CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 11

Posted Wednesday 14 November 2012. Due Wednesday 21 November 2012, 5pm.

This problem set counts for 75 points.
Late problem sets are due by 5pm on Monday 26 November 2012.

1. [30 points: 7 points each for (a), (b), (c); 9 points for (d)]
   This question concerns the homogeneous wave equation on an unbounded spatial domain:
   $$u_{tt}(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty, \quad t > 0.$$
   Find the solution $u(x,t)$ to this equation with the following initial conditions:
   (a) $u(x,0) = 2\sin(x)e^{-x^2}, \quad u_t(x,0) = 0;$
   (b) $u(x,0) = 0, \qquad u_t(x,0) = -\dfrac{2x}{(1+x^2)^2};$
   (c) $u(x,0) = 2\sin(x)e^{-x^2}, \quad u_t(x,0) = -\dfrac{2x}{(1+x^2)^2}.$
   (d) Produce a plot (or plots) showing your solution to part (c) over $-10 \le x \le 10$ at times $t = 0, 1, 2, 3, 4, 5$.

2. [10 points]
   Consider the wave equation on an infinite spatial domain, $x \in (-\infty, \infty)$, but now with a forcing term:
   $$u_{tt}(x,t) = u_{xx}(x,t) + f(t), \qquad x \in (-\infty, \infty)$$
   subject again to the initial data
   $$u(x,0) = \psi(x), \qquad u_t(x,0) = \gamma(x).$$
   Verify that the formula
   $$u(x,t) = \widehat{u}(x,t) + t\int_0^t f(\tau)\,\mathrm{d}\tau - \int_0^t \tau f(\tau)\,\mathrm{d}\tau$$
   is a solution to this inhomogeneous wave equation, where $\widehat{u}(x,t)$ denotes the solution of the homogeneous problem with the same initial data.

3. [35 points: 5 points each for (a) and (b); 10 points for (c); 15 points for (d)]
   Consider the wave equation posed on the infinite domain $x \in (-\infty, \infty)$:
   $$u_{tt}(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty, \quad t > 0 \qquad (*)$$
   with initial conditions $u(x,0) = \psi(x)$ and $u_t(x,0) = \gamma(x)$.
   At a given point $(\widetilde{x}, \widetilde{t})$, with $\widetilde{x} \in (-\infty, \infty)$ and $\widetilde{t} > 0$, the solution $u(\widetilde{x}, \widetilde{t})$ of the wave equation is only affected by some portion of the initial data. In other words, $u(\widetilde{x}, \widetilde{t})$ is only influenced by $\psi(x)$ and $\gamma(x)$ for $x \in [a,b]$, where a and b will depend upon $\widetilde{x}$ and $\widetilde{t}$. This interval $[a,b]$ is called the *domain of dependence* of the solution at $(\widetilde{x}, \widetilde{t})$.

(a) Determine the domain of dependence of the solution to the wave equation $(*)$ at $(\widetilde{x}, \widetilde{t}) = (0, 1)$.

Now consider the heat equation on an unbounded domain:

$$u_t(x,t) = u_{xx}(x,t), \qquad -\infty < x < \infty$$

with initial data

$$u(x,0) = \psi(x).$$

Like d'Alembert's solution, there exists a formula for the solution of the heat equation on this domain: for all $t > 0$,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \psi(s)\, \mathrm{d}s.$$

(b) What is the domain of dependence of this solution to the heat equation at $(\widetilde{x}, \widetilde{t}) = (0, 1)$? Contrast the physical implications of the domains of dependence for the heat and wave equations.

(c) Consider the *wave* equation with discontinuous initial data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0; \end{cases} \qquad \gamma(x) = 0.$$

On one plot, superimpose solutions to this equation at the four times $t = 0, 1/2, 1, 2$. (Notice how the discontinuity in the initial data is propagated in time.)

(d) Now consider the *heat* equation with the same starting data

$$\psi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

Using the formula for $u(x,t)$ given above, produce solutions to this equation at the four times $t = 0, 0.01, 0.1, 1$. What happens to the discontinuity for $t > 0$?

Important hint: You will need to compute some nasty integrals here that you cannot work out entirely by hand. To produce your plots, use MATLAB's `erfc` command. For example,

$$\frac{2}{\sqrt{\pi}} \int_{\mathtt{z}}^{\infty} e^{-y^2}\, \mathrm{d}y = \mathtt{erfc(z)}.$$