L26 - 07/11/2024

His approach was based on infinite series. term by term diff. & int. & a method for inverting the power series.

Using this, he constructed Taylor series expansions for $\log(1+x)$, $\sin(x)$, $\cos(x)$, $\sin^{-1}(x)$, e^x.

Examples

1. $$\log(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x 1 - t + t^2 - t^3 \dots dt$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} \dots$$

(found by Mercator & Kerala School)

2. $$\tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x 1 - t^2 + t^4 - t^6 \dots dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} \dots$$

3. Inverting / 'Extracting the Root'

$$y = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \quad \ldots$$

We wish to express x as follows

$$x = b_0 + b_1 y + b_2 y^2 + \ldots \qquad \text{(Ansatz)}$$

Substituting x in the previous expansion,

$$y = (b_0 + b_1 y \cdots) - \frac{1}{2}(b_0 + b_1 y \ldots)^2 + \frac{1}{3}(b_0 + b_1 y \ldots)^3 \ldots$$

By comparing coeffs. of powers of y,

$$y^0 : \quad 0 = b_0 - \frac{b_0^2}{2} + \frac{b_0^3}{3} \ldots = \log(1+b_0)$$

(if $|b_0| < 1$)

$\Rightarrow \quad b_0 = 0$

y: $\quad 1 = b_1 \quad \Rightarrow \quad b_1 = 1$

y^2: $\quad 0 = b_2 - b_1^2/2 \quad \Rightarrow \quad b_2 = 1/2$

y^3: $\quad 0 = b_3 - b_2 b_1 + b_1^3/3 \quad \Rightarrow \quad b_3 = 1/6$

So,
$$x = y + \frac{y^2}{2} + \frac{y^3}{6} + \dots$$
$$= \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} \dots$$
$$= e^y - 1$$

4. Binomial series

$$(1+a)^p = 1 + pa + \frac{p(p-1)}{2} a^2 + \dots$$

where $\quad p \in \mathbb{R}_{\geq 0}, \quad a \in \mathbb{R}$

Substitute $a = -t^2$, $p = -1/2$

we get,

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{t^2}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) t^4}{2} \cdots$$

$$\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = x + \frac{x^3}{2 \cdot 3} + \frac{1}{2} \cdot \frac{3}{4} x^5 \cdots$$

Inverting, we get the expansion of $\sin(x)$.

Leibniz

- Newton's 2 papers submitted to the Royal Society Proceedings were rejected in the 1670s

- In the meantime, a German mathematician philosopher, diplomat published 'Nova Methodus' laying out the foundations of calculus. ~ 1684

- In this paper, he lays down the sum, product & the quotient rule and introduced the notation dy/dx

To him, dy/dx was an actual quotient of infinitesimals

- Following this, in his 'De Geometrica' introduced '$\int$' & proved ~ 1686 the fundamental theorem of calculus

This was known to Newton & his teacher Barrow in a different form

For Leibniz $\int f(x)$ was a sum of terms representing infinitesimal rectangles of height $f(x)$ & width dx

He showed $\frac{d}{dx}\int f(x)\,dx = f(x)$