CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 13 · Solutions

Posted Monday 3 February 2014. Due 1pm Friday 14 February 2014.

13. [25 points]

Let the operator $L:\ C^2[0,1] \to C[0,1]$ be defined by

$$Lv = -v'' + 9v.$$

Let $u \in C^2[0,1]$ be the solution to the differential equation

$$-u''(x) + 9u(x) = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = \alpha$$

and

$$u(1) = \beta$$

where $f \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Note that

$$(Lu)(x) = -u''(x) + 9u(x)$$

for all $x \in [0,1]$. Let N be an integer which is such that $N \geq 2$ and let $h = \dfrac{1}{N+1}$ and $x_j = jh$ for $j = 0, \ldots, N+1$.

(a) Determine whether or not L is a linear operator.

(b) By using the approximation

$$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

for $j = 1, \ldots, N$ we can write

$$\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} \approx \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}$$

where $\mathbf{D} \in \mathbb{R}^{N \times (N+2)}$. What are the entries of the matrix $\mathbf{D}$? An acceptable way to present your final answer is

$$D_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise;} \end{cases}$$

with the question marks replaced with the correct values.

(c) We can use the differential equation and boundary conditions satisfied by u and the approximation from the previous part to write

$$\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ and $\mathbf{b} \in \mathbb{R}^N$. What are the entries of the matrix $\mathbf{A}$ and the vector $\mathbf{b}$? An acceptable way to present your final answer is

$$A_{jk} = \begin{cases} ? & \text{if } k = ?; \\ ? & \text{if } k = ? \text{ or } k = ?; \\ ? & \text{otherwise}; \end{cases}$$

and

$$b_j = \begin{cases} ? & \text{if } j = ?; \\ ? & \text{if } j = ?; \\ ? & \text{otherwise}; \end{cases}$$

with the question marks replaced with the correct values.

(d) Let $f(x) = 18$, $\alpha = \beta = 0$ and $N = 2$. Obtain approximations u_1 and u_2 to $u(x_1)$ and $u(x_2)$, respectively, by solving

$$\mathbf{A}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{b}$$

by hand.

Solution.

(a) [5 points] If $v \in C^2[0,1]$ and $w \in C^2[0,1]$ then

$$L(v+w) = -(v+w)'' + 9(v+w) = -v'' - w'' + 9v + 9w = -v'' + 9v - w'' + 9w = Lv + Lw$$

and so $L(v+w) = Lv + Lw$ for all $v, w \in C^2[0,1]$. If $v \in C^2[0,1]$ and $\gamma \in \mathbb{R}$ then

$$L(\gamma v) = -(\gamma v)'' + 9(\gamma v) = -\gamma v'' + 9\gamma v = \gamma(-v'' + 9v) = \gamma L v$$

and so $L(\gamma v) = \gamma L v$ for all $v \in C^2[0,1]$ and all $\gamma \in \mathbb{R}$. Consequently, L is a linear operator.

(b) [5 points] For $j = 1, 2, \ldots, N$, using the approximation

$$u''(x_j) \approx \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

yields that

$$\begin{aligned}(Lu)(x_j) &= -u''(x_j) + 9u(x_j) \approx -\frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + 9u(x_j) \\ &= -\frac{1}{h^2}u(x_{j-1}) + \left(\frac{2}{h^2} + 9\right)u(x_j) - \frac{1}{h^2}u(x_{j+1}).\end{aligned}$$

So,

$$
\begin{aligned}
&\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} \\
&\approx \begin{bmatrix} -\frac{1}{h^2}u(x_0) + \left(\frac{2}{h^2}+9\right)u(x_1) - \frac{1}{h^2}u(x_2) \\ -\frac{1}{h^2}u(x_1) + \left(\frac{2}{h^2}+9\right)u(x_2) - \frac{1}{h^2}u(x_3) \\ \vdots \\ -\frac{1}{h^2}u(x_{N-2}) + \left(\frac{2}{h^2}+9\right)u(x_{N-1}) - \frac{1}{h^2}u(x_N) \\ -\frac{1}{h^2}u(x_{N-1}) + \left(\frac{2}{h^2}+9\right)u(x_N) - \frac{1}{h^2}u(x_{N+1}) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix} \\
&= \mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}
\end{aligned}
$$

where $\mathbf{D} \in \mathbb{R}^{N\times(N+2)}$ is the matrix with entries

$$
D_{jk} = \begin{cases} \frac{2}{h^2}+9 & \text{if } k = j+1; \\ -\frac{1}{h^2} & \text{if } k = j \text{ or } k = j+2; \\ 0 & \text{otherwise.} \end{cases}
$$

(c) [10 points] Since

$$
(Lu)(x) = f(x), \quad 0 < x < 1
$$

we have that

$$
\begin{bmatrix} (Lu)(x_1) \\ (Lu)(x_2) \\ \vdots \\ (Lu)(x_{N-1}) \\ (Lu)(x_N) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix}
$$

for $j = 1, \ldots, N$. Using the approximation obtained in the previous part then yields that

$$
\mathbf{D} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix} \approx \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix}
$$

Moreover,

$$
\begin{aligned}
\mathbf{D}\begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ u(x_{N+1}) \end{bmatrix}
&= \mathbf{D}\left(\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} + \begin{bmatrix} u(x_0) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ u(x_{N+1}) \end{bmatrix}\right) \\
&= \mathbf{D}\left(\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix}\right) \\
&= \mathbf{D}\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} + \mathbf{D}\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix}
\end{aligned}
$$

since $u(x_0) = u(0) = \alpha$ and $u(x_{N+1}) = u(1) = \beta$. Furthermore,

$$
\begin{aligned}
&\mathbf{D}\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix}
-\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2}
\end{bmatrix}
\begin{bmatrix} 0 \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix}
\frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\
0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} \\
0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9
\end{bmatrix}
\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix}
\end{aligned}
$$

and so

$$\mathbf{A}\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix} \approx \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is the matrix with entries

$$A_{jk} = \begin{cases} \frac{2}{h^2}+9 & \text{if } k=j; \\ -\frac{1}{h^2} & \text{if } k=j-1 \text{ or } k=j+1; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mathbf{b} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} - \mathbf{D}\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix}.$$

Now,

$$\begin{aligned}
&\mathbf{D}\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix} \\
&= \begin{bmatrix}
-\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+9 & -\frac{1}{h^2}
\end{bmatrix}\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\alpha}{h^2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\frac{\beta}{h^2} \end{bmatrix}
\end{aligned}$$

and so

$$\mathbf{b} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} - \begin{bmatrix} -\frac{\alpha}{h^2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\frac{\beta}{h^2} \end{bmatrix} = \begin{bmatrix} f(x_1)+\frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N)+\frac{\beta}{h^2} \end{bmatrix}.$$

Hence $\mathbf{b} \in \mathbb{R}^N$ is the vector with entries

$$b_j = \begin{cases} f(x_1) + \frac{\alpha}{h^2} & \text{if } j = 1; \\ f(x_N) + \frac{\beta}{h^2} & \text{if } j = N; \\ f(x_j) & \text{otherwise.} \end{cases}$$

(d) [5 points] When $N = 2$, $h = \frac{1}{2+1} = \frac{1}{3}$ and so $h^2 = \frac{1}{3^2} = \frac{1}{9}$ and hence $\frac{1}{h^2} = 9$ and $\frac{2}{h^2} = 18$. Therefore,

$$\mathbf{A} = \begin{bmatrix} 18+9 & -9 \\ -9 & 18+9 \end{bmatrix} = \begin{bmatrix} 27 & -9 \\ -9 & 27 \end{bmatrix}.$$

Moreover, when $N = 2$, $f(x) = 18$ and $\alpha = \beta = 0$,

$$\mathbf{b} = \begin{bmatrix} 18 \\ 18 \end{bmatrix}.$$

Consequently, we have that

$$\begin{bmatrix} 27 & -9 \\ -9 & 27 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \end{bmatrix}$$

and hence

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{9-1} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 6+2 \\ 2+6 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 8 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$