

CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 12

Posted Friday 23 November 2012. Due Friday 30 November 2012, 5pm.

This problem set counts for 100 points, plus a 20 point bonus.

1. [50 points]

On Problem Set 10, you solved the heat equation on a two-dimensional square domain. Now we will investigate the wave equation on the same domain, a model of a vibrating membrane stretched over a square frame—that is, a square drum:

$$u_{tt}(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t),$$

with $0 \leq x \leq 1$, and $0 \leq y \leq 1$, and $t \geq 0$. Take homogeneous Dirichlet boundary conditions

$$u(x, 0, t) = u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0$$

for all x and y such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and all $t \geq 0$, and consider the initial conditions

$$u(x, y, 0) = u_0(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}(0) \psi_{j,k}(x, y), \quad u_t(x, y, 0) = v_0(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} b_{j,k}(0) \psi_{j,k}(x, y).$$

Here $\psi_{j,k}(x, y) = 2 \sin(j\pi x) \sin(k\pi y)$, for $j, k \geq 1$, are the eigenfunctions of the operator

$$Lu = -(u_{xx} + u_{yy}),$$

with homogeneous Dirichlet boundary conditions, as in Problem Set 10. You may use without proof that these eigenfunctions are orthogonal, and use the eigenvalues $\lambda_{j,k} = (j^2 + k^2)\pi^2$ computed for Problem Set 10.

(a) We wish to write the solution to the wave equation in the form

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}(t) \psi_{j,k}(x, y).$$

Show that the coefficients $a_{j,k}(t)$ obey the ordinary differential equation

$$a''_{j,k}(t) = -\lambda_{j,k} a_{j,k}(t)$$

with initial conditions

$$a_{j,k}(0), \quad a'_{j,k}(0) = b_{j,k}(0)$$

derived from the initial conditions u_0 and v_0 .

(b) Write down the solution to the differential equation in part (a).

(c) Use your solution to part (b) to write out a formula for the solution $u(x, y, t)$.

(d) Suppose the drum begins with zero velocity, $v_0(x, y) = 0$, and displacement

$$u_0(x, y) = 200xy(1-x)(1-y)(x-1/4)(y-1/4) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3 k^3 \pi^6} \psi_{j,k}(x, y).$$

Submit surface (or contour) plots of the solution at times $t = 0, 0.5, 1.0, 1.5, 2.5$, using $j = 1, \dots, 10$ and $k = 1, \dots, 10$ in the series.

2. [50 points]

Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x, t) = u_{xx}(x, t) - 2du_t(x, t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0, t) = u(1, t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We will look for solutions to the PDE of the form

$$u(x, t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

For simplicity, assume that $d \in (0, \pi)$.

- (a) From the differential equation and this form for $u(x, t)$, show that the coefficients $a_k(t)$ must satisfy the ordinary differential equation

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t).$$

- (b) Show that the following function satisfies the differential equation in part (a):

$$a_k(t) = C_1 \exp((-d + \sqrt{d^2 - k^2\pi^2})t) + C_2 \exp((-d - \sqrt{d^2 - k^2\pi^2})t)$$

for arbitrary constants C_1 and C_2 . (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

- (c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{k=1}^{\infty} b_k(0)\psi_k(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

- (d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x \sin(3\pi x)$, for which

$$b_k(0) = \frac{-6k\sqrt{2}(1 + (-1)^k)}{(k^2 - 9)^2\pi^2} \quad \text{for } k \neq 3, \quad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x, t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

- (e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d ?

bonus [20 points]

In class we will see that the wave equation $u_{tt} = u_{xx}$ can be written as a first order system. We will introduce the velocity variable

$$v(x, t) = u_t(x, t),$$

then notice that the wave equation is equivalent to the first order equation

$$\frac{\partial}{\partial t} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & 0 \end{bmatrix} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix}$$

where I is the identity operator (i.e., $Iv = v$ for any function v). The first row of this system gives $u_t = v$, while the second row gives $v_t = u_{xx}$, which is just another way of writing $u_{tt} = u_{xx}$.

We also have the usual initial conditions,

$$\begin{bmatrix} u(x, 0) \\ v(x, 0) \end{bmatrix} = \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix}.$$

This system can be analyzed in terms of its eigenvalues and eigenfunctions. In this problem, we will use the same approach to study the damped wave equation from Problem 2.

- (a) Verify that the damped wave equation $u_{tt} = u_{xx} - 2du_t$ is equivalent to the system

$$\frac{\partial}{\partial t} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & -2dI \end{bmatrix} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix}.$$

- (b) For any value of d , the operator

$$A(d) = \begin{bmatrix} 0 & I \\ \partial^2/\partial x^2 & -2dI \end{bmatrix}$$

has all its eigenvectors of the same form:

$$\Psi_{\pm k}(x) = \begin{bmatrix} \sin(k\pi x) \\ \lambda_{\pm k} \sin(k\pi x) \end{bmatrix}.$$

Use these eigenvectors to determine the eigenvalues $\lambda_{\pm k}$.

- (c) Plot the eigenvalues from part (b) in the complex plane for $d = 0, \pi/2, \pi$, and $3\pi/2$.
- (d) What value of $d \geq 0$ optimizes the *asymptotic decay* of the system? That is, what value of d moves *all* the eigenvalues as far to the left in the complex plane as possible? Said yet another way, which d minimizes the real part of the rightmost eigenvalue? Does larger damping always imply faster asymptotic decay?
- (e) Confirm this result by plotting the solution at $t = 1.5$ for $d = 2.5, 3.2, 20$ on the same plot with vertical axis $[-.002, .002]$, using the same values for u_0 and v_0 as in part (d) of Problem 2. (You should simply adapt the code you developed for Problem 2 to produce these plots.)