1.1

Performance Modelling for Computer System - Assignment 2

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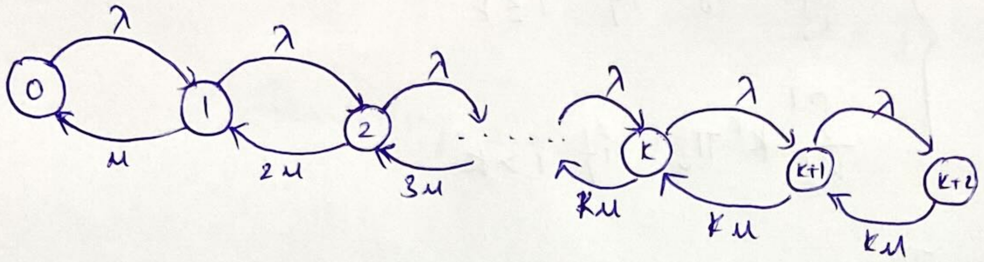
1. The state-space diagram of the given $M/M/K/\infty$ queue is:



The time-reversibility equations are as follows:

State	Time-Reversibility Eqn	Simplified Equation.
0	$\pi_0 \lambda = \pi_1 \mu$	$\pi_1 = \frac{\lambda}{\mu} \pi_0$
1	$\pi_1 \lambda = \pi_2 (2\mu)$	$\pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{2!} \pi_0$
2	$\pi_2 \lambda = \pi_3 (3\mu)$	$\pi_3 = \left(\frac{\lambda}{\mu}\right)^3 \frac{1}{3!} \pi_0$
K-1	$\pi_{K-1} \lambda = \pi_K (K\mu)$	$\pi_K = \left(\frac{\lambda}{k}\right)^K \frac{1}{K!} \pi_0$
K	$\pi_K \lambda = \pi_{K+1} (K\mu)$	$\pi_{K+1} = \left(\frac{\lambda}{\mu}\right)^{K+1} \left(\frac{1}{K!}\right) \cdot \frac{1}{K} \pi_0$
K+1	$\pi_{K+1} \lambda = \pi_{K+2} (K\mu)$	$\pi_{K+2} = \left(\frac{\lambda}{\mu}\right)^{K+2} \left(\frac{1}{K!}\right) \cdot \frac{1}{K^2} \cdot \pi_0.$

$$\therefore \pi_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0 & \text{if } i \le K \\ \left(\frac{\lambda}{\mu}\right)^i \frac{1}{K!} \cdot \left(\frac{1}{K}\right)^{i-K} \pi_0 & \text{if } i > K \end{cases}$$

1.2 Now, define $\rho = \frac{\lambda}{k\mu}$ as the system utilization.

So, π_i written in terms of ρ will be

$$\pi_i = \begin{cases} \frac{(k\rho)^i}{i!} \pi_0 & \text{if } i \le k \\ \frac{\rho^i}{k!} k^k \pi_0 & \text{if } i > k \end{cases}$$

To determine π_0:

$$\left[\sum_{i=0}^{k-1} \pi_i + \sum_{i=k}^{\infty} \frac{\rho^i}{k!} k^k \right] = 1$$

$$\Rightarrow \pi_0 \left[\sum_{i=0}^{k-1} \frac{(k\rho)^i}{i!} + \sum_{i=k}^{\infty} \frac{\rho^i}{k!} \cdot k^k \right] = 1$$

$$\Rightarrow \pi_0 \left[\sum_{i=0}^{k-1} \frac{(k\rho)^i}{i!} + \frac{k^k}{k!} \cdot \frac{\rho^k}{1-\rho} \right] = 1$$

$$\therefore \pi_0 = \left[\sum_{i=0}^{k} \frac{(k\rho)^i}{i!} + \frac{(k\rho)^k}{k!(1-\rho)} \right]^{-1}.$$

Next, we find the queueing probability P_Q, ~~and~~ expected probability of number of jobs in the queue $E(N_Q)$, Expected ~~number of~~ time spent in queue $E(T_Q)$ and Expected Response time $E(T)$.

All of these will help us in finding the mean number of jobs in the system, $E(N)$.

1.3

Queueing Probability P_Q:

$$P_Q = P\{\text{An arrival finds all servers busy}\}$$

$$= P\{\text{An arrival sees} \geq k \text{ jobs in the system}\}$$

$= \text{limiting probability that there are} \geq k \text{ jobs in the system.}$

$$= \sum_{i=k}^{\infty} \pi_i$$

$$= \frac{k^k}{k!}\pi_0 \sum_{i=k}^{\infty} \rho^i$$

$$= \frac{(k\rho)^k}{k!\,(1-\rho)}\pi_0, \text{ where } \pi_0 = \left[\sum_{i=0}^{k-1} \frac{(k\rho)^i}{i!} + \frac{(k\rho)^k}{k!\,(1-\rho)}\right]^{-1}.$$

Expected Number in the Queue $(E(N_Q))$:

$$E(N_Q)^{M/M/k} = \sum_{i=k}^{\infty} \pi_i (i-k)$$

$$= \pi_0 \sum_{i=k}^{\infty} \frac{\rho^i k^k}{k!}(i-k)$$

$$= \pi_0 \cdot \frac{\rho^k k^k}{k!} \cdot \sum_{i=0}^{\infty} \rho^i \cdot i$$

$$= \pi_0 \frac{\rho^k k^k}{k!} \cdot \frac{\rho}{(1-\rho)^2}$$

$$= \cancel{\rho_Q\,\pi_0}\; P_Q \cdot \frac{\rho}{1-\rho}.$$

1.4

Now,

$$E(T_Q) = \frac{1}{\lambda} \cdot E(N_Q) = \frac{1}{\lambda} \cdot P_Q \cdot \frac{\rho}{1-\rho}$$

$$E(T) = E(T_Q) + \frac{1}{\mu} = \frac{1}{\mu} + \frac{1}{\lambda} \cdot \frac{\rho}{1-\rho} \cdot P_Q.$$

$$E(N) = \lambda \cdot E(T) = P_Q \cdot \frac{\rho}{1-\rho} + K\rho$$

$\therefore$ Expected No. of jobs

$$E(N) = K\rho + P_Q \frac{\rho}{1-\rho} = \text{mean no. of jobs in the system.}$$

we see that as $\rho \to 1^{-}$, $E(N) \to \infty$

$\therefore$ For the system to be ~~sg~~ stable,

$$\rho < 1 \quad \text{or} \quad \frac{\lambda}{K\mu} < 1$$

or $\boxed{\lambda < K\mu}$ $\rightarrow$ stability condition.

2.1

2. M/M/2/∞ queue:

Inter-arrival ~~time~~ rate ~ $Exp(\lambda)$

service rate for server 1 ~ $Exp(\mu_1)$

Service rate for server 2 ~ $Exp(\mu_2)$

We let our descriptor be: $[x, y, k]$
where

x: State of server 1 (binary: 0 → Idle, 1 → Busy).

y: State of server 2 (binary: 0 → Idle, 1 → busy).

k: No. of jobs in the queue [No. of jobs is system − No. of jobs being serviced].

We will map our descriptor using the the following mapping:

$[x, y, k] \longrightarrow$ State $(2x+y+k)$.

∴ The map looks as follows:

$[0,0,0] \rightarrow$ state 0

$[0,1,0] \rightarrow$ state 1

$[1,0,0] \rightarrow$ state 2

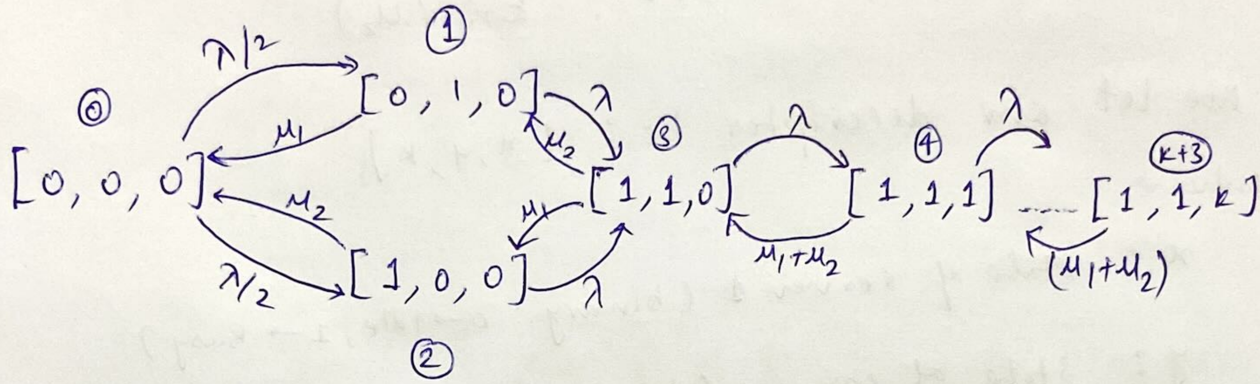
$[1,1,0] \rightarrow$ state 3

$[1,1,1] \rightarrow$ state 4

$[1,1,k] \rightarrow$ state k.

2.2

<u>ASSUMING</u> when the first job comes, it gets randomly assigned either to server 1 or 2, the state space diagram of our Markov chain will look as follows:



Looking at the Markov Chain, we get the Q-Matrix as following:

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \cdots \end{array} \\ \left[\begin{array}{ccccc} -\lambda & \lambda/2 & \lambda/2 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda) & 0 & & \lambda \\ \mu_2 & 0 & -(\mu_2+\lambda) & \lambda & 0 \cdots \\ 0 & \mu_1 & \mu_2 & -(\mu_1+\mu_2+\lambda) & \lambda \cdots \\ & & & & \end{array}\right. \end{array}$$

The dimensions of Q ~~are~~ is $|\mathbb{N} \times \mathbb{N}| = \aleph_0$

Now, using Time Reversibility Equations, we get:

State	Time Reversibility Eqn.	Simplified Eqn.
1	$\frac{\lambda}{2}\pi_0 = \mu_1\pi_1$	$\pi_1 = \frac{\lambda}{2\mu_1}\pi_0$
2	$\frac{\lambda}{2}\pi_0 = \mu_2\pi_2$	$\pi_2 = \frac{\lambda}{2\mu_2}\pi_0$
3	$\lambda\pi_1 = \mu_2\pi_3$	$\pi_3 = \frac{\lambda}{\mu_2}\pi_1 = \frac{\lambda^2}{2\mu_1\mu_2}\pi_0$

2.3

From the last equation, we get

$$\pi_n = \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{n-3} \cdot \left(\frac{\lambda^2}{2\mu_1\mu_2}\right) \cdot \pi_0 \quad n \geq 3$$

To get π_0, we have

$$\sum_{i=0}^{\infty} \pi_i = 1$$

$$\Rightarrow \pi_0 + \pi_1 + \pi_2 + \sum_{i=3}^{\infty} \pi_i = 1$$

$$\Rightarrow \pi_0 \left[1 + \frac{\lambda}{2\mu_1} + \frac{\lambda}{2\mu_2} + \sum_{i=3}^{\infty} \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{i-3} \cdot \frac{\lambda^2}{2\mu_1\mu_2} \right] = 1$$

$$\Rightarrow \pi_0 \left[1 + \frac{\lambda}{2\mu_1} + \frac{\lambda}{2\mu_2} + \frac{\lambda^2}{2\mu_1\mu_2} \cdot \frac{1}{1 - \frac{\lambda}{\mu_1+\mu_2}} \right] = 1$$

$$\Rightarrow \pi_0 \left[1 + \frac{\lambda}{2\mu_1} + \frac{\lambda}{2\mu_2} + \frac{\lambda^2}{2\mu_1\mu_2} \left[\frac{\mu_1+\mu_2}{\mu_1+\mu_2-\lambda} \right] \right] = 1$$

$$\Rightarrow \pi_0 = \left[1 + \frac{\lambda}{2\mu_1} + \frac{\lambda}{2\mu_2} + \frac{\lambda^2}{2\mu_1\mu_2} \left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2-\lambda} \right) \right]^{-1}$$

$\therefore$ The stationary distribution is:

$$\pi_i = \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{i-3} \cdot \frac{\lambda^2}{2\mu_1\mu_2} \cdot \pi_0 , \quad i \geq 3$$

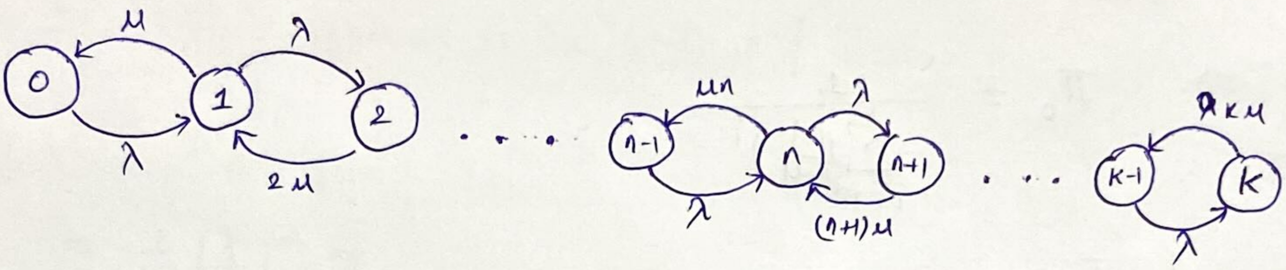
$$\pi_1 = \frac{\lambda}{2\mu_1} \pi_0 , \quad \pi_2 = \frac{\lambda}{2\mu_2} \pi_0$$

$$\pi_3 = \frac{\lambda^2}{2\mu_1\mu_2}$$

= Ans

3.1

3. The state space diagram for an M/M/K/k queue is as follows:



The Global balance equation for M/M/K/K will be:

$$\pi_n(\lambda + n\mu) = \pi_{n-1}\cdot\lambda + \pi_{n+1}(n+1)\mu, \text{ where } 0 < n < K \rightarrow (i).$$

The local balance equation will be (for state 0):

$$\pi_0 \cdot \lambda = \pi_1 \cdot \mu.$$

$$\therefore \pi_1 = \frac{\lambda}{\mu}\pi_0. \rightarrow (ii).$$

$$\pi_1(\lambda+\mu) = \pi_0\lambda + \pi_2(2\mu) \qquad [n=1 \text{ in } (i)]$$

$$\pi_1(\lambda+\mu) = \pi_1\mu + \pi_2(2\mu) \qquad [ii]$$

$$\pi_1 \cdot \lambda = \pi_2(2\mu)$$

$$\therefore \pi_2 = \frac{\lambda}{2\mu}\cdot\pi_1!$$

similarly, for π_i, we find

$$\pi_i = \frac{\lambda}{i\mu}\cdot\pi_{i-1}. \qquad i \le K.$$

$$\therefore \pi_i = \left(\frac{\lambda}{\mu}\right)^i \cdot \frac{1}{i\cdot(i-1)\cdots 1}\cdot\pi_0 = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\pi_0.$$

3.2

Now,

$$\sum_{i=0}^{K} \pi_i = 1$$

$$\Rightarrow \sum_{i=0}^{K} \frac{1}{i!} \cdot \left(\frac{\lambda}{\mu}\right)^i \pi_0 = 1$$

$$\Rightarrow \pi_0 = \frac{1}{\sum_{i=0}^{K} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i}.$$

Now, there is no closed form for the expression $\sum_{i=0}^{K} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i$, but we can express it as follows:

$$\sum_{i=0}^{K} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i = \frac{\Gamma(K+1, \lambda/\mu) \cdot e^{\lambda/\mu}}{\Gamma(K+1)}$$

where $\Gamma(x, y)$ is the incomplete Gamma function

and $\Gamma(x,y) = \int_y^\infty e^{-t}\, t^{x-1}\, dt$. and $\Gamma(x)$ is the classical Gamma function where $\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x-1}\, dt.$

and $\Gamma(x) = (x-1)!$ when $x \in \mathbb{N} \cup \{0\}$.

$$\therefore \pi_0 = \frac{\Gamma(K+1)}{\Gamma(K+1, \lambda/\mu)} e^{-\lambda/\mu}$$

$\therefore$ The stationary distribution for M/M/K/K is

$$\pi_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i \cdot \pi_0 \quad \forall\ i \le K.$$

3.3

The blocking probability = Probability that the arriving jobs ~~are~~ get blocked

= Probability that our system has k servers busy

= $P\{\text{system is in state } k\}$

= π_k

= $\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \cdot \pi_0$.

~~Q3A~~ 4.1

4. Given that $N(t) = \max(A(t) - D(t), 0)$,

To prove: $N(t)$ is a Markov Chain, where $A(t)$ = ~~Arrivals~~ Arrivals till time t & $D(t)$ = Departures till time t.

Consider the quantity / Probability:

$$P\{ N(t_{k-1}+\varepsilon) = j / N(0), N(t_1), \ldots, N(t_{k-1}) = l) \}$$

for some $\varepsilon > 0$ and $\forall_i$ $0 < t_i < \infty$ & $i = 0, 1, 2, \ldots, k-1$.

Case 1: If $j = l-1$ => in duration ε, the job currently recieving service finished ~~serv~~ being serviced.

$\therefore$ P(Remaining service for currect job $\le \varepsilon$)

= P(service time of a job $\le \varepsilon$)

This is the memoryless property ~~of~~, which satisfied only by exponential distribution.

$\therefore$ P(Service time of a job $\le \varepsilon$) = ~~$\mu e^{-\mu t}$~~ $\int_0^{\varepsilon} \mu e^{-\mu t} dt$

Case 2: If $j = l$,

This means there were no arrivals & departure in ε time

$\therefore$ $P\{$ no arrival & no departure in $\varepsilon\} = \int_0^{\varepsilon} (1-\lambda\varepsilon) e^{\mu\varepsilon} dt$.

Case 3: If $j = l+1$, then in ε time there is one arrival.

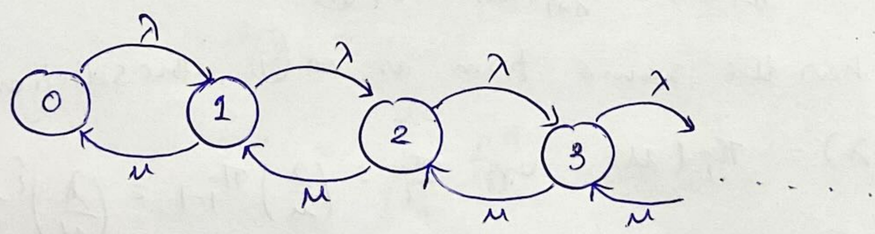
$\therefore$ $P\{$ one arrival in time $\varepsilon\} = \lambda\varepsilon$

$$\therefore P\{N(t_{k-1}+\varepsilon)=j / N(t_0), N(t_1), \cdots, N(t_{k-1})=l\}$$

$$= \begin{cases} \int_0^{\varepsilon} \mu e^{-\mu t} dt & \text{if } j = l-1 \\ \lambda \varepsilon & \text{if } j = l+1 \\ \int_0^{\varepsilon} (1-\lambda t) \cdot e^{\lambda t} dt & \text{if } j = l \end{cases}$$

$\therefore$ $N(t)$ is a Markovian Descriptor for $M/M/1/\infty$ queue.

Now, for an $M/M/1/\infty$ queue, the state space diagram is as follows:



looking the diagram, we see that:

$q_{01} = \lambda$, $q_{12} = \lambda$, $q_{23} = \lambda \ldots$

$q_{10} = \mu$, $q_{21} = \mu$, $q_{32} = \mu \ldots$

Assuming Q is conservative, we get:

$q_{00} = -\lambda$

$q_{11} = -(\lambda+\mu)$ $\qquad q_{ii} = -(\lambda+\mu)$ for $i > 0$ & $i \in \mathbb{Z}$.

$q_{22} = -(\lambda+\mu)$

4.13

$$\therefore\ Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline 0 & -\lambda & \lambda & 0 & 0 & 0 \\ 1 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 \\ 2 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 \\ 3 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda \\ 4 & 0 & 0 & 0 & \mu & -(\lambda+\mu) \\ \vdots & & & & & \end{array}$$

where the dimentions of Q are $|\mathbb{N} \times \mathbb{N}| = \aleph_0$.

To find stationary distribution, we find the balance equations for the first few states :

$$\pi_0(\lambda) = \pi_1(\mu) \quad \rightarrow (i)$$

$$\pi_1(\lambda+\mu) = \pi_0(\lambda) + \pi_2(\mu) \quad \rightarrow (ii).$$

$$\Rightarrow \pi_1 \lambda = \pi_2(\mu) \quad \rightarrow \{\text{using } (i) \ \& \ (ii)\}$$

The Global Balance equation is as follows:

$$\pi_n(\lambda+\mu) = \pi_{n-1}(\lambda) + \pi_{n+1}(\mu) \quad \rightarrow (iii)$$

and since it has the same form as (ii), the solution will be:

$$\pi_{i-1}(\lambda) = \pi_i(\mu). \quad \text{or} \quad \pi_i = \left(\frac{\lambda}{\mu}\right)\pi_{i-1} = \left(\frac{\lambda}{\mu}\right)^i \pi_0.$$

Now, since $\sum_{i=0}^{\infty} \pi_i = 1$

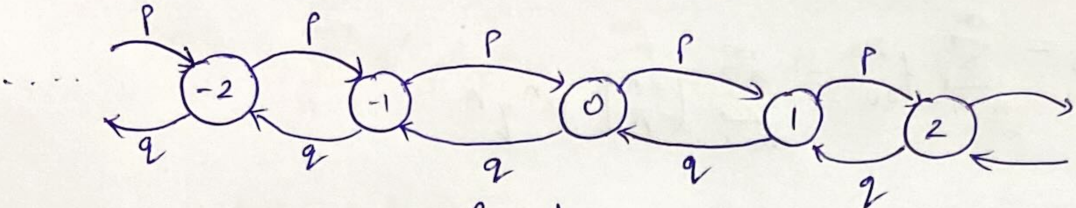
$$\sum_{i=0}^{\infty} \pi_i = \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \pi_0 = 1$$

$$= \frac{1}{1-\frac{\lambda}{\mu}} \pi_0 = 1 \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu}.$$

$$\therefore \pi_i = \left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right).$$

5.1

5. The state diagram for the described random walk is:



Random walk.

In the diagram, $q = 1-p$.

Since all the states are accessible to each other / they communicate, either all states are recurrent or all states are transient.

∴ To determine whether state 0 is transient or recurrent should suffice to determine whether the chain as a whole is recurrent or not.

For that, we need to calculate

$$F_{ii} = \sum_{i=1}^{\infty} f_{ii}^{n} \quad \text{for } i = 0$$

where ~~f_{00}~~ f_{ii}^{n} = Probability of starting in i and returning to state i ~~for the first time at n~~ in n-steps

$$\therefore F_{00} = \sum_{i=1}^{\infty} f_{00}^{n}$$

→ ~~We cannot~~ If F_{00} is finite, then the chain is transient otherwise recurrent.

5.2

Since we cannot go from state 0 to itself in odd number of steps, it follows that

$$F_{00} = \sum_{n=1}^{\infty} f_{00}^{2n} = \sum_{n=1}^{\infty} \binom{2n}{n} \cdot p^n q^n.$$

Consider,

$$\sum_{k=0}^{2n} \binom{2n}{k} = (1+1)^{2n} = 2^{2n} = 4^n.$$

Since the last term cannot be greater than its sum, we have

$$\binom{2n}{n} < 4^n$$

and since the last term always greater than the average we have

$$\frac{4^n}{2n+1} < \binom{2n}{n}.$$

$$\therefore \quad \frac{4^n}{2n+1} < \binom{2n}{n} < 4^n.$$

This is Misha/Lavrov's Lemma.

$$\therefore \quad \sum_{n=1}^{\infty} \frac{4^n}{(2n+1)} p^n q^n < F_{00} < \sum_{n=1}^{\infty} 4^n p^n q^n.$$

When $p = q = 1/2$, $LHS = \sum_{n=1}^{\infty} \frac{1}{2n+1}$ = Harmonic series $= \infty$

~~But if~~ $\cancel{p \neq q}$ $\therefore$ when $p = q = 1/2$, $F_{00} = \infty$ & the chain is recurrent.

But if $p \neq q$, then $\frac{p+q}{2} < \sqrt{pq} \Rightarrow \frac{1}{2} < \sqrt{pq}$ or $pq < \frac{1}{4}$.

$\therefore$ $RHS = \sum_{n=1}^{\infty} 4^n p^n q^n = \sum_{n=1}^{\infty} (4pq)^n = \frac{4pq}{1-4pq}$ since $4pq < 1$.

$\therefore$ $RHS =$ finite.

$\therefore$ when $p \neq q$, F_{00} is finite and therefore the chain would be transient.

The proof of the mean recurrence time of state 0 is given ~~at~~ in end, on page 5.4.

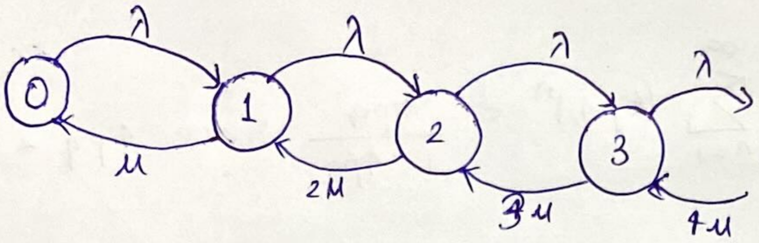
Now, since the mean recurrence time of state 0 is infinite, this means that state 0 is null recurrent.
And since state 0 is null recurrent, and since every state communicates with each other, they too are null recurrent. (Theorem 9.22 Mor Balter, pg. 162)

This implies $\forall\, i \in \mathbb{Z}$, $m_{ii} =$ mean recurrence time of state $i = \infty$.

$\therefore$ The chain is null recurrent and hence the mean recurrence time for the chain is infinite.

6.1

6. The state space diagram for M/M/∞ queue is



The local balance equation for state 0 is:

$$\pi_0(\lambda) = \pi_1(\mu) \quad \rightarrow (i)$$

The Global balance equation for state i, where $i > 0$ is:

$$\pi_i(\lambda + \mu i) = \pi_{i-1} \cdot \lambda + \pi_{i+1}((i+1)\mu) \quad \rightarrow (ii)$$

solving it for $i = 1, 2, \ldots$ we get the value of π_i as:

$$\pi_i = \frac{\lambda}{i\mu} \cdot \pi_{i-1} \quad \rightarrow (iii)$$

Example: For $i = 1$

$$\pi_1(\lambda + \mu) = \pi_0 \cdot \lambda + \pi_2 \cdot (2\mu)$$

$$\pi_1(\lambda + \mu) = \pi_1 \mu + \pi_2(2\mu) \quad [\text{using } (i)]$$

$$\lambda \pi_1 \cancel{\pi_1(\mu)} = 2\mu \pi_2$$

$$\Rightarrow \pi_2 = \cancel{2\mu} \frac{\lambda}{2\mu} \pi_1$$

similarly, we get/do this for all $i = 1, 2, \ldots$

$$\therefore \pi_i = \frac{\lambda}{i\mu} \cdot \pi_{i-1}$$

$$\therefore \pi_i = \left(\frac{\lambda}{i\mu}\right)\left(\frac{\lambda}{(i-1)\mu}\right) \cdot \left(\frac{\lambda}{(i-2)\mu}\right) \cdot \left(\frac{\lambda}{(i-3)\mu}\right) \cdots \left(\frac{\lambda}{\mu}\right) \pi_0.$$

$$= \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i \cdot \pi_0.$$

6.2

Since $\sum \pi_i = 1$, we get:

$$\sum_{i=0}^{\infty} \pi_i = 1$$

$$\Rightarrow \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i \pi_0 = 1$$

$$\Rightarrow \pi_0 \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i = 1$$

$$\Rightarrow \pi_0 \, e^{\lambda/\mu} = 1 \Rightarrow \boxed{\pi_0 = e^{-\lambda/\mu}}$$

$$\therefore \pi_i = \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i \cdot e^{-\lambda/\mu}$$

Now, since X is number of jobs at stationarity and since we know $\pi_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i e^{-\lambda/\mu}$, we know

$$X \sim \text{Poisson}\left(\frac{\lambda}{\mu}\right).$$

$\therefore$ $E(X) = \frac{\lambda}{\mu}$ = mean no. of jobs at stationarity.

since expectation of Poisson RV with parameter λ is λ.

5.4

Proof $M_{ii} = \infty$

The mean recurrence time to revisit ~~stat~~ state i from itself is infinite. i.e. $\mu_{ii} = \infty \ \forall\, i \neq 0, i \in \mathbb{Z}$

To prove this, we use the argument/Theorem used in Mor Balter Book on page 163, which is as follows:

Theorem: (9.23): For the symmetric random walk with $p = 1/2$, $m_{00} = \infty$, where m_{00} = mean time between visits to state 0.

Proof by contradiction: Assume $m_{00} \neq \infty$.

~~Then~~ Here, we define $m_{i,j}$ = mean time until we visit state j from i.

Now,

$$m_{00} = \frac{1}{2} m_{1,0} + \frac{1}{2} m_{-1,0} + 1$$

$$\therefore \quad m_{1,0} \neq \infty.$$

$$\therefore \quad m_{1,0} = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} m_{2,0}$$

$$= 1 + \frac{1}{2} m_{2,0}.$$

since $m_{2,1} = m_{1,0}$ and $m_{2,0} = m_{2,1} + m_{1,0}$

$$\therefore \quad m_{2,0} = 2 m_{1,0}$$

$$\therefore \quad m_{1,0} = 1 + \frac{1}{2}(2 m_{1,0})$$

$$m_{1,0} = 1 + m_{1,0}$$

~~which~~ which is a contradiction.

$\therefore$ $m_{0,0}$ is infinite