CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 10

Posted Wednesday 22 April 2015, due Friday 24 April 2015, 5pm. Accepted without late penalty until Wednesday 29 April 2015, 5pm. *This homework is extra credit.*

1. [20 points: 10 points each]
   Consider the wave equation
   $$u_{tt}(x,t) = u_{xx}(x,t)$$
   for $0 \le x \le 1$ and $t \ge 0$ subject to the *mixed* boundary conditions
   $$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0$$
   for all $t \ge 0$ and initial conditions
   $$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} a_n(0)\psi_n(x), \qquad u_t(x,0) = v_0(x) = \sum_{n=1}^{\infty} b_n\psi_n(x),$$
   where the functions ψ_n are the eigenfunctions
   $$\psi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}\,x)$$
   of the operator
   $$Lu = -u_{xx}$$
   with initial conditions $u(0) = u_x(1) = 0$ and eigenvalues $\lambda_n = (n-1/2)^2\pi^2$ for $n = 1, 2, \ldots$.
   (Recall that you computed these eigenvalues and eigenfunctions on Problem Set 5.)

   (a) We wish to write the solution to this wave equation in the form
   $$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$
   Show that the coefficients $a_k(t)$ obey the ordinary differential equation
   $$a_k''(t) = -\lambda_j a_k(t)$$
   subject to the initial values $a_k(0)$ and $a_k'(0) = b_k(0)$ obtained from u_0 and v_0, and write down the solution to the differential equation in part (a).

   (b) Use your solution to part (b) to write out a formula for the solution $u(x,t)$. Write a MATLAB program to compute solutions to this differential equation with initial conditions
   $$u_0(x) = 0, \qquad v_0(x) = x + \sin(\pi x).$$
   Submit your code, along with a surface plot showing the solution over the spatial interval $x \in [0,1]$ and the time interval $t \in [0,10]$.

2. [30 points: 6 points each]
Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. We will look for solutions to the PDE of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(x).$$

For simplicity, assume that $d \in (0,\pi)$.

(a) From the differential equation and this form for $u(x,t)$, show that the coefficients $a_k(t)$ must satisfy the ordinary differential equation

$$a_k''(t) = -\lambda_k a_k(t) - 2da_k'(t).$$

(b) Show that the following function satisfies the differential equation in part (a):

$$a_k(t) = C_1 \exp\big(\big(-d + \sqrt{d^2 - k^2\pi^2}\big)t\big) + C_2 \exp\big(\big(-d - \sqrt{d^2 - k^2\pi^2}\big)t\big)$$

for arbitrary constants C_1 and C_2. (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

(c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{k=1}^{\infty} b_k(0)\psi(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

(d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x\sin(3\pi x)$, for which

$$b_k(0) = \frac{-6k\sqrt{2}(1 + (-1)^k)}{(k^2 - 9)^2\pi^2} \quad \text{for } k \neq 3, \qquad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x,t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

(e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d?