CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 28

Posted Wednesday 9 October 2013. Due 1pm Friday 18 October 2013.

28. [25 points] Let the inner product $(\cdot,\cdot) : C[0,1] \times C[0,1] \to \mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

and let the norm $\|\cdot\| : C[0,1] \to \mathbb{R}$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let the linear operator $L : S \to C[0,1]$ be defined by

$$Lv = -v''$$

where

$$S = \{w \in C^2[0,1] : w(0) = w'(1) = 0\}.$$

Note that

$$(Lv,w) = (v,Lw) \text{ for all } v,w \in S.$$

(a) The operator L has eigenvalues λ_n with corresponding eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin\left(\left(n - \frac{1}{2}\right)\pi x\right)$$

for $n = 1, 2, \ldots$. Obtain a formula for the eigenvalues λ_n for $n = 1, 2, \ldots$.

(b) Use the spectral method to obtain a series solution to the problem of finding $\tilde{u} \in C^2[0,1]$ such that

$$-\tilde{u}''(x) = x + \sin(\pi x), \quad 0 < x < 1$$

and

$$\tilde{u}(0) = \tilde{u}'(1) = 0.$$

Note that, for $m, n = 1, 2, \ldots$,

$$(\psi_m, \psi_n) = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

(c) Plot the approximation to $\tilde{u}$ obtained by replacing the upper limit of the summation in your series solution with 20.

(d) By shifting the data, obtain a series solution to the problem of finding $u \in C^2[0,1]$ such that

$$-u''(x) = x + \sin(\pi x), \quad 0 < x < 1$$

and

$$u(0) = u'(1) = 1.$$

(e) Plot the approximation to u obtained by replacing the upper limit of the summation in your series solution with 20.