LII - 06/08/2024

So, $\forall x \in \mathbb{R}$, we can represent it by a seq. $(a_1, a_2, \cdots)$ s.t $a_i \neq 0$.

$$\Rightarrow \quad x^{-1} = (a_1^{-1}, a_2^{-1}, \cdots)$$

C: $x^{-1}$ is Cauchy

Pf – $\quad |a_m^{-1} - a_n^{-1}| = \dfrac{|a_n - a_m|}{|a_m||a_n|}$

$\because \exists\, c$ & $N_1$ s.t $\forall n \geq N_1$

$$|a_n| \geq c \Rightarrow \left|\frac{1}{a_n}\right| \leq \frac{1}{c}$$

$$\therefore \quad \frac{|a_n - a_m|}{|a_m||a_n|} \leq \frac{|a_n - a_m|}{c^2}$$

$\because x$ is Cauchy

$\therefore \forall \epsilon > 0$, $\exists N_2$ s.t $\forall m, n \geq N_2$

$$|a_m - a_n| < c^2 \epsilon$$

$\therefore \ \forall\, m, n \geqslant \max\{N_1, N_2\}$

$$|a_m^{-1} - a_n^{-1}| \leqslant \frac{|a_n - a_m|}{c^2}$$

$$< \frac{c^2 \epsilon}{c^2} = \epsilon$$

$\square$

## C: Inverse is well-defined

Pf - Consider $(a_n) \sim (b_n)$

We need to prove that

$$(a_n^{-1}) \sim (b_n^{-1})$$

$$|a_n^{-1} - b_n^{-1}| = \frac{|b_n - a_n|}{|a_n||b_n|}$$

$\because \ \exists\ c, d$ & $N_1, N_2$ s.t

$\forall\, n \geqslant N_1, \ |a_n| \geqslant c$ &

$\forall\, n \geqslant N_1, \ |b_n| \geqslant d$

$$\Rightarrow \left|\frac{1}{a_n}\right| \leqslant \frac{1}{c} \quad \& \quad \left|\frac{1}{b_n}\right| \leqslant \frac{1}{d}$$

$$\Rightarrow \frac{|b_n - a_n|}{|a_n||b_n|} \leq \frac{|b_n - a_n|}{cd}$$

$\because$ $(a_n) \sim (b_n)$

$\therefore$ $\forall \epsilon > 0$, $\exists N_3$ s.t. $\forall n \geq N_3$

$$|b_n - a_n| < cd\epsilon$$

$\therefore$ $\forall n \geq \max\{N_1, N_2, N_3\}$

$$|a_n^{-1} - b_n^{-1}| \leq \frac{|b_n - a_n|}{cd} \leq \frac{cd\epsilon}{cd} = \epsilon$$

$\square$

## Ordering

A real no. is +ve (−ve) if it is +vely (− vely) bounded away from 0.

i.e $\exists\ c \in \mathbb{Q}_{>0}$ & $N \in \mathbb{N}$ s.t

$\forall\ n \geq N \qquad a_n \geq c$

$\qquad\qquad\qquad (a_n \leq -c)$