CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 3

Posted Wednesday 5 September 2012. Due Wednesday 12 September 2012, 5pm.

1. [20 points]
   Determine whether each of the following functions $(\cdot,\cdot)$ determines an inner product on the vector space $\mathcal{V}$. If not, show **all the properties** of the inner product that are violated.

   (a) $\mathcal{V} = C^1[0,1]$, $(u,v) = \displaystyle\int_0^1 u(x)v'(x)\,dx$ (b) $\mathcal{V} = C[0,1]$: $(u,v) = \displaystyle\int_0^1 |u(x)||v(x)|\,dx$

   (c) $\mathcal{V} = C[0,1]$: $(u,v) = \displaystyle\int_0^1 u(x)v(x)e^{-x}\,dx$ (d) $\mathcal{V} = C[0,1]$: $(u,v) = \displaystyle\int_0^1 \big(u(x)+v(x)\big)\,dx$

2. [20 points]
   Suppose $\mathcal{V}$ is a vector space with an associated inner product. The angle $\angle(u,v)$ between u and $v \in \mathcal{V}$ is defined via the equation
   $$(u,v) = \|u\|\|v\|\cos\angle(u,v).$$
   Let $\mathcal{V} = C[0,1]$ and $(u,v) = \int_0^1 u(x)v(x)\,dx$. Compute $\cos\angle(x^n, x^m)$ between $u(x) = x^n$ and $v(x) = x^m$ for nonnegative integers m and n. What happens to $\angle(x^n, x^{n+1})$ as $n \to \infty$?

3. [25 points]
   Consider the polynomials $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = 3x^2 - 1$, which form a basis for the set of all quadratic polynomials. These polynomials are orthogonal in $C[-1,1]$ with the usual inner product
   $$(u,v) = \int_{-1}^1 u(x)v(x)\,dx.$$
   (You do not need to prove this.) In the parts below, "best approximation" is defined with respect to this inner product, and the norm it induces.

   Let $f(x) = \cos(\pi x)$.

   (a) Construct the best approximation $f_1(x) = c_1\phi_1(x)$ to $f(x)$ from span$\{\phi_1\}$ (i.e., determine c_1 to minimize $\|f - f_1\|$ in $C[-1,1]$).

   (b) Construct the best approximation $f_2(x) = c_1\phi_1(x) + c_2\phi_2(x)$ to $f(x)$ from span$\{\phi_1, \phi_2\}$.

   (c) Construct the best approximation $f_3(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ to $f(x)$ from span$\{\phi_1, \phi_2, \phi_3\}$.

   (d) Produce a plot that superimposes your best approximation from parts (a), (b), and (c) on top of a plot of $f(x)$.

please turn over

4. [35 points]
Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = jh$ for $j = 0, \ldots, N+1$.
We can approximate the differential equation

$$\frac{d^2}{dx^2} u = f(x), \quad x \in (0,1),$$

with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ by the matrix equation

$$\frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix},$$

where $u_j \approx u(x_j)$. (Entries of the matrix that are not specified are zero.)

(a) Suppose that $f(x) = 25\pi^2 \cos(5\pi x)$.
Compute and plot the approximate solutions obtained when $N = 8, 16, 32, 64, 128$.
You may superimpose these on one plot. To solve the linear systems, you may use MATLAB's 'backslash' command: `u = A\f`.
For each value of N compute the maximum error $|u_j - u(x_j)|$, given that the true solution is

$$u(x) = 1 - 2x - \cos(5\pi x).$$

Plot this error using a `loglog` plot with error on the vertical axis and N on the horizontal axis.

(b) Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

$$u(0) = 1, \quad u(1) = 2.$$

Compute and plot solutions for $N = 8, 32, 128$.

(c) Now suppose that we have mixed boundary conditions

$$u(0) = 1, \quad \frac{du}{dx}(1) = -5.$$

The matrix equation will now have the form

$$\frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \ddots & & \\ & & \ddots & \ddots & 1 & 0 \\ & & & 1 & -2 & \star \\ & & & \star & \star & \star \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} f(x_1) - * \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \\ \star \end{bmatrix}.$$

Specify values for the entries marked by $\star$ to impose the approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N-1} - 4u_N + 3u_{N+1}}{2h} = -5.$$

Compute and plot solutions for $N = 8, 32, 128$.
Optional: What happens to the overall accuracy of the approximate solution if, instead of the $O(h^2)$ accurate approximation given above, you only use the $O(h)$ approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N+1} - u_N}{h} ?$$