MA 110
Linear Algebra and Differential Equations
Lecture 07

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Basis and dimension of a subspace

Let V be a subspace of $\mathbb{R}^{n\times 1}$. Recall the following definitions.

Definition

A subset S of V is called a basis of V if S is linearly independent and S has maximum possible number of elements among linearly independent subsets of V.

Note that a basis of V has at most n elements, and any two bases of V have the same number of elements.

Definition

The dimension of V is defined as the number of elements in a basis of V. It is denoted by $\dim V$.

Definition

Let $S \subset \mathbb{R}^{n\times 1}$. The set of all linear combinations of elements of S is denoted by $\operatorname{span} S$ and called the span of S.

We proved the following useful characterization.

Proposition

Let V be a subspace of $\mathbb{R}^{n\times 1}$, and let $S \subset V$. Then S is a basis for $V \iff S$ is linearly independent and $\operatorname{span} S = V$.

Corollary

Let V be a subspace of $\mathbb{R}^{n\times 1}$. Every linearly independent subset of V can be enlarged to a basis for V.

Proof (Sketch). Begin with a linearly independent subset S of V. If $\operatorname{span} S = V$, then S is a basis for V. If not, then $\mathbf{x}_1 \in V$ such that $\mathbf{x}_1 \notin \operatorname{span} S$. Now $S_1 := S \cup \{\mathbf{x}_1\}$ is a linearly independent subset of V. Check if $\operatorname{span} S_1 = V$,. If yes, then S_1 is a basis for V. If not then continue the process with S replaced by S_1. This process must end after a finite number of steps (since $\dim V \leq n$), and so it will lead to an enlargement of S which is a basis of V. $\square$

Proposition

Let $S := \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ be a basis for a subspace V of $\mathbb{R}^{n\times 1}$, and let $\mathbf{x} \in V$. Then there are unique $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ such that $\mathbf{x} = \alpha_1\mathbf{c}_1 + \cdots + \alpha_r\mathbf{c}_r$.

Proof. Since V is a basis for V, we obtain $V = \operatorname{span} S$, and so the vector $\mathbf{x}$ is a linear combination of vectors in S, that is, there are scalars $\alpha_1, \ldots, \alpha_r$ such that $\mathbf{x} = \alpha_1\mathbf{c}_1 + \cdots + \alpha_r\mathbf{c}_r$. Now suppose $\mathbf{x} = \beta_1\mathbf{c}_1 + \cdots + \beta_r\mathbf{c}_r$ for some $\beta_1, \ldots, \beta_r \in \mathbb{R}$. Then

$$(\alpha_1 - \beta_1)\mathbf{c}_1 + \cdots + (\alpha_r - \beta_r)\mathbf{c}_r = \mathbf{0}.$$

Since the set S is linearly independent, it follows that $\alpha_1 - \beta_1 = \cdots = \alpha_r - \beta_r = 0$, that is, $\beta_1 = \alpha_1, \ldots, \beta_r = \alpha_r$. This proves the uniqueness. $\square$

Remark: All things we have defined above for column vectors can also be defined for row vectors.

Column Space and Null Space of a Matrix

Recall that the column space of an $m \times n$ matrix $\mathbf{A}$ is the space $\mathcal{C}(\mathbf{A})$ of all linear combinations of column vectors of $\mathbf{A}$. And the null space of $\mathbf{A}$ is $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^{n\times 1} : \mathbf{Ax} = \mathbf{0}\}$.

Proposition

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, and let $\operatorname{rank} \mathbf{A} = r$. Then $\dim \mathcal{C}(\mathbf{A}) = r$ and $\dim \mathcal{N}(\mathbf{A}) = n - r$.

Proof. Since the column rank of $\mathbf{A}$ is equal to r, there are r linearly independent columns $\mathbf{c}_{k_1}, \dots, \mathbf{c}_{k_r}$ of $\mathbf{A}$, and any other column of $\mathbf{A}$ is a linear combination of these r columns.

Let $\mathbf{x} \in \mathcal{C}(\mathbf{A})$. Then $\mathbf{x}$ is a linear combination of columns of $\mathbf{A}$, each of which in turn is a linear combination of $\mathbf{c}_{k_1}, \dots, \mathbf{c}_{k_r}$. Thus $\mathbf{x}$ is a linear combination of $\mathbf{c}_{k_1}, \dots, \mathbf{c}_{k_r}$. This shows that $\operatorname{span}\{\mathbf{c}_{k_1}, \dots, \mathbf{c}_{k_r}\} = \mathcal{C}(A)$. Hence $\{\mathbf{c}_{k_1}, \dots, \mathbf{c}_{k_r}\}$ is a basis for $\mathcal{C}(\mathbf{A})$ and $\dim \mathcal{C}(\mathbf{A}) = r$.

To find the dimension of $\mathcal{N}(\mathbf{A})$, let us transform $\mathbf{A}$ to a REF $\mathbf{A}'$ by EROs of type I and type II. Since the row rank of $\mathbf{A}$ is equal to r, the matrix $\mathbf{A}'$ has exactly r nonzero rows and exactly r pivotal columns.

Let the $n-r$ nonpivotal columns be denoted by $\mathbf{c}_{\ell_1}, \ldots, \mathbf{c}_{\ell_{n-r}}$. Then $x_{\ell_1}, \ldots, x_{\ell_{n-r}}$ are the free variables. For each $\ell \in \{\ell_1, \ldots, \ell_{n-r}\}$, there is a basic solution $\mathbf{s}_\ell$ of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$, and every solution of this homogeneous equation is a linear combination of these $n-r$ basic solutions. Let S denote the set of these $n-r$ basic solutions. Then span $S = \mathcal{N}(\mathbf{A})$.

We claim that the set S of the $n-r$ basic solutions is linearly independent. To see this, we note that each basic solution is equal to 1 in one of the free variables and it is equal to 0 in the other free variables. Let $\alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R}$ be such that

$$\mathbf{x} := \alpha_1 \mathbf{s}_{\ell_1} + \cdots + \alpha_{n-r} \mathbf{s}_{\ell_{n-r}} = \mathbf{0}.$$

For $j = 1, \ldots, n$, let x_j denote the jth entry of $\mathbf{x}$. Then for each $\ell \in \{\ell_1, \ldots, \ell_{n-r}\}$, we see that $\alpha_\ell \cdot 1 = x_\ell = 0$. Hence S is linearly independent. Thus S is a basis for $\mathcal{N}(\mathbf{A})$ and $\dim \mathcal{N}(\mathbf{A}) = n - r$, the number of elements in S. □

Given any $\mathbf{A} \in \mathbb{R}^{m \times n}$, the dimension of the null space $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$ is called the **nullity** of $\mathbf{A}$.

Theorem (Rank-Nullity Theorem)

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then $\operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = n$.

Proof. If $r = \operatorname{rank} \mathbf{A}$, then we have seen that

$$\dim \mathcal{C}(\mathbf{A}) = r \quad \text{and} \quad \dim \mathcal{N}(\mathbf{A}) = n - r.$$

This shows that $\operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = n$ □

Let us restate two earlier results which are in conformity with the Rank-Nullity Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. Then

$$\boxed{\mathbf{A} \text{ is invertible} \iff \text{nullity}\,\mathbf{A} = 0 \iff \text{rank}\,\mathbf{A} = n}.$$

Further, $\text{rank}\,\mathbf{A} = n \iff \mathcal{C}(\mathbf{A}) = \mathbb{R}^{n\times 1}$.

We are now in a position to state and prove a comprehensive result regarding solutions of a system of m linear equations in n unknowns that we started with, namely

$$\begin{array}{rcll} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 & \quad (1) \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 & \quad (2) \\ \vdots \qquad \vdots \qquad \vdots \qquad & \vdots & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m & \quad (m) \end{array}$$

As usual, we write this as $\mathbf{Ax} = \mathbf{b}$, where
$\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m\times n}$, $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{b} := \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}^{\mathsf{T}}$.

Theorem (Fundamental Theorem for Linear Systems: FTLS)

Let $m, n \in \mathbb{N}$ and $\mathbf{A}$ be an $m \times n$ matrix with real entries. Suppose $\boxed{\text{rank}\,\mathbf{A} = r}$.

(i) **Homogeneous Linear System**: $\mathbf{A}\mathbf{x} = \mathbf{0}$ **(H)**

The solution space $\{\mathbf{x} \in \mathbb{R}^{n\times 1} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$ of (H) is a subspace of $\mathbb{R}^{n\times 1}$ of dimension $\boxed{n - r}$.

In particular, $r = n$ if and only if $\mathbf{0}$ is the only solution of (H).

If $r < n$, then there are linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}$ of (H) and every solution of (H) is a unique linear combination of these $\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}$.

(ii) **General Linear System**: $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \in \mathbb{R}^{m\times 1}$ **(G)**

(G) has a solution if and only if $\text{rank}[\mathbf{A}|\mathbf{b}] = r$. In this case, let $\mathbf{x}_0$ be a particular solution of (G). If $\mathbf{x}$ is a solution of (G), then $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of (H) above.

Proof. (i) The solution space of the homogeneous linear system (H) is just the nullspace $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$, and we have seen that its dimension, that is, the nullity of $\mathbf{A}$, is equal to $n - r$.

We note that $r = n \iff n - r = 0$, that is, the dimension of $\mathcal{N}(\mathbf{A})$ is zero; in other words, $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$. This means that $\mathbf{0}$ is the only solution of (H).

Let now $r < n$. Then $\mathcal{N}(\mathbf{A})$ has a basis consisting of $n - r$ elements, say $\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}$. Hence every element of the solution space is a unique linear combination of the elements in this basis.

(ii) Let $\mathbf{b} \in \mathbb{R}^{n\times 1}$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be the n columns of $\mathbf{A}$. Then

$$\mathbf{Ax} = x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n \quad \text{for } \mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n\times 1}.$$

Hence $\mathbf{Ax} = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^{n\times 1}$ if and only if $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$, that is, $\mathbf{b} \in \mathcal{C}(\mathbf{A})$.

Since every column of $\mathbf{A}$ is also a column of the augmented matrix $[\mathbf{A}|\mathbf{b}]$, the column space $\mathcal{C}(\mathbf{A})$ of $\mathbf{A}$ is contained in the column space $\mathcal{C}([\mathbf{A}|\mathbf{b}])$ of $[\mathbf{A}|\mathbf{b}]$. It follows that $\mathbf{b} \in \mathcal{C}(\mathbf{A})$ if and only if $\mathcal{C}([\mathbf{A}|\mathbf{b}]) = \mathcal{C}(\mathbf{A})$, that is, the column rank of $[\mathbf{A}|\mathbf{b}]$ is equal to the column rank of $\mathbf{A}$. So $\operatorname{rank}[\mathbf{A}|\mathbf{b}] = \operatorname{rank} \mathbf{A} = r$.

Let $\mathbf{x}_0$ be a particular solution of (G), that is, let $\mathbf{x}_0 \in \mathbb{R}^{n\times 1}$ satisfy $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$. Then for any $\mathbf{x} \in \mathbb{R}^{n\times 1}$, we see that $\mathbf{A}\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$, that is, $\mathbf{x}$ is a solution of (G) if and only if $\mathbf{x}_h := \mathbf{x} - \mathbf{x}_0$ is a solution of (H). The proof is complete. $\square$

Remark

The above theorem is of immense theoretical importance. It tells us precisely when solutions exist, and also describes the nature of solutions of a linear system of equations.
For example, it says that when there is a nonzero solution of a homogeneous linear system, there are infinitely many solutions.

Further, when a homogeneous system has infinitely many solutions, it says that they can be described by a one parameter family, or a two parameter family etc.

It may seem that to implement the results of the above theorem, we must first find the rank of the coefficient matrix $\mathbf{A}$ of the linear system. This is not necessary.

We have already seen that we may directly proceed to find the solutions of the linear system by considering the augmented matrix $[\mathbf{A}|\mathbf{b}]$ and transform the coefficient matrix $\mathbf{A}$ to a row echelon form by the Gauss Elimination Method and then use Back Substitution . This process itself reveals all possibilities.

In particular, when the rank r of $\mathbf{A}$ is less than the number n of variables, we have shown how to construct a set S of basic solutions of an homogeneous linear system. This set S is in fact a basis of the solution space of the system. That is the reason for using the terminology 'basic solutions'.

Row Space and Column Space

Definition

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The row space of A, denoted $\mathcal{R}(\mathbf{A})$, is defined as the subspace of $\mathbb{R}^{1 \times n}$ spanned by the row vectors of $\mathbf{A}$.

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Here are some important observations.

- The row-rank of $\mathbf{A}$ is precisely the dimension of $\mathcal{R}(\mathbf{A})$.
- If $\mathbf{A}' \in \mathbb{R}^{m \times n}$ is obtained from $\mathbf{A}$ by an elementary row operation, then $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}')$.
- If $\mathbf{A}' \in \mathbb{R}^{m \times n}$ is in REF, then the pivotal rows of $\mathbf{A}'$ form a basis of $\mathcal{R}(\mathbf{A}')$.
- A basis of $\mathcal{R}(\mathbf{A})$ is given by the pivotal rows of its REF.
- If $\mathbf{A}'$ is obtained from $\mathbf{A}$ by an elementary row operation, then $\mathcal{C}(\mathbf{A})$ need not be equal to $\mathcal{C}(\mathbf{A}')$.
- However, the columns of $\mathbf{A}$ corresponding to the pivotal columns of its REF form a basis of $\mathcal{C}(\mathbf{A})$.

Example: Consider the 5×6 matrix $\mathbf{A}$ and its REF $\mathbf{A}'$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} \boxed{1} & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\operatorname{rank} \mathbf{A} = 3$. A basis of the row space $\mathcal{R}(\mathbf{A})$ is given by

$$\left\{ \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -2 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \right\}$$

whereas a basis for the column space $\mathcal{C}(\mathbf{A})$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 15 \\ 18 \end{bmatrix} \right\}.$$

Verify!