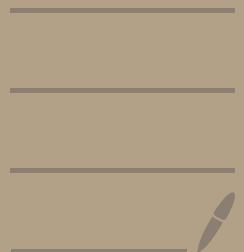


L19 - 15/10/2024



Infinity

The irrationality of $\sqrt{2}$

Thm The eqⁿ $x^2 - 2y^2 = 0$ does not have non-trivial solⁿs over the integers

$$\text{i.e. } \nexists (x_0, y_0) \in \mathbb{Z}^2 \setminus \{(0,0)\} \text{ s.t. } \underbrace{x_0^2 - 2y_0^2 = 0}_E$$

The proof introduces 2 useful & powerful methods.

- Proof by contradiction
- Method of infinite descent

Key steps

1. Define the notion of size of a sol^n non-(-ve) int.
2. for any $k \in \mathbb{Z}_{\geq 0}$, the no. of int. sol^n of E with $\text{size} \leq k$ is finite
3. Suppose E has a non-t'vl int. sol^n . Then, construct a non-t'vl int. sol^n of strictly smaller size.

Details

1. $N((x_0, y_0)) = x_0^2 + y_0^2$

2. Given k , s.t. $x_0^2 + y_0^2 \leq k$
then $x_0^2 \leq k$ & $y_0^2 \leq k$

So, $S_k = \{ (x, y) \in \mathbb{Z}^2 : x^2 + y^2 \leq k \}$

$$S_k \subseteq \{ (x, y) \in \mathbb{Z}^2 : \max\{|x|, |y|\} \leq \sqrt{k} \}$$

But, the set has size $\leq (2\sqrt{k})^2 = \underbrace{4k}_{\text{finite}}$

3. Suppose (x_0, y_0) is a non-trivial solⁿ to E

$$\text{Then, } x_0^2 - 2y_0^2 = 0$$

$$\Rightarrow x_0^2 = 2y_0^2$$

$\therefore x_0^2$ is even

$\therefore x_0$ is even i.e. $x_0 = 2z_0$

$$(2z_0)^2 - 2y_0^2 = 0$$

$$\Rightarrow 4z_0^2 - 2y_0^2 = 0$$

$$\Rightarrow y_0^2 - 2z_0^2 = 0$$

$\therefore (y_0, z_0)$ is a solⁿ of E

$$\begin{aligned} \text{But, } N((y_0, z_0)) &= y_0^2 + z_0^2 \\ &< y_0^2 + x_0^2 = N((x_0, y_0)) \end{aligned}$$