

# CAAM 336 · DIFFERENTIAL EQUATIONS

## Problem Set 1 · Solutions

Posted Wednesday 14, January 2015. Due Wednesday 21, January 2015, 5pm.

*A reminder from the course syllabus:* Mathematically rigorous solutions are expected; strive for clarity and elegance. You may collaborate on the problems, but your write-up must be your own independent work. Transcribed solutions and copied MATLAB code are both unacceptable. *You may not consult solutions from previous sections of this class.* Unless it is specified that a particular calculation must be performed ‘by hand,’ you are encouraged to use MATLAB’s Symbolic Math Toolbox (or Mathematica/Wolfram Alpha/Maple) to compute and simplify tedious integrals and derivatives on the problem sets. As always, you must document your calculations clearly.

1. [16 points]

For each of the following equations, specify whether each is (a) an ODE or a PDE; (b) determine its order; (c) specify whether it is linear or nonlinear. For those that are linear, specify whether they are (d) homogeneous or inhomogeneous, and (e) whether they have constant or variable coefficients.

$$(1.1) \quad \frac{dv}{dx} + \frac{2}{x}v = 0$$

$$(1.2) \quad \frac{\partial v}{\partial t} - 3\frac{\partial v}{\partial x} = x - t$$

$$(1.3) \quad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ 2u \frac{\partial u}{\partial x} \right] = 0$$

$$(1.4) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$(1.5) \quad \frac{d^2 y}{dx^2} - \mu(1 - y^2) \frac{dy}{dx} + y = 0$$

$$(1.6) \quad \frac{d^2}{dx^2} \left[ \rho(x) \frac{d^2 u}{dx^2} \right] = \sin(x)$$

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**Solution.**

(1.1) ODE, first order, linear, homogeneous, variable coefficient

The  $2/x$  factor in front of the  $v$  is the variable coefficient.

(1.2) PDE, first order, linear, inhomogeneous, constant coefficient

The  $x - t$  term on the right, which does not involve  $v$ , makes the equation inhomogeneous.

(1.3) PDE, second order, nonlinear

Using the product rule for partial derivatives, we can write this equation in the equivalent form

$$\frac{\partial u}{\partial t} - 2 \left( \frac{\partial u}{\partial x} \right)^2 - 2u \left( \frac{\partial^2 u}{\partial x^2} \right) = 0.$$

The second and third terms on the left hand side make this equation nonlinear.

(1.4) PDE, third order, nonlinear

The  $u(\partial u/\partial x)$  term makes this equation nonlinear. This is a version of the famous Korteweg-de Vries (KdV) equation that describes shallow water waves.

(1.5) ODE, second order, nonlinear

The  $(1 - y^2)(dy/dx)$  term makes this ODE nonlinear.

(1.6) ODE, fourth order, linear, inhomogeneous, variable coefficient

Using the product rule for partial derivatives, we can write this equation in the equivalent form

$$\frac{d^2 \rho}{dx^2} \frac{d^2 u}{dx^2} + 2 \frac{d\rho}{dx} \frac{d^3 u}{dx^3} + \rho(x) \frac{d^4 u}{dx^4} = \sin(x),$$

hence we can see that it is fourth order. This equation, attributed to Euler, describes the deflection of a one-dimensional beam with a static load of  $\sin(x)$ ;  $\rho(x)$  describes the elasticity of the material that constitutes the beam.

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2. [12 points]

Determine whether each of the following functions is a solution of the corresponding differential equation from question 1.

- (a) Is  $v(x) = 1/x^2$  a solution of (1.1) ?
  - (b) Is  $v(x, t) = t(t + x)$  a solution of (1.2) ?
  - (c) Is  $u(x, t) = xe^t$  a solution of (1.3) ?
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**Solution.**

- (a)  $v(x) = 1/x^2$  is a solution of (1.1).

To plug  $v(x) = 1/x^2$  into the left-hand side of (1.1), we compute  $dv/dx = d(x^{-2})/dx = -2x^{-3}$ . Substituting this formula, the left-hand side of (1.1) becomes

$$-2x^{-3} + 2x^{-1}x^{-2} = 0.$$

This agrees with the right-hand side of (1.1), so this  $v$  is a solution.

- (b)  $v(x, t) = t(t + x)$  is a solution of (1.2).

We compute  $\partial v/\partial t = 2t + x$  and  $\partial v/\partial x = t$ . Thus the left-hand side of (1.2) becomes

$$(2t + x) - 3(t) = x - t.$$

This agrees with the right-hand side of (1.2), so this  $v$  is a solution.

- (c)  $u(x, t) = xe^t$  is *not* a solution of (1.3).

We compute  $\partial u/\partial t = xe^t$  and  $\partial u/\partial x = e^t$ . From this it follows that

$$\frac{\partial}{\partial x} \left[ 2u \frac{\partial u}{\partial x} \right] = \frac{\partial}{\partial x} 2xe^{2t} = 2e^{2t}.$$

Thus the left-hand side of (1.3) is

$$xe^t - 2e^{2t},$$

which is nonzero in general, in disagreement with the right-hand side of (1.3).

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3. [34 points]

Consider the temperature function

$$u(x, t) = e^{-\kappa\theta^2 t/(\rho c)} \sin(\theta x)$$

for constant  $\kappa$ ,  $\rho$ ,  $c$ , and  $\theta$ .

(a) Show that this function  $u(x, t)$  is a solution of the homogeneous heat equation

$$\rho c \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \ell \text{ and all } t.$$

(b) For which values of  $\theta$  will  $u$  satisfy homogeneous Dirichlet boundary conditions at  $x = 0$  and  $x = \ell$ ?

(c) Suppose  $\kappa = 2.37 \text{ W/(cm K)}$ ,  $\rho = 2.70 \text{ g/cm}^3$ , and  $c = 0.897 \text{ J/(g K)}$  (approximate values for aluminum found on Wikipedia), and that the bar has length  $\ell = 10 \text{ cm}$ . Let  $\theta$  be such that  $u(x, t)$  satisfies homogeneous Dirichlet boundary conditions as in part (b) and  $u(x, t) \geq 0$  for all  $x$  and  $t$ .

Use MATLAB to plot the solution  $u(x, t)$  for  $0 \leq x \leq \ell$  and time  $0 \leq t \leq 20 \text{ sec}$ .

You may choose to do this in one of the following ways: (1) Plot the solution for  $0 \leq x \leq \ell$  at times  $t = 0, 4, 8, \dots, 20 \text{ sec.}$ , superimposing all six plots on the same axis (helpful commands: `linspace`, `plot`, `hold on`); (2) Create a three-dimensional plot of the data using `surf`, `mesh`, or `waterfall`. In either case, be sure to produce an attractive, well-labeled plot.

Solution.

(a) We compute

$$\frac{\partial u}{\partial t} = -\kappa\theta^2/(\rho c)e^{-\kappa\theta^2 t/(\rho c)} \sin(\theta x)$$

$$\frac{\partial u}{\partial x} = \theta e^{-\kappa\theta^2 t/(\rho c)} \cos(\theta x)$$

$$\frac{\partial^2 u}{\partial x^2} = -\theta^2 e^{-\kappa\theta^2 t/(\rho c)} \sin(\theta x).$$

With these formulas in hand it is easy to verify that

$$\rho c \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

(b) We wish to find the values of  $\theta$  that give homogeneous Dirichlet boundary conditions, i.e.,  $u(0, t) = u(\ell, t) = 0$  for all  $t$ . Since  $e^{-\kappa\theta^2 t/(\rho c)}$  is positive for all  $t$ , we can only get the homogeneous Dirichlet conditions when  $\sin(\theta x) = 0$ . For any  $\theta$ ,  $\sin(\theta \cdot 0) = 0$ , so the condition at  $x = 0$  is automatically satisfied. To get  $\sin(\theta\ell) = 0$ , we need  $\theta\ell$  to be an integer multiple of  $\pi$ , that is,

$$\theta\ell = \pi n, \quad n = 0, \pm 1, \pm 2, \dots,$$

or equivalently

$$\theta = \frac{\pi n}{\ell}, \quad n = 0, \pm 1, \pm 2, \dots$$

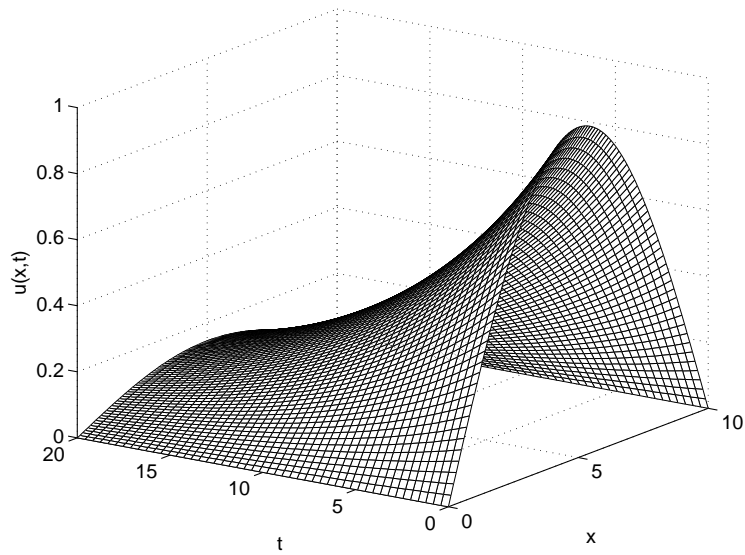
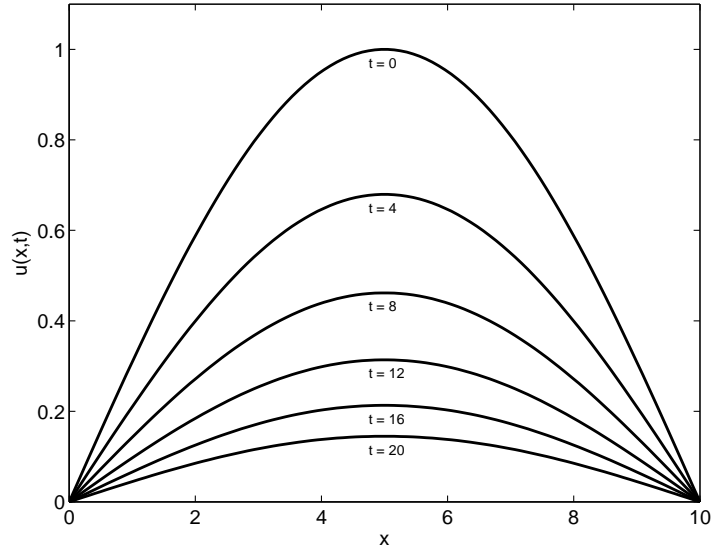
(Notice that if  $n = 0$  we have the trivial solution  $u(x, t) = 0$  for all  $x, t$ . If  $n = 1$ , we have a solution for which  $u(x, t) \geq 0$  for all  $x, t$ . For other values of  $n$  the solution will be *negative* for some  $x, t$ . If our temperature is measured in Kelvin this could be a problem! However, this heat equation takes the same form if we shift to Celsius units, so we needn't be so troubled by the negative values of temperature.)

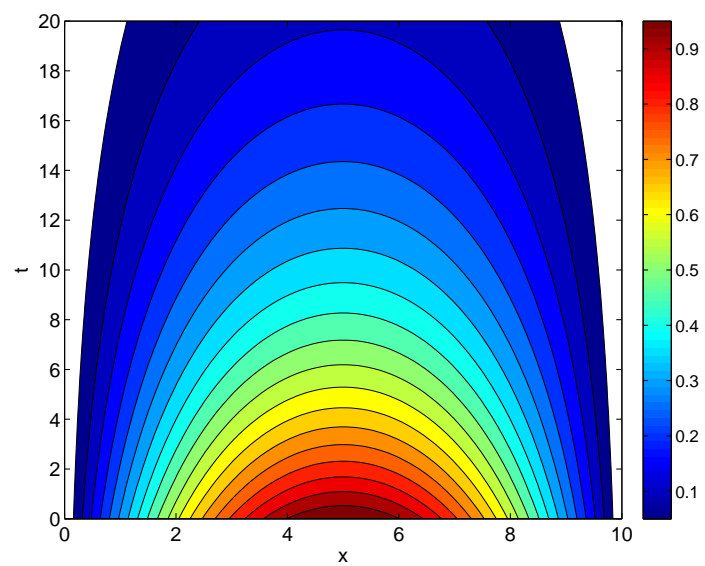
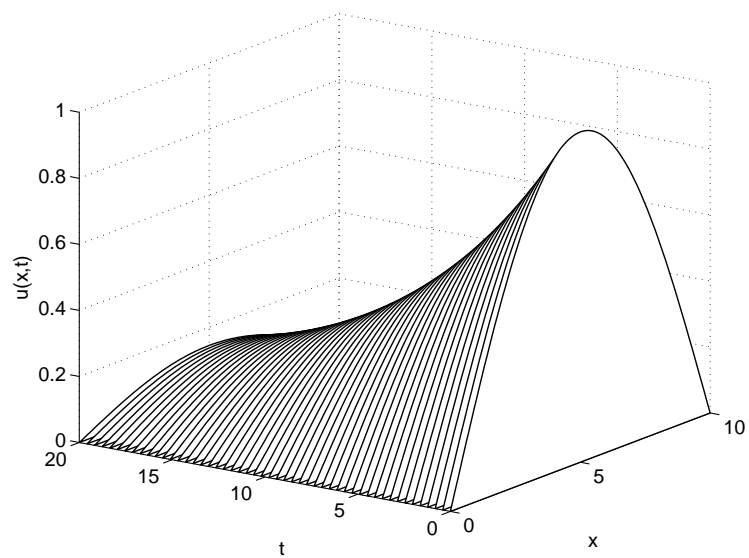
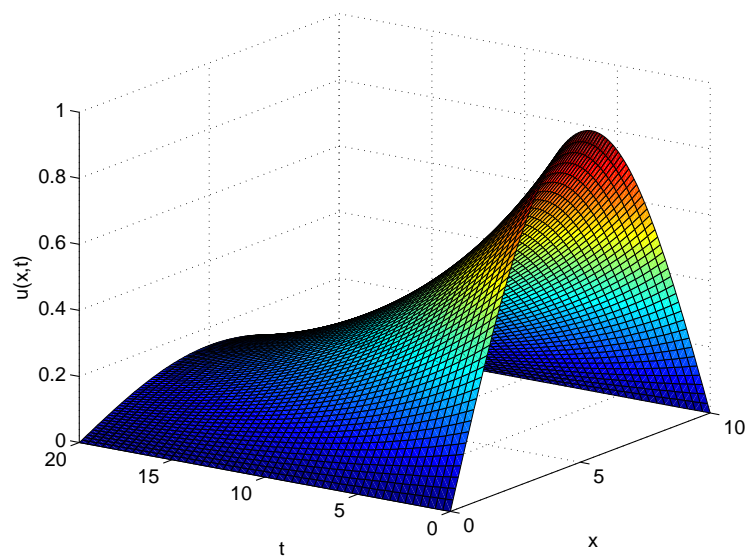
(c) Since  $n = 0$  is trivial, we shall take  $n = 1$  ( $\theta = \pi/\ell$ ) to obtain

$$\begin{aligned} u(x, t) &= e^{-\kappa\pi^2 t/(\ell^2 \rho c)} \sin(\pi x/\ell) \\ &= e^{-2.37\pi^2 t/(100 \cdot 2.70 \cdot 0.897)} \sin(\pi x/10). \end{aligned}$$

(d) Solutions are shown in the attached plots. Any of these style is acceptable. The MATLAB code that generated these plots follows.

**[GRADERS:** please make a note if students did not include their MATLAB code, but do not take off points for this first time. You should do so in the future, though!]





MATLAB code:

```
c = .897;
kappa = 2.37;
rho = 2.70;
l = 10;
theta = pi/l;

% first style: snapshots at t = 0, 4, 8, ..., 20

t = 0:4:20;
x = linspace(0,l,100);

figure(1), clf
for j=1:length(t)
    u = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x); % compute u(:,t(j))
    plot(x,u,'k-','linewidth',2), hold on
    text(4.75, max(u)-.03, sprintf('t = %d', t(j)))
end
axis([0 10 0 1.1])
set(gca,'fontsize',14)
xlabel('x')
ylabel('u(x,t)')
print -depsc2 checksol1

% generate data for 3-d plots

x = linspace(0,l,100);
t = linspace(0,20,50);
U = zeros(length(t), length(x));
for j=1:length(t)
    U(j,:) = exp(-kappa*theta^2*t(j)/(rho*c))*sin(theta*x);
end

% mesh plot
figure(2), clf
mesh(x,t,U,'edgecolor','k')
view(-55,20)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 checksol2

% surf plot
figure(3), clf
surf(x,t,U)
view(-55,20)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 checksol3

% waterfall plot
figure(4), clf
plt = waterfall(x,t,U);
set(plt,'edgecolor','k') % make the lines black
view(-55,20)
set(gca,'fontsize',14)
xlabel('x'), ylabel('t'), zlabel('u(x,t)')
print -depsc2 checksol4

% contour plot
figure(5), clf
[cs,h] = contourf(x,t,U,[.05:.05:.95],'k-');
set(gca,'fontsize',14)
xlabel('x'), ylabel('t')
colorbar
print -depsc2 checksol5
```

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4. [20 points]

Consider a bar of metal alloy manufactured such that its thermal conductivity is  $\kappa(x) = 1 + \alpha x$  for constant  $\alpha$  and  $0 \leq x \leq \ell$ . You may assume the heat equation for a non-uniform bar:

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u(x,t)}{\partial x}\right).$$

- (a) Determine a general formula for the steady-state temperature distribution of this bar, which should include two free constants. (Assume no additional source term  $f(x)$  is present.)
- (b) Find formulas for these free constants in the case that the ends of the bar are submerged in ice baths of  $\gamma$  deg on the left and  $\delta$  deg on the right.
- (c) Now find formulas for the free constants in the case that the left end has a fixed *heat flux* equal to  $\gamma$  (measured in  $\text{J}/(\text{m}^2 \cdot \text{sec})$ ) and the right end is submerged in an ice bath of  $\delta$  deg.

**Solution.**

- (a) Because the bar is at steady state, we have

$$\frac{\partial u}{\partial t}(x) = 0,$$

where we have dropped the  $t$  argument from  $u(x, t)$ , since  $u$  is independent of  $t$  in this context. The heat equation becomes

$$0 = \frac{d}{dx}\left(\kappa(x)\frac{d}{dx}u(x)\right).$$

In other words, the quantity on the right under the first derivative must be a constant. Integrate this equation once to obtain

$$\kappa(x)\frac{d}{dx}u(x) = C_1,$$

where the constant  $C_1$  will be determined later by the boundary conditions. Divide by  $\kappa(x)$  to obtain the equation

$$\frac{d}{dx}u(x) = \frac{C_1}{\kappa(x)}.$$

Substituting in the particular equation for this bar,  $\kappa(x) = 1 + \alpha x$ , we have

$$\frac{d}{dx}u(x) = \frac{C_1}{1 + \alpha x},$$

which we can integrate once to obtain the general form of the solution

$$u(x) = \frac{C_1}{\alpha} \log(1 + \alpha x) + C_2,$$

where  $C_1$  and  $C_2$  are constants. (As is common in higher mathematics, we use  $\log$  to denote the natural logarithm rather than  $\ln$ .)

- (b) If the ends are submerged in ice baths of  $\gamma$  and  $\delta$  degrees, we have the Dirichlet boundary conditions

$$u(0) = \gamma, \quad u(\ell) = \delta.$$

We must find  $C_1$  and  $C_2$  to satisfy these conditions. This gives two equations in two unknowns:

$$\gamma = \frac{C_1}{\alpha} \log(1 + \alpha \cdot 0) + C_2$$

$$\delta = \frac{C_1}{\alpha} \log(1 + \alpha \ell) + C_2.$$

Since  $\log(1 + \alpha \cdot 0) = \log(1) = 0$ , the first equation reduces to

$$\gamma = C_2.$$

Substituting this formula into the second equation and solving for  $C_1$  yields

$$C_1 = \frac{\alpha(\delta - \gamma)}{\log(1 + \alpha\ell)}.$$

Thus, the steady state solution with desired Dirichlet boundary conditions is

$$u(x) = (\delta - \gamma) \frac{\log(1 + \alpha x)}{\log(1 + \alpha\ell)} + \gamma.$$

(c) If the heat flux is fixed at  $\gamma \text{ J}/(\text{m}^2 \cdot \text{sec})$  on the left end of the bar, we have

$$q(0) = \gamma.$$

Fourier's law of heat conduction gives

$$q(0) = -\kappa(0) \frac{\partial u}{\partial x}(x, t),$$

and hence we have the Neumann boundary condition on the left end:

$$\frac{\partial u}{\partial x}(0) = -\frac{\gamma}{\kappa(0)} = -\frac{\gamma}{1} = -\gamma.$$

[**GRADERS:** If students directly assumed that  $\partial u(0)/\partial t = \gamma$  without appeal to Fourier's Law, please deduct 5 points.]

The ice bath on the right hand side gives the Dirichlet condition

$$u(\ell) = \delta.$$

To impose the Neumann condition, we must take a derivative of the general solution

$$u(x) = \frac{C_1}{\alpha} \log(1 + \alpha x) + C_2,$$

or just go back to the intermediate step in part (a), to obtain

$$\frac{\partial u}{\partial x}(x) = \frac{C_1}{1 + \alpha x},$$

and so

$$\frac{\partial u}{\partial x}(0) = C_1.$$

Thus our boundary conditions again impose two equations in two unknowns:

$$\begin{aligned} -\gamma &= C_1 \\ \delta &= \frac{C_1}{\alpha} \log(1 + \alpha\ell) + C_2. \end{aligned}$$

Substituting  $C_1 = -\gamma$  into the second equation and solving for  $C_2$  gives

$$C_2 = \delta + \frac{\gamma}{\alpha} \log(1 + \alpha\ell).$$

With these constants, the solution becomes

$$\begin{aligned} u(x) &= \frac{\gamma}{\alpha} (\log(1 + \alpha\ell) - \log(1 + \alpha x)) + \delta \\ &= \frac{\gamma}{\alpha} \log\left(\frac{1 + \alpha\ell}{1 + \alpha x}\right) + \delta \end{aligned}$$


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