L7 - 23/08/2024

Integers

Consider pairs of natural nos. written as

$$X = \{ a\text{--}b \mid a, b \in \mathbb{N} \}$$

Define $\sim$ on X as

$$(a\text{--}b) \sim (c\text{--}d)$$

if $a + d = b + c$

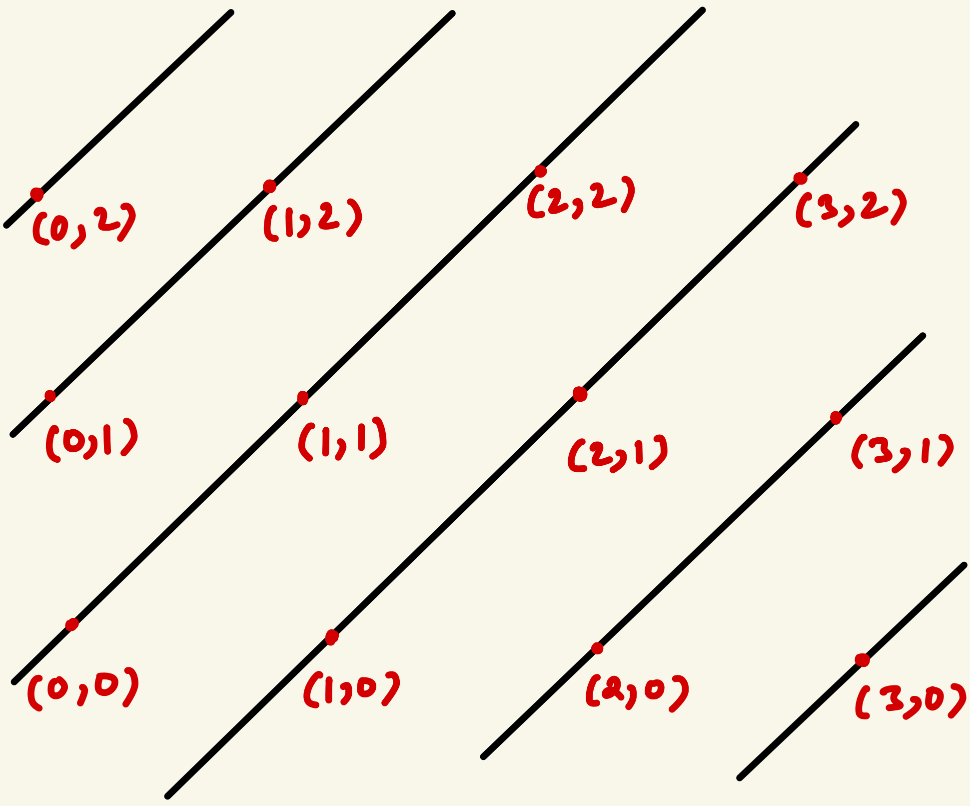
Clearly, $\sim$ is an eq. relⁿ.

Consider the eq. class of an element $(a\text{--}b) \in X$

$$EC(a\text{--}b) = \{ (c\text{--}d) \mid (a\text{--}b) \sim (c\text{--}d) \}$$

We can pictorially represent $\mathbb{N} \times \mathbb{N}$ as a square lattice

Over this lattice, all the equivalence classes of X can be represented by straight lines.



We consider each of these eq. classes as an <u>INTEGER</u>.

Given the set X and an eq. rel^n $\sim$ on X, let us define $X/\sim$ to be the set of eq. classes of X.

For any $x \in X$, $\quad EC(x) \subset X$

$$\Rightarrow EC(x) \in P(X)$$

$$\Rightarrow \underline{X/\sim \subset P(X)}$$

Consider the map

$$\begin{aligned} \pi : X &\longrightarrow X/\sim \\ x &\longmapsto EC(x) \end{aligned}$$

So, $\underline{\pi(x) := EC(x)}$

Claim – π is surjective

Pf – Given a $T \in X/\sim$, we need to find an x, s.t

$$EC(x) = T$$

But, by defⁿ of $X/\sim$, each of its elements is an eq. class of X.

So, $\exists\, y \in X$ s.t $T = EC(y)$

Choose $x = y$ $\square$

Q PT $\exists$ only one eq. relⁿ $\sim$ s.t $\pi : X \to X/\sim$ is injective and that $\sim$ is the identity relⁿ.

Pf - Consider an eq. relⁿ $\sim$ on X s.t $\pi : X \to X/\sim$ is injective.

$$x \mapsto EC(x)$$

Let $(a--b), (a'--b') \in X$ be s.t

$(a--b) \sim (a'--b')$ & $(a--b) \neq (a'--b')$

$$\Rightarrow EC(a--b) = EC(a'--b')$$
$$\Rightarrow \pi(a--b) = \pi(a'--b')$$

But, this is a contdⁿ since π is injective

$\therefore \nexists\, (a--b), (a'--b') \in X$ s.t

$(a--b) \sim (a'--b')$ & $(a--b) \neq (a'--b')$

$\therefore$ If $(a--b) \sim x$, then $x = a--b$

REMARK - Till now, we have only shown what $\sim$ cannot be.

This is because we haven't stated for which all $(a--b) \in X$ does $(a--b) \sim (a--b)$ hold yet.

$\because \sim$ is an equivalence rel^n.

$\therefore \quad (a--b) \sim (a--b) \quad \forall \ (a--b) \in X$

$\Rightarrow \sim$ is the identity rel^n.

REMARK - Showing that $\sim$ is the identity rel^n also proves it uniqueness.

Add^n of Integers

Let $X = \mathbb{N} \times \mathbb{N}$ and $\sim$ be an eq. rel^n on X.

We define add^n as

$$P : X/\sim \times X/\sim \to X/\sim$$

s.t for $\alpha, \beta \in X/\sim$, $\alpha = \pi(a - - b)$
$\beta = \pi(c - - d)$

$$P(\alpha, \beta) = \pi((a+c) - - (b+d))$$

Caveat: Is P well defined?

Notice that our defⁿ of P uses representatives of α & β (i.e $a--b$ and $c--d$ respectively)

We want the sum of 2 integers α, β to be independent of the choice of representatives, since both α & β correspond to more than one representative.

$$\because (a--b) \sim (a'--b') \Rightarrow \alpha = \pi(a--b) = \pi(a'--b')$$

In such a case, we call P to be well defined.

Claim - P is well defined.

Pf - Consider

$$\alpha = \pi(a--b) = \pi(a'--b')$$
$$\beta = \pi(c--d) = \pi(c'--d')$$

we need to show that

$$\pi((a+c)--(b+d)) = \pi((a'+c')--(b'+d'))$$

We know that

$(a--b) \sim (a'--b')$ & $(c--d) \sim (c'--d')$

$\Rightarrow a+b' = b+a'$ $\Rightarrow c+d' = d+c'$

$$\Rightarrow (a+b')+(c+d') = (b+a')+(d+c')$$
$$\Rightarrow (a+c)+(b'+d') = (b+d)+(a'+c')$$
$$\Rightarrow (a+c)--(b+d) \sim (a'+c')--(b'+d')$$
$$\Rightarrow \pi((a+c)--(b+d)) = \pi((a'+c')--(b'+d'))$$

$\square$

Multiplication of Integers

Let $X = \mathbb{N} \times \mathbb{N}$ and $\sim$ be an eq. rel^n on X.

We define multiplication as

$$M : X/\sim \times X/\sim \to X/\sim$$

s.t for $\alpha, \beta \in X/\sim$, $\alpha = \pi(a - - b)$
$\beta = \pi(c - - d)$

$$M(\alpha, \beta) = \pi((ac + bd) - - (bc + ad))$$

Claim – M is well defined.

Pf – Consider

$$\alpha = \pi(a--b) = \pi(a'--b')$$
$$\beta = \pi(c--d) = \pi(c'--d')$$

We need to show that

$$\pi((ac+bd)--(bc+ad))$$
$$= \pi((a'c'+b'd')--(b'c'+a'd'))$$

We know that

$(a--b) \sim (a'--b')$ & $(c--d) \sim (c'--d')$

$\Rightarrow a+b' = b+a'$ $\Rightarrow c+d' = d+c'$

Now, $(ac+bd+b'c'+a'd')+b'c$

$$= (a+b')c + bd + b'c' + a'd'$$
$$= (a'+b)c + bd + b'c' + a'd'$$
$$= bc + bd + b'c' + a'(d'+c)$$

$$= bc + bd + b'c' + a'(d + c')$$

$$= bc + (b + a')d + b'c' + a'c'$$

$$= bc + (b' + a)d + b'c' + a'c'$$

$$= bc + ad + b'(d + c') + a'c'$$

$$= bc + ad + b'(d' + c) + a'c'$$

$$= (bc + ad + a'c' + b'd') + b'c$$

$$\Rightarrow (ac + bd + b'c' + a'd') + b'c$$
$$= (bc + ad + a'c' + b'd') + b'c$$

$$\Rightarrow ac + bd + b'c' + a'd' = bc + ad + a'c' + b'd'$$

$$\Rightarrow (ac + bd) -- (bc + ad))$$
$$\sim (a'c' + b'd') -- (b'c' + a'd')$$

$$\Rightarrow \Pi((ac + bd) -- (bc + ad))$$
$$= \Pi((a'c' + b'd') -- (b'c' + a'd'))$$

$\square$

Negation of Integers

Let $X = \mathbb{N} \times \mathbb{N}$ and $\sim$ be an eq. rel^n on X.

We define negation as

$$N : X/\sim \;\longrightarrow\; X/\sim$$

s.t for $\alpha \in X/\sim$, $\alpha = \pi(a -- b)$

$$N(\alpha) = \pi(b -- a)$$

NOTE - We denote negation of x by $(-x)$.

Claim - N is well defined.

Pf - Consider

$$\alpha = \pi(a--b) = \pi(a'--b')$$

we need to show that

$$\pi(b--a) = \pi(b'--a')$$

We know that

$$(a--b) \sim (a'--b')$$

$$\Rightarrow a+b' = b+a'$$

$$\Rightarrow b+a' = a+b'$$

$$\Rightarrow (b--a) \sim (b'--a')$$

$$\Rightarrow \pi(b--a) = \pi(b'--a') \quad \square$$

Proposition – Let x, y, z be integers

Then
$$x+y=y+x$$
$$(x+y)+z=x+(y+z)$$
$$x+0=0+x=x$$
$$x+(-x)=(-x)+x=0$$
$$xy=yx$$
$$x1=1x=x$$
$$x(y+z)=xy+xz$$

NOTE – $\because$ We have proved that add^n, multiplication and negation are well defined, we don't need to prove these statements for multiple representatives.

If a statement holds for one representative, it holds for all the representatives.

Sub^n of Integers

Subtracting by an integer is the same as adding its negation.

$$x - y = x + (-y)$$