Proof 1

- $\bar{G}_i(u+v) = P(X(s) = i, s \in [t, t+u+v] | X(t) = i)$
- $\bar{G}_i(u+v) = P(X(s) = i, s \in [t+u, t+u+v]; X(p) = i, p \in [t, t+u] | X(t) = i)$
- $P(AB|C) = P(A|BC)P(B|C)$
- Due to Markov property we have $P(AB|C) = P(A|B)P(B|C)$
- $P(X(s) = i, s \in [t+u, t+u+v] | X(p) = i, p \in [t, t+u]) =$
- $P(X(s) = i, s \in [t+u, t+u+v] | X(t+u) = i) = \bar{G}_i(v)$
- $P(X(p) = i, p \in [t, t+u] | X(t = i)) = \bar{G}_i(u)$
- $\bar{G}_i(u+v) = \bar{G}_i(u)\bar{G}_i(v)$
- Only CCDF function which satisfies this equation is the exponential distribution. This requires a proof. We will skip this part.

Simpler Proof

- Let τ_i denote the time the CTMC spends in state i before moving out. Suppose the CTMC is in state i at time 0.

- What is $P(\tau_i > s + t | \tau_i > s)$?

- Note that $X(s) = i$ and therefore from the Markov property,

$$\begin{aligned}
P(\tau_i > s + t | \tau_i > s) &= P(X(u) = i, u \in [s, s + t] | X(t) = i, t \in [0, s]) \\
&= P(X(u) = i, u \in [s, s + t] | X(s) = i) \\
&= P((X(u) = i, u \in [0, t] | X(0) = i) \\
&= P(\tau_i > t).
\end{aligned}$$

- Since $P(\tau_i > s + t | \tau_i > s) = P(\tau_i > t)$, this implies the distribution has memoriless property and must be exponential.

Finite dimensional distributions

- Consider a DTMC $\{X_n, n \geq 0\}$ with tpm denoted by P.
- We assume M states and X_0 denotes the initial state.
- You can start in any starting state or may pick your starting state randomly.
- Let $\bar{\mu} = (\mu_1, \ldots, \mu_M)$ denote the initial distribution.
- How does one obtain the finite dimensional distribution $P(X_0 = x_0, X_1 = x_1, \ldots X_k = x_k)$?

Finite dimensional distributions

- Consider a CTMC $\{X_t, t \geq 0\}$ with t-time pm given by $P(t)$.
- We assume M states and X_0 denotes the initial state.
- Let $\bar{\mu} = (\mu_1, \ldots, \mu_M)$ denote the initial distribution.
- How does one obtain the finite dimensional distribution $P(X_0 = x_0, X_{t_1} = x_1, \ldots X_{t_k} = x_k)$?

Chapman Kolmogorov Equations for DTMC

- $P = [[p_{ij}]]$ denotes the one step transition probability matrix.
- Let $P^{(n)}$ denote the n-step transition probability matrix.
- CK equation tells us that $P^{(n+l)} = P^{(n)}P^{(l)}$.
- $p_{ij}^{(n+l)} = P(X_{n+l} = j | X_0 = i) = \sum_k P(X_{n+l} = j, X_n = k | X_0 = i)$
- $p_{ij}^{(n+l)} = \sum_k P(X_{n+l} = j | X_n = k, X_0 = i) P(X_n = k | X_0 = i)$
- $p_{ij}^{(n+l)} = \sum_k P(X_{n+l} = j | X_n = k) P(X_n = k | X_0 = i)$
- $p_{ij}^{(n+l)} = \sum_k p_{ik}^{(n)} p_{kj}^{(l)} = [P^{(n)}P^{(l)}]_{ij}$
- At which step did we use time homogeneity and the Markov property?

n step transition probabilities

- $P = [[p_{ij}]]$ denotes the one step transition probability matrix.
- Let $P^{(n)}$ denote the n-step transition probability matrix.
- From the CK equation we know that $P^{(n+l)} = P^{(n)}P^{(l)}$.
- It is easy to see that $P^{(n)} = P^{(n-1)}P$.
- For an M state DTMC, $p_{ij}^{(2)} = \sum_{k=1}^{M} p_{ik}p_{kj}$.
- This implies that that the n-step transition probability matrix can be obtained as $P^{(n)} = P^n$
- Given X_0 and P, you can generate n-step probabilities or $P_{X_0}(X_n)$

Chapman Kolmogorov Equations for CTMC

- Let $P(t)$ denote the t-time transition probability matrix.
- CK equation for a CTMC is $P(t+l) = P(t)P(l)$.
- $p_{ij}(t+l) = P(X(t+l) = j | X(0) = i)$
- $= \sum_k P(X(t+l) = j, X(t) = k | X(0) = i)$
- $= \sum_k P(X(t+l) = j | X(t) = k, X(0) = i) P(X(t) = k | X(0) = i)$
- $= \sum_k P(X(t+l) = j | X(t) = k) P(X(n) = k | X(0) = i)$
- $p_{ij}(t+l) = \sum_k p_{ik}(t) p_{kj}(l) = [P(t)P(l)]_{ij}$

What generates a CTMC ?

- $P(t+l) = P(t)P(l)$.
- In DTMC, we could use P to generate the chain on Matlab.
- What about CTMC ? Can we use $P(t)$?
- What is $\lim_{h\to 0} P(h)$?
- What is $\frac{dP(h)}{dh}$ evaluated at $h = 0$?

What generates a CTMC ?

- Lets look at $\frac{dP(h)}{dh}|_{h=0} = \lim_{h\to 0} \frac{P(h)-P(0)}{h} = \lim_{h\to 0} \frac{P(h)-I}{h}$.
- Define $Q := \lim_{h\to 0} \frac{P(h)-I}{h}$
- **Does it always exist ?** Yes! (Proposition 2.2 and 2.4 (Anderson))
- Q has terms of the form q_{ii} and q_{ij} for $i, j \in \{1, 2, \ldots, M\}$.
- $q_{ii} = \frac{dp_{ii}(h)}{dh}|_{h=0}$. Similarly $q_{ij} = \frac{dp_{ij}(h)}{dh}|_{h=0}$

What generates a CTMC ?

Theorem

Let $P(t)$ be a transition function. Then the generator matrix $Q = \lim_{h\to 0} \frac{P(h)-I}{h}$ exists.

Theorem

$P(Y_t > u | X(t) = i) = e^{-a_i u}$ *where* $a_i > 0$.

Theorem

(Proposition 2.8 Anderson)
$P(Y_t > u | X(t) = i) := e^{q_{ii} u}$, *i.e.,* $q_{ii} = -a_i$.
$P\left(X(t + Y_t) = j | X(t) = i\right) = \frac{q_{ij}}{|q_{ii}|}$.

Q generates the CTMC

- Cannot generate CTMC directly from $P(t)$.
- From $P(t)$, obtain Q using $Q = \frac{dP(h)}{dh}|_{h=0}$
- Consider Y_t when $X(t) = i$.
- Now use the following theorem for generating the CTMC on a computer

Theorem

(Proposition 2.8 Anderson: we won't see proof)
$P(Y_t > u|X(t) = i) := e^{q_{ii}u}$, *i.e.*, $q_{ii} = -a_i$.
$P\left(X(t + Y_t) = j|X(t) = i\right) = \frac{q_{ij}}{|q_{ii}|}$.

Properties of a conservative Q

Theorem

$P(Y_i > u|X(t) = i) := e^{q_{ii}u}$, *i.e.*, $q_{ii} = -a_i$.
$P\left(X(t + Y_t) = j|X(t) = i\right) = \frac{q_{ij}}{|q_{ii}|}$ *where* $q_{ij} \geq 0$.

- Suppose Q is conservative.
- Recall that q_{ii} is negative. A conservative Q implies $q_{ii} = -\sum_{j \neq i} q_{ij}$.
- $|q_{ii}|$ is the exponential rate at which you leave state i.
- q_{ij} is the exponential rate at which you leave state i to go to state j.
- minimum of exponentials is exponential with aggregated rate.
- This justifies the rate of leaving state i to be $\sum_{j \neq i} q_{ij}$.

Equivalent definition of a CTMC using Q

- Suppose Q is conservative.
- Then in the CTMC, you stay in state i for a random duration that has exponential($|q_{ii}|$) distribution.
- From i, you will move to state j with probability $\frac{q_{ij}}{|q_{ii}|}$.
- Equivalently, in state i, you have $M-1$ exponential(q_{ij}) clocks for $j = 1, 2, \ldots, i-1, i+1, \ldots M$.
- You move to that state whose clock rings first!