CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 12 · Solutions

Posted Friday 13 September 2013. Due 5pm Wednesday 25 September 2013.

12. [25 points]

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Prove there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that f is given by $f(\mathbf{u}) = \mathbf{Au}$. Hint: Each $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since f is linear, we have $f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2)$. Your formula for the matrix $\mathbf{A}$ may include the vectors $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$.

(b) Now we want to generalize the result in part (a): Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$.

(c) Now we want to generalize further: Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$.

(Thus any linear function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as a matrix-vector product.)

Solution.

(a) [10 points] We can write any $\mathbf{u} \in \mathbb{R}^2$ in the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Any matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ are the columns of $\mathbf{A}$. Now the matrix-vector product $\mathbf{Au}$ is a linear combination of the columns of $\mathbf{A}$:

$$\mathbf{Au} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2. \tag{$*$}$$

We are trying to find a formula for $\mathbf{A}$ such that $f(\mathbf{u}) = \mathbf{Au}$. Since f is a linear operator, we have that

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2). \tag{$**$}$$

Comparing $(*)$ and $(**)$, we see that

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) \end{bmatrix}$$

is such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^2$. Hence, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^2$.

(b) [10 points] Follow the same tack as in part (a). Let $\mathbf{e}_j \in \mathbb{R}^n$ be the vector whose jth entry is 1 and whose other entries are all 0. Write $\mathbf{u} \in \mathbb{R}^n$ as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{A} \in \mathbb{R}^{n \times n}$ as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ are the columns of $\mathbf{A}$. Comparing

$$\mathbf{Au} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots u_n\mathbf{a}_n$$

and

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2) + \cdots u_n f(\mathbf{e}_n),$$

we see that

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \cdots & f(\mathbf{e}_n) \end{bmatrix}$$

is such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$. Hence, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$.

(c) [5 points] Let $\mathbf{e}_j \in \mathbb{R}^n$ be the vector whose jth entry is 1 and whose other entries are all 0. Write $\mathbf{u} \in \mathbb{R}^n$ as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{A} \in \mathbb{R}^{m \times n}$ as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ are the columns of $\mathbf{A}$. Comparing

$$\mathbf{Au} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots u_n\mathbf{a}_n$$

and

$$f(\mathbf{u}) = u_1 f(\mathbf{e}_1) + u_2 f(\mathbf{e}_2) + \cdots u_n f(\mathbf{e}_n),$$

we see that

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \cdots & f(\mathbf{e}_n) \end{bmatrix}$$

is such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$. Hence, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ such that $f(\mathbf{u}) = \mathbf{Au}$ for all $\mathbf{u} \in \mathbb{R}^n$.