Open quantum systems and quantum thermodynamics

Mid-Sem 2022

Full marks 50 (10 × 5)

1. Consider the following operation

$$|0\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |1\rangle \to \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

with $|0\rangle = (1\ ,0)^T$ and $|1\rangle = (0\ ,1)^T$ (T is transposition). Find (i) the corresponding unitary matrix, (ii) the unitary matrix for the inverse operation.

2. Find the following matrices.

$$exp\,(t(|0\rangle\langle 1| + |1\rangle\langle 0|)) \text{ and } exp\,(t(|0\rangle\langle 1|)\,exp\,(t(|1\rangle\langle 0|)\,.$$

$|0\rangle\,,|1\rangle$ are the same as the previous questions. Are the two matrices same or not ?

3. Consider the following unitary transformation for a two qubit bi-partite system :

$$U_{AB}\,|0\ 0\rangle \to \frac{|0\ 0\rangle + |1\ 1\rangle}{\sqrt{2}}\ ;\ U_{AB}\,|1\ 1\rangle \to \frac{|0\ 0\rangle - |1\ 1\rangle}{\sqrt{2}}$$
$$U_{AB}\,|0\ 1\rangle \to \frac{|0\ 1\rangle + |1\ 0\rangle}{\sqrt{2}}\ ;\ U_{AB}\,|1\ 0\rangle \to \frac{|0\ 1\rangle - |1\ 0\rangle}{\sqrt{2}}.$$

Find the Kraus operators for the following CPTP map : $\Lambda(\rho_A) = Tr_B[U_{AB}\rho_A \otimes \rho_B U_{AB}^\dagger]$ with $\rho_B = |1\rangle\langle 1|$.

4. Check whether the following matrix is positive or not

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

$\begin{pmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \Big($

5. Check the following operation is complete positive or not

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \to \begin{pmatrix} i & f & c \\ h & e & b \\ g & d & a \end{pmatrix}.$$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \Rightarrow \frac{1}{2}\Big[|00\rangle\langle 00| + |00$

$+ |11\rangle\langle 00| +$

$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$

$|00\rangle = ($

$= ($

$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$|\psi\rangle\langle\psi|$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\ \frac{\langle 0| + \langle 1|}{\sqrt{2}}\Big)$

$|0\rangle\langle 1|1\rangle\langle 0| = |0\rangle\langle 0|$

$|1\rangle\langle 0|0\rangle\langle 1| = |1\rangle\langle 1|$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$|00\rangle + |11\rangle + |22\rangle$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

$$|\phi\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\0\\1\end{pmatrix}$$

$$|\phi\rangle\langle\phi| = \frac{1}{3}\begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix}$$

$$-\lambda(-\lambda(1-\lambda)) + 1(0 - 1 + \lambda)$$

$$\lambda^2(1-\lambda) + (\lambda - 1) = 0$$

$$(1-\lambda)(\lambda^2 - 1) = 0$$

$$\boxed{\lambda = 1, -1, 1}$$

$\therefore$ not a complete +ve map.

3. Consider the following unitary transformation for a two qubit bi-partite system :

$$U_{AB}|0\ 0\rangle \to \frac{|0\ 0\rangle+|1\ 1\rangle}{\sqrt{2}}\ ;\ U_{AB}|1\ 1\rangle \to \frac{|0\ 0\rangle-|1\ 1\rangle}{\sqrt{2}}$$
$$U_{AB}|0\ 1\rangle \to \frac{|0\ 1\rangle+|1\ 0\rangle}{\sqrt{2}}\ ;\ U_{AB}|1\ 0\rangle \to \frac{|0\ 1\rangle-|1\ 0\rangle}{\sqrt{2}}.$$

Find the Kraus operators for the following CPTP map : $\Lambda(\rho_A) = Tr_B[U_{AB}\rho_A \otimes \rho_B U_{AB}^\dagger]$ with $\rho_B = |1\rangle\langle 1|$.

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$
$$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

U_{AB}

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$|01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$|10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$|11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$U_{AB} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\\0 & 1 & -1 & 0\\1 & 0 & 0 & -1\end{pmatrix}$$

$$\rho_A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\rho_B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_A \otimes \rho_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

$$Tr_B \left(\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \right)$$

$$Tr_B \left(\frac{1}{2} \begin{pmatrix} 0 & c & 0 & d \\ 0 & a & 0 & b \\ 0 & a & 0 & b \\ 0 & -c & 0 & -d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \right)$$

$$Tr_B \left(\frac{1}{2} \begin{pmatrix} d & c & c & -d \\ b & a & a & -b \\ b & a & a & -b \\ -d & -c & -c & d \end{pmatrix} \right)$$

$$\Lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+d}{2} & \frac{c-b}{2} \\ \frac{b-c}{2} & \frac{a+d}{2} \end{pmatrix}$$

$$\begin{pmatrix} \Lambda\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \\ \Lambda\begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4}-\lambda & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4}-\lambda \end{pmatrix}$$

$$\left(\frac{1}{4}-\lambda\right)^2 = \left(\frac{1}{4}\right)^2$$

$$\frac{1}{4}-\lambda = \pm\frac{1}{4}$$

$$\lambda_3, \lambda_4 = 0, \frac{1}{2}$$

$$\left(\frac{1}{4}-\lambda\right)^2 = \left(\frac{1}{4}\right)^2$$

$$\frac{1}{4}-\lambda = \pm\frac{1}{4}$$

$$\lambda = \frac{1}{4} \mp \frac{1}{4}$$

$$\boxed{\lambda_1, \lambda_2 = 0, \frac{1}{2}}$$

$$\begin{pmatrix} -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$c, d \longrightarrow \text{free}$

$$\begin{pmatrix} -d \\ c \\ c \\ d \end{pmatrix}$$

$a = -d$

$b = c$

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$K_1 = \sqrt{\frac{1}{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad K_2 = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$K_1^\dagger K_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$