# Assignment 1

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Solutions below Questions

## ESC384 Assignment 1

Due Wednesday, 27 September 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

### Problem 1. Classification of PDEs (23%)

Consider the following PDEs:

(i) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(4x)$ in $\mathbb{R} \times \mathbb{R}_{>0}$

(ii) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\frac{u}{|1+x|}$ in $\mathbb{R} \times \mathbb{R}_{>0}$

(iii) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \sin(u) = 0$ in $\mathbb{R}^2 \times \mathbb{R}_{>0}$

(iv) $-2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$ in $\mathbb{R}^2$

For each PDE, answer the following questions and briefly justify your answers:

(a) (4%) State the order.

(b) (4%) State if the PDE is linear homogeneous, linear nonhomogeneous, or nonlinear.

(c) (5%) If the equation is second-order, classify the equation as elliptic, parabolic, or hyperbolic.
*Hint:* the standard analysis of the "$A$ matrix" applies even if the equation is nonlinear.

(d) (5%) State whether the following statement (S1) holds: if $v$ and $w$ satisfies the PDE, then $v+w$ also satisfies the PDE.

(e) (5%) State whether the following statement (S2) holds: if $u$ satisfies the PDE, then $w$ obtained by shifting $u$ by $\pi$ in the $x$ direction (i.e., $w(x,t) = u(x-\pi,t)$ in $\mathbb{R} \times \mathbb{R}_{>0}$ and $w(x,y,t) = u(x-\pi,y,t)$ in $\mathbb{R}^2 \times \mathbb{R}_{>0}$) also satisfies the PDE.

### Problem 2. Odd, even, and periodic functions (22%)

State if each of the following statements is true or false, and then prove the statement or provide a counterexample.

(a) (4%) If $f$ is odd and $g$ is even, then $f+g$ is odd.

(b) (5%) If $f : \mathbb{R} \to \mathbb{R}$ is even and continuously differentiable, then $g(x) \equiv f'(x)$ is odd.

(c) (5%) If $f : \mathbb{R} \to \mathbb{R}$ is even and integrable, then $g(x) \equiv \int_{\xi=0}^{x} f(\xi)d\xi$ is odd.

(d) (8%) If $f : \mathbb{R} \to \mathbb{R}$ is $z$-periodic and integrable, then $\int_{x=y}^{y+z} f(x)dx = \int_{x=0}^{z} f(x)dx$ for any $y \in \mathbb{R}$.



### Problem 3. Fourier series solution of ODE (20%)

We wish to solve an ODE boundary value problem

$$-u'' + u = f \quad \text{in } (0,L),$$
$$u(x=0) = u(x=L) = 0,$$

where $f$ is some square-integrable function, and $L > 0$ is the length of the domain. Our goal is to express the solution $u$ as a series; i.e., $u(x) \sim \sum_{n=1}^{\infty} \hat{u}_n \phi_n(x)$ where $(\phi_n)_{n=1}^{\infty}$ is an appropriate sequence of functions. Answer the following questions:

(a) (3%) Choose either (i) $\phi_n(x) = \sin(\lambda_n x)$ or (ii) $\phi_n(x) = \cos(\lambda_n x)$ and an appropriate $\lambda_n$ so that the boundary conditions are satisfied for any set of coefficients $\hat{u}_n$, $n = 1, 2, \dots$. (From hereon, $\phi_n(x)$ should be replaced by $\sin(\lambda_n x)$ or $\cos(\lambda_n x)$ with an appropriate $\lambda_n$.)

(b) (10%) Substitute the expression found in (a) to the ODE, perform the necessary differentiation (assuming the solution is sufficiently smooth), and invoke the orthogonality relationship to find the expression for the coefficient $\hat{u}_n$ in terms of the coefficient $\hat{f}_n = \frac{2}{L}\int_0^L f(x)\phi_n(x)dx$.

(c) (7%) Let $f = x$. Find coefficients $\hat{u}_n$, $n = 1, 2, \dots$, associated with the solution $u$.

### Problem 4. Convergence of Fourier series (35%)

Throughout this problem, let $f(x) = x^2$, $x \in [0,1]$. This problem requires the use MATLAB or another comparable programming language.

(a) (5%) Find, analytically, the Fourier cosine coefficients and the Fourier sine coefficients of $f$. We will hereafter refer to the cosine (resp. sine) series as $f^c$ (resp. $f^s$) and the associated $N$-term truncated series as $f_N^c$ (resp. $f_N^s$).

(b) (7%) Using MATLAB, plot, in a single figure, for $x \in [-1,2]$ (i) $f_{N=3}^c$, (ii) $f_{N=30}^c$, (iii) $f_{N=300}^c$, and (iv) the appropriate periodic extension of $f$ that the series approximates.

(c) (7%) Repeat (b) for the Fourier sine series.

(d) (6%) Report, in a table, the pointwise error in the truncated Fourier cosine and sine series $|f(x) - f_N^{c/s}(x)|$ at $x = 0.5$ for $N = 3$, 30, and 300. (There should be six entries in the table.) Repeat the exercise for $x = 0.9$. Discuss if the observed behavior is consistent with the expected pointwise convergence of the series.

(e) (6%) Report, in a table, the maximum error in the truncated Fourier cosine series $\max_{x\in[0,1]} |f(x) - f_N^c(x)|$ for $N = 3$, 30, and 300. (Approximate the "max" on a sufficiently fine set of evaluation points.) Discuss if the observed behavior is consistent with the expected uniform convergence behavior of the series.

(f) (4%) Would the maximum error in the truncated Fourier sine series $\max_{x\in[0,1]} |f(x) - f_N^s(x)|$ behave differently from that for the Fourier cosine series as $N \to \infty$? Describe the expected behavior.
*Hint:* we can find the answer analytically without carry out the computation.



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PDE's ESC384
Assignment 1

Q1

PDE's

1) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(4x)$

2) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{-u}{|1+x|}$

3) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \sin(u) = 0$

4) $-2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$

| | Order | Reason |
|---|---|---|
| a) i) | 2nd order | ($-\partial^2 u/\partial x^2$ term) |
| ii) | 2nd order | ($-\partial^2 u/\partial x^2$ term) |
| iii) | 2nd order | (all derivative terms) |
| iv) | 2nd order | (all terms) |

b) i) linear non-homogeneous
ii) linear homogeneous
iii) non-linear
iv) linear homogeneous

c) A matrices for all PDE's.
only considering 2nd order terms on matrix diagnol as eigenvalues with $\partial/\partial t^2$ term as the 0th coordinate, and other $\partial/\partial x^2$, $\partial/\partial y^2$ terms in each following column.

i) $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \boxed{\text{parabolic}}$

$\partial/\partial t^2 \rightarrow$ no term
$\partial/\partial x^2 \rightarrow -1$

one-zero eigenvalue only and all other eigenvalues have same sign, ∴ parabolic

ii) $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \boxed{\text{parabolic}}$

$\partial/\partial t^2 \rightarrow$ no term
$\partial/\partial x^2 \rightarrow -1$

one-zero eigenvalue only and al ot r eigenvalues have same sign, ∴ parabolic

iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \boxed{\text{hyperbolic}}$

$\partial/\partial t^2 \rightarrow 1$
$\partial/\partial x^2 \rightarrow -1$
$\partial/\partial y^2 \rightarrow -1$

all eigenvalues non-zero but one of them has a different size. ∴ hyperbolic

iv) $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \rightarrow \boxed{\text{elliptic}}$

$\partial/\partial y^2 \rightarrow -2$
$\partial/\partial x^2 \rightarrow -2$
does not exist in time domain

all eigenvalues non-zero and of the same sign. ∴ elliptic.

d) i) $\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = \sin(4x)$

If v and w satisfy the PDE, then

$$\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} = \sin(4x) \quad -(1)$$

$$\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = \sin(4x) \quad -(2)$$

$$(1)+(2) = \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} = 2\sin(4x) \quad (A)$$

but, $\frac{\partial(v+w)}{\partial t} + \frac{\partial^2(v+w)}{\partial x^2} = \sin(4x)$ (B), and

$$\frac{\partial(v+w)}{\partial t} + \frac{\partial^2(v+w)}{\partial x^2} = \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \quad (C)$$

since the LHS of A and RHS of C are the same

$\frac{\partial(v+w)}{\partial t} + \frac{\partial^2(v+w)}{\partial x^2}$ must be $= 2\sin(4x)$

but this contradicts (B) which is the actual value of the RHS.

Since $2\sin(4x) \neq \sin(4x)$, we can say that SI does not hold.

$\boxed{\therefore \text{ does not hold}}$

d) ii) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{-u}{|1+x|}$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{-v}{|1+x|} \quad -(1)$$

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = \frac{-w}{|1+x|} \quad -(2)$$

$$\frac{\partial(v+w)}{\partial t} - \frac{\partial^2(v+w)}{\partial x^2} = \frac{-(v+w)}{|1+x|} \quad (A)$$

$$\frac{\partial(v+w)}{\partial t} - \frac{\partial^2(v+w)}{\partial x^2} = \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} \quad -(3)$$

Adding (1), (2)

$$\frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} = \underbrace{\frac{-v}{|1+x|} - \frac{w}{|1+x|}}_{\hookrightarrow \frac{-(v+w)}{|1+x|}} \quad (B)$$

Since the RHS of (A) = LHS of (B), and the LHS and RHS of those equations respectively are equivalent.
Thus SI holds for this PDE.

$\boxed{\therefore \text{ holds}}$

d) iii) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -\sin(u)$

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = -\sin(v) \quad -(1)$$

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = -\sin(w) \quad -(2)$$

$$\frac{\partial^2(v+w)}{\partial t^2} - \frac{\partial^2(v+w)}{\partial x^2} - \frac{\partial^2(v+w)}{\partial y^2} = -\sin(v+w) \quad -(3)$$

$$\frac{\partial^2(v+w)}{\partial t^2} - \frac{\partial^2(v+w)}{\partial x^2} - \frac{\partial^2(v+w)}{\partial y^2} = \frac{\partial^2 v}{\partial t} + \frac{\partial^2 w}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 w}{\partial y^2} \quad (A)$$

Adding (1), (2)

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 w}{\partial y^2} = -\sin(v) - \sin(w) \quad (B)$$

But even though the LHS of B = RHS of A, the RHS and LHS of B and A respectively are not equivalent

since, $-\sin(v+w) \neq -\sin(v) - \sin(w)$

SI does not hold for this PDE.

$\boxed{\therefore \text{ does not hold}}$

d) iv) $-2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$

$$-2\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y} - 2\frac{\partial^2 v}{\partial y^2} = 0 \quad -(1)$$

$$-2\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x \partial y} - 2\frac{\partial^2 w}{\partial y^2} = 0 \quad -(2)$$

$$-2\frac{\partial^2(v+w)}{\partial x^2} + 2\frac{\partial^2(v+w)}{\partial x \partial y} - 2\frac{\partial^2(v+w)}{\partial y^2} = 0$$

and,

$$-2\frac{\partial^2(v+w)}{\partial x^2} + 2\frac{\partial^2(v+w)}{\partial x \partial y} - 2\frac{\partial^2(v+w)}{\partial y^2} = -2\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y} + \frac{2\partial^2 w}{\partial x \partial y} - \frac{2\partial^2 v}{\partial y^2} - \frac{2\partial^2 w}{\partial y^2} \quad (A)$$

Adding (1), (2)

$$\Rightarrow -\frac{2\partial^2 v}{\partial x^2} - \frac{2\partial^2 w}{\partial x^2} + \frac{2\partial^2 v}{\partial x \partial y} + \frac{2\partial^2 w}{\partial x \partial y} - \frac{2\partial^2 v}{\partial y^2} - \frac{2\partial^2 w}{\partial y^2} = 0 \quad (B)$$

Since, the RHS of (A) = LHS of (B), and the LHS of A is also equal to the RHS of B = 0.

so, SI holds for this PDE.

$\boxed{\therefore \text{ holds}}$

e)i) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(4x)$

given $u$ satisfies the PDE, and $w(x-\pi, t)$

when $w(x,y,t) = u(x-\pi, y, t)$

so, if we consider $z = x - \pi$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$, since as shifting a derivative does not affect it as the constant term '$-\pi$' goes to 0 anyway.

$\sin(4(x-\pi)) = \sin(4x - 4\pi)$, but we know $\sin(4\pi) = 0$ as it is $\pi$-periodic.

$\therefore \sin(4x - 4\pi) = \sin(4x - 0) = \sin(4x)$

which means we can say

$\frac{\partial u}{\partial t} - \frac{\partial^2 z}{\partial x^2} = \sin(4x)$ is a new way of writing the PDE, and since we know that $w(x,y,t) = u(x-\pi, y, t) = u(z, y, t)$, and that $z$-substitution does not affect the PDE, S2 holds for this PDE

$\boxed{\therefore \text{ holds}}$

e)ii) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{-u}{|1+x|}$

Again, $z = x - \pi$

$-\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 z}{\partial x^2}$, remains unaffected being a derivative term shift with constant '$-\pi$' going to 0.

$\frac{-u}{|1+x|} \neq \frac{-u}{|1+x-\pi|}$, this term changes and we don't know if $u$ is $\pi$ periodic

Meaning we can't say for sure if $u(x,y,t)$ is still a solution of $\frac{\partial u}{\partial t} - \frac{\partial z}{\partial x^2} = \frac{-u}{|1+x-\pi|}$

meaning we are not sure if $u(x-\pi, y, t)$ is a solution to the PDE, thus we conclude that S2 does not hold for sure.

$\boxed{\therefore \text{ does not hold}}$

e)iii) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \sin(u)$ → The PDE seems to be coordinate independent, S2 must hold.

$\partial/\partial t^2$, $\partial/\partial y^2$ terms unchanged as they don't get affected by a shift in $x$.

Let $z = x - \pi$,

$-\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 z}{\partial x^2}$, remains unaffected being a derivative term shift with constant '$-\pi$' going to 0.

$\sin(u) = \sin(z) = \sin(x - \pi)$, since $\sin$ is $\pi$-periodic the shift by $-\pi$ does not impact the term and it's unchanged.

Thus for all terms in the PDE, no change is observed with the shift, and $u(x,y,t) = w(x-\pi, y, t)$ still satisfies the PDE as a solution, and thus we conclude that S2 holds for this PDE

$\boxed{\therefore \text{ holds}}$

e) iv) $-2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$

Let $z = x - \pi$.

Since all terms are derivative terms, shifting them by $\pi$ with the $z$-substitution won't affect the PDE as the constant term '$-\pi$' in the derivative goes to 0 anyway.

$\therefore \quad -2\frac{\partial^2 u}{\partial x^2} = -2\frac{\partial^2 z}{\partial x^2}$

$2\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial^2 z}{\partial x \partial y}$

$-2\frac{\partial^2 u}{\partial y^2} = -2\frac{\partial^2 z}{\partial y^2}$

These new terms effectively form the same PDE $w(x-\pi, y, t) = u(x, y, t)$ which is just shifted by $\pi$, and S2 definitely holds for this PDE.

$\boxed{\therefore \text{ holds}}$

Q2

a) odd: $f(-x) = -f(x)$ — ①

even: $g(x) = g(-x)$ — ②

$(f+g)(-x) = f(-x) + g(-x)$

$-(f+g)(x) = -f(x) - g(x)$ — ③

subbing in ①, ②

$(f+g)(-x) = -f(x) + g(x)$

since this is $\neq$ ③, we can say that the statement is $\boxed{\text{false}}$

b) even: $f(x) = f(-x)$ — ①

continuously differentiable: $f'(x)$ is continuous on $I$

i.e. $\lim_{x \to c} f(x) = f(c)$ on $I$

and $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$

Then for $f'(x)$ to be odd.

$f'(-x) = -f'(x)$ — ②

differentiating both sides in ①

$f'(x) = -f'(-x)$

switching signs

$-f'(x) = f'(-x)$, which the same as ②, thus proved to be $\boxed{\text{true.}}$

c) even: $f(x) = f(-x)$ and integrable

$g(x) = \int^x f(\xi)\, d\xi = f(x)$

as (2), thus proved to be $\boxed{\text{true.}}$

c) even: $F(x) = F(-x)$ and integrable

$g(x) = \int_{\xi=0}^{x} F(\xi)\,d\xi = F(x)$

For $g(x)$ to be odd

$\Rightarrow g(-x) = -g(x)$

$\Rightarrow g(-x) = \int_{\xi=0}^{-x} F(\xi)\,d\xi = -\underbrace{\int_{\xi=0}^{x}}_{\text{switching limits to reverse sigs}} F(\xi)\,d\xi = -g(x)$

$\Rightarrow \int_{-x}^{\xi=0} F(\xi)\,d\xi$, now if we make $\xi$ -ve, it still gives us the same area under the integral as $F(\xi) = F(-\xi)$ even function.

Thus, $g(-x) = \int_{\xi=0}^{-x} F(\xi)\,d\xi = -\int_{\xi=0}^{x} F(\xi)\,d\xi = \int_{-x}^{\xi=0} F(\xi)\,d\xi = -g(x)$

Hence, $g(-x) = -g(x)$

Proved! $\boxed{\therefore \text{True}}$

d) z-periodic and integrable.

$\int_{x=y}^{y+z} F(x)\,dx = \int_{x=0}^{z} F(x) \quad \forall y \in \mathbb{R}$

we can split the LHS integral into 2 using laws of calculus on the bounds.

$\rightarrow \int_a^c F(x)\,dx = \int_a^b F(x)\,dx + \int_b^c F(x)\,dx$

$\Rightarrow \int_y^z F(x)\,dx + \int_z^{y+z} F(x)\,dx = \int_0^z F(x)\,dx \quad -(1)$

similarly, splitting the RHS integral with an arbitrary bound $y$ that cancels out.

$\Rightarrow \int_y^z F(x)\,dx + \int_z^{y+z} F(x)\,dx = \underbrace{\int_0^y F(x)\,dx + \int_y^z F(x)\,dx}_{(2)}$

subtracting (1)-(2)

$\Rightarrow \int_y^z F(x)\,dx + \int_z^{y+z} F(x)\,dx - \int_0^y F(x)\,dx - \int_y^z F(x)\,dx$

Terms 1 and 4 in the equation above are -ve's of each other and end up adding to 0.

$\Rightarrow \int_z^{y+z} F(x)\,dx - \int_0^y F(x)\,dx$

since we know the function is z-periodic

$\int_a^b F(x) = \int_a^{b-z} F(x+z) \rightarrow$ using this in the equation above on the first term

$\Rightarrow \int_z^{y+z-z} F(x+z)\,dx - \int_0^y F(x)\,dx \Rightarrow \int_0^y F(x+z)\,dx - \int_0^y F(x)\,dx$

Now, since the bounds are equal, we can combine the 2 integrals

$\Rightarrow \int_0^y (F(x+z) - F(x))\,dx \Rightarrow \int_0^y F(x+z-x)\,dx = \int_0^y F(z)\,dx = 0.$

Since, they are equal to 0, this must mean

$\int_{x=y}^{y+z} F(x)\,dx - \int_{x=0}^{z} F(x)\,dx = 0$, as these are the original integrals we rewrote in various ways throughout.

$\therefore \int_{x=y}^{y+z} F(x)\,dx = \int_{x=0}^{z} F(x)\,dx$, Thus proved!, $\boxed{\therefore \text{True.}}$

Q3

a) $\phi_n(x) = \sin(\lambda_n x)$ because the boundary conditions stating the function to be equal to 0 when $x=0$ and $x=L$. A $\cos(\lambda_n x)$ function would not be equal to 0, when $x=0$, sin $\cos(0) \neq 0$.

Now, that we know that $\phi_n(x)$ is of the form $\sin(\lambda_n x)$, we can use $x=L$ to find $\lambda_n$.

For $\sin(\lambda_n x)$ to be equal to 0 when $x=L$, $\sin(\lambda_n L) = 0$

$\therefore$, $\lambda_n L$ should be of the form $n\pi$ equating the two, we get

$\lambda_n = \frac{n\pi}{L}$

$\therefore$, $\lambda_n L$ should be of the form $n\pi$
equating the two, we get

$$\lambda_n = \frac{n\pi}{L}$$

Thus, $\boxed{\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$.

b) Now, using $\phi_n(x)$ from (a),
we get $\rightarrow$ $u(x) \sim \sum_{n=1}^{\infty} \hat{u}_n \sin\left(\frac{n\pi}{L}x\right)$

Plugging this into,

$-u'' + u = f$, we get

$$\sum_{n=1}^{\infty} \hat{u}_n \frac{n^2\pi^2}{L^2} \sin\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} \hat{u}_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

To invoke orthogonality relationship, we know

$$\int_{-1}^{1} \sin\left(\frac{m\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$

Thus multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$, and integrating over $(1,0)$, multiplying by 2.

$$2\int_0^1 \sin\left(\frac{m\pi}{L}x\right) \cdot \sum_{n=1}^{\infty} \hat{u}_n \frac{n^2\pi^2}{L^2} \sin\left(\frac{n\pi}{L}x\right) + 2\int_0^1 \sin\left(\frac{m\pi}{L}x\right) \cdot \sum_{n=1}^{\infty} \hat{u}_n \sin\left(\frac{n\pi}{L}x\right) = 2\int_0^1 \sin\left(\frac{m\pi}{L}x\right) f(x)$$

Using, the orthogonality relationship, we can simplify to

$$L \cdot \hat{u}_n \cdot \frac{n^2\pi^2}{L^2} + L\hat{u}_n = 2\sin\left(\frac{m\pi}{L}x\right) f(x)$$

Now, integrating both sides over $(0, L)$

$$\underbrace{L \cdot \hat{u}_n \cdot \frac{n^2\pi^2}{L^2} + L\hat{u}_n}_{\text{constant terms}} \int_0^L dx = \int_0^L 2f(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

Multiplying both sides by $1/L$

$$\Rightarrow \hat{u}_n \cdot \frac{n^2\pi^2}{L^2} + \hat{u}_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

RHS = $\hat{f}_n$ as given in the question, so replacing it we can get,

$$\hat{u}_n \frac{n^2\pi^2}{L^2} + \hat{u}_n = \hat{f}_n$$

$$\Rightarrow \hat{u}_n \left(\frac{n^2\pi^2}{L^2} + 1\right) = \hat{f}_n$$, and we know $\frac{n^2\pi^2}{L^2} = \lambda_n^2$

$$\therefore \boxed{\hat{u}_n = \frac{\hat{f}_n}{\lambda_n^2 + 1}}$$

c) If $f(x) = x$

Using relation from (b)

$$\hat{u}_n = \frac{\hat{f}_n}{\lambda_n^2 + 1}$$, where

$$\hat{f}_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$\Rightarrow \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi}{L}x\right) dx$$

using the integrals list, we know

$$\int x\sin(\alpha x)\, dx = \frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha}$$

subbing in $\alpha = n\pi$

$$\Rightarrow \frac{2}{L}\left[\frac{\sin(n\pi x)}{n^2\pi^2} - \frac{x\cos(n\pi x)}{n\pi}\Bigg|_0^L\right]$$

$$\Rightarrow \frac{2}{L}\left[\underbrace{\frac{\sin(n\pi L)}{n^2\pi^2}}_{\text{becomes 0 as } \sin(n\pi L) = 0} - \frac{L\cos(n\pi L)}{n\pi}\right]$$, other terms go to 0 due to $\sin(0)$ and $x$ in the denominator when $x = 0$.

, and $\cos(n\pi L) = L(-1)^n$

$$\hat{f}_n \Rightarrow \frac{2}{L}\left(\frac{-L^2(-1)^n}{n\pi}\right) \Rightarrow \frac{2}{L}\left(\frac{L^2(-1)^{n+1}}{n\pi}\right) \Rightarrow \frac{2L(-1)^{n+1}}{n\pi}$$

Now, finding $\hat{u}_n$ by plugging the $\hat{f}_n$ value we found into the expression for $\hat{u}_n$ found in (b).

$$\hat{u}_n = \frac{\hat{f}_n}{\lambda_n^2 + 1} \Rightarrow \frac{\frac{2L(-1)^{n+1}}{n\pi}}{\frac{n^2\pi^2}{L^2} + 1} \Rightarrow \frac{2L(-1)^{n+1}}{n\pi} \cdot \frac{L^2}{n^2\pi^2 + L^2}$$

$$\therefore \boxed{\hat{u}_n = \frac{2L^3(-1)^{n+1}}{n^3\pi^3 + L^2 n\pi}}, \quad n = 1, 2, 3 \ldots$$

Q4.

MATLAB IMPLEMENTATION

Code for all cosine series parts i.e Q4 b, d (cosine part), e

Code for all sine series parts. i.e Q4 c, d (sine part), f

$n^3\pi^3 + L^2 n\pi$

## Q4

(a) $f(x) = x^2 \ , \ x \in [0,1]$

Fourier cosine series

$$\frac{1}{2} f_0 + \sum_{n=1}^{\infty} \hat{f}_n \cos(n\pi x)$$

Fourier cosine coefficients

$$f_0 = 2\int_0^1 f(x)\,dx \Rightarrow 2\int_0^1 x^2 dx \Rightarrow \frac{2x^3}{3}\Big|_0^1 \Rightarrow \frac{2}{3}$$

$$\hat{f}_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx \Rightarrow 2\int_0^1 x^2\cos(n\pi x)\,dx$$

using integrals list, we know

$$2\int x^2\cos(\alpha x)\,dx = \left[\frac{2x\cos(\alpha x)}{\alpha^2} + \frac{(\alpha^2 x^2 - 2)\sin(\alpha x)}{\alpha^3}\right]\cdot 2$$

so, plugging in $\alpha$ as $n\pi$ be we get,

$$2\left[\frac{2x\cos(n\pi x)}{n^2\pi^2} + \frac{(n^2\pi^2x^2 - 2)\sin(n\pi x)}{n^3\pi^3}\Big|_0^1\right]$$

$$\Rightarrow 2\left[\frac{2\cos(n\pi)}{n^2\pi^2} + \underbrace{\frac{(n^2\pi^2 - 2)\sin(n\pi)}{n^3\pi^3}}\right]$$

, other terms becomes 0 due to $x$ and $\sin(n\pi x)$ in numerator.

$\sin(n\pi) = 0$ for any $n \in (0,1,2 \dots)$

and $\cos(n\pi) = (-1)^n$ for all $n \in (0,1,2,\dots)$

$$\Rightarrow \boxed{\hat{f}_n = \begin{cases} \dfrac{4}{n^2\pi^2}, & n \text{ is even} \\ \dfrac{-4}{n^2\pi^2}, & n \text{ is odd} \end{cases} \quad \text{OR} \quad \frac{4(-1)^n}{n^2\pi^2}}$$

Fourier sine series

$$\sum_{n=1}^{\infty} \hat{f}_n \sin(n\pi x)$$

Fourier sine coefficients

$$\hat{f}_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \Rightarrow 2\int_0^1 x^2\sin(n\pi x)\,dx$$

using integrals list, we know

$$2\int x^2\sin(\alpha x)\,dx = \left[\frac{2x\sin(\alpha x)}{\alpha^2} + \frac{(2-\alpha^2x^2)\cos(\alpha x)}{\alpha^3}\right]\cdot 2$$

so, plugging in $\alpha$ as $n\pi$ be we get,

$$2\left[\frac{2x\sin(n\pi x)}{n^2\pi^2} + \frac{(2-n^2\pi^2x^2)\cos(n\pi x)}{n^3\pi^3}\Big|_0^1\right]$$

$$\Rightarrow 2\left[\frac{(2-n^2\pi^2)\cos(n\pi)}{n^3\pi^3} - \frac{(2)\cos(0)}{n^3\pi^3}\right]$$

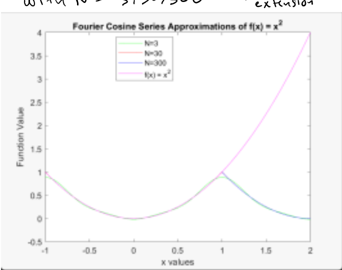
, other terms becomes 0 due to $x$ and $\sin(n\pi x)$ in numerator.

we know, $\cos(n\pi) = (-1)^n$ and $\cos(0) = 1$

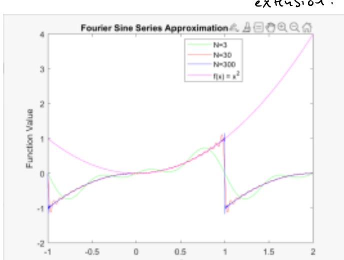
$$\Rightarrow 2\left[\frac{(2-n^2\pi^2)(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] = \frac{2}{n^3\pi^3}\left[(2-n^2\pi^2)(-1)^n - 2\right]$$

$$\boxed{\hat{f}_n = \begin{cases} \dfrac{-2}{n\pi}, & n \text{ is even} \\ \dfrac{2}{n\pi} - \dfrac{4}{n^3\pi^3}, & n \text{ is odd} \end{cases} \quad \text{OR} \quad 2\left(\frac{2-n^2\pi^2(-1)^n}{n^3\pi^3}\right) - 4}$$

(b) Plot for Fourier cosine series with $N = 3, 30, 300$ and periodic extension




(c) Plot for Fourier sine series with $N = 3, 30, 300$ and periodic extension.




(d) Truncated pointwise errors.

→ obtained from matlab, images on right. Refer to code for implementation.

# MATLAB IMPLEMENTATION

Code for all cosine series parts i.e Q4b, d (cosine part), e

```matlab
%Assignment 1 - Aaryan Nagpal, 1007792596, nagpalaa

%Question 4b - Plot for fourier cosine series approximations

%Define function, fourier cosine series x bounds, sum
x_val = linspace(-1,2,1000);
f = @(x_val) x_val.^2;
f_3_sum = 0;
f_30_sum = 0;
f_300_sum = 0;

%Initializing vector to hold fourier series value for n'th term
%(N=3,30,300)
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3 = zeros(1,length(x_val));
f_n_30 = zeros(1,length(x_val));
f_n_300 = zeros(1,length(x_val));

%Setting up truncated series to different N values
for n = 1:N_1
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_3 = f_n*cos(n*pi*x_val);
   f_3_sum = f_3_sum + f_3;
end
for n = 1:N_2
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_30 = f_n*cos(n*pi*x_val);
   f_30_sum = f_30_sum + f_30;
end
for n = 1:N_3
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_300 = f_n*cos(n*pi*x_val);
   f_300_sum = f_300_sum + f_300;
end

f_n_3  = 1/3+f_n_3+f_3_sum;
f_n_30  = 1/3+f_n_30+f_30_sum;
f_n_300  = 1/3+f_n_300+f_300_sum;

%Plotting and labelling
plot(x_val, f_n_3,'-g', LineWidth=0.5);
hold on
plot(x_val, f_n_30,'-r', LineWidth=0.5);
plot(x_val, f_n_300,'-b', LineWidth=0.5);
plot(x_val, f(x_val),'-m', LineWidth=0.5);
hold off
xlabel('x values');
ylabel('Function Value');
title('Fourier Cosine Series Approximations of f(x) = x^2');
legend('N=3', 'N=30', 'N=300', 'f(x) = x^2', 'Location', 'best');


%%
%Question 4 - part d (Fourier cosine pointwise error truncated)

% x = 0.5
%Define function, fourier cosine series x bounds, sum

f = @(x) x^2;
f_3_sum_x = 0;
f_30_sum_x = 0;
f_300_sum_x = 0;

%Initializing vector to hold fourier series value for n'th term
%(N=3,30,300)
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3_x = zeros(1,1);
f_n_30_x = zeros(1,1);
f_n_300_x = zeros(1,1);

%Setting up truncated series to different N values
x=0.5;
for n = 1:N_1
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_3_x = f_n*cos(n*pi*x);
   f_3_sum_x = f_3_sum_x + f_3_x;
end
for n = 1:N_2
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_30_x = f_n*cos(n*pi*x);
   f_30_sum_x = f_30_sum_x + f_30_x;
end
for n = 1:N_3
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_300_x = f_n*cos(n*pi*x);
   f_300_sum_x = f_300_sum_x + f_300_x;
end

f_n_3_x  = 1/3+f_n_3_x+f_3_sum_x;
f_n_30_x  = 1/3+f_n_30_x+f_30_sum_x;
f_n_300_x  = 1/3+f_n_300_x+f_300_sum_x;

%Error calculation for N=3,30,300 at x=0.5
%N=3
error_3_cos_x05 = abs(f(x)-f_n_3_x);
%N=30
error_30_cos_x05 = abs(f(x)-f_n_30_x);
%N=300
error_300_cos_x05 = abs(f(x)-f_n_300_x);

%%
% x = 0.9
%Define function, fourier cosine series x bounds, sum
f = @(x) x^2;
f_3_sum_x = 0;
f_30_sum_x = 0;
f_300_sum_x = 0;

%Initializing vector to hold fourier series value for n'th term
%(N=3,30,300)
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3_x = zeros(1,1);
f_n_30_x = zeros(1,1);
f_n_300_x = zeros(1,1);

%Setting up truncated series to different N values
x=0.9;
for n = 1:N_1
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_3_x = f_n*cos(n*pi*x);
   f_3_sum_x = f_3_sum_x + f_3_x;
end
for n = 1:N_2
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_30_x = f_n*cos(n*pi*x);
   f_30_sum_x = f_30_sum_x + f_30_x;
end
for n = 1:N_3
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_300_x = f_n*cos(n*pi*x);
   f_300_sum_x = f_300_sum_x + f_300_x;
end

f_n_3_x  = 1/3+f_n_3_x+f_3_sum_x;
f_n_30_x  = 1/3+f_n_30_x+f_30_sum_x;
f_n_300_x  = 1/3+f_n_300_x+f_300_sum_x;

%Error calculation for N=3,30,300 at x=0.9
%N=3
error_3_cos_x09 = abs(f(x)-f_n_3_x);
%N=30
error_30_cos_x09 = abs(f(x)-f_n_30_x);
%N=300
error_300_cos_x09 = abs(f(x)-f_n_300_x);

%%
%Question 4 - part e - Max error on x=[0,1] interval for Fourier cosine
%series approximation

%Define function, fourier cosine series x bounds, sum
x_val_2 = linspace(0,1,1000);
f = @(x_val_2) x_val_2.^2;
f_3_sum = 0;
f_30_sum = 0;
f_300_sum = 0;

%Initializing vector to hold fourier series value for n'th term
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3 = zeros(1,length(x_val_2));
f_n_30 = zeros(1,length(x_val_2));
f_n_300 = zeros(1,length(x_val_2));

%Setting up truncated series to different N values
for n = 1:N_1
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_3 = f_n*cos(n*pi*x_val_2);
   f_3_sum = f_3_sum + f_3;
end
for n = 1:N_2
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_30 = f_n*cos(n*pi*x_val_2);
   f_30_sum = f_30_sum + f_30;
end
for n = 1:N_3
   f_n = (4*(-1)^n)/(n^2*pi^2);
   f_300 = f_n*cos(n*pi*x_val_2);
   f_300_sum = f_300_sum + f_300;
end

f_n_3  = 1/3+f_n_3+f_3_sum;
f_n_30  = 1/3+f_n_30+f_30_sum;
f_n_300  = 1/3+f_n_300+f_300_sum;

%Truncated max error
%N=3
max_error_3 = max(abs(f(x_val_2)-f_n_3));
%N=30
max_error_30 = max(abs(f(x_val_2)-f_n_30));
%N=300
max_error_300 = max(abs(f(x_val_2)-f_n_300));
```

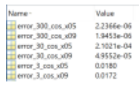
Code for all sine series parts. i.e Q4c, d (sine part), f

```matlab
%Assignment 1 - Aaryan Nagpal, 1007792596, nagpalaa

%Question 4c - Plot for fourier sine series approximations

%Define function, fourier sine series x bounds, sum
x_val = linspace(-1,2,1000);
f = @(x_val) x_val.^2;
f_3_sum = 0;
f_30_sum = 0;
f_300_sum = 0;

%Initializing vector to hold fourier series value for n'th term
%(N=3,30,300)
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3 = zeros(1,length(x_val));
f_n_30 = zeros(1,length(x_val));
f_n_300 = zeros(1,length(x_val));

%Setting up truncated series to different N values
for n = 1:N_1
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_3 = f_n*sin(n*pi*x_val);
   f_3_sum = f_3_sum + f_3;
end
for n = 1:N_2
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_30 = f_n*sin(n*pi*x_val);
   f_30_sum = f_30_sum + f_30;
end
for n = 1:N_3
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_300 = f_n*sin(n*pi*x_val);
   f_300_sum = f_300_sum + f_300;
end

f_n_3  = f_n_3+f_3_sum;
f_n_30  = f_n_30+f_30_sum;
f_n_300  = f_n_300+f_300_sum;

%Plotting and labelling
plot(x_val, f_n_3,'-g', LineWidth=0.5);
hold on
plot(x_val, f_n_30,'-r', LineWidth=0.5);
plot(x_val, f_n_300,'-b', LineWidth=0.5);
plot(x_val, f(x_val),'-m', LineWidth=0.5);
hold off
xlabel('x values');
ylabel('Function Value');
title('Fourier Sine Series Approximations of f(x) = x^2');
legend('N=3', 'N=30', 'N=300', 'f(x) = x^2', 'Location', 'best');


%%
%Question 4 - part d (Fourier sine pointwise error truncated)

% x = 0.5
%Define function, fourier sine series x bounds, sum

f = @(x) x^2;
f_3_sum_x = 0;
f_30_sum_x = 0;
f_300_sum_x = 0;

%Initializing vector to hold fourier series value for n'th term
%(N=3,30,300)
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3_x = zeros(1,1);
f_n_30_x = zeros(1,1);
f_n_300_x = zeros(1,1);

%Setting up truncated series to different N values
x=0.5;
for n = 1:N_1
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_3_x = f_n*sin(n*pi*x);
   f_3_sum_x = f_3_sum_x + f_3_x;
end
for n = 1:N_2
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_30_x = f_n*sin(n*pi*x);
   f_30_sum_x = f_30_sum_x + f_30_x;
end
for n = 1:N_3
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_300_x = f_n*sin(n*pi*x);
   f_300_sum_x = f_300_sum_x + f_300_x;
end

f_n_3_x  = f_n_3_x+f_3_sum_x;
f_n_30_x  = f_n_30_x+f_30_sum_x;
f_n_300_x  = f_n_300_x+f_300_sum_x;

%Error calculation for N=3,30,300 at x=0.5
%N=3
error_3_sin_x05 = abs(f(x)-f_n_3_x);
%N=30
error_30_sin_x05 = abs(f(x)-f_n_30_x);
%N=300
error_300_sin_x05 = abs(f(x)-f_n_300_x);

%%
% x = 0.9
%Define function, fourier sine series x bounds, sum
f = @(x) x^2;
f_3_sum_x = 0;
f_30_sum_x = 0;
f_300_sum_x = 0;

%Initializing vector to hold fourier series value for n'th term
%(N=3,30,300)
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3_x = zeros(1,1);
f_n_30_x = zeros(1,1);
f_n_300_x = zeros(1,1);

%Setting up truncated series to different N values
x=0.9;
for n = 1:N_1
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_3_x = f_n*sin(n*pi*x);
   f_3_sum_x = f_3_sum_x + f_3_x;
end
for n = 1:N_2
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_30_x = f_n*sin(n*pi*x);
   f_30_sum_x = f_30_sum_x + f_30_x;
end
for n = 1:N_3
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_300_x = f_n*sin(n*pi*x);
   f_300_sum_x = f_300_sum_x + f_300_x;
end

f_n_3_x  = f_n_3_x+f_3_sum_x;
f_n_30_x  = f_n_30_x+f_30_sum_x;
f_n_300_x  = f_n_300_x+f_300_sum_x;

%Error calculation for N=3,30,300 at x=0.9
%N=3
error_3_sin_x09 = abs(f(x)-f_n_3_x);
%N=30
error_30_sin_x09 = abs(f(x)-f_n_30_x);
%N=300
error_300_sin_x09 = abs(f(x)-f_n_300_x);

%%
%Question 4 - part f - Max error on x=[0,1] interval for Fourier
sine
%series approximation

%Define function, fourier sine series x bounds, sum
x_val_2 = linspace(0,1,100000);
f = @(x_val_2) x_val_2.^2;
f_3_sum = 0;
f_30_sum = 0;
f_300_sum = 0;

%Initializing vector to hold fourier series value for n'th term
N_1 = 3;
N_2 = 30;
N_3 = 300;
f_n_3 = zeros(1,length(x_val_2));
f_n_30 = zeros(1,length(x_val_2));
f_n_300 = zeros(1,length(x_val_2));

%Setting up truncated series to different N values
for n = 1:N_1
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_3 = f_n*sin(n*pi*x_val_2);
   f_3_sum = f_3_sum + f_3;
end
for n = 1:N_2
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_30 = f_n*sin(n*pi*x_val_2);
   f_30_sum = f_30_sum + f_30;
end
for n = 1:N_3
   f_n = (2/(n^3*pi^3))*((((2-(n^2*pi^2))*(-1)^n))-2);
   f_300 = f_n*sin(n*pi*x_val_2);
   f_300_sum = f_300_sum + f_300;
end

f_n_3  = f_n_3+f_3_sum;
f_n_30  = f_n_30+f_30_sum;
f_n_300  = f_n_300+f_300_sum;

%Truncated max error
%N=3
max_error_3 = max(abs(f(x_val_2)-f_n_3));
%N=30
max_error_30 = max(abs(f(x_val_2)-f_n_30));
%N=300
max_error_300 = max(abs(f(x_val_2)-f_n_300));
```

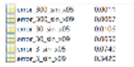
d) Truncated pointwise errors.

→ obtained from matlab, images on right. Refer to code for implementation.

| x | Number of Terms (N) | Fourier Cosine series pointwise truncated error | Fourier Sine series pointwise truncated error |
|---|---|---|---|
| 0.5 | 3 | 0.0180 | 0.0740 |
| | 30 | 2.1021e-04 | 0.0106 |
| | 300 | 2.2366e-06 | 0.0011 |
| 0.9 | 3 | 0.0172 | 0.3420 |
| | 30 | 4.9552e-05 | 0.0656 |
| | 300 | 1.9453e-06 | 0.0067 |



| Name | Value |
|---|---|
| error_300_cos_x05 | 2.2366e-06 |
| error_300_cos_x09 | 1.9453e-06 |
| error_30_cos_x05 | 2.1021e-04 |
| error_30_cos_x09 | 4.9552e-05 |
| error_3_cos_x05 | 0.0180 |
| error_3_cos_x09 | 0.0172 |



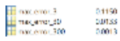
| | |
|---|---|
| error_300_sin_x05 | 0.0011 |
| error_300_sin_x09 | [illegible] |
| error_30_sin_x05 | 0.0106 |
| error_30_sin_x09 | [illegible] |
| error_3_sin_x05 | 0.0740 |
| error_3_sin_x09 | 0.3420 |

Yes, it is consistent with expected pointwise convergence of a series, as that suggests that as $n \to \infty$ for a piecewise continuously differentiable function, the fourier series approximation of a function $f(x)\ \forall x$, approaches $\frac{1}{2}|f(x^-)+f(x^+)|$. This trend is also shown in the table for both x values, where the difference between the fourier cosine/sine series approximation and the function value at those points approaches 0, suggesting that the fourier series approximation at that x goes to $\frac{1}{2}|f(x^-)+f(x^+)|$. Meaning that the approximation is pointwise convergent based on the error calculations.

e) Fourier cosine series max truncated error for $N = 3, 30, 300$, $x \in [0,1]$

→ obtained from matlab images on right. Refer to code for implementation.

| Number of Terms (N) | Fourier Cosine series max truncated error |
|---|---|
| 3 | 0.1150 |
| 30 | 0.0133 |
| 300 | 0.0013 |



| | |
|---|---|
| max_error_3 | 0.1150 |
| max_error_30 | 0.0133 |
| max_error_300 | 0.0013 |

Yes, the behaviour is consistent with the expected uniform convergence of a series as that states that for a continuous and piecewise continuously differentiable function, the max error (or the maximum absolute difference between the actual f(x) value for any x, and the fourier cos/sine series approximation for that x) would go to 0 as $n \to \infty$. As seen in the table above, the max error goes to 0 as n increases, allowing us to make the error ($\varepsilon$) arbitrarily small as $n \to \infty$, confirming uniform convergence of the series.

f) Fourier sine series max truncated error for $N = 3, 30, 300$, $x \in [0,1]$

→ obtained from matlab, images on right. Refer to code for implementation.

| Number of Terms (N) | Fourier sine series max truncated error |
|---|---|
| 3 | 1.0000 |
| 30 | 1.0000 |
| 300 | 1.0000 |

| | |
|---|---|
| max_error_3 | 1.0000 |
| max_error_30 | 1.0000 |
| max_error_300 | 1.0000 |

Yes, the behaviour of the max error in the fourier sine series would be different to that of the fourier cosine series as $N \to \infty$. The max error would always be 1 since $\sin(n\pi x) = 0$ when $x = 1$ no matter what the 'n' is, whereas the function value at $x = 1$, $f(x) = x^2 = 1$ always.

In the cosine series conversely, this error uniformly converges to 0 as $N \to \infty$. This is because the $\cos(n\pi x)$ part of the series can be simplified as $(-1)^n$, and this expression converges to 1 when $N \to \infty$, and cancels out the function value $f(x) = x^2 = 1$ at $x = 1$.