Linear Algebra Review

MLT

Karthik Thiagarajan

Vectors

Vectors

Features

- num of rooms
- size of house
- age of house
- parking facility

Data

ML: "Learning from data"

Vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$\mathbf{x} \in \mathbb{R}^d$

feature-vector

Algebra

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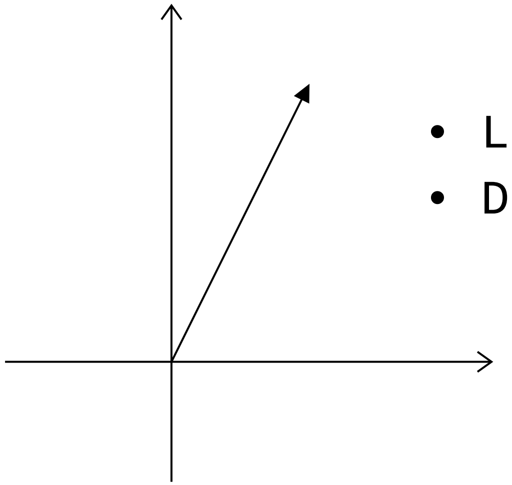
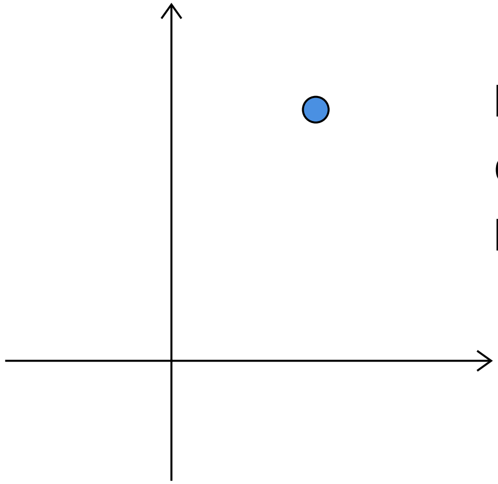


- Length (norm)
- Direction

Geometry



Data-point
or
Point

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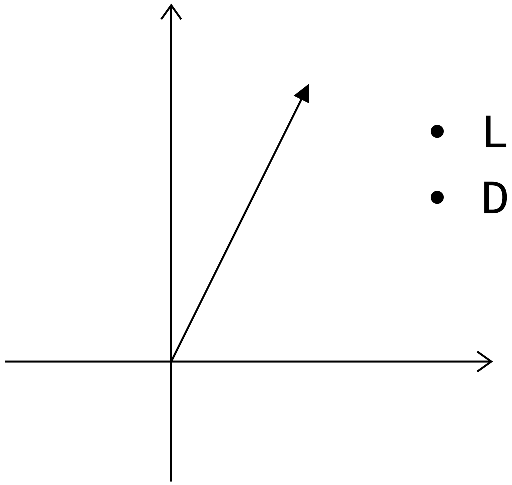

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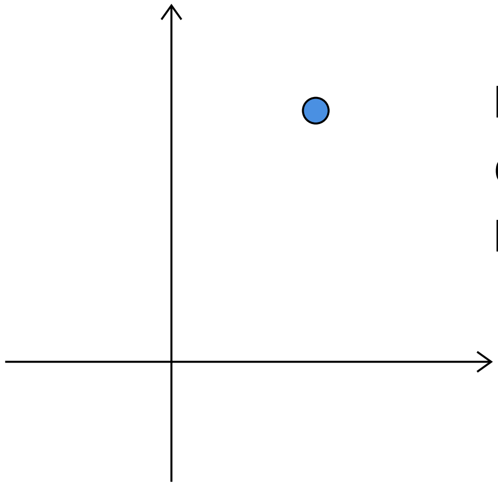

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$\mathbf{x} \in \mathbb{R}^d$

feature-vector

Algebra

[1, 2, 3, 4]

Array
or
List

Computation

$\mathbb{R}^d$: Conventions

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad \mathbf{x}^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

- $\mathbf{x}$ is a column vector by default.
- Its shape is $(d,\ 1)$.
- $\mathbf{x}^T$ is a row-vector.
- This is a convention.

$\mathbb{R}^d$: Conventions

$$D = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad \mathbf{x}^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix}$$

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- $\mathbf{X}$ is a data-matrix.
- Columns of $\mathbf{X}$ are data-points.
- Rows are the features.
- Its shape is (d, n).
- This is a convention.

$\mathbb{R}^d$: Basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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  - not unique
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- representation in a given basis
  - unique

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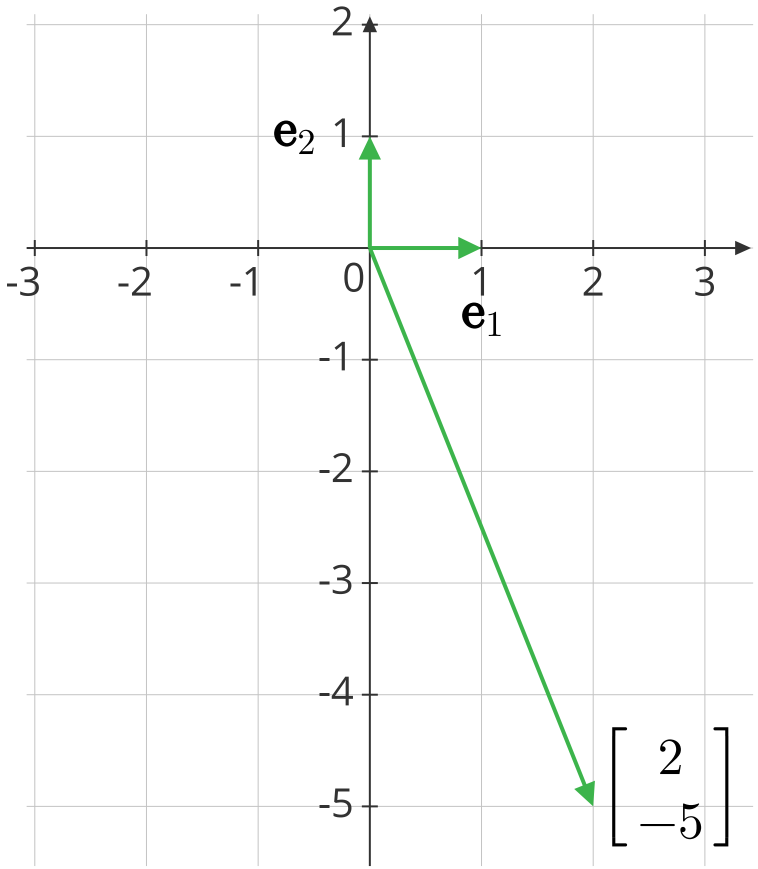

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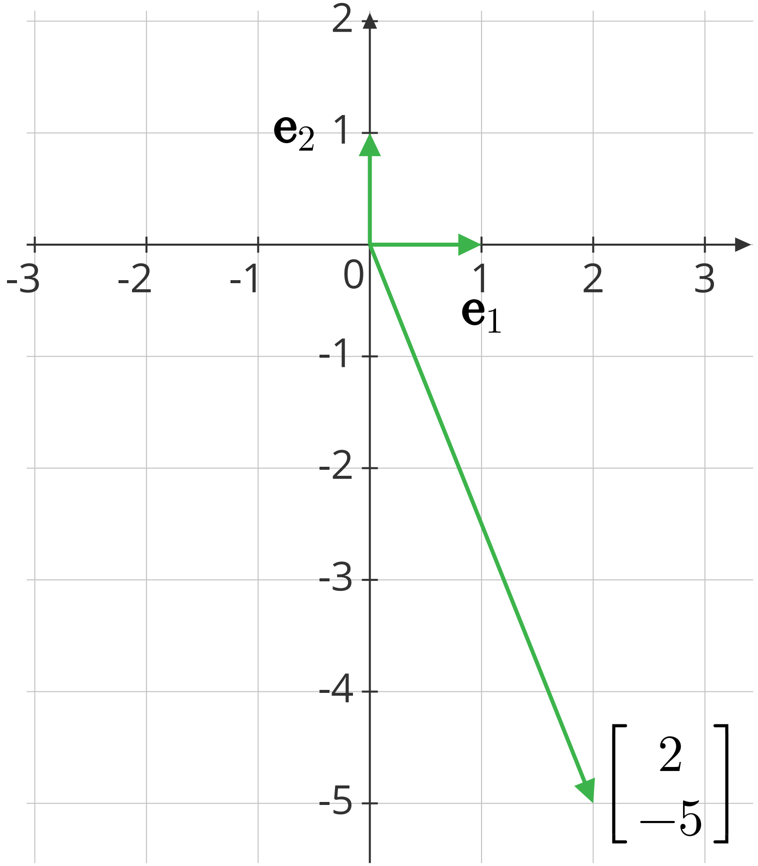

$$\begin{bmatrix} 2 \\ -5 \end{bmatrix} = 2 \cdot \mathbf{e_1} - 5 \cdot \mathbf{e_2}$$

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$\mathbb{R}^d$: Dot Product

Geometry:

- Lengths
- Angles (directions)

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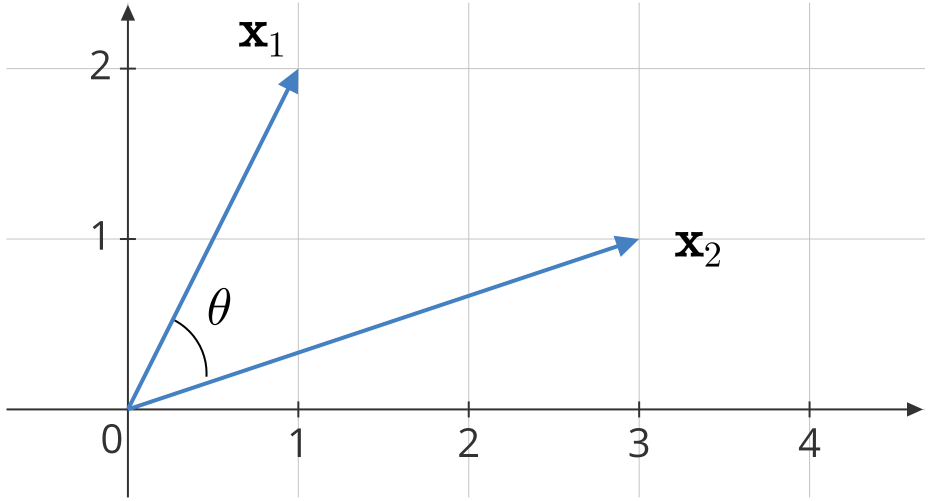

$\mathbb{R}^d$: Dot Product

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

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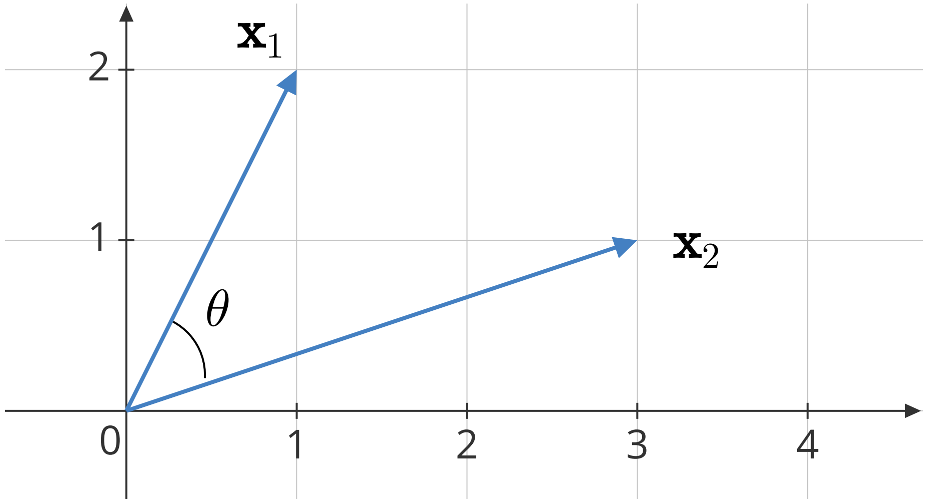

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Geometry:

- Lengths
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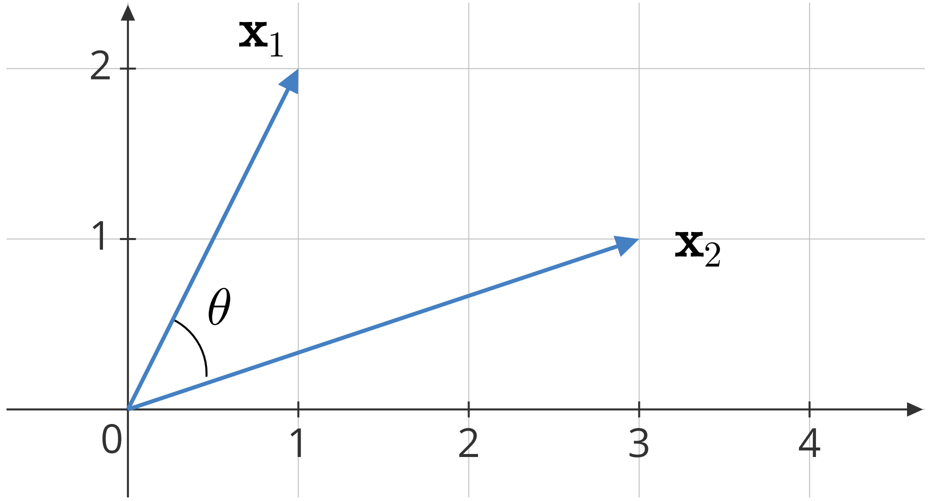
$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^T \mathbf{x}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 1 = 5$$



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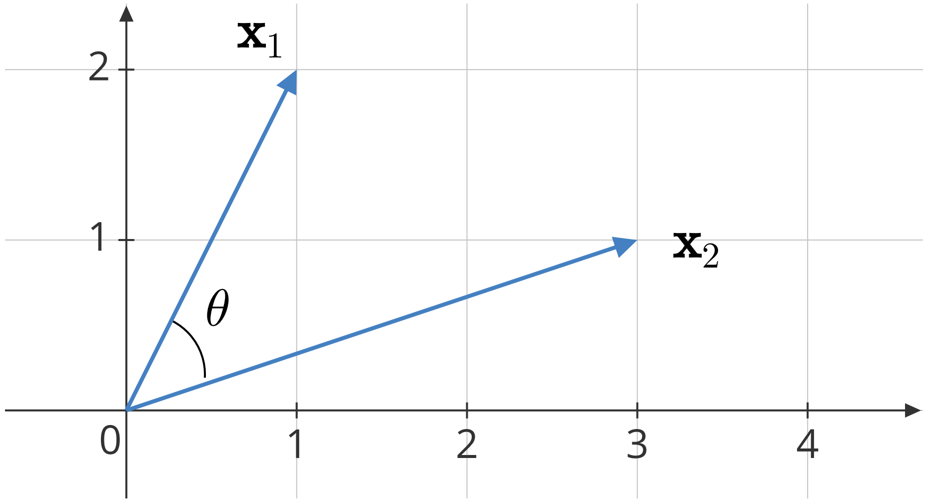

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$$||\mathbf{x}_1||^2 = \mathbf{x}_1^T \mathbf{x}_1 = 1^2 + 2^2 = 5$$

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$\mathbb{R}^d$: Dot Product

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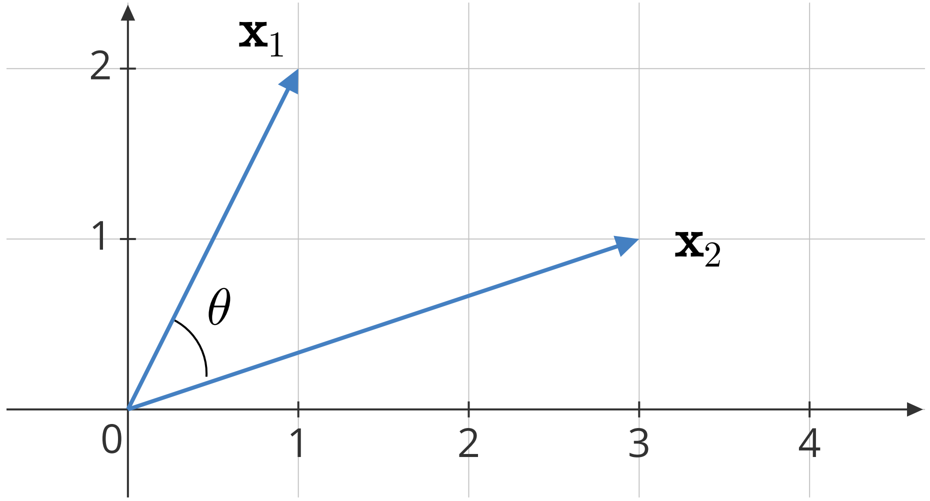

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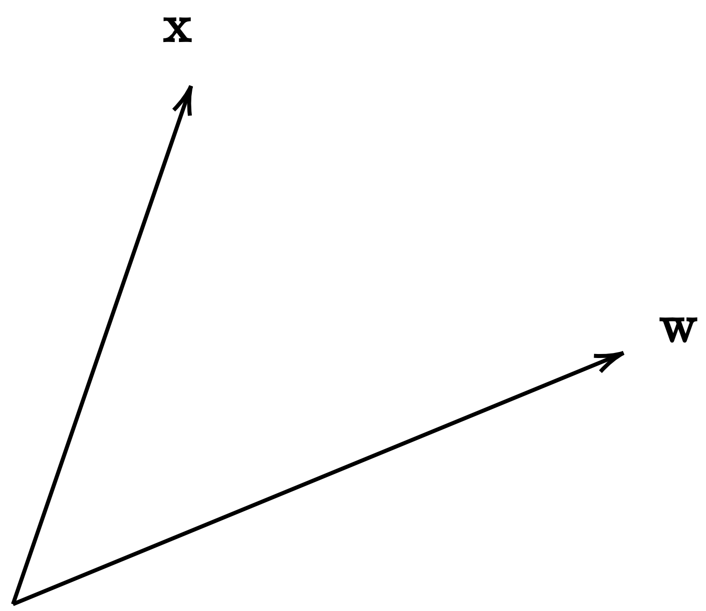
$$\cos\theta = \frac{\mathbf{x}_1^T \mathbf{x}_2}{||\mathbf{x}_1|| \cdot ||\mathbf{x}_2||} = \frac{5}{\sqrt{5} \cdot \sqrt{10}} = \frac{1}{\sqrt{2}}$$

$$\implies \theta = 45^\circ$$

$\mathbb{R}^d$: Projections



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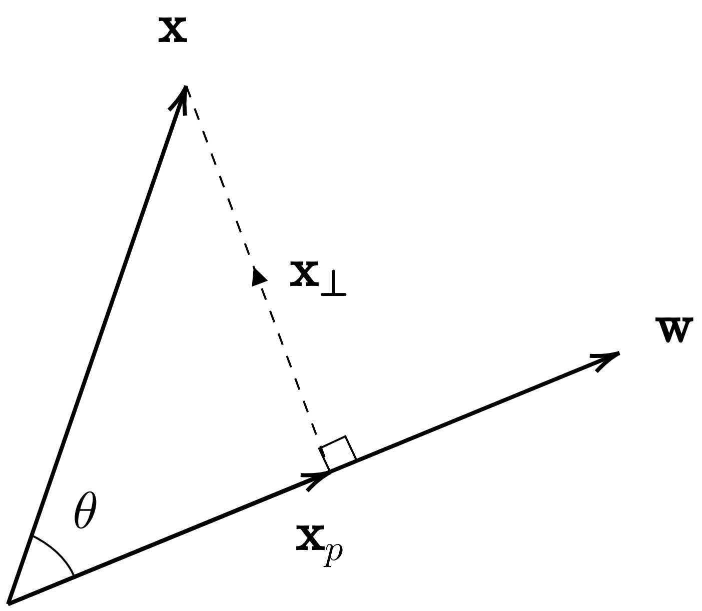

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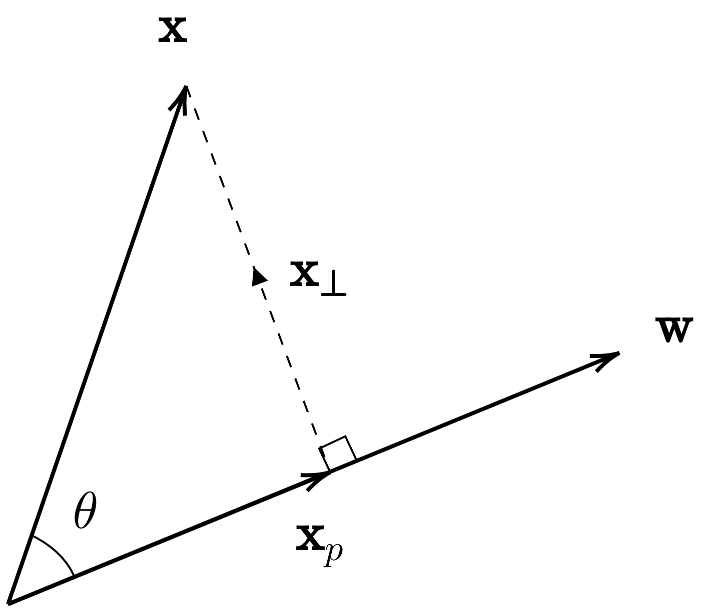
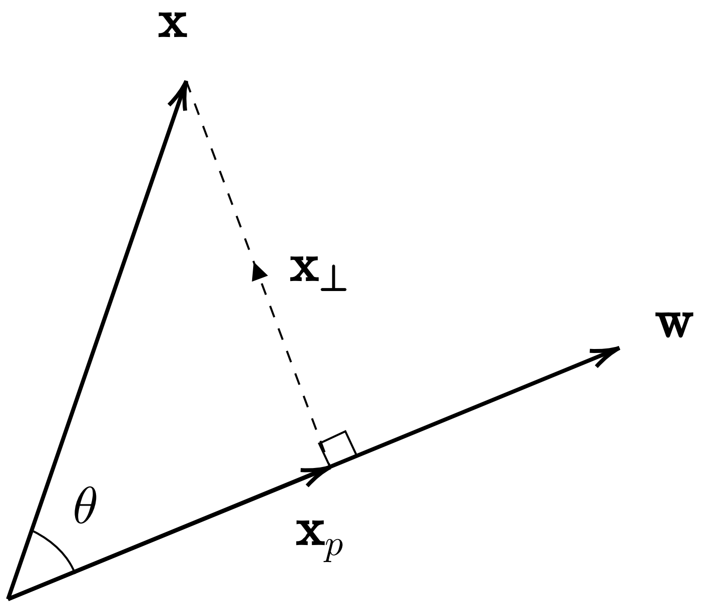


$\mathbf{x}_p = c\mathbf{w}$

$\mathbb{R}^d$: Projections

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_\perp$$



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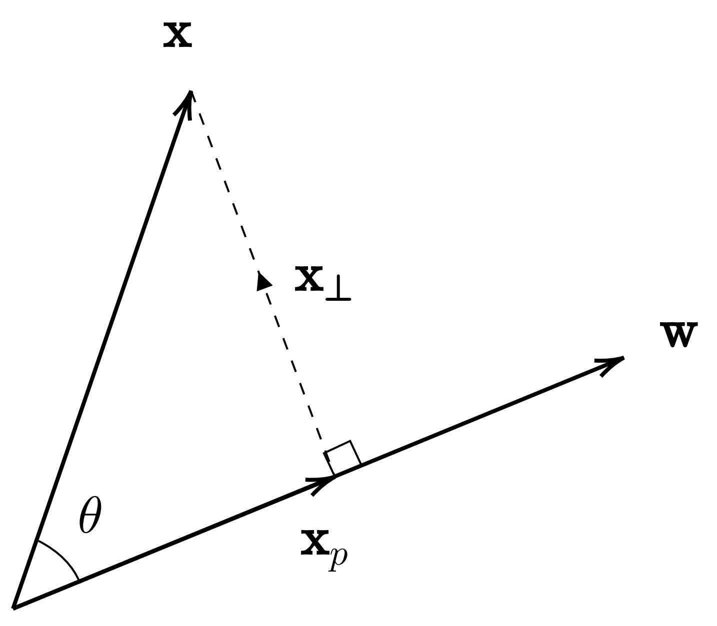

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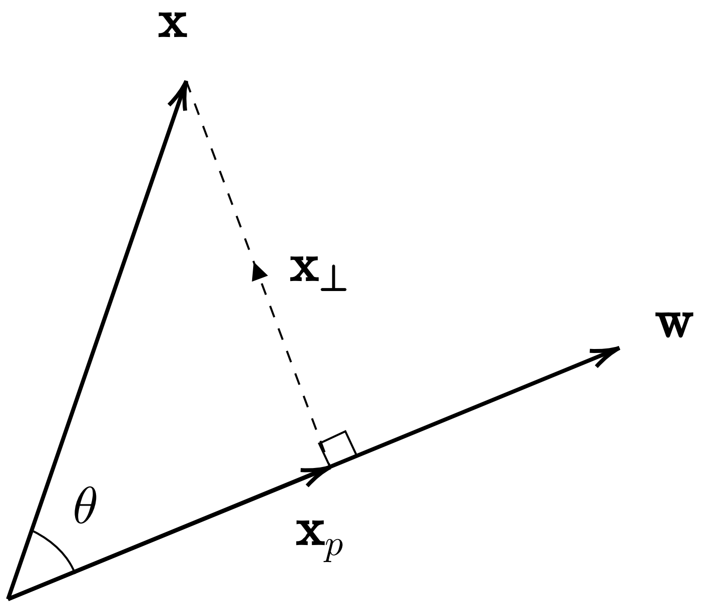

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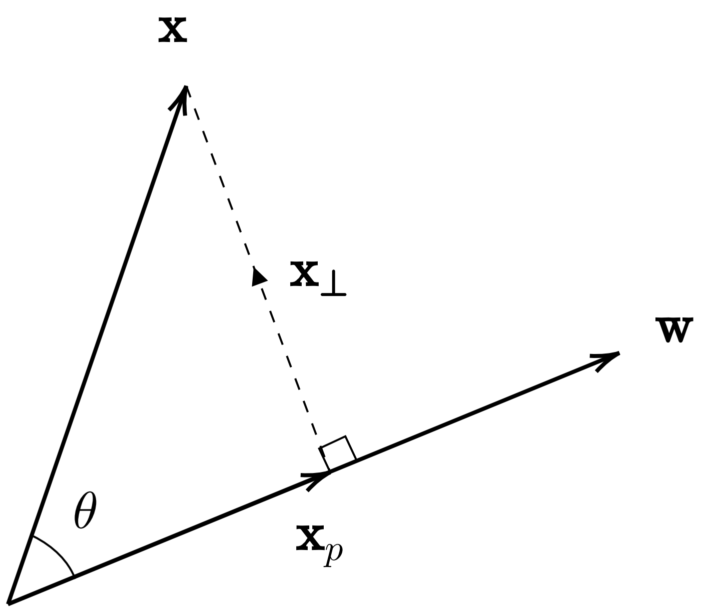

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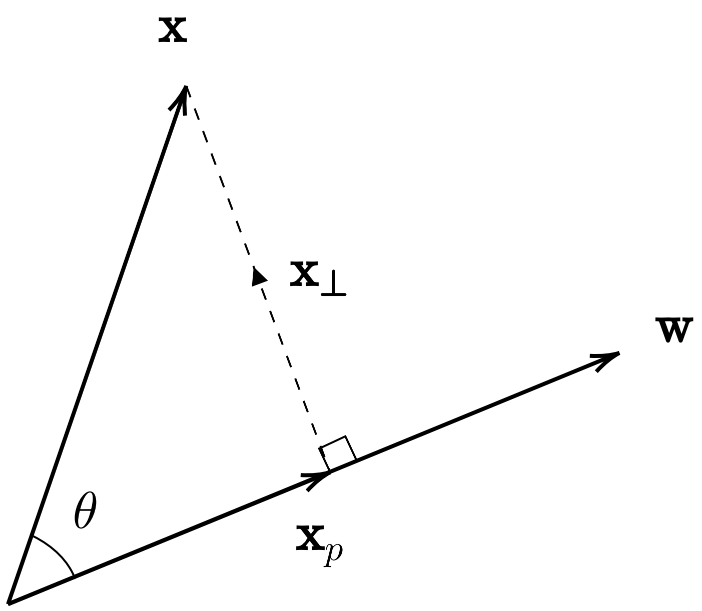

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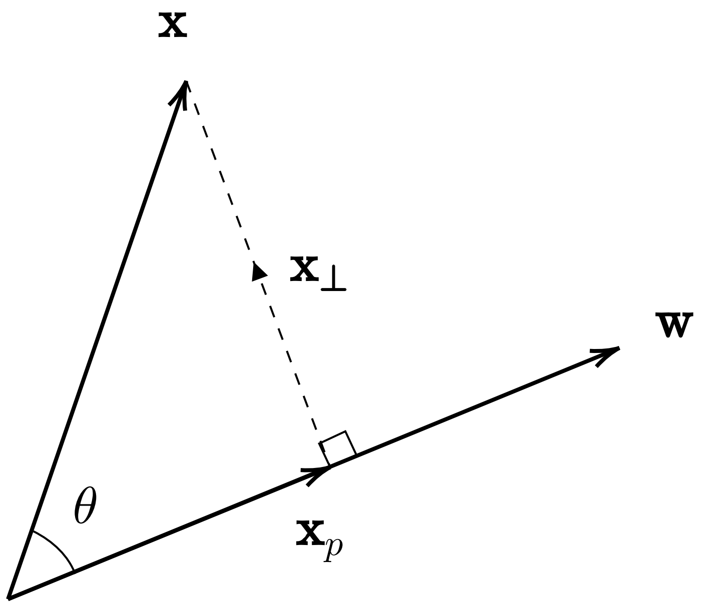

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Vector Projection

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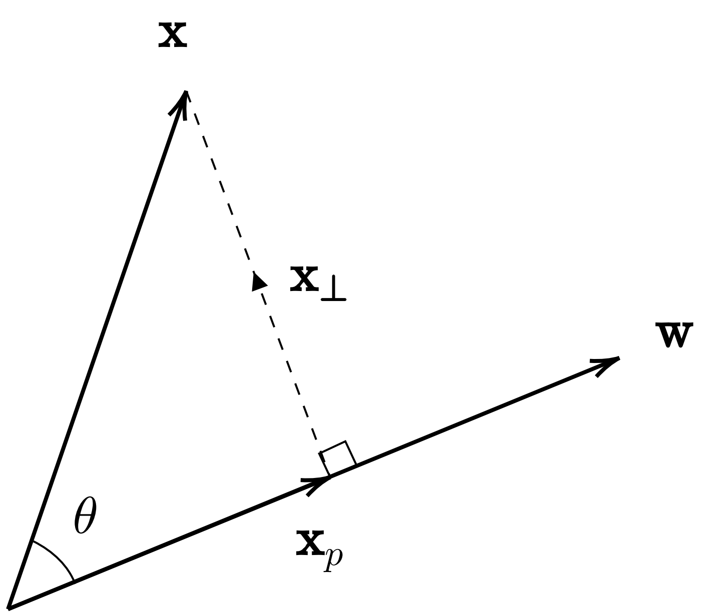

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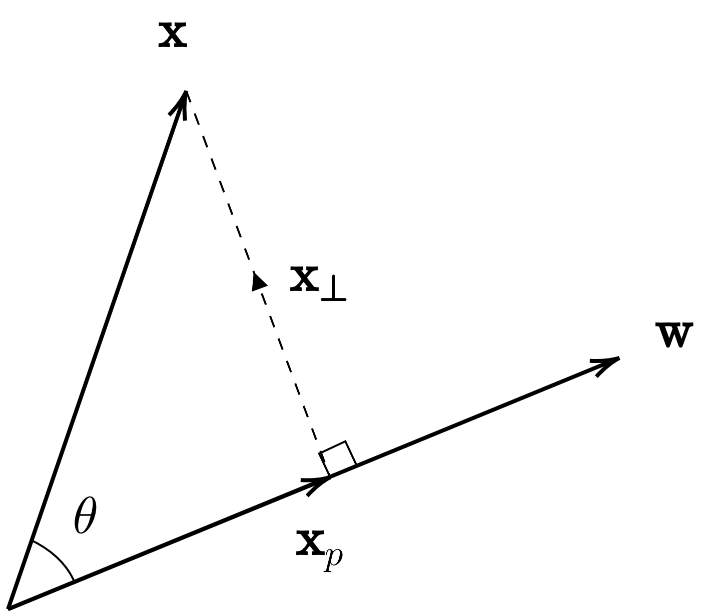
Vector Projection

$$\mathbf{x}_p = \frac{\mathbf{x}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}$$

Scalar Projection

$$||\mathbf{x}_p|| = \frac{\mathbf{x}^T\mathbf{w}}{||\mathbf{w}||}$$

$\mathbb{R}^d$: Projections



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Projection on unit-norm vector

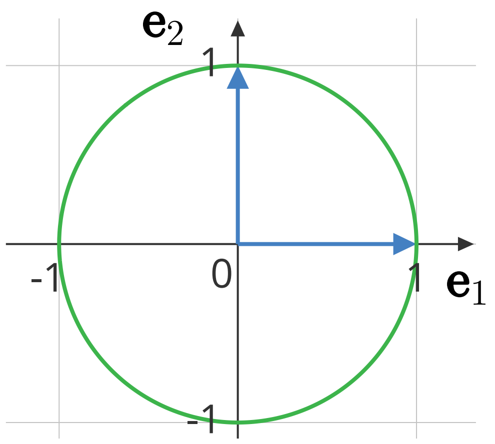
$||\mathbf{w}|| = 1$

$$\left(\mathbf{x}^T\mathbf{w}\right)\mathbf{w}$$

$\mathbb{R}^d$: Orthonormal Basis

- Orthogonal
- Unit norm

Basis-1



$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{e}_1^T \mathbf{e}_2 = 0$$

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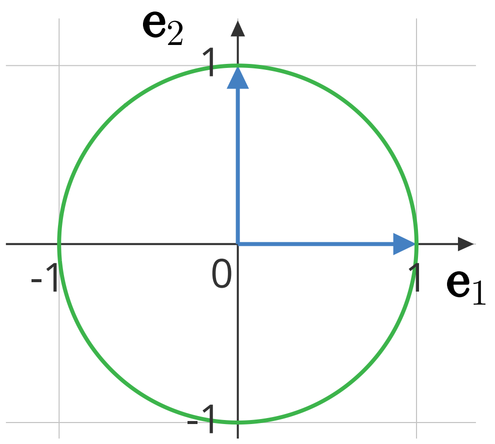

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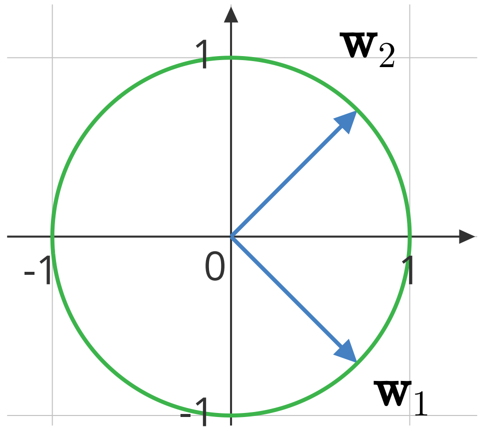
Basis-2



$$\mathbf{w}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{w}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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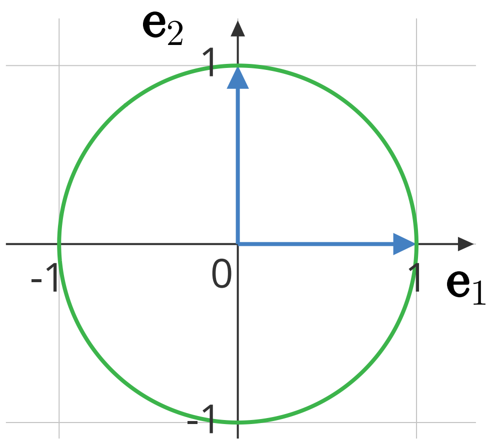

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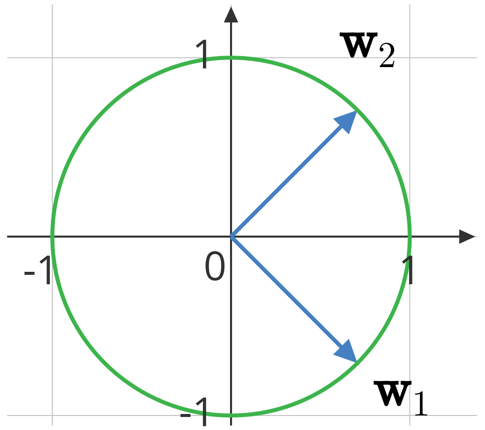
$$\text{Basis-1} \xleftrightarrow[\text{reflection}]{\text{rotation}} \text{Basis-2}$$

Basis-2



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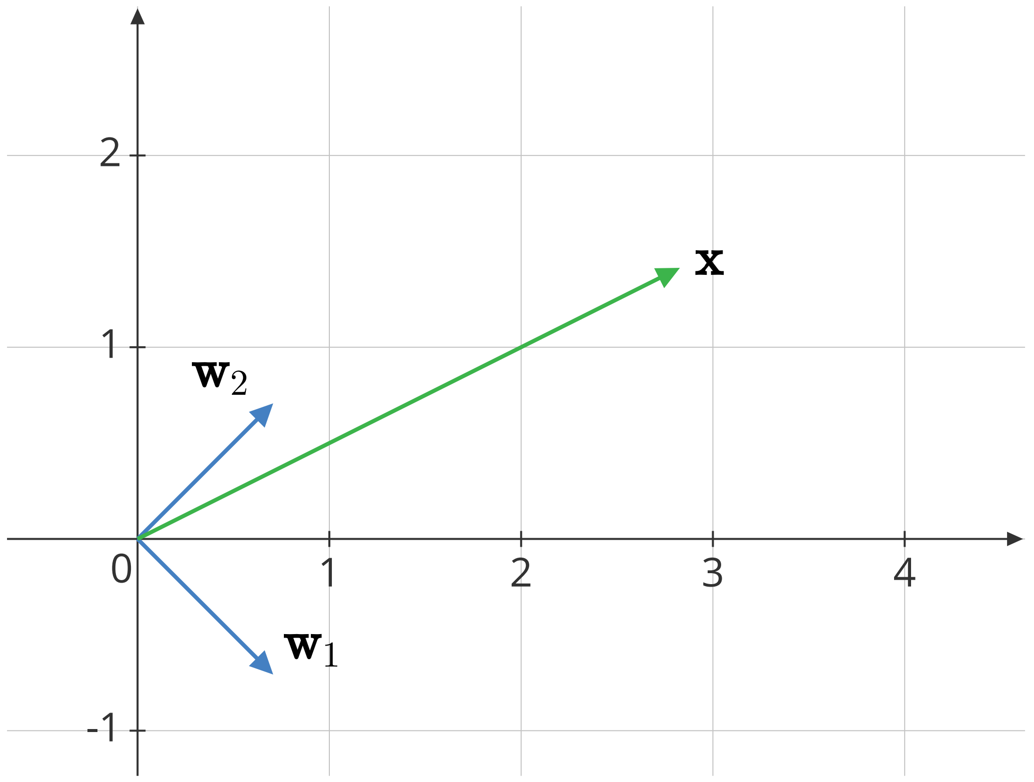

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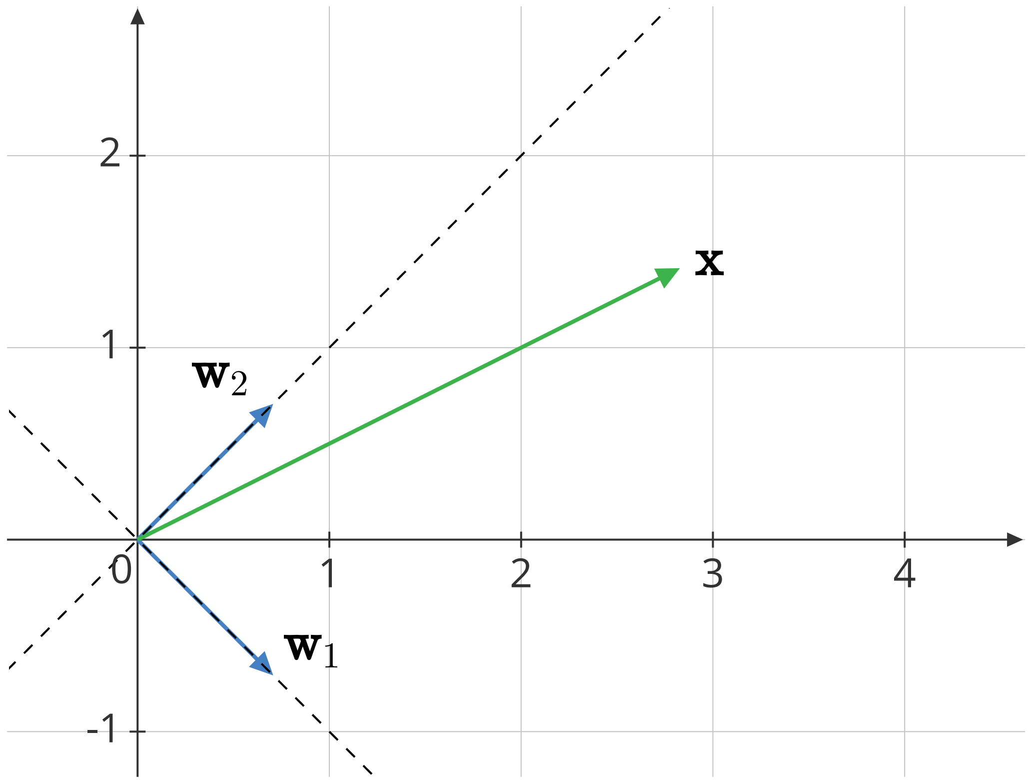

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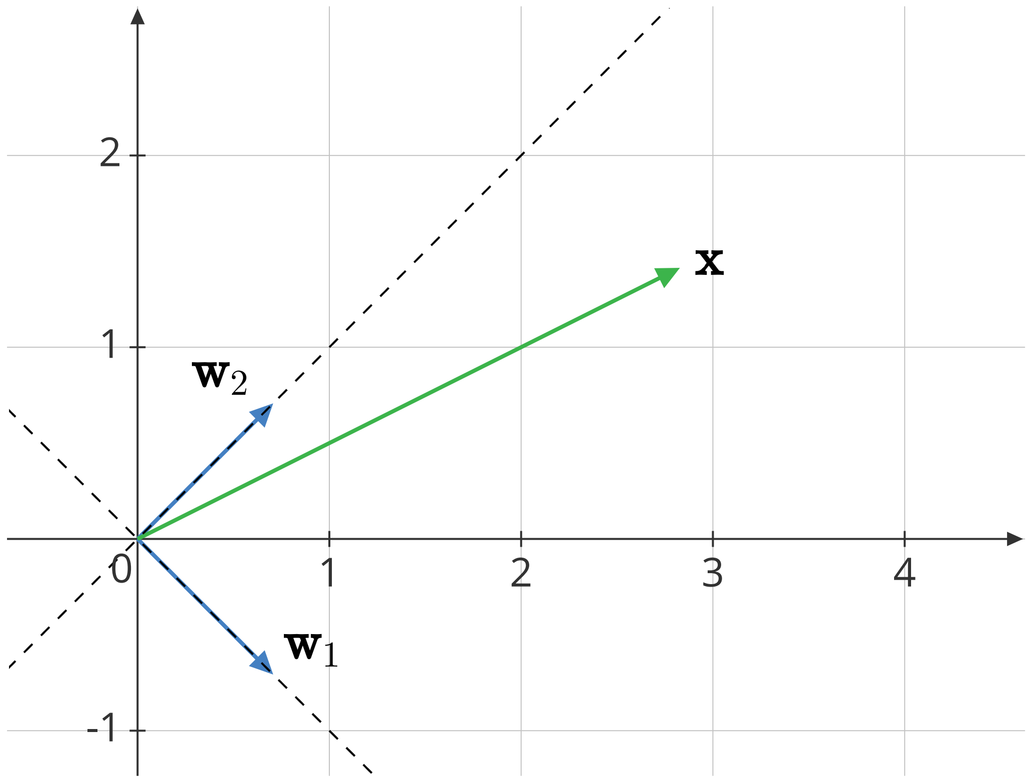

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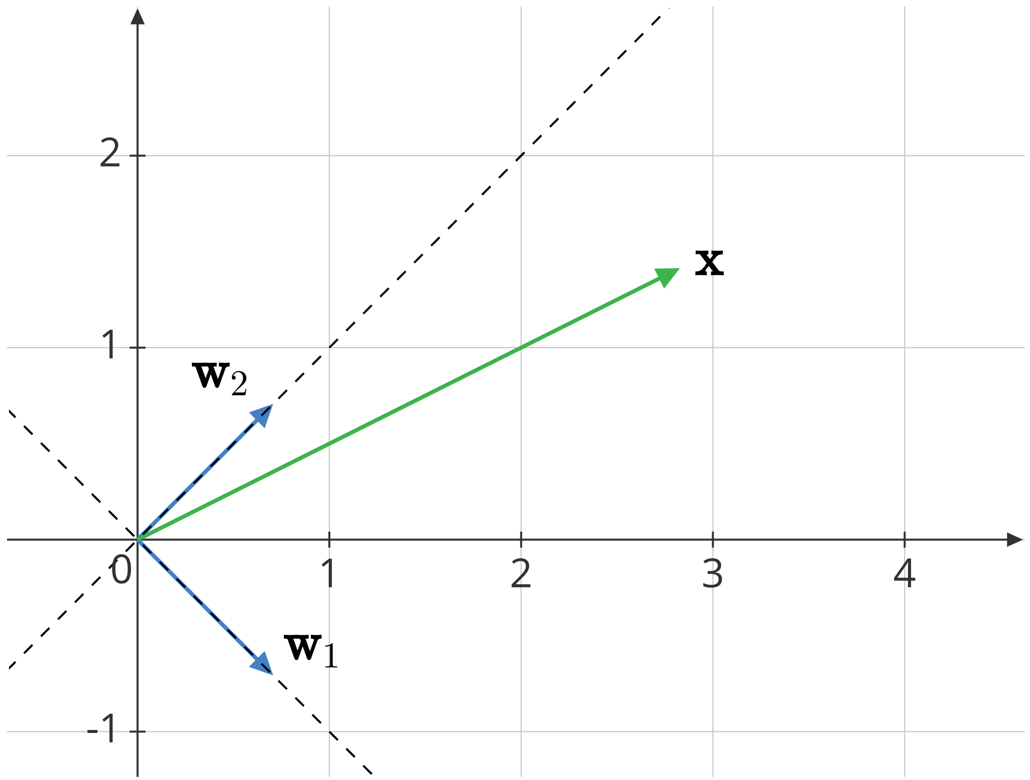

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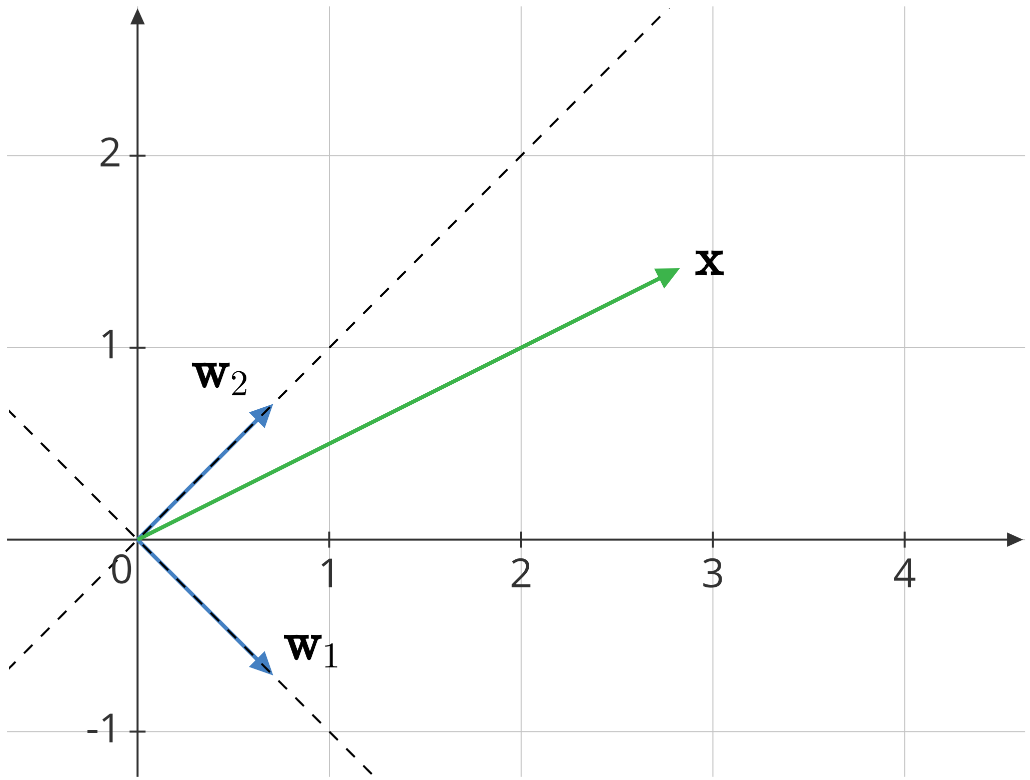

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$$\begin{aligned} \mathbf{x} &= 2\sqrt{2} \cdot \mathbf{e}_1 + \sqrt{2} \cdot \mathbf{e}_2 \\ &= \left(\mathbf{x}^T \mathbf{w}_1\right)\mathbf{w}_1 + \left(\mathbf{x}^T \mathbf{w}_2\right)\mathbf{w}_2 \end{aligned}$$

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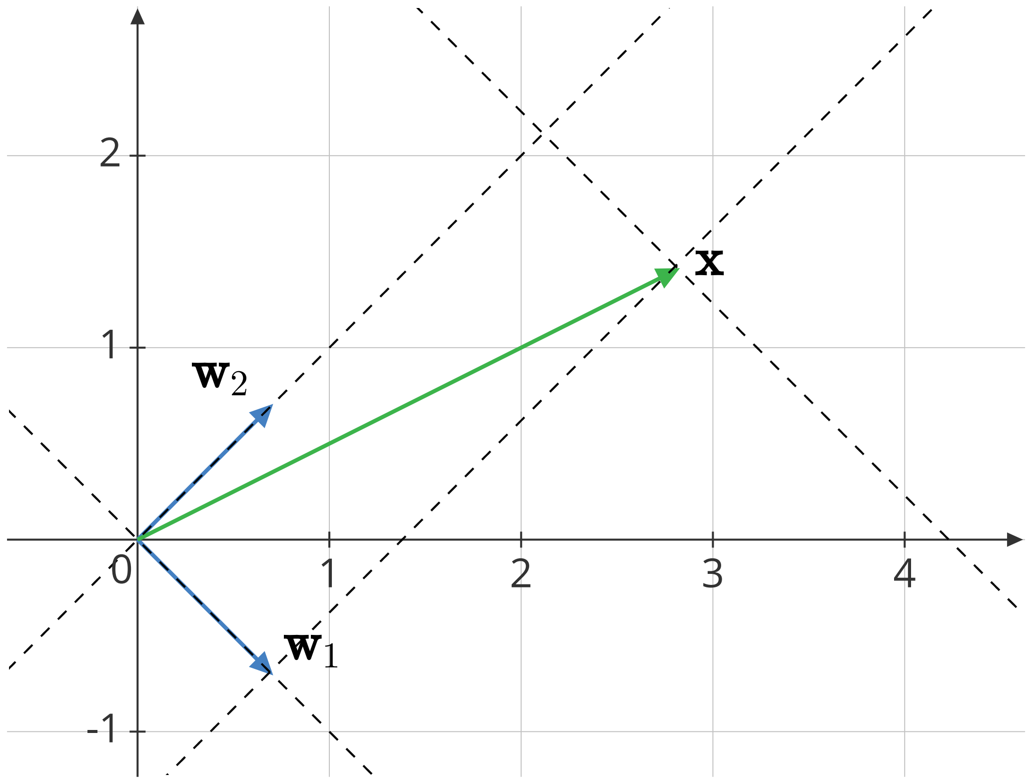

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$$\begin{aligned} \mathbf{x} &= 2\sqrt{2} \cdot \mathbf{e}_1 + \sqrt{2} \cdot \mathbf{e}_2 \\ &= \left(\mathbf{x}^T \mathbf{w}_1\right)\mathbf{w}_1 + \left(\mathbf{x}^T \mathbf{w}_2\right)\mathbf{w}_2 \\ &= 1\mathbf{w}_1 + 3\mathbf{w}_2 \end{aligned}$$

Orthogonal Matrices

$$\mathbf{Q} = \begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix}$$

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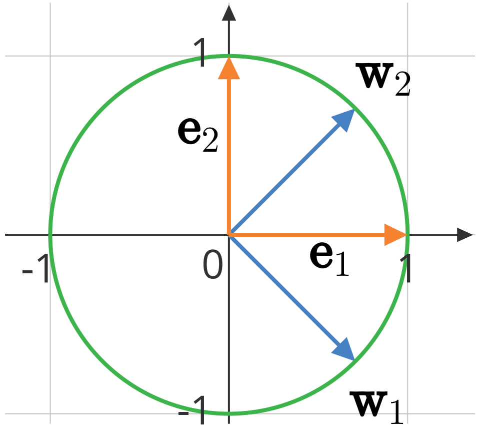
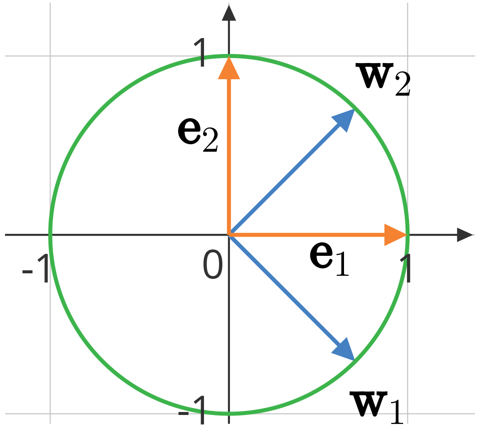


Orthogonal Matrices

$$\mathbf{Q} = \begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix}$$

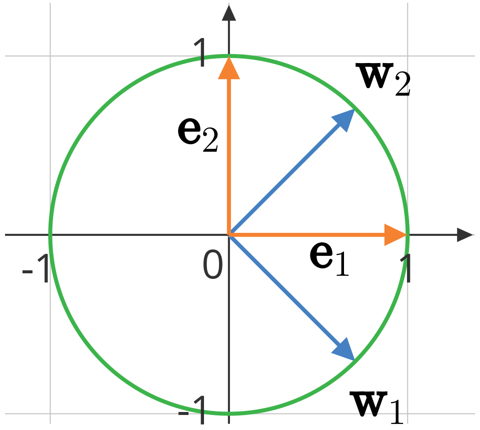
$$\mathbf{Q}^T\mathbf{Q} = \begin{bmatrix} - & \mathbf{w}_1^T & - \\ - & \mathbf{w}_2^T & - \end{bmatrix}\begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \mathbf{I}$$



Orthogonal Matrices

$$\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix}$$



$$\mathbf{Q}^T\mathbf{Q} = \begin{bmatrix} — & \mathbf{w}_1^T & — \\ — & \mathbf{w}_2^T & — \end{bmatrix}\begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix}$$

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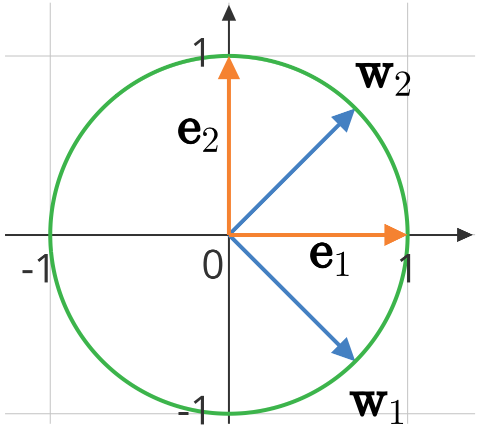


$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{w}_1 = \mathbf{Q}\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

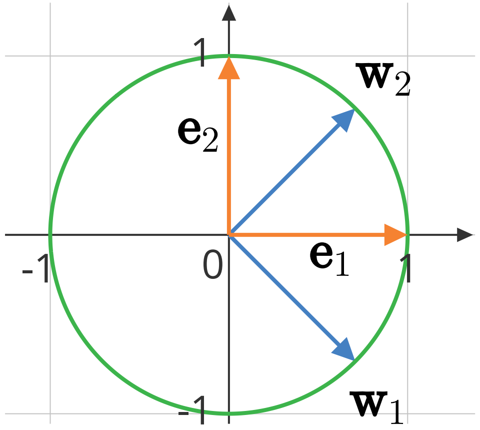
$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{w}_2 = \mathbf{Q}\mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Orthogonal Matrices

$$\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix}$$

$$\begin{aligned} \mathbf{Q}^T\mathbf{Q} &= \begin{bmatrix} — & \mathbf{w}_1^T & — \\ — & \mathbf{w}_2^T & — \end{bmatrix}\begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \mathbf{I} \end{aligned}$$



$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_1 = \mathbf{Q}\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \mathbf{Q}\mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(\mathbf{Q}\mathbf{x})^T(\mathbf{Q}\mathbf{y}) = \mathbf{x}^T\mathbf{Q}^T\mathbf{Q}\mathbf{y} = \mathbf{x}^T\mathbf{y}$$

Orthogonal matrix

- Preserves inner products
  - Preserves lengths
  - Preserves angles