Chapter 1: The Basics

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Vectors and Coordinate Systems

- Mathematicians define vectors as a set of numbers which transform in a particular way when the coordinate system is changed
- Physicists think of vectors as physical quantities with which a direction can be attached

Properties of vectors (continued)

- Consider two vectors B and C

- If these two vectors have the same length, and the same direction, they are equal

$$\mathbf{B} = \mathbf{C}$$

- If length of a vector is one unit, it is called a unit vector.
- The unit vector associated with a vector A is defined as

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|},$$

where $|\mathbf{A}|$ is the length (magnitude) of the vector. We will also use the notation $A = |\mathbf{A}|$

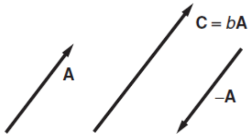
Properties of vectors....

- So that

$$\mathsf{A} = A\hat{\mathsf{A}}.$$

Algebraic operations on vectors

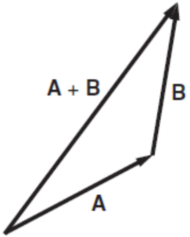
- One can multiply a vector by a scalar (number)



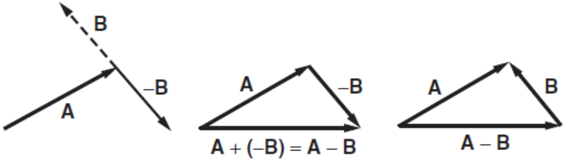
- The operation leads to a change in magnitude of the original vector if $b > 0$
- Will flip the direction as well, if $b < 0$
- Obviously, for $b = -1$, just a direction flip occurs

Adding and Subtracting Vectors

- Given two vectors A and B, one can add the two, which can be shown pictorially as



- Similarly, one can subtract one from the other



Algebraic properties of vectors

- Vector addition is commutative

$$A + B = B + A$$

- Associative law holds with respect to vector addition and scalar multiplication

$$\begin{aligned} A + (B + C) &= (A + B) + C \\ c(dA) &= (cd)A \end{aligned}$$

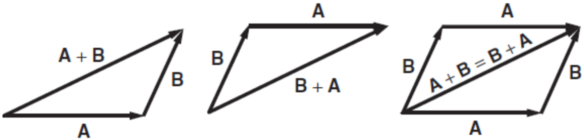
- Distributive law also holds

$$\begin{aligned} c(A + B) &= cA + cB \\ (c + d)A &= cA + dA \end{aligned}$$

above c and d are scalars.

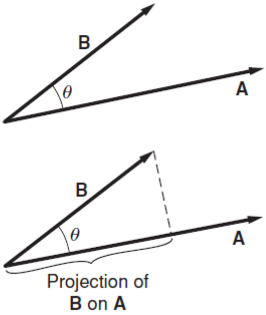
Algebraic properties contd...

- Pictorial proof of commutative law is as follows



Multiplication of vectors

- Can one also multiply two vectors?
- Yes, and in two possible ways!
- In one case, the end result is a scalar, so the product is called "scalar product" or "dot product".
- In the other case, the end result is a vector, and the product is called "cross product".
- Pictorially, the dot product can be shown as



Multiplication of vectors, contd...

- Mathematically it is defined as

$$\mathsf{A}\cdot\mathsf{B} = AB\cos\theta,$$

where θ is the angle between two vectors.

- Which can also be stated as

$$\begin{aligned}\mathsf{A}\cdot\mathsf{B} &= \text{projection of A on B}\\ &= \text{projection of B on A}\end{aligned}$$

- Naturally

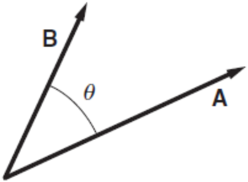
$$\mathsf{A}\cdot\mathsf{A} = AA\cos 0^{\circ} = A^2 = |\mathsf{A}|^2.$$

- This helps as define $|\mathsf{A}|$ as

$$A = |\mathsf{A}| = \sqrt{\mathsf{A}\cdot\mathsf{A}}.$$

Vector Cross Product

- Consider two vectors A and B, with an angle θ between them, as shown below



- The cross product of the two vectors yields a third vector C (say), and the operation is mathematically denoted as

$$\mathrm{C} = \mathrm{A} \times \mathrm{B}.$$

- The magnitude of C is given by

$$C = AB \sin\theta.$$

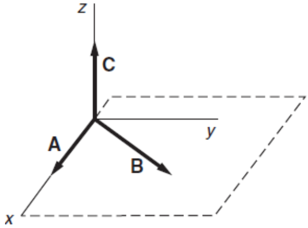
- And the direction of C is perpendicular to both A and B, given by the right-hand rule
- Above θ is taken to be the angle which is less than π.

Cross product continued...

- Easy to verify, that the cross product of a vector with itself is null vector

$$A \times A = 0$$

- As a matter of fact, cross product between any two parallel $(\theta = 0)$ and anti-parallel $(\theta = \pi)$, will always be zero.
- The direction of the cross product can be understood from the following figure



- A consequence of right-hand rule is

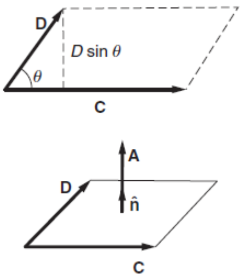
$$A \times B = -B \times A$$

Examples of dot and cross products in physics

- Work done W, due to a force F, causing displacement d, is given by
$$W = \mathsf{F} \cdot \mathsf{d}$$
- Torque $\boldsymbol{\tau}$, due to a force F, applied at a point whose position vector with respect to the reference point is r, is given by
$$\boldsymbol{\tau} = \mathsf{r} \times \mathsf{F}.$$
- Force F acting on a charged particle with charge q, moving with velocity v, exposed to a magnetic field B, is given by
$$\mathsf{F} = q\mathsf{v} \times \mathsf{B}.$$

Area as a cross product

- Even the surface area can be defined as a vector, in terms of a cross product
- Consider the parallelogram shown below



- Its area can be written as

$$
\begin{aligned}
A &= \text{base} \times \text{height} \\
&= CD \sin\theta \\
&= |\mathbf{C} \times \mathbf{D}|
\end{aligned}
$$

Area as vector....

- The direction is chosen to be one of the outward drawn normals $\hat{n}$, so that
$$A = |C \times D|\hat{n}.$$
- There is an ambiguity in the choice of $\hat{n}$, because there are two possibilities
- Choice doesn't matter as long as we are consistent with it

Coordinate Systems: Cartesian Coordinates

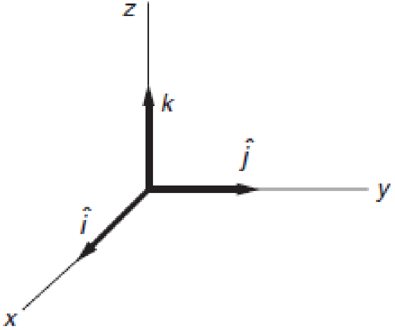
- So far we have described vectors in abstract terms
- But, we have to come up with a way of representing real-life vectors such as velocity, force, torque etc. in numerical terms
- For the purpose we need to define a coordinate system
- Then all the vectors can be represented with respect to the chosen coordinate system
- The simplest coordinate system in 3D is the Cartesian system
- Which is defined by three mutually perpendicular directions x, y, and z.
- A vector A, in this system, is defined by three components A_x, A_y, and A_z.

Cartesian Coordinate System contd.

- Symbolically we can represent the vector A, in terms of its components, as

$$\mathrm{A} = (A_x, A_y, A_z).$$

- We can also define a set of unit vectors $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$, in x, y, and z directions, respectively



Cartesian coordinates....

- Unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, are called basis vectors of the 3D Cartesian coordinate system
- Now we can express A, as a linear combination of these basis vectors

$$\mathrm{A} = A_x\hat{\mathrm{i}} + A_y\hat{\mathrm{j}} + A_z\hat{\mathrm{k}}.$$

- Because these basis vectors are perpendicular to each other, they satisfy

$$\begin{aligned} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} &= 1 \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} &= 0 \\ \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} &= 0 \\ \hat{i}\times\hat{j} = \hat{k};\ \hat{j}\times\hat{k} = \hat{i};\ \hat{k}\times\hat{i} &= \hat{j} \end{aligned}$$

- Using this, it is easy to verify the following

$$\mathrm{A}+\mathrm{B}=(A_x+B_x)\hat{\mathrm{i}}+(A_y+B_y)\hat{\mathrm{j}}+(A_z+B_z)\hat{\mathrm{k}}=\mathrm{B}+\mathrm{A}$$
$$\mathrm{A}\cdot\mathrm{B}=A_xB_x+A_yB_y+A_zB_z$$
$$|\mathrm{A}|=\sqrt{\mathrm{A}\cdot\mathrm{A}}=\sqrt{A_x^2+A_y^2+A_z^2}$$

- Similarly

$$\mathrm{A}\times\mathrm{B}=\begin{vmatrix}\hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

Position Vector...

- Position vector of a point P, with Cartesian coordinates (x, y, z), is given by

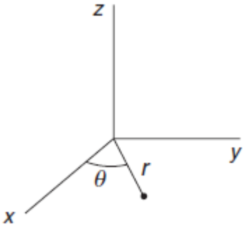
$$\mathrm{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$$

- Vector connecting two points (x, y, z), and $(x + dx, y + dy, z + dz)$, which are infinitesimally apart from each other, is given by

$$\mathrm{dr} = dx\hat{\mathrm{i}} + dy\hat{\mathrm{j}} + dz\hat{\mathrm{k}}$$

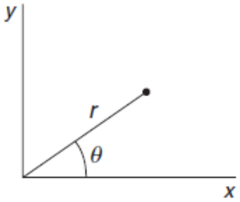
Coordinate System: Plane Polar Coordinates

- Whenever we are dealing with a system with circular symmetry, such as a particle executing circular motion, it is much more convenient to use a different set of coordinates called "Plane Polar Coordinates.



- It is a 2D coordinate system equivalent to Cartesian 2D:(x, y)
- Location of a point specified by (r, θ)
- r is distance from the origin
- θ is the angle which line joining the point to the origin, makes with the x axis.

Plane polar coordinates contd.

- 

  Clearly

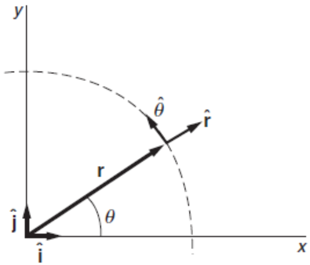
$$x = r\cos\theta$$
$$y = r\sin\theta$$

- Easy to deduce from above

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Plane polar coordinates ...

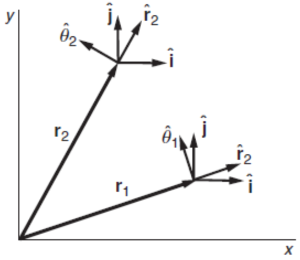
- Unit vectors denoted as $\hat{r}$ and $\hat{\boldsymbol{\theta}}$ are shown below



- Direction of $\hat{r}$ is the one in which r increases, but θ is held fixed.
- Similarly $\hat{\boldsymbol{\theta}}$ is in the direction in which θ increases, but r is held fixed
- Yet $\hat{r}$ and $\hat{\boldsymbol{\theta}}$ are mutually perpendicular, just like $\hat{i}$ and $\hat{j}$.
- Also note that unlike Cartesian coordinates, (r, θ) have different dimensions.
- r has dimensions of length, while θ is dimensionless

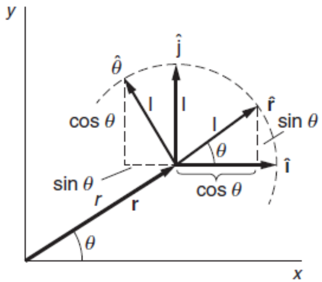
Plane Polar Coordinates....

- In Cartesian coordinates, directions of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are fixed in space, and same everywhere
- This is not true in plane polar coordinates



Relation between plane polar and Cartesian unit vectors

- Consider the figure below



- From above, it is easy to derive the relationship between two sets of unit vectors

$$\hat{\mathrm{r}} = \cos\theta\hat{\mathrm{i}} + \sin\theta\hat{\mathrm{j}}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathrm{i}} + \cos\theta\hat{\mathrm{j}}$$

Polar-Cartesian Relationship contd...

- And, the inverse relationship

$$\hat{\mathrm{i}} = \cos\theta\,\hat{\mathrm{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$
$$\hat{\mathrm{j}} = \sin\theta\,\hat{\mathrm{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$$

Polar-Cartesian Comparison

- Position vector of an arbitrary point P in two coordinate systems is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$
$$\mathbf{r} = r\hat{\mathbf{r}}$$

- Infinitesimal displacement $d\mathbf{r}$ is given by

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$$
$$d\mathbf{r} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}}$$