CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 9

Posted Wednesday 12 November, 2014. Due 5pm Wednesday 19 November, 2014.

*Please write your name and **residential college** on your homework.*

1. [40 points: 10 points each]

   (a) Consider the function $u_0(x) = \begin{cases} 1, & x \in [0, 1/3]; \\ 0, & x \in (1/3, 2/3); \\ 1, & x \in [2/3, 1]. \end{cases}$

   Recall that the eigenvalues of the operator $L : C_N^2[0,1] \to C[0,1]$,

   $$Lu = -u''$$

   are $\lambda_n = n^2\pi^2$ for $n = 0, 1, \ldots$ with associated (normalized) eigenfunctions $\psi_0(x) = 1$ and

   $$\psi_n(x) = \sqrt{2}\cos(n\pi x), \qquad n = 1, 2, \ldots.$$

   We wish to write $u_0(x)$ as a series of the form

   $$u_0(x) = \sum_{n=0}^{\infty} a_n(0)\psi_n(x),$$

   where $a_n(0) = (u_0, \psi_n)$.
   Compute these inner products $a_n(0) = (u_0, \psi_n)$ by hand and simplify as much as possible.
   For $m = 0, 2, 4, 80$, plot the partial sums

   $$u_{0,m}(x) = \sum_{n=0}^{m} a_n(0)\psi_n(x).$$

   (You may superimpose these on one single, well-labeled plot if you like.)

   (b) Write down a series solution to the homogeneous heat equation

   $$u_t(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad t \ge 0$$

   with Neumann boundary conditions

   $$u_x(0,t) = u_x(1,t) = 0$$

   and initial condition $u(x,0) = u_0(x)$.
   Create a plot showing the solution at times $t = 0, 0.002, 0.05, 0.1$.
   You will need to truncate your infinite series to show this plot.
   Discuss how the number of terms you use in this infinite series affects the accuracy of your plots.

   (c) Describe the behavior of your solution as $t \to \infty$.
   (To do so, write down a formula for the solution in the limit $t \to \infty$.)

   (d) How would you expect the solution to the inhomogeneous heat equation

   $$u_t(x,t) = u_{xx} + 1, \qquad 0 < x < 1, \quad t \ge 0$$

   with Neumann boundary conditions

   $$u_x(0,t) = u_x(1,t) = 0$$

   to behave as $t \to \infty$?

2. [30 points: 10 points each]
Consider the *fourth order* partial differential equation

$$u_t(x,t) = u_{xx}(x,t) - u_{xxxx}(x,t)$$

with so-called *hinged* boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$$

and initial condition (that should satisfy the boundary conditions)

$$u(x,0) = u_0(x).$$

(This equation is related to a model that arises in the study of thin films.)
To solve this PDE, we introduce the linear operator $L : C^4_H[0,1] \to C[0,1]$, where

$$Lu = -u'' + u''''$$

and

$$C^4_H[0,1] = \{u \in C^4[0,1], u(0) = u''(0) = u(1) = u''(1) = 0\}$$

is the set of C^4 functions that satisfy the hinged boundary conditions.

(a) The operator L has eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

Use this fact to compute a formula for the eigenvalues λ_n, $n = 1, 2, \ldots$.

(b) Suppose the initial condition $u_0(x)$ is expanded in the form

$$u_0(x) = \sum_{n=1}^{\infty} a_n(0)\psi_n(x).$$

Briefly describe how one can write the solution to the PDE $u_t = u_{xx} - u_{xxxx}$ as an infinite sum.

(c) Suppose the initial data is given by

$$u_0(x) = (x - x^2)\sin(3\pi x)^2,$$

with associated coefficients

$$a_n(0) = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3}, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Write a program (you may modify your earlier codes) to compute the solution you describe in part (b) up to seven terms in the infinite sum. At each time $t = 0$; 10^{-5}; 2×10^{-5}; 4×10^{-5}, produce a plot comparing the sum of the first 1, 5, and 7 terms of the series. For example, at time $t = 0$, your plot should appear as shown below. (Alternatively, you can produce attractive 3-dimensional plots over the time interval $t \in [0, 4 \times 10^{-5}]$ using 1, 5, and 7 terms in the series.)

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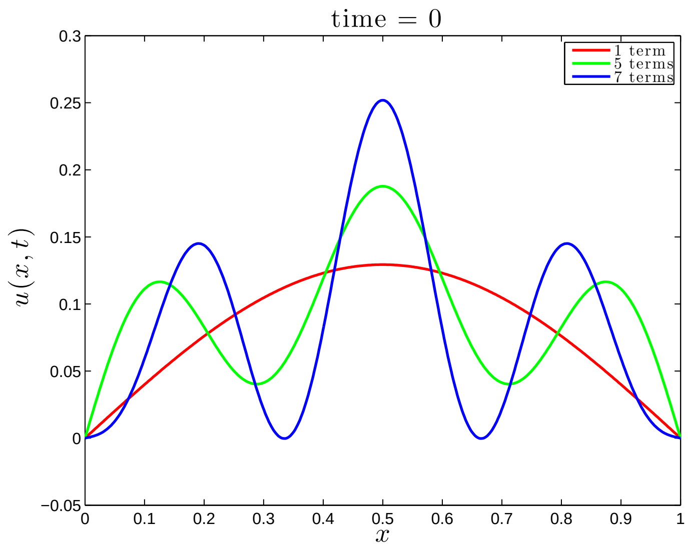

3. [30 points: 15 points each]
   We wish to approximate the solution to the heat equation

$$u_t(x,t) = u_{xx}(x,t) + 100tx, \qquad 0 \le x \le 1,\ t \ge 0$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = 0$$

using the finite element method (method of lines). Let $N \ge 1$, $h = 1/(N+1)$, and $x_k = kh$ for $k = 0, \ldots, N+1$. We shall construct approximations using the hat functions

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

The approximate solution shall have the form

$$u_N(x,t) = \sum_{k=1}^{N} a_k(t)\phi_k(x).$$

   (a) Write down the system of ordinary differential equations that determines the coefficients $a_k(t)$, $k = 1, \ldots, N$. Specify the entries in the mass and stiffness matrices and the load vector. (You may use results from previous homeworks and class as convenient.)

   (b) Write a MATLAB code that uses the backward Euler method to solve for the coefficients $a_k(t)$. Plot your approximate solution $u_N(x,t)$ at time $t = 1$. Choose N and Δt so that your solution appears to be accurate. Verify this accuracy by superimposing on your plot the computed solution at $t = 1$ obtained by using space and time steps that are ten times smaller.