CO2020

Homework 3

Q.1

(a) $u_{xx} + u_{yy} = 0$

This is a homogenous, linear PDE of order 2.

(b) $u x u_x + u^2 x y u_{xy} = 0$

This is a non homogenous, non linear PDE of second order.

(c) $|u_t|^2 + f(x,y)\, u = g(x,y)$

This is a non homogenous, non linear PDE of order one.

(d) $u_t + u u_x = \nu u_{xx}$

This is a ~~non~~ homogenous, non linear PDE of second order.

(e) $u_t + c u_x = 0$

This is a homogenous, linear PDE of first order

Q.2

(a) $x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = 0$

$$\Delta = \frac{(2xy)^2}{4} - x^2 y^2$$

$\Delta = 0 \Rightarrow$ This is a Parabolic PDEs

Characteristic(s) $\quad \frac{dy}{dx} = \frac{-xy \pm \sqrt{(-xy)^2 - x^2y^2}}{x^2}$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$\Rightarrow \ln y = -\ln x + \ln c$

$\Rightarrow \boxed{xy = c, \quad c > 0}$

This equation has only one real family of characteristic in the form of rectangular hyperbolas.

(b) $u_{xx} + (1+y^2)^2 u_{yy} - 2y(1+y^2)u_y = 0$

$$\Delta = 0^2 - 1 \times (1+y^2)^2$$

$$\Delta = -(1+y^2)^2$$

$\Delta < 0 \Rightarrow$ Elliptic equation

This equation has no real characteristic.

(c) $x^2 u_{xx} - 2xy\, u_{xy} - 3y^2 u_{yy} = u_y - u_x + f(x,y)u$

$$\Delta = (xy)^2 - (-x^2 \times 3y^2)$$

$$\Delta = 4x^2y^2$$

$\Rightarrow$ This equation is hyperbolic for $x > 0, y > 0$

2 family of real characteristics.

$$\frac{dy}{dx} = \frac{-xy \pm \sqrt{4x^2y^2}}{x^2} = \frac{-y}{x} \pm \frac{2y}{x}$$

$$\frac{dy}{dx} = \frac{-3y}{x}, \quad \frac{dy}{dx} = \frac{y}{x}$$

$\Rightarrow \ln y = -3\ln x + \ln c$, $\ln y = \ln x + \ln c$

$\Rightarrow \boxed{yx^3 = c, \quad c > 0}$, $\boxed{y = cx}$, $c > 0$

Q.3

Soln

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x)\frac{\partial u}{\partial x} = f(x,y) \quad -(1)$$

we have to transform (1) into laplace equation

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = A(x) f(x,y) \quad -(4)$$

$$\frac{d A(x)}{dx} = a(x)$$

Let $\tilde{u} = Au + g(x)$

$$\Rightarrow \frac{\partial \tilde{u}}{\partial x} = \frac{dA}{dx}.u + A\frac{\partial u}{\partial x} + g'(x) = au + A\frac{\partial u}{\partial x} + g'(x)$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = a\frac{\partial u}{\partial x} + \frac{da}{dx}u + A\frac{\partial^2 u}{\partial x^2} + \frac{dA}{dx}\frac{\partial u}{\partial x} + g''(x)$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{da}{dx}u + A\frac{\partial^2 u}{\partial x^2} + 2a\frac{\partial u}{\partial x} + g''(x) \quad -(2)$$

$$\frac{\partial^2 \tilde{u}}{\partial y^2} = A\frac{\partial^2 u}{\partial y^2} \quad -(3)$$

Substitute (2) and (3) in (4), we get

$$A\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{da}{dx}.u + 2a\frac{\partial u}{\partial x} + g''(x)$$

$$= A f(x,y)$$

$$\Rightarrow A\left(f(x,y) - a(x)\frac{\partial u}{\partial x}\right) + \frac{da}{dx}.u + 2a\frac{\partial u}{\partial x} + g''(x)$$

$$= Af(x,y)$$

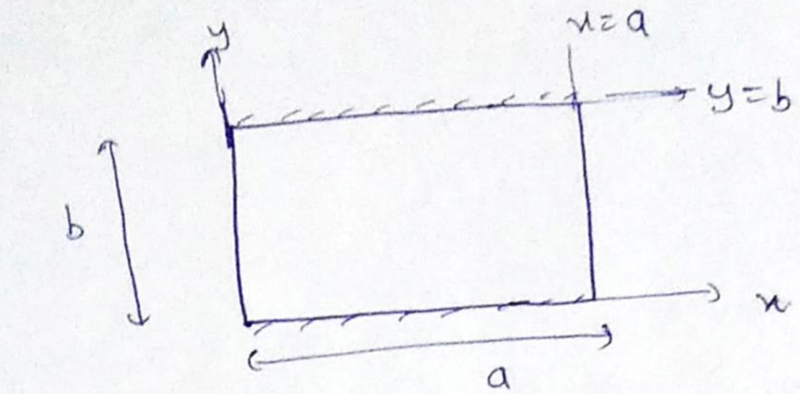
$$\Rightarrow \quad g''(x) = Aa\frac{\partial u}{\partial x} - 2a\frac{\partial u}{\partial x} - u\frac{da}{dx}$$

$$\Rightarrow \boxed{g(x) = \int\left[\int\left((A-2)a\frac{\partial u}{\partial x} - u\frac{da}{dx}\right)dx\right]dx}$$

$$\Rightarrow \boxed{\tilde{u} = Au + g(x)}$$

Q.4

Soln



Let us assume the temperature distribution $T(x,y)$ as

$$T(x,y) = f(x)\,G(y) + T_0$$

Applying boundary conditions:

(i) $T(0,y) = T_0$

$\Rightarrow f(0)G(y) + T_0 = T_0$

$\Rightarrow f(0)\,G(y) = 0 \quad \forall y \in [0,b]$

$\Rightarrow f(0) = 0$

(ii) $T(a,y) = T_0\left\{1 + 0.2\left(\frac{y}{b} - 0.5\right)\right\}$

$\Rightarrow f(a)\,G(y) + T_0 = T_0\left\{1 + 0.2\left[\frac{y}{b} - 0.5\right]\right\}$

$f(a)\,G(y) = 0.2T_0\left(\frac{y}{b} - 0.5\right)$

(iii) $G'(0) = 0 \;,\; G'(b) = 0$

Laplace Equation $\quad T_{xx} + T_{yy} = 0$

$\Rightarrow f''G + G''f = 0$

$$\Rightarrow \frac{f''}{f} = -\frac{G''}{G} = k$$

we can split this into following problems

① $G'' + kG = 0$

$G'(0) = 0$

$G'(b) = 0$

② $f'' - kf = 0$

$f(0) = 0$

$f(a)\, G(y) = 0.2T_0\left(\frac{y}{b} - 0.5\right)$

General solution for problem ① is:

$$G(y) = c_1 \cos\sqrt{k}\, y + c_2 \sin\sqrt{k}\, y$$

$$G'(y) = -c_1\sqrt{k}\sin\sqrt{k}\, y + c_2\sqrt{k}\cos\sqrt{k}\, y$$

$G'(0) = 0 \Rightarrow c_2 = 0$

$G'(b) = 0 \Rightarrow c_1\sqrt{k}\sin\sqrt{k}\, b = 0$

$$\Rightarrow \boxed{\sqrt{k} = \frac{n\pi}{b}}$$

General solution for problem ② is:

$$f(x) = c_3 e^{\sqrt{k}x} + c_4 e^{-\sqrt{k}x}$$

$f(0) = 0 \Rightarrow c_3 + c_4 = 0$

$$\Rightarrow f(x) = c_3\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

$$\boxed{f_n(x) = (c_3)_n \sinh\frac{n\pi x}{b}}$$

$$G_n = (C_1)_n \cos\frac{n\pi y}{b}$$

$$\Rightarrow \boxed{T(x,y) = T_0 + \sum C_n \cos\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{a}x\right)}$$

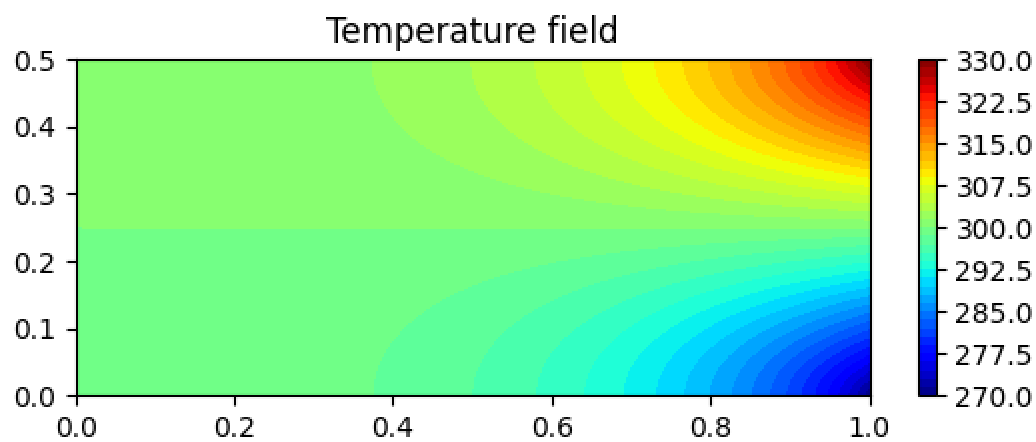
Applying $T(a,y) = T_0\left[1 + 0.2\left(\frac{y}{b} - 0.5\right)\right]$

Let $g(y) = 0.2T_0\left(\frac{y}{b} - 0.5\right)$

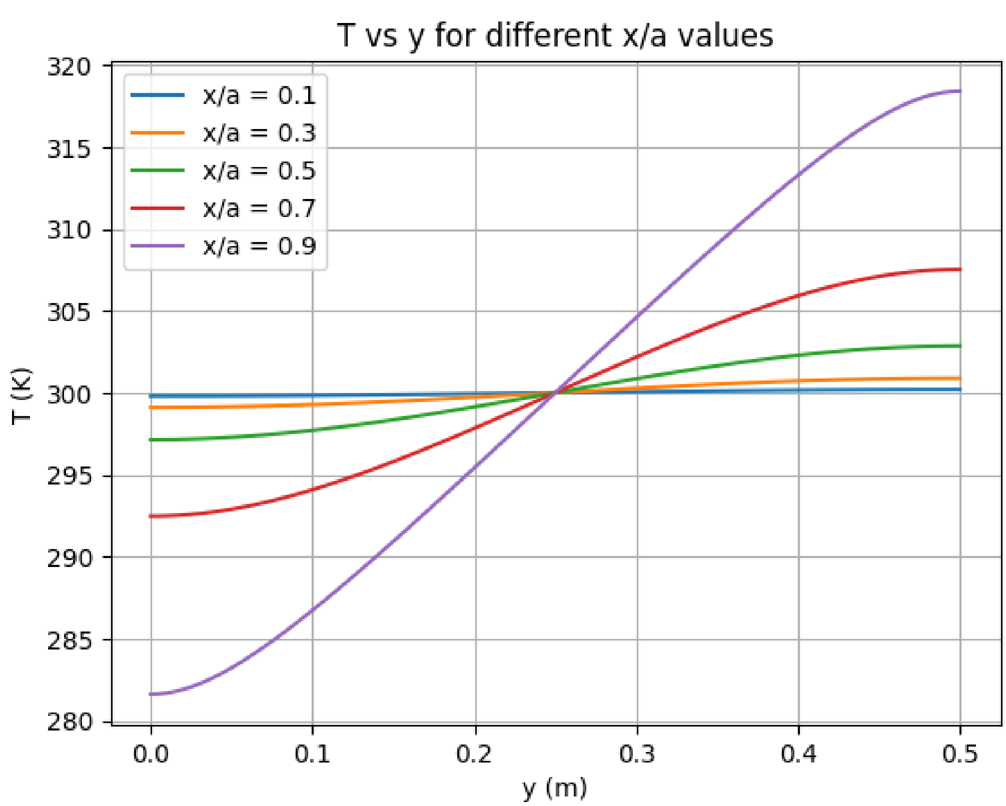
$$\Rightarrow g(y) = \sum\left(C_n \sinh\left(\frac{n\pi a}{b}\right)\right)\cos\frac{n\pi y}{b}$$

$$\Rightarrow \boxed{C_n = \frac{b_n}{\sinh\left(\frac{n\pi a}{b}\right)} \quad \text{where } b_n = \frac{2}{b}\int_0^b g(y)\cos\left(\frac{n\pi y}{b}\right)dy}$$

a. Contour plot in 2D domain:



b. Profiles of T vs y for x/a = 0.1, 0.3, 0.5, 0.7 and 0.9



c. Profiles of T vs x for y/b = 0.1, 0.5 and 0.9.