Lecture # 7

Bell states $\begin{cases} \dfrac{|00\rangle \pm |11\rangle}{\sqrt{2}} & |\psi_\pm\rangle \\[2em] \dfrac{|01\rangle \pm |10\rangle}{\sqrt{2}} & |\phi_\pm\rangle \end{cases}$

Topics for Quiz
partial tracing
partial measurement

why is outer product projection operator

3 conditions of a basis:
i) Orthogonal (perpendicular)
ii) Normalized (length = 1)
iii) Completeness ($\sum |\rangle\langle| = I$)

i) check if Bell states form basis
($|00\rangle = |0\rangle \otimes |0\rangle$)

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|\psi_\pm\rangle\langle\psi_\pm| = \begin{pmatrix} 1/2 & 0 & 0 & \pm 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

$$|\phi_\pm\rangle\langle\phi_\pm| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & \pm 1/2 & 0 \\ 0 & \pm 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|11\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$\therefore \sum |\rangle\langle| = \mathbb{I}$ (Identity)

$$|01\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle$$

$$= \text{Trace}(\hat{x} |\psi\rangle\langle\psi|)$$

$$= \text{Trace}(\hat{x} \rho)$$

Lecture #8

Projective measurement: $\hat{P}_\psi = |\psi\rangle\langle\psi|$

$\hat{P}_\psi^2 = \hat{P}_\psi \Leftarrow$ Idempotence

(More General Measurement op)
Projective Operator valued measurement (POVM): $\sum M_i = \mathbb{I}$

M_i = Positive semi definite Matrices

$\rho = \sum P_i |\psi_i\rangle\langle\psi_i| \rightarrow$ classical Noise

$U \rho \otimes |0\rangle\langle 0| U^\dagger$

$U = |\psi_1\rangle\langle 00| + |\psi_2\rangle\langle 01| + \ldots + p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ = Unitary (How?)

Instrument (Initially) = $|0\rangle\langle 0|$

$U_{SI} \, \rho_S \otimes |0\rangle\langle 0|_I \, U_{SI}^\dagger$

$\underbrace{\langle i|}_{\text{part}}$

$\mathbb{I}_S \otimes \langle i|_I \, U_{SI} \, \rho_S \otimes |0\rangle\langle 0|_I \, U_{SI}^\dagger \, \mathbb{I}_S \times |i\rangle_I$ $\quad K_i = \langle i|U|0\rangle$ | Measurement:

$\Rightarrow \sum_j \langle i$ $\qquad Tr(K_i \rho K_i^\dagger)$

$\Rightarrow Tr[K_i \rho K_i^\dagger] = Tr[K_i^\dagger K_i \rho] = Tr[M_i \rho]$ | $M_i = K_i^\dagger K$

Post Measurement State $\quad \tilde{\rho}_{K_i} = \dfrac{K_i \rho K_i^\dagger}{Tr[K_i \rho K_i^\dagger]}$

For probability conservation,

$\sum M_i = \mathbb{I}$

$\sum_i K_i^\dagger K_i = \mathbb{I}$ (To prove)

ρ_S = system

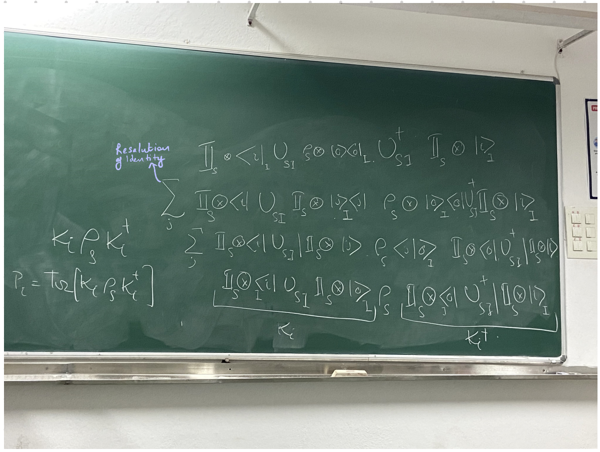
$|0\rangle\langle 0|_I$ = Instrument

U_{SI} = Global Unitary

$U_{SI}\, \rho_S \otimes |0\rangle\langle 0|_I\, U_{SI}^\dagger$

Projective measurement:

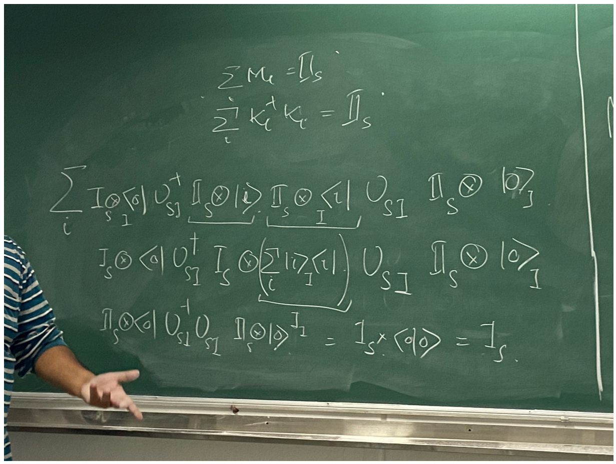
$\mathbb{I}_S \otimes \langle i|_I\, U_{SI}\, \rho_S \otimes |0\rangle\langle 0|_I\, U_{SI}^\dagger\, \mathbb{I} \otimes |i\rangle_I$



+ve

$(T \otimes I)(\quad) = (\quad)$

comp.



Lecture # 9

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{r}\cdot\vec{\sigma})$$

$|\vec{r}| = 1 \rightarrow$ Pure state

$|\vec{r}| < 1 \rightarrow$ mixed state

Purity $Tr(\rho^2)$

$\rho(0) \longrightarrow U\rho(0)U^{-1}$

$Tr(\rho(t)^2)$

$= Tr(U\rho(0)\cancel{U^{-1}U}\rho(0)U^{-1})$

$= Tr(\rho(0)^2) = 1.$

P_i's ^com^ permute, but not change their values under unitary evolution. (eigen values will not change)

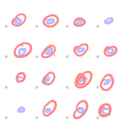
properties of density matrix

1. Trace preserving
2. Hermiticity
3. Positivity (completely)
4. Linearity

$\Lambda(\rho) = \sigma$

$\forall \rho$, σ is positive

$4 + 4(3+3)$



These 4 conditions are not enough to ensure if the transformation is physical

Sylvester's criteria: For a +ve semi-definite matrix, all of its is transpose a positive map? yes

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\ 1) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$$

$$\mathbb{I}\otimes|\psi\rangle\langle\psi| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix} = \begin{pmatrix}1/2 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & 1/2\\ 1/2 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & 1/2\end{pmatrix}$$

$$\mathbb{I}\otimes|\psi\rangle\langle\psi| = \begin{bmatrix}1/2 & 0 & 0 & 1/2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1/2 & 0 & 0 & 1/2\end{bmatrix}$$

$$\mathbb{I}\otimes T(|\psi\rangle\langle\psi|) = \begin{bmatrix}\begin{pmatrix}1/2 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 1/2\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 1/2 & 0\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 1/2\end{pmatrix}\end{bmatrix} \rightarrow \begin{bmatrix}1/2 & 0 & 0 & 0\\ 0 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\det\begin{bmatrix}0 & 1/2\\ 1/2 & 0\end{bmatrix} = -1/4$$

Not +ve

positivity is not enough

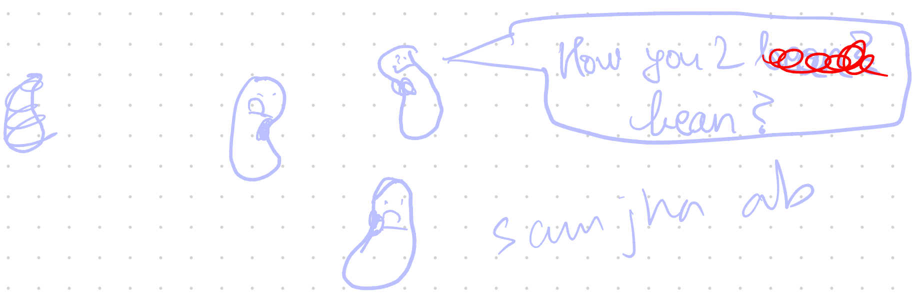
This means transpose is not a physical operator

choi-Jamilkowski Isomorphism

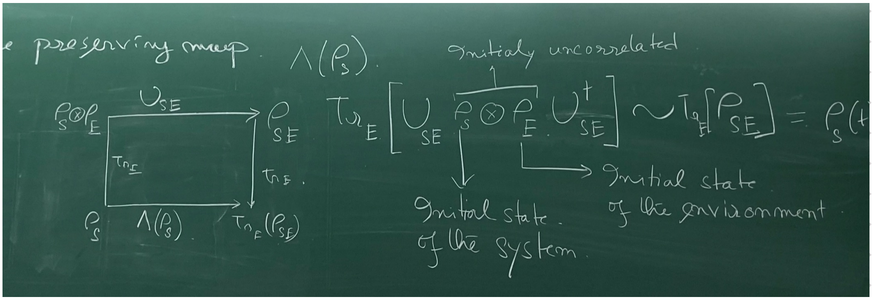
only consider maximally entangled state of $d\times d$ dimension

If $\mathbb{I}\otimes\Lambda|\psi_d\rangle\langle\psi_d|$ is completely positive, then Λ is a completely positive map. $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$

(Given its positive map first)



Preskill lecture notes



$$K_{ij} = \sqrt{\lambda_i}\; \mathbb{I}_s \otimes \langle j | U_{SE} | \mathbb{I}_s \otimes |i\rangle \qquad |\lambda_i\rangle\, \mathbb{I}_s \otimes \langle i | U_{SE}^\dagger\, \mathbb{I}_s \otimes |j\rangle = K_{ij}^\dagger$$

$$\sum_{ij} K_{ij}\, \rho_s\, K_{ij}^\dagger = \text{Krauss decomposition}$$

Operator sum representation

If we have have a CPTP map, then we always have a krauss decomposition

↳ Krauss Theorem

$$\rho_s(t) = \sum_{ij} K_{ij}(t)\, \rho_s(0)\, K_{ij}(t)^\dagger$$

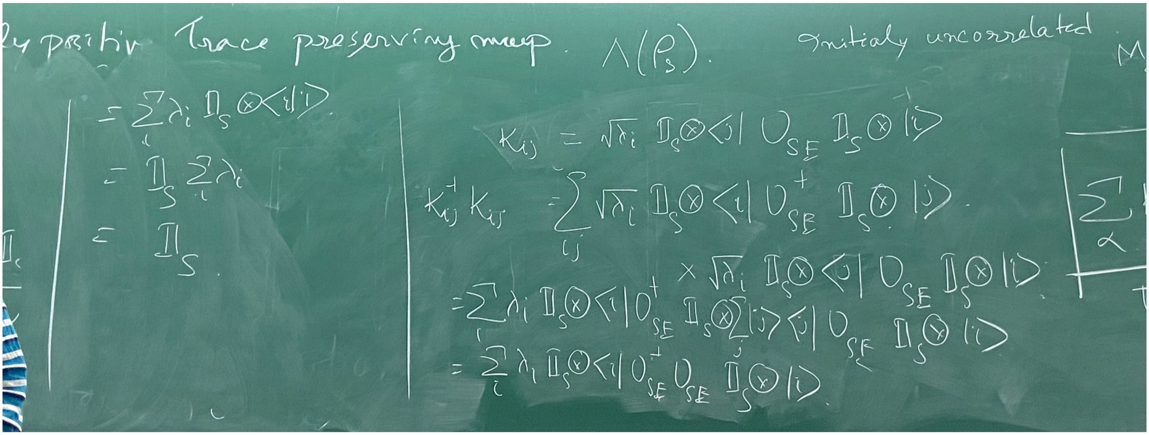
$$\Lambda(\rho) = \sum_{ij} K_{ij}\, \rho_s\, K_{ij}^\dagger$$

for Λ to be TP,

$$Tr(\Lambda(\rho)) = Tr\Big(\sum_{ij} K_{ij}\, \rho_s\, K_{ij}^\dagger\Big)$$

$$= Tr\Big(\sum_{ij} \rho_s\, K_{ij}^\dagger K_{ij}\Big)$$

$$= Tr\Big(\rho_s \sum_{ij} K_{ij}^\dagger K_{ij}\Big) \rightarrow \sum_{ij} K_{ij}^\dagger K_{ij} = \mathbb{I}$$



$$A = \sum \lambda_i |i\rangle\langle i|$$

$$\langle\psi| \lambda_i |i\rangle\langle i|\psi\rangle$$

$$\lambda_i \sum \langle\psi|i\rangle\langle i|\psi\rangle$$

$$\lambda_i \sum |\langle i|\psi\rangle|^2 \geq 0.$$

$+ve$

$$\mathcal{E}_1(\rho) = \sum_i K_i \rho K_i^{-1}$$

$\mathcal{E}_2 \to \mathcal{E}_1^{-1}$ s.t. $\mathcal{E}_2 \circ \mathcal{E}_1(\rho) = \rho$

Kraus Theorem: All CPTP maps have Kraus decomposition

Let $|\psi\rangle\langle\psi|$ be a pure state

$$\mathcal{E}_1(|\psi\rangle\langle\psi|) = \sum_a M_a |\psi\rangle\langle\psi| M_a^\dagger$$

$$\mathcal{E}_2 \circ \mathcal{E}_1(|\psi\rangle\langle\psi|) = \sum_{a,\mu} N_\mu M_a |\psi\rangle\langle\psi| M_a^\dagger N_\mu^\dagger = |\psi\rangle\langle\psi|$$

This can only happen when $N_\mu M_a = \lambda_{\mu a} \mathbb{I}$

$$\therefore \sum \lambda_{\mu a}^* \lambda_{\mu a} \mathbb{I} \rho \mathbb{I} = \rho$$

$$\Downarrow$$

$$\boxed{\sum \lambda_{\mu a}^* \lambda_{\mu a} = 1}$$

$$M_b^\dagger M_a = M_b^\dagger \sum_\mu N_\mu^\dagger N_\mu M_a$$

$$= \sum \lambda_{\mu b}^* \lambda_{\mu a} \mathbb{I}$$

$$= \sum \beta_{ba} \mathbb{I}$$

$$M_a = U_a \sqrt{M_a^\dagger M_a} \quad re^{i\theta}$$ → Polar decomposition

$$A = \sum \lambda_a |a\rangle\langle a|$$

$$A^{1/2} = \sum \sqrt{\lambda_a} |a\rangle\langle a|$$

$$\sqrt{A} \cdot \sqrt{A} = \sum_{a,b} \sqrt{\lambda_a} \sqrt{\lambda_b} |a\rangle\langle b| \langle a|b\rangle$$

$$= \sum_a \sqrt{\lambda_a} \sqrt{\lambda_a} |a\rangle\langle a| = \sum_a \lambda_a |a\rangle\langle a|$$

All normal matrices have a square root. ($AA^\dagger = A^\dagger A$)

$$M_a = U_a \sqrt{M_a^\dagger M_a}$$

$$= U_a \sqrt{\beta_{aa} \mathbb{I}}$$

$$= \sqrt{\beta_{aa}} \cdot U_a$$

$$\therefore \quad M_b^\dagger M_a = \sqrt{\beta_{bb}\beta_{aa}}\, U_b^\dagger U_a$$

$$\beta_{ba} \mathbb{I} = \sqrt{\beta_{bb}\beta_{aa}}\, U_b^\dagger U_a$$

$$\frac{\beta_{ba} U_b}{\sqrt{\beta_{bb}\beta_{aa}}} = U_a \quad \forall\, a, b \qquad \Big| \quad \frac{U_a}{\beta_{ba}} = \frac{U_b}{\sqrt{\beta_{bb}\beta_{aa}}} = U$$

Taken from Preskill lecture notes

Reversibility $\neq$ Invertibility

An invertible map has to be a CPTP to be reversible

channel state duality

$$\mathbb{I} \otimes \Lambda\, |\psi\rangle\langle\psi| = C_\Lambda \geq 0$$

(Somewhat like bijection)

$|\psi\rangle$ → maximally entangled state (choi)

(Isomorphism)

Λ → Quantum channel

C_Λ → Quantum State

Since there is an isomor., it is called channel state duality.

$$\Lambda\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}$$

$$\rho_{11}' = A\rho_{11} + B\rho_{22}$$

$$\rho_{22}' = (1-A)\rho_{11} + \rho_{22}$$

$$\rho_{12}' = \rho_{12}C$$

$$\rho_{21}' = \rho_{21}C^*$$

$$\mathbb{I}\otimes\Lambda|\psi\rangle\langle\psi|$$

$$\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix} \rightarrow \begin{pmatrix}A&0&0&C\\0&1-A&0&0\\0&0&B&0\\C^*&0&0&1-B\end{pmatrix} \rightarrow \text{Diagonalize}$$

$\rightarrow$ eigen vals / eigen vecs

$$\lambda_i \begin{pmatrix}a_i\\b_i\\c_i\\d_i\end{pmatrix}$$

$$K_i = \sqrt{\lambda_i}\begin{pmatrix}a_i & c_i\\ b_i & d_i\end{pmatrix}$$

we get this from choi isomorphism

from eigen value / eigenvector, we can "matrixify" it to get Kraus Operator

$\Lambda_d(\rho)$ $\qquad\qquad$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. Find choi state $\quad \mathbb{I} \otimes \Lambda_d(|\psi\rangle\langle\psi|) = C_{\Lambda_d}$
2. Diagonalize choi state $\quad C_{\Lambda} = \sum_i \lambda_i |i\rangle\langle i|$
3. Rearrange Eigen spectrum $\Rightarrow$

$$|i\rangle = \begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_{d^2}^i \end{pmatrix} \Rightarrow K_i^i = \sqrt{\lambda_i}\begin{pmatrix} a_1^i & \cdots & a_{d^2-d+1}^i \\ a_2^i & & \vdots \\ \vdots & & \vdots \\ a_d^i & & a_{d^2}^i \end{pmatrix}$$

Example: Qubit Dephasing Map

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$\Lambda(\rho)$

$\bar{\rho}_{11} = \bar{\rho}_{11}, \quad \bar{\rho}_{22} = \rho_{22}$

$\bar{\rho}_{12} = \rho_{12} e^{-\gamma t}$

$\bar{\rho}_{21} = \rho_{21} e^{-\gamma t}$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\Lambda} \begin{pmatrix} 1 & 0 & 0 & e^{-\gamma t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-\gamma t} & 0 & 0 & 1 \end{pmatrix}$$

$\lambda_1 = \lambda_2 = 0$

$\lambda_3 = (1 - e^{-\gamma t})$

$\lambda_4 = (1 + e^{-\gamma t})$

$$|1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$K_1 = \sqrt{\frac{1-e^{-\gamma t}}{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \sqrt{\frac{1+e^{-\gamma t}}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\frac{1-e^{-\gamma t}}{2}\right) + \left(\frac{1+e^{-\gamma t}}{2}\right) \cdot \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$\begin{pmatrix} +\rho_{11} & -\rho_{12} \\ -\rho_{21} & \rho_{22} \end{pmatrix} \qquad \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

?

$$\rho' = U\rho U^\dagger$$

$$U = e^{-\frac{i}{\hbar}Ht}$$

$$\downarrow$$

$$\frac{\partial\rho}{\partial t} = \frac{i}{\hbar}[\rho, H]$$

$$\rho' = \sum_i K_i \rho K_i^\dagger \longrightarrow \frac{\partial\rho}{\partial t} = \,?$$ (This may not exist)

We will consider the case where we can derive

Master Equation

Markovian Dynamics → commonly known

for a differential Equation (1st order), $\rho(t+dt)$ must be determined fully by $\rho(t)$ only

↳ Time local Master Equation

$$\begin{tikzcd}
\rho_S \otimes \rho_E \arrow[r] \arrow[r, bend left] \arrow[d, dash] & \rho_{SE} \arrow[d, dash] \\
\rho_S \arrow[r] & \rho_S = Tr_E(\rho_{SE})
\end{tikzcd}$$

$$\Lambda(t,0) = \Lambda(t,t')\circ\Lambda(t',t'')\circ\cdots\circ\Lambda(t''',0)$$

↳ Markovian Eg. Unitary

without Approx, eqn difficult to solve

$$\rho_{SE}(t) \approx Tr_E(\rho_{SE}(t)) \otimes \rho_E$$ → when $E >> S$ → stationary bath approx.

↓

1. stationary bath approx.
2. System-Environment coupling is weak

$$H_T = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_B + H_{SE}$$

$$H_S = \hbar\omega\sigma_z, \quad H_E = \hbar\omega\sigma_x$$

$$H_{SE} = (\sigma_z\otimes\sigma_x + \sigma_x\otimes\sigma_z)$$ (for qubit)

Heuristic Derivation of Master Equation (for understanding)

$$|\psi(t+dt)\rangle = e^{-iHdt/\hbar}|\psi(t)\rangle$$

$$= \left(\mathbb{I} - \frac{iHdt}{\hbar}\right)|\psi(t)\rangle$$

$$\rho(t+dt) = \rho(t) - \frac{i}{\hbar}dt[H, \rho(t)]$$

$$\rho(t+dt) = \mathcal{L}_{dt}(\rho(t))$$

$$\rho_{SE}(t+dt) = Tr_E[\rho_{SE}(t+dt)] \otimes \rho_E$$

$$L_a = \sqrt{\gamma_a}A_a$$

Unitary $\xrightarrow{\text{Gen.}}$ Kraus

Schrodinger $\xrightarrow{\text{Gen.}}$ Master Equation / lindblad master equation

Depolarising channels

Three possible quantum errors:

1. Bit flip error

$$|0\rangle \to |1\rangle, \quad |1\rangle \to |0\rangle$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_1 |\psi\rangle = |\phi\rangle$$

$$\sigma_1 \rho \sigma_1$$

2. Phase flip

$$|0\rangle \to |0\rangle$$

$$|1\rangle \to -|1\rangle$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\sigma_z \rho \sigma_z$$

3. Bit-phase flip error

$$|0\rangle \to i|1\rangle \qquad \sigma_y \rho \sigma_y$$

$$|1\rangle \to -i|0\rangle$$

$$\rho \qquad \sigma_1 \sigma_2 \sigma_3 \rho$$

Depolarizing means all of the three errors are there:

$$\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z \quad \Big| \quad \sum p_i \sigma_i \rho \sigma_i$$

$$(1 - \sum p_i)\rho + \sum_i p_i (\sigma_i \rho \sigma_i)$$ $\longrightarrow$ This is a depolarizing channel

$\Rightarrow$ Kraus operators: $K_0 = \sqrt{1 - \sum p_i}\, \mathbb{I}$

$$K_i = \sqrt{p_i}\, \sigma_i$$

Dephasing channel

↳ only phase flip → No energy loss

$p\,\rho + (1-p)\,\sigma_z\,\rho\,\sigma_z$

→ Depolarizing is dephasing along all three axes (x, y, z)

Section 3.4 | Preskill Lecture notes

Amplitude damping channel

$|0\rangle_S \otimes |0\rangle_E \longrightarrow |0\rangle_S \otimes |0\rangle_E$

$|1\rangle_S \otimes |0\rangle_E \longrightarrow \sqrt{1-p}\,|1\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E$

$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$

$$\gamma\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$$

$$= -2\gamma\begin{pmatrix} 0 & \rho_{12} \\ \rho_{21} & 0 \end{pmatrix} = \begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix}$$

$$\Rightarrow \quad \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0)\cdot e^{-2\gamma t} \\ \rho_{21}(0)\cdot e^{-2\gamma t} & \rho_{22}(0) \end{pmatrix}$$

$p(t) = e^{-2\gamma t}$

(prob)

Depolarising channel

$$\frac{d\rho}{dt} = \sum_{i=x,y,z} \gamma_i(\sigma_i \rho \sigma_i - \rho)$$

solving special case : $\gamma_x = \gamma_y = \gamma_z = \gamma$

$$\frac{d\rho}{dt} = \gamma \Bigg[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_{12} & a_{11} \\ -a_{22} & a_{21} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{pmatrix}$$

$$\gamma \Bigg[\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} + \begin{pmatrix} +a_{22} & +a_{21} \\ +a_{12} & +a_{11} \end{pmatrix} + \begin{pmatrix} a_{11} & -a_{11} \\ -a_{21} & a_{22} \end{pmatrix} \Bigg]$$

$$\begin{matrix} 2a_{22} - 2a_{11} & -a_{11} - 8a_{12} \\ 2a_{12} - 4a_{21} & 2a_{11} - 2a_{22} \end{matrix}$$

Amplitude dampening channel

$$\frac{d\rho}{dt} = \gamma\left(\sigma_- \rho \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right)$$ → will come in exam

Energy loss

$\sigma_- = |0\rangle\langle 1|$

$\sigma_+ = |1\rangle\langle 0|$

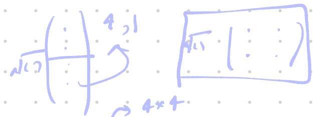
Generalized

$$\frac{d\rho}{dt} = \gamma(n+1)\left(\sigma_- \rho \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right) + \gamma(n)\left(\sigma_+ \rho \sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right)$$

$\gamma(n+1), \gamma(n)$ need to be positive for CPTP

$$\frac{d\rho}{dt} = \lambda(\rho) \quad \text{(Given)}$$

what are the Krauss operators for this master equation



1. Solve the master equation
2. Gives you a dynamical map
3. Find the choi states corresponding to the map
4. Choi state ⟶ Krauss operator

$(\mathbb{I} \otimes \Lambda(\rho))\,\Phi = \Phi_{choi}$ → 4×4

(find Eigenvector & eigenvalue, then

Krauss = unstack $(\sqrt{\lambda_i}\,|i\rangle)$)

Eg. Amplitude damping channel

$$\frac{d\rho}{dt} = \gamma\left(\sigma_- \rho \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right)$$

$\sigma_- = |0\rangle\langle 1|$ | $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$\sigma_+ = |1\rangle\langle 0|$ | $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}a_{12}&0\\a_{22}&0\end{pmatrix} = \begin{pmatrix}a_{22}&0\\0&0\end{pmatrix}$$

$$\sigma_- = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0&1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$\sigma_+ = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1&0\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\sigma_+\sigma_- = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&a_{12}\\0&a_{22}\end{pmatrix} \qquad \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} = \begin{pmatrix}0&0\\a_{21}&a_{22}\end{pmatrix}$$

$$\gamma \left(\begin{pmatrix} a_{22} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a_{12}/2 \\ a_{21}/2 & a_{22} \end{pmatrix} \right)$$

$$\frac{da_{11}}{dt} = \gamma a_{22} \qquad \frac{da_{22}}{dt} = -\gamma a_{22}$$

$$\frac{da_{12}}{dt} = -\frac{\gamma}{2} a_{12}/2$$

$$\frac{da_{21}}{dt} = -\frac{\gamma}{2} a_{21}/2$$

Given the Kraus operator, how do we find the master equation

$$\rho(t) = \Phi(\rho(0)) = \sum K_i(t)\, \rho(0)\, K_i(t)^\dagger \xrightarrow{\text{How?}} \dot{\rho}(t) = \mathcal{L}(\rho(t))$$

$$\frac{\mathbb{I}}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}$$

Lecture #15

1. Find F matrix

$$F_{kl} = Tr[G_k \phi(G_l)]$$

2. $L = \dot{F}F^{-1}$

$$L_{kl} = Tr[G_k \Lambda(G_l)]$$

$$\rho(t) = \phi(\rho(0))$$

$$\frac{d\rho}{dt} = \Lambda(\rho(t))$$

$$= \frac{-i}{\hbar}[\rho, H] + \sum \gamma_i \left(A_i \rho A_i^{-1} - \frac{1}{2}\{A_i^{-1}A_i, \rho\}\right)$$

Example:

$$G_0 = \frac{I}{\sqrt{2}}, \quad G_1 = \frac{\sigma_1}{\sqrt{2}}, \quad G_2 = \frac{\sigma_2}{\sqrt{2}}, \quad G_3 = \frac{\sigma_3}{\sqrt{2}}$$

$$F_{ij} = Tr(G_i \, \phi(G_j))$$

$$\rho(t) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \cdot e^{-\gamma t} \\ \rho_{21}(0) e^{-\gamma t} & \rho_{22}(0) \end{pmatrix}$$

$$F_{00} = Tr\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right) = 1$$

$$F_{01} = Tr\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & e^{-\gamma t}/\sqrt{2} \\ e^{-\gamma t}/\sqrt{2} & 0 \end{pmatrix} \right) = 0$$

$$F_{02} = 0 = F_{03}$$

$$F_{10} = 0, \quad F_{11} = Tr\begin{pmatrix} e^{-\gamma t}/2 & 0 \\ 0 & e^{-\gamma t}/2 \end{pmatrix} = e^{-\gamma t} \qquad F_{33} = 1$$

$$F_{12} = 0, \quad F_{13} = 0, \quad F_{20} = 0, \quad F_{21} = 0, \quad F_{22} = e^{-\gamma t}, \quad F_{23} = 0, \quad F_{30} = F_{31} = F_{32} = 0$$

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\gamma t} & 0 & 0 \\ 0 & 0 & e^{-\gamma t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow F^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\gamma t} & 0 & 0 \\ 0 & 0 & e^{\gamma t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\dot{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma e^{-\gamma t} & 0 & 0 \\ 0 & 0 & -\gamma e^{-\gamma t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L = \dot{F} F^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\dot{\rho}(t) = L_{11} \,\mathrm{Tr}\left(G_{11} \,\mathrm{Tr}(G_1 \rho(t)\right) G_1 + L_{22} \,\mathrm{Tr}(G_2 \rho(t)) G_2 = \begin{pmatrix} 0 & -\gamma \rho_{12} \\ -\gamma \rho_{21} & 0 \end{pmatrix}$$

$$y(t) = A \cos(\omega t)$$

$$y(t) = A e^{-\gamma t} \cos \omega t$$

$$= A e^{-\gamma t} \mathrm{Re}(e^{i\omega t})$$

$$= \mathrm{Re}(A e^{-\gamma t} e^{i\omega t})$$

If F is symmetric $\Rightarrow$ F is hermitian | L is always hermitian

Given diff equation, it definitely can be written in lindblad form.

$$\dot{\rho}(t) = \sum_K X_K(t) \rho(t) Y_K(t)^\dagger$$

Approximation : $\rho_{SB}(t) = Tr_B(\rho_{SB}(t)) \otimes \rho_B$

$$\Lambda(t_2, t_1) = \underbrace{\Lambda(t_2, t')}_{\text{CPTP}} \circ \Lambda(t', t_1) \rightarrow \text{divisible operation}$$

We consider divisible operation to be Markovian

$$\frac{d\rho}{dt} = \sum_i \underbrace{\gamma_i}_{\text{non-negative}} \left(A_i \rho A_i^{\dagger} - \frac{1}{2}\{A_i^{\dagger} A_i, \rho\}\right)$$

depolarizing map

$$\rho = \frac{\mathbb{I}}{d}(1 - p(t)) + p(t) \cdot \rho(0)$$

↳ corresponding Master equation

$$\frac{d\rho}{dt} = \sum_i \gamma (\sigma_i \rho \sigma_i - \rho)$$

$$p(t) = e^{-\gamma t}$$

Example of divisible map

But when $p(t) = \cos^2(\omega t)$, then the map is not divisible.

In general, analysing Markovianity / divisibility is tough. Actually

Markovian $\neq$ Divisible, but Divisible $\subset$ Markovian. but we will loosely say they're same.

If $\gamma_i(t) \geq 0$, then it is Markovian.

we restrict to operation that are invertible (have lindblad type master equation)

we have,

$$L = \dot{F} F^{-1} \Rightarrow \dot{F} = LF \Rightarrow F = e^{\int_0^t L(s) ds}$$

$$\Lambda(t+\varepsilon,t) \approx \rho(t) + \varepsilon \frac{d\rho(t)}{dt} = \rho + \varepsilon L\rho = (\mathbb{I} + \varepsilon L)\rho = \ell_\varepsilon \rho$$

$$f(x+a) = f(x) + f'(x)a + f''(x)\frac{a^2}{2!} + f'''(x)\frac{a^3}{3!} + \dots \quad \text{(Taylor series)}$$

If ℓ_ε is CPTP, then Λ is divisible

we check if ℓ_ε is CPTP using choi

if $\mathbb{I} \otimes \ell_\varepsilon |\phi\rangle\langle\phi|$ is positive $|\phi\rangle$ is maximally entangled.

we can prove that if $\gamma_i(t) \geq 0$, then $\mathbb{I} \otimes \ell_\varepsilon |\phi\rangle\langle\phi|$ is positive

Example: dephasing map

$$\Lambda\rho = \begin{pmatrix} \rho_{11} & \rho_{12}e^{-\gamma t} \\ \rho_{21}e^{-\gamma t} & \rho_{22} \end{pmatrix} \quad \Bigg| \quad \frac{d\rho}{dt} = \begin{pmatrix} 0 & -\gamma\rho_{12} \\ -\gamma\rho_{21} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} \qquad \ell_\varepsilon = (\mathbb{I} + \varepsilon L)$$

$$\begin{pmatrix} 1/2 & 0 & 0 & 1/2(1-\gamma\varepsilon) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2(1-\gamma\varepsilon) & 0 & 0 & 1/2 \end{pmatrix}$$

$$\frac{1}{4}\left[1 - (1-\gamma\varepsilon)^2\right] \geq 0 \quad \text{(principal minor)}$$

for it to be a valid positive semi definite map.

This is possible only when $\gamma \geq 0$

for Non-Markovian

$\gamma_i(t) < 0$ for some t possible

but $\int_0^T \gamma_i(t)\,dt \geq 0 \quad \forall T$ to preserve overall CPTP condition

The divisibility we saw is called CP-divisibility.

$C_{\mathcal{L}_\varepsilon} = \mathbb{I} \otimes \mathcal{L}_\varepsilon |\psi\rangle\langle\psi|$

$\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ $\rightarrow$ Trace norm

Hilbert-schmidt norm: $\sqrt{\mathrm{Tr}(A^\dagger A)}$

$$g(t) = \lim_{\varepsilon \to 0} \frac{\|C_{\mathcal{L}_\varepsilon}\|_1 - 1}{\varepsilon}$$

RHP measure of Non-Markovianity

$g(t) > 0$ iff not divisible

$g(t) = 0$ iff divisible

$\|A\|_1 = \sum_i |\lambda_i|$, $\lambda_i \rightarrow$ eigenvalue of A

$\mathcal{L}_\varepsilon$ may not be CP but it is trace preserving and Hermiticity preserving

$\mathcal{L}_\varepsilon(\rho) = \rho + \varepsilon\, \mathcal{L}(\rho)$

$\mathrm{Tr}\, \mathcal{L}(\rho) = 0$

$$\mathrm{Tr}[\mathcal{L}(\rho)] = \mathrm{Tr}\left[\sum_i r_i(t)\left(A_i \rho A_i^\dagger - \frac{1}{2}\{A_i^\dagger A_i, \rho\}\right)\right] = 0 \quad \text{(Check)}$$

can be proved using cyclicity of trace $|\ \mathrm{Tr}(AB) = \mathrm{Tr}(BA))$

due to probability conversation, $\mathrm{Tr}(\mathcal{L}(\rho))$ has to be 0.

$\therefore \mathrm{Tr}(\mathcal{L}_\varepsilon(\rho)) = \mathrm{Tr}(\rho + \varepsilon\, \mathcal{L}(\rho)) = \mathrm{Tr}(\rho)$.

for CP divisible map, $\|A\|_1 = \sum |\lambda_i| = \sum \lambda_i = 1$.

$\therefore g(t) = \lim_{\varepsilon \to 0} \frac{1-1}{\varepsilon} = 0$. (for CP divisible)

for non-CP divisible maps, $\sum \lambda_i = 1$, but some λ_i's may be -ve,

$\therefore \sum |\lambda_i| > 1$ & $\therefore g(t) > 0$.

$$I = \int_0^\infty q(t)\, dt$$

$$N = \frac{I}{1+I}, \quad 0 \le N \le 1$$

Detecting Non-Markovianity
(Information Backflow)

CP-TP Monotone

1. Trace distance

$$T(\rho_1, \rho_2) = \frac{1}{2} \| \rho_1 - \rho_2 \|_1$$

under CPTP divisible operations, this will be monotonically decreasing.

$$T(\Lambda(\rho_1), \Lambda(\rho_2)) \le T(\rho_1, \rho_2)$$

2. Relative entropy

$$S(\rho_1, \rho_2) = Tr[\rho_1 \ln(\rho_1) - \rho_1 \ln(\rho_2)] = Tr[\rho_1(\ln(\rho_1) - \ln(\rho_2))]$$ (similar to KL-divergence)

$T \rightarrow$ measurable physically

$$\frac{d}{dt} T(\Lambda(\rho_1), \Lambda(\rho_2)) \le 0$$

$$\sigma = \frac{d}{dt} T(\rho_1, \rho_2) > 0$$

$$\int_{\sigma(t)>0} \sigma(t)\, dt = R$$

$A = \frac{R}{1+R}$ if $A = 0$, it is not showing information backflow

but if $A > 0$, it is non-markovian

If graph monotone $\Rightarrow$ can't say anything about operation

If graph non-monotone $\rightarrow$ Non-Markovian operation vect → map

$|\psi\rangle\langle\psi| \longleftrightarrow$

CP<u>TP</u> → $Tr(\rho(t)) \to$ const

$Tr\left(\frac{d\rho}{dt}\right)$

$|\psi\rangle\langle\psi|$

Thermodynamics

Landi: Irreversible Entropy Production
(2021)

Entropy

Shannon Entropy : $-\sum p_i \log p_i$ for a classical process

In Quantum: $S(\rho) = -Tr(\rho \log \rho) = -\sum_i \lambda_i \log \lambda_i$ if $\rho = \sum \lambda_i |i\rangle\langle i|$ → Von Neumann Entropy

If ρ is pure, $S(\rho) = 1$ if ρ is $I/2$ (maximally entangled) then $S(\rho) = \log 2$ (for qubit)

for d dimen, $S(\rho) = \log d$

Relative Entropy: $S(\rho|\sigma) = Tr(\rho \log \rho - \rho \log \sigma]$

$\text{support}(\rho) \leq \text{support}(\sigma)$

Rel. Entropy is a pseudo distance measure [sym. prop & Triangle]

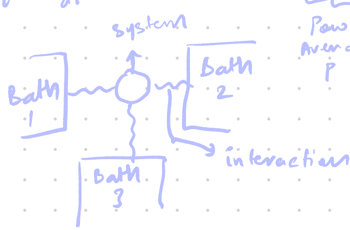
Review paper: Vladko Vedral

1^{st} law of Thermodynamics:

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[\rho(t), H(t)] + \sum \gamma_i \left(L_i \rho L_i^\dagger - \{L_i^\dagger L_i, \rho\}\right)$$

Average Energy: $\frac{dE}{dt} = \frac{d}{dt}\left(Tr(H(t)\rho)\right) = \underbrace{Tr(\dot{H}\rho)}_{\substack{\text{Power}\\ \text{Average}\\ P}} + \underbrace{Tr(H\dot{\rho})}_{J} \Rightarrow \frac{dE}{dt} = -P + J$

→ Heat current



$J_1 = Tr(H L_1 \rho)$

$J_2 = Tr(H L_2 \rho)$

$J_3 = Tr(H L_3 \rho)$

$J = P + \frac{dE}{dt}$

↳ 1^{st} law of thermodynamics

since $\frac{d\rho}{dt} = L_1\rho + L_2\rho_2 + L_3\rho$

$\therefore$ H (Total Heat current) $= H_1 + H_2 + H_3$ (sum of individual Heat current)

Thermal state

Mathematically: $\rho = \frac{e^{-\beta H}}{Z}$, $Z = Tr(e^{-\beta H})$, $\beta = \frac{1}{K_B T}$

↳ Boltzmann constant

Ergodic Hypothesis : Ensemble average = Time Average

$\rho_{th} = \frac{\sum e^{-\beta E_i}|i\rangle\langle i|}{Z}$

$S(\rho) = -Tr(\rho \ln \rho)$

$S(\rho|\sigma) = Tr(\rho \ln \rho - \rho \ln \sigma)$

$\frac{d}{dt}(S(\rho|\rho_{th})) = \frac{d}{dt}(Tr($

2nd Law: $\sigma \geq 0$

$$\sigma = -\frac{d}{dt}(S(\rho|\rho_{th})) = \frac{dS}{dt} + \beta J$$

Thermal operation, $\frac{d\rho_{th}}{dt} = 0$ → steady state

Thermal operation is such an operation under which steady ρ_{th} remains same.

depolarizing operation has only one steady state

But dephasing operation has infinite steady state.

Mutual Information

$$I = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

$$\rho_{A/B} = Tr_{B/A}(\rho_{AB})$$

Unital operations

$\sum_i K_i^\dagger K_i = \mathbb{I} \longrightarrow$ Trace preservation

$\sum_i K_i K_i^\dagger = \mathbb{I} \longrightarrow$ Unital operation
(classical counterpart
bi stochastic process)

$$\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$$

$$\Lambda\left(\frac{\mathbb{I}}{d}\right) = \sum K_i \frac{\mathbb{I}}{d} K_i^\dagger = \frac{\mathbb{I}}{d}\left(\sum K_i K_i^\dagger\right) = \frac{\mathbb{I}}{d} \text{ (for unital)}$$

$\frac{\mathbb{I}}{d} \rightarrow$ fixed point

$$\frac{d}{dt}\left(S\left(\rho \middle| \frac{\mathbb{I}}{d}\right)\right) \leq 0$$

$$S\left(\rho \middle| \frac{\mathbb{I}}{d}\right) = -S(\rho) + \ln d$$

$$\frac{d}{dt}\left(S\left(\rho \middle| \frac{\mathbb{I}}{d}\right)\right) = -\frac{d\,S(\rho)}{dt} \leq 0$$

$\therefore \frac{d\,S(\rho)}{dt} \geq 0 \rightarrow$ for unital op. entropy is monotonically increasing

$\therefore$ There is always information loss since all states gravitate towards maximally mixed states.

Unitary is trivally unital operation since it is entropy preserving and also $\mathbb{I}/d$ is a fixed point.

Thermal channel

$$\Lambda_{th}(\rho_S) = \mathrm{Tr}_E \left[\underbrace{E_{SE}}_{\text{Energy preserving unitary}} \rho_S \otimes \underbrace{\tau_E^{\beta}}_{\text{Thermal state of the bath with inverse temp } \beta = \frac{1}{k_B T}} E_{SE}^{\dagger} \right]$$

$$E = e^{-\frac{i}{\hbar} H_T t}$$

$$H_T = H_S + H_E + H_{SE}$$

Energy preservation:

$$[H_{SE}, H_S + H_E] = [H_{SE}, H_T] = 0$$

$$\Lambda_{th}(\tau_S^{\beta}) = \tau_S^{\beta}$$

thermal state of the same temp as the channel is the fixed point

$$\underbrace{\sigma = -\frac{d}{dt}(\rho_s \,|\, \tau_s^{\beta}) \geq 0}_{\text{spohn's theorem}}$$