ESC384 Assignment 4

Due Wednesday, 22 November 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Problem 1. Heat equation: finite difference (50%)

Consider the heat equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f \quad \text{in } (0,1) \times (0,T],\\
u(x=0,t) &= 0 \quad \text{on } \{x=0\} \times (0,T],\\
\frac{\partial u}{\partial x}(x=1,t) + u(x=1,t) &= 0 \quad \text{on } \{x=1\} \times (0,T],\\
u &= g \quad \text{on } (0,1) \times \{t=0\}
\end{aligned}
$$

for some initial condition function $g : (0,1) \to \mathbb{R}$ and source function $f : (0,1) \times (0,T] \to \mathbb{R}$. Note that a Robin boundary condition is imposed on the right boundary.

We will implement a finite difference heat equation solver in MATLAB. We will use $n+1$ grid points $0 \equiv x_0 < x_1 < \cdots < x_n \equiv 1$ to discretize the spatial domain, and $J+1$ time points $0 \equiv t^0 < t^1 \cdots < t^J = T$ to discretize the time interval. We assume both spatial and temporal points are equispaced. Our goal is to approximate the solution $u(x_i, t^j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, J$.

We first consider the semi-discrete form of the equation associated with the second-order accurate finite difference approximation in space.

(a) (10%) Find the semi-discrete equations for (i) the first unknown node $i = 1$, (ii) the last unknown node $i = n$, and (iii) all other unknown nodes $i \in [2, n-1]$. Also identify the equation for (iv) the initial condition. Express the answer in terms of $\tilde{x}_i$, $\tilde{u}_i(t)$, $\hat{f}_i(t) \equiv f(x_i, t)$, and $\hat{g}_i \equiv g(x_i)$.

(b) (6%) The semi-discrete equations found in (a) can be expressed as

$$
\begin{aligned}
\frac{d\tilde{u}}{dt}(t) + \hat{A}\tilde{u}(t) &= \hat{f}(t) \quad \text{in } \mathbb{R}^n,\\
\tilde{u}(t=0) &= \hat{g} \quad \text{in } \mathbb{R}^n,
\end{aligned}
$$

where $\tilde{u}(t) \in \mathbb{R}^n$, $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{f}(t) \in \mathbb{R}^n$, and $\hat{g} \in \mathbb{R}^n$. Find the expressions for the matrix $\hat{A}$ and vectors $\hat{f}(t)$ and $\hat{g}$.

We now consider the fully discrete form of the equation associated with the Crank-Nicolson approximation in time.

(c) (6%) The fully discrete equation associated with the semi-discrete equation found in (b) can be expressed as

$$\hat{C}\tilde{\tilde{u}}^j = \hat{D}\tilde{\tilde{u}}^{j-1} + \hat{F}(t^j, t^{j-1}), \quad j = 1, \dots, J.$$

where $\hat{C} \in \mathbb{R}^{n\times n}$, $\hat{D} \in \mathbb{R}^{n\times n}$, and $\hat{F}(t^j, t^{j-1}) \in \mathbb{R}^n$. Find the expressions for the matrices and vectors.

We now implement the finite difference solver in MATLAB.

(d) (15%) Starting with the template `heat_fd_temp.m`, implement the finite difference solver.

Note. Please include (i) a copy of the code in the PDF file and (ii) the source code in the zip file to facilitate the grading process.

Note 2. You do *not* have to use the template if you would rather code everything from scratch.

(e) (5%) Let

$$f(x,t) = \exp(-3x)\exp(t),$$
$$g(x) = \frac{1}{2}x(2-x) + \frac{1}{6\pi}\sin(3\pi x).$$

Invoke the solver for $T = 1$, $n = 16$, $J = 16$. Plot, in a single figure, the solution at the final time $t = 0$, $t = 1/16$, $t = 1/8$, $t = 1/4$, $1/2$, and $t = 1$.

(f) (8%) We wish to verify the convergence of the solver. To this end, for the functions f and g given in (e), compute the solution for $(n, J) = (8, 8), (16, 16), (32, 32)$, and $(64, 64)$, and then evaluate (the approximation of) the output $s \equiv u(x = 1, t = 1)$ for each of the four discretizations. Also evaluate the reference output s_{ref} associated with $(n, J) = (512, 512)$. Report, in a table, the error associated with the four different levels of discretizations. Report also the value of s_{ref} to at least six significant digits. Does the observed error behavior match your expectation?

Note. The table should have three columns with headings n, J, and $|s_{\text{ref}} - s|$. Please provide both the table and the value of s_{ref} in the hard copy of the assignment to facilitate the grading process.

Problem 2. Laplace's equation in an annular domain (30%)

Let Ω be an two-dimensional annular domain of the inner radius r_1 and the outer radius of r_2: i.e.,

$$\Omega \equiv \{x \in \mathbb{R}^2 \mid r_1 < \|x\| < r_2\}.$$

We consider Laplace's equation

$$-\Delta u = 0 \quad \text{in } \Omega,$$
$$u(r = r_1, \theta) = 0, \quad \theta \in [0, 2\pi),$$
$$u(r = r_2, \theta) = g(\theta), \quad \theta \in [0, 2\pi).$$

In words, the homogeneous Dirichlet boundary condition is imposed on the inner boundary, and a nonhomogeneous Dirichlet boundary condition is imposed on the outer boundary. Answer the following questions:

(a) (14%) Find a family of functions of the form $u_n(r,\theta) = r_n(r)\Theta_n(\theta)$ that satisfies (i) the PDE and (ii) the inner boundary condition (but not necessarily the outer boundary condition). Express the answer in terms of r, θ, r_1, r_2, and some unknown coefficients.

(b) (8%) Let $g(\theta) \equiv \cos(m\theta)$, where m is a positive integer. Find the associated solution u as a function of m.

Note. The integer m is positive; i.e., it does not take on 0.

(c) (8%) Let $r_1 = 1$ and $r_2 = 2$. Using MATLAB (or any other software), plot, in a single figure, $u(r, \theta = 0)$ over $r \in [r_1, r_2]$ for $m = 1$ and 10. Comment (i) if the relative decay in the two solutions away from the outer boundary is consistent with your expectation and (ii) if the maximum principle is satisfied.

Problem 3. Green's function (20%)

Consider Laplace's equation in the first quadrant $\Omega \equiv \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ with the bottom boundary $\Gamma_B \equiv \mathbb{R}_{>0} \times \{x_2 = 0\}$ and the left boundary $\Gamma_L \equiv \{x_1 = 0\} \times \mathbb{R}_{>0}$:

$$\begin{aligned} -\Delta u &= 0 \quad \text{on } \Omega, \\ u &= g_B \quad \text{on } \Gamma_B, \\ u &= g_L \quad \text{on } \Gamma_L. \end{aligned}$$

Answer the following questions.

(a) (10%) Find the Green's function $G : \Omega \times \Omega \to \mathbb{R}$ such that, for any given $\xi \in \Omega$,

$$\begin{aligned} -\Delta G(x,\xi) &= \delta(x - \xi) \quad \forall x \in \Omega, \\ G(x,\xi) &= 0 \quad \forall x \in \Gamma_B \cup \Gamma_L. \end{aligned}$$

Identify an appropriate number of image points, and express them as $\tilde{\xi}^{(1)} = (\tilde{\xi}_1^{(1)}, \tilde{\xi}_2^{(1)})$, $\tilde{\xi}^{(2)} = (\tilde{\xi}_1^{(2)}, \tilde{\xi}_2^{(2)})$, ... in terms of $\xi = (\xi_1, \xi_2)$. Then express the solution in terms of the fundamental solution Φ of Laplace's equation and the image points.

(b) (10%) Let $g_L = 0$. Find an integral expression for the solution $u(x = (x_1, x_2))$ in terms of g_B.

Hint. Note that the outward directional derivative on Γ_B is $\nu_B \cdot \nabla_\xi G(x,\xi) = -\frac{\partial G(x,\xi)}{\partial \xi_2}$. Also note that $-\frac{\partial \Phi(x-\xi)}{\partial \xi_2} = \frac{\partial}{\partial \xi_2}\left(\frac{1}{2\pi}\log(\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2})\right)$.

Hint 2. The final solution can be succinctly expressed in terms of Poisson's kernel for the upper half plane.

(c) (5%) Let $g_L = 0$, and

$$g_B(x_1) = \begin{cases} 1, & x_1 \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Find an integral representation of the solution. The solution may contain only an integral over a finite interval.