

CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 19 · Solutions

Posted Friday 14 February 2014. Due 1pm Friday 28 February 2014.

19. [25 points]

All parts of this question should be done by hand.

Let

$$C_z^1[-1, 1] = \left\{ v \in C^1[-1, 1] : \int_{-1}^1 v(x) dx = 0 \right\}.$$

Let $v_1 \in C_z^1[-1, 1]$, $v_2 \in C_z^1[-1, 1]$, and $f \in C_z^1[-1, 1]$ be defined by

$$v_1(x) = \frac{\sqrt{3}}{\sqrt{2}}x,$$

$$v_2(x) = \frac{\sqrt{3}}{\sqrt{2}}(3x^2 - x - 1),$$

and

$$f(x) = \frac{\sqrt{2}}{\sqrt{3}} \cos(\pi x)$$

for all $x \in [-1, 1]$. Let the inner product $(\cdot, \cdot) : C_z^1[-1, 1] \times C_z^1[-1, 1] \rightarrow \mathbb{R}$ be defined by

$$(u, v) = \int_{-1}^1 u(x)v(x) dx$$

and let the norm $\|\cdot\| : C_z^1[-1, 1] \rightarrow \mathbb{R}$ be defined by

$$\|u\| = \sqrt{(u, u)}.$$

Also, let the inner product $a(\cdot, \cdot) : C_z^1[-1, 1] \times C_z^1[-1, 1] \rightarrow \mathbb{R}$ be defined by

$$a(u, v) = \int_{-1}^1 (2 + x)u'(x)v'(x) dx$$

and let the norm $\|\cdot\|_a : C_z^1[-1, 1] \rightarrow \mathbb{R}$ be defined by

$$\|u\|_a = \sqrt{a(u, u)}.$$

Moreover, let the inner product $B(\cdot, \cdot) : C_z^1[-1, 1] \times C_z^1[-1, 1] \rightarrow \mathbb{R}$ be defined by

$$B(u, v) = a(u, v) + (u, v)$$

and the norm $\|\cdot\|_B : C_z^1[-1, 1] \rightarrow \mathbb{R}$ be defined by

$$\|u\|_B = \sqrt{B(u, u)}.$$

Note that $(v_1, v_1) = 1$; $(v_2, v_2) = \frac{17}{5}$; $(f, v_1) = 0$; $(f, v_2) = -\frac{12}{\pi^2}$; $a(v_1, v_1) = 6$; $a(v_2, v_2) = 66$; $a(f, v_1) = -2$ and $a(f, v_2) = -22$.

- (a) Use the fact that (\cdot, \cdot) and $a(\cdot, \cdot)$ are inner products on $C_z^1[-1, 1]$ to verify that $B(\cdot, \cdot)$ is an inner product on $C_z^1[-1, 1]$.

- (b) What is the best approximation to f from $\text{span}\{v_1\}$ with respect to the norm $\|\cdot\|$?
- (c) What is the best approximation to f from $\text{span}\{v_1\}$ with respect to the norm $\|\cdot\|_a$?
- (d) What is the best approximation to f from $\text{span}\{v_1\}$ with respect to the norm $\|\cdot\|_B$?
- (e) What is the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|_a$?
- (f) What is the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|$?

Solution.

- (a) [6 points] If $u \in C_z^1[-1, 1]$ and $v \in C_z^1[-1, 1]$ then

$$B(u, v) = a(u, v) + (u, v) = a(v, u) + (v, u) = B(u, v)$$

since $a(u, v) = a(v, u)$ and $(u, v) = (v, u)$ because $a(\cdot, \cdot)$ and (\cdot, \cdot) are inner products on $C_z^1[-1, 1]$. So

$$B(u, v) = B(u, v) \text{ for all } u, v \in C_z^1[-1, 1].$$

If $u, v, w \in C_z^1[-1, 1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned} B(\alpha u + \beta v, w) &= a(\alpha u + \beta v, w) + (\alpha u + \beta v, w) \\ &= \alpha a(u, w) + \beta a(v, w) + \alpha(u, w) + \beta(v, w) \\ &= \alpha(a(u, w) + (u, w)) + \beta(a(v, w) + (v, w)) \\ &= \alpha B(u, w) + \beta B(v, w) \end{aligned}$$

since $a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w)$ and $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$ because $a(\cdot, \cdot)$ and (\cdot, \cdot) are inner products on $C_z^1[-1, 1]$. So

$$B(\alpha u + \beta v, w) = \alpha B(u, w) + \beta B(v, w) \text{ for all } u, v, w \in C_z^1[-1, 1] \text{ and all } \alpha, \beta \in \mathbb{R}.$$

If $u \in C_z^1[-1, 1]$ then $B(u, u) \geq 0$ since $a(u, u) \geq 0$ and $(u, u) \geq 0$ because $a(\cdot, \cdot)$ and (\cdot, \cdot) are inner products on $C_z^1[-1, 1]$. Moreover, if $B(u, u) = 0$ then

$$a(u, u) + (u, u) = 0,$$

or equivalently,

$$a(u, u) = -(u, u)$$

and, since $a(u, u) \geq 0$ and $(u, u) \geq 0$, this can only hold when $a(u, u) = (u, u) = 0$ which then implies that $u = 0$ because of either the fact that $a(u, u) = 0$ only if $u = 0$ since $a(\cdot, \cdot)$ is an inner product on $C_z^1[-1, 1]$ or the fact that $(u, u) = 0$ only if $u = 0$ since (\cdot, \cdot) is an inner product on $C_z^1[-1, 1]$. So

$$B(u, u) \geq 0 \text{ for all } u \in C_z^1[-1, 1]$$

with

$$B(u, u) = 0 \text{ only if } u = 0.$$

Consequently, $B(\cdot, \cdot)$ is an inner product on $C_z^1[-1, 1]$.

(b) [2 points] The best approximation to f from $\text{span}\{v_1\}$ with respect to the norm $\|\cdot\|$ is

$$b_1(x) = \frac{(f, v_1)}{(v_1, v_1)} v_1(x) = 0$$

since $(f, v_1) = 0$.

(c) [2 points] The best approximation to f from $\text{span}\{v_1\}$ with respect to the norm $\|\cdot\|_a$ is

$$b_2(x) = \frac{a(f, v_1)}{a(v_1, v_1)} v_1(x) = \frac{-2}{6} v_1(x) = \frac{-1}{3} \frac{\sqrt{3}}{\sqrt{2}} x = \frac{-1}{\sqrt{6}} x.$$

(d) [3 points] The definition of $B(\cdot, \cdot)$ means that

$$B(v_1, v_1) = a(v_1, v_1) + (v_1, v_1) = 6 + 1 = 7$$

and

$$B(f, v_1) = a(f, v_1) + (f, v_1) = -2 + 0 = -2.$$

Therefore, the best approximation to f from $\text{span}\{v_1\}$ with respect to the norm $\|\cdot\|_B$ is

$$b_3(x) = \frac{B(f, v_1)}{B(v_1, v_1)} v_1(x) = \frac{-2}{7} \frac{\sqrt{3}}{\sqrt{2}} x = -\frac{\sqrt{6}}{7} x.$$

(e) [5 points] We first compute that

$$v_1'(x) = \frac{\sqrt{3}}{\sqrt{2}}$$

and

$$v_2'(x) = \frac{\sqrt{3}}{\sqrt{2}}(6x - 1)$$

and then compute that

$$\begin{aligned} a(v_1, v_2) &= a(v_2, v_1) = \int_{-1}^1 (2+x) v_1'(x) v_2'(x) dx \\ &= \int_{-1}^1 (2+x) \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{3}}{\sqrt{2}} (6x-1) dx \\ &= \frac{3}{2} \int_{-1}^1 (6x^2 + 11x - 2) dx \\ &= \frac{3}{2} \left[2x^3 + \frac{11}{2}x^2 - 2x \right]_{-1}^1 \\ &= \frac{3}{2} \left(2 + \frac{11}{2} - 2 - \left(-2 + \frac{11}{2} - 2 \right) \right) \\ &= 0. \end{aligned}$$

Therefore, v_1 and v_2 is orthogonal with respect to the inner product $a(\cdot, \cdot)$ and hence the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|_a$ is

$$b_4(x) = \frac{a(f, v_1)}{a(v_1, v_1)} v_1(x) + \frac{a(f, v_2)}{a(v_2, v_2)} v_2(x)$$

$$\begin{aligned}
&= \frac{-2}{6}v_1(x) + \frac{-22}{66}v_2(x) \\
&= -\frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}}x - \frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}}(3x^2 - x - 1) \\
&= \frac{1}{\sqrt{6}}(1 - 3x^2).
\end{aligned}$$

(f) [7 points] We first compute that

$$\begin{aligned}
(v_1, v_2) &= (v_2, v_1) = \int_{-1}^1 \frac{\sqrt{3}}{\sqrt{2}}x \frac{\sqrt{3}}{\sqrt{2}}(3x^2 - x - 1) dx \\
&= \frac{3}{2} \int_{-1}^1 (3x^3 - x^2 - x) dx \\
&= \frac{3}{2} \left[\frac{3}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{-1}^1 \\
&= \frac{3}{2} \left(\frac{3}{4} - \frac{1}{3} - \frac{1}{2} - \left(\frac{3}{4} + \frac{1}{3} - \frac{1}{2} \right) \right) \\
&= \frac{3}{2} \left(-\frac{2}{3} \right) \\
&= -1.
\end{aligned}$$

Therefore, v_1 and v_2 are not orthogonal with respect to the inner product (\cdot, \cdot) and hence the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|$ is

$$b_5(x) = c_1 v_1(x) + c_2 v_2(x)$$

where the coefficients $c_1, c_2 \in \mathbb{R}$ are such that

$$\begin{bmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_1, v_2) & (v_2, v_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (f, v_1) \\ (f, v_2) \end{bmatrix}$$

and hence are such that

$$\begin{bmatrix} 1 & -1 \\ -1 & \frac{17}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{12}{\pi^2} \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} c_1 - c_2 \\ -c_1 + \frac{17}{5}c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{12}{\pi^2} \end{bmatrix}.$$

Now, the first row implies that $c_1 = c_2$ which mean that the second row yields that $\left(\frac{17}{5} - 1\right)c_2 = -\frac{12}{\pi^2}$

which implies that $\frac{12}{5}c_2 = -\frac{12}{\pi^2}$ and hence $c_2 = -\frac{5}{\pi^2}$. Therefore, the best approximation to f from $\text{span}\{v_1, v_2\}$ with respect to the norm $\|\cdot\|$ is

$$b_5(x) = c_1 v_1(x) + c_2 v_2(x) = c_2(v_1(x) + v_2(x)) = -\frac{5}{\pi^2} \frac{\sqrt{3}}{\sqrt{2}}(x + 3x^2 - x - 1) = \frac{5}{\pi^2} \frac{\sqrt{3}}{\sqrt{2}}(1 - 3x^2).$$
