CAAM 336 · DIFFERENTIAL EQUATIONS

Examination 2

Posted Wednesday, 5 December 2012.

Due no later than 5pm on Wednesday, 12 December 2012.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points, plus a 5 point bonus.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Time started: ________________ Time completed: ________________
7. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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8. Staple this page to the front of your exam.

1. [34 points: (a), (c), (e) = 4 points; (b), (d) = 8 points; (f) = 6 points]

   Consider the fourth-order linear operator $L : C^4_H[0,1] \to C[0,1]$,

$$Lu = u_{xxxx} + \varepsilon u_{xx},$$

   where $C^4_H[0,1]$ consists of all $C^4[0,1]$ functions v with *hinged* boundary conditions

$$v(0) = v''(0) = v(1) = v''(1) = 0.$$

   Here ε is a constant and we use the inner product

$$(f,g) = \int_0^1 f(x)g(x)\,\mathrm{d}x.$$

   (a) The eigenfunctions of L are $\psi_k(x) = \sqrt{2}\sin(k\pi x)$, for $k = 1, 2, 3 \ldots$. Find the corresponding eigenvalues λ_k of L.

   We wish to solve the partial differential equation

$$u_t(x,t) = -(u_{xxxx}(x,t) + \varepsilon u_{xx}(x,t)),$$

   for $x \in [0,1]$ and $t \geq 0$, with hinged boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$$

   and initial condition

$$u(x,0) = u_0(x).$$

   (b) To compute the spectral method solution, we pose this equation as

$$u_t = -Lu$$

   and seek the solution in the form

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\psi_j(x).$$

   Identify the ordinary differential equation that a_j must satisfy, solve that differential equation, and give a simplified formula for $a_j(t)$ that depends only on u_0, λ_j, ψ_j, and t.

   (c) Under what conditions on ε can we ensure that the solution $u(x,t)$ decays to zero regardless of the initial data u_0?

Now consider the following partial differential equation, which is related to a model of vibrations of a stiff piano wire:

$$u_{tt}(x,t) = -(u_{xxxx}(x,t) + \varepsilon u_{xx}(x,t)),$$

again for $x \in [0,1]$ and $t \geq 0$, with hinged boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

For the rest of this question, assume that ε is chosen such that all eigenvalues are nonnegative: $\lambda_k \geq 0$ for all k.

(d) Now to compute the spectral method solution for this equation, we write

$$u_{tt} = -Lu$$

and again seek the solution in the form

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\psi_j(x).$$

For this second-order problem in time, identify the ordinary differential equation that a_j must satisfy, solve that differential equation, and give a simplified formula for $a_j(t)$ that depends only on u_0, v_0, λ_j, ψ_j, and t.

Be sure to handle the special case where there is a zero eigenvalue.

(e) How do solutions to this equation behave as $t \to \infty$ if there is no zero eigenvalue? Does this behavior change if there is a zero eigenvalue?

(f) Suppose we solve the problem $u_t(x,t) = -u_{xxxx}(x,t)$ using the finite element method, and when N elements are used, the eigenvalues λ of $\mathbf{M}^{-1}\mathbf{K}$ satisfy $0 < \lambda < 16N^4$, with largest eigenvalue $\lambda_N \approx 16N^4$.

We wish to solve the equation $\mathbf{c}'(t) = -\mathbf{M}^{-1}\mathbf{K}\mathbf{c}(t)$ using the *forward Euler method* with time step Δt. Let $t_k = (\Delta t)k$.

How should we pick the time step Δt so that the approximate solutions $\mathbf{c}_k \approx \mathbf{c}(t_k)$ decay to zero as $k \to \infty$?

If you double N, how should Δt be adjusted to maintain the decay to zero as $k \to \infty$?

How does your analysis change if we instead use the *backward Euler method*?

2. [22 points: (a) = 4 points; (b), (c) = 9 points]

The challenge problem on Problem Set 5 introduced the steady-state *convection–diffusion* equation, which arises often in fluid dynamics. Here we shall consider the time dependent version,

$$u_t(x,t) = u_{xx}(x,t) + cu_x(x,t) \tag{$*$}$$

for $x \in [0,1]$ and $t \geq 0$ with constant parameter c, boundary conditions $u(0,t) = u(1,t) = 0$, and initial condition $u(x,0) = u_0(x)$.

(a) The cu_x term prevents us from tackling this problem directly using the spectral method. Instead, write $u(x,t)$ in the form

$$u(x,t) = e^{-cx/2}w(x,t).$$

Show that w satisfies the equation

$$w_t(x,t) = w_{xx}(x,t) - \frac{c^2}{4}w(x,t). \tag{$**$}$$

Specify the correct initial and boundary conditions for w.

(b) We now seek a series solution for w.
Write the equation ($**$) in the form $w_t = -Lw$. Clearly define the operator L.
The eigenfunctions of your L should be the familiar $\psi_k(x) = \sqrt{2}\sin(k\pi x)$.
Use these to compute the eigenvalues of L. Write down a full series solution for w, and from this write down a series solution for u.

(c) We now wish to solve the problem ($*$) with inhomogeneous Dirichlet boundary conditions, $u(0,t) = \alpha$ and $u(1,t) = \beta e^{-c/2}$.

As before, we will write $u(x,t) = e^{-cx/2}w(x,t)$, where now w satisfies the inhomogeneous Dirichlet conditions $w(0,t) = \alpha$ and $w(1,t) = \beta$.

Explain how to find the solution in the form $w(x,t) = \widehat{w}(x,t) + g(x)$, where $\widehat{w}$ satisfies homogeneous Dirichlet boundary conditions and solves the problem $\widehat{w}_t = -L\widehat{w} + f$. Specify f, g, and the initial condition $\widehat{w}_0(x) = \widehat{w}(x,0)$. *You do not need to solve for* $\widehat{w}$.

3. [10 points: (a) = 6 points; (b) = 4 points]

Consider two metal bars in close proximity, so that the heat radiating off of each bar adds energy to the other. We model the heat in these bars through the coupled partial differential equations

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + v(x,t) \\ v_t(x,t) &= v_{xx}(x,t) + u(x,t) \end{aligned}$$

for $x \in [0,1]$ with homogeneous Dirichlet boundary conditions for u and v, and initial heat distributions $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$. [Polyanin]

(a) Show that if w and z satisfy the standard heat equations

$$w_t(x,t) = w_{xx}(x,t), \qquad z_t(x,t) = z_{xx}(x,t),$$

then

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(e^t w(x,t) + e^{-t} z(x,t)\Big) \\ v(x,t) &= \frac{1}{2}\Big(e^t w(x,t) - e^{-t} z(x,t)\Big) \end{aligned}$$

are solutions to the coupled partial differential equations.

(b) What initial conditions should w and z satisfy, in terms of u_0 and v_0?

What boundary conditions should w and z satisfy?

4. [34 points: (a), (b), (c), (e), (f) = 4 points; (d), (g) = 7 points]

On Problem Sets 10 and 12, we considered the spectral method solution of differential equations in two spatial dimensions. In this problem, we shall develop the finite element solution of such a problem.

For convenience we shall work on the square domain in two spatial dimensions given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Consider the differential equation

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad -1 \le x \le 1, \quad -1 \le y \le 1,$$

with homogeneous Neumann boundary conditions on all four sides of the domain:

$$u_x(-1,y) = u_x(1,y) = u_y(x,-1) = u_y(x,1) = 0.$$

Let $C^2[-1,1]^2$ denote the set of all continuous functions of $x \in [-1,1]$ and $y \in [-1,1]$ with all first and second derivatives continuous.

(a) Show that if $u \in C^2[-1,1]^2$ solves the differential equation, then

$$a(u,v) = (f,v) \qquad \text{for all } v \in C^2[-1,1]^2,$$

where

$$\begin{aligned} a(u,v) &= \int_{-1}^{1}\int_{-1}^{1} \Big(u_x(x,y)v_x(x,y) + u_y(x,y)v_y(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ (f,v) &= \int_{-1}^{1}\int_{-1}^{1} f(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Let $V_N = \mathrm{span}\{\phi_1, \ldots, \phi_N\}$. To solve the Galerkin problem, we seek

$$u_N = \sum_{j=1}^{N} c_j \phi_j(x,y)$$

such that

$$a(u_N, v) = (f,v) \qquad \text{for all } v \in V_N.$$

(b) Show how the constants $c_1, \ldots, c_N$ can be found by solving the matrix equation $\mathbf{Kc} = \mathbf{f}$, where the (j,k) entry of $\mathbf{K}$ equals $a(\phi_k, \phi_j)$. Be sure to specify the entries in $\mathbf{f}$.

(c) To expedite future calculations, compute these integrals:

$$I_1 = \int_{-1}^{1} (1-s)^2\,\mathrm{d}s, \qquad I_2 = \int_{-1}^{1} (1+s)^2\,\mathrm{d}s, \qquad I_3 = \int_{-1}^{1} (1-s)(1+s)\,\mathrm{d}s.$$

(d) Compute the matrix $\mathbf{K}$ for the case $N = 4$ with

$$\phi_1(x,y) = \tfrac{1}{4}(1-x)(1-y), \quad \phi_2(x,y) = \tfrac{1}{4}(1+x)(1-y),$$
$$\phi_3(x,y) = \tfrac{1}{4}(1-x)(1+y), \quad \phi_4(x,y) = \tfrac{1}{4}(1+x)(1+y).$$

Hint: The (1,1) entry of $\mathbf{K}$ is 2/3.

(e) Find a nonzero vector $\mathbf{w}$ such that $\mathbf{K}\mathbf{w} = \mathbf{0}$.

(f) From the vector $\mathbf{w}$ in part (e), construct the function

$$w(x,y) = \sum_{j=1}^{4} w_j \phi_j(x,y).$$

Simplify as much as possible.

How does this function w relate to the differential equation with Neumann boundary conditions? (You may draw inspiration from the analogous problem in one spatial dimension.)

(g) Now suppose we add an inhomogeneous Neumann condition on one side:

$$u_x(-1,y) = u_x(1,y) = u_y(x,-1) = 0, \qquad u_y(x,1) = g(x).$$

Explain how you should change the weak form that you derived in part (a) and the system $\mathbf{K}\mathbf{c} = \mathbf{f}$ derived in part (b) to accommodate these boundary conditions.

Be as explicit as possible.

bonus [5 points]

Consider the linear operator $L : C^2_P[-1,1] \to C[-1,1]$ given by $Lu = -u_{xx}$, where $C^2_P[-1,1]$ denotes the space of functions $v \in C^2[-1,1]$ that satisfy the *periodic* boundary conditions $v(-1) = v(1)$ and $v'(-1) = v'(1)$. We use the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x.$$

(a) Verify that the following are eigenfunctions of L:

$$\begin{aligned}\psi_{-k}(x) &= \cos(k\pi x), \qquad k = 1,2,\ldots,\\ \psi_0(x) &= 1/2\\ \psi_k(x) &= \sin(k\pi x), \qquad k = 1,2,\ldots.\end{aligned}$$

What are the corresponding eigenvalues?

(b) Show how to expand a function $u \in C^2_P[-1,1]$ in the basis of eigenfunctions, that is, specify formulas for coefficients c_j in the sum

$$u(x) = \sum_{j=-\infty}^{\infty} c_j \psi_j(x).$$

(c) Show that if u is an *odd function*, that is, $u(-x) = -u(x)$, then $c_j = 0$ for $j \le 0$.
Show that if u is an *even function*, that is, $u(-x) = u(x)$, then $c_j = 0$ for $j > 0$.

(d) Any function u can be written as the sum of an odd function and an even function:

$$u(x) = v(x) + w(x), \qquad v(x) = \frac{u(x) - u(-x)}{2}, \quad w(x) = \frac{u(x) + u(-x)}{2},$$

where $v \in C^2_P[-1,1]$ is odd and $w \in C^2_P[-1,1]$ is even.
Show that $v(0) = v(1) = 0$ (i.e., v satisfies homogeneous Dirichlet conditions),
while $w'(0) = w'(1) = 0$ (i.e., w satisfies homogeneous Neumann conditions).

You have just showed that any function $u \in C^2_P[-1,1]$ that satisfies $-u'' = f$ can be decomposed into two pieces, each of which satisfies the same differential equation on $[0,1]$: one with with homogeneous Dirichlet boundary conditions, the other with homogeneous Neumann boundary conditions.

The full series you wrote down in part (b) is generally referred to as the "Fourier series" for the periodic function u.