CO21BTECH11001

Homework 5

Q.1

(a) Continuity equation in x-y coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

In cylindrical co-ordinates: $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\cos\theta \quad -①$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) \quad -②$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial y}(\sin\theta) \quad -③$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial y}(r\cos\theta) \quad -④$$

$$u_r = u\cos\theta + v\sin\theta$$

$$u_\theta = v\cos\theta - u\sin\theta$$

$$\frac{\partial u_r}{\partial r} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial v}{\partial r}\sin\theta \quad -⑤$$

$$\frac{\partial u_\theta}{\partial \theta} = \frac{\partial v}{\partial \theta}\cos\theta - \sin\theta\, v - \frac{\partial u}{\partial \theta}\sin\theta - \cos\theta\, u \quad -⑥$$

⑤ + $\frac{1}{r}$ ⑥

$$\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial v}{\partial r}\sin\theta - \frac{(u\cos\theta + v\sin\theta)}{r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\cos\theta - \frac{\partial u}{\partial \theta}\sin\theta\right)$$

$$= \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta - \frac{u_r}{r} + \frac{1}{r}\left(\frac{\partial v}{\partial y} r\cos^2\theta + \frac{\partial u}{\partial x} r\sin^2\theta\right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{u_r}{r}$$

$$\Rightarrow \boxed{\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = 0}$$

(b) Laplace equation in x, y co-ordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$$

$$= \cos\theta \frac{\partial u}{\partial x} + \sin\theta \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r}\right)$$

$$+ \cos\theta\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r}\right)$$

$$= \cos^2\theta \frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$= -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) - r\sin\theta \frac{\partial u}{\partial y}$$

$$+ r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right)$$

$$- r\sin\theta \frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right)$$

$$= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}\right) + r^2\left(\sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y}\right.$$

$$\left. + \cos^2\theta \frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y}$$

$$+ \cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

CO2020 HW5

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Q.2
Solution:

a. 4th order centred-difference scheme for spatial discretization is:
For i > 1 and i < nx-2 :

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{(dx)^2}\left(\frac{4}{3}(T_{i+1} + T_{i-1}) - \frac{1}{12}(T_{i+2} + T_{i-2}) - \frac{5}{2}T_i\right)$$

For i == 1:

$$\frac{\partial^2 T}{\partial x^2} = (10T_{i-1} - 4T_{i+1} + 14T_{i+2} - 6T_{i+3} + T_{i+4} - 15T_i)/12$$

For i == nx-2:

$$\frac{\partial^2 T}{\partial x^2} = (10T_{i+1} - 4T_{i-1} + 14T_{i-2} - 6T_{i-3} + T_{i-4} - 15T_i)/12$$

b. When we perform stability analysis on 4th order scheme, we get:

$$\epsilon_j^{n+1} = \epsilon_j^{n}(1 - 5\lambda/2) + \frac{4\lambda}{3}(\epsilon_{j-1}^{(n)} + \epsilon_{j+1}^{(n)}) - \frac{\lambda}{12}(\epsilon_{j+2}^{(n)} + \epsilon_{j-2}^{(n)})$$

Therefore, we get

$$|1 - \frac{5\lambda}{2} + \frac{4\lambda}{3}(2cos(kh)) - \frac{\lambda}{12}(2cos(2kh))| < 1$$

Solving this we get

$$\lambda < \frac{3}{8}$$

To find maximum value of nx for a given time step

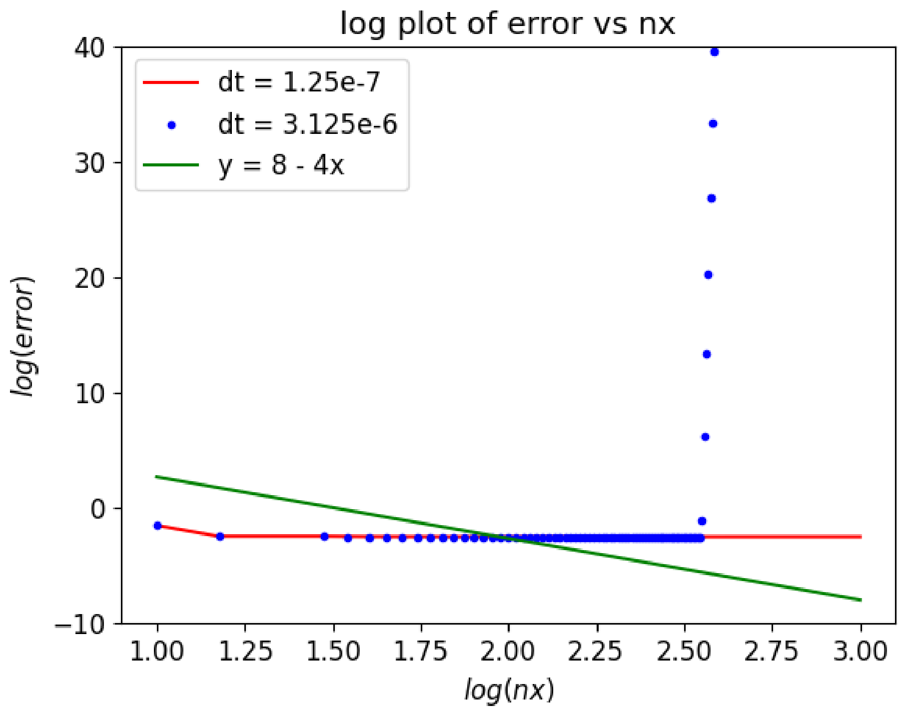
$$\frac{k\Delta t}{(\Delta x)^2} < \frac{3}{8}$$

$$\Delta x = \frac{L}{nx-1}$$

$$\frac{k\Delta t(nx-1)^2}{L^2} < \frac{3}{8}$$

$$nx < 1 + \sqrt{\frac{3L^2}{8k\Delta t}}$$

c. The log plot of error vs nx is following



We can see that error is decreasing with a slope of approximately -4, then the error becomes almost constant w.r.t. nx.

In case of $dt = 3.125e - 6$, since value of dt is more, the solution blows up after a value of $nx \cong 350$.

Whereas for a lower value of $dt = 1.25e - 7$, the solution is stable till nx = 1000.