SVD

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Let A be a $m \times n$ matrix. Then A can be written as –

$$A = U\sum V^T$$

where $\sum$ is a diagonal $m \times n$ matrix, and U and V are unitary matrices of dimensions $m \times m$ and $n \times n$ respectively.

We are trying to find out a set of orthonormal vectors in rowspace of A which when multiplied by A (along with some constant) gives a set of orthonormal vectors in columnspace of A.

i.e.

$$A[v_1\ v_2\ \dots v_n] = [u_1\ u_2\ \dots u]\begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \sigma_r & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where r is the number of positive eigen values of the matrix AA^T

$$A = [u_1\ u_2\ \dots u]\begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \sigma_r & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}[v_1\ v_2\ \dots v_n]^T$$

Now $A^T A = (U\sum V^T)^T(U\sum V^T) = V\sum^T\sum V^T$
Since $\sum^T\sum$ is also a diagonal matrix.
$\therefore$ Columns of V are eigen vectors of the matrix $A^T A$
Similarly columns of U are eigen vectors of the matrix AA^T

It can be proved that non-zero eigen values of $A^T A$ and AA^T are same (with the same multiplicity).

If $\lambda_1, \lambda_2, \dots, \lambda_r$ are +ve eigenvalues of $A^T A$ in decreasing order, then

$$\Sigma_{jj} = \sigma_j = \sqrt{\lambda_j} \qquad \text{for } j = 1, \dots, r$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$$

and all other elements are zero.

<u>Questions –</u>

1. SVD decomposition is possible for
(a) Square matrices
(b) Rectangular matrices
(c) Positive definite matrices
(d) Any matrix
Ans. (d)

2. What does U, $\sum$ and V signifies in SVD transform technique?
(a) Stretching, Rotation, Stretching
(b) Rotation, Stretching, Rotation
(c) Stretching, Stretching, Rotation
(d) Rotation, Rotation, Stretching
Ans. (b)

3. Which of the following expressions is/are correct?
(a) $AV = U\sum$
(b) $VA = U\sum$
(c) $CC^T = U\sum\sum^T U^T$
(d) $CC^T = U\sum^T\sum U^T$
Ans. a,c

4. What does the value of $\sigma_i s$ (singular values) signify?
   Ans. The singular values go down with each dimension, which tells us that each dimension is adding less and less value. Our goal is to stop adding dimensions to our approximation when the singular values become relatively negligible.

5. How is SVD used in machine learning?
   Ans. It is used as data reduction method in machine learning. It can also be used in image compression and denoising data.