Probability & Statistics
Tutorial 5

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Problem 1

Let X and Y be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is a? a=1

(b) What is the conditional PDF $f_{Y|X}(y|x)$ of Y give $X = x$? 2y

(c) What is the conditional expectation of Y given X?

(d) What is the expected value of Y?

Problem 2

Let X be the number of ice-cream cones a vendor sells on a day. If the average temperature of a summer day is a random variable Y (in Fahrenheit), where $Y \sim Uniform([95, 105])$. We also have $X \sim Poisson(Y^2/10 + Y/5 + 5)$.

(a) What is $E[X|Y]$?

(b) What is $E[X]$?

Problem 3

If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a dog, I stop and pet the dog and chat to the owner. The number Y of dogs I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the dog and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x}$$

(a) If I see a single dog, what is the expectation and variance of the time spent petting the dog and chatting to its owner?

(b) What is the conditional expectation of the total time spent petting dogs and chatting to their owners, as a function of Y?

(c) Using the law of iterated expectation calculate $E[X]$.

Problem 4

Let X and Y be independent Poisson random variables with parameter 1. Compute the following. (Give a correct formula involving sums — does not need to be in closed form.)

(a) The probability mass function for X given that $X + Y = 5$.

(b) The conditional expectation of Y^2 given that $X = 2Y$.

(c) The probability mass function for $X - 2Y$ given that $X > 2Y$.

(d) The probability that $X = Y$.

Problem 5

Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X > 1 \mid Y = y)$.

Problem 6

Show that

$$E[aY + bZ|X] = aE[Y|X] + bE[Z|X] \text{ for } a, b \in \mathbb{R}$$

Problem 7

Each child is equally likely to be male or female, independently of all other children.

(a) Show that, in a family of predetermined size, the expected number of boys equals the expected number of girls. Was the assumption of independence necessary?

(b) A randomly selected child is male; does the expected number of his brothers equal the expected number of his sisters? What happens if you do not require independence?

Problem 8

Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1 - p$.

(a) Show that the PDF of X is given by

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0 \\ (1-p)\lambda e^{-\lambda x}, & \text{if } x \geq 0 \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

Problem 9

Let X and Y be independent and have PMFs given by

$$p_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{3} & \text{if } x = 1, \\ \frac{1}{6} & \text{if } x = 2, \\ 0 & \text{otherwise} \end{cases}$$

Calculate the PMF of $W = X + Y$ by convolution.

Problem 10

Let Q be a continuous random variable with PDF:

$$f_Q(q) = \begin{cases} 6q(1-q) & \text{if } 0 \le q \le 1 \\ 0 & \text{otherwise} \end{cases}$$

This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$P(X = 1 \mid Q = q) = q$$

Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all q.

Problem 11

Shashank performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of Z fair coins.

1. Given that Shashank has just had a trial with Z tails, what is the probability that next two trials will also have this result?
2. Sandeep conducts an experiment like Shashank's, except that he uses M coins for the first trial, and then he obeys the following rule: Whenever all the coins land on the same side in a trial, Sandeep permanently removes one coin from the experiment and continues with the trials. He follows this rule until the $(M-1)^{th}$ time he removes a coin, at which point the experiment ceases. Find $E[X]$, where X is the number of trials in Sandeep's experiment.

Problem 12

Let X be a continuous random variable with PDF:

$$f_X(x) = \begin{cases} x^2\left(2x + \frac{3}{2}\right) & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find $Var(Y)$.