Tutorial 3 Solutions

Continuous Random Variables and MGF

Q1.

First, we note that $R_Y = [0, \infty)$. For $y \in [0, \infty)$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|} dx \\ &= \int_0^{\sqrt{y}} e^{-x} dx \\ &= 1 - e^{-\sqrt{y}}. \end{aligned}$$

Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Q2.

a. To find c, we can use $\int_{-\infty}^{\infty} f_X(u) du = 1$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(u) du \\ &= \int_{-1}^{1} cu^2 du \\ &= \frac{2}{3} c. \end{aligned}$$

Thus, we must have $c = \frac{3}{2}$.

b. To find EX, we can write

$$\begin{aligned} EX &= \int_{-1}^{1} u f_X(u) du \\ &= \frac{3}{2} \int_{-1}^{1} u^3 du \\ &= 0. \end{aligned}$$

In fact, we could have guessed $EX = 0$ because the PDF is symmetric around $x = 0$. To find Var(X), we have

$$\begin{aligned} \mathrm{Var}(X) &= EX^2 - (EX)^2 = EX^2 \\ &= \int_{-1}^{1} u^2 f_X(u) du \\ &= \frac{3}{2} \int_{-1}^{1} u^4 du \\ &= \frac{3}{5}. \end{aligned}$$

c. To find $P(X \geq \frac{1}{2})$, we can write

$$P(X \geq \frac{1}{2}) = \frac{3}{2} \int_{\frac{1}{2}}^{1} x^2 dx = \frac{7}{16}.$$

Q3.

(a) Find c.

Solution. We have to solve for c:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{\infty} cxe^{-x}dx = 1.$$

We use integration by parts, letting $u = x$, and $dv = e^{-x}\,dx$, making $du = dx$ and $v = -e^{-x}$ to obtain

$$c\left((x)(-e^{-x})\big|_2^{\infty} - \int_2^{\infty}(-e^{-x})\,dx\right).$$

We use L'Hopital's rule to evaluate the limit $\lim_{x\to\infty} -\frac{x}{e^{-x}} = \lim_{x\to\infty}\frac{1}{e^{-x}} = 0$. Thus

$$c\left((x)(-e^{-x})\big|_2^{\infty} + \int_2^{\infty}(e^{-x})\,dx\right) = 2e^{-2} - e^{-x}\big|_2^{\infty} = c(2e^{-2}+e^{-2}).$$

Therefore, $c = \frac{e^2}{3}$.

(b) Find $E[X]$.

Solution. We use integration by parts:

$$\begin{aligned}
E[X] &= \int_2^{\infty}\frac{e^2}{3}x^2e^{-x}\,dx \qquad u = \frac{e^2}{3}x^2, dv = e^{-x}\,dx, du = \frac{2}{3}e^2x\,dx, v = -e^{-x}\\
&= -\frac{e^2}{3}x^2e^{-x}\Big|_2^{\infty} + \int_2^{\infty}\frac{2}{3}e^2xe^{-x}\,dx\\
&= \frac{e^2}{3}\cdot 4e^{-2} + \frac{2e^2}{3}(-xe^{-x}\big|_2^{\infty} + \int_2^{\infty} e^{-x}\,dx)\\
&= \frac{4}{3} + \frac{4}{3} - \frac{2e^2}{3}e^{-x}\Big|_2^{\infty} = \frac{10}{3}
\end{aligned}$$

(Watch your signs! I didn't write out each sign step).

Q4.

If $Y \sim Geometric(p)$ and $q = 1 - p$, then

$$\begin{aligned} P(Y \leq n) &= \sum_{k=1}^{n} pq^{k-1} \\ &= p.\, \frac{1-q^n}{1-q} = 1 - (1-p)^n. \end{aligned}$$

Then for any $y \in (0, \infty)$, we can write

$$P(Y \leq y) = 1 - (1-p)^{\lfloor y \rfloor},$$

where $\lfloor y \rfloor$ is the largest integer less than or equal to y. Now, since $X = Y\Delta$, we have

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P\left(Y \leq \frac{x}{\Delta}\right) \\ &= 1 - (1-p)^{\lfloor \frac{x}{\Delta} \rfloor} = 1 - (1 - \lambda\Delta)^{\lfloor \frac{x}{\Delta} \rfloor}. \end{aligned}$$

Now, we have

$$\begin{aligned} \lim_{\Delta \to 0} F_X(x) &= \lim_{\Delta \to 0} 1 - (1 - \lambda\Delta)^{\lfloor \frac{x}{\Delta} \rfloor} \\ &= 1 - \lim_{\Delta \to 0} (1 - \lambda\Delta)^{\lfloor \frac{x}{\Delta} \rfloor} \\ &= 1 - e^{-\lambda x}. \end{aligned}$$

The last equality holds because $\frac{x}{\Delta} - 1 \leq \lfloor \frac{x}{\Delta} \rfloor \leq \frac{x}{\Delta}$, and we know

$$\lim_{\Delta \to 0^+} (1 - \lambda\Delta)^{\frac{1}{\Delta}} = e^{-\lambda}.$$

Q5.

We first find $P(X > t)$:

$$\begin{aligned} P(X > t) &= P(\text{No arrival in } [0, t]) \\ &= e^{-\lambda t} \frac{(\lambda t)^0}{0!} \\ &= e^{-\lambda t}. \end{aligned}$$

Thus, the CDF of X for $x > 0$ is given by

$$F_X(x) = 1 - P(X > x) = 1 - e^{-\lambda x},$$

which is the CDF of $Exponential(\lambda)$. Note that by the same argument, the time between the first and second customer also has $Exponential(\lambda)$ distribution. In general, the time between the k'th and $k+1$'th customer is $Exponential(\lambda)$.

Q6.

Let an event occur at time t_0

P(event occuring first time at t_1) $= p$

⋮

P(event occuring first time at $t=k$) $= q^{k-1} p$

If X = r.v. that represents time for next event to occur.

$P(X=k) = q^{k-1} p$.

$\Rightarrow X$ is geometrically distributed.

Q7.

From the point of view of waiting time until arrival of a customer, the memoryless property means that it does not matter how long you have waited so far. If you have not observed a customer until time a, the distribution of waiting time (from time a) until the next customer is the same as when you started at time zero. Let us prove the memoryless property of the exponential distribution.

$$\begin{aligned} P(X > x + a | X > a) &= \frac{P(X > x + a, X > a)}{P(X > a)} \\ &= \frac{P(X > x + a)}{P(X > a)} \\ &= \frac{1 - F_X(x + a)}{1 - F_X(a)} \\ &= \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} \\ &= e^{-\lambda x} \\ &= P(X > x). \end{aligned}$$

Q8.

$$PMF(x) = \begin{cases} 1/10 & x = -20 \\ 2/10 & x = -3 \\ 3/10 & x = 4 \\ 4/10 & x = 5 \\ 0 & \text{otherwise} \end{cases}$$

$$P(|X| \le 2) = 0$$

Q9.

i) $M_{kX}(t) = E[e^{(tk)X}] = \underline{M_X(kt)}$

ii) $M_{X+k}(t) = E[e^{t(x+k)}] = E[e^{tk}e^{tx}] = e^{tk}E[e^{tx}] = \underline{e^{tk}M_X(t)}$

iii) $M_Y(t) = E[e^{t(X_1+X_2+\cdots+X_n)}]$ independent? $= E[e^{tX_1}]E[e^{tX_2}]\cdots E[e^{tX_n}] = \underline{M_X(t)^n}$

iv) $M_Y(t) = E[e^{tY}] = \int e^{ty}p(y)\,dy = \int e^{tx}p(x+k)\,dx$ Let $x+k=z$, $x=z-k$

$= \int e^{t(z-k)}p(z)\,dz = \underline{e^{-kt}M_Z(t)}$

v) $M_Y(t) = E[e^{tY}] = \int e^{ty}p(y)\,dy = \int e^{tx}p(2x)\,dx$ Let $z=2x$, $dz=2dx$

$= \int e^{t\frac{z}{2}}p(z)\cdot\frac{1}{2}dz = \underline{\frac{1}{2}M_Z(t/2)}$

Q10.

$\Rightarrow F_T(t) = \Phi\left(\frac{t-10}{10}\right)$

To find $P(t \leq 59°F)$

$59°F = (59-32)\times\frac{5}{9} = 27\times\frac{5}{9} = 15°C$

$\Rightarrow P(t \leq 59°F) = P(t \leq 15°C)$

$= F_T(15)$

$= \Phi\left(\frac{15-10}{10}\right) = \Phi(0.5)$

$= 0.69146$

Q11.

$X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$

$$M_X(t) = e^{\mu_x t + \frac{1}{2}\sigma_x^2 t^2}$$

Now.

$$M_{X+Y}(t) = [e^{t(X+Y)}] = E[e^{tX} e^{tY}]$$

$$= E[e^{tX}] \; E[e^{tY}] \qquad \left(\begin{array}{l} \because X, Y \text{ are independent} \\ E[f(X)f(Y)] = E[f(X)]\,E[f(Y)] \end{array} \right)$$

$$= \left(e^{\mu_x t + \frac{1}{2}\sigma_x^2 t^2}\right)\left(e^{\mu_y t + \frac{1}{2}\sigma_y^2 t^2}\right)$$

$$= e^{(\mu_x + \mu_y)t + \frac{1}{2}(\sigma_x^2 + \sigma_y^2)t^2}$$

Also if $Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

then $M_Z(t) = e^{(\mu_x + \mu_y)t + \frac{1}{2}(\sigma_x^2 + \sigma_y^2)t^2}$

By uniqueness of Moment Generating Function if MGF's are same then PDF's are same too.

$\Rightarrow \because M_Z(t) = M_{X+Y}(t)$

$\Rightarrow X + Y = Z$

$\Rightarrow X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

Q12.

Let the starting time of both be S_1 and S_2.

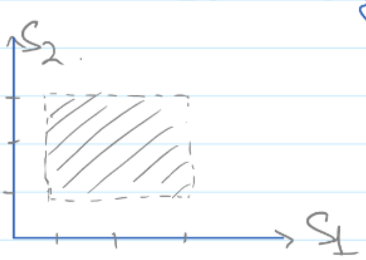
Now if $S_1 > S_2 + 0.5$, you will meet at point A itself since friend can travel 25 kms in 30 mins.

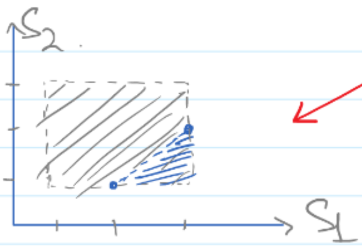
Similarly for all cases when $S_2 > S_1 + 0.5$, you will meet at point B.

The sample space for this would be

$$\Omega = [0,1] \times [0,1]$$

Since both can start any time in this interval.



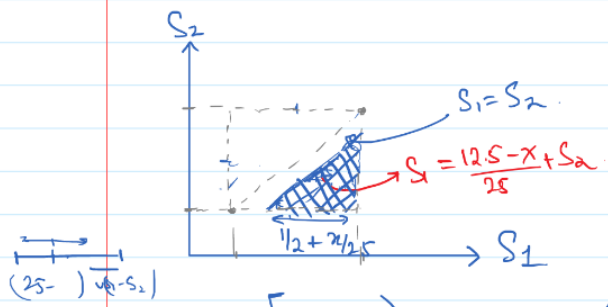


Example

Case: $S_2 < S_1 - 1/2$.

$X=0, \quad F_X(0)$

$$P(X \le 0) = \frac{Ar(\text{Blue})}{Ar(\text{grey})} = \boxed{\frac{1}{8}}$$



$S_2 \in [S_1 - 1/2, S_1]$

$$x = \frac{25 - v(S_1 - S_2)}{2}$$

$x \in [0, 12.5)$

$$(S_1 - S_2) = \frac{1}{2} - \frac{x}{25}$$

$$P(X \le x) = \frac{Ar(\text{Blue})}{Ar(\text{Grey})} = \left(\frac{12.5 + x}{25}\right)^2 \frac{1}{2} = \boxed{\left(\frac{1}{2} + \frac{x}{25}\right)^2 \cdot \frac{1}{2}}$$

Similarly

when $S_2 \in [S_1, S_1 + 1/2]$

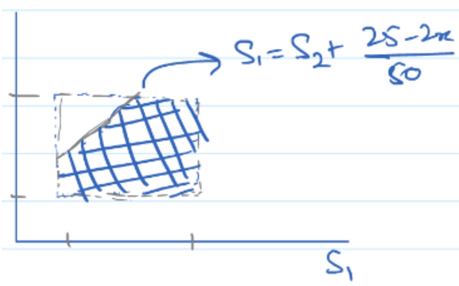
$x \in [12.5, 25]$

$$x = \vec{v}(S_2 - S_1) + \frac{25 - \vec{v}(S_2 - S_1)}{2}$$

$$x = \frac{25 + \vec{v}(S_2 - S_1)}{2}$$

$$S_1 = \frac{25 - 2x}{50} + S_2$$

$$\Rightarrow \quad S_1 = \frac{1}{2} - \frac{x}{25} + S_2$$



$$\boxed{P(X \leq x) = 1 - \left(\frac{3}{2} - \frac{x}{25}\right)^2 \cdot \frac{1}{2}}$$

$$\forall x \in [12.5, 25)$$

Finally,

$$F_X(x) = P(\{X \leq x\}) = \begin{cases} 0 & x \in (-\infty, 0) \\ \left(\frac{1}{2} + \frac{x}{25}\right)^2 \cdot \frac{1}{2} & x \in [0, 12.5) \\ 1 - \left(\frac{3}{2} - \frac{x}{25}\right)^2 \cdot \frac{1}{2} & x \in (12.5, 25) \\ 1 & x \in [25, \infty) \end{cases}$$

Discrete Random Variables

Question 13:

Here, the random variable Y is a function of the random variable X. This means that we perform the random experiment and obtain $X = x$, and then the value of Y is determined as $Y = (x+1)^2$. Since X is a random variable, Y is also a random variable.

a. To find R_Y, we note that $R_X = \{-2, -1, 0, 1, 2\}$, and

$$\begin{aligned} R_Y &= \{y = (x+1)^2 | x \in R_X\} \\ &= \{0, 1, 4, 9\}. \end{aligned}$$

b. Now that we have found $R_Y = \{0, 1, 4, 9\}$, to find the PMF of Y we need to find $P_Y(0), P_Y(1), P_Y(4)$, and $P_Y(9)$:

$$\begin{aligned} P_Y(0) &= P(Y=0) = P((X+1)^2 = 0) \\ &= P(X=-1) = \tfrac{1}{8}; \\ P_Y(1) &= P(Y=1) = P((X+1)^2 = 1) \\ &= P\big((X=-2) \text{ or } (X=0)\big); \\ &P_X(-2) + P_X(0) = \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{3}{8}; \\ P_Y(4) &= P(Y=4) = P((X+1)^2 = 4) \\ &= P(X=1) = \tfrac{1}{4}; \\ P_Y(9) &= P(Y=9) = P((X+1)^2 = 9) \\ &= P(X=2) = \tfrac{1}{4}. \end{aligned}$$

Again, it is always a good idea to check that $\sum_{y \in R_Y} P_Y(y) = 1$. We have

$$\sum_{y \in R_Y} P_Y(y) = \frac{1}{8} + \frac{3}{8} + \frac{1}{4} + \frac{1}{4} = 1.$$

Question 14:

a. We have $R_X = R_Y = \{1, 2, 3, 4, 5, 6\}$. Assuming the dice are fair, all values are equally likely so

$$P_X(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

Similarly for Y,

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

b. Since X and Y are independent random variables, we can write

$$\begin{aligned} P(X=2, Y=6) &= P(X=2)P(Y=6) \\ &= \tfrac{1}{6} \cdot \tfrac{1}{6} = \tfrac{1}{36}. \end{aligned}$$

c. Since X and Y are independent, knowing the value of Y does not impact the probabilities for X,

$$\begin{aligned} P(X > 3 | Y = 2) &= P(X > 3) \\ &= P_X(4) + P_X(5) + P_X(6) \\ &= \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{2}. \end{aligned}$$

d. First, we have $R_Z = \{2, 3, 4, \ldots, 12\}$. Thus, we need to find $P_Z(k)$ for $k = 2, 3, \ldots, 12$. We have

$$\begin{aligned} P_Z(2) &= P(Z=2) = P(X=1, Y=1) \\ &= P(X=1)P(Y=1) \text{ (since } X \text{ and } Y \text{ are independent)} \\ &= \tfrac{1}{6} \cdot \tfrac{1}{6} = \tfrac{1}{36}; \\ P_Z(3) &= P(Z=3) = P(X=1, Y=2) + P(X=2, Y=1) \\ &= P(X=1)P(Y=2) + P(X=2)P(Y=1) \\ &= \tfrac{1}{6} \cdot \tfrac{1}{6} + \tfrac{1}{6} \cdot \tfrac{1}{6} = \tfrac{1}{18}; \\ P_Z(4) &= P(Z=4) = P(X=1, Y=3) + P(X=2, Y=2) + P(X=3, Y=1) \\ &= 3 \cdot \tfrac{1}{36} = \tfrac{1}{12}. \end{aligned}$$

We can continue similarly:

$$\begin{aligned} P_Z(5) &= \tfrac{4}{36} = \tfrac{1}{9}; \\ P_Z(6) &= \tfrac{5}{36}; \\ P_Z(7) &= \tfrac{6}{36} = \tfrac{1}{6}; \\ P_Z(8) &= \tfrac{5}{36}; \\ P_Z(9) &= \tfrac{4}{36} = \tfrac{1}{9}; \\ P_Z(10) &= \tfrac{3}{36} = \tfrac{1}{12}; \\ P_Z(11) &= \tfrac{2}{36} = \tfrac{1}{18}; \\ P_Z(12) &= \tfrac{1}{36}. \end{aligned}$$

It is always a good idea to check our answers by verifying that $\sum_{z \in R_Z} P_Z(z) = 1$. Here, we have

$$\begin{aligned} \textstyle\sum_{z \in R_Z} P_Z(z) &= \tfrac{1}{36} + \tfrac{2}{36} + \tfrac{3}{36} + \tfrac{4}{36} + \tfrac{5}{36} + \tfrac{6}{36} \\ &+ \tfrac{5}{36} + \tfrac{4}{36} + \tfrac{3}{36} + \tfrac{2}{36} + \tfrac{1}{36} \\ &= 1. \end{aligned}$$

e. Note that here we cannot argue that X and Z are independent. Indeed, Z seems to completely depend on X, $Z = X + Y$. To find the conditional probability $P(X = 4|Z = 8)$, we use the formula for conditional probability

$$\begin{aligned} P(X=4|Z=8) &= \frac{P(X=4, Z=8)}{P(Z=8)} \\ &= \frac{P(X=4, Y=4)}{P(Z=8)} \\ &= \frac{P(X=4)P(Y=4)}{P(Z=8)} \text{ (since } X \text{ and } Y \text{ are independent)} \\ &\frac{\frac{1}{6} \cdot \frac{1}{6}}{\frac{5}{36}} \\ &= \tfrac{1}{5}. \end{aligned}$$

Question 15:

The CDF is defined by $F_X(x) = P(X \leq x)$. We have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 3 \\ P_X(3) = 0.3 & \text{for } 3 \leq x < 5 \\ P_X(3) + P_X(5) = 0.5 & \text{for } 5 \leq x < 8 \\ P_X(3) + P_X(5) + P_X(8) = 0.8 & \text{for } 8 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$$

Question 16:

Let's first make sure we understand what Var$(2X - Y)$ and Var$(X + 2Y)$ mean. They are Var(Z) and Var (W), where the random variables Z and W are defined as $Z = 2X - Y$ and $W = X + 2Y$. Since X and Y are independent random variables, then $2X$ and $-Y$ are independent random variables. Also, X and $2Y$ are independent random variables. Thus, by using Equation 3.7, we can write

$$\text{Var}(2X - Y) = \text{Var}(2X) + \text{Var(-Y)} = 4\text{Var}(X) + \text{Var(Y)} = 6,$$

$$\text{Var}(X + 2Y) = \text{Var}(X) + \text{Var(2Y)} = \text{Var}(X) + 4\text{Var(Y)} = 9.$$

By solving for Var(X) and Var(Y), we obtain Var$(X) = 1$ and Var$(Y) = 2$.

Question 17:

Note that

$$\begin{aligned} P(X > 0) &= P_X(1) + P_X(2) + P_X(3) + P_X(4) + \cdots, \\ P(X > 1) &= P_X(2) + P_X(3) + P_X(4) + \cdots, \\ P(X > 2) &= P_X(3) + P_X(4) + P_X(5) + \cdots. \end{aligned}$$

Thus

$$\begin{aligned} \textstyle\sum_{k=0}^{\infty} P(X > k) &= P(X > 0) + P(X > 1) + P(X > 2) + \ldots \\ &= P_X(1) + 2P_X(2) + 3P_X(3) + 4P_X(4) + \ldots \\ &= EX. \end{aligned}$$

Question 18:

$X \rightarrow$ Range $[0,3]$

Density $f_X(x) = kx^2$, $Y = X^3$

a) We know that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

We are given the Range of $X \Rightarrow [0,3]$.

Its PDF $= 0$ at all points apart from the range

$$\int_0^3 kx^2\,dx = 1 \Rightarrow \left.\frac{kx^3}{3}\right|_0^3 = 1$$

$$\Rightarrow \quad 9k = 1$$

$$k = \underline{\underline{1/9}}$$

CDF of $F_X(x) = \int_{-\infty}^{\infty} f_X(u)\,du$.

CDF of $F_X(x) = \int_{-\infty}^{\infty} f_X(u)\, du.$

Since, $f_X(u) = 0$ for all values apart in $[0,3]$, we get

$$\Rightarrow \int_0^x \frac{1}{9} u^2\, du \quad ; 0 \le x \le 3$$

$$\Rightarrow \left. \frac{u^3}{27} \right|_0^x \Rightarrow \underline{\underline{\frac{x^3}{27}}}$$

$$F_X(x) = \begin{cases} 0 & ; x < 0 \\ x^3/27 & ; 0 \le x \le 3 \\ 1 & ; x > 3 \end{cases}$$

b) We need to compute $E[Y]$.

$Y = X^3$ [given].

$$\Rightarrow E[Y] = E[X^3] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

[LOTUS]

We know that PDF $= 0$ except in $[0,3]$

$$\Rightarrow \int_0^3 x^3 \cdot \frac{x^2}{9}\, dx \Rightarrow \left. \frac{x^6}{54} \right|_0^3$$

$$\Rightarrow \frac{729}{54} \Rightarrow \frac{27}{2} = \underline{\underline{13.5}}$$

c) We need to compute Var(Y)

$Var(Y) = E[Y^2] - E[Y]^2$.

We know what $E[Y]$ is. We now calculate value of $E[Y^2]$.

$Y = X^3 \Rightarrow Y^2 = X^6$

$E[Y^2] = E[X^6] \Rightarrow \int x^6 f(x)\, dx$.

$= \int_0^3 x^6 \cdot \frac{x^2}{9}\, dx \Rightarrow \left.\frac{x^9}{81}\right|_0^3$

$\Rightarrow \underline{243}$

On substituting, we get $\Rightarrow$

$\therefore Var(Y) =$

$(243) - (13.5)^2$

$\Rightarrow 243 - 182.25$

$\Rightarrow \underline{\underline{60.75}}$.

d) To find: PDF $f_Y(y)$ for Y.

Using above derivations and formulas, we can say

$$F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3})$$

$$= \underline{\underline{F_X(y^{1/3})}}$$

$$F_X(y^{1/3}) = \begin{cases} 0 & , y^{1/3} < 0 \\ y/27 & , y^{1/3} \in [0,3] \\ 1 & , y^{1/3} > 3 \end{cases}$$

$$F_Y(Y) = \begin{cases} 0 & , y < 0 \\ y/27 & , y \in [0, 27] \\ 1 & , y > 27 \end{cases}$$

In order to find PDF we differentiate the CDF

$$f_Y(y) = \frac{d}{dy}(F_Y(y))$$

$$f_Y(y) = \begin{cases} 1/27 & ; 0 \le y \le 27 \\ 0 & ; \text{otherwise} \end{cases}$$

is the required PDF.