Bayes Classification

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  - Risk can also include different types of actions as well costs associated with different misclassification errors.
  - This is critical when some errors are more serious than others (e.g., intruder detection).

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- **Bayes Formula**

$$P(\omega_j \mid x) = \frac{p(\omega_j, x)}{p(x)} = \frac{p(x \mid \omega_j) P(\omega_j)}{p(x)}$$

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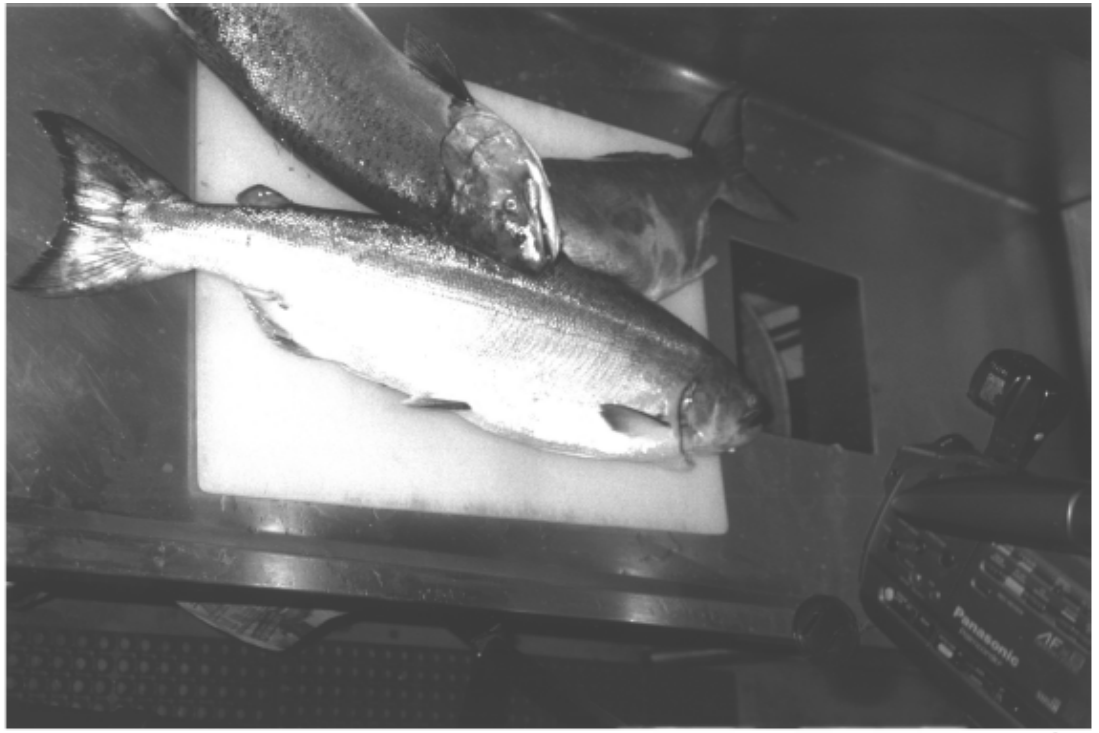
Likelihood Ratio

Threshold

Toy Example Walkthrough

- Image Based Fish Classification (Salmon v/s Sea Bass)



Prior knowledge based classification

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- State of nature (ω)
  - Let variable ω be the discrete random variable which can assume only two categorical values, namely, ω_1(i.e., sea bass) or ω_2 (i.e., solmon)
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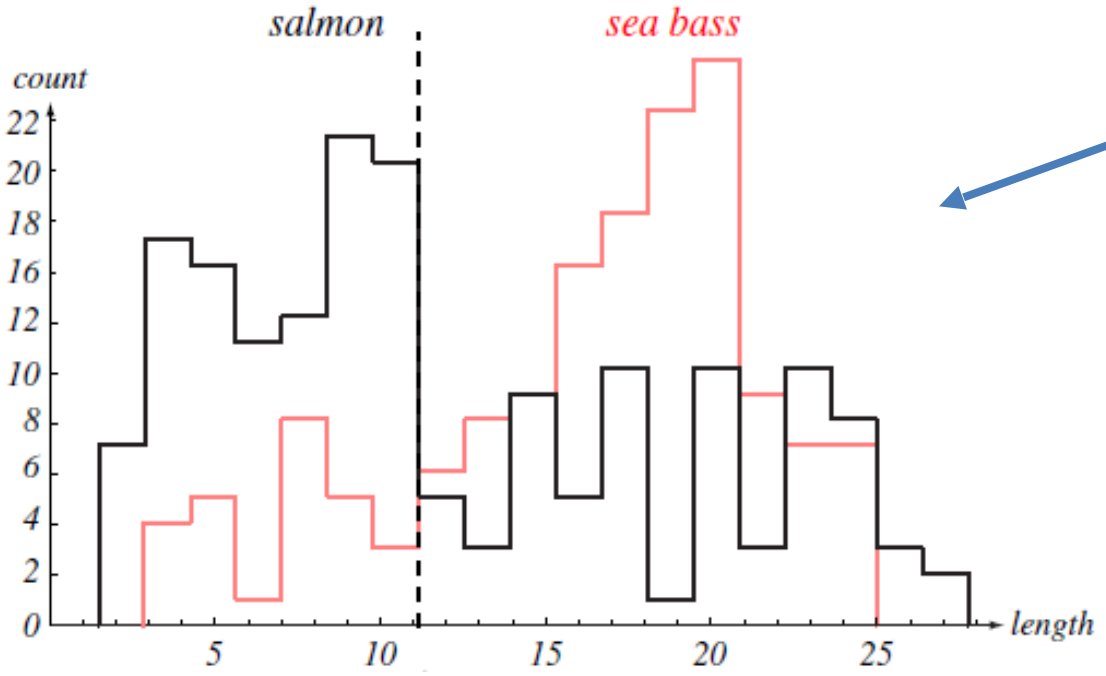
  - Will be making the same decision all times!
  - Favours the most likely class.
- But we can use more information !!

Features and Feature Spaces

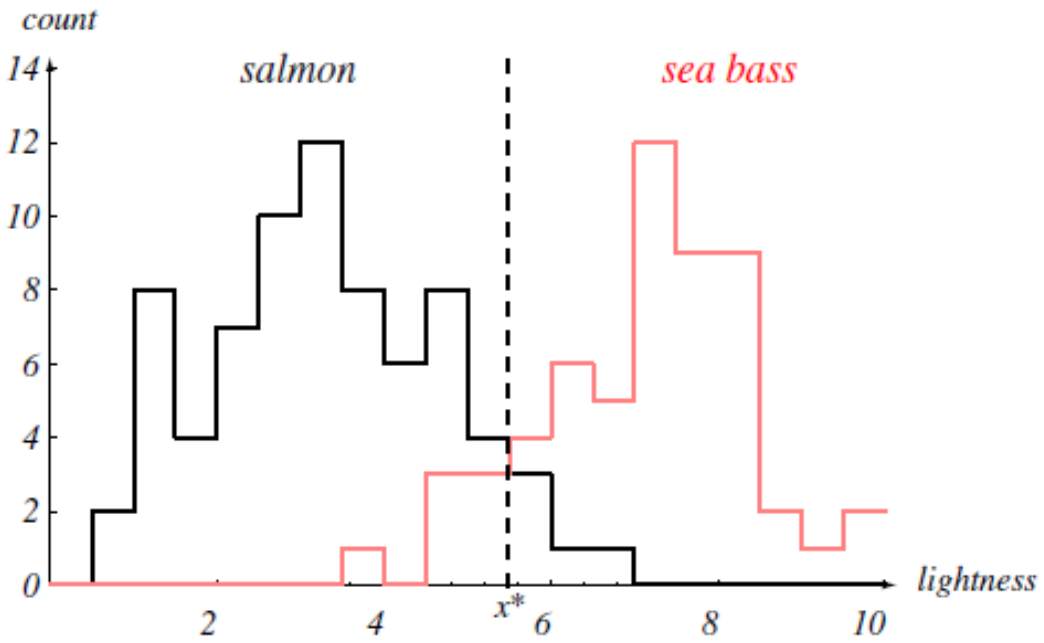
- A feature is an observable variable.
- A feature space is a set from which we can sample or observe values. Examples of features:
  - Length
  - Width
  - Lightness
  - Location of Dorsal Fin
- For simplicity, let's assume that our features are all continuous values.
- Denote a scalar feature as x and a vector feature as $\mathbf{x}$. For a d-dimensional feature space, $\mathbf{x} \in \mathbb{R}^d$.

Toy Examples Walkthrough



Length Histogram Feature

Lightness Histogram Feature



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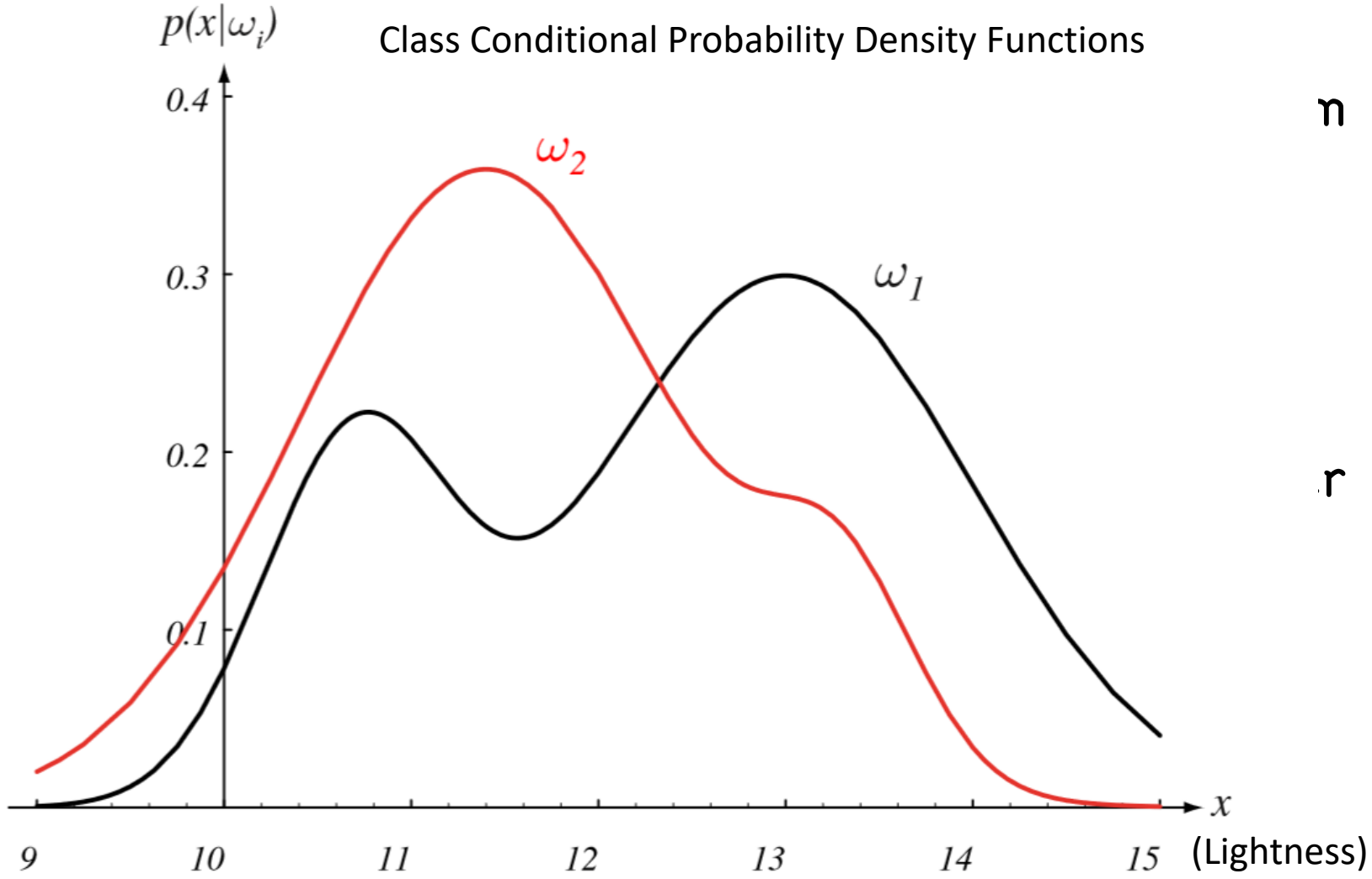
- These pdf curves are normalized such that area under each curve is strictly one.
- These density plots can very well be interpreted as continuous counterpart of histogram representation.

Toy Example Walkthrough

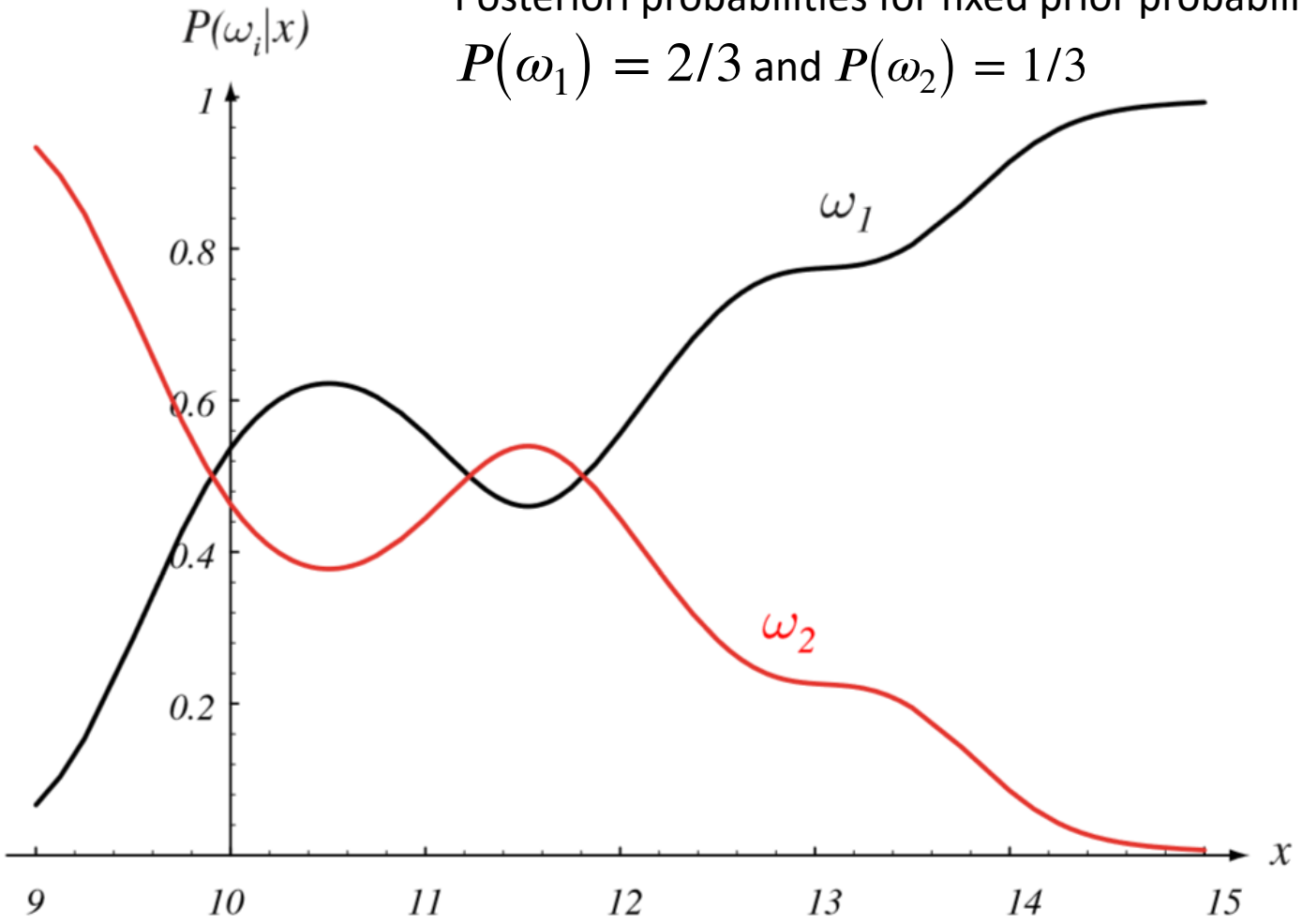
Class Conditional Probability Density Functions



Bayes Classifier

Posteriori probabilities for fixed prior probabilities

$P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$



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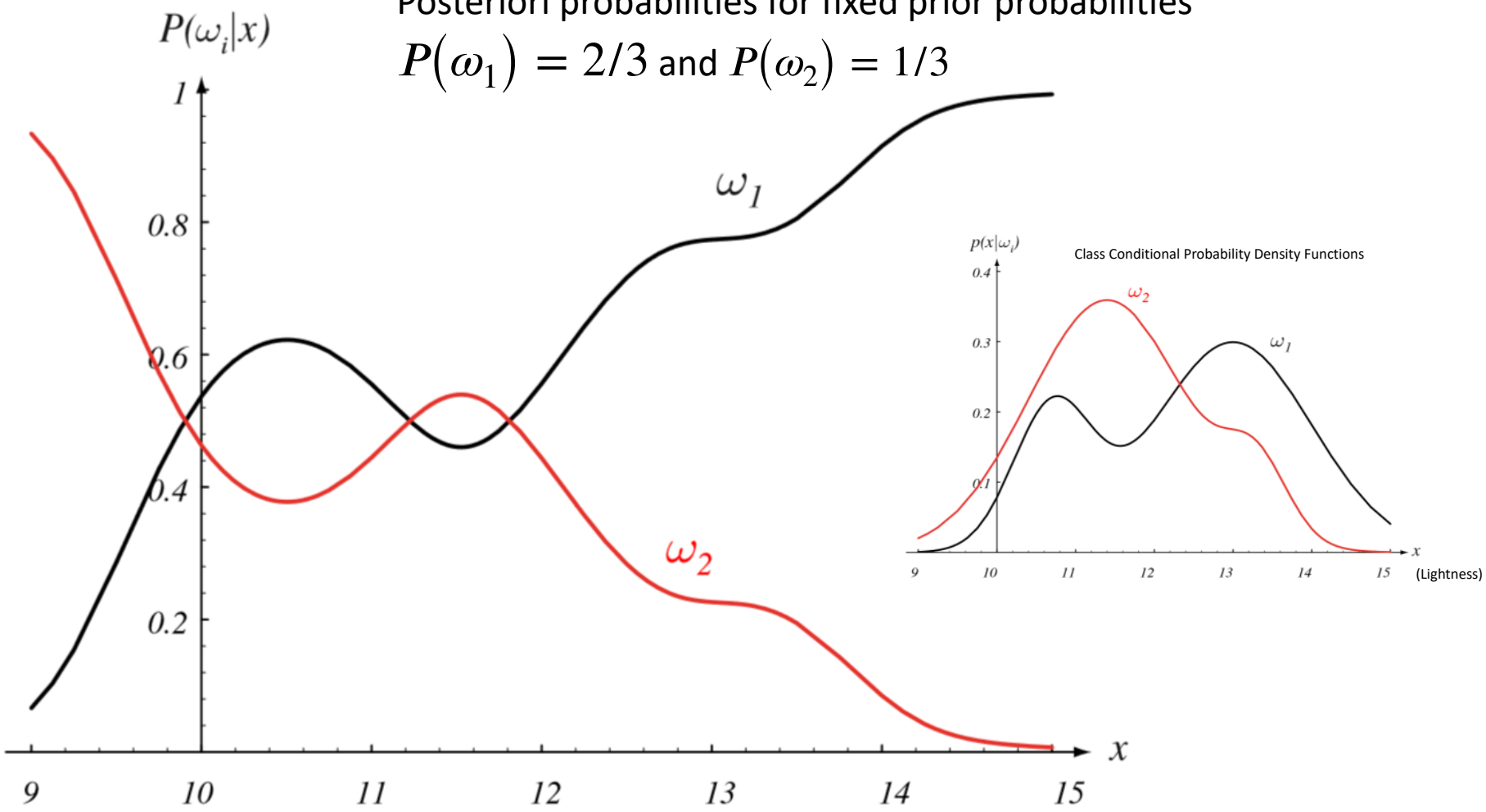
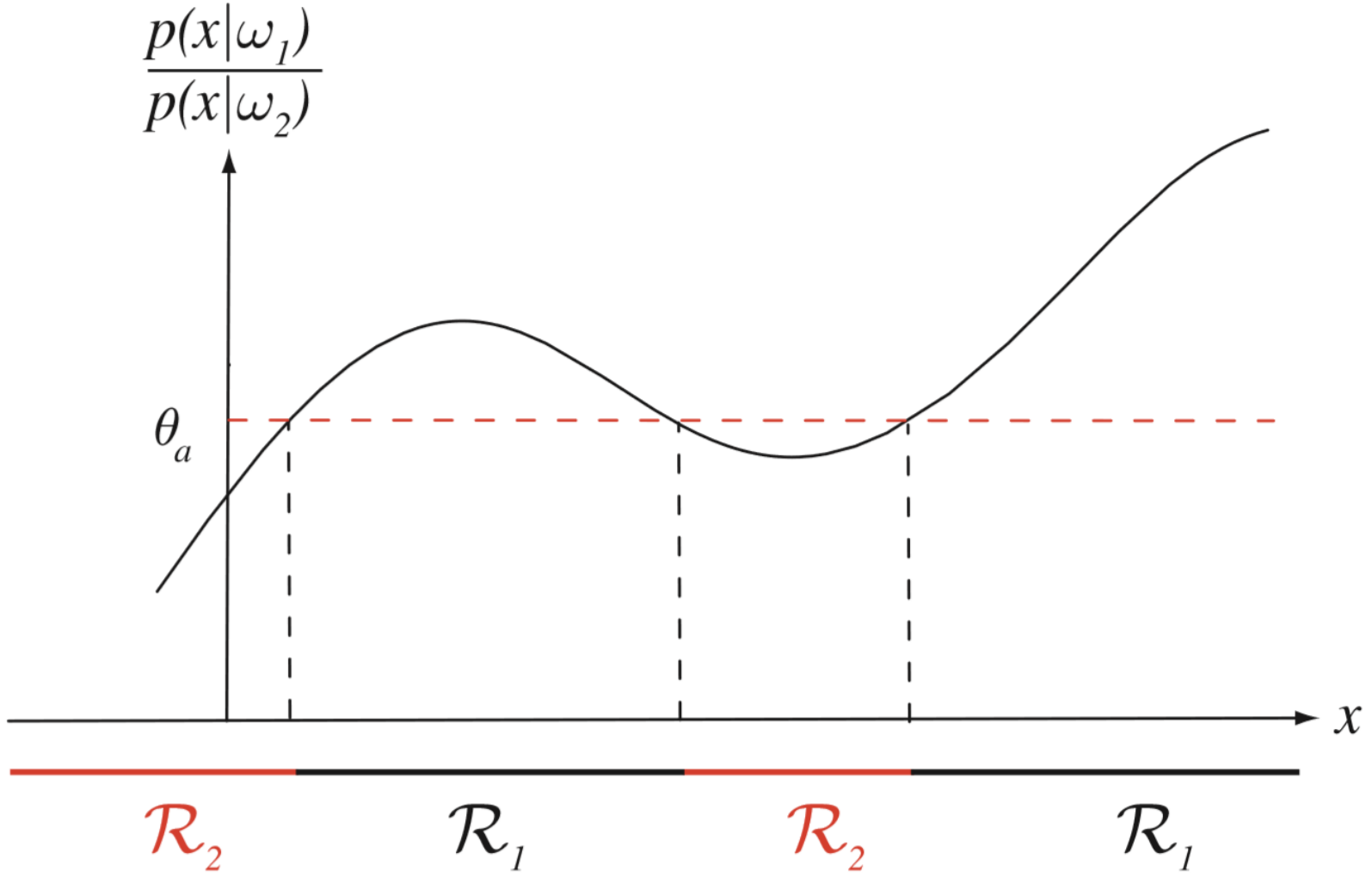


Bayesian Decision Theory



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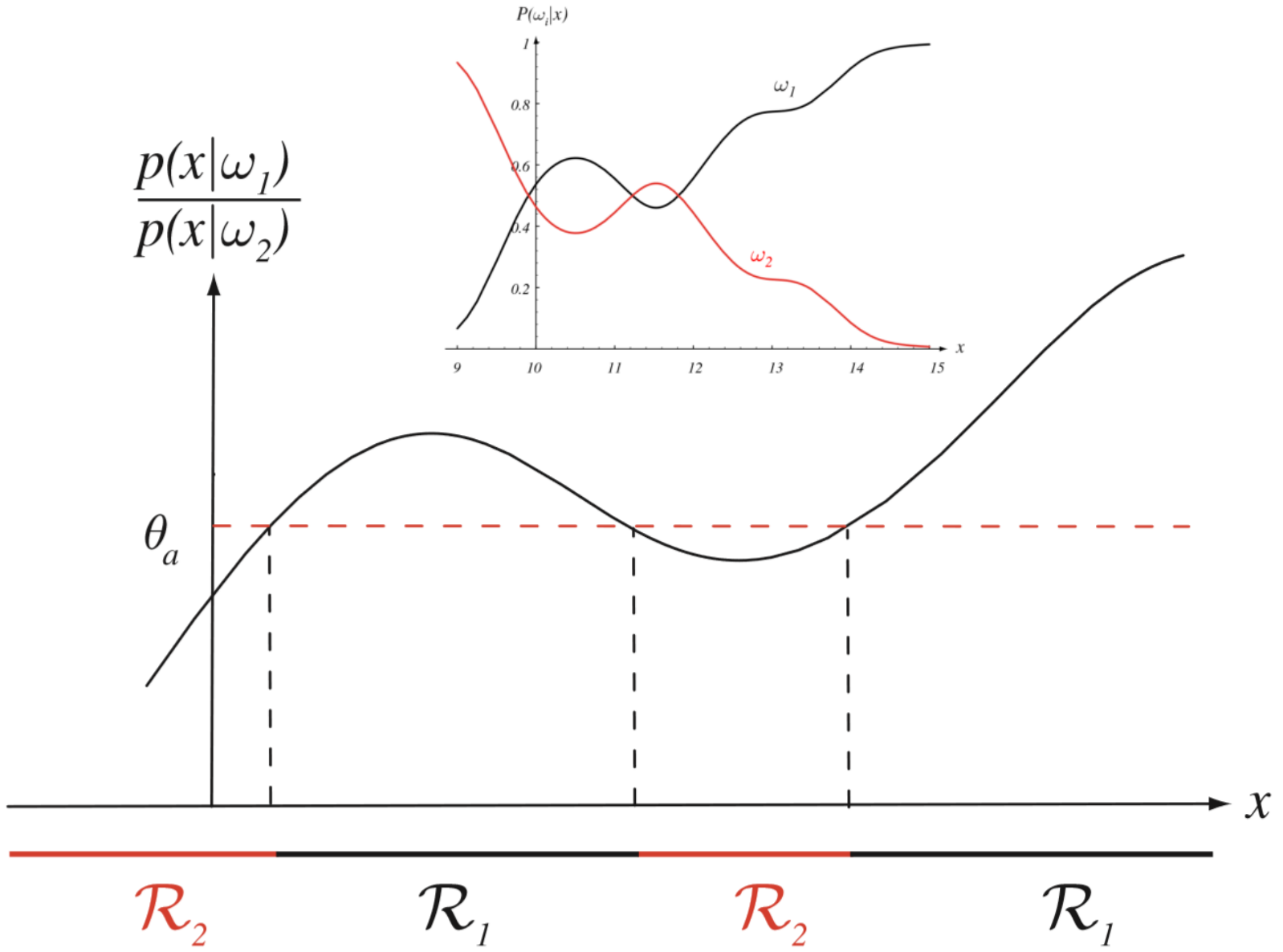

Bayesian Decision Theory for Multiclass Classification

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- Bayes formula can be generalized to:
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Questions

Optimality of Bayes Classifier

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$$P(error|\mathbf{x}) = \begin{cases} P(\omega_1|\mathbf{x}), & \text{if we decide } \omega_2 \\ P(\omega_2|\mathbf{x}), & \text{if we decide } \omega_1 \end{cases}$$

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- Classification is performed by minimizing $R(\alpha_i \,|\, \mathbf{x})$ instead of maximizing $P(\omega_i \,|\, \mathbf{x})$

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- The resulting minimum R* is called Bayes risk and is the best performance that can be achieved:

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- The overall risk in this case is the average probability error (which is minimized by the Bayes rule!).

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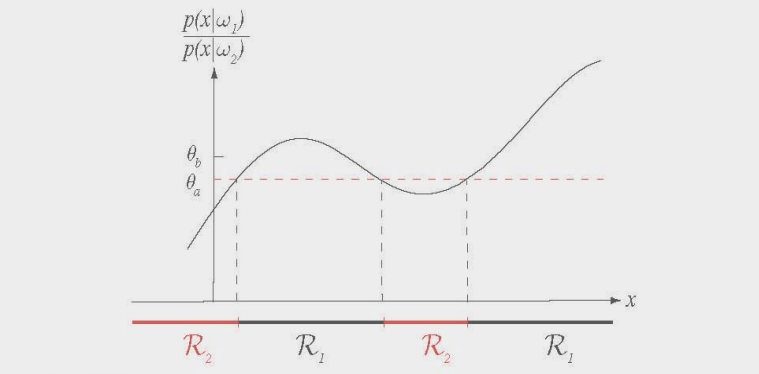

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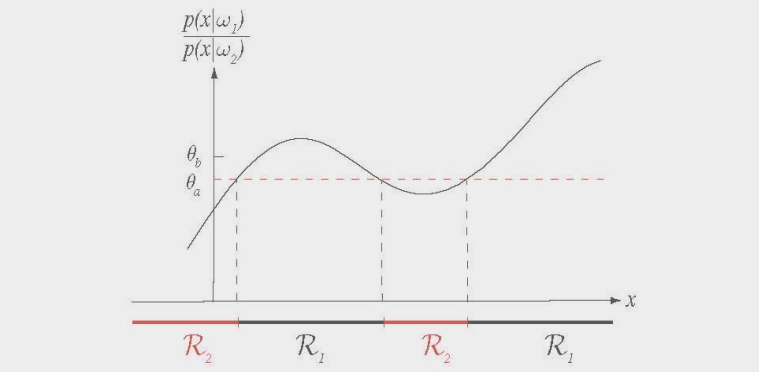

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$$\theta_a = P(\omega_2)/P(\omega_1)$$

$$\theta_b = \frac{P(\omega_2)(\lambda_{12}-\lambda_{22})}{P(\omega_1)(\lambda_{21}-\lambda_{11})}$$

Questions

Discriminants for the Bayes Classifier and Decision Surfaces

Discriminant Functions

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- Represent a classifier by a set of discriminant functions, one for each class:

$$g_i(\mathbf{x}), i = 1 \ldots c$$

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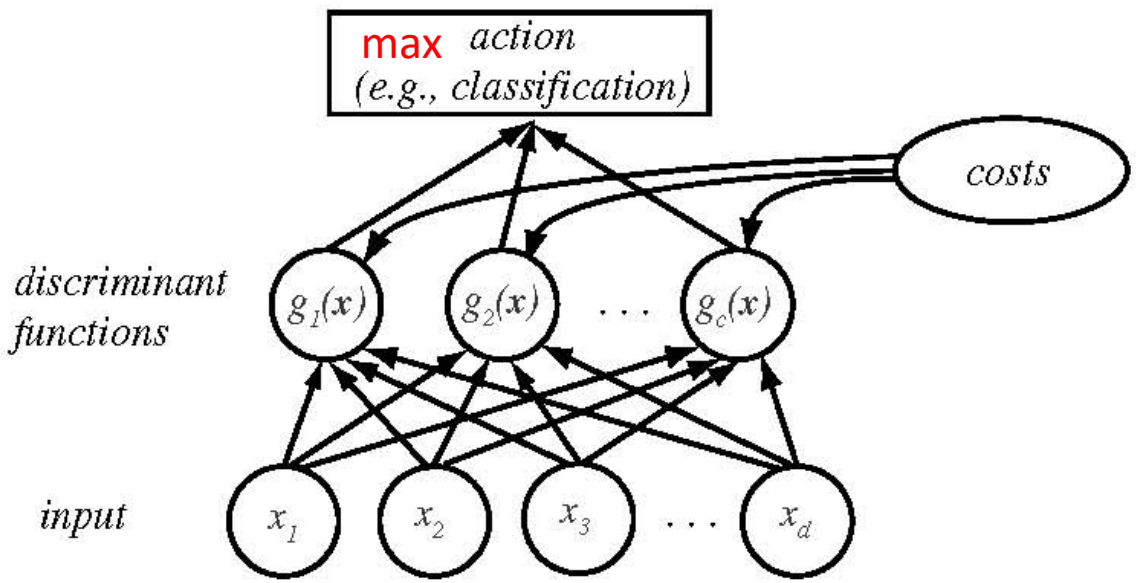

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- **We use this form extensively !!!**

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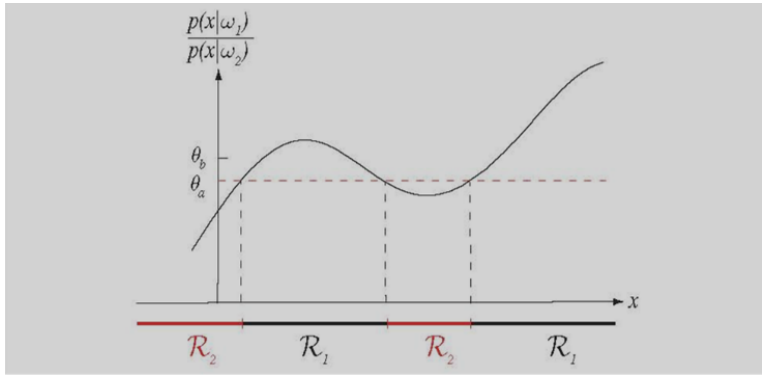

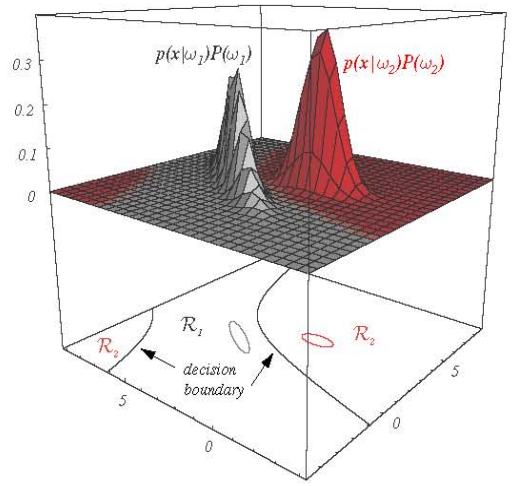

How does is the decision boundary defined?

$g_1(\mathbf{x})=g_2(\mathbf{x})$

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- Model complexity depends on the form of Σ_i; let's consider different cases to better understand this!

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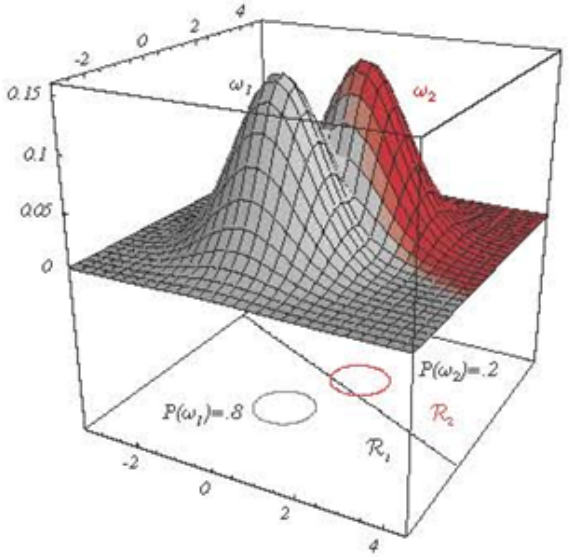

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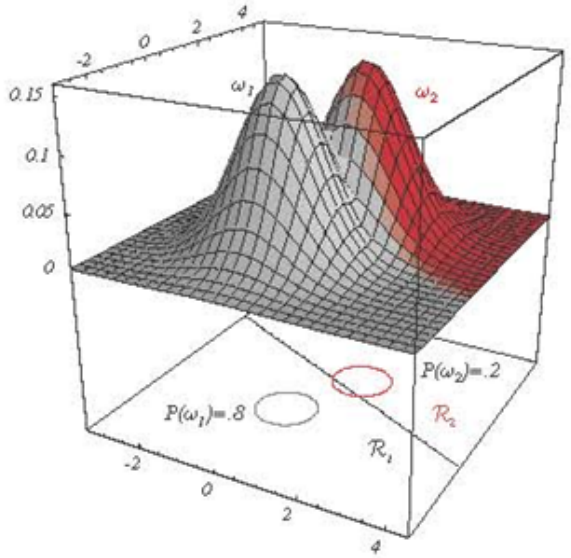

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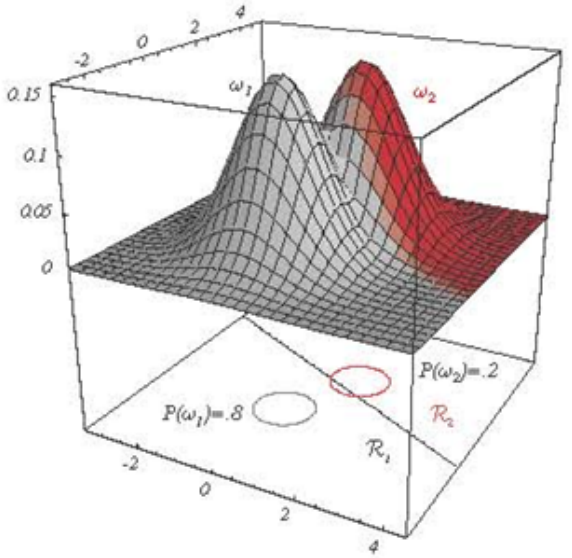

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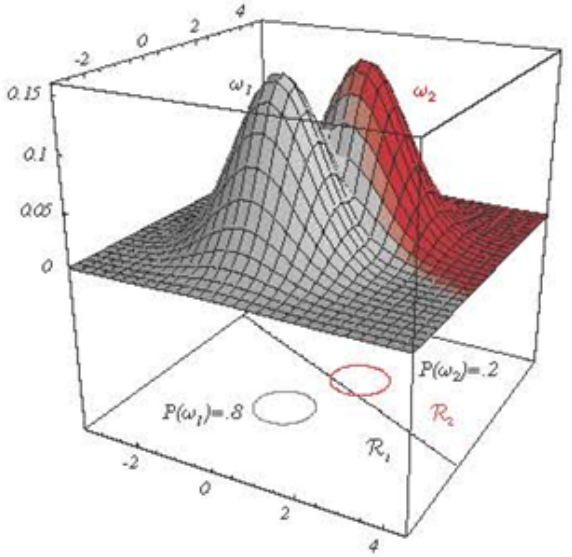

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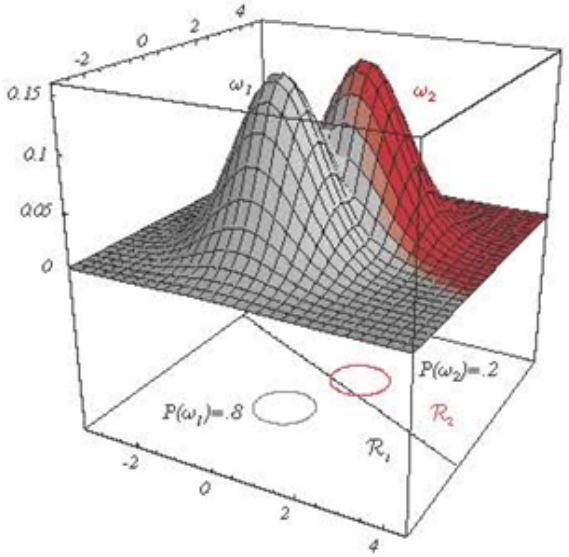

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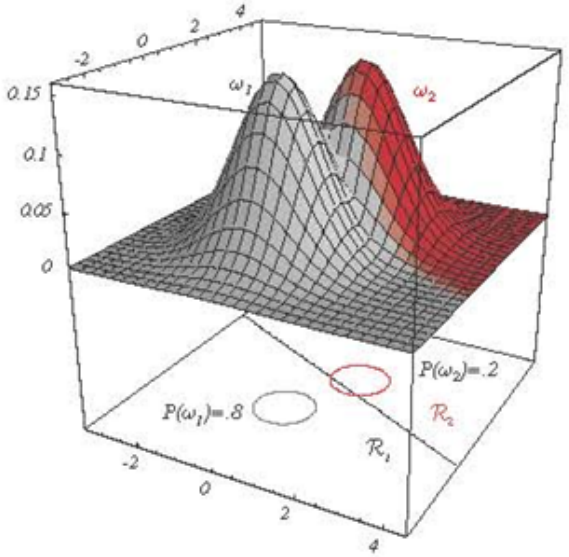
- This is actually a linear discriminant, let's see why!



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- What is the form of the **decision boundary** in this case?

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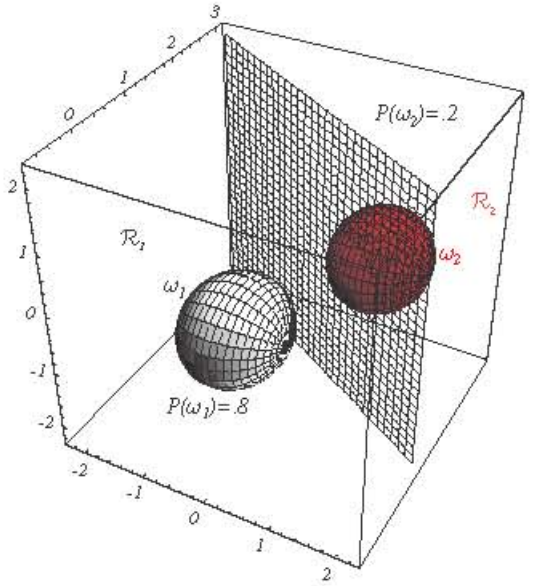

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- Very simple classifier, what does it do exactly?

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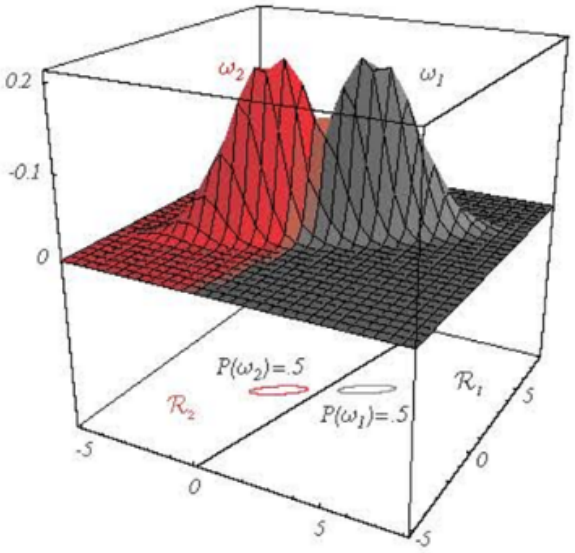

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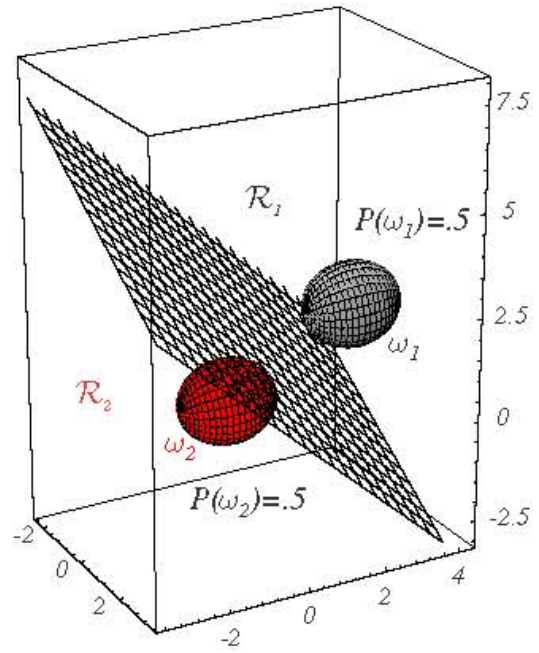

Questions