L8 - 28/08/2024

<u>NOTE</u> - From now on, we will use the not^n $[x]$ to refer to $\pi(x)$ or $EC(x)$.

There is a natural injective map from naturals to the integers.

$$\mathbb{N} \hookrightarrow \mathbb{Z}$$
$$n \longmapsto [n--0]$$

($\hookrightarrow$: denotes map is injective)

Pf - Consider $[m--0], [n--0] \in \mathbb{Z}$

s.t

$$[m--0] = [n--0]$$
$$\Rightarrow (m--0) \sim (n--0)$$
$$\Rightarrow m+0 = n+0$$
$$\Rightarrow m = n$$

$\square$

Proposition - Let x, y be integers s.t $xy = 0$.
Then $x = 0$ or $y = 0$

Pf - Let $x = [a -- b]$, $y = [c -- d]$
$a \neq b$ and $c \neq d$

$$xy = [a--b][c--d]$$
$$= [(ac+bd) -- (bc+ad)] = [0--0]$$

$$\Rightarrow \underline{ac+bd = bc+ad}$$

WLOG, let $a > b$ & $c > d$

$\Rightarrow \exists\ h, k > 0$ s.t

$$a = b + h \quad \& \quad c = d + k$$

$$\Rightarrow (b+h)(d+k) + bd = b(d+k) + (b+h)d$$

$$\Rightarrow hk = 0 \Rightarrow \underbrace{h = 0 \text{ or } k = 0}_{\text{contd}^n}$$

Similarly, we can prove for other cases $\square$

Corollary - (Cancellation Law)

Let x, y, z be integers s.t $z \neq 0$.

Then $xz = yz \Rightarrow x = y$

Pf - $xz - yz = (x-y)\underbrace{z}_{\neq 0} = 0$

$\Rightarrow x - y = 0$ (By prev. ppn)

$\Rightarrow x = y$ $\square$

Rationals

Consider the set $X = \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$.

We define an eq. rel^n $\sim$ on X s.t

for $(a//b), (c//d) \in X$

$$(a//b) \sim (c//d) \iff ad = bc$$

Pf – $\underline{R}$ – $ab = ba \Rightarrow (a//b) \sim (a//b)$

$\underline{S}$ – $ad = bc \Rightarrow cb = da$

So, $(a,b) \sim (c,d) \Rightarrow (c,d) \sim (a,b)$

$\underline{T}$ – $ad = bc$ & $cy = dx$

$\Rightarrow ady = bcy \Rightarrow bcy = bdx$

$\Rightarrow ady = bdx$

$\Rightarrow ay = bx \quad (\because d \neq 0)$

So, $(a,b) \sim (c,d)$ & $(c,d) \sim (x,y)$

$\Rightarrow (a,b) \sim (x,y)$

Hence, $\sim$ is an eq. rel^n $\square$

$$\boxed{\mathbb{Q} := X_{/\sim}}$$

Add^n

$$[a//b] + [c//d] := [(ad+bc)//bd]$$

Checking if add^n is well-defined

Consider $[\alpha//\beta] = [a//b] \Rightarrow \alpha b = \beta a$

$[\gamma//\delta] = [c//d] \Rightarrow \gamma d = \delta c$

We need to show that

$$[(\alpha\delta+\beta\gamma)//\beta\delta] = [(ad+bc)//bd]$$

Now, $(\alpha\delta+\beta\gamma)(bd) = \underbrace{\alpha b}_{\beta a}\delta d + \underbrace{\gamma d}_{\delta c}\beta b$

$= \beta\delta(ad+bc)$

$\Rightarrow (\alpha\delta+\beta\gamma)//\beta\delta \sim (ad+bc)//bd$

$\Rightarrow [(\alpha\delta+\beta\gamma)//\beta\delta] = [(ad+bc)//bd]$ $\square$

Multip^n

$$[a//b] \times [c//d] := [ac//bd]$$

Checking if multip^n is well-defined

Consider $[\alpha//\beta] = [a//b] \Rightarrow \alpha b = \beta a$

$[\gamma//\delta] = [c//d] \Rightarrow \gamma d = \delta c$

We need to show that

$$[\alpha\gamma//\beta\delta] = [ac//bd]$$

$$\alpha\gamma bd = \underbrace{(\alpha b)}_{\beta a}\underbrace{(\gamma d)}_{\delta c} = \beta\delta ac$$

$$\Rightarrow (\alpha\gamma//\beta\delta) \sim (ac//bd)$$

$$\Rightarrow [\alpha\gamma//\beta\delta] = [ac//bd]$$

$\square$

Neg^n

$$-[a//b] := [(-a)//b]$$

Sub^n

$$x - y := x + (-y)$$

There is a natural injective map from integers to rationals

$$\mathbb{Z} \hookrightarrow \mathbb{Q}$$

$$n \longmapsto [n//1]$$

Pf – Consider $[n//1]$, $[m//1] \in \mathbb{Q}$

s.t

$$[n//1] = [m//1]$$

$$\Rightarrow (n//1) \sim (m//1)$$

$$\Rightarrow n \cdot 1 = 1 \cdot m$$

$$\Rightarrow n = m \quad \square$$

Inverse

For $[a//b] \in \mathbb{Q}/\{0\}$

$$[a//b]^{-1} = [b//a]$$

NOTE - If $[a//b] \neq 0 \Rightarrow a \neq 0$

Pf - $a = 0 \Rightarrow a \cdot 1 = b \cdot 0$

$\Rightarrow (a//b) \sim (0//1)$

$\Rightarrow [a//b] = [0//1] = 0$

(Proof of contrapositive) $\square$

Proposition – Let x, y, z be rationals

Then
$$x + y = y + x$$
$$(x+y)+z = x+(y+z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$x1 = 1x = x$$
$$x(y+z) = xy + xz$$
$$xx^{-1} = x^{-1}x = 1$$

NOTE – Any set R having operations
$$+ : R \times R \to R \quad \& \quad \cdot : R \times R \to R$$
which obeys the laws of algebra for $\mathbb{Z}$ & $\mathbb{Q}$ forms a commutative ring & a field respectively.

Positive rational

A rational q is positive if $\exists$ positive a, b s.t

$$q = [a // b]$$

Lemma – If q is positive, then $\nexists$ c, d s.t $cd < 0$ and $q = [c // d]$

Pf – $\because$ q is positive

$\therefore$ $\exists$ a, b positive s.t $q = [a // b]$

$$\Rightarrow [a // b] = [c // d]$$

$$\Rightarrow (a // b) \sim (c // d)$$

$$\Rightarrow \quad ad = bc$$

WLOG, let $c < 0$ & $d > 0$.

$\Rightarrow$ LHS > 0 and RHS < 0 $\rightarrow$ Contd^n

$\square$

Reals

- **Absolute value** –

$$|\,|: \mathbb{Q} \to \mathbb{Q}$$

$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

- **Dist b/w rationals**

$$d(x,y) := |x-y| \quad ; \quad x, y \in \mathbb{Q}$$