L18 – 10/10/2024

Pythagoras Thm

- Arithmetic & Number Theory
- Geometry & Distance
- Infinity in Greek Math

Chord - Tangent Method

1. Start with a simple sol^n (seed)

2. Construct new sol^ns in terms of the seed.

So, start with sol^n $P(0,1)$.
Notice, L passing through P (except when tangent) intersects the circle at exactly 2 pts. (the other being (x_0, y_0))

L: $(y-1) = mx$ (for finite slope m)
$x = 0$

1. So, $y_0 - 1 = mx_0$

$$x_0^2 + y_0^2 = 1 \Rightarrow x_0^2 + (mx_0+1)^2 = 1$$

$$\Rightarrow (1+m^2)x_0^2 + 2mx_0 = 0$$

$$\Rightarrow x_0 = 0, \frac{-2m}{1+m^2}$$

$$y_0 = 1, \frac{1-m^2}{1+m^2}$$

Now, $m \in \mathbb{Q} \Rightarrow (x_0, y_0) \in \mathbb{Q}^2$

Conversely, if $(x_0, y_0) \in \mathbb{Q}^2$, the slope of line joining $(0,1)$ & (x_0, y_0) $m \in \mathbb{Q}$

2. $L: x = 0$ intersects the circle at $(0,1)$ & $(0,-1)$

Hence, the set of all rational pts. $C(\mathbb{Q})$ on the circle C is

$$\left\{ \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2} \right) \mid m \in \mathbb{Q} \right\} \cup \{(0,-1)\}$$

To enumerate all primitive Pythagorean triples, put $m = \frac{p}{q}$, $GCD(p,q) = 1$

$$\Rightarrow \left(\frac{-2m}{1+m^2}\right)^2 + \left(\frac{1-m^2}{1+m^2}\right)^2 = 1$$

$$\Rightarrow \left(\frac{2pq}{p^2+q^2}\right)^2 + \left(\frac{p^2-q^2}{p^2+q^2}\right)^2 = 1$$

$$\Rightarrow \underbrace{(2pq)}_{a}{}^2 + \underbrace{(p^2-q^2)}_{b}{}^2 = \underbrace{(p^2+q^2)}_{c}{}^2$$

Geometry

- The notion of distance in coordinate geometry is derived from the Pythagoras Thm

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$