CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 42 · Solutions

Posted Wednesday 13 November 2013. Due 5pm Wednesday 27 November 2013.

42. [25 points] Let

$$\Omega = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$$

and let $f(x, y, t) = (x - \frac{1}{2})^3(y - \frac{1}{2})e^{-t}$. Note that, for $m, n = 1, 2, \ldots$,

$$\int_0^1 \int_0^1 2f(x, y, t) \sin(m\pi x) \sin(n\pi y)\, dx\, dy = \frac{(1 + (-1)^m)(1 + (-1)^n)(m^2\pi^2 - 24)}{8m^3 n\pi^4} e^{-t}.$$

In this question we will consider the problem of finding the solution $u(x, y, t)$ to the heat equation

$$u_t(x, y, t) - (u_{xx}(x, y, t) + u_{yy}(x, y, t)) = f(x, y, t), \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

with homogeneous Dirichlet boundary conditions

$$u(x, 0, t) = u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

and initial condition

$$u(x, y, 0) = 0, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Let

$$C_D^2(\Omega) = \left\{v \in C^2(\Omega) : v(x, 0) = v(x, 1) = v(0, y) = v(1, y) = 0,\ 0 \le x \le 1,\ 0 \le y \le 1\right\}.$$

Let the linear operator $L : C_D^2(\Omega) \to C(\Omega)$ be defined by

$$(Lv)(x, y) = -(v_{xx}(x, y) + v_{yy}(x, y)).$$

The operator L has eigenvalues $\lambda_{j,k} \in \mathbb{R}$ and eigenfunctions

$$\psi_{j,k}(x, y) = 2\sin(j\pi x)\sin(k\pi y)$$

for $j, k = 1, 2, \ldots$, which are such that

$$L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$$

for $j, k = 1, 2, \ldots$. Recall that in Homework 40 you obtained a formula for $\lambda_{j,k}$ for $j, k = 1, 2, \ldots$.

(a) We can write

$$u(x, y, t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x, y)$$

and

$$f(x, y, t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(t)\psi_{j,k}(x, y)$$

where

$$c_{j,k}(t) = \int_0^1 \int_0^1 f(x, y, t)\psi_{j,k}(x, y)\, dx\, dy.$$

What ordinary differential equation and initial condition does $a_{j,k}(t)$ satisfy for $j, k = 1, 2, \ldots$?

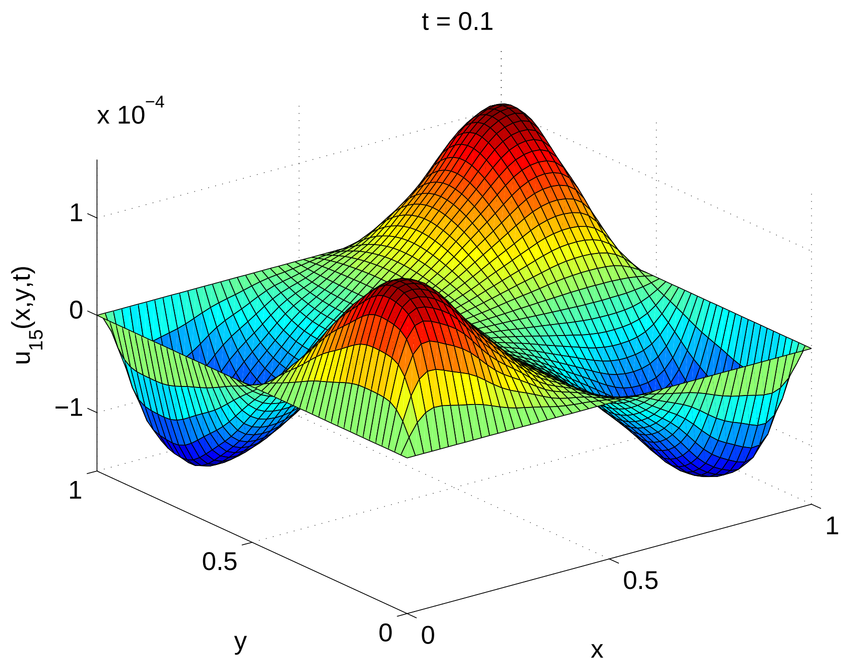
(b) Obtain an expression for $a_{j,k}(t)$ for $j, k = 1, 2, \ldots$.

(c) Use you answer to part (b) to write out a formula for $u(x, y, t)$.

(d) Plot

$$u_{15}(x, y, t) = \sum_{j=1}^{15} \sum_{k=1}^{15} a_{j.k}(t) \psi_{j,k}(x, y)$$

at the four times $t = 0, 0.005, 0.1, 2$. Use the command `zlim([-.00016 .00016])` so that the axes on all of your plots are the same. Your plot for $t = 0.1$ should resemble the plot below.



Solution.

(a) [7 points] Substituting the expressions for $u(x, y, t)$ and $f(x, y, t)$ into the partial differential equation yields

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a'_{j,k}(t) \psi_{j,k}(x, y) - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}(t) \left(- \left(L\psi_{j,k} \right)(x, y) \right) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(t) \psi_{j,k}(x, y)$$

and hence

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(a'_{j,k}(t) + \lambda_{j,k} a_{j,k}(t) \right) \psi_{j,k}(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(t) \psi_{j,k}(x, y).$$

We can then say that

$$\begin{aligned} &\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(a'_{j,k}(t) + \lambda_{j,k} a_{j,k}(t) \right) \int_0^1 \int_0^1 \psi_{j,k}(x, y) \psi_{m,n}(x, y) \, dx \, dy \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(t) \int_0^1 \int_0^1 \psi_{j,k}(x, y) \psi_{m,n}(x, y) \, dx \, dy, \end{aligned}$$

for $m, n = 1, 2, \ldots$, from which it follows that

$$a'_{m,n}(t) + \lambda_{m,n} a_{m,n}(t) = c_{m,n}(t),$$

for $m, n = 1, 2, \ldots$, since

$$\int_0^1 \int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{if } j \neq m \text{ or } k \neq n \end{cases}$$

for $j, k, m, n = 1, 2, \ldots$.
Also,

$$u(x, y, 0) = 0$$

means that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y) = 0$$

and so

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\int_0^1 \int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \int_0^1 \int_0^1 0\,dx\,dy,$$

for $m, n = 1, 2, \ldots$, from which it follows that

$$a_{m,n}(0) = 0,$$

for $m, n = 1, 2, \ldots$, since

$$\int_0^1 \int_0^1 \psi_{j,k}(x,y)\psi_{m,n}(x,y)\,dx\,dy = \begin{cases} 1 & \text{if } j = m \text{ and } k = n \\ 0 & \text{if } j \neq m \text{ or } k \neq n \end{cases}$$

for $j, k, m, n = 1, 2, \ldots$, and

$$\int_0^1 \int_0^1 0\,dx\,dy = 0.$$

Hence, for $j, k = 1, 2, \ldots$, $a_{j,k}(t)$ is the solution to the differential equation

$$a'_{j,k}(t) = -\lambda_{j,k}a_{j,k}(t) + c_{j,k}(t)$$

with initial condition

$$a_{j,k}(0) = 0.$$

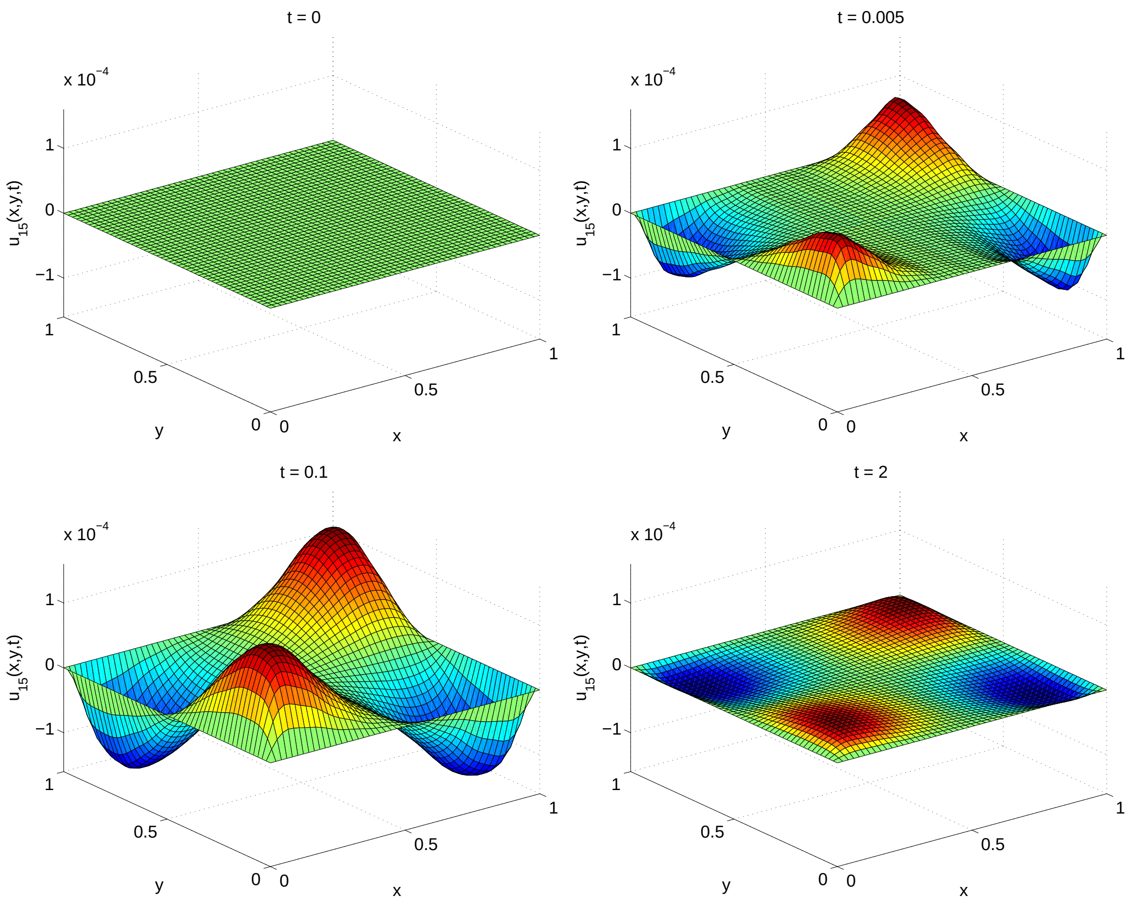
(b) [4 points] For $j, k = 1, 2, \ldots$,

$$\begin{aligned} a_{j,k}(t) &= \int_0^t e^{\lambda_{j,k}(s-t)} c_{j,k}(s)\,ds \\ &= \int_0^t e^{\lambda_{j,k}(s-t)} \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4} e^{-s}\,ds \\ &= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4} \int_0^t e^{(\lambda_{j,k}-1)s-\lambda_{j,k}t}\,ds \\ &= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4} \left[\frac{1}{\lambda_{j,k}-1} e^{(\lambda_{j,k}-1)s-\lambda_{j,k}t}\right]_{s=0}^{s=t} \\ &= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4} \left(\frac{1}{\lambda_{j,k}-1} e^{-t} - \frac{1}{\lambda_{j,k}-1} e^{-\lambda_{j,k}t}\right) \\ &= \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4\left(\pi^2\left(j^2+k^2\right)-1\right)} \left(e^{-t} - e^{-\pi^2\left(j^2+k^2\right)t}\right). \end{aligned}$$

(c) [4 points] We can write

$$
\begin{aligned}
u(x,y,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{8j^3k\pi^4\left(\pi^2\left(j^2+k^2\right)-1\right)}\left(e^{-t}-e^{-\pi^2(j^2+k^2)t}\right)\psi_{j,k}(x,y) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(1+(-1)^j)(1+(-1)^k)(j^2\pi^2-24)}{4j^3k\pi^4\left(\pi^2\left(j^2+k^2\right)-1\right)}\left(e^{-t}-e^{-\pi^2(j^2+k^2)t}\right)\sin(j\pi x)\sin(k\pi y).
\end{aligned}
$$

(d) [10 points] Plots at the requested times are shown below, followed by the code that generated them.




```
clear
clc
npts = 50;
x = linspace(0,1,npts);
y = linspace(0,1,npts);
[X,Y] = meshgrid(x,y);
tvec = [0 .005 .1 2];
for m=1:length(tvec)
    t = tvec(m);
    figure(1)
    clf
    U = zeros(npts,npts);
    n=15;
```

```
    for j=1:n
        for k=1:n
            cjk = (1+(-1)^j)*(1+(-1)^k)*(j^2*pi^2-24)/(8*j^3*k*pi^4); %  (x-1/2)^3 (y-
                1/2)
            lamjk = pi^2*(j^2+k^2);
            psijk = 2*sin(j*pi*X).*sin(k*pi*Y);
            U = U + (exp(-t)-exp(-lamjk*t))/(lamjk-1)*cjk*psijk;
        end
    end
    surf(X,Y,U)
    set(gca,'fontsize',16)
    xlabel('x')
    ylabel('y')
    zlabel('u_{15}(x,y,t)')
    title(sprintf(' t = %g', t))
    zlim([-.00016 .00016])
    eval(sprintf('print -depsc2 heat2d%d',m))
end
```