L2 - 07/08/2024

Peano Axioms

- **Notation** – $n{+}{+}$ denotes successor of n.

1. 0 is a natural no.
2. If n is a natural no., then $n{+}{+}$ is also a natural no.
3. 0 is NOT the successor of any natural no.
   i.e $\forall$ natural nos. n, $n{+}{+} \neq 0$
4. If $n{+}{+} = m{+}{+}$, then $n = m$
5. **Principle of Mathematical Induction**
   Let $P(n)$ be any ppt. pertaining to natural nos.
   If $P(0)$ is true & $P(n) \Rightarrow P(n{+}{+})$, the $P(n)$ is true $\forall$ natural nos.

Motivation

3. disqualifies number systems such as

$$0, 1, 2, 3, 0, 1, 2, 3, 0 \ldots$$

which loop back to 0.

4 disqualifies number systems such as

$$0, 1, 2, 3, 4, 4, 4 \ldots$$

which hit a ceiling.

<u>OR</u>

$$0, 1, 2, 3, 1, 1, 1 \ldots$$

which loop back to a non-zero natural no.

5. disqualifies number systems such as

$$0, 0.5, 1, 1.5, 2, 2.5 \ldots$$

which have 'extra' elements i.e which cannot be produced by axioms 1 – 4.

- Assumption – $\exists$ a number system $\mathbb{N}$ whose elements we shall call natural nos., for which Axioms 1 – 5 are true.

Recursive def^n

Let $fn : \mathbb{N} \to \mathbb{N}$ be a fn^n s.t

$a_0 = c$ for some natural no. c

& $a_{n++} := fn(a_n)$

Hence, we can assign a unique natural no. an to every natural no.

<u>Proof</u> - Let $P(n)$ be the proposition that an is unique.

<u>BC</u> - $a_0 = c$

None of the other def^{n}s $a_{n++} = fn(a_n)$ will redefine a_0 by Axiom 3.

So, $P(0)$ is true.

IH - Given a_n is unique,
PT a_{n++} is unique.

$$a_{n++} = f_n(a_n)$$

None of the other defn's
$a_{m++} = f_n(a_m)$ will redefine
a_{n++} by Axiom 4.

So, $P(n) \Rightarrow P(n++)$

$\because$ $P(0)$ is true & $P(n) \Rightarrow P(n++)$

$\therefore$ By PMI, $P(n)$ is true $\forall$ natural nos

$\square$