# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 25 · Solutions

Posted Friday 28 February 2014. Due 1pm Friday 14 March 2014.

25. [25 points]

Let the inner product $(\cdot,\cdot):\ C[0,1]\times C[0,1]\to\mathbb{R}$ be defined by

$$(v,w)=\int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\|:\ C[0,1]\to\mathbb{R}$ be defined by

$$\|v\|=\sqrt{(v,v)}.$$

Let $N$ be a posivitive integer and let $\psi_1,\ldots,\psi_N\in C[0,1]$ be such that $\{\psi_1,\ldots,\psi_N\}$ is orthonormal with respect to the inner product $(\cdot,\cdot)$. For $g\in C[0,1]$, let

$$g_N=\sum_{n=1}^{N}\alpha_n\psi_n$$

where $\alpha_n=(g,\psi_n)$. Note that $g_N$ is the best approximation to $g$ from span $\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$. Moreover, let $u\in C^2[0,1]$ be such that

$$-u''(x)=f(x),\quad 0<x<1$$

and

$$u(0)=u(1)=0$$

with $f\in C[0,1]$ being defined by $f(x)=1$ for all $x\in[0,1]$. Note that $u(x)=\dfrac{1}{2}x(1-x)$.

(a) Show that

$$\|g-g_N\|^2=\|g\|^2-\sum_{n=1}^{N}\alpha_n^2.$$

(b) For the remainder of this question we will just consider the case when

$$\psi_n(x)=\sqrt{2}\sin(n\pi x)\text{ for }n=1,2,\ldots.$$

The best approximation to $f$ from span$\{\psi_1,\ldots,\psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$f_N=\sum_{n=1}^{N}(f,\psi_n)\psi_n.$$

Produce a `loglog` plot of $\|f-f_N\|$ for $N=1,2,\ldots,1000000$. Note that, for $n=1,2,\ldots,$

$$(f,\psi_n)=\frac{\sqrt{2}}{n\pi}\left(1-(-1)^n\right).$$

(c) We can use the spectral method to conclude that the best approximation to $u$ from $\text{span}\{\psi_1, \dots, \psi_N\}$ with respect to the norm $\|\cdot\|$ is

$$u_N = \sum_{n=1}^{N} (u, \psi_n)\psi_n$$

where

$$(u, \psi_n) = \frac{(f, \psi_n)}{n^2\pi^2} = \frac{\sqrt{2}}{n^3\pi^3}\left(1 - (-1)^n\right).$$

Add a `loglog` plot of $\|u - u_N\|$ for $N = 1, 2, \dots, 1000000$ to the plot that you produced in part (b).

(Be aware that the norm of the error may appear to flatline or become imaginary around $10^{-8}$: this is a consequence of the computer's floating point arithmetic, and so you will not lose points because of this.)

---

Solution.

(a) [10 points] We have that

$$\begin{aligned}
\|g - g_N\|^2 &= (g - g_N, g - g_N) \\
&= \left(g - \sum_{n=1}^{N} \alpha_n \psi_n, g - \sum_{m=1}^{N} \alpha_m \psi_m\right) \\
&= \left(g - \sum_{n=1}^{N} \alpha_n \psi_n, g\right) - \sum_{m=1}^{N} \alpha_m \left(g - \sum_{n=1}^{N} \alpha_n \psi_n, \psi_m\right) \\
&= (g, g) - \sum_{n=1}^{N} \alpha_n (\psi_n, g) - \sum_{m=1}^{N} \alpha_m (g, \psi_m) + \sum_{m=1}^{N} \alpha_m \sum_{n=1}^{N} \alpha_n (\psi_n, \psi_m) \\
&= (g, g) - \sum_{n=1}^{N} \alpha_n (\psi_n, g) - \sum_{m=1}^{N} \alpha_m (g, \psi_m) + \sum_{n=1}^{N} \alpha_n^2 (\psi_n, \psi_n) \\
&= (g, g) - \sum_{n=1}^{N} \alpha_n (\psi_n, g) - \sum_{m=1}^{N} \alpha_m (g, \psi_m) + \sum_{n=1}^{N} \alpha_n^2 \\
&= (g, g) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n^2 \\
&= (g, g) - \sum_{n=1}^{N} \alpha_n^2 \\
&= \|g\|^2 - \sum_{n=1}^{N} \alpha_n^2,
\end{aligned}$$

where at each equal sign we have used: (1) the definition of the norm $\|\cdot\|$; (2) the definition of $g_N$; (3) linearity of the inner product in the second argument; (4) linearity of the inner product in the first argument; (5) the fact that $(\psi_n, \psi_m) = 0$ if $n \neq m$, for $m, n = 1, 2, \dots, N$, since $\{\psi_1, \dots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$; (6) the fact that $(\psi_n, \psi_n) = 1$, for $n = 1, 2, \dots, N$, since $\{\psi_1, \dots, \psi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$; (7) the fact that $(g, \psi_n) = (\psi_n, g) = \alpha_n$; (8) algebra; (9) the definition of the norm $\|\cdot\|$.

(b) [7 points] From part (a) it follows that

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} c_n^2,$$

where

$$c_n = \frac{\sqrt{2}}{n\pi}\left(1 - (-1)^n\right).$$

Moreover,

$$\|f\|^2 = \int_0^1 (f(x))^2 \, dx = \int_0^1 1^2 \, dx = [x]_0^1 = 1$$

and so

$$\|f - f_N\|^2 = 1 - \sum_{n=1}^{N} c_n^2.$$

The requested plot and the code used to produce it is shown in part (c).

(c) [8 points] From part (a) it follows that

$$\|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} d_n^2,$$

where

$$d_n = \frac{\sqrt{2}}{n^3\pi^3}\left(1 - (-1)^n\right).$$

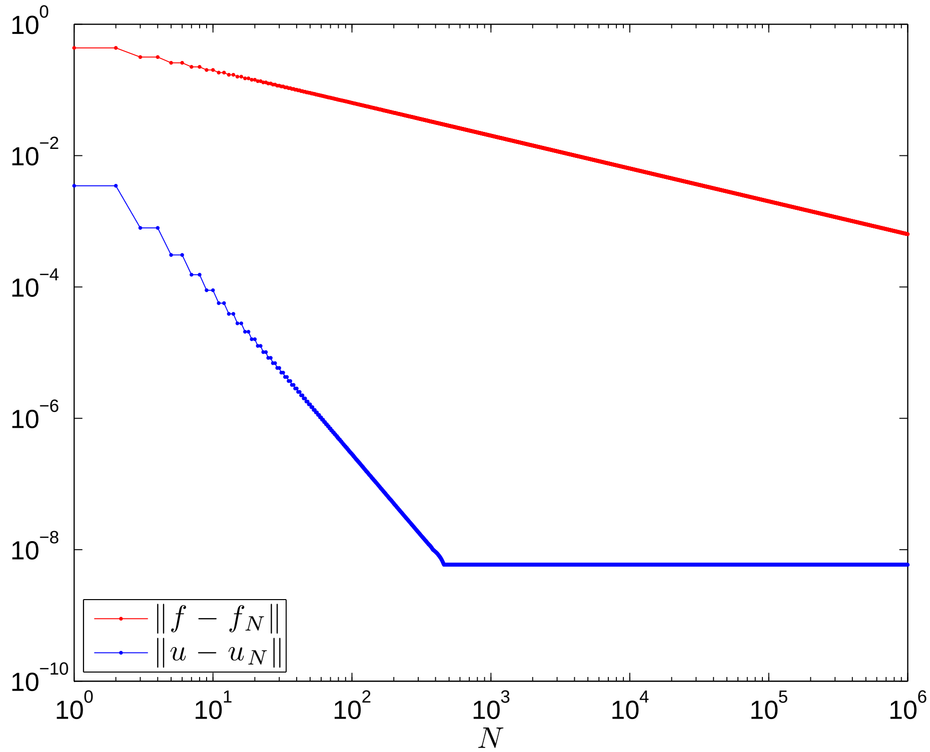
Moreover,

$$\begin{aligned}
\|u\|^2 &= \int_0^1 (u(x))^2 \, dx \\
&= \int_0^1 \frac{1}{4}x^2 (1-x)^2 \, dx \\
&= \frac{1}{4}\int_0^1 x^2 - 2x^3 + x^4 \, dx \\
&= \frac{1}{4}\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 \\
&= \frac{1}{4}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) \\
&= \frac{1}{4}\left(\frac{10}{30} - \frac{15}{30} + \frac{6}{30}\right) \\
&= \frac{1}{4}\frac{1}{30} \\
&= \frac{1}{120}
\end{aligned}$$

and so

$$\|u - u_N\|^2 = \frac{1}{120} - \sum_{n=1}^{N} d_n^2.$$

The requested plot is shown below.




The code that produced the plot is shown below.

```
n = [1:1e6]';
cn = (sqrt(2)/pi)*(1+(-1).^(n+1))./(n);
lamn = pi^2*n.^2;
normf2 = 1;
normu2 = 1/120;
figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')
hold on
loglog(n, sqrt(normu2-cumsum((cn./lamn).^2)),'b.-')

set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$','$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr
```