CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 30

Posted Friday 21 March 2014. Due 1pm Friday 28 March 2014.

30. [25 points]

Let N be a positive integer, let $h = \dfrac{1}{N+1}$ and let $x_k = kh$ for $k = 0, 1, \ldots, N+1$. Let the continuous piecewise linear hat functions $\widehat{\phi}_j \in C[0,1]$ be such that

$$\widehat{\phi}_j(x) = \begin{cases} \dfrac{(x - x_{j-1})}{h} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)}{h} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N$. Also, let the continuous piecewise quadratic functions $\phi_j \in C[0,1]$ be such that

$$\phi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ \dfrac{(x_j + x_{j+1} - 2x)(x_{j+1} - x)}{h^2} & \text{if } x \in [x_j, x_{j+1}), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N$ and let the continuous piecewise quadratic bubble functions $\psi_j \in C[0,1]$ be such that

$$\psi_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{h^2} & \text{if } x \in [x_{j-1}, x_j), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N+1$.

(a) What is

i. $\phi_j(x_k)$ for $j = 1, \ldots, N$ and $k = 0, \ldots, N+1$;

ii. $\phi_j\left(\dfrac{x_{k-1} + x_k}{2}\right)$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N+1$;

iii. $\psi_j(x_k)$ for $j = 1, \ldots, N+1$ and $k = 0, \ldots, N+1$;

iv. $\psi_j\left(\dfrac{x_{k-1} + x_k}{2}\right)$ for $j, k = 1, \ldots, N+1$.

(b) Show that $\{\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_{N+1}\}$ is linearly independent by showing that if $\alpha_j, \beta_j \in \mathbb{R}$ and $\displaystyle\sum_{j=1}^{N} \alpha_j \phi_j(x) + \sum_{j=1}^{N+1} \beta_j \psi_j(x) = 0$ for all $x \in [0,1]$ then $\alpha_k = 0$ for $k = 1, \ldots, N$ and $\beta_k = 0$ for $k = 1, \ldots, N+1$.

(c) Obtain an expression for

$$\phi_j + \frac{1}{2}(\psi_j + \psi_{j+1})$$

for $j = 1, \ldots, N$.

(d) For $j = 1, \ldots, N$, is $\widehat{\phi}_j \in \operatorname{span}\{\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_{N+1}\}$?

(e) Is $\{\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_{N+1}, \widehat{\phi}_1, \ldots, \widehat{\phi}_N\}$ linearly independent?