MA110/MA108 Differential Equations

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Class Information

- Instructor: Ronnie Sebastian
- Office: 211-F (second floor), Dept of Mathematics
- Office Hours: Tuesday 11.30am to 12.30pm or by appointment. Please send me an email if you would like to meet me.
- Email: ronnie@iitb.ac.in
- Reference Text: Elementary Differential Equations by William Trench available online (do a google search)
- One quiz of 40 to 50 percent weightage. This will be held on Wednesday 2nd April from 8.15 to 9.15 am.
- End Semester exam of 50 to 60 percent weightage.

First simple example of ODE

Consider the simplest example of an ordinary differential equation (ODE)

$$y'(x) + ay(x) = 0\,, \qquad\qquad a \in \mathbb{R}\,.$$

We can solve this differential equation as follows. Rewrite it as

$$\frac{y'(x)}{y(x)} = -a\,.$$

This can further be rewritten as

$$\frac{d}{dx}(\ln\, y(x)) = -a\,.$$

Thus, we get

$$\ln\, y(x) = -ax + c\,,$$

that is,

$$y(x) = e^c e^{-ax}\,.$$

First simple example of ODE

Note that the function $e^c e^{-ax}$ is defined on all of $\mathbb{R}$.
Thus, we conclude that the function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y(x) = e^c e^{-ax}$$

satisfies the differential equation

$$y' + ay = 0.$$

Note that using this method we only get solutions of the type Ce^{-ax} for C a positive real number.
On the other hand, notice that

$$\frac{d}{dx} \ln(-y) = \frac{-y'}{-y} = \frac{y'}{y}.$$

So we could have written the differential equation $y' + ay = 0$ as

$$\frac{d}{dx} \ln(-y) = -a.$$

Thus, we get

$$\ln(-y) = -ax + c\,,$$

that is,

$$y(x) = -e^c e^{-ax}\,.$$

Note that the function $-e^c e^{-ax}$ is defined on all of $\mathbb{R}$.

Thus, we conclude that the function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y(x) = -e^c e^{-ax}$$

satisfies the differential equation

$$y' + ay = 0.$$

Note that using this method we only get solutions of the type Ce^{-ax} for C a negative real number.

First simple example of ODE

Also, $y = 0$ is a solution to the differential equation.

We conclude that $y = Ce^{-ax}$ is a solution to the differential equation for any real number C. Moreover, all these solutions are defined on the whole of $\mathbb{R}$.

This raises a few natural questions:

- Given a differential equation, are there methods to find at least one solution?
- Will a particular method yield all possible solutions of the given differential equation?
- What is the domain of definition of the solutions? Can different solutions have different domains of definition?

In general we will have more complicated ordinary differential equations. Our aim will be to find functions $y(x)$ of one variable x, which satisfy this differential equation. In general, note that these functions y will not be defined on all of $\mathbb{R}$.

This course is about learning methods to solve some ODE's.

Some basic definitions related to ODE's

Definition

Let $y = y(x)$ be a function of x.
An Ordinary differential equation (ODE) is an equation involving atleast one derivative of y.
A precise definition may be as follows. We can take any function $F(s, t_0, t_1, \ldots, t_n)$ in $n+2$ variables, which depends on t_i for some $1 \leq i \leq n$, and consider the ODE $F(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}) = 0$.
Note, for example, that the function $F(s, t_0, t_1, \ldots, t_n) = s^2 + t_0^3$ does not depend on t_i for $i \geq 1$, and so we will not get an ODE on substitution as no derivative of y will appear.
The order of an ODE is the highest order of derivative of y occuring in the ODE.

Example

(1) $y' = x^2y^2 + x$ is a 1st order ODE.
(2) $y'' + 2xy' + y = \sin x$ is a 2nd order ODE.
(3) $y^{(4)} + 2xy^{(1)}y^{(2)} + y = \sin x$ is a 4th order ODE.

Open intervals

If $a < b$ are real numbers, then an open interval is a subset of $\mathbb{R}$ of the following type

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

$\mathbb{R} = (-\infty, \infty)$ is also an open interval.
$\mathbb{R} - \{0\}$ is not an open interval. It is the union of two open intervals

$$\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty).$$

Recall (from your calculus course) that to define the derivative of a function at a point x_0, we need it to be defined in a small "neighbourhood" of x_0. This is because the derivative is defined as a limit. Thus, it makes sense to talk of solutions of differential equations on open intervals or unions of open intervals.

Definition

An explicit solution of an ODE is a function $y = f(x)$ which satisfies the ODE on some open interval.

Second simple example of ODE

Next consider the following ODE

$$y' + ay = f(x)\,.$$

Since we know that the solution to the ODE $y' + ay = 0$ is e^{-ax}, let us try to look for a solution of the type $y = ue^{-ax}$.

Substituting into the differential equation, we get

$$u'e^{-ax} - aue^{-ax} + aue^{-ax} = f(x)\,,$$

that is,

$$u' = f(x)e^{ax}\,.$$

Thus,

$$u(x) = \int_{x_0}^{x} f(s)e^{as}ds + C\,,$$

$$y(x) = e^{-ax}\Big(\int_{x_0}^{x} f(s)e^{as}ds + C\Big)\,.$$

Clearly, this solution is defined on any open interval on which $f(x)$ is defined.

Example

Solve $y' + 2y = x^3 e^{-2x}$.

Note $y' + 2y = 0$ has a solution $y_1(x) = e^{-2x}$.

Look for solution of type $y = uy_1$. Then we get

$$\begin{aligned} u' y_1 &= x^3 e^{-2x} \,. \\ \implies u' &= x^3 \,. \\ \implies u(x) &= x^4/4 + C \,. \end{aligned}$$

Therefore,

$$y(x) = e^{-2x}(x^4/4 + C)$$

is a solution of ODE on $\mathbb{R}$.

Third example of ODE

Let us now consider the following, more difficult, ODE

$$y' + a(x)y = 0\,.$$

Notice that, unlike in the first two examples, $a(x)$ is now a function of x.

Let us assume that $a(x)$ is a function defined on the interval $I := (x_0 - \epsilon, x_0 + \epsilon)$. Define a function

$$g(x) := \exp\left(\int_{x_0}^{x} a(s)ds\right).$$

Third example of ODE

Now note that

$$\frac{d}{dx}(g(x)y) = g(x)(y' + a(x)y) = 0\,.$$

We have converted the ODE to something in "exact" form.
Thus, we get

$$y(x)\exp(\int_{x_0}^{x} a(s)ds) = C\,,$$

that is,

$$y(x) = C\exp(-\int_{x_0}^{x} a(s)ds)\,.$$

Clearly, this solution is defined on any open interval on which $a(x)$ is defined.

Example

Solve $y' - 2xy = 0$.

The function $a(x) = -2x$ is defined on all of $\mathbb{R}$.

Thus, the solution is given by

$$y(x) = C\exp(\int_0^x 2s\,ds) = Ce^{x^2}\,.$$

Linear ODE's

Definition

An ODE of order n is called linear if it can be written as

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x),$$

Linear first order differential equation

Consider the most general kind of linear first order differential equation. By definition, this looks like

$$p(x)y' + q(x)y = h(x)\,.$$

Dividing this by $p(x)$ let us rewrite this as

$$y' + a(x)y = f(x)\,.$$

Let us assume that $a(x)$ and $f(x)$ are defined on the interval $I := (x_0 - \epsilon, x_0 + \epsilon)$.

As in the constant coefficient case, let $y_1(x)$ be a solution to the **homogeneous** equation, that is, $y' + a(x)y = 0$, and we look for a solution of the type $y(x) = uy_1$.

Substituting into the differential equation we get

$$u'y_1 + uy_1' + auy_1 = f(x)\,,$$

that is, (since $uy_1' + auy_1 = u(y' + ay_1) = 0$)

$$u'y_1 = f(x)\,.$$

Thus,

$$u(x) = \int_{x_0}^{x} \frac{f(s)}{y_1(s)} ds + C\,.$$

and

$$y(x) = y_1(x)\Big(\int_{x_0}^{x} \frac{f(s)}{y_1(s)} ds + C \Big)\,.$$

Example

(1) Solve $y' - 2xy = 1$.
$y' - 2xy = 0$ has a solution $y_1(x) = e^{x^2}$.
The solution of ODE is $y = uy_1$, where

$$u'y_1 = 1 \implies u(x) = \int_0^x e^{-s^2} ds + C\,.$$

$$\implies y(x) = e^{x^2}\Big(\int_0^x e^{-s^2} ds + C\Big)\,.$$

(2) Solve $\boxed{y' - 2xy = 1,\ y(0) = y_0}$.
Write the solution of ODE as

$$y(x) = e^{x^2}\Big(\int_0^x e^{-s^2} ds + C\Big)\,.$$

Then $y(0) = y_0$ gives $C = y_0$.

Initial Value Problem (IVP)

Definition

An Initial value problem (IVP) for 1st order ODE is a problem of the form

$$y' = F(x, y), \quad y(x_0) = y_0.$$

A function $y = y(x)$ defined on some open interval (a, b) containing x_0 is a solution of the IVP if y satisfies the ODE on (a, b) and $y(x_0) = y_0$.

Existence and Uniqueness: Linear first order IVP

Theorem

Let $a(x)$ and $f(x)$ be continuous functions defined on an open interval $(c, d) \subset \mathbb{R}$.

Let $x_0 \in (c, d)$ be a point.

Then the differential equation $y' + a(x)y = f(x)$, along with the initial condition $y(x_0) = y_0$,

has a unique solution $y : (c, d) \to \mathbb{R}$.

Recall that we started the course with an example where a way to solve the ODE did not give all the solutions. The above Theorem should be contrasted with that example. This result is quite remarkable as it tells us that given an IVP, it has a unique solution.

Existence and Uniqueness: $y' = f(x,y)$ IVP

We have the following more general existence result.

Theorem (Existence and Uniqueness of solution : $y' = f(x,y)$)

Let $D = (a,b) \times (c,d)$ be an open rectangle containing the point (x_0, y_0) and consider the IVP

$$y' = f(x,y), \quad y(x_0) = y_0$$

(a) (Existence) Assume $f(x,y)$ is continuous on D. Then IVP has at least one solution on some interval $(a_1, b_1) \subset (a,b)$ containing x_0.

(b) (Uniqueness) If both $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution on some interval $(a', b') \subset (a,b)$ containing x_0.

(c) (Interval of validity) Given a solution of the IVP, the largest interval containing x_0 on which the solution is defined is called the interval of validity of the solution.

- Note the theorem says that for non-linear ODE, the solution and the interval where the solution exists, depends on the choice of our initial condition.
- Solutions of a non-linear ODE obtained using a particular method may not give the complete set of solutions.

  For example, we will see later that a family of solutions to the non-linear ODE $y' = 2xy^2$, is given by $y = -1/(x^2 + C)$. However, this does not give the solution $y \equiv 0$ for any value of C.

Example

Consider the IVP

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0$$

Here

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2}\,.$$

Thus,

$$\frac{df}{dy} = \frac{-2y}{1 + x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(1 + x^2 + y^2)^2} = \frac{-2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}\,.$$

Since $f(x, y)$ and df/dy are continuous for all $(x, y) \in \mathbb{R}^2$, by existence and uniqueness theorem, for any $(x_0, y_0) \in \mathbb{R}^2$, IVP has a unique solution on some open interval containing x_0.

Example (IVP)

Consider another example,

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0 .$$

Here $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. It is easily checked that

$$\frac{df}{dy} = \frac{-4x^2 y}{(x^2 + y^2)^2}$$

Note that f and df/dy are continuous for all $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$. Assume $(x_0, y_0) \neq (0, 0)$.

- There is an open rectangle R containing (x_0, y_0) but not containing $(0, 0)$.
- $f(x, y)$ and df/dy are continuous on R.
- By existence and uniqueness theorem, $(*)$ has a unique solution on some open interval containing x_0. $\square$

Example

Consider the IVP

$$y' = \frac{x+y}{x-y}, \quad y(x_0) = y_0.$$

Here

$$f(x,y) = \frac{x+y}{x-y}, \quad \text{and} \quad \frac{df}{dy} = \frac{2x}{(x-y)^2}.$$

Here $f(x,y)$ and df/dy are continuous everywhere except on the line $y = x$.

Assume $x_0 \neq y_0$.

- There is an open rectangle R containing (x_0, y_0) that does not intersect with the line $y = x$.
- $f(x,y)$ and df/dy are continuous on R.
- By existence and uniqueness theorem, $(*)$ has a unique solution on some open interval containing x_0. □

Example

Consider the IVP

$$y' = \frac{10}{3}\,xy^{2/5}, \quad y(x_0) = y_0.$$

Here

$$f(x,y) = \frac{10}{3}\,xy^{2/5} \;\text{ and }\; \frac{\partial f}{\partial y} = \frac{4}{3}\,xy^{-3/5}.$$

- Since $f(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$, this IVP has at least one solution for all $(x_0,y_0) \in \mathbb{R}^2$.
- If $y \neq 0$, then $f(x,y)$ and df/dy both are continuous for all $(x,y) \in \mathbb{R}^2$. Thus, if $y_0 \neq 0$, there is an open rectangle R containing (x_0,y_0) such that f and df/dy are continuous on R. Hence this IVP has a unique solution on some open interval containing x_0. $\square$

Example

Consider the following modification of the preceding example

$$y' = \frac{10}{3}\, x y^{2/5}, \quad y(0) = 0.$$

Note that $\dfrac{df}{dy} = \dfrac{4}{3}\, x y^{-3/5}$ is not continuous if $y = 0$.
Thus, this IVP may have more than one solution on every open interval containing $x_0 = 0$. Note that $y \equiv 0$ is one solution of this IVP.

Let us find a nonzero solution of this IVP.

$$\frac{y'}{y^{2/5}} = \frac{10}{3}\, x \implies \frac{5}{3}\, y^{3/5} = \frac{5}{3}\,(x^2 + C)$$
$$\implies y(x) = (x^2 + C)^{5/3}$$

Example (continued ...)

$$y(0) = 0 \implies C = 0\,.$$

Thus, the IVP

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = 0.$$

has at least two solutions, $y_1 \equiv 0$ and $y_1(x) = x^{10/3}$.

We can construct two more solutions of this IVP by "gluing" the above two. Define

$$y_2(x) = \begin{cases} x^{10/3} & , \quad 0 \leq x \\ 0 & , \quad x < 0 \end{cases}$$

$$y_3(x) = \begin{cases} 0 & , \quad 0 \leq x \\ x^{10/3} & , \quad x < 0 \end{cases}$$

One checks easily that the functions y_2 and y_3 are once differentiable with continuous derivative, and satisfy the given IVP.

Example

Similar to the preceding example, consider the following IVP

$$y' = \frac{10}{3}\,xy^{2/5}, \quad y(0) = -1$$
$$f(x,y), \qquad \frac{df}{dy} = \frac{4}{3}xy^{-3/5}$$

are continuous in an open rectangle containing $(0,-1)$. Hence the IVP has a unique solution on some open interval containing $x_0 = 0$.

Let us find the unique solution and its interval of validity.
Let $y \neq 0$ be the solution of $y' = (10/3)\,xy^{2/5}$. Then

$$y(x) = (x^2 + C)^{5/3}$$
$$y(0) = -1 \implies C = -1$$
$$\implies y(x) = (x^2 - 1)^{5/3}$$

Example (continued ...)

$$y(x) = (x^2 - 1)^{5/3}$$

is a solution of the given IVP on $(-\infty, \infty)$.

If we take any interval (a, b) with $a < -1 < 1 < b$, then we can define another solution

$$y_1(x) = \begin{cases} (x^2 - 1)^{5/3} & , \quad -1 \leq x \leq 1 \\ 0 & , \quad |x| > 1 \end{cases}$$

We have seen that if $y_0 \neq 0$, then the IVP

$$y' = (10/3)\, xy^{2/5}, \quad y(x_0) = y_0$$

has a unique solution on some open interval around x_0.

Let us check that $y(x)$ is the unique solution to the IVP on the interval $(-1, 1)$.

Example (continued ...)

- Suppose there is another solution $w(x)$ to the IVP. Then by the existence and uniqueness theorem, $y(x) = w(x)$ for all x in a neighbourhood (a, b) around 0. Choose a and b so that this interval is the largest.

  If $-1 < a$, then since both $y(x)$ and $w(x)$ are continuous, we get $y(a) = w(a) = A$. Note that $A \neq 0$ as $-1 < a < 1$.

  Now we apply the existence and uniqueness theorem to the IVP with the condition $y(a) = A$. This is possible as $A \neq 0$ and so $\frac{df}{dy}$ is continuous at (a, A). This will show that $y(x) = w(x)$ in a neighbourhood around a, which contradicts the assumption that (a, b) was the largest interval on which y and w agreed.

Example (continued ...)

The "reason" the above argument "works" is because $y(x) = (x^2 - 1)^{5/3}$ does not take zero value on $(-1, 1)$.

Exercise. Use the same reasoning above to show that the IVP

$$y' = \frac{10}{3}\,xy^{2/5}, \quad y(0) = 1$$

has a unique solution on the interval $(-\infty, \infty)$.

A non-linear differential equation

$$y' + p(x)y = f(x)y^r$$

where $r \in \mathbb{R} - \{0, 1\}$ is called the **Bernoulli Equation**. For $r = 0, 1$, it is linear, and we already saw how to solve it.

If y_1 is a non-zero solution of $y' + p(x)y = 0$, then putting $y = u(x)y_1$ in ODE, we get

$$\begin{aligned} & u'y_1 + uy_1' + puy_1 = fu^r y_1^r \,. \\ \implies & u'y_1 = fu^r y_1^r \\ \implies & \frac{u'}{u^r} = f(x)(y_1(x))^{r-1} \\ \implies & \frac{u^{-r+1}}{-r+1} = \int f(x)(y_1(x))^{r-1}\, dx + C \,. \end{aligned}$$

Example (Bernoulli Equation)

Consider

$$y' + y = xy^2\,.$$

Set $y = u(x)e^{-x}$, where $y_1 = e^{-x}$ is solution of homogeneous part. Substituting into the ODE we get

$$\begin{aligned}
& u'e^{-x} - ue^{-x} + ue^{-x} = u^2e^{-2x}x\,. \\
\implies & u'e^{-x} = u^2e^{-2x}x \\
\implies & \frac{u'}{u^2} = xe^{-x} \\
\implies & \frac{-1}{u} = -(1+x)e^{-x} + C \\
\implies & u = \frac{1}{(1+x)e^{-x} - C} \\
\implies & y = \frac{e^{-x}}{(1+x)e^{-x} - C} = \frac{1}{1+x-Ce^{x}}\,.
\end{aligned}$$

Example

Consider Bernoulli equation

$$xy' - 2y = \frac{x^2}{y^6} \implies y' - \frac{2}{x}y = \frac{x}{y^6}\,.$$

The solution to homogeneous part is $y_1 = x^2$. Set $y = u(x)y_1$,

$$\begin{aligned} &u'y_1 = x(uy_1)^{-6} \\ \implies &u^6u' = x(x^2)^5 = x^{11} \\ \implies &\frac{1}{7}u^7 = \frac{1}{12}x^{12} + C \\ \implies &\frac{1}{7}y^7 = \left[\frac{1}{12}x^{12} + C\right]y_1^7 = \left[\frac{1}{12}x^{12} + C\right]x^{14} \end{aligned}$$

How do we solve $y' + 2xy = 0$?

Let us rewrite this equation as $\dfrac{1}{y}\dfrac{dy}{dx} = -2x$.

We can solve these equations by separating the variables.

$$\begin{aligned} \frac{1}{y}\ dy &= -2x\ dx \\ \int \frac{1}{y}\ dy &= \int -2x\ dx \\ \ln|y| &= -x^2 + C \\ y &= C_1 e^{-x^2} \end{aligned}$$

Let $y' = f(x,y)$ be a differential equation.

Rewrite it as $M(x,y) + N(x,y)\,\dfrac{dy}{dx} = 0$. Such an M and N always exist by choosing $M = f$ and $N = 1$.

The equation is said to be **separable** if it is possible to choose M and N such that M is a function only in x and N is a function only in y. Assume the ODE is separable.

Let H_1 and H_2 be antiderivatives of M and N respectively. Then $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$. Then our ODE is

$$\begin{aligned} H_1'(x) + H_2'(y)\,\frac{dy}{dx} &= 0. \\ \frac{dH_1(x)}{dx} + \frac{dH_2(y)}{dy}\,\frac{dy}{dx} &= 0. \\ \frac{d}{dx}[H_1(x) + H_2(y(x))] &= 0. \end{aligned}$$

Thus we get that the solution to the ODE will satisfy the equation

$$H_1(x) + H_2(y(x)) = C.$$

In general, the separable variables method only gives us an **implicit solution** to the given ODE.

Example

Solve $y' = 2xy^2$.
Rewrite this ODE as

$$\frac{1}{y^2}y' = 2x\,.$$

Integrating, we get

$$\frac{-1}{y} = x^2 + C\,.$$

$$\implies y = \frac{-1}{x^2 + C}$$

Note that $y \equiv 0$ is also a solution, which cannot be obtained for any choice of C.
Also note that if we take $f(x,y) = 2xy^2$ then we see that both f and $\frac{df}{dy}$ are continuous everywhere. Thus, any IVP with this ODE will have a unique solution in a neighborhood of x_0.

Example

Solve IVP

$$y' = 2xy^2, \quad y(0) = y_0$$

and find the interval of validity.
A family of solutions is

$$y = \frac{-1}{x^2 + C}\,.$$

- If $y_0 = 0$, the solution is $y \equiv 0$ and the interval of validity is $\mathbb{R}$.
- If $y_0 \neq 0$, then $C = -\dfrac{1}{y_0}$. Hence $y = \dfrac{-y_0}{y_0 x^2 - 1}$.
- If $y_0 < 0$, the solution is defined for all x. Hence the interval of validity is $\mathbb{R}$.
- If $y_0 > 0$, the solution is valid when $x \in \mathbb{R} - \{\pm 1/\sqrt{y_0}\}$. Hence the interval of validity is $\left(\dfrac{-1}{\sqrt{y_0}}, \dfrac{1}{\sqrt{y_0}}\right)$.

Example

Solve IVP

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}; \quad y(0) = 1.$$

Write this as

$$\frac{1+2y^2}{y}dy = \cos x \; dx.$$

Integrating,

$$\ln|y| + y^2 = \sin x + c.$$

As $y(0) = 1$, we get $c = 1$.

Hence a solution to the IVP is

$$\ln|y| + y^2 = \sin x + 1.$$

It is not clear how to write this equation in the form $y = f(x)$.
Such a solution is called an **implicit solution**.
Note: $y \equiv 0$ is a solution to the ODE, but it is not a solution to the given IVP.

Converting to Separable Equation

Solve $xy' = y + x$. Rewrite it as $y' = y/x + 1$.

This is not separable, but we can make it separable by changing our variables.

Let $v = y/x$ or $y = vx$. Then $y' = v'x + v$

Given ODE is $v'x + v = v + 1$ or $v'x = 1$.

Apply separation of variables to get $y = (\ln|x| + C)x$.

Any non linear ODE $y' = q(y/x)$ can be converted to a separable equation by substituting $y = vx$.

Example

Solve $\quad x^2y' = y^2 + xy - x^2.$

$$\begin{aligned}
y' = \frac{y^2 + xy - x^2}{x^2} \quad &= \quad \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1 \\
\text{Substitute } \; y = vx & \\
v'x + v \quad &= \quad v^2 + v - 1 \\
\frac{v'}{v^2 - 1} \quad &= \quad \frac{1}{x} \\
\frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right)v' \quad &= \quad \frac{1}{x} \\
\frac{1}{2}\left(\ln|v-1| - \ln|v+1|\right) \quad &= \quad \ln|x| + C_1 \\
\frac{v-1}{v+1} = Cx^2 \quad &\Longrightarrow \quad v = \frac{1 + Cx^2}{1 - Cx^2}
\end{aligned}$$

Example (Continued ...)

$$y = x\,\frac{1 + Cx^2}{1 - Cx^2}$$

is a solution of

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

- **Question.** Are these all the solutions?
  Ans. No.
  Both $y = x$ and $y = -x$ are also solutions, but only $y = x$ can be obtained from the above solution (by putting $C = 0$).

  Consider the IVP $x^2y' = y^2 + xy - x^2$, $y(0) = 0$. Writing this IVP as $y' = f(x, y)$, we see that f is not continuous at $(0, y)$. Thus, we cannot apply the existence and uniqueness results.

Example (Continued ...)

Note that

$$y = x\,\frac{1 + Cx^2}{1 - Cx^2}$$

is continuous at $x = 0$. The derivative y' is also continuous at $x = 0$.

In fact, for arbitrary $C_1, C_2 \in \mathbb{R}$, the function

$$y(x) = \begin{cases} x\,\dfrac{1 + C_1x^2}{1 - C_1x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1 + C_2x^2}{1 - C_2x^2} & \text{if } x \geq 0 \end{cases}$$

is differentiable and satisfies the given IVP.

Example (continued ...)

Thus the IVP $\quad x^2y' = y^2 + xy - x^2, \quad y(0) = 0$
has infinitely many solutions

$$y(x) = \begin{cases} x\,\dfrac{1+C_1x^2}{1-C_1x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1+C_2x^2}{1-C_2x^2} & \text{if } x \geq 0 \end{cases}$$

one for each choice of C_1, C_2.

The interval of validity I of $y(x)$ depends on C_1, C_2.

- If $C_1 \leq 0$ and $C_2 \leq 0$, then $I = \mathbb{R}$.
- If $C_1 \leq 0$ and $C_2 > 0$, then $I = (-\infty, 1/\sqrt{C_2})$.
- If $C_1 > 0$ and $C_2 \leq 0$, then $I = (-1/\sqrt{C_1}, \infty)$.
- If $C_1 > 0$ and $C_2 > 0$, then $I = (-1/\sqrt{C_1}, 1/\sqrt{C_2})$.

Example

The ODE
$$y' = \frac{2x + y + 1}{x + 2y - 4}$$
can be converted to a separable equation, use substitution $X = x + 2$, $Y = y - 3$.

The same idea can be used to deal with ODE's
$$y' = \frac{ax + by + c}{a'x + b'y + c'}$$
when $ab' - a'b \neq 0$.

Example

Solve $\quad 3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0.$

Note $\quad 3x^2y^2 = \dfrac{d}{dx}(x^3y^2)$ and $2x^3y = \dfrac{d}{dy}(x^3y^2).$

Let $\quad G(x, y) = x^3y^2$. Then

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

$$\implies \frac{dG}{dx} + \frac{dG}{dy}\frac{dy}{dx} = 0$$

$$\implies \frac{d}{dx}G(x, y(x)) = 0$$

Therefore,

$$G(x, y) = C$$

is a solution of given ODE.

Note that unlike in the earlier cases, we may not be able to write y explicitly as a function of x here. Such a solution to an ODE ($G(x, y) = C$) is called an **implicit solution**.

Definition. A first order ODE written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is said to be **exact** if there exists a function G such that

$$\frac{dG}{dx} = M(x, y) \quad \text{and} \quad \frac{dG}{dy} = N(x, y).$$

If the ODE is exact, then

$$G(x, y) = C$$

is an implicit solution of ODE. (This is clear because the same argument as in the previous example applies.)

When is an ODE exact?

Theorem

Consider ODE $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$.

Assume functions M, N, $\dfrac{dM}{dy}$ *and* $\dfrac{dN}{dx}$ *be continuous in an open rectangle* $R := \{a < x < b,\ c < y < d\}$.

Then the above ODE is exact if and only if M *and* N *satisfies the condition* $\dfrac{dM}{dy} = \dfrac{dN}{dx}$.

In other words, there exists a function $G : R \to \mathbb{R}$ *such that* $\dfrac{dG}{dx} = M$ *and* $\dfrac{dG}{dy} = N$ *if and only if* $\dfrac{dM}{dy} = \dfrac{dN}{dx}$ *on* R.

Which of the following ODE's are exact?

1. $(2x+3)+(2y-2)y'=0$ Exact
2. $\dfrac{dy}{dx}=\dfrac{ax+by}{bx+cy}$. Not Exact
3. $(y/x+6x)+(\ln x-2)\dfrac{dy}{dx}=0 \quad x,y>0.$ Exact
4. $(3x^2y+2xy+y^3)+(x^2+y^2)\dfrac{dy}{dx}=0.$ Not Exact

Example

Solve $(2x+3)+(2y-2)y' = 0.$

The ODE is exact, so we need to find $\phi(x,y)$ such that

$$\frac{d\phi}{dx} = 2x+3 \quad \text{and} \quad \frac{d\phi}{dy} = 2y-2\,.$$

Integrating first equation gives

$$\phi(x,y) = x^2 + 3x + h(y)\,.$$

This gives

$$\frac{d\phi}{dy} = \frac{dh}{dy} = 2y-2 \implies h(y) = y^2 - 2y + C_1\,.$$

Therefore, an implicit solution to ODE is

$$\phi(x,y) = x^2 + 3x + y^2 - 2y = C\,.$$

Example

Solve $(y/x + 6x) + (\ln x - 2)\dfrac{dy}{dx} = 0 \quad x, y > 0$.

This is exact, so we need to find $\phi(x, y)$ such that

$$\frac{d\phi}{dx} = \frac{y}{x} + 6x \quad \text{and} \quad \frac{d\phi}{dy} = \ln x - 2\,.$$

Integrating the first equation gives

$$\phi(x, y) = y \ln |x| + 3x^2 + h(y)\,.$$

This gives

$$\frac{d\phi}{dy} = \ln |x| + \frac{dh}{dy} = \ln x - 2 \implies h(y) = -2y\,.$$

Therefore the solution is given by

$$\phi(x, y) = y \ln |x| + 3x^2 - 2y = C\,.$$

Method of integrating factor

Example

Solve

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)\frac{dy}{dx} = 0.$$

$$M = 3x^2y + 2xy + y^3, \quad N = x^2 + y^2$$

$$\frac{d}{dy}M = 3x^2 + 2x + 3y^2, \quad \frac{d}{dx}N = 2x$$

Therefore, the ODE is not exact.

Question. Can we multiply the ODE by a function $\mu(x, y)$ so that it becomes exact.

Assume

$$\mu(3x^2y + 2xy + y^3) + \mu(x^2 + y^2)\frac{dy}{dx} = 0$$

is exact.

Example (continued ...)

Then exactness condition gives

$$\frac{d}{dy}(\mu(3x^2y + 2xy + y^3)) = \frac{d}{dx}(\mu(x^2 + y^2)) \implies$$

$$\mu(3x^2 + 2x + 3y^2) + \frac{d\mu}{dy}(3x^2y + 2xy + y^3) = 2x\mu + \frac{d\mu}{dx}\,(x^2 + y^2)$$

Let us try to find a μ which is independent of y.

Then $d\mu/dy = 0$ and above equation becomes

$$\begin{aligned} 3\mu(x^2 + y^2) &= \frac{d\mu}{dx}(x^2 + y^2) \\ \implies \frac{d\mu}{dx} &= 3\mu \\ \implies \mu &= Ce^{3x} \end{aligned}$$

Example (continued ...)

The ODE now becomes

$$e^{3x}(3x^2y + 2xy + y^3) + e^{3x}(x^2 + y^2)\frac{dy}{dx} = 0.$$

Verify that this is exact. Hence there exists $\phi(x, y)$ such that

$$\frac{d\phi}{dx} = e^{3x}(3x^2y + 2xy + y^3) \text{ and } \frac{d\phi}{dy} = e^{3x}(x^2 + y^2)$$

$$\implies \phi(x, y) = e^{3x}x^2y + \frac{1}{3}\,e^{3x}y^3 + h(y)$$

$$\implies \frac{d\phi}{dy} = e^{3x}x^2 + e^{3x}y^2 + \frac{dh}{dy} = e^{3x}(x^2 + y^2)$$

$$\implies \frac{dh}{dy} = 0 \implies h(y) = C$$

$$\implies \phi(x, y) = e^{3x}(x^2y + \frac{1}{3}\,y^3) = C \text{ : implicit solution.}$$

Example (continued ...)

Question. Is $\phi(x,y) = e^{3x}(x^2y + \frac{1}{3}y^3) = C$ the solution to our original ODE?
How will the solutions to the two ODE's be related?

$$\begin{aligned}
&\phi'(x,y) = 0 \\
\Longrightarrow\ & e^{3x}2xy + 3e^{3x}x^2y + e^{3x}x^2y' + 3e^{3x}\ \frac{y^3}{3} + e^{3x}y^2y' = 0 \\
\Longrightarrow\ & e^{3x}(2xy + 3x^2y + x^2y' + y^3 + y^2y') = 0 \\
\Longrightarrow\ & 2xy + 3x^2y + x^2y' + y^3 + y^2y' = 0
\end{aligned}$$

since e^{3x} is non-zero for all $x \in \mathbb{R}$.
Thus every $y(x)$ which is a solution to the new exact equation is a solution to the original equation and vice versa.

- In general, if μ is an integrating factor, then solutions to $\mu M + \mu N y' = 0$ may not be the solutions to $M + Ny' = 0$.
- If $\mu(x, y(x))$ is non vanishing for all x in an open interval I, then the solution to exact ODE is a solution of original ODE on I.

Finding the integrating factors

Definition

We say $\mu(x, y)$ is a integrating factor of ODE

$$M(x, y) + N(x, y)y' = 0$$

if

$$\mu M + \mu N y' = 0$$

is exact, i.e.

$$\frac{d\mu}{dy} M + \mu \frac{dM}{dy} = \frac{d\mu}{dx} N + \mu \frac{dN}{dx}$$

that is,

$$\mu \left(M_y - N_x \right) = \frac{d\mu}{dx} N - \frac{d\mu}{dy} M$$

If the original equation

$$M(x, y) + N(x, y)y' = 0$$

was exact, then $\mu \equiv 1$ is an integrating factor. In general, there is no clear way to determine μ.

Case 1: If we assume that $\mu = \mu(x)$ is independent of y, then

$$\begin{aligned}
& \mu\,(M_y - N_x) = \frac{d\mu}{dx}\,N - \frac{d\mu}{dy}\,M \\
\implies \qquad & \mu\,(M_y - N_x) = \frac{d\mu}{dx}\,N \\
\implies \qquad & \frac{1}{\mu}\,\frac{d\mu}{dx} = \frac{M_y - N_x}{N} := p(x)
\end{aligned}$$

If $\dfrac{M_y - N_x}{N}$ is a function of x only

$$\implies \qquad \mu = e^{\int p(x)\ dx}$$

is an integrating factor.

Case 2: If we assume that $\mu = \mu(y)$ is independent of x, then

$$
\begin{aligned}
\mu\,(M_y - N_x) &= \frac{d\mu}{dx}\,N - \frac{d\mu}{dy}\,M \\
\implies \mu\,(M_y - N_x) &= -\frac{d\mu}{dy}\,M \\
\implies \frac{1}{\mu}\,\frac{d\mu}{dy} &= -\frac{M_y - N_x}{M} := -q(y)
\end{aligned}
$$

If $\dfrac{M_y - N_x}{M}$ is a function of y only

$$\implies \mu = e^{\int -q(y)\ dy}$$

is an integrating factor .

Case 3: If we assume that $\mu(x, y) = P(x)Q(y)$, then

$$\begin{aligned}
&\mu\left(M_y - N_x\right) = \frac{d\mu}{dx} N - \frac{d\mu}{dy} M \\
\Longrightarrow\ & P(x)Q(y)\left(M_y - N_x\right) = P'(x)Q(y)N - P(x)Q'(y)M \\
\Longrightarrow\ & M_y - N_x = \frac{P'}{P}N - \frac{Q'}{Q}M
\end{aligned}$$

If

$$M_y - N_x = p(x)N - q(y)M$$

$$\begin{aligned}
& \frac{P'}{P} = p(x), \quad \frac{Q'}{Q} = q(y) \\
\Longrightarrow \qquad & P(x) = e^{\int p(x)\,dx}, \quad Q(y) = e^{\int q(y)\,dy}
\end{aligned}$$

Then

$$\mu(x, y) = e^{\int p(x)\,dx} e^{\int q(y)\,dy}$$

is an integrating factor.

Example

Consider ODE

$$\cos x \cos y \; dx + (\sin x \cos y - \sin x \sin y + y) \; dy = 0.$$

Verify that this is not exact.

$$\begin{aligned}
& M = \cos x \cos y, \;\; N = \sin x \cos y - \sin x \sin y + y \\
& M_y - N_x = -\cos x \sin y - \cos x \cos y + \cos x \sin y \\
\implies & (N_x - M_y)/M = 1 \\
\implies & \text{ The integrating factor is } \mu = e^y \\
\implies & e^y \cos x \cos y \; dx + e^y(\sin x \cos y - \sin x \sin y + y) \; dy = 0
\end{aligned}$$

is exact. So there exists $\phi(x, y)$ such that

Example (continued ...)

$$\frac{d\phi}{dx} = e^y \cos x \cos y, \quad \frac{d\phi}{dy} = e^y(\sin x \cos y - \sin x \sin y + y)$$

Integrating first equation, we get

$$\begin{aligned}
\phi(x, y) &= e^y \sin x \cos y + h(y) \\
\implies \frac{d\phi}{dy} &= e^y \sin x \cos y - e^y \sin x \sin y + \frac{dh}{dy} \\
&= e^y(\sin x \cos y - \sin x \sin y + y) \\
\implies \frac{dh}{dy} &= ye^y \\
\implies h(y) &= e^y y + e^y + C \\
\phi(x, y) &= e^y(\sin x \cos y + y + 1) = C
\end{aligned}$$

is an implicit solution of ODE.

Example

Solve

$$(3x^2y^3 - y^2 + y)dx + (-xy + 2x)dy = 0\,.$$

$$\begin{aligned}
&M(x,y) = 3x^2y^3 - y^2 + y, \quad N(x,y) = -xy + 2x \\
&M_y - N_x = 3x^2 3y^2 - 2y + 1 + y - 2 = 9x^2y^2 - y - 1 \\
&\frac{-M_y + N_x}{M} \neq q(y), \quad \frac{M_y - N_x}{N} \neq p(x)
\end{aligned}$$

Can we write

$$M_y - N_x = p(x)N - q(y)M$$

for some $p(x)$ and $q(y)$?

Example (continued ...)

We have $M(x,y) = 3x^2y^3 - y^2 + y, \quad N(x,y) = -xy + 2x$

$M_y - N_x = 9x^2y^2 - y - 1$

Want $M_y - N_x = p(x)N - q(y)M$.

Choose $p(x) = -2/x$ and $q(y) = -3/y$. Then

$$p(x)N - q(y)M = M_y - N_x$$

The integrating factor is then given by

$$\mu(x,y) = e^{\int -2/x \ dx} e^{\int -3/y \ dy} = \frac{1}{x^2y^3}$$

We get an exact ODE

$$\frac{1}{x^2y^3}[(3x^2y^3 - y^2 + y) \ dx + (-xy + 2x) \ dy] = 0$$

Question. Is an integrating factor unique?
If μ is an integrating factor, then so is $c\mu$ for any constant $c \neq 0$.
What about upto constant multiple? No.

Example

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

is not exact.
Show that

$$\mu_1(x, y) = \frac{1}{xy(2x + y)}, \quad \mu_2(x) = x$$

both are integrating factors of ODE.

However one integrating factor may give a simpler ODE than the other.

Picard's Iteration Method

Picard's iteration method gives a rough guide to solving a given IVP. It is used in proving the existence and uniqueness theorem of the IVP $y' = f(t, y), \quad y(0) = 0.$

We will now give a rough sketch of the idea of the proof using this method. Note it is sufficient to assume the IVP is $y(0) = 0$, since the solution can obtained for any other initial condition by making appropriate substitution.

Suppose $y = \phi(t)$ is a solution to the IVP. Then,

$$\frac{d\phi}{dt} = f(t, \phi(t)), \quad \phi(0) = 0.$$

That is,

$$\phi(t) = \int_0^t f(s, \phi(s))ds, \quad \phi(0) = 0.$$

The previous equation is called an integral equation in the unknown function ϕ.

Conversely, if the integral equation holds, then by the Fundamental Theorem of Calculus,

$$y' = \frac{d\phi}{dt} = f(t, \phi(t)) = f(t, y).$$

Thus, solving the integral equation is equivalent to solving the IVP.

We define, iteratively, a sequence of functions $\phi_n(t)$ for every integer $n \geq 0$ as follows: Let

$$\begin{aligned} \phi_0(t) &\equiv 0 \\ \phi_1(t) &= \int_0^t f(s, \phi_0(s))ds \end{aligned}$$

The nth iterate is defined as,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds.$$

Note: Each ϕ_n satisfies the initial condition $\phi_n(0) = 0$. None of the ϕ_n may satisfy $y' = f(t, y)$. Suppose for some n, $\phi_{n+1} = \phi_n$. Then,

$$\phi_{n+1} = \phi_n = \int_0^t f(s, \phi_n(s))ds,$$

and this implies

$$\frac{d}{dt}(\phi_n(t)) = f(t, \phi_n(t))$$

is a solution of the given IVP.

In general, the sequence $\{\phi_n\}$ may not terminate. In fact, all the ϕ_n may not even be defined outside a small region in the domain.

However, it is possible to show that, if $f(x, y)$ and $\dfrac{df}{dy}$ is continuous in some open rectangle (hence continuous and bounded in a smaller closed rectangle), the sequence converges to a function

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

which will be the unique solution to the given IVP.

Example

Solve the IVP:

$$y' = 2t(1+y);\ y(0) = 0.$$

The corresponding integral equation is

$$\phi(t) = \int_0^t 2s(1+\phi(s))ds.$$

Let $\phi_0(t) \equiv 0$. Then,

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2sds = t^2, \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}, \\
\phi_3(t) &= \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}.
\end{aligned}$$

Example (continued ...)

We claim:

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(t) &= \int_0^t 2s(1+\phi_n(s))ds \\
&= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2n+2}}{(n+1)!}.
\end{aligned}$$

Hence $\phi_n(t)$ is the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$. Applying the ratio test, we get:

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Picard Iteration: Uniqueness

Let us give a very rough idea how to get uniqueness.

Suppose ϕ and ψ are solutions of $y' = f(x, y), y(0) = 0$. Thus, both these satisfy the integral equation as well. Then,

$$\phi(t) - \psi(t) = \int_0^t (f(s, \phi(s)) - f(s, \psi(s)))ds.$$

Thus,

$$|\phi(t) - \psi(t)| \leq \int_0^t |f(s, \phi(s) - f(s, \psi(s)|ds.$$

The crucial point is that there is a constant, say K, such that

$$|f(s, \phi(s)) - f(s, \psi(s))| \leq K|\phi(s) - \psi(s)|,$$

and this is since we assume $\dfrac{df}{dy}$ is continuous.

Let

$$U(t) = \int_0^t |\phi(s) - \psi(s)| ds.$$

Clearly, $U(0) = 0, U(t) \geq 0$.

Also, $U'(t) = |\phi(t) - \psi(t)|$.

So,

$$U'(t) - KU(t) \leq 0.$$

Thus:

$$[e^{-Kt}U(t)]' \leq 0.$$

Integrate from 0 to t and use $U(0) = 0$ to conclude $U(t) \leq 0$.

Thus,

$$U(t) \equiv 0,$$

and so

$$U'(t) \equiv 0.$$

Thus, $\phi(t) \equiv \psi(t)$.

If $a, b, c \in \mathbb{R}$ with $a \neq 0$, then

$$ay'' + by' + cy = F(x),$$

is called a **constant coefficient equation**.

We will begin with homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \quad (*)$$

Let us look for a solution of the type e^{mx}, where m is a constant. Then,

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0,$$

The quadratic polynomial

$$p(m) = am^2 + bm + c$$

is the **characteristic polynomial** of (*), and $p(m) = 0$ is the **characteristic equation**. Therefore, e^{mx} is a solution of (*) if and only if $p(m) = 0$.

The roots of the characteristic equation are given by

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider three cases:

Case 1: When $b^2 - 4ac > 0$. Then characteristic equation has two distinct real roots.

Case 2: When $b^2 - 4ac = 0$. Then characteristic equation has two repeated real roots.

Case 1: When $b^2 - 4ac < 0$. Then characteristic equation has two distinct complex roots which are conjugates.

Distinct real roots case

Example. Find general solution of $y'' + 6y' + 5y = 0 \;\; (1)$.

The characteristic polynomial is
$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$.

Thus roots of characteric equation are -1 and -5. Thus $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (1).

Note that any linear combination of the above two is also a solution

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

Ex. Solve IVP $y'' + 6y' + 5y = 0, \;\; y(0) = 3, y'(0) = 1.$

If we solve for c_1 and c_2 using the initial conditions, we get $c_1 = 4$ and $c_2 = -1$. Thus the solution to IVP is

$$y(x) = 4e^{-x} - e^{-5x}$$

A repeated real root case

Example. Find general solution of $y'' + 6y' + 9y = 0 \quad (1)$.

Characteristic polynomial $p(m) = m^2 + 6m + 9 = (m+3)^2$.

The characteristic equation has repeated roots $-3, -3$. Hence $y_1 = e^{-3x}$ is one solution. For other solution, let us try $y_2 = ue^{-3x}$.

$$\begin{aligned} y_2' &= u'e^{-3x} - 3ue^{-3x} \\ y_2'' &= u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x} \\ y_2'' + 6y' + 9y &= u''e^{-3x} = 0 \end{aligned}$$

This shows that $u(x) = c_1 x + c_2$. Thus, a solution to the ODE is $e^{-3x}(c_1 + c_2 x)$.

A repeated real root case

Ex. Solve IVP $y'' + 6y' + 9y = 0, \quad y(0) = 3, \; y'(0) = 1.$

We get $c_1 = 3$ and $1 = -3(3) + c_2$ gives $c_2 = 10$.

Thus solution of IVP is

$$y(x) = e^{-3x}(3 + 10x)$$

Ex. Find general solution of $y'' + 4y' + 13y = 0 \quad (1)$.

Characteristic polynomial is $m^2 + 4m + 13 = (m+2)^2 + 9$.

Roots of characteristic equation are $-2+3i$ and $-2-3i$.

It is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (1). Infact that is true. But they are complex valued solutions and we want real solutions. So let us write

$$e^{(-2+3i)x} = e^{-2x}(\cos 3x + i \sin 3x), \text{ and}$$
$$e^{(-2-3i)x} = e^{-2x}(\cos 3x - i \sin 3x)$$

Sum and difference gives $y_1 = e^{-2x}\cos 3x$ and $y_2 = e^{-2x}\sin 3x$ are fundamental solutions. Hence a solution is

$$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$$

Ex. Solve IVP $y'' + 4y' + 13y = 0, \;\; y(0) = 3, \; y'(0) = 1.$
$c_1 = 3$, $1 = -2(3) + 3c_2$ gives $c_2 = 7/3$.

Theorem

Let $p(m) = am^2 + bm + c$ *be the characteristic polynomial of*

$$ay'' + by' + cy = 0, \quad \textit{where} \;\; a, b, c \in \mathbb{R}, \;\; a \neq 0. \;\; \textit{Then}$$

1. *If* $p(m) = 0$ *has distinct real roots* m_1, m_2*, then the general solution is* $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
2. *If* $p(m) = 0$ *has repeated real roots* m_1, m_1*, then the general solution is* $y(x) = e^{m_1 x}(c_1 + c_2 x)$
3. *If* $p(m) = 0$ *has complex conjugate roots* $m_1 = \lambda + i\omega$ *and* $m_2 = \lambda - i\omega$*, where* $\omega > 0$*, then the general solution is*

$$y(x) = e^{\lambda x}[c_1 \cos(\omega x) + c_2 \sin(\omega x)]$$

Existence and Uniqueness: Linear Homogeneous 2^{nd} Order ODE

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = a, y'(x_0) = b,$$

where $p(x)$ and $q(x)$ are continuous on an interval I. Then there is a unique solution to the IVP on I.

Dimension Theorem

Theorem (Dimension Theorem)

If $p(x), q(x)$ are continuous on an open interval I, then the set of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

on I is a vector space of dimension 2. Any basis $\{y_1, y_2\}$ of solutions of (1) is called a **fundamental set of solutions** *of (1).*

• For example, the theorem says that once we know that e^x and e^{-x} are solutions of $y'' - y = 0$, any other solution will be of the form $y(x) = c_1e^x + c_2e^{-x}$. Here $\{e^x, e^{-x}\}$ is a fundamental set of solutions.

• Similarly, any solution of $y'' + y = 0$ are of the form $y(x) = c_1 \sin x + c_2 \cos x$. Here $\{\sin x, \cos x\}$ is a fundamental set of solutions.

Proof of Dimension Theorem

If y_1 and y_2 are solutions of (1), then $c_1y_1 + c_2y_2$ is also a solution of (1). To see this,

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) =$$

$$c_1[y_1'' + p(x)y_1' + q(x)y_1] + c_2[y_2'' + p(x)y_2' + q(x)y_2] = 0$$

Thus the solution space is a vector space.

We need to do the following two:
(i) we need to produce two linearly independent solutions, say f and g, and

(ii) show that any other solution is a linear combination of f and g.

Proof of Dimension Theorem Continued ...

(i) Proof of existence of f and g

Fix $x_0 \in I$. Let $y_1 = f(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 1, \ y'(x_0) = 0$$

y_1 exists on I by uniqueness theorem. Similarly, let $y_2 = g(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \ y'(x_0) = 1$$

We need to show that f, g are linearly independent. Assume

$$af(x) + bg(x) \equiv 0 \implies af'(x) + bg'(x) \equiv 0$$

for some scalars a and b. Evaluate at $x = x_0$, we get

$$a = 0, \quad b = 0$$

This proves f and g are linearly independent. Now we show **(ii)** that any solution is a linear combination of f and g.

Proof of Dimension Theorem Continued ...

Let $h(x)$ be an arbitrary solution of the given ODE. We want to find c and d in $\mathbb{R}$ such that

$$h(x) = cf(x) + dg(x)\,.$$

Note that this implies

$$h'(x) = cf'(x) + dg'(x) \quad \text{on} \quad I\,.$$

Therefore, evaluating at $x = x_0$ gives

$$h(x_0) = cf(x_0) + dg(x_0) = c \quad \text{and} \quad h'(x_0) = cf'(x_0) + dg'(x_0) = d$$

Let $\widetilde{h}(x) = h(x_0)f(x) + h'(x_0)g(x)$. Then $\widetilde{h}(x)$ is a solution of

$$y'' + p(x)y' + q(x)y = 0, \quad \widetilde{h}(x_0) = h(x_0),\ \widetilde{h}'(x_0) = h'(x_0) \quad (2)$$

Since $h(x)$ is also a solution of IVP (2), by uniqueness theorem, $\widetilde{h} = h$. Thus any solution is a linear combination of f and g. Therefore the solution space is 2-dimensional. $\square$

Given two solutions f and g of $y'' + p(x)y' + q(x)y = 0$. How to check whether f and g are linearly independent?

Definition

Let f and g be two differentiable functions on I. The **Wronskian** of $f(x)$ and $g(x)$ is a function defined by

$$W(f, g; x) := \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x).$$

Ex 1. Find Wronskian of e^x and e^{-x} at $x = 0$.

$W(e^x, e^{-x}, 0) = e^x(-e^{-x}) - e^{-x}e^x|_{x=0} = -2.$

2. $W(\sin x, \cos x, 0) = \sin x(-\sin x) - \cos x(\cos x)|_{x=0} = -1.$

Theorem (Abel's Formula)

Assume $p(x)$ and $q(x)$ are continuous on $I = (a, b)$. Let $f(x)$ and $g(x)$ be solutions of $y'' + p(x)y' + q(x)y = 0$.
Then Wronskian of $f(x)$ and $g(x)$ is given by

$$W(f, g; x) = W(f, g; x_0)\, e^{-\int_{x_0}^{x} p(t)dt},$$

for any $x_0 \in I$.

Proof. Set $W(f, g; x) = W(x)$. Then

$$W(x) = (fg' - f'g)(x).$$

Hence

$$W'(x) = (fg'' + f'g') - (f'g' + f''g)(x) = (fg'' - f''g)(x)$$

Wronskian and Linear Independence

Now f and g are solutions, hence

$$f'' = -p(x)f' - q(x)f, \quad g'' = -p(x)g' - q(x)g.$$

Thus,

$$\begin{aligned} W'(x) &= fg'' - f''g = f(-pg' - qg) - g(-pf' - qf) \\ &= -p(fg' - f'g) = -pW(x). \end{aligned}$$

Hence,

$$W(x) = ce^{-\int_{x_0}^{x} p(t)dt},$$

for a constant c. For $x = x_0$, we get $W(x_0) = c$. Hence,

$$W(f, g; x) = W(f, g; x_0)\, e^{-\int_{x_0}^{x} p(t)dt} \quad \text{on} \quad I.$$

• Thus $W(x_0) = 0$ for some $x_0 \in I \implies W(x) \equiv 0$ on I.
• Similarly, $W(x_0) \neq 0$ for some $x_0 \in I \implies W(x)$ does not take zero value on I.

Ex. Consider ODE $x^2y'' + xy' - 4y = 0$.
Here $y_1 = x^2$ and $y_2 = \dfrac{1}{x^2}$ are solutions. Compute the Wronskian $W(y_1, y_2; x)$.

Direct method: $W = y_1y_2' - y_1'y_2 = x^2(\dfrac{-2}{x^3}) - (2x)\dfrac{1}{x^2} = \dfrac{-4}{x}$.

Let's verify Abel's Formula: If x_0 and x both are in $(-\infty, 0)$ or in $(0, \infty)$, then

$$W(x) = W(x_0)\, exp\left[-\int_{x_0}^{x} p(t)\, dt\right] = W(x_0)\, exp\left[\int_{x_0}^{x} \frac{-1}{t}\, dt\right]$$

$$= W(x_0)\, exp\left[-(\ln|x| - \ln|x_0|)\right] = W(x_0)\, exp(\ln\frac{x_0}{x})$$

$$= \frac{-4}{x_0}\,\frac{x_0}{x} = -4/x$$

Wronskian and Linear Independence

Proposition

Suppose $f(x)$ and $g(x)$ are linearly dependent and differentiable on $I = (a, b)$. Then, $W(f, g; x) = 0$ for all $x \in I$.

Proof. As $f(x)$ and $g(x)$ are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0 \implies cf'(x) + dg'(x) = 0.$$

Hence,

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\begin{pmatrix} c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $W(f, g; x_0) = 0$ for all $x_0 \in I$. $\square$

Note: The converse is not true, i.e. it may happen that $W(f,g,x) \equiv 0$ on I, but f and g are linearly independent.

Ex. $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

then, check that $W(f,g;x) = 0$ for all $x \in \mathbb{R}$.
But f and g are linearly independent.

Show that $af + bg \equiv 0 \implies a = 0 = b$. $\square$

• In the next slide, we will see that the above cannot happen when f and g are solutions of an ODE.

Wronskian and Linear Independence

Theorem

Consider

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are continuous on $I = (a, b)$. Suppose f and g are solutions on I. Then f and g are linearly independent on I if and only if $W(f, g; x)$ has no zeros in I.

Proof. (i) ($\Rightarrow$). It is enough to show that if $W(x_0) = 0$ for some $x_0 \in I$, then f and g are linearly dependent.

Since f, g are linearly independent, $f(x_0) \neq 0$ for some $x_0 \in I$.

Choose an open interval J containing x_0 such that f does not take zero value on J. On J, we have:

$$\left(\frac{g}{f}\right)'(x) = \left(\frac{fg' - f'g}{f^2}\right)(x) = \frac{W(f, g; x)}{f^2(x)} = 0$$

since $W(x_0) = 0 \implies W(x) = W(x_0)e^{\int_{x_0}^{x} -p(t)dt} \equiv 0.$

$$\left(\frac{g}{f}\right)' \equiv 0 \quad \text{on} \quad J \implies \frac{g}{f} = k$$

a constant on J. Hence $g(x) = kf(x)$ on J.

But we want $g(x) = kf(x)$ on I. For this, consider the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, y'(x_0) = 0 \quad (*)$$

$y_1 \equiv 0$ and $y_2 = g - kf$ both are solutions of (*). By uniqueness theorem, $y_1 = y_2$ on I. Hence $g(x) = kf(x)$ on I.

Now we have to prove $(\Leftarrow)$. It is enough to show that if f and g are linearly dependent, then $W(f, g, ; x) \equiv 0$. This was proved earlier. $\square$

Remarks:

1. The continuity of $p(x)$ and $q(x)$ is required in the above theorem. Consider the ODE

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0.$$

   Then, x^2 and x^3 are linearly independent solutions, but $W(x^2, x^3; 0) = 0$.

2. Let $\{f, g\}$ be two linearly independent functions such that $W(f, g; a) = 0$ for some $a \in \mathbb{R}$. Then there is no ODE $y'' + p(x)y' + q(x)y = 0$ with $p(x)$ and $q(x)$ continuous in a neighborhood of a, such that f and g are solutions to this ODE.

Finding second solution by Variation of parameters

Consider the 2nd order linear ODE

$$y'' + p(x)y' + q(x)y = 0$$

Assume that we know one solution to the above ODE, call it $y_1(x)$.

Then we can try and look for a second solution of the type $y(x) = u(x)y_1(x)$.

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

Substituting into the differential equation and using that $y_1(x)$ is a solution to the ODE, we get that u satisfies

$$u''y_1 + 2u'y_1' + p(x)u'y_1 = 0\,.$$

We rewrite the equation in u as

$$\frac{d}{dx}\Big(\frac{u'}{u''}\Big) = -\Big(\frac{2y_1' + p(x)y_1}{y_1}\Big).$$

Therefore,

$$\ln|v'| = \ln\left(\frac{1}{f^2}\right) - \int p dx;$$

$$\implies v = \int \frac{e^{-\int p dx}}{f^2} dx.$$

Let us show that f and vf are linearly independent.
We can show non-vanishing of Wronskian at a point.

$$\begin{aligned} W(f, vf) &= f(v'f + f'v) - f'vf \\ &= f^2 v' \\ &= f^2 \frac{e^{-\int p\,dx}}{f^2} = e^{-\int p\,dx} \neq 0 \end{aligned}$$

Theorem

Let p, q be continuous on some open interval I.
If $y_1 \neq 0$ is a solution of

$$y'' + p(x)y' + q(x)y = 0$$

on I, then another solution of ODE on I is given by

$$\begin{aligned} y_2(x) &= v y_1(x) \\ &= \left(\int \frac{e^{-\int p dx}}{y_1^2} dx \right) y_1(x) \end{aligned}$$

Further, y_1 and y_2 are linearly independent on I.

Example

Find all solutions of $\quad x^2y'' + xy' - y = 0, \qquad x > 0.$
One solution $y_1 = x$ is given. The ODE in standard form is

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0.$$

Let $y_2 = vx$ be another solution. Then,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p dx}}{y_1^2} dx = \int \frac{e^{-\int (1/x) dx}}{x^2} dx \\ &= \int \frac{dx}{x^3} = -\frac{1}{2x^2}. \end{aligned}$$

Hence, $y_2 = \dfrac{-1}{2x}$. Therefore, the general solution is

$$y(x) = cx + \frac{d}{x}, \quad c, d \in \mathbb{R}.$$

Cauchy-Euler Equations

In general, the equation

$$x^2y'' + axy' + by = 0, \qquad x > 0$$

where $a, b \in \mathbb{R}$, is called a Cauchy-Euler equation.
This equation can be transformed into one with constant coefficients by change of variables on the interval $(0, \infty)$.

$$\begin{aligned}
t = \ln x \quad &\implies \quad x = e^t \\
\frac{dy}{dt} \quad &= \quad \frac{dy}{dx}\,\frac{dx}{dt} = \frac{dy}{dx}\,e^t \\
\frac{d^2y}{dt^2} \quad &= \quad \frac{d}{dt}\left[\frac{dy}{dx}e^t\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right]e^t + \frac{dy}{dx}\,e^t \\
&= \quad \frac{d}{dx}\left[\frac{dy}{dx}\right]e^{2t} + \frac{dy}{dt} \\
\frac{d^2y}{dx^2} \quad &= \quad \frac{1}{e^{2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)
\end{aligned}$$

We get a second order ODE with constant coefficients

$$y''(t) + (a-1)y'(t) + by(t) = 0\,.$$

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions to this equation, then the solutions to the Cauchy-Euler equation is given by

$$y(x) = c_1 y_1(\ln x) + c_2 y_2(\ln x)\,.$$

Theorem

Consider the Cauchy-Euler Equation

$$x^2y'' + axy' + by = 0, \qquad x > 0 \qquad (*)$$

Substituting $x = e^t$, *ODE becomes*

$$y''(t) + (a-1)y'(t) + by(t) = 0 \qquad (**)$$

Let m_1 *and* m_2 *be the roots of the char equation of* $(**)$

$$p(m) = m^2 + (a-1)m + b = 0.$$

Then the general solution of $(*)$ *is given as follows.*

1. *If* $m_1 \neq m_2 \in \mathbb{R}$, *then* $y(x) = c_1x^{m_1} + c_2x^{m_2}$.
2. *If* $m_1 = m_2 \in \mathbb{R}$, $y(x) = c_1x^m + c_2x^m \ln x$.
3. *If* $m_i = \lambda \pm i\omega$ *with* $\omega > 0$, *then*

$$y(x) = c_1x^\lambda \cos(\omega \ln x) + c_2x^\lambda \sin(\omega \ln x).$$

Example

Solve

$$x^2y'' + 7xy' + 5y = 0$$

Putting $t = \ln x$, we get

$$y''(t) + (7-1)y'(t) + 5y(t) = 0$$

It's characteristic equation is

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$$

Hence the general solution is

$$y(x) = c_1 \frac{1}{x} + c_2 \frac{1}{x^5}$$

Example

Solve

$$x^2y'' + 7xy' + 9y = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (7-1)m + 9 = (m+3)^2$$

Hence the general solution is

$$y(x) = c_1\frac{1}{x^3} + c_2\frac{1}{x^3}\ln x$$

Example

Solve

$$x^2y'' + 5xy' + 13y = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9$$

Hence the general solution is

$$y(x) = \frac{1}{x^2}[c_1 \cos(3\ln x) + c_2 \sin(3\ln x)]$$

Nonhomogeneous 2nd order linear ODE

Consider 2nd order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

with $p(x), q(x), r(x)$ continuous on open interval I.

The homogeneous part is

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

We have seen that solution space of (2) is a 2-dimensional vector space.

Suppose y_1 is a solution of (1) and y_2 is a solution of (2), then $y_1 + y_2$ is a solution of (1).

Similarly, if y, y_1 are solutions of (1), then $y - y_1$ is a solution of (2).

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

We can use variation of parameters to do the following.

If we can find two linearly independent solutions y_1 and y_2 of

$$y'' + p(x)y' + q(x)y = 0 \qquad (1).$$

then we can find a particular solution of

$$y'' + p(x)y' + q(x)y = r(x) \qquad (2)$$

where p, q, r are continuous of an open interval I.

Here, we try to find a partcular solution of (2) of the form

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)\,.$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

Now

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

gives

$$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2.$$

Let us assume that v_1 and v_2 satisfy

$$v_1' y_1 + v_2' y_2 = 0.$$

Then

$$y' = v_1 y_1' + v_2 y_2' \, .$$

Thus,

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

Substituting y, y', y'' in the given non-homogeneous ODE, we get:

$$(v_1y_1''+v_1'y_1'+v_2y_2''+v_2'y_2')+p(v_1y_1'+v_2y_2')+q(v_1y_1+v_2y_2) = r(x)$$

$$\iff v_1(y_1''+py_1'+qy_1)+v_2(y_2''+py_2'+qy_2)+v_1'y_1'+v_2'y_2' = r(x).$$

$$\iff v_1'y_1'+v_2'y_2' = r(x).$$

Recall that we also have

$$v_1'y_1+v_2'y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

Therefore,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \;\; v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Hence,

$$\begin{aligned} y &= v_1 y_1 + v_2 y_2 \\ &= y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx. \end{aligned}$$

Ex. Solve $y'' + 6y' + 5y = e^x$.

$y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are two linearly independent solutions of homogeneous part.

Wronskian of y_1 and y_2 is
$W(e^{-x}, e^{-5x}) = e^{-x}(-5e^{-5x}) - (-e^{-x})e^{-5x} = -4e^{-6x}$.

A particular solution y_p is given by variation of parameter:

$$y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx$$

$$= e^{-5x} \int \frac{e^{-x} e^{x}}{-4e^{-6x}}\, dx - e^{-x} \int \frac{e^{-5x} e^{x}}{-4e^{-6x}}\, dx$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

$$= -\frac{1}{4}\left[e^{-5x}\int e^{6x}\,dx - e^{-x}\int e^{2x}\,dx\right]$$

$$= -\frac{1}{4}\left[\frac{1}{6}e^{x} - \frac{1}{2}e^{x}\right] = \frac{1}{12}e^{x}\,.$$

Thus the general solution is $y(x) = \dfrac{1}{12}e^{x} + c_1 e^{-x} + c_2 e^{-5x}\,.$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

Ex. Solve $y'' + 6y' + 5y = e^{-x}$.

$y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are two linearly independent solutions of homogeneous part.

Wronskian $W(e^{-x}, e^{-5x}) = -4e^{-6x}$.

A particular solution y_p is given by

$$y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx$$

$$= e^{-5x} \int \frac{e^{-x} e^{-x}}{-4e^{-6x}}\, dx - e^{-x} \int \frac{e^{-5x} e^{-x}}{-4e^{-6x}}\, dx$$

$$= -\frac{1}{4} \left[e^{-5x} \int e^{4x}\, dx - e^{-x} \int dx \right]$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

$$= -\frac{1}{4}\left[e^{-x}(\frac{1}{4}-x)\right] = -\frac{1}{16}e^{-x}(1-4x)\,.$$

Thus the general solution is given by

$$y(x) = -\frac{1}{16}e^{-x}(1-4x)+c_1e^{-x}+c_2e^{-5x} = \frac{1}{4}xe^{-x}+c_1e^{-x}+c_2e^{-5x}\,.$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

Ex. Find a particular solution of $y'' + 4y = 3\cos 2t$.

$y_1 = \cos 2t,\ y_2 = \sin 2t$ are solutions of homogeneous part.

Wronskian $W(y_1, y_2) = 2$.

A particular solution y_p is given

$$y_p = y_2 \int \frac{y_1 r}{W(y_1, y_2)}\, dt - y_1 \int \frac{y_2 r}{W(y_1, y_2)}\, dt$$

$$= \sin 2t \int \frac{\cos 2t \cdot 3\cos 2t}{2} dt - \cos 2t \int \frac{\sin 2t \cdot 3\cos 2t}{2} dt$$

$$= \sin 2t \int \frac{3}{4}(1 + \cos 4t)\, dt - \cos 2t \int \frac{3}{4} \sin 4t\, dt$$

Nonhomogeneous 2nd order linear ODE: Variation of Parameters

$$
\begin{aligned}
&= \frac{3}{4}\sin 2t\left[t + \frac{1}{4}\sin 4t\right] - \frac{3}{4}\cos 2t(-\frac{1}{4}\cos 4t) \\
&= \frac{3}{4}t\sin 2t + \frac{3}{16}[\sin 2t \sin 4t + \cos 2t\cos 4t] \\
&= \frac{3}{4}t\sin 2t + \frac{3}{16}\cos 2t.
\end{aligned}
$$

n^{th} order differential equations

An n-order linear ODE is a differential equation of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x),$$

where a_i's and g are continuous on some open interval I.

An n-order linear ODE in standard form is given by

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

As in 2nd order case, an n-order linear ODE is said to be **homogeneous** if $r \equiv 0$ and **non-homogeneous** otherwise.

Existence and Uniqueness theorem

Def. An initial value problem for an n-order ODE is given by

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0 \qquad (1)$$

$$y(x_0) = k_0,\ y^1(x_0) = k_1,\ \ \ldots,\ y^{n-1}(x_0) = k_{n-1} \quad (2)$$

where p_i's are continuous on an open interval I and $x_0 \in I$.

Theorem (Existence and Uniqueness)

If $p_i(x)$ are continuous throughout an open interval I containing x_0, then the IVP defined above has a unique solution on I.

Define the vector space

$$C^n(I) = \{f : I \to \mathbb{R} \mid f, f^1, \dots, f^{(n)} \text{ are continuous}\}.$$

Let $f_1, f_2, \dots, f_n \in C^n(I)$. Define their **Wronskian,**

$$W(f_1, \dots, f_n; x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ \vdots & \vdots & \dots & \vdots \\ f_1^{(n)}(x) & f_2^{(n)}(x) & \dots & f_n^{(n)}(x) \end{vmatrix}$$

As in the second order case, we have the following theorem.

Theorem (Abel's theorem)

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)$$

where $p_1, \ldots, p_n$ *are continuous on an open interval* I*. Let* $y_1, \ldots, y_n$ *be solutions to the linear ODE* $Ly = 0$*.*
Let $x_0 \in I$*. Then the Wronskian of* $\{y_1, \cdots, y_n\}$ *is given by*

$$W(x) = W(x_0)\mathit{exp}\left(-\int_{x_0}^{x} p_1(t)\ dt\right)$$

where $x \in I$*. Thus* W *either has no zeros on* I *or* $W \equiv 0$ *on* I*.*

Then we have the following theorem, analogous to the second order case.

Theorem

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)I$$

where $p_1, \ldots, p_n$ are continuous on an open interval I. Let $y_1, \ldots, y_n$ be solutions to the linear ODE $Ly = 0$. Then the following statements are equivalent.

1. *The set $\{y_1, \ldots y_n\}$ is a fundamental set of solutions of $Ly = 0$ on I.*
2. *$\{y_1, \ldots, y_n\}$ is linearly independent on I.*
3. *The Wronskian of $\{y_1, \ldots, y_n\}$ is nonzero at some point on I.*
4. *The Wronskian of $\{y_1, \ldots, y_n\}$ is nonzero for all $x \in I$.*

Dimension Theorem - n^{th} order

We can prove the dimension theorem for, the nth order case, as in the second order case.

Theorem (Dimension Theorem)

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I$$

where p_i's are continuous on an open interval I. Then dimension of the vector space of solutions to the equation $Ly = 0$ is n.

Proof. Let $x_0 \in I$ and let $\{e_1, \ldots, e_n\}$ be the standard basis vectors of $\mathbb{R}^n$.
By existence and uniqueness theorem, the IVP

$$Ly = 0; \quad (y(x_0), y'(x_0), \ldots, y^{(n)}(x_0)) = e_i$$

has a unique solution y_i on I for all $i = 1, \ldots n$.

Dimension Theorem - n^{th} order

Clearly $W(y_1, \ldots, y_n; x_0) = 1$ is non-zero. Therefore $\{y_1, \ldots, y_n\}$ are linearly independent solutions.

Further, assume y is any solution to $Ly = 0$. Then

$$z(x) = y(x_0)y_1(x) + y'(x_0)y_2(x) + \ldots + y^{(n-1)}(x_0)y_n(x)$$

is a solution of $L(y) = 0$.

Since

$$z(x_0) = y(x_0),\ z'(x_0) = y'(x_0),\ \ \ldots,\ z^{(n-1)}(x_0) = y^{(n-1)}(x_0)$$

using the existence and uniqueness theorem to IVP, we get

$$y(x) \equiv z(x) \quad \text{on} \quad I$$

This proves the dimension theorem. $\square$

Constant Coefficient Differential Operators

Definition

Consider

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Thus for any function $f \in C^n(I)$,

$$L(f) = a_0 f^{(n)} + a_1 f^{(n-1)} + \ldots + a_{n-1} f' + a_n f.$$

Such an L is called a **constant coefficient differential operator**.

Ex. $D^2 - 5D + 6 = (D-3)(D-2)$ as a linear transformation from $\mathcal{C}^2(I) \to \mathcal{C}(I)$, i.e. for any $y \in \mathcal{C}^2(I)$,

$$\begin{aligned}
(D-3)(D-2)y &= (D-3)(y' - 2y) \\
&= (y'' - 5y' + 6y) \\
(D-2)(D-3)y &= (D-2)(y' - 3y) \\
&= y'' - 5y' + 6y
\end{aligned}$$

Hence $(D-2)(D-3) = (D-3)(D-2)$.

Ex. Check that $(D+1)(D^2+D+1) = (D^2+D+1)(D+1)$ as a function from $\mathcal{C}^3(I) \to \mathbb{R}$. i.e. for each $y \in \mathcal{C}^3(I)$,

$$(D+1)(D^2+D+1)y = (D^2+D+1)(D+1)y$$

Constant Coefficient Differential Operators

- The main point is that

$$D^r \circ D^s = D^s \circ D^r = D^{r+s}$$

Hence if $L = \sum_{i=0}^{n} a_{n-i} D^i$ and $M = \sum_{i=0}^{m} b_{m-i} D^i$ are differential operators with constant coefficient, then

$$L(M(f)) = M(L(f)),\ \forall f \in \mathcal{C}^{m+n} \implies LM = ML$$

Note that it is important that L and M are constant coefficient equations.

Ex. If $L = D + xI$ and $M = D + 1$, then

$$\begin{aligned} LM(f) &= L(f' + f) \\ &= (f'' + f') + x(f' + f) \\ ML(f) &= M(f' + xf) \\ &= (f'' + f + xf') + (f' + xf) \\ &= LM(f) + f \neq LM(f) \end{aligned}$$

Ex. Solve $y''' - 7y' - 6y = 0$.

We notice that this is same as the solution space of

$$Ly = (D^3 - 7D - 6I)y = 0$$

But $\quad L = (D^3 - 7D - 6I) = (D-3)(D+1)(D+2)$.

Now, note that if y is such that $(D+2)y = 0$, then $Ly = 0$.

$$L = (D-3)(D+1)(D+2)$$

$$= (D+2)(D+1)(D-3) = (D-3)(D+2)(D+1)$$

If $(D+1)y = 0$ or $(D-3)y = 0$, then $Ly = 0$.

This gives us that $f(x) = e^{-x}$, $g(x) = e^{-2x}$ and $h(x) = e^{3x}$ are solutions to $Ly = 0$.

By dimension theorem, if they are linearly independent, then they give basis for all solutions. We can check that these are linearly independent by checking that the Wronskian is nonzero.

Then $W(f,g,h;0) = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = -30 + 12 - 2 = -20 \neq 0.$

$\implies$ f, g and h are linearly independent solutions of the ODE $y''' - 7y' - 6y = 0$ on any interval I containing 0.

Constant Coefficient Differential Operators

Consider

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n, \quad a_i \in \mathbb{R}$$

We define the characteristic polynomial to be

$$P_L(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$

Theorem

Let L and M be two constant coefficient linear differential operators. Then,

1. $L = M$ *if and only if* $P_L(x) = P_M(x)$.
2. $P_{L+M}(x) = P_L(x) + P_M(x)$.
3. $P_{LM}(x) = P_L(x) \cdot P_M(x)$.

All the above statements are obvious.

Definition

Given a constant coefficient differential operator L we define $\mathrm{Ker}(L)$ to be the space of functions y such that $Ly = 0$.

By the dimension theorem, this is a vector space of dimension n, where n is the order of the differential operator.

We shall next investigate how to find $\mathrm{Ker}(L)$ for any L.

Let us begin with the simplest case, when

$$P_L(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n, \qquad a_i \in \mathbb{R}\,.$$

Assume that the roots of $P_L(x)$ are distinct and are all real. Then

$$P_L(x) = a_0(x - r_1)(x - r_2)\ldots(x - r_n)\,.$$

Let us assume that

$$r_1 < r_2 < \cdots < r_n \,.$$

Since $D - r_i$ commute with each other, and since $e^{r_i x}$ is a solution to $(D - r_i)y = 0$, we get that

$$\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\} \subset \mathrm{Ker}(L) \,.$$

We claim that these functions are linearly independent. If not, suppose there is a relation of the type

$$\lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x} + \cdots + \lambda_n e^{r_n x} = 0 \,, \qquad \lambda_i \in \mathbb{R} \,.$$

Not all the λ_i are 0. Let j be the largest such that $\lambda_j \neq 0$. Multiply the equation with -1 is necessary and assume that $\lambda_j > 0$. Also multiply the equation with $e^{-r_n x}$ to get

$$\lambda_1 e^{(r_1 - r_n)x} + \lambda_2 e^{(r_2 - r_n)x} + \cdots + \lambda_j = 0\,.$$

Since we have

$$r_1 - r_n < r_2 - r_n < \ldots < 0$$

and $\lambda_j > 0$, taking $x >> 0$ we get a contradiction.

Alternatively, we could have used Abel's theorem, but then we would have to show that the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix} \neq 0$$

Ex. Solve $y^{(3)} - 7y' + 6y = 0.$
Here,

$$L = D^3 - 7D + 6,$$

and hence

$$P_L(x) = x^3 - 7x + 6 = (x-1)(x-2)(x+3).$$

Therefore,

$$L = (D-1)(D-2)(D+3).$$

Note that

$$e^x \in \mathrm{Ker}(D-1),\ e^{2x} \in \mathrm{Ker}(D-2),\ e^{-3x} \in \mathrm{Ker}(D+3).$$

Thus, $e^x, e^{2x}, e^{-3x} \in \mathrm{Ker}(L)$. Hence, $\{e^x, e^{2x}, e^{-3x}\}$ is a basis of Ker L.
Thus, the general solution is of the form

$$c_1e^x + c_2e^{2x} + c_3e^{-3x}\,.$$

Q. What if $P_L(x)$ has some repeated real roots?

Let us begin with the following result.

Proposition

For any real number r, the functions

$$u_1(x) = e^{rx}, u_2(x) = xe^{rx}, \ldots, u_m(x) = x^{m-1}e^{rx}$$

are linearly independent and

$$u_1(x), \ldots, u_m(x) \in \mathrm{Ker}((D-r)^m).$$

Proof. That these functions are linearly independent is obvious, since $\{1, x, x^2, \ldots, x^m\}$ are linearly independent (e^{rx} is non-zero). We need to show that these functions are in Ker $(D-r)^m$.
When $m = 1$, we need to show

$$u_1(x) = e^{rx} \in \mathrm{Ker}((D-r)),$$

which is true, since

$$(D-r)(e^{rx}) = re^{rx} - re^{rx} = 0.$$

Suppose $m = 2$. Since u_1 is in Ker of $(D-r)$, it's in Ker of $(D-r)^2$. What about $u_2 = xe^{rx}$?

$$\begin{aligned}(D-r)^2(xe^{rx}) &= (D-r)(D-r)(xe^{rx}) \\ &= (D-r)(xre^{rx} + e^{rx} - rxe^{rx}) \\ &= (D-r)(e^{rx}) = 0.\end{aligned}$$

Use induction to prove general case.
Assume

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^{m-1}),$$

and we need to show that

$$u_1, u_2, \ldots, u_m \in \mathrm{Ker}((D-r)^m).$$

Clearly

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^{m-1}) \subseteq \mathrm{Ker}((D-r)^m).$$

To show that u_m is also in $\mathrm{Ker}((D-r)^m)$, compute
$(D-r)^m(x^{m-1}e^{rx})$

$$\begin{aligned}
&= (D-r)^{m-1}(D-r)(x^{m-1}e^{rx}) \\
&= (D-r)^{m-1}(x^{m-1}re^{rx} + (m-1)x^{m-2}e^{rx} - rx^{m-1}e^{rx}) \\
&= (D-r)^{m-1}((m-1)x^{m-2}e^{rx}) = 0.
\end{aligned}$$

Therefore, a basis for solution space of $(D-r)^m$ is

$$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}\,.$$

Thus, if

$$P_L(x) = (x-r_1)^{e_1}(x-r_2)^{e_2}\ldots(x-r_\ell)^{e_\ell},$$

where $\displaystyle\sum_{i=1}^{\ell} e_i = n$, then a basis of Ker L is given by

$$e^{r_1x}, \ldots, x^{e_1-1}e^{r_1x}, e^{r_2x}, \ldots, x^{e_2-1}e^{r_2x}, \ldots, e^{r_\ell x}, \ldots, x^{e_\ell-1}e^{r_\ell x}.$$

The above functions are linearly independent and since dim Ker $L=n$, these form a basis.
Ex: Check that the above functions are linearly independent by evaluating the Wronskian at 0.

Ex: Find the general solution of the ODE:

$$L(y) = (D^3 - D^2 - 8D - 12)(y) = 0.$$

We have

$$P_L(x) = x^3 - x^2 - 8x - 12 = (x-2)^2(x+3),$$

and therefore,

$$L = (D-2)^2(D+3).$$

Thus the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Ex: Find the general solution of the ODE:

$$L(y) = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

Now,

$$L = D^2(D-1)(D+1)^3.$$

Ker D^2 is $\{1, x\}$
Ker $(D-1)$ is $\{e^x\}$.
Ker $(D+1)^3$ is $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$.
Thus, the general solution is

$$c_1 + c_2x + c_3e^x + c_4e^{-x} + c_5xe^{-x} + c_6x^2e^{-x},$$

with $c_i \in \mathbb{R}$.

Assume $P_L(x)$ has some complex roots. In the 2nd order case, if $m_1 = a + \imath b, m_2 = a - \imath b$, then $y_1 = e^{ax} \cos bx$ and $y_2 = e^{ax} \sin bx$ were the basis for $N(L)$.

If $P_L(x)$ has a complex root $a + \imath b$, then it also has $a - \imath b$ as a root. Thus,

$$(x - (a + \imath b))(x - (a - \imath b)) = (x - a)^2 + b^2$$

is a factor of $P_L(x)$.

Null space of $(D - a)^2 + b^2$ has a basis

$$\{e^{ax} \cos bx, e^{ax} \sin bx\} \subset \mathrm{Ker}(L)$$

If $a \pm \imath b$ is a root of $P_L(x)$ of multiplicity m, then $((D-a)^2+b^2)^m$ is a factor of $P_L(x)$.

Can we find the null space of $((D-a)^2+b^2)^m$?

Ex. Check that

$$\begin{aligned} e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m-1}e^{ax}\cos bx,\\ e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m-1}e^{ax}\sin bx. \end{aligned}$$

is a basis for null space of $((D-a)^2+b^2)^m$.

Ex: Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(x-1)(x^2 - 4x + 5)^2.$$

The roots are

$$1, 2 \pm \imath, 2 \pm \imath.$$

Hence, the general solution is

$$y = c_1 e^x + e^{2x}[c_2 \cos x + c_3 \sin x + c_4 x \cos x + c_5 x \sin x],$$

where $c_i \in \mathbb{R}$.

Ex: Find the fundamental set of solutions to

$$D^3(D-2I)^2(D^2+4I)^2y = 0\,.$$

The fundamental set will be given by

$$\{1, x, x^2, e^{2x}, xe^{2x}, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x\}\,.$$

The Annihilator method or **method of undetermined coefficients** helps us in finding a particular solution of a non-homogeneous equation.

Example: Find a particular solution of

$$y^{(4)} - 16y = x^4 + x + 1 = r(x).$$

Here, $\quad L = D^4 - 16.$

Let us take $A = D^5$. Then $A(r(x)) = 0$.

We say A annihilates or kills $r(x)$.

A solution y of $L(y) = r(x)$ is also a solution of $A(L(y)) = 0$

$$D^5(D^4 - 16)y = 0.$$

$AL = D^5(D^4 - 16)$ has characteristic equation

$$x^5(x^4 - 16) = x^5(x-2)(x+2)(x^2+4).$$

A general solution of $(AL)(y) = 0$ is of the form

$$c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6e^{2x} + c_7e^{-2x} + c_8\cos 2x + c_9\sin 2x.$$

Here $c_6e^{2x} + c_7e^{-2x} + c_8\cos 2x + c_9\sin 2x$ is a solution of the homogeneous part $(D^4 - 16)y = 0$.

We want a particular solution y_p for $y^{(4)} - 16y = x^4 + x + 1$.

This implies that we can take $y_p = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$, since all the other terms are solutions to the corresponding homogenous ODE.

To find c_i's in $y_p = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$, solve $y_p^{(4)} - 16y_p = x^4 + x + 1$.

Then $24c_5 - 16(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4) = x^4 + x + 1$.

Equating the coefficients, we get

$$
\begin{aligned}
24c_5 - 16c_1 &= 1 \\
-16c_2 &= 1 \\
-16c_3 &= 0 \\
-16_c4 &= 0 \\
-16c_5 &= 1
\end{aligned}
$$

This gives $c_3 = c_4 = 0$, $c_5 = c_2 = -1/16$,
$-16c_1 = 1 - 24c_5 = 1 + 3/2 = 5/2$, hence $c_1 = -5/32$.
Therefore $y_p = -\dfrac{5}{32} - \dfrac{1}{16}x - \dfrac{1}{16}x^4$.

Ex. Solve $y^{(4)} - 4y'' = e^x + x^2$.
Let $L = D^4 - 4D^2 = D^2(D-2)(D+2)$. Let $z(x)$ and $w(x)$ be such that $Lz = e^x$ and $Lw = x^2$. Then $L(z+w) = e^x + x^2$.

Let us first solve $Lz = e^x$. We know that e^x is a solution of $My = (D - I)y = 0$.

Now, $MLz = (D-I)D^2(D-2I)(D+2I)z = 0$.

Clearly z satisfies this equation. Hence z will be of the form $z = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} + c_5 e^x$.

But $\{1, x, e^{2x}, e^{-2x}\}$ are all solution to $Ly = 0$ and therefore, $z = c_5 e^x$ for some $c_5 \in \mathbb{R}$.

Plugging $z = c_5 e^x$ into the equation $y^{(4)} - 4y'' = e^x$, we have $c_5 - 4c_5 = 1 \implies c_5 = -1/3$. Thus $\boxed{z = (-1/3)e^x}$.

Example. Let's solve $Lw = x^2$, where $L = (D^4 - 4)$. Note that x^2 is a solution to $Ny = D^3 y = 0$.

Then $NLw = D^3 D^2 (D - 2I)(D + 2I)w = 0$. Clearly $w = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 e^{2x} + c_7 e^{-2x}$.

But $c_1 + c_2x + +c_6e^{2x} + c_7e^{-2x}$ are solutions to $Ly = 0$.
Therefore, $w = c_3x^2 + c_4x^3 + c_5x^4$.

Substituting in $Lw = (D^4 - D^2)w = x^2$, we get

$$24c_5 - 2c_3 + 6c_4x + 12c_5x^2 = x^2$$

This implies, $24c_5 - 2c_3 = 0$, $c_4 = 0$, $c_5 = 1/12$ and $c_3 = 1$.

Therefore, $w = x^2 + \dfrac{1}{12}x^4$.

Hence a particular solution to $Ly = e^x + x^2$ is given by

$$y_p = z + w = -\frac{1}{3}e^x + x^2 + \frac{1}{12}x^4$$

Summary: Anhilator Method

- Given a linear differential operator L with constant coefficients, we want to solve $Ly = r(x)$.
- We find a particular solution as follows.
- We first find linear a differential operators M which have the property that $M(r(x)) = 0$.
- Find a basis for the solution space of $Ly = 0$. Extend this to a basis for the solution space of $MLy = 0$.
- Pick those elements in the basis which are not solutions to $Ly = 0$.
- Set y_p to be a linear combination of these particular basis elements and solve $Ly_p = r(x)$ for the constants.
- A general solution to $Ly = r$ is given by $y_p + z$, where z is a general solution to $Ly = 0$.

The variation of parameters method generalizes to nth order linear ODE $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$ where p_i's and r are continuous on I.

Let $y_1, \ldots, y_n$ be a basis of solutions of homogeneous part. Assume the particular solution y_p is given by

$$y_p = v_1(x)y_1 + v_2(x)y_2 + \ldots + v_n(x)y_n$$

Assume

$$\begin{array}{rlr} v_1'y_1 + \ldots + v_n'y_n & = 0 & (1) \\ v_1'y_1' + \ldots + v_1'y_n' & = 0 & (2) \\ \vdots & & \vdots \\ v_1'y_1^{(n-2)} + \ldots + v_n'y_n^{(n-2)} & = 0 & (n-1) \end{array}$$

Compute $y_p', \ldots, y_p^{(n)}$ and put in the ODE, we get

$$v_1'y_1^{(n-1)} + \ldots + v_n'y_n^{(n-1)} = r(x) \qquad (n)$$

Thus,

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for $v_1', v_2', \ldots, v_n'$, and thus get $v_1, v_2, \ldots, v_n$, and form $\boxed{y = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n}$, here v_i' is given by

$$v_i' = \frac{\begin{vmatrix} y_1 & \cdot & y_{i-1} & 0 & y_{i+1} & \cdot & y_n \\ y_1' & \cdot & y_{i-1}' & 0 & y_{i+1}' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & \cdot & y_{i-1}^{(n-1)} & r(x) & y_{i+1}^{(n-1)} & \cdot & y_n^{(n-1)} \end{vmatrix}}{W(y_1, \ldots, y_n; x)}$$

Ex: Solve $y^{(3)} - y^{(2)} - y^{(1)} + y = r(x)$.

Here $L = D^3 - D^2 - D + 1 = (D-1)^2(D+1)$.

Hence, a basis of solutions for the homoegenous part is $\{e^x, xe^x, e^{-x}\}$.

We need to calculate $W(x)$. Use Abel's formula:

$$W(x) = W(0)\, e^{-\int_0^x p_1(t)dt} = W(0) \cdot e^x.$$

$$W(x) = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x + xe^x & -e^{-x} \\ e^x & 2e^x + xe^x & e^{-x} \end{vmatrix}.$$

$$\implies W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4.$$

Hence,

$$W(x) = 4e^x.$$

$$W_1(x) = \begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & e^x + xe^x & -e^{-x} \\ r(x) & 2e^x + xe^x & e^{-x} \end{vmatrix} = -r(x)(2x+1).$$

Similarly,

$$\begin{aligned} W_2(x) &= 2r(x), \\ W_3(x) &= r(x)e^{2x} \end{aligned}$$

Therefore, a particular solution y_p is given by $y_p =$

$$e^x \int_0^x \frac{-r(t)(2t+1)}{4e^t} dt + xe^x \int_0^x \frac{2r(t)}{4e^t} dt + e^{-x} \int_0^x \frac{r(t)e^{2t}}{4e^t} dt.$$

Laplace Transforms

- Laplace tranform converts an IVP for a constant coefficient ODE, into an algebraic equation whose solution is used to solve the IVP.
- We have already seen some methods to solve IVP. Laplace transform is especially useful when we are dealing with discontinuous forcing functions $r(x)$.
- For example, when $r(x)$ is piece-wise continuous function, by earlier method, we need to solve IVP on each piece where $r(x)$ is continuous. Laplace transform gives solution in one step.

If g is integrable over the interval $[a, T]$ for every $T > a$, then the **improper integral** of g over $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt := \lim_{T\to\infty} \int_a^T g(t)\,dt\,.$$

We say that the improper integral **converges** if the limit exists and is finite;

Otherwise we say that the improper integral **diverges** or **does not exist**.

Example

1. Let $f(t) = e^{ct}$, $t \geq 0$ and $c \neq 0$ constant. Then

$$\int_0^\infty e^{ct}\, dt = \lim_{T\to\infty} \int_0^T e^{ct}\, dt = \lim_{T\to\infty} \frac{1}{c}(e^{cT} - 1)$$

   - the integral converges to $-1/c$ if $c < 0$;
   - the integral diverges if $c > 0$.
   - If $c = 0$, then $f(t) = 1$ and the integral again diverges.

2. Let $f(t) = 1/t$ for $t \geq 1$. Then

$$\int_1^\infty \frac{1}{t}\, dt = \lim_{T\to\infty} \int_1^T \frac{dt}{t} = \lim_{T\to\infty} \ln T$$

   the improper integral diverges.

Definition

Let $f(t)$ be defined for $t \geq 0$ and let s be a real number. The **Laplace transform** of f, denoted by $F(s)$, is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for those values of s for which the improper integral converges.

Note that s is a parameter and t is a variable of integration.

We will write it as

$$F = L(f) \quad \text{or} \quad F(s) = L(f(t)) \quad \text{or} \quad f(t) \leftrightarrow F(s).$$

Existence of Laplace transform

If $f(t) = e^{t^2}$, then
$$\int_0^\infty e^{-st} e^{t^2}\, dt = \infty$$
for every real s. Hence $f(t) = e^{t^2}$ does not have a Laplace transform.

In view of the above example, given a function f, we need to find for what values of s does $L(f)(s)$ make sense.

We will restrict our attention to the class of **piecewise continuous** functions.

Definition

A function $f : [0, T] \to \mathbb{R}$ is **piecewise continuous**, if

- $f(0+)$ and $f(T-)$ exists and are finite.
- there exists $0 = t_0 < t_1 < \ldots < t_{n+1} = T$ such that f is continuous on interval (t_{i-1}, t_i) for all i.
- $f(t_i+)$ and $f(t_i-)$ exists and are finite for $i = 1, \ldots, n$.

A function $f : [0, \infty) \to \mathbb{R}$ is called **piecewise continuous**, if it is so on $[0, T]$ for every $T > 0$.

Example

If

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ x^2 - 2, & 0 < x < 1 \\ 0, & 1 < x \leq 2 \end{cases}$$

Then f is a piecewise continuous function on $[-1, 2]$.
f has discontinuity at 0 and 1.

$$\begin{aligned} f(-1+) = 0 \quad &, \quad f(2-) = 0, \\ f(0+) = -2 \quad &, \quad f(0-) = 1, \\ f(1+) = 0 \quad &, \quad f(1-) = -1 \end{aligned}$$

Example

If

$$f(x) = \begin{cases} \dfrac{1}{x-1}, & 0 \leq x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

Then f is not piecewise continuous on $[0, 2]$, since f is discontinuous at 1, and $f(1+)$ does not exist (is not finite).

Example

If $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = \dfrac{1}{x}$, then f is continuous, but not piecewise continuous on $(0, 1)$.

Definition

A piecewise continuous function f is of **exponential order** s_0, if there exist constants M and s_0 such that

$$|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0.$$

Example

$f(t) = e^{t^2}$ is not of exponential order s_0 for any s_0.

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t\to\infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty$$

Theorem

If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then Laplace transform $L(f)$ is defined for $s > s_0$.

Proof.

Assume $|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0$.
We need to show that the integral

$$\int_0^\infty e^{-st} f(t)dt = \int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^\infty e^{-st} f(t)dt$$

converges for $s > s_0$.
The first integral exists and is finite, since f is piecewise continuous. For $t > t_0$,

$$|e^{-st} f(t)| \leq e^{-st} Me^{s_o t} = Me^{-(s-s_o)t}$$

Thus the second integral converges, since it is dominated by a convergent integral. Therefore $L(f)$ exists. $\square$

Example

- If f is bounded on $[t_0, \infty)$, say

$$|f(t)| \leq M, \quad t \geq t_0$$

  then f is of exponential order $s_0 = 0$.
- $\sin \omega t$ and $\cos \omega t$ are of exponential order 0.
  Thus $L(\sin \omega t)$ and $L(\cos \omega t)$ exists for $s > 0$.
- Show that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite, then f is of exponential order s_0.
- If $\alpha \in \mathbb{R}$ and $s_0 > 0$, then $\quad \lim_{t\to\infty} e^{-s_0 t} t^\alpha = 0$
  Hence t^α is of exponential order s_0 for any $s_0 > 0$.
- **Question.** Does this mean $L(t^\alpha)$ exists for any $\alpha \in \mathbb{R}$.
  No. We need piecewise continuity for $t \geq 0$.
- If $\alpha \geq 0$, then t^α is continuous on $(0, \infty)$, hence $L(t^\alpha)$ exists for $\alpha \geq 0$.

Example

Find Laplace transform of piecewise continuous function

$$
\begin{aligned}
f(t) &= \begin{cases} 1, & 0 \le t < 1 \\ e^{-t}, & t \ge 1 \end{cases} \\
L(f) &= F(s) = \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^1 e^{-st}dt + \int_1^\infty e^{-st}e^{-t}dt \\
&= -\frac{1}{s}e^{-st}|_0^1 + \frac{-1}{s+1}e^{-(s+1)t}|_1^\infty \\
&= \begin{cases} \dfrac{1-e^{-s}}{s} + \dfrac{e^{-(s+1)}}{s+1} & , \quad s > -1, s \neq 0 \\ 1 + \dfrac{1}{e} & , \quad s = 0 \end{cases}
\end{aligned}
$$

Example

Find the Lapalace transform $F(s)$ of $f(t) = 1$.

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \frac{1}{s}(1 - e^{-sT})$$

$F(s) \to \dfrac{1}{s}$ for $s > 0$ and diverges for $s < 0$.
For $s = 0$ also $F(s)$ diverges. We write this as

$$L(1) = \frac{1}{s}, \quad s > 0 \quad \text{or} \quad 1 \leftrightarrow \frac{1}{s}, \quad s > 0$$

Convention. Instead of writing $\lim_{T\to\infty}$ everytime, we will write directly as

$$\int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}|_0^\infty = \begin{cases} \dfrac{1}{s} & , \quad s > 0 \\ \infty & , \quad s < 0 \end{cases}$$

Example

Find Laplace transform of $f(t) = t$.

For $s \leq 0$, $F(s)$ diverges. For $s > 0$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} t \, dt \\ &= -\frac{1}{s} t e^{-st} |_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt \\ &= \frac{1}{s^2} \\ L(t) &= \frac{1}{s^2}, \quad s > 0 \end{aligned}$$

Exercise.

- $L(e^{at}) = \dfrac{1}{s-a}, \qquad s > a, \quad a \in \mathbb{R}.$
- $L(te^{at}) = \dfrac{1}{(s-a)^2}, \qquad s > a.$
- $L(t^n) = \dfrac{n!}{s^{n+1}}, \qquad s > 0, \quad n \geq 1$
- $L(\sin \omega t) = \dfrac{w}{s^2 + \omega^2}, \quad s > 0, \; \omega \in \mathbb{R}$

  $L(\cos \omega t) = \dfrac{s}{s^2 + \omega^2}, \quad s > 0$

Theorem (Linearity property)

Suppose $L(f_i)$ is defined for $s > s_i$ for $1 \leq i \leq n$.
Let s_0 be maximum of s_i's and $c_i \in \mathbb{R}$. Then

$$L(c_1 f_1 + \ldots + c_n f_n) = c_1 L(f_1) + \ldots + c_n L(f_n), \quad s > s_0$$

Example

(1) $L(e^{at}) = \dfrac{1}{s-a}$, $s > a$. Then for $b \neq 0$.

$$
\begin{aligned}
L(\cosh bt) &= L\left(\frac{e^{bt}+e^{-bt}}{2}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right) \\
&= \frac{s}{s^2-b^2}, \quad s > \max\{b,-b\} = |b|
\end{aligned}
$$

(2) $\quad L(\sinh bt) = L\left(\dfrac{e^{bt}-e^{-bt}}{2}\right)$

$$
\frac{1}{2}\left(\frac{1}{s-b}-\frac{1}{s+b}\right) = \frac{b}{s^2-b^2}, \quad s > |b|
$$

Theorem (First Shifting Theorem)

If $F(s) = L(f(t))$ *for* $s > s_0$, *then*

$$L(e^{at} f(t)) = F(s-a) \quad \textit{for } s > s_0 + a$$

Proof.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad s > s_0$$

$$\implies F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt, \quad s - a > s_0$$

$$\implies F(s-a) = L(e^{at} f(t)), \quad s > a + s_0$$

□

Example

1. $F(1) = \dfrac{1}{s},\ s > 0 \implies F(e^{at}) = \dfrac{1}{s-a},\ s > a.$
2. $F(t^n) = \dfrac{n!}{s^{n+1}},\ s > 0.$
3. $F(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}},\ s > a.$
4. $F(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2},\quad F(\cos \omega t) = \dfrac{s}{s^2+\omega^2},\quad s > 0.$
5. $F(e^{at} \sin \omega t) = \dfrac{\omega}{(s-a)^2+\omega^2},\ s > a.$
6. $F(e^{at} \cos \omega t) = \dfrac{s-a}{(s-a)^2+\omega^2},\ s > a.$
7. $L(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2-b^2},\ s > a+|b|.$
8. $L(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2-b^2},\ s > a+|b|.$

Inverse Laplace Transform

If $L(f(t)) = F(s)$ is the Laplace transform of f, then we say f is an **inverse Laplace transform** of F, and write

$$f = L^{-1}(F)$$

In order to solve an IVP using Laplace transform, we need to find inverse Laplace transforms.
In this course, we will use the table of Laplace transform to find inverse transform.

Theorem (Linearity Property)

If $F_1, \ldots, F_r$ are Laplace transforms of $f_1, \ldots, f_r$, i.e. $L^{-1}(F_i) = f_i$, then for $c_i \in \mathbb{R}$,

$$L^{-1}(c_1 F_1 + \ldots + c_r F_r) = c_1 L^{-1}(F_1) + \ldots + c_r L^{-1}(F_r).$$

Example

- $L^{-1}\left(\dfrac{1}{s^2-1}\right) = \sinh t,$
- $L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t.$
- $L(f) = F \implies L\left(e^{at}f(t)\right) = F(s-a).$
- $$\begin{aligned} & L^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right) \\ = \quad & L^{-1}\left(\frac{8}{s+5}\right) + L^{-1}\left(\frac{7}{s^2+3}\right) \\ = \quad & 8e^{-5t} + \frac{7}{\sqrt{3}}\sin\left(\sqrt{3}t\right) \end{aligned}$$

Example

$$
\begin{aligned}
& L^{-1}\left(\frac{3s+8}{s^2+2s+5}\right) \\
= \;& L^{-1}\left(\frac{3(s+1)+5}{(s+1)^2+4}\right) \\
= \;& e^{-t}L^{-1}\left(\frac{3s+5}{s^2+4}\right) \\
= \;& e^{-t}L^{-1}\left(\frac{3s}{s^2+4}\right)+e^{-t}L^{-1}\left(\frac{5}{s^2+4}\right) \\
= \;& e^{-t}\left[3\cos 2t+\frac{5}{2}\sin 2t\right]
\end{aligned}
$$

- If P, Q are polynomials with deg $P <$ deg Q, then L^{-1} of $P(s)/Q(s)$ is found, by finding partial fractions, using **Heaviside Method**.

Example

Find $L^{-1}(F(s))$, where $F(s) = \dfrac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)(s+1)}$.

The partial fraction of $F(s)$ is of the form

$$
\begin{aligned}
F(s) &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1} \\
A &= F(s)s|_{s=0} \\
&= \frac{6+(s+1)(s^2-5s+11)}{(s-1)(s-2)(s+1)}|_{s=0} \\
&= \frac{17}{2} \\
B &= F(s)(s-1)|_{s=1} \\
&= \frac{6+(s+1)(s^2-5s+11)}{s(s-2)(s+1)}|_{s=1} \\
&= \frac{6+2.7}{-2} = -10
\end{aligned}
$$

Example (continued ...)

$$
\begin{aligned}
C &= F(s)(s-2)|_{s=2} \\
&= \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s+1)}|_{s=2} \\
&= \frac{6+3.5}{6} = \frac{7}{2} \\
D &= F(s)(s+1)|_{s=-1} \\
&= \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)}|_{s=-1} \\
&= \frac{6}{-6} = -1 \\
L^{-1}\left(F(s)\right) &= L^{-1}\left(\frac{17}{2s} - \frac{10}{s-1} + \frac{7}{2(s-2)} - \frac{1}{s+1}\right) \\
&= \frac{17}{2} + -10e^{t} + \frac{7}{2}e^{2t} - e^{-t}
\end{aligned}
$$

Example

Let $F(s) = \dfrac{s^2 - 5s + 7}{(s+2)^3}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$
\begin{aligned}
F(s) &= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} \\
s^2 - 5s + 7 &= ((s+2)-2)^2 - 5((s+2)-2) + 7 \\
&= (s+2)^2 - 9(s+2) + 21 \\
&\Longrightarrow A = 1,\ B = -9,\ C = 21 \\
L^{-1}(F(s)) &= L^{-1}\left(\frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3}\right) \\
&= e^{-2t} L^{-1}\left(\frac{1}{s} - \frac{9}{s^2} + \frac{21}{s^3}\right) \\
&= e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right)
\end{aligned}
$$

Example

Let $F(s) = \dfrac{8+3s}{(s^2+1)(s^2+4)}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$\begin{aligned}\frac{8+3s}{(s^2+1)(s^2+4)} &= \frac{A+Bs}{s^2+1} + \frac{Cs+D}{s^2+4} \\ 8+3s &= (A+Bs)(s^2+4) + (C+Ds)(s^2+1)\end{aligned}$$

Equate the powers of s and solve to get A, B, C, D.

We have a simpler method in this particular case. Note

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

Example (continued ...)

Hence

$$
\begin{aligned}
F(s) &= \frac{8+3s}{(s^2+1)(s^2+4)} \\
&= \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right) \\
L^{-1}(F(s)) &= L^{-1}\left(\frac{8}{3(s^2+1)} + \frac{s}{s^2+1} - \frac{8}{3(s^2+4)} - \frac{s}{s^2+4}\right) \\
&= \left(\frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t\right)
\end{aligned}
$$

Laplace transform of Derivatives

Our goal is to apply Laplace transforms to differential equations. So we want to know the Laplace transform of derivative of a function.

Theorem

Let f be continuous on $[0, \infty)$ and of exponential order s_0. Let f' be piecewise continuous on $[0, \infty)$.
Then the Laplace transform for f' exists for $s > s_0$ and is given by

$$L(f') = sL(f) - f(0)$$

Proof.

If f' is piecewise continuous with $t_1 < t_2 < \ldots < t_n$ being the points of discontinuities in $[0, T]$, then

$$
\begin{aligned}
& \int_0^T e^{-st} f'(t)\, dt \\
= \quad & \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} e^{-st} f'(t) \; dt \\
= \quad & \sum_{i=1}^{n} \left[f(t) e^{-st} \Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} (-s) e^{-st} f(t) \; dt \right] \\
= \quad & f(t_n) e^{-st_n} - e^{-st_0} f(t_0) + s \int_{t_0}^{t_n} e^{-st} f(t) dt \\
= \quad & f(T) e^{-sT} - f(0) + s \int_0^T e^{-st} f(t) dt \\
\text{as } T \to \infty \quad = \quad & sL(f) - f(0)
\end{aligned}
$$

□

Example

Let us compute $L(\cos \omega t)$ using that

$$L(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

For $f(t) = \sin \omega t$, use $L(f') = sL(f) - f(0)$. Then

$$\begin{aligned} L(\omega \cos \omega t) &= s \frac{\omega}{s^2 + \omega^2} - 0 \\ \omega L(\cos \omega t) &= s \frac{\omega}{s^2 + \omega^2} \\ L(\cos \omega t) &= \frac{s}{s^2 + \omega^2} \end{aligned}$$

Example

Consider the IVP $\quad y' + y = 0, \quad y(0) = 5.$
We already know that the solution is given by $y = 5e^{-x}$.
Let us verify this using Laplace transform.

Let us assume that the given equation has a solution ϕ and it is of exponential order s_0 for some s_0.

$$\begin{aligned} L(\phi' + \phi) &= L(0) \\ \implies sL(\phi) - \phi(0) + L(\phi) &= 0 \\ \implies L(\phi) &= \frac{5}{s+1} \\ \implies \phi(x) &= 5e^{-x} \end{aligned}$$

Remark. Solving IVP with Laplace transform requires initial conditions at $t = 0$.

We have have the following result about $L(f^{(n)})$.

Theorem

Assume the following.

- $f,\ f',\ \ldots, f^{(n-1)}$ *are continuous on* $(0,\infty)$.
- $f^{(n)}$ *is piecewise continuous on* $[0,\infty)$.
- $f,\ f',\ \ldots, f^{(n-1)}$ *are of exponential order* s_0 *for some* s_0.

Then Laplace transforms of $f,\ f',\ \ldots, f^{(n-1)},\ f^{(n)}$ *exists,*

$$L(f^{(n)}) = s^n L(f) - f^{(n-1)}(0) - sf^{(n-2)} - \ldots - s^{n-1} f(0).$$

We do not need that $f^{(n)}$ *be of exponential order.*

Proof for $n=2$

$$L(f'') = sL(f') - f'(0) = s[sL(f) - f(0)] - f'(0)$$

Example

Consider the IVP

$$y'' + 4y = 3\sin t,\ y(0) = 1,\ y'(0) = -1$$

We know this equation has a unique solution ϕ on $\mathbb{R}$. Assume ϕ is of exponential order $s_0 \geq 0$ and apply Laplace transform on $[0, \infty)$. We get that for all $s > s_0$

$$\begin{aligned}
& L(\phi'') + 4L(\phi) = \frac{3}{s^2+1} \\
\implies \quad & (s^2 L(\phi) - s\phi(0) - \phi'(0)) + 4L(\phi) = \frac{3}{s^2+1} \\
\implies \quad & (s^2+4)L(\phi) - s + 1 = \frac{3}{s^2+1} \\
\implies \quad & L(\phi) = \frac{3}{(s^2+1)(s^2+4)} + \frac{s-1}{s^2+4}
\end{aligned}$$

Example (continued ...)

$$\begin{aligned} L(\phi) &= \frac{3}{(s^2+1)(s^2+4)} + \frac{s-1}{s^2+4} \\ &= \frac{1}{s^2+1} - \frac{2}{s^2+4} + \frac{s}{s^2+4} \\ \phi(t) &= L^{-1}\left(\frac{1}{s^2+1} - \frac{2}{s^2+4} + \frac{s}{s^2+4}\right) \\ &= \sin t - \sin 2t + \cos 2t \end{aligned}$$

From uniqueness theorem, this is the solution on all of $\mathbb{R}$.

Example

Solve IVP

$$y'' + 2y' + 2y = 1, \quad y(0) = -3, \; y'(0) = 1$$

The equation has a unique solution ϕ defined on all of $\mathbb{R}$. Assume ϕ is of exponential of order s_0. Then for all $s \geq s_0$,

$$\begin{aligned}
&L(\phi'') + 2L(\phi') + 2L(\phi) = L(1) \\
&(s^2 L(\phi) - s\phi(0) - \phi'(0)) + 2(sL(\phi) - \phi(0)) + 2L(\phi) = \frac{1}{s} \\
&(s^2 + 2s + 2)L(\phi) - (s+2)\phi(0) - \phi'(0) = \frac{1}{s} \\
&((s+1)^2 + 1)L(\phi) + 3(s+2) - 1 = \frac{1}{s} \\
&L(\phi) = \frac{1 - (3s+5)s}{((s+1)^2 + 1)s} := F(s)
\end{aligned}$$

Example (continued ...)

We want to compute $L^{-1}(F(s))$. We use partial fractions.

$$
\begin{aligned}
F(s) &= \frac{1-3s^2-5s}{((s+1)^2+1)s} \\
&= \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1} \\
1-3s^2-5s &= A((s+1)^2+1) + (B(s+1)+C)s \\
s=0 &\implies 1 = 2A \implies A = 1/2 \\
s=-1 &\implies 3 = A - C \implies C = -5/2 \\
s=1 &\implies -7 = 5A + 2B + C \implies B = -7/2 \\
L(\phi(t)) &= \frac{1}{2s} - \frac{7(s+1)}{2((s+1)^2+1)} - \frac{5}{2((s+1)^2+1)} \\
\phi(t) &= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t
\end{aligned}
$$

Example

More generally, to solve a constant coefficient IVP

$$y'' + py' + qy = r(t), \quad y(0) = a,\ y'(0) = b, \ \ p, q \in \mathbb{R}$$

let ϕ be the unique solution, which has a Laplace transform for all $s \geq s_0$. Applying Laplace transform, we get

$$(s^2 L(\phi) - s\phi(0) - \phi'(0)) + p(sL(\phi) - \phi(0)) + qL(\phi) = L(r)$$

$$\implies (s^2 + ps + q)L(\phi) = L(r) + sa + b + pa$$

We can simply this to an equation $L(\phi) = F(s)$ and compute the inverse Laplace transform of F, to get $\phi(t)$.

Remark. Although the unique solution exist on $\mathbb{R}$, Laplace transform gives solution only on $[0, \infty)$.

Unit Step Function

Let us consider IVP with constant coefficients, where the forcing function $r(t)$ is piecewise continuous.
To solve it using Laplace transform, we need to find Laplace transform of piecewise continuous functions.

Definition

The **unit (or Heaviside) step function** is defind as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Replacing t by $t - a$, we get

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Express the following functions in terms of unit step functions.

Example

Ramp Function

$$
\begin{aligned}
f(t) &= \begin{cases} 0, & 0 < t < a \\ t - a, & t > a \end{cases} \\
&= (t-a)u(t-a)
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= \begin{cases} \sin t, & 0 < t < t_0 \\ t, & t \geq t_0 \end{cases} \\
&= \sin t + u(t-t_0)(t - \sin t)
\end{aligned}
$$

Example

$$
\begin{aligned}
f(t) &= \begin{cases} \sin t, & 0 < t < t_0 \\ \cos t, & t_0 \leq t \leq t_1 \\ t, & t > t_1 \end{cases} \\
&= \sin t + u(t - t_0)(\cos t - \sin t) \\
&\quad + u(t - t_1)(t - \cos t)
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= \begin{cases} f_1, & 0 \leq t < t_1 \\ f_2, & t_1 \leq t < t_2 \\ . & . \\ f_n, & t_{n-1} \leq t \end{cases} \\
&= f_1 + u(t - t_1)(f_2 - f_1) + \dots \\
&\quad + u(t - t_{n-1})(f_n - f_{n-1})
\end{aligned}
$$

Writing a piecewise continuous function in terms of unit step functions simplifies the computation of its Laplace transform.

Theorem (Second Shifting Theorem)

Let $g(t)$ *be defined for* $t \geq 0$*.*
Assume $L(g(t+a))$ *exists for* $s > s_0$*, where* $a \geq 0$*.*
Then $L(u(t-a)g(t))$ *exists for* $s > s_0$*, and*

$$L(u(t-a)g(t)) = e^{-sa}L(g(t+a)).$$

Proof.

$$\begin{aligned} L(u(t-a)g(t)) &= \int_0^\infty e^{-st}u(t-a)g(t)\,dt \\ = \int_a^\infty e^{-st}g(t)\,dt &= \int_0^\infty e^{-s(x+a)}g(x+a)\,dx \\ &= e^{-sa}\,L(g(t+a)) \end{aligned}$$

Theorem (Second Shifting Theorem)

If $a \geq 0$ and $L(f)$ exists for $s > s_0$, then $L(u(t-a)f(t-a))$ exists for $s > s_0$ and

$$L(u(t-a)f(t-a)) = e^{-as}L(f(t)) = e^{-as}F(s).$$

Example

(1) $L(u(t-a)) = e^{-as}L(1) = \dfrac{e^{-as}}{s}.$

(2) $L(u(t-1)(t^2+1))$

$$\begin{aligned} &= e^{-s}L((t+1)^2+1) \\ &= e^{-s}L(t^2+2t+2) \\ &= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{2}{s}\right) \end{aligned}$$

Example

Find $L(f(t))$, where $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ -2t+1, & 2 \le t < 3 \\ 3t, & 3 \le t < 5 \\ t-1, & t \ge 5 \end{cases}$.

Write $f(t)$ in terms of unit step functions as

$$\begin{aligned} f(t) &= 1 + u(t-2)(-2t+1-1) + \\ &\quad u(t-3)(3t-(-2t+1)) + u(t-5)(t-1-3t) \\ &= 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1) \\ L(f) &= L(1) - 2e^{-2s}L(t+2) + e^{-3s}L(5(t+3)-1) \\ &\quad -e^{-5s}L(2(t+5)+1)) \end{aligned}$$

Example

$$
\begin{aligned}
L(f) &= L(1) - 2e^{-2s}L(t+2) + e^{-3s}L(5t+14) \\
&\quad -e^{-5s}L(2t+11) \\
&= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{14}{s}\right) \\
&\quad -e^{-5s}\left(\frac{2}{s^2} + \frac{11}{s}\right)
\end{aligned}
$$

Example

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \frac{\pi}{2} \\ \cos t - 3\sin t, & \dfrac{\pi}{2} \le t \end{cases}$$

Use

$$L(u(t-a)f(t)) = e^{-sa}L(f(t+a))$$

$$\begin{aligned} f(t) &= \sin t + u\left(t - \frac{\pi}{2}\right)(\cos t - 4\sin t) \\ L(f) &= L(\sin t) + e^{-\pi s/2} L\left(\cos\left(t + \frac{\pi}{2}\right) - 4\sin\left(t + \frac{\pi}{2}\right)\right) \\ &= \frac{1}{s^2+1} + e^{-\pi s/2}\left(\frac{-1-4s}{s^2+1}\right) \end{aligned}$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{s}$$

Use the fact that

$$\begin{aligned}
L(u(t-a)f(t-a)) &= e^{-as}L(f(t)) \\
F(s) &= \frac{1}{s} \\
\Longrightarrow \; f(t) &= 1 \\
\Longrightarrow \; f(t-2) &= 1 \\
L^{-1}(H) &= u(t-2)f(t-2) \\
&= u(t-2)
\end{aligned}$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{s^2}$$

Here

$$F(s) = \frac{1}{s^2} \implies f(t) = t$$

$$\begin{aligned} L^{-1}\left(\frac{e^{-2s}}{s^2}\right) &= u(t-2)f(t-2) \\ &= u(t-2)(t-2) \end{aligned}$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{s-3}$$

Here

$$F(s) = \frac{1}{s-3} \implies f(t) = e^{3t}$$

$$\begin{aligned} L^{-1}\left(\frac{e^{-2s}}{s-3}\right) &= u(t-2)f(t-2) \\ &= u(t-2)e^{3(t-2)} \end{aligned}$$

Example

Find inverse Laplace transform of

$$H(s) = \frac{e^{-2s}}{(s-3)^2}$$

Here

$$F(s) = \frac{1}{(s-3)^2} \implies f(t) = te^{3t}$$

$$\begin{aligned} L^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) &= u(t-2)f(t-2) \\ &= u(t-2)(t-2)e^{3(t-2)} \end{aligned}$$

Example

Find inverse Laplace transform of

$$F(s) = e^{-s}\frac{1}{2s} - e^{-2s}\frac{s+1}{(s+1)^2+1}.$$

$$L^{-1}\left(\frac{1}{2s}\right) = \frac{1}{2}$$

$$L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) = e^{-t}\sin t$$

$$\begin{aligned} L^{-1}(F(s)) &= \frac{1}{2}u(t-1) - u(t-2)e^{-(t-2)}\sin(t-2) \\ &= \begin{cases} 0, & 0 \le t < 1 \\ 1/2, & 1 \le t < 2 \\ -e^{-(t-2)}\sin(t-2) + \dfrac{1}{2}, & t \ge 2 \end{cases} \end{aligned}$$

IVP with piecewise continuous forcing functions

Theorem

Let f be a piecewise continuous function with jump discontinuities at $t_1, t_2, \ldots, t_n$.
Let k_0 and k_1 be arbitrary real numbers.
Consider the ODE

$$ay'' + by' + cy = f(t) \qquad (*), \quad a, b, c \in \mathbb{R}$$

Then there is a unique function y defined on $[0, \infty)$ such that

1. $y(0) = k_0$ *and* $y'(0) = k_1$.
2. y *and* y' *are continuous on* $[0, \infty)$.
3. y'' *is defined on every open sub-interval* I *of* $[0, \infty)$ *that does not contain any of the points* $t_1, \ldots, t_n$.
4. y *satisfies* $(*)$ *on every such sub-interval* I *of* $(0, \infty)$.
5. y'' *has left and right limits at* $t_1, \ldots, t_n$.

Example

Solve the IVP

$$y'' + y = \begin{cases} 1, & 0 \le t < \pi/2 \\ -1, & \pi/2 \le t < \infty \end{cases}$$
$$y(0) = 2 \quad , \quad y'(0) = -1$$

Let $y_1(t)$ be the solution of

$$y'' + y = 1,\ y(0) = 2,\ y'(0) = -1$$

Then

$$y_1(t) = 1 + \cos t - \sin t$$

Compute

$$y_1(\pi/2) = 0, \quad y_1'(\pi/2) = -1$$

Example (continued ...)

Let $y_2(t)$ be solution of

$$y'' + y = -1, \quad y(\pi/2) = 0, \; y'(\pi/2) = -1$$

Then

$$y_2(t) = -1 + \cos t + \sin t$$

The solution of oroginal IVP is

$$y(t) = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \frac{\pi}{2} \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases}$$

Example

Let us solve the same problem using Laplace transform.

$$
\begin{aligned}
y'' + y &= \begin{cases} 1, & 0 \le t < \pi/2 \\ -1, & \pi/2 \le t < \infty \end{cases} \\
y(0) = 2 \quad &, \quad y'(0) = -1
\end{aligned}
$$

$$f(t) = 1 + (-1-1)u\left(t - \frac{\pi}{2}\right) = 1 - 2u\left(t - \frac{\pi}{2}\right)$$

Assume that the ODE has a solution ϕ such that ϕ and ϕ' are continuous. Then

$$
\begin{aligned}
L(\phi'') + L(\phi) &= L(f(t)) \\
s^2 L(\phi) - \phi'(0) - s\phi(0) + L(\phi) &= L\left(1 - 2u(t - \frac{\pi}{2})\right) \\
(s^2+1)L(\phi) + 1 - 2s &= \frac{1}{s} - 2e^{-\pi s/2}\frac{1}{s}
\end{aligned}
$$

Example (continued ...)

$$
\begin{aligned}
L(\phi) &= (1-2e^{-\pi s/2})\frac{1}{s(s^2+1)} + \frac{2s-1}{s^2+1} \\
L(\phi) &= \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - 2e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{2s-1}{s^2+1} \\
\phi(t) &= 1 - \cos t - 2u\left(t-\frac{\pi}{2}\right) + 2u\left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right) \\
&\quad + 2\cos t - \sin t \\
&= 1 + \cos t - \sin t - 2u\left(t-\frac{\pi}{2}\right)(1-\sin t) \\
&= \begin{cases} 1+\cos t - \sin t, & 0 \le t < \frac{\pi}{2} \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases}
\end{aligned}
$$

Check that ϕ and ϕ' are continuous and ϕ'' has left and right limit at $\pi/2$.

Example

Solve the IVP

$$y'' + y = f(t), \quad y(0) = 0, \; y''(0) = 0$$

$$\begin{aligned}
f(t) &= \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4} \\ \cos 2t, & \dfrac{\pi}{4} \le t < \pi \\ 0, & t \ge \pi \end{cases} \\
&= u\left(t - \frac{\pi}{4}\right)\cos 2t - u(t-\pi)\cos 2t \\
L(f) &= L\left(u\left(t - \frac{\pi}{4}\right)\cos 2t\right) - L(u(t-\pi)\cos 2t) \\
&= e^{-\pi s/4} L\left(\cos 2\left(t + \frac{\pi}{4}\right)\right) - e^{-\pi s} L\left(\cos 2(t+\pi)\right) \\
&= e^{-\pi s/4} L(-\sin 2t) - e^{-\pi s} L(\cos 2t) \\
&= -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}
\end{aligned}$$

Example (continued ...)

$$y'' + y = f(t), \quad y(0) = 0, \; y'(0) = 0$$

$$\implies \quad L(y)(s^2+1) = -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}$$

$$\begin{aligned}
L(y) &= \frac{1}{s^2+1}\left[-\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}\right] \\
&= e^{-\pi s/4}H_1(s) + e^{-\pi s}H_2(s) \\
H_1(s) &= \frac{-2}{(s^2+1)(s^2+4)} = \frac{-2}{3(s^2+1)} + \frac{2}{3(s^2+4)} \\
h_1(t) &= \frac{-2}{3}\sin t + \frac{1}{3}\sin 2t \\
H_2(s) &= \frac{-1}{(s^2+1)(s^2+4)} = \frac{-1}{3(s^2+1)} + \frac{1}{3(s^2+4)} \\
h_2(t) &= \frac{-1}{3}\cos t + \frac{1}{3}\cos 2t
\end{aligned}$$

Example (continued ...)

$$
\begin{aligned}
L(y(t)) &= e^{-\pi s/4} H_1(s) + e^{-\pi s} H_2(s) \\
y(t) &= u\left(t - \frac{\pi}{4}\right) h_1\left(t - \frac{\pi}{4}\right) + u(t-\pi) h_2(t-\pi) \\
&= u\left(t - \frac{\pi}{4}\right)\left[\frac{-2}{3}\sin\left(t - \frac{\pi}{4}\right) + \frac{1}{3}\sin 2\left(t - \frac{\pi}{4}\right)\right] \\
&\quad + u(t-\pi)\left[\frac{-1}{3}\cos(t-\pi) + \frac{1}{3}\cos 2(t-\pi)\right] \\
&= u(t-\pi/4)\left[\frac{-\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t\right] \\
&\quad + \frac{1}{3} u(t-\pi)(\cos t + \cos 2t)
\end{aligned}
$$

Example (continued ...)

$$y(t) = \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4} \\ \dfrac{-\sqrt{2}}{3}(\sin t - \cos t) - \dfrac{1}{3}\cos 2t, & \dfrac{\pi}{4} \le t < \pi \\ \dfrac{-\sqrt{2}}{3}\sin t + \dfrac{1+\sqrt{2}}{3}\cos t, & t \ge \pi \end{cases}$$

Check that y, y' are continuous and y'' has left and right limits at $\pi/4$ and π.

Consider IVP

$$ay'' + by' + cy = f(t),\ y(0) = 0,\ y'(0) = 0$$

Taking Laplace transform gives

$$\begin{aligned} (as^2 + bs + c)Y(s) &= F(s) \\ \implies \quad Y(s) &= \frac{F(s)}{as^2 + bs + c} \end{aligned}$$

We were finding $y(t) = L^{-1}(Y(s))$, for known forcing function, by partial fraction method.

Question. What if $f(t)$ is unknown function?
Can we get a formula for

$$y(t) = L^{-1}(F(s)G(s))$$

in terms of $f(t)$?

Example

Consider IVP

$$y' - ay = f(t), \quad y(0) = 0, \qquad a \in \mathbb{R}$$

$$\begin{aligned}
e^{-at}(y' - ay) &= f(t)e^{-at} \\
(e^{-at}y)' &= f(t)e^{-at} \\
e^{-at}y(t) &= \int_0^t e^{-a\tau} f(\tau)\, d\tau \\
y(t) &= e^{at} \int_0^t e^{-a\tau} f(\tau)\, d\tau \\
&= \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau
\end{aligned}$$

Example (continued ...)

Let us use Laplace transform to solve same IVP.

$$\begin{aligned}
y' - ay &= f(t), \quad y(0) = 0 \\
(s-a)Y(s) &= F(s) \\
\implies Y(s) &= F(s)\frac{1}{s-a} \\
&= F(s)G(s), \qquad g(t) = e^{at} \\
\implies y(t) &= L^{-1}(F(s)G(s)) \\
&= \int_0^t f(\tau)g(t-\tau)\,d\tau
\end{aligned}$$

Definition (Convolution)

The convolution $f * g$ of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

We saw that $g(t) = e^{at}$, then

$$L^{-1}(F(s)G(s)) = f * g, \qquad F(s)G(s) = L(f * g)$$

This is true in general.

Show the followings.

1. $f * g = g * f$.
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$
5. $f * 1 \neq f$,
   e.g. $\sin t * 1 = 1 - \cos t$.

Theorem (Convolution Theorem)

If $L(f) = F(s)$ *and* $L(g) = G(s)$, *then* $L(f * g)$ *exists, and*

$$L(f * g) = L\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right) = F(s)G(s)$$

Proof.

Let us assume that the Laplace transform of $f * g$ exists.
We will prove the formula. □

continued ...

$$L(f*g) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\ d\tau\right)\ dt$$

Reversing the order of integration gives us the following

$$\begin{aligned}
&= \int_0^\infty f(\tau)\left(\int_\tau^\infty e^{-st}g(t-\tau)\ dt\right)d\tau \\
&= \int_0^\infty f(\tau)\left(\int_0^\infty e^{-s(x+\tau)}g(x)\ dx\right)d\tau \\
&= \int_0^\infty f(\tau)\left(\int_0^\infty e^{-s(x+\tau)}g(x)\ dx\right)d\tau \\
&= \int_0^\infty f(\tau)e^{-s\tau}(G(s))d\tau \\
&= G(s)F(s)
\end{aligned}$$

$\square$

Example

Let us verify the result for $f(t) = e^{at}$ and $g(t) = e^{bt}$.

$$
\begin{aligned}
F(s)G(s) &= \left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right) = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \\
(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\ d\tau \\
&= \int_0^t e^{a\tau}e^{b(t-\tau)}\ d\tau = \int_0^t e^{(a-b)\tau}e^{bt}\ d\tau \\
&= e^{bt}\left(\frac{e^{(a-b)t}}{a-b} - \frac{1}{a-b}\right) = \frac{e^{at}}{a-b} - \frac{e^{bt}}{a-b} \\
L(f * g) &= \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \\
&= F(s)G(s) \\
&= L(f)L(g)
\end{aligned}
$$

Example

$$y'' + 3y' + 2y = f(t), \quad y(0) = 0,\ y'(0) = 0$$

Applying Laplace transform, we get

$$\begin{aligned}
(s^2 + 3s + 2)L(y) &= F(s) \\
Y(s) &= F(s)G(s) \\
&= \frac{F(s)}{(s+2)(s+1)} \\
&= F(s)\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\
y(t) &= (f * g)(t) \\
&= f * (e^{-t} - e^{-2t}) \\
&= \int_0^t f(t-\tau)(e^{-\tau} - e^{-2\tau})\, d\tau
\end{aligned}$$

Example

Give a formula for the solution of the IVP.

$$
\begin{aligned}
y'' + 4y &= f(t), \quad y(0) = a,\ y'(0) = b \\
(s^2+4)Y(s) &= F(s) + b + as \\
Y(s) &= \frac{1}{s^2+4}F(s) + \frac{b+as}{s^2+4} \\
L^{-1}\left(\frac{1}{s^2+4}\right) &= \frac{1}{2}\sin 2t \\
L^{-1}\left(\frac{b+as}{s^2+4}\right) &= \frac{b}{2}\sin 2t + a\cos 2t \\
y(t) &= \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\, d\tau + \frac{b}{2}\sin 2t + a\cos 2t
\end{aligned}
$$

Example

Give a formula for the solution of the IVP.

$$y'' + 2y' + 2y = f(t), \quad y(0) = a,\ y'(0) = b$$

Taking Laplace transform gives,

$(s^2 + 2s + 2)Y(s) = F(s) + b + as + 2a$. Therefore,

$$Y(s) = \frac{1}{s^2 + 2s + 2} F(s) + \frac{b + a + a(s+1)}{s^2 + 2s + 2}$$

$$L^{-1}\left(\frac{1}{s^2 + 2s + 2}\right) = e^{-t}\sin t,$$

Hence

$$y(t) = \int_0^t f(t-\tau)e^{-\tau}\sin\tau\, d\tau + e^{-t}\left[(b+a)\sin t + a\cos t\right]$$

Evaluating Convolution Integrals

Def. An integral of the form $\int_0^t f(\tau)g(t-\tau)\,d\tau$ is called a **convolution integral**.

Example

Evaluate the integral using convolution theorem.

$$\begin{aligned}
h(t) &= \int_0^t (t-\tau)^5\tau^7\,d\tau \\
h(t) &= t^5 * t^7 \\
H(s) &= L(t^5)L(t^7) \\
&= \frac{5!\,7!}{s^6 s^8} \\
h(t) &= L^{-1}\left(\frac{5!\,7!}{s^{14}}\right) = \frac{5!\,7!}{13!}t^{13}
\end{aligned}$$

Example

Evaluate the following integral

$$h(t) = \int_0^t \sin a(t-\tau)\cos b\tau \, d\tau, \quad |a| \neq |b|$$

Note that $h(t) = (\sin at) * (\cos bt)$. Hence

$$\begin{aligned} H(s) &= L(\sin at) L(\cos bt) \\ &= \frac{a}{s^2+a^2}\frac{s}{s^2+b^2} \\ &= \frac{a}{b^2-a^2}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) \end{aligned}$$

Therefore,

$$h(t) = \frac{a}{b^2-a^2}(\cos at - \cos bt)$$

Volterra Integral Equations

An integral equation of the form

$$y(t) = f(t) + \int_0^t k(t-\tau)y(\tau)\,d\tau$$

is called a **Volterra integral equation**.
Here $f(t)$ and $k(t)$ are known functions and y is unknown.

We can solve them using convolution theorem.
Taking Laplace transform, we get

$$\begin{aligned} Y(s) &= F(s) + K(s)Y(s) \\ \implies Y(s) &= \frac{F(s)}{1-K(s)} \end{aligned}$$

Example

Solve the integral equation

$$y(t) = 1 + 2\int_0^t e^{-2(t-\tau)} y(\tau)\, d\tau$$

Taking Laplace transform, we get

$$\begin{aligned}
Y(s) &= \frac{1}{s} + \frac{2}{s+2} Y(s) \\
Y(s)\left(1 - \frac{2}{s+2}\right) &= \frac{1}{s} \\
Y(s)\frac{s}{s+2} &= \frac{1}{s} \\
Y(s) &= \frac{1}{s} + \frac{2}{s^2} \\
\implies y(t) &= 1 + 2t
\end{aligned}$$

Additional Properties of Laplace Transform

Assume $L(f(t))$ is defined for $s > s_0$, then

- $L\left(e^{-at}f(t)\right) = F(s+a), \quad s > s_0 + a.$
- $L(u(t-a)f(t-a)) = e^{-as}F(s), \quad s > s_0, a > 0.$
- $L\left(\int_0^t f(\tau)\,d\tau\right) = \dfrac{F(s)}{s}, \quad s > \max\{0, s_0\}.$
- $L(tf(t)) = -F^{(1)}(s), \quad s > s_0.$
- $L\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(s')ds', \quad s > s_0.$

- f : piecewise continuous and of exponential order. Then
  $(i) \lim_{s\to\infty} F(s) = 0$,
  $(ii) \lim_{s\to\infty} sF(s)$ is bounded.
- f, f' : piecewise continuous and of exponential order. Then
  $\lim_{s\to\infty} sF(s) = f(0)$.
- If f is piecewise continuous and periodic of period T, then
  $L(f(t)) = \dfrac{1}{1-e^{-sT}} \int_0^T f(T)e^{-st}dt$, $s > 0$

Theorem

If $F(s)$ exists for $s > s_0$, then

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}, \quad s > \mathit{max}\{0, s_0\}$$

Proof.

$$\begin{aligned} L\left(\int_0^t f(\tau)d\tau\right) &= L(f * 1) \\ &= L(f)L(1) \\ &= \frac{F(s)}{s} \end{aligned}$$

for $s > \max\{0, s_0\}$. □

Example

Compute $L^{-1}\left(\dfrac{1}{s^{n+1}}\right)$.

$$
\begin{aligned}
L(t) &= \frac{1}{s^2} \\
L\left(\int_0^t t\,dt\right) &= \frac{L(t)}{s} = \frac{1}{s^3} \\
\implies L(t^2) &= \frac{2}{s^3} \\
L\left(\int_0^t t^2\,dt\right) &= \frac{L(t^2)}{s} = \frac{2}{s^4} \\
\implies L(t^3) &= \frac{3!}{s^4} \quad \ldots \\
L(t^n) &= \frac{n!}{s^{n+1}}
\end{aligned}
$$

Example

Find

$$L^{-1}\left(\frac{1}{s^2(s^2+1)}\right)$$

Since $L(\sin t) = \dfrac{1}{s^2+1}$,

$$\begin{aligned} L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) &= \int_0^t \int_0^t \sin t \, dt \\ = \int_0^t (1-\cos t)\, dt &= t - \sin t \end{aligned}$$

Theorem

If $F(s)$ exists for $s > s_0$, then

$$L(tf(t)) = -\frac{dF(s)}{ds}, \quad s > s_0.$$

In general, $\quad L(t^k f(t)) = (-1)^k F^{(k)}(s), \quad s > s_0, \; k > 0.$

Proof.

$$\begin{aligned} \frac{dF(s)}{ds} &= \frac{d}{ds}\left(\int_0^\infty f(t)e^{-st}\,dt\right) \\ = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)\,dt &= \int_0^\infty -te^{-st}f(t)\,dt \\ &= -L(tf(t)). \end{aligned}$$

How to justify the interchanging of differentiation and integration? □

Differentiation under the Integral sign

Suppose we need to differentiate the function

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

with respect to x. Assume $a(x)$ and $b(x)$ and their derivatives are continuous for $x_0 \le x \le x_1$. Further $f(x,t)$ and $\frac{\partial}{\partial x} f(x,t)$ are continuous (in both t and x) in some open rectangle containing $x_0 \le x \le x_1$ and $a(x) \le t \le b(x)$.

Then for $x_0 \le x \le x_1$:

$$\frac{d}{dx} F(x) = f(x, b(x))\, b'(x) - f(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt\,.$$

Search for "Leibniz Integral Rule".

Example

Find $L^{-1}\left(\dfrac{s}{(s^2+4)^2}\right)$.

If $F(s) = \dfrac{1}{s^2+4}$, then $f(t) = \dfrac{1}{2}\sin 2t$. Hence

$L(tf(t)) = -\dfrac{dF(s)}{ds} = \dfrac{2s}{(s^2+4)^2}$.

Therefore, $L^{-1}\left(\dfrac{s}{(s^2+4)^2}\right) = \dfrac{1}{4}t\sin 2t$.

Exercise. Find $L^{-1}\left(\dfrac{s}{(s^2+4)^3}\right)$.

Theorem

Assume

- $F(s)$ *exists for* $s > s_0$,
- $\lim_{t\to 0} \dfrac{f(t)}{t}$ *exists.*

Then $L\left(\dfrac{f}{t}\right)$ *exists, and*

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s')\,ds', \quad s > s_0$$

Proof.

$$
\begin{aligned}
\int_s^\infty F(s')\,ds' &= \int_s^\infty \left(\int_0^\infty f(t)e^{-s't}\,dt\right)\,ds' \\
&= \int_0^\infty f(t)\left(\int_s^\infty e^{-s't}\,ds'\right)\,dt \\
&= \int_0^\infty \frac{f(t)}{t}\,e^{-st}\,dt \\
&= L\left(\frac{f(t)}{t}\right)
\end{aligned}
$$

By Fubini's Theorem

$$
\int\int |f(x,y)|\,dx\,dy < \infty \implies \int\int f\,dx\,dy = \int\int f\,dy\,dx
$$

$\square$

Example

Find

$$L^{-1}(F(s)), \quad F(s) = \ln\left(\frac{s-a}{s-b}\right), \quad a \neq b \in \mathbb{R}$$

$F(s)$ exists for $s > s_0 = max\{|a|, |b|\}$

$$\begin{aligned}
\frac{dF(s)}{ds} &= \frac{1}{s-a} - \frac{1}{s-b} \\
&:= G(s) \quad \text{exists } s > s_0 \\
g(t) &= L^{-1}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \\
&= e^{at} - e^{bt}, \\
\lim_{t\to 0} \frac{g(t)}{t} &= \lim_{t\to 0} \frac{e^{at} - e^{bt}}{t} \\
&= a - b \quad \text{exists}
\end{aligned}$$

Example (continued ...)

By the theorem,

$$
\begin{aligned}
L\left(\frac{g(t)}{t}\right) &= \int_s^\infty G(s')\,ds' \\
&= \int_s^\infty \left(\frac{1}{s'-a} - \frac{1}{s'-b}\right)\,ds' \\
&= \ln\left(\frac{s'-a}{s'-b}\right)\Big|_s^\infty \\
&= -\ln\left(\frac{s-a}{s-b}\right) L^{-1}\left(\ln\left(\frac{s-a}{s-b}\right)\right) \\
&= -\frac{g(t)}{t} \\
&= \frac{e^{bt}-e^{at}}{t}
\end{aligned}
$$

Theorem

If f is piecewise continuous and of exponential order, then

$$(i)\ \lim_{s\to\infty} F(s) = 0, \quad (ii)\ \lim_{s\to\infty} sF(s) < \infty.$$

Proof. $|f(t)| \le Me^{s_0 t}$ for $t \ge t_0$. Further we may assume $|f(t)| \le K$ for $t \in [0, t_0]$. Hence

$$\begin{aligned}
|F(s)| &= \left|\int_0^\infty f(t)e^{-st}\,dt\right| \le \int_0^\infty |f(t)|e^{-st}\,dt \\
&= \int_0^{t_0} |f(t)|e^{-st}\,dt + \int_{t_0}^\infty |f(t)|e^{-st}\,dt \\
&\le \int_0^{t_0} Ke^{-st}\,dt + \int_{t_0}^\infty Me^{-(s-s_0)t}\,dt \\
&= K\frac{1-e^{-st_0}}{s} + \frac{M}{s-s_0}, \quad \text{for all } s > s_0
\end{aligned}$$

$$\implies \lim_{s\to\infty} F(s) = 0, \text{ and } \lim_{s\to\infty} sF(s) = K + M < \infty$$

Question. Does there exist a function $f(t)$ which is piecewise continuous and of exponential order, such that $L(f(t)) = 1$?
No. Since then $\lim_{s\to\infty} F(s) = 0$.

May be there exist some function $f(t)$ which is either not piecewise continuous or not of exponential order, and $L(f(t)) = 1$. Answer is Yes. Dirac delta function or impluse function has this property.

Exercise Find L^{-1} of (i) $\left(\dfrac{1}{s}\tanh s\right)$,
(ii) $\ln\left(\dfrac{s^2+1}{s^2+s}\right)$,
(iii) $\ln\left(1 \pm \dfrac{1}{s^2}\right)$.

Find if $\lim_{s\to\infty} sF(s) \to f(0)$. If not, then state why.

Theorem

Assume f and f' both are piecewise continuous and of exponential order. Then

$$\lim_{s\to\infty} sF(s) = f(0).$$

Proof.

$$L(f'(t)) = sL(f(t)) - f(0)$$

Since f and f' both are piecewise continuous and of exponential order, we get

$$\lim_{s\to\infty} L(f'(t)) = 0, \text{ and } \lim_{s\to\infty} sF(s) < \infty$$

Therefore,

$$\lim_{s\to\infty} sF(s) = f(0)$$

Example

Let $f(t) = L^{-1}\left(\frac{1 - s(5+3s)}{s((s+1)^2+1)}\right)$. Find $f(0)$.

We can find $f(t)$ by partial fraction. Hence we know that f and f' are continuous and of exponential order. Therefore,

$$\begin{aligned} f(0) &= \lim_{s\to\infty} sF(s) \\ &= \lim_{s\to\infty} \frac{1 - s(5+3s)}{((s+1)^2+1)} \\ &= \lim_{s\to\infty} \frac{1 - 5s - 3s^2}{s^2+2s+2} \\ &= -3 \end{aligned}$$

Theorem

If f is piecewise continuous and periodic of period T, then

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(T)e^{-st}\,dt, \;\; s > 0$$

Proof.

$$\begin{aligned}
L(f(t)) &= \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \dots \\
&= \int_0^T f(t)e^{-st}\,dt + \int_0^T f(t+T)e^{-s(t+T)}\,dt + \dots \\
&= \int_0^T f(t)e^{-st}\,dt \,\left(1 + e^{-sT} + e^{-2sT} + \dots\right) \\
&= \frac{1}{(1-e^{-sT})} \int_0^T f(t)e^{-st}\,dt\,, \;\; s > 0
\end{aligned}$$

Example

Find the Laplace transform of periodic function

$$
\begin{aligned}
f(t) &= \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}, \quad f(t+2) = f(t) \\
L(f(t)) &= \frac{1}{(1-e^{-2s})} \int_0^2 f(t) e^{-st}\, dt \\
&= \frac{1}{(1-e^{-2s})} \int_0^1 t e^{-st}\, dt \\
&= \frac{1}{(1-e^{-2s})} \left[t \frac{e^{-st}}{-s} |_0^1 - \int_0^1 \frac{e^{-st}}{-s}\, dt \right] \\
&= \frac{1}{(1-e^{-2s})} \left[\frac{e^{-s}}{-s} - \frac{1}{s^2}(e^{-s} - 1) \right]
\end{aligned}
$$

1. $L\left(e^{-at}f(t)\right) = F(s+a), \quad s > s_0 + a.$
2. $L(u(t-a)f(t-a)) = e^{-as}F(s), \quad s > s_0, a > 0.$
3. $L\left(\int_0^t f(\tau)\,d\tau\right) = \dfrac{F(s)}{s}, \quad s > \max\{0, s_0\}.$
4. $L(tf(t)) = -F^{(1)}(s), \quad s > s_0.$
5. $L\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(s')ds', \quad s > s_0.$
6. Assume f is piecewise continuous and of exponential order. Then
   $(i)\lim_{s\to\infty} F(s) = 0, \qquad (ii)\lim_{s\to\infty} sF(s)$ is bounded.
7. Assume f and f' both are piecewise continuous and of exponential order. Then $\lim_{s\to\infty} sF(s) = f(0)$.
8. If f is piecewise continuous and periodic of period T, then
   $L(f(t)) = \dfrac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt, \; s > 0$

Exercise. Find Inverse Laplace transform and varify, whether $\lim_{s\to\infty} sF(s) = f(0)$. If not, then state why it is not.

$$(i) \quad \frac{s}{(s^2+a^2)^2}, \qquad (ii) \quad \frac{s}{(s^2-a^2)^2}$$

$$(iii) \quad \frac{s^2}{(s+1)^3}, \qquad (iv) \quad \frac{1}{\sqrt{s+1}}$$

$$(v) \quad \frac{s}{(s-a)^{3/2}}, \qquad (vi) \quad \frac{s}{(s+4)^6}$$

$$(vii) \quad \frac{e^{-s}}{s^5}, \qquad (viii) \quad \frac{e^{-2s}}{(s+1)^2}$$

$$(ix) \quad \ln\left(1+\frac{a^2}{s^2}\right), \qquad (x) \quad \ln\left(1-\frac{a^2}{s^2}\right),$$

$$(xi) \quad \frac{1}{s(1-e^{-s})}, \qquad (xii) \quad \frac{1}{s(1+e^{-s})}$$

$$(xiii) \quad \frac{s}{(s^2+1)^{3/2}}, \qquad (xiv) \quad \frac{1}{(s+1)(1-e^{-2s})}$$

Let us consider ODE

$$ay'' + by' + cy = f(t), \quad a, b, c \in \mathbb{R}$$

where $f(t)$ represents an impulsive force that is very large for a very short time and zero otherwise.
If f is an integrable function and $f(t) = 0$ outside $[t_0, t_0 + h]$, then

$$I = \int_{t_0}^{t_0+h} f(t)\, dt$$

is called the **total impulse** of f.
In the idealized situation, h is so small that total impulse is assumed to be applied instantaneously at $t = t_0$. In this case f is an impulse function.

Let $\delta(t-t_0)$ denote the impulse function with total impulse $=1$, applied at $t=t_0$.
For $t_0=0$, $\delta(t)$ is the Dirac δ-function.
Note $\delta(t-t_0)$ is not a function in the standard sense, since our definition implies

$$\delta(t-t_0)=0, \quad t\neq t_0, \quad \text{and} \quad \int_{t_0}^{t_0}\delta(t-t_0)\,dt=1$$

Let's try to define the meaning of solution of IVP

$$ay''+by'+cy=\delta(t-t_0), \quad y(0)=0,\ y'(0)=0, t_0>0$$

Theorem

Fix $t_0 \geq 0$. For each $h > 0$, let y_h be the solution of IVP

$$ay_h'' + by_h' + cy_h = f_h(t), \quad y_h(0) = 0, \ y_h'(0) = 0$$

$$f_h(t) = \begin{cases} 0, & 0 \leq t < t_0 \\ 1/h, & t_0 \leq t < t_0 + h \\ 0, & t \geq t_0 + h \end{cases}$$

Then f_h has total impulse 1 and the solution of IVP

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \ y'(0) = 0, t_0 > 0$$

$$\begin{aligned} y(t) &= \lim_{h \to 0+} y_h(t) = u(t - t_0)w(t - t_0) \\ w(t) &= L^{-1}\left(\frac{1}{as^2 + bs + c}\right) \end{aligned}$$

Proof of the theorem.

$$ay_h'' + by_h' + cy_h = f_h(t)$$

$$(as^2 + bs + c)Y_h(s) = F_h(s)$$

$$\begin{aligned}
Y_h(s) &= \frac{F_h(s)}{as^2 + bs + c} \\
y_h(t) &= \int_0^t w(t-\tau) f_h(\tau)\, d\tau \\
&= \begin{cases} 0, & 0 \le t < t_0 \\ \frac{1}{h}\int_0^t w(t-\tau)\, d\tau, & t_0 \le t \le t_0 + h \\ \frac{1}{h}\int_{t_0}^{t_0+h} w(t-\tau)\, d\tau, & t > t_0 + h \end{cases} \\
y_h(t) &= 0 \text{ for all } h \text{ if } 0 \le t \le t_0 \\
\Longrightarrow\ y(t) &= \lim_{h \to 0+} y_h(t) = 0 \ \text{ if } 0 \le t \le t_0
\end{aligned}$$

We will show that $\lim_{h\to 0+} y_h(t) = w(t - t_0), \quad \text{if} \quad t > t_0$

Proof continued ...

Suppose $t > t_0$ is fixed. Then

$$
\begin{aligned}
y_h(t) &= \frac{1}{h}\int_{t_0}^{t_0+h} w(t-\tau)\,d\tau, \quad \text{if} \;\; t - t_0 > h \\
w(t-t_0) &= \frac{1}{h}\int_{t_0}^{t_0+h} w(t-t_0)\,d\tau \\
y_h(t) - w(t-t_0) &= \frac{1}{h}\int_{t_0}^{t_0+h} (w(t-\tau) - w(t-t_0))\,d\tau \\
|y_h(t) - w(t-t_0)| &\leq \frac{1}{h}\int_{t_0}^{t_0+h} |w(t-\tau) - w(t-t_0)|\,d\tau \\
&\leq M_h \\
&:= \text{max}_{t\in[t_0,t_0+h]} \; |w(t-\tau) - w(t-t_0)|, \\
\lim_{h\to 0+} M_h &= 0, \;\; \text{since w is continuous} \\
\lim_{h\to 0+} y_h(t) &= w(t-t_0), \quad \text{if} \quad t > t_0
\end{aligned}
$$

Note that

$$w(t) = L^{-1}\left(\frac{1}{as^2+bs+c}\right)$$

is the solution of the IVP

$$aw'' + bw' + cw = 0,\ w(0) = 0,\ w'(0) = \frac{1}{a}$$

Infact if m_1, m_2 are roots of the characteristic polynomial $p(m) = am^2 + bm + c$, then $w(t)$ is defined on $(-\infty, \infty)$, and is given by

- $w = \dfrac{e^{m_2 t} - e^{m_1 t}}{a(m_2 - m_1)}$, if $m_1 \neq m_2$ are real.
- $w = \dfrac{1}{a} t e^{m_1 t}$, if $m_1 = m_2$ are real.
- $w = \dfrac{1}{a\omega} e^{\lambda t} \sin \omega t$, if $m_i = \lambda \pm i\omega$.

- For $t_0 > 0$, the solution $\boxed{y(t) = u(t - t_0)w(t - t_0)}$ of the IVP

$$ay'' + by' + cy = \delta(t - t_0),\ y(0) = 0,\ y'(0) = 0, t_0 > 0$$

is defined on $(-\infty, \infty)$ and has the following properties.

$$\begin{aligned} y(t) &= 0 \quad \text{for all } t < t_0 \\ ay'' + by' + cy &= 0 \ t \in (-\infty, t_0) \cup (t_0, \infty) \\ y'(t_0-) = 0, & \qquad y'(t_0+) = \frac{1}{a} \end{aligned}$$

- When $t_0 = 0$, $y'(0-)$ is not defined, so in this case

$$y(t) = u(t)w(t)$$

is a solution of

$$ay'' + by' + cy = \delta(t),\ y(0) = 0,\ y'(0+) = 0$$

Example

Solve

$$y'' + 2y' + y = \delta(t - t_0),\ y(0) = 0,\ y'(0) = 0$$

Here

$$\begin{aligned} w(t) &= L^{-1}\left(\frac{1}{s^2 + 2s + 1}\right) \\ &= L^{-1}\left(\frac{1}{(s+1)^2}\right) \\ &= e^{-t}t \end{aligned}$$

Therefore, the solution is given by

$$\begin{aligned} y(t) &= u(t - t_0)w(t - t_0) \\ &= u(t - t_0)e^{-(t-t_0)}(t - t_0) \end{aligned}$$

Example

Solve

$$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t-1), \quad y(0) = -3, \ y'(0) = 2$$

If $y_1(t)$ is a solution of

$$\begin{aligned} y'' + 6y' + 5y &= 3e^{-2t}, \quad y(0) = -3, \ y'(0) = 2 \\ y_1(t) &= -\frac{5}{2}e^{-t} + \frac{1}{2}e^{-5t} - e^{-2t} \end{aligned}$$

The solution of IVP is

$$y(t) = y_1 + y_2$$

where y_2 is a solution of

$$y'' + 6y' + 5y = 2\delta(t-1), \ y(0) = 0, \ y'(0) = 0$$

Example (continued ...)

Hence

$$\begin{aligned} w(t) &= 2L^{-1}\left(\frac{1}{s^2+6s+5}\right) \\ &= \frac{1}{2}\left(\frac{1}{s+1}-\frac{1}{s+5}\right) \\ &= \frac{1}{2}(e^{-t}-e^{-5t}) \end{aligned}$$

$$y(t) = -\frac{5}{2}e^{-t}+\frac{1}{2}e^{-5t}-e^{-2t}+\frac{1}{2}u(t-1)\left(e^{-(t-1)}-e^{-5(t-1)}\right)$$

Example

Solve the following ODEs.

- $y'' + 3y' + 2y = 6e^{2t} + 2\delta(t-1), \quad y(0) = 2,\ y'(0) = -6.$

  Solve two ODE's

$$\begin{aligned} y'' + 3y' + 2y &= 6e^{2t}, \quad y(0) = 2,\ y'(0) = -6 \\ y'' + 3y' + 2y &= 2\delta(t-1), \quad y(0) = 0,\ y'(0) = 0 \end{aligned}$$

- $y'' + y = \sin 3t + 2\delta(t - \pi/2), \qquad y(0) = 1,\ y'(0) = -1.$

  Solve two ODE's

$$\begin{aligned} y'' + y &= \sin 3t, \quad y(0) = 1,\ y'(0) = -1 \\ y'' + y &= 2\delta(t - \pi/2), \quad y(0) = 0,\ y'(0) = 0 \end{aligned}$$

Example

Solve the following ODE's.

- $y'' + 2y' + 2y = \delta(t-\pi) - 3\delta(t-2\pi),\ y(0) = -1,\ y'(0) = 2.$
- $y'' + 4y = f(t) + \delta(t-2\pi), \quad y(0) = 0,\ y'(0) = 1$

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 2, & t \ge \pi/2 \end{cases}$$

- $y'' + 4y' + 4y = -\delta(t), \qquad y(0) = 1,\ y'(0+) = 5.$
- Find a solution not involving unit step function which represents y on each suninterval of $[0,\infty)$ on which the forcing function is zero.

  (a) $y'' - y = \sum_{k=1}^{\infty} \delta(t-k), \qquad y(0) = 0,\ y'(0) = 1$

  (b) $y'' - 3y' + 2y = \sum_{k=1}^{\infty} \delta(t-k),\ y(0) = 0,\ y'(0) = 1$