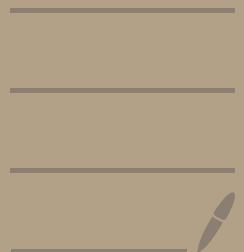


L2 - 07/08/2024



# Peano Axioms

. Notation -  $n++$  denotes  
successor of  $n$ .

1.  $0$  is a natural no.

2. If  $n$  is a natural no., then  
 $n++$  is also a natural no.

3.  $0$  is NOT the successor of  
any natural no.

i.e  $\forall$  natural nos.  $n$ ,  $n++ \neq 0$

4. If  $n++ = m++$ , then  $n = m$

5. Principle of Mathematical Induction

Let  $P(n)$  be any ppt. pertaining to  
natural nos.

If  $P(0)$  is true &  $P(n) \Rightarrow P(n++)$ ,  
then  $P(n)$  is true  $\forall$  natural nos.

## Motivation

3. disqualifies number systems  
such as

$0, 1, 2, 3, 0, 1, 2, 3, 0 \dots$

which loop back to 0.

4. disqualifies number systems  
such as

$0, 1, 2, 3, 4, 4, 4 \dots$

which hit a ceiling.

OR

$0, 1, 2, 3, 1, 1, 1 \dots$

which loop back to a  
non-zero natural no.

5. disqualifies number systems  
such as

$0, 0.5, 1, 1.5, 2, 2.5 \dots$

which have 'extra' elements  
i.e. which cannot be produced  
by axioms 1-4.

. Assumption -  $\exists$  a number system  
 $\mathbb{N}$  whose elements we shall  
call natural nos., for which  
Axioms 1-5 are true.

## Recursive def<sup>n</sup>

Let  $f_n: \mathbb{N} \rightarrow \mathbb{N}$  be a fn<sup>n</sup> s.t  
 $a_0 = c$  for some natural no.  $c$   
&  $a_{n++} := f_n(a_n)$

Hence, we can assign a unique natural no.  $a_n$  to every natural no.

Proof - Let  $P(n)$  be the proposition that  $a_n$  is unique.

BC -  $a_0 = c$   
None of the other def<sup>n</sup>,  
 $a_{n++} = f_n(a_n)$  will redefine  $a_0$   
by Axiom 3.

So,  $P(0)$  is true.

IH - Given  $a_n$  is unique,  
PT  $a_{n++}$  is unique.

$$a_{n++} = f_n(a_n)$$

None of the other def<sup>n</sup>s  
 $a_{m++} = f_n(a_m)$  will redefine  
 $a_{n++}$  by Axiom 4.

$$\text{So, } P(n) \Rightarrow P(n++)$$

$\therefore P(0)$  is true &  $P(n) \Rightarrow P(n++)$

$\therefore$  By PMI,  $P(n)$  is true  $\forall$  natural  
nos □