Assignment 5

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MIE 376 — Assignment 5 — Winter 2024

In this assignment, you solve problems for Bi-Level Linear Programming (BLLP). You need to formulate the mathematical model and solve it with Python and Gurobi.

1. Submission Instructions

Submit a PDF file describing a Bi-Level Linear Programming (BLLP) model and reporting the solution to the problem instance. Also, submit a program (a Python script or a Jupyter notebook) using Gurobi to solve the problem instance. In the formulation, clearly define decision variables and state the objective function and constraints. For the problem instance, report the values of the objective and decision variables.

SOLUTION BELOW AND ON PYTHON - GUROBI

2. Problem

An upper-level problem of a BLLP problem is given with:

$$\begin{aligned} \max\ & x + 2y^* \\ s.t.\ & 2x - 3y^* \geq -12 \\ & x + y^* \leq 14 \end{aligned}$$

Here, x is the decision variable of the upper-level problem, and y^* is derived from the optimal solution of the following lower-level problem with given x:

$$\begin{aligned} \max\ & y \\ s.t.\ & 3x - y \geq 3 \\ & 3x + y \leq 30 \end{aligned}$$

Question 1: use the KKT condition to reformulate the problem and then solve it using Python and Gurobi.

Question 2: try to apply an iterative approach to solve the problem. Initially, solve the upper-level problem by treating y as a decision variable, without considering the lower-level problem. Then, use the obtained x value to solve the lower-level problem for y. In subsequent iterations, iteratively update x using the upper-level problem and y using the lower-level problem based on the previous iteration.

Solve them using Python and Gurobi, describe the outcome (the objective and decision variables) of each iteration and report whether it converges to the optimal solution (as Question 1) or not.

SOLUTION BELOW AND ON PYTHON - GUROBI

Problem 1 - KKT condition problem formulation

Followers Problem:

$\max\ y$ $\Rightarrow$ $\max\ y$] Followers problem objective function.

decision variable for Followers problem

s.t. $3x - y \geq 3$ / s.t. $y - 3x \leq -3$

$3x + y \leq 30$ / $3x + y \leq 30$] Followers problem constraints.

Leaders Problem

NOTE: $y^* = y$ from Followers problem decision variable.

decision variable for leaders problem.

$\max\ x + 2y^*$] → leaders problem objective function.

s.t. $2x - 3y^* \geq -12$

$x + y^* \leq 14$] leaders problem constraints.

NOTE: This is just the seperate leader and follower problems, we combine them in the answer to formulate a KKT-condition based BLLP problem.

KKT conditions and formulation steps.

Step 1: Primal and Dual Feasibility on Followers problem

$-3x + y + s_1 = -3 \quad (\lambda_1)$

$3x + y + s_2 = 30 \quad (\lambda_2)$] Adding dual variables λ_1, λ_2, and slack variables s_1, s_2 to create equalities

$s_1, s_2 \geq 0$] → constraints

$(DF: \nabla F = \sum_j \lambda_j \nabla g_j)$] → Dual Feasibility setup for max function

$\frac{\partial y}{\partial y} = \frac{\partial(-3x+y+s_1)}{\partial y}\lambda_1 + \frac{\partial(3x+y+s_2)}{\partial y}\lambda_2 \Rightarrow 1 = \lambda_1 + \lambda_2$] dual feasibility constraint for dual variables.

Step 2: Complementary Slackness on followers problem.

$\lambda_1 s_1 = 0, \quad \lambda_2 s_2 = 0$] complementary slackness conditions.

Step 3: Adding constraints from followers problem to leaders problem and solving the overall BLLP as one.

$\max\ x + 2y$] → objective function

decision variables.

$$\begin{aligned} s.t.\ & 2x - 3y \geq -12 \\ & x + y \leq 14 \\ & \lambda_1 + \lambda_2 = \\ & -3x + y + s_1 = 3 \\ & 3x + y + s_2 = 30 \\ & \lambda_1 s_1 = 0 \\ & \lambda_2 s_2 = 0 \\ & s_1, s_2, \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

Constraints

NOTE: This is the actual BLLP we will solve to obtain values for decision variables and objective function based on followers output to leaders problem.

Problem instance Solution

$x = 8, \quad y = 6$

max objective : 20] objective and decision variable values.

$\lambda_1 = 0, \quad \lambda_2 = 1$

$s_1 = 15, \quad s_2 = 0$] values of dual variables and complementary slackness variables in optimal solution.

BLLP solution for Final problem.

Problem 2 - Iterative approach

Step 1: upper level problem, y as decision variable. (Iteration 1)

$\max\ x + 2y$] step 1 objective function, decision variables 'y' and 'x'

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STEP 1: upper level problem, y as decision variable. (Iteration 1)

max $x+2y$] step 1 objective function, decision variables 'y' and 'x'

s.t. $2x-3y \geq -12$
$x+y \leq 14$] step 1 constraints.

Problem solution: $x=6$, $y=8$, objective $=22$] step 1 problem instance solution.

Take obtained x-value $=6$.] using $x=6$ from step 1 in step 2.

STEP 2: Lower level problem, solving for y using x from upper-level problem. (Iteration 2)

max y] step 2 objective function, decision variable 'y'.

s.t. $3x-y \geq 3$
$3x+y \leq 30$] step 2 constraints.

→ known

$\Rightarrow 3(6)-y \geq 3 \Rightarrow -y \geq -15 \Rightarrow y \leq 15$
$3(6)+y \leq 30 \Rightarrow y \leq 12 \qquad y \leq 12$] Simplified problem version constraints.

decision variable

Problem solution: $x=6$, $y=12$, objective $=12$] step 2 problem instance solution.

Take obtained y-value $=12$] using $y=12$ from step 2 in step 3

STEP 3: Repeat step 1 with y-value from lower level problem to obtain x. (Iteration 3)

Repeating upper-level problem with x from lower level problem.

max $x+2y$] step 3 objective function, decision variable 'x'

s.t. $2x-3y \geq -12$
$x+y \leq 14$] step 3 constraints.

$\Rightarrow 2x-3(12) \geq -12 \Rightarrow 2x \geq 24$
$x+12 \leq 14 \qquad x \leq 2$] Simplified problem version constraints.

decision variable

Problem solution: infeasible/unbounded model] step 3 problem instance solution.

Reason: if $y=12$, then $x+y \leq 14 \rightarrow$ so $x \leq 2$.
but $2x-3y \geq -12$ is not satisfied unless $x \geq 12$
So there is no intersection and x can't be solved for,
so the model is infeasible here.] Reasoning for infeasibility of model in iteration 3 when solving upper-level problem.

Conclusion: So, the final solution in this iteration based approach does not converge to the KKT-Formulation based BLLP solution from problem 1. This method leads to infeasibility, while the KKT-method gives $x=8$, $y=6$ and objective $=20$.] No solution convergence conclusion.

NOTE: solution does not converge.