CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 26

Posted Friday 28 February 2014. Due 1pm Friday 14 March 2014.

26. [25 points]

Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let the linear operator $L: C_D^2[0,1]\to C[0,1]$ be defined by

$$Lv=-v''$$

where

$$C_D^2[0,1]=\{w\in C^2[0,1]: w(0)=w(1)=0\}.$$

Recall that the operator L has eigenvalues

$$\lambda_n=n^2\pi^2$$

with corresponding eigenfunctions

$$\psi_n(x)=\sqrt{2}\sin(n\pi x)$$

for $n=1,2,\ldots$. Let N be a positive integer, let $f\in C[0,1]$ be defined by $f(x)=8x^2(1-x)$ and let u be the solution to

$$Lu=f.$$

(a) Compute the best approximation f_N to f from span $\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$.

(b) Use the spectral method to compute the best approximation u_N to u from span $\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$.

(c) Produce a plot comparing f to f_N for $N=1,2,3,4,5,6$.

(d) Plot the approximations u_N to u that you obtained using the spectral method for $N=1,2,3,4,5,6$.

(e) Write down the series solution to

$$Lu=f$$

that is obtained using the spectral method.

(f) By shifting the data and then using a series solution that you have obtained previously in this question, obtain a series solution to the problem of finding $\tilde{u}\in C^2[0,1]$ such that

$$-\tilde{u}''(x)=f(x),\quad 0<x<1;$$

$$\tilde{u}(0) = -\frac{1}{4}$$

and

$$\tilde{u}(1) = \frac{1}{4}.$$

(g) Let $\tilde{u}_N$ be the series solution that you obtained in part (f) but with ∞ replaced by N. Plot $\tilde{u}_N$ for $N = 1, 2, 3, 4, 5, 6$.