# One-Way Functions (II)

601.642/442: Modern Cryptography

Fall 2020

# Recap: One Way Functions

$$\begin{tikzcd} x \arrow[r, bend left, "\text{EASY}"] & f(x) \arrow[l, bend left, "\text{HARD}"] \end{tikzcd}$$

- A function is one-way if it "easy to compute," but "hard to invert"
- Necessary for the existence of most cryptographic primitives (e.g., multi-message encryption, digital signatures)
- Also sufficient for some cryptographic primitives (e.g., pseudorandom generators, secret-key encryption, digital signatures).

# Recap: One Way Functions (Definition)

**Definition (One Way Function)**

A function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is a one-way function (OWF) if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm $\mathcal{C}$ s.t. $\forall x \in \{0,1\}^*$,
$$\Pr\left[\mathcal{C}(x) = f(x)\right] = 1.$$
- **Hard to invert:** there exists a negligible function $\nu : \mathbb{N} \rightarrow \mathbb{R}$ s.t. for every non-uniform PPT adversary $\mathcal{A}$ and $\forall n \in \mathbb{N}$:
$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') = f(x)\right] \leqslant \nu(n).$$

- The above definition is also called **strong** one-way functions.

# Recap: Factoring Assumption

**Definition (Factoring Assumption)**

For every non-uniform PPT adversary $\mathcal{A}$, there exists a negligible function $\nu$ such that

$$\Pr\left[p \xleftarrow{\$} \Pi_n; q \xleftarrow{\$} \Pi_n; N = pq : \mathcal{A}(N) \in \{p, q\}\right] \leqslant \nu(n).$$

# Multiplication Function

- Recall: multiplication function $f_\times : \mathbb{N}^2 \to \mathbb{N}$.

$$f_\times(x,y) = \begin{cases} \perp & \text{if } x = 1 \vee y = 1 \\ x \cdot y & \text{otherwise} \end{cases}$$

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- This is already useful!

- Usually called a **weak** OWF.

# Noticeable Functions

These are functions that are **at most polynomially small**.

**Definition (Noticeable Function)**

A function $\nu(n)$ is noticeable if $\exists c, n_0$ such that $\forall n > n_0$, $\nu(n) \geqslant \frac{1}{n^c}$.

# Weak One Way Functions

**Definition (Weak One Way Function)**

A function $f : \{0,1\}^* \to \{0,1\}^*$ is a *weak one-way function* if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm $\mathcal{C}$ s.t. $\forall x \in \{0,1\}^*$,
$$\Pr\left[\mathcal{C}(x) = f(x)\right] = 1.$$
- **Somewhat hard to invert:** there is a noticeable function $\varepsilon : \mathbb{N} \to \mathbb{R}$ s.t. for every non-uniform PPT $\mathcal{A}$ and $\forall n \in \mathbb{N}$:
$$\Pr\left[x \leftarrow \{0,1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') \neq f(x)\right] \geqslant \varepsilon(n).$$

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- Proof Idea: The fraction of prime numbers between 1 and $2^n$ is noticeable!
- **Chebyshev's theorem**: An $n$ bit number is a prime with probability $\frac{1}{2n}$

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- But if $\Pr[(x, y) \in \mathsf{GOOD}]$ is noticeable, then overall, the adversary can only invert with some bounded noticeable probability.
- Formally: Let $q(n) = 8n^2$. Will show that no non-uniform PPT adversary can invert $f_\times$ with probability greater than $1 - \frac{1}{q(n)}$

# Proof via Reduction

**Goal:** Given an adversary $\mathcal{A}$ that breaks weak one-wayness of $f_\times$ with probability *at least* $1 - \frac{1}{q(n)}$, we will construct an adversary $\mathcal{B}$ that breaks the factoring assumption with noticeable probability

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**factoring Challenger**

$p, q \leftarrow_\$ \Pi_n$

$z := p \cdot q$

$\xrightarrow{\quad z \quad}$

$\mathcal{B}$

$x, y \leftarrow_\$ \{0,1\}^n$

if $x$ and $y$ both primes

$\quad \tilde{z} := z$

else

$\quad \tilde{z} := x \cdot y$

$\xrightarrow{\quad \tilde{z} \quad}$ $\mathcal{A}$ $\xrightarrow{\quad \tilde{\omega} \quad}$

if $x$ and $y$ both primes

$\quad \omega := \tilde{\omega}$

else

$\quad$ output $\perp$

$\xleftarrow{\quad \omega \quad}$

if $\omega \in \{p, q\}$ output 1

# Proof via Reduction

Why did we randomly choose $(x, y)$ instead of passing the input $z$ directly to $\mathcal{A}$?

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The input of $\mathcal{B}$ is a product of two random $n$-bit **primes** while that of $\mathcal{A}$ is the product of two random $n$-bit **numbers**. Passing the input directly to $\mathcal{A}$ would not emulate the distribution of the inputs given to $\mathcal{A}$.

# Analysis of $\mathcal{B}$

- Since $\mathcal{A}$ is non-uniform PPT, so is $\mathcal{B}$ (using polynomial-time primality testing)

$$\begin{aligned}\Pr[\mathcal{B}\text{ fails}] &= \Pr[\mathcal{B}\text{ passes input to }\mathcal{A}]\cdot\Pr[\mathcal{A}\text{ fails to invert }f_\times] \\ &\quad + \Pr[\mathcal{B}\text{ fails to pass input to }\mathcal{A}] \\ &\leqslant \Pr[\mathcal{A}\text{ fails to invert }f_\times] + \Pr[\mathcal{B}\text{ fails to pass input to }\mathcal{A}] \\ &\leqslant \frac{1}{8n^2} + \left(1-\frac{1}{4n^2}\right) \leqslant \left(1-\frac{1}{8n^2}\right)\end{aligned}$$

- $\mathcal{B}$ succeeds with probability at least $\frac{1}{8n^2}$: Contradiction to factoring assumption!

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- **Yao's Hardness Amplification:** YES!

# Weak to Strong OWFs

**Theorem (Yao)**

*Strong OWFs exist if and only weak OWFs exist*

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**Theorem (Yao)**

*Strong OWFs exist if and only weak OWFs exist*

- This is called hardness amplification: convert a somewhat hard problem into a really hard problem
- Intuition: Use the weak OWF *many* times
- Think: Is $f(f(\ldots f(x)))$ a good idea?

# Weak to Strong OWFs

**Theorem**

*For any weak one-way function $f : \{0,1\}^n \to \{0,1\}^n$, there exists a polynomial $N(\cdot)$ s.t. the function $F : \{0,1\}^{n \cdot N(n)} \to \{0,1\}^{n \cdot N(n)}$ defined as*

$$F(x_1, \ldots, x_N(n)) = (f(x_1), \ldots, f(x_N(n)))$$

*is strongly one-way.*

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- In order to successfully invert the new OWF, adversary must invert ALL the $N$ outputs of the weak OWF
- If $N$ is sufficiently large and the inputs are chosen independently at random, then the probability of inverting all of them should be small

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- Nevertheless, it can be shown via a non-trivial proof that hardness does amplify for one-way functions (albeit not all the way to exponentially small inversion probability – there are counterexamples to this!)
- In fact, hardness amplification is not a general phenomenon; for other cases such as interactive arguments (we will study later), hardness does not amplify in general

# Weak to Strong OWFs: Example

- We will show that Yao's hardness amplification works for $f_\times$
- The general case requires a different and careful proof; see lecture notes for details

# Hardness Amplification for $f_\times$

**Theorem**

*Assume the factoring assumption and let $m = 4n^3$. Then, $\mathcal{F} : \left(\{0,1\}^{2n}\right)^m \rightarrow \left(\{0,1\}^{2n}\right)^m$ is a strong OWF:*

$$\mathcal{F}\big((x_1, y_1), \ldots, (x_m, y_m)\big) = \big(f_\times(x_1, y_1), \ldots, f_\times(x_m, y_m)\big).$$

- **Intuition:** Recall that by Chebyshev's Thm, a pair of random $n$-bit numbers are both primes with prob $\frac{1}{4n^2}$

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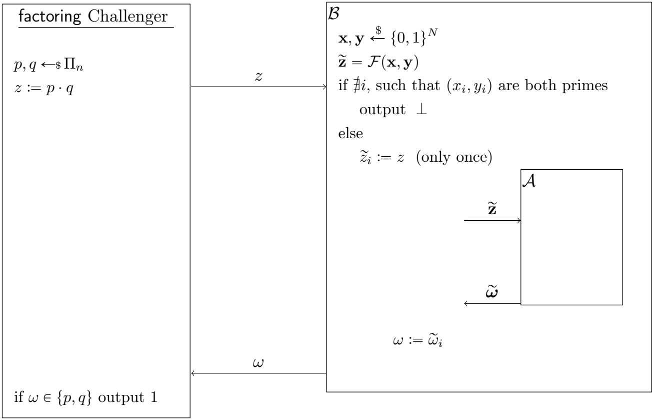
- **Intuition:** Recall that by Chebyshev's Thm, a pair of random $n$-bit numbers are both primes with prob $\frac{1}{4n^2}$
- When we choose $m = 4n^3$ pairs, then the prob that no pair consists of primes is at most $e^{-n}$, which is negligible

# Hardness Amplification for $f_\times$: Proof Details

- Let $N = 2n \cdot 4n^3 = 8n^4$. Let $(\mathbf{x}, \mathbf{y}) = (x_1, y_1), \ldots, (x_m, y_m)$
- Suppose $\mathcal{F}$ is not a strong OWF. Then, $\exists$ a non-uniform PPT adversary $\mathcal{A}$ that inverts $\mathcal{F}$ with prob at least $\varepsilon(2n)$ for some non-negligible function $\varepsilon(\cdot)$
- We will use $\mathcal{A}$ to construct a non-uniform PPT adversary $\mathcal{B}$ that breaks the factoring assumption

# Hardness Amplification for $f_\times$: Reduction



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- Overall, $\mathcal{B}$ fails with prob at most $(1 - \varepsilon(2n)) + e^{-n} < (1 - \frac{\varepsilon(2n)}{2})$
- Thus, $\mathcal{B}$ succeeds with prob at least $\frac{\varepsilon(2n)}{2}$, which is a contradiction to the factoring assumption.