CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 4 · Solutions

Posted Wednesday 17 September 2014. Due 5pm Wednesday 24 September 2014.

1. [30 points - 5 points each]
   For each part, if the set is not a vector space, please show what properties of a vector space are violated. Otherwise, show that all properties of a vector space are satisfied.

   (a) Demonstrate whether or not the set $S_1 = \left\{ \mathbf{x} \in \mathbb{R}^2 : \ x_2 = x_1^3 \right\}$ is a subspace of $\mathbb{R}^2$.

   (b) Demonstrate whether or not the set $S_2 = \left\{ \mathbf{x} \in \mathbb{R}^3 : \ 3x_1 + 2x_2 + x_3 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

   (c) Demonstrate whether or not the set $S_3 = \{ f \in C[0,1] : \ f(x) \geq 0 \text{ for all } x \in [0,1] \}$ is a subspace of $C[0,1]$.

   (d) Demonstrate whether or not the set $S_4 = \left\{ f \in C[0,1] : \ \max\limits_{x\in[0,1]} f(x) \leq 1 \right\}$ is a subspace of $C[0,1]$.

   (e) Demonstrate whether or not the set $S_5 = \left\{ f \in C^2[0,1] : \ f(1) = 1 \right\}$ is a subspace of $C^2[0,1]$.

   (f) Demonstrate whether or not the set $S_6 = \left\{ f \in C^2[0,1] : \ f(1) = 0 \right\}$ is a subspace of $C^2[0,1]$.

Solution.

(a) The set S_1 is not a subspace of $\mathbb{R}^2$.
The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set S_1, yet $2\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not, since $2 \neq 2^3 = 8$. Consequently, the set S_1 is not a subspace of $\mathbb{R}^2$.

(b) The set S_2 is a subspace of $\mathbb{R}^3$.
The set S_2 is a subset of $\mathbb{R}^3$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a member of the set S_2. Now, suppose $\mathbf{x}$ and $\mathbf{y}$ are members of the set S_2. Then $3x_1 + 2x_2 + x_3 = 0$ and $3y_1 + 2y_2 + y_3 = 0$. Adding these two equations gives

$$3(x_1 + y_1) + 2(x_2 + y_2) + (x_3 + y_3) = 0,$$

and hence $\mathbf{x} + \mathbf{y}$ is also in the set S_2. Multiplying $3x_1 + 2x_2 + x_3 = 0$ by an arbitrary constant $\alpha \in \mathbb{R}$ gives

$$3(\alpha x_1) + 2(\alpha x_2) + \alpha x_3 = 0,$$

and hence $\alpha\mathbf{x}$ is also in the set S_2. Consequently, the set S_2 is a subspace of $\mathbb{R}^3$.

(c) The set S_3 is not a subspace of $C[0,1]$.
Let $f(x) = 1$ for $x \in [0,1]$. Then f is in the set S_3, but a scalar multiple, $-1 \cdot f(x) = -1$ for $x \in [0,1]$, takes negative values and thus violates the requirement for membership in the set S_3. Consequently, the set S_3 is not a subspace of $C[0,1]$.

(d) The set S_4 is not a subspace of $C[0,1]$.
Let $f(x) = 1$ for $x \in [0,1]$. Then f is in the set S_4, but a scalar multiple, $2 \cdot f(x) = 2$ for $x \in [0,1]$, takes values greater than one and thus violates the requirement for membership in the set S_4. Consequently, the set S_4 is not a subspace of $C[0,1]$.

(e) The set S_5 *is not* a subspace of $C^2[0,1]$.
The function z defined by $z(x) = 0$ for $x \in [0,1]$ is not in the set S_5 since $z(1) = 0$ and thus violates the requirement for membership in the set S_5. Consequently, the set S_5 is not a subspace of $C^2[0,1]$.

(f) The set S_6 subspace of $C^2[0,1]$.
The set S_6 is a subset of $C^2[0,1]$ and the function z defined by $z(x) = 0$ for $x \in [0,1]$ is in the set S_6. If f and g are in the set S_6, then $f(1) = g(1) = 0$, so

$$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and hence $f + g$ is in the set S_6. Also, if f is in the set S_6 and $\alpha \in \mathbb{R}$, then

$$(\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0$$

and hence αf is in the set S_6. Consequently, the set S_6 is a subspace of $C^2[0,1]$.

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2. [25 points - 5 points each]

Demonstrate whether or not each of the following is a linear operator.

(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and fixed nonzero vector $\mathbf{b} \in \mathbb{R}^m$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

(c) $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ defined by $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$.

(d) $L : C^1[0,1] \to C[0,1]$ defined by $(Lu)(x) = u(x)u'(x)$.

(e) $L : C^2[0,1] \to C[0,1]$ defined by $(Lu)(x) = u''(x) - \sin(x)u'(x) + \cos(x)u(x)$.

Solution.

(a) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$f(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + \mathbf{b}$$

but

$$f(\mathbf{u}) + f(\mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{b} + \mathbf{A}\mathbf{v} + \mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + 2\mathbf{b}$$

and so $f(\mathbf{u} + \mathbf{v})$ does not equal $f(\mathbf{u}) + f(\mathbf{v})$ when $\mathbf{b} \neq \mathbf{0}$. Hence, f *is not* a linear operator.

(b) Suppose $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

$$f(\alpha\mathbf{x}) = (\alpha\mathbf{x})^T(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^T\mathbf{x}$$

and

$$\alpha f(\mathbf{x}) = \alpha\mathbf{x}^T\mathbf{x}.$$

However, if $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\alpha = 2$ then $\mathbf{x}^T\mathbf{x} = 1$ and so

$$f(\alpha\mathbf{x}) = 2^2 = 4$$

but

$$\alpha f(\mathbf{x}) = 2.$$

Hence, f *is not* a linear operator.

(c) Suppose $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n\times n}$. Then

$$f(\mathbf{X}+\mathbf{Y}) = \mathbf{A}(\mathbf{X}+\mathbf{Y}) + (\mathbf{X}+\mathbf{Y})\mathbf{B} = \mathbf{AX}+\mathbf{XB}+\mathbf{AY}+\mathbf{YB} = f(\mathbf{X}) + f(\mathbf{Y}),$$

and if $\alpha \in \mathbb{R}$, then

$$f(\alpha\mathbf{X}) = \mathbf{A}(\alpha\mathbf{X}) + (\alpha\mathbf{X})\mathbf{B} = \alpha(\mathbf{AX}+\mathbf{XB}) = \alpha f(\mathbf{X}).$$

Hence, f *is* a linear operator.

(d) Suppose that $u \in C^1[0,1]$ and $\alpha \in \mathbb{R}$. Then

$$\alpha(Lu)(x) = \alpha u(x)u'(x)$$

and

$$(L(\alpha u))(x) = (\alpha u)(x)(\alpha u)'(x) = \alpha^2 u(x)u'(x).$$

However, if $u(x) = x$ and $\alpha = 2$ then

$$\alpha(Lu)(x) = 2x$$

but

$$(L(\alpha u))(x) = 2^2 x = 4x.$$

Hence, L *is not* a linear operator.

(e) Suppose that $u, v \in C^2[0,1]$. Then

$$\begin{aligned}
(L(u+v))(x) &= (u+v)''(x) - \sin(x)(u+v)'(x) + \cos(x)(u+v)(x) \\
&= u''(x) - \sin(x)u'(x) + \cos(x)u(x) + v''(x) - \sin(x)v'(x) + \cos(x)v(x) \\
&= (Lu)(x) + (Lv)(x),
\end{aligned}$$

and for all $\alpha \in \mathbb{R}$,

$$\begin{aligned}
(L(\alpha u))(x) &= (\alpha u)''(x) - \sin(x)(\alpha u)'(x) + \cos(x)(\alpha u)(x) \\
&= \alpha\left(u''(x) - \sin(x)u'(x) + \cos(x)u(x)\right) \\
&= \alpha(Lu)(x).
\end{aligned}$$

Hence, L *is* a linear operator.

3. [24 points - 10 points for (a), 14 points for (b)]

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since f is a linear operator, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

(b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

Solution.

(a) We can write any $\mathbf{u} \in \mathbb{R}^2$ in the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Any matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ are the columns of $\mathbf{A}$. Now the matrix-vector product $\mathbf{A}\mathbf{u}$ is a linear combination of the columns of $\mathbf{A}$:

$$\mathbf{A}\mathbf{u} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2. \tag{$*$}$$

We are trying to find a formula for $\mathbf{A}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Since f is a linear operator, we have that

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2). \tag{$**$}$$

Comparing $(*)$ and $(**)$, we see that

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) \end{bmatrix}$$

is such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$. Hence, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$.

(b) Follow the same tack as in part (a). Let $\mathbf{e}_j \in \mathbb{R}^n$ be the vector whose jth entry is 1 and whose other entries are all 0. Write $\mathbf{u} \in \mathbb{R}^n$ as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{A} \in \mathbb{R}^{n\times n}$ as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ are the columns of $\mathbf{A}$. Comparing

$$\mathbf{A}\mathbf{u} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots u_n\mathbf{a}_n$$

and

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2) + \cdots u_n f(\mathbf{e}_n),$$

we see that

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \cdots & f(\mathbf{e}_n) \end{bmatrix}$$

is such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$. Hence, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$.

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4. [21 points - 7 points each]
In this problem we'll consider a linear operator mapping to and from a very specific vector space, and use it to explore what an operator inverse can look like.

Consider the V defined as

$$V = \left\{ u(x) = \sum_{j=1}^{N} c_j \sin(j\pi x), \quad c_j \in \mathbb{R} \right\}.$$

In other words, V is the set of all functions that are linear combinations of a finite number of different sine functions. This means that, for each $u \in V$, there is a set of coefficients $c_1, \ldots, c_N$ that is also associated with u.

(a) Show that V is a subspace of the vector space C_D^2, where

$$C_D^2 = \{u(x) \in C^2[0,1], \quad u(0) = u(1) = 0\}.$$

(b) Let the operator L be defined as

$$Lu = -\frac{\partial^2 u}{\partial x^2}$$

Show that, for $u \in V$, $Lu \in V$. This shows that L can be viewed as

$$L : V \to V,$$

a map from V to V.

(c) We can define the operator $\tilde{L} : V \to V$ as

$$\tilde{L}u = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x).$$

Show that both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$.

Since both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$, we can refer to $\tilde{L}$ as the inverse L^{-1} of $L : V \to V$.

Solution.

Let V be defined as

$$V = \left\{ u \in C_D^2[0,1] : u(x) = \sum_{j=1}^{N} c_j \sin(j\pi x), \quad c_j \in \mathbb{R} \right\}.$$

(a) Suppose $u(x) \in V$. Then,

$$u(0) = \sum_{j=1}^{N} c_j \sin(0), \qquad u(1) = \sum_{j=1}^{N} c_j \sin(j\pi)$$

Since $\sin(0) = 0$, $u(0) = 0$. Similarly, since j is an integer in the above expression, $\sin(j\pi 1) = \sin(j\pi) = 0$, since sine evaluated at any multiple of π is zero, and $u(1) = 0$ as well. Since the sum of sines is a continuous function, any $u(x) \in V$ is also contained in $C_D^2[0,1]$, and V is contained in $C_D^2[0,1]$.

Next, we just need to show that V itself is a vector space. If $u(x)$ and $v(x)$ are in the set of V, then we have the representations

$$u(x) = \sum_{j=1}^{N} d_j \sin(j\pi x), \qquad v(x) = \sum_{j=1}^{N} e_j \sin(j\pi x)$$

for $d_j, e_j \in \mathbb{R}$. We wish to show that, for any $\alpha, \beta \in \mathbb{R}$, the sum $w(x) = \alpha u(x) + \beta v(x)$ is contained in V as well.

$$\begin{aligned} w(x) = \alpha u(x) + \beta v(x) &= \alpha \sum_{j=1}^{N} d_j \sin(j\pi x) + \beta \sum_{j=1}^{N} e_j \sin(j\pi x) \\ &= \sum_{j=1}^{N} \underbrace{\alpha d_j}_{\tilde{d}_j} \sin(j\pi x) + \sum_{j=1}^{N} \underbrace{\beta e_j}_{\tilde{e}_j} \sin(j\pi x) \\ &= \sum_{j=1}^{N} \underbrace{(\tilde{d}_j + \tilde{e}_j)}_{c_j} \sin(j\pi x) \\ &= \sum_{j=1}^{N} c_j \sin(j\pi x) \end{aligned}$$

Therefore $w \in V$, and since V is contained in $C_D^2[0,1]$ as well, V is a subspace of $C_D^2[0,1]$.

(b) The operator L be defined as

$$Lu = -\frac{\partial^2 u}{\partial x^2}.$$

We are going to show that for all $u \in V$, $Lu \in V$ as well. Let

$$u = \sum_{j=1}^{N} c_j \sin(j\pi x) \in V$$

for $c_j \in \mathbb{R}$. Then

$$\begin{aligned}
Lu = -\frac{\partial^2 u}{\partial x^2} &= -\frac{\partial^2}{\partial x^2}\left(\sum_{j=1}^{N} c_j \sin(j\pi x)\right) \\
&= -\frac{\partial}{\partial x}\left[\sum_{j=1}^{N} c_j \frac{\partial(\sin(j\pi x))}{\partial x}\right] \\
&= -\frac{\partial}{\partial x}\left[\sum_{j=1}^{N} c_j (j\pi)\cos(j\pi x)\right] \\
&= -\sum_{j=1}^{N} c_j (j\pi) \frac{\partial(\cos(j\pi x))}{\partial x} \\
&= \sum_{j=1}^{N} c_j (j\pi)^2 \sin(j\pi x).
\end{aligned}$$

If we define $\tilde{c}_j = c_j(j\pi)^2 \in \mathbb{R}$ then

$$Lu = \sum_{j=1}^{N} \tilde{c}_j \sin(j\pi x)$$

which is also in V for all $\tilde{c}_j \in \mathbb{R}$. This shows that L can be viewed as

$$L : V \to V,$$

a map from V to V.

(c) We define the operator $\tilde{L} : V \to V$ as

$$\tilde{L}u = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x).$$

We will show that $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$. First, let us show $L\tilde{L}u = u$. From part (b) we can deduce that if $u = \sum_{j=1}^{N} c_j \sin(j\pi x)$

$$Lu = L\left(\sum_{j=1}^{N} c_j \sin(j\pi x)\right) = \sum_{j=1}^{N} c_j (j\pi)^2 \sin(j\pi x)$$

Then

$$\begin{aligned}
L\tilde{L}u = L(\tilde{L}u) &= L\left(\sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x)\right) \\
&= \sum_{j=1}^{N} \frac{c_j (j\pi)^2}{(j\pi)^2} \sin(j\pi x) \\
&= \sum_{j=1}^{N} c_j \sin(j\pi x) \\
&= u
\end{aligned}$$

Now, let us show $\tilde{L}Lu = u$ as well. Remember that from problem if $u = \sum_{j=1}^{N} c_j \sin(j\pi x)$ then $\tilde{L}u$ is given as follows

$$\tilde{L}u = \tilde{L}\left(\sum_{j=1}^{N} c_j \sin(j\pi x)\right) = \sum_{j=1}^{N} \frac{c_j}{(j\pi)^2} \sin(j\pi x)$$

Then, similarly

$$\begin{aligned}
\tilde{L}Lu = \tilde{L}(Lu) &= \tilde{L}\left(\sum_{j=1}^{N} c_j (j\pi)^2 \sin(j\pi x)\right) \\
&= \sum_{j=1}^{N} \frac{c_j (j\pi)^2}{(j\pi)^2} \sin(j\pi x) \\
&= \sum_{j=1}^{N} c_j \sin(j\pi x) \\
&= u
\end{aligned}$$

Since both $L\tilde{L}u = u$ and $\tilde{L}Lu = u$ for any $u \in V$, we can refer to $\tilde{L}$ as the inverse L^{-1} of $L : V \to V$.