

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 42

Posted Friday 28 March 2014. Due 1pm Friday 18 April 2014.

42. [25 points]

Let

$$\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

and let $f \in C(\Omega)$ be defined by $f(x, y) = x(1 - y)$. In this question we will consider the problem of finding the solution $u(x, y)$ to the steady-state heat equation

$$-(u_{xx}(x, y) + u_{yy}(x, y)) = f(x, y), \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

with homogeneous Dirichlet boundary conditions

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

Let

$$C_D^2(\Omega) = \{v \in C^2(\Omega) : v(x, 0) = v(x, 1) = v(0, y) = v(1, y) = 0, 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Let the linear operator $L : C_D^2(\Omega) \rightarrow C(\Omega)$ be defined by

$$(Lv)(x, y) = -(v_{xx}(x, y) + v_{yy}(x, y)).$$

Let the inner product $(\cdot, \cdot) : C(\Omega) \times C(\Omega) \rightarrow \mathbb{R}$ be defined by

$$(v, w) = \int_0^1 \int_0^1 v(x, y)w(x, y) dx dy.$$

(a) Show that L is symmetric by showing that

$$(Lv, w) = (v, Lw) \text{ for all } v, w \in C_D^2(\Omega).$$

(b) The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions

$$\psi_{j,k}(x, y) = 2 \sin(j\pi x) \sin(k\pi y)$$

for $j, k = 1, 2, \dots$, which are such that

$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$

for $j, k = 1, 2, \dots$. Obtain a formula for $\lambda_{j,k}$ for $j, k = 1, 2, \dots$.

(c) Let

$$C_D^2[0, 1] = \{v \in C^2[0, 1] : v(0) = v(1) = 0\}$$

and let the linear operator $L_1 : C_D^2[0, 1] \rightarrow C[0, 1]$ be defined by

$$L_1 w = -w''.$$

Use what you know about the eigenfunctions of L_1 to compute $(\psi_{j,k}, \psi_{m,n})$ for $j, k, m, n = 1, 2, \dots$.

(d) The solution to $Lu = f$ that we obtain using the spectral method is

$$u(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})} \psi_{j,k}(x, y).$$

Plot

$$u_N(x, y) = \sum_{j=1}^N \sum_{k=1}^N \frac{1}{\lambda_{j,k}} \frac{(f, \psi_{j,k})}{(\psi_{j,k}, \psi_{j,k})} \psi_{j,k}(x, y)$$

for $N = 1, 2, 3, 4, 5, 10$. Note that, for $j, k = 1, 2, \dots$,

$$(f, \psi_{j,k}) = 2 \frac{(-1)^{j+1}}{jk\pi^2}.$$

Also note that to plot $\psi_{1,1}(x, y) = 2 \sin(\pi x) \sin(\pi y)$ you could use

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x = linspace(0,1,50);
y = linspace(0,1,50);
[X,Y] = meshgrid(x,y);
psill = 2*sin(pi*X).*sin(pi*Y);
surf(X,Y,psill)
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