Last time we saw examples of using the "Solution steps" 4-7 to solve boundary value problems with homogeneous dirichlet and homogeneous mixed boundary conditions. We also introduced the concept of "Shifting the data" for solving boundary value problems with inhomogeneous boundary conditions.

Lets recall the method:

Suppose we want to solve a type I or type II inhomogeneous problem:

A) <u>type I</u>: $Lu = f$, $u(0) = a$, $u(1) = b$ <u>type II</u>: $Lu = f$, $u(0) = a$, $\frac{\partial u}{\partial x}(1) = b$

Here we have assumed that $l = 1$ for simplicity.

The overall idea is to consider the auxilliary problem:

B) <u>type I</u>: $Lv = 0$, $u(0) = a$, $u(1) = b$ <u>type II</u>: $Lv = 0$, $u(0) = a$, $\frac{\partial u}{\partial x}(0) = 1$

Then if we consider $w = u - v$ where u solves (A) and v solves (B) then

<u>type I</u>

$$Lw = Lu - Lv = f - 0 = f$$
$$w(0) = u(0) - v(0) = a - a = 0$$
$$w(1) = u(1) - v(1) = b - b = 0$$
$$\downarrow$$

<u>type II</u>

$$Lw = Lu - Lv = f - 0 = f$$
$$w(0) = u(0) - v(0) = a - a = 0$$
$$\frac{\partial w}{\partial x}(0) = \frac{\partial u}{\partial x}(1) - \frac{\partial v}{\partial x}(1) = b - b = 0$$
$$\downarrow$$

C)
$$\begin{cases} Lw = f \\ w(0) = w(1) = 0 \end{cases} \qquad \begin{cases} Lw = f \\ w(0) = 0, \quad \frac{\partial w}{\partial x} = 1 \end{cases}$$

So that (C) is a standard homogeneous problem of type I or II - We can use the spectral method to find the solution to (C) so if we can somehow find a function v solving (B) then we can <u>find</u> a solution to (A) by $u = w + v$.

One very natural question to ask: is $u(x) = w(x) + v(x)$ unique or is it just one of many options? Is it <u>the</u> solution or just <u>a</u> solution?

Suppose that $u(x)$ and $r(x)$ <u>both</u> solve the inhomogeneous problem (A). Consider the function $p(x) = u(x) - r(x)$. What does $p(x)$ satisfy?

(D) <u>type I</u>: $Lp = Lu - Lr = f - f = 0$

$p(0) = u(0) - r(0) = a - a = 0$

$p(1) = u(1) - r(1) = b - b = 0$

$\downarrow$

$$\begin{cases} Lp = 0 \\ p(0) = p(1) = 0 \end{cases}$$

<u>type II</u>: $Lp = Lu - Lr = f - f = 0$

$p(0) = u(0) - r(0) = a - a = 0$

$\partial p/\partial x(0) = \partial u/\partial x(0) - \partial r/\partial x(0) = b - b = 0$

$\downarrow$

$$\begin{cases} Lp = 0 \\ p(0) = 0, \quad \partial p/\partial x(1) = 0 \end{cases}$$

<u>Key IDEA:</u> Equations (D) shows that the difference, $p(x)$, between the two solutions to (A):

i) lives in the vector space $C^2_D[0,1]$ (for type I problems) or in $C^2_M[0,1]$ (for type II problems)

ii) is in the kernel of the differential operator L.

- We know that (for $L = \frac{-\partial^2}{\partial x^2}$) the nullspace $N(L)$ of L in $C^2_D[0,1]$ and $C^2_M[0,1]$ is trivial! That is $N(L) = \{0\}$. Since $p \in N(L)$ and $p \in C^2_D[0,1]$ (type I) or $p \in C^2_M[0,1]$ (type II) it follows that $\underline{p = 0}$. This means that $u(x) - r(x) = 0$ or that $u(x) = r(x)$.

- So the solution to (A) which we found by "Shifting the data" given by $u(x) = V(x) + w(x)$ is <u>unique</u>.

- Step-by-Step procedure for "Shifting the data"

① Identify what type of inhomogeneous problem you have at hand

② write down the corresponding "Kernel problem" as in (B) and Solve it to find $V(x)$.

③ write down the corresponding "homogeneous problem" as in (C) and solve it to find $w(x)$. (note: we have seen a step-by-step way to do this via the spectral method)

④ write down the solution: $u(x) = w(x) + V(x)$.

Note: ONE way to solve ② for $V(x)$ is to determine what a "typical" function in the nullspace of L would look like and then apply boundary conditions.

Example #1: Consider the BVP given by:

$$-\frac{\partial^2}{\partial x^2}u = x(1-x) \qquad u(0) = -1 \quad u(1) = 3$$

this is an inhomogeneous type I problem. (1)

(2) the corresponding "kernel problem" is:

$$-\frac{\partial^2}{\partial x^2}v = 0 \qquad u(0) = -1, \quad u(1) = 3$$

We know that, in general, the nullspace (or kernel) of L consists of functions of the form $v(x) = c + dx$. $v(0) = -1 \rightarrow c = -1$ and $v(1) = 3 \rightarrow -1 + d = 3 \rightarrow d = 4$. Hence $v(x) = 4x - 1$

(3) the corresponding homogeneous equation is:

$$-\frac{\partial^2}{\partial x^2}w = x(1-x) \qquad w(0) = 0, \quad w(1) = 0$$

We know we can solve this equation using the spectral method. We did this as an example in class and the full solution is contained in the corresponding extra study notes.

We found:

$$w(x) = \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{(n\pi)^5} \sin(n\pi x)$$

(4) The solution to the inhomogeneous boundary value problem is therefore:

$$u(x) = w(x) + v(x)$$
$$= \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{(n\pi)^5} \sin(n\pi x) + 4x - 1$$

Example #2: Consider the boundary value problem:

$$-\frac{\partial^2}{\partial x^2}u = 1 \qquad u(0) = 2 \qquad \frac{\partial u}{\partial x}(1) = -2$$

this is an inhomogeneous boundary value problem of type II (mixed boundary conditions) (1)

(2) The associated "kernel problem" is given by:

$$-\frac{\partial^2}{\partial x^2}v = 0 \qquad v(0) = 2, \quad \frac{\partial v}{\partial x}(1) = -2$$

The kernel of $L = -\frac{\partial^2}{\partial x^2}$ is given by the general function: $v(x) = c + dx$. Applying the boundary conditions gives $c = v(0) = 2$, $d = \frac{\partial v}{\partial x}(1) = -2$ so that $v(x) = 2 - 2x$

(3) The associated homogeneous equation is given by:

$$-\frac{\partial^2}{\partial x^2}w = 1 \qquad w(0) = 0, \quad \frac{\partial w}{\partial x}(1) = 0$$

this can be solved using the spectral method. This problem was solved

In class and its full solution can be found in the posted notes.

The solution to this problem is:

$$w(x) = \sum_{n=1}^{\infty} \frac{16(-1)^n}{(2n-1)^3\pi^3} \sin\left((2n-1)\pi x\right)$$

④ The solution to the inhomogeneous boundary value problem is therefore:

$$u(x) = w(x) + v(x)$$

$$= \sum_{n=1}^{\infty} \frac{16(-1)^n}{(2n-1)^3\pi^3} \sin\left((2n-1)\pi x\right) + 2-2x$$