Pseudorandomness (IV)

601.642/442: Modern Cryptography

Fall 2020

Pseudorandomness: Next-Bit Test

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- **Next Bit Test**: for a truly random sequence of bits, it is not possible to predict the "next bit" in the sequence with probability better than 1/2 even given all previous bits of the sequence so far
- A sequence of bits *passes the next bit test* if no efficient adversary can predict "the next bit" in the sequence with probability better than 1/2 even given all previous bits of the sequence so far

Next-bit Unpredictability

Definition (Next-bit Unpredictability)

An ensemble of distributions $\{X_n\}$ over $\{0,1\}^{\ell(n)}$ is next-bit unpredictable if, for all $0 \leqslant i < \ell(n)$ and n.u. PPT $\mathcal{A}$, $\exists$ negligible function $\nu(\cdot)$ s.t.:

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Theorem (Completeness of Next-bit Test)

If $\{X_n\}$ is next-bit unpredictable then $\{X_n\}$ is pseudorandom.

Next-bit Unpredictability $\implies$ Pseudorandomness

$$H_n^{(i)} := \left\{ x \leftarrow X_n, u \leftarrow U_n : x_1 \ldots x_i u_{i+1} \ldots u_{\ell(n)} \right\}$$

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- Exercise: Do the full formal proof

PRG with 1-bit stretch

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    $\Rightarrow f(s)$ is uniformly distributed

PRG with 1-bit stretch

- Let $f : \{0,1\}^* \rightarrow \{0,1\}^*$ be a **OWP**
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Theorem (PRG based on OWP)

G is a pseudorandom generator with 1-bit stretch.

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- Proof Idea: Use next-bit unpredictability. Since first n bits of the output are uniformly distributed (since f is a permutation), any adversary for next-bit unpredictability with non-negligible advantage $\frac{1}{p(n)}$ must be predicting the $(n+1)$th bit with advantage $\frac{1}{p(n)}$. Build an adversary for hard-core predicate to get a contradiction.

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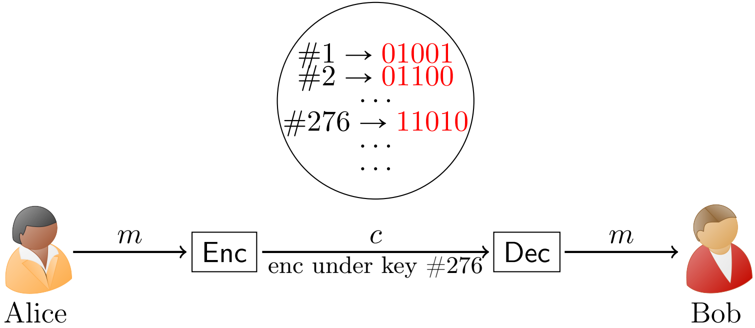
Idea: Functions that index exponentially long pseudorandom strings

Motivation

- Imagine if Alice and Bob had an exponential amount of shared randomness - not just a short key.
- They could split it up into λ-bit chunks and use each one as a one-time pad whenever they want to send an encrypted message of length λ.



- Although Alice publicly announces which location/chunk was used as each OTP key, Eve doesn't know the value at that location.

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- The size of the table for $F = 2^n \times n = n2^n$
- Total number of functions mapping n bits to n bits $= 2^{n2^n}$

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- M's output distribution identical to that of F.

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  - but actually needs much fewer bits to describe/store/query?

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- **Idea**: D can only query the function on inputs of its choice, and see the output.

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- PRF Security: No D can win with probability better than $1/2$.

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$$|\Pr[D \text{ wins } \mathsf{GuessGame}] - 1/2| \leqslant \nu(n).$$

where $\mathsf{GuessGame}$ is defined below

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Remark: note that for any b only one of the two functions is ever used.

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- **Simpler problem:** build PRF for just 1-bit inputs using PRG

From PRG to PRF with 1-bit input

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  - For general case: Apply the "double and choose" idea repeatedly!

PRF from PRG

Theorem (Goldreich-Goldwasser-Micali (GGM))

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- Construction for n-bit inputs $x = x_1 x_2 \ldots x_n$

$$F_k(x) = G_{x_n}\big(G_{x_{n-1}}\big(\ldots\big(G_{x_1}(k)\big)_{..}\big)$$

PRF from PRG (contd.)

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- At level ℓ, 2^ℓ nodes, one for each path, denoted by $k_{x_1 \ldots x_\ell}$

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- **Observation:** Efficient adversary can only make polynomial queries
- Thus, only need to change polynomial number of nodes in the tree

Proof Strategy (contd.)

Two layers of hybrids:

- First, define hybrids over the n levels in the tree. For every i, H_i is such that the nodes up to level i are random, but the nodes below are pseudorandom.

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- Now, hybrid over the nodes in level $i+1$ that are "affected" by adversary's queries, replacing each node one by one with random
- Use hybrid lemma again to identify one node that is changed from pseudorandom to random and break PRG's security to get a contradiction

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- Will use two key points to ensure this:
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- Think: Formal proof?

Concluding Remarks

- **Efficient PRFs from concrete assumptions:**
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- **Related-key Security:** Evaluation of $F_s(x)$ does not help in predicting $F_{s'}(x)$ [Bellare-Cash10]
- **Key-homomorphic PRFs:** Given $f_s(x)$ and $f_{s'}(x)$, compute $f_{g(s,s')}(x)$ [Boneh-Lewi-Montgomery-Raghunathan13]