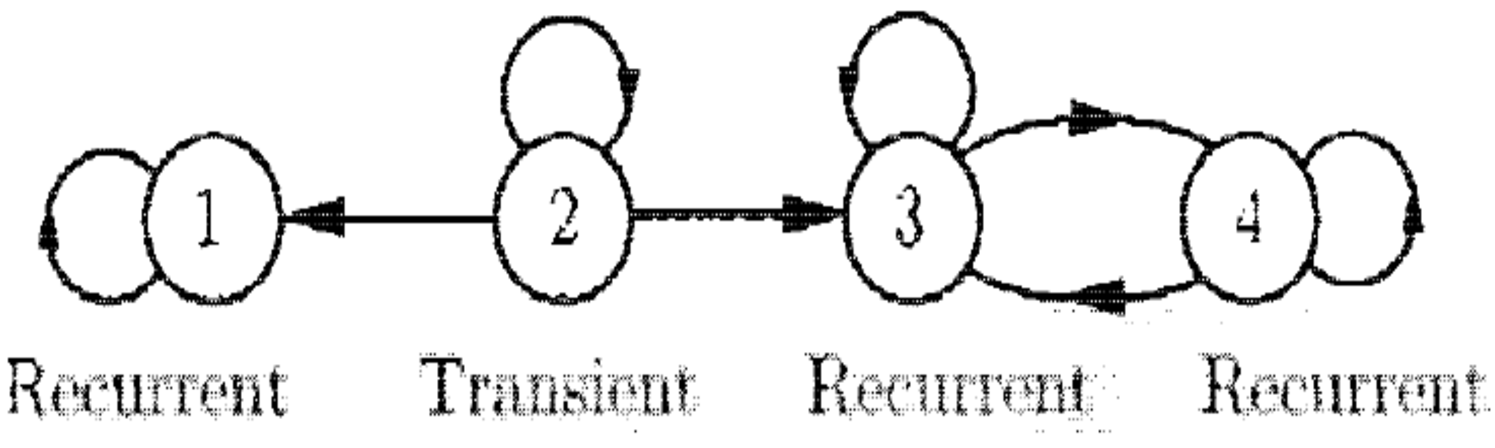
More on Transience and Recurrence

Recall: Recurrent and Transient states

- We say that a state i is recurrent if
  $F_{ii} = P($ ever returning to i having started in i $) = 1$.
- F_{ii} is not easy to calculate. (We will see this today)
- If a state is not recurrent, it is transient.
- For a transient state i, $F_{ii} < 1$.
- If $i \leftrightarrow j$ and i is recurrent, then j is recurrent.



First passage probabilities

- Consider a DTMC with state space denoted by $\mathcal{S}$.
- $f_{ij}^n := P(X_n = j, X_k \neq j \text{ for } 1 \leq k \leq n-1 | X_0 = i)$. $(f_{ij}^0 = 0)$.
- $\{f_{ij}^n, n \geq 0, i, j \in \mathcal{S}\}$ is called the first passage probabilities
- For a fixed n, f^n denotes the matrix $[[f_{ij}^n]]$.
- For a fixed ij pair, $\{f_{ij}^n, n \geq 0\}$ represents a probability mass function on $\mathbb{Z}_+$.
- This mass function can be degenerate which means it can have a point mass at infinity.
- Define: $F_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$.

First passage probabilities

- Define: $F_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$.
- F_{ij} has the interpretation of the probability of ever visiting state j when starting from state i.
- If $F_{ij} = p < 1$, then there is a finite probability $1 - p$ with which you may not ever reach j when starting from state i.
- In this case $\{f_{ij}^n, n \geq 0\}$ is not a proper mass function since it does not sum to 1.
- If $F_{ij} = 1$, then the from i you can certainly reach j.
- In this case $\{f_{ij}^n, n \geq 0\}$ is a proper mass function.
- Let T_{ij} denote the first passage time from i to j.
- T_{ij} has the probability mass function $\{f_{ij}^n, n \geq 0\}$. In other words, $P(T_{ij} = k) = f_{ij}^k$.

First recurrence probabilities

- Define: $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{n}$.
- f_{ii}^{n} : probability of starting in i and returning to state i for the first time at time n
- F_{ii} has the interpretation of the probability of ever returninig to state i.
- If $F_{ii} = p < 1$, then there is a finite probability $1 - p$ with which you may not return to state i .
- If $F_{ii} = 1$, then the from i you can certainly return to i.
- For any $i \in \mathcal{M}$, the first return time T_{ii} has the probability mass function $\{f_{ii}^{n}, n \geq 0\}$.

Mean passage and recurrence times

- Let μ_{ij} be the mean (first passage) time from state i to j.
- $\mu_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{n}$
- Let μ_{ii} be the mean recurrence time at state i.
- $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{n}$
- All the above definitions have an equivalent counterpart in a CTMC.
- For eg: f_{ij}^{t} has a natural interpretation. We wont go further into this.

Transient and recurrent states

- Suppose for a state i we have $F_{ii} = 1$. Then we say that state i is recurrent.
- Once in i, you are certain to come back to i.
- If $\mu_{ii} = \infty$, it is called null recurrent. The chain is bound to return to state i, but possibly after an infinite time.
- $\mu_{ii} < \infty$, it is called positive recurrent.
- If all states of the Markov chain are (null /positive) recurrent, it is called as a (null /positive) recurrent Markov chain.
- Null recurrence is possible in infinite state space models.
- State i is transient if $F_{ii} < 1$.
- You may not return back to i with a finite probability.

Transient and recurrent states

- Consider a DTMC and consider $X_0 = i$.
- Let us count the number of times the chain is in state i.
- Let I_n denote an indicator variable which is 1 if $X_n = i$ and 0 if $X_n \neq i$.
- $\sum_{n=1}^{\infty} I_n$ counts the number of times state i was visited.
- $E[I_n] = P(X_n = i | X_0 = i) = p_{ii}^{(n)}$.
- The mean total number of visits to state i is given by $\sum_{n=1}^{\infty} p_{ii}^{n}$
- Convergence or divergence of this sum also defines transient or recurrent states.

Recurrent criteria

- The mean total number of visits to state i is given by $\sum_{n=1}^{\infty} p_{ii}^{n}$
- Suppose the chain visits state i only exactly n times.
- The P(exactly n visits to i) $= F_{ii}^{n}(1 - F_{ii})$.
- For a recurrent state, $F_{ii} = 1$. Hence P(exactly n visits to i) $=$ 0.
- P(exactly infinite visits to i) $= 1$.
- Mean total number of visits is also infinite and hence $\sum_{n=1}^{\infty} p_{ii}^{n}$ diverges.

Transient state criteria

- The mean total number of visits to state i is given by $\sum_{n=1}^{\infty} p_{ii}^{n}$
- Suppose the chain visits state i only exactly n times.
- The P(exactly n visits to i) $= F_{ii}^{n}(1 - F_{ii})$.
- For transient state i, $F_{ii} < 1$.
- The P(exactly n visits to i) $= F_{ii}^{n}(1 - F_{ii})$. Compare this with geometric random variable.
- Mean total number of visits to state i is $\frac{F_{ii}}{1-F_{ii}}$ which is finite.
- Hence for transient state $\sum_{n=1}^{\infty} p_{ii}^{n}$ must converge.