# Crash of '87 - Was it Expected?
# Aggregate Market Fears and Long Range Dependence

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**Abstract**

We develop a dynamic framework to identify aggregate market fears ahead of a major market crash through the skewness premium of European options. Our methodology is based on measuring the distribution of a skewness premium through a $q$-Gaussian density and a maximum entropy principle. Our findings indicate that the October 19th, 1987 crash was predictable from the study of the skewness premium of deepest out-of-the-money options about two months prior to the crash.

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## 1.   Introduction

At a given point in time, the sentiment of a financial market can be summarized through the aggregation of the subjective beliefs of its participants, namely, aggregate market expectations. If each participant acts independently with one's own belief, there would be a wide dispersion of beliefs. If however, market participants have a highly dependent set of beliefs, this leads to a smaller belief dispersion and less independence in the direction of individual trading decisions.

The presence of independence necessitates the law of large numbers and the population mean of beliefs can be captured accurately under such settings. In the presence of cascading (lack of independence between beliefs), and therefore belief clustering, the independence between trader beliefs fades away, the dispersion becomes much smaller and the aggregate average beliefs diverge much further away from the population fundamentals. Both crashes and bubbles may be interpreted as examples of the lack of belief heterogeneity and its lack of dispersion. In such extreme market conditions, the majority of the market participants take one side of the market which leads to large changes in return and high volatility. Therefore, lack of belief heterogeneity necessitates large market movements and high volatility.[1]

Financial stability, therefore, requires belief heterogeneity, even when those beliefs are formed in a bounded sense. In this paper, we examine the degree of belief heterogeneity of the S&P-500 market participants before the crash of October 19th, 1987. Our approach is to measure the belief heterogeneity through an entropic measure of dispersion.[2] It is a bounded measure and therefore the lower bound refers to a total lack of belief dispersion whereas the upper bound is the maximum dispersion at a point in time.

Our approach can also be related to market microstructure[3] in the sense that we deal with dispersed information and investigate how it is aggregated and reflected in the price. However, in contrast to market microstructure, we are not interested in market institutions (mechanisms) and order flows, but attempt to understand how shifts in the heterogeneity aspect of the private/public information can be directly linked to price.

Our entropic measure of market belief dispersion is calculated from option prices, or, more specifically, from the percentage deviation of call and put prices. We argue that option

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[1]The transition from belief heterogeneity to belief homogeneity leads to sequences of high volatility, or volatility clustering. Obviously, this may happen either in an upward or downward market. This clustering property was first noted in Mandelbrot (1963) in his study of cotton prices and in the long memory in Mandelbrot (1971). These findings remained dormant until the early 1980s for volatility clustering until Engle (1982) and Bollerslev (1986) proposed the ARCH and GARCH processes.

[2]Gell-Mann and Tsallis (2004) has an extensive survey of this methodology and its interdisciplinary applications.

[3]See, for instance, Lyons (2001).

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prices can offer insight into the aggregate market expectations. For instance, if the prices of deep out-of-the-money put options are relatively large compared to deep out-of-the-money call options, this may imply that the market expects a large downward movement in the price of the underlying. Similarly, pricing deep out-of-the-money puts significantly below deep out-of-the-money calls can be understood as an expectation of an upward movement. Bates (1991, 2000) finds that out-of-the-money American put options on S&P-500 Index futures were unusually expensive relative to out-of-the-money calls before the October 1987 crash.[4] Surprisingly, the relative prices of puts peaked in August 1987 and then returned to "normal" levels where they stayed until the crash. Bates (1991) concludes that "if there was a rational bubble in the stock market, one would have to conclude that it burst in mid-August, not in mid-October." Therefore, market expectations (reflected in American option prices) immediately preceding the crash did not predict any unusual shifts. Noteworthy, to investigate the issue further, we do not study option prices directly as in Bates (1991), but use them to calculate a time-dependent entropic measure that can potentially capture long-range, time-dependent, aggregate market expectations in the S&P-500 market. Also, we utilize both transactions data for American options on S&P-500 Index futures contracts[5] as well as daily S&P-500 Index European options.[6] In a related work, Grech and Mazur (2004) analyze the behavior of the Hurst exponent for the Dow Jones Industrial Average Index prior to the crash of 1987 and find that changes in the "excitation state" of the market were to some extent predictable.

From a methodological perspective, our so-called time-dependent entropic measure, Tsallis (1988) entropy, that is derived from a time-series of put and call option prices presents a new empirical technique for the analysis of dynamic predictive ability in finance.[7] By monitoring a time-dependent entropy, one can gain insight into the evolution of the belief heterogeneity of market participants and obtain an early indication of upcoming crises or bubbles. We focus on two sets of underlying signals for both American (hourly data) and European (daily data) options: 1) average percentage deviation of call prices from put prices[8], and 2) percentage deviation of call prices from put prices for the deepest out-of-the-money options. The results for the American options in general, corroborate the evidence by Bates (1991) where there are sharp declines in the entropy level in June-August

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[4]October 1986 - February 1987 and June - August 1987.

[5]We thank David S. Bates for sharing his data with us.

[6]This is the same data set that was used in Garcia and Gençay (2000). Bates (1991) also argues that European options may shed more light on the size of the percentage deviation of call prices from put prices before the crash.

[7]Stutzer (2000) and Stutzer and Kitamura (2002) study option and asset pricing through an entropic methodology.

[8]The call and put prices are matched in terms of the time to maturity and strike price.

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1987 and immediately following the crash.

European options generate more striking findings by identifying strong "abnormal" shifts in the S&P-500 market participants' beliefs prior to the crash. More precisely, when the average percent deviation is utilized, we find that the entropy declines[9] from a relatively stable level of 0.76 to 0.28 on October 16, 1987. Similarly, for the deepest out-of-the-money options, the entropy drops from 0.92 to 0.02 on August 25, 1987 and remains at a low level until the day of the crash.

This paper is organized as follows. In the next section, we motivate our approach and present a statistical framework for the dynamic entropy. Section 3 explains the implementation of the time-dependent entropy. Section 4 explains the link between market expectations and options, and presents our findings. We conclude afterwards.

## 2.   Distribution of asset returns and $q$-Gaussianity

The principle reason for the work on asset pricing is to model the underlying distribution of returns (and volatility) and generate testable hypotheses along with reliable forecasts. The distribution of returns is therefore one of the most fundamental properties of markets and its functional form is still a topic of active debate. On one side, central-limit theorem arguments suggest Gaussian distribution. While the normal distribution provides a good approximation for the center of the distribution for low frequency series, there are substantial deviations from normality for higher frequencies.

Almost half a century ago, Mandelbrot (1963) and Fama (1965) presented evidence in favor of a stable Lévy distribution. The stable Lévy distributions emerge from a generalization of a central limit theorem for random variables in which their second moment does not exist. The Lévy distributions are characterized by a parameter $1 \leq \alpha \leq 2$ where $\alpha = 2$ corresponds to a special case of a normal distribution. For $\alpha < 2$, however, the stable Lévy distributions are so fat-tailed that their second and all higher moments are infinite. Based on daily prices in different markets, Mandelbrot (1963) and Fama (1965) measured $\alpha \approx 1.7$, suggesting that the second moment of the return distribution is not finite. More recent studies, such as Dacorogna *et al.* (2001) disputed this finding subject to large returns asymptotically following a power law behavior. With a tail index value of $\alpha > 2$, the second moment is well-defined and this is incompatible with the Lévy distribution. This indicates that generalization of the central limit theorem with long tails may not be the correct explanation.

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[9]The entropy is bounded between [0,5/3].

Examination of extensive high-frequency data sets, as carried out in Dacorogna *et al.* (2001), indicate $\alpha \approx 3$. Thus, the mean and variance are well-defined, the kurtosis clearly diverges, and the behavior of the skewness is at the margin. There is no overwhelming evidence that the return distribution is from a stable class and the fact that the distribution's shape changes with the time scale makes it clear that the random process underlying prices must have a nontrivial temporal structure.

We will motivate our approach of modeling aggregate belief heterogeneity (aggregate market expectations) from an entropic perspective, relating how the number of states (or regimes) in a market translate themselves into a probability distribution of the aggregate market sentiment. One well-known entropy is the Shannon information measure ($S_S$):

$$
\begin{aligned}
S_S(f(x)) &= \int f(x) \ln\left[\frac{1}{f(x)}\right] dx \\
&= -\int f(x) \ln\left[f(x)\right] dx,
\end{aligned}
\tag{1}
$$

or, in discrete setting $S_S$ is,

$$
S_S = -\sum_{i=1}^{n} p_i \ln p_i, \qquad \sum_{i=1}^{n} p_i = 1
\tag{2}
$$

where the number of states $i = 1, \ldots, n$, $p_i$ is the probability of outcome $i$, and $n$ is the number of states. Namely, the entropy is the sum over the product of the probability of outcome ($p_i$) times the logarithm of the inverse of $p_i$. This is also called $i$'s surprisal and the entropy of $x$ is the expected value of its outcome's surprisal. It is worthwhile to note that if two states $A$ and $B$ are independent from one another, $p(A \cup B) = p(A)p(B)$, then $S_S$ is additive $S_S(A \cup B) = S_S(A) + S_S(B)$.

Tsallis (1988) entropy ($S_q$) is a generalization to a non-additive measure

$$
S_q(f(x)) = \frac{1 - \int f(x)^q dx}{q - 1}
\tag{3}
$$

where $q$ is a measure of non-additivity such that $S_q(A \cup B) = S_q(A) + S_q(B) - (1 - q)S_q(A)S_q(B)$. Larger values for $q$ emphasize long-range interactions between regimes (states) and can be interpreted as a long memory parameter. Tsallis entropy recovers the Shannon entropy when $q \to 1$ such that $\lim_{q \to 1} S_q = S_S$.

The maximum entropy principle for Tsallis entropy under the constraints

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$$\int f(x)dx = 1, \quad \frac{\int x^2 f(x)^q dx}{\int f(y)^q dy} = \sigma^2 \tag{4}$$

yields[10] the $q$-Gaussian probability density function

$$f(x) = \frac{\exp_q(-\beta_q x^2)}{\int \exp_q(-\beta_q x^2)dx} \propto [1 + (1-q)(-\beta_q x^2)]^{\frac{1}{1-q}} \tag{5}$$

where $\beta_q$ is a function of $q$ and $\exp_q(x)$ is the $q$-exponential function defined by

$$\exp_q(x) = \begin{cases} [1+(1-q)x]^{\frac{1}{1-q}} & \text{if } 1+(1-q)x > 0 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

For $q \to 1$, $q$-Gaussian distribution[11] Equation (5) recovers the usual Gaussian distribution.

To illustrate the empirical relevance of the $q$-Gaussian probability distribution we will estimate daily, weekly and monthly S&P-500 Index returns for the period 1990-2000 with the $q$-Gaussian density. We first define returns as

$$R_t = \ln S_t - \ln S_{t-1} \tag{7}$$

where $S$ denotes the S&P-500 Index over daily, weekly and monthly time intervals.

We then define normalized returns as

$$r_t = \frac{R_t - \mu_R}{\sigma_R} \tag{8}$$

where $\mu_R$ denotes the mean of $R_t$ and $\sigma_R$ is the standard deviation of $R_t$.

Figure 1 displays empirical histograms for the daily (circles), weekly (triangles) and monthly (squares) normalized returns, together with the estimated $q$-Gaussian distribution. Clearly, the plain vanilla Gaussian distribution is unable to approximate the tails at all three time scales. However, the fit of the $q$-Gaussian probability distribution captures the frequency of extreme events together with ordinary frequencies satisfactorily. The estimation involves the optimal $q$ estimation which minimizes the sum of the squared errors of the logarithms of the $q$-Gaussian probability density and the data-implied empirical density. It is found that the optimal $q$=1.62 (0.03) for the daily data, $q$=1.64 (0.02) for the weekly

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[10] See, for instance, Suyari (2006).

[11] For a zero mean process with unitary variance $\beta_q = 1/(5 - 3q)$ with $q < 5/3$.

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data, and $q$=1.63 (0.01) for the monthly data.[12] Noteworthy, the $q$-Gaussian probability distribution describes the returns measured over three different time intervals well and $q$ values are stable across three data frequencies.

## 3. Implementation

Implementation of $S_q$ for a time series of observations $x_t, t = 1, \ldots, T$ involves the following steps.[13] Define a moving window $(X)$ with $K$ observations

$$X = \{x_{t,k}, k = 1, \ldots, K\} \tag{9}$$

to calculate the underlying discrete probability distribution by an equipartitioning of $X$ into $n$ states, $x_0 < x_1 < \ldots < x_n$. Here, $x_0 = \min[X]$ and $x_n = \max[X]$. Next, let us introduce the set $\{I_i = [x_{i-1}, x_i), i = 1, \ldots, n\}$ of disjoint intervals such that

$$D = \bigcup_{i=1}^{n} I_i \tag{10}$$

where $D = x_n - x_0$ is the range of X.

Let $p_i$ be the probability that $x_t$ belongs to the interval $I_i$. $p_i$ is defined as the ratio between the number of observations found within $I_i$ and the total number of observations $(K)$. Then, $S_q$ in its discrete version can be written as

$$S_q = \frac{1 - \sum_{i=1}^{n} p_i{}^q}{q - 1}. \tag{11}$$

Through a moving window, the evolution of $S_q$ for $x_t$ is calculated over time. We use the percentage deviation of call from put prices for the underlying signal $x_t$. The calculation of a time-dependent entropy is influenced by the following considerations (Thakor and Tong, 2004):[14]

1. *Number of states.* With too few states, one may not be able to characterize the underlying market sentiment reliably, and with too many states, tracking fine changes in the dispersion of beliefs becomes difficult. Without loss of generality, we set $n$=10.

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[12]The numbers in parentheses are standard errors. We estimate standard deviations through a block bootstrap procedure with 1000 replications.

[13]See Gamero *et al.* (1997) or Tong *et al.* (2002) for more information.

[14]Overviews of non-additive entropy can be found in Gell-Mann and Tsallis (2004) and Anteneodo and Tsallis (2003). Tabak and Cajueiro (2006) use $S_q$ to assess efficiency of major world currencies. Queiros *et al.* (2006) study probability distributions of stock market returns and volumes within the non-additive entropy framework.

2. *Partitioning method.* There are two different methods for partitioning the range of a signal: (a) fixed partitioning (equipartition is performed on all available data) and (b) adaptive partitioning (equipartition is performed on each moving-window of data, i.e., it changes over time). The adaptive partitioning approach can track transient changes in a signal better than the fixed partitioning and is more suitable for our application.

3. *Estimation of q.* The entropic index $q$ is the degree of long-memory in the data. Gell-Mann and Tsallis (2004) estimate $q \approx 1.4$ for high-frequency financial data (returns and volumes) and stress that as the data frequency decreases, $q$ approaches unity. Larger $q$ values $(1 < q \leq 2)$ emphasize highly volatile activities in the signal when a time-dependent entropy is plotted against time, i.e., the entropy is more sensitive to possible disturbances in the probability distribution function. When $q=2$, the expression for $S_q$ simplifies to $S_q = 1 - \sum_{i=1}^{n} p_i{}^2$. In this paper, we find the optimal $q$ for each time series by minimizing the sum of the squared errors of the logarithms of the $q$-Gaussian probability density and the data-implied empirical density.

4. *Sliding step ($\Delta$) and moving window size (K).* The sliding step (the number of observations by which the moving window is shifted forward across time) and moving window size (the number of observations used in calculating the entropy) determine the time resolution of $S_q$. If the focus is on tracking the local changes, the sliding step is set to be very small (e.g., one observation: $\Delta=1$). Non-overlapping windows $(\Delta \geq K)$ are useful only when one is interested in monitoring the general trend of $x_t$. To get a reliable probability distribution function, $K$ should not be too small. We set $\Delta=1$ and $K=120$.

An example of dynamic entropy is illustrated in Figure 2 for a GARCH (1,1) process where $x_t = z_t \sigma_t$, $\sigma_t^2 = c + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$ where $z_t$ is an identically and independently distributed normal random variable with zero mean and unit variance. The entropy is calculated with $q = 2$, $n = 10$, $K = 200$ days and 2,000 observations. The range of entropy is [0, 0.9] for $q = 2$. In the top panel, the volatility persistence is set to be minimal with coefficients $c = 0.5$, $\alpha = 0.05$ and $\beta = 0.05$, and leads to stable entropy across time. The spread of volatility across time does not exhibit large bursts and entropy fluctuates around a tight band of 0.85. The bottom level corresponds to a very persistent volatility with $c = 0.5$, $\alpha = 0.5$ and $\beta = 0.4999$. Outbursts of volatility are evident with 40% high peaks and it is clustered. The entropy falls dramatically in such high volatility periods and the dips in the entropy are evident relative to the top panel. In this example, the drop in the entropy

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in high volatility periods is contemporaneous with almost no lead time for predictability. In our analysis, we will demonstrate that it is possible to predict such high volatile periods with plenty of lead-time using option based indicators and the dynamic entropy principle.

## 4.   Market Expectations and Options

The options are priced at the discounted expected value of their future payoffs using a risk-neural distribution. An equivalent risk-neural distribution is derived by using the actual distribution of the underlying asset price and summarizes the prices of Arrow-Debreu state-contingent claims.

In a broader context, option prices represent the market's aggregate expectations. Hence, a set of put and call option prices can reveal an aggregate view of the market price at a given maturity. For instance, if the prices of deep out-of-the-money put options on S&P-500 are relatively large compared to deep out-of-the-money call options, this may imply that the market expects a large downward movement in the price of the underlying. Similarly, pricing deep out-of-the-money puts significantly below deep out-of-the-money calls can be understood as an expectation of an upward movement.

Bates (1991, 2000) investigated the S&P-500 Index options on futures to gather evidence on whether the October 1987 crash was expected. He documented that from October 1986 to August 1987, the market experienced a 42% upsurge, but gradually declined until the crash. This was also reflected in the percentage deviation of put prices from call prices, named as the *skewness premium*:[15]

$$x \equiv \frac{P(S, T, K_c)}{C(S, T, K_p)} - 1 \tag{12}$$

where $S$ is the price of the underlying (S&P-500 Index for European options or S&P-500 futures price for American options), $T$ is maturity, $K_c$ and $K_p$ are strike prices for call and put options, respectively.

Bates (1991) found that although the skewness premium[16] was negative from October 1986 until August 1987, it returned to normal levels two months prior to the October 1987 crash. The strong indication of downside risk peaked in August 1987 when it was about -25% for the out-of-the-money options (i.e., put options were 25% more expensive than call options), but returned to "normal" levels where it stayed until the crash. Secondly, Bates

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[15]We were inspired by Bates (1991), but our definition of the skewness premium is slightly different and we use it for all available American and European options (not just for the 0-4% out-of-the-money options).

[16]Bates (1991) used $[C(S, T, K_c)/P(S, T, K_p)] - 1$.

(1991) fitted a jump-diffusion model to option prices to capture the subjective probability distribution of S&P-500 futures implicit in their call and put option prices. He noticed that the implicit distribution became negatively skewed in October 1986, as the S&P-500 futures price started increasing, reaching the negative skewness peak in the June-August 1987 period. His conclusion was that although the October 19, 1987 crash had been expected months prior it actually took place, its timing was not predictable because market indicators returned to normal levels two months prior to the crash. Bates (1991) notes that there were no fears of a market crash even late on Friday afternoon of October 16th. These findings may lend to the interpretation that the crash was expected to take place in August 1987 as a self-fulfilling prophecy (or a "rational bubble").

Rappoport and White (1994) examined whether the crash of 1929 was also expected. As opposed to Bates (1991), they assessed crash fears through return volatilities implied by the option (approximated by brokers' loans) prices calculated using the Black-Scholes and the knockout option pricing models. Their evidence shows that implied volatilities rose sharply over a year prior to the crash, i.e., crash fears increased until the moment of the crash. It is noteworthy that Rappoport and White (1994) and Bates (1991) are consistent with each other in the sense that an increase in the market's expectation of a downward movement in prices coincides with a booming market.

Our study is a novel refinement of this literature of the predictability of a crash through an entropic analysis of market heterogeneity embedded in option prices. Since the Bates (1991) framework loses its predictive power two months prior to the October 1987 crash, our contribution will be to investigate early warning signals which remain strong until the day of the crash.

We will calculate the time-dependent entropy with two types of skewness premium measures. The first one is the average skewness premium for options with the same time-to-maturity and strike price. The second one is the skewness premium for the deepest available out-of-the-money option pairs.

## 4.1 Bates (1991) Data - American Options

The asymmetry of the skewness premium, as a proxy for the aggregate market expectations, is used to calculate the time-dependent entropy for hourly American options for 1987. More specifically, the Bates (1991) options data is used to first calculate the average and the deepest out-of-the-money skewness premia. Subsequently, the non-additive entropy is maximized to estimate the probability distribution of the skewness premium to estimate $q$ (long-memory parameter) for the $q$-Gaussian distribution. We minimize the sum of the squared errors between the $q$-Gaussian probability density and the empirical density to es-

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timate $q$. For the average skewness premium, the long-memory parameter is $\hat{q} = 1.61$ (0.12) and the one for the deepest out-of-the-money option pairs is $\hat{q} = 1.60$ (0.14) where the bootstrap standard errors are in parentheses. The entropy is bounded between [0, 1.25] for $q = 1.60$ where the lower boundary implies belief homogeneity of market participants. On the other hand, the upper boundary indicates the maximum dispersion of beliefs amongst the market participants.

An early substantive shift in the aggregate beliefs of the market participants before the crash of 1987, namely, a sharp decrease in the time-dependent entropy, is an early warning of the crash. If the entropy stays at such low levels until the time of the crash, this would represent evidence for the upcoming crash. Figure 3 (top panel) presents the entropy with the hourly average skewness premium (for the American options) before the crash.[17] On May 27th, the entropy drops to 0.78 and recovers to 0.84 on September 8th. On October 16th, the entropy value is 0.98. On October 19th, it drops to 0.77 by the mid-day trading and to 0.41 in the last hour of the trading. On October 20th, the entropy starts at 0.22 and drops to 0.05 in the last hour of the trading.

The informative content of the skewness premium for the deepest out-of-the-money options is somewhat stronger (Figure 3, bottom panel). Although there was a sharp decrease in the entropy between May 1987 - August 1987, this does not constitute a clear warning signal because the entropy recovers in the last hours of the trading on Friday before the crash. On May 19th, the entropy is at 0.46 and goes to 0.51 on August 26th. On October 16th, it starts at 0.58, dropping to 0.48 in the last hour of the trading. On the day of the crash it falls from 0.37 to 0.01 in the last hours of the trading and stays in the range of 0.01 - 0.02 on October 20th.

Overall, our findings for the average skewness premium with American options corroborate Bates (1991) where the crash fears were not strong immediately before the crash. In addition, the deepest out-of-the-money American options provide slightly more useful early information about the crash. Since deep out-of-the money European options at a given maturity are more informative relative to the American options about market expectations, the next section extends the analysis to the skewness premium of the European options.

## 4.2 European Options

In this part of the paper, we analyse the daily S&P-500 Index European put and call option prices from the Chicago Board Options Exchange. The data set[18] contains options across

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[17]The size of the moving window is $K$=500 hours ($\approx$ 60 days). Changing the size of the moving window does not change the main message of our findings.

[18]This is the data set used in Garcia and Gençay (2000).

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different strike prices and maturities over the period from January 1987 to December 1988. The estimate of the long-memory parameter $(\hat{q})$ for the daily average skewness premium and the skewness premium for the deepest out-of-the-money options are 1.51 $(0.05)$[19] and 1.53 $(0.04)$, respectively.

First, we study the entropy with the daily S&P-500 Index without use of any options market information.[20] The index itself does not provide any advance warnings and there is no significant change in the entropy level prior to the 1987 crash (Figure 4). More precisely, the entropy declined by about 17% on the day of the crash and subsequently returned to the pre-crash levels of March - April 1988. Table 1 illustrates how the probabilities of the states which were roughly evenly distributed on October 14, 1987, converged to states $s_6 - s_{10}$ on October 19, 1987. This indicates the lack of belief heterogeneity during and following the crash, and early warning signals are not prevalent when the entropy is based on the S&P-500 Index.

Consequently, we extend the set of underlying signals to potentially more useful two variants of the daily skewness premium. As the top panel of Figure 5 indicates, for the average skewness premium, there appears to be no strong indication of crash fears until a few days prior to the crash. On October 15th, the entropy is at 0.76, drops to 0.28 on October 16th, and to 0.06 on the day of the crash. From the study of the state probabilities it is apparent that they are not evenly distributed and occupy mostly states $s_1 - s_3$ before the crash (Table 2). Furthermore, the most striking result is the convergence of the probabilities to state $s_1$ on October 16, 1987. This is also reflected in the entropy that drops by more than 63% one day before the crash.

Finally, we investigate the evolution of the entropy based on the skewness premium for the deepest out-of-the-money options. These findings are the most striking where an early indication of the lack of belief dispersion surfaced on August 25, 1987 (Figure 5, bottom panel), two months prior to the crash. Moreover, after this substantial decrease from 0.92 to 0.02, the entropy remained at a relatively stable level of 0.27 until the time of the crash and only going down to 0.26 on the day of the crash. The distribution of the probabilities for the states also converged to state $s_1$ on August 25, 1987 (Table 3).[21] In this particular case, the deepest out-of-the-money options are more informative regarding the extreme expectations, and the entropy provides a useful platform to measure the concentration of

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[19]The numbers in parentheses are the bootstrap standard errors. We use one leave-out bootstrap with replacement for a window size of $K$ observations.

[20]The moving window is $K$=120 days, $\hat{q}$=1.62 which was estimated in Section 2, and the number of states is $n$=10.

[21]It is important to note that our results are not sensitive to various reasonable choices for the size of the moving window $K$ (40-140 days) and the number of states $n$ (7-10).

these expectations in a particular direction.

## 5. Conclusions

By monitoring a time-dependent entropy, one can gain insight into the evolution of the aggregate market expectations and obtain an early indication of upcoming crises or bubbles. We focus on two sets of underlying signals for both American (hourly data) and European (daily data) options. The results for the American options in general corroborate the evidence by Bates (1991) with sharp declines in the entropy immediately following the crash. In the case of American options, the skewness of the corresponding risk-neutral probability distribution is not directly linked to the relative prices of out-of-the-money put and call options. We extend our investigation to European options and this generates more striking findings where we are able to identify strong "abnormal" shifts in the S&P-500 market participants' aggregate beliefs roughly two months prior to the October 1987 crash.

| Date | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ | $s_{10}$ | TE |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|-----|
| Oct 14, 87 | 0.05 | 0.08 | 0.11 | 0.04 | 0.11 | 0.18 | 0.15 | 0.12 | 0.07 | 0.09 | 1.18 |
| Oct 15, 87 | 0.04 | 0.08 | 0.11 | 0.05 | 0.11 | 0.18 | 0.15 | 0.12 | 0.07 | 0.09 | 1.18 |
| Oct 16, 87 | 0.04 | 0.08 | 0.10 | 0.05 | 0.11 | 0.18 | 0.15 | 0.12 | 0.07 | 0.09 | 1.18 |
| Oct 19, 87 | 0.00 | 0.00 | 0.00 | 0.00 | 0.03 | 0.12 | 0.15 | 0.29 | 0.25 | 0.16 | 1.00 |
| Oct 20, 87 | 0.00 | 0.00 | 0.00 | 0.00 | 0.03 | 0.12 | 0.15 | 0.29 | 0.25 | 0.16 | 1.01 |
| Oct 21, 87 | 0.00 | 0.00 | 0.01 | 0.00 | 0.03 | 0.11 | 0.15 | 0.29 | 0.25 | 0.16 | 1.01 |
| Oct 22, 87 | 0.00 | 0.00 | 0.02 | 0.00 | 0.03 | 0.10 | 0.15 | 0.29 | 0.25 | 0.16 | 1.01 |

Table 1: DISTRIBUTION OF BELIEF HETEROGENEITY - PRICE

The time-dependent Tsallis entropy (TE) is calculated with a moving window of 120 days for the S&P-500 Index. $s_1,\ldots,s_{10}$ denote non-overlapping intervals (states). The lower boundary of $s_1$ is the minimum of the moving window. Accordingly, the upper boundary of $s_{10}$ is the maximum of the moving window. Belief probabilities ($p_i$) are calculated from the ratio between the number of observations in each interval and the total number of observations in the moving window. The maximum entropy (belief heterogeneity) corresponds to equal probability of 10% for each state. The minimum entropy (belief homogeneity) occurs when all observations concentrate in one particular state such that one state receives 100% of the probability. In this particular case above, belief distribution is more evenly distributed on October 14, 1987 which becomes more concentrated towards October 22, 1987 in states $s_7, s_8, \ldots, s_{10}$. The increased concentration from October 14th to 22nd leads to a reduction in the entropy. The entropy is bounded between $[0, 1.23]$ for $q = 1.62$.

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| Date | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ | $s_{10}$ | TE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Oct 14, 87 | 0.56 | 0.32 | 0.07 | 0.02 | 0.03 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.75 |
| Oct 15, 87 | 0.56 | 0.31 | 0.07 | 0.03 | 0.03 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.76 |
| Oct 16, 87 | 0.89 | 0.09 | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.28 |
| Oct 19, 87 | 0.98 | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.06 |
| Oct 20, 87 | 0.97 | 0.02 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.09 |
| Oct 21, 87 | 0.95 | 0.01 | 0.02 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.01 | 0.11 |
| Oct 22, 87 | 0.95 | 0.01 | 0.03 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.13 |

Table 2: DISTRIBUTION OF BELIEF HETEROGENEITY - AVERAGE SKEWNESS PREMIUM
The time-dependent Tsallis entropy (TE) is calculated with a moving window of 120 days for the S&P-500 average skewness premium for European options. $s_1,\ldots,s_{10}$ denote non-overlapping intervals (states). The lower boundary of $s_1$ is the minimum of the moving window. Accordingly, the upper boundary of $s_{10}$ is the maximum of the moving window. Belief probabilities ($p_i$) are calculated from the ratio between the number of observations in each interval and the total number of observations in the moving window. The maximum entropy (belief heterogeneity) corresponds to equal probability of 10% for each state. The minimum entropy (belief homogeneity) occurs when all observations concentrate in one particular state such that one state receives 100% of the probability. In this particular case above, belief distribution is concentrated in states $s_1, s_2$ and $s_3$ on October 14, 1987. The concentration increases towards $s_1$ as October 22nd approaches, leading the entropy to fall to 0.13 from 0.75. The entropy is bounded between [0, 1.23] for $q = 1.62$.

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| Date | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ | $s_{10}$ | TE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Aug 20, 87 | 0.53 | 0.19 | 0.06 | 0.04 | 0.04 | 0.03 | 0.00 | 0.07 | 0.01 | 0.03 | 0.92 |
| Aug 21, 87 | 0.53 | 0.19 | 0.05 | 0.04 | 0.04 | 0.03 | 0.00 | 0.07 | 0.02 | 0.03 | 0.91 |
| Aug 24, 87 | 0.53 | 0.20 | 0.04 | 0.04 | 0.04 | 0.03 | 0.00 | 0.07 | 0.02 | 0.03 | 0.92 |
| Aug 25, 87 | 0.99 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.02 |
| Aug 26, 87 | 0.99 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.04 |
| Aug 27, 87 | 0.99 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.06 |
| Sep 16 - Oct 16, 87 | 0.89 | 0.05 | 0.01 | 0.02 | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.27 |
| Oct 19, 87 | 0.89 | 0.05 | 0.01 | 0.03 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.26 |

Table 3: DISTRIBUTION OF BELIEF HETEROGENEITY - DAILY SKEWNESS PREMIUM FOR THE DEEPEST OUT-OF-THE-MONEY EUROPEAN OPTIONS

The time-dependent Tsallis entropy (TE) is calculated with a moving window of 120 days for the S&P-500 average skewness premium. $s_1, \ldots, s_{10}$ denote non-overlapping intervals (states). The lower boundary of $s_1$ is the minimum of the moving window. Accordingly, the upper boundary of $s_{10}$ is the maximum of the moving window. Belief probabilities ($p_i$) are calculated from the ratio between the number of observations in each interval and the total number of observations in the moving window. The maximum entropy (belief hetero-geneity) corresponds to equal probability of 10% for each state. The minimum entropy (belief homogeneity) occurs when all observations concentrate in one particular state such that one state receives 100% of the probability. In this particular case above, belief distribution is spread out in states $s_1$ to $s_{10}$ on August 20, 1987. The concentration increases towards $s_1$ on August 25th, leading the entropy to fall to 0.02 from 0.92. The concentration in $s_1$ remains until the day of the crash. The entropy is bounded between [0, 1.23] for $q = 1.62$.
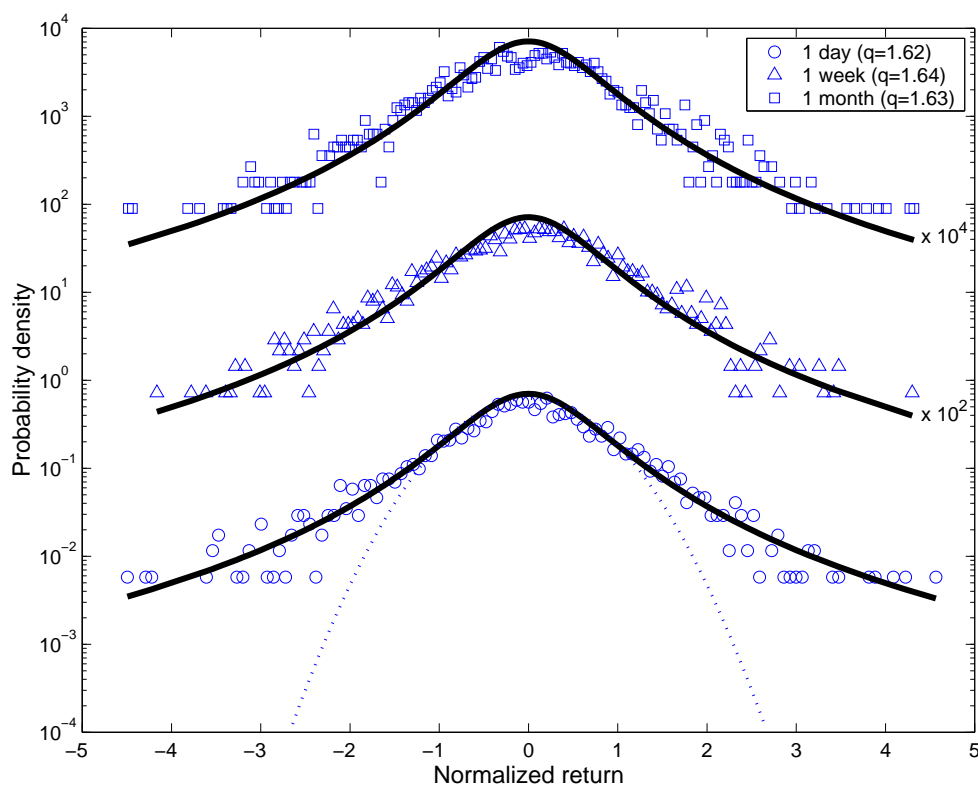
15

Figure 1: EMPIRICAL HISTOGRAMS FOR THE DAILY, WEEKLY AND MONTHLY S&P-500 INDEX NORMALIZED RETURNS (1990-2000)

Normalized returns are plotted on the x-axis and the probability density on the y-axis as a log-linear plot. The estimation of the optimal $q$ involves minimizing the sum of the squared errors of the logarithms of the $q$-Gaussian probability density (solid line) and the data-implied empirical density (circles for the daily, triangles for the weekly, and squares for the monthly data). The curves for the weekly and monthly data have been shifted vertically by $10^2$ and $10^4$, respectively. The dotted line is the Gaussian distribution.

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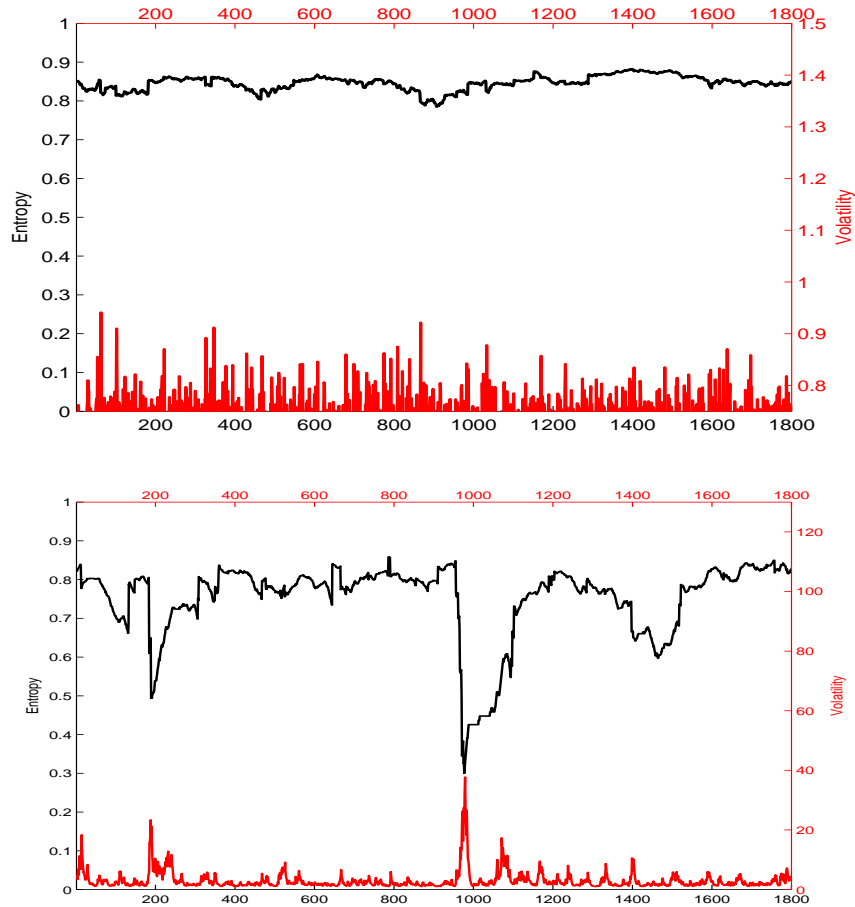
Figure 2: TSALLIS ENTROPY WITH GARCH(1,1) PROCESS

For each panel, entropy values are on the left side of the vertical axis and the corresponding conditional volatility is on the right vertical axis. The GARCH (1,1) process is $x_t = z_t \sigma_t$, $\sigma_t^2 = c + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$ where $z_t$ is an identically and independently distributed normal random variable with zero mean and unit variance. The entropy is calculated with $q = 2$, $n = 10$, $K = 200$ days and 2,000 observations. The range of entropy is $[0, 0.9]$ for $q = 2$. In the top panel, the volatility persistence is set to be minimal with coefficients $c = 0.5$, $\alpha = 0.05$ and $\beta = 0.05$ which leads to stable entropy across time. The spread of volatility across time does not exhibit large bursts and entropy fluctuates around a tight band of 0.85. The bottom level corresponds to a very persistent volatility with $c = 0.5$, $\alpha = 0.5$ and $\beta = 0.4999$. Outbursts of volatility are evident with 40% high peaks and it is clustered. The entropy falls dramatically in such high volatility periods and the dips in the entropy are evident relative to the top panel. In this example, the drop in the entropy in high volatility periods is contemporaneous with almost no lead time for predictability. In subsequent figures, we will demonstrate that it is possible to predict such high volatile periods with plenty of lead-time using option based indicators and the entropy principle of belief heterogeneity.
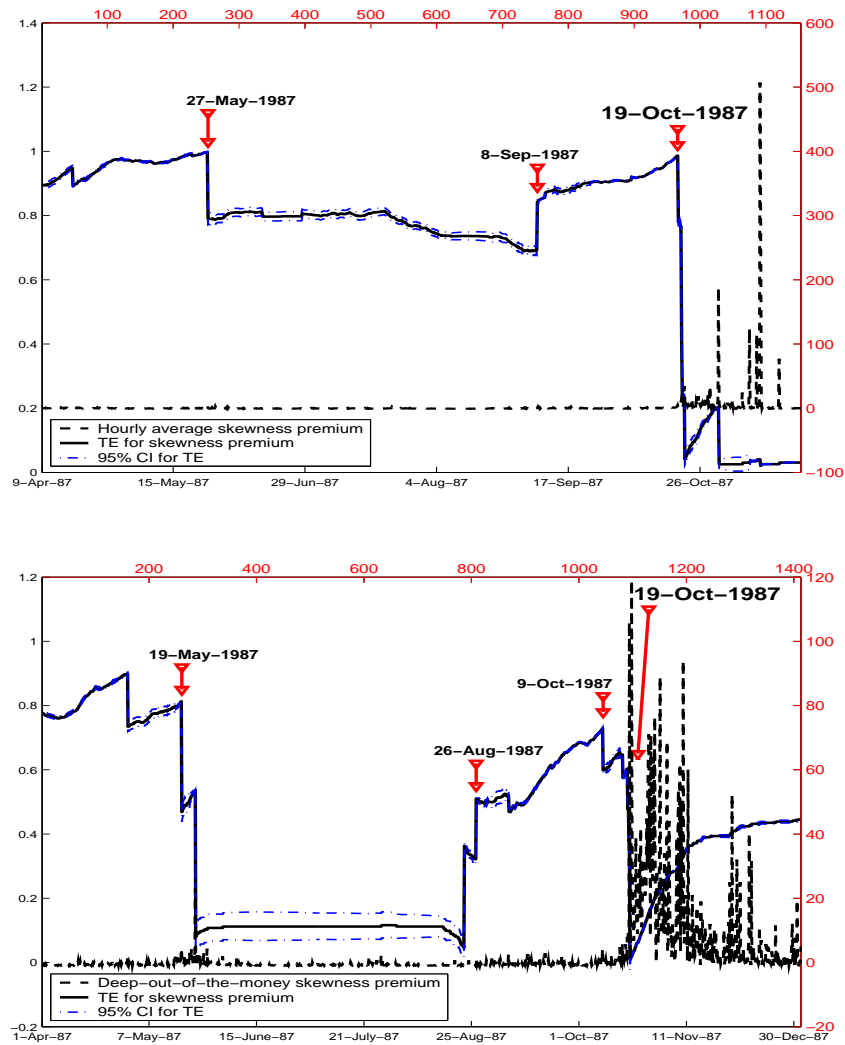
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Figure 3: HOURLY SKEWNESS PREMIUM AND DYNAMIC ENTROPY

For each panel, entropy values are on the left side of the vertical axis and the hourly skewness premium is on the right vertical axis. The size of the moving window is $K$=500 hours ($\approx$ 60 days). In the top panel, the hourly average skewness premium ($x$) is plotted with the dotted line and time-dependent discrete Tsallis entropy ($TE$) for $x$ is the solid line. The entropy dips on May 27, 1987 and recovers on September 8, 1987. In the bottom panel, hourly skewness premium ($x$) for the deepest (available)-out-of-the-money call and put options for 1987 is plotted with the dotted line and time-dependent discrete Tsallis entropy ($TE$) for $x$ is the solid line. The entropy dips on May 20, 1987 and starts recovering on August 31, 1987. There is another sudden small drop a week before the crash.
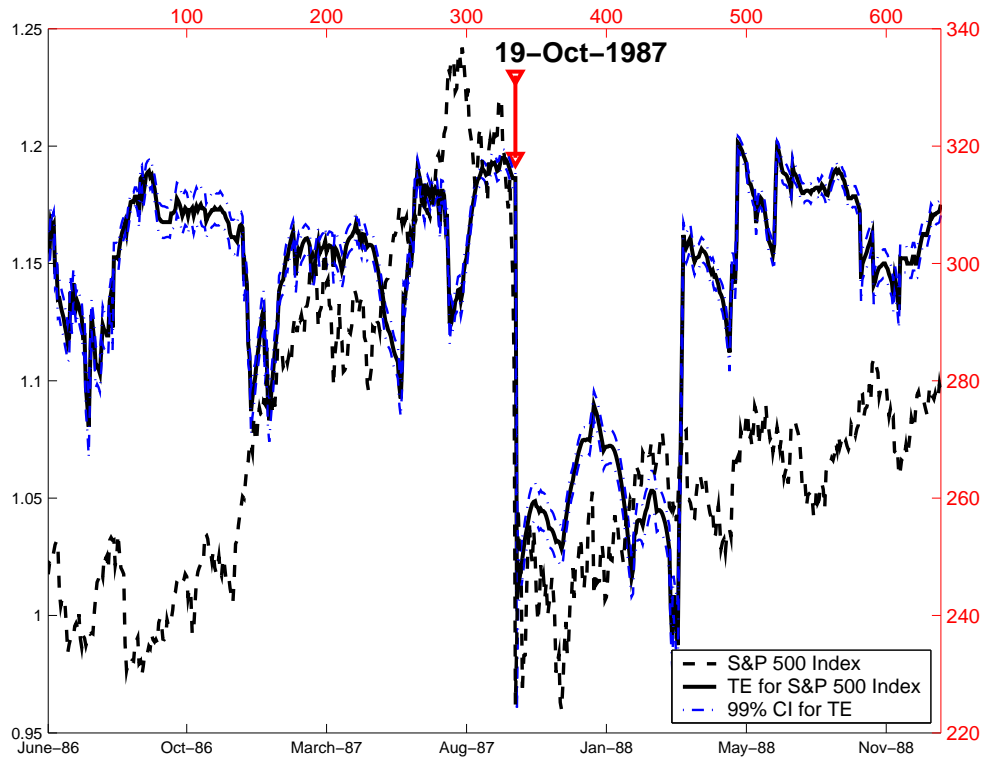
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Figure 4: DAILY S&P-500 INDEX AND DYNAMIC ENTROPY

Tsallis entropy ($TE$) values are on the left side of the vertical axis (solid line) and the daily S&P-500 Index for 1986-1988 is on the right vertical axis (dotted line). The time-dependent, discrete entropy is calculated from the daily S&P-500 Index based on the size of the moving window of $K$=120 days. The entropy does not provide any lead time for predicting the October 19th crash. The sudden drop in the entropy is simultaneous with the drop in the index.
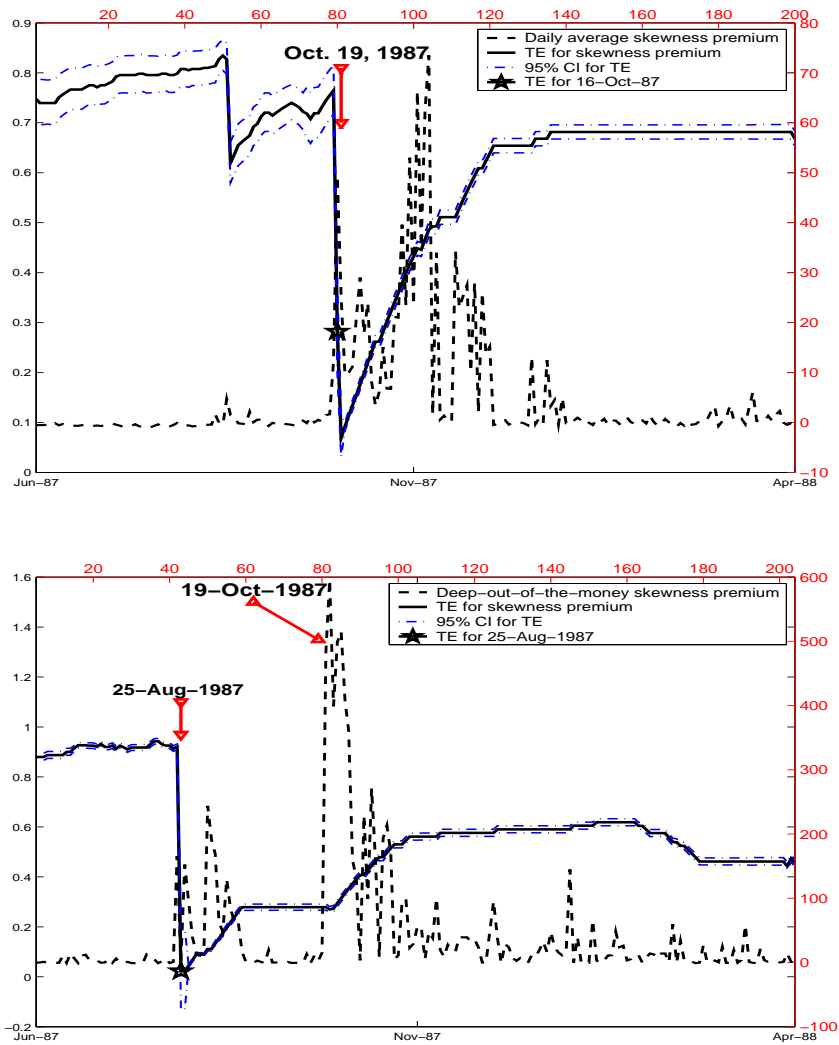
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Figure 5: DAILY SKEWNESS PREMIUM AND DYNAMIC ENTROPY

For each panel, entropy values are on the left side of the vertical axis and the daily skewness premium is on the right vertical axis. The size of the moving window is $K$=120 days. In the top panel, daily average skewness premium $(x)$ is plotted with the dotted line and time-dependent discrete Tsallis entropy $(TE)$ for $x$ is the solid line. The entropy dips on October 16, 1987 and its level on that day is marked with the star. In the bottom panel, daily skewness premium $(x)$ for the deepest (available)-out-of-the-money call and put options is plotted with the dotted line and time-dependent discrete Tsallis entropy $(TE)$ for $x$ is the solid line. The entropy dips on August 25, 1987 and its level on that day is marked with the star.

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