Basis, Dimension and linear independence (Chapter 3.3)

Important observation:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a matrix and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ a vector.

Then $A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{31}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ a_{32}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \\ a_{33}x_3 \end{bmatrix}$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$\Rightarrow$ The multiplication $A\vec{x}$ multiplies the Colums of A by the elements of the vector $\vec{x}$. In particular if we let $\vec{a}_1, \vec{a}_2, \vec{a}_3$ denote the columns of A then $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$.

Definition: Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ all be vectors in V. Let $\beta_1, \beta_2, \dots, \beta_n$ be scalars. Then the quantity $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \dots + \beta_n\vec{v}_n$ is called a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

Definition: Suppose that the set $B = \{v_1, v_2, \dots, v_n\}$ of vectors in the vector space V has the property that every vector $\vec{w}$ in V can be expressed uniquely as a linear combination of the vectors $v_1, v_2, \dots, v_n$. Then the collection $B = \{v_1, v_2, \dots, v_n\}$ is called a basis and we say that V has dimension n or is n-dimensional

Example: $B_{\mathbb{R}^2} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. So is $\hat{B}_{\mathbb{R}^2} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\}$.

Notice: This example shows that a vector space can have more then one basis. However, it can be shown that if $B_1 = \{v_1, v_2, \dots, v_n\}$ and $B_2 = \{w_1, w_2, \dots, w_j\}$ are two different

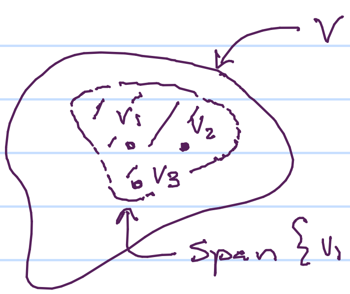
bases of V then $n=j$. That is $\text{size}(B_1) = \text{Size}(B_2)$
So that the concept of the "dimension of V" is well defined.

Ex: Let $\mathbb{P}_k = \{$ polynomials $p(x)$ with degree$(p) \le k\}$
You can show that $\mathbb{P}_k$ is a vector space and that
$B = \{1, x, x^2, \dots, x^k\}$ is a basis. Thus
dimension $(\mathbb{P}_k) = k+1$

Ex: Two bases for $\mathbb{P}_2$ are $B_1 = \{1, x, x^2\}$, $B_2 = \{1, x - 1/2, x^2 - x + 1/6\}$

Defn: Let V be a vector space of dimension n and let $0 < m \le n$
Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be vectors in V. Then the span of
$\{v_1, v_2, \dots, v_m\}$ is the set of all linear combinations of these
vectors. i.e. $\text{Span}\{v_1, v_2, \dots, v_m\} = \{\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m \mid \beta_i \in \mathbb{R}\}$.

Intuition: The span of $\{v_1, v_2, \dots, v_m\}$ is everything in V that you
can "reach" by using linear combinations of $\{v_1, v_2, \dots, v_m\}$
Notice that if $m < n$ then $\text{span}\{v_1, v_2, \dots, v_m\}$ is a proper
subset of V (e.g. it is not all of V. Can you see why?)



Definition: Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be vectors in the vector space V
then this collection of vectors is said to be linearly independent
if $\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \dots + \beta_m \vec{v}_m = \vec{0}$ implies that $\beta_1 = 0, \beta_2 = 0, \dots, \beta_m = 0$
(ie, the only way to get the zero vector as a linear
combination is multiplying everything by zero).

Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$ is linearly independent but $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\}$ is not

since $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = 0$

Here is an important result from linear algebra:

Theorem: Suppose that A is an $n \times n$ matrix. Then A is invertible if and only if the columns of A form a basis. Here invertible means that "for every b in the set $R(A)$ there exists one and only one x satisfying $Ax = b$".

This result means that the notion of a basis is connected intimately with the idea of invertible matrices.