CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 49

Posted Wednesday 27 November 2013. Due 1pm Friday 6 December 2013.

49. [25 points] Let $H_D^1(0,1) = \{v \in H^1(0,1) : v(0) = 0\}$. Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\phi_j \in H_D^1(0,1)$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N$ and

$$\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $V_N = \text{span}\{\phi_1, \ldots \phi_{N+1}\}$, let $u_0 \in H_D^1(0,1)$ be such that

$$u_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1/4], \\ 4x - 1 & \text{if } x \in (1/4, 1/2], \\ 3 - 4x & \text{if } x \in (1/2, 3/4], \\ 0 & \text{if } x \in (3/4, 1], \end{cases}$$

and let

$$u_{0,N}(x) = \sum_{j=1}^{N+1} u_0(x_j)\phi_j(x).$$

Note that $u_0 = u_{0,N}$ if and only if $u_0 \in V_N$.

(a) Write a MATLAB function for $u_0(x)$. It should take in as input x. It should return the value $u_0(x)$. It should also be able to take in a vector for $\mathbf{x} = (\hat{x}_1, \ldots, \hat{x}_m)$ and return the vector $u_0(\mathbf{x}) = (u_0(\hat{x}_1), \ldots, u_0(\hat{x}_m))$. Use your function to produce a plot of u_0. For this figure and the ones that you have to produce in part (b), use the command `set(gca,'XTick',[0 0.25 0.5 0.75 1])` to change the location of the tick marks on the x-axis.

(b) Write a MATLAB function for $u_{0,N}(x)$. It should take in as input x and N. It should return the value $u_{0,N}(x)$. It should also be able to take in a vector for $\mathbf{x} = (\hat{x}_1, \ldots, \hat{x}_m)$ and return the vector $u_{0,N}(\mathbf{x}) = (u_{0,N}(\hat{x}_1), \ldots, u_{0,N}(\hat{x}_m))$. On the same figure, plot u_0 as well as $u_{0,N}$ for $N = 3, 4, 5, 6$. On another figure, plot u_0 as well as $u_{0,N}$ for $N = 47, 48, 49, 50$.

(c) For which 2 of the 8 values of N that you plotted for in part (b) is $\max\limits_{x \in [0,1]} |u_0(x) - u_{0,N}(x)|$ the smallest? Use the fact that

$$\begin{aligned} &\text{span}\{\phi_1, \ldots \phi_{N+1}\} \\ = &\{v \in C[0,1] : v(0) = 0,\ v(x) = a_j x + b_j, \text{ where } a_j, b_j \in \mathbb{R}, \text{ if } x \in [x_{j-1}, x_j], \text{ for } j = 1, \ldots, N+1\}, \end{aligned}$$

as well as information given previously in the question, to explain your answer.