Assignment 3

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SOLUTIONS BELOW QUESTIONS ↓

Aaryan Nagpal
1007792596
nagpalaa

ESC384 Assignment 3 Due Wednesday, 25 October 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Important note: in preparation for the midterm exam on October 27th, the solution to this assignment will be posted on Quercus shortly after the submission deadline of October 25th at 9:10am. Hence, no late submissions *will be accepted for this assignment* (i.e., a late submission will result in a mark of 0%), with exception of those due to medical or family emergencies approved by the University through a term-work petition.

Problem 1. Heat equation: mixed boundary condition (26%)

Consider the homogeneous heat equation with homogeneous mixed boundary conditions:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } (0,1) \times \mathbb{R}_{>0},$$
$$\frac{\partial u}{\partial x}(x=0,t) = 0, \quad \forall t \in \mathbb{R}_{>0},$$
$$u(x=1,t) = 0, \quad \forall t \in \mathbb{R}_{>0},$$
$$u(x,t=0) = 1-x, \quad \forall x \in (0,1).$$

Answer the following question:

(a) (26%) Find a series representation of the solution u. The final expression should not contain any integrals.

Hint. The final expression for the (generalized) Fourier coefficients should be simple. If the final expression is complicated, double check the arithmetic.

Hint 2. You could validate the Fourier coefficients by evaluating the series for a sufficiently large number of terms using MATLAB. (This is not required.)

Problem 2. Nonhomogeneous heat equation (36%)

Consider a nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } (0,1) \times \mathbb{R}_{>0},$$
$$\frac{\partial u}{\partial x}(x=0,t) = 0, \quad \forall t \in \mathbb{R}_{>0},$$
$$u(x=1,t) = 0, \quad \forall t \in \mathbb{R}_{>0},$$
$$u(x,t=0) = 1-x, \quad \forall x \in (0,1).$$

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for some source term $f : (0,1) \to \mathbb{R}_{>0} \to \mathbb{R}$. Answer the following questions:

(a) (16%) Find a series representation of the solution u. The expression may contain a time integral with an integrand that depends on the time-dependent (generalized) Fourier coefficients of f.

Note. The final expression may be decomposed into multiple pieces (e.g., $u(x,t) = \sum_{n=1}^{\infty} \hat{u}(t)\cos(\cdots)$, $\hat{u}(t) = \ldots$, $\hat{f}(t) = \ldots$), but please make sure to identify all pieces in the final answer and (say) box all pieces to facilitate the grading process. Please follow this rule throughout this assignment.

(b) (10%) Let f be a time-independent discontinuous source function

$$f(x,t) = \begin{cases} 0, & x \le 1/2, \\ 1, & x > 1/2. \end{cases}$$

Find a series representation of the solution. The final expression may *not* contain an integral.

Hint. A sequence $a_1 = 1, a_2 = 1, a_3 = -1, a_4 = -1, a_5 = 1, a_6 = 1, a_7 = -1, \ldots$ can be compactly expressed as $a_n = (-1)^{\lfloor (n-1)/2 \rfloor}$, where $\lfloor \cdot \rfloor$ is the flooring function.

(c) (4%) Find the steady-state solution $u^s : (0,1) \to \mathbb{R}$ of the problem solved in (b).

Note. The steady-state solution is the solution obtained after a very long time (i.e., $t \to \infty$). By definition, it must be independent of time; not all initial value problems have a steady state solution.

(d) (6%) Prove or disprove each of the following statements about the steady state solution $u^s : (0,1) \to \mathbb{R}$ found in (c) associated with the discontinuous source function:

(i) u^s is continuously differentiable;

(ii) u^s depends on the initial condition.

Problem 3. Fundamental solution and method of reflection (38%)

(a) (9%) Verify that the fundamental solution

$$\Phi(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

satisfies the homogeneous heat equation in $\mathbb{R} \times \mathbb{R}_{>0}$; i.e., $\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = 0$.

(b) (9%) Recall that the general solution to the heat equation on the half line with a homogeneous Dirichlet boundary condition is

$$u_D(x,t) = \int_0^{\infty} (\Phi(x-\xi,t) - \Phi(x+\xi,t)) g(\xi) d\xi,$$

where Φ is the fundamental solution, and $g : \mathbb{R}_{>0} \to \mathbb{R}$ specifies the initial condition. Show that the solution satisfies (i) the heat equation $\frac{\partial u_D}{\partial t} - \frac{\partial^2 u_D}{\partial x^2} = 0$ in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and (ii) the homogeneous Dirichlet boundary condition at $x = 0$ for all $t > 0$.

Note. Assume that the order of differentiation and integration may be interchanged.

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(c) (5%) Recall that the general solution to the heat equation on the half line with a homogeneous Neumann boundary condition is

$$u_N(x,t) = \int_0^{\infty} (\Phi(x-\xi,t) + \Phi(x+\xi,t)) g(\xi) d\xi.$$

Show that the solution satisfies the homogeneous Neumann boundary condition at $x = 0$ for all $t > 0$.

(d) (15%) Let $g : \mathbb{R}_{>0} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 1-x, & x \le 1, \\ 0, & x > 1. \end{cases}$$

For each of the following statements, state whether the statement holds and justify your answer:

(i) $u_N(x,t) \ge u_D(x,t)$ for all $x \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{>0}$.

(ii) $u_N(x,t) > u_D(x,t)$ for all $x \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{>0}$.

(iii) $\int_0^{\infty} u_N(x,t)dx = \int_0^{\infty} g(x)dx$ for all $t \in \mathbb{R}_{>0}$.

(iv) $\int_0^{\infty} u_D(x,t)dx = \int_0^{\infty} g(x)dx$ for all $t \in \mathbb{R}_{>0}$.

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Aaryan Nagpal
1007792596
nagpalaa

ESC384 PDE's Assignment 3
SOLUTIONS

Q1 (a) Given homogeneous mixed B.C. for homogeneous Heat Eqn.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } (0,1) \times \mathbb{R}_{>0}$$
$$\frac{\partial u}{\partial x}(x=0,t) = 0 \quad \forall t \in \mathbb{R}_{>0}$$
$$u(x=1,t) = 0 \quad \forall t \in \mathbb{R}_{>0}$$
$$u(x,t=0) = 1-x \quad \forall x \in (0,1)$$

Finding solutions $u_n(x,t) = \psi_n(x) T_n(t)$
substituting into PDE, we get

$$\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} \Rightarrow \psi_n T_n' = \psi_n'' T_n$$

divide by $\psi_n T_n$ $\Rightarrow \frac{T_n'}{T_n} = \frac{\psi_n''}{\psi_n} = -\alpha_n$

For the boundary conditions, we can express them as

$$\frac{\partial u}{\partial x}(x=0,t) = 0 \Rightarrow \psi_n'(0) = 0$$
$$u(x=1,t) = 0 \Rightarrow \psi_n(1) = 0$$

Now, solving the spatial problem,

Spatial problem

$$-\psi_n'' = -\alpha_n \psi_n \quad \text{in } (0,1)$$
$$\psi_n'(x=0) = 0$$
$$\psi_n(x=1) = 0$$

This problem is setup in a way that the coefficients follow a Strum-liouville problem, so we know that the eigenvalues $(-\alpha_n) \geq 0$ to satisfy the ODE. $p = -1, w = 1, \alpha_2 = -1, \beta_1 = 1$, all else $= 0$.

Case : $\alpha_n = 0$

eqn $\rightarrow$ $-\psi_n'' = 0$

General sol : $\psi_n(x) = A_n(x) + B_n$

B.C at $x=0$, $\psi_n'(x=0) = A_n \overset{\text{want}}{=} 0$, $\Rightarrow A_n = 0$

B.C at $x=1$ $\psi_n(x=1) = B_n \overset{\text{want}}{=} 0$, $\Rightarrow B_n = 0$

$\Rightarrow \psi_n = 0$ (ie. trivial soln)

Case : $\alpha_n > 0$

General form of solution

$$\psi_n(x) = a_n \cos(\sqrt{-\alpha_n} x) + b_n \sin(\sqrt{-\alpha_n} x)$$

Now at B.C. $x=0$

$$\psi_n'(x=0) = 0 \Rightarrow -a_n \sin(\underbrace{\sqrt{-\alpha_n} \cdot 0}_{0 \ (\sin(0)=0)}) \cdot (\sqrt{-\alpha_n}) + b_n \cos(\underbrace{\sqrt{-\alpha_n} \cdot 0}_{1 \ (\cos(0)=1)}) (\sqrt{-\alpha_n}) \overset{\text{want}}{=} 0$$

$\Rightarrow b_n(\sqrt{\alpha_n}) \overset{\text{want}}{=} 0$, $\therefore$ either b_n or $\alpha_n = 0$

Lets say for now that $\alpha_n = 0$, now B.C at $x = 1$.

$$\Rightarrow \psi_n(x=1) = 0 \Rightarrow a_n \cos(\sqrt{0} \cdot x) + b_n \sin(\underbrace{\sqrt{0} \cdot x}_{0}) \overset{\text{want}}{=} 0$$

$\hookrightarrow \sin(0) = 0$

$\therefore$ we have, $a_n \underbrace{\cos(0)}_{1, \ \cos(0)=1} \overset{\text{want}}{=} 0$

$\Rightarrow \psi_n(x=1)=0 \Rightarrow a_n\cos(\sqrt{0}\cdot x) + b_n \underbrace{\sin(\sqrt{0}\cdot x)}_{\sin(0)=0} \overset{\text{want}}{=} 0$

$\therefore$ we have, $a_n \underset{\cos(0)=1}{\cos(0)} \overset{\text{want}}{=} 0$

but, $a_n = 0$. $\rightarrow$ once again trivial solution

Lets try, $b_n = 0$ at $x=1$ now

$\Rightarrow \psi_n(x=1) = 0 \Rightarrow a_n\cos(\sqrt{-\alpha_n}\cdot 1) + \overset{0}{b_n}\sin(\sqrt{-\alpha_n}\cdot 1) \overset{\text{want}}{=} 0$

$\Rightarrow a_n \cos(-\sqrt{\alpha_n}) \overset{\text{want}}{=} 0$

$\therefore -\sqrt{\alpha_n} = \left(n-\frac{1}{2}\right)\pi$

or $\alpha_n = -\left(n-\frac{1}{2}\right)^2\pi^2$

Thus we have, $\psi_n(x) = \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)$, $n = 1,2,3\ldots$

Now, solving the <u>Temporal ODE</u>

$T_n' = -\alpha_n T_n$

From lecture we have,

$\Rightarrow T_n(t) = \exp(-\alpha_n t) = \exp\left(-\left(n-\frac{1}{2}\right)^2\pi^2 t\right)$

General soln :

$$u(x,t) = \sum_{n=1}^{\infty} a_n \underbrace{\exp\left(-\left(n-\tfrac{1}{2}\right)\pi^2 t\right)}_{T_n(t)} \underbrace{\cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)}_{\psi_n(x)}$$

$\psi_n(x) \hookrightarrow$ From spatial problem part.

satisfies the PDE and B.C for any $\{a_n\}$

I.C. $u(x,t=0) \Rightarrow \sum_{n=1}^{\infty} a_n\cos\left(\left(n-\frac{1}{2}\right)\pi x\right) \overset{\text{want}}{=} g(x)$

we have $g(x) = 1-x$

We know $\left\{\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\right\}_{n=1}^{\infty}$ forms an orthogonal basis because it a soln to a S.L. problem.

$\therefore$ Deduce co-efficients using orthogonality

- multiply by $\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)$ and integrate over $0,1$

$$\int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right) \sum_{n=1}^{\infty} a_n\cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)dx \overset{\text{want}}{=} \int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right) g(x)\,dx$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \underbrace{\int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right)\cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)dx}_{=0 \text{ if } m\neq n \text{ orthogonality relationship}} = \int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right) g(x)\,dx$$

$$\Rightarrow a_m \underbrace{\int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right)^2 dx}_{=\frac{1}{2} \text{ (as obtained in the integral list)}} = \int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right) g(x)\,dx$$

$\therefore$ we have $\frac{a_m}{2} = \int_0^1 \cos\left(\left(m-\frac{1}{2}\right)\pi x\right) g(x)\,dx$

$g(x) \hookrightarrow (1-x)$ as in problem

$$a_m = 2\int_0^1 \cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right)(1-x)\,dx$$

$$\Rightarrow 2\left(\int_0^1 \cos\left(\left(m-\tfrac{1}{2}\pi x\right)\right)dx - \int_0^1 x\cos\left(\left(m-\tfrac{1}{2}\right)\pi x\right)dx\right)$$

Using the integral list we get,

$$2\left[\left(\frac{\sin\left(\left(m-\frac{1}{2}\right)\pi x\right)}{\left(m-\frac{1}{2}\right)\pi}\right)\Bigg|_0^1 - \frac{\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)}{\left(m-\frac{1}{2}\right)^2\pi^2}\Bigg|_0^1 - \frac{x\sin\left(\left(m-\frac{1}{2}\right)\pi x\right)}{\left(m-\frac{1}{2}\right)\pi}\Bigg|_0^1\right]$$

$\cos\left(\left(m-\frac{1}{2}\right)\pi\right) = 0$, $m = 1,2,3\ldots$

$$= 2\left[\frac{\overset{(-1)^{m+1}}{\sin\left(\left(m-\frac{1}{2}\right)\pi\right)}}{\left(m-\frac{1}{2}\right)\pi} - \frac{\overset{0}{\cos\left(\left(m-\frac{1}{2}\right)\pi\right)} - \overset{1}{\cos\left(\left(m-\frac{1}{2}\right)\pi\cdot 0\right)}}{\left(m-\frac{1}{2}\right)^2\pi^2} - \frac{\overset{(-1)^{m+1}}{\sin\left(\left(m-\frac{1}{2}\right)\pi\right)}}{\left(m-\frac{1}{2}\right)\pi}\right]$$

$$= \left[\cancel{\frac{(-1)^{m+1}}{\left(m-\frac{1}{2}\right)\pi}} - \frac{-1}{\left(m-\frac{1}{2}\right)^2\pi^2} - \cancel{\frac{(-1)^{m+1}}{\left(m-\frac{1}{2}\right)\pi}}\right]\cdot 2$$

$a_m = \frac{2}{\left(m-\frac{1}{2}\right)^2\pi^2}$, $\therefore a_n = \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2}$, $n = 1,2,3\ldots$

$$\left[\frac{}{(m-\frac{1}{2})\pi} \quad (m-\frac{1}{2})^2\pi^2 \quad (m-\frac{1}{2})\pi \right]$$

$$a_m = \frac{2}{(m-\frac{1}{2})^2\pi^2}, \quad \therefore a_n = \frac{2}{(n-\frac{1}{2})^2\pi^2}, \quad n = 1,2,3\dots$$

Thus, the final solution is of the form

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \exp\left(-\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\cos\left(\left(n-\frac{1}{2}\right)\pi x\right) \\ &= \sum_{n=1}^{\infty} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right) \cdot \frac{2}{(n-\frac{1}{2})^2\pi^2}\left(\exp\left(-\left(n-\frac{1}{2}\right)\pi^2 t\right)\right) \end{aligned}}$$

Q2 (a) Given non-homogeneous mixed B.C. for homogeneous Heat Eqn.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } (0,1)\times\mathbb{R}_{>0}$$

$$\frac{\partial u}{\partial x}(x=0,t) = 0 \quad \forall t \in \mathbb{R}_{>0}$$

$$u(x=1,t) = 0 \quad \forall t \in \mathbb{R}_{>0}$$

$$u(x,t=0) = 1-x \quad \forall x \in (0,1)$$

f = source term $\in (0,1) : \mathbb{R}_{>0} \to \mathbb{R}$

we know from Q1. a that solution exists and is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \hat{u}_n(t)\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)$$

substituting into the PDE, we obtain

$$\sum_{n=1}^{\infty} \hat{u}_n'(t)\cos\left(\left(n-\frac{1}{2}\right)\pi x\right) - \left(-\left(n-\frac{1}{2}\right)^2\pi^2 \hat{u}_n(t)\left(\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\right)\right) = f$$

$$\Rightarrow \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\left(\hat{u}_n'(t) + \left(n-\frac{1}{2}\right)^2\pi^2 \hat{u}_n(t)\right) = f$$

Now, multiplying both sides by $\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)$ and integrating over 0,1 to invoke orthogonality relationship

i.e 0 if $m \neq n$

$$\Rightarrow \int_0^1 \underbrace{\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)}_{\substack{=0 \text{ if } m\neq n \\ =1 \text{ o.w.}}}\left(\hat{u}_m'(t) + \left(m-\frac{1}{2}\right)^2\pi^2 \hat{u}_m(t)\right)dx = \int_0^1 f(x,t)\cdot\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)dx$$

$$\Rightarrow \underbrace{\left(\hat{u}_m'(t) + \left(m-\frac{1}{2}\right)^2\pi^2 \hat{u}_m(t)\right)}_{\substack{\text{constant or} \\ \text{no } x\text{-terms}}}\int_0^1 \cos\left(\left(m-\frac{1}{2}\right)\pi x\right)^2 dx = \int_0^1 f(x,t)\cdot\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)dx$$

$$\Rightarrow \hat{u}_m'(t) + \left(m-\frac{1}{2}\right)^2\pi^2 \hat{u}_m(t) = \underbrace{2\int_0^1 \cos\left(\left(m-\frac{1}{2}\right)\pi x\right)\cdot f(x,t)\,dx}_{\text{define this term as } \hat{f_n}(t)} \quad — (1)$$

Now, the initial condition says,

$$u(x,t=0) = 1-x \quad \forall x \in (0,1)$$

i.e $\sum_{n=1}^{\infty} u_n(t=0)\cos\left(\left(n-\frac{1}{2}\right)\pi x\right) = 1-x$ where $u_n(t=0)$ are the Fourier cosine coefficients of $g(x) = 1-x$.

$\therefore$ we have now,

$$u_n(t=0) = 2\int_0^1 (1-x)\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)dx = g_n$$

Using the integrating factor $\exp\left(\left(n-\frac{1}{2}\right)\pi^2 t\right)$ and multiplying to (1) (both sides)

$$\Rightarrow \frac{d}{dt}\left[\exp\left(\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\hat{u}_n(t)\right] = \exp\left(\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\hat{f}_n(t)$$

integrating from 0 to t

$$\Rightarrow \frac{d}{dt}\int_0^t \left[\exp\left(\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\hat{u}_n(t)\right] = \int_0^t \exp\left(\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\hat{f}_n(\tau)$$

$$\Rightarrow \left(\exp\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\hat{u}_n(t) - \hat{u}_n(0) = \int_0^t \exp\left(-\left(n-\frac{1}{2}\right)^2\pi^2(t-\tau)\right)\hat{f}_n(\tau)\,d\tau$$

$$\Rightarrow \left(\exp\left(\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)\hat{u}_n(t) - \hat{u}_n(0)\right) = \int_0^t \exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2(t-\tau)\right)\hat{F}_n(\tau)\,d\tau$$

moving everything but $\hat{u}_n(t)$ to one side

$$\Rightarrow \hat{u}_n(t) = \int_0^t \exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2(t-\tau)\right)\hat{F}_n(\tau)\,d\tau + \frac{u_n(0)}{\left(\exp\left(\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)\right)}$$

we found in part (a) that $u_n(0) = \dfrac{2}{\left(n-\frac{1}{2}\right)^2\pi^2}$

plugging in to get $\hat{u}_n(t)$

$$\Rightarrow \hat{u}_n(t) = \int_0^t \exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2(t-\tau)\right)\hat{F}_n(\tau)\,d\tau + \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2}\exp\left(-\left(n-\tfrac{1}{2}\right)\pi^2 t\right)$$

we also found $\hat{F}_n(t)$ earlier in this part

$$\hat{F}_n(t) = 2\int_0^1 \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)\cdot f(x,t)\,dx$$

and, finally $u(x,t) = \sum_{n=1}^{\infty} u_n \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)$

$\therefore$ we have found the series representation solution of u.

b) Given time-independent source function

$$F(x,t) = \begin{cases} 0, & x \le 1/2 \\ 1, & x > 1/2 \end{cases}$$

To get the series representation of the discontinuous function, we can use

$$\hat{F}_n(t) = 2\int_0^1 \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)F(x)\,dx = 2\int_{1/2}^1 \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)dx$$

using the integral list we have,

$$\frac{2}{\left(n-\frac{1}{2}\right)\pi}\sin\left(\left(n-\tfrac{1}{2}\right)\pi x\right)\Big|_{1/2}^{1} = 2\left[\frac{\overbrace{\sin\left(\left(n-\tfrac{1}{2}\right)\pi\right)}^{(-1)^{n+1}} - \overbrace{\sin\left(\left(n-\tfrac{1}{2}\right)\tfrac{\pi}{2}\right)}^{-1/\sqrt{2}\,(-1)^{\lfloor (n-1)/2\rfloor}}}{\left(n-\frac{1}{2}\right)\pi}\right]$$

$$= \frac{2}{\left(n-\frac{1}{2}\right)\pi}\left[(-1)^{n+1} - \frac{1}{\sqrt{2}}(-1)^{\lfloor (n-1)/2\rfloor}\right]$$

Also, due to the coefficients being time-independent, we have

$$u_n = \int_{\tau=0}^{t} \exp\left(-\left(n-\tfrac{1}{2}\right)\pi^2(t-\tau)\right)F_n(\tau)\,d\tau + \underbrace{g_n \exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)}$$

$\hookrightarrow b_n$: Fourier coefficient at time t.

$$\Rightarrow \frac{\hat{F}_n}{\left(n-\frac{1}{2}\right)^2\pi^2}\left[\exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2(t-\tau)\right)\right]\Big|_0^t + \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2}\exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)$$

$$\hat{u}_n(t) \Rightarrow \frac{\hat{F}_n}{\left(n-\frac{1}{2}\right)^2\pi^2}\left[1-\exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)\right] + \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2}\exp\left(-\left(-n-\tfrac{1}{2}\right)^2\pi^2 t\right)$$

$$\hat{u}_n(t) = \frac{2}{\left(n-\frac{1}{2}\right)\pi}\left[(-1)^{n+1} - \frac{1}{\sqrt{2}}(-1)^{\lfloor (n-1)/2\rfloor}\right]\cdot\frac{1}{\left(n-\frac{1}{2}\right)^2\pi^2}\left[1-\exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)\right] + \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2}\exp\left(-\left(n-\tfrac{1}{2}\right)^2\pi^2 t\right)$$

The $\hat{u}_n(t)$, $\hat{F}_n(t)$ value expressions found above satisfy the

The $\hat{u_n}(t)$, $\hat{F_n}(t)$ value expressions found above satisfy the condition $u(x,t) = \sum_{n=1}^{\infty} \hat{u_n}(t) \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)$

Thus confirmed.

(c) steady state solution as $t \to \infty$

Taking the limit of $u(x,t)$, we get

$$\lim_{t\to\infty} u(x,t) = \sum_{t\to\infty} \hat{u_n}(t) \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)$$

$$\Rightarrow \lim_{t\to\infty} \frac{\hat{F_n}}{\left(n-\frac{1}{2}\right)^2\pi^2} \underbrace{\left(1-\exp\left(-\left(n-\frac{1}{2}\right)^2\pi^2 t\right)\right)}_{\text{goes to 0 as } t\to\infty \text{ in limit}} + \lim_{t\to\infty} \underbrace{\frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2} \exp\left(-\left(n-\frac{1}{2}\right)\pi^2 t\right)}_{\text{goes to 0 as } t\to\infty \text{ in limit}}$$

$$\therefore \Rightarrow \frac{\hat{F_n}}{\left(n-\frac{1}{2}\right)^2\pi^2}$$

Therefore we can say,

$$u^s(x) = \sum_{n=1}^{\infty} \hat{u}_n^s \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)$$

$$u_n^s = \frac{\hat{F_n}}{\left(n-\frac{1}{2}\right)^2\pi^2} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)$$

steady state solution: $\boxed{u_n^s = \frac{2}{\left(n-\frac{1}{2}\right)^3\pi^3}\left[(-1)^{n+1} - \frac{1}{\sqrt{2}}(-1)^{\lfloor (n-1)/2 \rfloor}\right]}$

(d) (i) checking if u^s is continuously differentiable

we can either check if derivative of u^s is continuous or if $|nu_n^s|$ is converges since for $\sin \leq 1$

$$\frac{\partial}{\partial x} u_n^s = -\pi \sum_{n=1}^{\infty} u_n^s \left(n-\frac{1}{2}\right) \sin\left(n-\frac{1}{2}\right)\pi x, \text{ and}$$

and $|nu_n^s| \geq nu_n^s \geq \frac{1}{2}u_n^s$ for $n = 1,2,3\ldots$

$$\therefore |nu_n^s| = \left|\frac{2n}{\left(n-\frac{1}{2}\right)^3\pi^3}\left[(-1)^{n+1} - \frac{1}{\sqrt{2}}(-1)^{\lfloor (n-1)/2 \rfloor}\right]\right| \leq \frac{2\sqrt{2}\,n}{\left(n-\frac{1}{2}\right)^3\pi^3}$$

we can set this less than equal to $\frac{2\sqrt{2}\,n}{\left(n-\frac{1}{2}\right)^3\pi^3}$

Now, using a comparison test, we can compare this to $\frac{1}{n}\frac{2\sqrt{2}}{\pi^3}$ to check if the series converges

$$\Rightarrow \frac{2\sqrt{2}\,n}{\left(n-\frac{1}{2}\right)^3\pi^3} \leq \frac{1}{n}\frac{2\sqrt{2}}{\pi^3}$$, cancelling common terms, moving n to RHS denominator

Now since $\frac{1}{\left(n-\frac{1}{2}\right)^3} \leq \frac{1}{n^2}$, and we know $1/n^2$ converges

now since the inequality holds and $1/n^2$ converges,

$\therefore$ $1/(n-1/2)^3$ must also converge.

$\therefore$ by comparison test we have proved $|nu_n^s|$ converges, and thus $\boxed{u^s \text{ must be continuously differentiable}}$, proved!

(ii) Checking if u^s depends on the initial condition

The final solution has the part $u_n(t=0)$ in the form of the coefficient below, and that is the part that comes from the initial condition.

$$\Rightarrow \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2} \exp\left(-\left(n-\frac{1}{2}\right)^2\pi^2 t\right)$$, but as we found in part (c), as $t\to\infty$

the coefficient below, and that is the part that comes from the initial condition.

$$\Rightarrow \frac{2}{(n-\frac{1}{2})^2\pi^2} \exp\left(-\left(n-\tfrac{1}{2}\right)^2 \pi^2 t\right)$$, but as we found in part (c), as $t \to \infty$

the exponent term goes to 0 and the whole part vanishes from u_n^s, and since $\hat{u}_n^s$ is a part of u^s, there is no affect from it on the u^s term, so it es not depend on the initial condition.

Q3 a) Homogeneous heat eqn: $\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = 0$ in $\mathbb{R} \times \mathbb{R}_{>0}$

Fundamental solution given: $\Phi(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$

Plugging in the solution

$$\frac{\partial \Phi}{\partial t} = \frac{-1}{4t^{3/2}\sqrt{\pi}} \exp\left(\frac{-x^2}{4t}\right) + \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)\left(\frac{x^2}{4t^2}\right)$$

product rule ↗ chain rule ↑

Now, Finding $\frac{\partial^2 \Phi}{\partial x^2}$

First, $\frac{\partial \Phi}{\partial x} = \underbrace{\frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)}_{=\Phi(x,t)}\left(\frac{-2x}{4t}\right)$ — ①

we can write this as $\Rightarrow \Phi(x,t)\left(\frac{-2x}{4t}\right)$

then, $\frac{\partial^2 \Phi}{\partial x^2} = \underbrace{\Phi'(x,t)}_{\text{subbing in ① here}}\left(\frac{-2x}{4t}\right) + \Phi(x,t)\left(\frac{-2}{4t}\right)$

$$\Rightarrow \Phi(x,t)\left(\frac{-2x}{4t}\right)^2 + \Phi(x,t)\left(\frac{-2}{4t}\right)$$

Further, $\frac{\partial \Phi}{\partial t} = \underbrace{\frac{-1}{4t^{3/2}\sqrt{\pi}} \exp\left(\frac{x^2}{4t}\right)}_{\text{can be written as}} + \underbrace{\frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)}_{\Rightarrow \Phi(t)}\left(\frac{x^2}{4t^2}\right)$

$$\frac{-1}{2t} \cdot \underbrace{\frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)}_{\Rightarrow \Phi(x,t)}$$

Thus we have, $\Phi(x,t)\left(\frac{-1}{2t}\right) + \Phi(x,t)\left(\frac{x^2}{4t^2}\right)$

Now, $\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = 0 \Rightarrow \Phi(x,t)\left(\frac{-1}{2t} + \frac{x^2}{4t^2}\right) - \Phi(x,t)\left(\left(\frac{-2x}{4t}\right)^2 - \frac{2}{4t}\right) = 0$

$$\therefore \Phi(x,t)\left(\frac{-1}{2t} + \frac{x^2}{4t^2}\right) = \Phi(x,t)\left(\left(\frac{-2x}{4t}\right)^2 - \frac{2}{4t}\right)$$

Now comparing the parts in the brackets

$$\Rightarrow \frac{-1}{2t} + \frac{x^2}{4t^2} = \left(\frac{-2x}{4t}\right)^2 - \frac{2}{4t}$$

$$\frac{-1}{2t} + \frac{x^2}{4t^2} = \frac{4x^2}{16t^2} - \frac{2}{4t}$$

$\frac{2}{4t} \hookrightarrow -1/2t$ (divide by 2)

$$\Rightarrow \frac{-1}{2t} + \frac{x^2}{4t^2} = \frac{x^2}{4t^2} - \frac{1}{2t}$$

$\therefore \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = 0$ is satisfied by the Fundamental Solution.

proved!

b) We are given $u_D(x,t) = \int_0^\infty \left(\Phi(x-\xi,t) - \Phi(x+\xi,t)\right) g(\xi)\, d\xi$

i) we need to prove

b) We are given $u_D(x,t) = \int_0 \left(\Phi(x-\xi,t) - \Phi(x+\xi,t)\right) g(\xi)\, d\xi$

i) we need to prove

$$\frac{\partial u_D}{\partial t} - \frac{\partial^2 u_D}{\partial x^2} = 0$$

$$\frac{\partial}{\partial t} \int_0^\infty \left(\Phi(x-\xi,t) - \Phi(x+\xi,t)\right) g(\xi)\, d\xi - \frac{\partial^2}{\partial x^2} \left(\int_0^\infty \left(\Phi(x-\xi,t) - \Phi(x+\xi,t)\right) g(\xi)\, d\xi \right.$$

→ (as allowed in note)

Taking the integral outside and rearranging, we get (using superposition principle).

$$\Rightarrow \int_0^\infty \left[\underbrace{\frac{\partial}{\partial t}\left[\left(\Phi(x-\xi,t)\right) - \frac{\partial^2}{\partial x^2}\left[\left(\Phi(x-\xi,t)\right)\right]}_{=0\text{ , Fundamental soln of heat equation plugged in heat eqn}} g(\xi)\, d\xi \right] + \int_0^\infty \left[\underbrace{\frac{\partial}{\partial t}\left[\left(\Phi(x+\xi,t)\right) - \frac{\partial^2}{\partial x^2}\left[\left(\Phi(x+\xi,t)\right)\right]}_{=0\text{ , Fundamental soln of heat equation plugged in heat eqn}} g(\xi)\, d\xi \right]$$

$$\Rightarrow \int_0^\infty 0 \cdot g(\xi)\, d\xi + \int_0^\infty 0 \cdot g(\xi)\, d\xi = 0 + 0 = 0$$

$$\boxed{\text{Proved, } \frac{\partial u_D}{\partial t} - \frac{\partial^2 u_D}{\partial x^2} = 0}$$

ii) Given homogeneous Dirichlett B.C at $x=0$

Condition: $u_D(x=0,t)$

Plugging into the function we have,

$$\int_0^\infty \left(\Phi(\overset{0}{x}-\xi,t) - \Phi(\overset{0}{x}+\xi,t)\right) g(\xi)\, d\xi$$

$$\Rightarrow \int_0^\infty \left(\Phi(-\xi,t) - \Phi(\xi,t)\right) g(\xi)\, d\xi$$

Plugging in $\overset{\text{Fundamental soln}}{\Phi(x,t)} = \Phi(-\xi,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-\xi^2}{4t}\right)$

and $\Phi(\xi,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{\xi^2}{4t}\right)$

This becomes,

$$\int_0^\infty \underbrace{\left(\frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-\xi^2}{4t}\right) - \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{\xi^2}{4t}\right)\right)}_{=0\text{ , terms cancel}} g(\xi)\, d\xi$$

$$\Rightarrow \int_0^\infty 0 \cdot g(\xi)\, d\xi = 0$$

∴ homogenous Dirichlett B.C at $x=0$ is satisfied.

c) Neumann B.C (homogeneous) at $x=0$

In Neumann B.C we first need to take the derivative of $u_N(x,t)$ wrt x to evaluate at $x=0$

$$u_N(x,t) = \int_0^\infty \left(\Phi(x-\xi,t) - \Phi(x+\xi,t)\right) g(\xi)\, d\xi$$

First plug in $\Phi(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$ → fundamental soln.

$$\Rightarrow u_N(x) = \int_0^\infty \left(\frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-(x-\xi)^2}{4t}\right) - \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-(x+\xi)^2}{4t}\right)\right) g(\xi)\, d\xi$$

$$\therefore u_N'(x,t) = \int_0^\infty \left(\frac{-(x-\xi)\exp\left(\frac{-(x-\xi)^2}{4t}\right)}{4\sqrt{\pi}\, t^{3/2}} - \frac{(x+\xi)\exp\left[-\frac{(x+\xi)^2}{4t}\right]}{4\sqrt{\pi}\, t^{3/2}} \right) g(\xi)\, d\xi$$

Plug in $x=0$ in the expression, we get

$$\Rightarrow \int_0^\infty \underbrace{\left(\frac{-\xi}{4\sqrt{\pi}\, t^{3/2}} \exp\left(-\frac{(-\xi)^2}{4t}\right) - \frac{\xi}{4\sqrt{\pi}\, t^{3/2}} \exp\left(-\frac{(\xi)^2}{4t}\right)\right)}_{=0\text{ , terms cancel out}} g(\xi)\, d\xi$$

$$\Rightarrow \int_0^\infty 0 \cdot g(\xi)\, d\xi = 0$$

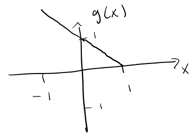
∴ homogeneous Neumann B.C at $x=0$ satisfied.

$\Rightarrow \int_0 0 \cdot g(\xi)\,d\xi = 0$

$\therefore$ homogeneous Neumann B.C at $x=0$ satisfied.

d) i) $g(x) = \begin{cases} 1-x, & x \le 1 \\ 0, & x > 1 \end{cases}$

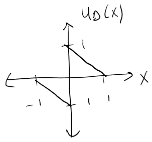
g(x) graph $\rightarrow$



Also graphing the Neumann and Dirichlett B.C solutions using the method of reflection.

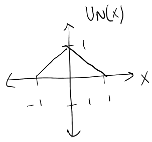
we know from lecture that Dirichlett B.C uses odd method of reflection

$\therefore u_D(x) \Rightarrow$



we know from lecture that Neumann B.C uses even method of reflection

$\therefore u_N(x) \Rightarrow$



we can check if $u_N(x,t) \geq u_D(x,t)$ $\forall x \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$

so comparing the graphs with $x \in \mathbb{R}$, and checking how they affect the solutions on $x \in \mathbb{R}_{>0}$.

so by checking the positivity observation on the graphs we conclude that $u_N(x,t)$ will always be +ve when $x \in \mathbb{R}_{>0}$ and also when $x \in \mathbb{R}$, since the $x<0$ part is just the even reflection of $x \geq 0$. However, in the case of $u_D(x,t)$, on $x \in \mathbb{R}_{>0}$ $\longrightarrow$ the solution is always +ve, but the odd reflection $x<0$ part is always -ve and decreases (on $x \in \mathbb{R}$ but not $x \notin \mathbb{R}_{>0}$) when x increases further. Therefore, the statement $\boxed{u_N(x,t) \geq u_D(x,t) \ \forall x \in \mathbb{R}_{>0}, t \in \mathbb{R}_{>0} \text{ is true and holds}}$

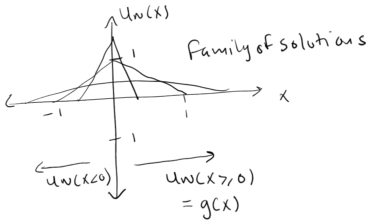
ii) we can check if $u_N(x,t) > u_D(x,t)$ $\forall x \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{>0}$

For this case we need to discuss the infinite propagation speed condition. Since any disturbance effects the solution at every point $(t>0)$, and the infinite speed causes it to immediately happen on the $u_N(x,t)$ solution for $x \in \mathbb{R}_{>0}$ due to the $x \in \mathbb{R}_{>0}$ and the initial condition (+ve) even reflection part. As for the $u_D(x,t)$ solution, the propagation happens on $x \notin \mathbb{R}_{>0}$ due to $x \in \mathbb{R}_{>0}$ and by the initial condition (-ve) where the odd reflection is shown. The -ve propagation in $u_D(x,t)$ as compared to the +ve one in $u_N(x,t)$ is what makes the $u_N(x,t)$ solutions larger.

$\boxed{\text{Therefore, } u_N(x,t) > u_D(x,t) \ \forall \text{ all } x \in \mathbb{R}_{>0}, t \in \mathbb{R}_{>0} \text{ is true and holds.}}$

iii) Checking $\int_0^\infty u_N(x,t)\,dx = \int_0^\infty g(x)\,dx$ $\forall$ all $t \in \mathbb{R}_{>0}$

This case requires the conservation of thermal energy condition for a fundamental solution on the entire axis $(-\infty, \infty)$.



Taking the integral to get thermal energy we get

$\int_0^\infty u_N(x,t)\,dx = 1$ on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$, this is equivalent to

$\int_0^\infty u_N(x,t)\,dx = \int_{-\infty}^0 u_N(x,t)\,dx$

due to symmetry as seen in the graph

$\therefore$ we have $\int^\infty g(x)\,dx$ = total thermal energy on $\mathbb{R}_{>0} \times \{t=0\}$

due to symmetry as seen in the graph

$\therefore$ we have $\int_0^\infty g(x)dx$ = total thermal energy on $\mathbb{R}_{>0} \times \{t=0\}$

and since in homogeneous Neumann case the heat flux input is 0,

$\Rightarrow$ thus we have by conservation of thermal energy on $\mathbb{R}_{>0}, t>0$

$\int_0^\infty g(x)\,dx = \int_0^\infty u_N(x,t)\,dx$, $\therefore$ proved! True

iv) Checking $\int_0^\infty u_D(x,t)\,dx = \int_0^\infty g(x)dx \quad \forall t \in \mathbb{R}_{>0}$

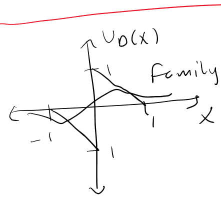
Using the same thermal energy consevation condition as above we have, $\int_0^\infty u_D(x,t)dx$ = conserved on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$

We also know that $\int_0^\infty g(x)dx$ = total thermal energy on $\mathbb{R}_{>0} \times \{t=0\}$,

this is also conserved as t increases since in the homogeneus Dirichlet case there is not a heat flux input which would disrupt it i.e. heat input flux $= 0$.

But due to the odd reflection as shown in the graph, the energy needs to go from $x \in \mathbb{R}_{>0}$ to $x \notin \mathbb{R}_{>0}$ as time increases to maintain this conserved state, and thus over time the energy in $x \in \mathbb{R}_{>0}$ for $\int_0^\infty u_D(x,t)\,dx$ decreases as compared to $\int_0^\infty g(x)dx$ on $x \in \mathbb{R}_{>0} \times \{t=0\}$ at time 0, where the max energy is conserved and present

As a result of conservation of thermal energy being needed to be conserved



we thus have

$\int_0^\infty u_D(x,t)dx < \int_0^\infty g(x)dx$

as t increases from $t=0$

Thus the statement is False.