Worksheet 2: Some More Probability

Name: Due September 18, 2022

1. (Checking definitions) Show that a convex combination $\alpha\mathbb{P} + (1-\alpha)\mathbb{P}'$ of probability measures $\mathbb{P}, \mathbb{P}'$ and $\alpha \in [0,1]$, is a probability measure.

2. (Practice with principles) Let $\mathcal{X} = \mathbb{R}$ and define $\mathbb{P}_{\mathcal{X}}(S) = \sum_{j=1}^{\infty} \frac{1}{2^j} \mathbb{1}_{r_j \in S}$ for $S \subset \mathbb{R}$, where $\{r_j\}_{j=1}^{\infty} = \mathbb{Q}$ is an enumeration of the rational numbers. Show that $\mathbb{P}_{\mathcal{X}}$ defines a probability measure on $\mathbb{R}$. (You may need to recall rearrangement.)

3. (Function of random variable) Let $\mathcal{X}^2 = [0,1]^2$ with independent uniform measure $\mathbb{P}_{\mathcal{X}^2}([a,b] \times [c,d]) = (d-c)(b-a)$ for $a < b, c < d \in [0,1]$. Consider random variable $+ : \mathcal{X}^2 \to \mathbb{R}$ defined by $(x_1, x_2) \mapsto x_1 + x_2$ (see fig. 1, here $n = 2$). Using the *law of total expectation (probability)*, express both $\mathbb{P}_{\mathcal{X}^2}(x_1 + x_2 \leq t)$ as well as the pdf (you may draw a picture).

4. (Loose concentration) Let $(\mathcal{X} = \{0,1\}, \mathbb{P}_{\mathcal{X}}(\{1\}) = p)$ be Bernoulli p. Show that you may estimate p to precision $\varepsilon > 0$ with probability at least $1-\delta$ by taking at least $m \geq \frac{p(1-p)}{\delta\varepsilon^2}$ samples $x_1, \ldots, x_m$. (Define random variable $s_m : \mathcal{X}^m \to \mathbb{R}$ by $(x_1, \ldots, x_m) \mapsto \frac{1}{m}\sum_{j=1}^{m} x_j$, the empirical mean, and be sure to define your measure $\mathbb{P}_{\mathcal{X}^m}$.)

5. (How fat can tails be?) For probability space $(\mathcal{X} = \mathbb{R}, \mathbb{P}_{\mathcal{X}})$ with mean 0 and finite variance (wlog suppose that $\sigma = 1$), prove the following tail probability bound

$$\mathbb{P}_{\mathcal{X}}(|x| > k) \leq \frac{1}{k^2}. \tag{1}$$

Construct a measure/random variable for which $\mathbb{P}_{\mathcal{X}}(x > t) > \frac{1}{t^p} \cdot \mathbb{1}_{t>1}$ for $p \in (1, 2)$, and justify why *this* bound does not contradict the one in (1).

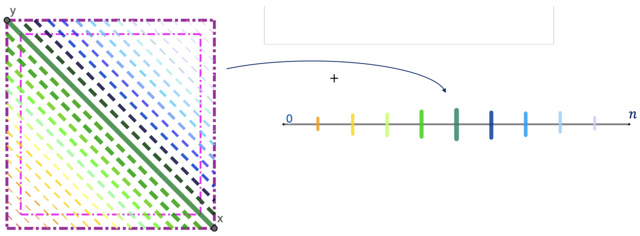

Figure 1: Geometry of LLN

6. (LLN) Let $\mathcal{X} = [0, 1]$ with uniform measure $\mathbb{P}_{\mathcal{X}}([a, b]) = b - a$ for $0 \leq a \leq b \leq 1$, and $\mathbb{P}_{\mathcal{X}^m} = \prod_{j=1}^{m} \mathbb{P}_{\mathcal{X}}$ be independent. Using the relevant concentration bound (a stronger one than Chebyshev!), directly verify this instance of the Law of Large Numbers, namely that

$$\mathbb{P}_{\mathcal{X}^m}\left(\left|\frac{1}{m}\sum_{j=1}^{m} x_j - \frac{1}{2}\right| > \varepsilon\right) \xrightarrow{m\to\infty} 0,$$

give a bound on the rate of convergence, and shade in the figure region of probability concentration.

7. (Strong concentration: Boosting) Let $\mathcal{X} = \{0, 1\}$ with $\mathbb{P}_{\mathcal{X}}(\{1\}) = 1/2 + \varepsilon$ ("slightly biased coin"), and define random variable $M_m : \mathcal{X}^m \to \mathbb{R}$ by $M_m(x_1, \ldots, x_m) := \mathbb{1}_{S_m \geq 1/2}$, where $S_m := \frac{1}{m}\sum_{i=1}^{m} x_i$ denotes the empirical mean. Show that we may guarantee $M_m = 1$ with probability at least $1 - \delta$ as long as $m > \frac{1}{2\varepsilon^2} \log\left(\frac{1}{\delta}\right)$. Don't forget to define your measure. (Notice that dependence on $1/\delta$ is a logarithmic relation, rather than linear as in problem #4; you will use a different concentration inequality in this problem.)