_1(0) \\ \alpha_2(0) \end{bmatrix}.$$

(g) How does $\boldsymbol{\alpha}(t)$ behave as $t \to \infty$ when $\alpha_0(0) = \alpha_1(0) = \alpha_2(0) = 1$?

(h) When $u_0(x) = 1$ the true solution is $u(x,t) = 1$. What is the error $u - u_N$ when $u_0(x) = 1$?

3. Let us consider the problem of finding $u(x,y)$ such that

$$-u_{xx}(x,y) - u_{yy}(x,y) = f(x,y), \quad -1 < x < 1, \quad -1 < y < 1,$$

with Neumann boundary conditions

$$u_x(-1,y) = u_x(1,y) = 0, \quad -1 \le y \le 1$$

and

$$u_y(x,-1) = 0, \; u_y(x,1) = g(x) \quad -1 \le x \le 1.$$

(a) Show that if $u \in C^2[-1,1]^2$ is the solution to the above problem then

$$a(u,v) = l(v) \text{ for all } v \in C^2[-1,1]^2$$

where

$$a(w,v) = \int_{-1}^{1}\int_{-1}^{1} w_x(x,y)\, v_x(x,y) + w_y(x,y)\, v_y(x,y) \; dxdy$$

and

$$l(v) = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\, v(x,y)\, dxdy + \int_{-1}^{1} g(x)\, v(x,1) \; dx.$$

(b) Let $V_N = \operatorname{span}\{\phi_1, \ldots, \phi_N\}$. Suppose that we want to find

$$u_N(x,y) = \sum_{j=1}^{N} \alpha_j \phi_j(x,y)$$

such that

$$a(u_N, v) = l(v) \text{ for all } v \in V_N.$$

We can obtain the coefficients α_j by solving the linear system of equations

$$\boldsymbol{K}\boldsymbol{\alpha} = \boldsymbol{b}.$$

where the matrix $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ has entries

$$K_{ij} = a(\phi_j, \phi_i)$$

for $i, j = 1, \ldots, N$. What are the entries of the vector $\boldsymbol{b}$?

(c) Compute $\int_{-1}^{1} (1-s)^2\, ds$, $\int_{-1}^{1} (1+s)^2\, ds$ and $\int_{-1}^{1} (1-s)(1+s)\, ds$.

(d) Construct the matrix $\boldsymbol{K}$ when $N = 4$ and

$$\phi_1 = \frac{1}{4}(1-x)(1-y)$$

$$\phi_2 = \frac{1}{4}(1+x)(1-y)$$

$$\phi_3 = \frac{1}{4}(1-x)(1+y)$$

$$\phi_4 = \frac{1}{4}(1+x)(1+y)$$

for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

(e) Simplify

$$\sum_{j=1}^{4} \phi_j(x,y).$$

(f) The matrix $\boldsymbol{K}$ is such that

$$\boldsymbol{K}\boldsymbol{v}_0 = \mathbf{0}$$

where

$$\boldsymbol{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Consequently,

$$\boldsymbol{K}\boldsymbol{\alpha} = \boldsymbol{b}$$

will have no solutions if $\boldsymbol{v}_0^T\boldsymbol{b} \neq 0$. Show that if $l(1) \neq 0$ then $\boldsymbol{v}_0^T\boldsymbol{b} \neq 0$.