CAAM 336 · DIFFERENTIAL EQUATIONS

Recitation Example 2

Will be worked through on 16 September 2013.

2. Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_j = jh$ for $j = 0, \ldots, N+1$. We can approximate the differential equation

$$\frac{d^2}{dx^2}u = f(x), \quad x \in (0,1),$$

with mixed boundary conditions

$$u(0) = \alpha, \quad \frac{du}{dx}(1) = \beta$$

by a matrix equation having the form

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & \ddots & & & \\ & & \ddots & \ddots & 1 & 0 & \\ & & & 1 & -2 & \star & \\ & & & \star & \star & \star & \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} f(x_1) - * \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \\ \star \end{bmatrix}$$

where $u_j \approx u(x_j)$.

(a) Specify values for the entries marked by $*$ to impose

$$u(0) = u_0 = \alpha$$

and values for the entries marked by $\star$ to impose the approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N-1} - 4u_N + 3u_{N+1}}{2h} = \beta.$$

(b) Compute and plot the approximate solutions obtained using the above method when $\alpha = 1$ and $\beta = -5$ for $N = 8, 32, 128$.

(c) What happens to the overall accuracy of the approximate solution if, instead of the $O(h^2)$ accurate approximation given above, you only use the $O(h)$ approximation

$$\frac{du}{dx}(1) \approx \frac{u_{N+1} - u_N}{h}\ ?$$