LIO - 04/09/2024

Real nos. are eq. classes of Cauchy seq. of rationals.

· **Bounded seq.** – (a_n) is bounded, if $\exists$ integer M s.t

$$|a_n| \leq M \quad \forall\, n \in \mathbb{Z}_{\geq 1}$$

L: Every Cauchy seq. is bounded

Pf – Given a Cauchy seq (a_n),

$\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t $\forall\, m, n \geq N$

$$|a_m - a_n| < \epsilon$$

So, for $\epsilon = 1$ & $n = N$, we have

$$|a_m - a_N| < 1$$

$$\Rightarrow a_N - 1 < a_m < a_N + 1$$

Let

$$M = \max\{|a_1|, |a_2|, \ldots |a_{N-1}|, |a_N - 1|, |a_N + 1|\}$$

$$\Rightarrow \quad |a_m| < M \quad \forall\, m \in \mathbb{Z}_{\geq 1}$$

$\square$

Add^n

Given Cauchy seq. (a_n) & (b_n)

$$(a_n) + (b_n) := (a_n + b_n)$$

C: $(a_n + b_n)$ is Cauchy

Pf — $\because$ (a_n) and (b_n) are Cauchy

$\therefore$ $\forall\, \epsilon > 0$, $\exists\, N_1$ and N_2 s.t

$\forall\, m, n \geq N_1$, $\quad |a_m - a_n| < \epsilon/2$

$\forall\, m, n \geq N_2$, $\quad |b_m - b_n| < \epsilon/2$

Consider $\quad N = \max\{N_1, N_2\}$

So, $\forall\, m, n \geq N$

$$|(a_m + b_m) - (a_n + b_n)| = |(a_m - a_n) + (b_m - b_n)|$$

$$\leq \underbrace{|a_m - a_n|}_{< \epsilon/2} + \underbrace{|b_m - b_n|}_{< \epsilon/2} < \epsilon \quad \square$$

C. Add^n is well-defined.

Pf – Consider $(A_n) \sim (a_n)$ & $(B_n) \sim (b_n)$

We need to prove that

$$(A_n + B_n) \sim (a_n + b_n)$$

So, $\forall \epsilon > 0$, $\exists$ N_1 and N_2 s.t

$\forall n \geq N_1$, $\quad |A_n - a_n| < \epsilon/2$

$\forall n \geq N_2$, $\quad |B_n - b_n| < \epsilon/2$

Consider $N = \max\{N_1, N_2\}$

So, $\forall n \geq N$

$$|(A_n + B_n) - (a_n + b_n)| = |(A_n - a_n) + (B_n - b_n)|$$

$$\leq \underbrace{|A_n - a_n|}_{< \epsilon/2} + \underbrace{|B_n - b_n|}_{< \epsilon/2} < \epsilon \quad \square$$

Multi^n

Given Cauchy seq. (a_n) & (b_n)

$$(a_n) \times (b_n) := (a_n b_n)$$

C: $(a_n b_n)$ is Cauchy

Pf – $\because$ (a_n) and (b_n) are Cauchy

$\therefore$ 1. $\exists$ a, b s.t $|a_n| \leq a$, $|b_n| \leq b$
$\forall n \in \mathbb{Z}_{\geq 1}$

2. $\forall \epsilon > 0$, $\exists$ N_1 and N_2 s.t

$\forall m, n \geq N_1$, $|a_m - a_n| < \epsilon/2b$

$\forall m, n \geq N_2$, $|b_m - b_n| < \epsilon/2a$

Consider $N = \max\{N_1, N_2\}$

So, $\forall m, n \geq N$

$$|a_m b_m - a_n b_n| = |a_m b_m - a_n b_m + a_n b_m - a_n b_n|$$

$$\leq |b_m||a_m - a_n| + |a_n||b_m - b_n|$$

$$\leq b\,\underbrace{|a_m - a_n|}_{< \epsilon/2b} + a\,\underbrace{|b_m - b_n|}_{< \epsilon/2a}$$

$$< \epsilon \quad \square$$

C. Multi^n is well-defined.

Pf – Consider $(A_n) \sim (a_n)$ & $(B_n) \sim (b_n)$

We need to prove that

$$(A_n B_n) \sim (a_n b_n)$$

$\because$ (A_n) and (B_n) are Cauchy

$\therefore$ $\exists$ A, B s.t $|A_n| \leq A$, $|B_n| \leq B$ $\forall\, n \in \mathbb{Z}_{\geq 1}$

2. $\forall\, \epsilon > 0$, $\exists$ N_1 and N_2 s.t

$\forall\, m, n \geq N_1$, $\quad |a_m - a_n| < \epsilon/2b$

$\forall\, m, n \geq N_2$, $\quad |b_m - b_n| < \epsilon/2a$

$\because$ $(A_n) \sim (a_n)$ & $(B_n) \sim (b_n)$

$\therefore$ $\forall \epsilon > 0$, $\exists$ N_1 and N_2 s.t

$\forall n \geqslant N_1$, $\quad |A_n - a_n| < \epsilon/2B$

$\forall n \geqslant N_2$, $\quad |B_n - b_n| < \epsilon/2A$

Consider $\quad N = \max\{N_1, N_2\}$

So, $\forall n \geqslant N$

$$
\begin{aligned}
|A_nB_n - a_nb_n| &= |A_nB_n - A_nb_n + A_nb_n - a_nb_n| \\
&\leqslant |B_n||A_n - a_n| + |A_n||B_n - b_n| \\
&\leqslant B\underbrace{|A_n - a_n|}_{< \epsilon/2B} + a\underbrace{|B_n - b_n|}_{< \epsilon/2A} \\
&< \epsilon \quad \square
\end{aligned}
$$

There is a natural injective map from rationals to the reals.

$$\mathbb{Q} \hookrightarrow \mathbb{R}$$

$$q \mapsto (q, q, q \cdots)$$

Pf - Let $a, b \in \mathbb{Q}$, $a \neq b$

$\therefore$ Both (a_n) & (b_n) are const. seq.

$$|a_n - b_n| = |a - b|$$

Consider $\epsilon = \dfrac{|a-b|}{2}$

$$\Rightarrow |a_n - b_n| > \epsilon$$

$\therefore \exists \ \epsilon > 0$ s.t $\forall n \in \mathbb{N}$,

$$|a_n - b_n| > \epsilon$$

$\therefore$ $(a_n) \not\sim (b_n)$

$\square$

Inverse

Given $x \in \mathbb{R}$, $x \neq 0$, we would like to define its inverse x^{-1} as follows

$$x = (x_1, x_2, \ldots)$$
$$x^{-1} = (x_1^{-1}, x_2^{-1}, \ldots)$$

But, one of the x_i might be 0.

So, first, we need to modify finitely many terms of the seq. s.t this does <u>NOT</u> occur.

<u>L</u>: Let $x \in \mathbb{R}$, $x \neq 0$.
Suppose $x = (x_1, x_2, \ldots)$.

Then $\exists\, c \in \mathbb{Q}$ & $N \in \mathbb{N}$ s.t $\forall\, n \geq N$

$$|x_n| \geq c$$

Pf - Suppose $\nexists$ such c.

$\because$ (x_n) is Cauchy

$\therefore$ $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall m, n \geq N$

$$|x_m - x_n| < \epsilon/2$$

Consider $\epsilon = 2c$

$\because$ The hypothesis does <u>NOT</u> hold

$\therefore$ $\exists n_0 \geq N$ s.t

$$|x_{n_0}| < c \Rightarrow |x_{n_0}| < \epsilon/2$$

So, $\forall n \geq N$,

$$|x_n| = |x_n - x_{n_0} + x_{n_0}|$$

$$\leq \underbrace{|x_n - x_{n_0}|}_{< \epsilon/2} + \underbrace{|x_{n_0}|}_{< \epsilon/2}$$

$$< \epsilon$$

$$\Rightarrow |x_n - 0| < \epsilon \Rightarrow (x_n) \sim (0_n)$$

$$\Rightarrow \underbrace{x = 0}_{\text{contd}^n} \quad \square$$

As stated previously, we will modify the first $(n_0 - 1)$ terms of the seq.

$$x' = (1, 1, \ldots, 1, x_{n_0}, x_{n_0+1}, \ldots)$$

$\because (x_n) \sim (x'_n)$

$\therefore x = x'$

Hence, we can now define x^{-1} as

$$x^{-1} = (1, 1, \ldots, 1, x_{n_0}^{-1}, x_{n_0+1}^{-1}, \ldots)$$

Q Let $x = (a_n)$ be Cauchy.

Let $N \in \mathbb{Z}_{\geq 1}$ and

$$x' = (b_1, b_2, \dots, b_{N-1}, a_N, a_{N+1}, \dots)$$

$b_i \in \mathbb{Q}$

Show that x' is Cauchy

& $x' \sim x$

Pf 1. Let x'_n be the nth term of the seq. corresponding to x'.

$\because$ (a_n) is Cauchy

$\therefore$ $\forall \epsilon > 0$, $\exists N_0 \in \mathbb{Z}_{\geq 1}$ s.t. $\forall m, n \geq N_0$

$$|a_m - a_n| < \epsilon$$

$\because$ $\forall n \geq N$, $x'_n = a_n$

$\therefore$ $\forall m, n \geq \max\{N, N_0\}$, $|x'_m - x'_n| < \epsilon$

Hence, x' is Cauchy. $\square$

2. Let x'_n be the nth term of the seq. corresponding to x'.

$\therefore$ $\forall n \geq N$, $x'_n = a_n$

$\therefore$ $\forall \epsilon > 0$, $\forall n \geq N$

$$|x'_n - a_n| = 0 < \epsilon$$

Hence, $x' \sim x$ $\square$