→ CAAM 336 FINAL REVIEW ←

→ THINGS FROM THE 1st HALF TO REMEMBER:

- Eigenvalues / Eigenfunctions
  - ↳ be able to compute them etc.
- Best Approximation from a subspace;
  especially notion that $\varphi_1, \varphi_2, \varphi_3 \ldots$ forms a basis for V of eigenfunctions, then $f \in V$ can be written as:

$$\boxed{f = \sum_{k=1}^{\infty} \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)} \varphi_k}$$

- Weak form of diff eq. & Finite Element Method (steady state)
  - ↳ Final may cover this topic begun just before midterm

→ 2nd HALF OF COURSE: TIME DEPENDENT PROBLEMS

GENERIC MATRIX PROBLEM:

$u'(t) = Au(t)$

solution: $\boxed{u(t) = e^{tA} u(0)}$

↳ Matrix exponential e^{tA}

- If $A = V \Lambda V^{-1}$, then

$$e^{tA} = V e^{t\Lambda} V^{-1}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{bmatrix}$$

OR

$$e^{tA} = I + (tA) + \frac{1}{2}(tA)^2 + \frac{1}{3!}(tA)^3 + \ldots$$

How does $u(t)$ behave as $t \to \infty$?

- How does e^{tA} behave as $t \to \infty$?
  - If ALL eigenvalues of A have neg. real part, ↳ ($Re(\lambda_j) < 0 \ \forall j$)
    THEN... $e^{tA} \to 0$ (decay)

    (If A is symmetric, λ_j is real,
    so $Re(\lambda_j) = \lambda_j$)

  - If ANY eigenvalue of A has positive real part,
    THEN... e^{tA} blows up as $t \to \infty$

  - If $Re(\lambda_j) \leq 0 \ \forall \lambda_j$, and $Re(\lambda_k) = 0$ for some k,
    THEN... e^{tA} neither decays nor blows up!

Think about . . .

$A = -M^{-1}K$ from heat equation

or $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}$ from wave equation

PROBLEMS WITH FORCING FUNCTIONS:

$u'(t) = Au(t) + f(t)$

General solution: $\boxed{u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)} f(s)\,ds}$

APPROXIMATE SOLUTION IN TIME:

$u'(t) = Au(t)$; $u_k \approx u(\Delta t\, k)$

- Forward Euler:

$u_{k+1} = u_k + \Delta t\,(Au_k)$

$\Rightarrow \boxed{u_k = (I + \Delta t A)^k u_0}$

- Backward Euler:

$u_{k+1} = u_k + \Delta t A\, u_{k+1}$

$(I - \Delta t A)\, u_{k+1} = u_k$ → "linear system"

↳ MATLAB: $u_{k+1} = (I - \Delta t A) \backslash u_k$

$u_{k+1} = (I - \Delta t A)^{-1} u_k$

$\Rightarrow \boxed{u_k = ((I - \Delta t A)^{-1})^k u_0}$

Both methods give problems like:

$u_{k+1} = B^k u_0$, for some matrix B.

- When does $B^k \to 0$ as $k \to \infty$?

When all eigenvalues β of B satisfy $|\beta| < 1$

- When does B^k blow up as $k \to \infty$?

If $|\beta| > 1$ for <u>any</u> eigenvalue β of B.

ok. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A

Fw Euler

$B = I + \Delta t A$ → eigs of B: $\beta_j = 1 + \Delta t \lambda_j$

Bw Euler

$B = (I - \Delta t A)^{-1}$ → eigs of B: $\beta_j = \frac{1}{1 - \Delta t \lambda_j}$

Suppose A is symmetric w/ negative eigenvalues, $\lambda_j < 0 \ \forall j$

THEN $e^{tA} \to 0$ as $t \to \infty$

Thus, $u(t) = e^{tA} u(0)$ = exact solution decays!

We want the numerical approximation u_k to decay as well.

- FW Euler decays provided: $|1 + \Delta t \lambda_j| < 1 \ \forall j$

$$-1 < 1 + \Delta t \lambda_j < 1$$

$$-2 < 1 + \Delta t \lambda_j < 0 \qquad \forall j$$

$$\Rightarrow \frac{-2}{\lambda_j} > \Delta t > 0 \Big\} \text{ We must pick } \Delta t \text{ s.t. } 0 < \Delta t < \frac{-2}{\lambda_j}$$

↳ If some λ_j are large & negative this imposes a severe restriction on Δt

- BW Euler decays provided: $\left|\frac{1}{1 - \Delta t \lambda_j}\right| < 1$

which holds for any Δt!

SOLUTION OF TIME DEPENDENT PROBLEMS BY EIGENFUNCTIONS:

L = linear operator, symmetric

$\lambda_1, \lambda_2 \ldots$ eigenvalues, real

$\varphi_1, \varphi_2 \ldots$ eigenfunctions, orthogonal.

We wish to solve $u_t = -Lu$

① Write the solution in the form $u(x,t) = \sum_{j=1}^{\infty} a_j(t) \varphi_j(x)$

② Determine ODE's that the coefficients a_j must satisfy

Substitute $u(x,t)$ into $u_t = -Lu$

$$\frac{\partial}{\partial t}\left(\sum_{j=1}^{\infty} a_j(t) \varphi_j(x)\right) = -L\left(\sum_{j=1}^{\infty} a_j(t)\varphi_j(x)\right)$$

$$\sum_{j=1}^{\infty} a_j'(t)\varphi_j(x) = -\sum_{j=1}^{\infty} a_j(t) L\varphi_j(x)$$

$$= -\sum_{j=1}^{\infty} a_j(t)\lambda_j \varphi_j(x)$$

TAKE IP w/ φ_k:

$$\left(\sum a_j'(t)\varphi_j(x), \varphi_k(x)\right) = -\left(\sum a_j(t)\lambda_j \varphi_j(x), \varphi_k(x)\right)$$

$$\sum a_j'(t)(\varphi_j, \varphi_k) = -\sum a_j(t)\lambda_j(\varphi_j, \varphi_k)$$

AND... $(\varphi_j, \varphi_k) = 0$ if $j \neq k$ so:

$$\boxed{a_k'(t) = -\lambda_k a_k(t)}$$

③ Solve the ODE's w/ initial condition from $u(x,0) = u_0$

$$a_j(0) = \frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)} \Big\} \text{ Best approx!}$$

SOLUTION: $\boxed{a_k(t) = e^{-\lambda_j t} a_k(0)}$ $\Rightarrow$ $\boxed{u_{x,t}(x,t) = \sum_{j=1}^{\infty}\left(e^{-\lambda_j t}\frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)}\varphi_j\right)}$

Example: Heat equation w/ Neumann B.C.'s

$L: C^2_N[0,1] \to C[0,1]$

$Lu = -u''$, $u'(0) = u'(1) = 0$, Eigenvalues: $\lambda_0 = 0$

$-\psi''(x) = \lambda\psi(x)$

$\Rightarrow \psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

$\psi'(0) = 0 \quad \psi'(1) = 0$

$\psi'(x) = \sqrt{\lambda}A\cos(\sqrt{\lambda}x) - \sqrt{\lambda}B\sin(\sqrt{\lambda}x)$

$0 = \psi'(0) = \sqrt{\lambda}A \Rightarrow A = 0$

$0 = \psi'(1) = -\sqrt{\lambda}B\sin(\sqrt{\lambda}) \Rightarrow \sin(\sqrt{\lambda}) = 0$

$\Rightarrow \sqrt{\lambda} = n\pi \longrightarrow \lambda_n = n^2\pi^2$!

λ_0 is special:

$-\psi''(x) = 0$

$\Rightarrow \psi(x) = A + Bx \longrightarrow \psi'(x) = B$

$0 = \psi'(0) = B \Rightarrow B = 0$

$0 = \psi'(1) = B \Rightarrow B = 0$

Take $\psi(x) = A = 1$ for $\lambda = 0$!

So: $\lambda_j = j^2\pi^2$, $\psi_j(x) = \sqrt{2}\cos(j\pi x)$

$\lambda_0 = 0$, $\psi_0(x) = 1$

Soln: $u(x,t) = \sum e^{-\lambda_j t}\dfrac{(u_0, \psi_j)}{(\psi_j, \psi_j)}\psi_j(x)$

$$= e^{-\lambda_0 t}\frac{(u_0, \psi_0)}{(\psi_0, \psi_0)}\psi_0 + \sum_{j=1}^{\infty} e^{-\lambda_j t}\frac{(u_0, \psi_j)}{(\psi_j, \psi_j)}\psi_j$$

$$= \underbrace{\boxed{\frac{(u_0, \psi_0)}{(\psi_0, \psi_0)} \cdot 1}}_{\text{Constant! } \int_0^1 u_0(x)\,dx} + \underbrace{\boxed{\sum e^{-\lambda_j t}\frac{(u_0, \psi_j)}{(\psi_j, \psi_j)}\psi_j}}_{\to 0 \text{ as } t \to \infty \text{ since } \lambda_j = j^2\pi^2 > 0}$$

FINITE ELEMENT METHOD FOR $u_t = -Lu$ (HEAT EQUATION)

(Turn PDE into system of ODE's)

1. Weak Form

Ex. $u_t = u_{xx}$, $u(0,t) = u(1,t) = 0$

Pick $V = C^2_0[0,1]$, For $v \in V$ ...

$\frac{\partial}{\partial t}(u, v) = (u_t, v) = (u_{xx}, v)$

IBP $= (u_x, v_x) = -\int_0^1 u_x(x,t)v_x(x)\,dx = -a(u,v)$ energy inner product

$$\boxed{\frac{\partial}{\partial t}(u,v) = -a(u,v) \quad \forall v \in V}$$

WEAK FORM

② Galerkin Approximation

$V_N = \text{span}\{\phi_1, \dots, \phi_N\} \subseteq V$

$$\boxed{\begin{array}{l} \text{Find } u_N \in V_N \times C[0,\infty] \text{ s.t.} \\ \frac{\partial}{\partial t}(u_N, v) = -a(u_N, v) \quad \forall\, v \in V_N \end{array}}$$

③ Turn into a linear algebra problem:

write $u_N(x,t) = \sum_{j=1}^{N} c_j(t)\,\varphi_j(x)$

Test at $v = \phi_k$, $k = 1, \dots, N$

$$\longrightarrow \sum_{j=1}^{N} c_j'(t)\,(\phi_j, \phi_k) = -\sum_{j=1}^{N} c_j(t)\, a(\phi_j, \phi_k)$$

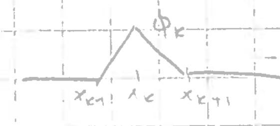
$$\underbrace{\begin{bmatrix} (\phi_1,\phi_1) & \cdots & (\phi_N,\phi_1) \\ \vdots & \ddots & \\ (\phi_1,\phi_N) & & (\phi_N,\phi_N) \end{bmatrix}}_{\substack{M \\ \text{(Mass)}}} \underbrace{\begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_N' \end{bmatrix}}_{c'} = -\underbrace{\begin{bmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_N,\phi_1) \\ \vdots & & \\ a(\phi_1,\phi_N) & & a(\phi_N,\phi_N) \end{bmatrix}}_{\substack{K \\ \text{(stiffness)}}} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}}_{c}$$

④ Solve for $c(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_N(t) \end{bmatrix}$

$$\boxed{c'(t) = -M^{-1}K\,c(t)} \longrightarrow \boxed{c(t) = e^{-M^{-1}Kt}\,c(0)}$$ Approx in space, exact in time.

If $\phi_1, \dots, \phi_N$ = hat functions: $c_j(0) = u_0(x_j)$

$$\phi_j(x_k) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$



$$u_N(x_k, t) = \sum_{j=1}^{N} c_j(t)\,\phi_j(x_k) = c_k(t)$$

$$c_k(0) = u_N(x_k, 0) = u_0(x_k)$$

<u>Alternative</u>: Fw Euler: eigs of $M^{-1}K \approx j^2\pi^2$, $j = 1, \dots, N$

<u>Roughly</u> $\Delta t \in \left(0, \frac{2}{N^2\pi^2}\right)$

BW Euler: <u>ANY</u> $\Delta t > 0$ is stable

rule of thumb: if you double N, cut Δt to $\frac{\Delta t}{4}$

WAVE EQUATION ($u_{tt} = -Lu$)

L is symmetric linear operator
$\lambda_1, \lambda_2 \dots$ eigenvalues (real)
$\varphi_1, \varphi_2 \dots$ eigenfunctions orthogonal

① Write $u(x,t) = \sum a_j(t)\varphi_j(t)$
② Substitute into PDE to get ODE for coefficients

$$\sum a_j''(t)\varphi_j(x) = -\sum a_j(t) L\varphi_j(x)$$
$$= -\sum a_j(t)\lambda_j \varphi_j(x)$$

inner product $\Rightarrow$

$$\boxed{a_k''(t) = -\lambda_k a_k(t)}$$

③ Solve ODE's for a_k

$a_k(t) = A_k \sin(\sqrt{\lambda_k}\, t) + B_k \cos(\sqrt{\lambda_k}\, t)$
with initial conditions:

$$u_0(x) = u(x,0) = \sum \frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)}$$

$$v_0(x) = u_t(x,0) = \sum \frac{(v_0, \varphi_j)}{(\varphi_j, \varphi_j)}$$

$$\rightarrow u_0(x) = \sum_{j=1}^{\infty} A_j \sin(\sqrt{\lambda_j}\, 0) + B_j \cos(\sqrt{\lambda_j}\, 0)\varphi_j$$
$$= \sum_{j=1}^{\infty} B_j \varphi_j(x) \Rightarrow \boxed{B_j = \frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)}}$$

$$\rightarrow v_0(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\,(A_j \cos\sqrt{\lambda_j}\, 0) - B_j \sin(\sqrt{\lambda_j}\, 0))\varphi_j(x)$$
$$= \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, A_j \varphi_j(x) \Rightarrow \boxed{A_j = \frac{1}{\sqrt{\lambda_j}} \frac{(v_0, \varphi_j)}{(\varphi_j, \varphi_j)}} \quad \lambda_j \neq 0$$

$u(x,t) =$

$$\sum_{j=1}^{\infty} \underbrace{\left(\frac{1}{\sqrt{\lambda_j}} \frac{(v_0, \varphi_j)}{(\varphi_j, \varphi_j)} \sin(\sqrt{\lambda_j}\, t) + \frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)} \cos(\sqrt{\lambda_j}\, t) \right)}_{\text{oscillator!}} \varphi_j(x)$$

FINITE ELEMENT METHOD w/ WAVE EQ

Find $u \in V \times C(0,t)$

Weak form: $\frac{\partial^2}{\partial t^2}(u,v) = -a(u,v) \quad \forall\, v \in V$

Galerkin: $u_N(x,t) = \sum c_j(t)\,\varphi_j(x)$

Find u_N s.t. $\frac{\partial^2}{\partial t^2}(u_N, v) = -a(u_N, v)$

$\forall\, v \in V_N = \text{span}\{\phi_1, \phi_2 \dots \phi_N\}$

$\rightarrow \boxed{Mc'' = -Kc}$

$\frac{\partial}{\partial t}\begin{bmatrix} c \\ c' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}\begin{bmatrix} c \\ c' \end{bmatrix}$ Solve w/ Mix exponential

FW Euler

BW Euler

Trapazoid

If eigs of $M^{-1}K$ are real positive then...

eigs of $\begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}$ are purely imaginary

d'ALEMBERT'S SOLUTION ON UNBOUNDED DOMAINS

$u_{tt}(x,t) = c^2 u_{xx}(x,t) \quad x \in (-\infty, \infty)$

$u(x,0) = \varphi(x) \quad u_t(x,0) = \gamma(x)$

$$\boxed{u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \gamma(s)\,ds}$$

Also review domain of dependence!