Assignment 2

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SOLUTIONS BELOW QUESTIONS ↓

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ESC384 Assignment 2 Due Wednesday, 11 October 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Problem 1. Properties of Fourier series (48%)

Throughout this problem, function $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = \pi x$.

(a) (4%) Find the Fourier cosine series of f.

(b) (6%) Let f_N be the N-term truncated Fourier cosine series of f. State whether each of the following statements holds. Briefly justify your answers.

(i) $\int_0^1 (f(x) - f_N(x))^2 dx \to 0$ as $N \to \infty$.
(ii) $\max_{x \in [0,1]} |f(x) - f_N(x)| \to 0$ as $N \to \infty$.
(iii) For any fixed $x \in [0, 1]$, $|\frac{1}{2}(f(x^-) + f(x^+)) - f_N(x)| \to 0$ as $N \to \infty$.

(c) (6%) Using the Fourier cosine series found in (a), evaluate $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

Hint. First find the relationship between the Fourier cosine coefficients a_{2k-1}, $k = 1, 2, \ldots$, and the summand. Then evaluate the Fourier cosine series at a well-chosen x, and invoke a convergence property of the Fourier series. Note that you can also verify your solution using MATLAB. (This family of problem is known as Basel problem; it was first proposed in 1650 and was solved by Euler in 1734.)

(d) (4%) Find the derivative of the Fourier cosine series found in (a).

(e) (4%) The series found in (d) is the Fourier sine series of some function g on $[0, 1]$. Identify the function g, and verify that the Fourier sine series of g is the same as the series in (d).

(f) (6%) Sketch at least two periods of the Fourier series found in (d). If the function is discontinuous, clearly indicate the value of the function at discontinuities using ◦ or •.

(g) (6%) Let g_N be the N-term partial sum of the series found in (d). State whether each of the statements in (b) holds (where f and f_N and replaced by g and g_N, respectively). Briefly justify your answers.

(h) (6%) Evaluate the integral

$$I(\{a_n\}_{n=0}^{\infty}) = \int_0^1 \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\right)^2 dx$$

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in terms of a_n, $n = 0, 1, 2, \ldots$. The final expression should not contain an integral.

Hint. First rewrite the integrand as $(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x))(\frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(m\pi x)) = \frac{1}{4}a_0^2 + a_0 \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n,m=1}^{\infty} a_n a_m \cos(n\pi x)\cos(m\pi x)$. Then carry out the integration term by term.

(i) (6%) Evaluate $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$.

Hint. Substitute the Fourier cosine coefficients found in (b) to the expression found in (h) and manipulate the relationship.

Problem 2. Sturm-Liouville problem (24%)

Consider an eigenproblem

$$-(x\phi_n')' = \lambda_n \frac{1}{x}\phi_n \quad \text{in } (1, 2),$$
$$\phi_n(1) = 0,$$
$$\phi_n(2) = 0.$$

(This eigenproblem arises when we consider Laplace's equation in the polar coordinate system.) Answer the following questions:

(a) (4%) Verify that the eigenproblem is a regular Sturm-Liouville problem. Identify p, q, w, α_1, α_2, β_1, and β_2.

(b) (10%) The general solution of the ODE is of the form

$$\phi_n(x) = a_n \cos(\mu_n \log(x)) + b_n \sin(\mu_n \log(x)),$$

for some a_n, b_n, and μ_n. Find the eigenfunctions and eigenvalues of the eigenproblem.

Note. The "log" is the natural log.

(c) (5%) Evaluate

$$\int_1^2 \frac{1}{x}\phi_n(x)\phi_m(x)dx \quad m \neq n.$$

Note. You need not show all work, but justify your answer.

(d) (5%) Evaluate

$$\int_1^2 x\phi_n'(x)\phi_m'(x)dx \quad m \neq n.$$

Hint. Use a combination of the solution to (c) and the boundary value eigenproblem.

Problem 3. Heat equation (28%)

Consider an initial-boundary value problem associated with the heat equation on $\Omega \equiv (0, 1)$,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega \times \mathbb{R}_{>0},$$
$$u = g \quad \text{on } \Omega \times \{t = 0\},$$

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for some initial condition function $g : \Omega \to \mathbb{R}$. For part (a)–(c), we impose a Dirichlet boundary conditions of the form

$$u = u_L \quad \text{on } \{x = 0\} \times \mathbb{R}_{>0},$$
$$u = u_R \quad \text{on } \{x = 1\} \times \mathbb{R}_{>0},$$

for (time-independent) constants u_L and u_R. Prove or provide a counterexample to each of the following statements:

(a) (5%) If $g \geq 0$, $u_L \geq 0$, and $u_R \geq 0$, then $u \geq 0$.

Note. The statement $u \geq 0$ should be interpreted as "u is non-negative everywhere", i.e., $u(x,t) \geq 0 \ \forall (x,t) \in \Omega \times \mathbb{R}_{>0}$. The physical question is this: if the initial and boundary temperatures are non-negative everywhere, then will the temperature in the body be non-negative everywhere at anytime?

(b) (5%) Let u_1 and u_2 be the solutions associated with two distinct initial conditions g_1 and g_2, respectively, and the same boundary condition. If $g_1 \geq g_2$, then $u_1 \geq u_2$.

Note. Again, the statement $u_1 \geq u_2$ should be interpreted as $u_1(x,t) \geq u_2(x,t) \ \forall (x,t) \in \Omega \times \mathbb{R}_{>0}$. The physical question is this: if the initial temperature is higher everywhere for one case, then will the temperature remain higher everywhere at anytime for the case?

(c) (6%) Let u_1 and u_2 be the solutions associated with two distinct initial conditions g_1 and g_2, respectively, and the same boundary condition. Let $D(t) \equiv \max_{x \in \Omega} |u_1(x,t) - u_2(x,t)|$ be the maximum difference in the solutions at time t. Then $D(t)$ is a non-increasing function of t.

Note. The physical question is this: does the difference in the solutions decay or grow over time?

For part (d)–(f), we impose a boundary condition of the form

$$\nu \frac{\partial u}{\partial x} = q,$$

where ν is the outward pointing normal (i.e., $\nu = -1$ at $x = 0$ and $\nu = 1$ at $x = 1$), and q is the heat flux from the surroundings to the body. We also define the (mathematical) energy as

$$E(t) = \frac{1}{2}\int_\Omega u(x,t)^2 dx.$$

For each of the following cases, choose one of the following statements that is true: (i) E is a non-increasing function; (ii) E is a non-decreasing function; (iii) E may increase or decrease. Justify your answer.

(d) (4%) The boundary $\partial\Omega$ is insulated.

(e) (4%) The boundary $\partial\Omega$ is exposed to a different medium at temperature $u^{\text{env}} = 0$ such that the rate of heat transfer from the surroundings to the body is $q = u^{\text{env}} - u = -u$.

(f) (4%) The boundary $\partial\Omega$ is subject to radiative heat transfer such that $q = -\sigma u^4$, where $\sigma > 0$. Assume that $u > 0$ on the boundary. (This is called the Stefan-Boltzmann law of radiation.)

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PDE's ESC384
Assignment 2

Q1)

a)

Fourier cosine series

$$f(x) \simeq \frac{1}{2}\hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n \cos(n\pi x)$$

$$\hat{f}_0 = 2\int_0^1 \pi x \, dx \Rightarrow 2\pi \cdot \frac{x^2}{2}\Big|_0^1 = \pi$$

$$\hat{f}_n = 2\int_0^1 \cos(n\pi x)(\pi x)\,dx$$

[using integral list
$\int x\cos(\alpha x)\,dx = \frac{\cos(\alpha x)}{\alpha^2} + \frac{x\sin(\alpha x)}{\alpha}$]

$$= 2\pi\left[\frac{\cos(n\pi x)}{n^2\pi^2} + \frac{x\sin(n\pi x)}{n\pi}\Big|_0^1\right]$$

(also $\cos(n\pi) = (-1)^n$)

$$= 2\pi\left[\frac{\cos(n\pi)}{n^2\pi^2} + \frac{\sin(n\pi)}{n\pi} - \frac{\cos(0)}{n^2\pi^2}\right]$$

$\sin(n\pi) = 0$
$\therefore 0.$

$$\Rightarrow 2\pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] \Rightarrow \frac{2}{n^2\pi}\left((-1)^n - 1\right) = \begin{cases} 0, & n \text{ is even} \\ \frac{-4}{n^2\pi}, & n \text{ is odd} \end{cases}$$

Fourier cosines series $\rightarrow \frac{1}{2}(\pi) - \frac{4}{\pi}\cos(\pi x) - \frac{4}{9\pi}\cos(3\pi x) - \cdots - \frac{4}{n^2\pi}\cos(n\pi x)$

simplifies to $\rightarrow$ $\boxed{\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)\pi x)}{(2m-1)^2}}$

b) i) $\int_0^1 (F(x) - F_N(x))^2 dx \rightarrow 0$ as $N \rightarrow \infty$

The expression in the integral $F(x) - F_N(x)$, where
$F_N(x)$ = Fourier cosine series N'th truncation. matches the theorem below,

Theorem: L^2 convergence

Let F be a square integrable function with a period 2,
ie $\int_{-1}^{1} F(x)^2 dx < \infty$

Then, $\int_{-1}^{1} (F(x) - S_N(x))^2 dx \rightarrow 0$ as $N \rightarrow \infty$, where S_N is the partial sum to N'th ie $\int_{-1}^{1} F(x)^2\,dx < \infty$

Then, $\int_{-1}^{1} (F(x) - S_N(x))^2\,dx \to 0$ as $N \to \infty$, where S_N is the partial sum to N'th term.

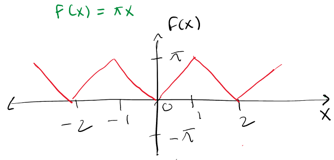
we know $F(x) = \int_{-1}^{1} \pi^2 x^2\,dx \Rightarrow \frac{2\pi^2}{3} < \infty$, which is square integrable,

thus by L^2 convergence theorem, the statement $\boxed{\text{holds true}}$

ii) $\max_{x\in[0,1]} |F(x) - F_N(x)| \to 0$ as $N \to \infty$

even-extension

$F(x) = \pi x$



∴ clearly continuous, and piecewise continuously differentiable.

The setup of this expression is similar to the theorem below,

Theorem: Uniform convergence

Let F be a continuous and piecewise continuously differentiable function with a period 2. Then,

$\max_{x\in\mathbb{R}} |F(x) - S_N(x)| \to 0$ as $N \to \infty$, where S_N is the partial sum to N'th term.

we know that $F(x) = \pi x$ and its even-periodic extension, [πx = constant, even extension as cosine series]

is continuous everywhere on $x \in [0,1]$ and also piecewise continuously differentiable on the interval.

Therefore, the statement $\boxed{\text{holds true}}$ by uniform convergence theorem.

iii) $x \in (0,1)$ $\left|\frac{1}{2}F(x^-) + \frac{1}{2}F(x^+) - F_N(x)\right| \to 0$ as $N \to \infty$

The setup of this expression is similar to the theorem and corollary below,

Theorem: Pointwise Convergence

Let F be piecewise continuously differentiable function with a period 2.

Then, for any $x \in \mathbb{R}$

$S_N(x) \to \frac{1}{2}[F(x^-) + F(x^+)]$ as $N \to \infty$

Corollary: if F is continuous at x, then $S_N(x) \to F(x)$,] [in part b(ii) we showed $|F(x) - S_N(x)| \to 0$, ∴ $S_N(x) \to F(x)$ and also, $|S_N(x) - F(x)| \to 0$]

we already know from b(ii) that $F(x) = \pi x$ is continuous and graph and piecewise continuously differentiable, on $x \in [0,1]$ and all even periodic extensions, thus by pointwise convergence theorem, the statement $\boxed{\text{holds true}}$.

c) As confirmed in (b), since the series is uniformly convergent we can assume that

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos((2m-1)\pi x)}{(2m-1)^2} = F(x)$$

now to use this to evaluate $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \cdots$, we need to make F(x) equivalent in expression.

To do so, we can set $x = 0$, and get $\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)\pi\cdot 0)}{(2m-1)^2}$ [$\cos 0 = 1$]

$$\pi\cdot 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} \Rightarrow \frac{\pi}{2}\cdot\frac{\pi}{4} = \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}$$

$$\boxed{\therefore \text{ we get } \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}, \ \therefore \text{ evaluated.}}$$

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%ESC384 Assignment 2, Aaryan Nagpal, 1007792596, nagpalaa
%Question 1c - Checking if series summation is equal to analytical
%calculation
syms k
series = symsum(1/((2*k-1)^2),k,1,Inf);
display(series)
sum = pi^2/8;
tf = isequal(series, sum);
display(tf)
%since the output is logical 0 or true, it can be confirmed!

#output

series =
pi^2/8

tf =
  logical
   0
```

d) $$\frac{d}{dx}\left[\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)\pi x)}{(2m-1)^2}\right] = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\pi(2m-1)\sin((2m-1)\pi x)}{(2m-1)^2}$$ (chain rule)

$$\Rightarrow \boxed{\sum_{m=1}^{\infty}\frac{4\sin((2m-1)\pi x)}{(2m-1)}}$$

e) Fourier sine series $= \sum_{1}^{\infty} \hat{F}_n \sin(n\pi x)$

$m=1 \quad (2m-1)$

(e) Fourier sine series $= \sum_{n=1}^{\infty} \hat{f}_n \sin(n\pi x)$

$$\sum_{m=1}^{\infty} \frac{4\sin((2m-1)\pi x)}{(2m-1)} = \sum_{n=1}^{\infty} \hat{f}_n \sin(n\pi x)$$

$\therefore \hat{f}_n = \frac{4}{(2m-1)}$, also $\hat{f}_n = 2\int_0^1 \sin(n\pi x)\, f(x)\, dx$

Let us assume $f(x) = \pi$ is the function as that is the derivative of the cosine series function $f(x) = \pi x$.

$$\therefore \hat{f}_n = 2\int_0^1 \sin(n\pi x)\,\pi\, dx = -\frac{2\pi \cos(n\pi x)}{n\pi}\Big|_0^1 = -\frac{2}{n}\cos(n\pi) + \frac{2}{n}\cos(0)$$

$$= -\frac{2}{n}\left((-1)^n - 1\right) \qquad [\cos(n\pi) = (-1)^n]$$

$$= \begin{cases} 0, & n \text{ is even} \\ \frac{4}{n}, & n \text{ is odd} \end{cases}$$

since we chose n to be 2m-1 earlier, we can say that the coefficient $\hat{f}_n = \frac{4}{n} = \frac{4}{2m-1}$, thus $f(x) = \pi$ is the correct function.

going the other way $\Leftarrow$

sine series $= \sum_{n=1}^{\infty} \hat{f}_n \sin(n\pi x)$, $\hat{f}_n = 2\int_0^1 \pi \sin(n\pi x)\, dx$

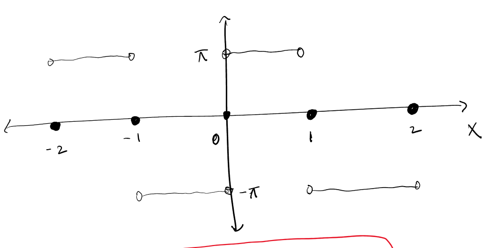
$\hat{f}_n = \begin{cases} 0, & n \text{ is even} \\ \frac{4}{n}, & n \text{ is odd} \end{cases}$, $\therefore$ series $= \sum_{n=1,3,..}^{\infty} \frac{4}{n}\sin(n\pi x)$

$$= 4\sin(\pi x) + \frac{4}{3}\sin(3\pi x) + \cdots \frac{4}{n}\sin(n\pi x)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{4}{2k-1}\sin((2k-1)\pi x)$$

As this exactly matches the derivative in (d), we can say that we have found $\boxed{f(x) = \pi}$ as the correct function for the Fourier sine series.

(f) $f(x) = \sum_{m=1}^{\infty} \frac{4\sin((2m-1)\pi x)}{(2m-1)}$ sketch for 2-periods



• → continuity

o → discontinuity

2-period sketch of F(x)

(g) (i) $\int_0^1 (g(x) - g_N(x))^2\, dx \to 0$ as $N \to \infty$

The expression in the integral $g(x) - g_N(x)$, where $g_N(x)$ = Fourier sine series N'th truncation. matches the theorem below, Theorem: L^2 convergence

Let F be a square integrable function with a period 2,

i.e $\int_{-1}^{1} F(x)^2\,dx < \infty$

Then, $\int_{-1}^{1} (F(x) - S_N(x))^2\,dx \to 0$ as $N \to \infty$, where S_N is the partial sum to N'th term.

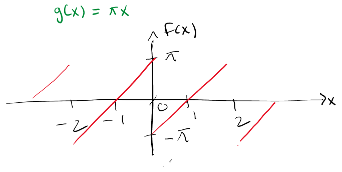
we know $g(x) = \int_{-1}^{1} \pi^2 x^2\,dx \Rightarrow \frac{2\pi^2}{3} < \infty$, which is square integrable,

thus by L^2 convergence theorem, the statement $\boxed{\text{holds true}}$

ii) $\max_{x\in[0,1]} |g(x) - g_N(x)| \to 0$ as $N \to \infty$

odd extension

$g(x) = \pi x$



∴ clearly discontinuous, but piecewise continuously differentiable.

The setup of this expression is similar to the theorem below,

Theorem: Uniform convergence

Let F be a continuous and piecewise continuously differentiable function with a period 2. Then,

$\max_{x\in\mathbb{R}} |F(x) - S_N(x)| \to 0$ as $N \to \infty$, where S_N is the partial sum to N'th term.

we know that $g(x) = \pi x$ but its odd-periodic extension, [πx = constant, odd-extension as sine series] is not continuous everywhere on $x \in [0,1]$, however it is still piecewise continuously differentiable on the interval.

Therefore, the statement violates the uniform convergence theorem, $\boxed{\text{and is False}}$.

iii) $x \in [0,1]$ $\left|\frac{1}{2}g(x^-) + \frac{1}{2}g(x^+) - g_N(x)\right| \to 0$ as $N \to \infty$

The setup of this expression is similar to the theorem below,

Theorem: Pointwise convergence

Let F be piecewise continuously differentiable function with a period 2.

Then, for any $x \in \mathbb{R}$

$S_N(x) \to \frac{1}{2}[F(x^-) + F(x^+)]$ as $N \to \infty$

we already know from F(ii) and graph that $g(x) = \pi x$ is piecewise continuously differentiable, on $x \in [0,1]$ and all odd-periodic extensions, thus by pointwise convergence theorem, the statement $\boxed{\text{holds true}}$.

h) $I(a_n{}_{n=0}^{\infty}) = \int_0^1 \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\right)^2 dx$

using the hint given, we can simplify to

$$\underbrace{\int_0^1 \frac{1}{4}a_0^2\,dx}_{= \frac{1}{4}a_0^2} + \underbrace{\int_0^1 a_0 \sum_{n=1}^{\infty} a_n \cos(n\pi x)\,dx}_{1} + \underbrace{\int_0^1 \sum_{m,n=1}^{\infty} a_n a_m \cos(n\pi x)\cos(m\pi x)\,dx}_{2}$$

$= \int_0^1 \sum_{m,n=1}^{\infty} a_n^2 \cos^2(n\pi x)\,dx$ (as $m=n$)

using integral list

$\int \cos(\alpha x)\,dx = \frac{\sin(\alpha x)}{\alpha}$, $\int \cos^2(\alpha x)\,dx = \frac{x}{2} - \frac{\sin(2\alpha x)}{2\alpha}$

Term 1 : $a_0 \sum_{n=1}^{\infty} a_n \left.\frac{\sin(n\pi x)}{n\pi}\right|_0^1 \to a_0 \sum_{n=1}^{\infty} a_n \left[\frac{\sin(n\pi)}{n\pi} - \frac{\sin(0)}{n\pi}\right]$

0

$\sin(n\pi) = 0$, $\sin 0 = 0$

$\therefore$ term 1 also becomes 0.

Term 2 : $\sum_{n=1}^{\infty} a_n^2 \left. \frac{x}{2} - \frac{\sin(2n\pi x)}{2n\pi} \right|_0^1 \rightarrow \sum_{n=1}^{\infty} a_n^2 \left[\frac{1}{2} - \underbrace{\frac{\sin(2n\pi)}{2n\pi}}_{=0} \right]$ (all other terms go to 0)

$\sin(2n\pi) = 0$ as sin is 2π periodic.

$\therefore$ term 2 $= \frac{1}{2} \sum_{n=1}^{\infty} a_n^2$

Thus, $\boxed{I\left((a_n)_{n=0}^{\infty}\right) = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2}$

i) To find $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = ?$

Using $\hat{f_n}$, f_0 from (a) and expression from (h)

$f_0 = \pi$, $\hat{f_n} = \frac{-4}{n^2\pi}$, $I\left((a_n)_{n=0}^{\infty}\right) = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2$

$\Rightarrow \frac{1}{4}(\pi)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{-4}{n^2\pi}\right)^2$

$\Rightarrow \frac{\pi^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{16}{n^4\pi^2}\right)$, also we can sub-in $n = 2k-1$ as we did in (b)

$\Rightarrow \frac{\pi^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{16}{(2k-1)^4\pi^2}\right) \Rightarrow \frac{\pi^2}{4} + \sum_{k=1}^{\infty} \frac{8}{(2k-1)^4\pi^2}$

Now since we proved in (b) that the fourier series expression is L^2 convergent, we can say that

$$\int_0^1 (f(x))^2 dx = I\left((a_n)_{n=0}^{\infty}\right) = \frac{\pi^2}{4} + \sum_{k=1}^{\infty} \frac{8}{(2k-1)^4\pi^2}$$

$$\int_0^1 \pi^2 x^2 dx \Rightarrow \frac{\pi^2}{3} = \frac{\pi^2}{4} + \sum_{k=1}^{\infty} \frac{8}{(2k-1)^4\pi^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{8}{(2k-1)^4\pi^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^2 \cdot \pi^2}{12 \cdot 8} = \frac{\pi^4}{96}$$

$$\therefore \boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}}$$

Q2. a) $-(x\phi_n')' = \lambda_n \frac{1}{x} \phi_n$ in $(1,2)$

$\phi_n(1) = 0$

$\phi_n(2) = 0$

Form of a Strum-Liouville problem

$/ -(p\psi')' + q\psi = \lambda w \psi$ in (a,b) $\rightarrow$ ODE

$\phi_n(2) = 0$

Form of a Strum - Liouville problem

$$\begin{cases} -(p\psi')' + q\psi = \lambda w \psi \quad \text{in } (a,b) & \to \text{ODE} \\ \alpha_1 \psi(a) - \alpha_2 \psi'(a) = 0 & \to \text{Left B.C} \\ \beta_1 \psi(b) + \beta_2 \psi'(b) = 0 & \to \text{Right B.C} \end{cases}$$

where, (1) p, p', q, w are continuous functions on $[a,b]$.

(2) $p > 0$, $w > 0$ on $[a,b]$

(3) $\alpha_1^2 + \alpha_2^2 > 0$

(4) $\beta_1^2 + \beta_2^2 > 0$

In our case, $p = x$, $q = 0$, $w = \frac{1}{x}$

$\alpha_1 = 1, \alpha_2 = 0$, $\beta_1 = 1, \beta_2 = 0$

b) General ODE form sol for such problems

$\to \phi_n(x) = a_n \cos(\mu_n \ln(x)) + b_n \sin(\mu_n \ln(x))$

Plugging in $-(x\phi_n')' - \lambda_n \frac{1}{x}\phi_n = 0$

$$\phi_n''(x) = \frac{-a_n \mu_n^2 \cos(\mu_n \ln(x))}{x^2} - \frac{b_n \mu_n^2 \sin(\mu_n \ln(x))}{x^2}$$

$$(-x\phi_n'(x))' = \frac{a_n \mu_n^2 \cos(\mu_n \ln(x))}{x} + \frac{b_n \mu_n^2 \sin(\mu_n \ln(x))}{x}$$

$$\lambda_n = \frac{x \cdot (-x\phi_n')'}{\phi \lambda_n} \Rightarrow \frac{x\left(a_n \mu_n^2 \cos(\mu_n \ln(x)) + b_n \mu_n^2 \sin(\mu_n \ln(x))\right)}{x \quad (a_n \cos(\mu_n \ln(x)) b + \sin(\mu_n \ln(x)))}$$

$\lambda_n = \mu_n^2$

and $\mu_n = \sqrt{\lambda_n}$, subbing this in while solving w.r.t B.C's.

Case 1: $\lambda_n > 0$

we know general sol is of the form above

for B.C. at $x = 1$

$$\phi_n(1) = a_n \cos(\sqrt{\lambda_n} \ln(1)) + b_n \sin(\sqrt{\lambda_n} \ln(1)) \overset{\text{want}}{=} 0$$

since $\ln(1) = 0$, the $b_n \sin(\sqrt{\lambda_n} \ln(1))$ term disappears as $\sin(0) = 0$.

$\therefore \quad \phi_n(1) = a_n \cos(0) \overset{\text{want}}{=} 0$

$\therefore a_n = 0$.

For B.C at $x = 2$

$$\phi_n(2) = \underbrace{a_n \cos(\sqrt{\lambda_n} \ln(2))}_{a_n = 0} + b_n \sin(\sqrt{\lambda_n} \ln(2)) \overset{\text{want}}{=} 0$$

$\phi_n(2) = b_n \sin(\sqrt{\lambda_n} \ln(2)) = 0$

but taking $b_n = 0$ would give us the trivial solution, $\therefore \sin(\sqrt{\lambda_n} \ln(2))$ must be 0.

we know $\sin(0) = n\pi$

$$\therefore \sqrt{\lambda_n} \ln(2) = n\pi \Rightarrow \lambda_n = \frac{n^2\pi^2}{\ln(2)^2}$$

eigenpairs

Therefore, eigenfunction:

$$\Rightarrow \phi_n(x) = \sin(\mu_n \ln(x))$$

$$\Rightarrow \sin\left(\frac{n\pi}{\ln(2)} \cdot \ln(x)\right) \quad \left[\text{as } \mu_n = \sqrt{\lambda_n} \text{ from earlier}\right]$$

eigenvalues: $\lambda_n = \dfrac{n^2\pi^2}{\ln(2)^2}$

c) To Find $\displaystyle\int_1^2 \frac{1}{x} \phi_n(x) \phi_m(x)\, dx$, $m \neq n$

we know $\phi_n(x) = \sin\left(\frac{n\pi}{\ln(2)} \cdot \ln(x)\right)$ from (b)

$$\Rightarrow \int_1^2 \frac{1}{x} \sin\left(\frac{n\pi}{\ln(2)} \cdot \ln(x)\right) \sin\left(\frac{m\pi}{\ln(2)} \cdot \ln(x)\right) dx$$

since $m \neq n$, invoking orthogonality relationship

we know $\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = 0$, $m \neq n$

in our case we can clearly see the integral is in the form of a orthogonality ivoking integral and thus it evaluates to 0. This is also true because the Strum Liouville problem eigenfunctions always satisfy the orthogonality relationship

$\therefore \boxed{\int_1^2 \frac{1}{x}\phi_n(x)\,\phi_m(x)\,dx = 0}$

d) To Find $\int_1^2 x\,\phi_n'(x)\,\phi_m'(x)\,dx$

Using integration by parts,

$u = x\phi_n'(x)$, $dv = \phi_m'(x)$

$du = (x\phi_n'(x))'$, $v = \phi_m(x)$

$\Rightarrow uv - \int v\,du$

$\Rightarrow \underbrace{x\phi_n'(x)\,\phi_m(x)}_{①} - \int_1^2 \underbrace{\phi_m(x)\,(x\phi_n'(x))'\,dx}_{②}$

①: $\frac{n\pi}{\ln(2)} \underbrace{\cos\left(\frac{n\pi}{\ln(2)}\ln(x)\right)\cdot\cos\left(\frac{m\pi}{\ln(2)}\ln(x)\right)}_{\text{For } m\neq n,\ =0 \text{ by invoking orthogonality relationship.}}$

$\therefore$ ① $= 0$.

②: we can write $(x\phi_n'(x))'$ as

$$\frac{n\pi}{\ln(2)}\left(\cos\left(\frac{n\pi}{\ln(2)}\ln(x)\right)'\right) = -\frac{n\pi}{\ln(2)}\cdot\frac{n\pi}{x\ln(2)}\left(\sin\left(\frac{n\pi}{\ln(2)}\ln(x)\right)\right)$$

$$= -\frac{n^2\pi^2}{\ln(2)^2}\cdot\frac{1}{x}\left(\sin\left(\frac{n\pi}{\ln(2)}\ln(x)\right)\right) = -\frac{n^2\pi^2}{\ln(2)^2}\,\frac{1}{x}\,\phi_n(x) \quad -③$$

Subbing in ② we get

$-\left(\frac{-n^2\pi^2}{\ln(2)^2}\underbrace{\int_1^2 \frac{1}{x}\phi_n(x)\,\phi_m(x)\,dx}_{\text{same function as part (c) which we found to be } 0}\right)$, $\therefore$ ② $= 0$ as the integral in it evaluates to 0.

Since both ① and ② are 0, the integration by parts becomes 0.

$\therefore \boxed{\int x\phi_n'(x)\,\phi_m'(x),\ m\neq n \text{ evaluates to } 0.}$

Q3 a) Given IBVP

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ in $\Omega \times \mathbb{R}_{>0}$

$u = g$ on $\Omega \times \{t=0\}$

$u = u_L$ on $\{x=0\} \times \mathbb{R}_{>0}$

$u = u_R$ on $\{x=1\} \times \mathbb{R}_{>0}$

Given $g, u_L, u_R \geq 0$.

To prove $u \geq 0$ we can use the minimum principle from lecture

: minimum principle

$\min_{(x,t)\in\overline{\Omega\times I}} u(x,t) = \min_{(x,t)\in\Gamma} u(x,t)$

i.e minimum value is attained somewhere on Γ. (space-time boundary)

In our case, Γ is defined as

$$\Gamma = \underbrace{(\Omega \times \{t=0\})}_{\text{initial time}} \cup \underbrace{(\partial\Omega \times I)}_{\text{spatial boundary}}$$

and we know based on the conditions given i.e $g, U_L, U_R \geq 0$ that everywhere on the space-time boundary $u \geq 0$, $\therefore$ $\underbrace{\min_{(x,t)\in\overline{\Omega \times I}} u(x,t) \geq 0}$

$\Downarrow$

temperature anywhere in the body.

Proved!

b) Given $g_1 \geq g_2$, prove $u_1 \geq u_2$.

We can subtract the 2 equations

$$\rightarrow \frac{\partial(u_1-u_2)}{\partial t} + \frac{\partial^2(u_1-u_2)}{\partial x^2} = 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}$$

$$u_1 - u_2 = g_1 - g_2 \quad \text{on } \Omega \text{ in } \{t=0\}$$

using the minimum principle again

: minimum principle

$$\min_{(x,t)\in\overline{\Omega \times I}} u(x,t) = \min_{(x,t)\in\Gamma} u(x,t) \geq 0$$

i.e minimum value is attained somewhere on Γ. (space-time boundary)

In our case,

$$\min_{(x,t)\in\overline{\Omega \times I}} (u_1-u_2)(x,t) = \min_{(x,t)\in\Gamma} (u_1-u_2)(x,t) \geq 0$$

but we know $u_1 - u_2 = g_1 - g_2$ (Initial condition on subtracting)

$$\therefore \min_{(x,t)\in\Omega} (g_1-g_2) \geq 0 \quad \text{inside the body}$$

since the above statement is true, we can deduce that $g_1 - g_2 = u_1 - u_2 \geq 0$ must be true.

initial condition $g_1 \geq g_2$

thus for boundary conditions,

$\boxed{u_1 \geq u_2}$, proved!

c) Given: $D(t) \equiv \max_{x\in\Omega} |u_1(x,t) - u_2(x,t)|$ is the max difference between solutions at time t. Sols = u_1, u_2, I.C. = g_1, g_2.

To show: $D(t) \leq 0$, as a function of time.

since the max temp in a system at any time t, due to a flux would always be at the boundary, and nowhere in the body can the temperature exceed this, we can say that

$$\max_{x\in\Omega} |u_1(x,t) - u_2(x,t)| = \max_{x\in\overline{\Omega \times I}} |u_1(x,t) - u_2(x,t)|$$

and for any time dt after t, or at time $t+dt$, we can say that, [dt is infinitely small]

$$\max_{x\in\Omega} |u_1(x,t) - u_2(x,t)| \geq \underbrace{\max_{x\in\Omega} |u_1(x,t+dt) - u_2(x,t+dt)|}_{\text{let this be } D(t+dt)}$$

since the enclosed system $\overline{\Omega \times I}$ remains the same across all time between t and $t+dt$, we can say that

$D(t) \geq D(t+dt)$, where dt is an infinitely small time period after t, i.e $t+dt$.

We also know that u_1, u_2 are both solutions to the PDE, and that $u_1 - u_2$ is also a solution by superposition principle, ... becomes 0 with their respective initial conditions

We also know that u_1, u_2 are both solutions to the PDE, and that $u_1 - u_2$ is also a solution by superposition principle, thus $|u_1 - u_2|$ becomes 0 with their respective initial conditions g_1, g_2. Boundary conditions are also equal.

Therefore $\boxed{D(t) \leq 0 \text{ and a non-increasing function in time.}}$

Proved!

d) $E(t) \equiv \frac{1}{2} \int_\Omega u(x,t)^2 \, dx$

B.C. $\nu \frac{\partial u}{\partial x} = q$, since in this case the boundary is insulated $q = 0$.

$$\frac{dE(t)}{dt} = \frac{1}{2} \int_\Omega \frac{d}{dt} (u(x,t))^2 \, dx \Rightarrow 2 \cdot \frac{1}{2} \int_\Omega u(x,t) \cdot \frac{du(x,t)}{dt} \, dx$$

Now, recognizing that $\int_\Omega u(x,t) \cdot \frac{du(x,t)}{dt} \, dx$ is actually a form of the heat equation.

Integrating by parts, (as done in lecture)

$$\Rightarrow -\int_\Omega \underbrace{\frac{du(x,t)}{dt} \cdot \frac{du(x,t)}{dt}}_{\text{square function } \therefore \text{ always increasing,}} \, dx + \int_{\partial\Omega} \underbrace{u(x,t) \cdot \left(\nu \cdot \frac{du(x,t)}{dt} \right) ds}_{0 \text{ by B.C as } \nu \frac{du(x,t)}{dt} = q \text{ and } q = 0}$$

but $-$ve sign means always non-increasing

$\therefore$ $\boxed{E \text{ is } \leq 0 \text{ or non increasing}}$ when $\partial\Omega$ (boundary) is insulated.

e) We can use the same integrated by parts form to deduce the nature of E when $q = \underbrace{u^{env}}_{=0} - u = -u$ when the boundary is exposed to a new medium.

$$\Rightarrow \frac{dE}{dt} = -\underbrace{\int_\Omega \left(\frac{\partial u(x,t)}{dt}\right)^2 dx}_{-\text{ve square function, always non-increasing}} + \int_{\partial\Omega} u(x,t) \cdot \underbrace{\left(\nu \frac{du(x,t)}{dt}\right)}_{= q = -u} ds$$

$$\Rightarrow \underbrace{-\int u(x,t) \cdot u(x,t) \, ds}_{-\text{ve square, non-increasing function.}}$$

$\therefore$ $\boxed{E \text{ is } \leq 0 \text{ or non-increasing}}$ when $\partial\Omega$ is exposed to a new medium and rate of heat transfer $= -u(x,t)$.

F) Radiative heat transfer, $q = -\sigma u^4$, $\sigma > 0$, $u > 0$ on boundary

Once again, using the integrated by parts form of $\frac{dE}{dt}$ from (d).

$$\frac{dE}{dt} = -\underbrace{\int_\Omega \left(\frac{\partial u(x,t)}{dt}\right)^2 dx}_{-\text{ve square function, always non-increasing}} + \int_{\partial\Omega} u(x,t) \cdot \underbrace{\left(\nu \frac{du(x,t)}{dt}\right)}_{q = -\sigma u^4} ds$$

$$\Rightarrow -\int u(x,t)^5 \cdot \sigma \, ds$$

but given $u > 0$, $u(x,t)^5 > 0$, and the $-$ve sign makes the integral ≤ 0, because $\sigma > 0$.

$\therefore$ $\boxed{E \text{ is } \leq 0 \text{ or non increasing}}$ as both integrals are non-increasing, thus when heat transfer is radiative, i.e. $q = -\sigma u^4$, $E \leq 0$. (non-increasing).