# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 41

Posted Wednesday 13 November 2013. Due 5pm Wednesday 27 November 2013.

41. [25 points] Let $u_0(x) = (x - x^2)\,(\sin(3\pi x))^2$. Note that, for $n = 1, 2, \ldots$,

$$\int_0^1 \sqrt{2}u_0(x)\sin(n\pi x)\,dx = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Consider the problem of finding the solution $u(x,t)$ to the the fourth order partial differential equation

$$u_t(x,t) = u_{xx}(x,t) - u_{xxxx}(x,t), \quad 0 < x < 1, \quad t > 0$$

with so-called hinged boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0, \quad t \geq 0$$

and initial condition

$$u(x,0) = u_0(x), \quad 0 < x < 1.$$

This equation is related to a model that arises in the study of thin films. Let

$$C_H^4[0,1] = \{v \in C^4[0,1] : v(0) = v''(0) = v(1) = v''(1) = 0\}.$$

Let the linear operator $L : C_H^4[0,1] \to C[0,1]$ be defined by

$$Lv = -v'' + v''''.$$

(a) The operator $L$ has eigenvalues $\lambda_n \in \mathbb{R}$ and eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

for $n = 1, 2, \ldots$, which are such that

$$L\psi_n = \lambda_n \psi_n$$

for $n = 1, 2, \ldots$. Obtain a formula for $\lambda_n$ for $n = 1, 2, \ldots$.

(b) We can write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

What ordinary differential equation and initial condition does $a_n(t)$ satisfy for $n = 1, 2, \ldots$?

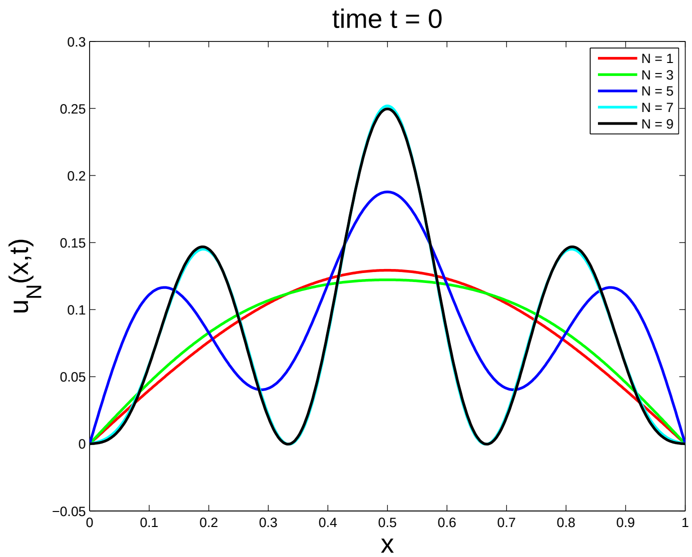
(c) Obtain an expression for $a_n(t)$ for $n = 1, 2, \ldots$.

(d) Use you answer to part (c) to write out a formula for $u(x,t)$.

(e) Let

$$u_N(x,t) = \sum_{n=1}^{N} a_n(t)\psi_n(x).$$

For each time $t = 0, 10^{-5}, 2 \times 10^{-5}, 4 \times 10^{-5}$, produce a plot comparing $u_1(x,t)$, $u_3(x,t)$, $u_5(x,t)$, $u_7(x,t)$ and $u_9(x,t)$. For example, at time $t = 0$, your plot should appear as shown below.



time t = 0
N = 1
N = 3
N = 5
N = 7
N = 9
$u_N(x,t)$
x
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1