$$(a_n)$$

<u>Definition</u>: Let $x \in \mathbb{R}$. We say x is positive if $\exists\ c \in \mathbb{Q}_{>0}$ and N such that $\forall\ n \geq N$, $a_n \geq c$. We say x is negative if $\exists\ c \in \mathbb{Q}_{>0}$ and N such that $\forall\ n \geq N$, $a_n \leq -c$.

<u>Lemma 1</u>: Let $x \in \mathbb{R}$ and $x \neq 0$. If $x = (a_n)$, then $\exists\ c \in \mathbb{Q}_{>0}$ and N such that $\forall\ n \geq N$, $|a_n| \geq c$.

The above Lemma was proved in class.

<u>Lemma</u>: Let $x \in \mathbb{R}$ and $x \neq 0$. The notion of being positive or negative is well-defined, that is, independent of the choice of representative.

<u>Proof</u>: Let us assume that $x = (a_n) \sim (b_n)$. First assume that for (a_n) we have $c \in \mathbb{Q}_{>0}$ and N such that $\forall\ n \geq N$, $a_n \geq c$. We need to show that for (b_n) there is c' and N' such that $\forall\ n \geq N'$ we have $b_n \geq c'$.

Since $(a_n) \sim (b_n)$, for $\epsilon = c/2$, there is N_1 such that $|a_n - b_n| \leq \epsilon$. This implies that $-\epsilon \leq b_n - a_n \leq \epsilon$. This shows that $a_n - \epsilon \leq b_n$ $\forall\ n \geq N_1$. If we take $N' = \max\{N, N_1\}$ then we get $\frac{c}{2} = c - \epsilon \leq a_n - \epsilon \leq b_n$. Thus, taking $c' = \frac{c}{2}$ and N' we get what we wanted to prove.

Next consider the case $a_n \leq -c$ for $n \geq N$. Then we need to show that $\exists\ c' \in \mathbb{Q}_{>0}$ and N' such that $\forall\ n \geq N'$ we have $b_n \leq -c'$. This is done similarly, and is left as an exercise. This completes the proof of the lemma.

<u>Remarks</u>: (1) If x is positive then $x \neq 0$. Similarly, if x is negative then $x \neq 0$. (2) x cannot be both +ve and −ve. (3) If x is +ve then $-x$ is −ve. If x is −ve then $-x$ is +ve. The proof of this remark is left as an exercise.

Proposition: Let $x \in \mathbb{R}$ and $x \neq 0$. Then either is positive or it is negative.

Proof: Let us assume that x is not negative. Then we need to show that it is positive. Let us represent $x = (a_n)$. By Lemma 1, there is $c \in \mathbb{Q}_{>0}$ and N such that $\forall\ n \geq N$, $|a_n| \geq c$.

Let us take $\epsilon = c/2$. Since (a_n) is a Cauchy seq there is N_1 such that $\forall\ n, m \geq N_1$ we have $|a_n - a_m| \leq \epsilon = c/2$.

Since (a_n) is not negative, if we fix c, then there is no N_2 such that $\forall\ n \geq N_2$ we have $a_n \leq -c$. In other words, given any N_2, there is an $m \geq N_2$ such that $a_m > -c$. Thus, $\exists\ M \geq \max\{N, N_1\}$ such that $a_M > -c$.

We claim that $\forall\ n \geq M$, $a_n \geq c$. First note that $|a_M| \geq c$ and $a_M > -c \implies a_M \geq c$. For any $n \geq M$, we have $|a_n - a_M| \leq \frac{c}{2} \implies -\frac{c}{2} \leq a_n - a_M \leq \frac{c}{2} \implies a_M - \frac{c}{2} \leq a_n$.

As $a_M \geq c \implies \frac{c}{2} \leq a_M - \frac{c}{2} \leq a_n \implies a_n > 0$. As $|a_n| \geq c$
$\implies a_n \geq c$.

This proves that x is positive.

Proposition: The following are easily proved using definitions:

1. If x is +ve then x^{-1} is +ve.
2. If x, y have the same parity then xy is +ve
3. If x, y have different parity then xy is $-$ve.

Proof. Write $x = (a_n)$ with $a_n \neq 0$. Then $x^{-1} = (a_n^{-1})$. Since x is +ve, $\exists\ c \in \mathbb{Q}_{>0}$ and N such that $a_n \geq c$ $\forall\ n \geq N$. As (a_n) is Cauchy $\implies |a_n| \leq M$. Thus, $\forall\ n \geq N$, we have $c \leq a_n \leq M$.
$\implies \forall n \geq N$ we have $\frac{1}{a_n} \geq \frac{1}{M}$

For (2), write $x=(a_n)$ and $y=(b_n)$. Then $xy=(a_nb_n)$. There exist $\underbrace{c,c'}_{\uparrow\, \mathbb{Q}_{>0}}, N$ such that $\forall n\geq N$ $a_n\geq c$, $b_n\geq c'$ $\Rightarrow a_nb_n\geq cc'$. This proves $xy>0$. Similarly, do (3).

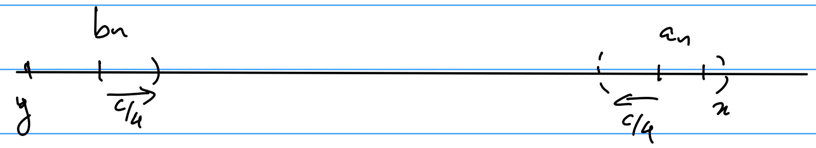
<u>Definition</u>: We say $x>y$ if $x-y$ is +ve. We say $x<y$ if $x-y$ is −ve.

<u>Lemma</u>: If $x>y>0$ then $y^{-1}>x^{-1}$.
<u>Proof</u>: Since xy is +ve, $y^{-1}-x^{-1}$ has the same parity as $(y^{-1}-x^{-1})xy = x-y$, which is +ve.

<u>Proposition</u>: Let $x=(a_n)$. If $a_i\geq 0$, then $x\geq 0$.
<u>Proof</u>: If $x<0$, then x is −ve $\Rightarrow$ $\exists$ $c\in\mathbb{Q}_{>0}$ and N such that $\forall n\geq N$ $a_n\leq -c$ which is a contradiction.

<u>Corollary</u>: If $x=(a_n)$ and $y=(b_n)$ and $a_n\geq b_n$ $\forall n$, then $x\geq y$.
<u>Proof</u>: $x-y=(a_n-b_n)$ and now apply the previous proposition to get $x-y\geq 0 \Rightarrow x\geq y$.

<u>Proposition</u>: Let $x>y$. Then we can find $q\in\mathbb{Q}$ such that $x>q>y$.
<u>Proof</u>: Let $x=(a_n)$ and $y=(b_n)$. Then $x-y=(a_n-b_n)>0$. Thus, $\exists$ $c\in\mathbb{Q}_{>0}$ and N such that $\forall n\geq N$ we have $a_n-b_n\geq c$.



$\exists$ N_1 such that $\forall$ $n,m\geq N_1$ $|a_n-a_m|\leq c/4$ and $|b_n-b_m|\leq c/4$.
Let $q=\dfrac{a_{N_1}+b_{N_1}}{2}$. Then for all $m\geq N_1$ we have

$$\begin{aligned} q-b_m &= \tfrac{1}{2}(b_{N_1}-b_m)+\tfrac{1}{2}(a_{N_1}-b_m)\\ &= \tfrac{1}{2}(b_{N_1}-b_m)+\tfrac{1}{2}(a_{N_1}-b_{N_1})+\tfrac{1}{2}(b_{N_1}-b_m) = (b_{N_1}-b_m)+\tfrac{1}{2}(a_{N_1}-b_{N_1})\\ &\geq -c/4+c/2 = c/4.\end{aligned}$$

$\Rightarrow \quad q - y > 0.$

$$a_m - q = \frac{1}{2}(a_m - a_{N_1}) + \frac{1}{2}(a_m - b_{N_1}) = (a_m - a_{N_1}) + \frac{1}{2}(a_{N_1} - b_{N_1})$$
$$\geq -c/4 + c/2 = c/4$$

$\Rightarrow \quad x > q.$

Least Upper Bound: Let $E \subset \mathbb{R}$ be a subset. For simplicity we shall assume that E is bounded, that is, $\exists$ integer M such that every $x \in E$ satisfies $-M \leq x \leq M$.

Definition (Upper Bound): A real number α is said to be an upper bound for E if $\forall x \in E$ we have $x \leq \alpha$. A real number β is said to be a least upper bound for E if β is an upper bound for E and given any other upper bound α for E, we have $\beta \leq \alpha$.

Proposition: Let $E \subset \mathbb{R}$ be a subset, then E can have atmost one least upper bound.

Proof: Suppose β_1 and β_2 are two least upper bounds for E, then we have $\beta_1 \leq \beta_2$ and $\beta_2 \leq \beta_1$. Thus, $\beta_1 = \beta_2$.

Theorem: Let $E \subset \mathbb{R}$ be a bounded subset. Then E has a unique least upper bound.

Proof: Since E is bounded, we have that $\forall x \in E$, $-M \leq x \leq M$. Thus, M is an upper bound for E. For each $n \geq 1$, we consider the finite collection of rationals $\{-M + \frac{i}{2^n} \mid i = 0, \ldots M2^{n+1}\}$. There is a unique i such that $-M + \frac{i}{2^n}$ is an upper bound for E and $-M + \frac{i-1}{2^n}$ is not an upper bound for E. Let us denote this i by $i(n)$.

Note that $-M + \frac{i(n)}{2^n} = -M + \frac{2i(n)}{2^{n+1}}$, is an upper bound for E.

Thus, $i(n+1) \le 2i(n)$. Also note that $-M + \frac{i(n)-1}{2^n} = -M + \frac{2i(n)-2}{2^{n+1}}$,

which is not an upper bound for E. Thus, $i(n+1) - 1 \geq 2i(n) - 2$, that is, $i(n+1) \geq 2i(n) - 1$. Thus, $2i(n) - 1 \le i(n+1) \le 2i(n)$.

① We claim that the sequence $-M + \frac{i(n)}{2^n}$ is a Cauchy seq.

$$\left| -M + \frac{i(n)}{2^n} - \left(-M + \frac{i(m)}{2^m}\right)\right| = \left|\frac{i(n)}{2^n} - \frac{i(m)}{2^m}\right| \qquad (m > n)$$

$$\le \left|\frac{i(n)}{2^n} - \frac{i(n+1)}{2^{n+1}}\right| + \cdots \left|\frac{i(m-1)}{2^{m-1}} - \frac{i(m)}{2^m}\right|$$

$$\le \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} = \frac{(1-1/2)(\quad)}{(1-1/2)} = \frac{\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}}{1/2} = \frac{1}{2^n} - \frac{1}{2^m}$$

$$\le \frac{1}{2^n}.$$

Thus, if N is such that $\frac{1}{2^N} \le \epsilon$, then $\forall\, m \geq n \geq N$ we have $\left| -M + \frac{i(n)}{2^n} - \left(-M + \frac{i(m)}{2^m}\right)\right| \le \frac{1}{2^n} \le \frac{1}{2^N} \le \epsilon$. This proves Cauchy.

Let us call the real number represented by this sequence β. We claim β is an upper bound for E. First we need a Lemma.

Let $x \in \mathbb{R}$ and let $y = (a_n)$. Suppose $x \le a_n\ \forall n$, then $x \le y$. If not, then $y < x$. Let q be such that $y < q < x$. Then $q < x \le a_n\ \forall n$. By the above Corollary we $q \le y$, which is a contradiction.

If $x \in E$, then $x \le -M + \frac{i(n)}{2^n} \quad \forall n \Rightarrow x \le \beta$.

Next we claim that β is the least upper bound. Suppose α is an upper bound for E. By construction, $-M + \frac{i(n)-1}{2^n}$ is not an upper bound for E. Thus, $\exists\, x \in E$ such that

$$-M + \frac{i(n)-1}{2^n} < x \leq \alpha .$$

It is easy to check, using the same method as above, that $\left(-M + \frac{i(n)-1}{2^n}\right)$ is a Cauchy seq. It is also easy to check that this seq is equivalent to the seq $\left(-M + \frac{i(n)}{2^n}\right)$.

Again, we have the following Lemma. Let $x \in \mathbb{R}$ and $y = (a_n)$. Suppose $a_n \leq x \ \forall n$, then $y \leq x$.

Applying this to $\beta = \left(-M + \frac{i(n)-1}{2^n}\right)$ and α, we get that $\beta \leq \alpha$.

This completes the proof of the Theorem.

The least upper bound is often called the supremum, and denoted $\sup E$. Similar to the least upper bound we have the greatest lower bound. The definition and existence are similar. In fact, using the observation $\sup(-E) = -\inf E$ we get another proof of existence. In other words show that $-\sup(-E)$ is a greatest lower bound for E.

Before we proceed, let us prove a few useful results.

① Recall that given $x \in \mathbb{R}$ we defined

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Lemma: If x is represented by a Cauchy seq (q_n) of rationals, then $|x|$ is represented by the Cauchy seq $(|q_n|)$.

Proof: Let us first show that the sequence $(|q_n|)$ is Cauchy. By triangle inequality we have

$$|q_n| = |q_m + (q_n - q_m)| \leq |q_m| + |q_n - q_m|$$

$$\Rightarrow \quad |q_n| - |q_m| \leq |q_n - q_m|.$$

Similarly, switching n, m we get $|q_m| - |q_n| \leq |q_m - q_n| = |q_n - q_m|$. Thus, $\big||q_n| - |q_m|\big| \leq |q_n - q_m|$.

Given $\epsilon \in \mathbb{Q}_{>0}$, $\exists N$ such that $\forall n, m \geq N$ we have $|q_n - q_m| \leq \epsilon$. Thus, $\big||q_n| - |q_m|\big| \leq |q_n - q_m| \leq \epsilon$.
Thus, $(|q_n|)$ is a Cauchy seq.

If $x > 0$, then $\exists$ $c \in \mathbb{Q}_{>0}$ and N such that $\forall n \geq N$, $q_n \geq c$. Thus, we also have that $\forall n \geq N$, $|q_n| = q_n \geq c$. This shows that the sequences $(|q_n|)$ and (q_n) are the same for $n \geq N$ and so they represent the same rational number. Since $x > 0$, $|x| = x$. Thus, it follows that $(|q_n|)$ represents x, that is, $(|q_n|)$ represents $|x|$ if $x > 0$.

If $x < 0$, then $\exists$ $c \in \mathbb{Q}_{>0}$ and N such that $\forall n \geq N$, $q_n \leq -c$. Thus, we have $\forall n \geq N$, $|q_n| = -q_n \geq c$. Thus, the seq $(|q_n|)$ and $(-q_n)$ different only at finitely many places and so they represent the same number. Since $(-q_n)$ represents $-x$, it follows $(|q_n|)$ represents $-x = |x|$. Thus, $(|q_n|)$ represents $|x|$ if $x < 0$.

If $x = 0$, then this means that the sequences $(0, 0, \ldots)$ and $(q_1, q_2, \ldots)$ are equivalent. That is, for $\epsilon \in \mathbb{Q}_{>0}$, there is N such that

$\forall n \geq N$, we have $|q_n - 0| = |q_n| \leq \epsilon$. But this shows that the sequence $(|q_1|, |q_2|, \dots)$ is also equivalent to $(0, 0, \dots)$. Thus, $(|q_n|)$ represents $0 = |x|$. This completes the proof of the Lemma.

<u>Lemma</u>: Let $x \in \mathbb{R}$ and assume that x is represented by the Cauchy sequence (q_n), $q_i \in \mathbb{Q}$. Then for every $\epsilon \in \mathbb{Q}_{>0}$, $\exists N$ such that $\forall n \geq N$, $|x - q_n| \leq \epsilon$.

<u>Proof</u>: Let $\epsilon \in \mathbb{Q}_{>0}$. Then $\exists N$ such that $\forall n, m \geq N$ we have $|q_n - q_m| \leq \epsilon$. Let us fix an $n \geq N$. The number $x - q_n \in \mathbb{R}$ is represented by the equivalence class of the Cauchy seq

$$(q_1 - q_n, q_2 - q_n, \dots).$$

By the previous Lemma, the number $|x - q_n|$ is represented by the Cauchy seq $(|q_1 - q_n|, |q_2 - q_n|, \dots)$. Recall that if two Cauchy seq differ at finitely many places, then they define the same number. Thus, consider the sequence $(0, 0, \dots 0, |q_{N+1} - q_n|, |q_{N+2} - q_n|, \dots)$. This also represents $|x - q_n|$. Let us call this seq (b_m). Then for every m, we have $b_m \leq \epsilon$ $\longrightarrow$ if $m \leq N$ then $b_m = 0$

$\longrightarrow$ $m > N$, then use (q_n) is Cauchy.

Recall we proved that if $y = (a_m)$ and $a_m \leq q$, $q \in \mathbb{Q}$, then $y \leq q$. Using this we get $|x - q_n| \leq \epsilon$. This happens for all $n \geq N$. Thus, the proof of the Lemma is complete.

<u>Proposition 5.5.12</u>: There is a real number x such that $x^2 = 2$.

We saw that there were gaps in $\mathbb{Q}$, and the above proposition shows that $\mathbb{R}$ fills at least one of these. Are there gaps in $\mathbb{R}$? We can define Cauchy sequences of reals as follows. Say that a seq (x_n) is Cauchy if

for every $\epsilon \in \mathbb{Q}_{>0}$, $\exists N$ such that $\forall n, m \geq N$, $|x_n - x_m| \leq \epsilon$.

We define an equivalence relation on the set of Cauchy sequences of reals (note that we need triangle inequality for reals to do this, but this easily follows from the one for rationals).

There is a natural inclusion $\mathbb{R} \hookrightarrow$ Equivalences classes of Cauchy seq.

<u>Theorem</u>: This map is surjective.

<u>Proof</u>: Let (x_n) be a Cauchy seq of reals. For each n, choose a rational q_n such that $x_n < q_n < x_n + \frac{1}{n}$. We claim that (q_n) form a Cauchy seq. To see this, consider

$$|q_n - q_m| = |q_n - x_n + x_n - x_m + x_m - q_m| \leq |q_n - x_n| + |x_n - x_m| + |x_m - q_m|.$$
$$\leq \frac{1}{n} + |x_n - x_m| + \frac{1}{m}$$

Given $\epsilon \in \mathbb{Q}_{>0}$, choose N such that $\frac{1}{N} \leq \frac{\epsilon}{3}$ and $\forall\, n,m \geq N$ $|x_n - x_m| \leq \frac{\epsilon}{3}$.

Then $\forall\, n,m \geq N$ we have $|q_n - q_m| \leq \frac{1}{n} + \frac{\epsilon}{3} + \frac{1}{m} \leq \frac{2}{N} + \frac{\epsilon}{3} \leq \epsilon$.

Let $x \in \mathbb{R}$ be the class of the sequence (q_n). We want to show that the Cauchy seq $(x, x, \ldots) \sim (x_1, x_2, \ldots)$. Since $(q_n) = x$, by earlier Lemma, we get for $\epsilon/2$, $\exists\, N_1$ such that $\forall n \geq N_1$ $|x - q_n| \leq \epsilon/2$. Let N_2 be such that $\frac{1}{N_2} \leq \frac{\epsilon}{2}$. If $n \geq N_2$,

then $|x_n - q_n| \leq \frac{1}{n} \leq \frac{1}{N_2} \leq \epsilon/2$. Thus, $|x - x_n| \leq |x - q_n| + |q_n - x_n|$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon$$
$$\forall\, n \geq \max\{N_1, N_2\}.$$

This proves the sequences are equivalent and completes the proof of the Theorem.

The above theorem shows that there are no "gaps" in $\mathbb{R}$, that is, $\mathbb{R}$ is complete.

<u>Corollary/Definition</u>: Given a Cauchy seq (x_n) of reals, the above theorem shows that there is a unique real number L (as the map $\mathbb{R} \hookrightarrow E C$ is an inclusion) such that $(x_n) \sim (L, L, \ldots)$. This number L will be called the limit of (x_n) and we write $\lim_{n\to\infty} x_n = L$.

<u>Lemma</u>: Let (x_n) be a Cauchy seq of reals. Let M be a real number. Then $\lim x_n = M \iff \lim (x_n - M) = 0 \iff \lim |x_n - M| = 0$.

<u>Proof</u>: Def of $\lim x_n = M$ is for every $\epsilon \in \mathbb{Q}_{>0}$ $\exists N$ such that

$$\forall n \geq N \quad |x_n - M| \leq \epsilon.$$

Def of $\lim |x_n - M| = 0$ is for every $\epsilon \in \mathbb{Q}_{>0}$, $\exists N$ such that

$$\forall n \geq N \quad \big||x_n - M| - 0\big| \leq \epsilon \iff |x_n - M| \leq \epsilon$$

Def of $\lim (x_n - M) = 0$ is for every $\epsilon \in \mathbb{Q}_{>0}$, $\exists N$ such that

$$\forall n \geq N \quad |(x_n - M) - 0| \leq \epsilon \iff |x_n - M| \leq \epsilon.$$

Let (x_n) be a seq of real numbers, not necessarily Cauchy. Suppose $\exists$ a real number L such that $\forall \epsilon \in \mathbb{Q}_{>0}$ $\exists N(\epsilon)$ such that $\forall n \geq N$ we have $|x_n - L| \leq \epsilon$, then we say that x_n converges to L, and that x_n is a convergent seq.

<u>Lemma</u>: A convergent seq is Cauchy.

<u>Proof</u>: (x_n) is convergent. Thus, $\exists N$ such that $\forall n \geq N$ $|x_n - L| \leq \epsilon/2$. $\Rightarrow$ $|x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |x_m - L|$

$$\leq \epsilon \quad \forall n, m \geq N.$$

Clearly, the Cauchy seq (x_n) satisfies $\lim x_n = L$.

The theorem we proved showed that Cauchy sequences are convergent.

<u>Convergence of Monotone Sequences</u>: Let (x_n) be a sequence of reals, assume $x_1 \leq x_2 \leq \cdots$ and x_n's are bounded above by M. Then (x_n) is a convergent sequence.

<u>Proof</u>: Let $L = \sup\{x_n\}$. We claim that $\lim x_n = L$. Let $\epsilon \in \mathbb{Q}_{>0}$. Since L is the lub $\Rightarrow$ $L-\epsilon$ is not an upper bound for $\{x_n\}$. Thus, $\exists\ x_N$ such that $L-\epsilon < x_N \leq L$. Thus, for every $m \geq N$ we have $L-\epsilon < x_N \leq x_m \leq L$. Thus, $|L - x_m| \leq |L-(L-\epsilon)| = \epsilon \quad \forall\ m \geq N$.
This proves that $\lim x_n = L$.

Similar to the above, we have the following: If $x_1 \geq x_2 \geq \cdots$ and x_n are bounded below by M, then (x_n) is a convergent sequence. In this case it converges to $\inf\{x_n\}$.

<u>lim sup and lim inf</u>: Let (x_n) be a sequence of reals, not necessarily Cauchy. Let E_N be the set $E_N = \{x_N, x_{N+1}, \ldots\}$. Recall that $\sup E$ is the least upper bound of E. If $E \subset F$, then it is clear that $\sup E \leq \sup F$, as every upper bound for F is an upper bound for E.

Since $E_1 \supset E_2 \supset \cdots \Rightarrow \sup E_1 \geq \sup E_2 \geq \cdots$.

Let us assume that (x_n) is bounded. Then we get that $\exists\ M$ and $-M \leq x_n \leq M \quad \forall\ n$. Thus, $-M \leq x_N \leq \sup E_N$. By the monotone convergence theorem we get that this seq converges. The limit of this sequence is denoted $\limsup E$.

Thus, $\quad \sup E_1 \geq \sup E_2 \geq \cdots \geq \limsup E$.

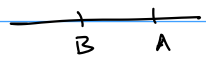
Similarly, if $E \subset F$ then we get $\inf F \le \inf E$ and so
$\inf E_1 \le \inf E_2 \le \dots$.

As the sequence (x_n) is bounded above by M, we have $\inf E_N \le x_N \le M$.
Thus, the seq $\inf E_1 \le \dots$ is bounded above by M and so there is a limit which we denote $\liminf E$. Thus,

$$\inf E_1 \le \inf E_2 \le \dots \le \liminf E.$$

<u>Claim</u>: $\liminf E \le \limsup E$.

To prove the claim we need the following lemma for reals.
Let (x_n) and (y_n) be Cauchy sequences of reals such that $\forall n$ $x_n \le y_n$. Then $\lim x_n \le \lim y_n$. The proof is by contradiction. Assume $\lim y_n = B < \lim x_n = A$.



Let $\epsilon \in \mathbb{Q}_{>0}$, then $\exists$ N such that $\forall$ $n \ge N$ $|y_n - B| \le \epsilon$, and so $y_n \le B + \epsilon$ and $|x_n - A| \le \epsilon$ and so $x_n \ge A - \epsilon$. Then $y_n - x_n \le B - A + 2\epsilon$.
Since $B - A < 0$, we may choose ϵ small so that $B - A < -2\epsilon < 0$ and so we get $B - A + 2\epsilon < 0$ for all $n \ge N$. This contradicts $y_n - x_n \ge 0$.

Fix m, then $\forall$ $n \ge m$ we have $E_n \subset E_m \Rightarrow$
$\inf E_1 \dots \le \inf E_{n-1} \le \inf E_n \le \sup E_n \le \sup E_m$. Thus, for fixed m, we see that

$$\inf E_n \le \sup E_m \quad \forall n \Rightarrow \liminf E \le \sup E_m.$$

Letting the m vary we get $\liminf E \le \dots \sup E_{m+2} \le \sup E_{m+1} \le \dots$

$$\Rightarrow \liminf E \le \limsup E.$$

<u>Proposition</u>: $E = \{x_n\}$ is a convergent seq $\Leftrightarrow \liminf E = \limsup E$.

<u>Proof</u>: Let us assume that $\lim x_n = L$. Given $\epsilon \in \mathbb{Q}_{>0}$, $\exists$ N such that $\forall$ $n \ge N$ we have $|x_n - L| \le \epsilon$. Thus, $x_n \le L + \epsilon$ $\forall$ $n \ge N$. Then, $\sup E_N \le L + \epsilon \Rightarrow \limsup E \le \sup E_N \le L + \epsilon$. This happens for every $\epsilon \in \mathbb{Q}_{>0}$. This shows that $\limsup E \le L$. Similarly, $x_n \ge L - \epsilon$.
Thus, $\inf E_N \ge L - \epsilon \Rightarrow \liminf E \ge \inf E_N \ge L - \epsilon$. This happens

for all $\epsilon \in \mathbb{Q}_{>0} \Rightarrow \liminf E \geq L$. Thus, we have

$$L \leq \liminf E \leq \limsup E \leq L \Rightarrow \text{All are equal.}$$

Conversely, suppose $\liminf E = \limsup E = L$. Then we have two sequences

$$\inf E_1 \leq \inf E_2 \leq \dots \inf E_n \leq \dots \leq L \leq \dots \sup E_n \leq \dots \leq \sup E_2 \leq \sup E_1$$

and both converging to L. Thus, for every $\epsilon \in \mathbb{Q}_{>0}$ $\exists N$ such that $\forall n \geq N$ we have $|L - \inf E_n| \leq \epsilon$ and $|L - \sup E_n| \leq \epsilon$.

Note that $\inf E_n \leq x_n \leq \sup E_n$. Thus, $L - \inf E_n \geq L - x_n \geq L - \sup E_n$

$\Rightarrow \epsilon \geq L - x_n \geq -\epsilon \Rightarrow |L - x_n| \leq \epsilon \;\; \forall n \geq N$. Thus, $\lim x_n = L$.

<u>Series</u>: Given a sequence (x_n) of real numbers, we can form another sequence s_n as follows. Define $s_n := x_1 + \dots + x_n$. We may ask if the sequence (s_n) is Cauchy, or equivalently, if it converges. For this sequence to be Cauchy, applying the definition we get that for every $\epsilon \in \mathbb{Q}_{>0}$, there is N such that $\forall n, m \geq N$ we have $|s_n - s_m| \leq \epsilon$, that is, $\forall n, m \geq N$, we have

$$\left| \sum_{i=n+1}^{m} x_i \right| \leq \epsilon .$$

The sequence s_n is called the sequence of partial sums. If (s_n) is a Cauchy sequence then we say the series $\sum_{i \geq 1} x_i$ converges to the limit $\lim s_n = L$.

$\sum_{i=1}^{\infty} x_i$

Our final aim in this part of the course is to show that $\mathbb{R}$ is not countable.

<u>Lemma</u>: Let X be a set. Recall the set $\mathcal{P}(X)$ whose elements are subset of X. Then X is not in bijection with $\mathcal{P}(X)$.

$x \in X$ (not A)

Proof: Let us assume that there is a bijection $f: X \to \mathcal{P}(X)$. Consider the subset $A := \{x \in A \mid x \notin f(x)\}$. Since f is a bijection, there is $y \in X$ such that $f(y) = A$. If $y \notin f(y) = A$, then by the defining property of A, we see that $y \in A$, which is a contradiction. On the other hand, if $y \in f(y) = A$, then again, by the defining property of A, we get that $y \notin f(y) = A$, a contradiction. Thus, there is no such f.

This shows that the "size" of $\mathcal{P}(X)$ is strictly larger than the "size" of X. Obviously X can be put into $\mathcal{P}(X)$, the simplest way being $X \hookrightarrow \mathcal{P}(X)$ $\quad x \mapsto \{x\}$.

We will now define an embedding $\mathcal{P}(\mathbb{N}) \hookrightarrow \mathbb{R}$, which will show that the "size" of $\mathbb{R}$ is strictly greater than the size of $\mathbb{N}$. Given a subset $A \subset \mathbb{N}$, define the number α_A as follows. If $A = \phi$ then define $\alpha_A = 0$. If not, then define $A_m = \{n \in A \mid n \leq m\}$. Define $\alpha_{A_m} = \sum_{n \in A_m} 10^{-n}$. This is clearly a monotone sequence as $\alpha_{A_1} \leq \alpha_{A_2} \leq \dots$

$$\alpha_{A_m} = \sum_{n \in A_m} 10^{-n} \leq \sum_{n=0}^{m} 10^{-n} = \left(1 - \frac{1}{10^{m+1}}\right)\frac{10}{9} \leq \frac{10}{9}.$$

Thus, this sequence is also bounded above and so converges to a number which we take to be α_A.

Thus, we have defined a map $\mathcal{P}(\mathbb{N}) \to \mathbb{R}$. We need to check this is an inclusion. Suppose $A \neq B$. Then either $A \not\subseteq B$ or $B \not\subseteq A$. Thus, there is some n such that $n \in A \setminus B$ or $n \in B \setminus A$. Choose the smallest such n, that is, choose the smallest element in $(A \setminus B) \cup (B \setminus A)$. Call this element n_0. Let us assume that $n_0 \in A \setminus B$. Then if $n < n_0$, we have $n \in A \iff n \in B$.

For $m \geq n_0$, let us consider $|\alpha_{A_m} - \alpha_{B_m}|$. From the definition we have

$$|\alpha_{A_m} - \alpha_{B_m}| = \left| \sum_{j \in A_m} 10^{-j} - \sum_{j \in B_m} 10^{-j} \right|$$

$$= \left| \sum_{j \in A_{n_0-1}} 10^{-j} + 10^{-n_0} + \sum_{j \in A_m \setminus A_{n_0}} 10^{-j} - \left(\sum_{j \in B_{n_0-1}} 10^{-j} + \sum_{j \in B_m \setminus B_{n_0}} 10^{-j} \right) \right|$$

$$= \left| 10^{-n_0} + \sum_{j \in A_m \setminus A_{n_0}} 10^{-j} - \sum_{j \in B_m \setminus B_{n_0}} 10^{-j} \right|$$

$$\geq \left(10^{-n_0} + \sum_{j \in A_m \setminus A_{n_0}} 10^{-j} \right) - \sum_{j \in B_m \setminus B_{n_0}} 10^{-j}$$

$$\geq 10^{-n_0} - \sum_{j \in B_m \setminus B_{n_0+1}} 10^{-j} \geq 10^{-n_0} - \sum_{j = n_0+1}^{m} 10^{-j}$$

$$= 10^{-n_0} - 10^{-(n_0+1)} \frac{10}{9} \left(1 - 10^{-(m-n_0-1)}\right)$$

$$\geq 10^{-n_0} - \frac{10^{-n_0}}{9} \quad \text{for } m > n_0+1.$$

Thus, taking limit $m \to \infty$ we get $|\alpha_A - \alpha_B| \geq 10^{-n_0} \frac{8}{9} > 0$.

Thus, $\alpha_A \neq \alpha_B$. This shows that $\#(\mathbb{R}) \geq \#(\mathcal{P}(\mathbb{N})) > \#(\mathbb{N})$. Thus, $\mathbb{R}$ is not countable.

Finally, let us show that between any two real numbers $x < y$ there is an irrational. It is easily checked that the set $S = \{a \in \mathbb{R} \mid x < a < y\}$ is in bijection with $\mathbb{R}$. Since $\mathbb{Q}$ is countable, it follows that $S \cap \mathbb{Q}$ is countable. Thus, there is an element of S which is not in $\mathbb{Q}$.