160B Homework 3

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Part 1: Solution to the Heat Equation

Question 0: Provide a solution to the diffusion equation using any preferred method (e.g. Fourier transforms, separation of variables, other methods). You are allowed to consult external sources for this problem, but make sure you understand the idea behind the solution method.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x} \text{ in } \mathbb{R}\ u(t,x) \text{ where } t \in \mathbb{R}.$$

with the initial condition $u(0,x) = f(x)$.

Solution: I will be solving this using the Fourier Transformation.

Some of the facts that will be used throughout this problem are listed below:

- $\partial_t u = D\Delta u$ where Δ is the Laplace operator.
- $(f * g)(x) \equiv \int_{\mathbb{R}^n} f(y)g(x-y)m(dy)$ where $m(dy) \equiv \frac{dy}{(2\pi)^n}$
- $\mathcal{F}f(\xi) \equiv \widehat{f}(\xi) \equiv \int_{\mathbb{R}^n} f(x)e^{-i(x,\xi)}m(dx)\ \forall f \in L^1(\mathbb{R}^n)$
- Fix $\xi \in \mathbb{R}^n, x \mapsto e^{-i(\xi,x)}\ (\mathbb{R}^n,+) \xrightarrow{Homo} (\pi,\cdot)$ where π is the unit circle.
- $f \in L^1, f' \in L^1 \Rightarrow \widehat{f'}(\xi) = i\xi\widehat{f}(\xi)$
- $f \in L^1, \widehat{f} \in L^1 \Rightarrow f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{i(\xi,x)}m(d\xi)\ \therefore \mathcal{F}^{-1}\mathcal{F}f$
- $f \in L^1, g \in L^1, h \equiv f * g \Rightarrow \widehat{h}(\xi) = \widehat{f * g}(\xi) = \widehat{f}(\xi)\cdot\widehat{g}(\xi)$

Taking the Fourier on both sides,

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \mathcal{F}\left(D\frac{\partial^2 u}{\partial x}\right) = D\mathcal{F}\left(\frac{\partial^2 u}{\partial x}\right)$$

Applying the transformation function

$$u(x,t) = \widehat{u}(\xi,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}dx$$

Note from the facts that differentiation on the real space corresponds to multiplication by $(i\xi)$ in the Fourier space.

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = i\xi\widehat{u}(\xi,t)$$

Then,

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2}\right) = (i\xi)(i\xi)\widehat{u}(\xi, t) = -\xi^2\widehat{u}(\xi, t)$$

Substituting back,

$$\implies \frac{\partial \widehat{u}}{\partial t} = -D\xi^2\widehat{u}$$

Note that this is a first order linear differential equation which is separable. The general form looks something like $y = y(0)e^{-At}$ where A is a constant. Thus,

$$\widehat{u} = \widehat{f}(\xi, 0)e^{-D\xi^2 t}$$

We can see as $t \to \infty, e^{-D\xi^2 t}$ gets smaller and smaller decaying $\widehat{f}(\xi, t)$. Large ξdecays faster than small ξ.

Now that we have the desired equation, we need to take the inverse Fourier. Note,

$$\widehat{u}(\xi, t) = \widehat{f}(\xi)e^{-D||\xi||^2 t}$$

To use the facts stated above, we want $e^{-D||\xi||^2 t}$ to be $\widehat{g}(\xi)$. Let $\widehat{\varphi}_t(\xi) = e^{-D||\xi||^2 t}$. Then,

$$\varphi_t(x) = \mathcal{F}^{-1}\mathcal{F}\varphi_t(x)$$

From the Fourier transformation table, this solves to:

$$= \frac{1}{\sqrt{4\pi Dt}^n}e^{-\frac{||x||^2}{4Dt}}$$

This also known as the heat kernel which would be the Gaussian kernal for $D = \frac{1}{2}$ We know,

$$\widehat{u}(t, \xi) = \widehat{f}(\xi) \cdot \widehat{\varphi}(\xi) = \widehat{f * g}(\xi)$$

Then,

$$u(t, x) = (f * \varphi_t)(x)$$

Note that as t gets closer and closer to zero, it becomes more and more like the initial condition, $f(x)$.

$$u(t, x) = f * \varphi_t(x)$$

$$= \int_{\mathbb{R}^n} f(y)\varphi_t(x - y)m(dy)$$

$$= \frac{1}{\sqrt{4\pi Dt}^n}\int_{\mathbb{R}^n} f(y)e^{-\frac{||x-y||^2}{4Dt}}m(dy)$$

Setting $D = \frac{1}{2}$, we get

$$= \frac{1}{\sqrt{2\pi t}^n}\int_{\mathbb{R}^n} f(y)e^{-\frac{||x-y||^2}{2t}}m(dy)$$

Using the initial data which says $f(0) = 1$, we get,

$$\implies \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi t}}e^{-\frac{||x||^2}{2t}}dt$$

Note that this is just the density function of the Gaussian distribution. Therefore, generalizing for any f, the solution to the heat equation looks like:

$$f(x) = \frac{1}{\sqrt{4\pi Dt}^n}\int_{\mathbb{R}^n} f(y)e^{-\frac{||x-y||^2}{4Dt}}dy$$

Part 2: Simulation of Maximums of the Brownian Motion

Question 00: Simulate the following:

a) Simulate for $n = 10^2, 10^3, 10^4$ the scaled random walk and obtain its maximum value over $[0, 1]$. For each value of n, obtain the maximum value for 10 different paths (i.e. 10 different simulations for each fixed n, hence you should obtain 3Ö10 = 30 values of the maximum).

b) Repeat a) for the time intervals $[0, 5]$ and $[0, 100]$. Compare and discuss the results in a) and b).

Solution: Follow the R code below for this particular solution

```
set.seed(123)

simulated_max <- function(n, Time){
  dt <- Time / n
  W <- c(0, cumsum(rnorm(n, mean = 0, sd = sqrt(dt))))
  return(max(W))
}


maxvals100 <- replicate(10, simulated_max(100, 1))
maxvals1000 <- replicate(10, simulated_max(1000, 1))
maxvals10000 <- replicate(10, simulated_max(10000, 1))

maxvals100_5 <- replicate(10, simulated_max(100, 5))
maxvals1000_5 <- replicate(10, simulated_max(1000, 5))
maxvals10000_5 <- replicate(10, simulated_max(10000, 5))

maxvals100_100 <- replicate(10, simulated_max(100, 100))
maxvals1000_100 <- replicate(10, simulated_max(1000, 100))
maxvals10000_100 <- replicate(10, simulated_max(10000, 100))

# Round to 3 decimal places
maxvals100 <- round(maxvals100, 3)
maxvals1000 <- round(maxvals1000, 3)
maxvals10000 <- round(maxvals10000, 3)

maxvals100_5 <- round(maxvals100_5, 3)
maxvals1000_5 <- round(maxvals1000_5, 3)
maxvals10000_5 <- round(maxvals10000_5, 3)

maxvals100_100 <- round(maxvals100_100, 3)
maxvals1000_100 <- round(maxvals1000_100, 3)
maxvals10000_100 <- round(maxvals10000_100, 3)
```

```
## Max values for n = 100 and time = 1:

## 1.03 0 1.213 0.353 1.264 0.442 0.107 1.521 1.082 0.975

## Max values for n = 1000 and time = 1:

## 1.385 0.176 0.195 0.137 1.133 0.198 0.994 1.421 0.297 1.256
```

```
## Max values for n = 10000 and time = 1:

## 0.538 0.237 0.497 1.907 0.824 0.813 1.4 2.381 0.062 1.685

## Max values for n = 100 and time = 5:

## 3.314 1.164 1.042 4.712 0.494 0.883 1.614 0.138 0.099 3.216

## Max values for n = 1000 and time = 5:

## 0.536 0.275 2.671 2.645 1.633 2.471 1.231 0.957 2.307 1.681

## Max values for n = 10000 and time = 5:

## 2.202 3.29 0.57 4.775 0.583 2.277 3.196 1.947 2.345 5.133

## Max values for n = 100 and time = 100:

## 4.046 0 5.83 2.888 17.614 9.834 5.947 9.303 7.266 0

## Max values for n = 1000 and time = 100:

## 0.977 4.477 9.833 2.868 4.301 4.612 2.862 8.532 6.122 0.054

## Max values for n = 10000 and time = 100:

## 9.871 2.589 2.036 3.755 16.67 7.112 5.319 3.594 4.506 5.665
```

As we can see these results are quite different. The maximum for the different intervals take the values not that high. Why might that be the reason? The Brownian motion is a very jagged motion and it is very finicky. Furthermore, the maximum values of the motion will be achieved at whole numbers because of the linear interpolation. This means that we have at most only 100 values to test where this motion peaked. If we increased our interval, we might get better maximum values.

Part 3: Textbook Problems 8.1-8.3, 8.5, 8.9-8.11, 8.16

Question 8.1: Show that

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},$$

satisfies the partial differential heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

Solution: Let's start by taking the partial with respect to t.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}\left[\frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\right]$$

$$= \frac{1}{\sqrt{2\pi t}}\frac{2x^2}{4t^2} e^{-\frac{x^2}{2t}} + e^{-\frac{x^2}{2t}}\left(\frac{-\pi(2\pi t)^{\frac{-1}{2}}}{2\pi t}\right)$$

$$= \frac{1}{2t^2\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} - \frac{1}{2t\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

$$\frac{\partial f}{\partial t} = \frac{1}{2\sqrt{2\pi t}}\left(\frac{e^{\frac{-x^2}{2t}}}{t^2} - \frac{e^{\frac{-x^2}{2t}}}{t}\right)$$

Moving on to performing the second order partial with respect to x.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\left[\frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\right]$$

$$= \frac{1}{\sqrt{2\pi t}} \cdot \frac{-x}{t} e^{-\frac{x^2}{2t}}$$

Taking the second order partial,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}\left[\frac{1}{\sqrt{2\pi t}} \cdot \frac{-x}{t} e^{-\frac{x^2}{2t}}\right]$$

$$= (2\pi t)^{-\frac{1}{2}}\left(-\frac{1}{t}\cdot\frac{-1}{t} e^{\frac{-x^2}{2t}} + e^{\frac{-x^2}{2t}}\left(\frac{-1}{t}\right)\right)$$

$$\frac{\partial^2 f}{\partial x^2} = (2\pi t)^{-\frac{1}{2}}\left(\frac{1}{t^2} e^{-\frac{x^2}{2t}} - \frac{e^{-\frac{x^2}{2t}}}{t}\right)$$

Multiplying the second order partial with respect to x by $\frac{1}{2}$ we get the following equality,

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \frac{1}{2\sqrt{2\pi t}}\left(\frac{e^{\frac{-x^2}{2t}}}{t^2} - \frac{e^{\frac{-x^2}{2t}}}{t}\right) = \frac{\partial f}{\partial t}$$

Question 8.2: For the Standard Brownian motion, find

a) $\mathbb{P}(B_2 \leq 1)$

b) $\mathbb{E}(B_4|B_1 = x)$

c) $Corr(B_{t+s}, B_s)$

d) $Var(B_4|B_1)$

e) $\mathbb{P}(B_3 \leq 5|B_1 = 2)$

Solution:

a)

$$\mathbb{P}(B_2 \leq 1) = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{x^2}{2\cdot 2}} dx = 0.7602499$$

We can do this in R as well:

```
a2 <- pnorm(1, 0, sqrt(2))
```

The desired probability is 0.7602499.

b) By linearity,

$$\begin{aligned}
\mathbb{E}(B_4|B_1 = x) &= \mathbb{E}[B_4 - B_1 + x|B_1 = x] \\
&= x + \mathbb{E}[B_4 - B_1] \\
&= x + 0 = x
\end{aligned}$$

c) Recall that the formula for the correlation is:

$$Corr(X, Y) = \frac{Cov(X, Y)}{SD(X)SD(Y)}$$

Thus,

$$Corr(B_{t+s}, B_s) = \frac{Cov(B_{t+s}, B_s)}{SD(B_{t+s})SD(B_s)}$$

$$Corr(B_{t+s}, B_s) = \frac{min\{t+s, s\}}{\sqrt{t+s}\sqrt{s}} = \frac{s}{\sqrt{t+s}\sqrt{s}} = \sqrt{\frac{s}{t+s}}$$

d)

$$Var(B_4|B_1) = Var(B_4 - B_1 + B_1|B_1) = Var(B_4 - B_1) = Var(B_3) = 3$$

e)

$$\mathbb{P}(B_3 \leq 5|B_1 = 2) = \mathbb{P}(B_3 - B_1 \leq 5 - 2|B_1 = 2) = \mathbb{P}(B_2 \leq 3)$$

$$\mathbb{P}(B_2 \leq 3) = \int_{-\infty}^{3} \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{x^2}{2\cdot 2}} dx = 0.9830526$$

```
e2 <- pnorm(3,0,sqrt(2))
```

The desired probability is 0.9830526.

Question 8.3: For standard Brownian motion started at $x = -3$, find

a) $\mathbb{P}(X_1 + X_2 > -1)$

b) The conditional density of X_2 given $X_1 = 0$.

c) $Cov(X_3, X_4)$

d) $\mathbb{E}(X_4|X_1)$.

Solution: a) We can write $X_t = B_t - 3$ Then,

$$X_1 + X_2 = B_1 - 3 + B_2 - 3$$
$$= B_1 + B_2 - 6 \sim \text{ Normal with mean} - 6$$

Then the Variance follows,

$$Var(B_1 + B_2 - 6) = Var(B_1) + Var(B_2) + 2Cov(B_1, B_2)$$
$$= 1 + 2 + 2(2 - 1)$$
$$= 5$$

So the Probability is:

$$\mathbb{P}(B_1 + B_2 - 6 > -1) = \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi(5)}} e^{-\frac{(x+6)^2}{2(5)}} = 0.01267366$$

In R, the code is as follows:

```
tempa3 <- pnorm(5/sqrt(5), 0, 1)
a3 <- 1 - tempa3
```

The desired probability is 0.0126737.

b) We want the conditional density of X_2 given $X_1 = 0$. Note that this is just saying we want the distribution of $X_2 - X_1$ which is just $B_2 - B_1$ and that's just the distribution of B_1

$$B_1 \sim Norm(0, 1)$$

Then,

$$f(X_2|X_1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \text{ for } -\infty < x < \infty$$

c) Note,

$$X_3 = B_3 - 3 \text{ and } X_4 = B_4 - 3$$

So,

$$Cov(X_3, X_4) = Cov(B_3 - 3, B_4 - 3)$$
$$= \mathbb{E}(B_3 - 3, B_4 - 3) - \mathbb{E}(B_3 - 3)\mathbb{E}(B_4 - 3)$$
$$= \mathbb{E}(B_3 - 3(B_4 - B_3 + B_3 - 3))$$

Note,

$$= \mathbb{E}(B_3(B_4 - B_3) + B_3^2 - 3B_3 - 3(B_4 - B_3) - 3B_3 + 9)$$
$$= \mathbb{E}(B_3(B_4 - B_3)) + \mathbb{E}(B_3^2) - \mathbb{E}(6B_3) - \mathbb{E}(B_4 - B_3) + \mathbb{E}(9)$$
$$= 0 + Var(B_3) - 6(0) - 3(0) + 0 = 3$$

Another way of seeing the same thing is:

$$Cov(X_3, X_4) = Cov(X_3, X_4 - X_3 + X_3)$$

Recall,

$$Cov(A, B + A) = Cov(A, B) + Cov(A, A) = Cov(A, B) + Var(A)$$

Then,

$$Cov(X_3, X_4 - X_3 + X_3) = Cov(X_3, X_4 - X_3) + Var(X_3)$$
$$Cov(X_3, X_4 - X_3) + Var(X_3) = 0 + 3 = 3$$

d)

$$\mathbb{E}[X_4|X_1] = \mathbb{E}[X_4 - X_1 + X_1|X_1] = \mathbb{E}[X_4 - X_1] + X_1 = X_1$$

Question 8.5: Consider a standard Brownian motion. Let $0 < s < t$.

a) Find the joint density of (B_s, B_t).

b) Show that the conditional distribution of B_s given $B_t = y$ is normal, with mean and variance

$$\mathbb{E}(B_s|B_t = y) = \frac{sy}{t} \text{ and } Var(B_s|B_t = y) = \frac{s(t-s)}{t}$$

Solution: We know that B_s and B_t are standard Brownian motion, thus, they have a normal distribution. By the theorem proved in class and in the textbook, the joint density function of two normally distributed random variables is also normal. Consider,

$$B_s = x \text{ and } B_t = y$$

Then,

$$f(x,y) = f_s(x) \cdot f_{t-s}(y-x)$$

Plugging the parameters in the density of a Gaussian,

$$\frac{1}{\sqrt{2\pi s}} e^{\frac{-x^2}{2s}} \cdot \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-x^2}{2(t-s)}}$$

$$= \frac{1}{2\pi\sqrt{s(t-s)}} exp\left(\frac{-x^2(t-s) - s(y-x)^2}{2s(t-s)}\right)$$

$$= \frac{1}{2\pi\sqrt{s(t-s)}} exp\left(\frac{-x^2t + x^2s - s(y^2 - 2xy + x^2)}{2s(t-s)}\right)$$

$$= \frac{1}{2\pi\sqrt{s(t-s)}} exp\left(\frac{-x^2t - sy^2 + 2sxy}{2s(t-s)}\right)$$

Note that this is a density function of a Gaussian.

b) To make this into a form where we can clearly see the expectation and variance from the density function, we don't want there to be a coefficient in front of x^2, thus, we divide by t on both the numerator and denominator.

$$= \frac{\frac{1}{t}}{\frac{2\pi\sqrt{s(t-s)}}{t}} exp\left(\frac{\frac{-x^2t - sy^2 + 2sxy}{t}}{\frac{2s(t-s)}{t}}\right)$$

Note that $s, t,$ and y are constants. Everything other than x is just a constant. Thus,

$$f(x,y) = \frac{1}{2\pi\sqrt{\frac{s(t-s)}{t}}} \cdot exp\left(\frac{-(\frac{s}{t} - \frac{s^2}{t^2})y^2}{\frac{2s(t-s)}{t}}\right) \cdot exp\left(\frac{-(x - \frac{sy}{t})^2}{\frac{2s(t-s)}{t}}\right)$$

Note that $exp\left(\frac{-(x-\frac{sy}{t})^2}{\frac{2s(t-s)}{t}}\right)$ is the kernel of the normal distribution from the joint density function. Thus, the kernel implies,

$$B_s|_{B_t=y} \sim Norm\left(\frac{sy}{t}, \frac{s(t-s)}{t}\right)$$

Thus,

$$\mathbb{E}(B_s|B_t = y) = \frac{sy}{t} \text{ and } Var(B_s|B_t = y) = \frac{s(t-s)}{t}$$

Question 8.9: Let $W_t = B_{2t} - B_t$ for $t \geq 0$.

a) Is $(W_t)_{t\geq 0}$ a Gaussian process?

b) Is $(W_t)_{t\geq 0}$ a Brownian motion process?

Solution:

a) Let B_t be a standard Brownian motion. This implies that $B_t \sim Normal$. B_t is a Gaussian process. Since it's Brownian, it has independent increments, meaning for any $t \geq 0$. If $t \geq 0$, then $2t \geq 0$, implies that B_{2t} is also Gaussian. Thus, W_t is just a linear combination of two Gaussian, by theorem proved in class, W_t is a Gaussian too.

b) Is W_t Brownian? To check this we need to check whether the expectation of $W_t = 0$ and if the $Cov(W_s, W_t) = min\{s, t\}$. Checking the expectation:

$$\mathbb{E}[W_t] = \mathbb{E}[B_{2t} - B_t]$$

Since B_{2t} and B_t are independent, we can distribute the expectation.

$$\mathbb{E}[B_{2t} - B_t] = \mathbb{E}[B_{2t}] - \mathbb{E}[B_t]$$

Note, that this is just

$$\mathbb{E}[B_{2t}] - \mathbb{E}[B_t] = 0 - 0 = 0$$

So the expectation checks out. As long as the covariance does too, we can say it's brownian.

Consider the counterexample W_7 and W_8 we want the $Cov(W_7, W_8) = min\{7, 8\} = 7$ for W_t to be Brownian. Note,

$$W_7 = B_{2\cdot 7} - B_7 \text{ and } W_8 = B_{2\cdot 8} - B_8$$

then,

$$Cov(W_7, W_8) = Cov(B_{14} - B_7, B_{16} - B_8)$$

$$Cov(B_{14} - B_7, B_{16} - B_8) = Cov(B_{14}, B_{16}) - Cov(B_{14}, B_8) - Cov(B_7, B_16) + Cov(B_7, B_8)$$

$$= 14 - 8 - 7 + 7 = 6$$

Note that $6 \neq min\{7, 8\}$. Thus W_t is not a Brownian motion.

Question 8.10: Let $(B_t)_{t\geq 0}$ be a Brownian motion started at x. Let

$$X_t = B_t - t(B_1 - y), \text{ for } 0 \leq t \leq 1.$$

The process $(X_t)_{t\geq 0}$ is a Brownian bridge with start in x and end in y. Find the mean and covariance functions.

Solution: Note that,

$$\mathbb{E}[X_t] = \mathbb{E}[B_t - t(B_1 - y)]$$
$$= \mathbb{E}[B_t] - t\mathbb{E}[B_1 - y]$$

Note that a Brownian motion that starts at x has expectation of x. Thus,

$$\mathbb{E}[B_t] - t\mathbb{E}[B_1 - y] = x - t[\mathbb{E}[B_1] - y]$$

$$\mathbb{E}[X_t] = x - t(x - y)$$

Then, the covariance function is as follows:

$$Cov(X_s, X_t) = Cov(B_s - s(B_1 - y), B_t - t(B_1 - y))$$

$$= Cov(B_s, B_t) - tCov(B_s, B_1 - y) - sCov(B_1 - y, B_t) + stCov(B_1 - y, B_1 - y)$$
$$= s - ts - st + stVar(B_1 - y)$$
$$s - ts - st + st(1) = s - st$$

Question 8.11: A standard Brownian motion cross the t-axis at times $t = 2$ and $t = 5$. Find the probability that the process exceeds level $x = 1$ at time $t = 4$.

Solution: The desired probability is $\mathbb{P}(B_4 \geq 1 | B_2 = 0, B_5 = 0)$. Let,

$$\widetilde{B}_t = B_{t-2} \text{ since we are starting at 2}$$

Then,

$$\mathbb{P}(\widetilde{B}_2 \geq 1 | \widetilde{B}_0 = 0, \widetilde{B}_3 = 0)$$

This is just,

$$\mathbb{P}(\widetilde{B}_2 \geq 1 | \widetilde{B}_3 = 0$$

Using the result from exercise $8.5b$, this is a joint density function.

$$\widetilde{B}_2|_{\widetilde{B}_3=0} \sim N\left(\frac{2(0)}{3}, \frac{2(3-2)}{3}\right) \sim N(0, \frac{2}{3})$$

We can calculate the probability by,

```
a11 <- 1 - pnorm(1, 0, sqrt(2/3), 3)
```

Thus, the desired probability is 0.1103357.

Question 8.16: Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be independent, standard Brownian motions. Show that $Z_t = a(X_t - Y_t)$ defines a standard Brownian motion for some a. Find a.

Solution: We know that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are standard Brownian motion. This implies that they are both Gaussian Processes. Note that, $Z_t = a(X_t - Y_t)$ is a linear combination of X_t and Y_t which makes it a Gaussian Process. Check that the expectation is 0.

$$\mathbb{E}[Z_t] = \mathbb{E}[a(X_t - Y_t)]$$

$$= a\mathbb{E}[X_t - Y_t]$$

$$= a\left(\mathbb{E}[X_t] - \mathbb{E}[Y_t]\right)$$

$$\mathbb{E}[Z_t] = a(0-0) = 0$$

Checking the Covariance,

$$Cov(Z_s, Z_t) = Cov(a(X_s - Y_s), a(X_t - Y_t))$$

$$Cov(a(X_s - Y_s), a(X_t - Y_t)) = a^2 Cov(X_s - Y_s, X_t - Y_t)$$

$$= a^2[Cov(X_s, X_t) - Cov(X_s, Y_t) - Cov(Y_s, X_t) + Cov(Y_s, Y_t)]$$

$$= a^2[min\{s,t\} - 0 - 0 + min\{s,t\}]$$

Thus,

$$Cov(Z_s, Z_t) = a^2 2min\{s,t\}$$

If we want Z_t to be a Brownian motion, we need the Covariance to be the minimum of s and t. This implies,

$$2a^2 = 1$$

$$a = \pm\frac{1}{\sqrt{2}}$$

Lemma: Let X_t be a standard Brownian motion. If $a > 0$ and $Y_t = a^{\frac{-1}{2}} X_{at}$, then Y_t is a standard Brownian motion.

Proof The continuity of paths and independence of increments remain unchanged after the scaling. Since $X_0 = 0$ we know that $Y_0 = 0$. Now we must observe that $Y_{t+\epsilon} - Y_t = a^{\frac{-1}{2}}\left(X_{a(t+\epsilon)} - X_{at}\right)$ which gives us the necessary variance and distribution. Thus, Y_t is a standard Brownian motion. ■

Part 4: Simulation of Brownian Paths

Question 000: Recall that we have shown in the lecture that $u(t,x) = \mathbb{E}[f(x + B_t)]$, where B is a standard Brownian motion, solves the heat equation above with the initial condition $u(0,x) = f(x)$.

a) Simulate 10000 paths of B and compute approximately the expectation above in the one-dimensional case (i.e. x is a real number, and you can set $D = \frac{1}{2}$) by computing the arithmetic average $\frac{1}{n}\sum_{i=1}^{n} f(x + B_t^i)$, where $n = 10000$, and $f(x) = 1$ if x is in $[10, \infty)$ and 0 otherwise, i.e. the indicator function of $[10, \infty)$. Choose a grid value for t and x to obtain a visually accurate plot of $u(t,x)$. The plot will be 3-dimensional: two axes for t and x, and the vertical axis for the corresponding values of u.

b) What is the interpretation of $u(t,x)$ as a function of t? What does the choice of f in a) as the indicator function of a set mean?

c) Repeat a) for two other choices of f. You are free to choose your favourite function. Display a plot for each case.

Solution: a) Simulating 10000 paths of the brownian motion to approximate the expectation in the one dimensional case with the indicator function of $[10, \infty)$.

```r
set.seed(123)
# Ensure that you have the 'plotly' package loaded
library(plotly)

# Parameters
n_paths <- 10000  # Number of Brownian motion paths
t_values <- seq(0, 2, length.out = 20)  # Grid for time (t) from 0 to 2
x_values <- seq(5, 15, length.out = 20) # Grid for spatial points (x)

# Function to compute u(t, x)
compute_u <- function(x, t, n_paths) {
  B_t <- rnorm(n_paths, mean = 0, sd = sqrt(t))  # Simulate B_t
  return(mean((x + B_t) >= 10))  # Indicator function: 1 if x + B_t >= 10
}

# Compute u(t, x) for all grid values
u_matrix <- outer(x_values, t_values, Vectorize(function(x, t) compute_u(x, t, n_paths)))

# Create the surface plot
plot <- plot_ly(x = ~x_values, y = ~t_values, z = ~u_matrix) %>%
  add_surface(colorscale = "Viridis") %>%
  layout(title = "Approximate Solution u(t, x)",
         scene = list(xaxis = list(title = "x"),
                      yaxis = list(title = "t"),
                      zaxis = list(title = "u(t, x)")))

# Save the plot as a static image (PNG) using `orca`
orca(plot, "surface_plot.png")
knitr::include_graphics("surface_plot.png")
```

b) The function $u(t,x)$ represents the probability that a Brownian path, starting at a point x will be at least 10 units away from x after time t. As t increases, Brownian motion has more time to fluctuate, and the probability of the event $x + B_t \geq 10$ changes. Therefore, we can interpret $u(t,x)$ can be interpreted as the probability of some event occurring at time t for a given initial position x.

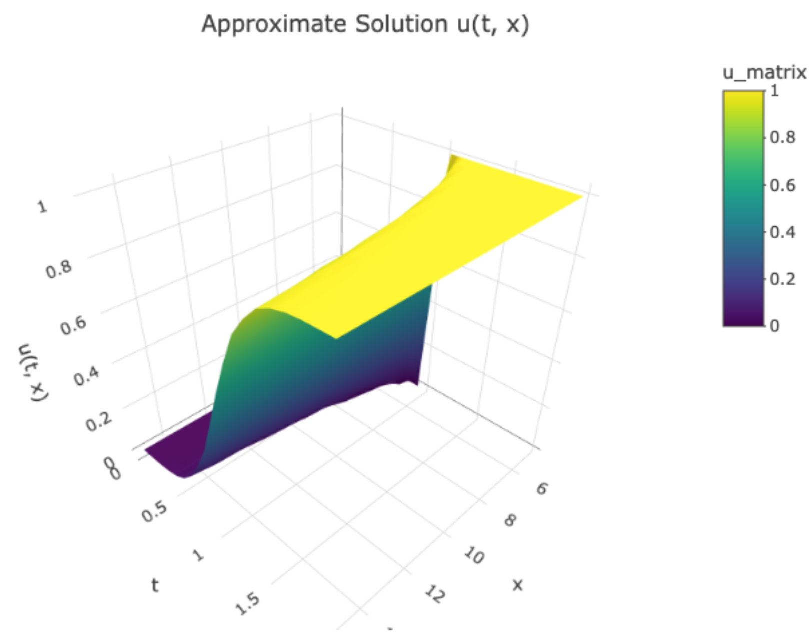


Figure 1: This surface plot shows the approximate solution of u(t,x) as a function of time t, and spatial position x. The plot represents the probability of a Brownian path being at least 10 units away from the initial position after time t.

For a small t the Brownian motion has less of a chance to get away from the initial starting position. As t increases, there's more time for the Brownian path to move further, increasing the probability of $x + B_t \geq 10$.

The choice of the indicator function represents that $f(x)$ is a binary function taking values 1 or 0. It can be defined as:

$$f(x) = \begin{cases} 1 & \text{if } x + B_t \geq 10, \\ 0 & \text{otherwise} \end{cases}$$

By taking the expectation of this over the multiple paths of the Brownian motion we estimate the probability of the event occurring. The choice of the indicator function means that we are treating the problem in terms of event occurrence (1 for the event happening, 0 for the event not happening). This kind of approach is often used in Monte Carlo simulations to estimate probabilities of events in stochastic processes.

c) For this part I will choose the two basic trigonometric functions *sin* and *cos*. For $f(x) = sin(x + B_t)$ the plot is as follows:

```r
set.seed(123)
# Ensure that you have the 'plotly' package loaded
library(plotly)

# Parameters
n_paths <- 10000  # Number of Brownian motion paths
t_values <- seq(0, 2, length.out = 20)  # Grid for time (t) from 0 to 2
x_values <- seq(5, 15, length.out = 20) # Grid for spatial points (x)

# Function to compute u(t, x)
compute_u_sin <- function(x, t, n_paths) {
  B_t <- rnorm(n_paths, mean = 0, sd = sqrt(t))  # Simulate B_t
```

```
  return(mean(sin(x + B_t)))  # Indicator function: 1 if x + B_t >= 10
}

# Compute u(t, x) for all grid values
u_matrix_sin <- outer(x_values, t_values, Vectorize(function(x, t) compute_u_sin(x, t, n_paths)))

# Create the surface plot
plot_sin <- plot_ly(x = ~x_values, y = ~t_values, z = ~u_matrix_sin) %>%
  add_surface(colorscale = "Viridis") %>%
  layout(title = "Approximate Solution u(t, x)",
         scene = list(xaxis = list(title = "x"),
                      yaxis = list(title = "t"),
                      zaxis = list(title = "u(t, x)")))

# Save the plot as a static image (PNG) using `orca`
orca(plot_sin, "surface_plot_sin.png")
knitr::include_graphics("surface_plot_sin.png")
```

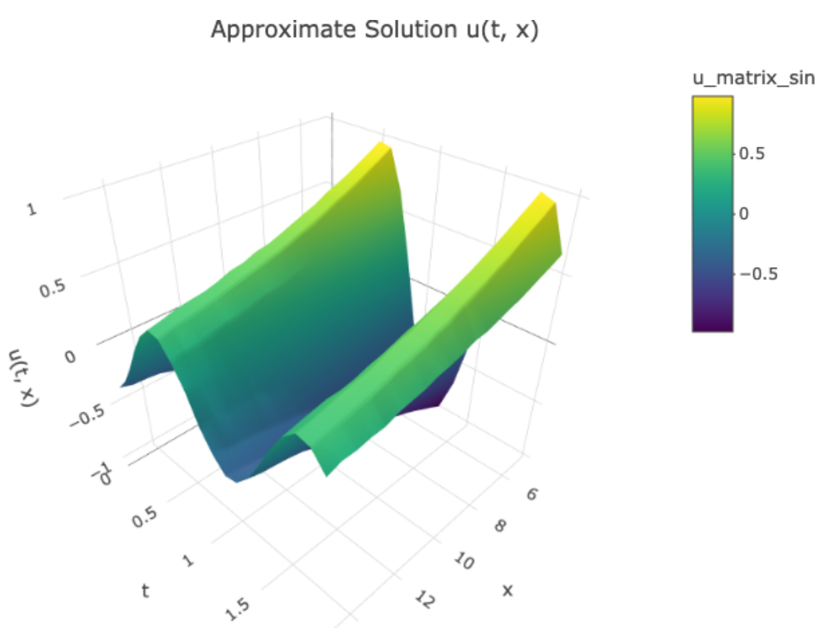


Figure 2: This surface plot shows the approximate solution $u(t,x) = sin(x + B_t)$, where B_t represents Brownian motion at time t. The plot represents the behavior of the sine function for different spatial positions x and times t, providing insight into how the sine function evolves with Brownian paths.

Note that the sine function behaves kinda like the sine wave in the traditional 2-dimensional plane. However, the function is smooth giving the approximation for the $u(t,x)$ function. As t increases the variance of B_t, which follows the normal distribution with mean 0 and variance t, increases, causing greater deviations in the sine function. Since $u(t,x)$ is computed as the expectation over multiple realizations of Brownian motion, the result is a smoothed version of the stochastic sine function. This approach effectively removes extreme fluctuations and provides a more stable approximation. The interaction between the sine function and Brownian motion results in an evolving oscillatory pattern that is smooth yet influenced by randomness.

Now for the $f(x) = cos(x + B_t)$ function, the plot is as follows:

```r
set.seed(123)
# Ensure that you have the 'plotly' package loaded
library(plotly)

# Parameters
n_paths <- 10000  # Number of Brownian motion paths
t_values <- seq(0, 2, length.out = 20)  # Grid for time (t) from 0 to 2
x_values <- seq(5, 15, length.out = 20) # Grid for spatial points (x)

# Function to compute u(t, x)
compute_u_cos <- function(x, t, n_paths) {
  B_t <- rnorm(n_paths, mean = 0, sd = sqrt(t))  # Simulate B_t
  return(mean(cos(x + B_t)))  # Indicator function: 1 if x + B_t >= 10
}

# Compute u(t, x) for all grid values
u_matrix_cos <- outer(x_values, t_values, Vectorize(function(x, t) compute_u_cos(x, t, n_paths)))

# Create the surface plot
plot_sin <- plot_ly(x = ~x_values, y = ~t_values, z = ~u_matrix_cos) %>%
  add_surface(colorscale = "Viridis") %>%
  layout(title = "Approximate Solution u(t, x)",
         scene = list(xaxis = list(title = "x"),
                      yaxis = list(title = "t"),
                      zaxis = list(title = "u(t, x)")))

# Save the plot as a static image (PNG) using `orca`
orca(plot_sin, "surface_plot_cos.png")
knitr::include_graphics("surface_plot_cos.png")
```

From the plots above, we can see that the cosine and the sine function graphs look pretty similar to the traditional cosine wave in 2dimensions. Maybe a logarithmic or an exponential function would have a different graph. These are the nice functions in math, so these would tend to have a nice smooth-looking graphs with nice plots. As t increases the variance of B_t increases just like in the sine function. Since we are calculating the average over multiple paths of the Brownian motion, we are removing the extremities where the function might fluctuate a lot to give us a smooth looking function giving us a more stable approximation. The color gradient for all the functions above help visualize the values of $u(t, x)$ with different shades indicating different magnitudes for the function. Lighter shades correspond to higher values of $u(t, x)$ while the darker shades represent lower values. Similar to the sine function, the interaction between the cosine function and Brownian motion produces an evolving oscillatory pattern that remains smooth while incorporating random perturbations, reflecting the balance between deterministic cosine behavior and stochastic influences.

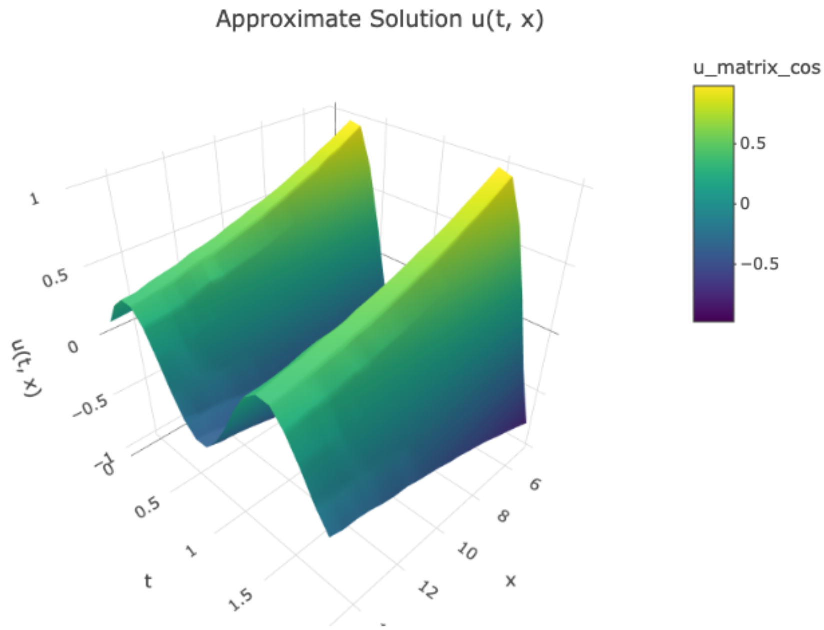


Figure 3: This surface plot shows the approximate solution $u(t,x) = cos(x + B_t)$, where B_t represents Brownian motion at time t. The plot represents the behavior of the sine function for different spatial positions x and times t, providing insight into how the cosine function evolves with Brownian paths.

Part 5: Textbook Problems 8.18-8.22

Question 8.18: Show that the first hitting time T_a has the same distribution as a^2T_1.

Solution: *Proof:* We want to show that these two probabilities are the same:

$$\mathbb{P}(a^2T_1 \leq t) = \mathbb{P}(T_1 \leq \frac{t}{a^2})$$

Note,

$$\mathbb{P}(T_1 \leq \frac{t}{a^2}) = f_{T_1}(\frac{t}{a^2}) = \frac{|1|}{\sqrt{2\pi\left(\frac{t}{a^2}\right)^3}} e^{\frac{-1}{2\left(\frac{t}{a^2}\right)}}$$

This simplifies to:

$$\begin{aligned} \frac{|1|}{\sqrt{2\pi\left(\frac{t}{a^2}\right)^3}} e^{\frac{-1}{2\left(\frac{t}{a^2}\right)}} &= \frac{a^3}{t\sqrt{2\pi t}} e^{-a^2} 2t \\ &= \frac{|a|}{t\sqrt{2\pi t}} e^{-a^2} 2t \\ &= f_{T_a}(t) \blacksquare \end{aligned}$$

Question 8.19: Find the mean and variance of the maximum value of standard Brownian motion on $[0, t]$.

Solution: Let B be a one dimensional Brownian motion. Then,

$$M_t := \max_{0 \le s \le t} B_s$$

From the theorem in the textbook we know that $M \overset{d}{=} |B|$. This implis,

$$\begin{aligned}
\mathbb{E}[M] = \mathbb{E}[|B_t|] &= \int_{\mathbb{R}^n} x f_x\left(|B_t|\right) \\
&= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} dx \\
&= 2 \int_0^{\infty} \frac{x}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} dx \\
&= \frac{-2}{\sqrt{2\pi}} \sqrt{t} \int_0^{\infty} \frac{-x}{\sqrt{t} \cdot \sqrt{t}} e^{\frac{-x^2}{2t}} dx \\
&= -\sqrt{\frac{2}{\pi}} \sqrt{t} \Big[e^{-\frac{x^2}{et}} \Big|_0^{\infty}
\end{aligned}$$

Note, that $\Big[e^{-\frac{x^2}{et}} \Big|_0^{\infty} = -1$. Thus,

$$\mathbb{E}[M] = \sqrt{\frac{2t}{\pi}}$$

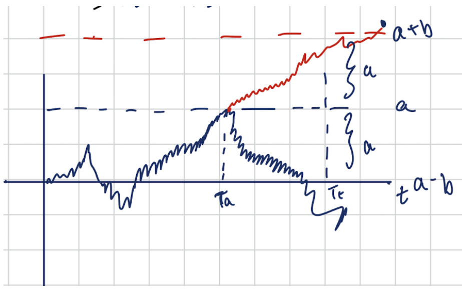
Moving on to finding the Variance of M.

$$\begin{aligned}
Var(M_t) &= \mathbb{E}[M_t^2] - \mathbb{E}[M_t]^2 \\
&= t - \left(\sqrt{\frac{2t}{\pi}} \right)^2 \\
Var(M_t) &= t - \frac{2t}{\pi}
\end{aligned}$$

Question 8.20: Use the reflection principle to show

$$\mathbb{P}(M_t \geq a, B_t \leq a-b) = \mathbb{P}(B_t \geq a+b), \text{ for } a, b > 0.$$

Solution: For this question, refer to the image below.



Let $M_t = \max\limits_{0\leq s\leq t} B_s$, the maximum value of the Brownian motion on $[0,t]$. We know $a, b > 0$. If at time t, B_t exceeds a then the maximum value on $[0,t]$ is greater than a. That is:

$$\{B_t > a\} \implies \{M_t > a\}.$$

This gives,

$$\begin{aligned}\{M_t > a\} &= \{M_t > a, B_t > a\} \cup \{M_t > a, B_t \leq a\}\\ &= \{B_t > a\} \cup \{M_t > a, B_t \leq a\}\end{aligned}$$

As the union is disjoint, $\mathbb{P}(M_t > a) = \mathbb{P}(B_t > a) + \mathbb{P}(M_t > a, B_t \leq a)$ This gives,

$$\mathbb{P}(M_t \geq a, B_t \leq a-b) = \mathbb{P}(M_t \geq a, B_t \geq a+b)$$

Since $a < a+b$ and that B_t has to be more than $a+b$. Thus,

$$\mathbb{P}(M_t \geq a, B_t \leq a-b) = \mathbb{P}(M_t \geq a, B_t \geq a+b) = \mathbb{P}(B_t \geq a+b)$$

Question 8.21: From standard Brownian motion, let X_t be the process defined by

$$X_t = \begin{cases} B_t & \text{if } t < T_a, \\ a & \text{if } t \geq T_a \end{cases}$$

where T_a is the first hitting time of $a > 0$. The process $(X_t)_{t \geq 0}$ is called *Brownian motion absorbed at a.* The distribution of X_t has discrete and continuous parts.

a) Show

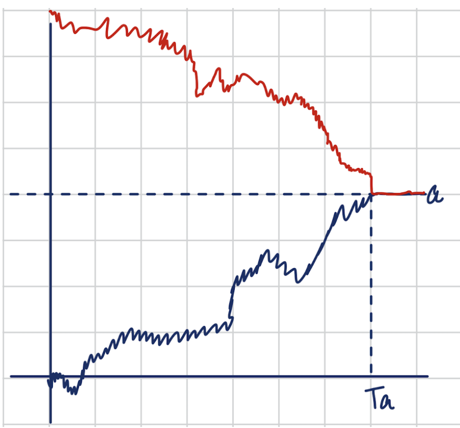
$$\mathbb{P}(X_t = a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}}\, dx$$

b) For $x < a$, show

$$\mathbb{P}(X_t \leq x) = \mathbb{P}(B_t \leq x) - \mathbb{P}(B_t = x - 2a) = \frac{1}{\sqrt{2\pi t}} \int_{x-2a}^{x} e^{-\frac{z^2}{2t}}\, dz.$$

Hint: Use the result of Exercise 8.20.

Solution: For part a, follow the picture below



$$\mathbb{P}(X_t = a) = \mathbb{P}(X_t = a, t \geq T_a)$$

Let $M_t := \max\limits_{0 \leq s \leq t} B_s$ then $M_t \geq a$.

$$\mathbb{P}(X_t = x) = \mathbb{P}(M_t \geq a)$$

Note,

$$\mathbb{P}(M_t \geq a) = \mathbb{P}(B_t \geq a) + \mathbb{P}(M_t \geq a, B_t \leq a)$$

Note that a is the absorption level of the Brownian motion. Then, $\mathbb{P}(B_t \geq a) = \mathbb{P}(X_t = a)$. Then,

$$\mathbb{P}(M_t \geq a) = \mathbb{P}(X_t = a) + \mathbb{P}(\widetilde{B}_t \geq a) \text{ where } \widetilde{B}_t = \begin{cases} B_t & \text{if } 0 \leq t \leq T_a, \\ 2a - B_t & \text{if } t \geq T_a \end{cases} \text{ by the reflection principle.}$$

$$\implies \mathbb{P}(M_t \geq a) = \mathbb{P}(X_t = a) + \mathbb{P}(B_t > a)$$

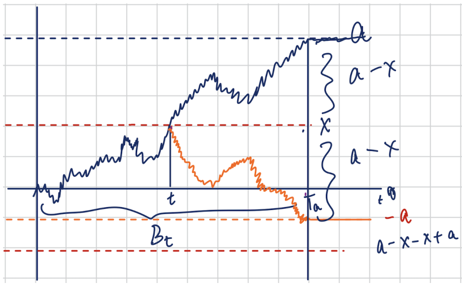
$$\implies \mathbb{P}(M_t \geq a) = \mathbb{P}(X_t = a) + \mathbb{P}(X_t = a)$$

$$\implies \mathbb{P}(M_t \geq a) = 2\mathbb{P}(B_t = a) = 2f_{T_a}(x) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}}\, dx.$$

Thus, we show that

$$\mathbb{P}(X_t = a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}}\, dx$$

b) For part b, we consider this image:



Note that the reflection can be represented as:

$$\widetilde{B}_t = \begin{cases} B_t & \text{if } 0 \le t \le T_a, \\ 2a - B_t & \text{if } t \ge T_a \end{cases}$$

We know,

$$\{M_t > a\} = \{M_t > a, B_t > a\} \cup \{M_t > a, B_t \le a\}$$
$$= \{B_t > a\} \cup \{M_t > a, B_t \le a\}$$

Note that the union is disjoint. This gives,

$$\mathbb{P}(M_t > a) = \mathbb{P}(B_t > a) + \mathbb{P}(M_t > a, B_t \le a)$$

Furthermore, the only way for $X_t \le x$ is true is if the original Brownian motion was still below x and had not yet hit a. Thus, we can write,

$$\mathbb{P}(X_t \le x) = \mathbb{P}(B_t \le x, T_a > t) \text{ where } t \text{ is the first hitting time of } x.$$

For $x < a$, if $B_t \le x$, then the process must not have hit a yet, meaning $T_a > t$. Then,

$$\mathbb{P}(X_t \le x) = \mathbb{P}(B_t \le x, M_t < a)$$

$$\mathbb{P}(B_t \le x, T_a > t) = \mathbb{P}(B_t \le x) - \mathbb{P}(B_t \le x, T_a \le t)$$

We essentially want all possible paths that are below x but not necessarily till 0. This gives,

$$\mathbb{P}(X_t \le x) = \mathbb{P}(B_t \le x) = \mathbb{P}(B_t \le x, M_t \ge a - (a - x))$$

where $a - (a - x)$ is the chunk between the red dotted line, x, and the blue dotted line, a. This gives us all the paths between x and a.

$$\mathbb{P}(X_t \le x) = \mathbb{P}(B_t \le x) - \mathbb{P}(B_t \le x, M_t \ge a - (a - x))$$
$$= \mathbb{P}(B_t \le x) - \mathbb{P}(B_t \ge 2a - B_t)$$

Note, in $2a - B_t$, the $B_t = x$ because $B_t \le x$ i.e. the orange line. Then,

$$\mathbb{P}(X_t \le x) = \mathbb{P}(B_t \le x) - \mathbb{P}(B_t \ge 2a - x)$$
$$= \mathbb{P}(B_t \le x) - \mathbb{P}(B_t \le x - 2a)$$

Since B_t is a Brownian motion, we can use the density function of a Gaussian. Thus,

$$\mathbb{P}(B_t \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz$$

And,

$$\mathbb{P}(B_t \leq x - 2a) = \int_{-\infty}^{x-2a} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\, dz$$

This gives,

$$\mathbb{P}(B_t \leq x) - \mathbb{P}(B_t \leq x - 2a) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\, dz - \int_{-\infty}^{x-2a} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\, dz$$

$$\mathbb{P}(B_t \leq x) - \mathbb{P}(B_t \leq x - 2a) = \int_{x-2a}^{x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\, dz$$

This is exactly what we have been trying to show. Thus,

$$\mathbb{P}(X_t \leq x) = \frac{1}{\sqrt{2\pi t}} \int_{x-2a}^{x} e^{-\frac{z^2}{2t}}\, dz$$

Question 8.22: Let Z be the smallest zero of the Brownian motion past t. Show that

$$\mathbb{P}(Z \leq z) = \frac{2}{\pi} arccos\left(\frac{t}{z}\right), \text{ for } z > 0.$$

Solution: For $0 \leq t \leq z$, let Z be the probability that standard Brownian motion has at least one zero in (t, z). Then,

$$Z_{t,z} = \frac{2}{\pi} arccos\left(\sqrt{\frac{t}{z}}\right).$$

with $t = 0$, the result gives,

$$Z = \frac{2}{\pi} arccos\,(0) = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$$

Let

$$F(\tau) = \frac{2}{\pi} arcsin\left(\sqrt{\tau}\right) \text{ for } 0 \leq \tau \leq 1$$

be the arcsine function. Then the density function would be:

$$f(\tau) - F'(\tau) = \frac{1}{\pi\sqrt{\tau(1-\tau)}}, \text{ for } 0 \leq \tau \leq 1$$

Proof: Conditioning on B_t,

$$Z_{t,z} = \mathbb{P}(B_s = 0 \text{ for some } s \in (t, z))$$
$$= \int_{-\infty}^{\infty} \mathbb{P}(B_s = 0 \text{ for some } s \in (t, z) | B_t = x) \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} dx$$

Assume that $B_t = x < 0$, the probability that $B_s = 0$ for some $s \in (t, z)$ is the probability for the process started in x, the maximum on $(0, z - t) > 0$. This implies that the later is equal to the probability that for process started in 0, the maximum on $(0, z - t) > x$. That is,

$$\mathbb{P}(B_s = 0 \text{ for some } s \in (t, z) | B_t = x) = \mathbb{P}(M_{z-t} > x).$$

For $x > 0$, consider the reflected process $-B_s$ started in $-x$. Then,

$$Z_{t,z} = \int_{-\infty}^{\infty} \mathbb{P}(M_{z-t} > |x|) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$
$$= \int_{-\infty}^{\infty} \int_{0}^{z-t} \frac{1}{\sqrt{2\pi s^3}} |x| e^{-\frac{x^2}{2s}} ds \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

Changing the lower bound,

$$\int_{-\infty}^{\infty} \int_{0}^{z-t} \frac{1}{\sqrt{2\pi s^3}} |x| e^{-\frac{x^2}{2s}} ds \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \frac{2}{2\pi} \int_{0}^{z-t} \frac{1}{\sqrt{ts^3}} \int_{0}^{\infty} x e^{\frac{-x^2(t+s)}{2ts}} dx ds$$
$$= \frac{1}{\pi} \int_{0}^{z-t} \frac{1}{\sqrt{ts^3}} \int_{0}^{\infty} e^{\frac{-r(t+s)}{ts}} dr ds$$
$$= \frac{1}{\pi} \int_{0}^{z-t} \frac{1}{\sqrt{ts^3}} \left(\frac{ts}{t+s}\right) ds$$

Let $x = \frac{t}{t+s}$, Then:

$$Z_{t,z} = \frac{1}{\pi} \int_{\frac{t}{z}}^{1} \frac{1}{\sqrt{x(1-x)}} dx$$

Note, that the above is an arcsine probability. Therefore,

$$Z_{t,z} = \frac{2}{\pi} \left(arcsin(\sqrt{1}) - arcsin\left(\sqrt{\frac{t}{z}}\right)\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2} - arcsin\left(\sqrt{\frac{t}{z}}\right)\right)$$

$$= 1 - \frac{2}{\pi}arcsin\left(\sqrt{\frac{t}{z}}\right)$$

$$Z_{t,z} = \frac{2}{\pi}arccos\left(\sqrt{\frac{t}{z}}\right) \blacksquare$$