CAAM 336 · DIFFERENTIAL EQUATIONS

Midterm Examination 2

Posted Monday, 9 April 2012.

Due no later than 4pm on Monday, 16 April 2012.

Instructions:

1. Time limit: **4 uninterrupted hours**. You may fill out this page before you start the timer.
2. There are five questions worth a total of 100 points. Please do not look at the questions until you begin the exam.
3. You may use a crib sheet of relevant facts from the course that might help you in answering some of the questions. This should be on one side of a standard 8.5 by 11 inch sheet of paper. Please staple your crib sheet to the exam when you turn it in.
4. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
5. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
6. Print your name and section number (or instructor) on the line below:
7. Name: ______________________ Section: ______________________
8. Time started: ______________________ Time completed: ______________________
9. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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10. Staple this page to the front of your exam along with your crib sheet.

In the problems below, the spatial grid is $x_j = j * h$ for $j = 0, 1, 2, \ldots, n, n+1$, with $h = \ell/(n+1)$ and n is the number of interior points ($1 \le j \le n$). The $n+2$ hat functions are defined as

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/h, & x \in [x_{j-1}, x_j); \\ (x_{j+1} - x)/h, & x \in [x_j, x_{j+1}); \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, n$.

$$\phi_0(x) = \begin{cases} (x_1 - x)/h, & x \in [0, x_1); \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_{n+1}(x) = \begin{cases} (x - x_n)/h, & x \in [x_n, \ell]; \\ 0, & \text{otherwise} \end{cases}$$

All of the problems below use the inner product

$$(u, v) = \int_0^\ell u(x)v(x)\; dx.$$

Problems

1. [15 points] **The Weak Form**
   Consider the linear differential equation,

$$-\frac{d}{dx}\left[\kappa(x)\frac{du(x)}{dx}\right] + r(x)u(x) = f(x), \quad 0 < x < \ell$$
$$\frac{du(0)}{dx} = u(\ell) = 0$$

   where $\kappa \in C[0, \ell]$ is positive and $r \in C[0, \ell]$ is nonnegative.

   (a) [5 pts] Derive the weak form of this differential equation and define the associated bilinear operator $a : U \times U \to \mathbb{R}$ where U is an appropriately defined vector space. Give the definition of this space U.

   (b) [5 pts] Prove that a defines an inner product on U.

   (c) [5 pts] Use item (b) to show that the linear operator $\mathcal{L}$ defined by $\mathcal{L}u = -\frac{d}{dx}\left[\kappa(x)\frac{du(x)}{dx}\right] + r(x)u(x)$ for all $u \in U$ is symmetric and positive definite.

2. [25 points]

Consider the following BVP with mixed in-homogeneous boundary conditions

$$\begin{aligned} -\frac{d}{dx}\left[\kappa(x)\frac{du(x)}{dx}\right] &= f(x), \quad 0 < x < \ell \\ u(0) = \alpha, \frac{du(\ell)}{dx} &= \beta \end{aligned}$$

where $\kappa \in C[0, \ell]$ is positive.

(a) [8 pts] Formulate the weak form of this mixed boundary conditions BVP. To what space $\mathcal{V}$ should the test functions $v(x)$ belong?

(b) [8 pts] Show that the weak form is equivalent to the strong form in the case $\alpha = \beta = 0$.

(c) [9pts] Let $\ell = \pi/4$ and let $\kappa(x) = \cos(x)$. Explain how to set up the finite element approximation to this equation using the "hat" functions. Which "hat" functions will you need to use for this? You should produce a system of linear equations involving the coefficients of the corresponding "hat" functions needed for the approximate solution u_n. This system will be of the form

$$K\underline{\mathrm{u}} = \underline{\mathrm{f}}$$

as usual. What will be the dimensions of the matrix K and the vectors $\underline{\mathrm{u}}$ and $\underline{\mathrm{f}}$? Give explicit formulas for the elements $K(i, j)$ of the matrix K and the entries $\underline{\mathrm{f}}(i)$ of the vector $\underline{\mathrm{f}}$.

3. [10 points]

Consider the inhomogeneous heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial u(x)}{\partial x}\right] &= f(x,t), \quad 0 < x < \ell,\ 0 < t \\ u(x, 0) &= 0, \quad 0 < x < \ell \\ u(0, t) &= \alpha, \quad t > 0 \\ u(\ell, t) &= \beta, \quad t > 0. \end{aligned}$$

Using the results of Problem (2c) above, explain how to construct the method of lines solution to this problem. Recall that the method of lines approach transforms a partial differential equation into a system of ordinary differential equations for the time dependent coefficients of the basis functions that approximate the solution. Be sure to specify the mass matrix and stiffness matrix in this equation.

4. [25 points]

Consider the system of differential equations

$$\frac{du}{dt} = Au, \quad u(0) = u_0$$

where $A \in \mathbb{R}^{n\times n}, \quad u(t), u_0 \in \mathbb{R}^n$.

(a) [3 pts] Give the forward and backwards Euler's methods for this equation.

(b) [4 pts] Explain how to use these two methods to compute a numerical approximation to the matrix exponential e^A.

(c) [5 pts] Let

$$A = -\begin{bmatrix} 1+a & 1-a \\ 1-a & 1+a \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

for $a > 0$. Explain why $u(t) \to 0$ as $t \to \infty$. What values of Δt will assure that $u^{(j)} \to 0$ for Forward Euler. What values of Δt will assure that $u^{(j)} \to 0$ for Backwards Euler. Here, $u^{(j)}$ is the $j = th$ iterate of the corresponding method. Justify your answers.

Hint: Recall that to find eigenvalues of a matrix, you solve for the roots of the characteristic polynomial $\det(\lambda I - A) = 0$.

(d) [5 pts] If $a = 10^4$ how many time steps (iterations) N must be taken to reach the time $T = 1$ with Forward Euler and Backards Euler applied to the problem in item (c) with an accuracy of $\|u(1) - u^{(N)}\| < c/100$ where we assume the accuracy of both methods is $c\Delta t$. *This means both methods have an accuracy $\|u(t_n) - u^{(n)}\| < c\Delta t$, where $u^{(j)} \approx u(t_j)$ is the $j-th$ step of the iteration, with $t_n = T$ (the final time) and $\Delta t = T/n$.*

(e) [5 pts] Suppose now that A is time dependent so that

$$\frac{du(t)}{dt} = A(t)u(t), \quad u(0) = u_0$$

(the entries of $A(t)$ are functions of t). Give the forward and backwards Euler's methods for this equation and compare computational costs of the two methods for large n. Assume the cost of matrix-vector product Au is $c_1 n^2$ and the cost of solving $Ax = b$ is $c_2 n^3$.

(f) [3 pts] When would you choose Forward Euler over Backwards Euler in the previous item?

5. [25 points] Consider the heat equation with inhomogeneous time dependent boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(x,t), \quad 0 < x < 1, \ t > 0, \\ u(x,0) &= \psi(x), \quad 0 < x < 1 \\ u(0,t) &= g(t), \quad t > 0 \\ u(1,t) &= h(t), \quad t > 0 \end{aligned}$$

(a) [12 pts] Define $p(x,t) = g(t) + x(h(t) - g(t))$ and let $v(x,t) = u(x,t) - p(x,t)$. What IBVP does v satisfy?

(b) [13 pts] The Fourier series solution will be of the form

$$u(x,t) = \sum_{j=1}^{\infty} \mu_j(t)\psi_j(x)$$

where $\psi_j(x)$ are eigenfunctions of the spatial operator.

i. What are the functions $\psi_j(x)$?

ii. Explain how to construct the Fourier series solution to the given problem. Fully specify the equations the time-dependent coefficients $\mu_j(t)$ must satisfy. In your solution, you just need to write out the explicit formulas for the integrals involving the spatial variable x required to get the Fourier coefficients. You do not need to evaluate any of the integrals.