L5 - 16/08/2024

Set

Axioms

1. Sets are objects

If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

2. Empty Set ($\emptyset$)

There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

3. Singleton sets & pair sets

If a is an object, then there exists a set {a} whose only element is a, i.e., for every object y, we have $y \in \{a\}$ iff $y=a$; we refer to {a} as the singleton set whose element is a.

Furthermore, if a and b are objects, then there exists a set {a, b} whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.

4. Pairwise Union

Given any two sets A, B, there exists a set A ∪ B, called the union A ∪ B of A and B, whose elements consists of all the elements which belong to A or B or both. In other words, for any object x,

$$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$

5. Axiom of specification

Let A be a set, and for each x ∈ A, let P(x) be a property pertaining to x (i.e., P(x) is either a true statement or a false statement). Then there exists a set, called {x ∈ A : P(x) is true} (or simply {x ∈ A : P(x)} for short), whose elements are precisely the elements x in A for which P(x) is true. In other words, for any object y,

$$y \in \{x \in A : P(x)\} \iff (y \in A \ \& \ P(y))$$

6. Axiom of Replacement

Let A be a set. For any object x ∈ A, and any object y, suppose we have a statement P(x,y) pertaining to x and y, such that for each x ∈ A there is at most one y for which P(x,y) is true. Then there exists a set {y : P (x, y) is true for some x ∈ A}, such that for any object z,

$$z \in \{P(x,y) \text{ is true for some } x \in A\}$$
$$\iff P(x,z) \text{ is true for some } x \in A$$

7. Axiom of Infinity

There exists a set N, whose elements are called natural numbers, as well as an object 0 in N, and an object n++ assigned to every natural number n ∈ N, such that the Peano axioms (Axioms 2.1 - 2.5) hold.

Q. PT $(A \cup B) \cup C = A \cup (B \cup C)$

Pf - Let Ω be a set s.t $A, B, C \subset \Omega$

Consider $x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

C1 - $x \in C \Rightarrow x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$

C2 - $x \in (A \cup B) \Rightarrow x \in A$ or $x \in B$

C2.1 - $x \in B \Rightarrow x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$

C.2.2 - $x \in A \Rightarrow x \in A \cup (B \cup C)$

$\therefore (A \cup B) \cup C \subseteq A \cup (B \cup C)$

By similar logic, we can show that $A \cup (B \cup C) \subset (A \cup B) \cup C$

$\because (A \cup B) \cup C \subseteq A \cup (B \cup C)$ &

$A \cup (B \cup C) \subseteq (A \cup B) \cup C$

$\therefore (A \cup B) \cup C = A \cup (B \cup C)$ $\square$

<u>Proposition</u> - Sets are partially ordered by set inclusion

i.e

1. $A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C$
2. $A \subseteq B$ & $B \subseteq A \Rightarrow A = B$
3. $A \subsetneq B$ & $B \subsetneq C \Rightarrow A \subsetneq C$

Proposition - Sets form a boolean algebra

Let A, B, C be sets, and let X be a set containing A, B, C as subsets.

(a) (Minimal element) We have $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.

(b) (Maximal element) We have $A \cup X = X$ and $A \cap X = A$.

(c) (Identity) We have $A \cap A = A$ and $A \cup A = A$.

(d) (Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(e) (Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

(f) (Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) (Partition) We have $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \varnothing$.

(h) (De Morgan laws) We have $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.

Pf - b) $A \cap X = A$

Consider $x \in A \cap X$

$\Rightarrow x \in A$ and $x \in X$

$\Rightarrow x \in A$

$\therefore A \cap X \subseteq A$

Consider $x \in A$.

Since, $A \subset X \Rightarrow (x \in A \Rightarrow x \in X)$

$\Rightarrow x \in X$

$\Rightarrow x \in A$ and $x \in X$

$\Rightarrow x \in A \cap X$

$\therefore A \subseteq A \cap X$

$\therefore A = A \cap X$ $\square$

f) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Consider $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in (B \cap C)$

<u>CI</u> - $x \in A$

$\Rightarrow x \in A \cup B$, $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

<u>CII</u> - $x \in (B \cap C)$

$\Rightarrow x \in B$ and $x \in C$

$\Rightarrow x \in (A \cup B)$ $\Rightarrow x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Consider $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

<u>C I</u> - $x \in A$ and $x \in A \Rightarrow x \in A \cup (B \cap C)$

<u>C II</u> - $x \in A$ and $x \in C \Rightarrow x \in A \cup (B \cap C)$

<u>C III</u> - $x \in B$ and $x \in A \Rightarrow x \in A \cup (B \cap C)$

<u>C IV</u> - $x \in B$ and $x \in C \Rightarrow x \in (B \cap C)$

$\Rightarrow x \in A \cup (B \cap C)$

$$\therefore (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

$$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \square$$

g) $A \cup (X \setminus A) = X$

Consider $x \in A \cup (X \setminus A)$

$\Rightarrow x \in A$ or $x \in (X \setminus A)$

<u>CI</u> – $x \in A$

But $A \subset X \Rightarrow (x \in A \Rightarrow x \in X)$

$\Rightarrow x \in X$

<u>CII</u> – $x \in (X \setminus A) \Rightarrow x \in X$ and $x \notin A$

$\Rightarrow x \in X$

$\therefore A \cup (X \setminus A) \subseteq X$

Consider $x \in X$

<u>CI</u> – $x \in A \Rightarrow x \in A \cup (X \setminus A)$

<u>CII</u> – $x \notin A \Rightarrow x \in (X \setminus A)$

$\Rightarrow x \in A \cup (X \setminus A)$

$\therefore X \subseteq A \cup (X \setminus A)$

$\therefore A \cup (X \setminus A) = X \quad \square$

g) $A \cap (X \setminus A) = \emptyset$

Consider $x \in A \cap (X \setminus A)$

$\Rightarrow x \in A$ and $x \in (X \setminus A)$

$\Rightarrow x \in X$ and $x \notin A$

$\Rightarrow x \in A$ and $x \in X$ and $x \notin A$

Contd^n

$\therefore \nexists\, x \in A \cap (X \setminus A)$

$\therefore A \cap (X \setminus A) = \emptyset \;\square$

h) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

Consider $x \in X \setminus (A \cup B)$

$\Rightarrow x \in X$ and $x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in (X \setminus A)$ $\Rightarrow x \in (X \setminus B)$

$\Rightarrow x \in (X \setminus A) \cap (X \setminus B)$

$\therefore X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$

Consider $x \in (X \setminus A) \cap (X \setminus B)$

$\Rightarrow x \in (X \setminus A)$ and $x \in (X \setminus B)$

$\Rightarrow x \in X$ and $x \notin A$

and $x \in X$ and $x \notin B$

$\Rightarrow x \in X$ and $x \notin (A \cup B)$ $\left(\because x \notin A \text{ and } x \notin B\right)$

$\Rightarrow x \in X \setminus (A \cup B)$

$\therefore (X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$

$\therefore X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ $\square$

Functions

Cartesian Product

Let A, B be 2 sets.
Then the cartesian product $A \times B$ is a set defined as

$$A \times B = \{(x,y) : x \in A,\ y \in B\}$$

NOTE – In general $(x,y) \neq (y,x)$

Function

$f : X \rightarrow Y$ defined by P on the domain X and range Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x,f(x))$ is true.
Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x,y) \text{ is true}$$

Graph

Given any fxⁿ, we can draw its graph $\Gamma_f \subset (X \times Y)$ as

$$\Gamma_f = \{(x, y) \in (X \times Y) \mid y = f(x)\}$$
$$= \{(x, f(x))\}$$

NOTE - 1. $\Gamma_f \xrightarrow{i} (X \times Y) \xrightarrow{P_X} X$

$$(x, f(x)) \mapsto (x, f(x)) \mapsto x$$

Then fxⁿ $P_X i : \Gamma_f \to X$ is bijective

Pf - Injectivity

Consider $x_1, x_2 \in X$ s.t $x_1 = x_2$

$$\Rightarrow P_x i((x_1, f(x_1))) = P_X i((x_2, f(x_2)))$$

$$\Rightarrow (x_1, f(x_1)) = (x_2, f(x_2))$$

$\because$ $x_1 = x_2 \Rightarrow (x_1, f(x_1)) = (x_2, f(x_2))$

$\therefore$ $P_x i$ is injective

Surjectivity

Let $x \in X$.

Consider $a = (x, f(x)) \in (X \times Y)$

$$P_X i(a) = P_X i((x, f(x))) = x$$

$\therefore \forall x \in X, \exists a \in (X \times Y)$ s.t. $P_X i(a) = x$

$\therefore P_X i$ is surjective

$\therefore P_X i$ is bijective. $\square$

2. 2 fxⁿs with the same domain and range $f, g : X \to Y$ are equal iff $f(x) = g(x) \ \forall x \in X$

Composition - $X \xrightarrow{f} Y \xrightarrow{g} Z$

$g \circ f : X \to Z$ is the fn given by

$$(g \circ f)(x) = g(f(x))$$

NOTE - Composition is Associative

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

$$(h \circ g) \circ f = h \circ (g \circ f)$$
$$= h(g(f(x)))$$

Inverse - If $f : X \to Y$ is bijective,

$\exists\, g : Y \to X$ s.t

$$g(y) = x$$

NOTE - $g \circ f = Id_X \qquad f \circ g = Id_Y$

where Id_D is the identity fn $x \mapsto x$ on domain D.

Q. Suppose $f : X \to Y$ is only surjective.

Define $g : Y \to X$ s.t $g(y) = x$ taking any x that maps to y.

Is $g \circ f = Id_X$ or $f \circ g = Id_Y$?

Pf - <u>1.</u> <u>$g \circ f \neq Id_X$</u>

eg - $f : \{0, 1\} \to \{1\}$

$$x \longmapsto 1$$

$$g : \{1\} \longrightarrow \{0, 1\}$$

Let $g(1) = 1$

Consider $g \circ f(0) = g(f(0)) = g(1)$

$$= 1 \neq 0$$

2. $f \circ g = Id_Y$

Consider $f: X \to Y$ & $g: Y \to X$

Define $g(y) = x_0$

We can do so since $\exists$ such $x_0 \in X$ by surjectivity of f.

$$\Rightarrow f(x_0) = y$$

Consider
$$f \circ g(y) = f(g(y)) = f(x_0) = y$$

$\because f \circ g(y) = y \quad \forall \, y \in Y$

$\therefore f \circ g = Id_Y$ $\square$