CS 3.307
Performance Modeling for Computer Systems

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Markov Process

- There are two versions of Markov chains- Discrete time and Continuous time.
- A stochastic process $\{X_n, n \in \mathbb{Z}_+\}$ is a discrete time Markov chain if for any $n_1 < n_2 < \ldots < n_k < n$,

$$P(X_n = j | X_{n_1} = x_1, .., X_{n_k} = i) = P(X_n = j | X_{n_k} = i)$$

- $\{X(t), t \geq 0\}$ is a Markov process (ctmc) if for $t_1 < t_2 < \ldots t_n < t$,

$$P(X(t) = j | X(t_1) = x_1, .., X(t_n) = i) = P(X(t) = j | X(t_n) = i)$$

- This is known as the Markov property.
- State space in both cases can be integers or general ($\mathbb{R}^d$)
- We will stick with integer or finite state space

Example: Coin with memory!

- In a Markovian coin with memory, the outcome of the next toss depends on the current toss.
- $X_n = 1$ for heads and $X_n = -1$ otherwise. $\mathcal{S} = \{+1, -1\}$.
- Sticky coin : $P(X_{n+1} = 1 | X_n = 1) = 0.9$ and $P(X_{n+1} = -1 | X_n = -1) = 0.8$ for all n.
- Flippy Coin: $P(X_{n+1} = 1 | X_n = 1) = 0.1$ while $P(X_{n+1} = -1 | X_n = -1) = 0.3$ for all n.
- This can be represented by a transition diagram (see board)
- The transition probability matrix P for the two cases is $P_s = \begin{bmatrix} 0.9 & .1 \\ 0.2 & 0.8 \end{bmatrix}$ and $P_f = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}$
- The row corresponds to present state and the column corresponds to next state.

Running example: Dice with memory!

- In a markovian dice with memory, the outcome of the next roll depends on the current roll.
- $X_n = i$ for $i \in \mathcal{S}$ where $\mathcal{S} = \{1, \dots, 6\}$.
- Example transition probability matrix

$$P = \begin{bmatrix} 0.9 & .1 & 0 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

- In the ctmc counterpart for these examples, imagine the coin tosses itself/ dice rolls itself after waiting in the state for a random time that is exponentially distributed. (more later)

Time-homogenous Markov Process

- A DTMC is said to be time homogeneous if the one step transition probabilities are same at all time.
- $P(X_{n+1} = j | X_n = i) = P(X_{n+1+s} = j | X_{n+s} = i) := p_{ij}$
- One step transition probability matrix $P = [[p_{ij}]]$
- $i, j \in \mathcal{S}$ which is countable and $|\mathcal{S}| \leq \infty$

For a CTMC ...

- For a time homogeneous CTMC, we have

$$\begin{aligned} P(X(t)=j|X(t_n)=i) &= P(X(t+s)=j|X(t_n+s)=i) \\ &= P(X(t-t_n)=j|X(0)=i). \end{aligned}$$

- We have a transition probability matrix with entries $p_{ij}(t)$, i.e., $P(t)=[[p_{ij}(t)]]$.

DTMC – Time spent in a state

- For a time homogeneous DTMC, we have a transition probability matrix with entries p_{ij}, i.e., $P = [[p_{ij}]]$.
- Let $Y_n = inf\{s > 0 : X_{n+s} \neq X_n\}$
- Y_n is the remaining time that the process spends in whichever state it is in, at time n.
- Consider a Markov coin, its state transition matrix and diagram
- Y_n is geometric random variable.
- What would be the time spent in a state for a continuous time Markov chain ?

CTMC – Time spent in a state

- For a time homogeneous CTMC, we have a transition probability matrix with entries $p_{ij}(t)$, i.e., $P(t) = [[p_{ij}(t)]]$.
- Let $Y_t = inf\{s > 0 : X(t+s) \neq X(t)\}$
- Y_t is the remaining time that the process spends in whichever state it is in, at time t.
- Intuitively, the time spent in a state should depend only on what state it is in, and not on the previous state.

Theorem

$$P(Y_t > u | X(t) = i) := \bar{G}_i(u) = e^{-a_i u}$$

for all $i \in \mathcal{S}$ *and* $t \geq 0, u \geq 0$ *and for some real number* $a_i \in [0, \infty]$.

Proof 1

- $\bar{G}_i(u+v) = P(X(s) = i, s \in [t, t+u+v] | X(t) = i)$
- $\bar{G}_i(u+v) = P(X(s) = i, s \in [t+u, t+u+v]; X(p) = i, p \in [t, t+u] | X(t) = i)$
- $P(AB|C) = P(A|BC)P(B|C)$
- Due to Markov property we have $P(AB|C) = P(A|B)P(B|C)$
- $P(X(s) = i, s \in [t+u, t+u+v] | X(p) = i, p \in [t, t+u]) =$
- $P(X(s) = i, s \in [t+u, t+u+v] | X(t+u) = i) = \bar{G}_i(v)$
- $P(X(p) = i, p \in [t, t+u] | X(t = i)) = \bar{G}_i(u)$
- $\bar{G}_i(u+v) = \bar{G}_i(u)\bar{G}_i(v)$
- Only CCDF function which satisfies this equation is the exponential distribution. This requires a proof. We will skip this part.