CO21BTECH11001

Homework 5

Q.1

(a) Continuity equation in x-y coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

In cylindrical co-ordinates: $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\cos\theta \quad -①$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) \quad -②$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial y}(\sin\theta) \quad -③$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial y}(r\cos\theta) \quad -④$$

$$u_r = u\cos\theta + v\sin\theta$$

$$u_\theta = v\cos\theta - u\sin\theta$$

$$\frac{\partial u_r}{\partial r} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial v}{\partial r}\sin\theta \quad -⑤$$

$$\frac{\partial u_\theta}{\partial \theta} = \frac{\partial v}{\partial \theta}\cos\theta - \sin\theta\, v - \frac{\partial u}{\partial \theta}\sin\theta - \cos\theta\, u \quad -⑥$$

$⑤ + \frac{1}{r}⑥$

$$\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial v}{\partial r}\sin\theta - \frac{(u\cos\theta + v\sin\theta)}{r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\cos\theta - \frac{\partial u}{\partial \theta}\sin\theta\right)$$

$$= \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta - \frac{u_r}{r} + \frac{1}{r}\left(\frac{\partial v}{\partial y} r\cos^2\theta + \frac{\partial u}{\partial x} r\sin^2\theta\right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{u_r}{r}$$

$$\Rightarrow \boxed{\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = 0}$$

(b) Laplace equation in x, y co-ordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r}\right)$$

$$+ \cos\theta\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) - r\sin\theta \frac{\partial u}{\partial y}$$

$$+ r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right)$$

$$- r\sin\theta \frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right)$$

$$= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}\right) + r^2\left(\sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y}\right.$$

$$\left. + \cos^2\theta \frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y}$$

$$+ \cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$