

# CAAM 336 · DIFFERENTIAL EQUATIONS

## Homework 40

Posted Wednesday 13 November 2013. Due 5pm Wednesday 20 November 2013.

40. [25 points] Let

$$\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

and let  $f \in C(\Omega)$  be defined by  $f(x, y) = x(1 - y)$ . In this question we will consider the problem of finding the solution  $u(x, y)$  to the steady-state heat equation

$$-(u_{xx}(x, y) + u_{yy}(x, y)) = f(x, y), \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

with homogeneous Dirichlet boundary conditions

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

Let

$$C_D^2(\Omega) = \{v \in C^2(\Omega) : v(x, 0) = v(x, 1) = v(0, y) = v(1, y) = 0, 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Let the linear operator  $L : C_D^2(\Omega) \rightarrow C(\Omega)$  be defined by

$$(Lv)(x, y) = -(v_{xx}(x, y) + v_{yy}(x, y)).$$

Let the inner product  $(\cdot, \cdot) : C(\Omega) \times C(\Omega) \rightarrow \mathbb{R}$  be defined by

$$(v, w) = \int_0^1 \int_0^1 v(x, y)w(x, y) dx dy.$$

(a) Show that  $L$  is symmetric by showing that

$$(Lv, w) = (v, Lw) \text{ for all } v, w \in C_D^2(\Omega).$$

(b) The operator  $L$  has eigenvalues  $\lambda_{j,k} \in \mathbb{R}$  and eigenfunctions

$$\psi_{j,k}(x, y) = 2 \sin(j\pi x) \sin(k\pi y)$$

for  $j, k = 1, 2, \dots$ , which are such that

$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$

for  $j, k = 1, 2, \dots$ . Obtain a formula for  $\lambda_{j,k}$  for  $j, k = 1, 2, \dots$

(c) Let

$$C_D^2[0, 1] = \{v \in C^2[0, 1] : v(0) = v(1) = 0\}$$

and let the linear operator  $L_1 : C_D^2[0, 1] \rightarrow C[0, 1]$  be defined by

$$L_1 w = -w''.$$

Use what you know about the eigenfunctions of  $L_1$  to compute  $(\psi_{j,k}, \psi_{m,n})$  for  $j, k, m, n = 1, 2, \dots$

(d) The solution to  $Lu = f$  that we obtain using the spectral method is

$$u(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})} \psi_{j,k}(x, y).$$

Plot

$$u_N(x, y) = \sum_{j=1}^N \sum_{k=1}^N \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})} \psi_{j,k}(x, y)$$

for  $N = 1, 2, 3, 10$ . Note that, for  $j, k = 1, 2, \dots$ ,

$$(f, \psi_{j,k}) = 2 \frac{(-1)^{j+1}}{jk\pi^2}.$$

Also note that to plot  $\psi_{1,1}(x, y) = 2 \sin(\pi x) \sin(\pi y)$  you could use

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x = linspace(0,1,50);  
y = linspace(0,1,50);  
[X,Y] = meshgrid(x,y);  
ps11 = 2*sin(pi*X).*sin(pi*Y);  
surf(X,Y,ps11)
```