ESC384 Assignment 2 Due Wednesday, 11 October 2023, at 9:10am

The solution to the assignment must be uploaded to Quercus as a single PDF file at the specified time. For problems that require coding, please include a copy of the code in the aforementioned PDF file and also upload the source code as a single ZIP file to facilitate the grading process. In summary, there should be two separate files uploaded to Quercus: (i) a single PDF file with the entire solution; (ii) a single ZIP file with all the source code. Everything that you would like to have marked should be in the PDF file; TAs will nominally only look at the PDF file (and not the ZIP file). Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Problem 1. Properties of Fourier series (48%)

Throughout this problem, function $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = \pi x$.

(a) (4%) Find the Fourier cosine series of f.

(b) (6%) Let f_N be the N-term truncated Fourier cosine series of f. State whether each of the following statements holds. Briefly justify your answers.

  (i) $\int_0^1 (f(x) - f_N(x))^2 dx \to 0$ as $N \to \infty$.

  (ii) $\max_{x\in[0,1]} |f(x) - f_N(x)| \to 0$ as $N \to \infty$.

  (iii) For any fixed $x \in [0, 1]$, $|\frac{1}{2}(f(x^-) + f(x^+)) - f_N(x)| \to 0$ as $N \to \infty$.

(c) (6%) Using the Fourier cosine series found in (a), evaluate $\sum_{k=1}^\infty \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

Hint. First find the relationship between the Fourier cosine coefficients a_{2k-1}, $k = 1, 2, \ldots$, and the summand. Then evaluate the Fourier cosine series at a well-chosen x, and invoke a convergence property of the Fourier series. Note that you can also verify your solution using MATLAB. (This family of problem is known as Basel problem; it was first proposed in 1650 and was solved by Euler in 1734.)

(d) (4%) Find the derivative of the Fourier cosine series found in (a).

(e) (4%) The series found in (d) is the Fourier sine series of some function g on $[0, 1]$. Identify the function g, and verify that the Fourier sine series of g is the same as the series in (d).

(f) (6%) Sketch at least two periods of the Fourier series found in (d). If the function is discontinuous, clearly indicate the value of the function at discontinuities using $\circ$ or $\bullet$.

(g) (6%) Let g_N be the N-term partial sum of the series found in (d). State whether each of the statements in (b) holds (where f and f_N and replaced by g and g_N, respectively). Briefly justify your answers.

(h) (6%) Evaluate the integral

$$I((a_n)_{n=0}^\infty) = \int_0^1 \left(\frac{1}{2}a_0 + \sum_{n=1}^\infty a_n \cos(n\pi x) \right)^2 dx$$

in terms of a_n, $n = 0, 1, 2, \ldots$. The final expression should not contain an integral.

Hint. First rewrite the integrand as $\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\right)\left(\frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(m\pi x)\right) = \frac{1}{4}a_0^2 + a_0 \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n,m=1}^{\infty} a_n a_m \cos(n\pi x)\cos(m\pi x)$. Then carry out the integration term by term.

(i) (6%) Evaluate $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$.

Hint. Substitute the Fourier cosine coefficients found in (b) to the expression found in (h) and manipulate the relationship.

Problem 2. Sturm-Liouville problem (24%)

Consider an eigenproblem

$$\begin{aligned} -(x\phi_n')' &= \lambda_n \frac{1}{x}\phi_n \quad \text{in } (1,2), \\ \phi_n(1) &= 0, \\ \phi_n(2) &= 0. \end{aligned}$$

(This eigenproblem arises when we consider Laplace's equation in the polar coordinate system.) Answer the following questions:

(a) (4%) Verify that the eigenproblem is a regular Sturm-Liouville problem. Identify p, q, w, α_1, α_2, β_1, and β_2.

(b) (10%) The general solution of the ODE is of the form

$$\phi_n(x) = a_n \cos(\mu_n \log(x)) + b_n \sin(\mu_n \log(x)),$$

for some a_n, b_n, and μ_n. Find the eigenfunctions and eigenvalues of the eigenproblem.

Note. The "log" is the natural log.

(c) (5%) Evaluate

$$\int_1^2 \frac{1}{x}\phi_n(x)\phi_m(x)dx \quad m \neq n.$$

Note. You need not show all work, but justify your answer.

(d) (5%) Evaluate

$$\int_1^2 x\phi_n'(x)\phi_m'(x)dx \quad m \neq n.$$

Hint. Use a combination of the solution to (c) and the boundary value eigenproblem.

Problem 3. Heat equation (28%)

Consider an initial-boundary value problem associated with the heat equation on $\Omega \equiv (0,1)$,

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \\ u &= g \quad \text{on } \Omega \times \{t = 0\}, \end{aligned}$$

for some initial condition function $g : \Omega \to \mathbb{R}$. For part (a)–(c), we impose a Dirichlet boundary conditions of the form

$$\begin{aligned} u &= u_{\mathrm{L}} \quad \text{on } \{x = 0\} \times \mathbb{R}_{>0}, \\ u &= u_{\mathrm{R}} \quad \text{on } \{x = 1\} \times \mathbb{R}_{>0}, \end{aligned}$$

for (time-independent) constants u_{L} and u_{R}. Prove or provide a counterexample to each of the following statements:

(a) (5%) If $g \geq 0$, $u_{\mathrm{L}} \geq 0$, and $u_{\mathrm{R}} \geq 0$, then $u \geq 0$.

Note. The statement $u \geq 0$ should be interpreted as "u is non-negative everywhere", i.e., $u(x,t) \geq 0 \ \forall (x,t) \in \Omega \times \mathbb{R}_{>0}$. The physical question is this: if the initial and boundary temperatures are non-negative everywhere, then will the temperature in the body be non-negative everywhere at anytime?

(b) (5%) Let u_1 and u_2 be the solutions associated with two distinct initial conditions g_1 and g_2, respectively, and the same boundary condition. If $g_1 \geq g_2$, then $u_1 \geq u_2$.

Note. Again, the statement $u_1 \geq u_2$ should be interpreted as $u_1(x,t) \geq u_2(x,t) \ \forall (x,t) \in \Omega \times \mathbb{R}_{>0}$. The physical question is this: if the initial temperature is higher everywhere for one case, then will the temperature remain higher everywhere at anytime for the case?

(c) (6%) Let u_1 and u_2 be the solutions associated with two distinct initial conditions g_1 and g_2, respectively, and the same boundary condition. Let $D(t) \equiv \max_{x \in \Omega} |u_1(x,t) - u_2(x,t)|$ be the maximum difference in the solutions at time t. Then $D(t)$ is a non-increasing function of t.

Note. The physical question is this: does the difference in the solutions decay or grow over time?

For part (d)–(f), we impose a boundary condition of the form

$$\nu \frac{\partial u}{\partial x} = q,$$

where ν is the outward pointing normal (i.e., $\nu = -1$ at $x = 0$ and $\nu = 1$ at $x = 1$), and q is the heat flux from the surroundings to the body. We also define the (mathematical) energy as

$$E(t) \equiv \frac{1}{2} \int_\Omega u(x,t)^2 dx.$$

For each of the following cases, choose one of the following statements that is true: (i) E is a non-increasing function; (ii) E is a non-decreasing function; (iii) E may increase or decrease. Justify your answer.

(d) (4%) The boundary $\partial\Omega$ is insulated.

(e) (4%) The boundary $\partial\Omega$ is exposed to a different medium at temperature $u^{\text{env}} = 0$ such that the rate of heat transfer from the surroundings to the body is $q = u^{\text{env}} - u = -u$.

(f) (4%) The boundary $\partial\Omega$ is subject to radiative heat transfer such that $q = -\sigma u^4$, where $\sigma > 0$. Assume that $u > 0$ on the boundary. (This is called the Stefan-Boltzmann law of radiation.)