CVaR Lab 6

- rets $\in {}^{T}\mathbb{R}^{n}$, use each return
- observation as a scenario

$$\text{rets} = \begin{bmatrix} (r^{(1)})^{T} \\ \vdots \\ (r^{(T)})^{T} \end{bmatrix} \qquad r^{(i)} \in {}^{n}\mathbb{R}^{1},\ (r^{(i)})^{T} \in {}^{1}\mathbb{R}^{n}$$

- is this the only way to define scenarios?

  $\longrightarrow$ No! but it is what we are going to do

- α – CVaR minimization ; α is confidence

$$\mathcal{P}\left\{\begin{array}{lll} \min\limits_{\gamma, x, z} & \gamma + \frac{1}{(1-\alpha)T}\sum_{s=1}^{T} z_s & \\ & z_s \geq 0 \quad \forall s = 1, \dots T & (C1) \\ & z_s \geq -(r^{(s)})^{T}x - \gamma \quad \forall s = 1 \dots T & (C2) \\ & \mathbb{1}^{T}x = 1 & (C3) \\ & \mu^{T}x \geq R & (C4) \end{array}\right.$$

Note : $\qquad x \in \mathbb{R}^{n},\ z \in \mathbb{R}^{T},\ \gamma \in \mathbb{R}$

(1) at optimality $\gamma = \underline{VaR_\alpha}$

(2) (1) $\Rightarrow$ $z_s \geq 0$, $z_s \geq -(r^{(s)})^{T}x - VaR_\alpha$

since we are minimizing $\frac{1}{(1-\alpha)T}\sum z_s$ in the objective, at optimality @ least one of (C1) or (C2) is tight for each scenario

$\Rightarrow z_s = \max\{0, -(r^{(s)})^{T}x - VaR_\alpha\} \ \forall s$

i.e z_s = the loss in excess of VaR_α
@ optimality! Cool!

(3) the objective and constraints are all linear in $x, z, \gamma \Rightarrow$

ok lets solve. Matlab linprog solves

$$\mathcal{M} \begin{cases} \min_{\tilde{x}} \; c^T \tilde{x} \\ A_{eq} \tilde{x} = b_{eq} \\ A \tilde{x} \leq b \\ lb \leq \tilde{x} \leq ub \end{cases}$$

matlab doc calls this f (for c)

linprog

- we have to structure the inputs into $\mathcal{M}$ so that solving $\mathcal{M}$ solves $\mathcal{P}$

- lets arrange our x, z, and γ so that

$$\tilde{x} = \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix} \in \mathbb{R}^{n+T+1}$$

lets proceed in the order given in the sample code lower bounds: x is free, $z \geq 0$, γ is free

i.e $\quad -\infty \leq x_i \quad \forall\, i = 1 \dots n$

$0 \leq z_s \quad \forall\, s = 1 \dots T$

$-\infty \leq \gamma$

$$\text{inf}(n,1) = \begin{bmatrix} \infty \\ \vdots \\ \infty \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

i.e $\begin{bmatrix} -\text{inf}(n,1) \\ 0_{T\times 1} \\ -\text{inf} \end{bmatrix} \leq \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix}$

this is in Matlab

upper bounds: there are none. You can set that to the empty [ ]

Equality (C4) is the only one

$$\mathbb{1}^T x = 1 \iff \underline{\mathbb{1}^T x + 0^T z + 0\gamma = 1}$$

$$\iff [\mathbb{1}^T \;\; 0^T \;\; 0] \underbrace{\begin{bmatrix} x \\ z \\ \gamma \end{bmatrix}}_{\tilde{x}} = 1$$

i.e

$$\underbrace{\underline{[\mathbb{1}_{1\times n} \;\; 0_{1\times T} \;\; 0_{1\times 1}]}}_{A_{eq} \in \mathbb{R}^{1 \times (n+T+1)}} \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix} = \underbrace{1}_{b_{eq}}$$

Inequalities

capital S in code

$$z_s \geq \quad \forall\, s = 1 \ldots T \quad (C2)$$

$$\mu^T x \geq R \quad (C4)$$

$$\underbrace{A\tilde{x}} \leq \underbrace{b}$$

$$-(r^{(s)})^T x \;-\; z_s \;-\; \gamma \;\leq\; 0 \qquad \underline{\forall s = 1 \ldots T}$$

$$\Big\{ -\mu^T x \;\leq\; -R$$

vertical concat

$$\underbrace{\begin{bmatrix} -(r^{(1)})^T x \\ -(r^{(2)})^T x \\ \vdots \\ -(r^{(T)})^T \end{bmatrix}}_{x} \underbrace{\begin{matrix} -z_1 & & \\ & z_2 & \\ & & -z_S \end{matrix}}_{z} \underbrace{\begin{matrix} -\gamma \\ -\gamma \\ -\gamma \end{matrix}}_{\gamma} \leq \begin{bmatrix} 0_{T\times 1} \\ -R \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix}$$

$$\Big[\underset{T\times n}{-\text{rets}},\ \underset{T\times T}{\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}},\ \underset{T\times 1}{\begin{bmatrix} -1 \\ -1 \\ \vdots \end{bmatrix}} \Big] \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix} \leq 0_{T\times 1}$$

Next, do C4

$$-\mu^T x \leq -R$$

$$\Updownarrow$$

$$-\mu^T x + 0^T z + 0\gamma \leq -R$$

$$\Updownarrow$$

$$\begin{bmatrix} -\mu^T + 0_{1\times T} & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix} \leq -R$$

$$\begin{bmatrix} -\text{rets}, & \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}, & \begin{bmatrix} -1 \\ -1 \\ \vdots \end{bmatrix} \\ -\mu^T + 0_{1\times T} & & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0_{T\times 1} \\ -R \end{bmatrix}}_{b}$$

$$\Downarrow$$

$$A = \begin{bmatrix} -\text{rets}, & \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}, & \begin{bmatrix} -1 \\ -1 \\ \vdots \end{bmatrix} ; & \mu^T & 0_{1\times T} & 0_{1\times 1} \end{bmatrix}$$

objective:

$$\gamma + \frac{1}{(1-\alpha)T} \underbrace{\sum_{s=1}^{T} z_s}_{1^T z}$$

$$= 0^T x + \frac{1}{(1-\alpha)T} 1^T z + 1_{1\times 1}\, \gamma$$

$$= \begin{bmatrix} 0^T & \frac{1}{(1-\alpha)T} 1^T & 1_{1\times 1} \end{bmatrix} \begin{bmatrix} x \\ z \\ \gamma \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0_{1\times n} & \frac{1}{(1-\alpha)T}\,1_{1\times T} & 1_{1\times 1} \end{bmatrix}}_{c}{}_{\boxed{1}\cancel{\times(n+T+1)}} \begin{bmatrix} x \\ z \\ y \end{bmatrix}_{\cancel{(n+T+1)\times}\boxed{1}}$$