Basics of Provable Security (II) & Computational Intractability

601.642/442: Modern Cryptography

Fall 2020

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- In other words, for a given m, $\Pr[c = \mathsf{Enc}(k, m)] = 1/2^n$.
- Hence, the ciphertexts are uniformly distributed.

The Hybrid Technique

Example (Double OTP)

Prove uniform ciphertext security of the following scheme:

- $\mathsf{KeyGen}(1^n) : k_1 \xleftarrow{\$} \{0,1\}^n$, $k_2 \xleftarrow{\$} \{0,1\}^n$ and output (k_1, k_2)
- $\mathsf{Enc}((k_1, k_2), m) : c_1 = k_1 \oplus m$, $c_2 = k_2 \oplus c_1$ and output c_2.
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- $\mathsf{Dec}((k_1, k_2), c) : c_1 = k_2 \oplus c$, $m = k_1 \oplus c_1$ and output m.

We need to show that for each m, the following distributions are identical:

1. $\{c_2 = k_2 \oplus c_1; k_1 \leftarrow \mathsf{KeyGen}(1^n), k_2 \leftarrow \mathsf{KeyGen}(1^n), c_1 = k_1 \oplus m\}$
2. $\left\{c_2 \xleftarrow{\$} \{0,1\}^n\right\}$

Proving Security using the Hybrid Technique

We consider the following set of distributions called **hybrids**.

$\mathcal{H}_1$: $\{c_2 = k_2 \oplus c_1; k_1 \leftarrow \mathsf{KeyGen}(1^n), k_2 \leftarrow \mathsf{KeyGen}(1^n), c_1 = k_1 \oplus m\}$

$\mathcal{H}_2$: $\left\{c_2 \xleftarrow{\$} \{0,1\}^n; k_1 \leftarrow \mathsf{KeyGen}(1^n), c_1 = k_1 \oplus m\right\}$

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Our goal is to show that $\mathcal{H}_1$ and $\mathcal{H}_3$ are identical distributions. We will do this in two steps by using the "intermediate" hybrid $\mathcal{H}_2$.

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$\mathcal{H}_1$ is identical to $\mathcal{H}_2$ because of the uniform ciphertext security of OTP.

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The Hybrid Technique is very common in cryptographic proofs and we will see it again and again throughout the course.

Encryption: One-Time Perfect Security

Lets consider an alternate idea of security for encryption schemes.

- The secret key should be kept hidden from Eve.
- The key is only used to encrypt one plaintext.
- ~~The ciphertexts look like random values to Eve.~~
- Encryptions of m_0 look like encryptions of m_1 to Eve.

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An encryption scheme is a good one if encryptions of m_0 look like encryptions of m_1 to Eve, when each key is secret and used to encrypt only one plaintext, even when Eve chooses both m_0 and m_1.

Encryption: One-Time Perfect Security

One-Time Perfect Security

We say that an encryption scheme is one-time perfectly secure if $\forall m_0, m_1 \in \mathcal{M}$ chosen by Eve, the following distributions are identical:

1. $\mathcal{D}_1 := \{c := \mathsf{Enc}(k, m_0); k \leftarrow \mathsf{KeyGen}(1^n)\}$
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As earlier, from adversary's viewpoint, the ciphertext carries no information about the plaintext.

Insecure Encryption

Insecure Encryption Scheme

An encryption scheme is does not satisfy one-time perfect security, if $\exists m_0, m_1 \in \mathcal{M}$, such that the following distributions are not identical:

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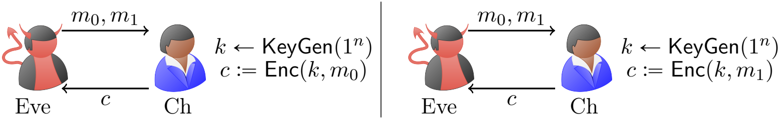
For $m_0 = 0^n$, $m_1 = 1^n$

$$\Pr[c = 0^n | \mathcal{D}_1] = 1$$
$$\Pr[c = 0^n | \mathcal{D}_2] = 1/2^n$$

Clearly the two distributions are not identical in this case.

Encryption: One-Time Perfect Security

Consider the following two interactions between Eve and a challenger.



- Interaction with a challenger helps us model what Eve can see during encryption, and what remains hidden.
- We say that an encryption scheme is secure if for any (m_0, m_1) chosen by Eve, the above two scenarios seem identical to Eve.

Comparing Both Security Notions

Theorem

If an encryption scheme achieves one-time uniform ciphertext security, then it also achieves one-time perfect security.

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We are given that for each $m \in \mathcal{M}$ (where $\mathcal{M}$ is the message space), the following distributions are identical:

1. $\mathcal{D}_1 := \{c := \mathsf{Enc}(k, m); k \leftarrow \mathsf{KeyGen}(1^n)\}$
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We want to show that for each $m_0, m_1 \in \mathcal{M}$, the following distributions are also identical:

1. $\mathcal{D}'_1 := \{c := \mathsf{Enc}(k, m_0); k \leftarrow \mathsf{KeyGen}(1^n)\}$
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Proof (Using hybrid technique): Consider the following distributions:

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- $\mathcal{H}_1 \equiv \mathcal{H}_2$: because of one-time uniform ciphertext security.

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Corollary

One-time pad satisfies one-time perfect security.

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Does this scheme satisfy one-time perfect security? Why?

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Does it also satisfy one-time uniform ciphertext security? Why not?

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  - Key must be as long as plaintext (necessary for perfect security). Think Why?
  - A key cannot be used to encrypt more than one plaintext (see HW1).

Computational Intractability

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Why?

It is like a knob that allows the user to tune the security to any desired level. Increasing n makes the difficulty of a brute force attack grow exponentially fast.

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2^{128} already seems like a lot.

Must we use (say) a 500-bit key to encrypt 500-bit messages, as in one-time pad? Or can we somehow use a smaller (say 128-bit) key to encrypt long messages and still get meaningful security?

Computational Infeasibiity

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- "Modern" cryptography is based on this principle, where security is based on intractable computations.
- If his letters hadn't been kept classified until 2012, they might have accelerated the development of modern cryptography.

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- In order to make security definitions that say *only feasible attacks are ruled out*, we need a concrete way to draw the line between **feasible attacks** (which we want to protect against) and **infeasible attacks** (which we agreed we don't need to care about).

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- Nevertheless, the reason why polynomial time is very useful is because of **closure property**: *repeating a poly-time algorithm polynomial times is still polynomial time!*

Some Examples

Efficient Algorithms known	**Efficient Algorithms not known**
Computing GCDs	Factoring Integers
Arithmetic mod N	Discrete Logarithm
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- In this class, we will mostly focus on algorithms on classical computers. Indeed, even in the second category, most problems, except last one are known to have efficient quantum algorithms.

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- While an attack with success probability 2^{-128} should not really count as an attack, one with success probability $1/2$ should. Where should we draw the line?