Lecture 2: Introduction to Hilbert space

- In Classical mechanics, a particle's motion is governed by Newton's Laws.
- The equations of motion dictated by Newton's laws are second order ordinary differential equations.
- The state of the motion is given by the position $\hat{x}(t)$ and momentum $\hat{P}(t)$, where "t" is time which comes as a parameter.
- At any given time instant "t", if we know the pair ($\hat{x}(t)$, $\hat{P}(t)$) , we know everything about the particle in consideration.
- The co-ordinate space consisting all the position and momentum components is called the phase space.
- In general, for a N particle system, the phase space is 6 dimensional, with 3N position and 3N momentum co-ordinates.
- So the bottom line in classical mechanics is to know the instantaneous position $(\hat{x}(t), \hat{P}(t))$ in phase space, which determines the state of motion.
- The trajectory in the phase space is governed by the equation of motion.

Quantum Mechanics

- In Quantum mechanics, the basic question remains the same: **What is the state of motion ?**
- The state of motion cannot be determined by the point in phase space.
- We have to consider uncertainty principle.
- Position and momentum cannot be determined with perfect accuracy simultaneously.
- Also the entity obeying quantum mechanics do not obey Newton’s Laws.
- Therefore we need a new “space” and a new “Principles of motion”

Why do we need Hilbert space

- In Quantum mechanics, everything we know about a particle is encoded in a vector ψ in a space called Hilbert space

- This vector is called the State vector.

- The state vector evolves in time according to the “Schrödinger equation”

- The observables are represented by certain operators, acting on the Hilbert space.

- The operators are linear maps $O: H \to H$, which means they map a vector ψ into another vector ϕ in the same Hilbert space.

Metric Space

A metric space is a space X together with a distance function $d: X \times X \rightarrow \boldsymbol{R}$ such that:

I) $d(x, y) \geq 0$

II) $d(x, y) = 0 \ \ iff \ \ x = y$

III) $d(x, y) = d(y, x)$ (Symmetric property)

IV) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Hilbert Space

- Hilbert space is a vector space $\boldsymbol{H}$ over $\boldsymbol{C}$ (complex vector space), equipped with a complete inner product.
- Saying that Hilbert space is a vector space means that it is a set on which we have an operation `+` of addition obeying
  - Commutativity: $\psi + \phi = \phi + \psi$.
  - Associativity: $\psi + (\phi + \chi) = (\psi + \phi) + \chi$.
  - Identity: There exists $o \in \boldsymbol{H}$ such that $\psi + o = \psi$.
  - Here,$o \rightarrow Null\ vector$ For all $\psi, \phi, \chi \in \boldsymbol{H}$.
- Multiplication by a complex scaler:

  The multiplication operation is

  i. Distributive over $\boldsymbol{H}$: $c(\psi + \phi) = c\psi + c\phi$.

  ii. Distributive over $\boldsymbol{C}$: $(a + b)\psi = a\psi + b\psi$.

Hilbert Space-Inner Product

- Any Hilbert Space $\boldsymbol{H}$ is equipped with an inner product $(\,,\,)$.
- This is a map $(\,,\,): \boldsymbol{H} \times \boldsymbol{H} \rightarrow \boldsymbol{C}$ that obeys
  - Conjugate symmetry: $(\psi\,,\phi) = (\phi\,,\psi)^*$.
  - Linearity : $(\phi, a\psi) = a(\phi, \psi)$.
  - Additivity: $(\phi, \psi + \chi) = (\phi, \psi) + (\phi, \chi)$.

- Points to remember:
  - Inner product is anti linear in first argument: $(a\phi, \psi) = a^*(\phi, \psi)$.
  - $(\psi, \psi) = (\psi, \psi)^*$ This property gives a norm.

Norm

- Whenever we have an inner product, we can define a norm of the form:

$$|\psi| = \sqrt{\psi, \psi}$$

- These properties ensure that the Cauchy-Schwarz inequality holds true

$$|\phi, \psi|^2 \leq (\phi, \phi)(\psi, \psi)$$

- As a consequence of this, the triangle inequality also holds.

Linear Independence and more…

- Linear independence: A set of vectors $\{\phi_1, \phi_2, \ldots., \phi_n\}$ are linearly independent, if and only if the only solution to $c_1\phi_1 + c_2\phi_2 + \cdots + c + c_n\phi_n = 0$ for $c_i \in \boldsymbol{C}$ is $c_1 = c_2 = \cdots = c_n = 0$.
- The dimension of the vector space is the largest possible number of linearly independent vectors we can find.
- If there is no such number, the vector space is infinite dimensional.
- Orthogonality: An orthogonal set of vectors $\{\phi_1, \phi_{2,\ldots,}\phi_n\}$ is defined by $\left(\phi_i, \phi_j\right) = 0\ for\ i \neq j\ and\ \left(\phi_i, \phi_j\right) = constant\ for\ i = j\ \forall\ i, j$.
- Normalized vectors: $(\phi_i, \phi_i) = 1$
- An orthonormal set of vectors $\{\phi_1, \phi_{2,\ldots,}\phi_n\}$ forms a basis of n dimensional Hilbert space if every vector ψ can be uniquely expressed as $\psi = \sum_\alpha c_\alpha \phi_\alpha$ with some complex coefficients c_α .

$$(\phi_\alpha, \psi) = (\phi_\alpha, \textstyle\sum_a c_a \phi_a) = \sum_a c_a (\phi_\alpha, \phi_a) = c_\alpha$$

Cauchy-Schwarz inequality $|(x, y)|^2 \leq (x, x)(y, y)$

- Suppose x is not a scaler multiple of y and they are both non-zero.
- Because for the previous case, the equality always holds.
- $x - \alpha y$ is then always non zero for any complex α.
- Consider $|x - \alpha y|^2 > 0$
- Expanding we get $|x|^2 - \alpha(x, y) - \alpha^*(y, x) + \alpha\alpha^*|y|^2 > 0$.
- Let $\alpha = \mu t$ with $t\ real\ and\ |\mu| = 1$ and $\mu = |\mu| \exp(i\theta)\,, where\ (x, y) = |(x, y)|\exp(-i\theta)$
- Therefore $\mu(x, y) = |(x, y)|$.
- Then $|x|^2 - 2t|(x, y)| + t^2|y|^2 > 0$.
- The minimum of LHS occurs when $-2|(x, y)| + 2t|y|^2 = 0$ giving $t = \frac{|(x,y)|}{|y|^2}$.
- Putting this value of t in the inequality, we get the desired result.

Triangle inequality : $|v + w| \leq |v| + |w|$

$$(|v| + |w|)^2 - |v + w|^2 = |v|^2 + |w|^2 + 2|v||w| - |v|^2 - |w|^2 - (v, w) - (w, v)$$

$$\Rightarrow 2|v||w| - 2Re(v, w) \geq 2|v||w| - 2(v, w) \geq 0.$$

Thoughts to take home…

- Consider Cartesian Co-ordinate system in three dimension.
- Verify all the properties of a vector space
- What will be the inner product ?
- Verify the Cauchy-Schwarz inequality and Triangle inequality.