ESC384 Assignment 3

Due Wednesday, 25 October 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Important note: in preparation for the midterm exam on October 27th, the solution to this assignment will be posted on Quercus shortly after the submission deadline of October 25th at 9:10am. Hence, *no late submissions will be accepted for this assignment* (i.e., a late submission will result in a mark of 0%), with exception of those due to medical or family emergencies approved by the University through a term-work petition.

Problem 1. Heat equation: mixed boundary condition (26%)

Consider the homogeneous heat equation with homogeneous mixed boundary conditions:

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in } (0,1) \times \mathbb{R}_{>0}, \\
\frac{\partial u}{\partial x}(x=0,t) &= 0, \quad \forall t \in \mathbb{R}_{>0}, \\
u(x=1,t) &= 0, \quad \forall t \in \mathbb{R}_{>0}, \\
u(x,t=0) &= 1-x, \quad \forall x \in (0,1).
\end{aligned}$$

Answer the following question:

(a) (26%) Find a series representation of the solution u. The final expression should not contain any integrals.

Hint. The final expression for the (generalized) Fourier coefficients should be simple. If the final expression is complicated, double check the arithmetic.

Hint 2. You could validate the Fourier coefficients by evaluating the series for a sufficiently large number of terms using MATLAB. (This is not required.)

Problem 2. Nonhomogeneous heat equation (36%)

Consider a nonhomogeneous heat equation

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f \quad \text{in } (0,1) \times \mathbb{R}_{>0}, \\
\frac{\partial u}{\partial x}(x=0,t) &= 0, \quad \forall t \in \mathbb{R}_{>0}, \\
u(x=1,t) &= 0, \quad \forall t \in \mathbb{R}_{>0}, \\
u(x,t=0) &= 1-x, \quad \forall x \in (0,1),
\end{aligned}$$

for some source term $f : (0,1) \times \mathbb{R}_{>0} \to \mathbb{R}$. Answer the following questions:

(a) (16%) Find a series representation of the solution u. The expression may contain a time integral with an integrand that depends on the time-dependent (generalized) Fourier coefficients of f.

Note. The final expression may be decomposed into multiple pieces (e.g., $u(x,t) = \sum_{n=1}^{\infty} \hat{u}(t)\cos(\cdots)$, $\hat{u}(t) = \ldots$, $\hat{f}(t) = \ldots$), but please make sure to identify all pieces in the final answer and (say) box all pieces to facilitate the grading process. Please follow this rule throughout this assignment.

(b) (10%) Let f be a time-independent discontinuous source function

$$f(x,t) = \begin{cases} 0, & x \le 1/2, \\ 1, & x > 1/2. \end{cases}$$

Find a series representation of the solution. The final expression may *not* contain an integral.

Hint. A sequence $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, $a_4 = -1$, $a_5 = 1$, $a_6 = 1$, $a_7 = -1$, $\ldots$ can be compactly expressed as $a_n = (-1)^{\lfloor (n-1)/2 \rfloor}$, where $\lfloor \cdot \rfloor$ is the flooring function.

(c) (4%) Find the steady-state solution $u^{\mathrm{s}} : (0,1) \to \mathbb{R}$ of the problem solved in (b).

Note. The steady-state solution is the solution obtained after a very long time (i.e., $t \to \infty$). By definition, it must be independent of time; not all initial value problems have a steady state solution.

(d) (6%) Prove or disprove each of the following statements about the steady state solution $u^{\mathrm{s}} : (0,1) \to \mathbb{R}$ found in (c) associated with the discontinuous source function:

(i) u^{s} is continuously differentiable;

(ii) u^{s} depends on the initial condition.

Problem 3. Fundamental solution and method of reflection (38%)

(a) (9%) Verify that the fundamental solution

$$\Phi(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

satisfies the homogeneous heat equation in $\mathbb{R} \times \mathbb{R}_{>0}$: i.e., $\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = 0$.

(b) (9%) Recall that the general solution to the heat equation on the half line with a homogeneous Dirichlet boundary condition is

$$u_D(x,t) = \int_0^\infty (\Phi(x-\xi,t) - \Phi(x+\xi,t)) g(\xi) d\xi,$$

where Φ is the fundamental solution, and $g : \mathbb{R}_{>0} \to \mathbb{R}$ specifies the initial condition. Show that the solution satisfies (i) the heat equation $\frac{\partial u_D}{\partial t} - \frac{\partial^2 u_D}{\partial x^2} = 0$ in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and (ii) the homogeneous Dirichlet boundary condition at $x = 0$ for all $t > 0$.

Note. Assume that the order of differentiation and integration may be interchanged.

(c) (5%) Recall that the general solution to the heat equation on the half line with a homogeneous Neumann boundary condition is

$$u_N(x,t) = \int_0^\infty (\Phi(x-\xi,t) + \Phi(x+\xi,t))g(\xi)d\xi.$$

Show that the solution satisfies the homogeneous Neumann boundary condition at $x = 0$ for all $t > 0$.

(d) (15%) Let $g : \mathbb{R}_{>0} \to \mathbb{R}$ be given by

$$g(x) \equiv \begin{cases} 1-x, & x \leq 1, \\ 0, & x > 1. \end{cases}$$

For each of the following statements, state whether the statement holds and justify your answer:

(i) $u_N(x,t) \geq u_D(x,t)$ for all $x \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{>0}$.

(ii) $u_N(x,t) > u_D(x,t)$ for all $x \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{>0}$.

(iii) $\int_0^\infty u_N(x,t)dx = \int_0^\infty g(x)dx$ for all $t \in \mathbb{R}_{>0}$.

(iv) $\int_0^\infty u_D(x,t)dx = \int_0^\infty g(x)dx$ for all $t \in \mathbb{R}_{>0}$.