

# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 22 · Solutions

Posted Monday 30 September 2013. Due 5pm Wednesday 9 October 2013.

22. [25 points] Suppose that

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

- (a) Compute by hand the eigenvalues and eigenvectors of this matrix.
- (b) Choose three eigenvectors of  $\mathbf{A}$ , one corresponding to each eigenvalue of  $\mathbf{A}$ , and verify by hand that these eigenvectors are orthogonal.
- (c) Solve the linear system  $\mathbf{Ax} = \mathbf{b}$  using the spectral method, where

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

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Solution.

- (a) [12 points] For this matrix  $\mathbf{A}$  we have

$$\lambda \mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix},$$

and hence the characteristic polynomial is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - 3)(\lambda^2 - 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).$$

The eigenvalues of  $\mathbf{A}$  are the roots of the characteristic polynomial, which we label

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 3.$$

To compute the eigenvectors associated with the eigenvalue  $\lambda_1 = -1$ , we seek  $\mathbf{u} = (u_1, u_2, u_3)^T$  that makes the following vector zero:

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -u_2 + u_3 \\ u_2 - u_3 \end{bmatrix}.$$

To make this vector zero we need to set  $u_1 = 0$  and  $u_3 = u_2$ . Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue  $\lambda_1 = -1$ .

To compute the eigenvectors associated with the eigenvalue  $\lambda_2 = 1$  we now seek  $\mathbf{u} = (u_1, u_2, u_3)^T$  that makes the following vector zero:

$$(\lambda_2 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ u_2 + u_3 \\ u_3 + u_2 \end{bmatrix}.$$

To make this vector zero we need to set  $u_1 = 0$  and  $u_3 = -u_2$ . Thus any vector of the form

$$\begin{bmatrix} 0 \\ u_2 \\ -u_2 \end{bmatrix}, \quad u_2 \neq 0$$

is an eigenvector associated with the eigenvalue  $\lambda_2 = 1$ .

To compute the eigenvectors associated with the eigenvalue  $\lambda_3 = 3$  we now seek  $\mathbf{u} = (u_1, u_2, u_3)^T$  that makes the following vector zero:

$$(\lambda_3 \mathbf{I} - \mathbf{A})\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2 + u_3 \\ u_2 + 3u_3 \end{bmatrix}.$$

To make the second component zero we need  $u_2 = -u_3/3$ , while to make the third component zero we need  $u_3 = -u_2/3$ . The only way to accomplish both is to set  $u_2 = u_3 = 0$ . Thus any vector of the form

$$\begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, \quad u_1 \neq 0$$

is an eigenvector associated with the eigenvalue  $\lambda_3 = 3$ .

(b) [3 points] We choose the eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We can compute that

$$\mathbf{u}_1^T \mathbf{u}_2 = \mathbf{u}_2^T \mathbf{u}_1 = 0 \cdot 0 + (1/\sqrt{2}) \cdot (1/\sqrt{2}) + (1/\sqrt{2}) \cdot (-1/\sqrt{2}) = 0,$$

$$\mathbf{u}_1^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_1 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (1/\sqrt{2}) \cdot 0 = 0,$$

and

$$\mathbf{u}_2^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_2 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (-1/\sqrt{2}) \cdot 0 = 0.$$

(c) [10 points] We first note that, for  $j = 1, 2, 3$ ,  $\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j$  and  $\mathbf{u}_j^T \mathbf{u}_j = 1$ . Since  $\mathbf{A} = \mathbf{A}^T$ , the spectral method then yields that

$$\mathbf{x} = \sum_{j=1}^3 \frac{1}{\lambda_j} \frac{\mathbf{u}_j^T \mathbf{b}}{\mathbf{u}_j^T \mathbf{u}_j} \mathbf{u}_j = \sum_{j=1}^3 \frac{\mathbf{u}_j^T \mathbf{b}}{\lambda_j} \mathbf{u}_j.$$

We can compute that

$$\mathbf{u}_1^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (1/\sqrt{2}) \cdot 3 = \sqrt{2},$$

$$\mathbf{u}_2^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (-1/\sqrt{2}) \cdot 3 = -2\sqrt{2},$$

and

$$\mathbf{u}_3^T \mathbf{b} = 1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 3 = 2,$$

and hence

$$\mathbf{x} = \frac{\sqrt{2}}{-1} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{-2\sqrt{2}}{1} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -3 \\ 1 \end{bmatrix}.$$

We can multiply  $\mathbf{Ax}$  out to verify that the desired  $\mathbf{b}$  is obtained.

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