L19 - 15/10/2024

Infinity

The irrationality of $\sqrt{2}$

Thm The eq^n $x^2 - 2y^2 = 0$ does not have non-trivial sol^{n}s over the integers

i.e $\nexists\ (x_0, y_0) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ s.t $\underbrace{x_0^2 - 2y_0^2 = 0}_{E}$

The proof introduces 2 useful & powerful methods.

- Proof by contradiction
- Method of infinite descent

Key steps

1. Define the notion of $\underbrace{\text{size}}_{\text{non-(-ve) int.}}$ of a sol^n

2. For any $k \in \mathbb{Z}_{\geq 0}$, the no. of int. sol^{n}s of E with size $\leq k$ is finite

3. Suppose E has a non-t'vl int. sol^n. Then, construct a non-t'vl int. sol^n of strictly smaller size.

Details

1. $$N((x_0, y_0)) = x_0^2 + y_0^2$$

2. Given k, s.t $x_0^2 + y_0^2 \leq k$
   then $x_0^2 \leq k$ & $y_0^2 \leq k$

So, $$S_k = \{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 \leq k\}$$

$$S_k \subseteq \{(x,y) \in \mathbb{Z}^2 : \max\{|x|, |y|\} \leq \sqrt{k}\}$$

But, the set has size $\leq (2\sqrt{k})^2 = \underbrace{4k}_{\text{finite}}$

3. Suppose (x_0, y_0) is a non-t'vl sol^n to E

Then, $x_0^2 - 2y_0^2 = 0$

$\Rightarrow x_0^2 = 2y_0^2$

$\therefore$ x_0^2 is even

$\therefore$ x_0 is even i.e $x_0 = 2z_0$

$$(2z_0)^2 - 2y_0^2 = 0$$

$$\Rightarrow 4z_0^2 - 2y_0^2 = 0$$

$$\Rightarrow y_0^2 - 2z_0^2 = 0$$

$\therefore$ (y_0, z_0) is a sol^n of E

But, $N((y_0, z_0)) = y_0^2 + z_0^2$

$< y_0^2 + x_0^2 = N((x_0, y_0))$