For an ergodic process,

$\bar{X}_{time-avg} = \bar{X}_{ensemble}$.

Lecture #13: (Lecture Recording)

Renewal theorem:

N(t) is the counting process associated with the renewal process.

$$L(t) = \begin{cases} \infty & \text{where the event happened} \\ 0 & \text{otherwise} \end{cases}$$

L(t) ~ dirac delta function

Dirac delta function $\delta(x)$

$$\delta(x) \triangleq \begin{cases} +\infty & x=0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

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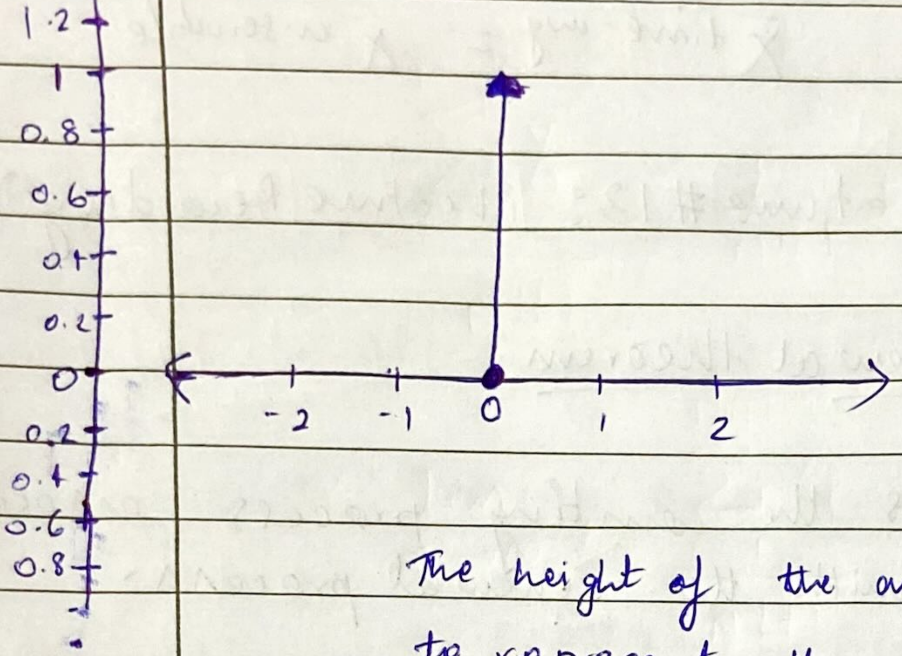
Schematic representation of dirac-delta



The height of the arrow is used to represent the area under the curve/function.

So $L(t)$ consists of ∞ many dirac delta functions summed & shifted from their origin ($t=0$) to other times where the events have happended.

$$\therefore \quad N(t) = \int_0^t L(t)\,dt.$$

$$\bar{X}^{\text{time-avg}} = \lim_{t\to\infty} \frac{N(t,\omega)}{t} = \lim_{t\to\infty} \frac{\int_0^t L(t,\omega)\,dt}{t}$$

Lemma :

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{E(X_1)} \quad \text{almost surely}$$

Meaning, $\lim_{t \to \infty} \frac{N(t, \omega)}{t} = \frac{1}{E(X_1)}$

$\forall \omega \in \Omega$ ~~where~~ such that $P(\omega) = 1$

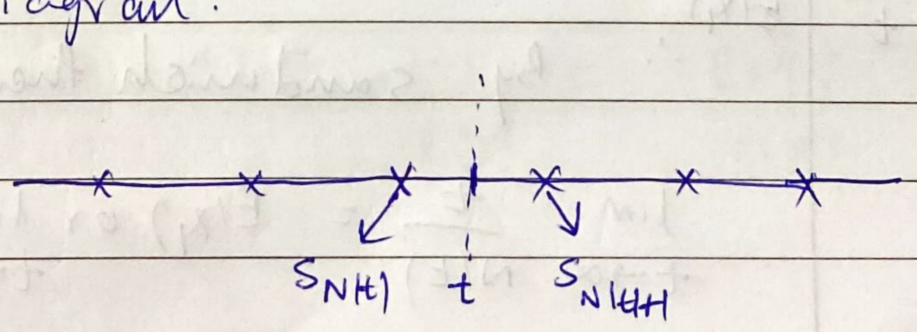
~~i.e. where the time average does~~

i.e. the realization on which the time average does not equal to $\frac{1}{E(X_1)}$, the probability of that realization happening is 0.

Proof: Let S_n be the time of the n^{th} renewal.

Then $S_{N(t)} \leq t < S_{N(t)+1}$

we can see this by the following diagram:



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$$\therefore \sum_{i=1}^{N(t)} s_i \leq t \leq \sum_{i=1}^{N(t)+1} s_i$$

$$\Rightarrow \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \leq \frac{t}{N(t)} \leq \frac{\sum_{i=1}^{N(t)+1} X_i}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

as $\lim_{t \to \infty}$, $\frac{\sum_{i=1}^{N(t)} X_i}{N(t)} = E(X_i) = E(X_1)$

By strong Law of Large Numbers

and similarly,

$$\lim_{t \to \infty} \frac{\sum_{i=1}^{N(t)+1} X_i}{N(t)+1} = E(X_i) = E(X_1)$$

$$\frac{N(t)+1}{N(t)} \to 1 \text{ as } t \to \infty$$

$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{E(X_1)}$$

$\therefore$ By sandwich theorem,

$$\lim_{t \to \infty} \frac{t}{N(t)} = E(X_1) \text{ or } \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{E(X_1)}$$

Renewal Reward Theorem

→ Consider a renewal process with interarrival time X_i's and suppose a random reward Y_i is earned at the time of the i^{th} arrival.

Let $Y(t)$ be the total reward earned till time t. Then

$$Y(t) = \sum_{i=1}^{N(t)} Y_i$$

Lemma:

$$\lim_{t \to \infty} \frac{Y(t)}{t} = \frac{E(Y)}{E(X)} \text{ almost surely.}$$

Proof:

$$\lim_{t \to \infty} \frac{Y(t)}{t} = \lim_{t \to \infty} \frac{Y(t)}{N(t)} \cdot \frac{N(t)}{t}$$

$$\Rightarrow \lim_{t \to \infty} \frac{\sum_{i=1}^{N(t)} Y_i}{N(t)} \cdot \lim_{t \to \infty} \frac{N(t)}{t}$$

$$\Rightarrow E(Y) \cdot \frac{1}{E(X)} = \frac{E(Y)}{E(X)}.$$