Assignment 3

February 1, 2024   5:30 PM

MIE 376 — Assignment 3 — Winter 2024

In this assignment, you solve problems for duality and sensitivity analysis. You need to
(1) Formulate the dual problem of an LP model and solve it with Python and Gurobi.
(2) Perform the sensitivity analysis both manually and using Gurobi.

1. Submission Instructions

Submit a PDF file describing a linear programming (LP) model and a solution to the problem instance. Also, submit a program (a Python script or a Jupyter notebook) using Gurobi to solve the problem instance. In the LP formulation, clearly define decision variables and state the objective function and constraints. For the problem instance, report the values of the objective and decision variables.

2. Problem

SOLUTION BELOW AND PYTHON FILE ↓

For the given LP problem,

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ s.t.\ & x_1 + 2x_2 \le 6 \\ & 2x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Question 1: formulate the dual problem and then solve both the primal and dual problems using Python and Gurobi.

Question 2: we can address the primal problem using the simplex method. Its standard form is as follows:

SOLUTION BELOW AND PYTHON FILE ↓

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ s.t.\ & x_1 + 2x_2 + x_3 = 6 \quad (c_1) \\ & 2x_1 + x_2 + x_4 = 8 \quad (c_2) \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Here, x_3 and x_4 serve as slack variables. At the final iteration of the simplex, it is established that x_1 and x_2 are basis, with $x_1 = \frac{10}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4$ and $x_2 = \frac{4}{3} - \frac{2}{3}x_3 + \frac{1}{3}x_4$. Utilizing this information, along with your solutions from Q1, conduct a sensitivity analysis both manually and using Gurobi and complete the following tables.

Variables:

Variable	Objective Coefficient	Optimal Value	Reduced Cost	Allowable Decrease	Allowable Increase
x_1	2	10/3	0	−1/2	4
x_2	3	4/3	0	−2	1

Constraints:

Constraints	RHS	Shadow Price	Allowable Decrease	Allowable Increase
c_1	6	4/3	−2	10
c_2	8	1/3	−5	4

Q2. Tables, solutions below ↓

Variables

Variable	Objective Coefficient	Optimal value	Reduced Cost	Allowable Decrease	Allowable Increase
x_1	2	10/3	0	−1/2	4
x_2	3	4/3	0	−2	1

Constraints

Constraints	RHS	Shadow Prices	Allowable Decrease	Allowable Increase
2	6	4/3	−2	10
3	8	1/3	−5	4

Q1. Primal

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ & x_1 + 2x_2 \le 6 \\ & 2x_1 + x_2 \le 8 \\ & x_i \ge 0 \end{aligned}$$

Primal solution
max objective = 10.667
$x_1 = 3.333$
$x_2 = 1.333$

↝ problem instance solution

setting up the dual

Step #1
Express max {≤, =} as min {≥, =}

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ & x_1 + 2x_2 \le 6 \\ & 2x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Step #2
Create dual variable for every constraint

Step #3
Dual variables are ≥ 0, except equality constraint variable is urs
$y_1, y_2 \ge 0$

Step #4
Dual objective uses Primal rhs coefficient → {6, 8} + opposite objective → {max}.

$\min\ 6y_1 + 8y_2$

Step #5
Dual constraints
- one constraint per primal variable → 2 primal vars (x_1, x_2)
- opposite direction for non-negative primal.
- use primal objective coefficients for dual rhs → {2, 3} become rhs

} equality for primal urs

$$\Rightarrow \quad \begin{aligned} y_1 + 2y_2 &\ge 2 \\ 2y_1 + y_2 &\ge 3 \end{aligned}$$

Combining everything → Decision variables.

Dual:

$$\begin{aligned} \min\ & 6y_1 + 8y_2 \\ & y_1 + 2y_2 \ge 2 \\ & 2y_1 + y_2 \ge 3 \\ & y_1, y_2 \ge 0 \end{aligned}$$

] → objective Function
] → constraints
] → bounds.

Dual Solution
Min objective = 10.667
$y_1 = 1.333$
$y_2 = 0.333$

problem instance solution.

Q2. Sensitivity Analysis

$x_1 = \boxed{\frac{10}{3}} + \frac{1}{3}x_3 - \frac{2}{3}x_4$ → optimal x_1 value

$x_2 = \boxed{\frac{4}{3}} - \frac{2}{3}x_3 + \frac{1}{3}x_4$ → optimal x_2 value

] optimal values for x_1, x_2 in the dual formulation.

$z = 2x_1 + 3x_2 = \underbrace{\frac{20}{3} + \frac{2}{3}x_3 - \frac{4}{3}x_4}_{2x_1} + \underbrace{4 - 2x_3 + x_4}_{3x_2}$

$= \boxed{\frac{32}{3}} - \frac{4}{3}x_3 - \frac{1}{3}x_4$ ← optimal objective value

] writing the objective function 'z', in terms of x_1 and x_2 final equations as given in the question.

Reduced costs = coefficients of basic variables (x_1, x_2) in z-eqn $= \boxed{0, 0}$] obtaining reduced costs

$z = (2+\delta_1)x_1 + 3x_2 = (2+\delta_1)\left(\frac{10}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4\right) + 3\left(\frac{4}{3} - \frac{2}{2}x_3 + \frac{1}{3}x_4\right)$] changes to objective coefficient x_1: $(2+\delta)$

optimal objective value ← $\frac{\;}{3}$ $\frac{\;}{3}x_3$ $\frac{\;}{3}x_4$

Reduced costs = coefficients of basic variables (x_1, x_2) in z-eqn $= \boxed{0, 0}$] obtaining reduced costs

$$z = (2+\delta_1)x_1 + 3x_2 = (2+\delta_1)\left(\frac{10}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4\right) + 3\left(\frac{4}{3} - \frac{2}{3}x_3 + \frac{1}{3}x_4\right)$$

$$= \frac{20}{3} + \frac{2}{3}x_3 - \frac{4}{3}x_4 + \frac{10}{3}\delta_1 + \frac{1}{3}x_3\delta_1 - \frac{2}{3}x_4\delta_1 + 4 - 2x_3 + x_4$$

$$= \left(\frac{32}{3} + \frac{10}{3}\delta_1\right) + \left(\frac{1}{3}\delta_1 - \frac{4}{3}\right)x_3 + \left(-\frac{1}{3} - \frac{2}{3}\delta_1\right)x_4$$

] changes to objective coefficient $x_1 : (2+\delta)$ getting the new 'z' equation with δ accounted for, only in terms of non-basic variables x_3, x_4

To satisfy feasibility conditions for maximizing, we must have :
$\left(\frac{1}{3}\delta_1 - \frac{4}{3}\right) \leq 0 \Rightarrow \delta_1 \leq 4$
$\left(-\frac{1}{3} - \frac{2}{3}\delta_1\right) \leq 0 \Rightarrow \delta_1 \geq -1/2$

↗ allowable decrease
$\boxed{-1/2 \leq \delta \leq 4}$
↳ allowable increase

] allowable change to x_1 while maintaining optimality

$$z = 2x_1 + (3+\delta_2)x_2 = 2\left(\frac{10}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4\right) + (3+\delta_2)\left(\frac{4}{3} - \frac{2}{3}x_3 + \frac{1}{3}x_4\right)$$

$$= \frac{20}{3} + \frac{2}{3}x_3 - \frac{4}{3}x_4 + 4 - 2x_3 + x_4 + \frac{4}{3}\delta_2 - \frac{2}{3}x_3\delta_2 + \frac{1}{3}x_4\delta_2$$

$$= \left(\frac{32}{3} + \frac{4}{3}\delta_2\right) + \left(-\frac{2}{3}\delta_2 - \frac{4}{3}\right)x_3 + \left(-\frac{1}{3} + \frac{1}{3}\delta_2\right)x_4$$

] changes to objective coefficient $x_2 : (3+\delta)$ getting the new 'z' equation with δ accounted for, only in terms of non-basic variables x_3, x_4

To satisfy feasibility conditions for maximizing, we must have :
$\left(-\frac{2}{3}\delta_2 - \frac{4}{3}\right) \leq 0 \Rightarrow \delta_2 \geq -2$
$\left(-\frac{1}{3} + \frac{1}{3}\delta_2\right) \leq 0 \Rightarrow \delta_2 \leq 1$

↗ allowable decrease
$\boxed{-2 \leq \delta \leq 1}$
↳ allowable increase

] allowable change to x_2 while maintaining optimality

$$z = 2x_1 + 3x_2 = \frac{20}{3} + \frac{2}{3}x_3 - \frac{4}{3}x_4 + 4 - 2x_3 + x_4$$

$$= \frac{32}{3} - \frac{4}{3}x_3 - \frac{1}{3}x_4$$

] writing the objective function 'z', in terms of x_1 and x_2 final equations as given in the question.

shadow prices = coefficients of non-basic variables $\{x_3, x_4\}$ in z-equation (abs. value) $= \boxed{\frac{4}{3}, \frac{1}{3}}$] obtaining shadow prices

$x_1 + 2x_2 + x_3 = 6 + \delta \Rightarrow x_1 + 2x_2 + \underbrace{(x_3 - \delta)}_{= x_3'} = 6$] changing the rhs value by δ for constraint 1

$x_1 = \frac{10}{3} + \frac{1}{3}x_3' - \frac{2}{3}x_4$ $\quad z = \frac{32}{3} - \frac{4}{3}x_3' - \frac{1}{3}x_4$
$x_2 = \frac{4}{3} - \frac{2}{3}x_3' + \frac{1}{3}x_4$

] writing x_1, x_2, z in terms of the basic variables $x_3' = x_3 - \delta$ and x_4.

$x_1 = \frac{10}{3} + \frac{1}{3}(x_3 - \delta) - \frac{2}{3}x_4$ $\quad z = \frac{32}{3} - \frac{4}{3}(x_3 - \delta) - \frac{1}{3}x_4$
$x_2 = \frac{4}{3} - \frac{2}{3}(x_3 - \delta) - \frac{1}{3}x_4$

] plugging $x_3' = x_3 - \delta$ back in to obtain equations

$\Rightarrow x_1 = \left(\frac{10}{3} - \frac{1}{3}\delta\right) + \frac{1}{3}x_3 - \frac{2}{3}x_4$ $\quad z = \left(\frac{32}{3} + \frac{4}{3}\delta\right) - \frac{4}{3}x_3 - \frac{1}{3}x_4$
$x_2 = \left(\frac{4}{3} + \frac{2}{3}\delta\right) - \frac{2}{3}x_3 - \frac{1}{3}x_4$

] combing the constant terms with δ in equations for x_1, x_2, z.

we still must have : $\left(\frac{10}{3} - \frac{1}{3}\delta\right) \geq 0 \Rightarrow \delta \leq 10$
$\left(\frac{4}{3} + \frac{2}{3}\delta\right) \geq 0 \Rightarrow \delta \geq -2$

] checking the value of change δ can take to still have the same basis.

↗ allowable decrease
$\Rightarrow \boxed{-2 \leq \delta \leq 10}$
↳ allowable increase

] allowable changes in value of rhs value of constraint to maintain the same basis.

$2x_1 + x_2 + x_4 = 8 + \delta \Rightarrow 2x_1 + x_2 + \underbrace{(x_4 - \delta)}_{= x_4'} = 8$] changing the rhs value by δ for constraint 1

$x_1 = \frac{10}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4'$ $\quad z = \frac{32}{3} - \frac{4}{3}x_3 - \frac{1}{3}x_4'$
$x_2 = \frac{4}{3} - \frac{2}{3}x_3 + \frac{1}{3}x_4'$

] writing x_1, x_2, z in terms of the basic variables $x_3' = x_3 - \delta$ and x_4.

$x_1 = \frac{10}{3} + \frac{1}{3}x_3 - \frac{2}{3}(x_4 - \delta)$ $\quad z = \frac{32}{3} - \frac{4}{3}x_3 - \frac{1}{3}(x_4 - \delta)$
$x_2 = \frac{4}{3} - \frac{2}{3}x_3 + \frac{1}{3}(x_4 - \delta)$

] plugging $x_3' = x_3 - \delta$ back in to obtain equations

$x_1 = \left(\frac{10}{3} + \frac{2}{3}\delta\right) + \frac{1}{3}x_3 - \frac{2}{3}x_4$ $\quad z = \left(\frac{32}{3} + \frac{1}{3}\delta\right) - \frac{4}{3}x_3 - \frac{1}{3}x_4$
$x_2 = \left(\frac{4}{3} - \frac{1}{3}\delta\right) - \frac{2}{3}x_3 + \frac{1}{3}x_4$

] combing the constant terms with δ in equations for x_1, x_2, z.

we still must have : $\left(\frac{10}{3} + \frac{2}{3}\delta\right) \geq 0 \Rightarrow \delta \geq -5$
$\left(\frac{4}{3} - \frac{1}{3}\delta\right) \geq 0 \Rightarrow \delta \leq 4$

] checking the value of change δ can take to still have the same basis.

↗ allowable decrease
$\Rightarrow \boxed{-5 \leq \delta \leq 4}$
↳ allowable increase

] allowable changes in value of rhs value of constraint to maintain the same basis.

↳ allowable increase