PH111: Tutorial Sheet 1

This tutorial sheet contains some problems related to vectors and kinematics, employing Cartesian coordinate system.

1. Consider two vectors $\mathbf{A} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Find a third vector $\mathbf{C}$ (say), which is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

2. Again, consider the vectors $\mathbf{A}$ and $\mathbf{B}$ of the previous problem. Find the angle between them.

3. Find a unit vector, which lies in the xy plane, and which is perpendicular to $\mathbf{A}$ of previous problems. Similarly, find a unit vector which is perpendicular to $\mathbf{B}$, and lies in the xz plane.

4. Calculate $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A})$, for the vectors of the previous problem. Does the result obtained hold only for the given $\mathbf{A}$ and $\mathbf{B}$ vectors, or will it hold for any general vectors $\mathbf{A}$ and $\mathbf{B}$.

5. Consider two distinct general vectors $\mathbf{A}$ and $\mathbf{B}$. Show that $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$ implies that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

6. Position of a particle in the xy plane is given by

$$\mathbf{r}(t) = A\left(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}\right),$$

   where α and A are constants. Calculate the velocity and acceleration of the particle, as functions of time t. Plot the vectors corresponding to $\mathbf{r}(0)$ and $\mathbf{v}(0)$.

7. Acceleration of a particle in the xy plane is given by $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$, where $\mathbf{r}(t)$ denotes its position, and ω is a constant. If $\mathbf{r}(0) = a\hat{\mathbf{j}}$, and $\mathbf{v}(0) = a\omega\hat{\mathbf{i}}$ ($\mathbf{v}$ is the velocity), integrate the equation of motion to obtain the expression for $\mathbf{r}(t)$, in Cartesian coordinates.

8. Rate of change of acceleration of a particle is called "jerk" ($\mathbf{j}(t)$) in physics. If the jerk of a particle is given by

$$\mathbf{j}(t) = a\hat{\mathbf{i}} + bt\hat{\mathbf{j}} + ct^2\hat{\mathbf{k}},$$

   where a, b, and c are constants. Assuming that at time $t = 0$, particle was located at the origin, and its velocity and acceleration were zero, obtain its position $\mathbf{r}(t)$, as a function of time, in Cartesian coordinates.

9. A rocket of mass M accelerates in free space by expelling hot gas in the backward direction. The speed of the exhaust gas depends on the energy released in the combustion process and can be taken to be a constant, say $|u|$ w.r.t to the rocket. Assume that in time Δt the rocket loses an amount of mass $\Delta m = -\frac{dM}{dt}\Delta t$, where $\frac{dM}{dt}$ denotes the rate of change of the mass of the rocket. Answer the following questions.

(a) If the instantaneous velocity of the rocket is v, in an inertial frame, what is the velocity of the exhaust in that frame? Write down the total momentum of the system as seen from that inertial frame at t and $t + \Delta t$.

(b) From this obtain a differential equation connecting the changes in mass and velocity.

(c) Show that the final velocity increases only as the *logarithm* of the amount of fuel.

(d) What is the significance of such a dependence ?

PH 111: Tutorial Sheet 1
Solutions

This tutorial sheet contains some problems related to vectors and kinematics, employing Cartesian coordinate system.

1. Consider two vectors $\mathbf{A} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Find a third vector $\mathbf{C}$ (say), which is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

   Soln: By definition, vectors $\pm(\mathbf{A} \times \mathbf{B})$ will be perpendicular both to $\mathbf{A}$ and $\mathbf{B}$. Thus, the answer is

$$\pm(\mathbf{A} \times \mathbf{B}) = \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 3 \\ 1 & 1 & -2 \end{vmatrix} = \pm\left(-\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right)$$

2. Again, consider the vectors $\mathbf{A}$ and $\mathbf{B}$ of the previous problem. Find the angle between them.

   Soln: By definition

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{-5}{\sqrt{14 \times 6}} = -\frac{5}{2\sqrt{21}} = -0.5455447255$$
$$\implies \theta = \cos^{-1}(-0.5455447255) = 123.0618^{\circ}$$

3. Find a unit vector, which lies in the xy plane, and which is perpendicular to $\mathbf{A}$ of previous problems. Similarly, find a unit vector which is perpendicular to $\mathbf{B}$, and lies in the xz plane.

   Soln: Let the (non-unit) vector $\perp$ to $\mathbf{A}$ be $\mathbf{C} = \pm(c\hat{\mathbf{i}} + d\hat{\mathbf{j}})$, where c and d are constants to be determined Thus

$$\mathbf{A} \cdot \mathbf{C} = \pm(2c - d) = 0$$
$$\implies d = 2c$$
$$\implies \mathbf{C} = \pm c(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$$
$$\implies \mathbf{C} = \pm\frac{c(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})}{\sqrt{c^2 + 4c^2}} = \pm\frac{1}{\sqrt{5}}\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right)$$

   Following a similar procedure, we obtain unit vector $\hat{\mathbf{D}}$, which lies in the xz plane, and is perpendicular to $\mathbf{B}$

$$\hat{\mathbf{D}} = \pm\frac{1}{\sqrt{5}}\left(2\hat{\mathbf{i}} + \hat{\mathbf{k}}\right)$$

4. Calculate $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A})$, for the vectors of the previous problem. Does the result obtained hold only for the given $\mathbf{A}$ and $\mathbf{B}$ vectors, or will it hold for any general vectors $\mathbf{A}$ and $\mathbf{B}$.

Soln: In the previous problem we computed

$$\mathbf{B} \times \mathbf{A} = \left(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}}\right)$$
$$\therefore \mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) = 2 + 7 - 9 = 0$$

The result is general because by definition $\mathbf{B} \times \mathbf{A}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, therefore

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) = \mathbf{B} \cdot (\mathbf{B} \times \mathbf{A}) = \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$$

5. Consider two distinct general vectors $\mathbf{A}$ and $\mathbf{B}$. Show that $|\mathbf{A}+\mathbf{B}| = |\mathbf{A}-\mathbf{B}|$ implies that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

   Soln: We have

$$\begin{aligned}
|\mathbf{A}+\mathbf{B}| &= |\mathbf{A}-\mathbf{B}| \\
\implies |\mathbf{A}+\mathbf{B}|^2 &= |\mathbf{A}-\mathbf{B}|^2 \\
\implies (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{B}) &= (\mathbf{A}-\mathbf{B}) \cdot (\mathbf{A}-\mathbf{B}) \\
\implies A^2 + B^2 + 2\mathbf{A} \cdot \mathbf{B} &= A^2 + B^2 - 2\mathbf{A} \cdot \mathbf{B} \\
\implies 4\mathbf{A} \cdot \mathbf{B} &= 0 \\
\implies \mathbf{A} &\perp \mathbf{B}
\end{aligned}$$

6. Position of a particle in the xy plane is given by

$$\mathbf{r}(t) = A\left(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}\right),$$

   where α and A are constants. Calculate the velocity and acceleration of the particle, as functions of time t. Plot the vectors corresponding to $\mathbf{r}(0)$ and $\mathbf{v}(0)$.

   Soln: By definition

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = A\alpha\left(e^{\alpha t}\hat{\mathbf{i}} - e^{-\alpha t}\hat{\mathbf{j}}\right)$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = A\alpha^2\left(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}\right) = \alpha^2\mathbf{r}$$

   Note that

$$\mathbf{r}(0) \cdot \mathbf{v}(0) = A^2\alpha\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right) \cdot \left(\hat{\mathbf{i}} - \hat{\mathbf{j}}\right) = 0,$$

   so $\mathbf{r}(0)$ and $\mathbf{v}(0)$ can be plotted as two mutually perpendicular vectors.

7. Acceleration of a particle in the xy plane is given by $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$, where $\mathbf{r}(t)$ denotes its position, and ω is a constant. If $\mathbf{r}(0) = a\hat{\mathbf{j}}$, and $\mathbf{v}(0) = a\omega\hat{\mathbf{i}}$ ($\mathbf{v}$ is the velocity), integrate the equation of motion to obtain the expression for $\mathbf{r}(t)$, in Cartesian coordinates. What is the the curve along which the particle is moving?

Soln: The acceleration equation $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$ is equivalent to the following two differential equations in Cartesian coordinates

$$\frac{d^2x}{dt^2} = -\omega^2 x$$
$$\frac{d^2y}{dt^2} = -\omega^2 y.$$

We will integrate the x equation, and the same procedure will apply to the y equation. Multiply on both sides by $2\frac{dx}{dt}$

$$\begin{aligned}
2\frac{dx}{dt}\frac{d^2x}{dt^2} &= -\omega^2 2x\frac{dx}{dt} \\
\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 &= -\omega^2\frac{dx^2}{dt} \\
\implies \left(\frac{dx}{dt}\right)^2 &= -\omega^2x^2 + c^2 \quad (c \text{ is a constant}) \\
\implies \frac{dx}{dt} &= \pm\sqrt{c^2 - \omega^2x^2} \\
\pm\frac{dx}{\omega\sqrt{\alpha^2 - x^2}} &= dt \\
\int\frac{dx}{\sqrt{\alpha^2 - x^2}} &= \pm\,\omega\int dt + C \quad (\alpha \text{ and } C \text{ are constants}) \\
\sin^{-1}\left(\frac{x}{\alpha}\right) &= C \pm \omega t \\
x &= \alpha\sin\left(C \pm \omega t\right) \\
x &= A\sin\omega t + B\cos\omega t.
\end{aligned}$$

Similarly,

$$y = C\sin\omega t + D\cos\omega t,$$

above A, B, C, and D are constants of integration to be determined by initial conditions, which are

$$\begin{aligned}
x(0) &= 0 \\
v_x(0) &= a\omega \\
y(0) &= a \\
v_y(0) &= 0.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
v_x(t) &= \frac{dx}{dt} = A\omega\cos\omega t - B\omega\sin\omega t \\
v_y(t) &= \frac{dy}{dt} = C\omega\cos\omega t - D\omega\sin\omega t,
\end{aligned}$$

application of x initial conditions gives

$$
\begin{aligned}
x(0) &= B = 0 \\
v_x(0) &= A\omega = a\omega \\
\implies A &= a
\end{aligned}
$$

and y initial conditions yield

$$
\begin{aligned}
y(0) &= D = a \\
v_y(0) &= C\omega = 0 \\
\implies C &= 0.
\end{aligned}
$$

Thus the final solution is

$$
\begin{aligned}
x(t) &= a \sin \omega t \\
y(t) &= a \cos \omega t \\
\mathbf{r}(t) &= a \left(\sin \omega t \hat{\mathbf{i}} + \cos \omega t \hat{\mathbf{j}} \right).
\end{aligned}
$$

And the curve is a circle of radius a, centered at the origin, because

$$x^2 + y^2 = a^2(\cos^2 \omega t + \sin^2 \omega t) = a^2.$$

8. Rate of change of acceleration of a particle is called "jerk" ($\mathbf{j}(t)$) in physics. If the jerk of a particle is given by

$$\mathbf{j}(t) = a\hat{\mathbf{i}} + bt\hat{\mathbf{j}} + ct^2\hat{\mathbf{k}},$$

where a, b, and c are constants. Assuming that at time $t = 0$, particle was located at the origin, and its velocity and acceleration were zero, obtain its position $\mathbf{r}(t)$, as a function of time, in Cartesian coordinates.

Soln: We have

$$
\begin{aligned}
\frac{da_x}{dt} &= a \\
\frac{da_y}{dt} &= bt \\
\frac{da_z}{dt} &= ct^2
\end{aligned}
$$

from which we have

$$
\begin{aligned}
da_x &= adt \\
\implies \int da_x &= a \int dt + A \\
a_x &= at + A
\end{aligned}
$$

Similarly

$$a_y = \frac{bt^2}{2} + B$$
$$a_z = \frac{ct^3}{3} + C,$$

where A, B, and C are constants. Using the fact that $a_x(0) = a_y(0) = a_z(0) = 0$, we obtain $A = B = C = 0$, so that

$$a_x(t) = \frac{dv_x}{dt} = at$$
$$a_y(t) = \frac{dv_y}{dt} = \frac{bt^2}{2}$$
$$a_z(t) = \frac{dv_x}{dt} = \frac{ct^3}{3}$$

which on integration, and using the initial condition $v(0) = 0$, obtain

$$v_x(t) = \frac{dx}{dt} = \frac{at^2}{2}$$
$$v_y(t) = \frac{dy}{dt} = \frac{bt^3}{6}$$
$$v_z(t) = \frac{dz}{dt} = \frac{ct^4}{12}.$$

On integrating these equations, and employing the initial condition $\mathbf{r}(0) = 0$, we obtain the final result

$$x(t) = \frac{at^3}{6}$$
$$y(t) = \frac{bt^4}{24}$$
$$z(t) = \frac{ct^5}{60}$$

so that

$$\mathbf{r}(t) = \frac{at^3}{6}\hat{\mathbf{i}} + \frac{bt^4}{24}\hat{\mathbf{j}} + \frac{ct^5}{60}\hat{\mathbf{k}}.$$

9. A rocket of mass M accelerates in free space by expelling hot gas in the backward direction. The speed of the exhaust gas depends on the energy released in the combustion process and can be taken to be a constant, say $|u|$ w.r.t to the rocket. Assume that in time Δt the rocket loses an amount of mass $\Delta m = -\frac{dM}{dt}\Delta t$, where $\frac{dM}{dt}$ denotes the rate of change of the mass of the rocket. Answer the following questions.

   (a) If the instantaneous velocity of the rocket is v, in an inertial frame, what is the velocity of the exhaust in that frame? Write down the total momentum of the system as seen from that inertial frame at t and $t + \Delta t$.
   Soln: In an inertial frame the velocity of the exhaust gas is $v - |u|$

(b) From this obtain a differential equation connecting the changes in mass and velocity.

(c) Show that the final velocity increases only as the *logarithm* of the amount of fuel.

(d) What is the significance of such a dependence ?

Soln: In an inertial frame the velocity of the exhaust gas is $v - |u|$. Since no external forces acts on the rocket, the change in momentum must be zero.

$$\begin{aligned}
\delta P &= \underbrace{M(t+\delta t)v(t+\delta t)}_{\text{rocket}-\text{gas}} + \underbrace{\left(-\frac{dM}{dt}\delta t\right)(v-|u|)}_{\text{ejected gas}} - \underbrace{M(t)v(t)}_{P(t)} \\
&= M\frac{dv}{dt}\delta t + \frac{dM}{dt}\delta t\,|u| \\
0 &= M\frac{dv}{dt} + \frac{dM}{dt}\,|u| \\
\int_{M_i}^{M_f}\frac{dM}{M} &= -\frac{1}{|u|}\int_{v_i}^{v_f} \\
v_f - v_i &= |u|\ln\frac{M_i}{M_f}
\end{aligned}$$

Note that if the rocket wants to attain twice the velocity it needs to have about seven times more fuel. It means that velocity is a very slowly increasing function of the amount of fuel burnt. Typically almost 90% of the initial (lift-off) mass of the rocket is just fuel. Rest is the satelite's mass (payload).

PH111: Tutorial Sheet 2

This tutorial sheet contains problems related to plane-polar coordinate system.

1. Using known results from the Cartesian coordinate system, and the relation between plane-polar and Cartesian basis vectors, calculate: (a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}$, (b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}}$, and (c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$.

2. A particle is moving along a circular path of radius a, with angular velocity given by $\omega(t) = \omega_0 + \alpha t$, where ω_0 and α are constants. Obtain the radial and tangential components of its velocity and acceleration.

3. A particle is moving along the line $y = a$, with the velocity $\mathbf{v} = u\hat{\mathbf{i}}$, where u is a constant. Express its velocity in plane polar coordinates.

4. A particle moves in such a way that $\dot{\theta} = \omega$ (constant), and $r = r_0 e^{\beta t}$, where r_0 and β are constants. Write down its velocity and acceleration in plane polar coordinates. For what values of β will the radial acceleration of the particle by zero?

5. Consider a circle of radius a, with the origin of the plane polar coordinate system placed at a point on the circumference. The particle is moving along the circle with a constant speed u.

   (a) What is the equation of the circle in this coordinate system?

   (b) What is the value of $\dot{\theta}$ in terms of u and a?

   (c) Write down the velocity of the particle in plane-polar coordinate system.

   (d) What is the acceleration of the particle in plane-polar coordinate system?

6. A particle moves along the curve $r = A\theta$, with $A = 1/\pi$ m/rad, and $\theta = \alpha t^2$, where α is a constant. Obtain the expressions for the velocity and acceleration of this particle in plane polar coordinates.

   (a) Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

   (b) At what angles do radial and tangential components of the acceleration have equal magnitude?

PH111: Tutorial Sheet 2
Solutions

This tutorial sheet contains problems related to plane-polar coordinate system.

1. Using known results from the Cartesian coordinate system, and the relation between plane-polar and Cartesian basis vectors, calculate: (a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}$, (b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}}$, and (c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$
Soln: Here we use $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$, so that

   (a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) \times \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) = \cos^2\theta\hat{\mathbf{k}} + \sin^2\theta\hat{\mathbf{k}} = \hat{\mathbf{k}}$

   (b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}} = \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) \times \hat{\mathbf{k}} = \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{i}} = \hat{\mathbf{r}}$

   (c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\mathbf{k}} \times \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) = \cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{i}} = \hat{\boldsymbol{\theta}}$

2. A particle is moving along a circular path of radius a, with angular velocity given by $\omega(t) = \omega_0 + \alpha t$, where ω_0 and α are constants. Obtain the radial and tangential components of its velocity and acceleration.
Soln: Here we have

$$\begin{aligned} r &= a \\ \dot{r} &= 0 \\ \ddot{r} &= 0, \end{aligned}$$

and

$$\begin{aligned} \dot{\theta} &= \omega(t) = \omega_0 + \alpha t \\ \ddot{\theta} &= \alpha, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = a\left(\omega_0 + \alpha t\right)\hat{\boldsymbol{\theta}} \\ \mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = -a\left(\omega_0 + \alpha t\right)^2\hat{\mathbf{r}} + a\alpha\hat{\boldsymbol{\theta}} \end{aligned}$$

3. A particle is moving along the line $y = a$, with the velocity $\mathbf{v} = u\hat{\mathbf{i}}$, where u is a constant. Express its velocity in plane polar coordinates.
Soln: Because $\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$, we get

$$\mathbf{v} = u\cos\theta\hat{\mathbf{r}} - u\sin\theta\hat{\boldsymbol{\theta}}$$

4. A particle moves in such a way that $\dot{\theta} = \omega$ (constant), and $r(t) = r_0 e^{\beta t}$, where r_0 and β are constants. Write down its velocity and acceleration in plane polar coordinates.

For what values of β will the radial acceleration of the particle be zero?
Soln: Here we have

$$\dot{\theta} = \omega$$
$$\ddot{\theta} = 0$$
$$\dot{r} = r_0 \beta e^{\beta t}$$
$$\ddot{r} = r_0 \beta^2 e^{\beta t},$$

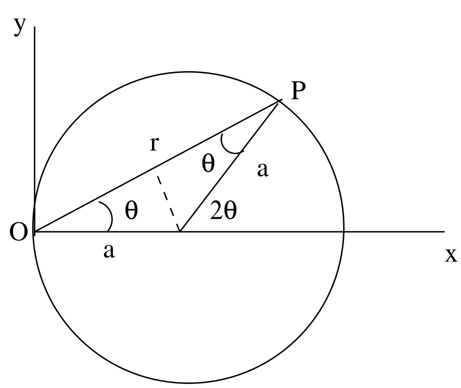
so that

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r_0 e^{\beta t}\left(\beta\hat{\mathbf{r}} + \omega\hat{\boldsymbol{\theta}}\right)$$
$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = r_0 e^{\beta t}\left(\beta^2 - \omega^2\right)\hat{\mathbf{r}} + 2r_0\omega\beta e^{\beta t}\hat{\boldsymbol{\theta}}.$$

It is obvious from the expression of acceleration that its radial component will vanish when $\beta = \pm\omega$.

5. Consider a circle of radius a, with the origin of the plane polar coordinate system placed at a point on the circumference. The particle is moving along the circle with a constant speed u. Assume that $\theta(t = 0) = 0$

   (a) What is the equation of the circle in this coordinate system?
   Soln:



   It is obvious from the figure above that equation of the circle is

$$r = 2a\cos\theta$$

   (b) What is the value of $\dot{\theta}$ in terms of u and a?
   Soln: From the figure it is obvious that

$$2\dot{\theta} = \frac{u}{a}$$
$$\implies \dot{\theta} = \frac{u}{2a}$$
$$\theta = \frac{ut}{2a}$$

(c) Write down the velocity of the particle in plane-polar coordinate system.
Soln: From equation of the circle we obtain

$$\dot{r} = \frac{d}{dt}(2a\cos\theta) = -2a\sin\theta\dot{\theta} = -u\sin\theta,$$

so that

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = -u\sin\theta\hat{\mathbf{r}} + 2a\cos\theta\left(\frac{u}{2a}\right)\hat{\boldsymbol{\theta}} = -u\sin\left(\frac{ut}{2a}\right)\hat{\mathbf{r}} + u\cos\left(\frac{ut}{2a}\right)\hat{\boldsymbol{\theta}}$$

(d) What is the acceleration of the particle in plane-polar coordinate system?
Soln: We have

$$\begin{aligned}
\ddot{\theta} &= \frac{d}{dt}\left(\frac{u}{2a}\right) = 0 \\
\ddot{r} &= \frac{d}{dt}(-u\sin\theta) = -u\cos\theta\dot{\theta} = -\frac{u^2}{2a}\cos\theta,
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} \\
&= \left(-\frac{u^2}{2a}\cos\theta - 2a\cos\theta\frac{u^2}{4a^2}\right)\hat{\mathbf{r}} - \frac{u^2}{a}\sin\theta\hat{\boldsymbol{\theta}} \\
&= -\frac{u^2}{a}\cos\left(\frac{ut}{2a}\right)\hat{\mathbf{r}} - \frac{u^2}{a}\sin\left(\frac{ut}{2a}\right)\hat{\boldsymbol{\theta}}
\end{aligned}$$

6. A particle moves along the curve $r = A\theta$, with $A = 1/\pi$ m/rad, and $\theta = \alpha t^2$, where α is a constant. Obtain the expressions for the velocity and the acceleration of this particle in plane polar coordinates.
Soln: We have

$$\begin{aligned}
r &= A\theta = A\alpha t^2 \\
\Longrightarrow \dot{r} &= A\dot{\theta} = 2A\alpha t \\
\Longrightarrow \ddot{r} &= 2A\alpha,
\end{aligned}$$

and

$$\begin{aligned}
\dot{\theta} &= 2\alpha t \\
\ddot{\theta} &= 2\alpha.
\end{aligned}$$

Therefore, the expression for velocity is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = 2A\alpha t\hat{\mathbf{r}} + 2A\alpha^2 t^3\hat{\boldsymbol{\theta}}.$$

Expression for acceleration is given below.

(a) Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.
Soln: Now

$$\begin{aligned}\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} \\ &= 2A\alpha\left(1 - 2\alpha^2 t^4\right)\hat{\mathbf{r}} + A\alpha^2\left(8t^2 + 2t^2\right)\hat{\boldsymbol{\theta}} \\ & 2A\alpha\left(1 - 2\alpha^2 t^4\right)\hat{\mathbf{r}} + 10A\alpha^2 t^2\hat{\boldsymbol{\theta}}\end{aligned}$$

It is obvious from above that the radial component of the acceleration vanishes for $t^2 = 1/\alpha\sqrt{2}$, for which $\theta = \alpha(1/\alpha\sqrt{2}) = 1/\sqrt{2}$.

(b) At what angles do radial and tangential components of the acceleration have equal magnitude?
Soln: The two components will be equal in magnitude when

$$\begin{aligned}2A\alpha\left(1 - 2\alpha^2 t^4\right) &= \pm 10A\alpha^2 t^2 \\ 2\alpha^2 t^4 \pm 5\alpha t^2 - 1 &= 0.\end{aligned}$$

This equation has two possible solutions for t^2

$$\begin{aligned}t^2 &= \frac{\pm 5 + \sqrt{33}}{4\alpha} \\ \Longrightarrow\ \theta = \alpha t^2 &= \frac{\pm 5 + \sqrt{33}}{4}\ \text{radians}\end{aligned}$$

PH111: Tutorial Sheet 3

This tutorial sheet contains problems related to work-energy theorem, conservative force, and potential energy.

1. Mass m rotates on a frictionless table, held to circular path by a string which passes through a hole in the table. The string is slowly pulled through the hole so that the radius of the circle changes from R_0 to R_1. Show that the work done in pulling the string equals the increase in kinetic energy of the mass.

2. A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed towards the origin with the magnitude B, and and inverse law repulsive force of magnitude A/x^2.

   (a) Find the potential energy function $V(x)$

   (b) Plot the potential energy as a function of x, and the total energy of the system, assuming that the maximum kinetic energy is $K_0 = \frac{1}{2}mv_0^2$.

   (c) What is the point of equilibrium, i.e., the point where net force acting on the particle is zero.

3. A particle of mass M is held fixed at the origin. The gravitational potential energy of another particle of m, in the field of the first mass, is given by

$$V(\mathbf{r}) = -\frac{GMm}{r},$$

   where G is the gravitational constant, and r is the distance of mass m from the origin.

   (a) What is the force acting on the particle of mass m?

   (b) Calculate the curl of this force.

4. Consider a 2D force field $\mathbf{F} = A(y^2\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}})$. Calculate the work done by this force in going around a closed path which is a square made up of sides of length a, lying in the xy-plane, with two of its vertices located at the origin, and point (a, a). Find the answer by doing the line integral, as well as by using the Stokes' theorem. The path is traversed in a counter clockwise manner.

5. Find the forces for the following potential energies

   (a) $V(x, y, z) = Ax^2 + By^2 + Cz^2$

   (b) $V(x, y, z) = A\ln(x^2 + y^2 + z^2)$

   (c) $V(r, \theta) = A\cos\theta/r^2$ (r and θ are plane polar coordinates)

   Above, A, B, and, C are constants.

6. Determine whether each of the following forces is conservative. Find the potential energy function if it exists. A, α, β are constants.

(a) $\mathbf{F} = A(3\hat{\mathbf{i}} + z\hat{\mathbf{j}} + y\hat{\mathbf{k}})$

(b) $\mathbf{F} = Axyz(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

(c) $F_x = A\sin(\alpha y)\cos(\beta z)$, $F_y = -Ax\alpha\cos(\alpha y)\cos(\beta z)$, $F_z = Ax\sin(\alpha y)\sin(\beta z)$

PH111: Tutorial Sheet 3
Solutions

This tutorial sheet contains problems related to work-energy theorem, conservative force, and potential energy.

1. Mass m rotates on a frictionless table, held to circular path by a string which passes through a hole in the table. The string is slowly pulled through the hole so that the radius of the circle changes from R_0 to R_1. Show that the work done in pulling the string equals the increase in kinetic energy of the mass.

   Soln: We know that the radial component of acceleration is given by

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

   Here the mass is being pulled very slowly so $\ddot{r} \approx 0$, therefore, the only acceleration experienced by the mass is the centripetal acceleration

$$a_r \approx -r\dot{\theta}^2 = -r\omega^2.$$

   If the particle is pulled slowly, the force applied is the tension in the string which should be the centripetal force

$$F = T = -m\omega^2 r,$$

   so that the work done will be

$$W = \int_{R_0}^{R_1} F dr = -m \int_{R_0}^{R_1} \omega^2 r dr.$$

   r dependence of ω can be calculated using the fact that the angular momentum of the particle about the center of the circle will be conserved because F being radial, does not impart any torque to the particle, with respect to the center. Assuming that the initial angular velocity of the mass was ω_0 corresponding to the radius R_0, conservation of angular momentum ($I\omega = mr^2\omega$), implies

$$mR_0^2\omega_0 = mr^2\omega$$
$$\implies \omega(r) = \frac{R_0^2\omega_0}{r^2},$$

   using this we have

$$W = -\int_{R_0}^{R_1} \omega^2 r dr = -m\omega_0^2 R_0^4 \int_{R_0}^{R_1} \frac{rdr}{r^4} = -m\omega_0^2 R_0^4 \int_{R_0}^{R_1} \frac{dr}{r^3} = \frac{1}{2} m\omega_0^2 R_0^4 \left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right)$$

   Increase in kinetic energy of the particle will be

$$\Delta K = \frac{1}{2} I_1 \omega_1^2 - \frac{1}{2} I_0 \omega_0^2 = \frac{1}{2}\left\{ mR_1^2 \left(\frac{R_0^4\omega_0^2}{R_1^4}\right) - mR_0^2\omega_0^2 \right\} = \frac{1}{2} m\omega_0^2 R_0^4 \left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right),$$

   which is same as the expression for W above.

2. A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed towards the origin with the magnitude B, and and inverse law repulsive force of magnitude A/x^2.

   (a) Find the potential energy function $V(x)$
   Soln: The force is given by

$$\mathbf{F} = \left(-B + \frac{A}{x^2}\right)\hat{\mathbf{i}}, \text{ for } x \geq 0.$$

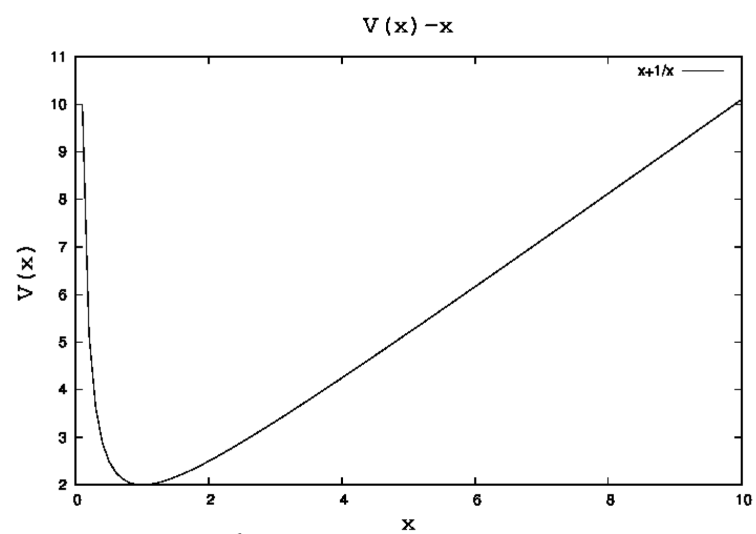
   If $U(x)$ is the potential energy, then for $x \geq 0$

$$-\frac{dV}{dx} = -B + \frac{A}{x^2}$$
$$\implies V(x) = Bx + \frac{A}{x} + C,$$

   where C is a constant, which we can set to zero so

$$V(x) = Bx + \frac{A}{x}$$

   (b) Plot the potential energy as a function of x, and the total energy of the system, assuming that the maximum kinetic energy is $K_0 = \frac{1}{2}mv_0^2$.
   Soln: The plot of the potential energy for $A = B = 1$ is



   The minimum of potential energy can be obtained

$$\frac{dV}{dx} = B - \frac{A}{x^2} = 0$$
$$\implies x = \sqrt{\frac{A}{B}},$$

   for which $V = B\sqrt{\frac{A}{B}} + A\sqrt{\frac{B}{A}} = 2\sqrt{AB}$. Because total energy is conserved, therefore, $E = K + V = \text{constant}$. Thus, when we have maximum kinetic energy, we will have minimum potential energy, implying that $E = K_{max} + V_{min} = \frac{1}{2}mv_0^2 + 2\sqrt{AB}$. Therefore, total energy of the particle can be represented by a horizontal line in the plot, corresponding to $V(x) = \frac{1}{2}mv_0^2 + 2\sqrt{AB}$.

(c) What is the point of equilibrium, i.e., the point where net force acting on the particle is zero.
Soln: Point of equilibrium is obtained by

$$F = -B + \frac{A}{x^2} = 0$$
$$\implies x = \sqrt{\frac{A}{B}},$$

which is the same point where the potential energy is minimum.

3. A particle of mass M is held fixed at the origin. The gravitational potential energy of another particle of m, in the field of the first mass, is given by

$$V(\mathbf{r}) = -\frac{GMm}{r},$$

where G is the gravitational constant, and r is the distance of mass m from the origin.

(a) What is the force acting on the particle of mass m?
Soln: Here

$$\begin{aligned} V(\mathbf{r}) &= -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}} \\ \implies \mathbf{F}(\mathbf{r}) &= -\nabla V(\mathbf{r}) \\ &= GMm\left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{k}}\right) \\ &= -GMm\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\right) = -\frac{GMm\mathbf{r}}{r^3} = -\frac{GMm\hat{\mathbf{r}}}{r^2} \end{aligned}$$

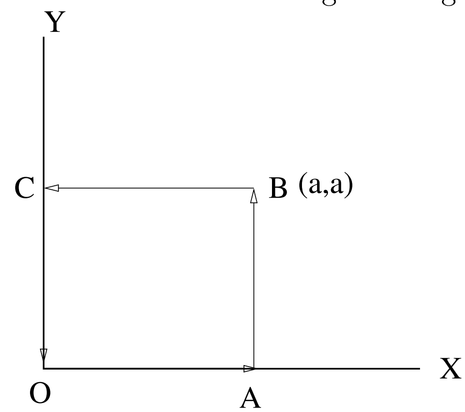
(b) Calculate the curl of this force.
Soln: Because for this force a potential energy function exists, its curl must vanish. We calculate it as

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix} \\ &= -3\left(\frac{yz - yz}{(x^2+y^2+z^2)^{5/2}}\right)\hat{\mathbf{i}} + \cdots = 0 \end{aligned}$$

4. Consider a 2D force field $\mathbf{F} = A(y^2\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}})$. Calculate the work done by this force in going around a closed path which is a square made up of sides of length a, lying in

the xy-plane, with two of its vertices located at the origin, and point (a, a). Find the answer by doing the line integral, as well as by using the Stokes' theorem. The path is traversed in a counter clockwise manner.

Soln: Let us first calculate the line integral along the given path



The work done will be

$$W = \oint \mathbf{F} \cdot d\mathbf{r} = \int_{OA} \mathbf{F} \cdot d\mathbf{r} + \int_{AB} \mathbf{F} \cdot d\mathbf{r} + \int_{BC} \mathbf{F} \cdot d\mathbf{r} + \int_{CO} \mathbf{F} \cdot d\mathbf{r}$$
$$= A(0) \int_0^a dx + 2A(a^2) \int_0^a dy + Aa^2 \int_a^0 dx + 2(0) \int_a^0 dy = Aa^3$$

To verify Stokes theorem we need to compute

$$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Here

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ay^2 & 2Ax^2 & 0 \end{vmatrix} = (4Ax - 2Ay)\hat{\mathbf{k}}$$

and

$$d\mathbf{S} = dxdy\hat{\mathbf{k}},$$

so that

$$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 4A \int_0^a xdx \int_0^a dy - 2A \int_0^a dx \int_0^a ydy$$
$$= 4A(\frac{a^2}{2})a - 2Aa(\frac{a^2}{2}) = Aa^3.$$

Thus we get the same value of work done by computing the line integral, and by using the Stokes theorem.

5. Find the forces for the following potential energies

   (a) $V(x, y, z) = Ax^2 + By^2 + Cz^2$
   Soln:

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$
$$= -2Ax\hat{\mathbf{i}} - 2By\hat{\mathbf{j}} - 2Cz\hat{\mathbf{k}}$$

(b) $V(x,y,z) = A\ln(x^2+y^2+z^2)$
Soln:

$$\mathbf{F} = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}} = \frac{2A}{(x^2+y^2+z^2)}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

(c) $V(r,\theta) = A\cos\theta/r^2$ (r and θ are plane polar coordinates)

Above, A, B, and, C are constants.
Soln:

$$\begin{aligned} V &= \frac{A\cos\theta}{r^2} = \frac{Ax}{r^3} = \frac{Ax}{(x^2+y^2)^{3/2}} \\ \implies \mathbf{F} &= -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} = -A\left(\frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}\right)\hat{\mathbf{i}} + \frac{A3xy}{(x^2+y^2)^{5/2}}\hat{\mathbf{j}} \\ &= \frac{A(2x^2-y^2)}{(x^2+y^2)^{5/2}}\hat{\mathbf{i}} + \frac{A3xy}{(x^2+y^2)^{5/2}}\hat{\mathbf{j}} \end{aligned}$$

6. Determine whether each of the following forces is conservative. Find the potential energy function, if it exists. A, α, β are constants.

(a) $\mathbf{F} = A(3\hat{\mathbf{i}} + z\hat{\mathbf{j}} + y\hat{\mathbf{k}})$
Soln: First we compute the curl of $\mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3A & Az & Ay \end{vmatrix} = 0.$$

Therefore, the force is conservative and it is possible to obtain a potential energy function for it using

$$\begin{aligned} -\frac{\partial V}{\partial x} &= 3A \\ -\frac{\partial V}{\partial y} &= Az \\ -\frac{\partial V}{\partial z} &= Ay. \end{aligned}$$

On integrating the first equation above we have

$$V(x,y,z) = -3Ax + f(y,z),$$

which on substitution in the second equation yields

$$\begin{aligned} -\frac{\partial f}{\partial y} &= Az \\ \implies f(y,z) &= -Ayz + C \\ \implies V(x,y,z) &= -3Ax - Ayz + C, \end{aligned}$$

where C is a constant. Note that this expression for V satisfies the third equation above, implying that the solution is complete.

(b) $\mathbf{F} = Axyz(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
Soln:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axyz & Axyz & Axyz \end{vmatrix} = A(xz - xy)\hat{\mathbf{i}} + A(xy - yz)\hat{\mathbf{j}} \hat{+} A(yz - xz)\hat{\mathbf{i}} \neq 0,$$

therefore, a potential energy function does not exist for this force.

(c) $F_x = A\sin(\alpha y)\cos(\beta z)$, $F_y = -Ax\alpha\cos(\alpha y)\cos(\beta z)$, $F_z = Ax\sin(\alpha y)\sin(\beta z)$
Soln:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A\sin(\alpha y)\cos(\beta z) & -Ax\alpha\cos(\alpha y)\cos(\beta z) & Ax\sin(\alpha y)\sin(\beta z) \end{vmatrix} \\ &= A(x\alpha\cos(\alpha y)\sin(\beta z) - x\alpha\beta\cos(\alpha y)\sin(\beta z))\hat{\mathbf{i}} \\ &+ A(-x\beta\sin(\alpha y)\sin(\beta z) - x\alpha\cos(\alpha y)\sin(\beta z))\hat{\mathbf{j}} \\ &+ A(0 - \alpha\cos(\alpha y)\cos(\beta z)\hat{\mathbf{k}} \\ &\neq 0, \end{aligned}$$

hence, for this force also, no potential energy function exists.

PH111: Tutorial Sheet 4

This tutorial sheet contains problems related to vector nature of angular velocity, non-inertial frames of reference, and pseudo forces.

1. A particle is rotating in the xy-plane, along a circular path in counter-clockwise direction, with angular speed ω, about the z-axis.

   (a) Write down the angular velocity of the particle in the vector form, i.e., in terms of components and unit vectors.

   (b) If the particle is moving along a circle of radius a, write down its position vector $\mathbf{r}(t)$, as a function of time, assuming that $\mathbf{r}(0) = a\hat{\mathbf{i}}$

   (c) Express its velocity both in Cartesian, and plane polar coordinates

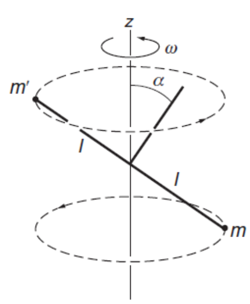
   (d) Compute the acceleration of the particle both in Cartesian, and plane polar coordinates

2. A vector $\mathbf{A}$ of magnitude a is rotating in the yz plane in a counter-clockwise manner, with a uniform angular velocity ω. It is given that $\mathbf{A}(t = 0) = a\hat{\mathbf{j}}$.

   (a) Obtain $\mathbf{A}(t)$, as a function of time.

   (b) Show that $\frac{d\mathbf{A}}{dt}$ calculated directly, and computed using $\boldsymbol{\omega} \times \mathbf{A}$, are the same.

3. Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis that rotates at angular speed ω around the z axis. The rod is skewed at angle α, as shown in the figure.



   (a) Calculate the angular momentum $\mathbf{L}(t)$ of the system, in Cartesian coordinates.

   (b) Verify that $\frac{d\mathbf{L}}{dt}$ is same as $\boldsymbol{\omega} \times \mathbf{L}$.

4. A cylinder of mass M and radius R rolls without slipping on a plank which is moving with an acceleration $\mathbf{A}$. Calculate the acceleration of the cylinder by analyzing the problem both in the inertial frame and the non-inertial frames. You can use the fact that moment of inertial of a cylinder about its axis is $\frac{1}{2}MR^2$.

5. A bead of mass m slides without friction on a horizontal rigid wire rotating at constant angular speed ω about the z axis.

(a) Find the distance of the bead from the axis of rotation $r(t)$, as a function of time given that $r(0) = 0$, and $\dot{r}(0) = v_0$.

(b) What is the force exerted on the bead by the wire.

PH 111: Tutorial Sheet 4
Solutions

This tutorial sheet contains problems related to vector nature of angular velocity, non-inertial frames of reference, and pseudo forces.

1. A particle is rotating in the xy-plane, along a circular path in counter-clockwise direction, with angular speed ω, about the z-axis.

   (a) Write down the angular velocity of the particle in the vector form, i.e., in terms of components and unit vectors.
   Soln: Obviously
   $$\boldsymbol{\omega} = \omega \hat{\mathbf{k}}$$

   (b) If the particle is moving along a circle of radius a, write down its position vector $\mathbf{r}(t)$, as a function of time, assuming that $\mathbf{r}(0) = a\hat{\mathbf{i}}$
   Soln: Obviously
   $$\mathbf{r}(t) = a\cos\omega t\hat{\mathbf{i}} + a\sin\omega t\hat{\mathbf{j}}$$

   (c) Express its velocity both in Cartesian, and plane polar coordinates
   Soln: Velocity can be computed easily
   $$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -a\omega\sin\omega t\hat{\mathbf{i}} + a\omega\cos\omega t\hat{\mathbf{j}} = \mathrm{a}\omega\hat{\boldsymbol{\theta}}(t)$$
   Also easy to verify that
   $$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}(t)$$

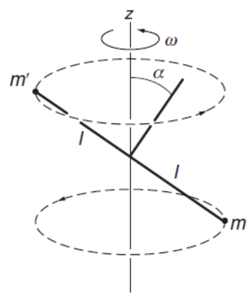
   (d) Compute the acceleration of the particle both in Cartesian, and plane polar coordinates
   Soln: We have
   $$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -a\omega^2(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}}) = -\omega^2\mathbf{r} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

2. A vector $\mathbf{A}$ of magnitude a is rotating in the yz plane in a counter-clock-wise manner, with a uniform angular velocity ω. It is given that $\mathbf{A}(t = 0) = a\hat{\mathbf{j}}$.

   (a) Obtain $\mathbf{A}(t)$, as a function of time.
   Soln: $\mathbf{A}(t)$ can be written as
   $$\mathbf{A}(t) = a\cos\omega t\hat{\mathbf{j}} + a\sin\omega t\hat{\mathbf{k}}.$$

   (b) Show that $\frac{d\mathbf{A}}{dt}$ calculated directly, and computed using $\boldsymbol{\omega} \times \mathbf{A}$, are the same.
   Soln: Clearly
   $$\frac{d\mathbf{A}}{dt} = -a\omega\sin\omega t\hat{\mathbf{j}} + a\omega\cos\omega t\hat{\mathbf{k}},$$
   because $\boldsymbol{\omega} = \omega\hat{\mathbf{i}}$, easy to see that
   $$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}$$

3. Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis that rotates at angular speed ω around the z axis. The rod is skewed at angle α, as shown in the figure.



(a) Calculate the angular momentum $\mathbf{L}(t)$ of the system, in Cartesian coordinates.
Soln: We will calculate the angular momentum of the system using $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We assume that the origin lies on the middle of the rod, and note that the rod makes an angle of $90^\circ - \alpha$ from the vertical, and the two masses move around circles of radii $l \cos\alpha$. With this, the position vectors of the upper mass $\mathbf{r}_1$, and of the lower mass $\mathbf{r}_2$ can be written as

$$\mathbf{r}_1 = l\cos\alpha\left(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}}\right) + l\sin\alpha\hat{\mathbf{k}}$$
$$\mathbf{r}_2 = -l\cos\alpha\left(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}}\right) - l\sin\alpha\hat{\mathbf{k}}$$

Above we used that $\mathbf{r}_2 = -\mathbf{r}_1$. Their momenta will be

$$\mathbf{p}_1 = m\frac{d\mathbf{r}_1}{dt} = ml\omega\cos\alpha(-\sin\omega t\hat{\mathbf{i}} + \cos\omega t\hat{\mathbf{j}})$$
$$\mathbf{p}_2 = m\frac{d\mathbf{r}_2}{dt} = -ml\omega\cos\alpha(-\sin\omega t\hat{\mathbf{i}} + \cos\omega t\hat{\mathbf{j}}) = -\mathbf{p}_1$$

So the total angular momentum will

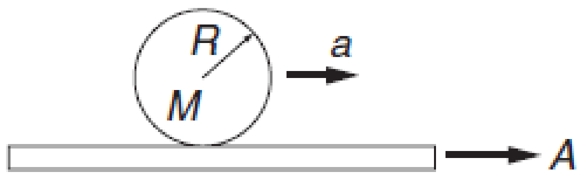
$$\begin{aligned}\mathbf{L} &= \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = 2\mathbf{r}_1 \times \mathbf{p}_1\\ &= 2ml^2\omega\cos^2\alpha\cos^2\omega t\hat{\mathbf{k}} + 2ml^2\omega\cos^2\alpha\sin^2\omega t\hat{\mathbf{k}}\\ &- 2ml^2\omega\cos\alpha\sin\alpha\sin\omega t\hat{\mathbf{j}} - 2ml^2\omega\cos\alpha\sin\alpha\cos\omega t\hat{\mathbf{i}}\\ &= 2ml^2\omega\cos\alpha\left(-\sin\alpha\cos\omega t\hat{\mathbf{i}} - \sin\alpha\sin\omega t\hat{\mathbf{j}} + \cos\alpha\hat{\mathbf{k}}\right)\end{aligned}$$

(b) Verify that $\frac{d\mathbf{L}}{dt}$ is same as $\boldsymbol{\omega} \times \mathbf{L}$.
Soln: Using the expression above

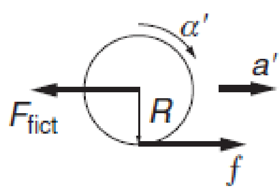
$$\frac{d\mathbf{L}}{dt} = 2ml^2\omega^2\cos\alpha\left(\sin\alpha\sin\omega t\hat{\mathbf{i}} - \sin\alpha\cos\omega t\hat{\mathbf{j}}\right).$$

Given the fact that $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, it is easy verify that $\boldsymbol{\omega} \times \mathbf{L}$ leads to the same expression as $\frac{d\mathbf{L}}{dt}$, above.

4. A cylinder of mass M and radius R rolls without slipping on a plank which is moving with an acceleration **A**. Calculate the acceleration of the cylinder by analyzing the problem both in the inertial frame and the non-inertial frames. You can use the fact that moment of inertial of a cylinder about its axis is $\frac{1}{2}MR^2$.
Soln: The situation can be shown as follows



Non-Intertial Frame Analysis: In the nonintertial frame, there are two forces acting on the cylinder: frictional force f and pseudo force $F_{fict} = mA$, as shown below. a' is the acceleration of the cylinder with respect to plank, and α' is the corresponding angular acceleration.



Then clearly, the equations of motion are

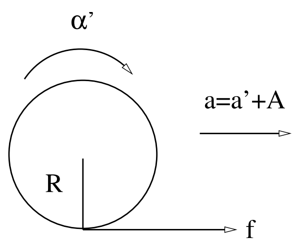
$$fR = -I\alpha' = \frac{1}{2}MR^2\alpha'$$
$$f - F_{fict} = Ma' \implies f - MA = Ma',$$

along with the constraint equation $a' = \alpha' R$, for the case of rolling without slipping. These equations can be readily solved to give

$$a' = -\frac{MA}{M + M/2} = -\frac{2}{3}A.$$

So that the acceleration with respect to the inertial frame is $a = a' + A = \frac{1}{3}A$.
Inertial Frame Analysis: In the inertial frame, the cylinder is moving with acceleration $a = a' + A$, with respect to the ground, and only one force, i.e., force of friction f is acting on it, and providing it with torque for rotation about its axis, as shown below. a' is the acceleration of the cylinder with respect to plank, and α' is the corresponding

angular acceleration.



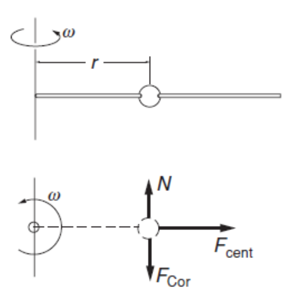
Therefore, the equations of motion are

$$fR = -I\alpha' = \frac{1}{2}MR^2\alpha'$$
$$f = M(a' + A),$$

along with the constraint equation $a' = \alpha' R$, for the case of rolling without slipping. Because, we obtain the same set of equations as before, therefore, it will lead to the same solution.

5. A bead of mass m slides without friction on a horizontal rigid wire rotating at constant angular speed ω about the z axis.

   (a) Find the distance of the bead from the axis of rotation $r(t)$, as a function of time given that $r(0) = 0$, and $\dot{r}(0) = v_0$.
   Soln: In the non-inertial frame rotating with the rod, situation is as below



In the rotating frame, the pseudo forces are: centrifugal force F_{cent} and Coriolis force F_{Cor}, as shown. N is the reaction force applied by the rod on the bead. It is obvious, that in the non-inertial frame, on which we use plane polar coordinates, we obtain

$$\begin{aligned}\mathbf{F}_{cent} &= -m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) = -m\omega^2 r(\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{r}})) \\ &= m\omega^2 r\hat{\mathbf{r}},\end{aligned}$$

above we used $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$. Similarly, Coriolis force is

$$\mathbf{F}_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}_{rot}.$$

But velocity of the bead with respect to the rotating frame clearly is strictly in the $\hat{\mathbf{r}}$ direction and given by $\dot{r}\hat{\mathbf{r}}$, so that

$$\mathbf{F}_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}_{rot} = -2m\omega\dot{r}\hat{\boldsymbol{\theta}}.$$

Now, equation of motion for the radial motion in non-inertial frame is

$$\begin{aligned} m\ddot{r}\hat{\mathbf{r}} &= F_{cent} = m\omega^2 r\hat{\mathbf{r}} \\ \implies \ddot{r} - \omega^2 r &= 0 \\ \implies r(t) &= Ae^{\omega t} + Be^{-\omega t} \end{aligned}$$

On applying the initial conditions $r(0) = 0$, and $\dot{r}(0) = v_0$, we obtain

$$\begin{aligned} A + B &= 0 \\ A\omega - B\omega &= v_0 \\ \implies A &= \frac{v_0}{2\omega} = -B \end{aligned}$$

so that

$$r(t) = \frac{v_0}{2\omega}\left(e^{\omega t} - e^{-\omega t}\right)$$

(b) What is the force exerted on the bead by the wire.
Soln: In the non-inertial frame, the equation of motion in $\hat{\boldsymbol{\theta}}$ direction is

$$\begin{aligned} &N - F_{Cor} = 0 \\ &\implies N = F_{Cor} = 2m\omega\dot{r}(t) = 2m\omega\left\{\frac{v_0}{2\omega}\omega\left(e^{\omega t} + e^{-\omega t}\right)\right\} = m\omega v_0\left(e^{\omega t} + e^{-\omega t}\right), \end{aligned}$$

thus N is the required force applied by the rod on the bead.

PH111: Tutorial Sheet 5

This tutorial sheet contains problems related to the central force motion

1. In the lectures, we argued that the effective potential for the central force problem is

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r),$$

where $V(r)$ is the potential energy corresponding to the central force, and L is the angular momentum. Consider the case of gravitational motion so that $V(r) = -\frac{C}{r}$, with $C > 0$. Plot the effective potential as a function of r, and argue based upon the plot that for $E \geq 0$, orbits will be unbound, while for $E < 0$, we will obtain bound orbits, where E is the total energy of the system.

2. Suppose a satellite is moving around a planet in a circular orbit of radius r_0. Due to a collision with another object, satellite's orbit gets perturbed. Show that the radial position of the satellite will execute simple harmonic motion with $\omega = \frac{L}{mr_0^2}$, where L is the initial angular momentum of the satellite.

3. In this problem we will explore an alternative way of obtaining the equation of the curve corresponding to the central force orbits.

   (a) Make a change of variable $u = \frac{1}{r}$ and show that the $u - \theta$ differential equation for a central force $\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$ is

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{u^2L^2} f(\frac{1}{u})$$

   (b) Integrate this differential equation for the case of gravitational force ($f(r) = -\frac{C}{r^2}$), and show that it leads to the same orbit as obtained in the lectures

$$r = \frac{r_0}{1 - \epsilon \cos\theta}$$

4. A particle of mass m is moving under the influence of a central force $\mathbf{F}(\mathbf{r}) = -\frac{C}{r^3}\hat{\mathbf{r}}$, with $C > 0$. Find the nonzero values of angular momentum L for which the particle will move in a circular orbit.

5. A geostationary orbit is one in which a satellite moves in a circular orbit at the given height in the equatorial plane, so that its angular velocity of rotation around earth is same as earth's angular velocity, thereby, making it look stationary when seen from a point on equator. Assuming that the earth's rotational velocity, and radius, respectively, are $\Omega_e = \frac{2\pi}{86400}$ rad/s, and$R_e = 6400$ km, calculate the altitude of the satellite, and its orbital velocity.

6. A space company wants to launch a satellite of mass $m = 2000$ kg, in an elliptical orbit around earth, so that the altitude of the satellite above earth at perigee is 1100 kms, and at apogee it is 35,850 kms. Assuming that the launch takes place at the equator,

calculate: (a) energy of the satellite in the elliptical orbit, (b) energy required to launch the satellite, (c) eccentricity of the orbit, (d) angular momentum of the satellite, and (e) speeds of the satellite at apogee and perigee. Use the values of R_e and Ω_e specified in the previous problem.

7. The ultimate aim of the space company of the previous problem is to put the satellite in a geostationary orbit. Therefore, after launching it in the elliptical orbit, the company wants to transfer it in a geostationary orbit by firing rockets at the apogee to increase its speed to the required one. How much change in speed is needed to put the satellite in the geostationary orbit, and how much energy will be required to achieve that change?

PH 111: Tutorial Sheet 5
Soln slides

Problem 1

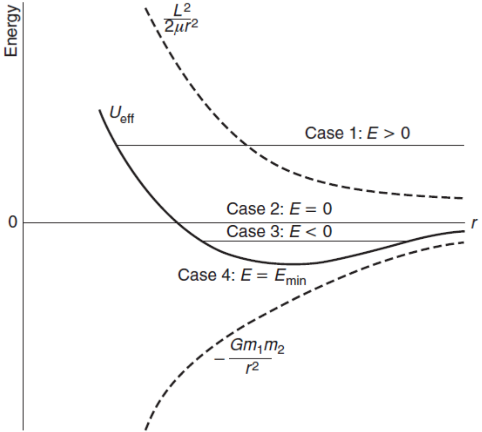
1. In the lectures, we argued that the effective potential for the central force problem is

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r),$$

where $V(r)$ is the potential energy corresponding to the central force, and L is the angular momentum. Consider the case of gravitational motion so that $V(r) = -\frac{C}{r}$, with $C > 0$. Plot the effective potential as a function of r, and argue based upon the plot that for $E \geq 0$, orbits will be unbound, while for $E < 0$, we will obtain bound orbits, where E is the total energy of the system.

Problem 1: Soln

1. A representative plot of $V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$, where $V(r) = -\frac{Gm_1 m_2}{r} = -\frac{C}{r}$, with $C > 0$ is presented below. Plots of $-\frac{C}{r}$ and $\frac{L^2}{2\mu r^2}$, are also presented in the same figure. Note that in the figure, what we call V_{eff}, has been denoted as U_{eff}.



Problem 1: Soln...

1. Two important general points can be made:
   1. centrifugal potential energy $\frac{L^2}{2\mu r^2}$ is always a positive quantity, while the gravitational potential energy $-\frac{C}{r}$ is always a negative quantity. Therefore, effective potential energy $V_{eff}(r)$, which is a sum of both, has both positive and negative values, and has a minimum with respect to r.
   2. For any value of total energy, the particle cannot be in the region where $V_{eff}(r) > E$, because then to keep total energy $E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$ constant, kinetic energy $\frac{1}{2}\mu\dot{r}^2$ will have to be negative, which means imaginary value of velocity $\dot{r}$. Because of this, particle will turn back from the points r for which $E = V_{eff}(r)$. These points are called "turning points".

Problem 1: Soln

Let us calculate the minimum of $V_{eff}(r)$

$$\frac{\partial V_{eff}(r)}{\partial r} = -\frac{L^2}{\mu r^3} + \frac{C}{r^2} = 0$$

$$\implies r_{min} = \frac{L^2}{\mu C}$$

$$\implies V_{eff}^{min} = V_{eff}(r_{min}) = \frac{L^2}{2\mu r_{min}^2} - \frac{C}{r_{min}} = \frac{L^2}{2\mu}\left(\frac{\mu^2 C^2}{L^4}\right) - C\frac{\mu C}{L^2}$$

$$= -\frac{\mu C^2}{2L^2}$$

Problem 1: Soln...

- Let us consider four possible cases:
- **Case I, E>0:** From the graph it is obvious that for this case we have only one turning point, therefore, the motion will be unbound. We know from lectures that the orbit here is hyperbola.
- **Case II, E=0:** Again from the graph above it is clear that we have only one turning point for this case, implying that the motion is unbound. From the lectures we know that the orbit for this case is a parabola.
- **Case III, 0>E>V_{eff}^{min}:** For this case, clearly there are two turning points, and because, due to conservation of angular momentum, the motion is confined in a plane for central force motion, this clearly implies a bound orbit. For planar motion, only bound orbit with two turning points is an ellipse. Thus motion is along an elliptic orbit.

Problem 1: Soln...

- **Case IV, E=**V_{eff}^{min}: Clearly, here there is only one possible value of radial distance $r = r_{min}$. Because $E = \frac{1}{2}\mu\dot{r}^2 + V_{eff} = V_{eff}^{min} \implies \dot{r} = 0$, which means that there is no radial motion for this case. Only orbit which satisfies this condition is a circle. Another way to approach this problem is by force considerations. If the particle is executing circular motion, then the centripetal force is provided by the gravitational force

$$\frac{\mu v^2}{r} = \frac{C}{r^2}$$
$$\implies r = \frac{C}{\mu v^2}$$

dividing previous equation by r^2on both sides

$$\frac{1}{r} = \frac{C}{\mu v^2 r^2} = \frac{C\mu}{\mu^2 v^2 r^2} = \frac{C\mu}{L^2}$$
$$\implies r = \frac{L^2}{C\mu} = r_{min}$$

Problem 1: Soln...

- Case IV: Contd.: $r_{min} = \frac{L^2}{C\mu}$ is the same result, as derived above.
- Note that we have used the value of the orbital angular momentum to be $L = \mu v r$.

- **Problem 2:** Suppose a satellite is moving around a planet in a circular orbit of radius r_0. Due to a collision with another object, satellite's orbit gets perturbed. Show that the radial position of the satellite will execute simple harmonic motion with $\omega = \frac{L}{mr_0^2}$, where L is the initial angular momentum of the satellite.
- **Soln:** Because it is a small perturbation, we can Taylor expand the potential energy of the satellite around $r_0 = r_{min}$

$$V_{eff}(r) = V_{min} + (r - r_{min}) \left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_{min}} + \frac{1}{2} \left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_{min}} + \cdots$$

Problem 2: Soln...

- Note that

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_{min}} = 0$$

$$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_{min}} = \frac{3L^2}{\mu r_{min}^4} - \frac{2C}{r_{min}^3} = \frac{3L^2}{\mu}\left(\frac{\mu^4 C^4}{L^8}\right) - 2C\left(\frac{\mu^3 C^3}{L^6}\right)$$

$$= \frac{\mu^3 C^4}{L^6}$$

- But above we showed $C = \frac{L^2}{\mu r}_{min}$, therefore,

$$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_{min}} = \frac{\mu^3}{L^6}\left(\frac{L^8}{\mu^4 r_{min}^4}\right) = \frac{L^2}{\mu r_{min}^4}$$

- but $r_{min} = r_0$, therefore, after perturbation

$$V_{eff}(r) \approx V_{min} + \frac{L^2}{2\mu r_0^4}(r - r_0)^2$$

Problem 2: Soln....

- Thus, the radial equation of motion of the perturbed orbit

$$\mu \ddot{r} = -\frac{\partial V_{eff}(r)}{\partial r} = -\frac{L^2}{\mu r_0^4}(r - r_0)$$

Define $x = r - r_0$, we obtain from above

$$\ddot{x} + \omega^2 x,$$

where $\omega = \frac{L}{\mu r_0^2}$. Given the fact that $\mu = \frac{mM}{m+M} \approx m$, because $m \ll M$, where M is the mass of the planet. Thus $\omega = \frac{L}{m r_0^2}$, and equation above denotes simple harmonic motion about $r = r_0$, with frequency ω.

- **Problem 3:** In this problem we will explore an alternative way of obtaining the equation of the curve corresponding to the central force orbits.
- (a) Make a change of variable $u = \frac{1}{r}$ and show that the $u - \theta$ differential equation for a central force F(r) $= f(r)\hat{\text{r}}$ is

$$\frac{d^2 u}{d\theta^2} + u = -\frac{\mu}{u^2 L^2} f(\frac{1}{u})$$

- **Soln:** The radial equation is

$$\mu \frac{d^2 r}{dt^2} - \mu r \dot{\theta}^2 = f(r), \tag{1}$$

- while the angular equation leads to

$$\mu r^2 \dot{\theta} = L \tag{2}$$

$$\dot{\theta} = \frac{L}{\mu r^2} \tag{3}$$

Substituting Eq. 3 in Eq. 1, we have

$$\mu \frac{d^2 r}{dt^2} - \frac{L^2}{\mu r^3} = f(r) \tag{4}$$

- Substitute $r = \frac{1}{u}$ in Eq. 4

$$\mu \frac{d^2}{dt^2}\left(\frac{1}{u}\right) - \frac{L^2 u^3}{\mu} = f(\frac{1}{u}) \tag{5}$$

Now,

$$\frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} \tag{6}$$

Problem 3: Soln

- Using Eq. 3 in Eq. 6, we obtain

$$\frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{d\theta}\frac{L}{\mu r^2} = -\frac{Lu^2}{\mu u^2}\frac{du}{d\theta} = -\frac{L}{\mu}\frac{du}{d\theta} \tag{7}$$

- Similarly

$$\frac{d^2}{dt^2}\left(\frac{1}{u}\right) = \frac{d}{dt}\left\{\frac{d}{dt}\left(\frac{1}{u}\right)\right\} = \frac{d}{d\theta}\left\{\frac{d}{dt}\left(\frac{1}{u}\right)\right\}\frac{d\theta}{dt} \tag{8}$$

- Using Eqs. 3 and 7 in 8, we have

$$\frac{d^2}{dt^2}\left(\frac{1}{u}\right) = \frac{d}{d\theta}\left\{-\frac{L}{\mu}\frac{du}{d\theta}\right\}\frac{Lu^2}{\mu} = -\frac{L^2u^2}{\mu^2}\frac{d^2u}{d\theta^2} \tag{9}$$

1. Substituting Eq 9 in Eq. 5, we obtain the desired result

$$-\frac{L^2u^2}{\mu}\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{\mu} = f(\frac{1}{u})$$

$$\Longrightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2u^2}f(\frac{1}{u})$$

- **(b)** Integrate this differential equation for the case of gravitational force ($f(r) = -\frac{C}{r^2}$), and show that it leads to the same orbit as obtained in the lectures

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

- **Soln:** For gravitational force $f(u) = -Cu^2$, so that

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu C}{L^2} = \frac{1}{r_0}.$$

Define $u' = u - \frac{1}{r_0}$, so that

$$\frac{d^2u'}{d\theta^2} + u' = 0$$

Problem 3: Soln

- Which implies

$$
\begin{aligned}
u'(\theta) = A\sin\theta + B\cos\theta &= \frac{1}{F}\cos(\theta-\theta_0) \\
\implies \frac{1}{r} - \frac{1}{r_0} &= \frac{1}{F}\cos(\theta-\theta_0) \\
\frac{1}{r} &= \frac{1}{r_0} + \frac{1}{F}\cos(\theta-\theta_0),
\end{aligned}
$$

- where F is a constant with dimensions of length. With suitable choice of θ_0, this equation can be put in the form

$$r = \frac{r_0}{1-\varepsilon\cos\theta}$$

Problem 4: Soln

- **Problem 4:** A particle of mass m is moving under the influence of a central force $\mathrm{F(r)} = -\frac{C}{r^3}\hat{\mathrm{r}}$, with $C > 0$. Find the nonzero values of angular momentum L for which the particle will move in a circular orbit.
- **Soln:** For this, the potential energy can be obtained as

$$V(r) = -\int_{\infty}^{r} F(r')dr' = C\int_{\infty}^{r} \frac{dr'}{r'^3} = -\frac{C}{2r^2}\bigg|_{\infty}^{r} = -\frac{C}{2r^2}.$$

- The effective potential energy for this case

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} - \frac{C}{2r^2}.$$

Problem 4: Soln...

- We know that for the circular orbit, the total energy must be equal to the minimum of the effective potential energy, which can be found by

$$\frac{\partial V_{eff}(r)}{\partial r} = -\frac{L^2}{\mu r^3} + \frac{C}{r^3} = 0$$
$$\implies L = \sqrt{\mu C}.$$

- Thus, if the system has this angular momentum, circular orbit of any radius is possible.

Problem 5: Soln

- **Problem 5:** A geostationary orbit is one in which a satellite moves in a circular orbit at the given height in the equatorial plane, so that its angular velocity of rotation around earth is same as earth's angular velocity, thereby, making it look stationary when seen from a point on equator. Assuming that the earth's rotational velocity, and radius, respectively, are $\Omega_e = \frac{2\pi}{86400}$ rad/s, and$R_e = 6400$ km, calculate the altitude of the satellite, and its orbital velocity.
- **Soln:** The radius of the circular orbit is obtained by the force condition

$$\frac{GM_e m}{r^2} = \frac{mv^2}{r}$$
$$\implies r = \frac{GM_e}{v^2}$$

Problem 5: Soln...

- For geostationary satellite $v = \Omega_e r$, therefore,

$$r = \frac{GM_e}{\Omega_e^2 r^2}$$
$$\implies r = \left(\frac{GM_e}{\Omega_e^2}\right)^{1/3}$$

- But $r = h + R_e$, where h is the needed altitude, and R_e is the radius of the earth, and $GM_e = gR_e^2$, therefore

$$h = \left(\frac{gR_e^2}{\Omega_e^2}\right)^{1/3} - R_e.$$

  Using the values $g = 9.8$ m/s^2, $R_e = 6.4 \times 10^6$ m, and $\Omega_e = \frac{2\pi}{86400}$ s^{-1}, we obtain $h \approx 35850$ km.
- And orbital speed of the satellite $v = r\Omega_e = (35850 + 6400) \times 10^6 \times \frac{2\pi}{86400} = 3070\, m/s$

- **Problem 6:** A space company wants to launch a satellite of mass $m = 2000$ kg, in an elliptical orbit around earth, so that the altitude of the satellite above earth at perigee is 1100 kms, and at apogee it is 35,850 kms. Assuming that the launch takes place at the equator, calculate: (a) energy of the satellite in the elliptical orbit, (b) energy required to launch the satellite, (c) eccentricity of the orbit, (d) angular momentum of the satellite, and (e) speeds of the satellite at apogee and perigee. Use the values of R_e and Ω_e specified in the previous problem.
- **Soln:** (a) We showed in the lectures that for the gravitational potential energy of the form

$$V(r) = -\frac{C}{r},$$

the energy of a mass moving in an elliptical orbit is

$$E = -\frac{C}{A},$$

where A is the major axis of the ellipse.

Problem 6: Soln...

- In this case $C = GM_e m = R_e^2 gm$, where m is the mass of the satellite. This elliptical orbit is about earth, with earth's center as one of its foci. Thus, A will be sum of earth's diameter, altitude at perigee, and altitude at apogee

$$A = (1100 + 2 \times 6400 + 35,850) \times 10^3 = 5 \times 10^7 m.$$

Therefore,

$$E_{orb} = -\frac{9.8 \times (6.4 \times 10^6)^2 \times 2000}{5 \times 10^7} = -1.61 \times 10^{10} J$$

- **Soln (b):** The energy of the satellite just before the launch is nothing but its gravitational potential energy at the surface of the earth, and kinetic energy due to rotation of the earth at the equator

$$
\begin{aligned}
E_{ground} = V(r) + K &= -\frac{GM_e m}{R_e} + \frac{1}{2}m(\Omega_e R_e)^2 \\
&= -mgR_e + \frac{1}{2}m(\Omega_e R_e)^2 \\
&= -2000 \times 9.9 \times 6.4 \times 10^6 \\
&+ 0.5 \times 2000 \times (6.4 \times 10^6)^2 \times (\frac{2\pi}{86400})^2 \\
&= -1.25 \times 10^{11} J.
\end{aligned}
$$

Therefore, energy required to launch the satellite will be

$$\Delta E = E_{orb} - E_{ground} = 1.09 \times 10^{11} J$$

- **Soln (c):** We showed in the class that the radial distances from the focus corresponding to perigee (r_{min}) and apogee (r_{max}) are given by

$$r_{min} = \frac{r_0}{1+\varepsilon}$$
$$r_{max} = \frac{r_0}{1-\varepsilon}$$

These equations lead to

$$r_0 = r_{min}(1+\varepsilon) = r_{max}(1-\varepsilon)$$
$$\implies \varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{(35850+6400)-(1100+6400)}{(35850+6400)+(1100+6400)} = 0.7$$

- **Soln (d):** To obtain the angular momentum we use the formula for eccentricity derived in the lectures

$$\varepsilon^2 = 1 + \frac{2E_{orb}L^2}{mC^2},$$

which on using various values yields

$$L = 1.43 \times 10^{14} kg\text{-}m^2/s$$

- **Soln (e):** We know that at perigee and apogee the velocity of the satellite will be perpendicular to the radial distance from the earth's center, thus

$$L = mr_p v_p = mr_a v_a,$$

where subscripts p and a denote, perigee and apogee respectively.

- **Soln (e) contd:** Here $m = 2000$ kg, $r_p = r_{min} = 1100 + 6400 = 7.5 \times 10^6\ m$, $r_o = r_{max} = 35850 + 6400 = 4.225 \times 10^7\ m$. With this we obtain

$$v_a = \frac{L}{mr_a} = 1690\, m/s$$
$$v_p = \frac{L}{mr_p} = 9530\, m/s$$

Problem 7: Soln

- **Problem 7:** The ultimate aim of the space company of the previous problem is to put the satellite in a geostationary orbit. Therefore, after launching it in the elliptical orbit, the company wants to transfer it in a geostationary orbit by firing rockets at the apogee to increase its speed to the required one. How much change in speed is needed to put the satellite in the geostationary orbit, and how much energy will be required to achieve that change?
- **Soln:** Recalling that in problem 5 we obtained that the radius of the geostationary orbit is $R_{geo} = 35850\,km + 6400\,km = 4.225 \times 10^7\,m$, which is identical to the radial distance at the apogee r_o for the elliptical orbit. Thus, it is best to fire the rockets at the apogee of the elliptical orbit, to provide it the energy needed for a geostationary orbit.

Problem 7: Soln...

- Now, energy required will be

$$\Delta E = -\frac{C}{A_{geo}} - E_{orb},$$

where E_{orb} is the energy of the elliptical orbit computed in the last problem, while A_{geo} is the major axis corresponding to the geostationary orbit. But, because geostationary orbit is a circular one, therefore, its major axis is nothing but its diameter, so that $A_{geo} = 2R_{geo} = 8.45 \times 10^7 m$. Using this we obtain

$$\Delta E = 6.6 \times 10^9 \, J$$

Problem 7: Soln...

- To compute the change in speed, we note that change in energy ΔE, changes only the kinetic energy of the satellite because during the rocket firing, the location of the satellite does not change, and hence, it potential energy remains constant. Thus, if v_f is the final speed of the satellite after the rocket is fired, we have

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_a^2 = \Delta E = 6.6 \times 10^9$$

$$\implies v_f = \sqrt{\frac{2\Delta E + mv_a^2}{m}}$$

Above v_a is the speed of the satellite at the apogee, calculated in the previous problem. Using values of various quantities, we obtain the required change in speed

$$\Delta v = v_f - v_a = \sqrt{\frac{2 \times 6.6 \times 10^9 + 2000 \times (1690)^2}{2000}} - 1690$$

$$= 3070 - 1690 = 1110\, m/s$$

PH111: Tutorial Sheet 6

This tutorial sheet deals with problems related to the special theory of relativity.

1. The time interval between two ticks of two identical clocks is 2.0 sec. One of the two clocks is set in motion, so that its speed relative to the observer, who holds the other clock is $0.6c$. What is the time interval between the ticks of the moving clock as measured by the observer with the stationary clock?

2. The incoming primary cosmic rays create μ-mesons in the upper atmosphere. The lifetime of μ-mesons at rest is 2 μs. If the mean speed of μ-mesons is $0.998c$, what fraction of the mu-mesons created at a height of 20 km reach the sea level?

3. Two observers A and B are close to a point where lightning strikes the earth. According to A, a second lightning strikes t_0 seconds later at a distance d from him. B, on the other hand finds the two events to be simultaneous. Find his velocity with respect to A. Also find the distance between the two lightnings as seen by B. Assume earth to be inertial frame of reference.

4. Observer A is at rest in frame S' moving horizontally past an inertial frame S at a speed of $0.6c$. A boy in the frame S, drops a ball, which according to the clock of observer A, falls for 1.5sec. How long will the ball fall for an observer at rest in S frame ?

5. A meter stick is positioned so that it makes an angle 30^o with the x-axis in its rest frame. Determine its length and its orientation as seen by an observer who is moving along x-axis with a speed of $0.8c$.

6. A rod flies with constant velocity past a mark, which is stationary in reference frame S. In reference frame S, it takes 20 ns for the rod to fly past the mark. In the reference frame S', which is fixed with respect to the rod, the mark moves past the rod for 25 ns. Find the length of the rod in S and S' and the speed of S' with respect to S.

7. A rod of length 60 cm in its rest frame is traveling along its length with a speed of $0.6c$ in the frame S. A particle moving in the opposite direction to the rod, with a speed $0.6c$ in S, passes the rod. How much time will the particle take to cross the rod

   (a) in the frame S.

   (b) in the rest frame of the particle.

8. Two spacehips pass each other, travelling in opposite directions. The speed of ship B, measured by a passenger in ship A is $0.96c$. This passenger has measured the length of the ship A as 100 mt and determines that the ship B is 30 mt long. What are the lengths of the two ships as measured by a passenger in ship B ?

9. An observer O is at the origin of an inertial frame. He notices a vehicle A to pass by him in $+x$ direction with constant speed. At this instant, the watch of the observer O and the watch of the driver of A show time equal to zero. 50 μs after A passed by, O sees another vehicle B pass by him, also in $+x$ direction and again with constant speed. After sometime B catches A and sends a light signal to O, which O receives at 200 μs according to his watch. The driver of B notices that, in his frame, the time between passing O and catching A is 90 μs Assume that drivers A and B are at the origins of their respective frames. Find

(a) the speeds of B and A, in the frame of O.

(b) position of A in O's frame when B passes O.

(c) the position of O in the frame of A, when B passes O.

10. An intertial frame S' moves relative to another frame S with a velocity $v_1\hat{\imath} + v_2\hat{\jmath}$ in such a way that the x and x' axes, y and y' axes and z and z' axes are always parallel. Let the time $t = t' = 0$ when the origins of the two frames are co-incident. Find the Lorenz transformation relating the co-ordinates and time of S' to those in S.

11. An observer sees two spaceships flying in opposite directions with speeds $0.99c$. What is the speed of one spaceship as viewed by the other?

12. Two identical spaceships, each 200 m long, pass one another traveling in opposite directions. If the relative velocity of the two space ships is $0.58c$: (a) how long does it take for the other ship to pass by as measured by a passenger in one of the ships, and (b) if these spaceships are moving with velocities $\pm u$ with respect to a frame S along the x-direction, what are their lengths as measured by an observer in S.

13. A rod of proper length l is oriented parallel to x-axis in a frame S, and is moving with a speed u along the same direction. Find its length in a frame S' which is moving with a speed v with respect to S, also along the x direction.

PH 111: Tutorial Sheet 6
Soln slides

Problem 1

- **Prob1:** The time interval between two ticks of two identical clocks is 2.0 sec. One of the two clocks is set in motion, so that its speed relative to the observer, who holds the other clock is $0.6c$. What is the time interval between the ticks of the moving clock as measured by the observer with the stationary clock?
- **Soln:** In all the problems of this tutorial sheet we use the notation $\beta = v/c$. Clearly, in the frame of the observer with stationary clock

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}} \quad \text{Clock is always at } x' = 0$$

$$t = \frac{2}{\sqrt{1-\beta^2}} = \frac{2}{\sqrt{1-0.36}} = 2.5\,\text{s}$$

Problem 2

- **Prob 2:** The incoming primary cosmic rays create μ-mesons in the upper atmosphere. The lifetime of μ-mesons at rest is 2 μs. If the mean speed of μ-mesons is $0.998c$, what fraction of the μ-mesons created at a height of 20 km reach the sea level?
- **Soln:** The lifetime of μ-mesons in rest frame = 2 μs. To travel 20 km at $0.998c$ requires

$$\triangle t = \frac{20}{0.998 \times 3 \times 10^5} \text{ s} \;= 66.8 \times 10^{-6} \text{ s}$$

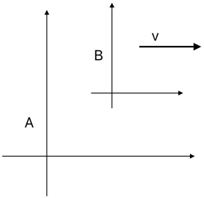
The lifetime τ will appear to be $\frac{\tau}{\sqrt{1-\beta^2}}$

The fraction f that will survive

$$f = \exp\left(-\frac{\Delta t\sqrt{1-\beta^2}}{\tau}\right) = \exp\left(-\frac{66.8\sqrt{1-0.998^2}}{2}\right) = 0.12$$

Problem 3

- **Prob 3:** Two observers A and B are close to a point where lightning strikes the earth. According to A, a second lightning strikes t_0 seconds later at a distance d from him. B, on the other hand finds the two events to be simultaneous. Find his velocity with respect to A. Also find the distance between the two lightnings as seen by B. Assume earth to be an inertial frame of reference.
- **Soln:**



Assume for the first lighting is at $(0, 0)$ for both. For A the second event is at $x_A = d, t_A = t_0$.

For B

$$t_B = \frac{t_A - vx_A/c^2}{\sqrt{1-\beta^2}} = \frac{t_0 - vd/c^2}{\sqrt{1-\beta^2}}$$

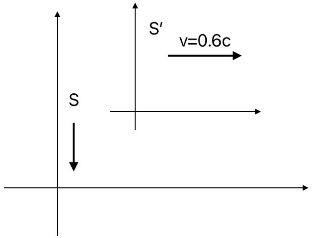
But B finds the events to be simultaneous implying $t_B = 0 \Rightarrow v = c^2 t_0/d$, so that

$$x_B = \frac{x_A - vt}{\sqrt{1-\beta^2}} = \frac{d - \frac{c^2 t_o^2}{d}}{\sqrt{1 - \frac{c^4 t_o^2}{c^2 d^2}}} = \frac{d^2 - c^2 t_o^2}{\frac{d\sqrt{d^2 - c^2 t_o^2}}{d}}$$

$$x_B = \sqrt{d^2 - c^2 t_o^2}$$

Problem 4

- **Prob 4:** Observer A is at rest in frame S' moving horizontally past an inertial frame S at a speed of $0.6c$. A boy in the frame S, drops a ball, which according to the clock of observer A, falls for 1.5 sec. How long will the ball fall for an observer at rest in S frame?



- **Soln:** We have

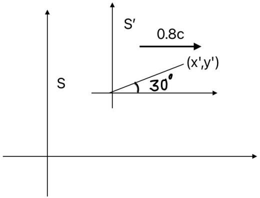
$$t' = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$$

Because the ball is always at $x = 0$

$$t = t'\sqrt{1-\beta^2} = 1.5 \times \sqrt{1-(0.6)^2} = 1.2\text{s}$$

Problem 5

- **Prob 5:** A meter stick is positioned so that it makes an angle 30^o with the x-axis in its rest frame. Determine its length and its orientation as seen by an observer who is moving along x-axis with a speed of $0.8c$.
- **Soln:** Let us take S' to be the rest frame of the meter stick, with its one end at the origin $(x' = 0, y' = 0)$, and the other end at the point (x', y') given by



$$x' = \cos 30^\circ = \sqrt{3}/2$$
$$y' = \sin 30^\circ = 1/2$$

Prob 5, soln...

Assume that the origins coincided at $t = t' = 0$. Now the coordinates of the two ends with respect to the S will be related by

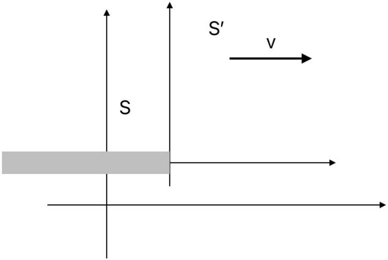
$$\begin{aligned} x_1' &= \gamma(x_1 - vt) = 0 \\ x_2' &= \gamma(x_2 - vt) = x' \\ y_1 &= y_1' = 0 \\ y_2 &= y_2' = y' \end{aligned}$$

Angle ϕ will be given by

$$\tan\phi = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1/2}{\frac{\sqrt{3}}{2}\sqrt{1-(0.8)^2}}$$

so that $\tan\phi = \frac{5}{3\sqrt{3}} \implies \phi = 43^\circ 54'$. We can also directly argue that the x component of length will be Lorentz contracted, while the y component will be unchanged leading again to the result $\tan\phi = (1/2)/\left\{(\sqrt{3}/2)\sqrt{1-(0.8)^2}\right\}$.

- **Prob 6:** A rod flies with constant velocity past a mark, which is stationary in reference frame S. In reference frame S, it takes 20 ns for the rod to fly past the mark. In the reference frame S', which is fixed with respect to the rod, the mark moves past the rod for 25 ns. Find the length of the rod in S and S' and the speed of S' with respect to S.
- **Soln:**



In S' the rod is at rest between $x' = -L_0$ and $x' = 0$.

Problem 6: Soln

The tip coincides with the origin at $t = 0$

$$x' = -L_0 \text{ coincides with } x = 0 \text{ at } t_0 = 20 \text{ ns}$$

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \Rightarrow -L_0 = \frac{vt_0}{\sqrt{1-\beta^2}}$$

In S', the time to cross the origin

$$0 = x = \frac{x' + vt'x'}{\sqrt{1-\beta^2}} \Rightarrow vt' = \frac{vt_0}{\sqrt{1-\beta^2}} \Rightarrow \sqrt{1-\beta^2} = \frac{20}{25} \quad \boxed{\beta = 0.6\mathrm{c}}$$

$$\Longrightarrow L_0 = 0.6 \times 3 \times 10^8 \times 25 \times 10^{-9} = 4.5\mathrm{m}.$$

S will measure this length to be $4.5\sqrt{1-\beta^2} = 3.6\mathrm{m}$.

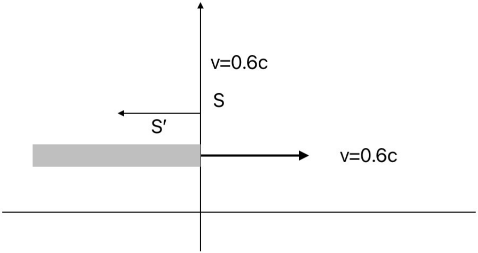
Alternative Solution: If the proper length of the rod is L_0 as seen by an observer in S', the contracted length $L = L_0\sqrt{1-\beta^2}$ will be seen by an observer in S. If S' moves with speed v, with respect to S, we have

$$\frac{L}{v} = 20 \times 10^{-9}\mathrm{s}$$
$$\frac{L_0}{v} = 25 \times 10^{-9}\mathrm{s}$$
$$\implies \frac{L}{L_0} = \frac{20 \times 10^{-9}\mathrm{s}}{25 \times 10^{-9}\mathrm{s}} = \frac{4}{5} = \sqrt{1-\beta^2}$$
$$\implies \beta = \frac{3}{5} \implies v = 0.6c$$

Now

$$L = 20 \times 10^{-9} v = 20 \times 10^{-9} \times 0.6 \times 3 \times 10^{-9} = 3.6\mathrm{m}$$
$$L_0 = \frac{L}{\sqrt{1-\beta^2}} = \frac{3.6}{\frac{4}{5}} = 4.5\mathrm{m}$$

- **Prob 7:** A rod of length 60 cm in its rest frame is traveling along its length with a speed of $0.6c$ in the frame S. A particle moving in the opposite direction to the rod, with a speed $0.6c$ in S, passes the rod. How much time will the particle take to cross the rod
  (a) as seen in frame S
  (b) in the rest frame of the particle.
- **Soln:**



Let the rod be at rest in S' frame, with its tip at $x' = 0$ and the back end at $x' = -L_0$

Prob 7 soln...

In S, the back end position x is computed as

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \Rightarrow x = -L_0\sqrt{1-\beta^2}, \text{ at } t = 0$$

At $t = 0$ the particle is at $x = 0$ in S frame. The time (in S frame) at which the particle will cross the back end of the rod

$$t = \frac{L_0\sqrt{1-\beta^2}}{2v} = \frac{L_0}{2c}\sqrt{\frac{1-\beta^2}{\beta^2}}$$

$$= \frac{60}{100} \times \frac{1}{2\times 3\times 10^8}\sqrt{\frac{64}{36}} = \boxed{\frac{4}{3}\times 10^{-9}\text{ seconds}}$$

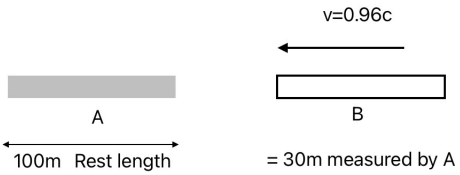
The crossing happens at $x = -vt$ with $v = 0.6c$

The time in particle's frame

$$t'' = \frac{t + vx/c^2}{\sqrt{1-\beta^2}} = \frac{t\left(1 - v^2/c^2\right)}{\sqrt{1-\beta^2}} = t\sqrt{1-\beta^2}$$

$$= \frac{4}{3}\times 10^{-9}\sqrt{1-(0'6)^2} = \frac{4}{3}\times 10^{-9}\cdot\frac{4}{5} = \boxed{\frac{16}{15}\times 10^{-9}\text{s}}$$

- **Prob8:** Two spaceships pass each other, traveling in opposite directions. The speed of ship B, measured by a passenger in ship A is $0.96c$. This passenger has measured the length of the ship A as 100 mt and determines that the ship B is 30 mt long. What are the lengths of the two ships as measured by a passenger in ship B ?
- **Soln:**



A finds B is 30m long. If actual length in B's frame is L_0, we have

$$L_0\sqrt{1-\beta^2} = 30 \Rightarrow L_0 = \frac{30}{\sqrt{1-(0.96)^2}} = 107.14\,m$$

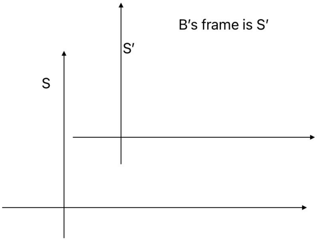
Problem 8: Soln

Similarly B will find the length of A as

$$100\sqrt{1-\beta^2} = 100\sqrt{1-(0.96)^2} = 28\,m$$

- **Prob 9:** An observer O is at the origin of an inertial frame. He notices a vehicle A to pass by him in $+x$ direction with constant speed. At this instant, the watch of the observer O and the watch of the driver of A show time equal to zero. 50 μs after A passed by, O sees another vehicle B pass by him, also in $+x$ direction and again with constant speed. After sometime B catches A and sends a light signal to O, which O receives at 200 μs according to his watch. The driver of B notices that, in his frame, the time between passing O and catching A is 90 μs Assume that drivers A and B are at the origins of their respective frames. Find
(a) the speeds of B and A, in the frame of O
(b) position of A in O's frame when B passes O
(c) the position of O in the frame of A, when B passes O.

Soln:



At $t = 0$ A crosses the origin with speed v_A in frame S
$\Rightarrow$ his position $x_A = v_A t$
At $t = t_B$ B crosses the origin with speed v_B
$\Rightarrow$ his position $x_B = v_B(t - t_B)$ $t_B = 50\mu\mathrm{s}$
B catches up with A when

$$v_A t = v_B(t - t_B) \Rightarrow t = \frac{v_B t_B}{v_B - v_A}$$

Problem 9: Soln....

He releases a light signal that reaches origin at $200\mu\text{s}$

$$\therefore t + \frac{1}{c} v_A t = 200\mu\text{s}, \qquad \frac{v_B t_B}{v_B - v_A} \times \left(1 + \frac{v_A}{c}\right) = 200\mu\text{s}$$

$$\Rightarrow \frac{1}{1 - v_A/v_B} \left(1 + \frac{v_A}{c}\right) = 4 \ldots \ (1)$$

In the frame of B, he catches up with A 90 μs after crossing the origin of S

In his frame this happens at $x' = 0,\ t' = 90\,\mu s$

In the S frame the interval was $(t - t_B)$ since B crossed the origin

$$t - t_B = \frac{v_B t_B}{v_B - v_A} - t_B = \frac{t' + v_B x'/c^2}{\sqrt{1 - v_B^2/c^2}}$$

$$\left(\frac{v_B}{v_B - v_A} - 1\right) t_B = \frac{t' + 0}{\sqrt{1 - v_B^2/c^2}}$$

$$\frac{1}{\frac{v_B}{v_A} - 1} = \frac{90}{50} \times \frac{1}{\sqrt{1 - v_B^2/c^2}}$$

$$1+\beta_A = 4\left(1-\frac{\beta_A}{\beta_B}\right)$$
$$\frac{\beta_B}{\beta_A} - 1 = \frac{5}{9}\sqrt{1-\beta_B^2}$$

Solve for β_A and β_B from there two

$$\beta_A\left(1+\frac{4}{\beta_B}\right) = 3 \Rightarrow \beta_A = \frac{3\beta_B}{4+\beta_B}$$
$$\Rightarrow \frac{\beta_B}{\beta_A} - 1 = \frac{4+\beta_B}{3} - 1 = \frac{1+\beta_B}{3}$$

Put this in (2)

$$\frac{(1+\beta_B)^2}{9} = \frac{25}{81}\left(1-\beta_B^2\right)$$
$$9(1+\beta_B) = 25(1-\beta_B)$$
$$34\beta_B = 16$$

Or

$$\boxed{\beta_B = \frac{8}{17}}$$

$$\beta_A = \frac{3\beta_B}{4+\beta_B} = \frac{3.8}{17\left(4+\frac{8}{17}\right)}$$

$$= \frac{24}{17} \times \frac{17}{76} = \frac{6}{19} \Rightarrow \boxed{\beta_A = \frac{6}{19}}$$

The position of A in S frame when B crosses origin

$$x_A = v_A t_B = \frac{6}{19} \cdot 3 \times 10^8 \cdot 50 \times 10^{-6}$$

$$= \frac{90}{19} \times 10^3 \mathrm{m} \Rightarrow \boxed{\frac{90}{19}\ \mathrm{km}}$$

Prob 9 soln...

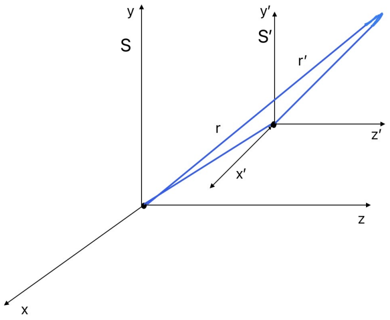
The position of the origin of S in $A's$ frame when B crosses O

$$\begin{cases} x & = 0 \\ t & = t_B \\ x'_A & = \frac{x - v_A t}{\sqrt{1-\beta_A^2}} \end{cases}$$

$$\begin{aligned} x'_A = \frac{0 - v_A t_B}{1 - \beta_A^2} &= -\frac{\beta_A}{\sqrt{1-\beta_A^2}}(ct_B) \\ &= -\frac{6}{19}\frac{1}{\sqrt{1-(6/19)^2}} \cdot 3 \times 10^8 \cdot 50 \times 10^{-6} \\ &= -\frac{6}{19} \cdot \frac{19}{\sqrt{325}} \cdot 15 \times 10^3 \\ &= -\frac{6 \times 15 \times 3}{5\sqrt{13}} \cdot 10^3 \mathrm{m} = \boxed{-4.992\,\mathrm{km}} \end{aligned}$$

Problem 10

- **Prob 10:** An intertial frame S' moves relative to another frame S with a velocity $v_1\hat{i} + v_2\hat{j}$ in such a way that the x and x' axes, y and y' axes and z and z' axes are always parallel. Let the time $t = t' = 0$ when the origins of the two frames are co-incident. Find the Lorenz transformation relating the co-ordinates and time of S' to those in S.
- **Soln:**



Solve the general problem as follows.

Problem 10, soln...

Write a position vector $\vec{r}$ in the S frames as a sum of two vectors, one projected along the direction of the velocity $\vec{v}$, and the other one perpendicular to it.

$$\vec{r}_{\parallel} = (\vec{r}.\hat{n})\hat{n} = \frac{(r\cdot\vec{v})\vec{v}}{v^2}$$
$$\vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel}$$

The Lorenz transformation only affects $\vec{r}_{\parallel}$, therefore, we obtain

$$\vec{r}\,' = \vec{r}_{\perp} + \gamma\left(\vec{r}_{\parallel} - \vec{v}t\right)$$
$$\vec{r}\,' = \vec{r} + (\gamma - 1)\vec{r}_{\parallel} - \gamma\vec{v}t$$

above $\gamma = 1/\sqrt{1-\beta^2}$. And the analogue of the time coordinate transformation

$$t' = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$$

is obtained by replacing vx by $\vec{v}\cdot\vec{r}$

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right)$$

Taking $\vec{v} = v_1\hat{i} + v_2\hat{j} = v\cos\theta\hat{i} + v\sin\theta\hat{j}$, with $v = \sqrt{v_1^2 + v_2^2}$, we have

$$\vec{r}_{\parallel} = \frac{(\vec{r}\cdot\vec{v})\vec{v}}{v^2} = \frac{(xv\cos\theta + yv\sin\theta)\vec{v}}{v^2}$$

we obtain from $\vec{r}\,' = \vec{r} + (\gamma-1)\vec{r}_{\parallel} - \gamma\vec{v}t$

$$\begin{aligned}
x' &= x + (\gamma-1)\frac{(xv\cos\theta + yv\sin\theta)v\cos\theta}{v^2} - \gamma v\cos\theta t \\
&= x\left[\sin^2\theta + \gamma\cos^2\theta\right] + y(\gamma-1)\sin\theta\cos\theta - \gamma v\cos\theta t \\
y' &= y + (\gamma-1)(x\cos\theta + y\sin\theta)\sin\theta - \gamma v\sin\theta t \\
&= x(\gamma-1)\sin\theta\cos\theta + y\left[\cos^2\theta + \gamma\sin^2\theta\right] - \gamma v\sin\theta t. \\
z' &= z \\
t' &= \gamma\left[t - (x\cos\theta + y\sin\theta)\frac{v}{c^2}\right].
\end{aligned}$$

Problem 11

- **Prob 11:** An observer sees two spaceships flying in opposite directions with speeds $0.99c$. What is the speed of one spaceship as viewed by the other?
- **Soln:** Suppose spaceship 1 is moving in the -ve x direction and while spaceship 2 is moving in the +ve x direction. For finding the speed of space ship 2 with respect to 1, we will use the formula

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}.$$

Here $u_x = 0.99c$, $v = -0.99c$ so that

$$u'_x = \frac{0.99c - (-0.99c)}{1 - \frac{0.99c*(-0.99c)}{c^2}} = \frac{1.98c}{1 + 0.99^2} = 0.9999495c < c$$

The velocity of ship 1 w.r.t. 2, will be just the opposite of this.

- **Prob 12:** Two identical spaceships, each 200 m long, pass one another traveling in opposite directions. If the relative velocity of the two space ships is $0.58c$: (a) how long does it take for the other ship to pass by as measured by a passenger in one of the ships, and (b) if these spaceships are moving along the x-direction with velocities $\pm u$ with respect to a frame S, what are their lengths as measured by an observer in S.
- **Soln:** (a) An observer in one spaceship will see the Lorentz contracted length of the other spaceship, given by

$$L = L_0\sqrt{1 - v^2/c^2} = 200\sqrt{1 - 0.58^2} = 162.92329482 \text{ m}$$

Time measured by an observer of one spaceship for the other spaceship to pass it will be L divided by the relative speed $0.58c$

$$\Delta t = 162.92329482/(0.58 \times 3 \times 10^8) = 0.936\,\mu\text{s}$$

Problem 12, soln...

(b) Clearly, from the rule of addition of velocities, we have

$$\frac{u-(-u)}{1-u(-u)/c^2} = 0.58c$$
$$\implies 0.58u^2/c - 2u + 0.58c = 0$$

$$\implies u = \frac{2 \pm \sqrt{4-4(0.58)^2}}{2 \times (0.58/c)} = c\left\{\frac{1 \pm \sqrt{1-0.58^2}}{0.58}\right\} = c\,\frac{1 \pm 0.814616}{0.58}$$
$$\implies u = 3.128648c \text{ or } 0.319628c$$

The only acceptable value is $u = 0.319628c$. Thus the equal contracted lengths of the two space ships with respect to S

$$L' = L_0\sqrt{1-u^2/c^2} = 200\sqrt{1-0.319628^2} = 189.509 \text{ m}$$

Problem 13

- **Prob 13:** A rod of proper length l is oriented parallel to x-axis in a frame S, and is moving with a speed u along the same direction. Find its length in a frame S' which is moving with a speed v with respect to S, also along the x direction.
- **Soln:** The relative velocity of the rod with respect to S' is

$$u' = \frac{u - v}{1 - uv/c^2} = \frac{c^2(u - v)}{(c^2 - uv)}$$

Therefore, the contracted length of the rod l' with respect to S'

$$l' = l\sqrt{1 - u'^2/c^2} = l\sqrt{1 - \frac{c^2(u - v)^2}{(c^2 - uv)^2}} = \frac{l\sqrt{(c^2 - uv)^2 - c^2(u - v)^2}}{(c^2 - uv)}$$

$$= \frac{l}{(c^2 - uv)}\sqrt{c^4 - 2c^2uv + u^2v^2 - c^2u^2 - c^2v^2 + 2c^2uv}$$

$$= \frac{l\sqrt{c^4 - c^2u^2 - c^2v^2 + u^2v^2}}{(c^2 - uv)} = \frac{l\sqrt{(c^2 - u^2)(c^2 - v^2)}}{(c^2 - uv)}$$