L9 - 30/08/2024

Let us assume all pts on the 'line' are rational nos.

Ppⁿ - Let $\epsilon > 0$ be a rational.
Then, $\exists\, x \in \mathbb{Q}$ s.t

$$x^2 < 2 < (x+\epsilon)^2$$

Pf - Assume if $x^2 < 2$ for some rational x, then $(x+\epsilon)^2 < 2$ $\forall$ rational $\epsilon > 0$.

Now, $0 \in \mathbb{Q}$ & $0^2 < 2 \Rightarrow (0+\epsilon)^2 < 2$
$\Rightarrow \epsilon^2 < 2$

Also, $\epsilon \in \mathbb{Q}$ & $\epsilon^2 < 2 \Rightarrow (\epsilon+\epsilon)^2 < 2$
$\Rightarrow (2\epsilon)^2 < 2$

- We should have considered $(x+\epsilon)^2 \leq 2$, but we can safely reject $(x+\epsilon)^2 = 2$ since we have proved earlier that $\sqrt{2}$ is irrational.

By ind^n, we can show $\underbrace{(n\epsilon)^2 < 2}_{\text{Contd}^n}$. $\square$

<u>REMARK</u> - In Tao's Analysis I, the following lemma has been proved

If $\alpha, \beta \in \mathbb{Q}_{>0}$, $\exists\, n \in \mathbb{Z}_{\geq 1}$ s.t $n\alpha > \beta$

The statement $(n\epsilon)^2 < 2$ contradicts this lemma for $\alpha = \epsilon^2$ & $\beta = 2$

- **Sequences** – A seq. of rational nos. is a subset of $\prod_{n=1}^{\infty} \mathbb{Q}$

- **Cauchy seq.** – A seq. $(a_n)_{n \geq 1}$ of rationals is said to be Cauchy if $\forall$ rational $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t $\forall\, m, n \geq N$, $|a_m - a_n| < \epsilon$

We will now define an eq. rel^n on the set of Cauchy seq. of rationals.

Let (a_n), (b_n) be 2 Cauchy seq. of rationals and $\sim$ be an eq. rel^n on the set of Cauchy seq. of rationals s.t

$$(a_n) \sim (b_n)$$

$$\Updownarrow$$

$\forall$ rational $\epsilon > 0$, $\exists\, N \in \mathbb{Z}_{\geq 1}$ s.t

$\forall\, n \geq N$, $|a_n - b_n| < \epsilon$

Pf - $\underline{R}$ - $|a_n - a_n| = 0 < \epsilon$

$\underline{S}$ - $|b_n - a_n| = |a_n - b_n| < 0$

$\underline{T}$ - Given $|a_n - b_n| < \epsilon \quad \forall\, n \geq N_1$

$|b_n - c_n| < \epsilon \quad \forall\, n \geq N_2$

Let $N = \max(N_1, N_2)$.

$\forall\, n \geq N$,

$$|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)|$$
$$\leq \underbrace{|a_n - b_n|}_{< \epsilon/2} + \underbrace{|b_n - c_n|}_{< \epsilon/2} < \epsilon$$

$\square$