Worksheet 5: Universal Approximation

Name: Due October 25, 2023

In this worksheet, we will constructively tie up loose ends from universal approximation theorem, to the point of rendering lemmas 11.1 and 11.2 from the notes almost superfluous.

1. Let $K \subset \mathbb{R}$ be a compact interval, and $\mathcal{O} = \{U_\alpha\}_{\alpha \in A}$ an open cover. Show that there is refinement $\mathcal{O}' = \{I_\beta\}_{\beta \in B}$ consisting of open intervals I_β—i.e. for each $\beta \in B$, there is $\alpha \in A$ for which $I_\beta \subset U_\alpha$—with the property that each $x \in K$ is contained in at most two I_βs from $\mathcal{O}'$. (It may be helpful, though not necessary, to remind yourself of Heine-Borel.)

2. Let $I_0 \cap I_1 \neq \varnothing$ be two intervals, with $I_0 = (-\infty, b_0)$, $I_1 = (a_1, \infty)$ with $a_1 < b_0$. Explicitly detail how to construct sigmoidals $\sigma_0(t) := \frac{1}{1+e^{-w_0 t + b_0}}$ and $\sigma_1(t) := \frac{1}{1+e^{-w_1 t + b_1}}$ so that

   (a) $\sigma_0 + \sigma_1 \equiv 1$ and

   (b) $\sigma_0^{-1}(0, \varepsilon/2) \subset I_1 \setminus I_0$ and $\sigma_0^{-1}(1 - \varepsilon/2, 1) \subset I_0 \setminus I_1$.

   In showing the (a), make sure to (also) express σ_1 in terms of w_0 and b_0. State and conclude the analog of (b) for σ_1.

3. Using the preceding problem, explain how to construct smooth step function σ_0' and σ_1' (not derivatives) satisfying

   (a) $\sigma_0' + \sigma_1' \equiv 1$ and

   (b) $\sigma_0'^{-1}(I_0 \setminus I_1) \equiv 1$ and $\sigma_0'^{-1}(I_1 \setminus I_0) \equiv 0$.

   You may use the fact *that* any open cover of $\mathbb{R}$ has a partition of unity subordinate to it.

4. For a given open cover $\mathcal{O}$ of intervals as in 1. above, explain how you would construct a partition of unity $\{\rho_\beta\}_{\beta \in B}$ subordinate to $\mathcal{O}$ such that to each ρ_β there is sigmoidal $\sigma_\beta := \frac{1}{1+e^{-w_\beta(\cdot)+b_\beta}}$ for which $\|\rho_\beta - \sigma_\beta\|_\infty < \varepsilon$.

5. Conclude: for any open cover $\mathcal{O}$ of K, there is partition of unity $\{\rho_\beta\}_{\beta \in B}$ subordinate to $\mathcal{O}$ which the class of sigmoidals $\Sigma := \{\sigma_{w,b} = 1/(1+e^{-w(\cdot)+b}) : w, b \in \mathbb{R}\}$ expresses.