Assignment 4

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MIE 376 Assignment 4 Winter 2024

In this assignment, you solve problems for Stochastic Linear Programming and Quadratic Programming. You need to

(1) Formulate the Stochastic Programming model and solve it with Python and Gurobi.
(2) Formulate the Quadratic Programming model and solve it with Python and Gurobi.

1. Submission Instructions

Submit a PDF file describing stochastic programming (SP) and quadratic programming (QP) models and the solutions to the problem instance. Also, submit a program (a Python script or a Jupyter notebook) using Gurobi to solve the problem instance. In the formulation, clearly define decision variables and state the objective function and constraints. For the problem instance, report the values of the objective and decision variables.

2. Problem

Question 1: You are a production manager at SweetFlow Syrups, a company specializing in producing a variety of flavored syrups. SweetFlow produces three types of syrups: vanilla (V), maple (M) and cherry (C). The company faces significant demand variability, especially due to seasonal changes and market trends. To manage this uncertainty effectively, you decide to use stochastic programming for production planning.

SOLUTION BELOW AND PYTHON FILE

Your task is to develop a monthly production plan to maximize the total expected profit over the next month, considering the uncertainties in demand. You have historical data that suggests the demand and cost scenarios can be modeled with discrete distributions.

Storage: Notably, any syrup produced in excess of the demand within the month must be disposed of, as the company's storage capabilities do not permit carrying over inventory to the next month.

Raw Material Cost (per liter): Raw Syrup ($0.5/liter) and Flavor Concentrates ($2/liter for vanilla and maple, $2.5/liter for cherry).

Pricing (per liter): $6/liter for vanilla, $7/liter for maple and $8/liter for cherry.

Demand Scenarios: The demand of vanilla syrup can be high (900 liters with 40% probability) and low (500 liters with 60% probability). The demand of maple syrup can be high (600 liters with 50% probability) and low (300 liters with 50% probability). The demand of cherry syrup can be high (1100 liters with 70% probability) and low (700 liters with 30% probability). The probability of demands for different syrups is independent.

Raw Syrup Production (per month): Due to equipment limitations and labor constraints, the maximum production of raw syrup is 1800 liters per month.

Minimum Production Requirement: SweetFlow has contracts with suppliers that require at least 300 litres production amount for each syrup to maintain their business relationships.

Formulate the stochastic programming problem and then solve it using Python and Gurobi.

Question 2: Non-Negative Least Squares (NNLS)

SOLUTION BELOW AND PYTHON FILE

Background:

The Least Squares problem is a fundamental optimization problem in mathematics, statistics, and engineering, primarily used for fitting a model to data. It finds the best-fitting curve or line to a set of points by minimizing the sum of the squares of the differences between the observed values and those predicted by the model. NNLS is a variant of the Least Squares problem that adds a non-negativity constraint on the solution. This constraint is crucial in many real-world applications where the quantities being estimated are inherently non-negative, such as concentrations of substances, pixel values in images, or amounts of materials.

Problem Statement:

You are provided with a set of measurements obtained from a sensor array that captures information about signal $\boldsymbol{x}$. Each signal has been transformed through a known measurement process $\bar{\boldsymbol{b}} = \boldsymbol{A}\boldsymbol{x}$, resulting in a measurement $\boldsymbol{b}$ that are lower in dimensionality than the original signal $\boldsymbol{x}$. Your task is to reconstruct the original signal $\boldsymbol{x}$ using NNLS optimization, ensuring all signal values are non-negative.

Dataset:

Let's define the measurement matrix $\boldsymbol{A}$ and the measured projections $\boldsymbol{b}$ with numerical details:

$$\boldsymbol{A} = \begin{Bmatrix} 1.0 & 0.5 & 0.3 & 0.2 & 0.1 & 0.7 & 0.2 & 0.1 \\ 0.5 & 1.0 & 0.4 & 0.3 & 0.2 & 0.1 & 0.3 & 0.9 \\ 0.4 & 0.4 & 1.0 & 0.8 & 0.6 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0.8 & 1.0 & 0.8 & 0.6 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.5 & 0.8 & 1.0 & 0.8 & 0.6 & 0.4 \end{Bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0.3 \\ 1.0 \\ 0.8 \\ 0.7 \\ 0.5 \end{bmatrix}$$

Question:

Formulate the quadratic problem to reconstruct the original signal $\boldsymbol{x}$ and then solve it using Python and Gurobi.

Problem 1 : 2 Stage-Stochastic LP

Deterministic is not solved but only for referecence

stochastic below is solved with scenarios considered.

Deterministic Formulation - stage 1

NOTE: This deterministic step is only to build off when adding scenarios in the stochastic problem

Decision Variables

6 decision variables,
3 for flavour concentrate

... $i \in \{$vanilla $= V$, maple $= M$, cherry $= C\}$

Decision Variables

Let f_i be the litres of Flavour concentrates of type i used in production, $i \in \{vanilla = v, maple = m, cherry = c\}$

Let P_i be the litres of syrup of type i produced, $i \in \{vanilla = v, maple = m, cherry = c\}$, and

6 decision variables, 3 for flavour concentrate production demands, 3 for syrup production by type.

Constraints and Bounds

$\sum_i P_i \leq 1800$] constraint for max production limit of 1800L

$P_i \geq 300$] constraint for min production limit for each syrup at 300L] also act as bounds

$f_i \geq P_i \quad \forall i$] constraint to ensure flavour concentrate availability is greater than syrup production for each type.

Objective Function:

$$\max_{P_i} \; 6P_v + 7P_m + 8P_c - (2.5f_v + 2.5f_m + 3.0f_c)$$] deterministic objective function.]

subtracting costs from maximized production to get profits.

note: added 0.5 to flavour concentrate costs/liter to account for 0.5 \$/liter raw syrup cost for all syrup production.

↓ actual solution.

Stochastic Formulation - stage 2 : Solved using Gurobi-Python.

Decision Variables

Let f_i be the litres of Flavour concentrates of type i used in production, $i \in \{vanilla = v, maple = m, cherry = c\}$

Let P_{ij} be the litres of syrup of type i produced in scenario j, $i \in \{vanilla = v, maple = m, cherry = c\}$, and $j \in \{low = l, high = h\}$

9 decision variables, 3 for flavour concentrate production demands, 6 for syrup production by scenario.

Constraints and Bounds

$P_{vl} + P_{ml} + P_{ch} \leq 1800$
$P_{vl} + P_{mh} + P_{ch} \leq 1800$
$P_{vh} + P_{ml} + P_{cl} \leq 1800$
$P_{vh} + P_{mh} + P_{cl} \leq 1800$
$P_{vh} + P_{ml} + P_{ch} \leq 1800$
$P_{vl} + P_{mh} + P_{cl} \leq 1800$
$P_{vl} + P_{ml} + P_{cl} \leq 1800$
$P_{vh} + P_{mh} + P_{ch} \leq 1800$

scenario specific maximum production limit of 1800 litres for all syrups combined.

8 scenarios for all combination of low and high production among the syrup types.

$f_v \geq P_{vj} \geq 300$
$f_m \geq P_{mj} \geq 300$ where $j \in \{low = l, high = h\}$
$f_c \geq P_{cj} \geq 300$

implemented as lower bound constraint.

minimum 300 L production constraint on each syrup regardless of scenario. Also ensuring flavour concentrate quantity is more than production demand for each syrup.

$P_{vl} \leq 500, \quad P_{vh} \leq 900$
$P_{ml} \leq 300, \quad P_{mh} \leq 600$
$P_{cl} \leq 700, \quad P_{ch} \leq 1100$

bounding production limits on each low, high scenario for all syrup types as provided.

Objective Function

$max\_lll = 0.6 \times 0.5 \times 0.3 \times (6P_{vl} + 7P_{ml} + 8P_{cl})$

$max\_llh = 0.6 \times 0.5 \times 0.7 \times (6P_{vl} + 7P_{ml} + 8P_{ch})$

$max\_lhh = 0.6 \times 0.5 \times 0.7 \times (6P_{vl} + 7P_{mh} + 8P_{ch})$

$max\_hlh = 0.4 \times 0.5 \times 0.7 \times (6P_{vh} + 7P_{ml} + 8P_{ch})$

$max\_lhl = 0.6 \times 0.5 \times 0.3 \times (6P_{vl} + 7P_{mh} + 8P_{cl})$

$max\_hll = 0.4 \times 0.5 \times 0.3 \times (6P_{vh} + 7P_{ml} + 8P_{cl})$

$max\_hhl = 0.4 \times 0.5 \times 0.3 \times (6P_{vh} + 7P_{mh} + 8P_{cl})$

$max\_hhh = 0.4 \times 0.5 \times 0.7 \times (6P_{vh} + 7P_{mh} + 8P_{ch})$

$max\ costs = 2.5f_v + 2.5f_m + 3.0f_c$

Stepwise objectives for costs and production of all syrup types in each scenario.

maximizing by using scenario probability X (production x price) for all syrups and scenarios.

note: added 0.5 to flavour concentrate costs/liter to account for 0.5 \$/liter raw syrup cost for all syrup production.

max profit max_lll + max_llh + max_lhh + max_hlh + max_lhl + max_hll + max_hhl + max_hhh − max_costs] Final objective function.] subtracting costs from maximized production to get profits.

Problem Instance Solution

Max Profit = $7380.00
Litres of vanilla syrup produced: 500L
Litres of maple syrup produced: 300L
Litres of cherry syrup produced: 1000L
] gurobi solution output.

Problem 2

Standard Form of Quadratic Programming Minimization Problem

$$\min_x \frac{1}{2} x^T Q x + q^T x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0$$
] standard QP setup with non-negative x.

In our case: given $\vec{A}$ = 5x8 matrix for sensor measurements.
$\vec{b}$ = 5x1 vector for sensor projections.
To find x, s.t $\vec{A}\vec{x} = \vec{b}$, ∴ x must be a 8x1 vector of transformations.
] dimensional analysis for $\vec{x}$ using the $\vec{A}, \vec{b}$ given

Also given: Problem is NNLS, so to formulate the least squares problem we have: $\min_x (\vec{A}\vec{x} - \vec{b})^2$] least squares minimization setup.
s.t $\vec{x} \geq 0$] non-negativity constraint
] Problem formulation

$$(\vec{A}\vec{x} - \vec{b})^2 = \vec{x}\vec{A}^T\vec{A}\vec{x} - 2\vec{b}^T\vec{A}\vec{x} + \vec{b}^T\vec{b}$$

Now setting this up in terms of a standard quadratic programming minimization problem, we have
] Least squares formulation.

$$\underbrace{\frac{1}{2} x^T Q x}_{\text{on comparing}} + \underbrace{q^T x}_{\text{on comparing}} = \underbrace{\vec{x}\vec{A}^T\vec{A}\vec{x}}_{} \underbrace{- 2\vec{b}^T\vec{A}\vec{x}}_{} + \underbrace{\vec{b}^T\vec{b}}_{\text{no comparison terms}}$$

↳ ie. does not affect minimization problem due to its constant nature.

↳ clearly $q^T = -2\vec{b}^T\vec{A}$

clearly, $Q = 2\vec{A}^T\vec{A}$
] Obtaing Q, q^T using simple comparison with QP form of given problem.

So we have, $\min_x \frac{1}{2} x^T Q x + q^T x$ where $Q = 2\vec{A}^T\vec{A}$, $q^T = -2\vec{b}^T\vec{A}$] objective function
s.t $x \geq 0$] constraint.

This setup will minimize the $\vec{x}$ vector with least squares optimization while keeping all $x \in \vec{x}$ non-negative as required.] outcome.

Single row example for what the above setup is doing

row 1: $\min_{x \in \vec{x}} \big(\underbrace{1.0x_1 + 0.5x_2 + 0.3x_3 + 0.2x_4 + 0.1x_5 + 0.7x_6 + 0.2x_7 + 0.1x_8}_{} - 0.3\big)^2$

↳ $\vec{A}$ values for row1 sensor measurements.

↳ $\vec{b}$ value for row 1 sensor projection

s.t $x \in \vec{x} \geq 0$

but doing so for all rows at once!
] output of NNLS QP for 1 row, example.

Problem instance Solution

$$\vec{x} = \begin{bmatrix} 0 \\ 0.24 \\ 0.59 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.54 \end{bmatrix}, \quad \text{min least squares error: } 0.01$$

] gurobi solution output $\vec{x}$ vector in transformation function, and the minimized NNLS error. rounded to 2 decimals as required.