L6 - 21/08/2024

Support of Fxn

For sets X & Y, we can define Y^X to be the set of all maps

$f : X \to Y$

Consider $Y = \{0, 1\}$

Then, $\{0,1\}^X$ is the set of all maps $f : X \to \{0,1\}$

Given such a map, we can define a subset of X as

$$\underbrace{S_f}_{\text{Support of } f} := \{x \in X \mid f(x) = 1\}$$

Power Set

Let Y be a set.

Its Power Set $P(Y)$ is defined to be the set of all subsets of Y.

eg – $Y = \{a, b\}$

$$\Rightarrow P(Y) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Thm

There is a natural bijection

$$\varphi : \{0,1\}^X \longrightarrow P(X)$$

$$f \longmapsto S_f$$

Pf - Injectivity

Given an $f : X \to \{0,1\}$,
consider an $g : X \to \{0,1\}$ s.t

$$S_g = S_f$$

$$\Rightarrow \quad g(x) = \begin{cases} 1, & x \in S_f \\ 0, & \text{otherwise} \end{cases}$$

$\because \quad S_f = \{x \in X \mid f(x) = 1\}$

$\Rightarrow \quad g = f \qquad \left(\begin{array}{l} \because \text{value of } f, g \\ \text{is same } \forall\, x \in X \end{array} \right)$

$\therefore \quad S_f = S_g \Rightarrow g = f$

(i.e. S_f completely determines f)

$\therefore \quad \varphi$ is injective.

Surjectivity

Let $T \subset X$ (or equiv. $T \in P(X)$)

Consider $\underbrace{\chi_T}_{\substack{\text{Characteristic} \\ \text{fn}^n \text{ of } T}} : X \to \{0,1\}$ s.t

$$\chi_T(x) = \begin{cases} 1, & x \in T \\ 0, & \text{otherwise} \end{cases}$$

$$S_{\chi_T} = \{x \in X \mid \chi_T(x) = 1\}$$

$$= \{x \in X \mid x \in T\}$$

$$\therefore \; S_{\chi_T} = T \Rightarrow \varphi(\chi_T) = T$$

$\therefore \; \forall \, T \in P(X), \; \exists \, \chi_T \in \{0,1\}^X$ s.t

$$\varphi(\chi_T) = T$$

$\therefore \; \varphi$ is surjective

$\therefore \; \varphi$ is bijective

Rel^n on a set X

Subset of $X \times X$

$$R \subset X \times X$$

- ## Equivalence Rel^n

$R \subset X \times X$ s.t it is

- **Reflexive** $\forall x,\ (x,x) \in R$
- **Symmetric** $(x,y) \in R \Rightarrow (y,x) \in R$
- **Transitive** $(x,y) \in R$ and $(y,z) \in R$
  $\Rightarrow (x,z) \in R$

eg- Let $R_d \subset \mathbb{Z} \times \mathbb{Z}$

$$R_d = \{(a,b) : d \mid a-b,\ d \in \mathbb{Z}_{>0}\}$$

R ✓ S ✓ T ✓ $\Rightarrow$ R_d is equiv. rel^n on $\mathbb{Z}$

NOTE – (Informal)

We can denote a rel^n R in the following manner.

If $(x, y) \subset R \Rightarrow x \sim y$

So, if $\sim$ is 'a rel^n on X',

$\sim$ is eq. rel^n if it satisfies

$\underline{R}$ – $\forall x,\ x \sim x$

$\underline{S}$ – $x \sim y \Rightarrow y \sim x$

$\underline{T}$ – $x \sim y$ and $y \sim z \Rightarrow x \sim z$

• Equivalence class

For an equivalence rel^n R on X, we can define subsets of X. called equivalence classes as follows.

Given $x \in X$, $EC(x) \subset X$ s.t

$$EC(x) = \{ y \in X : (x,y) \in R \}$$

NOTE - All the eq. classes of X are mutually exclusive & collectively exhaustive.

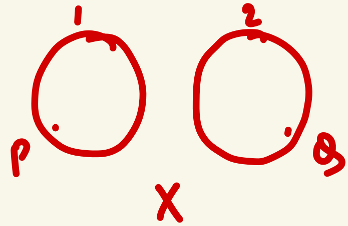
i.e

$$X = \coprod_{i \subset I} \overbrace{X_i}^{\text{Equivalence class}}$$

($\coprod$ - disjoint union)

$\llcorner$ index set

Q. Let $X \subset \mathbb{R}^2$



For $x, y \in X$, we say $x \sim y$ if we can join x & y by a cont. path.

Is $\sim$ an eq. rel^n?

A. R ✓ S ✓ T ✓ $\Rightarrow$ Eq. rel^n

This eq. rel^n has exactly 2 eq. classes (namely, the 2 discs)

$EC(P) = \text{Disc } 1$

$EC(Q) = \text{Disc } 2$

Q. Let $\sim$ be an eq. relⁿ on X.

For $x, y \in X$, PT

1. $EC(x) \cap EC(y) = \emptyset$ OR $EC(x) = EC(y)$
2. $EC(x) = EC(y) \Leftrightarrow x \sim y$

Pf –

1. Consider $z \in EC(x) \cap EC(y)$

$\Rightarrow x \sim z$ and $y \sim z$

To show $EC(x) = EC(y)$,
we first show $EC(x) \subseteq EC(y)$

1.1 $\underline{EC(x) \subseteq EC(y)}$

Consider $t \in EC(x) \Rightarrow x \sim t$

$\Rightarrow t \sim x$ (S)

$t \sim x$ and $x \sim z \Rightarrow t \sim z$ (T)

$t \sim z$ and $\underset{(S)}{z \sim y} \Rightarrow t \sim y$ (T)

$\Rightarrow y \sim t$ (S)

$$\Rightarrow t \in EC(y)$$

$\because$ $t \in EC(x) \Rightarrow t \in EC(y)$

$\therefore$ $EC(x) \subseteq EC(y)$

Similarly, we can show that

$$EC(y) \subseteq EC(x)$$

$\therefore$ $EC(x) = EC(y)$ $\square$

2.

2.1 $\underline{EC(x) = EC(y) \Rightarrow x \sim y}$

$EC(x) = EC(y)$

$\Rightarrow \exists\ z \in EC(x) \cap EC(y)$

$\Rightarrow x \sim z$ and $y \sim z$

$\Rightarrow x \sim z$ and $z \sim y$ (S)

$\Rightarrow x \sim y$ (T)

2.2 $x \sim y \Rightarrow EC(x) = EC(y)$

Let $z \in EC(x)$

$\Rightarrow x \sim z$

$\Rightarrow z \sim x$ (S)

$z \sim x$ and $x \sim y$

$\Rightarrow z \sim y$ (T)

$\Rightarrow y \sim z$ (S)

$\Rightarrow z \in EC(y)$ (S)

$\therefore EC(x) \subseteq EC(y)$

Similarly, we can prove that

$$EC(y) \subseteq EC(x)$$

$\therefore EC(y) = EC(x)$