CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 12 · Solutions

Posted Monday December 1, 2014. Due Friday 5 December 2014, 5pm. Accepted without penalty until Monday, December 8, 5pm.

This problem set counts for 50 points, plus a 25 point bonus problem.

1. [50 points]
On Problem Set 10, you solved the heat equation on a two-dimensional square domain. Now we will investigate the wave equation on the same domain, a model of a vibrating membrane stretched over a square frame—that is, a square drum:

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t),$$

with $0 \le x \le 1$, and $0 \le y \le 1$, and $t \ge 0$. Take homogeneous Dirichlet boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$$

for all x and y such that $0 \le x \le 1$ and $0 \le y \le 1$ and all $t \ge 0$, and consider the initial conditions

$$u(x,y,0) = u_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y), \qquad u_t(x,y,0) = v_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}(0)\psi_{j,k}(x,y).$$

Here $\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$, for $j,k \ge 1$, are the eigenfunctions of the operator

$$Lu = -(u_{xx} + u_{yy}),$$

with homogeneous Dirichlet boundary conditions, as in Problem Set 10. You may use without proof that these eigenfunctions are orthogonal, and use the eigenvalues $\lambda_{j,k} = (j^2+k^2)\pi^2$ computed for Problem Set 10.

(a) We wish to write the solution to the wave equation in the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$

Show that the coefficients $a_{j,k}(t)$ obey the ordinary differential equation

$$a''_{j,k}(t) = -\lambda_{j,k}a_{j,k}(t)$$

with initial conditions

$$a_{j,k}(0), \qquad a'_{j,k}(0) = b_{j,k}(0)$$

derived from the initial conditions u_0 and v_0.

(b) Write down the solution to the differential equation in part (a).

(c) Use your solution to part (b) to write out a formula for the solution $u(x,y,t)$.

(d) Suppose the drum begins with zero velocity, $v_0(x,y) = 0$, and displacement

$$u_0(x,y) = 200xy(1-x)(1-y)(x-1/4)(y-1/4) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}\psi_{j,k}(x,y).$$

Submit surface (or contour) plots of the solution at times $t = 0, 0.5, 1.0, 1.5, 2.5$, using $j = 1,\ldots,10$ and $k = 1,\ldots,10$ in the series.

Solution. This question follows the same pattern as the first problem on this problem set.

(a) Substitute the formula

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x,y).$$

into the two dimensional wave equation to obtain

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{d^2 a_{j,k}}{dt^2}(t)\psi_{j,k}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\psi_{j,k}(x,y),$$

which implies

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{d^2 a_{j,k}}{dt^2}(t)\psi_{j,k}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} -a_{j,k}(t)\lambda_{j,k}\psi_{j,k}(x,y).$$

Take the inner product of both sides with $\psi_{m,n}$ and use the orthogonality of the eigenfunctions to obtain

$$a''_{j,k}(t) = -\lambda_{j,k}a_{j,k}(t).$$

The initial value $a_{j,k}(0)$ is simply the (j,k) coefficient in the expansion of the initial condition,

$$u(x,y,0) = u_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\psi_{j,k}(x,y).$$

As the differential equation describing $a_{j,k}$ is of second order, we require a second initial condition to determine a unique solution. Since

$$u_t(x,y,0) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a'_{j,k}(0)\psi_{j,k}(x,y),$$

and

$$v_0(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}\psi_{j,k}(x,y),$$

the second initial condition we require is simply the (j,k) coefficient of v_0:

$$a'_{j,k}(0) = b_{j,k}.$$

(b) As we have seen often in this class, the equation $a''_{j,k}(t) = -\lambda_{j,k}a_{j,k}(t)$ has solutions of the form

$$a_{j,k}(t) = A\sin(\sqrt{\lambda_{j,k}}t) + B\cos(\sqrt{\lambda_{j,k}}t),$$

with A and B determined by the initial conditions. In particular, evaluating the general solution at $t = 0$ immediately gives

$$B = a_{j,k}(0).$$

Computing the derivative

$$a'_{j,k}(t) = A\sqrt{\lambda_{j,k}}\cos(\sqrt{\lambda_{j,k}}t) - B\sqrt{\lambda_{j,k}}\sin(\sqrt{\lambda_{j,k}}t)$$

and evaluating it at $t = 0$ then gives

$$A = \frac{b_{j,k}(0)}{\sqrt{\lambda_{j,k}}}.$$

(c) From this formula for $a_{j,k}(t)$ we obtain the solution

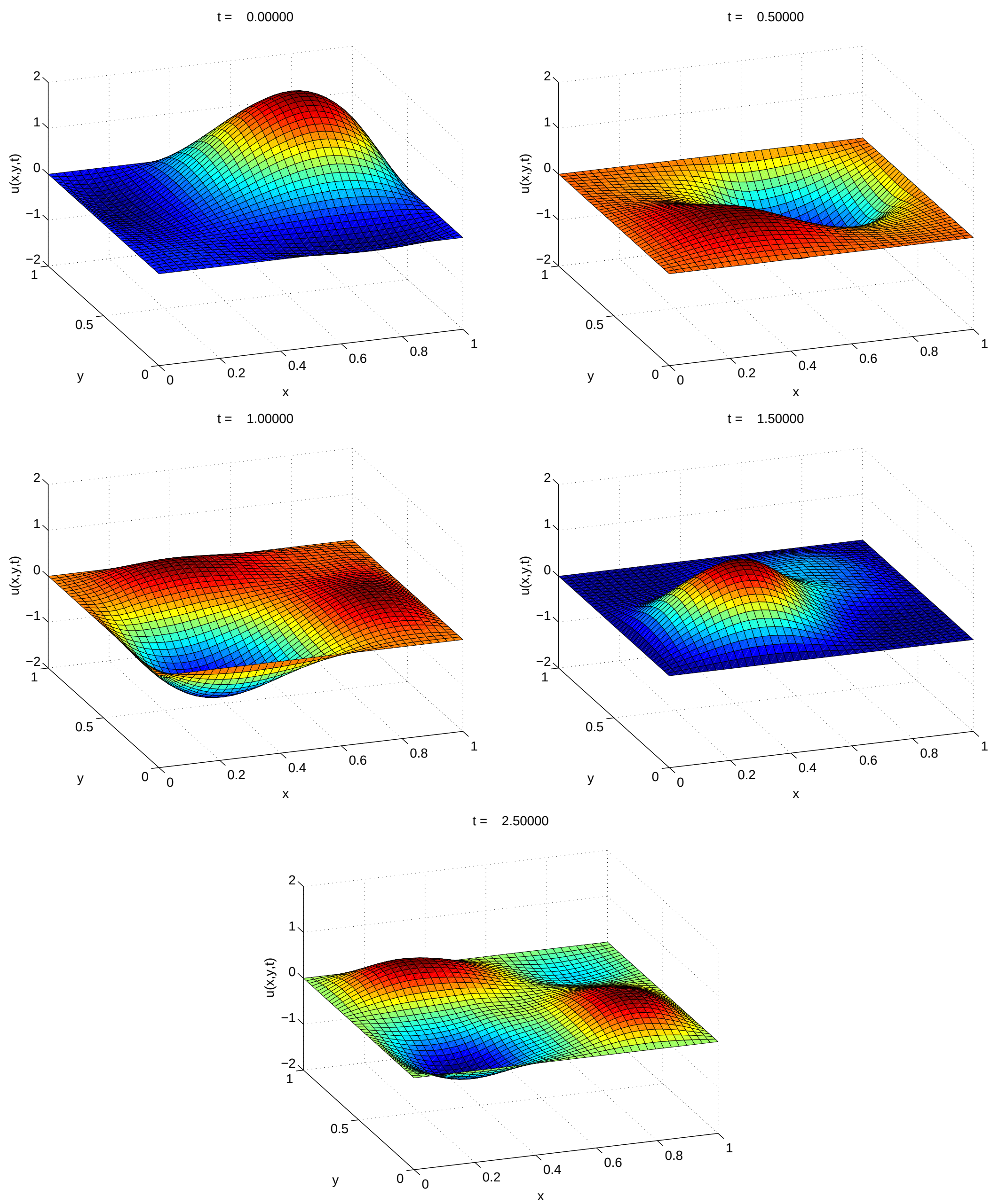
$$u(x,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big[\Big(\frac{b_{j,k}(0)}{\sqrt{\lambda_{j,k}}}\Big)\sin(\sqrt{\lambda_{j,k}}t) + a_{j,k}(0)\cos(\sqrt{\lambda_{j,k}}t)\Big]\big(2\sin(\sqrt{\lambda_{j,k}}x)\sin(\sqrt{\lambda_{j,k}}y)\big).$$

(d) Zero initial velocity implies that $b_{j,k}(0) = 0$ for all (j,k) pairs. The solution thus simplifies to

$$u(x,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(0)\cos(\sqrt{\lambda_{j,k}}t)\big(2\sin(\sqrt{\lambda_{j,k}}x)\sin(\sqrt{\lambda_{j,k}}y)\big).$$

with

$$a_{j,k}(0) = \frac{100(5+7(-1)^j)(5+7(-1)^k)}{j^3k^3\pi^6}.$$



The code that produced these plots follows.

```
N = 40;
x = linspace(0,1,N); y = linspace(0,1,N);
[X,Y] = meshgrid(x,y);
nmax = 10;

tvec = 0:.05:4;
tvec = [0 0.5 1.0 1.5 2.5];
for m=1:length(tvec)
   t = tvec(m)
   figure(1), clf
   U = zeros(N,N);
   for j=1:nmax
      for k=1:nmax
         ajk = 100*(5+7*(-1)^j)*(5+7*(-1)^k)/(j^3*k^3*pi^6);
         lamjk = (j^2+k^2)*(pi^2);
         psijk = 2*sin(j*pi*X).*sin(k*pi*Y);
         U = U + ajk*cos(sqrt(lamjk)*t)*psijk;
      end
   end
   surf(X,Y,U)
   axis([0 1 0 1 -2 2]) %, caxis([-8 8])
   set(gca,'fontsize',18)
   xlabel('x'), ylabel('y'), zlabel('u(x,y,t)')
   title(sprintf('t = %10.5f\n', t))
   view(-20, 30)
   eval(sprintf('print -depsc2 wave2d_%d',m))
end
```

2. [Bonus: 25 points] Consider the heterogeneous wave equation with

$$\begin{aligned}
\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= f(x,t) \\
u(x,0) &= \psi(x) \\
\frac{\partial u}{\partial t}(x,0) &= 0 \\
u(0,t) &= 0 \\
\frac{\partial u}{\partial x}(1,t) &= 0
\end{aligned}$$

In seismic imaging problems (i.e. sonar, radar, finding oil, etc), the wave equation can be used to simulate a sound wave propagating in the x direction through a medium. Here, we assume $\rho(x)$ is the density of the medium, and that it only changes in the direction of propagation.

(a) Formulate a weak form for the above equation.

(b) Let $f = 0$ and $\rho(x) = 1$, which reduces to the standard wave equation with $c = 1$. For a pulse initial condition

$$\psi(x) = xe^{-100x^2}$$

compute the finite element solution (using the matrix exponential) with $N = 63$ and $dt = .015$. Create a 3D plot of the solution by using `surf` to plot the solution at equally spaced times from 0 to the final time $T = 2$. Note: If you compute your solution at points x_j and times t_i,

$$x_j = 0, h, \ldots, 1-h, 1, \qquad t_i = 0, dt, \ldots, 2$$

and form a matrix

$$U_{ij} = u(x_j, t_i)$$

then you may use `surf(X,T,U)` to compute a 3D plot of the solution, where `X` and `T` are vectors of the points x_i and times t_j. You may also wish to use the command `shading interp` to remove mesh lines from the 3D solution plot.

(c) Let $\rho(x)$ now be a discontinuous function

$$\rho(x) = \begin{cases} k_1, & x < .5 \\ k_2, & x \geq .5. \end{cases}$$

For N odd, give a formula depending on j and/or x_j for the entries of the mass matrix.

(d) Take $k_1 = .25$ and $k_2 = 1$. Compute the finite element solution using the same initial condition and N, dt as in (b). What effect does the discontinuity have on the behavior of the solution over x and t?

Solution.

(a) Multiplying the equation and integrating the spatial derivative by parts gives

$$\int_0^1 \rho(x)\frac{\partial^2 u}{\partial t^2} v(x)dx + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \left[\frac{\partial u}{\partial x} v\right]_0^1.$$

The boundary term vanishes after taking into account the boundary condition on $\frac{\partial u}{\partial x}(1,t) = 0$, and the condition on $v(x)$ that $v(0) = 0$.

(b) The solution for $k_1 = k_2 = 1$ is given as follows

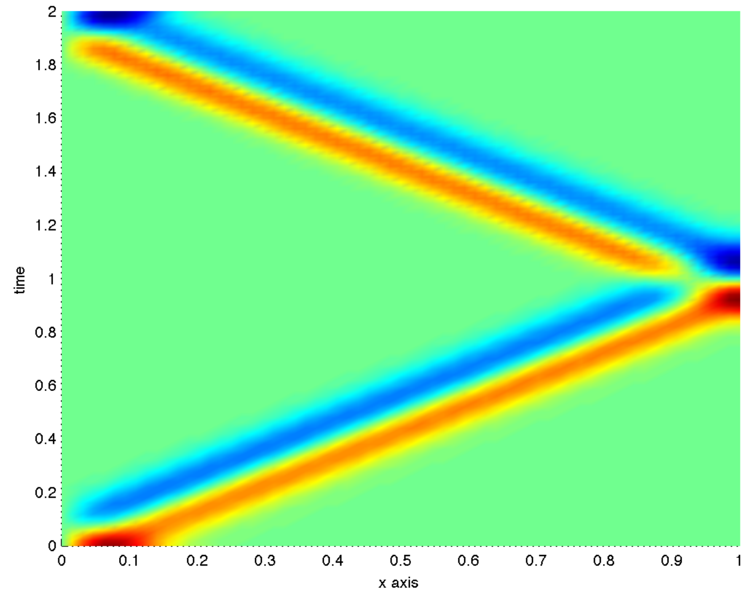


Figure 1: Top-down view of the solution for $k_1 = k_2 = 1$

(c) If N is odd, then there lies exactly one point x_m such that $x_m = .5$. Then, the mass matrix will have modified terms

$$M_{i,i} = \begin{cases} k_1 \frac{4h}{6}, & i < m \\ (k_1 + k_2)\frac{h}{3}, & i = m \\ k_2 \frac{4h}{6}, & i > m \end{cases}, \qquad M_{i,i-1} = \begin{cases} k_1 \frac{h}{6}, & i \leq m \\ k_2 \frac{h}{6}, & i > m \end{cases},$$

and $M_{i-1,i}$ can be found using symmetry of M.

(d) The solution shows a unique physical feature: plotting the solution over time shows that a wave propagates up to the discontinuity in $\rho(x)$, then produces two waves — one traveling forward at a faster speed, while the other reflects off of the discontinuity and propagates backwards.

This phenomena is observed due to the face that $\rho(x)$ is representative of density — if density is discontinuous, it is as if there are two types of media that a wave is propagating through. For example, if you propagate a wave through water and the wave hits rock, the wave may propagate through the rock, but it will also reflect back into the water.

The code that produced these plots follows.

```
% demo of the finite element method for the problem d2u/dt2 - d2u/d2x = 0
% with zero Dirichlet boundary conditions

N = 63; % num interior points
h = 1/(N+1);
x = [1:N+1]*h;

% construct the stiffness matrix (integrals done by hand)
k1 = 1; k2 = 1;
maindiag = 2*ones(N+1,1);
offdiag  = -ones(N,1);
K = diag(maindiag) + diag(offdiag,1) + diag(offdiag,-1);

% adjust for Neumann BC on right
```

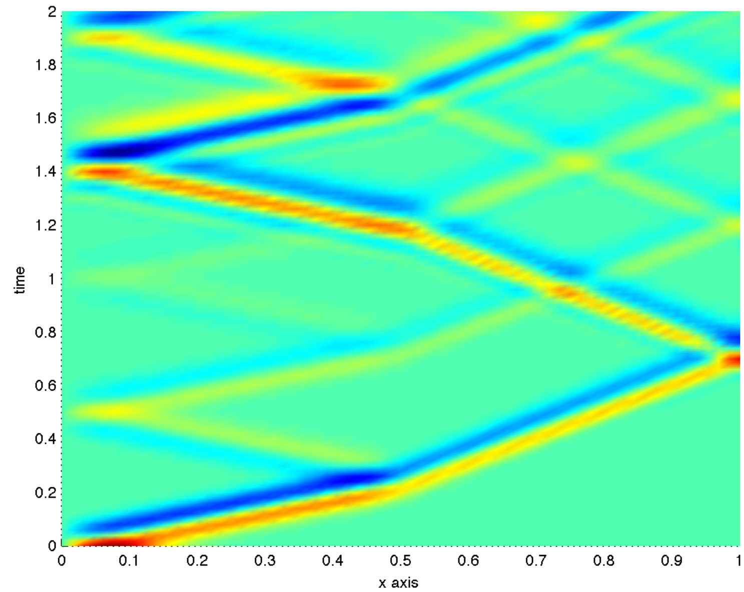


Figure 2: Top-down view of the solution for $k_1 = .25, k_2 = 1$

```
K(N+1,N) = -1;K(N,N+1) = -1;
K(N+1,N+1) = 1;
K = K/h;

% construct mass matrix
middle = ceil(N/2); % middle point index
k1 = .25;
k2 = 1;
maindiag = 2*h/3*ones(N+1,1);
maindiag(1:middle) = k1*maindiag(1:middle);
maindiag(middle) = h/3*(k1+k2);
maindiag(middle+1:N+1) = k2*maindiag(middle+1:N+1);

offdiag  = (h/6)*ones(N,1);
offdiag(1:middle) = k1*offdiag(1:middle);
offdiag(middle+1:N) = k2*offdiag(middle+1:N);

M = diag(maindiag) + diag(offdiag,1) + diag(offdiag,-1);
% adjust for Neumann BC
M(N+1,N+1) = M(N+1,N+1)/2;

u = x(:).*exp(-(x(:)).^2*100);
u = u(:);

% solve using matrix exponential
U0 = [zeros(N+1,1);u];
O = zeros(N+1);
A = [O -eye(N+1);
    M\K O];

uMat = [];
tVec = 0:.015:2;
for t = tVec
    U = expm(-A*t)*U0;
    u = U(N+2:end);
    plot([0;x(:)],[0;u],'.-');
```

```
    uMat = [uMat; [0;u]'];
    axis([0 1 -.1 .1])
    drawnow
end

surf([0;x(:)],tVec,uMat)
shading interp
```