CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Examination 1 with Solutions

1. [25 points: (a) = 5, (b),(c) = 10]

   The derivation of the heat equation given at the beginning of the course arrived at the condition

$$\rho c \frac{\partial u}{\partial t}(x,t) + \frac{\partial q}{\partial x} = f(x)$$

   where $q(x,t)$ is the heat flux per unit volume, ρ is the density of the material and c is the specific heat. The relationship of *Fourier's Law*, which says that

$$q(x,t) = \kappa \frac{\partial u}{\partial x}(x,t),$$

   was used to derive the heat equation.

   Suppose that an alien substance has been found, for which Fourier's law has been shown not to hold, and that scientists have determined that a slightly different relation

$$q(x,t) = \kappa \frac{\partial u}{\partial x}(x,t) + \beta u(x,t)$$

   describes the heat flux more accurately.

   (a) Write the modified, time-dependent heat equation for the newly discovered substance.

   (b) Suppose a bar of length $L = 1$ of the alien substance is manufactured with $\kappa = 3$, $\beta = 2$. Determine a general formula for the steady-state termperature distribution of this bar, which should include two free constants (assume the source term $f(x) = 0$).
   Hint: You may wish to use the idea of "integrating factors". From ordinary differential equations, for any constant μ we have

$$\frac{\partial}{\partial x}\left(ue^{\mu x}\right) = \frac{\partial u}{\partial x}e^{\mu x} + \mu u e^{\mu x}.$$

   You may wish to manipulate the steady state equation such that it resembles the form of the hint, and then proceed in deriving the solution.

   (c) Find formulas for these free constants in the case where the following two conditions hold: the left end, at $x = 0$, has a fixed *heat flux* equal to 0, i.e.

$$q(0) = \kappa \frac{\partial u}{\partial x}(0) + \beta u(0) = 0$$

   and the temperature at $x = 1$ is held constant at δ degrees, so that

$$u(1) = \delta.$$

Solution

(a) Using the form $\rho c \frac{\partial u}{\partial t}(x,t) + \frac{\partial q}{\partial x} = f(x)$ with the heat flux given as $q(x,t) = \kappa \frac{\partial u}{\partial x}(x,t) + \beta u(x,t)$ gives the final modified heat equation :

$$\rho c \frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}\left(\kappa \frac{\partial u}{\partial x}(x,t) + \beta u(x,t)\right) = f(x)$$

(b) We want to find a general solution, involving two free constants, to the time-independent problem

$$\frac{\partial}{\partial x}\left(3\frac{\partial u}{\partial x}(x) + 2u(x)\right) = 0$$

Integrating both sides once with respect to x gives

$$3\frac{\partial u}{\partial x}(x,t) + 2u(x,t) = C_1$$

Dividing both sides by 3 and (in order to use the hint) multiplying each side by $e^{\frac{2}{3}x}$ gives

$$\frac{\partial u}{\partial x} e^{\frac{2}{3}x} + \frac{2}{3} u e^{\frac{2}{3}x} = C_1 e^{\frac{2}{3}x}$$

Which, again by the hint, can be rewritten as

$$\frac{\partial}{\partial x}\left(u e^{\frac{2}{3}x}\right) = C_1 e^{\frac{2}{3}x}$$

Integrating both sides with respect to x and dividing by $e^{\frac{2}{3}x}$ gives

$$u(x) = \frac{3}{2}C_1 + C_2 e^{-\frac{2}{3}x}$$

(c) The condition on the heat flux, with $\kappa = 3$ and $\beta = 2$ for this problem, gives

$$0 = q(0) = 3\frac{\partial u}{\partial x}(0) + 2u(0)$$

Using the general form from part (b) the expressions $\frac{\partial u}{\partial x}(0) = -\frac{2}{3}C_2$ and $u(0) = \frac{3}{2}C_1 + C_2$. It follows that $0 = q(0) = -3\left(\frac{2}{3}C_2\right) + 2\left(\frac{3}{2}C_1 + C_2\right) = -2C_2 + 3C_1 + 2C_2$ so that $C_1 = 0$ follows from this condition. The second boundary condition says that $u(L) = \delta$. Using $C_1 = 0$ in the general form from (b) gives $\delta = u(L) = C_2 e^{-\frac{2}{3}L}$ which determines $C_2 = \delta e^{\frac{2}{3}L}$. The solution to the modified, steady state heat equation with $\kappa = 3$, $\beta = 2$ and the given boundary conditions is therefore

$$u(x) = \delta e^{\frac{2}{3}(L-x)}$$

2. [25 points: (a) = 5, (b),(c) = 10]

In this problem, we will derive a finite difference discretization for slight variations on the steady state heat equation on the interval $[0,1]$. The motivation for finite difference equations is to satisfy an approximations to the differential equation at a finite number of points x_i by replacing $\frac{\partial^2 u(x_i)}{\partial x^2}$ with a central finite difference approximation

$$\frac{\partial^2 u(x_i)}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

where u_i approximates $u(x_i)$ at one the 5 points with spacing $h = 1/4$

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad x_4 = 1.$$

(a) Derive the finite difference equations for

$$-\frac{\partial^2 u(x)}{\partial x^2} = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = a, \quad u(1) = b,$$

where a, b are some non-zero (but unspecified) values. Write down the resulting matrix system for the unknowns u_1, u_2, u_3. Please write your answers in terms of h.

(b) Consider now the case of Neumann boundary conditions on both sides

$$u'(0) = u'(1) = 0.$$

We may approximate these boundary conditions with the forward difference

$$u'(0) = u'(x_0) \approx \frac{u_1 - u_0}{h} = 0$$

and the backward difference formula

$$u'(1) = u'(x_4) \approx \frac{u_4 - u_3}{h} = 0.$$

Using these approximations, write down the finite difference equations, and construct explicitly the corresponding matrix system. Please write your answers in terms of h, rather than using the specific values of x_i.

(c) Using the specific values of h and x_i given at the beginning of the problem and specific $f(x) = e^x$, set up the matrix equation $Au = b$ for part (b). Verify that $\mathbf{e}$ is in the null space of A, where $\mathbf{e} = (1,1,1)^T$ is the vector of all ones. What does this imply about the solution to the system?

(d) Derive the finite difference equations for the following modification of the heat equation:

$$\cos(x)u - \frac{\partial^2 u(x)}{\partial x^2} = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = u(1) = 0.$$

Write down the finite difference equations and the corresponding matrix system for the unknowns u_1, u_2, u_3. Please write your answers in terms of h, rather than using the specific values of x_i.

Solution.

(a) The finite difference approximation for second derivative $u''(x)$ around the point x_i can be written by central difference difference scheme

$$\frac{\partial^2 u}{\partial x^2}(x_i) = u''(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}.$$

for $i = 1, 2, 3$. This leads to three equations:

$$\begin{aligned} -\frac{u_2 - 2u_1 + u_0}{h^2} &= f(x_1) \\ -\frac{u_3 - 2u_2 + u_1}{h^2} &= f(x_2) \\ -\frac{u_4 - 2u_3 + u_2}{h^2} &= f(x_3). \end{aligned}$$

Since $u(x_0) = u(0) = a$ and $u(x_4) = u(1) = b$, we may substitute those values in to get

$$\begin{aligned} \frac{-u_2 + 2u_1}{h^2} &= f(x_1) + \frac{a}{h^2} \\ \frac{-u_3 + 2u_2 - u_1}{h^2} &= f(x_2) \\ \frac{2u_3 - u_2}{h^2} &= f(x_3) + \frac{b}{h^2}. \end{aligned}$$

The resulting matrix system is

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f(x_1) + \frac{a}{h^2} \\ f(x_2) \\ f(x_3) + \frac{b}{h^2} \end{bmatrix}$$

(b) If we approximate Neumann boundary conditions using finite differences, we have

$$u_0 = u_1, \qquad u_3 = u_4.$$

Then, we may substitute these values into the equations:

$$\begin{aligned} \frac{-u_2 + 2u_1 - u_0}{h^2} &= f(x_1) \\ \frac{-u_3 + 2u_2 - u_1}{h^2} &= f(x_2) \\ \frac{-u_4 + 2u_3 - u_2}{h^2} &= f(x_3). \end{aligned}$$

This reduces to

$$\begin{aligned} \frac{-u_2 + u_1}{h^2} &= f(x_1) \\ \frac{-u_3 + 2u_2 - u(x_1)}{h^2} &= f(x_2) \\ \frac{u_3 - u_2}{h^2} &= f(x_3). \end{aligned}$$

The corresponding matrix system can be written

$$\frac{1}{h^2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}.$$

(c) Substituting in the specific value of $h = 1/4$ (or $1/h = 4$)

$$16\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} e^{x_1} \\ e^{x_2} \\ e^{x_3} \end{bmatrix}.$$

If we multiply the system matrix by $\mathbf{e} = (1,1,1)^T$, we can see that

$$16\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1-1) \\ 4(-1+2-1) \\ 4(1-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This implies that the system matrix has a non-trivial nullspace, so that the solution to the finite difference matrix equation is non-unique.

(d) The finite difference equations for this modification of the heat equation give

$$\begin{aligned} \cos(x_1)u(x_1) + \frac{-u(x_2) + 2u(x_1) - u(x_0)}{h^2} &= f(x_1) \\ \cos(x_2)u(x_2) + \frac{-u(x_3) + 2u(x_2) - u(x_1)}{h^2} &= f(x_2) \\ \cos(x_3)u(x_3) + \frac{-u(x_4) + 2u(x_3) - u(x_2)}{h^2} &= f(x_3). \end{aligned}$$

Applying zero Dirichlet boundary conditions removes u_4 and u_0 from the above equations, and the resulting matrix equation gives

$$\left(\begin{bmatrix} \cos(x_1) & 0 & 0 \\ 0 & \cos(x_2) & 0 \\ 0 & 0 & \cos(x_3) \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}.$$

3. [25 points: (a), (b) = 10,(c)=5]

   In class we considered inner products for vectors in $\mathbb{R}^n$ and functions in $C[0,1]$, and used these inner products to generate best approximations. This problem will introduce you to an inner product of *matrices* in $\mathbb{R}^{n\times n}$. (Related math arises in many modern applications; e.g., at the roots of how Netflix recommends movies.)

   Let $V = \mathbb{R}^{2\times 2}$, the set of all 2×2 matrices, and define the inner product for matrices as

$$(\mathbf{A},\mathbf{B}) = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}b_{jk},$$

   where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

   (a) Verify that $(\cdot,\cdot)$ is an inner product on $V = \mathbb{R}^{2\times 2}$.

   (b) Given that V (the set of all possible 2×2 matrices) is a vector space, show that the set V_s of all symmetric 2×2 matrices

$$V_s = \left\{ \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad a_{12} = a_{21} \right\}$$

   is a subspace of V.

   (c) Consider the operators $S : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ and $K : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$

$$S(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T, \qquad K(\mathbf{A}) = \mathbf{A} - \mathbf{A}^T,$$

   where $\mathbf{A}^T$ is the matrix transpose

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

   Show that both of these operators are linear operators.

   (d) Show that, for any $\mathbf{A} \in \mathbb{R}^{2\times 2}$,

$$S(K(\mathbf{A})) = K(S(\mathbf{A})) = \mathbf{0}$$

   where $\mathbf{0}$ is the 2×2 zero matrix. Use this fact to explain how the range and null space of S and range and null space of K are related.

Solution.

(a) We need to show 2 properties:

- $(\alpha\mathbf{A}+\beta\mathbf{B},\mathbf{C}) = \alpha(\mathbf{A},\mathbf{C})+\beta(\mathbf{B},\mathbf{C})$: we may show this using the formula

$$\begin{aligned}(\alpha\mathbf{A}+\beta\mathbf{B},\mathbf{C}) &= \sum_{j=1}^{2}\sum_{k=1}^{2}(\alpha a_{jk}+\beta b_{jk})c_{jk}\\ &= \sum_{j=1}^{2}\sum_{k=1}^{2}(\alpha a_{jk}c_{jk}+\beta b_{jk}c_{jk})\\ &= \alpha\sum_{j=1}^{2}\sum_{k=1}^{2}a_{jk}c_{jk}+\beta\sum_{j=1}^{2}\sum_{k=1}^{2}b_{jk}c_{jk}\\ &= \alpha(\mathbf{A},\mathbf{C})+\beta(\mathbf{B},\mathbf{C}).\end{aligned}$$

- $(\mathbf{A},\mathbf{A})\geq 0$, and $(\mathbf{A},\mathbf{A})=0$ only if $\mathbf{A}$ is the zero matrix. This may be shown by again using the formula

$$(\mathbf{A},\mathbf{A}) = \sum_{j=1}^{2}\sum_{k=1}^{2}a_{jk}^2 \geq 0,$$

because a_{jk}^2 is positive. If $(\mathbf{A},\mathbf{A})=0$, then a_{jk}^2 must equal 0, and then the matrix $\mathbf{A}=0$.

(b) Let $\mathbf{A},\mathbf{B}\in V_s$. We just need to show that $\alpha\mathbf{A}+\beta\mathbf{B}\in V_s$, or that $\alpha\mathbf{A}+\beta\mathbf{B}$ is a symmetric matrix. Note that

$$\alpha\mathbf{A}+\beta\mathbf{B} = \begin{bmatrix}\alpha a_{11}+\beta b_{11} & \alpha a_{12}+\beta b_{12}\\ \alpha a_{21}+\beta b_{21} & \alpha a_{22}+\beta b_{22}\end{bmatrix}.$$

Since $\mathbf{A},\mathbf{B}\in V_s$, $a_{12}=a_{21}$ and $b_{12}=b_{21}$, so

$$\alpha\mathbf{A}+\beta\mathbf{B} = \begin{bmatrix}\alpha a_{11}+\beta b_{11} & \alpha a_{12}+\beta b_{12}\\ \alpha a_{21}+\beta b_{21} & \alpha a_{22}+\beta b_{22}\end{bmatrix} = \begin{bmatrix}\alpha a_{11}+\beta b_{11} & \alpha a_{12}+\beta b_{12}\\ \alpha a_{12}+\beta b_{12} & \alpha a_{22}+\beta b_{22}\end{bmatrix}.$$

The result is a symmetric matrix, so $\alpha\mathbf{A}+\beta\mathbf{B}\in V_s$, and V_s is a subspace of $\mathbb{R}^{2\times 2}$.

(c) For S and K to be linear operators, we need

$$S(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha S(\mathbf{A})+\beta S(\mathbf{B}),\quad K(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha K(\mathbf{A})+\beta K(\mathbf{B}).$$

You may prove this element-wise or using matrices: for S, we have

$$S(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha\mathbf{A}+\alpha\mathbf{A}^T+\beta\mathbf{B}+\beta\mathbf{B}^T = \alpha(\mathbf{A}+\mathbf{A}^T)+\beta(\mathbf{B}+\mathbf{B}^T) = \alpha S(\mathbf{A})+\beta S(\mathbf{B}).$$

and for K we have

$$K(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha\mathbf{A}-\alpha\mathbf{A}^T+\beta\mathbf{B}-\beta\mathbf{B}^T = \alpha(\mathbf{A}-\mathbf{A}^T)+\beta(\mathbf{B}-\mathbf{B}^T) = \alpha K(\mathbf{A})+\beta K(\mathbf{B}).$$

(d) We can show the result by noting that $(\mathbf{A}^T)^T=\mathbf{A}$:

$$S(K(\mathbf{A})) = S(\mathbf{A}-\mathbf{A}^T) = (\mathbf{A}-\mathbf{A}^T)+(\mathbf{A}-\mathbf{A}^T)^T = \mathbf{A}-\mathbf{A}^T+\mathbf{A}^T-\mathbf{A} = 0$$

$$K(S(\mathbf{A})) = K(\mathbf{A}+\mathbf{A}^T) = (\mathbf{A}+\mathbf{A}^T)-(\mathbf{A}+\mathbf{A}^T)^T = \mathbf{A}+\mathbf{A}^T-\mathbf{A}^T-\mathbf{A} = 0$$

What this implies is that the range of S (and possible matrix $\mathbf{A}$ such that $\mathbf{A}=S(\mathbf{B})$ for some $\mathbf{B}$) is contained in the null space of K, and the range of K is contained in the null space of S (the two are actually equal, which also receives full credit as an answer).

4. [25 points: (a) = 10, (b), (c), (d) = 5]

Consider the vector space V of polynomials of degree less than or equal to three. This vector space is the span of the monomials $\{1, x, x^2, x^3\}$. Let $(\cdot,\cdot): \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be an inner product on V defined by

$$(f, g) = \int_0^1 f(x)g(x)\,dx$$

(a) Consider the subspace $W \subset V$ of linear functions given by the span of $\{1, x\}$. Find the best approximation $m(x) \in W$ to the function $g(x) \in V$ defined by

$$g(x) = 2x^3 - x^2 - x + \frac{1}{2}$$

by solving the 2×2 Gram system $Gx = b$. You may use the fact that the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(b) Let $p(x) = a + bx$ be an arbitrary element of the subspace of linear functions W. Show explicitly the orthogonality condition of the best approximation $m(x)$ holds by computing $(g(x) - m(x), p(x))$ and showing that it must be zero.

(c) Consider the linearly independent set of functions $\{u_1, u_2\}$ in W given by $u_1(x) = x$ and $u_2(x) = x + 2$. Apply the Gram-Schmidt procedure to construct an alternate, orthogonal basis for W. Your basis does not need to be orthonormal.

Recall: given a vector v_1, the Gram-Schmidt procedure modifies the vector v_2 as $v_2 \to v_2 - \mathrm{proj}_{v_1}(v_2) = v_2 - \frac{(v_2, v_1)}{(v_1, v_1)} v_1$

(d) Consider another subspace Y of V defined by the span of the following functions

$$\mathrm{span}\left\{7, 2x, x^2 + 5x, 13x^3\right\}.$$

What is the best approximation $y(x)$ to $g(x)$ in Y, where

$$g(x) = 2x^3 - x^2 - x + \frac{1}{2}.$$

Justify your answer without using any computations.

Solution

(a) To solve this problem we first set up the associated Gram system $Gx = b$ where

$$G = \begin{bmatrix} (1,1) & (x,1) \\ (1,x) & (x,x) \end{bmatrix}, \quad b = \begin{bmatrix} (1,g) \\ (x,g) \end{bmatrix}$$

Using the definition of the inner prodct to evaluate each of the above vector entries yields

$$G = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}, \quad b = \begin{bmatrix} 1/6 \\ 1/15 \end{bmatrix}$$

Using the hint we can solve $Gx = b$ by multiplying both sides by G^{-1} giving $x = G^{-1}b$ which gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} 1/6 \\ 1/15 \end{bmatrix} = \begin{bmatrix} 4/15 \\ -1/5 \end{bmatrix}$$

So that the best approximation to $g(x)$ in W is $m(x) = \left(\frac{4}{15}\right) 1 + \left(-\frac{1}{5}\right) x = -\frac{1}{5}x + \frac{4}{15}$

(b) We have $g(x) - m(x) = 2x^3 - x^2 - \frac{4}{5}x + \frac{7}{30}$. To show that $g(x)-m(x)$ is orthogonal to any polynomial of the form $p(x) = ax + b$ in W it sufficies to show that $(g(x) - m(x), 1) = 0$ and $(g(x) - m(x), x) = 0$ since $(g(x) - m(x), ax+b) = a(g(x) - m(x), x) + b(g(x) - m(x), 1)$. So that the general result follows from the two integrals below

$$(g(x) - m(x), 1) = \int_0^1 2x^3 - x^2 - \frac{4}{5}x + \frac{7}{30}\, dx = \left. \frac{2x^4}{4} - \frac{x^3}{3} - \frac{4x^2}{10} + \frac{7x}{30} \right|_0^1 = 0$$

$$(g(x) - m(x), x) = \int_0^1 2x^4 - x^3 - \frac{4}{5}x^2 + \frac{7}{30}x\, dx = \left. \frac{2x^5}{5} - \frac{x^4}{4} - \frac{4x^3}{15} + \frac{7x^2}{60} \right|_0^1 = 0$$

(c) Take the first vector of the new basis to be $w_1 = x$. Applying the formula given on the exam the second vector is given by the following calculation

$$w_2 = (x+2) - \frac{(x+2, x)}{(x,x)}x = (x+2) - \frac{4/3}{1/3}x = -3x + 2$$

You can check that $w_1 = x$ and $w_2 = -3x + 2$ satisfy $(w_1, w_2) = 0$. Therefore $\{x, -3x+2\}$ is an alternate basis for $W = \text{span}\{1, x\}$ consisting of orthogonal vectors.

(d) The vectors $\{7, 2x, x^2 + 5x, 13x^3\}$ are all linearly independent and there are four of them. Therefore $Y = \text{span}\{7, 2x, x^2 + 5x, 13x^3\}$ is a 4-dimensional subspace of the vector space $V = \text{span}\{1, x, x^2, x^3\}$. However, V is also 4-dimensional so that Y must be **all** of V. So $g(x) \in V$ implies $g(x) \in Y$ and so the best approximation to $g(x)$ in Y is, of course, $g(x)$ itself. That is, $m(x) = g(x)$.