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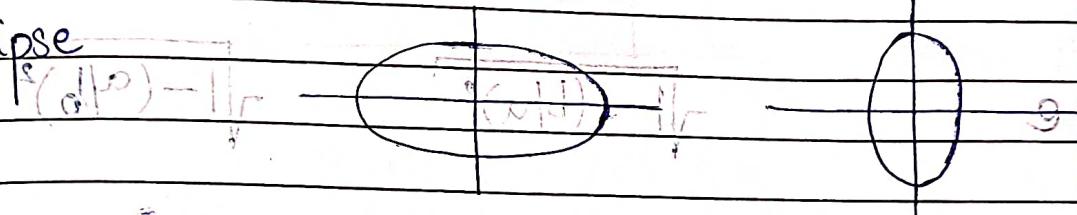
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Ellipse

General Defn. —

Locus of a pt. s.t. the ratio of its dist. from a fix. pt & a fix. line is constant. It is less than 1.

Ellipse



$$\text{Eqn } O = X \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$O = Y \quad (a > b) = X \quad (a < b) = Y$$

Centre $(0,0)$ $(0,0)$

Foci $(-ae, 0); (ae, 0)$ $M_f((0, -be), (0, be))$

Directrices $x = (-a/e); x = a/e$ $y = (-b/e); y = b/e$

Vertices $(-a, 0); (a, 0)$ $(0, -b); (0, b)$

Major Axis $2a$ $2b$

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$$e = \sqrt{1 - \left(\frac{\text{chota}}{\text{Bada}}\right)^2}$$

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Minor Axis

2b

2a

L.R.

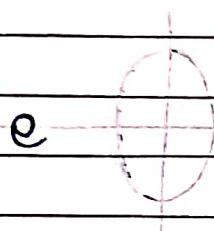
2b^2/a

2a^2/b

L.R.

x = (-ae); x = ae; y = (-be); y = be

Endpts. for min. axis sq. o b. w/o a
 of L.R. for x-axis $(-ae, -b^2/a)$ $(-\frac{a^2}{b}, -be)$
 for y-axis $(-ae, b^2/a)$ $(\frac{a^2}{b}, -be)$
 $(ae, -b^2/a)$ $(-\frac{a^2}{b}, be)$
 $(ae, b^2/a)$ $(\frac{a^2}{b}, be)$



$$\sqrt{1 - (b/a)^2}$$

$$\sqrt{1 - (a/b)^2}$$

Major Axis

y = 0

x = 0

Minor Axis

x = 0 ($< p$)

y = 0

(0,0)

(0,0)

(0,0)

Second Defn of Ellipse

Locus of pt. s.t. sum of dist. from two fix. pts. is const. if this const. is greater than dist. b/w the fix. pts.

ds

ds

fix A & B



Proof :

$$PS = e \cdot RL$$

$$PS' = e \cdot PL'$$

$$\Rightarrow (PS + PS') =$$

$$= e \cdot (PL + PL')$$

$$= e \cdot (MQ + MQ') = e \cdot (CQ - CM + CQ' + CM)$$

$$= e \cdot (Q'C + CQ) = 2e \cdot QC = 2e \cdot a/c$$

 \Rightarrow

$$PS + PS' = 2a$$

$$S = \frac{a}{c} \cos(\theta) = p$$

$$(S, 0) : (S-a, 0) = 2a$$

Parametric Coordinates

for horiz. ellipse,

$$\text{Auxiliary circle: } x^2 + y^2 = a^2$$

$O = S + b\theta i - xP - yQ$

P & Q are
Corresponding pts.

$$P(a \cos(\theta), b \sin(\theta))$$

Auxiliary Circle

Ellipse

Eccentric Angle

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DATE _____
PAGE _____Exercise

- 1) find Major Axis, Minor Axis, foci, vertices, eccentricity of ellipse $3x^2 + 2y^2 = 6$.

$$3x^2 + 2y^2 = 6 \quad (29+29)$$

$$(MP + QP + PQ - QM) \cdot 9 = (QM + QM) \cdot 9$$

A) $a = \sqrt{2}$, $b = \sqrt{3}$, $e = 1/\sqrt{3}$

$$\text{Vertices} = (0, -\sqrt{3}) ; (0, \sqrt{3})$$

$$\text{Foci} = (0, -1) ; (0, 1)$$

$$\text{Directrices} = y = -3 ; y = 3$$

$$\text{Vertices} = (0, -\sqrt{3}) ; (0, \sqrt{3})$$

$$|\text{Major axis}| = 2\sqrt{3} \quad |\text{Minor axis}| = 2\sqrt{2}$$

- 2) find centre, length of axes & eccentricity of $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.

A). $2(x-1)^2 + 3(y-2)^2 = 13$

$$\text{Centre} = (1, 2) \quad a = \sqrt{2}, b = \sqrt{3}$$

$$|\text{Major Axis}| = \sqrt{2} \quad e = 1/\sqrt{3}$$

$$|\text{Minor Axis}| = 2\sqrt{3}$$

3) find eqns of ellipse with foci $(2, 3)$ & $(-2, 3)$ whose semi minor axis is $\sqrt{5}$.

A) $b = \sqrt{5}$, Centre $\equiv (0, 3)$, $ae = 2$.

$$e^2 = 1 - \left(\frac{b}{a}\right)^2 \Rightarrow a = 3$$

Eqn: $\frac{(x-0)^2}{9} + \frac{(y-3)^2}{5} = 1$

$$\boxed{\frac{x^2}{9} + \frac{(y-3)^2}{5} = 1}$$

4) find eqns of ellipse with centre $(1, 2)$; one focus at $(6, 2)$ it passing thru $(4, 6)$

A) Ellipse $\equiv \frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$

$$5 = ae \Rightarrow a^2 = 25 + b^2 \quad \text{--- (1)}$$

Also, $\frac{9}{a^2} + \frac{16}{b^2} = 1 \Rightarrow 9b^2 + 16a^2 = a^2b^2$

$$\begin{aligned} \textcircled{1} \curvearrowright \textcircled{2} \Rightarrow 9b^2 + 16(25 + b^2) \\ = b^2(25 + b^2) \end{aligned} \quad \text{--- (2)}$$

$$\Rightarrow b^4 = 16 \cdot 25$$

$$\Rightarrow b^2 = 20$$

$$\text{et } a^2 = 45$$

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$$\Rightarrow (x-1)^2/45 + (y-2)^2/20 = 1$$

- Q = 90 $(x, 0) \equiv \text{orthocentre}$ $\angle A = \alpha$ (A)
- 5) If a chord joining 2 pts whose eccentric angles are $\alpha < 0$ & $\beta = 1$ cuts the major axis of an ellipse at a dist. $(d, 0)$ away from centre, then show that

$$l = \tan(\alpha/2) \tan(\beta/2) \equiv \frac{(d-a)}{d+a}$$

A) Chord: $(x/a)^2 + (y/b)^2 = \sec^2(\alpha/2)$

Thru $(d, 0) \Rightarrow (d/a) = \frac{\sec \alpha/2 \sec \beta/2 + \tan \alpha/2 \tan \beta/2}{\sec \alpha/2 \sec \beta/2 - \tan \alpha/2 \tan \beta/2}$

$$\Rightarrow l = \frac{s(\alpha/2)}{a} + \frac{s(\beta/2)}{b} \equiv \text{orthocentre} \quad (\text{A})$$

$\therefore (s_d + s_b) = l$

$$(s_d + s_b) d + s_d b \leftarrow l = s_d d + s_b b \quad \text{orthocentre}$$

$$(s_d + s_b) s_d = (s_d d + s_b b) \leftarrow (s_d + s_b) s_d = s_d d$$

$$(s_b - s_d) b = 0 \leftarrow s_b = s_d$$

Formulae

$$\frac{(x^2 - x_0^2)}{a^2} + \frac{(y^2 - y_0^2)}{b^2} = 1 \quad \text{Eqn of ellipse}$$

1) ~~Tangent~~ Tangent —
(at pt. on curve)

$$T=0$$

2) Tangent —
(pt. outside curve) $x_m = 1$ $y_m = 1$

$$SS_1 = T^2$$

3) ~~Chord of Contact~~ Chord of Contact —

$$T=0$$

4) Chord with given midpt —

$$T=S_1$$

fr. prop. in $x + x_m = \mu$

at $\frac{x}{a} + \frac{x_m}{a} = \frac{\mu}{a}$ (8)

5) Eqn of Tangent —
(at pt. on curve)

$$(x_1, y_1) \Rightarrow \frac{x}{a^2} + \frac{y}{b^2} = 1 \quad \text{Eqn of tangent}$$

$$(a \cos \theta, b \sin \theta) \Rightarrow \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \quad \text{Eqn of tangent}$$

6) Eqn of a Normal — $(y) + (\beta + \alpha) \cos(\alpha)(x)$
(at pt. on curve)

$$(x_1, y_1) \Rightarrow \frac{(x - x_1)}{(x_1/a^2)} = \frac{(y - y_1)}{(y_1/b^2)}$$

$$(a \cos \theta, b \sin \theta) \Rightarrow \left(\frac{ax}{\cos \theta}, \frac{by}{\sin \theta} \right) = (a^2 - b^2)$$

$$[0 = T]$$

→ fahim mario & fahim broad
(guru m. fig to)

- 7) Tangent with $T = 12^\circ$
given slope —

$$y = mx + \sqrt{a^2m^2 + b^2}$$

Pls. of Contact — $\left(\frac{\pm a^2(m)}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}} \right)$

- 8) Condⁿ for Tangent

$$y = mx + c \text{ is tangent} \Leftrightarrow$$

$$c^2 = a^2m^2 + b^2$$

9) Director(P,P) + (1, x) $x^2 + y^2 = a^2 + b^2$

- 10) Chord joining $P(\alpha)$ & $Q(\beta)$ —

$$\left(\frac{x}{a} \right) \cos \left(\frac{\alpha + \beta}{2} \right) + \left(\frac{y}{b} \right) \sin \left(\frac{\alpha + \beta}{2} \right) = 1 \cos \left(\frac{\alpha - \beta}{2} \right)$$

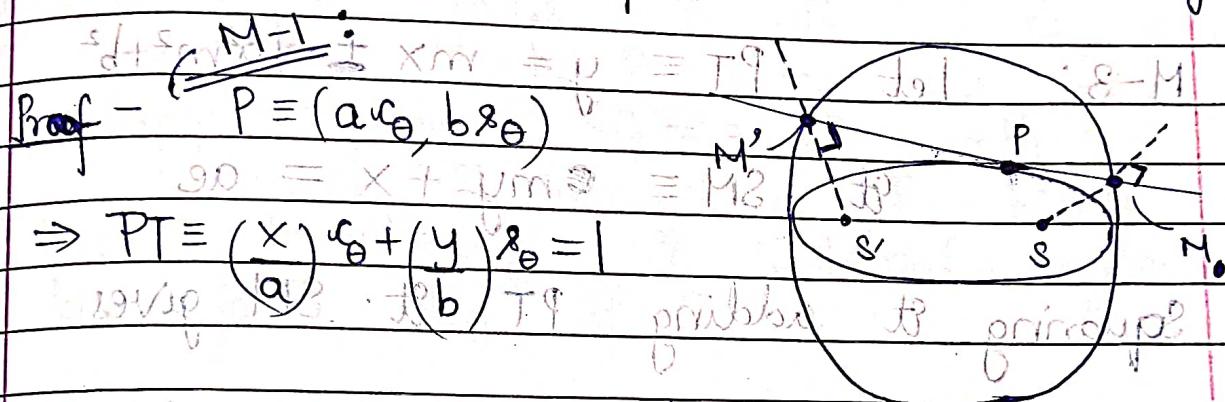
$$\left(\frac{b^2 - b^2}{a^2 + b^2} \right) = \left(\frac{x - x}{a^2 + b^2} \right) \Leftrightarrow \left(\frac{b^2 - b^2}{a^2 + b^2} \right)$$

for focal chord, $\tan\left(\frac{\alpha}{2}\right) \tan\left(\frac{\beta}{2}\right) = \left(\frac{e-1}{e+1}\right) \text{ or } \left(\frac{e+1}{e-1}\right)$

Prop's of Ellipse

if $M_2 R T_2 \leq s_0 = 2p + ex$

- Meeting pt. of ls from foci on any tangent to ellipse, lie on auxiliary circle.



$$M: ax^2 - 2ae \equiv ty \leftarrow \left(\frac{ex_0}{a} + 1\right) + \frac{y}{b} = 1$$

$$\left(\frac{v_0}{a}\right) \left(\frac{v_0}{b}\right) \left(\frac{v_0^2/a^2 + v_0^2/b^2}{1+e^2}\right) = 1$$

$$\Rightarrow M \equiv \left(ae + \frac{v_0}{a(1+e^2)}, \frac{v_0}{a\sqrt{1-e^2}(1+e^2)} \right)$$

Now, $(X_M)^2 + (Y_M)^2 = a^2 c^2$

$$d_1^2 = M_2 \cdot M_2$$

~~(M-2)~~ In (1), M-1) and $PT \equiv (x/a)^{s_0} + (y/b)^{s_0} = 1$

~~(2)~~ $\Rightarrow (1+s_0)(2-s_0) = 1$

Let $SM \equiv \frac{x^{s_0}}{b} - \frac{y^{s_0}}{a} = \frac{(ae)^{s_0}}{b}$

Squaring it adding PT & SM gives

$$x^2 + y^2 = a^2 \Rightarrow PT \cap SM \text{ lies}$$

now we just need to find eqn aux. (i)

① ~~fixed~~ now find ~~eqn~~ ~~of~~ ~~Prepared~~

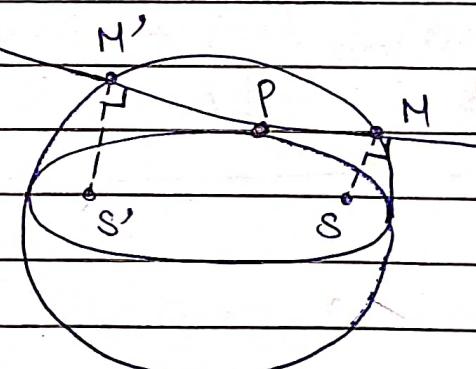
M-3: Let $PT \equiv y = mx \pm \sqrt{a^2m^2 + b^2}$

Let $SM \equiv my + x = ae$

Squaring it adding PT & SM gives

$$(1) = x^2 + y^2 = a^2 \Rightarrow (PT \cap SM \text{ lies : M})$$

2) Product of dist. from foci on any tangent to ellipse $= (a^2) + (b^2)$



$$SM \cdot S'M' = b^2$$

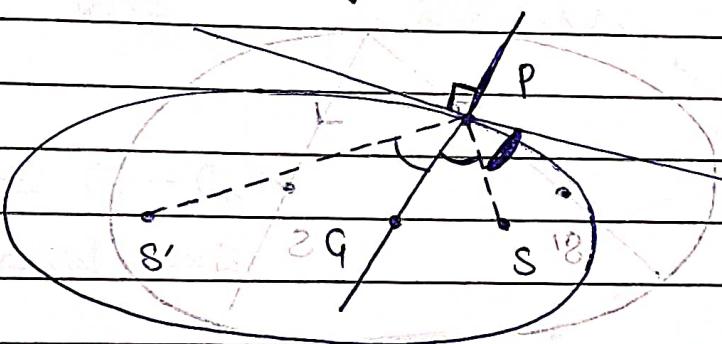
Proof: $PT = (x/a) \cos \theta + (y/b) \sin \theta = 1$

$$SM = b \frac{1 - e \cos \theta}{\sqrt{1 - e^2 \cos^2 \theta}}, S'M' = b \frac{1 + e \cos \theta}{\sqrt{1 - e^2 \cos^2 \theta}}$$

\Rightarrow

$$SM \cdot S'M' = b^2$$

3) If S' be focus & G be pt. m where normal at P meets axis of ellipse, then $SG = e \cdot SP$ and tangent at P bisects the external & internal angles b/w focal dist. of P .



Proof: $P = (a \cos \theta, b \sin \theta) \Rightarrow PG = (ax) - (by) = (a^2 - b^2)$

$$\Rightarrow G = (a \cos \theta, b \sin \theta, 0) \Rightarrow SG = (ae \cos \theta, ae \sin \theta, 0)$$

Now, $SP = e PM = e \left(\frac{a - a \cos \theta}{e} \right) \Rightarrow SP = (a - a \cos \theta e)$

$$\Rightarrow SG = e \cdot SP$$

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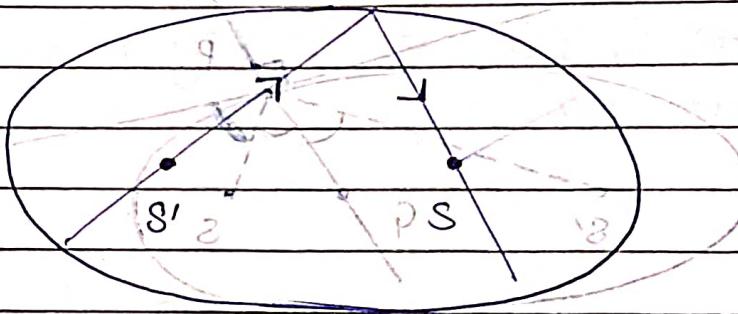
Similarly, $\angle S'PQ = \angle SPQ = 90^\circ$ (given)

By Angle Bisector Theorem, $(PG \parallel PQ)$ is the angle bisector of $\angle S'PS$

$$\Rightarrow \angle S'PG = \angle SPG = 90^\circ$$

- 4) An incoming flight may passes through one focus of ellipse, then it will get reflected to other focus.

Proof - Using 3rd property



- 5) If tangent at P meets directrix in F, then PF will subtend a right angle at corresponding focus.

$$(90^\circ - \theta) = 90^\circ \Leftrightarrow (90^\circ - \theta) \theta = 90^\circ \theta = 90^\circ$$

Proof: $PT = (x/a) v_0 + (y/b) s_0 = 1$

$F = \left(\frac{a}{e}, \left(\frac{b}{es_0} \right) (e - v_0) \right); P = (av_0, bs_0); S = (ae, 0)$

$m_{SF} = \frac{y_0 - b s_0}{x_0 - a v_0} = \frac{(e - v_0)}{\left(\frac{a}{e} - e \right)} = \frac{(e - v_0)}{\left(\frac{a - ae}{e} \right)} = \frac{(e - v_0)}{a - ae} = \frac{v_0 - e}{-e}$

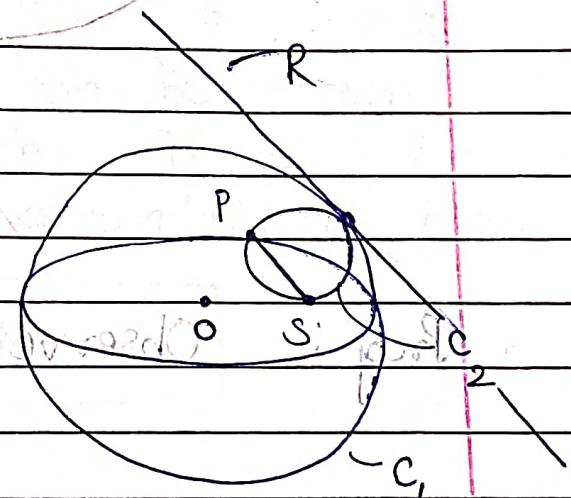
$\Rightarrow m_{SF} \cdot m_{PF} = (-1)$

6) Tangents at extremities of L.R. on corresponding directrix.

7) The circle on any focal dist. as diameter touches the aux. circle.

Proof: $R = c_1 - c_2$
(radical axis)

$\Rightarrow R = a(e + v_0)x + (b s_0)y = a^2 e v_0$



If dist. b/w O & R = $R^2 = a^2 + a^2(v_0)^2 = T^2$

\Rightarrow Radical axis is tangent \Rightarrow Circles touch.

$(R - d) = (a^2 v_0^2) - (a^2 x^2) \equiv R^2 - d^2 \equiv T^2 \equiv 0^2$

$$d = \sqrt{\frac{a^2 e v_0}{\sqrt{a^2(e+v_0)^2 + b^2 s_0^2}}} = \sqrt{\frac{a^2 e v_0}{\sqrt{a^2 + 2a^2 e v_0 + a^2(e+v_0)^2}}} = \frac{a^2 e v_0}{e v_0 + 1}$$

$$1 = \frac{a^2}{c^2}(\frac{d}{|d|}) + \frac{b^2}{c^2}(\frac{a}{|a|})^2 = T^2$$

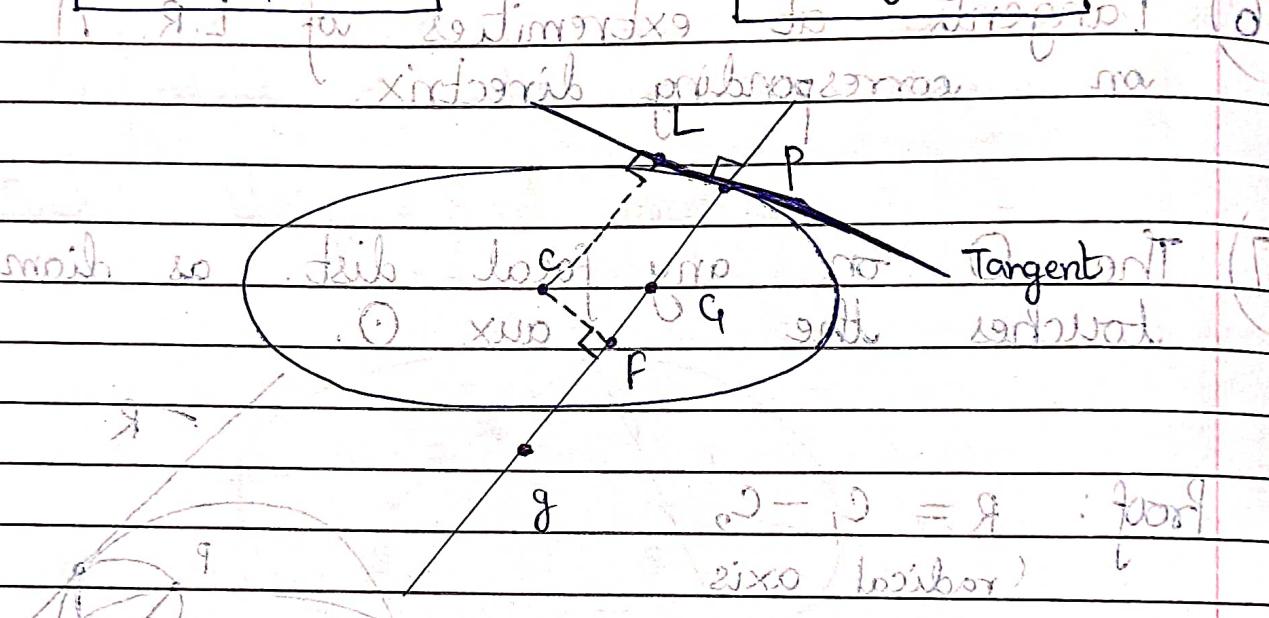
$$(0, b) - \text{if } (\text{and } d > 0) \Rightarrow 1 = \left(\left(\frac{a^2 - b^2}{c^2} \right) \frac{d}{|d|} \right)^2 = 1$$

- 8) If normal at any pt. P meets major axis (a^2) and minor axis (b^2) at G and $(g_1 - g_2)$ resp., then if CF is \perp upon this normal from centre C, then

$$PF \cdot PG = b^2$$

at

$$PF \cdot Pg = a^2$$



Proof : Observe $PF = \sqrt{(1 + \text{dist. } a^2 \text{ from } OC)} = \sqrt{a^2 + b^2}$

$$PT = \sqrt{a^2} \cdot v_{C_0} + \frac{y}{b} \cdot s_{C_0} = \sqrt{a^2 + b^2}$$

Now $\frac{y}{b} = \tan \theta$ \leftarrow formed in sixo insibor

$$PG = Pg = \sqrt{\left(\frac{ax}{v_{C_0}}\right)^2 + \left(\frac{by}{s_{C_0}}\right)^2} = \sqrt{a^2 + b^2}$$

$$\begin{aligned} PT &= \sqrt{a^2 + b^2} \\ &= \sqrt{s_{C_0}^2 + \left(\frac{ax}{v_{C_0}} + \frac{by}{s_{C_0}}\right)^2} \\ &= \sqrt{s_{C_0}^2 + \left(\frac{a^2 + b^2}{v_{C_0}^2}\right)} = \sqrt{a^2 + b^2} \end{aligned}$$

$$PF \equiv CL \Rightarrow \sqrt{\left(\frac{xc_0}{a}\right)^2 + \left(\frac{sc_0}{b}\right)^2} = ab \sqrt{1 - (ec_0)^2}$$

$$q \equiv (ae^2 c_0, 0)$$

$\text{INT}(L_{\infty}) = \text{pt}$

$$g \equiv (0, (-a^2/b)s_0 e^2)$$

$$PG = \sqrt{(ac_0)^2(1-e^2)^2 + (bs_0)^2} = b\sqrt{1-(ec_0)^2}$$

(ExNP mat. If ad \neq 0) $\Rightarrow S-M$

Q also known as $(\theta)^T$ prop - After rotation

Q form \equiv $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$

$$\begin{pmatrix} 1-\frac{s^2}{e^2} \\ \frac{sc}{e^2} \end{pmatrix} = Q \quad \begin{pmatrix} 1+\frac{s^2}{e^2} \\ \frac{sc}{e^2} \end{pmatrix} = Q^2$$

$$\frac{s^2}{e^2} - s_0^2 = a^2 s_0^2 - b^2 s_0^2$$

9) In general, 4 normals can be drawn from any pt. on ellipse. If $\alpha, \beta, \gamma, \delta$ be eccentric angles of co-normal pts., then

$$\boxed{\alpha + \beta + \gamma + \delta = -(2n+1)\pi}$$

M-1

$$\text{Proof: Normal} \equiv (ax/c_0) - (by/s_0) = (a^2 - b^2)$$

$$\left(\frac{-s}{a}\right) \left(\frac{s}{a} - \frac{s}{b}\right) = (1 - \frac{s^2}{e^2})(1 + \frac{s^2}{e^2}) - (1 - \frac{s^2}{e^2})(1 + \frac{s^2}{e^2})$$

$$\text{This pass thru } (h, k) \Rightarrow \frac{(ah)}{(c_0)} - \frac{(bk)}{(s_0)} = (a^2 - b^2)$$

$$(1 - \frac{s^2}{e^2})(\frac{s}{a} - \frac{s}{b}) - (1 - \frac{s^2}{e^2})(\frac{s}{a} + \frac{s}{b})$$

$$(1 - \frac{s^2}{e^2})(\frac{s}{a} - \frac{s}{b}) =$$

Create 4-th degree eqn in $\alpha_{(1/2)}$

\Rightarrow It has 4 roots.

By trigonometry show,

$$\sum \alpha = (2n+1)\pi$$

by using $\tan(\sum(\alpha/2)) = 0$

M-2: Let 'z' be pt. on aux. circle with arg (θ) . Assume aux. \odot = unit \odot

$$\Rightarrow v_\theta = \frac{(z^2+1)}{2z} \quad \text{et} \quad s_\theta = \frac{(z^2-1)}{2zi}$$

$$\text{Now, } ax s_\theta - by c_\theta = a^2 - b^2$$

work 3d now tomorrow : Increase nI
is it normal to the ellipse? Let's it must pass through fix! (h, k) and 's' is ad

$$\Rightarrow \frac{ah}{2z} - \frac{bk}{2zi} = a^2 - b^2$$

$$\Rightarrow (ah) \left(\frac{z^2-1}{2z} \right) - (bk) \left(\frac{z^2+1}{2z} \right) = (a^2 - b^2) \left(\frac{z^4 - 1}{4z^2} \right)$$

$$\Rightarrow (ah)(2z)(z^2-1) - (bk)(2z)(z^2+1)$$

$$= (a^2 - b^2)(z^4 - 1)$$

$$\Rightarrow z^4 [a^2 - b^2] + z^3 [2bki - 2ah] + z [2bki + 2ah]$$

$$+ [b^2 - a^2] = 0$$

This is (4th) degree in z . $(2+a)$ is

\Rightarrow It has 4 roots \Rightarrow 4 normals thru (h, k)

Let z_1, z_2, z_3, z_4 be roots.

Let they be corresponding pts of pts on ellipse, each with eccentric angles $\alpha, \beta, \gamma, \delta$.

$$\Rightarrow \arg(z_1) = \alpha, \arg(z_2) = \beta, \arg(z_3) = \gamma, \arg(z_4) = \delta.$$

By quartic eqn, $z_1 z_2 z_3 z_4 = (-1)$

$$\Rightarrow e^{i(\alpha+\beta+\gamma+\delta)} = e^{i(2n+1)\pi}$$

$$\Rightarrow \alpha + \beta + \gamma + \delta = (2n+1)\pi$$

$$O = [(x+r)^2 + (y+s)^2 - (x-t)^2] / (8) \leftarrow$$

$$O = (x+r)^2 + (y+s)^2 + (x-t)^2$$

10) If α, β, γ are [are angles] of 3 pts. on ellipse, the normals at which are concurrent then

$$\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$$

(Proof): In prev. method, we had

$$\begin{aligned} z_1 z_2 &= 0 \Rightarrow e^{i(\alpha+\beta)} + e^{i(\beta+\gamma)} + e^{i(\gamma+\alpha)} \\ &\quad + e^{i(\delta+\alpha)} + e^{i(\alpha+\gamma)} + e^{i(\gamma+\delta)} = 0 \\ \Rightarrow \delta_{(\alpha+\beta)} + \delta_{(\beta+\gamma)} + \delta_{(\gamma+\alpha)} + \delta_{(\delta+\alpha)} + \delta_{(\alpha+\gamma)} + \delta_{(\gamma+\delta)} &= 0 \end{aligned}$$

By prev. prop., $\alpha+\beta+\gamma+\delta = (2n+1)\pi$

$$(-) \Rightarrow \delta_{(\alpha+\beta)} = \sin((2n+1)\pi - (\gamma+\delta))$$

$$\Rightarrow \delta_{(\alpha+\beta)} = \delta_{(\gamma+\delta)}$$

$$\pi(1+n\pi) = \beta + \gamma + \alpha + \delta$$

$$\delta_{(\alpha+\beta)} + \delta_{(\beta+\gamma)} + \delta_{(\gamma+\alpha)} + \delta_{(\alpha+\delta)} + \delta_{(\beta+\delta)} + \delta_{(\gamma+\delta)} = 0$$

$$\Rightarrow (2) [\delta_{(\alpha+\beta)} + \delta_{(\beta+\gamma)} + \delta_{(\gamma+\alpha)}] = 0$$

$$\Rightarrow \delta_{(\alpha+\beta)} + \delta_{(\beta+\gamma)} + \delta_{(\gamma+\alpha)} = 0$$

In general ellipse,

$$r = \frac{d}{1 + e \cos \theta} \Leftrightarrow d + e r \cos \theta + e m = \frac{d}{e}$$

$$A \neq B \Leftrightarrow \frac{(\text{Dist. from})^2}{\text{Major Axis}} + \frac{(\text{Dist. from})^2}{\text{Minor Axis}} = 1$$

$$\frac{A^2}{(s_r - s_d)} + \frac{B^2}{(s_o - s_r)} = m$$

$$A \neq B \Leftrightarrow \frac{(\text{Dist. from})^2}{\text{Minor Axis}} + \frac{(\text{Dist. from})^2}{\text{Major Axis}} = 1$$

$$\frac{A^2}{(s_o - s_d)} + \frac{B^2}{(s_r - s_o)} = m$$

$$(1 - \frac{P_1 H}{P_2 H} + \frac{x_1}{x_2}) = \frac{(P_1 H)^2}{P_2 H} \Leftrightarrow (P_1 H)^2 = P_2 H (P_1 H - x_1)$$

$$2(1 - \frac{P_1 H}{P_2 H} + \frac{x_1}{x_2}) = \frac{(P_1 H)^2}{P_2 H} + \frac{(P_2 H)^2}{P_1 H}$$

$$0 = \frac{(P_1 H - x_1 P_1 H)}{P_2 H} + \frac{(x_2 P_2 H - P_2 H)}{P_1 H} \Leftrightarrow$$

$$0 = \frac{P_1 H - x_1 P_1 H}{P_2 H} + \frac{x_2 P_2 H - P_2 H}{P_1 H}$$

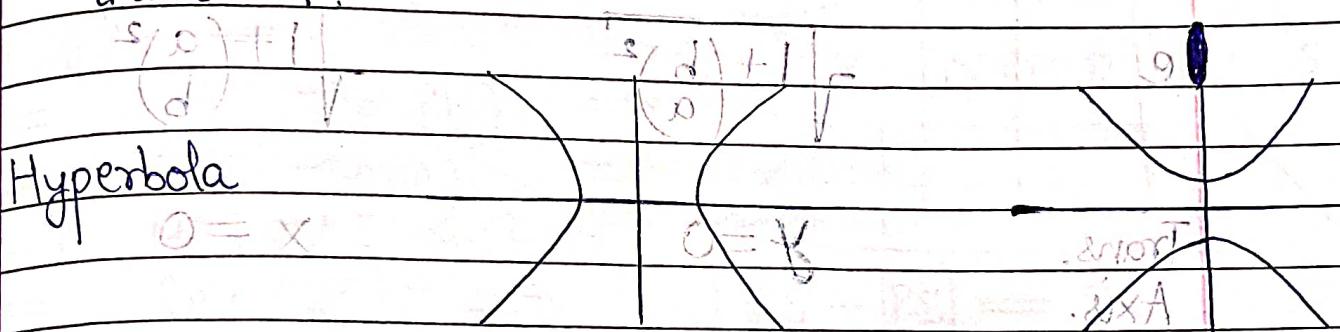
$$\frac{s_1}{s_2} + \frac{x_1}{x_2} = \frac{s_2}{s_1} + \frac{x_2}{s_1} \Leftrightarrow$$

$$s_1 + s_2 = s_1 x_2 + s_2 x_1 \Leftrightarrow$$

Hyperbola

General Defn -

Locus of a pt. of g.t. ratio of its dist. of a fix. pt. & a fix. line is constant & greater than 1.



$$\text{Eqn} \quad O = V \quad x^2/a^2 - y^2/b^2 = 1 \quad y^2/b^2 - x^2/a^2 = 1$$

Centre $(0,0)$

$(0,0)$

Foci $(-ae, 0); (ae, 0)$ $(0, -be); (0, be)$

Directrices $x = (-a/e); x = a/e$ $y = (-b/e); y = b/e$

Vert. axis, horiz. axis, and horiz. axis

Vertices $(\pm a, 0); (0, \pm b)$

Transverse Axis

$$-2ad/V - (a/x) : \text{abdry} 2b/V$$

Ax

$$(1) = (d/b) - (a/x) : \text{abdry H. abd}$$

Conjugate Axis

$$2b$$

$$10 \quad 2a$$

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$$e = \sqrt{1 + \left(\frac{-ve}{+ve}\right)^2}$$

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LR.

$$2b^2/a$$

$$2a^2/b$$

L.R.

$$x = (-ae); x = ae$$

$$y = (-be); y = be$$

Endpts
of L.R.

$$(-ae, -b^2/a)$$

$$(-ae, b^2/a)$$

$$(ae, -b^2/a)$$

$$(ae, b^2/a)$$

$$(-a^2/b, -be)$$

$$(a^2/b, -be)$$

$$(-a^2/b, be)$$

$$(a^2/b, be)$$

e

$$\sqrt{1 + \left(\frac{b}{a}\right)^2}$$

$$\sqrt{1 + \left(\frac{a}{b}\right)^2}$$

Trans.

$$y = 0$$

$$x = 0$$

Axis.

Conj.
Axis.

$$x = 0$$

$$y = 0$$

$$(0,0)$$

$$(0,0)$$

$$(0,0)$$

$$(ad, 0) : (ad, 0)$$

$$(0, ad) : (0, ad -)$$

Conjugate Hyperbola

$d = p$: $3d = p$ $3p = x$: $(3p -) = x$

Hyperbola obtained by interchanging trans. conj. axis of (given) hyperbola

$$\text{Hyperbola} : (x/a)^2 - (y/b)^2 = 1$$

$$\text{Conj. Hyperbola} : (x/a)^2 - (y/b)^2 = (-1)$$

Second Defⁿ of Hyperbola

Locus of pt. s.t. diff. of its dist. from two fix pts. is const. & this const. is smaller than dist. b/w the fix pts.

$$\text{Proof: } PS = e \cdot PL$$

$$PS' = e \cdot PL'$$

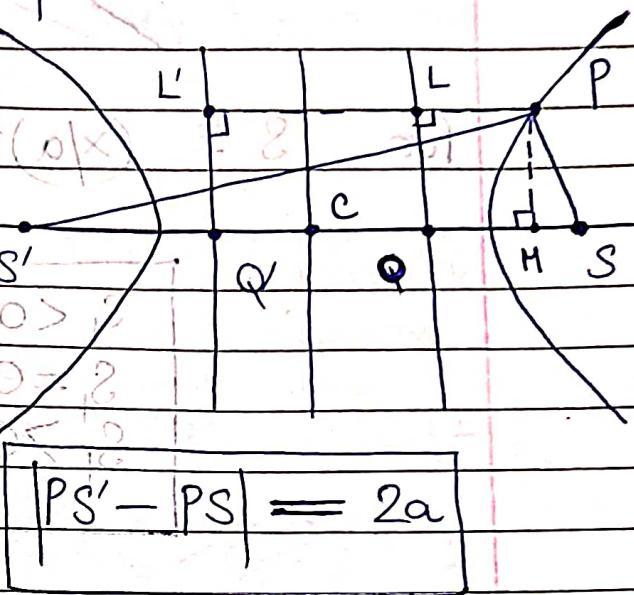
$$\Rightarrow (PS - PS')$$

$$= |e \cdot (PL - PL')| \Leftrightarrow 0 > \beta$$

$$= e \cdot (MQ - MQ') \Leftrightarrow 0 = \beta$$

$$= e \cdot (MC + CQ - MC + CQ') \Leftrightarrow 0 < \beta$$

$$= |2e \cdot CQ' \Rightarrow |$$



$$|PS' - PS| = 2a$$

Parametric Coordinates

for horiz. hyperbola,

$$P \equiv (a \sec \theta, b \tan \theta)$$

$$Q \equiv (a \cos \theta, b \sin \theta)$$

$$QM \equiv x \cos \theta + y \sin \theta = a$$

$$\Rightarrow M(a \sec \theta, 0)$$

for non-horiz. Hyperbola

(gen) no fix foci

Impractical

oblique

angle

eccentric

angle

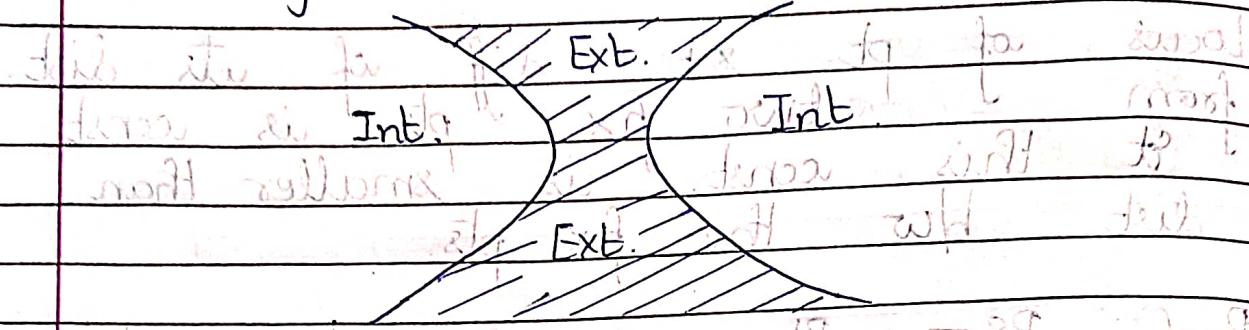
Auxiliary

circle

Eccentric

angle

Post. of Pt. w.r.t. to Ellipse



$$\text{For } S \equiv \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 = 29$$

$s_1 < 0$	\Leftrightarrow	Pt. OUTSIDE
$s_1 = 0$	\Leftrightarrow	Pt. ON-CURVE
$s_1 > 0$	\Leftrightarrow	Pt. INSIDE
$ s_1 = 29 - 29 $	\Leftrightarrow	$ 29 - 29 = 0$

Formulae

1) Tangent

(at pt. on Curve)

$$T = \sqrt{1 - \frac{s^2}{a^2}}$$

2) Tangent

(pt. outside curve)

$$S = T^2$$

3) Chord of Contact -

$$T = 0$$

1) Chord with given midpt. + T = S

$$T = S$$

5) Eqⁿ of Tangent (at pt. on curve) $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$

$$(x_1, y_1) \Rightarrow \left(\frac{xx_1}{a^2}\right) - \left(\frac{yy_1}{b^2}\right) = 1$$

~~Face~~ (aS₀, b₀) $\Rightarrow \left(\frac{xS_0}{a}\right) - \left(\frac{y_{b_0}}{b}\right) = 1$

6) Eqⁿ of Normal -

(at pt. on curve) $\left(\frac{a^2x}{x_1}\right) + \left(\frac{b^2y}{y_1}\right) = a^2 + b^2$

$$(x_1, y_1) \Rightarrow \cancel{\left(\frac{x-x_1}{a^2}\right)} = \cancel{\left(\frac{y-y_1}{b^2}\right)}$$

~~Face~~ (aS₀, b₀) $\Rightarrow \left(\frac{ax}{S_0}\right) + \left(\frac{by}{b_0}\right) = a^2 + b^2$

7) Tangent with given slope -

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

(8) $\text{Cof} = g \Leftrightarrow (g = d) \quad (d = S_0)$

8) Director

$$x^2 + y^2 = \pm a^2 + b^2$$

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9) Chord joining $P(\alpha)$ & $Q(\beta)$ = $b \omega$

$$\left(\frac{x}{a} \right) \cos\left(\frac{\alpha-\beta}{2}\right) - \left(\frac{y}{b} \right) \sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

$$1 = (\omega \omega) - (xx) \Leftarrow (N, \infty)$$

Asymptotes

The straight line, to which the tangent to the curve tends as the pt. of contact tends to approach $x = \infty$. $y = \frac{b}{a}x$

To find eqn of asymptotes of

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right) \rightarrow A^2 x$$

$$\text{Eqn of asymptote} = \pm x \Rightarrow x = \pm \frac{b}{a}x$$

Let $(\pm x_0, \pm y_0)$ be a pt. on it.

$$PT \equiv \left(\frac{x}{a} \right)_0 - \left(\frac{y}{b} \right)_0 = 1$$

As $\theta \rightarrow \pm \pi/2$, $P \rightarrow \infty$.

$$PT \equiv \left(\frac{x}{a} \right)_0 - \left(\frac{y}{b} \right)_0 = \infty$$

$$\left(\frac{x}{a} \right)_0 - \left(\frac{y}{b} \right)_0 = \infty \quad (\theta \rightarrow \pm \pi/2)$$

$$\left(\frac{x}{a} \right)_0 - \left(\frac{y}{b} \right)_0 = 0 \Rightarrow y = \left(\frac{\pm b}{a} \right)x$$

Eqn of Asymptotes

Joint Eqⁿ of Asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$0 = l + \frac{px}{a} - \frac{y}{b} : \text{graph A}$$

1) In general, asymptotes of a given hyperbola pass thru its centre.

2) Angle b/w Asymptotes = $\tan^{-1} \left| \frac{2ab}{a^2 - b^2} \right|$

3) Angle b/w asymptotes of Rect. Hyperbola = 90°

4) Eqⁿ of asymptotes is same for a given hyperbola & its conj. hyperbola.

5) Eqⁿ of hyperbola & joint eqⁿ of asymptotes differ. only by a const. term.

$$S = V - VS \quad \text{const. diff. and brf}$$

6) Any line parallel to Γ = asymptote of hyperbola would meet the curve only at 1 pt.

$$0 = (\Gamma - p + xS)(S - V - VS) : \text{graph A}$$

7) Const. terms of eqⁿ of hyperbola, joint eqⁿ of asymptotes & eqⁿ of conj. hyperbola make ind. A.P.

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Q) Find asymptotes of $xy - 3y + 2x = 0$

A) Asymptotes : $xy - 2x - 3y + \lambda = 0$

$$\Delta = 0 \Rightarrow \begin{vmatrix} 0 & 1/2 & (-1) \\ 1/2 & 0 & (-3/2) \\ (-1) & (-3/2) & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & -3 \end{vmatrix} = 0$$

$$0 = -2 - 3 \Rightarrow -2 - 3 = 2\lambda \Rightarrow \lambda = -1$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ -2 & -3 & 2\lambda - 6 \end{vmatrix} = 0$$

$$\Rightarrow (2\lambda - 6) - 6 = 0 \Rightarrow \lambda = 6$$

Q) Find hyp. with asymp. $2x - y = 3$

st $3x + y = 7$ which passes thru $(1, 1)$

A) Asymp : $(2x - y - 3)(3x + y - 7) = 0$

$$\Rightarrow 6x^2 - y^2 - xy - 23x + 4y = (-21)$$

Since a hyp. thru $(1, 1)$

\Rightarrow Hyp :

$$6x^2 - y^2 - xy - 23x + 4y = 0$$

(+15)

Q) Asymp. of hyp. with centre (1, 2)
 are \parallel to $2x + 3y = 0$ & $3x + 2y = 0$.
 If find hyp. thru its eqn.

A) Asymp : $(2x + 3y - 8)(3x + 2y - 7) = 0$

$$\Rightarrow 6x^2 + 6y^2 + 13xy - 38x - 37y = (-56)$$

Since hyp. thru (5, 3).

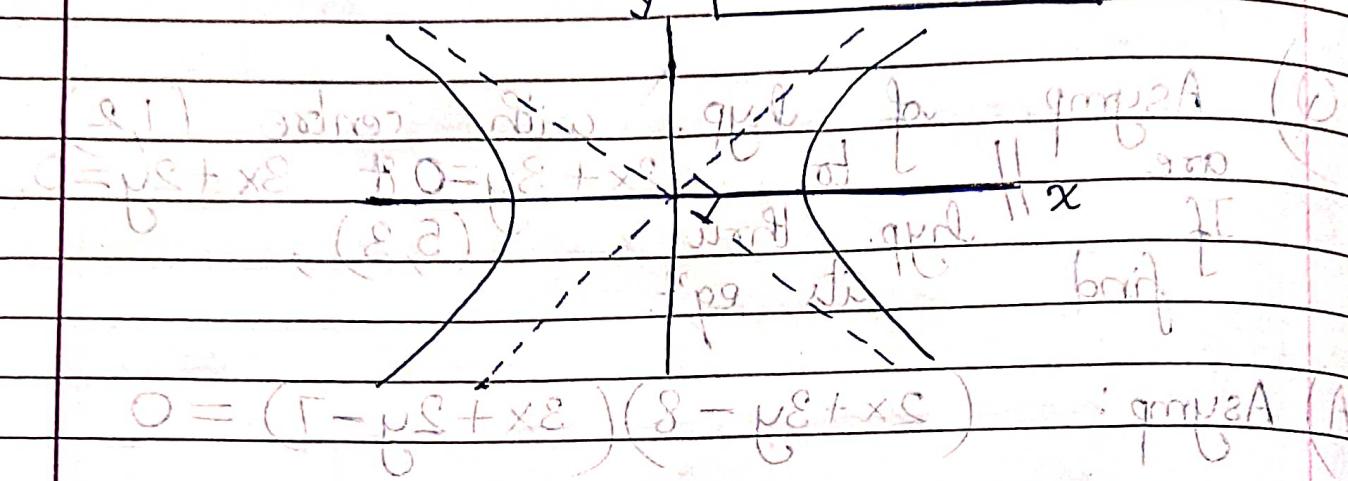
$$\Rightarrow \text{Hyp : } 6x^2 + 6y^2 + 13xy - 38x - 37y = 98$$

$$3(jV + X) = (jP + X)$$

Rectangular Hyperbola

Eqn of Rect. Hyp:

$$x^2 - y^2 = a^2$$

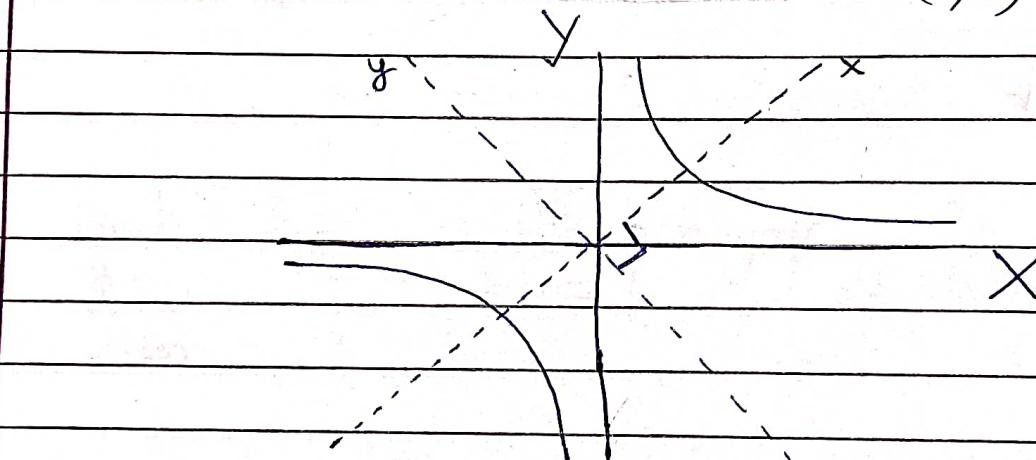


Most popular form:

$$xy = c^2$$

where $c^2 = (a^2)$

In fig above keep hyp. fix: $\sqrt{a^2}$
rotate axes about (0,0) by $-\frac{\pi}{4}$



Using, $(x+yi) = (x+Yi)c$

1) Parametric Coor:

$$P \equiv (vt), \frac{v}{t}$$

$$2) \text{Tangent at } P \equiv \left(x(vt) + y(vt) \right) = v^2$$

$$\Rightarrow \text{Slope of Tangent} = \left(-\frac{1}{t^2} \right)$$

$$3) \text{Normal at } P \equiv \left(y - \frac{v}{t} \right) = t^2(x - vt)$$

$$\Rightarrow \text{Slope of Normal} = t^2$$

Q) Let A, B, C be vertices of \triangle , with
 90° at B. P.T. tangent at B
 will do ~~hypotenuse~~ AC \perp

$$A \equiv (vt_1, \frac{v}{t_1}) \quad C \equiv (vt_2, \frac{v}{t_2})$$

$$B \equiv (vt_3, \frac{v}{t_3}) \quad (x - \mu) \quad \therefore \text{to form } \triangle$$

$$\angle ABC = 90^\circ \Rightarrow \left(\frac{1/t_2 - 1/t_3}{t_2 - t_3} \right) \left(\frac{1/t_1 - 1/t_3}{t_1 - t_3} \right) = (-1) \Rightarrow \frac{t_1^2}{t_2 t_3} = (-1)$$

$$(x - \mu) \left(\frac{1}{t_2 t_3} \right) = (x - \mu) \quad \therefore \text{to form } \triangle$$

$$\text{Slope of } AC \equiv (-1/t_2 t_3) \rightarrow \text{Tangent } \perp \text{ to } AC$$

$$\text{Slope of tangent at } B = (-1/t_1^2) \rightarrow$$

Q) 1) Show that normals to $xy = c^2$ at pt. 'E' meets curve again at pt. 'E₁' then $t_1 t^3 = -1$

\star 2) Normals sat P, Q, R on $xy = c^2$ intersect at pt. of hyp. is centroid of $\triangle PQR$.

A) 1) $(vt, vc/t)$ \Rightarrow Normal: $(y - vc) = t^2(x - vt)$

Thru $(vt_1, vc/t_1) \Rightarrow (c)\left(\frac{1}{t_1} - 1\right) = t^2(c)(t_1 - v)$

$$\Rightarrow t_1 t^3 = -1$$

$s_2 = px$

2) $P \equiv (vt_1, vc/t_1)$, $Q \equiv (vt_2, vc/t_2)$, $R \equiv (vt_3, vc/t_3)$

Normal at P: $(y - vc/t_1) = t_1^2(x - vt_1)$

Normal at Q: $(y - vc/t_2) = t_2^2(x - vt_2)$

Normal at R: $(y - vc/t_3) = t_3^2(x - vt_3)$

$\Rightarrow t_1 t_2 t_3 = -1$ \Rightarrow G.A. for $\triangle PQR$

If this pass thru (h, k) ,

$$ct^4 - kt^3 + kt - c = 0 \quad t_1, t_2, t_3, t_4$$

Now,

$$t_1 t_2 = (t_4) \circlearrowleft + (t_1 t_2 + t_2 t_3 + t_3 t_4) = 0$$

$$\sum t_1 t_2 t_3 = (t_4)(t_1 t_2 + t_2 t_3 + t_3 t_1) + t_1 t_2 t_3 = 0 \quad (A)$$

$$t_4 = (-1)(t_1 t_2 t_3) \circlearrowleft = (-1)^2 + 2$$

Let all these meet at $(ct_4, c/t_4)$.

$$\text{By prev. Q; } t_4 t_1^3 = (-1)$$

$$t_4 t_2^3 = (-1) \Leftarrow$$

$$t_4 t_3^3 = (-1)$$

$$\Rightarrow t^3 = \frac{(-1)}{t_4} \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix}$$

$$\sum(t_i) = 0$$

$$\sum(1/t_i) = 0$$

$$0 + \boxed{\text{Centroid} \equiv (0,0)} \Leftarrow$$

$$\text{Centroid} \equiv (0,0)$$

$$x_2 - x_1 \Leftarrow (x_1 - x_2)$$

Q) P.T. the locus of a pt. which moves s.t. sum of slopes of normals from it to $xy = c^2$ is equal to the sum of coordinates of foot of normals is a parabola

A) Let pt be (h, k) . Let feet of normals be $(vt, \frac{v}{t})$; $(vct_3, \frac{v}{t_3})$; $(vct_4, \frac{v}{t_4})$.

$$\text{(Normal)} \Rightarrow |(y - \frac{v}{t})| = (vt^2)(\cancel{v} - vct)$$

$$\Rightarrow vt^4 - vht^3 + kt - v = 0$$

$$\text{We need, } t_1^2 = \cancel{\sum(\frac{1}{t_i})}$$

$$\Rightarrow 0 = (\cancel{t_1})^2 = \cancel{t_1} + 2 \sum t_1 t_2$$

$$\Rightarrow (h/c)^2 = \cancel{t_1} + 0$$

$$\Rightarrow \begin{pmatrix} h^2 \\ c^2 \end{pmatrix} = \begin{pmatrix} -k/c \\ -v/c \end{pmatrix} \Rightarrow$$

$$h^2 = c^2 k$$

Intersection of Circle & Rect. Hyp.

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and a rect. hyp. whose centre is (α, β) , intersecting each other at 4 pts.

$A(x_1, y_1)$; $B(x_2, y_2)$; $C(x_3, y_3)$; $D(x_4, y_4)$

Then,

$$\frac{\sum x_i}{4} = (\alpha + (-g))$$

$$\frac{\sum y_i}{4} = (\beta + (-f))$$

i.e. $\boxed{\text{A.M. of } x_i = \text{A.M. of } y_i}$ (Centres)

Proof: Let no. of pts. $= n$. Then $x^2 + y^2 + 2gx + 2fy + c = 0$ & hyp. $: xy = \lambda$.

$$\cap \Rightarrow x^2 + \lambda^2/x^2 + 2gx + 2f\lambda/x + c = 0$$

$$\Rightarrow x^4 + 2gx^3 + cx^2 + 2f\lambda x + \lambda^2 = 0$$

$$\Rightarrow \boxed{\sum x_i = (-2g)} \quad \text{Similarly doing for } y \text{ gives}$$

$$\Rightarrow \boxed{\sum y_i = (-2f)}$$

Q) A rect. hyp. with centre in C is inscribed by a circle with radius 'r' in 4 pts

$$P, Q, R, S: CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2$$

$$P, Q, R, S: CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2$$

Q) A circle with centre $(3\alpha, 3\beta)$ and a variable radius cuts $x^2 - y^2 = 9a^2$ at P, Q, R, S

P.T. centroid S of $\triangle PQS$ is

$$(x - 2\alpha)^2 - (y - 2\beta)^2 = a^2$$

Q) 5 pts. are given on a line of length a . 4 of these are made thru 4 of these, one at a time

P.T. centre of hyp. lies on a circle of radius $a/2$

A) Let $O = x + yK$ be the hyp. $O = x^2 + y^2$ $x^2 + y^2 = r^2$ Let $C = (\alpha, \beta)$

Let $xy = \lambda^2$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let pts. of A be $(\lambda t_i, \lambda |b_i|)$

$$\Rightarrow \text{Req.} = (\lambda^2) \left(\sum t_i^2 + \sum (|b_i|^2) \right)$$

$$\text{Now, } \frac{\lambda^2 t^2 + g^2}{t^2} + 2g\lambda t + 2f\lambda + c = 0 - t_i$$

$$\Rightarrow (\lambda^2) t^4 + (2g\lambda) t^3 + (c) t^2 + (2f\lambda) t + \lambda^2 = 0$$

$$\sum t_i^2 = (\sum t_i)^2 - 2 \sum t_i t_j \quad \Leftarrow t_i$$

$$t^2 = \left(\frac{-2g\lambda}{\lambda^2}\right)^2 - 2\left(\frac{c}{\lambda^2}\right) \Rightarrow \sum t_i^2 = \left(\frac{2}{\lambda^2}\right)(2g^2 - c)$$

$$\sum (|b_i|)^2 = (\sum (|b_i|))^2 - 2 \left(\sum (|b_i| t_j) \right) \quad \text{A}$$

$$= \left(\frac{-2f\lambda}{\lambda^2}\right)^2 - 2\left(\frac{c}{\lambda^2}\right) \Rightarrow \sum (|b_i|)^2 = \left(\frac{2}{\lambda^2}\right)(2f^2 - c)$$

Now and by adding we get

$$\Rightarrow \text{Req.} = (2) \left(2g^2 + 2f^2 - 2c \right) \Rightarrow \text{Req.} = 4r^2$$

$$(2g)(2f) = \left[\left(\sum t_i \right) \left(\sum |b_i| \right) - \sum t_i |b_i| \right] - \text{work}$$

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A)

Let n pts. be (x_i, y_i)

$$\Rightarrow \left(\begin{matrix} G \cdot f \\ \Delta PQ's \end{matrix} \right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Now, $\sum x_i = 2(3\alpha) = 6\alpha$

$$\sum y_i = 2(3\beta) = 6\beta$$

$$\Rightarrow G = \left(2\alpha - \frac{x_4}{3}, 2\beta - \frac{y_4}{3} \right)$$

$\Rightarrow G$ lies on $(x-2\alpha)^2 + (y-2\beta)^2 = a^2$

A)

Let pts. be $ae^{i\theta_k}, (k \in \{1, \dots, 5\})$

$$\text{By prop., } \left(\frac{z_5}{2} \right) \equiv \left(\frac{a}{2} \right) \left(\sum_{k=1}^4 e^{i\theta_k} \right)$$

where z_5 is centre of hyp. thru $ae^{i\theta_k}$

Centre of circle = O .

$$\text{Now, } - \left[z_5 - \left(\frac{a}{2} \right) \left(\sum_{k=1}^4 e^{i\theta_k} \right) \right] = \left(\frac{a}{2} \right) (e^{i\theta_5})$$



Centres of hyp. lie on \odot

$$\left| z - \left(\frac{a}{2} \right) \sum_{k=1}^5 (e^{i\theta_k}) \right| = \left(\frac{a}{2} \right)$$