# Quicksort

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Programming, Data Structures and Algorithms using Python

Week 3

# Shortcomings of merge sort

- Merge needs to create a new list to hold the merged elements
  - No obvious way to efficiently merge two lists in place
  - Extra storage can be costly
- Inherently recursive
  - Recursive calls and returns are expensive

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  - Consider an input of the form `[0,2,4,6,1,3,5,9]`

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- Merging happens because elements in the left half need to move to the right half and vice versa
  - Consider an input of the form `[0,2,4,6,1,3,5,9]`
- Can we divide the list so that everything on the left is smaller than everything on the right?
  - No need to merge!

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- How do we find the median?
  - Sort and pick up the middle element
  - But our aim is to sort the list!
- Instead pick some value in `L` — pivot
  - Split `L` with respect to the pivot element

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- Input list

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- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments
- After partitioning, exchange the pivot with the last element of the Lower segment

# Quicksort code

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Classify the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
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def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
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      (lower,upper) = (lower+1,upper+1)
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  (L[l],L[lower-1]) = (L[lower-1],L[l])
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  # Recursive calls
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- The partitioning strategy we described is not the only one used in the literature
  - Can build the lower and upper segments from opposite ends and meet in the middle

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- By partitioning the list carefully, we avoid a merge step
  - This allows an in place sort
- We can also provide an iterative implementation to avoid the cost of recursive calls
- The partitioning strategy we described is not the only one used in the literature
  - Can build the lower and upper segments from opposite ends and meet in the middle
- Need to analyse the complexity of quick sort

# Analysis of Quicksort

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- Worst case? Pivot is maximum or minimum
  - Partitions are of size $0$, $n-1$
  - $T(n) = T(n-1) + n$
  - $T(n) = n + (n-1) + \cdots + 1$
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- Already sorted array: worst case!

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  - Analyze behaviour over permutations of $\{1, 2, \ldots, n\}$
  - Each input permutation equally likely
- Expected running time is $O(n \log n)$

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- Instead, choose pivot position randomly at each step

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# Randomization

- Any fixed choice of pivot allows us to construct worst case input
- Instead, choose pivot position randomly at each step
- Expected running time is again $O(n \log n)$

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def quicksort(L,l,r):  # Sort L[l:r]
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- Can explicitly keep track of left and right endpoints of each segment to be sorted

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# Quicksort in practice

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# Quicksort in practice

- In practice, quicksort is very fast
- Very often the default algorithm used for in-built sort functions
  - Sorting a column in a spreadsheet
  - Library sort function in a programming language

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- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$
- Randomly choosing the pivot is a good strategy to beat worst case inputs
- Quicksort works in-place and can be implemented iteratively
- Very fast in practice, and often used for built-in sorting functions
  - Good example of a situation when the worst case upper bound is pessimistic

# Sorting: Concluding Remarks

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Programming, Data Structures and Algorithms using Python

Week 3

# Stable sorting

- Often list values are tuples
  - Rows from a table, with multiple columns / attributes
  - A list of students, each student entry has a roll number, name, marks, ...

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# Stable sorting

- The quicksort implementation we described is not stable
  - Swapping values while partitioning can disturb existing sorted order
- Merge sort is stable if we merge carefully
  - Do not allow elements from the right to overtake elements on the left
  - While merging, prefer the left list while breaking ties

# Other criteria

- Minimizing data movement
  - Imagine each element is a heavy carton
  - Reduce the effort of moving values around

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  - Retrieve in parts from the disk and write back
- Other $O(n \log n)$ algorithms exist — heapsort
- Sometimes hybrid strategies are used
  - Use divide and conquer for large $n$
  - Switch to insertion sort when $n$ becomes small (e.g., $n < 16$)

# Lists and Arrays

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Programming, Data Structures and Algorithms using Python

Week 3

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    - Lists
    - Arrays
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  - Fixed size
  - Allocate a contiguous block of memory
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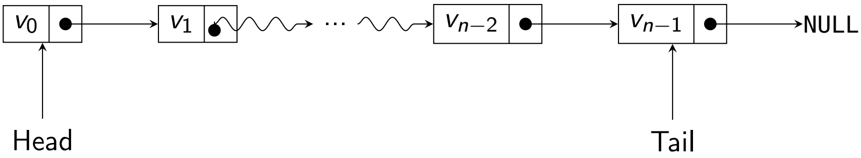

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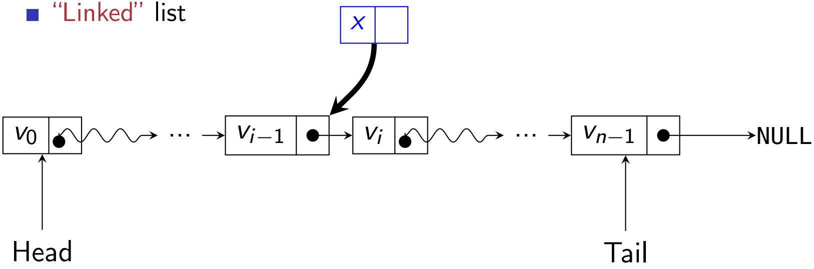

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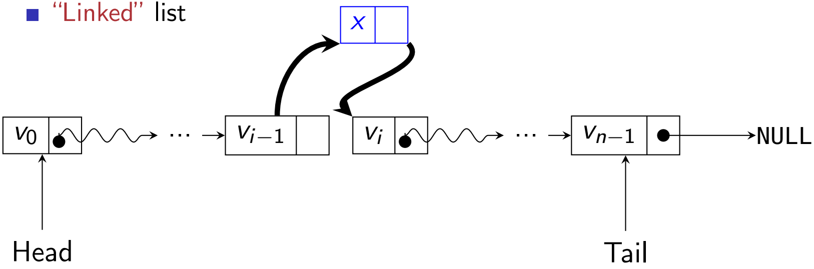

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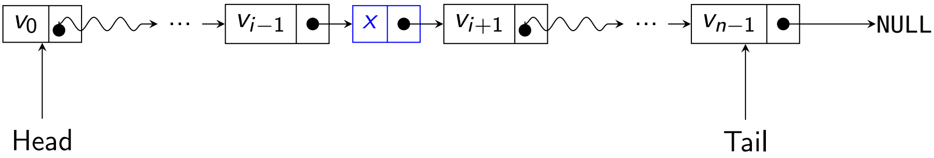

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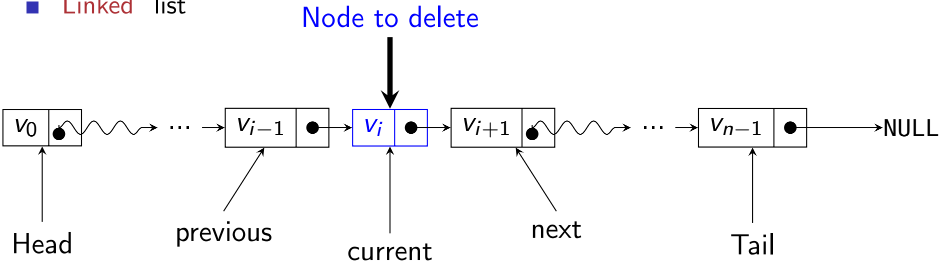

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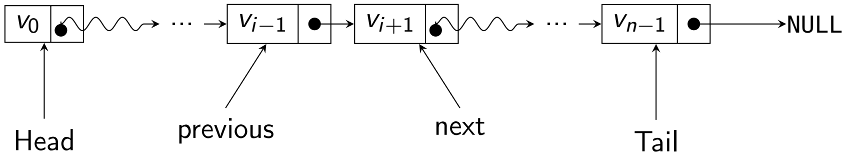

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- Fixed size, declared in advance
- Allocate a contiguous block of memory
  - $n$ times the storage for a single value

| Index | Value |
|---|---|
| `A[0]` | $v_0$ |
| `A[1]` | $v_1$ |
| $\vdots$ | $\vdots$ |
| `A[i]` | $v_i$ |
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  - Compute offset to `A[i]` from `A[0]`
  - Accessing `A[i]` takes constant time, independent of `i`
- Inserting and deleting elements is expensive
  - Expanding and contracting requires moving $O(n)$ elements in the worst case

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# Operations

- Exchange `A[i]` and `A[j]`
  - Constant time for arrays
  - $O(n)$ for lists
- Delete `A[i]`, insert `v` after `A[i]`
  - Constant time for lists if we are already at `A[i]`
  - $O(n)$ for arrays
- Need to keep implementation in mind when analyzing data structures
  - For instance, can we use binary search to insert in a sorted sequence?
  - Either search is slow, or insertion is slow, still $O(n)$

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- Lists are flexible but accessing an element is $O(n)$
- Arrays support random access but are difficult to expand, contract
- Algorithm analysis needs to take into account the underlying implementation
- How does it work in Python?
  - Is the built-in list type in Python really a "linked" list?
  - Numpy library provides arrays — are these faster than lists?

# Designing a flexible list

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Programming, Data Structures and Algorithms using Python

Week 3

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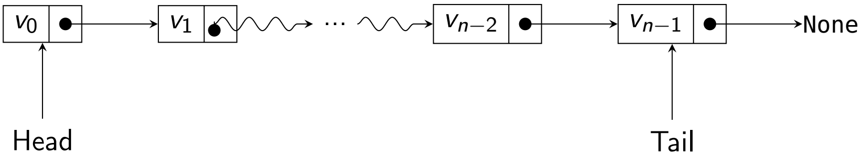

# Implementing lists in Python

- Python class `Node`

```
class Node:
    def __init__(self, v = None):
        self.value = v
        self.next = None
        return

    def isempty(self):
        if self.value == None:
            return(True)
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# Appending to a list

- Add v to the end of list l
- If l is empty, update l.value from None to v
- If at last value, l.next is None
  - Point next at new node with value v
- Otherwise, recursively append to rest of list

```
def append(self,v):
    # append, recursive
    if self.isempty():
        self.value = v
    elif self.next == None:
        self.next = Node(v)
    else:
        self.next.append(v)
    return
```

# Appending to a list

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- If `l` is empty, update `l.value` from `None` to `v`
- If at last value, `l.next` is `None`
  - Point `next` at new node with value `v`
- Otherwise, recursively append to rest of list
- Iterative implementation
  - If empty, replace `l.value` by `v`
  - Loop through `l.next` to end of list
  - Add `v` at the end of the list

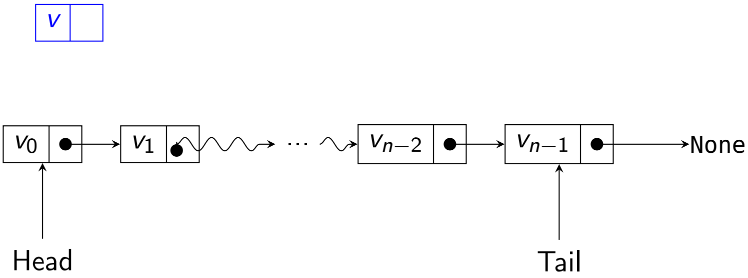
```
def appendi(self,v):
    # append, iterative
    if self.isempty():
        self.value = v
        return

    temp = self
    while temp.next != None:
        temp = temp.next

    temp.next = Node(v)
    return
```

# Insert at the start of the list

- Want to insert $v$ at head
- Create a new node with $v$



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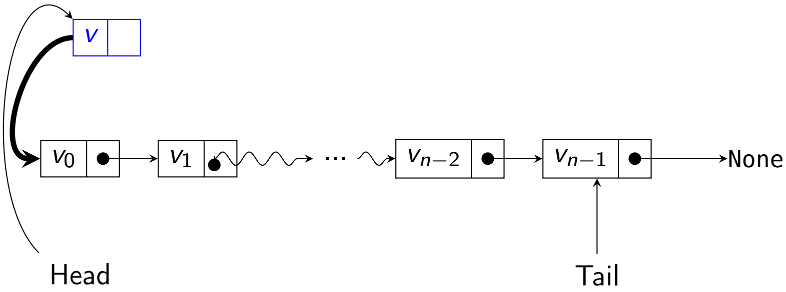

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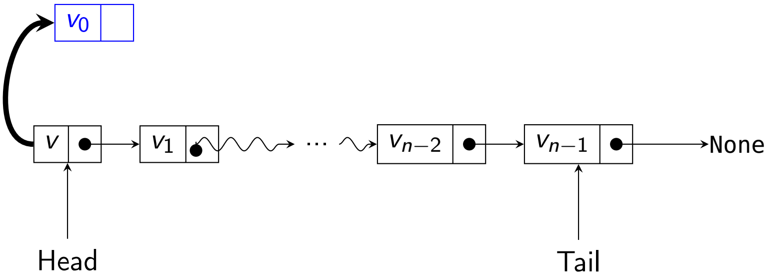

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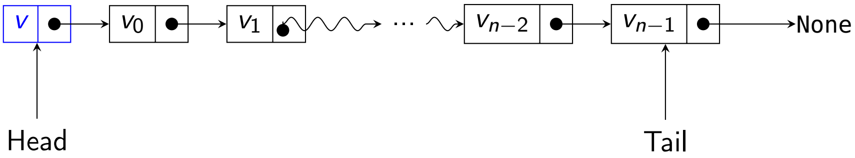

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```
def insert(self,v):
    if self.isempty():
        self.value = v
        return

    newnode = Node(v)

    # Exchange values in self and newnode
    (self.value, newnode.value) =
        (newnode.value, self.value)

    # Switch links
    (self.next, newnode.next) =
        (newnode, self.next)

    return
```

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- Recursive implementation

```
def delete(self,v):
# delete, recursive

    if self.isempty():
        return

    if self.value == v:
        self.value = None
        if self.next != None:
            self.value = self.next.value
            self.next = self.next.next
        return
    else:
        if self.next != None:
            self.next.delete(v)
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- Exercise: write an iterative version

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# Summary

- Use a linked list of nodes to implement a flexible list
- Append is easy
- Insert requires some care, cannot change where the head points to
- When deleting, look one step ahead to bypass the node to be deleted

# Lists and Arrays in Python

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Programming, Data Structures and Algorithms using Python

Week 3

# Lists and arrays in Python

- Sequences can be stored as lists or arrays
- Lists are flexible but accessing an element is $O(n)$
- Arrays support random access but are difficult to expand, contract
- Algorithm analysis needs to take into account the underlying implementation
- How does it work in Python?
  - Is the built-in list type in Python really a "linked" list?
  - Numpy library provides arrays — are these faster than lists?

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  - `l.append()` and `l.pop()` are constant time, amortised — $O(1)$
  - Insertion/deletion require time $O(n)$
- Effectively, Python lists behave more like arrays than lists

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- Mutability aliases different values
- Instead, use list comprehension

```
zeromatrix = [ [ 0 for i in range(3) ]  for j in range(3) ]
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- Can operate on a matrix as a whole
  - `C = 3*A + B`
  - `C = np.matmul(A,B)`
  - Very useful for data science

# Summary

- Python lists are not implemented as flexible linked structures
- Instead, allocate an array, and double space as needed
- Append is cheap, insert is expensive
- Arrays can be represented as multidimensional lists, but need to be careful about mutability, aliasing
- Numpy arrays are easier to use

# Implementing dictionaries

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- A dictionary allows access through arbitrary keys
  - A collection of key-value pairs
  - Random access — access time is the same for all keys
- How is a dictionary implemented?

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  - SHA-256 is an industry standard hashing function whose range is 256 bits
    - Use to hash large files — avoid uploading duplicates to cloud storage

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- Dictionary keys in Python must be immutable
    - If value changes, hash also changes!

# Summary

- A dictionary is implemented as a hash table
  - An array plus a hash function
- Creating a good hash function is important (and hard!)
- Need a strategy to deal with collisions
  - Open addressing/closed hashing — probe for free space in the array
  - Open hashing — each slot in the hash table points to a list of key-value pairs
  - Many heuristics/optimizations possible for dea