# MA 110
# Linear Algebra and Differential Equations
# Lecture 09

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Spring 2025

# Review of last lecture

We mainly discussed the notion of determinant of a square matrix $\mathbf{A} = (a_{jk})$ of size $n \times n$ and observed the following.

- Defined inductively by $\det \mathbf{A} = a_{11}$ if $n = 1$ and for $n \geq 2$,

  $$\det \mathbf{A} := a_{11} M_{11} - a_{12} M_{12} + \cdots + (-1)^{1+n} a_{1n} M_{1n}.$$

  where $M_{jk}$ denotes the the $(j, k)$th minor of $\mathbf{A}$., which is the determinant of the $(n-1)\times(n-1)$ submatrix of $\mathbf{A}$ obtained by deleting the $j$th row and the $k$th column of $\mathbf{A}$,
- $\det \mathbf{A}$ has a similar expansion along any of its rows, and also a similar expansion along any of its columns.
- $\det \mathbf{A}^{\mathsf{T}} = \det \mathbf{A}$.
- The determinant of a triangular matrix is the product of its diagonal entries. In particular, $\det \mathbf{I} = 1$.
- The determinant function $\mathbf{A} \longmapsto \det \mathbf{A}$ from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$:is multilinear and alternating in columns (as well as rows).
- $\mathbf{A}$ is invertible $\iff \det \mathbf{A} \neq 0$.

# Determinant and Rank

We now relate the rank of a matrix with determinants of its submatrices.

**Lemma**

*Let* $\mathbf{A}$ *be an* $m \times n$ *matrix, and* $r \in \mathbb{N}$*. Then*

$$\operatorname{rank} \mathbf{A} \geq r \iff \exists \text{ an } r \times r \text{ submatrix } \mathbf{B} \text{ of } \mathbf{A} \text{ with } \det \mathbf{B} \neq 0$$

Proof. Suppose $\operatorname{rank} \mathbf{A} \geq r$. Since $\operatorname{rank} \mathbf{A}$ equals the column rank of $\mathbf{A}$, there are $r$ linearly independent columns of $\mathbf{A}$. Let $\mathbf{C}$ denote the $m \times r$ submatrix of $\mathbf{A}$ consisting of these $r$ columns. Then the column rank of $\mathbf{C}$ is $r$, and so the row rank of $\mathbf{C}$ is also $r$. Hence there are $r$ linearly independent rows of $\mathbf{C}$. Let $\mathbf{B}$ denote the $r \times r$ submatrix of $\mathbf{C}$ consisting of these $r$ rows. These $r$ rows of $\mathbf{B}$ are linearly independent, and so $\operatorname{rank} \mathbf{B} = r$. Hence $\mathbf{B}$ is invertible, and so $\det \mathbf{B} \neq 0$.

Conversely, suppose $\mathbf{B}$ is an $r \times r$ submatrix of $\mathbf{A}$ such that $\det \mathbf{B} \neq 0$. Then $\mathbf{B}$ is invertible, and so $\operatorname{rank} \mathbf{B} = r$. Hence the $r$ rows of $\mathbf{B}$, and consequently, the corresponding $r$ rows of $\mathbf{A}$ are linearly independent. Hence $\operatorname{rank} \mathbf{A} \geq r$. $\square$

**Corollary (Determinantal Characterization of Rank)**

Let $\mathbf{A}$ be an $m \times n$ matrix, and $r \in \mathbb{N}$. Then $r = \operatorname{rank} \mathbf{A}$ if and only if the following two conditions are satisfied.

(i) there is an $r \times r$ submatrix $\mathbf{B}$ of $\mathbf{A}$ such that $\det \mathbf{B} \neq 0$,

(ii) $\det \mathbf{C} = 0$ for every $(r+1)\times(r+1)$ submatrix $\mathbf{C}$ of $\mathbf{A}$.

Proof. This is an immediate consequence of the above lemma.

We remark that although the above result is of theoretical interest, it does not give a practically useful method for finding the rank of a matrix $\mathbf{A}$. On the other hand, transformation of $\mathbf{A}$ to a Row Echelon Form quickly reveals its rank.

## Example

Let $\mathbf{A} := \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$. Since $\det \begin{bmatrix} 3 & 0 \\ -6 & 42 \end{bmatrix} \neq 0$ and since the determinants of all $3 \times 3$ submatrices of $\mathbf{A}$ are equal to 0, we see that $\operatorname{rank} \mathbf{A} = 2$.

This also follows by noting that $\mathbf{A}$ can be transformed by EROs to

$$\mathbf{A}' = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in REF, and by noting that $\operatorname{rank} \mathbf{A}$ is equal to the row rank of $\mathbf{A}'$, which is 2.

We now consider another classical method of finding solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is invertible.

## Proposition (Cramer's Rule)

Let $\mathbf{A} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$ be invertible, and let $\mathbf{b} \in \mathbb{R}^{n\times 1}$. For $k = 1, \ldots, n$, let $\mathbf{B}_k := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{c}_n \end{bmatrix}$ be the matrix obtained by replacing the $k$th column $\mathbf{c}_k$ of $\mathbf{A}$ by the right side $\mathbf{b}$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then the unique solution $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$ of the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by

$$x_k := \frac{\det \mathbf{B}_k}{\det \mathbf{A}} \quad \text{for } k = 1, \ldots, n.$$

Proof. If $\mathbf{A}\mathbf{x} = \mathbf{b}$, then $\mathbf{b} = x_1\mathbf{c}_1 + \cdots + x_k\mathbf{c}_k + \cdots + x_n\mathbf{c}_n$. By the first two crucial properties of the determinant function, $\det \mathbf{B}_k = \det \begin{bmatrix} \mathbf{c}_1 & \cdots & x_1\mathbf{c}_1 + \cdots + x_k\mathbf{c}_k + \cdots + x_n\mathbf{c}_n & \cdots & \mathbf{c}_n \end{bmatrix}$ $= x_k \det \mathbf{A}$ for $k = 1, \ldots, n$. Since $\mathbf{A}$ is invertible, $\det \mathbf{A} \neq 0$. Hence the result. $\square$

Let $\mathbf{A} := \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 4 & -5 \end{bmatrix}$, $\mathbf{b} := \begin{bmatrix} 13 \\ 11 \\ -31 \end{bmatrix}$. Then $\det \mathbf{A} = -60$.

Also,

$$\det \mathbf{B}_1 = \det \begin{bmatrix} 13 & -2 & 1 \\ 11 & 1 & 4 \\ -31 & 4 & -5 \end{bmatrix} = -60,$$

$$\det \mathbf{B}_2 = \det \begin{bmatrix} 3 & 13 & 1 \\ -2 & 11 & 4 \\ 1 & -31 & -5 \end{bmatrix} = 180,$$

$$\det \mathbf{B}_3 = \det \begin{bmatrix} 3 & -2 & 13 \\ -2 & 1 & 11 \\ 1 & 4 & -31 \end{bmatrix} = -240.$$

Hence the unique solution of the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by $x_1 := 1,\ x_2 = -3,\ x_3 = 4$, that is, $\mathbf{x} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^{\mathsf{T}}$.

Note: Cramer's Rule is rarely used for solving linear systems; the preferred method is the GEM. But Cramer's Rule is of theoretical interest, especially in solutions of differential eqns.

# Formula for the Inverse of a Matrix

Let $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{n \times n}$ with $n \geq 2$. Recall that $\mathbf{A}_{jk}$ denotes the submatrix of $\mathbf{A}$ obtained by deleting the $j$th row and the $k$th column of $\mathbf{A}$, and $M_{jk} := \det \mathbf{A}_{jk}$, the $(j,k)$th minor of $\mathbf{A}$, for $j,k = 1,\ldots,n$. We define $C_{jk} := (-1)^{j+k} M_{jk}$, $j,k = 1,\ldots,n$. It is called the **cofactor** of the entry $a_{jk}$. Then the expansion of $\det \mathbf{A}$ in terms of the $k$th column is given by

$$\det \mathbf{A} = \sum_{\ell=1}^{n} a_{\ell k} C_{\ell k}, \quad \text{where } k \in \{1,\ldots,n\}.$$

Define $\mathbf{C} := [C_{jk}] \in \mathbb{R}^{n \times n}$. It is called the **cofactor matrix** of $\mathbf{A}$.

**Theorem**

Let $\mathbf{A}$ be a square matrix. Then $\mathbf{C}^{\mathsf{T}}\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^{\mathsf{T}}$. In particular, if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \mathbf{C}^{\mathsf{T}} / \det \mathbf{A}.$$

Proof. Let $\mathbf{D} := \mathbf{C}^{\mathsf{T}}\mathbf{A} = [d_{jk}]$ say. By the definition of matrix multiplication, the $(j,k)$th entry of $\mathbf{D}$ is $d_{jk} = \sum_{\ell=1}^{n} C_{\ell j} a_{\ell k}$.

If $j = k$, then $d_{kk} = \sum_{\ell=1}^{n} C_{\ell k} a_{\ell k} = \det \mathbf{A}$, being the expansion in terms of its $k$th column of $\mathbf{A}$.

Let now $j \neq k$. Write $\mathbf{A} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_j & \cdots \mathbf{c}_n \end{bmatrix}$ in terms of its columns, and let $\mathbf{B}$ denote the matrix obtained by replacing the $j$th column $\mathbf{c}_j$ by the $k$th column $\mathbf{c}_k$ of $\mathbf{A}$, that is, $\mathbf{B} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix} = [b_{jk}]$, say. Then $\det \mathbf{B} = 0$ since two columns are identical. Expanding $\det \mathbf{B}$ in terms of its $j$th column, $\det \mathbf{B} = \sum_{\ell=1}^{n} b_{\ell j} C_{\ell j} = \sum_{\ell=1}^{n} a_{\ell k} C_{\ell j}$. Thus $d_{jk} = \sum_{\ell=1}^{n} C_{\ell j} a_{\ell k} = \det \mathbf{B} = 0$ if $j \neq k$.

This shows that $\mathbf{C}^{\mathsf{T}}\mathbf{A} = (\det \mathbf{A})\mathbf{I}$. Similarly, we can prove $\mathbf{A}\mathbf{C}^{\mathsf{T}} = (\det \mathbf{A})\mathbf{I}$, and so $\mathbf{C}^{\mathsf{T}}\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^{\mathsf{T}}$. In case $\det \mathbf{A} \neq 0$, we see that $\mathbf{A}$ is invertible, and $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathsf{T}}$. □

Remark: $\mathbf{C}^{\mathsf{T}}$ is sometimes called the adjugate of $\mathbf{A}$ and denoted by $\mathrm{adj}(\mathbf{A})$. Thus, $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} adj(\mathbf{A})$ when $\det \mathbf{A} \neq 0$.

# Multiplicativity of Determinant Function

**Proposition**

Let $\mathbf{A}$, $\mathbf{B}$ be $n \times n$ matrices. Then $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$.

Proof. Suppose first $\mathbf{A}$ is not invertible. Then $(\det \mathbf{A}) = 0$. Also, $\mathbf{AB}$ is not invertible; otherwise there would be $\mathbf{C}$ such that $(\mathbf{AB})\mathbf{C} = \mathbf{I}$, that is, $\mathbf{A}(\mathbf{BC}) = \mathbf{I}$, which is impossible since $\mathbf{A}$ is not invertible. Hence $\det(\mathbf{AB}) = 0 = (\det \mathbf{A})(\det \mathbf{B})$.

Next, suppose $\mathbf{A}$ is invertible. Then we can transform $\mathbf{A}$ to a diagonal matrix $\mathbf{A}'$ (having nonzero diagonal elements) by elementary row transformations of type I and type II. Then $\det \mathbf{A}' = \det \mathbf{A}$ if an even number of row interchanges are involved, and $\det \mathbf{A}' = -\det \mathbf{A}$ otherwise.

We observe that the same elementary row operations transform $\mathbf{AB}$ to $\mathbf{A}'\mathbf{B}$.

To see this, we can use Q. 2.3 in Tut Sheet 2: Making an elementary row operation is equivalent to multiplying on the left by the corresponding elementary matrix! And of course $\mathbf{E}(AB) = (EA)B$ where $E$ is any elementary matrix.

Hence $\det \mathbf{A'B} = \det \mathbf{AB}$ if an even number of row interchanges are involved, and $\det \mathbf{A'B} = -\det \mathbf{AB}$ otherwise.

Thus it is enough to show that

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}) \quad \text{when } \mathbf{A} \text{ is a diagonal matrix.}$$

But this is easily seen because

$$\mathbf{A} := \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix} \text{ and } \mathbf{B} := \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} \implies \mathbf{AB} := \begin{bmatrix} \alpha_1\mathbf{b}_1 \\ \alpha_2\mathbf{b}_2 \\ \vdots \\ \alpha_n\mathbf{b}_n \end{bmatrix},$$

where $\mathbf{b}_1, \ldots, \mathbf{b}_n$ denote the rows of $\mathbf{B}$. Hence

$$\det(\mathbf{AB}) = \alpha_1\alpha_2\cdots\alpha_n \det \mathbf{B} = (\det \mathbf{A})(\det \mathbf{B}). \qquad \square$$

## Corollary

If $\mathbf{A}$ is invertible, then $\det \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}$.

Proof. $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \implies (\det \mathbf{A})(\det \mathbf{A}^{-1}) = \det \mathbf{I} = 1.$ □

Example

Let $\mathbf{A} := \begin{bmatrix} 13 & 0 & 0 \\ 11 & 1 & 0 \\ -31 & 22 & -5 \end{bmatrix}$. Then $\det \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} = -\dfrac{1}{65}$.