CAAM 336 · DIFFERENTIAL EQUATIONS

Midterm Examination 1

Posted Friday, 17 February 2012.

Due no later than 4pm on Friday, 24 February 2012.

Instructions:

1. Time limit: **4 uninterrupted hours**. You may fill out this page before you start the timer.
2. There are five questions worth a total of 100 points. Within a question, each part has equal weight. Please do not look at the questions until you begin the exam.
3. You may use a crib sheet of relevant facts from the course that might help you in answering some of the questions. This should be on one side of a standard 8.5 by 11 inch sheet of paper. Please staple your crib sheet to the exam when you turn it in.
4. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
5. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
6. Print your name and section number (or instructor) on the line below:
7. Name: ____________________ Section: ____________________
8. Time started: ____________________ Time completed: ____________________
9. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

   ____

   ____

   ____
10. Staple this page to the front of your exam along with your crib sheet.

1. [10 points] For each of the following equations, specify whether it is (a) an ODE or a PDE; (b) determine its order; (c) specify whether it is linear or nonlinear. For those that are linear specify whether they are (d) homogeneous or inhomogeneous, and (e) whether they have constant or variable coefficients.

   (1.1) $\dfrac{\partial u}{\partial t} = \epsilon \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} + u$

   (1.2) $\dfrac{d^2 u}{dx^2} - \dfrac{2}{x^2} u = 0$

   (1.3) $\dfrac{d^2}{dx^2}\left[\sin(x)\dfrac{d^2 u}{dx^2}\right] = \cos(x)$

   For items 1.1 - 1.3 above you may just give an answer and do not need to justify it.

   Which equations listed above is solved by the function $u(x) = 1/x$? Please do justify your answer to this part.

2. [10 points] Determine whether each of the following functions $(\cdot\ ,\ \cdot)$ determines an inner product on the vector space $\mathcal{V}$. If not, **show all the properties** of the inner product that are violated.

   (a) $\mathcal{V} = C^2[0,1],\ (u,v) = \int_0^1 [u(x)v(x) + u'(x)v'(x) + u''(x)v''(x)]dx$

   (b) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and define $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{A} \mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$

   (c) $\mathcal{V} = C^1[0,1],\ (u,v) = \int_0^1 u(x)v(x)\rho(x)dx$ where $\rho(x) > 0,\ x \in [0,1]$ and ρ is continuous.

3. [20 points]

   (a) Let $\hat{K}$ be a linear operator on a vector space $\mathcal{V}$ and suppose that $\hat{K}v = \hat{\lambda}v$ with $v \neq 0$ so that $\hat{\lambda}$ is an eigenvalue of $\hat{K}$ with corresponding eigenvector v. Show that the operator K defined by

   $$Ku = \hat{K}u + \mu u, \quad u \in \mathcal{W}, \quad \mu \text{ a scalar},$$

   is a linear operator on $\mathcal{W}$ and that v is an eigenvector for K corresponding to the eigenvalue $\lambda = \hat{\lambda} + \mu$.

   Now, consider the operator $L : C_M^2[0,1] \to C[0,1]$ where $C_M^2[0,1] = \{u \in C^2[0,1] : u(0) = u'(1) = 0\}$.

   $$Lu(x) = -\frac{d^2 u(x)}{dx^2} + u(x).$$

   (b) Show that L is symmetric and positive definite.

   (c) Calculate the eigenvalues λ_k and corresponding eigenfunctions ψ_k of L.

   For Parts (b) and (c) you may use any relevant facts that were presented in class or on the first five homework assignments.

4. [30 points]
   Let L be the linear operator defined in Problem 4 above. Let

   $$\mathcal{F}_n = \text{Span}\{\psi_1, \psi_2, \ldots, \psi_n\}.$$

(a) Show $\mathcal{F}_n$ is a subspace of $C[0,1]$.

(b) Show that $\{\psi_1, \psi_2, \ldots, \psi_n\}$ is an orthogonal basis for the subspace $\mathcal{F}_n$.

(c) Let $f(x) = (2-x)x$. Assuming items (a) and (b) are true, explain how to compute the best approximation $f_n \in \mathcal{F}_n$ to f. Explicitly give the integral formulas for the coefficients needed for the best approximation. DO NOT evaluate those integrals, just give the formulas.

(d) Explain how to compute the approximate solution u_n to the equation

$$-\frac{d^2u}{dx^2}(x) + u(x) = f(x), \quad u(0) = u'(1) = 0$$

using the spectral method.

(e) Explain how to modify the approximate solution u_n given in item (d) so in order to solve

$$-\frac{d^2u}{dx^2}(x) + u(x) = f(x), \quad u(0) = 0.05, u'(1) = 0.1 \quad .$$

5. [30 points]
Consider the equation

$$-u''(x) = f(x), \quad x \in [0,1]$$

with a homogeneous Dirichlet boundary condition on the left,

$$u(0) = 0$$

and a homogeneous *mixed* condition on the right,

$$u(1) - 2u'(1) = 0.$$

Define the linear operator $L : V \to C[0,1]$ via $Lu = -u''$ with

$$V = \{u \in C^2[0,1] : u(0) = u(1) - 2u'(1) = 0\}.$$

(a) Show that L is symmetric.

(b) Compute the eigenfunctions of L corresponding to positive eigenvalues, and show that these positive eigenvalues λ must satisfy the equation

$$\tan(\sqrt{\lambda}) = 2\sqrt{\lambda}.$$

(c) What is the null space of L? Is $\lambda = 0$ an eigenvalue of L?

(d) Draw a plot of $g(\theta) = \tan(\theta)$ for $\theta \in [0, 4\pi]$ and superimpose a plot of $h(\theta) = 2\theta$.
Mark the points $(\theta_k, \tan(\theta_k))$ where these two plots intersect and note that these must give the positive eigenvalues $\lambda_k = \theta_k^2$.

(e) Use your plot in (d) to argue that L has infinitely many positive eigenvalues, with

$$\left(\frac{2k-1}{2}\right)^2 \pi^2 < \lambda_k < \left(\frac{2k+1}{2}\right)^2 \pi^2.$$

(f) Write down the spectral method solution to the equation

$$-u''(x) = f(x), \quad u(0) = u(1) - u'(1) = 0.$$

(You do not need to work out numerical values for the inner products.)