CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 42 · Solutions

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

42. [25 points]

Let
$$\Omega = \{(x, y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$
and let $f \in C(\Omega)$ be defined by $f(x, y) = x(1 - y)$. In this question we will consider the problem of finding the solution $u(x, y)$ to the steady-state heat equation
$$-(u_{xx}(x, y) + u_{yy}(x, y)) = f(x, y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$
with homogeneous Dirichlet boundary conditions
$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$
Let
$$C^2_D(\Omega) = \left\{v \in C^2(\Omega) : v(x, 0) = v(x, 1) = v(0, y) = v(1, y) = 0,\, 0 \le x \le 1,\, 0 \le y \le 1\right\}.$$
Let the linear operator $L : C^2_D(\Omega) \to C(\Omega)$ be defined by
$$(Lv)\,(x, y) = -\left(v_{xx}(x, y) + v_{yy}(x, y)\right).$$
Let the inner product $(\cdot, \cdot) : C(\Omega) \times C(\Omega) \to \mathbb{R}$ be defined by
$$(v, w) = \int_0^1 \int_0^1 v(x, y)w(x, y)\, dx\, dy.$$

(a) Show that L is symmetric by showing that
$$(Lv, w) = (v, Lw) \text{ for all } v, w \in C^2_D(\Omega).$$

(b) The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions
$$\psi_{j,k}(x, y) = 2 \sin(j\pi x) \sin(k\pi y)$$
for $j, k = 1, 2, \ldots$, which are such that
$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$
for $j, k = 1, 2, \ldots$. Obtain a formula for $\lambda_{j,k}$ for $j, k = 1, 2, \ldots$.

(c) Let
$$C^2_D[0, 1] = \left\{v \in C^2[0, 1] : v(0) = v(1) = 0\right\}$$
and let the linear operator $L_1 : C^2_D[0, 1] \to C[0, 1]$ be defined by
$$L_1 w = -w''.$$
Use what you know about the eigenfunctions of L_1 to compute $(\psi_{j,k}, \psi_{m,n})$ for $j, k, m, n = 1, 2, \ldots$.

(d) The solution to $Lu = f$ that we obtain using the spectral method is

$$u(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

Plot

$$u_N(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y)$$

for $N = 1, 2, 3, 4, 5, 10$. Note that, for $j, k = 1, 2, \ldots,$

$$(f,\psi_{j,k}) = 2\frac{(-1)^{j+1}}{jk\pi^2}.$$

Also note that to plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$ you could use

```
x = linspace(0,1,50);
y = linspace(0,1,50);
[X,Y] = meshgrid(x,y);
psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,psi11)
```

Solution.

(a) [6 points] If $v \in C_D^2(\Omega)$ and $w \in C_D^2(\Omega)$ then

$$\begin{aligned}
(Lv,w) &= -\int_0^1\int_0^1 (v_{xx}(x,y)+v_{yy}(x,y))\,w(x,y)\,dx\,dy \\
&= -\int_0^1\int_0^1 v_{xx}(x,y)w(x,y)\,dx\,dy - \int_0^1\int_0^1 v_{yy}(x,y)w(x,y)\,dx\,dy \\
&= -\int_0^1\int_0^1 v_{xx}(x,y)w(x,y)\,dx\,dy - \int_0^1\int_0^1 v_{yy}(x,y)w(x,y)\,dy\,dx \\
&= -\int_0^1 \left([v_x(x,y)w(x,y)]_{x=0}^{x=1} - \int_0^1 v_x(x,y)w_x(x,y)\,dx \right) dy \\
&\quad -\int_0^1 \left([v_y(x,y)w(x,y)]_{y=0}^{y=1} - \int_0^1 v_y(x,y)w_y(x,y)\,dy \right) dx \\
&= -\int_0^1 \left([v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1} + \int_0^1 v(x,y)w_{xx}(x,y)\,dx \right) dy \\
&\quad -\int_0^1 \left([v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1} + \int_0^1 v(x,y)w_{yy}(x,y)\,dy \right) dx \\
&= -\int_0^1 \left([v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1} \right) dy - \int_0^1\int_0^1 v(x,y)w_{xx}(x,y)\,dx\,dy \\
&\quad -\int_0^1 \left([v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1} \right) dx - \int_0^1\int_0^1 v(x,y)w_{yy}(x,y)\,dy\,dx \\
&= -\int_0^1 \left([v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1} \right) dy - \int_0^1\int_0^1 v(x,y)w_{xx}(x,y)\,dx\,dy \\
&\quad -\int_0^1 \left([v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1} \right) dx - \int_0^1\int_0^1 v(x,y)w_{yy}(x,y)\,dx\,dy \\
&= -\int_0^1 \left([v_x(x,y)w(x,y)]_{x=0}^{x=1} - [v(x,y)w_x(x,y)]_{x=0}^{x=1} \right) dy
\end{aligned}$$

$$
\begin{aligned}
& - \int_0^1 \left([v_y(x,y)w(x,y)]_{y=0}^{y=1} - [v(x,y)w_y(x,y)]_{y=0}^{y=1} \right) dx \\
& - \int_0^1 \int_0^1 v(x,y) \left(w_{xx}(x,y) + w_{yy}(x,y) \right) dx\, dy \\
= & - \int_0^1 \left(v_x(1,y)w(1,y) - v_x(0,y)w(0,y) - v(1,y)w_x(1,y) + v(0,y)w_x(0,y) \right) dy \\
& - \int_0^1 \left(v_y(x,1)w(x,1) - v_y(x,0)w(x,0) - v(x,1)w_y(x,1) + v(x,0)w_y(x,0) \right) dx \\
& + (v, Lw) \\
= & \ (v, Lw)
\end{aligned}
$$

since $w(1,y) = w(0,y) = v(1,y) = v(0,y) = w(x,1) = w(x,0) = v(x,1) = v(x,0) = 0$ because $v, w \in C_D^2(\Omega)$. Consequently, $(Lv, w) = (v, Lw)$ for all $v, w \in C_D^2(\Omega)$.

(b) [4 points] We can compute that, for $j, k = 1, 2, \ldots$,

$$
\begin{aligned}
\left(L\psi_{j,k} \right)(x,y) &= -\frac{\partial^2}{\partial x^2} \left(2\sin(j\pi x)\sin(k\pi y) \right) - \frac{\partial^2}{\partial y^2} \left(2\sin(j\pi x)\sin(k\pi y) \right) \\
&= 2j^2\pi^2 \sin(j\pi x)\sin(k\pi y) + 2k^2\pi^2 \sin(j\pi x)\sin(k\pi y) \\
&= 2\left(j^2 + k^2 \right) \pi^2 \sin(j\pi x)\sin(k\pi y) \\
&= \left(j^2 + k^2 \right) \pi^2 \psi_{j,k}(x,y).
\end{aligned}
$$

Hence,

$$
\lambda_{j,k} = (j^2 + k^2)\pi^2 \text{ for } j, k = 1, 2, \ldots.
$$

(c) [7 points] The operator L_1 has eigenfunctions ψ_p for $p = 1, 2, 3, \ldots$ which are such that

$$
\psi_p(s) = \sqrt{2}\sin(p\pi s)
$$

for $p = 1, 2, 3, \ldots$ and

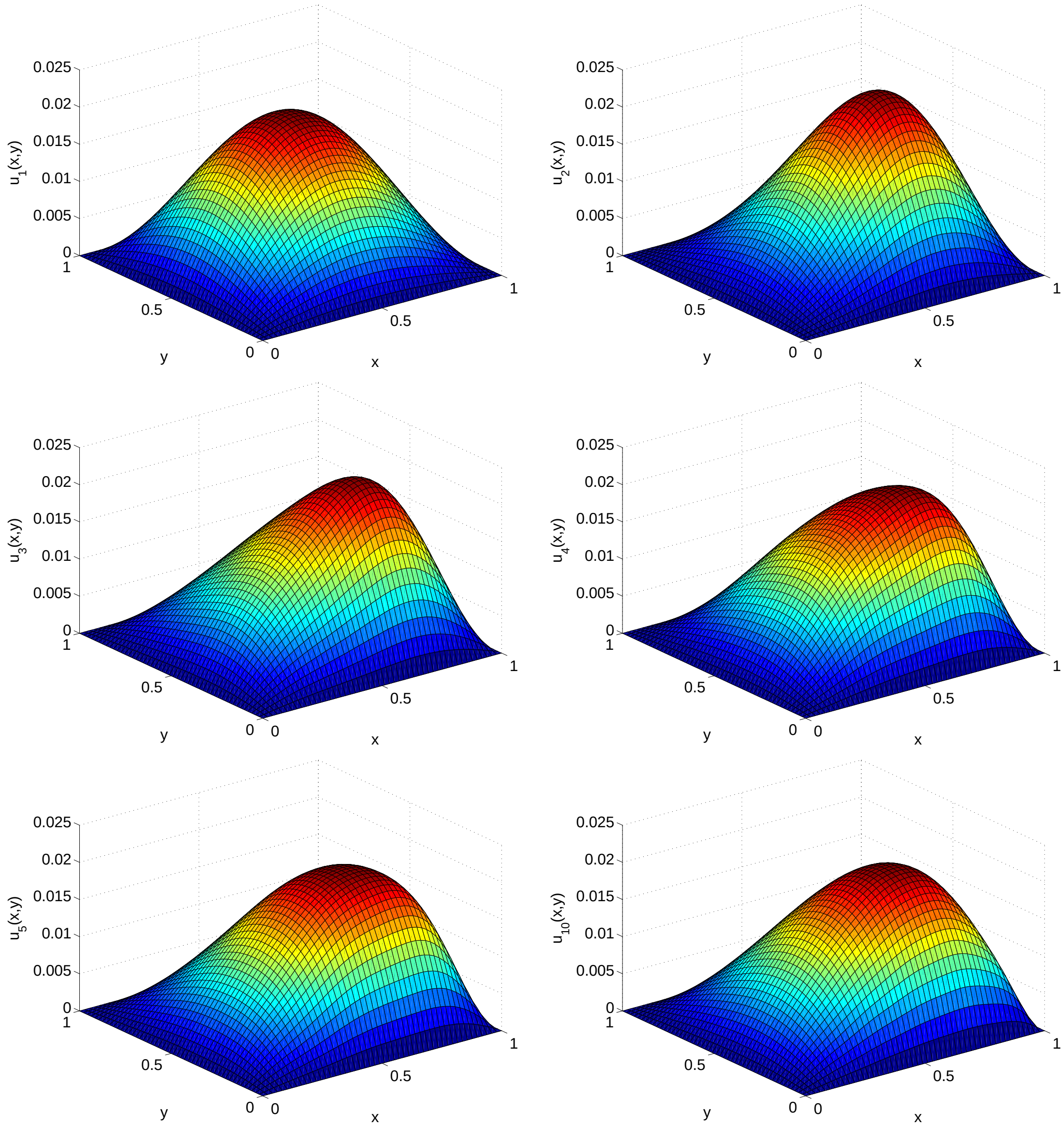
$$
\int_0^1 \psi_p(s)\psi_q(s)\, ds = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}
$$

for $p, q = 1, 2, 3, \ldots$. Therefore,

$$
\begin{aligned}
(\psi_{j,k}, \psi_{m,n}) &= \int_0^1 \int_0^1 2\sin(j\pi x)\sin(k\pi y) 2\sin(m\pi x)\sin(n\pi y)\, dx\, dy \\
&= \int_0^1 \int_0^1 \psi_j(x)\psi_k(y)\psi_m(x)\psi_n(y)\, dx\, dy \\
&= \int_0^1 \psi_k(y)\psi_n(y) \int_0^1 \psi_j(x)\psi_m(x)\, dx\, dy \\
&= \begin{cases} \displaystyle\int_0^1 \psi_k(y)\psi_n(y)\, dy & \text{if } j = m \\ 0 & \text{if } j \neq m \end{cases} \\
&= \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

for $j, k, m, n = 1, 2, 3, \ldots$.

(d) [8 points] The requested plots and the code used to produce them are below.




```
clear
clc
npts = 50;
x = linspace(0,1,npts);
y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
for n=1:10
    figure(1)
    clf
    U = zeros(npts,npts);
    for j=1:n
        for k=1:n
            U = U + 4*(-1)^(j+1)/(j*k*pi^2)*sin(j*pi*X).*sin(k*pi*Y)/(j^2+k^2)/(pi^2);
        end
    end
    surf(X,Y,U), drawnow
    set(gca,'fontsize',16)
```

```
    xlabel('x')
    ylabel('y')
    zlabel(['u_{' num2str(n) '}(x,y)'])
    if ismember(n,[1 2 3 4 5 10]),
        eval(sprintf('print -depsc2 twoD%d', n))
    end
    pause
end
```