CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 44 · Solutions

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

44. [25 points]

Let
$$\Omega = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$
and let $u_0(x,y) = 200xy(1-x)(1-y)(x-\frac{1}{4})(y-\frac{1}{4})$. Note that, for $m,n = 1,2,3,\ldots$,
$$\int_0^1 \int_0^1 2u_0(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy = \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}.$$
In this question we will consider the problem of finding the solution $u(x,y,t)$ to the wave equation
$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$
with homogeneous Dirichlet boundary conditions
$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$
and initial conditions
$$u(x,y,0) = u_0(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$
and
$$u_t(x,y,0) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$
Let
$$C_D^2(\Omega) = \left\{v \in C^2(\Omega) : v(x,0) = v(x,1) = v(0,y) = v(1,y) = 0,\ 0 \le x \le 1,\ 0 \le y \le 1\right\}.$$
Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by
$$(Lv)(x,y) = -\left(v_{xx}(x,y) + v_{yy}(x,y)\right).$$
The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions
$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$
for $j,k = 1,2,\ldots$, which are such that
$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$
for $j,k = 1,2,\ldots$. Recall that in Homework 42 you obtained a formula for $\lambda_{j,k}$ for $j,k = 1,2,\ldots$.

(a) We can write
$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$
What ordinary differential equation and initial conditions does $a_{j,k}(t)$ satisfy for $j,k = 1,2,\ldots$?

(b) Obtain an expression for $a_{j,k}(t)$ for $j,k = 1,2,\ldots$.

(c) Use you answer to part (b) to write out a formula for $u(x,y,t)$.

(d) Plot

$$u_{10}(x,y,t) = \sum_{j=1}^{10}\sum_{k=1}^{10} a_{j,k}(t)\psi_{j,k}(x,y)$$

at times $t = 0, 0.5, 1.0, 1.5, 2.5, 3$. Use the command `zlim([-2 2])` so that the axes on all of your plots are the same.

Solution.

(a) [7 points] Substituting the expression for $u(x,y,t)$ into the partial differential equation yields

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a''_{j,k}(t)\psi_{j,k}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\left(-\left(L\psi_{j,k}\right)(x,y)\right)$$

and hence

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a''_{j,k}(t)\psi_{j,k}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(-\lambda_{j,k}\right) a_{j,k}(t)\psi_{j,k}(x,y).$$

We can then say that

$$\begin{aligned}&\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a''_{j,k}(t)\int_0^1\int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy\\ =&\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(-\lambda_{j,k}\right) a_{j,k}(t)\int_0^1\int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy\end{aligned}$$

for $m,n = 1,2,\ldots$, from which it follows that

$$a''_{m,n}(t) = -\lambda_{m,n}a_{m,n}(t),$$

for $m,n = 1,2,\ldots$, since

$$\int_0^1\int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \left\{\begin{array}{ll} 1 & \text{if } j = m \text{ and } k = n\\ 0 & \text{if } j \neq m \text{ or } k \neq n\end{array}\right.$$

for $j,k,m,n = 1,2,\ldots$.
Also,

$$u(x,y,0) = u_0(x,y)$$

means that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y) = u_0(x,y)$$

and so

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\int_0^1\int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \int_0^1\int_0^1 u_0(x,y)\psi_{m,n}(x,y)\,dx\,dy,$$

for $m,n = 1,2,\ldots$, from which it follows that

$$a_{m,n}(0) = \int_0^1\int_0^1 u_0(x,y)\psi_{m,n}(x,y)\,dx\,dy,$$

for $m,n = 1,2,\ldots$, since

$$\int_0^1\int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \left\{\begin{array}{ll} 1 & \text{if } j = m \text{ and } k = n\\ 0 & \text{if } j \neq m \text{ or } k \neq n\end{array}\right.$$

for $j, k, m, n = 1, 2, \ldots$.

Moreover,

$$u_t(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a'_{j,k}(t) \psi_{j,k}(x, y).$$

Hence,

$$u_t(x, y, 0) = 0$$

means that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a'_{j,k}(0) \psi_{j,k}(x, y) = 0$$

and so

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a'_{j,k}(0) \int_0^1 \int_0^1 \psi_{j,k}(x, y) \psi_{m,n}(x, y) \, dx \, dy = \int_0^1 \int_0^1 0 \, dx \, dy,$$

for $m, n = 1, 2, \ldots$, from which it follows that

$$a'_{m,n}(0) = 0,$$

for $m, n = 1, 2, \ldots$, since

$$\int_0^1 \int_0^1 \psi_{j,k}(x, y) \psi_{m,n}(x, y) \, dx \, dy = \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{if } j \neq m \text{ or } k \neq n \end{cases}$$

for $j, k, m, n = 1, 2, \ldots$, and

$$\int_0^1 \int_0^1 0 \, dx \, dy = 0.$$

Hence, for $j, k = 1, 2, \ldots$, $a_{j,k}(t)$ is the solution to the differential equation

$$a''_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t)$$

with initial conditions

$$a_{j,k}(0) = \int_0^1 \int_0^1 u_0(x, y) \psi_{j,k}(x, y) \, dx \, dy$$

and

$$a'_{j,k}(0) = 0.$$

(b) [4 points] For $j, k = 1, 2, \ldots$, the differential equation $a''_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t)$ has solutions of the form

$$a_{j,k}(t) = A_{j,k} \sin\left(\sqrt{\lambda_{j,k}} t\right) + B_{j,k} \cos\left(\sqrt{\lambda_{j,k}} t\right)$$

with the constants $A_{j,k}$ and $B_{j,k}$ being determined by the initial conditions. Evaluating the general solution at $t = 0$ gives

$$B_{j,k} = a_{j,k}(0)$$

and so

$$a_{j,k}(t) = A_{j,k} \sin\left(\sqrt{\lambda_{j,k}} t\right) + \frac{100(5 + 7(-1)^j)(5 + 7(-1)^k)}{j^3 k^3 \pi^6} \cos\left(\sqrt{\lambda_{j,k}} t\right)$$

since

$$a_{j,k}(0) = \int_0^1 \int_0^1 u_0(x, y) \psi_{j,k}(x, y) \, dx \, dy = \frac{100(5 + 7(-1)^j)(5 + 7(-1)^k)}{j^3 k^3 \pi^6}.$$

Computing the derivative

$$a'_{j,k}(t) = A_{j,k} \sqrt{\lambda_{j,k}} \cos(\sqrt{\lambda_{j,k}} t) - \frac{100(5 + 7(-1)^j)(5 + 7(-1)^k)}{j^3 k^3 \pi^6} \sqrt{\lambda_{j,k}} \sin(\sqrt{\lambda_{j,k}} t)$$

and evaluating it at $t = 0$ then gives

$$a'_{j,k}(0) = A_{j,k}\sqrt{\lambda_{j,k}}$$

and so

$$A_{j,k} = 0$$

since

$$a'_{j,k}(0) = 0.$$

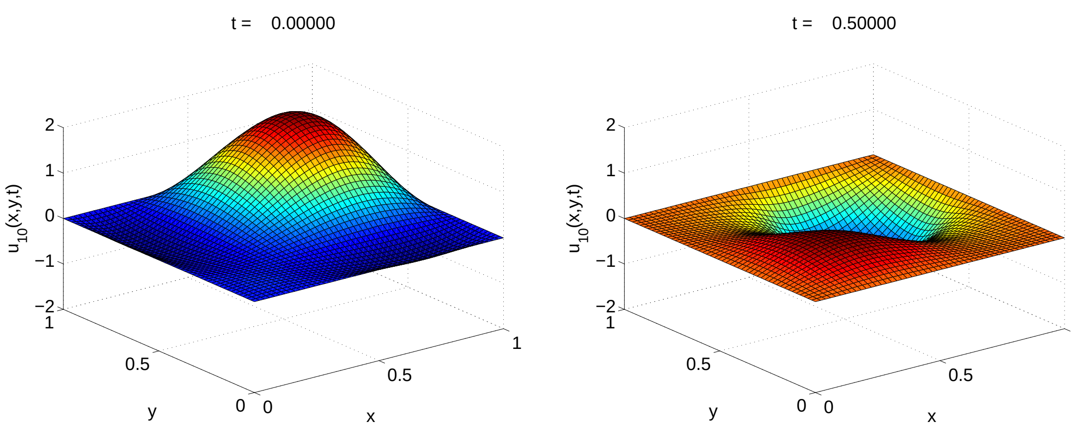
Therefore, for $j, k = 1, 2, \ldots$,

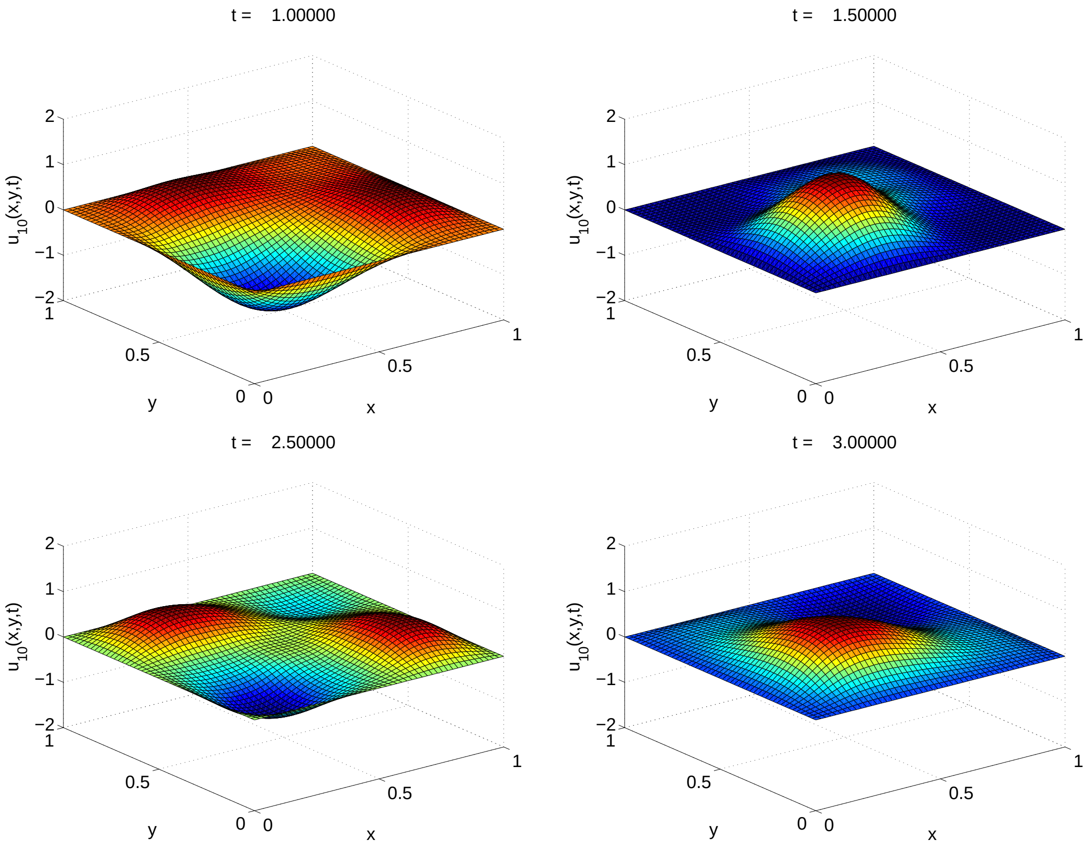
$$\begin{aligned} a_{j,k}(t) &= \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\cos\left(\sqrt{\lambda_{j,k}}t\right) \\ &= \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\cos\left(\pi\sqrt{j^2+k^2}t\right). \end{aligned}$$

(c) [4 points] We can write

$$\begin{aligned} u(x,y,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y) \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\cos\left(\pi\sqrt{j^2+k^2}t\right)\psi_{j,k}(x,y) \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{200(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\cos\left(\pi\sqrt{j^2+k^2}t\right)\sin(j\pi x)\sin(k\pi y). \end{aligned}$$

(d) [10 points] The requested plots are below.





The code that produced these plots follows.

```
clear
clc
N = 50;
x = linspace(0,1,N);
y = linspace(0,1,N);
[X,Y] = meshgrid(x,y);
nmax = 10;
tvec = [0 0.5 1 1.5 2.5 3];
for m=1:length(tvec)
    t = tvec(m);
    figure(1)
    clf
    U = zeros(N,N);
    for j=1:nmax
        for k=1:nmax
            ajk = 100*(5+7*(-1)^j)*(5+7*(-1)^k)/(j^3*k^3*pi^6);
            lamjk = (j^2+k^2)*(pi^2);
            psijk = 2*sin(j*pi*X).*sin(k*pi*Y);
            U = U + ajk*cos(sqrt(lamjk)*t)*psijk;
        end
    end
    surf(X,Y,U)
    axis([0 1 0 1 -2 2])
    set(gca,'fontsize',18)
    xlabel('x')
    ylabel('y')
    zlabel('u_{10}(x,y,t)')
    title(sprintf('t = %10.5f\n', t))
    eval(sprintf('print -depsc2 wave2d_%d',m))
 end
```