CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 18 · Solutions

Posted Friday 14 February 2014. Due 1pm Friday 21 February 2014.

18. [25 points]

All parts of this question should be done by hand.

Let the inner product $(\cdot,\cdot) : \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined by

$$(\mathbf{A},\mathbf{B}) = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}b_{jk},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

With this inner product we associate the norm $\|\cdot\| : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\|\mathbf{A}\| = \sqrt{(\mathbf{A},\mathbf{A})}$. In the parts below best approximation is defined to be best approximation with respect to this norm.

(a) Verify that $(\cdot,\cdot)$ is an inner product on $\mathbb{R}^{2\times 2}$.

(b) Consider the subspace of $\mathbb{R}^{2\times 2}$ consisting of symmetric matrices:

$$V_3 = \{\mathbf{A} \in \mathbb{R}^{2\times 2} : \mathbf{A} = \mathbf{A}^T\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Compute the best approximation $\mathbf{M}_3$ to the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

from the subspace V_3.

(c) Now let $\mathbf{M} \in \mathbb{R}^{2\times 2}$ be any 2×2 matrix, and let $\mathbf{M}_3$ be its best approximation from V_3. Explain why the error $\mathbf{M} - \mathbf{M}_3$ must *always* have zero diagonal entries.

(d) Carefully consider the subspace

$$\widehat{V}_3 = \{\mathbf{A} \in \mathbb{R}^{2\times 2} : \mathbf{A} = \mathbf{A}^T\} = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}.$$

What is the best approximation $\widehat{\mathbf{M}}_3$ to the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

from the subspace $\widehat{V}_3$?

Solution.

(a) [6 points] If $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\mathbf{B} \in \mathbb{R}^{2\times 2}$ then

$$(\mathbf{A},\mathbf{B}) = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}b_{jk} = \sum_{j=1}^{2}\sum_{k=1}^{2} b_{jk}a_{jk} = (\mathbf{B},\mathbf{A})$$

and so $(\mathbf{A},\mathbf{B}) = (\mathbf{B},\mathbf{A})$ for all $\mathbf{A},\mathbf{B} \in \mathbb{R}^{2\times 2}$.
Also, if $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathbb{R}^{2\times 2}$ and $\alpha,\beta \in \mathbb{R}$ then

$$\begin{aligned}
(\alpha\mathbf{A}+\beta\mathbf{B},\mathbf{C}) &= \sum_{j=1}^{2}\sum_{k=1}^{2} (\alpha a_{jk} + \beta b_{jk})\, c_{jk} \\
&= \sum_{j=1}^{2}\sum_{k=1}^{2} (\alpha a_{jk}c_{jk} + \beta b_{jk}c_{jk}) \\
&= \sum_{j=1}^{2}\sum_{k=1}^{2} \alpha a_{jk}c_{jk} + \sum_{j=1}^{2}\sum_{k=1}^{2} \beta b_{jk}c_{jk} \\
&= \alpha\sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}c_{jk} + \sum_{j=1}^{2}\beta\sum_{k=1}^{2} b_{jk}c_{jk} \\
&= \alpha(\mathbf{A},\mathbf{C}) + \beta(\mathbf{B},\mathbf{C})
\end{aligned}$$

and so $(\alpha\mathbf{A}+\beta\mathbf{B},\mathbf{C}) = \alpha(\mathbf{A},\mathbf{C}) + \beta(\mathbf{B},\mathbf{C})$ for all $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathbb{R}^{2\times 2}$ and all $\alpha,\beta \in \mathbb{R}$.
Moreover, if $\mathbf{A} \in \mathbb{R}^{2\times 2}$ then

$$(\mathbf{A},\mathbf{A}) = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}^2 \geq 0$$

and if $(\mathbf{A},\mathbf{A}) = 0$ then since $a_{jk} \in \mathbb{R}$, $a_{11}^2 = a_{12}^2 = a_{21}^2 = a_{22}^2 = 0$ from which it follows that $a_{11} = a_{12} = a_{21} = a_{22} = 0$. So, $(\mathbf{A},\mathbf{A}) \geq 0$ for all $\mathbf{A} \in \mathbb{R}^{2\times 2}$ with $(\mathbf{A},\mathbf{A}) = 0$ only if $\mathbf{A} = \mathbf{0}$.
Consequently, $(\cdot,\cdot)$ is an inner product on $\mathbb{R}^{2\times 2}$.

(b) [7 points] Let $\mathbf{w}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since, $(\mathbf{w}_1,\mathbf{w}_2) = (\mathbf{w}_1,\mathbf{w}_3) = (\mathbf{w}_2,\mathbf{w}_3) = 0$, the best approximation to $\mathbf{M}$ from V_3 is

$$\mathbf{M}_3 = \sum_{j=1}^{3} \frac{(\mathbf{M},\mathbf{w}_j)}{(\mathbf{w}_j,\mathbf{w}_j)}\mathbf{w}_j.$$

Now, $(\mathbf{M},\mathbf{w}_1) = 1$, $(\mathbf{M},\mathbf{w}_2) = 9$ and $(\mathbf{M},\mathbf{w}_3) = 10$. Also, $(\mathbf{w}_1,\mathbf{w}_1) = 1$, $(\mathbf{w}_2,\mathbf{w}_2) = 1$ and $(\mathbf{w}_3,\mathbf{w}_3) = 2$. Therefore,

$$\mathbf{M}_3 = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{9}{1}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{10}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 9 \end{bmatrix}.$$

(c) [6 points] If $\mathbf{M}_3$ is the best approximation to $\mathbf{M}$ from V_3 with respect to the norm $\|\cdot\|$ then the error $\mathbf{M} - \mathbf{M}_3$ will be such that

$$(\mathbf{M} - \mathbf{M}_3, \mathbf{v}) = 0 \text{ for all } \mathbf{v} \in V_3.$$

So, $(\mathbf{M} - \mathbf{M}_3, \mathbf{w}_1) = 0$ and $(\mathbf{M} - \mathbf{M}_3, \mathbf{w}_2) = 0$. Hence, since $(\mathbf{M} - \mathbf{M}_3, \mathbf{w}_1) = (\mathbf{M} - \mathbf{M}_3)_{11}$ and $(\mathbf{M} - \mathbf{M}_3, \mathbf{w}_2) = (\mathbf{M} - \mathbf{M}_3)_{22}$, the diagonal entries of $\mathbf{M} - \mathbf{M}_3$ will always be zero.

The above is all that is required for full credit. However, alternatively, we could have computed that

$$
\begin{aligned}
M_3 &= \sum_{j=1}^{3} \frac{(\mathbf{M}, \mathbf{w}_j)}{(\mathbf{w}_j, \mathbf{w}_j)} \mathbf{w}_j \\
&= \frac{m_{11}}{1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{m_{22}}{1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{m_{12} + m_{21}}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} m_{11} & \dfrac{m_{12} + m_{21}}{2} \\ \dfrac{m_{12} + m_{21}}{2} & m_{22} \end{bmatrix}
\end{aligned}
$$

where

$$
M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}
$$

and hence concluded that the error

$$
\mathbf{M} - \mathbf{M}_3 = \begin{bmatrix} 0 & \dfrac{m_{12} - m_{21}}{2} \\ \dfrac{m_{21} - m_{12}}{2} & 0 \end{bmatrix}.
$$

(d) [6 points] First observe that $V_3 = \widehat{V}_3$. Hence, since the best approximation to $\mathbf{M}$ from V_3 with respect to the norm $\|\cdot\|$ is the same regardless of which basis for $\widehat{V}_3$ is used to compute it,

$$
\widehat{\mathbf{M}}_3 = \begin{bmatrix} 1 & 5 \\ 5 & 9 \end{bmatrix}.
$$

The above is all that is required for full credit. However, alternatively, we could have computed that

$$
\widehat{\mathbf{M}}_3 = \sum_{j=1}^{3} c_j \widehat{\mathbf{w}}_j = \begin{bmatrix} 1 & 5 \\ 5 & 9 \end{bmatrix}
$$

by assembling and solving

$$
\begin{bmatrix} (\widehat{\mathbf{w}}_1, \widehat{\mathbf{w}}_1) & (\widehat{\mathbf{w}}_2, \widehat{\mathbf{w}}_1) & (\widehat{\mathbf{w}}_3, \widehat{\mathbf{w}}_1) \\ (\widehat{\mathbf{w}}_1, \widehat{\mathbf{w}}_2) & (\widehat{\mathbf{w}}_2, \widehat{\mathbf{w}}_2) & (\widehat{\mathbf{w}}_3, \widehat{\mathbf{w}}_2) \\ (\widehat{\mathbf{w}}_1, \widehat{\mathbf{w}}_3) & (\widehat{\mathbf{w}}_2, \widehat{\mathbf{w}}_3) & (\widehat{\mathbf{w}}_3, \widehat{\mathbf{w}}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} (\mathbf{M}, \widehat{\mathbf{w}}_1) \\ (\mathbf{M}, \widehat{\mathbf{w}}_2) \\ (\mathbf{M}, \widehat{\mathbf{w}}_3) \end{bmatrix}
$$

where

$$
\widehat{\mathbf{w}}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix}, \ \widehat{\mathbf{w}}_2 = \begin{bmatrix} \dfrac{1}{\sqrt{10}} & \dfrac{2}{\sqrt{10}} \\ \dfrac{2}{\sqrt{10}} & \dfrac{1}{\sqrt{10}} \end{bmatrix}, \text{ and } \widehat{\mathbf{w}}_3 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ 0 & -\dfrac{1}{\sqrt{2}} \end{bmatrix}.
$$