Hard Core Predicates

601.642/442: Modern Cryptography

Fall 2020

Last Time

- Proof via Reduction: $f_\times$ is a weak OWF
- Amplification: From weak to strong OWFs

Today

- Hard Core Predicate
- 1-bit stretch PRGs from hard core predicate.

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Is there any non-trivial (non-identity) function of x, even 1 bit, that OWFs hide?

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  - you can always guess the bit with probability 1/2.

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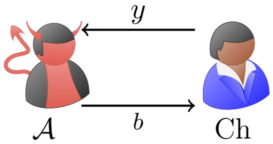

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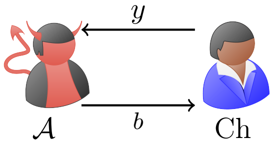

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We want that for all n.u. PPT adversary $\mathcal{A}$, the adversary wins with probability only at most negligible more than 1/2.

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Theorem (Goldreich-Levin)

Let f be a OWF. Define function

$$g(x, r) = (f(x), r)$$

where $|x| = |r|$. Then g is a OWF and

$$h(x, r) = \langle x, r\rangle$$

is a hard-core predicate for f

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- **Main challenge**: Adversary $\mathcal{A}$ for h only outputs 1 bit. Need to build an inverter $\mathcal{B}$ for f that outputs n bits.

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  - Yes! Levin gives us a method!

Back to PRGs

(How to construct PRGs with 1-bit stretch)

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- **Next Bit Test**: for a truly random sequence of bits, it is not possible to predict the "next bit" in the sequence with probability better than 1/2 even given all previous bits of the sequence so far
- A sequence of bits *passes the next bit test* if no efficient adversary can predict "the next bit" in the sequence with probability better than 1/2 even given all previous bits of the sequence so far

Next-bit Unpredictability

Definition (Next-bit Unpredictability)

An ensemble of distributions $\{X_n\}$ over $\{0,1\}^{\ell(n)}$ is next-bit unpredictable if, for all $0 \leqslant i < \ell(n)$ and n.u. PPT $\mathcal{A}$, $\exists$ negligible function $\nu(\cdot)$ s.t.:

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Theorem (Completeness of Next-bit Test)

If $\{X_n\}$ is next-bit unpredictable then $\{X_n\}$ is pseudorandom.

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- <u>Exercise</u>: Do the full formal proof

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    $\Rightarrow f(s)$ is uniformly distributed

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- Let $h : \{0,1\}^* \rightarrow \{0,1\}$ be a hardcore predicate for f
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Theorem (PRG based on OWP)

G is a pseudorandom generator with 1-bit stretch.

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- Proof Idea: Use next-bit unpredictability. Since first n bits of the output are uniformly distributed (since f is a permutation), any adversary for next-bit unpredictability with non-negligible advantage $\frac{1}{p(n)}$ must be predicting the $(n+1)$th bit with advantage $\frac{1}{p(n)}$. Build an adversary for hard-core predicate to get a contradiction.