# ESC384 Assignment 5

Due Wednesday, 6 December 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

## Problem 1. Finite difference method for $L$-shaped domain (30%)

In this problem, we consider an eigenproblem on the $L$-shaped domain

$$\Omega \equiv (0,1)^2 \setminus ([0,1/2] \times [1/2,1]).$$

This is a unit square domain with the top left subdomain missing. (Note in particular the location of the missing subdomain is different from the $L$-shaped domain in the course notes.) The eigenproblem is given by the following: find eigenfunctions $\phi_n$ and the associated eigenvalues $\lambda_n$, $n = 1, 2, \ldots$, such that

$$\begin{aligned} -\Delta \phi_n &= \lambda_n^2 \phi_n \quad \text{in } \Omega, \\ \phi_n &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Physically, the $n$-th eigenfunction and eigenvalue represents the vibration mode and the frequency of the $L$-shaped "drum" (i.e., membrane). It can be shown all eigenvalues are (strictly) positive. Unfortunately the eigenproblem on the $L$-shaped domain is not solvable by analytical means. We hence consider a finite difference approximation. The discretized problem is of the form

$$\hat{A}\tilde{\phi}_n = \tilde{\lambda}_n^2 \tilde{\phi}_n,$$

where $\hat{A}$ is the finite difference matrix associated with the Laplacian, and $\tilde{\phi}_n$ and $\tilde{\lambda}_n$ are the finite difference approximation of the $n$-th eigenfunction and eigenvalue. Answer the following questions:

(a) (14%) Complete the `make_matrix.m` function, which `ldrum.m` calls. See the template for detailed instructions.

*Note.* Please include a hard copy of the code in the assignment and also upload the code to Quercus to facilitate the grading process.

*Note 2.* You do *not* have to use the template if you would rather code everything from scratch.

(b) (7%) We wish to verify the code is working correctly. For the square domain case (i.e., `ldomain = false` in `ldrum.m`), the analytical value of the first and fourth eigenvalues are $\lambda_1 = \sqrt{2\pi^2}$ and $\lambda_4 = \sqrt{8\pi^2}$. Compute the finite difference approximation of the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_4$ for the grid spacing (i.e., the reciprocal of `ns1`) of 1/8, 1/16, and 1/32. Report the error in the eigenvalues in a table, and verify the error converges. What is the observed convergence rate?

*Note.* The table should have $n_s$, $|\lambda_1 - \tilde{\lambda}_1|$, and $|\lambda_4 - \tilde{\lambda}_4|$ as the column heading. Please include the table in the hard copy of the assignment.

(c) (7%) We now solve the eigenproblem on the $L$-shaped domain (i.e., set `ldomain = true`). Using whatever the grid spacing (i.e., `ns1`) necessary, compute the first four eigenvalues accurately to at least 0.1. Report (i) the grid spacing used and (ii) the four eigenvalues. Also (iii) plot, in four separate figures, the associated eigenfunctions in two-dimensional plot (i.e., set `use_pretty_plot = false`).

*Hint.* Try different values of grid spacing until the eigenvalues are "sufficiently converged".

(d) (2%) Plot the first eigenfunction of the $L$-shaped domain using `use_pretty_plot = true`. Where have you seen the figure?

## Problem 2. Wave equation: separation of variables (40%)

Consider the initial-boundary value problem on $\Omega \equiv (0, 1)$:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \\ \frac{\partial u}{\partial x} &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ u &= g \quad \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial u}{\partial t} &= h \quad \text{on } \Omega \times \{t = 0\}. \end{aligned}$$

Answer the following questions.

(a) (10%) Find a family of separable solutions $u_n(x, t) = \phi_n(x) T_n(t)$, each of which satisfies the PDE and boundary condition, but not necessarily the initial conditions. Each separable solution may contain unknown coefficients.

(b) (5%) Find a series representation of the solution in terms of appropriate (generalized) Fourier coefficients of $g$ and $h$.

(c) (5%) Let $g(x) = 3\cos(2\pi x)$ and $h(x) = \cos(5\pi x)$. Find the solution $u(x, t)$.

(d) (5%) Let $h = 0$. Find the traveling wave (i.e., d'Alembert's) form of the solution in terms of an appropriate periodic extension of $g$.

*Hint:* $\cos(a)\cos(b) = \frac{1}{2}(\cos(a + b) + \cos(a - b))$.

For the following questions, consider the initial condition

$$g(x) = \begin{cases} 0, & x \in [0, 3/8), \\ 8(x - 3/8), & x \in [3/8, 1/2), \\ 8(-x + 5/8), & x \in [1/2, 5/8), \\ 0, & x \in [5/8, 1], \end{cases} \qquad \text{and} \qquad h(x) = 0.$$

(The function $g$ is a "hat" function with a peak height of 1 at $x = 1/2$ and a half-width of 1/8.)

(e) (7%) Sketch the solution for time $t = 0$, 1/4, 1/2, 3/4, and 1.

(f) (8%) Recall the solution of the wave equation with homogeneous Dirichlet boundary conditions. Sketch the solution for time $t = 0$, $1/4$, $1/2$, $3/4$, and $1$. Compare and comment on the way the wave reflects at the boundaries for the homogeneous Neumann and homogeneous Dirichlet boundary conditions.

*Note.* You do not need to re-derive the solution to the wave equation with homogeneous Dirichlet boundary conditions.

## Problem 3. Finite difference method for the wave equation (30%)

We consider the solution of the wave equation using the finite difference method. For homogeneous Dirichlet boundary conditions, the semi-discrete form of the equation is given by

$$\begin{aligned} \frac{d^2\tilde{u}}{dt^2} &= -\hat{A}\tilde{u}, \quad t \in (0, t_f], \\ \tilde{u}(t=0) &= \tilde{u}^0, \\ \frac{d\tilde{u}}{dt}(t=0) &= \tilde{v}^0, \end{aligned}$$

where the $\hat{A}$ is the finite difference matrix associated with the Laplacian, as constructed in Problem 1. To apply the Crank-Nicolson time discretization, we first rewrite the equation in the first-order form: let $\tilde{v} \equiv \frac{d\tilde{u}}{dt}$ so that

$$\begin{aligned} \frac{d\tilde{u}}{dt} &= \tilde{v}, \quad t \in (0, t_f], \\ \frac{d\tilde{v}}{dt} &= -A\tilde{u}, \quad t \in (0, t_f], \\ \tilde{u}(t=0) &= \tilde{u}^0, \\ \frac{d\tilde{u}}{dt}(t=0) &= \tilde{v}^0, \end{aligned}$$

which can be expressed more compactly as

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} &= \begin{pmatrix} 0 & I \\ -\hat{A} & 0 \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \quad t \in (0, t_f], \\ \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}_{t=0} &= \begin{pmatrix} \tilde{u}^0 \\ \tilde{v}^0 \end{pmatrix}. \end{aligned} \tag{1}$$

Answer the following questions:

(a) (6%) Apply the Crank-Nicolson method to (1) to obtain the fully discrete equation for $\tilde{\tilde{u}}^j$ and $\tilde{\tilde{v}}^j$, so that $\tilde{\tilde{u}}^j \approx \tilde{u}(t^j)$ and $\tilde{\tilde{v}}^j \approx \tilde{v}(t^j)$ for time indices $j = 1, \dots, J$.

*Hint.* Recall that the general form of the Crank-Nicolson method for $\frac{d\tilde{w}}{dt} = f(\tilde{w}, t)$ is $\frac{\tilde{w}^j - \tilde{w}^{j-1}}{\Delta t} = \frac{1}{2}(f(\tilde{w}^j, t^j) + f(\tilde{w}^{j-1}, t^{j-1}))$. Set $\tilde{w} \equiv \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$, identify the associated $f(\tilde{w}, t)$ as $B\tilde{w}$ for some matrix $B$, and apply the Crank-Nicolson method.

(b) (15%) Implement the Crank-Nicolson method by completing `wave.m` for `ns1 = 128` (i.e., $127^2$ interior nodes), and the initial conditions vectors $\tilde{u}^0 \in \mathbb{R}^{127^2}$ and $\tilde{v}^0 \in \mathbb{R}^{127^2}$ provided in `ic.mat`. Choose the time step of $\Delta t = 0.005$. Plot the solution at $t = 0.0$, $t = 0.15$, and $t = 0.3$.

(c) (4%) Provide the top-down view of the solution at the final time $T = 0.3$ by using `view(0, 90)`. You should see a six-letter message; what is it?

(d) (5%) This "message encryption by PDE" relies on certain properties of the wave equation. I encrypted the message by marching backward in time, and you decrypted the message by marching forward in time. Discuss whether this would be possible using the heat equation (instead of the wave equation).

*Hint.* Solution to any PDE can be thought of as a linear combination of various modes. Information is "lost" over time if the modes are no longer discernible. How do the modes of wave and heat equations evolve over time?