CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 10 · Solutions

Posted Wednesday 7 November 2012. Due Wednesday 14 November 2012, 5pm.

1. [50 points: 8 points each for (a), (b), (d), (e); 4 points for (c); 14 points for (f)]
This problem and the next study the heat equation in two dimensions. We begin with the steady-state problem. In place of the one dimensional equation, $-u'' = f$, we now have

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$ for all $0 \le x \le 1$ and $0 \le y \le 1$. The associated operator L is defined as

$$Lu = -(u_{xx} + u_{yy}),$$

acting on the space $C_D^2[0,1]^2$ consisting of twice continuously differentiable functions on $[0,1] \times [0,1]$ with homogeneous boundary conditions. We can solve the differential equation $Lu = f$ using the spectral method just as we have seen in class before. This problem will walk you through the process; you may consult Section 8.2 of the text for hints.

(a) Show that L is symmetric, given the inner product

$$(v,w) = \int_0^1 \int_0^1 v(x,y) w(x,y)\, dx\, dy.$$

(b) Verify that the functions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

are eigenfunctions of L for $j,k = 1,2,\ldots$.
(To do this, you simply need to show that $L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$ for some scalar $\lambda_{j,k}$.)

(c) What is the eigenvalue $\lambda_{j,k}$ associated with $\psi_{j,k}$?

(d) Compute the inner product $(\psi_{j,k}, \psi_{j,k}) = \|\psi_{j,k}\|^2$.

(e) Let $f(x,y) = x(1-y)$. Compute the inner product $(f, \psi_{j,k})$.

(f) The solution to the diffusion equation is given by the spectral method, but now with a double sum to account for all the eigenvalues:

$$u(x,y) = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})}\, \psi_{j,k}(x,y).$$

In MATLAB plot the partial sum

$$u_{10}(x,y) = \sum_{j=1}^{10} \sum_{k=1}^{10} \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})}\, \psi_{j,k}(x,y).$$

Hint for 3d plots: To plot $\psi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$, you could use

```
x = linspace(0,1,40); y = linspace(0,1,40);
[X,Y] = meshgrid(x,y);
Psi11 = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,Psi11)
```

Solution.

(a) To show that L is symmetric, we must show that $(Lu, v) = (u, Lv)$ for all $u, v \in C_D^2[0,1]^2$. We can establish this result by integrating by parts twice in each spatial dimension:

$$\begin{aligned}
(Lu, v) &= -\int_0^1 \int_0^1 \big(u_{xx}(x,y) + u_{yy}(x,y)\big)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= -\int_0^1 \Big(\int_0^1 u_{xx}(x,y)v(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y - \int_0^1 \Big(\int_0^1 u_{yy}(x,y)v(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= \int_0^1 \Big(-\Big[u_x(x,y)v(x,y)\Big]_{x=0}^1 + \Big[u(x,y)v_x(x,y)\Big]_{x=0}^1 - \int_0^1 u(x,y)v_{xx}(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y \\
&\quad + \int_0^1 \Big(-\Big[u_y(x,y)v(x,y)\Big]_{y=0}^1 + \Big[u(x,y)v_y(x,y)\Big]_{y=0}^1 - \int_0^1 u(x,y)v_{yy}(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= -\int_0^1 \Big(\int_0^1 u(x,y)v_{xx}(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y - \int_0^1 \Big(\int_0^1 u(x,y)v_{yy}(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= -\int_0^1 \int_0^1 u(x,y)\big(v_{xx}(x,y) + v_{yy}(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= (u, Lv).
\end{aligned}$$

More generally, you can appeal to Green's Second Identity, which amounts to integration by parts in higher dimensions. Let $\Omega := [0,1] \times [0,1] \subset \mathbb{R}^2$ denote the domain $0 \le x \le 1$ and $0 \le y \le 1$, and $\partial\Omega$ its boundary. We write $-Lu = -u_{xx} - u_{yy} = -\Delta u$. Green's Second Identity gives

$$\int_\Omega \big((\Delta u)v - u(\Delta v)\big)\,\mathrm{d}V = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,\mathrm{d}S,$$

where $\partial u/\partial n$ and $\partial v/\partial n$ denote derivatives with respect to the outward-pointing normal direction. Thus

$$(Lu, v) = -\int_\Omega (\Delta u)v\,dV = -\int_\Omega u(\Delta v)\,dV + \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,\mathrm{d}S = -\int_\Omega u(\Delta v)\,dV = (u, Lv),$$

since u and v are zero on the boundary $\partial\Omega$.

(b) We simply compute

$$\begin{aligned}
L\psi_{j,k} = -\Big(\frac{\partial^2 \psi_{j,k}}{\partial x^2} + \frac{\partial^2 \psi_{j,k}}{\partial y^2}\Big) &= -\frac{\partial}{\partial x^2}(\sin(j\pi x)\sin(k\pi y) - \frac{\partial}{\partial y^2}(\sin(j\pi x)\sin(k\pi y) \\
&= j^2\pi^2 \sin(j\pi x)\sin(k\pi y) + k^2\pi^2 \sin(j\pi x)\sin(k\pi y) \\
&= (j^2 + k^2)\pi^2 \sin(j\pi x)\sin(k\pi y) \\
&= \lambda_{j,k}\psi_{j,k}(x,y).
\end{aligned}$$

One can also notice that $\psi_{j,k}(x,y)$ satisfies the necessary boundary conditions

$$\psi_{j,k}(0,y) = \psi_{j,k}(1,y) = \psi_{j,k}(x,0) = \psi_{j,k}(x,1) = 0.$$

[GRADERS: please deduct 2 points if the student forgot to check the boundary conditions.]
Thus $\psi_{j,k}(x,y) = \sin(j\pi x)\sin(k\pi x)$ is an eigenfunction for the operator L.

(c) The computation in part (b) reveals the eigenvalue to be $\lambda_{j,k} = (j^2 + k^2)\pi^2$.

(d) The inner product computation reduces to the product of single integrals:

$$\begin{aligned}(\psi_{j,k}, \psi_{j,k}) &= 4\int_0^1 \int_0^1 \sin(j\pi x)^2 \sin(k\pi y)^2 \, dx\, dy \\ &= 4\Big(\int_0^1 \sin(j\pi x)^2 \, dx\Big)\Big(\int_0^1 \sin(k\pi y)^2 \, dy\Big) \\ &= 1.\end{aligned}$$

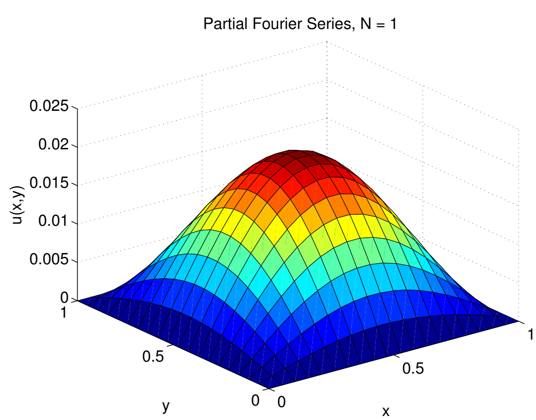
(e) This inner product also breaks into the products of two integrals that are straightforward to compute:

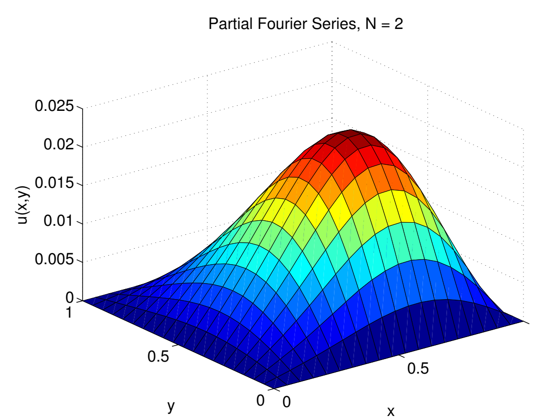
$$\begin{aligned}(f, \psi_{j,k}) &= 2\int_0^1 \int_0^1 x(1-y)\sin(j\pi x)\sin(k\pi y)\, dx\, dy \\ &= 2\Big(\int_0^1 x\sin(j\pi x)\, dx\Big)\Big(\int_0^1 (1-y)\sin(k\pi y)^2 \, dy\Big) \\ &= 2\Big(\frac{(-1)^{j+1}}{j\pi}\Big)\Big(\frac{1}{k\pi}\Big) \\ &= 2\frac{(-1)^{j+1}}{jk\pi^2}.\end{aligned}$$

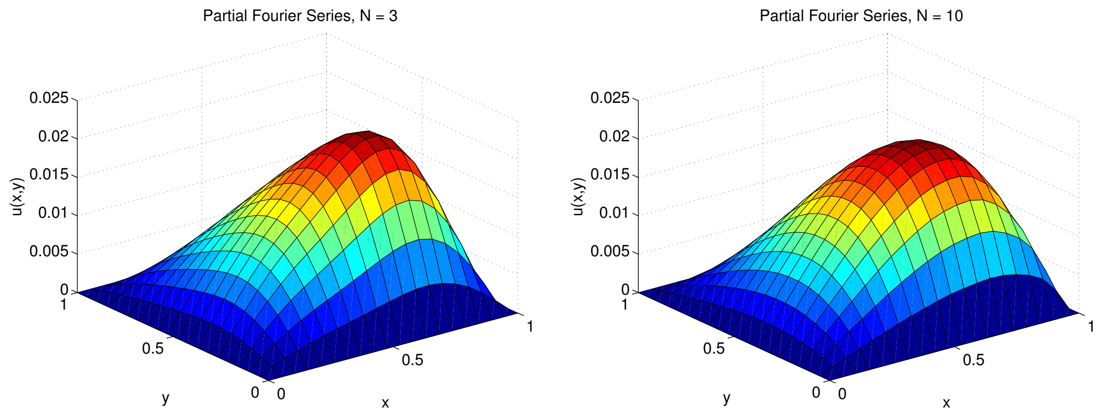
(f) The code below plots the partial sum

$$u_N(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{(f, \psi_{j,k})}{\lambda_{j,k}(\psi_{j,k}, \psi_{j,k})}\psi_{j,k}(x,y)$$

for various values of N, as shown in the plots below.








```
npts = 20;
x = linspace(0,1,npts); y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
for n=1:10
   figure(1), clf
   U = zeros(N,N);
   for j=1:n
      for k=1:n
         U = U + 4*(-1)^(j+1)/(j*k*pi^2)*sin(j*pi*X).*sin(k*pi*Y)/(j^2+k^2)/(pi^2);
      end
   end
   surf(X,Y,U), drawnow
   set(gca,'fontsize',16)
   xlabel('x')
   ylabel('y')
   zlabel('u(x,y)')
   title(sprintf(' Partial Fourier Series, N = %d', n))
   if ismember(n,[1 2 3 10]),
      eval(sprintf('print -depsc2 twoD%d', n))
   end
   pause
end
```

please see the next page...

2. [50 points: 20 points for (a); 10 points for (b); 20 points for (c)]
We now consider the time-dependent heat equation in two dimensions,

$$u_t(x,y,t) = (u_{xx}(x,y,t) + u_{yy}(x,y,t)) + f(x,y,t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions $u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$ for all $0 \le x \le 1$, $0 \le y \le 1$, and $t \ge 0$, and initial condition $u(x,y,0) = u_0(x,y)$. We can consider this problem in the abstract setting of $u_t = -Lu + f$, where, as in the previous problem,

$$Lu = -(u_{xx} + u_{yy}),$$

acting on the space $C_D^2[0,1]^2$. Recall that the eigenvalues $\lambda_{j,k}$ and associated eigenfunctions $\psi_{j,k}$ of this operator were studied in the previous problem.

(a) The solution to the two-dimensional heat equation takes the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big(e^{-\lambda_{j,k}t}a_{j,k}(0) + \int_0^t e^{-\lambda_{j,k}(t-\tau)}c_{j,k}(\tau)\,\mathrm{d}\tau\Big)\psi_{j,k}(x,y).$$

Give a brief derivation of this equation, explaining what the values $a_{j,k}(0)$ and $c_{j,k}(\tau)$ denote, and what ordinary differential equation needs to be solved for each (j,k) pair. (You do not need to derive the solution to that equation from scratch; it should take a familiar form, and you can just quote the solution for equations of this form.)

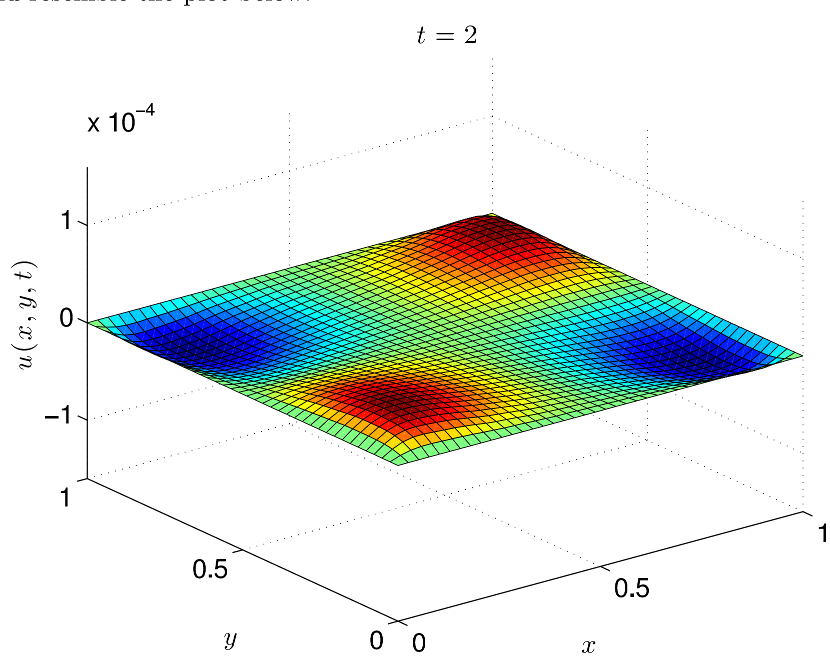
(b) Suppose $u_0(x,y) = 0$ and $f(x,y,t) = (x-1/2)^3(y-1/2)\mathrm{e}^{-t}$. Simplify the formula in part (a) as much as possible. That is, write out $a_{j,k}(0)$, $c_{j,k}(t)$, and compute a formula for

$$\int_0^t e^{-\lambda_{j,k}(t-\tau)}c_{j,k}(\tau)\,\mathrm{d}\tau.$$

(c) Plot the partial Fourier series solution

$$u_{15}(x,y,t) = \sum_{j=1}^{15}\sum_{k=1}^{15}\Big(e^{-\lambda_{j,k}t}a_{j,k}(0) + \int_0^t e^{-\lambda_{j,k}(t-\tau)}c_{j,k}(\tau)\,\mathrm{d}\tau\Big)\psi_{j,k}(x,y)$$

at the four times $t = 0, 0.005, 0.1, 2$ for the values of u_0 and f given in part (b). Your solution for $t = 0.1$ should resemble the plot below.



Solution.

(a) We seek a solution $u(x,y,t)$ of the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y),$$

given a forcing function of the form

$$f(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x,y).$$

Substituting these formulas into the differential equation $u_t = u_{xx} + f$ yields

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a'_{j,k}(t)\psi_{j,k}(x,y) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi''_{j,k}(x,y) + + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x,y)\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} -\lambda_{j,k}a_{j,k}(t)\psi_{j,k}(x,y) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x,y)\end{aligned}$$

Take an inner product of both sides with $\psi_{m,n}$ and use the orthonormality of the eigenfunctions to obtain the ordinary differential equation

$$a'_{m,n}(t) = -\lambda_{m,n}a_{m,n}(t) + c_{m,n}(t).$$

This differential equation is accompanied by the initial condition on $a_{m,n}(0)$ obtained from the initial data for the problem,

$$a_{m,n}(0) = (u_0, \psi_{m,n}) = \int_0^1\int_0^1 2u_0(x,y)\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y.$$

The contribution from $c_{m,n}(t)$ can be computed *a priori* from the expansion of the forcing data in the eigenfunctions, i.e.,

$$c_{m,n}(t) = (f(x,y,t), \psi_{m,n}(x)) = \int_0^1\int_0^1 2f(x,y,t)\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y.$$

(b) Since $u_0(x) = 0$ for all $x \in [0,1]$, we simply have $a_{m,n}(0) = 0$.
The computation of $c_{m,n}(t)$ requires a bit more work. We heed to compute

$$\begin{aligned}c_{m,n}(t) &= \int_0^1\int_0^1 2f(x,y,t)\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\int_0^1 2(x-1/2)^3(y-1/2)\mathrm{e}^{-t}\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \mathrm{e}^{-t}\int_0^1\int_0^1 2(x-1/2)^3(y-1/2)\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The integral can be computed using symbolic integration. From Mathematica, we find that

$$\int_0^1\int_0^1 2(x-1/2)^3(y-1/2)\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y = \frac{(1+(-1)^m)(1+(-1)^n)(m^2\pi^2-24)}{8m^3n\pi^4},$$

giving

$$c_{m,n}(t) = \frac{(1+(-1)^m)(1+(-1)^n)(m^2\pi^2 - 24)}{8m^3 n\pi^4}\mathrm{e}^{-t}.$$

The differential equation $a'_{m,n}(t) = -\lambda_{m,n} a_{m,n}(t) + c_{m,n}(t)$ has the exact solution

$$a_{m,n}(t) = \mathrm{e}^{-\lambda_{m,n} t} a_{m,n}(0) + \int_0^t \mathrm{e}^{-\lambda_{m,n}(t-\tau)} c_{m,n}(\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{-\lambda_{m,n}(t-\tau)} c_{m,n}(\tau)\,\mathrm{d}\tau,$$

where the last equality holds for the initial condition $u_0(x,y) = 0$ for all $x, y \in [0,1]$. Notice that

$$\int_0^t \mathrm{e}^{-\lambda_{m,n}(t-\tau)} \mathrm{e}^{-\tau}\,\mathrm{d}\tau = \mathrm{e}^{-\lambda_{m,n} t}\Big[\frac{\mathrm{e}^{(\lambda_{m,n}-1)\tau}}{\lambda_{m,n}-1}\Big]_{\tau=0}^{\tau=t} = \mathrm{e}^{-\lambda_{m,n} t}\frac{\mathrm{e}^{(\lambda_{m,n}-1)t} - 1}{\lambda_{m,n}-1} = \frac{\mathrm{e}^{-t} - \mathrm{e}^{\lambda_{m,n} t}}{\lambda_{m,n}-1}.$$

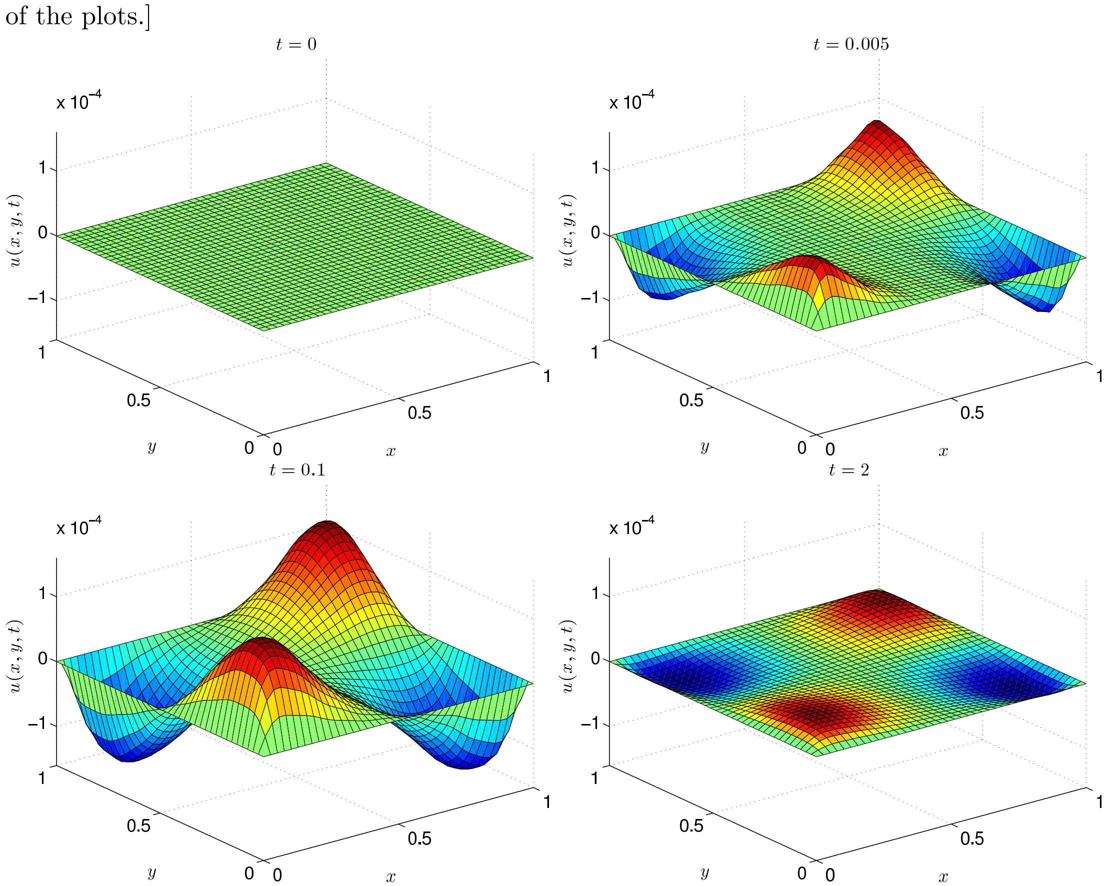
Hence, we have

$$\int_0^t e^{-\lambda_{m,n}(t-\tau)} c_{m,n}(\tau)\,\mathrm{d}\tau = \Big(\frac{(1+(-1)^m)(1+(-1)^n)(m^2\pi^2 - 24)}{8m^3 n\pi^4}\Big)\Big(\frac{\mathrm{e}^{-t} - \mathrm{e}^{-\lambda_{m,n} t}}{\lambda_{m,n}-1}\Big).$$

Finally, we can simplify the true solution as

$$u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{(1+(-1)^m)(1+(-1)^n)(m^2\pi^2 - 24)}{8m^3 n\pi^4}\Big)\Big(\frac{\mathrm{e}^{-t} - \mathrm{e}^{-\lambda_{m,n} t}}{\lambda_{m,n}-1}\Big)\big(2\sin(m\pi x)\sin(n\pi y)\big).$$

(c) Plots at the requested times are shown below, followed by the code that generated them. [GRADERS: If students do not simplify the exponentials as given in the formula for $u(x,t)$ above, it is possible that errors will prevent an accurate plot for the later times below. For example, it could be that $e^{\lambda_{m,n} t}$ evaluates in MATLAB as `Inf`, while $e^{-\lambda_{m,n} t}$ evaluates as `0`, with `Inf` $\times$ `0` = `NaN`, an unplottable quantity. Please deduct 5 points (once) if this is a problem for any of the plots.]



```
npts = 40;
x = linspace(0,1,npts); y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
tvec = [0 .005 .1 2];
for m=1:length(tvec)
   t = tvec(m);
   figure(1), clf
   U = zeros(npts,npts);
   n=15;
   for j=1:n
      for k=1:n
         cjk = (1+(-1)^j)*(1+(-1)^k)*(j^2*pi^2-24)/(8*j^3*k*pi^4); %  (x-1/2)^3 (y-1/2)
         lamjk = pi^2*(j^2+k^2);
         psijk = 2*sin(j*pi*X).*sin(k*pi*Y);
         U = U + (exp(-t)-exp(-lamjk*t))/(lamjk-1)*cjk*psijk;
      end
   end
   surf(X,Y,U)
   set(gca,'fontsize',16)
   xlabel('x')
   ylabel('y')
   zlabel('u(x,y,t)')
   title(sprintf(' t = %g', t))
   zlim([-.00016 .00016])
   eval(sprintf('print -depsc2 heat2d%d',m))
end
```