CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 36

Posted Friday 1 November 2013. Due 5pm Wednesday 13 November 2013.

36. [25 points] Let $H_D^1\left(0,1\right)=\left\{w\in H^1\left(0,1\right)\colon w(1)=0\right\}$ and let the inner product $(\cdot,\cdot)\colon L^2\left(0,1\right)\times L^2\left(0,1\right)\to\mathbb{R}$ be defined by

$$(v,w) = \int_0^1 v(x)w(x) \, dx$$

and the inner product $a(\cdot,\cdot):H_{D}^{1}\left(0,1\right)\times H_{D}^{1}\left(0,1\right)\to\mathbb{R}$ be defined by

$$a(v,w) = \int_0^1 v'(x)w'(x) dx.$$

Also, let $f \in L^2(0,1)$, let N be a positive integer, and let V_N be a subspace of $H_D^1(0,1)$. Moreover, let $u \in H_D^1(0,1)$ be such that

$$a(u,v)=\left(f,v\right)$$
 for all $v\in H_{D}^{1}\left(0,1\right)$

and let $u_N \in V_N$ be such that

$$a(u_N, v) = (f, v)$$
 for all $v \in V_N$.

(a) Show that

$$a(u - u_N, u - u_N) = a(u, u) - a(u_N, u_N).$$

(b) Let $\phi_0, \phi_1, \dots, \phi_N \in V_N$ and let $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ be the matrix with entries $K_{jk} = a(\phi_{k-1}, \phi_{j-1})$ for $j, k = 1, \dots, N+1$. Also, let

$$u_N = \sum_{j=0}^{N} \alpha_j \phi_j$$

where, for k = 1, ..., N + 1, $\alpha_{k-1} \in \mathbb{R}$ is the kth entry of the vector $\mathbf{c} \in \mathbb{R}^{N+1}$. Show that

$$\mathbf{c}^T \mathbf{K} \mathbf{c} = a(u_N, u_N).$$