L1 - 31/07/2024

Instructor - Prof. Ronnie Sebastain

Reference book - Analysis I by Terence Tao (first 8 chapters)

Mandatory attendance

Quiz - One per week - Friday (40% weightage)

Mid Sem & End Sem - 30% each

Q. Show $\varphi: \mathbb{N} \to \mathbb{Z}_{\geq 0}$, $\varphi(n) = n-1$ is injective & surjective

Proof: 1. Injectivity

$$\varphi(n_1) = \varphi(n_2) \quad ; \quad n_1, n_2 \in \mathbb{N}$$

$$\Rightarrow \quad n_1 - 1 = n_2 - 1$$

$$\Rightarrow \quad n_1 = n_2$$

$\because \varphi(n_1) = \varphi(n_2) \Rightarrow n_1 = n_2$

$\therefore \varphi$ is injective.

2. Surjectivity

Consider $m \in \mathbb{Z}_{\geq 0}$ & $n = m+1$

Clearly $n \in \mathbb{N}$.

$$\varphi(n) = n-1 = (m+1)-1 = m$$

$\therefore \forall m \in \mathbb{Z}_{\geq 0}, \exists n \in \mathbb{N}$ s.t

$$\varphi(n) = m$$

$\therefore \varphi$ is surjective. $\square$

Q. Prove $\nexists\, \alpha \in \mathbb{Q}$ s.t $\alpha^2 = 2$.

Proof - Let $\alpha = \dfrac{p}{q}$; $p, q \in \mathbb{Z}$, $q \neq 0$

s.t $GCD(p,q) = 1$

By defn, $\alpha^2 = 2 \Rightarrow \left(\dfrac{p}{q}\right)^2 = 2$

$\Rightarrow p^2 = 2q^2$

Now, $2 \mid RHS \Rightarrow 2 \mid LHS \Rightarrow 2 \mid p^2$

$\Rightarrow 2 \mid p$

$\Rightarrow \exists\, k \in \mathbb{Z}$ s.t $p = 2k$

Substituting back, $p^2 = 2q^2$

$$\Rightarrow (2k)^2 = 2q^2$$

$$\Rightarrow 2k^2 = q^2$$

Now, $2 \mid LHS \Rightarrow 2 \mid RHS \Rightarrow 2 \mid q^2$

$$\Rightarrow 2 \mid q$$

$\therefore$ $2 \mid p$ & $2 \mid q$

$\therefore$ $2 \mid GCD(p,q) \Rightarrow \underbrace{2 \mid 1}_{\text{Contd}^n}$

$\therefore$ $\nexists\, \alpha \in \mathbb{Q}$ s.t $\alpha^2 = 2$ $\square$