

Math of Quantum Mechanics

Quantum Sense



Saturday

Maths of Quantum Mechanics

- Quantum Sense

Chapter 1: Why linear algebra

(Motivation)

→ In classical physics, physical quantities are :-

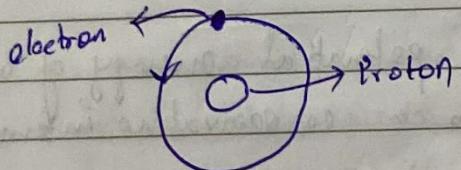
- i) single valued (position, velocity, etc.)
- ii) continuous
→ Many smoothly in time

∴ A way to model these quantities is with a ~~gta~~ continuous function.

This may seem obvious, but a model like this breaks down ~~when we~~ in the quantum world.

classical v/s Quantum worlds

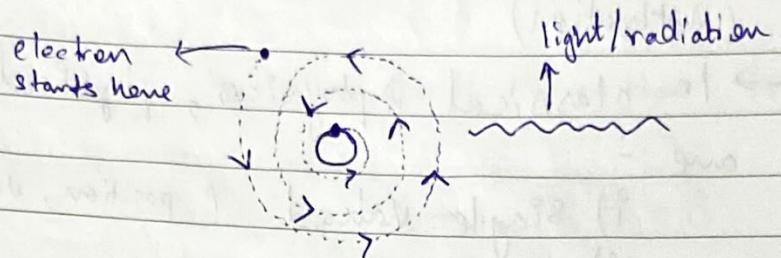
Consider a hydrogen atom :



From Maxwell's law of electrodynamics, we know that a particle (charged) emits radiation when in acceleration.

∴ The electron orbiting the proton is under

Centripetal acceleration, it will emit radiation & light, thus lose energy and fall into the proton: (eventually)



Now, potential energy of a system of point charges is defined as the work required to assemble this system of charges by bringing them close together, as in the system from an infinite distance.

Work & Work energy theorem.

Work-energy theorem: The work done by on the particle by the resultant force is equal to the change in its KE.

General Definition of PE :

The change in potential energy of a system corresponding to a conservative internal force is defined as :

$$U_f - U_i = -W = - \int_{i}^{f} \vec{F} \cdot d\vec{r}$$

where W is the work done by the internal force on the system as the system

passed from initial configuration i to final configuration f .

∴ Assuming only internal conservative forces act on a system between the parts of the system and there are no external forces and work done by them, then we have

$$U_f - U_i = -W = -(K_f - K_i)$$

$$\Rightarrow U_f + K_f = U_i + K_i$$

If we define $U+K$ as the total mechanical energy of the system, then we see that

the total mechanical energy of the system remains constant, given all internal forces are conservative and no external force is acting or doing work on the system.

This is called the law of conservation of mechanical energy.

→ Potential energy can only be defined for conservative forces, therefore the conservation law for mechanical energy only hold for conservative forces within the system.

→ Though the work-energy theorem still holds.

Now, the centripetal force ~~in the~~ acting on the electron will be

$$\frac{m_e v_e^2}{r_h}$$

where m_e = mass of electron

v_e = velocity of electron

and ~~r_h~~ r_h = radius of hydrogen atom.

$$\therefore \frac{m_e v_e^2}{r_h} = \frac{1}{4\pi\epsilon_0} \cdot \frac{e \cdot e}{r_h^2}$$

where $\frac{1}{4\pi\epsilon_0}$ is coulomb constant, e = charge of electron/proton.

$$\therefore m_e v_e^2 = \frac{e^2}{4\pi\epsilon_0 r_h} \rightarrow \textcircled{i}$$

Now, the angular momentum of an electron on an orbit is

$$m_e v_e r_h = \frac{n h}{2\pi}$$

where h = plank's constant.

$$\therefore v_e = \frac{n h}{2\pi r_h m_e} \rightarrow \textcircled{i}$$

$$\therefore K_e = \text{kinetic energy of electron} = \frac{1}{2} m_e v_e^2$$

Now, the potential energy of the system will be

$$U_f - U_i^0 = - \int_{\infty}^{r_h} \vec{F} \cdot d\vec{r} = - \int_{\infty}^{r_h} \frac{1}{4\pi\epsilon_0} \cdot \frac{e \cdots e}{r^2} + \hat{F} \cdot \hat{dr} dr$$

$\hat{F} \cdot \hat{dr} = 0$ since they are opp. to each other

$$\therefore U_f - U_i^0 = + \int_{\infty}^{r_h} \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} dr$$

$$= - \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{r} \Big|_{\infty}^{r_h}$$

$$U_f - U_i^0 = - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_h}.$$

If we consider $U_i^0 = 0$ (when the charges are at ∞),

$$\text{we'll get } U_f = - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_h}$$

$$= - m_e V e^2.$$

$$\text{and } K_f = \frac{1}{2} m_e V e^2$$

$$\therefore T = K_f + U_f = - \frac{1}{2} m_e V e^2.$$

we also get from ①

$$m_e V e^2 = \frac{n^2 h^2}{4\pi^2 r_h^2 m_e} = \frac{e^2}{4\pi\epsilon_0 r_h} \quad (\text{from ②})$$

$$\therefore r_h = \frac{n^2 h^2 \epsilon_0}{\pi m_e e^2}$$

$$\therefore U_f = -\frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{\frac{n^2 \cdot h^2 \epsilon_0}{4\pi m_e e^2}}$$

$$= -\frac{e^2 m_e}{4\epsilon_0^2 n^2 h^2}$$

Now, for $n=1$, if we put values for

$$m_e = 9.1 \times 10^{-31} \text{ kg}$$

$$\epsilon_0 = 8.8542 \times 10^{-12} \text{ N}^{-1} \text{ C}^{-2}$$

$$e = 1.6 \times 10^{-19} \text{ C}$$

$$h = 6.626 \times 10^{-34} \text{ joule-seconds}^{-1}$$

we get

$$U_f = -\frac{(1.6 \times 10^{-19})^4 \times 9.1 \times 10^{-31}}{4 \times (8.8542 \times 10^{-12})^2 \times (6.626 \times 10^{-34})^2}$$

$$= -0.0433 \times 10^{-19 \times 4 - 31 + 2 + 68}$$

$$= -0.0433 \times 10^{-15} \text{ Joules}$$

$$= -4.33 \times 10^{-17} \text{ joules}$$

$$= -4.33 \times 10^{-17} \times \frac{1}{1.6 \times 10^{-19}} \text{ eV}$$

$$= -\frac{4.33}{1.6} \times 10^2 \text{ eV}$$

$$= -27.074 \text{ eV}$$

$$\therefore \text{Total energy of the system} = \frac{1}{2} \times U_f = T$$

$$= -13.537 \text{ eV}$$

Also, $K_e = 13.537 \text{ eV}$ = kinetic energy of electron.

Experimentally, it has been found that

$$K_e = 13.06 \text{ eV} =$$

∴ As the electron falls, spiralling into the proton, we can calculate the electron's energy as it emits radiation by placing a detector.

Quantum World

We run the experiment and try to determine electron's energy.

We notice that

→ Electron's kinetic energy takes a certain set of values, never measuring anything in between

→ The values that we measure is random. However, some energy values have higher probability than others.

Summary :

- Experiments show that physical quantities can sometimes be discrete
- Values we measure are random, but probabilistic; there is a probability attached to each value
- Continuous functions cannot be used to model the quantum world.

Now, let us try to mathematically model the results of our experiment.

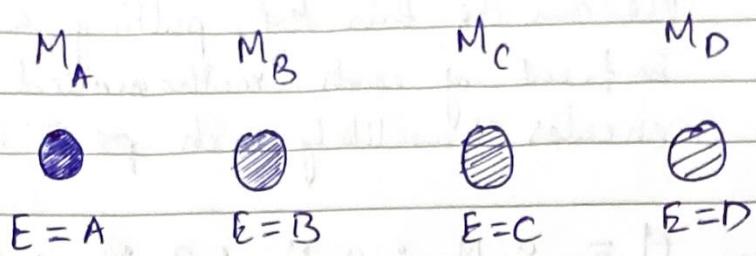
→ Randomness

Before we make a measurement, the particle somehow seems to hold the information on every possible outcome we could get.

Let's say we know for sure that our particle has among A , and let us represent the particle (more precisely, its state) by some mathematical object M_A .

M_A can be anything : A function, a ring, a manifold, something that we have yet to determine

So we have a mathematical object for every possible outcome we could get for our particle.



Somehow, our particle is represented by an amalgamation of all of these mathematical objects; (before we make the measurement)

→ So we need to put all these mathematical objects together somehow.

- be an unknown mathematical operation.
- can be anything — \times , dot product, $+$ etc.

∴ We can represent our particle then by an aggregate mathematical object.

$$M = M_A \cdot M_B \cdot M_C \cdot M_D$$

M is a mathematical object that describes our particle before the measurement.

→ Now, we also need to codify the idea that some outcomes are more likely than others.

So each mathematical object needs to carry with it the probability of getting that particular outcome.

We can do this by putting a number in front of each mathematical object that encodes how likely each possibility is to occur.

$$M = a_A M_A \cdot a_B M_B \cdot a_C M_C \cdot a_D M_D$$

\downarrow encodes probability.

M looks very much like a linear combination of sort. (It can be other things as well, but a linear ~~or~~ combination is something it can be)

Let's assume it is a linear combination.

$$\therefore M = a_A M_A + a_B M_B + a_C M_C + a_D M_D$$

~~if~~

So $M = A$ mathematical object describing our particle is a linear combination of some sort of all outcome possibilities, which we'll assume are represented by some sort of vector.

(This may seem like a big leap in logic, but we'll see how we came to those conclusions)

The Discretevous Problem

We should now ask: How should we represent our physical quantities?

We know that continuous functions won't work.

We need a mathematical object that allows us to sometimes extract discrete values.

One possible guess is: Linear Operators/Matrices

Why? Because Matrix contains a discrete set of numbers, so maybe we can extract our physical quantities from this discrete set.

Now, we have a solid guess into how we want to model quantum mechanics!

1) Particles are represented by a linear combination of vectors in some space vector space.

2) Physical quantities are represented by linear operators within that space.

Sunday

Chapter 2: What are kets & wavefunctions

→ What is a vector space?

Mathematical definition of vector space:

Definition: for any vectors $\vec{u}, \vec{v}, \vec{w}$ and scalars a, b :

i) $(\vec{u} + \vec{v})$ is also a vector vi) $-\vec{u}$ exists

ii) $-a\vec{u}$ is also a vector vii) $a(b\vec{u}) = (ab)\vec{u}$

iii) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ viii) $1\vec{u} = \vec{u}$

iv) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ ix) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

v) 0 vector exists

x) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

→ This definition makes no mention of lists of numbers or arrows!

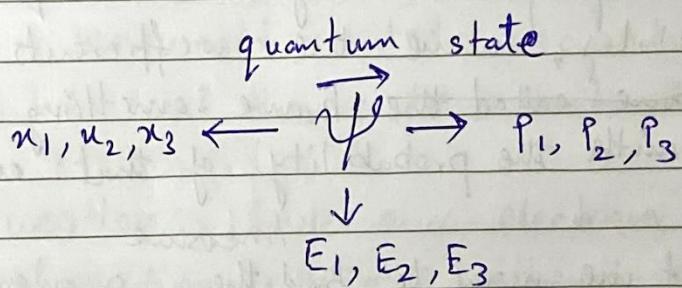
* ∵ Any set of objects that satisfies these rules, forms a vector space

∴ With the right set of objects, it could be fruits, functions or our familiar columns and arrows.

→ Linear Algebra and vector spaces are all about structure and patterns, not about the type of object you're using (or imagining)

It just so happens that the particular structure of a vector space is exactly what we need to describe the quantum world.

- As mentioned before, we want our particle to be represented by a vector in a vector space.
 - A vector would then represent a Quantum State.
- A quantum state is a mathematical object that holds all the physical properties of our particle.



From $\vec{\Psi}$, we should be able to extract the possible possible positions, momenta & energy energies etc. at any particular moment in time,

AS WELL AS all the probabilities associated with it.

Instead of using a typical vector symbol, we'll be using a different symbol for vectors in QM.

$$2) \vec{\Psi} \rightarrow \text{Ket} | \Psi \rangle$$

We will understand why this representation is useful in later chapters.

Therefore, our quantum state Ψ will be a linear combination of Energy Kets representing each energy outcome.

$$\Psi | \Psi \rangle = c_1 | E_1 \rangle + c_2 | E_2 \rangle + c_3 | E_3 \rangle$$

This Linear combination is function called a superposition of all these quantum states, where the coefficients (c_1, c_2, c_3, \dots) are called the have something to do with the probability of that outcome.

If we want to find the angular momentum, we would have a list of quantum state Kets representing each possible angular momentum outcome

$$| \Psi \rangle = c_1 | L_1 \rangle + c_2 | L_2 \rangle + c_3 | L_3 \rangle + c_4 | L_4 \rangle$$

We would have a different linear combination outcome, but same present quantum state.

The expression outcome here assumes the outcomes are finite, but we can have infinite outcomes of energy / any physical quantity as well even if it's discrete.

So the quantum state $|\Psi\rangle$ can be a linear combination of infinitely many kets.

$$|\Psi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle + c_4|E_4\rangle + \dots$$

↓
superposition of infinitely many outcomes.

→ There are physical quantities that are not discrete.

Like position: we do not have experimental evidence showing that it is discrete.

∴ we would want to encapsulate this idea into our theory / mathematics modeled.

of question: Say our electron can have any position possible (although some may be more likely than others).

How do we represent the quantum state of our electron using the kets of possible possible position?

Answer: A typical sum as follows; ~~we~~

$$\sum_i c_i |n_i\rangle$$

~~we~~ won't work since it is not possible to ~~list~~ list the whole number line as a discrete list.

∴ We need a mathematical operation that has the power ~~to~~ to

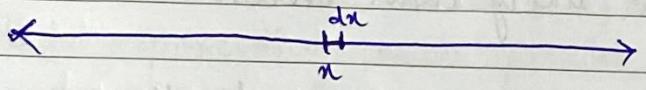
"sum over a continuous set"

$$\int \rightarrow \text{Integral} \}$$

Integral is the mathematical operation that does exactly this!

When finding the area under the curve, the integral basically sums the ~~or~~ infinite strips that the area is divided into to ~~and~~ get the area.

So, we will use the power of the integral to write a linear combination when we have a continuous set of possible outcomes.



Let the particle be at position ~~at~~ n and so it is in state $|n\rangle$.

Let $\rho(n)$ be the probability density function indicating with how much ~~likely~~ likelihood the particle can be at any given position.

The probability of it being at n will be equal to $p = \rho(n) dn$.

∴ Our linear combination will be

$$|\psi\rangle = \int_{-\infty}^{\infty} p \cdot |n\rangle = \int_{-\infty}^{\infty} c(n) \cdot |n\rangle dn$$

Chapter 3 : Why do we need a Hilbert space?

Now, we look at why letting there ~~be~~ infinite ~~be~~ a linear combination of infinite kets can cause a problem.

Let's assume that we have $|\psi\rangle$ as follows:

$$|\psi\rangle = a_1 |E_1\rangle + a_2 |E_2\rangle + \dots$$

We have to now find what is dimension of the vector space in which $|\psi\rangle$ lies?

Dimension of a vector space is the number of vectors needed to form a basis.

Let's assume the kets $|E_i\rangle \forall i \geq 1$ form the basis of the vector space (we'll prove this later)

Then the dimension of our vector space would be infinity.

But having an infinite dimensional vector space can cause problems.

For example:

We can verify that set of all possible polynomials form a vector space.

(Based on the listed conditions in chapter 2,

we can check if polynomials satisfy all the conditions to form a vector space).

Now, the basis of this vector space will be:

$$\{1, n, n^2, \dots\}$$

Now, any linear combination of the basis vector should give us a vector in the vector space, but we see this being violated as follows:

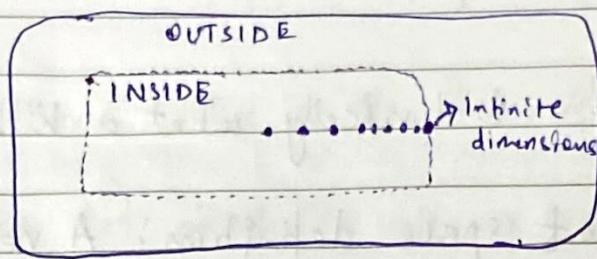
$$1 + \frac{n}{1!} + \frac{n^2}{2!} + \frac{n^3}{3!} + \dots \infty = e^n.$$

We see LHS consists of only polynomials, albeit infinity, but the RHS is e^n , which is not polynomial (clearly from macLaurin series)

So having a linear combination of basis that gave a vector that is outside of a vector space.

∴ infinite dimension can cause problems

We can think of this with the following diagram:



As we increase the number of inner regions, marks the elements that are in our vector space, and as we increase dimensions to infinity, we push the limits of what our vector space can consist of to sit right on the boundary of what defined our vector space with.

The dotted line represents only elements that are inside the boundary (and not on the boundary) are in a vector space.

This means our quantum state $|1\rangle$ that is represented by a linear combination of infinitely many energy kets may not be a quantum state at all when it is out of our vector space.

In order to avoid this, we add an extra rule in our quantum vector space:

Extra Rule: Every convergent sum of vectors must converge to an element inside the vector space.

$$\sum_i^{\infty} |E_i\rangle = |\Psi\rangle$$

This is basically what a Hilbert space is.

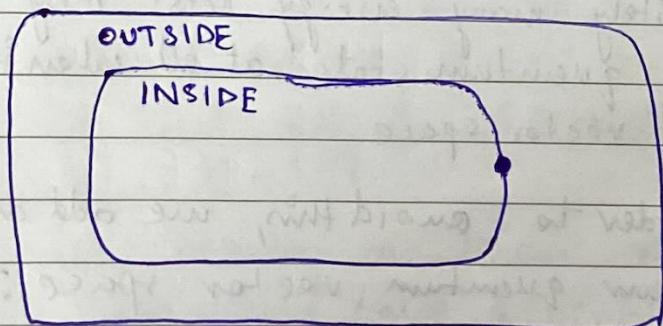
~~Hilbert~~ Hilbert space definition: A vector space equipped with an inner product that is Cauchy complete.

Here, inner product is essentially a generalized dot product. (will be discussed in next chapter)

And Cauchy complete means:

Every convergent sequence of vectors (e.g. partial sums of infinite linear combinations) converges to an element inside the vector space.

We can think of Cauchy completeness with the same diagram as before



INSIDE Region
consists of the elements inside our vector space (Hilbert space)

Basically, ~~Hilbert~~ Hilbert space ~~consists~~ consists of the elements that were sitting right on the boundary.

Monday

we represent the Hilbert space with H ,
and use H to our quantum vector space

All our quantum states live in H .

$$\therefore |\psi\rangle \in H$$

chapter 4: What is an inner product?

→ The dot product allows us to define a
notion of angle and orthogonality between
two vectors

$$\begin{aligned} \bullet \vec{v} \cdot \vec{w} &= v_1 w_1 + v_2 w_2 + \dots \\ &= |\vec{v}| |\vec{w}| \cos \theta \end{aligned}$$

→ ^{com} also be used to define length as

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$$

we want to generalize dot product so that
we can use this in our quantum vector
space, not just on arrows & columns of numbers

At its core, the inner product is a map that
takes in two vectors and spits out a number

we can represent it as

$$\text{InProd}(\psi), (\phi) = c \text{ (real)}$$

This notation is a bit unlegant, so we use the following:

$$\text{Innerproduct: } \langle \psi | \phi \rangle = c \text{ (real)}$$

we can also represent sum & of two kets
& scalar multiplication as follows:

$$|\psi\rangle + |\phi\rangle = |\psi + \phi\rangle$$

$$a|\psi\rangle = |a\psi\rangle$$

we want the inner product in order to define a notion of length, which should be a real positive number.

In order to do that, we define some rules:

$$\langle \psi | \phi + \varsigma \rangle = \langle \psi | \phi \rangle + \langle \psi | \varsigma \rangle$$

$$\langle \phi | a\psi \rangle = a \langle \phi | \psi \rangle$$

for kets $|\psi\rangle, |\phi\rangle, |\varsigma\rangle$ and scalar a

we can make the inner product commutative, i.e.

$$\langle \psi | \phi \rangle = \langle \phi | \psi \rangle$$

but this will cause issues as follows:

$$\text{assume } \langle \psi | \psi \rangle = 1$$

let's find the length of $i\psi$ as:

$$\begin{aligned} \langle i\psi | i\psi \rangle &= i \langle i\psi | \psi \rangle = i \langle \psi | i\psi \rangle \\ &= i^2 \langle \psi | \psi \rangle \\ &= -1. \end{aligned}$$

As we wish to define length of a ket as the square root of the inner product with itself, we see this will cause problems as

$$\langle i\psi | i\psi \rangle = \sqrt{-1} = i$$

which doesn't make any sense.

∴ we ~~make~~ modify the commutative property as follows:

$$\langle \psi | \phi \rangle = \star \times \langle \phi | \psi \rangle^*$$

where the * stands for complex conjugate.

Now,

$$\begin{aligned}\langle i\phi | i\phi \rangle &= i \langle i\phi | \phi \rangle = i \langle \phi | i\phi \rangle^* \\ &= i i^* \langle \phi | \phi \rangle \\ &= -i^2 \times 1 \\ &= 1.\end{aligned}$$

∴ Adding the complex conjugate rule fixes this problem

We add a final condition while defining the inner product:

If $|\psi\rangle \neq 0$, then $\langle \psi | \psi \rangle > 0$

OR

Only the zero vector has zero length.

Formal definition of an inner product:

An inner product $\langle \psi | \phi \rangle$ is a map from vectors to scalars that satisfies the following:

- $\langle \psi | \xi + \phi \rangle = \langle \psi | \xi \rangle + \langle \psi | \phi \rangle$
- $\langle \psi | a\phi \rangle = a \langle \psi | \phi \rangle$
- $\langle \psi | \phi \rangle = \langle \phi | \psi \rangle^*$
- for $|\psi| \neq 0$, $\langle \psi | \psi \rangle > 0$.

Now, we define the magnitude of a ket as:

$$\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$$

and two vectors are orthogonal if

$$\langle \psi | \phi \rangle = 0.$$

→ Use of inner product:

Assume orthonormal basis $\{ |E_i\rangle \}$, meaning

$$\langle E_i | E_j \rangle = 1 \Rightarrow \langle E_i | E_j \rangle = 0, \text{ for all } i \neq j.$$

We can combine these into one condition called Kronecker delta as follows:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

$$\langle E_i | E_j \rangle = \delta_{ij}$$

\therefore For any arbitrary & S $| \psi \rangle$, we have

$$| \psi \rangle = \sum_i c_i | E_i \rangle$$

If we want to find the ^{second} coefficient of the & S, we can take the inner product with $| E_2 \rangle$ and get:

$$\begin{aligned} \langle E_2 | \psi \rangle &= \cancel{\langle E_2 | \sum_i c_i | E_i \rangle} \quad \langle E_2 | (\sum_i c_i | E_i \rangle) \\ &= \sum_i c_i \langle E_2 | E_i \rangle \\ &= \sum_i c_i \delta_{2i} \\ &= c_2 \delta_{22} = c_2. \end{aligned}$$

Let's now evaluate the inner product of two arbitrary kets ψ & ϕ :

$$\begin{aligned} &\cancel{\langle \psi | \phi \rangle} \quad \cancel{\langle \psi | \phi \rangle} \\ &\Rightarrow \cancel{\langle \sum_i a_i | \sum_j b_j \rangle} \\ &\Rightarrow \left(\sum_i \langle a_i | E_i \rangle \right) \cdot \left(\sum_j \langle b_j | E_j \rangle \right) \\ &\Rightarrow \left(\sum_i \sum_j \langle a_i | E_i \rangle \langle b_j | E_j \rangle \right) \\ &\Rightarrow \sum_i \sum_j a_i^* b_j \langle E_i | E_j \rangle. \end{aligned}$$

$$= \sum_i \sum_j a_i^* b_j \delta_{ij}$$

$$= \sum_i a_i^* b_i \delta_{ii}$$

$$\therefore \langle \psi | \phi \rangle = \sum_i a_i^* b_i \cdot 1 = \sum_i a_i^* b_i$$

This would be simply a dot product if a_i 's & b_i 's were not complex.

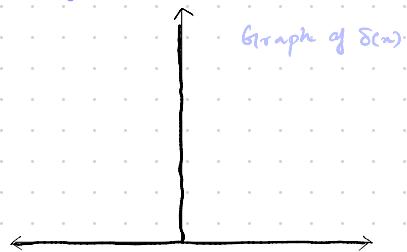
\therefore we find inner product to a generalization of dot product.

Running Notes

Chapter 5: What are dirac deltas & wave function inner products?

The "big spike" interpretation of dirac delta.

$$\rightarrow \delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x=0 \end{cases}$$
$$\rightarrow \int_{-\infty}^{\infty} \delta(x) dx = 1$$



$$\lim_{a \rightarrow \infty} \frac{1}{a\pi/2\pi} e^{-x^2/a^2} \rightarrow \text{Aka. Dirac delta.}$$

Defining dirac delta like this causes 'some' problems, so we define dirac delta as follows:

The dirac delta is a special object that satisfies the following property:

$$f(c) = \int_{-\infty}^{\infty} f(x) \delta(cx) dx \xrightarrow{\text{Dirac delta}}$$

instead of Kronecker delta, we have dirac delta:
 $\langle n_i | n_j \rangle = \delta(n_i - n_j)$.

$$\begin{aligned}
 \langle \psi | \phi \rangle &= \left(\int \psi(x) \phi(x) dx \right) \left(\int \phi(y) |y\rangle dy \right) \\
 &= \iint \psi(x)^* \phi(y) dx |y\rangle dy \\
 &= \iint \psi(x)^* \phi(y) \delta(x-y) dx dy \\
 &= \int \psi(x)^* \phi(x) dx \rightarrow \text{inner product for} \\
 &\quad \text{wave function.}
 \end{aligned}$$

Chapter 6: What are bras & bra-ket notation?

$$\langle \psi | \rightarrow \text{bra}$$

Dual space definition:

Given a vector space V , the dual space V^* is the vector space of all linear functionals in V
 \downarrow
(maps taking vectors to numbers)

All linear functionals $L \in V^*$

$\langle \psi |$ is a linear functional acting on ket $| \phi \rangle$ to give a number
 $\langle \psi | \in H^*$ (Hilbert dual space)

RIESZ REPRESENTATION THEOREM

for any linear functional L_ϕ , the action of L_ϕ is equivalent to taking the inner product with some unique vector $\vec{\phi}$

$$\leftarrow \langle \psi | |\phi \rangle = \langle \psi | \phi \rangle$$

$$L_x \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \vec{v}$$

Power of bra-ket notation (Resolution of identity)

$$| \psi \rangle = \sum c_i | A_i \rangle$$

we know: $c_i = \langle A_i | \psi \rangle$ $\hat{P} = \sum \langle A_i | \psi \rangle | A_i \rangle = (\sum \langle A_i | A_i \rangle) | \psi \rangle \rightarrow$ why not this?

$$| \psi \rangle = \sum \langle A_i | \psi \rangle | A_i \rangle = \sum | A_i \rangle \langle A_i | \psi \rangle = \sum | A_i \rangle \langle A_i | | \psi \rangle$$

$$\therefore I = \sum | A_i \rangle \langle A_i |$$

Because $\langle A_i | \psi \rangle$
is a scalar

Chapter 7: How are observables operator?



LINEAR OPERATOR DEFINITION

a linear operator is a map on a vector space that preserves the linear structure of the space

properties:

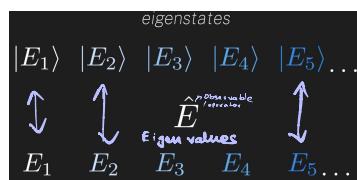
$$\hat{M}(|\psi\rangle + |\phi\rangle) = \hat{M}|\psi\rangle + \hat{M}|\phi\rangle$$

$$\hat{M}(c|\psi\rangle) = c\hat{M}|\psi\rangle$$

eigenvalues	definite states	eigen vectors
$L_1 = 1.41 \text{ N}\cdot\text{m}\cdot\text{s}$		$ L_1\rangle$
$L_2 = 2.44 \text{ N}\cdot\text{m}\cdot\text{s}$		$ L_2\rangle$
$L_3 = 3.46 \text{ N}\cdot\text{m}\cdot\text{s}$		$ L_3\rangle$
$L_4 = 4.47 \text{ N}\cdot\text{m}\cdot\text{s}$		$ L_4\rangle$
$L_5 = 5.47 \text{ N}\cdot\text{m}\cdot\text{s}$		$ L_5\rangle$
:		:

A linear operator is a map, while a matrix is an abstract representation of that map in a particular basis

why eigen values & eigen vectors?
(Answer at the end of the note)



Based on intuition, we can guess the properties an observable should have:

- 1) Eigenvalues of operators/observables are real
- 2) Observables eigenstate must span the entire Hilbert space.

$$|\Psi\rangle = \sum_i c_i |E_i\rangle \Leftrightarrow \text{Any QS can be represented as a linear combination of eigenstates}$$

3) Eigen states must be mutually orthogonal.

→ If they aren't, we can decompose of eigenstate as a linear combination of other eigen state, which means our states are not definite (contradiction).

An observable's eigen states must form an orthonormal basis.

Why do eigenvalues correspond to observable quantities?

Suppose we don't know quantum mechanics yet and we want to calculate the expectation value of an observable A . Could be momentum, spin whatever. It is given by

$$\mathbb{E}(A) = \sum_i a_i p(a_i)$$

Where a_i are the possible outcomes and $p(a_i)$ are the probabilities of those outcomes. When the outcome is continuous this becomes an integral.

To each of these states we can associate a vector $|a_i\rangle$ and it is possible to make these states orthonormal such that $\langle a_i | a_j \rangle = \delta_{ij}$. Quantum mechanics is linear so if we have two solutions $|a_1\rangle, |a_2\rangle$ then the state $|\psi\rangle = \alpha|a_1\rangle + \beta|a_2\rangle$ is also a valid solution. How do we interpret this new state? It is a postulate (Born rule) that the probability of finding a_1 is given by $p(a_1) = |\alpha|^2$. This means we have to normalize $|\psi\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$ in order for it to be a valid state.

If we then define Dirac notation as usual we get $\alpha = \langle a_1 | \psi \rangle$ and $\alpha^* = \langle \psi | a_1 \rangle$ which you can check using orthonormality. After some manipulation we can get the expectation value in the following form

$$\begin{aligned}\mathbb{E}(A) &= |\alpha|^2 a_1 + |\beta|^2 a_2 \\ &= \alpha^* \alpha a_1 + \beta^* \beta a_2 \\ &= \langle \psi | a_1 \rangle \langle a_1 | \psi \rangle a_1 + \langle \psi | a_2 \rangle \langle a_2 | \psi \rangle a_2 \\ &= \langle \psi | \left(\sum_i |a_i\rangle \langle a_i| a_i \right) | \psi \rangle\end{aligned}$$

If we then define $\hat{A} = \sum_i |a_i\rangle \langle a_i|$ then we get $\mathbb{E}(A) = \langle \psi | \hat{A} | \psi \rangle$.

So what's the link with eigenvectors/eigenvalues? It turns out that according to the [spectral theorem](#) that any Hermitian matrix can be written as $\hat{A} = \sum_i |\lambda_i\rangle \langle \lambda_i| \lambda_i$ where λ_i are its eigenvalues and $|\lambda_i\rangle$ its eigenvectors. Notably these eigenvectors form an orthonormal basis. This implies that only the eigenvectors of \hat{A} can give the outcome of a measurement. This is because $|a_i\rangle \langle a_i|$ is a projection along $|a_i\rangle$. Any vectors that are orthogonal to $|a_i\rangle$ will be projected out. If a state is orthogonal to all eigenvectors of \hat{A} , which means it can't be written as a sum of eigenvectors, then it will automatically give zero contribution in the expectation value because it is projected out.

As a final note I would like to add that my reasoning has been a bit backwards from how you would usually do it but I hope this made it more clear why this eigenvalue/eigenvector construction actually makes a lot of sense.

Chapter 8: Why is probability equal to amplitude squared?

$$|\Psi(n)|^2 \stackrel{?}{=} \square \square$$

Idea: If the Qs is closer to a certain eigen basis, the probability of getting the value corresponding to that eigen basis will be large.

- The larger the component along an eigenvector, the larger the probability of getting that outcome.

- To find the projection along $|E_4\rangle$,

take inner product along the unit vector

$$\text{projection: } \langle E_4 | \Psi \rangle = \langle E_4 | \sum_i c_i | E_i \rangle = \sum_i c_i s_{4i} = c_4.$$

- We know that the probability is proportional to the projection (c_i^2), so a good guess for the probability function will be $P(E=E_i) = |c_i|^2$, where $|\cdot|$ is the modulus of a complex number.

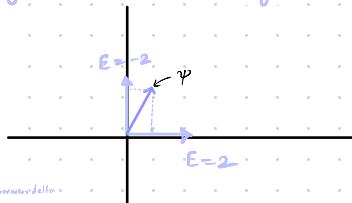
- Though this a good guess, we will run into some problem if we consider this as our probability function.

Ex: $|\Psi\rangle = \frac{1}{\sqrt{2}}|E_1\rangle - \frac{1}{\sqrt{2}}|E_2\rangle$ satisfies, but when representing $|\Psi\rangle$ in angular momentum basis we might have something like: $|\Psi\rangle = \frac{e^{i\theta}}{\sqrt{2}}|L_1\rangle + \frac{1}{\sqrt{2}}|L_2\rangle$, which does not preserve probability.

∴ we must find a probability function $p(c_i)$ that satisfies the following constraints:

$$\sum_i p(c_i) = 1 \quad (\text{conservation of probability})$$

$$\langle \Psi | \Psi \rangle = \sum_i c_i^2 = k^2 \text{ (constant)} \quad [\text{length of our Qs should remain constant in any basis}]$$



• we note that our coefficients are independent of each other, but finding all but one finds the last one as well.

• we find all but one coeff. and express the last one using all other coeffs.

$$\text{let } c_1 = \sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}$$

$$f(c_1) + f(c_2) + \dots + f(\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}) = 1$$

$$\Rightarrow \frac{df(c_1)}{dc_1} + f'(\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}) \cdot \frac{-c_1}{\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}} = 0$$

$$\Rightarrow \frac{1}{c_1} \frac{df(c_1)}{dc_1} = \frac{f'(\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)})}{\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}}$$

Similarly doing this for c_2 , we get

$$\frac{df(c_2)}{dc_2} + f'(\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}) \cdot \frac{-c_2}{\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}} = 0$$

$$\Rightarrow \frac{1}{c_2} \frac{df(c_2)}{dc_2} = \frac{f'(\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)})}{\sqrt{k^2 - (c_1^2 + c_2^2 + \dots + c_m^2)}}$$

$$\Rightarrow \frac{1}{c_1} \frac{df(c_1)}{dc_1} = \frac{1}{c_2} \frac{df(c_2)}{dc_2}$$

since c_1 & c_2 are independent,

$$\frac{d}{dc_1}(c_2) = 0$$

$$\frac{d}{dc_1} \left(\frac{1}{c_1} \frac{df(c_1)}{dc_1} \right) = 0$$

$$\therefore \frac{1}{c_1} \frac{df(c_1)}{dc_1} = \lambda, \text{ for some arbitrary constant } \lambda$$

And since the choice of selecting c_1 was arbitrary, we can replace it with c .

$$\Rightarrow \frac{df(c)}{dc} = \lambda c$$

$$\Rightarrow df(c) = \lambda c dc$$

$$\Rightarrow \int df(c) = \int \lambda c dc \Rightarrow f(c) = \frac{\lambda c^2}{2} + \mu$$

\Rightarrow Now, when $c=0$, $f(c)=0$ (projection = 0 \Rightarrow probability = 0)
and when $c=k$, $f(c)=1$

$$\Rightarrow 0 = \frac{\lambda(0)^2}{2} + \mu \Rightarrow \boxed{\mu = 0}$$

$$\Rightarrow 1 = \frac{\lambda k^2}{2} \Rightarrow \lambda = \frac{2}{k^2}$$

$$\therefore f(c) = \left(\frac{2}{k^2}\right) \cdot \frac{c^2}{2} + 0 = \frac{c^2}{k^2}$$

$P(E) \propto c_i^2$
 If we fix the length of our quantum state $|\psi\rangle$ to 1, then $K=1$

$$\therefore [P(E) = c_i^2]$$

The probability of $E = E_i$ will be

$$P(E = E_i) = |\langle E_i | \psi \rangle|^2 = c_i^2 \text{ (Born Rule)}$$

As for the continuous case,

$$\langle x | \psi \rangle = \langle x | \int \psi(x) \cdot n \rangle dx = \int \psi(x) \langle x | n \rangle dx = \int \psi(x) \delta(x-n) dx \\ = \psi(n)$$

The probability density of the particle being in an arbitrary position x is $|\psi(x)|^2$

This is called the Born rule, that is often stated as an axiom or a postulate in QM.

Gleason's Theorem \rightarrow Rigorous Mathematical Theorem

Establishing Born Rule
 Valid for $\dim(\mathcal{V}) \geq 3$ due to constraints on c_i^2 's

Though the Born Rule is applicable in 2D quantum vector spaces

Homework: Verify that

$$1) \langle E \rangle = \langle \psi | \hat{E} | \psi \rangle = \sum_i p_i E_i$$

$\xrightarrow{\text{Average}}$

$$2) \langle n \rangle = \langle \psi | \hat{n} | \psi \rangle = \int n p(x) dx, \text{ where } p(x) = \text{probability density function}$$

$$1) \langle \psi | \left(\sum_i E_i | E_i \rangle \underbrace{\langle E_i | \psi \rangle}_{c_i^2} \right)$$

$$\left(\sum_i E_i \langle \psi | E_i \rangle c_i^2 \right)$$

$$\left(\sum_i E_i c_i^* c_i \right) = \sum_i |c_i|^2 E_i = \sum_i p(E_i) E_i = \langle E \rangle$$

$$2) \langle \psi | n | \psi \rangle = \langle \psi | \left(\int n(x) \psi(x) dx \right) | \psi \rangle$$

$$= \int n \cdot \langle \psi | x \rangle \cdot \langle x | \psi \rangle \cdot dx$$

$\xrightarrow{\text{probability density}}$

$$= \int n \cdot \psi^*(x) \psi(x) dx = \int |\psi(x)|^2 n dx$$

$$= \int n f(x) dx = \langle n \rangle$$

$\xrightarrow{f(x) = |\psi(x)|^2}$

Chapter 9: What are hermitian operators?

$$\hat{M}|\psi\rangle = |\hat{M}\psi\rangle \text{ (New Notation)}$$

$$\langle\psi|\hat{M}|\phi\rangle \Rightarrow \langle\psi|\hat{M}\phi\rangle$$

We want an operator \hat{N} that satisfies the following:

$$\langle\hat{N}\psi|\phi\rangle = \langle\psi|\hat{M}\phi\rangle$$

The matrix that satisfies this property is called the hermitian adjoint & is denoted by:

$$\hat{N} = \hat{M}^+$$

∴ Any hermitian adjoint would satisfy the following property:

$$\langle\psi|\hat{M}\phi\rangle = \langle\hat{M}^+\psi|\phi\rangle$$

Properties of hermitian adjoint: (can be proved using bra-ket notation)

- $(\hat{A}^\dagger)^\dagger = \hat{A}$
- $(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$
- $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$ (try verifying this one)

For any scalar c ,

$$\langle\psi|c\phi\rangle = c\langle\psi|\phi\rangle = \langle c^*\psi|\phi\rangle.$$

$$c^\dagger = c^*$$

Now, let $c = \langle\psi|\phi\rangle$

$$\begin{aligned}\langle\psi|\phi\rangle^\dagger &= (\langle\psi|\phi\rangle)^* \\ &= \langle\phi|\psi\rangle\end{aligned}$$

Also,

$$\begin{aligned}\langle\psi|\phi\rangle^\dagger &= (\langle\psi||\phi\rangle)^\dagger \\ &= |\phi\rangle^* \langle\psi|\end{aligned}$$

This seems to point to the fact that

$$\langle\psi|^* = |\psi\rangle$$

ADDENDUM: In more physics-friendly notation, here's what's going on. Let H be your (finite-dimensional) Hilbert space, and let $|\alpha\rangle \in H$. You can interpret $|\alpha\rangle$, in a completely natural way, as defining a linear transformation $\Phi(|\alpha\rangle) : \mathbb{C} \rightarrow H$ by $\Phi(|\alpha\rangle)|\lambda\rangle := \lambda|\alpha\rangle$. The Hermitian conjugate of $\Phi(|\alpha\rangle)$, then, is a linear transformation $\Phi(|\alpha\rangle)^\dagger : H \rightarrow \mathbb{C}$, so that $\Phi(|\alpha\rangle)^\dagger$ is simply a bra vector. The computation above then shows that $\Phi(|\alpha\rangle)^\dagger = \langle\alpha|$. Thus, as long as you're fine with identifying $|\alpha\rangle$ with $\Phi(|\alpha\rangle)$ (which is actually completely rigorous), you do indeed have that $\langle\alpha| = |\alpha\rangle^*$.

<https://math.stackexchange.com/questions/345910/is-a-bra-the-adjoint-of-a-ket>

lets try to find the hermitian adjoint of an observable consider: $\hat{E}|\psi\rangle$

$$\begin{aligned}\hat{E}|\psi\rangle &= \hat{E} \sum_i c_i |E_i\rangle \\&= \sum_i c_i \hat{E} |E_i\rangle \\&= \sum_i c_i E_i |E_i\rangle \quad (\text{since } |E_i\rangle \text{ is an eigen vector of } \hat{E}) \\&= \sum_i \langle E_i | \psi \rangle E_i |E_i\rangle \\&= \sum_i E_i |E_i\rangle \langle E_i | \psi \rangle \\&= (\sum_i E_i |E_i\rangle \langle E_i|) |\psi\rangle\end{aligned}$$

$$\therefore \hat{E} = \sum_i E_i |E_i\rangle \langle E_i|$$

$$\begin{aligned}\text{Now, } \hat{E}^\dagger &= (\sum_i E_i |E_i\rangle \langle E_i|)^+ \\&= \sum_i (E_i |E_i\rangle \langle E_i|)^+ \\&= \sum_i \langle E_i |^+ |E_i\rangle^+ E_i^+ \xrightarrow{E_i^* = E_i \text{ (since real)}} \\&= \sum_i E_i |E_i\rangle \langle E_i| = \hat{E}.\end{aligned}$$

$\therefore \hat{E}^\dagger = \hat{E} \iff$ Hermitian adjoint of a physical observable is itself.

Operators who are their own hermitian adjoints are called

Hermitian Operators | $\hat{A}^\dagger = \hat{A} \iff \hat{A}$ is hermitian operator

All observables are Hermitian operators

$$\therefore \langle \psi | \hat{E} \phi \rangle = \langle \hat{E} \psi | \phi \rangle$$

Take the following matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Start computing its first few powers by hand: A^2 , A^3 , etc. What pattern do you see? Can you explain why this pattern shows up? This might make you curious to know if there's an efficient way to compute arbitrary powers of this matrix, A^n for any number n .

Given that two eigenvectors of this matrix are

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix}.$$

see if you can figure out a way to compute A^n by first changing to an eigenbasis, compute the new representation of A^n in that basis, then converting back to our standard basis. What does this formula tell you?

$$M = [\vec{v}_1 \vec{v}_2]$$

$$A^{\vec{v}}$$

$$M = \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}$$

$$\begin{aligned} M^{-1} &= \frac{1}{|M|} \circ \begin{bmatrix} 1-\sqrt{5} & -(1+\sqrt{5}) \\ -2 & 2 \end{bmatrix} \\ &= \frac{1}{4\sqrt{5}} \begin{bmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ 2 & -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} M \vec{v} \\ M^{-1} \vec{v} \end{aligned}$$

$\vec{v} = \text{A vector described in some other basis}$

$M \vec{v} = \text{same vector described in our basis}$

$AM \vec{v} = \text{Transformation in our basis}$

$M^{-1}AM \vec{v} = \text{same transformed vector, but now described in other basis}$

$M^{-1}AM = \text{Transformation described in other's basis}$

Hence $M = \text{change of basis matrix}$

(Has basis vectors of other system ; as expressed in our basis, as columns)

$$M^{-1}AM = \frac{1}{4\sqrt{5}} \begin{bmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}$$

$$\begin{aligned} &-(\sqrt{5}+1)(\sqrt{5}-3) \\ &-2+2\sqrt{5} \\ &(\sqrt{5}+1)(\sqrt{5}+3) \end{aligned}$$

$$8+4\sqrt{5}$$

$$5+1-2\sqrt{5}$$

$$2\sqrt{5}-6$$

$$4\sqrt{5}-8$$

$$= \frac{1}{4\sqrt{5}} \begin{bmatrix} 4+8+4\sqrt{5} & 4\sqrt{5}-8 \\ 2+2 & 2-2 \end{bmatrix} \begin{bmatrix} \Phi_1^n & 0 \\ 0 & \Phi_2^n \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 3+\sqrt{5} & \sqrt{5}-2 \\ 1 & -1 \end{bmatrix}$$

$$\stackrel{\text{Actual}}{=} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 0 & -\Phi^{-1} \end{bmatrix}$$

$$M \rightarrow [i, j] \rightarrow [\vec{v}_1, \vec{v}_2]$$

$$M^{-1} \rightarrow [\vec{v}_1, \vec{v}_2] = [i, j]$$

$$M(M^{-1}AM)^n M^{-1} \quad M = \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}$$

$$M^{-1} = \frac{1}{|M|} \text{adj}(M)$$

$$|M| = .2(1-\sqrt{5}) - .2(1+\sqrt{5}) \\ = -4\sqrt{5}$$

$$M^{-1} = \frac{1}{-4\sqrt{5}} \begin{bmatrix} 1-\sqrt{5} & -1+\sqrt{5} \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}} \begin{bmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ 2 & -2 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}} \begin{bmatrix} -2\phi_2 & 2\phi_1 \\ 2 & -2 \end{bmatrix}$$

$$\frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & 2 \\ 2\phi_1 & 2\phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^n & 0 \\ 0 & \phi_2^n \end{bmatrix} \begin{bmatrix} -2\phi_2 & 2\phi_1 \\ 2 & -2 \end{bmatrix} \quad \phi_1\phi_2 = -1$$

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^n & 0 \\ 0 & \phi_2^n \end{bmatrix} \begin{bmatrix} -\phi_2 & \phi_1 \\ 1 & -1 \end{bmatrix}$$

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^{n-1} & \phi_1^{n+1} \\ \phi_2^n & -\phi_2^n \end{bmatrix}$$

$$\frac{1}{\sqrt{5}} \begin{bmatrix} \phi_2^n + \phi_1^{n-1} & \phi_1^{n+1} - \phi_2^n \\ \phi_1^n + \phi_2^{n+1} & \phi_1^{n+2} - \phi_2^{n+1} \end{bmatrix} \quad \begin{array}{l} \phi_1\phi_2 = -1 \\ \phi_1^{-1}\phi_2^{-1} = -1 \\ -\phi_2^{n-1} \cdot \phi_1 + \phi_1^{n-1} \end{array}$$

$$A^n = \begin{bmatrix} \phi^n & 0 \\ 0 & (-\phi)^n \end{bmatrix}$$

$$\begin{aligned} M A^n &= \begin{bmatrix} 2 & 2 \\ \frac{2\phi}{1+\sqrt{5}} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & (-\phi)^n \end{bmatrix} \\ &= \begin{bmatrix} 2\phi^n & -2(-\phi)^n \\ 2\phi^{n+1} & 2(-\phi)^{n-1} \end{bmatrix} \end{aligned}$$

Chapter 10: what's the commutator and the uncertainty principle?

$$\hat{A}\hat{B} = \hat{B}\hat{A} \Leftrightarrow \hat{A}, \hat{B} \text{ commute}$$

We define an operator as follows:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Let \hat{A} & \hat{B} be two physical observable that commute

Let $|\alpha\rangle$ be eigenvector of \hat{A} , then $\hat{A}|\alpha\rangle = \lambda|\alpha\rangle$

$$\hat{B}\hat{A}|\alpha\rangle = \hat{B}\lambda|\alpha\rangle = \hat{A}\hat{B}|\alpha\rangle$$

$\hat{B}|\alpha\rangle$ eigenvec of \hat{B} with eigen value λ

Degeneracy: If for the same eigen value λ of an operator, there are multiple linearly independent eigenvectors, then we call this eigenvalue degenerate.

∴ There can be two cases here, non-degenerate & degenerate eigenvalues.

• Non Degenerate Eigenvalue

In this case, $\hat{B}|\alpha\rangle$ must be in the span of $|\alpha\rangle$.

$$\therefore \hat{B}|\alpha\rangle = \mu|\alpha\rangle \Rightarrow |\alpha\rangle \text{ is an eigenvector of both } \hat{A} \text{ & } \hat{B}$$

Non-degenerate eigenvectors of \hat{A} must also be eigenvectors of \hat{B}

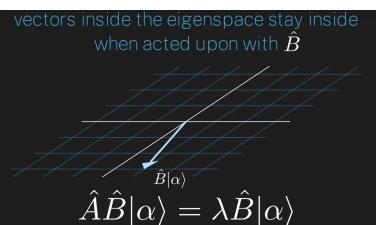
• Degenerate Eigenvalue

ASSUMPTION: 2-Fold Degeneracy (2[↑] eigenvectors w/ same eigenvalues) + \mathbb{R}^3 space

The two degenerate eigenvectors form a degenerate eigenspace

$\hat{A}\hat{B}|\alpha\rangle = \lambda\hat{B}|\alpha\rangle$, since $|\alpha\rangle$ is an eigenvector of \hat{A} , it must lie in the 2-eigenspace

∴ vectors inside the eigenspace stay inside when acted upon with \hat{B}



Consider what would happen if \hat{B} acts on a vector $|p\rangle$ perpendicular to the λ -eigenspace $\hat{B}|p\rangle$

Let $|\alpha\rangle$ be an arbitrary vector in the λ -eigen space

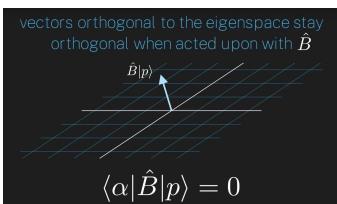
$$\text{Then } \langle \alpha | \hat{B} | p \rangle = \langle \alpha | \hat{B} p \rangle$$

$$= \langle \hat{B} \alpha | p \rangle \quad (\text{since } \hat{B} \text{ is an observable} \Leftrightarrow \text{a hermitian})$$

$$= 0 \quad (\text{since } \hat{B}\alpha \text{ is a vector within the } \lambda\text{-eigen space})$$

Since $|\alpha\rangle$ was an arbitrary vector with the λ -eigen space, this means \hat{B} keeps $|p\rangle$ orthogonal to the λ -eigen space

$$\therefore \hat{B}|p\rangle = c|p\rangle \quad (\text{for some } c)$$



Let $|\beta\rangle$ be an eigenvector of \hat{B} in \mathbb{R}^2 that is neither in the λ -eigen space nor perpendicular to it.

Then $|\beta\rangle = |\beta_{||}\rangle + |\beta_{\perp}\rangle$, where $|\beta_{||}\rangle = \text{projection in } \lambda\text{-eigen space}$

$|\beta_{\perp}\rangle = \text{projection orthogonal to the eigen space}$

$$\text{since } \hat{B}|\beta\rangle = \mu|\beta\rangle$$

$$= \mu|\beta_{||}\rangle + \mu|\beta_{\perp}\rangle$$

$$= \hat{B}|\beta_{||}\rangle + \hat{B}|\beta_{\perp}\rangle$$

$$\therefore \hat{B}|\beta_{||}\rangle = \mu|\beta_{||}\rangle \quad \& \quad \hat{B}|\beta_{\perp}\rangle = \mu|\beta_{\perp}\rangle$$

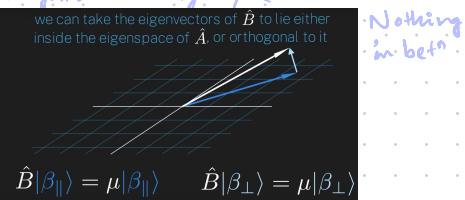
Now, the eigenspace of \hat{A} forms a Hilbert space, and for \hat{B} to be an observable, the eigenvectors of \hat{B} must span this λ -eigen space.

And therefore they must lie in the λ -eigen space (otherwise they won't be able to span it)

since every vector in the λ -eigen space is the eigenvector of \hat{A} , the eigenvectors of \hat{B}

are also eigenvalues of \hat{A} with eigenvalue λ

\hat{A} & \hat{B} share eigenvectors even in the degenerate case



Now, for observables $\hat{A} + \hat{B}$, any arbitrary eigenvector $|a\rangle$ is also an eigenvector of $\hat{A} + \hat{B}$.
 i.e. The set $\{|a_i\rangle\}$ form a eigenbasis for both \hat{A} & \hat{B} . $\{|a_i\rangle\}$ forms a simultaneous eigenbasis ^{of momentum}
 This is important because when making a measurement, $|a\rangle$ collapses to say $|P\rangle$ and we make a measurement by calculating its eigenvalue
 and if Energy and momentum operators commute, this means they both have a simultaneous eigenbasis, meaning as $|P\rangle$ collapses to a definite state $|P\rangle$, it also collapses to $|E\rangle$.

In other words, our particle can be in a definite state of both observables if they commute, and therefore we can simultaneously know the value of $\hat{A} + \hat{B}$.

Now, if $[\hat{A}, \hat{B}] \neq 0$, then they cannot have a simultaneous eigen basis.

We can prove this by contradiction by assuming a simultaneous eigenbasis $\{|c_j\rangle\}$. Then:

$$\hat{A} = \sum_i A_i |c_i\rangle \langle c_i| \text{ and } \hat{B} = \sum_i B_i |c_i\rangle \langle c_i|$$

We can then show that $\hat{A}\hat{B} = \hat{B}\hat{A}$ leading to a contradiction (To prove)

For example: If $[\hat{P}, \hat{E}] \neq 0$, then even after measuring & collapse of one observable; say momentum, into its eigenstate $|P\rangle$, you may still be in Energy superposition i.e.

$$|P\rangle = c_1 |E_1\rangle + c_2 |E_2\rangle + c_3 |E_3\rangle + \dots$$

and therefore you will be uncertain about the energy measurement until you make a dedicated energy measurement.

But when you make this dedicated energy measurement, you may now be thrown into momentum superposition i.e.

$$|E_4\rangle = c_1 |P\rangle + c_2 |P_2\rangle + \dots$$

You may not be able to measure both at the same time as collapsing of the eigenstate of one can put you into a superposition of the other.

As for the position operator \hat{x} and momentum operator \hat{p} , they do not commute i.e. $[\hat{x}, \hat{p}] \neq 0$, in fact

$$[\hat{x}, \hat{p}] = i\hbar \mathbb{I}$$

This means that they do not share an eigenbasis, in fact, they do not share any eigen vectors.

Therefore, whenever you are in a position definite state, you will always be in a momentum superposition state, and whenever you are in a momentum definite state, you will always be in position superposition state!

This is the root of Heisenberg uncertainty principle:

$$\sigma_x \sigma_p \geq \hbar/2$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$ x_1\rangle$	$ p_1\rangle$
$ p_2\rangle p_3\rangle \dots$	$ x_1\rangle x_2\rangle x_3\rangle \dots$

The Heisenberg uncertainty principle is actually derived from the Generalized Uncertainty principle:

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right|$$

There are other (and stronger) uncertainty relation like the Maccane-Pati uncertainty relation which is as follows:

$$\sigma_A^2 + \sigma_B^2 \geq \pm i \langle \Psi | (\hat{A}, \hat{B}) | \Psi \rangle + |\langle \Psi | (\hat{A} + i\hat{B}) | \Psi \rangle|^2$$



why don't energy & momentum visit each other? Because they're uncertain about their relation!

Chapter 11: What are unitary operators?

$$\hat{U}^+ = \hat{U}^{-1}$$

- Is there an operator that preserves the inner product i.e. $\langle \psi | \phi \rangle = \langle \hat{U}\psi | \hat{U}\phi \rangle$
we want an operator that preserves lengths of the vectors and the angles between them
One transformation that satisfy this is the rotation (if you consider the traditional dot product)

But Rotations are just one type of operation that preserve the inner product,
In general, operators that preserve the inner product are called Unitary operators.

$$\begin{aligned}\langle \psi | \phi \rangle &= \langle \hat{U}\psi | \hat{U}\phi \rangle \\ &= \langle \hat{U}^+ \hat{U}\psi | \phi \rangle \\ &\Rightarrow \hat{U}^+ \hat{U} = \mathbb{I} \\ &\Rightarrow \hat{U}^+ = \hat{U}^{-1}\end{aligned}$$

- Unitary operators are defined as follows:

$$\hat{U}^+ = \hat{U}^{-1}$$

- Unitary operators are just generalized Rotations!

Let $|w\rangle$ & λ be the ^(normalized) eigen vector and eigen value associated with \hat{U} . Then,

$$\hat{U}|w\rangle = \lambda|w\rangle$$

$$\begin{aligned}\langle w | w \rangle &= \langle \hat{U}w | \hat{U}w \rangle \\ &= \langle \lambda w | \lambda w \rangle \\ &= \lambda\bar{\lambda}^* \langle w | w \rangle \\ \lambda\bar{\lambda}^* &= |\lambda|^2 = 1 \Rightarrow |\lambda| = 1\end{aligned}$$

- Eigen values of unitary operator must have magnitude = 1

This makes sense since \hat{U} are just generalized rotations and therefore the transformation does not stretch or squish any vectors in the space.

Unitary Operators are helpful since they conserve inner product which is used for calculating probabilities. This means Unitary operators conserve probabilities! While describing a particle, we may want to Rotate it ($\hat{R}(\theta)$), Translate it ($\hat{T}(\vec{a})$) or describe how it evolves in time ($\hat{U}(t)$). And since all these properties seem to conserve probabilities, they would have to be Unitary!

Chapter 12: What are generators in classical physics?

$L(t, \vec{x}(t), \dot{\vec{x}}(t))$ = lagrangian, function of time, position, and velocity

& analogous to $\langle \psi | \psi \rangle$ in classical physical; central object of classical physics

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) \quad (\text{pattern shown while deriving the form of Quantum Path Integral})$$