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DISCRETE STOCHASTIC PROCESSES

DISCRETE STOCHASTIC PROCESSES

by

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Printed on acid-free paper.

To Marie, and all the joy we share

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Preface

The purpose of this text is to help students develop the understanding and intuition necessary to apply stochastic process models to problems in engineering, science, and operations research. Although the ultimate objective is applications, there is relatively little description of actual applications. Rather, there are many simple examples designed to build insight about the structure of stochastic processes and about the generic effect of these phenomena in real systems. I am convinced that the “case study” method, in which many applications are studied, is inappropriate for understanding stochastic processes (and similarly inappropriate for any field that has a rich and highly cohesive mathematical structure).

When we try to either design new kinds of systems or understand physical phenomena, we usually employ a variety of stochastic process models to gain understanding about different tradeoffs and aspects of the problem. Creating these models requires deep understanding both of the application area and of the structure of stochastic processes. The application areas are too broad, and the structure too deep, to do all this in one text. My experience indicates that engineers rarely have difficulty applying well understood theories and tools to well understood application areas. The difficulty comes when the mathematical structure is not understood on both an intuitive and mathematical level. The “back of the envelope calculations” that we so prize as engineers are the result of this deep understanding of both application areas and conceptual structure.

I try to present the structural aspects of stochastic processes in the simplest possible light here, thus helping students develop insight. This requires somewhat more abstraction than engineers are used to, but much less than often appears in mathematics texts. It also requires students to spend less time doing complex calculations and more time drawing illustrative diagrams and thinking. The proofs and explanations here are meant to be read, not because students might doubt the result, but to enhance understanding. In order to use these results in modeling real situations, students must understand the robustness of the results at an intuitive level, and this is gained only by understanding why the results are true.

Students learn about new concepts in many ways, partly by learning facts, partly by doing exercises, and partly by successively refining a structural picture of what the subject is about. The combination of all of these leads to understanding and the ability to create models for real problems. This ability to model, however, requires much more than the “plug and chug” of matching exercises to formulas and theorems. The ability to model is based on an intuitive level understanding, backed by mathematics.

Stochastic processes is the branch of probability dealing with probabilistic systems that evolve in time. By discrete stochastic processes, I mean processes in which changes occur only at discrete times separated by either deterministic or random intervals. In particular, we do not treat noise processes such as Gaussian processes. This distinction between discrete and non-discrete processes is somewhat artificial, but is dictated by two practical considerations. The first is that many engineering graduate students take a course in Gaussian processes, second moment theory, detection, and estimation (the material in the course is more standard than the title). Such a course has much cohesion, fits nicely into one academic term, and has relatively little conceptual overlap with the material here. The second consideration is that extensions of the material here to continuous processes often obscure the probabilistic ideas with mathematical technicalities.

The mathematical ideas are presented here without measure theory, and with only minimal mathematical analysis. The material has been used for many terms in a course primarily for engineering and operations research graduate students, most of whom have an engineering background. The material does require more patience than engineering students are used to, but that is intentional, since the purpose is to develop understanding rather than computational ability.

In the proofs and explanations, I have favored clarity over formal rigor and simplicity over generality. Occasionally proofs are done only for the case of discrete random variables or only for variables with a density. In most of these cases, students with a little more mathematical background can easily fill in the additional formalism for a general proof. Numerous examples are given showing how results fail to hold when all the conditions are not satisfied. Understanding is often as dependent on a collection of good counterexamples as on knowledge of theorems. In engineering, there is considerable latitude in generating mathematical models for real problems. Thus it is more important to have a small set of well understood tools than a large collection of very general but less intuitive results.

Most results here are quite old and well established, so I have not made any effort to attribute results to investigators, most of whom are long dead or retired. The organization of the material owes a great deal, however, to Sheldon Ross's book, *Stochastic Processes*, [Ros83] and to William Feller's classic books, *Probability Theory and its Applications, vols. 1 and 2*, [Fel68] and [Fel66].

The stimulating environment at M.I.T., with its constant healthy tug of war between theory and applications, has been an ideal setting for writing the text. I am deeply indebted to graduate students who studied from these notes over the last eight years. Many sections have been rewritten many times until the students here found them understandable. My conviction is that if the students here cannot understand it, it shouldn't be in a text book. Special thanks go to Serap Savari and Irvin Schick who have each gone over the entire manuscript. Each have a wonderful talent for spotting things that are unclear, awkward, or simply wrong. In addition, Dimitri Bertsekas, Al Drake, Doug Gallager, Diane Ho, Armann Ingolfsson, Dick Larson, Muriel Medard,

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Robert G. Gallager

Chapter 1

Introduction and Probability Review

1.1 INTRODUCTION

A stochastic process (or random process) is a probabilistic experiment or model that evolves in time. That is, each sample point (i.e., possible outcome) of the experiment is a function of time called a sample function. The sample space is the set of possible sample functions, and the events are subsets of sample functions. Finally, there is a rule for determining the probabilities of the various events. As an example, we might be concerned with arrivals to some system. The arrivals might be incoming jobs for a computer system, arriving packets for a communication system, patients in a health care system, or orders for some merchandising warehouse.

EXAMPLE 1: Suppose, to look at a particularly simple case, that arrivals occur only at integer instants of time, at most one arrival occurs at any given integer time, and the time between successive arrivals is one with probability 1/2 and two with probability 1/2; the time between arrival n and arrival $n+1$ is independent of all previous inter-arrival intervals. Finally, the first arrival occurs at time 1 or 2, with equal probability.

A sample point for such a process could be regarded as the set of instants at which arrivals occur. Alternatively, we could view a sample function as $\{n(t); t \geq 0\}$, where $n(t)$ is the number of arrivals up to and including time t (see figure 1.1). One can then ask questions such as: for any given t , what is the probability distribution of the number of arrivals from 0 to t ? what is the probability of an arrival at t ? is it meaningful to talk about the time average arrival rate, $n(t)/t$, in the limit $t \rightarrow \infty$?

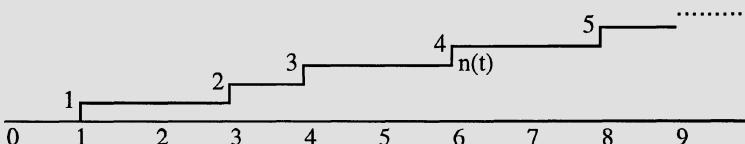


Figure 1.1. A sample function of the arrival process in example 1.

A more general form of the above simple process is to allow the inter-arrival intervals (i.e., the times between successive arrivals) to be independent, identically distributed random variables with an arbitrary discrete or continuous distribution function. Such stochastic processes are called renewal processes and play a central role in the theory of stochastic processes.

Renewal processes are examples of *discrete stochastic processes*. The distinguishing characteristic of such processes is that interesting things (arrivals, departures, changes of state) occur at discrete instants of time separated by deterministic or random intervals. Discrete stochastic processes are to be distinguished from noise-like stochastic processes in which changes are continuously occurring and the sample functions are continuously varying functions of time. The description of discrete stochastic processes above is not intended to be precise. The various types of stochastic processes developed in subsequent chapters are all discrete in the above sense, however, and we refer to these processes, somewhat loosely, as discrete stochastic processes in what follows.

Discrete stochastic processes find wide and diverse applications in operations research, communication, control, computer systems, management science, etc. Paradoxically, we shall spend relatively little of our time discussing these particular applications, and rather spend our time developing results and insight about these processes in general. We shall discuss a number of examples drawn from the above fields, but the examples will be “toy” examples, designed to enhance understanding of the material rather than to really understand the application areas.

One possible approach to applying probabilistic models to an application area is to first gather statistics about the area, then construct the most accurate model possible from those statistics, and then to analyze the model. Unfortunately, this approach almost invariably fails, the reason being that real problems are too complicated to model completely. A better approach is to choose a simple and well understood model that is intuitively related to the application area. Analysis and understanding of the model, coupled with knowledge and some simple statistics about the application area, then suggest refinements of the model. Both the original choice of model and successive refinements require thorough understanding and insight about a broad class of potential models. This means that the key to successful modeling is the acquisition of insight and deep understanding about broad classes of models. It is for this reason that we stress models here rather than application areas.

The above discussion of modeling is applicable to any class of problems amenable to mathematical analysis. It is particularly applicable to stochastic processes for a number of reasons. First, the range of applications is diverse. Second, many of the most important practical results depend on various limiting operations; understanding these results requires time and patience, more than is required in many engineering subjects where many applications lie close to the surface. Third, there are many paradoxical results that are counter-intuitive until they are thoroughly understood.

1.2 PROBABILITY REVIEW

A *probability model* (or probability experiment) has three ingredients—a sample space, a set of events, and a rule for assigning probabilities to events. The sample space is the

set of possible outcomes (sample points) of the experiment. When the experiment is performed, one (and only one) of these outcomes occurs. An event is a subset of the sample space and the probability of that event is a real number between 0 and 1. Note that the technical meanings of outcome and event are not quite the same as their ordinary usage in English. An outcome completely determines the result of the experiment, specifying a single sample point, whereas an event partially determines the result, specifying only that the outcome lies in the subset of the sample space corresponding to the event.

For the simplest type of experiment, the number of sample points is finite or countably infinite (i.e., the sample points can be listed $(\omega_1, \omega_2, \dots)$). Each sample point, or, more precisely, each event consisting of a single sample point, has some probability associated with it. For an arbitrary event, the probability of that event is the sum of the probabilities of the sample points making up the event. All the probabilities are non-negative and the sum of the probabilities of all sample points is 1.

EXAMPLE 2: Consider an experiment of two flips of a coin. The sample space consists of four sample points, $\{(HH), (HT), (TH), (TT)\}$ and the probability of each sample point is taken to be $1/4$. Each subset of the sample points corresponds to an event. For example, the event that the first coin flip is heads is the subset $\{(HH), (HT)\}$ and has probability $1/2$. The event that at least one flip results in heads is the subset $\{(HH), (HT), (TH)\}$ and has probability $3/4$.

Note that by choosing the sample space as above, we have ruled out the possibility of the coin getting lost before the second toss or of coming to rest on its edge. By choosing the probabilities as above, we have ruled out the possibility of a bias toward heads or tails. That is, once the probability model is selected, all questions about the physical situation have been resolved (at least for as long as one continues to use that model).

It would seem that the class of experiments involving only a finite number of sample points is trivial, involving nothing more than defining the probability of each sample point and then adding these probabilities appropriately. What makes some of these problems non-trivial is the very large number of sample points and their combinatorial aspects.

EXAMPLE 3: Suppose we modify example 1 to consider only the first 1000 arrivals for the process. Then a sample point becomes a sequence of 1000 arrival instants and there are 2^{1000} sample points. Finding the probability of an arrival at time 1000 by adding up the probabilities of individual sample points is highly impractical.

For some sample spaces, it is not only impractical but also meaningless to determine the probability of events from the probabilities of individual sample points. This is illustrated in the next example, and is one reason why we focus on the probabilities of events rather than sample points.

EXAMPLE 4: Consider a sample space consisting of the set of real numbers between 0 and 1, and assume that the outcome is uniformly distributed over this interval. Each

sample point then has zero probability, and there is no way to add up these zero probability single point events in a meaningful way. It is reasonable in this case to take the probability of any interval as the size of that interval, and to take the probability of a union of disjoint intervals as the sum of the sizes of the intervals.

For experiments such as example 4, one must assign probabilities to events directly. Along with the restriction that the probability of each event must lie between 0 and 1 and that the probability of the sample space itself is 1, there is the final restriction that for any sequence of disjoint events E_1, E_2, \dots , the probability of the union of these events is given by

$$P(\cup_i E_i) = \sum_i P(E_i) \quad (1)$$

These three restrictions can be regarded as the axioms of probability.¹ For a countable² set of outcomes, (1) specifies the probability of an event as the sum of the included sample point probabilities. For more general cases, as in example 4, one must find a way to assign probabilities to enough events that (1) suffices to determine the probabilities of all events.

We are not interested here in showing how all of the rules for manipulating probabilities follow from these axioms, but we give three examples to indicate why (1) says more than is immediately apparent. First, for any event A, let A^c be the complementary event (i.e., the set of all sample points not in A). Then A and A^c are disjoint and their union is the entire space, which has probability 1. Thus, $P(A^c) = 1 - P(A)$. Second, if A and B are non-disjoint (i.e., contain sample points in common), then $A \cup B = AB^c \cup AB \cup A^cB$ (where, for example, AB^c denotes the intersection of subsets A and B^c) and these events are all disjoint. Thus $P(A \cup B) = P(AB^c) + P(AB) + P(A^cB) = P(A) + P(A^cB)$. Third, since $B = AB \cup A^cB$, we can rewrite $P(A \cup B)$ as $P(A) + P(B) - P(AB)$, which gives a convenient relation between the probabilities of unions and intersections.

A sample point of the stochastic process described in example 1 is a time function $n(t)$ ($0 \leq t < \infty$) which is constant except for unit increases at integer times separated by one or two. Each sample point corresponds to an infinite sequence of binary choices between inter-arrival intervals of 1 or 2, and thus each sample point has probability 0. The independent identically distributed (IID) inter-arrival intervals, which are 1 or 2 with equal probability, allow one to calculate the probability of any event of interest. For example, the probability of an arrival at time 2 is 3/4 (with probability 1/2, the first arrival occurs then, and with probability 1/4, the second arrival occurs then).

Note that many questions concerning the process of example 1 can be answered by considering the process of example 3, thus avoiding the necessity of dealing with sample points that have zero probability. We shall see, as we proceed, that using such artifices is unnatural and counterproductive—it is like avoiding the use of calculus by approximating all functions as discrete functions.

A subtle question arises both in example 1 and example 4; namely, can we define probabilities for all subsets of the sample space, consistent with the axioms above? The

answer, unfortunately, is no, and understanding this requires mathematics beyond the scope of this text. Fortunately, the subsets that cause problems are so bizarre that they do not arise in any of the practical or conceptual problems of interest here. The set of events is now defined as those subsets for which probabilities exist. These subsets are always constructed in such a way that countable unions and intersections of events are also events. Thus one cannot create a subset of undefined probability in the course of ordinary analysis.

There is another simpler type of subtlety that arises in examples 1 and 4; the probability of each sample point is 0, and thus, from Eq. (1), any event containing only a finite or countably infinite set of sample points has probability zero. Typically there are also many other kinds of events that have zero probability. We shall find that many of the most important results in stochastic processes are true for all sample points except some such set of probability zero. An event including all sample points except perhaps some set of probability zero is referred to as an *event of probability 1*. Fortunately, we can work with these special events of zero probability or probability 1 without a great deal of mathematical sophistication.

1.3 CONDITIONAL PROBABILITIES

For any two events A and B (with $P(B)>0$), the *conditional probability* of A, conditional on B, is defined by $P(A|B) = P(AB)/P(B)$. One visualizes an experiment that has been partly carried out with B as the result. $P(A|B)$ is then the probability of A within the sample space restricted to event B. It is important to recognize that anything we know about probability can also be applied to such a restricted probability space.

Two events, A and B, are said to be *independent* if $P(AB) = P(A)P(B)$. For $P(B)>0$, this is equivalent to $P(A|B) = P(A)$; i.e., A and B are independent if the observation of B does not change the probability of A. Such intuitive statements about “observation” and “occurrence” are helpful in reasoning probabilistically, but also can lead to great confusion. For $P(B)>0$, $P(A|B)$ is perfectly well defined without any notion of B being observed “before” A. For example, $P(A|B)P(B) = P(B|A)P(A)$ is simply a consequence of the definition of conditional probability and has nothing to do with causality or observations. This issue caused immense confusion in probabilistic arguments before the theory was placed on a firm mathematical foundation.

The notion of independence is of vital importance in defining, and reasoning about, probability models. One needs a way to assign probabilities to all events, and this is often done by assuming independence between events. In example 1, we assumed such independence without even defining it. Often, when events are not independent, they are conditionally independent, where A and B are said to be *conditionally independent* given C if $P(AB|C) = P(A|C)P(B|C)$. Most of the stochastic processes to be studied here are characterized by particular forms of independence or conditional independence.

1.4 RANDOM VARIABLES

The outcome of a probabilistic experiment often contains a collection of numerical values such as temperatures, voltages, numbers of arrivals or departures in various

intervals, etc. Each such numerical value varies depending on the particular outcome of the experiment, and thus is a mapping from the set of sample points to the set of numerical values. These variable numerical outcomes, as mapped from the sample points, are called random variables. More precisely, a *random variable* X is defined as a function that maps each point ω in the sample space S (or, more generally, each point except perhaps those in some subset of probability 0) into the set of finite real numbers. Thus, for any random variable X and any real number x , there is an event $X \leq x$; this event is the subset of S whose elements are mapped by X into real numbers less than or equal to x , i.e.,

$$P(X \leq x) = P(\{\omega \in S : X(\omega) \leq x\}) \quad (2)$$

Note that $P(X \leq x)$ is a function of the real variable x . It is monotonic nondecreasing, from 0 to 1, as x goes from $-\infty$ to $+\infty$. It is called the *distribution function* of the random variable X and will usually be denoted by $F_X(x)$ (see figure 1.2). Note that x is the argument of the function, whereas the subscript X denotes the particular random variable under consideration; if the random variable is clear from the context, we will omit the subscript. We shall always denote random variables by capital letters; this convention is almost universally observed in the field of probability.

As mentioned above, it is permissible for a random variable X to be undefined over some set of sample points comprising an event of zero probability. We shall see many examples of this later, and the student is best advised to postpone concern about it until faced with such examples. These events of zero probability have no effect on the distribution function of X , and, as will be more clear later, no effect on anything we do with random variables.

The concept of a random variable must sometimes be extended to complex random variables and vector random variables. A *complex random variable* is a mapping from the sample space to the set of finite complex numbers, and a *vector random variable* is a mapping from the sample space to the set of finite vectors in some finite dimensional vector space. Another extension is that of defective random variables. X is *defective* if there is a set of sample points of *positive* probability for which the mapping is either undefined or defined to be either $+\infty$ or $-\infty$. When we refer to random variables in this text (without any modifier such as complex, vector, or defective), we explicitly restrict attention to the original definition, i.e., a function from the sample space (except a subset of zero probability) to the finite real numbers.

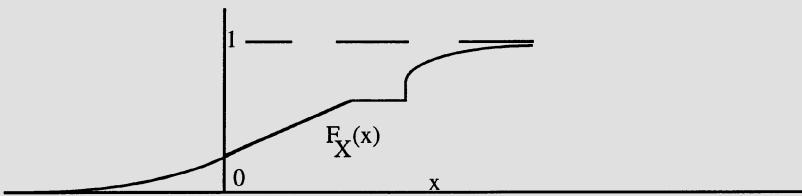


Figure 1.2. Example of a distribution function for a random variable that is neither continuous nor discrete.

If the distribution function $F_X(x)$ of a random variable X has a derivative, the derivative is called the *probability density* (or just density) of X and denoted by $f_X(x)$; for sufficiently small δ , $\delta F_X(x)$ approximates the probability that X is mapped into a value between x and $x+\delta$. If the density exists and is finite everywhere, we say that the random variable is continuous. Similarly, if X has a countable number of possible outcomes, x_1, x_2, \dots , the probability of each outcome, $\{P(x_i); i \geq 1\}$, is called the *probability mass function* (PMF) of X and denoted by $\{P_X(x_i); i \geq 1\}$; such a random variable is called discrete. Finally, the *distribution* of a random variable is any rule from which the distribution function can be determined; thus the distribution of X is specified by the density or the PMF or the distribution function.

Elementary probability courses usually work primarily with the density and the PMF, since they are convenient for computational exercises. We will work mostly with the distribution function here. This is partly to avoid saying everything thrice, once for discrete variables, once for variables with a density, and once for other variables, and partly because it is the distribution function that is important in limiting arguments such as going to steady state and dealing with time averages.

Often we must deal with multiple random variables in a single probability experiment. For example 1 above, each inter-arrival interval can be regarded as a random variable, taking on the value 1 or 2 with equal probability. If X_1, X_2, \dots, X_n are n random variables, their joint distribution function is defined by

$$F_{X_1, \dots, X_n}(x_1, x_2, \dots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \quad (3)$$

For a vector random variable \mathbf{X} with components X_1, \dots, X_n , or a complex random variable \mathbf{X} with real and imaginary parts X_1, X_2 , the distribution function is also defined by (3). We will often leave out the subscript X_1, \dots, X_n in such expressions if the meaning is clear from the context. Note that the expression $(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n)$ is an event and the corresponding probability is nondecreasing in each argument x_i . Also the distribution function of any subset of the random variables is obtained by setting the other arguments to $+\infty$. For example, the distribution of a single variable (often called a marginal distribution) is given by

$$F_{X_i}(x_i) = F_{X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n}(\infty, \dots, \infty, x_i, \infty, \dots, \infty)$$

The joint probability density $f(x_1, \dots, x_n)$, if it exists, is given by the derivative $\frac{\partial^n F(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_n}$. Similarly the joint PMF is given by $P(X_1=x_1, \dots, X_n=x_n)$.

The random variables X_1, \dots, X_n are said to be *independent* if, for all x_1, \dots, x_n ,

$$F(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(X_i \leq x_i) \quad (4)$$

Another way to state this is that X_1, \dots, X_n are independent if the events $X_i \leq x_i$ for $1 \leq i \leq n$ are independent for all choices of x_1, \dots, x_n . Similarly, events A_1, \dots, A_k and random

variables X_1, \dots, X_n are independent if the events $A_1, \dots, A_k, X_1 \leq x_1, \dots, X_n \leq x_n$ are independent for all choices of x_1, \dots, x_n . If the density or mass function exists, (4) is equivalent to a product form for the density or mass function. A set of random variables is said to be pairwise independent if each pair of random variables in the set is independent. As shown in exercise 1.9, pairwise independence does not imply that the entire set is independent.

1.5 EXPECTATIONS

The *expected value* (or the mean) of a continuous or discrete random variable respectively is defined to be

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx ; \quad E[X] = \sum_x xP_X(x) \quad (5)$$

We shall denote the expected value of a random variable X both by $E[X]$ and by \bar{X} . In order to avoid rewriting things for the discrete, continuous, and mixed cases, we shall generally express the expected value as a Stieltjes integral,

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} xdF_X(x) \quad (6)$$

The Stieltjes integral can be defined by

$$\int_{-\infty}^{\infty} g(x) dF(x) = \lim_{\delta \rightarrow 0} \sum_{n=-\infty}^{\infty} \sup_{n\delta < x \leq n\delta + \delta} g(x) [F(n\delta + \delta) - F(n\delta)] \quad (7)$$

This integral is said to exist only if the limit exists and if the limit remains the same if \sup^3 above is replaced by \inf . It can be seen that if a density exists, $F(n\delta + \delta) - F(n\delta) \approx \delta f(n\delta)$, so that, aside from limiting questions, (7) is the same as $\int g(x) f(x) dx$. For our purposes, we shall not often be concerned with peculiar functions for which these limit questions are serious. Thus, we will regard (6) mostly as shorthand for the expressions in (5). One can also view $dF_X(x)$ as standing for $(dF_X(x)/dx)dx$ or $f_X(x)dx$ and then allow $f_X(x)$ to include impulse functions to deal with the discrete case. The integral in (7) (and similarly the integral and sum in (5)) are said to exist only if both the integral from 0 to ∞ and also the integral from $-\infty$ to 0 exist. Thus $E[X]$ exists if and only if $E[|X|]$ exists. This is illustrated later in example 6.

For a non-negative random variable (i.e., a random variable for which $F_X(x) = 0$ for $x < 0$), the expectation can be found in a particularly simple way, namely as the integral of the complementary distribution function, where the *complementary distribution function* of a random variable is defined as $P(X > x) = 1 - F_X(x)$.

$$E[X] = \int_0^{\infty} [1 - F_X(x)] dx \quad (8)$$

This relationship is even more important for conceptual purposes than for computational purposes. To derive it for a discrete random variable, consider sketching the complementary distribution function, $P(X>x) = 1 - F_X(x)$ as in figure 1.3. The figure shows that $\sum_i a_i P_X(a_i)$ is equal to the area under the curve $1-F_X(x)$ from $x=0$ to ∞ .

For an arbitrary distribution, we can visualize quantizing the distribution function, using the above argument, and then passing to the limit of arbitrarily fine quantizing. Since there are no mathematical subtleties in integrating a non-negative decreasing function, many people prefer defining expectation in this way (see exercise 1.1 for the generalization to arbitrary rather than non-negative random variables). We shall use (8) frequently throughout the text.

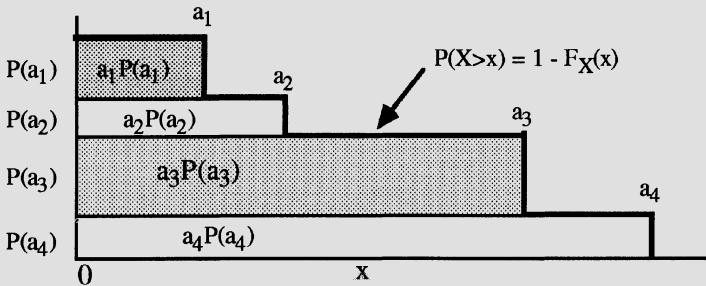


Figure 1.3. For this example, X takes on four possible values, $a_1 < a_2 < a_3 < a_4$, with $a_i \geq 0$. Thus $P(X>x) = 1$ for all $x \leq a_1$. For $a_1 < x \leq a_2$, $P(X>x) = 1 - P_X(a_1)$, and $P(X>x)$ has similar drops as x reaches a_2 , a_3 , and a_4 . $E[X]$, from (2), is $\sum_i a_i P_X(a_i)$, which is the sum of the rectangles in the figure. This is also the area under the curve $1-F_X(x)$, verifying (8).

Random variables are often defined in terms of each other. For example if g is a function from real numbers to real numbers and X is a random variable, then $Y = g(X)$ is the random variable that maps each sample point ω into $g(X(\omega))$. As indicated in exercise 1.6, one can find the expected value of Y (if it exists) in either of the following ways:

$$E[Y] = \int_{-\infty}^{\infty} y dF_Y(y) = \int_{-\infty}^{\infty} g(x) dF_X(x) \quad (9)$$

Particularly important examples of such expected values are the moments $E[X^n]$ of a random variable X (which is simply the expected value of the random variable X^n) and the central moments $E[(X - \bar{X})^n]$ of X where \bar{X} is the mean $E[X]$. The second central moment is called the *variance*, denoted $VAR(X)$ or σ_X^2 . It is given by

$$VAR(X) = E[(X - \bar{X})^2] = E[X^2] - \bar{X}^2 \quad (10)$$

The *standard deviation* of X , σ_X , is the square root of the variance and provides a measure of the dispersion of the random variable around the mean. Thus the mean is a

rough measure of typical values for the outcome of the random variable, and σ_X is a measure of how close one expects to come to that typical value. There are other measures of typical value (such as the median and the mode) and other measures of dispersion, but the mean and standard deviation have a number of special properties that make them important. One of these (see exercise 1.10) is that $E(X)$ is the value of z that minimizes $E[(X-z)^2]$.

If X and Y are random variables, then the sum $Z=X+Y$ is also a random variable. If X and Y are independent, then the distribution function of Z is given by

$$F_Z(z) = \int_{-\infty}^{\infty} F_X(z-y)dF_Y(y) = \int_{-\infty}^{\infty} F_Y(z-x)dF_X(x) \quad (11)$$

If X and Y both have densities, this can be rewritten as

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_Y(z-x)f_X(x)dx \quad (12)$$

Eq. (12) is the familiar convolution equation from linear systems, and we similarly refer to (11) as the convolution of distribution functions (although it has a different functional form from (12)). If X and Y are non-negative random variables, then the integrands in (11) and (12) are non-zero only between 0 and z , so we often use 0 and z as the limits in (11) and (12). Note, however, that if $P(Y=0) \neq 0$, then the lower limit must be 0, i.e., in terms of (11), it must include the jump in $F_Y(y)$ at $y=0$, and in terms of (12), it must include the impulse at $y=0$. One can avoid confusion of this type by always keeping infinite limits until actually calculating something.

If X_1, X_2, \dots, X_n are independent random variables, then the distribution of the random variable $S_n = X_1 + X_2 + \dots + X_n$ can be found by first convolving the distributions of X_1 and X_2 to get the distribution of S_2 , and then for each $i \geq 2$, convolving the distributions of S_i and X_{i+1} to get the distributions of S_{i+1} . The distributions can be convolved in any order to get the same resulting distribution.

Whether or not X_1, X_2, \dots, X_n are independent, the expected value of $S_n = X_1 + X_2 + \dots + X_n$ satisfies

$$E[S_n] = E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n] \quad (13)$$

This says that the expected value of a sum is equal to the sum of the expected values whether or not the random variables are independent (see exercise 1.3). The following example shows how this can be a valuable problem solving aid with appropriate choice of random variables.

EXAMPLE 5: In packet networks, a packet can be crudely modeled as a string of IID binary digits with $P(0) = P(1) = 1/2$. Packets are usually separated from each other by a special bit pattern, 0111110, called a flag. If this special pattern appears within a packet, it could be interpreted as a flag indicating the end of the packet. To prevent this prob-

lem, an extra binary digit of value 0 is inserted after each appearance of 011111 in the original string (this can be deleted after reception). Suppose we want to find the expected number of inserted bits in a string of length n . For each position $i \geq 6$ in the original string, define X_i as a random variable whose value is 1 if an insertion occurs after the i^{th} data bit. The total number of insertions is then just the sum of X_i from $i=6$ to n inclusive. Since $E[X_i] = 2^{-6}$, the expected number of insertions is $(n-5)2^{-6}$. Note that the positions in which the insertions occur are highly dependent, and the problem would be quite difficult if one didn't use (13) to avoid worrying about the dependence.

If the random variables X_1, \dots, X_n are independent, then, as shown in exercise 1.3 and 1.7, the variance of $S_n = X_1 + \dots + X_n$ is given by

$$\sigma_{S_n}^2 = \sum_{i=1}^n \sigma_{X_i}^2 \quad (14)$$

If X_1, \dots, X_n are also identically distributed, then $\sigma_{S_n}^2 = n\sigma_{X_i}^2$ and the standard deviation of S_n is $\sigma_{S_n} = \sqrt{n}\sigma_{X_i}$. It is important to remember that the standard deviation of S_n increases with n , although only with the square root of n .

Similarly, if X_1, \dots, X_n are independent, exercise 1.3 shows that

$$E\left[\prod_{i=1}^n X_i\right] = \prod_{i=1}^n E[X_i] \quad (15)$$

1.6 TRANSFORMS

The *moment generating function* for a random variable X is given by

$$g_X(r) = E[e^{rX}] = \int_{-\infty}^{\infty} e^{rx} dF_X(x) \quad (16)$$

In (16), we can view $g_X(r)$ as a function of a complex variable r . If r is pure imaginary, then the magnitude of e^{rx} is 1 for all x , and the magnitude of $g_X(r)$ is at most one. For values of r with a positive real part, $g_X(r)$ only exists if $1-F_X(x)$ approaches 0 at least exponentially as $x \rightarrow \infty$. Similarly, for values of r with a negative real part, $g_X(r)$ exists only if $F_X(x)$ approaches 0 at least exponentially as $x \rightarrow -\infty$. If $g_X(r)$ exists in a region of real r around 0, then derivatives of all orders also exist, given by

$$\frac{\partial^n g_X(r)}{\partial r^n} = \int_{-\infty}^{\infty} x^n e^{rx} dF_X(x) ; \quad \left. \frac{\partial^n g_X(r)}{\partial r^n} \right|_{r=0} = E[X^n] \quad (17)$$

This shows that finding the moment generating function often provides a convenient way to calculate the moments of a random variable. Another convenient feature of moment generating functions is their use in dealing with sums of independent random variables. For example, suppose $S = X_1 + X_2 + \dots + X_n$. Then

$$g_S(r) = E[e^{rS}] = E\left[\exp\left(\sum_{i=1}^n rX_i\right)\right] = E\left[\prod_{i=1}^n \exp(rX_i)\right] = \prod_{i=1}^n g_{X_i}(r) \quad (18)$$

In the last step here, we have used (15). There are many other similar types of transforms. For example, if we replace e^r with z , we get the z transform; this is mainly useful for integer valued random variables, but if one transform can be evaluated, the other can be found immediately. If we use $j\omega$ in place of r , where $j=\sqrt{-1}$ and ω is real, we get the characteristic function; it is the inverse Fourier transform of the density function of X and (as pointed out before), it always exists. Finally, if we use $-s$, viewed as a complex variable, in place of r , we get the two sided Laplace transform of the density of the random variable. Note that for all of these transforms, multiplication in the transform domain corresponds to convolution of the distribution functions or densities, and summation of independent random variables. It is the simplicity of taking products of transforms that make transforms so useful in probability theory. We will use transforms sparingly here, since, along with simplifying computational work, they frequently obscure underlying probabilistic insights.

1.7 WEAK LAW OF LARGE NUMBERS

The laws of large numbers are a collection of results in probability that describe the behavior of the arithmetic average of n random variables, for n large. Under fairly general assumptions, the standard deviation of the average goes to 0 with increasing n , and, in various ways, depending on the assumptions, the sample average approaches the mean. These results are central to the study of stochastic processes because they allow us to relate time averages (i.e., the average over time of individual sample paths) to ensemble averages (i.e., the mean of the value of the process at a given time). In this and the next section, we discuss two of these results, the weak and the strong law of large numbers for independent identically distributed random variables. The strong law requires considerable patience to understand, but it will be used frequently throughout the text. We start with some basic inequalities which are useful in their own right and which lead us directly to the weak law of large numbers.

BASIC INEQUALITIES: The Markov inequality states that if a non-negative random variable Y has a mean $E[Y]$, then the probability that the outcome exceeds any given number y satisfies

$$P(Y \geq y) \leq \frac{E[Y]}{y} \quad \text{Markov Inequality} \quad (19)$$

Figure 1.4 derives this result using the fact (see figure 1.3) that the mean of a non-negative random variable is the integral of its complementary distribution function, i.e., of the area under the curve $P(Y>z)$ or equivalently under the curve $P(Y\geq z)$.

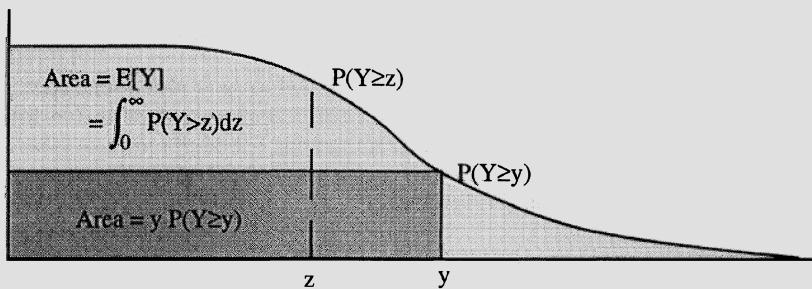


Figure 1.4. Demonstration that $yP(Y \geq y) \leq E[Y]$.

As an example of this inequality, assume that the average height of a population of people is 1.6 meters. Then the Markov inequality states that at most one half of the population have a height exceeding 3.2 meters. We see from the example that this inequality is typically very weak. However, for any $y > 0$, we can consider a random variable that takes on the value y with probability ϵ and the value 0 with probability $1-\epsilon$; this random variable satisfies the Markov inequality at the point y with equality. Another application of figure 1.4 is the observation that, for any given non-negative random variable with finite mean, (i.e., with finite area under the curve $P(Y \geq y)$),

$$\lim_{y \rightarrow \infty} y P(Y \geq y) = 0 \quad (20)$$

This is shown in exercise 1.18 and will be useful shortly in the proof of theorem 1.

Next, let Z be an arbitrary random variable with finite mean $E[Z]$ and finite variance σ_Z^2 , and define Y as the non-negative random variable $Y = (Z - E[Z])^2$. Thus $E[Y] = \sigma_Z^2$. Applying (19),

$$P\left(\left(Z - E[Z]\right)^2 \geq y\right) \leq \frac{\sigma_Z^2}{y}$$

Replacing y with ϵ^2 (for any $\epsilon > 0$) and noting that the event $(Z - E[Z])^2 \geq \epsilon^2$ is the same as $|Z - E[Z]| \geq \epsilon$, this becomes the well known Chebyshev inequality,

$$P(|Z - E[Z]| \geq \epsilon) \leq \frac{\sigma_Z^2}{\epsilon^2} \quad \text{Chebyshev inequality} \quad (21)$$

Note that the Markov inequality bounds just the upper tail of the distribution function and applies only to non-negative random variables, whereas the Chebyshev inequality bounds both tails of the distribution function. The more important difference, however,

is that the Chebyshev bound goes to zero inversely with the square of the distance from the mean, whereas the Markov bound goes to zero inversely with the distance from 0 (and thus asymptotically with distance from the mean).

There is another variant of the Markov inequality, known as an exponential bound or a Chernoff bound, in which the bound goes to 0 exponentially with distance from the mean. Let $Y = \exp(rZ)$ for some arbitrary random variable Z that has a moment generating function, $g_Z(r) = E[\exp(rZ)]$ over some open interval of real values of r including $r=0$. Then, for r in that interval, (19) becomes

$$P(\exp(rZ) \geq y) \leq \frac{g_Z(r)}{y}$$

Letting $y = \exp(ra)$ for some constant a , we have the two inequalities,

$$P(Z \geq a) \leq g_Z(r) \exp(-ra); \text{ for } r \geq 0. \quad (\text{Exponential bound}) \quad (22)$$

$$P(Z \leq a) \leq g_Z(r) \exp(-ra); \text{ for } r \leq 0. \quad (\text{Exponential bound}) \quad (23)$$

These bounds can be optimized over r to get the strongest bound; the bound in (22), however, is exponentially decreasing in a for fixed $r > 0$, and the bound in (23) is exponentially decreasing in $-a$ for fixed $r < 0$. These bounds will be used extensively in chapter 7 and are useful in information theory, detection theory, and random walk theory.

WEAK LAW ASSUMING A FINITE VARIANCE: Let X_1, X_2, \dots, X_n be IID random variables with a finite mean \bar{X} and finite variance σ^2 , let $S_n = X_1 + \dots + X_n$, and consider the sample average S_n/n . We saw in (14) that $\sigma_{S_n}^2 = n\sigma^2$. Thus the variance of S_n/n is

$$\text{VAR}(S_n/n) = E\left[\left(\frac{S_n - n\bar{X}}{n}\right)^2\right] = \frac{1}{n^2}E[(S_n - n\bar{X})^2] = \frac{\sigma^2}{n} \quad (24)$$

This says that the standard deviation of the sample average S_n/n is σ/\sqrt{n} . Thus, the standard deviation of S_n approaches ∞ as $n \rightarrow \infty$, whereas the standard deviation of S_n/n approaches 0 as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} E[(S_n/n - \bar{X})^2] = 0$, S_n/n is said to *converge to $E[X]$ in the mean square sense*.

Applying the Chebyshev inequality in (21) to the sample average S_n/n , we have

$$P\left(\left|\frac{S_n}{n} - \bar{X}\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2} \quad (25)$$

For any $\varepsilon > 0$, we can pass to the limit $n \rightarrow \infty$, getting

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{S_n}{n} - \bar{X}\right| \geq \varepsilon\right) = 0 \quad (26)$$

Eqs. (25) and (26) are alternate forms of the weak law of large numbers for IID random variables $\{X_i\}$ that have a mean and variance. For any $\epsilon > 0$, (26) says that $\lim_{n \rightarrow \infty} P(A_n) = 0$ where A_n is the event that the sample average S_n/n differs from the true mean by more than ϵ . This means (see figure 1.5) that the distribution function of S_n/n approaches a unit step function with the step at \bar{X} as $n \rightarrow \infty$. Because of (26), S_n/n is said to *converge to $E[X]$ in probability*.

To get a better appreciation of (25) and (26), consider figures 1.6, 1.7, and 1.8. These figures show the distribution functions of S_n , $(S_n - n\bar{X})/\sqrt{n}$, and $(S_n - n\bar{X})/\sqrt{n}$ as a function of n for a typical random variable X that is binary with $P(X=0) = 3/4$, $P(X=1) = 1/4$. There are two important effects to be observed in figure 1.6. First, $E[S_n]$ equals $n\bar{X}$, which is linear in n . Second, the standard deviation of S_n is $\sqrt{n}\sigma$, which gives rise to a spreading of the distribution with \sqrt{n} . In figure 1.7, note that the standard deviation of S_n/n decreases as $1/\sqrt{n}$, and note the corresponding compression of the distribution function with increasing n . Note finally that with the normalization of figure 1.8, the distribution function neither compresses or expands, but simply becomes smoother with increasing n .

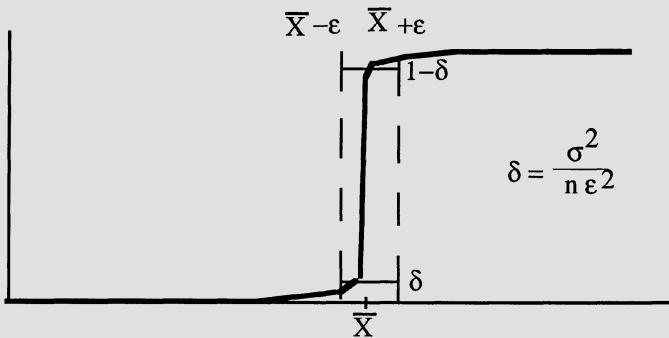


Figure 1.5. Approximation of a distribution function by a step function. Note the rectangle of width 2ϵ and height $1-2\delta$ in the figure. Eq. (25) asserts that the distribution function of S_n/n enters this rectangle from below and exits from above, thus approximating a unit step.

THE CENTRAL LIMIT THEOREM: The law of large numbers does *not* say that S_n is close to $n\bar{X}$ with high probability as $n \rightarrow \infty$. In fact, the standard deviation of S_n is $\sigma\sqrt{n}$, which increases with n . It can be seen that the standard deviation of S_n/\sqrt{n} is σ and does not vary with n . In fact, the celebrated central limit theorem states that if $\sigma^2 < \infty$, then

$$\lim_{n \rightarrow \infty} \left[P\left(\frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq y \right) \right] = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (27)$$

Note that the random variable $(S_n - n\bar{X})/(\sqrt{n}\sigma)$ in (27) has mean 0 and variance 1 for all n . The central limit theorem says that this random variable tends to the distribution on the right hand side of (27) as $n \rightarrow \infty$. This is the *normal* distribution, or *normalized Gaussian* distribution. The theorem is illustrated by figure 1.8. The difference between the right side of (27) and the term in brackets on the left goes to 0 as $1/\sqrt{n}$ if X has a third moment and it goes to zero, but perhaps slowly, if X does not have a third moment. For large negative y , the right side of (27) is very close to 0, so, for moderate values of n , the ratio of the left to right side of (27) might be (and typically is) far from 1, even though the difference between left and right is very small. For large values of y , both sides of (27) are close to 1, so that again (27) does not yield a very good approximation for reasonable values of n . It is for this reason that the word central appears in the name “central limit theorem.”

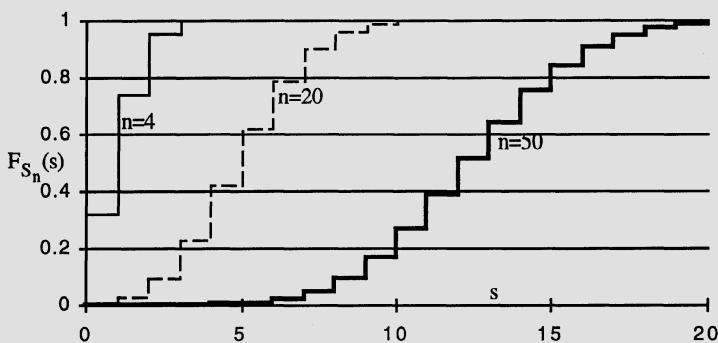


Figure 1.6. The distribution function of S_n as a function of n . In the figure, S_n is the sum of n binary random variables with $P(1) = 1/4$, $P(0) = 3/4$.

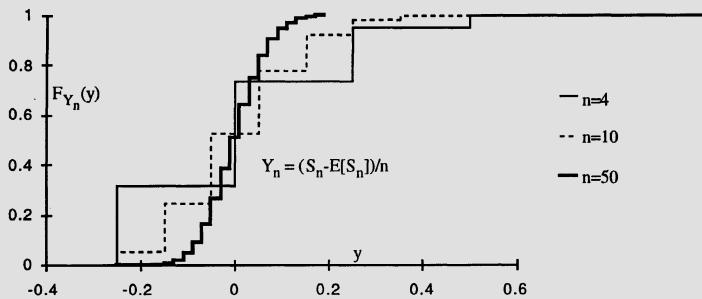


Figure 1.7. The same distribution as figure 1.6, scaled differently to focus on sample average.

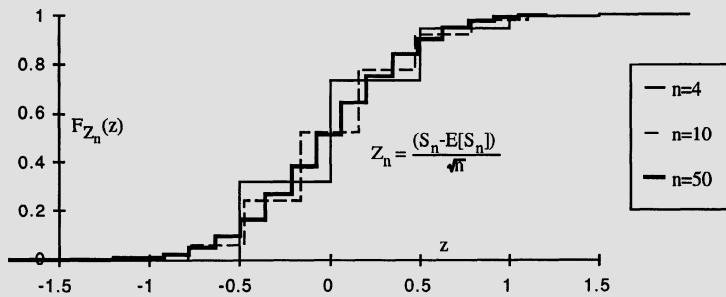


Figure 1.8. The same distribution as figure 1.6 scaled for constant standard deviation

The central limit theorem (CLT) helps explain why Gaussian random variables play such a central role in probability theory. In fact, many of the cookbook formulas of elementary statistics are based on the tacit assumption that the underlying variables are Gaussian, and the CLT helps explain why these formulas often give reasonable results.

One should be careful to avoid reading more into the CLT than it says. For example, the normalized sum, $[S_n - n\bar{X}] / (\sqrt{n} \sigma)$ need not have a *density* that is approximately Gaussian (in fact, if the underlying variables are discrete, the normalized sum is also, and does not have a density at all). What is happening is that the normalized sum can have very detailed fine structure; this does not disappear as n increases, but becomes “integrated out” in the distribution function. We will not use the CLT extensively here, and will not prove it (See Feller⁴ for a thorough and careful exposition on various forms of the central limit theorem). Giving a proof from first principles is quite tricky; many elementary texts on probability give “heuristic” proofs indicating that the normalized sum has a density that tends to Gaussian (thus indicating that both the heuristics and the manipulations are wrong).

Since the central limit theorem gives such explicit information on the behavior of S_n as $n \rightarrow \infty$, one wonders why the law of large numbers should be studied at all. There are three answers—the first is that sometimes one doesn’t care about the added information in (27), and that the added information obscures some issues of interest. The next is that (27) is only meaningful if the variance σ^2 of X is finite, whereas, as we soon show, (26) holds whether or not the variance is finite. One might think that variables with infinite variance are of purely academic interest. Unfortunately, it is precisely in renewal theory that random variables without variances are sometimes important.

The third reason for interest in laws of large numbers is that they hold in many situations more general than that of sums of IID random variables.⁵ These more general laws of large numbers can usually be interpreted as saying that some time average (i.e., an average over time of a sample function of a stochastic process) approaches the expected value of the process at a given time. Since expected values only exist within a probability model, but time averages can be evaluated from a sample function of the

actual process being modeled, *the relationship between time averages and expected values is the link between a model and the reality being modeled.*

RELATIVE FREQUENCY AND INDICATOR FUNCTIONS: We next show that (25) and (26) can be applied to the relative frequency of an event as well as to the sample average. For any event A, we define I_A , the *indicator function* of A, to be a random variable that has the value 1 for all sample points in A and has the value 0 otherwise. Thus $P(A) = E[I_A]$. Indicator functions are useful because they allow us to apply many of the results we know about random variables to events. For the situation here, we view an experiment containing an event A as being independently repeated n times (i.e., all events in each repetition are independent of events in each other repetition). Let A_i be the event A in the i^{th} repetition. The relative frequency of A over the n experiments is then the number of times that A occurs divided by n. This is best expressed as

$$\text{relative frequency of } A = \frac{\sum_{i=1}^n I_{A_i}}{n} \quad (28)$$

Thus the relative frequency is the sample average of the indicator functions, and, from (25)

$$P\left(\left| \text{relative frequency of } A - P[A] \right| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2} \quad (29)$$

where σ^2 is the variance of I_A , which is $P(A)[1-P(A)]$.

DIGRESSION: One usually motivates the use of probability theory for real world applications by first making an analogy between the probability of an event A (theory) and relative frequency (in the real world). One also makes an analogy between independence (theory) and physical independence of experiments (real world). It then appears that (29) justifies the connection between probability and relative frequency.

A sceptic, listening to this argument, will rightfully call “FOUL,” because of the somewhat circular nature of the argument. That is, the real world must “act like” the probability model in performing the n independent experiments, and we cannot precisely define what that means for the real world.

For example, we could determine, for a sample of integrated circuits, the relative frequency of those that work correctly. However, different integrated circuits in the sample might be manufactured on different machines, at different times of the day, with different operators, etc. Asserting that these are independent and identically distributed depends on understanding both the physical processes and the theory.

What the law of large numbers does is to relate relative frequency *within the theory* to the probability of an event *within the theory*. It provides a type of consistency, within

the theory, between relative frequency and probability. This is all that can be expected. Theorems establish rigorous results within a model of reality, but cannot prove things about the real world itself. These theorems, however, provide us with the framework to understand and interpret the behavior of the real world. They both allow us to improve our models, and to predict future behavior to the extent that the models reflect reality.

WEAK LAW WITH INFINITE VARIANCE: We now establish the law of large numbers without assuming a finite variance.

THEOREM 1: WEAK LAW OF LARGE NUMBERS: Let $S_n = X_1 + \dots + X_n$ where X_1, X_2, \dots are IID random variables with a finite mean $E[X]$. Then for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{S_n}{n} - E[X]\right| \geq \epsilon\right) = 0 \quad (30)$$

Proof*: We use a truncation argument; such arguments are used frequently in dealing with random variables that have infinite variance. Let b be a real number (which we later take to be increasing with n), and for each variable X_i , define a new random variable \tilde{X}_i (see figure 1.9) by

$$\begin{aligned} \tilde{X}_i &= X_i && \text{if } |X_i - E[X]| \leq b \\ \tilde{X}_i &= E[X] + b && \text{if } X_i - E[X] \geq b \\ \tilde{X}_i &= E[X] - b && \text{if } X_i - E[X] \leq -b \end{aligned} \quad (31)$$

The variables \tilde{X}_i are IID and we let $E[\tilde{X}]$ be the mean of \tilde{X}_i . As shown in exercise 1.10, the variance of \tilde{X} can be upper bounded by the second moment around any other value, so $\sigma_{\tilde{X}}^2 \leq E[(\tilde{X} - E[\tilde{X}])^2]$. This can be further upper bounded by

$$\sigma_{\tilde{X}}^2 \leq E[(\tilde{X} - E[\tilde{X}])^2] = \int_{-\infty}^{\infty} (x - E[\tilde{X}])^2 dF_{\tilde{X}}(x) \leq b \int_{-\infty}^{\infty} |x - E[\tilde{X}]| dF_{\tilde{X}}(x)$$

The last inequality follows since $|x - E[\tilde{X}]| \leq b$ over the range of \tilde{X} . We next use the fact that $F_{\tilde{X}}(x) = F_X(x)$ for $E[X] - b < x < E[X] + b$ to upper bound the final integral.

$$\sigma_{\tilde{X}}^2 \leq b \int_{-\infty}^{\infty} |x - E[\tilde{X}]| dF_X(x) = b \alpha \quad \text{where } \alpha = \int_{-\infty}^{\infty} |x - E[X]| dF_X(x) \quad (32)$$

The quantity α in (32) is the mean of $|X - E[X]|$ and must exist since we assume that X has a mean (see example 6). Now, letting $\tilde{S}_n = \tilde{X}_1 + \dots + \tilde{X}_n$ (and using $\epsilon/2$ in place of ϵ), (25) becomes

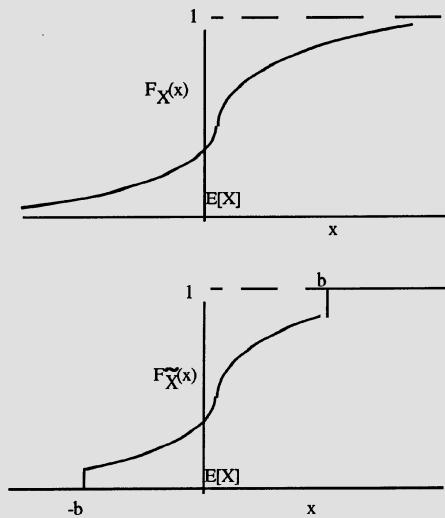


Figure 1.9. Truncated variable \tilde{X} .

$$P\left(\left|\frac{\tilde{S}_n}{n} - E[\tilde{X}]\right| \geq \frac{\epsilon}{2}\right) \leq \frac{4 \sigma_{\tilde{X}}^2}{n \epsilon^2} \leq \frac{4b\alpha}{n\epsilon^2}$$

As b increases, $E[\tilde{X}]$ approaches $E[X]$. Thus for sufficiently large b , $|E[\tilde{X}] - E[X]| < \epsilon/2$ and

$$P\left(\left|\frac{\tilde{S}_n}{n} - E[X]\right| \geq \epsilon\right) \leq \frac{4b\alpha}{n\epsilon^2} \quad (33)$$

Now \tilde{S}_n and S_n have the same value for sample points where $|X_i - E[X]| \leq b$ for all i , $1 \leq i \leq n$. Thus, using the union bound (which says that the probability of a union of events is less than or equal to the sum of the probabilities of the individual events),

$$P(\tilde{S}_n \neq S_n) \leq n P(|X - E[X]| > b) \quad (34)$$

The event $\{|(S_n/n) - E[X]| \geq \epsilon\}$ can only occur if either $|(\tilde{S}_n/n) - E[X]| \geq \epsilon$ or if $\tilde{S}_n \neq S_n$. Thus, combining (33) and (34), and letting $\delta = b/n$, we have

$$P\left(\left|\frac{S_n}{n} - E[X]\right| \geq \epsilon\right) \leq \frac{4\delta\alpha}{\epsilon^2} + \frac{1}{\delta} \left[\delta n P\left(|X - E[X]| > \delta n\right) \right] \quad (35)$$

Since (33) and (34) are valid for arbitrary n and sufficiently large $b > 0$, (35) is valid for arbitrary $\delta > 0$ and sufficiently large n . For any given $\epsilon > 0$, we now choose δ to make the first term on the right of (35) as small as desired. From (20), the final term in brackets in (35) can be made arbitrarily small by choosing n large enough, and thus the right hand side of (35) can be made arbitrarily small by choosing n large enough, thus completing the proof.

EXAMPLE 6: The Cauchy random variable Z has the probability density $f_Z(z) = 1/\pi(1+z^2)$. The mean of Z does not exist, and Z has the very peculiar property that $[Z_1+Z_2+\dots+Z_n]/n$ has the same density as Z for all n . Thus the law of large numbers does not hold for the Cauchy distribution, which is not surprising since the mean doesn't exist. Recall that the mean of a random variable exists only if

$$\int_{-\infty}^0 x dF_X(x) > -\infty \quad \text{and} \quad \int_0^\infty x dF_X(x) < \infty, \quad \text{or equivalently,} \quad (36)$$

$$\int_{-\infty}^\infty |x| dF_X(x) < \infty$$

From symmetry, we note that, for the Cauchy distribution, the integral $\int_{z=-b}^b dF_Z(z)$ is zero for all b , and thus the integral exists in the Cauchy principal value sense, but not in the ordinary sense of (36). In this text, the existence of integrals always refers to existence in the ordinary rather than Cauchy principal value sense.

1.8 STRONG LAW OF LARGE NUMBERS

We next discuss the strong law of large numbers. We will not prove this result here, but will prove a slightly weaker form of it after discussing martingales in Chapter 7.

THEOREM 2: STRONG LAW OF LARGE NUMBERS (Version 1): Let $S_n = X_1 + \dots + X_n$ where X_1, X_2, \dots are IID random variables with a finite mean \bar{X} . Then for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P\left(\sup_{m \geq n} \left| \frac{S_m}{m} - \bar{X} \right| > \epsilon\right) = 0 \quad (37)$$

The notation *sup* above stands for supremum (see note 2). The supremum of a set $\{Y_i; i \geq 1\}$ of random variables is a random variable. For each sample point ω in the sample

space, $Y_i(\omega)$ is the sample value of Y_i for that ω , and $\sup_i Y_i(\omega)$ is the supremum of those sample values. Thus, $\sup_i Y_i$ is the random variable that maps each sample point ω into $\sup_i Y_i(\omega)$.

We can then interpret the event, $\{\sup_{m \geq n} |S_m/m - \bar{X}| > \epsilon\}$ as the event that $|S_m/m - \bar{X}|$ exceeds ϵ for at least one value of $m \geq n$. The theorem states that the probability of this event approaches 0 with increasing n . Thus, the theorem states that, for large n , it is unlikely that the sequence $|S_m/m - \bar{X}|$ for $m \geq n$ ever exceeds ϵ . An alternative way to express this limit is the statement that for each $\epsilon > 0$ and each $\delta > 0$, there is an integer $n(\epsilon, \delta)$ such that

$$P\left(\sup_{m \geq n(\epsilon, \delta)} \left| \frac{S_m}{m} - \bar{X} \right| > \epsilon\right) \leq \delta \quad (38)$$

The weak law states that it is increasingly unlikely (with increasing n) for S_n/n to deviate from the mean by any given $\epsilon > 0$. The strong law states that the sequence $S_1, S_2/2, S_3/3, \dots$ has the property that it is increasingly unlikely for *any* term in the sequence beyond n to deviate by more than ϵ from the mean. The following example illustrates the difference between the strong and weak law. Note that in this example, the random variables X_i ; $i \geq 1$ are neither independent nor equally distributed. Since both the strong and weak law hold for IID variables, we cannot illustrate the difference for the IID case.

EXAMPLE 7: Let $\{S_n; n \geq 1\}$ be a sequence of independent random variables for which S_n has value n with probability $1/n$ and value 0 otherwise (see figure 1.10). Then $E[S_n] = 1$ and $E[S_n/n] = 1/n$. If one wishes, S_n can be interpreted as a sum of dependent random variables, $X_1 + \dots + X_n$ where $X_1 = S_1$ and $X_i = S_i - S_{i-1}$, but the argument really just concerns the sequence $\{S_n; n \geq 1\}$. For any ϵ , $0 < \epsilon < 1$, the Markov inequality shows that $P(S_n/n > \epsilon) \leq (n\epsilon)^{-1}$, so $\lim_{n \rightarrow \infty} P(S_n/n > \epsilon) = 0$. Next, we see that $\sup_{m \geq n} S_m/m$ takes on only the values 0 and 1, and takes on the value 0 only if $S_m/m = 0$ for all $m \geq n$. This is an event of probability $\prod_{m \geq n} (1-1/m)$, which (see exercise 1.19) can be seen to be zero for all n . Thus

$$\lim_{n \rightarrow \infty} P\left(\frac{S_n}{n} > \epsilon\right) = 0 \text{ but } \lim_{n \rightarrow \infty} P\left(\sup_{m \geq n} \frac{S_m}{m} > \epsilon\right) = 1$$

This says that S_n/n is likely to be small for large n (and in fact likely to be 0), but that if one waits long enough beyond n , a sample sequence has a value exceeding ϵ (and in fact equal to 1) with probability one.

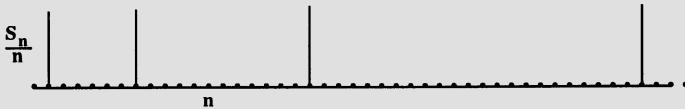


Figure 1.10. Sample function of $\{S_n/n; n \geq 1\}$.

One sees from this example that the strong law is saying something about a sample outcome of an infinite sequence of random variables. If one views X_i as a time sequence, then the sample output from the sequence of sample averages S_n/n can be viewed as a sequence of more and more elaborate attempts to estimate the mean. There is clearly some advantage to being able to say that this sequence of attempts not only gets close to the mean with high probability but also stays close to the mean.

Despite the above rationalization, the difference between the strong and weak law almost appears to be mathematical nit picking. On the other hand, we shall discover, as we use these results, that the strong law is often much easier to use than the weak law. The useful form of the strong law, however, is the following theorem. The statement of this theorem is deceptively simple, and it will take some care to understand what the theorem is saying.

THEOREM 3: STRONG LAW OF LARGE NUMBERS (Version 2): Let $S_n = X_1 + \dots + X_n$ where X_1, X_2, \dots are IID random variables with a finite mean \bar{X} . Then with probability 1,

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \bar{X} \quad (39)$$

For each sample point ω , $S_n(\omega)/n$ is a sequence of real numbers that might or might not have a limit. If this limit exists for all sample points, then $\lim_{n \rightarrow \infty} S_n/n$ is a random variable that maps each sample point ω into $\lim_{n \rightarrow \infty} S_n(\omega)/n$. Usually this limit does not exist for all sample points, but the theorem implicitly asserts that the limit does exist for all sample points except a set of probability 0. Thus $\lim_{n \rightarrow \infty} S_n/n$ is still regarded as a random variable. The theorem asserts not only that $\lim_{n \rightarrow \infty} S_n(\omega)/n$ exists for all sample points except a set of zero probability, but also asserts that the limit is equal to \bar{X} for all sample points except a set of probability 0. A sequence of random variables S_n/n that converges in the sense of (39) is said to *converge with probability 1*.

EXAMPLE 8: Suppose the X_i are Bernoulli with equiprobable ones and zeros. Then $\bar{X} = 1/2$. We can easily construct sequences for which the sample average is not 1/2; for example the sequence of all zeros, the sequence of all ones, sequences with 1/3 zeros and 2/3 ones, and so forth. The theorem says, however, that collectively those sequences have zero probability.

Proof of theorem 3: We assume theorem 2 (which we do not prove until Chapter 7) in order to prove theorem 3. Consider the event illustrated in figure 1.11 in which tighter and tighter bounds are placed on successive elements of the sequence $\{S_n/n; n \geq 1\}$. In particular, for some increasing set of positive integers n_1, n_2, \dots , we consider the bound $|S_n/n - \bar{X}| \leq 2^{-k}$ for $n_k \leq n < n_{k+1}$. For any sample point ω , if $\{S_n(\omega); n \geq 1\}$ satisfies all these constraints, then $\lim_{n \rightarrow \infty} S_n(\omega)/n = \bar{X}$. The probability of the complementary set of sample points for which one of these bounds is *unsatisfied* is given by

$$P\left(\bigcup_{k \geq 1} \left[\bigcup_{n_k \leq n < n_{k+1}} \left(\left| \frac{S_n}{n} - \bar{X} \right| > 2^{-k} \right) \right] \right) = P\left(\bigcup_{k \geq 1} \left[\bigcup_{n_k \leq n} \left(\left| \frac{S_n}{n} - \bar{X} \right| > 2^{-k} \right) \right] \right) \quad (40)$$

$$\leq \sum_{k=1}^{\infty} P\left(\bigcup_{n \geq n_k} \left(\left| \frac{S_n}{n} - \bar{X} \right| > 2^{-k} \right) \right) \quad (41)$$

$$= \sum_{k=1}^{\infty} P\left(\sup_{n \geq n_k} \left| \frac{S_n}{n} - \bar{X} \right| > 2^{-k} \right) \quad (42)$$

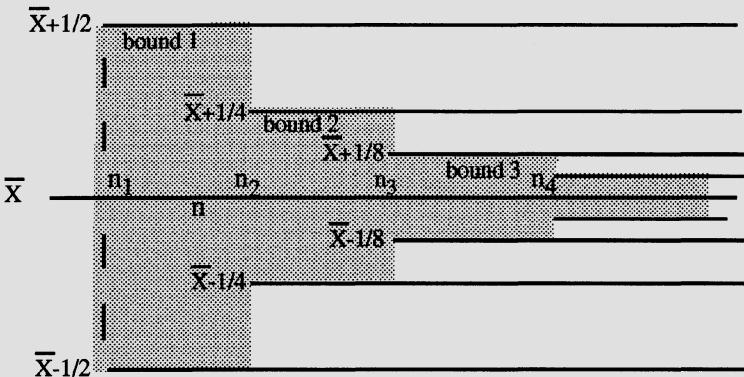


Figure 1.11. Illustration of the union of events in (40); the k^{th} sub-event in (40) is the set of sample points for which S_n/n falls outside of the k^{th} bound for some n , $n_k \leq n < n_{k+1}$, i.e., for which $|S_n/n - \bar{X}| > 2^{-k}$ for some $n_k \leq n < n_{k+1}$.

The first equality is most easily visualized in figure 1.11; if $|S_n/n - \bar{X}| > 2^{-k}$ for one value of k , then $|S_n/n - \bar{X}| > 2^{-k}$ for all $k \geq k$. In going from (40) to (41), we have used the union bound; this says that the probability of a union of events is less than or equal to the sum of the probabilities of the individual events. Finally (42) follows because the supremum of a sequence exceeds 2^{-k} if and only if one of the elements exceeds 2^{-k} .

From (38), for any $\epsilon, \delta > 0$, there is an $n(\epsilon, \delta)$ such that $P(\sup_{n \geq n(\epsilon, \delta)} |S_n/n - \bar{X}| > \epsilon) \leq \delta$. For given δ_0 , we then choose n_k in the bound above as $n_k = n(2^{-k}, \delta_0 2^{-k})$, i.e., so that $P(\sup_{n \geq n_k} |S_n/n - \bar{X}| > 2^{-k}) \leq \delta_0 2^{-k}$. Substituting this in (42), we have

$$\sum_{k=1}^{\infty} P\left(\sup_{n \geq n_k} \left| \frac{S_n}{n} - \bar{X} \right| > 2^{-k} \right) \leq \sum_{k=1}^{\infty} \delta_0 2^{-k} = \delta_0 \quad (43)$$

It follows that the set of sample points that do not violate these bounds has probability at least $1 - \delta_0$, and as we have seen, $\lim_{n \rightarrow \infty} S_n/n = \bar{X}$ for each of these sample points. Since this is true for any $\delta_0 > 0$, the set of sample points for which $\lim_{n \rightarrow \infty} S_n/n = \bar{X}$ must have probability 1, completing the proof.

Note that as δ_0 is decreased, the integers n_1, n_2, \dots become larger, thus enlarging the set of sample points that fall within the given bounds. Thus if $\{S_n(\omega)/n; n \geq 1\}$ converges very slowly to \bar{X} for a given ω , then a very small value of δ_0 is required for $\{S_n(\omega)/n; n \geq 1\}$ to stay within the bounds of figure 1.11.

1.9 SUMMARY

This chapter has provided a brief review of elementary probability theory, starting with the basic ingredients of sample space, events, and probabilities of events, then moving to random variables, and then to laws of large numbers. The emphasis has been on understanding the underlying structure of the field rather than reviewing details and

TABLE OF STANDARD RANDOM VARIABLES

The following table summarizes the properties of some common random variables. If a density or PMF is specified only in a given region, it is assumed to be zero elsewhere.

Name	Density or PMF	Mean	Variance	Generating function
<i>(Continuous rv, $f_x(x)$)</i>				
Exponential	$\lambda \exp(-\lambda x); x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-r}$
Erlang	$\frac{\lambda^n x^{n-1} \exp(-\lambda x)}{(n-1)!}; x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-r}\right)^n$
Gaussian	$\frac{\exp(-x^2/2\sigma^2)}{\sqrt{2\pi} \sigma}$	0	σ^2	$\exp(r^2\sigma^2/2)$
Uniform	$\frac{1}{a}; 0 \leq x \leq a$	$\frac{a}{2}$	$\frac{a^2}{12}$	$\frac{\exp(ra) - 1}{ra}$
<i>(Integer rv, $P_N(n)$)</i>				
Bernoulli	$P_N(0)=1-p; P_N(1)=p$	p	$p(1-p)$	$1-p + pe^r$
Binomial	$\frac{k!}{n!(k-n)!} p^n (1-p)^{k-n}; 0 \leq n \leq k$	kp	$kp(1-p)$	$[1-p + pe^r]^k$
Geometric	$(1-p)p^n; n \geq 0$	$\frac{p}{1-p}$	$\frac{p}{(1-p)^2}$	$\frac{1-p}{1-pe^r}$
Poisson	$\frac{\lambda^n \exp(-\lambda)}{n!}; n \geq 0$	λ	λ	$\exp[\lambda(e^r - 1)]$

problem solving techniques. The strong law of large numbers requires mathematical maturity, and might be postponed to Chapter 3 when it is first used.

There are too many texts on elementary probability to mention here, and most of them serve to give added understanding and background to the material here. [Ros94] and [Dra67] are both quite readable. [Kol50] is of historical interest (and is also readable) as the translation of the 1933 book that first put probability on a firm mathematical basis. [Fel68] is an extended and elegant treatment of elementary material from a mature point of view.

EXERCISES

1.1) The text shows that, for a non-negative random variable X with distribution function $F_X(x)$, $E[X] = \int_0^\infty [1 - F_X(x)]dx$.

a) Write this integral as a sum for the special case in which X is a non-negative integer random variable.

b) Generalize the above integral for the case of an arbitrary (rather than non-negative) random variable Y with distribution function $F_Y(y)$; use a graphical argument.

c) Find $E[|Y|]$ by the same type of argument.

d) For what value of α is $E[|Y-\alpha|]$ minimized? Use a graphical argument again.

1.2) Let X be a random variable with distribution function $F_X(x)$. Find the distribution function of the following random variables.

a) The maximum of n IID random variables with distribution function $F_X(x)$.

b) The minimum of n IID random variables with distribution $F_X(x)$.

c) The difference of the random variables defined in (a) and (b); assume X has a density $f_X(x)$.

1.3) a) Let X_1, X_2, \dots, X_n be random variables with expected values $\bar{X}_1, \dots, \bar{X}_n$. Prove that $E[X_1 + \dots + X_n] = \bar{X}_1 + \dots + \bar{X}_n$. Do not assume that the random variables are independent.

b) Now assume that X_1, \dots, X_n are statistically independent and show that the expected value of the product is equal to the product of the expected values.

c) Again assuming that X_1, \dots, X_n are statistically independent, show that the variance of the sum is equal to the sum of the variances.

1.4) Suppose X_1, X_2, X_3, \dots is a sequence of continuous IID random variables. X_n , for a given $n > 1$, is called a local minimum of the sequence if $X_n \leq X_{n+1}, X_n \leq X_{n-1}$. Find the probability that X_n is a local minimum. Hint: No computation is necessary—use symmetry.

1.5) Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent identically distributed (IID) continuous random variables with the common probability density function $f_X(x)$; note that $P(X = \alpha) = 0$ for all α and that $P(X_1 = X_2) = 0$.

- a) Find $P(X_1 \leq X_2)$ (give a numerical answer, not an expression; no computation is required and a one or two line explanation should be adequate).
- b) Find $P(X_1 \leq X_2; X_1 \leq X_3)$ (in other words, find the probability that X_1 is the smallest of X_1, X_2, X_3 ; again, think—don’t compute).
- c) Let the random variable N be the index of the first r.v. in the sequence to be less than X_1 ; that is, $P(N=n) = P(X_1 \leq X_2; X_1 \leq X_3; \dots; X_1 \leq X_{n-1}; X_1 > X_n)$. Find $P(N > n)$ as a function of n . Hint: Generalize part (b).
- d) Show that $E[N] = \infty$.

- 1.6 a)** Assume that X is a discrete random variable taking on values a_1, a_2, \dots , and let $Y = g(X)$. Let $b_i = g(a_i)$, $i \geq 1$ be the i th value taken on by Y . Show that $E[Y] = \sum_i b_i P_Y(b_i) = \sum_i g(a_i) P_X(a_i)$.
- b) Let X be a continuous random variable with density $f_X(x)$ and let g be differentiable and monotonic increasing. Show that $E[Y] = \int y f_Y(y) dy = \int g(x) f_X(x) dx$.

- 1.7 a)** Show that, for uncorrelated random variables, the expected value of the product is equal to the product of the expected values (X and Y are uncorrelated if

$$E[(X-E[X])(Y-E[Y])] = 0.$$

- b) Show that if X and Y are uncorrelated, then the variance of $X + Y$ is equal to the variance of X plus the variance of Y .
- c) Show that if X_1, \dots, X_n are uncorrelated, the the variance of the sum is equal to the sum of the variances.
- d) Show that independent random variables are uncorrelated.
- e) Let X, Y be identically distributed ternary valued random variables with the probability assignment $P(X = 1) = P(X = -1) = 1/4$; $P(X = 0) = 1/2$. Find a *simple* joint probability assignment such that X and Y are uncorrelated but dependent.
- f) You have seen that the moment generating function of a sum of independent random variables is equal to the product of the individual moment generating functions. Give an example where this is false if the variables are uncorrelated but dependent.

- 1.8)** Suppose X has the Poisson PMF, $P(X = n) = \lambda^n \exp(-\lambda)/n!$ for $n \geq 0$ and Y has the Poisson PMF, $P(Y = n) = \mu^n \exp(-\mu)/n!$ for $n \geq 0$. Find the distribution of $Z = X + Y$ and find the conditional distribution of Y conditional on $Z = n$.

- 1.9) a)** Suppose X, Y and Z are binary random variables, each taking on the value 0 with probability 1/2 and the value 1 with probability 1/2. Find an example in which X, Y, Z are statistically dependent but are *pairwise* statistically independent (i.e., X, Y are statistically independent, X, Z are statistically independent, and Y, Z are statistically independent). Give $P_{XYZ}(x,y,z)$ for your example.

- b) Is pairwise statistical independence enough to ensure that

$$E\left[\prod_{i=1}^n X_i\right] = \prod_{i=1}^n E[X_i]$$

for a set of random variables X_1, \dots, X_n ?

1.10) Show that $E[X]$ is the value of z that minimizes $E[(X-z)^2]$.

1.11) A computer system has n users, each with a unique name and password. Due to a software error, the n passwords are randomly permuted internally (i.e., each of the $n!$ possible permutations are equally likely). Only those users lucky enough to have had their passwords unchanged in the permutation are able to continue using the system.

a) What is the probability that a particular user, say user 1, is able to continue using the system?

b) What is the expected number of users able to continue using the system? Hint: Let X_i be a random variable with the value 1 if user i can use the system and 0 otherwise.

1.12) Suppose the random variable X is continuous and has the distribution function $F_X(x)$. Consider another random variable $Y = F_X(X)$. That is, for any sample point α such that $X(\alpha) = x$, we have $Y(\alpha) = F_X(x)$. Show that Y is uniformly distributed in the interval 0 to 1.

1.13) Let Z be an integer valued random variable with the PMF $P_Z(n) = 1/k$ for $0 \leq n \leq k-1$. Find the mean, variance, and moment generating function of Z . Hint: The elegant way to do this is to let U be a uniformly distributed continuous random variable over $(0,1]$ that is independent of Z . Then $U+Z$ is uniform over $(0,k]$. Use the known results about U and $U+Z$ to find the mean, variance, and mgf for Z .

1.14) Let $\{X_n ; n \geq 1\}$ be a sequence of independent but not identically distributed random variables. We say that the weak law of large numbers holds for this sequence if for all $\epsilon > 0$

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{S_n}{n} - \frac{E[S_n]}{n}\right| \geq \epsilon\right) = 0 \quad \text{where } S_n = X_1 + X_2 + \dots + X_n \quad (\text{a})$$

a) Show that (a) holds if there is some constant A such that $\text{VAR}(X_n) \leq A$ for all n .

b) Suppose that $\text{VAR}(X_n) \leq A n^{1-\alpha}$ for some $\alpha < 1$ and for all n . Show that (a) holds in this case.

1.15) Let $\{X_i ; i \geq 1\}$ be IID Bernoulli random variables. Let $P(X_i=1) = \delta$, $P(X_i=0) = 1-\delta$. Let $S_n = X_1 + \dots + X_n$. Let m be an arbitrary but fixed positive integer. Think! then evaluate the following and explain your answers:

a) $\lim_{n \rightarrow \infty} \sum_{i: n\delta-m \leq i \leq n\delta+m} P(S_n=i)$

b) $\lim_{n \rightarrow \infty} \sum_{i: 0 \leq i \leq n\delta+m} P(S_n=i)$

c) $\lim_{n \rightarrow \infty} \sum_{i: n(\delta-1/m) \leq i \leq n(\delta+1/m)} P(S_n=i)$

1.16) Let $\{X_i; i \geq 1\}$ be IID random variables with mean 0 and infinite variance. Assume that $E[|X_i|^{1+h}] = \beta$ for some given h , $0 < h < 1$ and some given β . Let $S_n = X_1 + \dots + X_n$.

a) Show that $P(|X_i| \geq y) \leq \beta y^{-1-h}$

b) Let $\{\tilde{X}_i; i \geq 1\}$ be truncated variables $\tilde{X}_i = \begin{cases} b; & X_i \geq b \\ X_i; & -b \leq X_i \leq b \\ -b; & X_i \leq -b \end{cases}$

Show that $E[\tilde{X}_i^2] \leq \frac{2\beta b^{1-h}}{1-h}$

Hint: For a non-negative r.v. Z , $E[Z^2] = \int_0^\infty 2z P(Z \geq z) dz$ (you can establish this, if you wish, by integration by parts).

c) Let $\tilde{S}_n = \tilde{X}_1 + \dots + \tilde{X}_n$. Show that $P(S_n \neq \tilde{S}_n) \leq n \beta b^{-1-h}$

d) Show that $P\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) \leq \beta \left[\frac{2b^{1-h}}{(1-h)n \varepsilon^2} + \frac{n}{b^{1+h}}\right]$

e) Optimize your bound with respect to b . How fast does this optimized bound approach 0 with increasing n ?

1.17) A town starts a mosquito control program and we let the random variable Z_n be the number of mosquitos at the end of the n^{th} year ($n = 0, 1, 2, \dots$). Let X_n be the growth rate of mosquitos in year n ; i.e., $Z_n = X_n Z_{n-1}$; $n \geq 1$. Assume that $\{X_n; n \geq 1\}$ is a sequence of IID random variables with the PMF $P(X=2) = 1/2$; $P(X=1/2) = 1/4$; $P(X=1/4) = 1/4$. Suppose that Z_0 , the initial number of mosquitos, is some known constant and assume for simplicity and consistency that Z_n can take on non-integer values.

a) Find $E[Z_n]$ as a function of n and find $\lim_{n \rightarrow \infty} E[Z_n]$.

b) Let $W_n = \log_2 X_n$. Find $E[W_n]$ and $E[\log_2 (Z_n/Z_0)]$ as a function of n .

c) There is a constant α such that $\lim_{n \rightarrow \infty} (1/n)[\log_2 (Z_n/Z_0)] = \alpha$ with probability

1. Find α and explain how this follows from the strong law of large numbers

d) Using (c), show that $\lim_{n \rightarrow \infty} Z_n = \beta$ with probability 1 for some constant β and evaluate β .

e) Explain carefully how the result in (a) and the result in (d) are possible. What you should learn from this problem is that the expected value of the log of a product of IID random variables is more significant than the expected value of the product itself.

1.18) Use figure 1.4 to verify Eq. (20). Hint: Show that $y P(Y \geq y) \leq \int_{z \geq y} zdF_Y(z)$ and show that $\lim_{y \rightarrow \infty} \int_{z \geq y} zdF_Y(z) = 0$ if $E[Y]$ is finite.

1.19) Show that $\Pi_{m \geq n} (1-1/m) = 0$. Hint: Note that $(1-1/m) = \exp(\ln(1-1/m)) \leq \exp(-1/m)$.

NOTES

1. One must add the axioms of set theory to this and specify the class of events.
2. A set is countable if it has a finite number of elements or if its elements can be put into one to one correspondence with the positive integers. See any elementary text on set theory.
3. The *sup*, or *supremum*, of a set of numbers is the smallest number greater than or equal to all members of the set. It is essentially the maximum of the set, but takes care of situations where the max doesn't exist. For example, the sup of real numbers x satisfying $x < 2$ is 2, whereas there is no maximum x less than 2. The *inf*, or *infimum*, is defined similarly as the largest number less than or equal to all members of the set. It is essentially the minimum of the set.
4. Feller, *An Introduction to Probability Theory and its Application*, vol. I and II, Wiley, 1968 and 1966.
5. Central limit theorems also hold in many of these more general situations, but they usually do not have quite the generality of the laws of large numbers.
6. Proofs and sections marked with an asterisk, while instructive, can be omitted without loss of continuity.

Chapter 2

Poisson Processes

2.1 INTRODUCTION

A Poisson process is a simple and widely used stochastic process for modeling the times at which arrivals enter a system. We usually look at arrivals after some starting time, say $t=0$. Figure 2.1 illustrates some of the different ways to characterize random arrivals over the positive time axis. The sequence of times at which arrivals occur is denoted by the random variables $\{S_1, S_2, \dots\}$. We usually refer to a point on the time axis at which something happens as an *epoch*, and thus we refer to S_n as the epoch of the n^{th} arrival, or the n^{th} arrival epoch. In principle, an arrival process can be characterized by a rule specifying the joint distribution functions of $\{S_1, S_2, \dots, S_n\}$ for all $n \geq 1$, but usually these distribution functions are derived in terms of other random variables.

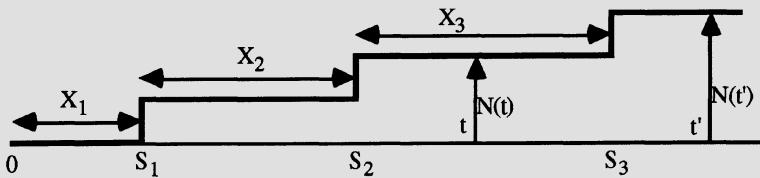


Figure 2.1. An arrival process and its arrival epochs (S_1, S_2, \dots), its inter-arrival intervals (X_1, X_2, \dots), and its counting process ($\{N(t); t \geq 0\}$).

An arrival process over the positive time axis can also be described by the inter-arrival intervals, denoted $\{X_1, X_2, \dots\}$. For $n \geq 2$, X_n is the interval between the $n-1^{\text{st}}$ and the n^{th} arrival epoch, i.e., $X_n = S_n - S_{n-1}$. By convention, $X_1 = S_1$. It follows that the n^{th} arrival epoch can be expressed in terms of the inter-arrival intervals as

$$S_n = \sum_{i=1}^n X_i \quad (1)$$

A rule specifying the joint distribution function of $\{X_1, \dots, X_n\}$ for all $n \geq 1$ specifies the arrival process. Renewal processes, the topic of Chapter 3, are usually specified di-

rectly in this way. In particular, a *renewal process* is an arrival process in which the inter-arrival intervals are independent identically distributed (IID) positive random variables. Thus, a renewal process is specified via the distribution function of the inter-arrival intervals. A *Poisson process* is a renewal process in which the inter-arrival intervals have an exponential distribution function; i.e., each X_i has the density $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.¹ The parameter λ is called the *rate* of the process. The following section demonstrates the special properties that follow from the exponential density function.

Figure 2.1 also shows a family of random variables $\{N(t); t \geq 0\}$ where $N(t)$, for each $t > 0$, is the number of arrivals in the interval from 0 to t . Whether the end points are included in these intervals is sometimes important, and we use parentheses to represent intervals without end points and square brackets to represent inclusion of the end point. Thus (a,b) denotes the interval $\{t: a < t < b\}$, and $[a,b]$ denotes $\{t: a < t \leq b\}$. The counting random variables $N(t)$ for each $t > 0$ are then defined as the number of arrivals in the interval $(0,t]$. $N(0)$ is defined to be 0 with probability 1, which means that we are considering only arrivals at strictly positive times.

A *counting process* $\{N(t), t \geq 0\}$ is a family of non-negative integer valued random variables, one for each real number $t \geq 0$, with the properties that $N(\tau) \geq N(t)$ for all $\tau \geq t$ (i.e., $N(\tau) - N(t)$ is a non-negative random variable) and $N(0) = 0$ with probability 1. We interpret the random variable $N(t)$, for each $t > 0$, as the number of arrivals in $(0,t]$, and thus, the formal definition of counting process above is simply another way to characterize an arrival process. In summary, then, an arrival process can be specified by the joint distributions of the arrival epochs, or of the inter-arrival intervals, or of the counting random variables. In principle, specifying any one of these specifies the others also. We shall often refer to arrival processes as counting processes, since stochastic processes are often characterized by a family of random variables, $\{N(t); t \geq 0\}$, and a counting process is then a special case of a stochastic process in which $N(t)$ is non-negative, integer, and non-decreasing in t .

For any given integer $n \geq 1$ and time $t \geq 0$, the n^{th} arrival epoch, S_n , and the counting random variable, $N(t)$, are related by

$$\{S_n \leq t\} = \{N(t) \geq n\} \quad (2)$$

To see this, note that $\{S_n \leq t\}$ is the event that the n^{th} arrival occurs by time t . This event implies that $N(t)$, the number of arrivals up to and including t , must be at least n ; i.e., it implies the event $\{N(t) \geq n\}$. Similarly, $\{N(t) \geq n\}$ implies $\{S_n \leq t\}$, yielding the equality in (2). This equation is essentially obvious, but is one of those peculiar obvious things that is often difficult to see. One should be sure to understand it, since it is fundamental in going back and forth between arrival epochs and counting random variables.

Although we have referred to these processes as arrival processes, we could equally well be modeling departures from a system, or any other sequence of incidents. Although it is quite common, especially in the simulation field, to refer to incidents or arrivals as events, we shall avoid that here. The n^{th} arrival epoch S_n is a random vari-

able and $\{S_n \leq t\}$, for example, is an event. This would make it confusing to also refer to the n^{th} arrival itself as an event.

2.2 DEFINITION AND PROPERTIES OF THE POISSON PROCESS

A *Poisson process* is a renewal process (i.e., an arrival process with IID inter-arrival intervals) in which the distribution function of the inter-arrival intervals is the exponential distribution, $F_X(x) = 1 - e^{-\lambda x}$. The parameter λ is called the **rate** of the process. We shall see later that for any interval of size t , λt is the expected number of arrivals in that interval; this is why λ is interpreted as the arrival rate. There are two other common definitions of a Poisson process; we discuss these later and see that all three definitions are equivalent. The definition here leads naturally into the renewal processes discussed in the next chapter.

MEMORYLESS PROPERTY: What makes the Poisson process unique among renewal processes is the memoryless property of the exponential distribution. A non-negative random variable X is said to possess the memoryless property if, for every $x > 0$ and $t > 0$,

$$P(X > t+x | X > t) = P(X > x) \quad (3)$$

If X is interpreted as the waiting time until some given incident, then (3) states that, given that the incident has not occurred by time t , the distribution of the remaining waiting time is the same as it was originally. The left side of (3) can be rewritten as $P(X > t+x)/P(X > t)$. For the exponential distribution, this is $\exp[-\lambda(t+x)]/\exp[-\lambda t] = \exp[-\lambda x]$. Thus an exponentially distributed random variable satisfies the memoryless property, and it is not too hard to convince oneself that no other positive random variable satisfies this property.

We now use the memoryless property to find the distribution of the first arrival in a Poisson process after some given time $t > 0$. We not only find this distribution, but also show that this first arrival after t is independent of all arrivals up to and including t . Note that t is an arbitrarily selected constant here; it is not a random variable. Let Z_1 be the duration of the interval from t until the next arrival after t . First we find $P(Z_1 > x | N(t)=0)$.

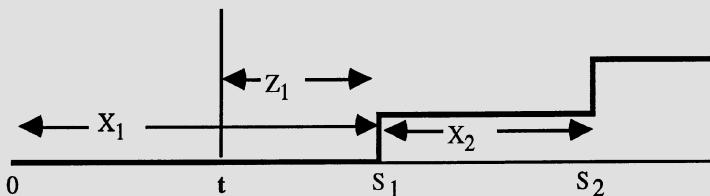


Figure 2.2. Given $N(t)=0$, Z_1 is the interval from t to S_1 ; i.e., $Z_1 = X_1 - t$.

The condition $N(t)=0$ means that the first arrival occurs after time t and thus is equivalent to $X_1 > t$ (see figure 2.2). Given this condition, $Z_1 = X_1 - t$, so $P(Z_1 > x \mid N(t)=0) = P(X_1 > t+x \mid X_1 > t)$. From the memoryless property of X_1 , we then have

$$P(Z_1 > x \mid N(t)=0) = P(X_1 > x) = \exp[-\lambda x] \quad (4)$$

Next consider the condition that there are n arrivals in $(0, t]$ and the n^{th} occurred at epoch $S_n = \tau \leq t$. The event $\{S_n = \tau, N(t)=n\}$ is the same as the event $\{S_n = \tau, X_{n+1} > t - \tau\}$. It can also be seen with the help of figure 2.3 that the event $\{S_n = \tau, N(t)=n, Z_1 > x\}$ is the same as $\{S_n = \tau, X_{n+1} > x + t - \tau\}$. It follows that

$$P(Z_1 > x \mid N(t)=n, S_n = \tau) = \frac{P(S_n = \tau, X_{n+1} > x + t - \tau)}{P(S_n = \tau, X_{n+1} > t - \tau)}$$

Since X_{n+1} is independent of all earlier inter-arrival intervals, and thus of S_n , the right hand side above is just $P(X_{n+1} > x + t - \tau \mid X_{n+1} > t - \tau)$. Using the memoryless property in (3), this yields

$$P(Z_1 > x \mid N(t)=n, S_n = \tau) = \exp[-\lambda x] \quad (5)$$

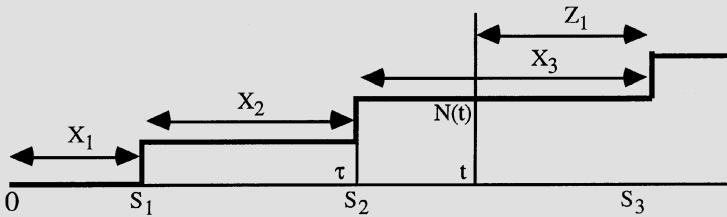


Figure 2.3. Given $N(t)=2$, Z_1 is equal to $S_3 - t$. Thus, given $N(t)=2$ and $S_2 = \tau$, Z_1 is equal to $X_3 - (t - \tau)$. The event $\{N(t)=2, S_2 = \tau\}$ is the same as the event $\{S_2 = \tau, X_3 > t - \tau\}$ and the event $\{N(t)=2, S_2 = \tau, Z_1 > x\}$ is the same as $\{S_2 = \tau, X_3 > x + t - \tau\}$.

Since this result does not depend on τ , we see that $P(Z_1 > x \mid N(t)=n) = \exp[-\lambda x]$, and conditional on $N(t)=n$, Z_1 is independent of S_n . Since the result also does not depend on n , $P(Z_1 > x) = \exp[-\lambda x]$ and Z_1 is independent of $N(t)$. The same argument applies if, in (5), we condition not only on S_n but also on S_1, \dots, S_{n-1} . Since this is equivalent to conditioning on $N(\tau)$ for all τ in $(0, t]$, we have

$$P(Z_1 > x \mid \{N(\tau), 0 \leq \tau \leq t\}) = \exp(-\lambda x) \quad (6)$$

The following theorem states this in words.

THEOREM 1: For a Poisson process of rate λ , and any given time $t > 0$, the interval from t until the first arrival after t is a random variable with the distribution function 1-

$\exp[-\lambda x]$. This random variable is independent of all arrival epochs before time t and independent of $N(\tau)$ for all $\tau \leq t$.

The length of our derivation of (6) somewhat hides its conceptual simplicity. Z_1 , conditional on the time τ of the last arrival before t , is simply the remaining time until the next arrival, which, by the memoryless property, is independent of $\tau \leq t$, and hence also independent of everything before t .

Next consider subsequent inter-arrival intervals after a given time t . For $m \geq 2$, let Z_m be the inter-arrival interval from the $m-1^{\text{st}}$ arrival epoch after t to the m^{th} arrival epoch after t . Given $N(t) = n$, we see that $Z_m = X_{m+n}$, and therefore Z_1, Z_2, \dots , are IID exponentially distributed random variables, conditional on $N(t)=n$ (see exercise 2.5). Since this joint distribution is independent of n , we see that Z_1, Z_2, \dots are unconditionally IID and also independent of $N(t)$. It should also be clear that Z_1, Z_2, \dots are independent of $\{N(\tau); 0 < \tau \leq t\}$.

The above argument shows that the portion of a Poisson process starting at some time $t > 0$ is a probabilistic replica of the process starting at 0; that is, the time until the first arrival after t is an exponentially distributed random variable with parameter λ , and all subsequent arrivals are independent of this first arrival and of each other and have the same exponential distribution. Let us define $\tilde{N}(t, t') = N(t') - N(t)$ as the number of arrivals in the interval $(t, t']$ for $t' \geq t$. What we have just shown is that the random variable $\tilde{N}(t, t')$ has the same distribution as $N(t'-t)$. A counting process with the property that $\tilde{N}(t, t')$ has the same distribution function as $N(t'-t)$ for all $t' \geq t$ and all $t > 0$ is said to have the *stationary increment property*. This means that the distribution of the number of arrivals in an interval depends on the size of the interval but not on its starting point.

We have shown also that $\tilde{N}(t, t')$ is independent of all arrival epochs in $(0, t]$ and thus independent of the number of arrivals in all earlier intervals. The number of arrivals in any earlier interval is also independent of those in yet earlier intervals. Thus, given any $k > 0$ and any times $0 < t_1 < t_2 < \dots < t_k$, $\{N(t_1), \tilde{N}(t_1, t_2), \tilde{N}(t_2, t_3), \dots, \tilde{N}(t_{k-1}, t_k)\}$ is a set of statistically independent random variables. A counting process with this property is said to have the *independent increment property*.

DISTRIBUTION FUNCTIONS FOR S_n AND $N(t)$: Recall from (1) that S_n is the sum of n IID random variables each with the density function $f(x) = \lambda \exp[-\lambda x]$, $x \geq 0$. Also recall that the density of the sum of two independent random variables can be found by convolving their densities, and thus the density of S_2 can be found by convolving $f(x)$ with itself, S_3 by convolving the density of S_2 with $f(x)$, and so forth. The result, for $t \geq 0$, is

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \quad (7)$$

This density function is called the *Erlang density* or the *gamma density*. It could be integrated to obtain the distribution function of S_n , and then (2) could be used to find

the distribution function of $N(t)$. A simpler and more insightful derivation comes from recognizing that for small δ , δ times the density in (7) approximates the probability that the n^{th} arrival occurs between t and $t+\delta$. This is also the probability that both $N(t)=n-1$ and a new arrival occurs between t and $t+\delta$. These latter events are independent, and a new arrival occurs between t and $t+\delta$ with probability $1-\exp(-\lambda\delta)$. Expanding $\exp(-\lambda\delta)$ in a power series, $1 - \lambda\delta + (\lambda\delta)^2/2 - \dots$, we see that $1-\exp(-\lambda\delta) \approx \lambda\delta$ with an error proportional to δ^2 . Thus,

$$\delta f_{S_n}(t) \approx \lambda\delta P(N(t)=n-1) \quad (8)$$

The error on each side above is proportional to δ^2 , so cancelling δ and passing to the limit $\delta \rightarrow 0$, we get $P(N(t)=n-1) = (\lambda t)^{n-1} e^{-\lambda t} / (n-1)!$. Replacing $n-1$ with n , we get

$$P(N(t)=n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad (9)$$

This is the *Poisson probability mass function*, and $N(t)$ for any given t is a *Poisson random variable*. The mean of $N(t)$ is λt , the variance is λt , and the moment generating function is $\exp[\lambda t(e^t-1)]$. The above derivation should illustrate the close connection between the Poisson PMF and the Erlang density.

DEFINITION 2 OF A POISSON PROCESS: An alternate definition of a Poisson process is any counting process that satisfies (9) and has the independent and stationary increment properties. We have just shown that these properties follow from the original definition of a Poisson process as a renewal process with exponential inter-arrival intervals. Exercise 2.3 demonstrates that IID exponential inter-arrival intervals follow from this definition, so the two definitions are equivalent.

Next consider the number of arrivals in some very small interval $(t, t+\delta]$. Since $\tilde{N}(t, t+\delta)$ has the same distribution as $N(\delta)$, we can use (9) to get

$$\begin{aligned} P(\tilde{N}(t, t+\delta)=0) &= \exp[-\lambda\delta] \approx 1-\lambda\delta \\ P(\tilde{N}(t, t+\delta)=1) &= \lambda\delta \exp[-\lambda\delta] \approx \lambda\delta \\ P(\tilde{N}(t, t+\delta)\geq 2) &= 1 - (1+\lambda\delta) \exp[-\lambda\delta] \approx 0 \end{aligned}$$

The errors in the above approximations are proportional to δ^2 and thus go to 0 faster than δ as $\delta \rightarrow 0$. If we break the time axis into disjoint segments of very small width δ , then a negligible proportion of intervals contain more than one arrival, and the number of arrivals in each interval is independent of all other intervals. In what follows, we use the notation $o(\delta)$ to indicate any function $f(\delta)$ that goes to zero faster than δ as δ approaches zero, i.e., such that $\lim_{\delta \rightarrow 0} f(\delta)/\delta = 0$. Thus the approximations above can be written more precisely as

$$\begin{aligned}
 P(\tilde{N}(t,t+\delta)=0) &= 1-\lambda\delta + o(\delta) \\
 P(\tilde{N}(t,t+\delta)=1) &= \lambda\delta + o(\delta) \\
 P(\tilde{N}(t,t+\delta)\geq 2) &= o(\delta)
 \end{aligned} \tag{10}$$

DEFINITION 3 OF A POISSON PROCESS: Finally, a Poisson process can be defined as any counting process that satisfies (10) and has the stationary and independent increment properties. The essence of the argument that this is equivalent to our original definition is that for any inter-arrival interval X , $F_X(x+\delta) - F_X(x)$ is the probability of an arrival in an appropriate infinitesimal interval of width δ , which by (10) is $\lambda\delta+o(\delta)$. Turning this into a differential equation (see exercise 2.4), we get the desired exponential inter-arrival intervals. This third definition best captures the intuition behind a Poisson process; it is a process in which the arrivals are independent of each other and uniformly distributed. Section 2.6 will illustrate the meaning of this more clearly.

What (10) accomplishes, in addition to the assumption of independent and stationary increments, is the prevention of bulk arrivals. For example, consider a counting process in which arrivals always occur in pairs, and the intervals between successive pairs are IID and exponentially distributed with parameter λ (see figure 2.4). For this process, $P(\tilde{N}(t,t+\delta)=1) = 0$, and $P(\tilde{N}(t,t+\delta)=2) = \lambda\delta+o(\delta)$, thus violating (10). This process has stationary and independent increments, however, since the process formed by viewing a pair of arrivals as a single incident is a Poisson process.

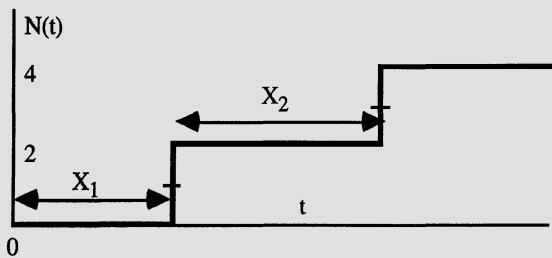


Figure 2.4. A counting process modeling bulk arrivals.

Definition 3 is a little strange and abstract in that it specifies properties of the Poisson process without explicitly spelling out the distributions of any of the random variables involved. A more concrete way of viewing definition 3 follows from defining a “baby Bernoulli process” for a given increment size δ and a given rate λ . Let $\{Z_i; i \geq 1\}$ be a sequence of IID binary random variables with $P(Z_i=0) = 1-\lambda\delta$ and $P(Z_i=1) = \lambda\delta$. Consider a counting process $\{N(t); t \geq 0\}$ in which Z_i for each $i \geq 1$ denotes the number of arrivals in the interval $((i-1)\delta, i\delta]$. Thus $N(k\delta) = Z_1+Z_2+\dots+Z_k$ has a binomial distribution. If one considers only increments starting and ending with multiples of δ , the baby Bernoulli process has stationary and independent increments. Also, it is shown

in exercise 2.6 that, for t a multiple of δ , $P(N(t)=n) = [1+\epsilon(\delta)](\lambda t)^n \exp(-\lambda t) / n!$ where $\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0$. In other words, one can view a Poisson process as a limiting form of a baby Bernoulli process in the limit as the increment size δ goes to 0. One can also conclude from this that the distribution of a sum of a large number of binary random variables, each taking on the value 1 with small probability, tends to the Poisson PMF.

2.3 COMBINATIONS AND SUBDIVISIONS OF INDEPENDENT POISSON PROCESSES

Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson counting processes² of rates λ_1 and λ_2 respectively. We want to look at the sum process where $N(t) = N_1(t) + N_2(t)$ for all $t \geq 0$. In other words, $\{N(t), t \geq 0\}$ is the process consisting of all arrivals to both process 1 and process 2. We shall show that $\{N(t), t \geq 0\}$ is a Poisson counting process of rate $\lambda = \lambda_1 + \lambda_2$. We show this in three different ways, first using definition 3 of a Poisson process (since that is most natural for this problem), then using definition 2, and finally definition 1. We then draw some conclusions about the way in which each approach is helpful. Since $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are independent and both possess the stationary and independent increment properties, it follows from the definitions that $\{N(t); t \geq 0\}$ also possesses the stationary and independent increment properties. Using the approximations in (10) for the individual processes, we see that

$$\begin{aligned} P(\tilde{N}(t, t+\delta) = 0) &= P(\tilde{N}_1(t, t+\delta) = 0) P(\tilde{N}_2(t, t+\delta) = 0) \\ &= (1 - \lambda_1 \delta)(1 - \lambda_2 \delta) \approx 1 - \lambda \delta \end{aligned}$$

where in the last step, we have dropped the term $\lambda_1 \lambda_2 \delta^2$. In the same way, it can be shown that $P(\tilde{N}(t, t+\delta) = 1) \approx \lambda \delta$ and $P(\tilde{N}(t, t+\delta) \geq 2) \approx 0$, both with errors proportional to δ^2 . It follows that $\{N(t), t \geq 0\}$ is a Poisson process.

In the second approach, we have $N(t) = N_1(t) + N_2(t)$. Since $N(t)$, for any given t , is the sum of two independent Poisson random variables (RV), it is also a Poisson RV with mean $\lambda t = \lambda_1 t + \lambda_2 t$. If the reader is not aware that the sum of two independent Poisson RV is Poisson, it can be derived by discrete convolution of the two PMF's (see exercise 1.8). More elegantly, one can observe that we have already implicitly shown this fact. That is, if we break an interval I into disjoint subintervals, I_1 and I_2 , the number of arrivals in I (which is Poisson) is the sum of the number of arrivals in I_1 and in I_2 (which are independent Poisson). Finally, since $N(t)$ is Poisson for each t , and since the stationary and independent increment properties are satisfied, $\{N(t); t \geq 0\}$ is a Poisson process.

In the third approach, X_1 , the first inter-arrival interval for the sum process, is the minimum of X_{11} , the first inter-arrival interval for the first process, and X_{21} , the first inter-arrival interval for the second process. Thus $X_1 > t$ if and only if both X_{11} and X_{21} exceed t , so

$$P(X_1 > t) = P(X_{11} > t) P(X_{21} > t) = \exp(-\lambda_1 t - \lambda_2 t) = \exp(-\lambda t)$$

Using the memoryless property, each subsequent inter-arrival interval can be analyzed in the same way.

The first approach above was the most intuitive for this problem, but it required constant care about the order of magnitude of the terms being neglected. The second approach was the simplest analytically (after recognizing that sums of independent Poisson RVs are Poisson), and required no approximations. The third approach was very simple in retrospect, but not very natural for this problem.

If we add many independent Poisson processes together, it is clear, by adding them one at a time, that the sum process is again Poisson. What is more interesting is that when many independent counting processes (not necessarily Poisson) are added together, the sum process often tends to be approximately Poisson if the individual processes have small rates compared to the sum. To obtain some crude intuition about why this might be expected, note that the inter-arrival intervals for each process (assuming no bulk arrivals) will tend to be large relative to the mean inter-arrival interval for the sum process. Thus arrivals that are close together in time will typically come from different processes. The number of arrivals in an interval large relative to the combined mean inter-arrival interval, but small relative to the individual inter-arrival intervals, will be the sum of the number of arrivals from the different processes; each of these is 0 with large probability and 1 with small probability, so the sum will be approximately Poisson.

SUBDIVIDING A POISSON PROCESS: Next we look at how to split $\{N(t), t \geq 0\}$, a Poisson counting process of rate λ , into two processes, 1 and 2. Suppose that each arrival in $\{N(t), t \geq 0\}$ is sent to process 1 with probability p and to process 2 with probability $1-p$ (see figure 2.5). Each arrival is divided independently of each other arrival and independently of the arrival epochs. We shall show that processes 1 and 2 are each Poisson, with rates λp and $\lambda(1-p)$ respectively, and that furthermore the two processes are independent. Note that, conditional on the original process, the two new processes are not independent; in fact one completely determines the other. Thus this independence might be a little surprising.

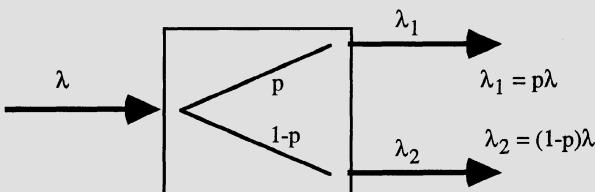


Figure 2.5. Each arrival is independently sent to process 1 with probability p and to process 2 otherwise.

First consider a small increment $(t, t+\delta]$. The original process has an arrival in this incremental interval with probability $\lambda\delta$ (ignoring δ^2 terms as usual), and thus process 1 has an arrival with probability $\lambda\delta p$ and process 2 with probability $\lambda\delta(1-p)$. Because of the independent increment property of the original process and the independence of

the division of each arrival between the two processes, the new processes each have the independent increment property, and from above have the stationary increment property. Thus each process is Poisson. Note now that we cannot verify that the two processes are independent from this small increment model. We would have to show that the number of arrivals for process 1 and 2 are independent over $(t, t+\delta]$. Unfortunately, leaving out the terms of order δ^2 , there is at most one arrival to the original process and no possibility of an arrival to each new process in $(t, t+\delta]$. If it is impossible for both processes to have an arrival in the same interval, they cannot be independent. It is possible, of course, for each process to have an arrival in the same interval, but this is a term of order δ^2 . Thus, without paying attention to the terms of order δ^2 , it is impossible to demonstrate that the processes are independent.

To demonstrate that process 1 and 2 are independent, we first calculate the joint PMF for $N_1(t), N_2(t)$ for arbitrary t . Conditioning on a given number of arrivals $N(t)$ for the original process, we have

$$P(N_1(t)=m, N_2(t)=k \mid N(t)=m+k) = \frac{(m+k)!}{m!k!} p^m (1-p)^k \quad (11)$$

Eq.(11) is simply the binomial distribution, since, given $m+k$ arrivals to the original process, each independently goes to process 1 with probability p . Since the event $\{N_1(t)=m, N_2(t)=k\}$ is a subset of the conditioning event above,

$$P(N_1(t)=m, N_2(t)=k \mid N(t)=m+k) = \frac{P(N_1(t)=m, N_2(t)=k)}{P(N(t)=m+k)}$$

Combining this with (11), we have

$$P(N_1(t)=m, N_2(t)=k) = \frac{(m+k)!}{m!k!} p^m (1-p)^k \frac{(\lambda t)^{m+k} e^{-\lambda t}}{(m+k)!} \quad (12)$$

Rearranging terms, we get

$$P(N_1(t)=m, N_2(t)=k) = \frac{(p\lambda t)^m e^{-\lambda t}}{m!} \frac{[(1-p)\lambda t]^k e^{-\lambda(1-p)t}}{k!} \quad (13)$$

This shows that $N_1(t)$ and $N_2(t)$ are independent. To show that the processes are independent, we must show that for any $k>1$ and any set of times $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$, the sets $\{N_1(t_i); 1 \leq i \leq k\}$ and $\{N_2(t_j); 1 \leq j \leq k\}$ are independent of each other. It is equivalent to show that the sets $\{\tilde{N}_1(t_{i-1}, t_i); 1 \leq i \leq k\}$ and $\{\tilde{N}_2(t_{j-1}, t_j); 1 \leq j \leq k\}$ (where t_0 is 0) are independent. The argument above shows this independence for $i=j$, and for $i \neq j$, the independence follows from the independent increment property of $\{N(t); t \geq 0\}$.

EXAMPLES USING INDEPENDENT POISSON PROCESSES: We have observed that if the arrivals of a Poisson process are split into two new arrival processes, each arrival of the original process independently going into the first of the new processes with some fixed probability p , then the new processes are Poisson processes and are independent. The most useful consequence of this is that any two independent Poisson processes can be viewed as being generated from a single process in this way. Thus, if one process has rate λ_1 and the other has rate λ_2 , they can be viewed as coming from a process of rate $\lambda_1 + \lambda_2$. Each arrival to the combined process then goes to the first process with probability $p = \lambda_1 / (\lambda_1 + \lambda_2)$ and to the second process with probability $1-p$.

The above point of view is very useful if we want to find the answer to questions such as: what is $P(S_{1k} < S_{2j})$ where S_{1k} is the epoch of the k^{th} arrival to the first process and S_{2j} is the epoch of the j^{th} arrival to the second process? The question can be rephrased in terms of a combined process to ask: out of the first $k+j-1$ arrivals to the combined process, what is the probability that k or more of them are switched to the first process? (Note that if k or more of the first $k+j-1$ go to the first process, at most $j-1$ go to the second, so the k^{th} arrival to the first precedes the j^{th} arrival to the second; similarly if fewer than k of the first $k+j-1$ go to the first process, then the j^{th} arrival to the second process precedes the k^{th} arrival to the first.) Since each of these first $k+j-1$ arrivals are switched independently with the same probability p , the answer is

$$P(S_{1k} < S_{2j}) = \sum_{i=k}^{k+j-1} \frac{(k+j-1)!}{i!(k+j-1-i)!} p^i (1-p)^{k+j-1-i} \quad (14)$$

As an example of this, suppose a queueing system has arrivals according to a Poisson process (process 1) of rate λ . There is a single server who serves arriving customers in order with a service time distribution $F(y) = 1 - \exp[-\mu y]$. Thus during periods when the server is busy, customers leave the system according to a Poisson process (process 2) of rate μ . Thus, if j or more customers are waiting at a given time, then (14) gives the probability that the k^{th} subsequent arrival comes before the j^{th} departure.

2.4 NON-HOMOGENEOUS POISSON PROCESSES

The Poisson process, as we defined it, is characterized by a constant arrival rate λ . It is often useful to consider a more general type of process in which the arrival rate varies as a function of time. A non-homogeneous Poisson process with time varying arrival rate $\lambda(t)$ is defined³ as a counting process $\{N(t); t \geq 0\}$ which has the independent increment property and, for all $t \geq 0, \delta \geq 0$, also satisfies:

$$\begin{aligned} P(\tilde{N}(t, t+\delta) = 0) &= 1 - \delta\lambda(t) + o(\delta) \\ P(\tilde{N}(t, t+\delta) = 1) &= \delta\lambda(t) + o(\delta) \\ P(\tilde{N}(t, t+\delta) \geq 2) &= o(\delta) \end{aligned} \quad (15)$$

where $\tilde{N}(t, t+\delta) = N(t+\delta) - N(t)$.

The non-homogeneous Poisson process does not have the stationary increment property.

One common application occurs in optical communication where a non-homogeneous Poisson process is often used to model the stream of photons from an optical modulator; the modulation is accomplished by varying the photon intensity $\lambda(t)$. We shall see another application shortly in the next example. Sometimes a Poisson process, as we defined it earlier, is called a homogeneous Poisson process.

We can use a “baby Bernoulli process” again to approximate a non-homogeneous Poisson process. To see how to do this, assume that $\lambda(t)$ is bounded away from zero and rewrite (15), substituting ε for $\delta\lambda(t)$ (thus using increments whose sizes are inversely proportional to $\lambda(t)$):

$$\begin{aligned} P\left[\tilde{N}\left(t, t+\frac{\varepsilon}{\lambda(t)}\right) = 0\right] &= 1 - \varepsilon + o(\varepsilon) \\ P\left[\tilde{N}\left(t, t+\frac{\varepsilon}{\lambda(t)}\right) = 1\right] &= \varepsilon + o(\varepsilon) \\ P\left[\tilde{N}\left(t, t+\frac{\varepsilon}{\lambda(t)}\right) \geq 2\right] &= o(\varepsilon) \end{aligned} \quad (16)$$

Now partition the time axis into increments such that an increment starting at time t has size $\varepsilon/\lambda(t)$ (see figure 2.6). More precisely, if we define $m(t)$ as

$$m(t) = \int_0^t \lambda(\tau) d\tau \quad (17)$$

then the i^{th} increment ends at that t for which $m(t) = ei$.

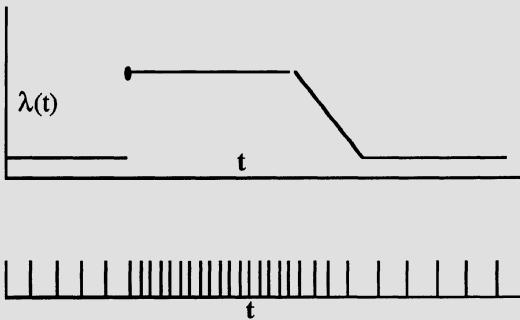


Figure 2.6. Partitioning the time axis into increments each with an expected number of arrivals equal to ε .

As before, let $\{Z_i; i \geq 1\}$ be a sequence of IID binary random variables with $P(Z_i=1) = \varepsilon$ and $P(Z_i=0) = 1-\varepsilon$. Consider the counting process $\{N(t); t \geq 0\}$ in which Z_i , for each $i \geq 1$, denotes the number of arrivals in the interval $(t_{i-1}, t_i]$, where t_i satisfies $m(t_i) = ie$. Thus, $N(t_k) = Z_1 + Z_2 + \dots + Z_k$. As shown in exercise 2.6, for $t = t_k$,

$$P(N(t)=n) = [1+v(\varepsilon)][m(t)]^n \exp[-m(t)]/n! \quad (18)$$

where $v(\varepsilon)$ is a function of ε satisfying $\lim_{\varepsilon \rightarrow 0} v(\varepsilon) = 0$.

Similarly, for any interval $(t, \tau]$, taking $\tilde{m}(t, \tau) = \int_t^\tau \lambda(u)du$, and taking $t = t_k$, $\tau = t_i$ for some k, i , we get

$$P(\tilde{N}(t, \tau)=n) = \frac{[1+v(\varepsilon)][\tilde{m}(t, \tau)]^n \exp[-\tilde{m}(t, \tau)]}{n!} \quad (19)$$

Going to the limit $\varepsilon \rightarrow 0$, the counting process $\{N(t); t \geq 0\}$ above approaches the non-homogeneous Poisson process under consideration, and we have the following theorem:

THEOREM 2: For a non-homogeneous Poisson process with right continuous arrival rate $\lambda(t)$ bounded away from zero, the distribution of $\tilde{N}(t, \tau)$, the number of arrivals in $(t, \tau]$, satisfies

$$P(\tilde{N}(t, \tau)=n) = \frac{[\tilde{m}(t, \tau)]^n \exp[-\tilde{m}(t, \tau)]}{n!} \quad \text{where } \tilde{m}(t, \tau) = \int_t^\tau \lambda(u)du \quad (20)$$

Hence, one can view a non-homogeneous Poisson process as a (homogeneous) Poisson process over a non-linear time scale. That is, let $\{N^*(s); s \geq 0\}$ be a (homogeneous) Poisson process with rate 1. The non-homogeneous Poisson process is then given by $N(t) = N^*(m(t))$ for each t .

EXAMPLE—THE M/G/ ∞ QUEUE: Queueing theorists use a standard notation of characters separated by slashes to describe different kinds of queueing systems. The first character describes the arrival process to the queue. M stands for memoryless and thus means a Poisson process; D stands for deterministic and means that the inter-arrival interval is fixed and non-random; G stands for general inter-arrival distribution (sometimes GI is used to explicitly indicate that the inter-arrival intervals are independent). The second character describes the service process. The same letters are used, with M indicating the exponential service time distribution. The following character gives the number of servers. Thus, M/G/ ∞ indicates a queue with Poisson arrivals, a general service distribution, and an infinite number of servers. Similarly, the example following Eq. (14) considered an M/M/1 queue. Since the M/G/ ∞ queue has an infinite number of servers, no arriving customers are ever queued. Each arrival immediately starts to be served by some server, and it is assumed that the service time Y_i of each

customer i is IID with some distribution function $G(y)$; the service time is the interval from start to completion of service and is assumed to be independent of arrival epochs as well as other service times. We would like to find the distribution function of the number of customers being served at a given epoch τ .

Let $\{N(t); t \geq 0\}$ be the counting process of customer arrivals. Consider the arrival times of those customers that are still in service at some time τ . In some arbitrarily small interval $(t, t+\delta]$, the probability of an arrival is $\delta\lambda + o(\delta)$ and the probability of 2 or more arrivals is negligible (i.e., $o(\delta)$). The probability that an arrival occurred in $(t, t+\delta]$ and that that customer is still being served at time $\tau > t$ is then $\delta\lambda[1-G(\tau-t)] + o(\delta)$. Consider a counting process $\{N_1(t); 0 \leq t \leq \tau\}$ where $N_1(t)$ is the number of arrivals between 0 and t that are still in service at τ . This counting process has the independent increment property. To see this, note that the overall arrivals in $\{N(t); t \geq 0\}$ have the independent increment property; also the arrivals in $\{N(t); t \geq 0\}$ have independent service times, and thus are independently in or not in $\{N_1(t); 0 \leq t < \tau\}$. Thus $\{N_1(t); 0 \leq t < \tau\}$ is a non-homogeneous Poisson process with rate $\lambda[1-G(\tau-t)]$ at time $t \leq \tau$. The expected number of arrivals still in service at time τ is then

$$m(\tau) = \lambda \int_{t=0}^{\tau} [1-G(\tau-t)]dt = \lambda \int_{t=0}^{\tau} [1-G(t)]dt \quad (21)$$

and the PMF of the number in service at time τ is given by

$$P(N_1(\tau)=n) = \frac{m(\tau)^n \exp(-m(\tau))}{n!} \quad (22)$$

Note that as $\tau \rightarrow \infty$, the integral in (21) approaches the mean of the service time distribution (i.e., it is the integral of the complementary distribution function, $1-G(t)$, of the service time). This means that in steady state (as $\tau \rightarrow \infty$), the distribution of service depends on the service time distribution only through its mean. This problem can be used to model situations such as the number of phone calls taking place at a given epoch, since arrivals of new calls can be modeled as a Poisson process and the holding time of each call can be modeled as a random variable independent of other holding times and of call arrival times. Finally, as shown in figure 2.7, we can regard $\{N_1(t); 0 \leq t \leq \tau\}$ as a splitting of the arrival process $\{N(t); t \geq 0\}$. By the same type of argument as in section 2.3, the number of customers who have completed service by time τ is independent of the number still in service.

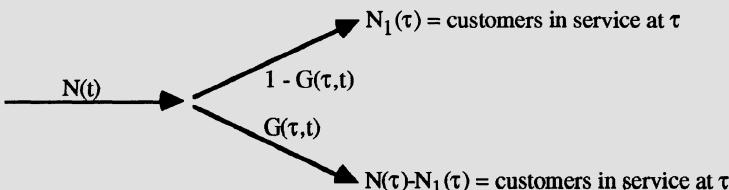


Figure 2.7. Poisson arrivals $\{N(t); t \geq 0\}$ can be considered to be split in a non-homogeneous way. An arrival at t is split with probability $1-G(\tau,t)$ into process of customers still in service at τ .

2.5 ORDER STATISTICS AND CONDITIONAL ARRIVAL EPOCHS

A diverse range of problems involving Poisson processes are best tackled by conditioning on a given number of arrivals in the interval $(0, t]$, i.e., by the condition $N(t) = n$. Because of the incremental view of the Poisson process as independent and stationary arrivals in each tiny interval of the time axis, we would guess that the arrivals should have some sort of uniform distribution given $N(t) = n$. To understand this clearly, we calculate the joint density function of S_1, S_2, \dots, S_n conditional on $N(t) = n$, i.e., $f(s_1, s_2, \dots, s_n | N(t) = n)$. We look at a small cube, δ on a side, of n dimensional space at some given location $0 < s_1 < s_2 < \dots < s_n < t$ (see figure 2.8). For δ small, we have

$$\begin{aligned} f(s_1, s_2, \dots, s_n | N(t) = n) \delta^n &\approx P(s_1 < S_1 \leq s_1 + \delta, \dots, s_n < S_n \leq s_n + \delta | N(t) = n) \\ &= \frac{P(s_1 < S_i \leq s_i + \delta \text{ for } 1 \leq i \leq n, N(t) = n)}{P(N(t) = n)} \end{aligned} \quad (23)$$

with an error $o(\delta^n)$.

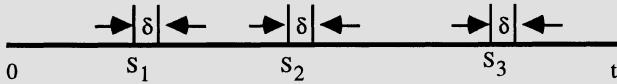


Figure 2.8. Intervals for arrival density.

The probability in the numerator above is the joint probability of no arrivals in the intervals $(0, s_1], (s_1 + \delta, s_2], \dots, (s_n + \delta, t]$ and of precisely one arrival in each of the intervals $(s_i, s_i + \delta]$ for $1 \leq i \leq n$. Since the number of arrivals in each of these subintervals is independent, we can simply multiply the corresponding probabilities, i.e.,

$$\begin{aligned} P[s_1 < S_i \leq s_i + \delta, 1 \leq i \leq n \ \& \ N(t) = n] &= P[N(s_1) = 0] P[\tilde{N}(s_1, s_1 + \delta) = 1] P[\tilde{N}(s_1 + \delta, s_2) = 0] \\ &\quad P[\tilde{N}(s_2, s_2 + \delta) = 1] \dots P[\tilde{N}(s_n, s_n + \delta) = 1] P[\tilde{N}(s_n + \delta, t) = 0] \end{aligned}$$

Note that $P[\tilde{N}(s_i, s_i + \delta) = 1] = \lambda\delta \exp[-\lambda\delta]$ and $P[\tilde{N}(s_i + \delta, s_{i+1}) = 0] = \exp[-\lambda(s_{i+1} - s_i - \delta)]$. There is one exponential term for each subinterval of $(0, t]$, and when we multiply all those exponentials together, we simply get $\exp[-\lambda t]$. There are also n terms $(\lambda\delta)$, so the numerator of (23) is $(\lambda\delta)^n \exp[-\lambda t]$. Dividing this by the denominator of (23),

$$f(s_1, \dots, s_n | N(t) = n) \delta^n \approx (\lambda\delta)^n n! / (\lambda t)^n$$

Dividing by δ^n and passing to the limit $\delta \rightarrow 0$,

$$f(s_1, \dots, s_n | N(t) = n) = n! / t^n \quad \text{for } 0 < s_1 < s_2 < \dots < s_n < t \quad (24)$$

Note that this doesn't depend on the values of s_1, \dots, s_n , except for the constraint that $s_1 < s_2 < \dots < s_n$. It is instructive to compare this with the joint distribution of n IID

uniformly distributed random variables, U_1, \dots, U_n on $(0, t]$. For any point $U_i = u_i$, $0 < u_i \leq t$, $1 \leq i \leq n$, this joint density is

$$g(u_1, \dots, u_n) = 1/t^n \quad \text{for } 0 < u_i \leq t, 1 \leq i \leq n$$

Both densities are uniform over the volume where they are non-zero, but as illustrated in figure 2.9 for $n=2$, the volume for the latter is $n!$ larger than the volume for the former.

To explain this more fully, we can define a set of random variables S_1, \dots, S_n as the order statistics function of U_1, \dots, U_n ; that is

$$S_1 = \min(U_1, \dots, U_n); S_2 = \text{2}^{\text{nd}} \text{ smallest}(U_1, \dots, U_n); \text{etc.}$$

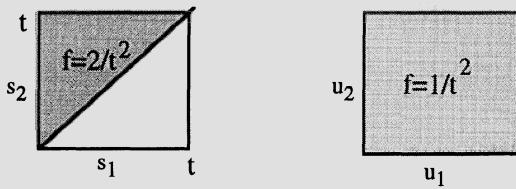


Figure 2.9. Density for order statistics and for uniform distribution.

Since the $n!$ permutations of a set of distinct values u_1, \dots, u_n all map into the same ordered values (and this remains true for sufficiently small volume elements), we see that these order statistics have the same probability density function as the arrival epochs S_1, \dots, S_n conditional on $N(t)=n$. Thus anything we know (or can discover) about order statistics is valid for arrival epochs given $N(t)=n$ and vice versa.

Next we want to find the marginal distribution functions of the individual S_i conditional on $N(t)=n$. Starting with S_1 , and viewing it as the minimum of the IID uniformly distributed variables U_1, \dots, U_n , we recognize that $S_1 > \tau$ if and only if $U_i > \tau$ for all i , $1 \leq i \leq n$. Thus,

$$P(S_1 > \tau | N(t)=n) = \left[\frac{t-\tau}{t} \right]^n \quad (25)$$

Since this is the complement of the distribution function of S_1 , conditional on $N(t)=n$, we can integrate it to get the conditional mean of S_1 ,

$$E[S_1 | N(t)=n] = t/(n+1) \quad (26)$$

We come back later to the distribution functions of S_2, \dots, S_n , and first look at the marginal distributions of the inter-arrival intervals. We want to find $P(X_i > \tau | N(t)=n)$, and note that the condition can be rewritten as

$$P(X_i > \tau | N(t) = n) = P\left(X_i > \tau | \sum_{j=1}^n X_j \leq t, \sum_{j=1}^{n+1} X_j > t\right) \quad (27)$$

We observe that, a priori, the X_i are IID. We also note that the condition in (27) does not distinguish between the ordering of X_1, \dots, X_n . Thus (27) implies a symmetry between the variables X_1, \dots, X_n , and thus these variables must be identically distributed conditional on $N(t)=n$ (although of course they are not independent given the condition). Since $X_1=S_1$ and each $X_i, 1 \leq i \leq n$ has the same distribution, we see from (25) that

$$P(X_i > \tau | N(t) = n) = \left[\frac{t-\tau}{t} \right]^n \quad (28)$$

From the same symmetry argument, it is seen that the joint distribution function of X_1, \dots, X_n , given $N(t)=n$, is independent of the order of the variables. This means that the arguments of the distribution function can be arbitrarily permuted without changing the value, i.e., that

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n) = F_{X_1, \dots, X_n}(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$$

where $\pi(1), \dots, \pi(n)$ is a permutation of the integers 1, ..., n. This means, for example, that the probability density where $X_1=7, X_2=1, X_3=4$ is equal to the probability density where $X_1=4, X_2=7, X_3=1$.

Now think in terms of n uniform IID random variables again and note the symmetry between entering the interval from the left and from the right. Define $X_{n+1}^* = t - S_n$ to be the interval from the largest of the IID variables to t, the right end of the interval. From the right-left symmetry, we see that X_{n+1}^* has the same distribution function as X_1 . This gives us a partial check on our work, since the interval $(0, t]$ is divided into $n+1$ intervals of sizes $X_1, X_2, \dots, X_n, X_{n+1}^*$, and each of these has a mean size $t/(n+1)$. We also see, from the right-left symmetry between X_1 and X_{n+1}^* , that the joint distribution of X_1, \dots, X_n , given both $X_1=x$ and $N(t)=n$, is the same as the joint distribution of X_n, X_{n-1}, \dots, X_2 , given both $X_{n+1}^*=x$ and $N(t)=n$. From this it follows that the joint distribution function of $X_1, X_2, \dots, X_n, X_{n+1}^*$ is also independent of the order of the variables.

Next consider the distribution function of X_{i+1} ($i < n$) conditional both on $N(t)=n$ and $S_i=s_i$ (or conditional on any given values for X_1, \dots, X_i summing to s_i). We see that X_{i+1} is just the wait until the first arrival in the interval $(s_i, t]$ given that this interval contains $n-i$ arrivals. From the same argument as used in (25), we have

$$P(X_i > \tau | N(t) = n, S_i = s_i) = \left[\frac{t-s_i-\tau}{t-s_i} \right]^{n-i} \quad (29)$$

Since S_{i+1} is $X_{i+1} + S_i$, this immediately gives us the conditional distribution of S_{i+1}

$$P(S_{i+1} > s_{i+1} \mid N(t)=n, S_i = s_i) = \left[\frac{t-s_{i+1}}{t-s_i} \right]^{n-i} \quad (30)$$

We note that this is independent of S_1, \dots, S_{i-1} . As a check, one can find the conditional densities from (30) and multiply them all together to get back to (24) (see Exercise 2.21). One can also find the distribution of each S_i conditioned on $N(t)=n$ but unconditioned on S_1, S_2, \dots, S_{i-1} . The density for this is calculated by looking at n uniformly distributed random variables in $(0,t]$. The probability that one of these lies in the interval $(x, x+dt]$ is $(n dt)/t$. Out of the remaining $n-1$, the probability that $i-1$ lie in the interval $(0,x]$ is given by the binomial distribution with probability of success x/t . Thus the desired density is

$$\begin{aligned} f_{S_i}(x \mid N(t)=n) dt &= \frac{x^{i-1}(t-x)^{n-i}(n-1)!}{t^{n-1}(n-i)!(i-1)!} \frac{n dt}{t} \\ f_{S_i}(x \mid N(t)=n) &= \frac{x^{i-1}(t-x)^{n-i}n!}{t^n(n-i)!(i-1)!} \end{aligned} \quad (31)$$

2.6 SUMMARY

We started the chapter with three equivalent definitions of a Poisson process—first as a renewal process with exponentially distributed inter-renewal intervals, second as a stationary and independent increment counting process with Poisson distributed arrivals in each interval, and third essentially as a limit of a Bernoulli process. We saw that each definition provided its own insights into the properties of the process. We emphasized the importance of the memoryless property of the exponential distribution, both as a useful tool in problem solving and as an underlying reason why the Poisson process is so simple.

We next showed that the sum of independent Poisson processes is again a Poisson process. We also showed that if the arrivals in a Poisson process were independently routed to different locations with some fixed probability assignment, then the arrivals at each of these locations formed independent Poisson processes. This ability to view independent Poisson processes either independently or as a splitting of a combined process is a powerful technique to find almost trivial solutions to many problems.

It was next shown that a non-homogeneous Poisson process could be viewed as a (homogeneous) Poisson process on a non-linear time scale. This allows all the properties of (homogeneous) Poisson properties to be applied directly to the non-homogeneous case. The simplest and most useful result from this is (20), showing that the number of arrivals in any interval has a Poisson PMF. This result was used to show that the number of customers in service at any given time τ in an $M/G/\infty$ queue has a Poisson PMF with a mean approaching λ times the expected service time as $\tau \rightarrow \infty$.

Finally we looked at the distribution of arrivals conditional on n arrivals in the interval $(0,t]$. It was found that these arrivals had the same joint distribution as the order statistics of n uniform IID random variables in $(0,t]$. By using symmetry and going

back and forth between the uniform variables and the Poisson process arrivals, we found the distribution of the interarrival times, of the arrival epochs, and of various conditional distributions.

EXERCISES

2.1) a) Find the Erlang densities $f_{S_n}(t)$ by convolving $f_X(x) = \lambda e^{-\lambda x}$ with itself n times.

b) Find the moment generating function of X (or find the Laplace transform of $f_X(x)$), and use this to find the moment generating function (or Laplace transform) of $S_n = X_1 + X_2 + \dots + X_n$. Invert your result to find $f_{S_n}(t)$.

c) Find the mean, variance, and moment generating function of $N(t)$, as given by (9). Show that the sum of two independent Poisson random variables is again Poisson.

2.2) The purpose of this exercise is to give an alternate derivation of the Poisson distribution for $N(t)$, the number of arrivals in a Poisson process up to time t ; let λ be the rate of the process.

a) Find the conditional probability $P(N(t) = n | S_n = \tau)$ for all $\tau \leq t$.

b) Using the Erlang density for S_n , use (a) to find $P(N(t) = n)$.

2.3) Assume that a counting process $\{N(t); t \geq 0\}$ has the independent and stationary increment properties and satisfies (9) (for all $t > 0$).

a) Let X_1 be the epoch of the first arrival and X_n be the interarrival time between the $n-1^{\text{st}}$ and the n^{th} arrival. Show that $P(X_n > x) = e^{-\lambda x}$.

b) Let S_{n-1} be the epoch of the $n-1^{\text{st}}$ arrival. Show that $P(X_n > x | S_{n-1} = \tau) = e^{-\lambda x}$.

c) Show that, for each $n > 1$, $P(X_n > x) = e^{-\lambda x}$ and X_n is independent of S_{n-1} .

d) Argue that X_n is independent of X_1, X_2, \dots, X_{n-1} .

2.4) Assume that a counting process $\{N(t); t \geq 0\}$ has the independent and stationary increment properties and satisfies (for all $t > 0, \delta > 0$)

$$P(\tilde{N}(t, t+\delta) = 0) = 1 - \lambda\delta + o(\delta)$$

$$P(\tilde{N}(t, t+\delta) = 1) = \lambda\delta + o(\delta)$$

$$P(\tilde{N}(t, t+\delta) > 1) = o(\delta)$$

a) Let $F_0(\tau) = P(N(\tau) = 0)$ and show that $F'_0(\tau) = -\lambda F_0(\tau)$.

b) Show that X_1 , the time of the first arrival, is exponential with parameter λ .

c) Let $F_n(\tau) = P(\tilde{N}(t, t+\tau) = 0 | S_{n-1} = t)$ and show that $F'_n(\tau) = -\lambda F_n(\tau)$.

d) Argue that X_n is exponential with parameter λ and independent of earlier arrival times.

2.5) Let $t > 0$ be an arbitrary time, let Z_1 be the duration of the interval from t until the next arrival after t , and let Z_m , for each $m > 1$, be the interarrival time from the epoch of the $m-1^{\text{th}}$ arrival after t until the m^{th} arrival.

- a) Given that $N(t) = n$, explain why $Z_m = X_{m+n}$ for $m > 1$ and $Z_1 = X_{n+1} - t + S_n$.
 b) Conditional on $N(t)=n$ and $S_n=t$, show that Z_1, Z_2, \dots are IID.
 c) Show that Z_1, Z_2, \dots are IID.

2.6) Consider a “baby Bernoulli” approximation to a Poisson process. X_i is the number of arrivals in the interval $(i\delta-\delta, i\delta]$, and we assume $\{X_i; i \geq 1\}$ is IID with $P(X_i=1) = \delta\lambda$, and $P(X_i=0) = 1-\delta\lambda$. Let $N(k\delta) = X_1 + X_2 + \dots + X_k$ be the number of arrivals in $(0, k\delta]$ according to the baby Bernoulli approximation.

a) Show that $P(N(k\delta)=n) = \binom{k}{n}(\lambda\delta)^n(1-\lambda\delta)^{k-n}$

b) Let $t = k\delta$, and consider holding t fixed as $\delta \rightarrow 0$ and $k \rightarrow \infty$. Show that for any given n ,

$$P(N(t)=n) = (1+v(\delta)) \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

where $v(\delta)$ is a function of δ satisfying $\lim_{\delta \rightarrow 0} v(\delta) = 0$.

Hint: Show that $\frac{k!}{(k-n)!} = k^n \exp\left[\sum_{i=1}^{n-1} \ln(1 - \frac{i}{k})\right] = (1+v(\delta)) k^n$

Show that $(1-\lambda\delta)^{k-n} = \exp[(k-n)\ln(1-\lambda\delta)] = (1+v(\delta)) \exp(-\lambda t)$.

2.7) Let $\{N(t); t \geq 0\}$ be a Poisson process of rate λ .

- a) Find the joint probability mass function (PMF) of $N(t), N(t+s)$ for $s > 0$.
 b) Find $E[N(t) \cdot N(t+s)]$ for $s > 0$.
 c) Find $E[\tilde{N}(t_1, t_3) \cdot \tilde{N}(t_2, t_4)]$ where $\tilde{N}(t, \tau)$ is the number of arrivals in $(t, \tau]$ and $t_1 < t_2 < t_3 < t_4$.

2.8) An experiment is independently performed N times where N is a Poisson random variable of mean λ . Let $\{a_1, a_2, \dots, a_K\}$ be the set of elementary outcomes of the experiment and let P_k , $1 \leq k \leq K$, denote the probability of a_k .

- a) Let N_i denote the number of experiments performed for which the output is a_i . Find the PMF for N_i ($1 \leq i \leq K$). Hint: no calculation is necessary.
 b) Find the PMF for $N_1 + N_2$.
 c) Find the conditional PMF for N_1 given that $N = n$.
 d) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.
 e) Find the conditional PMF for N given that $N_1 = n_1$.

2.9) Starting from time 0, northbound buses arrive at 77 Mass. Avenue according to a Poisson process of rate λ . Passengers arrive according to an independent Poisson process of rate μ . When a bus arrives, all waiting customers instantly enter the bus and subsequent customers wait for the next bus.

- a) Find the PMF for the number of customers entering a bus (more specifically, for any given m , find the PMF for the number of customers entering the m^{th} bus).

- b) Find the PMF for the number of customers entering the m^{th} bus given that the interarrival interval between bus $m-1$ and bus m is x .
- c) Given that a bus arrives at time 10:30 PM, find the PMF for the number of customers entering the next bus.
- d) Given that a bus arrives at 10:30 PM and no bus arrives between 10:30 and 11, find the PMF for the number of customers on the next bus.
- e) Find the PMF for the number of customers waiting at some given time, say 2:30 PM (assume that the processes started infinitely far in the past). Hint: Think of what happens moving backward in time from 2:30 PM.
- f) Find the PMF for the number of customers getting on the next bus to arrive after 2:30. Hint: this is different from part (a); look carefully at part (e).
- g) Given that I arrive to wait for a bus at 2:30 PM, find the PMF for the number of customers getting on the next bus.

2.10) Eq. (31) in chapter 2 gives $f_{S_i}(x \mid N(t) = n)$, the density of random variable S_i conditional on $N(t) = n$ for $n \geq i$. Multiply this expression by $P(N(t) = n)$ and sum over n to find $f_{S_i}(x)$; verify that your answer is indeed the Erlang density.

2.11) Consider generalizing the bulk arrival process in figure 2.4. Assume that the epochs at which arrivals occur form a Poisson process $\{N(t); t \geq 0\}$ of rate λ . At each arrival epoch, S_n , the number of arrivals, Z_n , satisfies $P(Z_n=1) = p$, $P(Z_n=2) = 1-p$. The variables Z_n are IID.

- a) Let $\{N_1(t); t \geq 0\}$ be the counting process of the epochs at which single arrivals occur. Find the PMF of $N_1(t)$ as a function of t . Similarly, let $\{N_2(t); t \geq 0\}$ be the counting process of the epochs at which double arrivals occur. Find the PMF of $N_2(t)$ as a function of t .
- b) Let $\{N_B(t); t \geq 0\}$ be the counting process of the total number of arrivals. Give an expression for the PMF of $N_B(t)$ as a function of t .

2.12) a) For a Poisson counting process of rate λ , find the joint probability density of S_1, S_2, \dots, S_{n-1} conditional on $S_n = t$. Use the same technique for the condition $S_n = t$ as in (24) for the condition $N(t) = n$.

- b) Find $P(X_1 > \tau \mid S_n = t)$.
- c) Find $P(X_i > \tau \mid S_n = t)$ for $1 \leq i \leq n$.
- d) Find the density $F_{S_i}(x \mid S_n = t)$ for $1 < i < n-1$.
- e) Give an explanation for the striking similarity between the condition $N(t) = n-1$ and the condition $S_n = t$.

2.13) a) For a Poisson process of rate λ , find $P(N(t) = n \mid S_1 = \tau)$ for $t > \tau$, $n > 1$.

- b) Using this, find $f_{S_1}(\tau \mid N(t) = n)$
- c) Check your answer against (25).

2.14) Consider a counting process in which the rate is a random variable Λ with probability density $f_\Lambda(\lambda) = \alpha e^{-\alpha\lambda}$ for $\lambda > 0$. Conditional on a given sample value λ for the

rate, the counting process is a Poisson process of rate λ (i.e., nature first chooses a sample value λ and then generates a sample function of a Poisson process of that rate λ).

- a) What is $P(N(t)=n \mid \Lambda=\lambda)$, where $N(t)$ is the number of arrivals in the interval $(0, t]$ for some given $t > 0$?
- b) Show that $P(N(t)=n)$, the unconditional PMF for $N(t)$, is given by

$$P(N(t)=n) = \frac{\alpha t^n}{(t+\alpha)^{n+1}}$$

- c) Find $f_{\Lambda}(\lambda \mid N(t)=n)$, the density of λ conditional on $N(t)=n$.
- d) Find $E[\Lambda \mid N(t)=n]$ and interpret your result for very small t with $n=0$ and for very large t with n large.
- e) Find $E[\Lambda \mid N(t)=n, S_1, S_2, \dots, S_n]$. Hint: Consider the distribution of $S_1 \dots S_n$ conditional on $N(t)$ and Λ . Find $E[\Lambda \mid N(t)=n, N(\tau)=m]$ for some $\tau < t$.

- 2.15) a)** Use Eq. (31) of chapter 2 to find $E[S_i \mid N(t)=n]$. Hint: In integrating $x f_{S_i}(x \mid N(t)=n)$, compare this integral with $f_{S_{i+1}}(x \mid N(t)=n+1)$ and use the fact that the latter expression is a probability density.

- b)** Find the second moment and the variance of S_i conditional on $N(t)=n$. Hint: Extend the previous hint.

- c)** Assume that n is odd, and consider $i=(n+1)/2$. What is the relationship between S_i , conditional on $N(t)=n$, and the sample median of n IID uniform random variables.

- d)** Give a weak law of large numbers for the above median.

- 2.16)** Suppose cars enter a one-way infinite highway at a Poisson rate λ . The i th car to enter chooses a velocity V_i and travels at this velocity. Assume that the V_i 's are independent positive random variables having a common distribution F . Derive the distribution of the number of cars that are located in the interval $(0, a)$ at time t .

- 2.17)** Consider an $M/G/\infty$ queue, i.e., a queue with Poisson arrivals of rate λ in which each arrival i , independent of other arrivals, remains in the system for a time X_i , where $\{X_i; i \geq 1\}$ is a set of IID random variables with some given distribution $F(x)$.

You may assume that the number of arrivals in any interval $(t, t+\varepsilon)$ that are still in the system at some later time $\tau \geq t+\varepsilon$ is *statistically independent* of the number of arrivals in that same interval $(t, t+\varepsilon)$ that have departed from the system by time τ .

- a)** Let $N(\tau)$ be the number of customers in the system at time τ . Find the mean, $m(\tau)$, of $N(\tau)$ and find $P(N(\tau)=n)$.

- b)** Let $D(\tau)$ be the number of customers that have departed from the system by time τ . Find the mean, $E[D(\tau)]$, and find $P(D(\tau)=d)$.

- c)** Find $P(N(\tau)=n, D(\tau)=d)$.

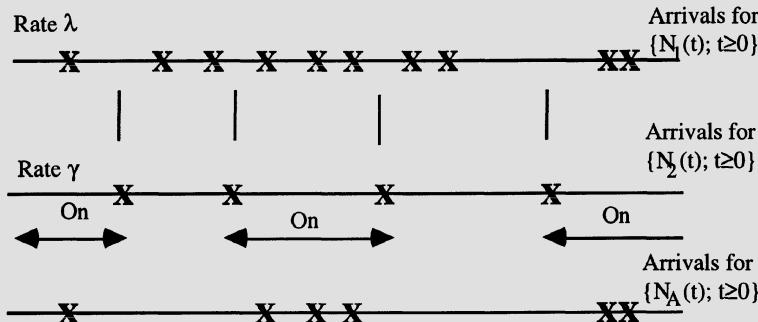
- d)** Let $A(\tau)$ be the total number of arrivals up to time τ . Find $P(N(\tau)=n \mid A(\tau)=a)$.

- e)** Find $P(D(\tau+\varepsilon)-D(\tau)=d)$.

2.18) The voters in a given town arrive at the place of voting according to a Poisson process of rate $\lambda = 100$ voters per hour. The voters independently vote for candidate A and candidate B each with probability $1/2$. Assume that the voting starts at time 0 and continues indefinitely.

- a) Conditional on 1000 voters arriving during the first 10 hours of voting, find the probability that candidate A receives n of those votes.
- b) Again conditional on 1000 voters during the first 10 hours, find the probability that candidate A receives n votes in the first 4 hours of voting.
- c) Let T be the epoch of the arrival of the first voter voting for candidate A. Find the density of T .
- d) Find the PMF of the number of voters for candidate B who arrive before the first voter for A.
- e) Define the n^{th} voter as *a reversal* if the n^{th} voter votes for a different candidate than the $n-1^{\text{st}}$. For example, in the sequence of votes A A B A A B B, the third, fourth, and sixth voters are reversals; the third and sixth are A to B reversals and the fourth is a B to A reversal. Let $N(t)$ be the number of reversals up to time t (t in hours). Is $\{N(t); t \geq 0\}$ a renewal process? Is it a delayed renewal process? Explain.
- f) Find the expected time (in hours) between reversals.
- g) Find the probability density of the time between reversals.
- h) Find the density of the time from one A to B reversal to the next A to B reversal.

2.19) Let $\{N_1(t); t \geq 0\}$ be a Poisson counting process of rate λ . Assume that the arrivals from this process are switched on and off by arrivals from a second independent Poisson process $\{N_2(t); t \geq 0\}$ of rate γ .



Let $\{N_A(t); t \geq 0\}$ be the switched process; that is $N_A(t)$ includes the arrivals from $\{N_1(t); t \geq 0\}$ during periods when $N_2(t)$ is even and excludes the arrivals from $\{N_1(t); t \geq 0\}$ while $N_2(t)$ is odd.

- a) Find the PMF for the number of arrivals of the first process, $\{N_1(t); t \geq 0\}$, during the n^{th} period when the switch is on.
- b) Given that the first arrival for the second process occurs at epoch τ , find the conditional PMF for the number of arrivals of the first process up to τ .
- c) Given that the number of arrivals of the first process, up to the first arrival for the second process, is n , find the density for the epoch of the first arrival from the second process.
- d) Find the density of the interarrival time for $\{N_A(t); t \geq 0\}$.

2.20) Let us model the chess tournament between Fisher and Spassky as a stochastic process. Let $X_i, i \geq 1$, be the duration of the i^{th} game and assume that $\{X_i; i \geq 1\}$ is a set of IID exponentially distributed rv's each with density $f(x) = \lambda e^{-\lambda x}$. Suppose that each game (independently of all other games, and independently of the length of the games) is won by Fisher with probability p , by Spassky with probability q , and is a draw with probability $1-p-q$. The first player to win n games is defined to be the winner, but we consider the match up to the point of winning as being embedded in an unending sequence of games.

- a) Find the distribution of time, from the beginning of the match, until the completion of the first game that is won (i.e., that is not a draw). Characterize the process of the number $\{N(t); t \geq 0\}$ of games won up to and including time t . Characterize the process of the number $\{N_F(t); t \geq 0\}$ of games won by Fisher and the number $\{N_S(t); t \geq 0\}$ won by Spassky.
- b) For the remainder of the problem, assume that the probability of a draw is zero; i.e., that $p+q=1$. How many of the first $2n-1$ games must be won by Fisher in order to win the match?
- c) What is the probability that Fisher wins the match? Your answer should not involve any integrals. Hint: Consider the unending sequence of games and use part (b).
- d) Let T be the epoch at which the match is completed (i.e., either Fisher or Spassky wins). Find the distribution function of T .
- e) Find the probability that Fisher wins and that T lies in the interval $(t, t+\delta)$ for arbitrarily small δ .

2.21) Using (30), find the conditional density of S_{i+1} , conditional on $N(t)=n$ and $S_i=s_i$ and use this to find the joint density of S_1, \dots, S_n conditional on $N(t)=n$. Verify that your answer agrees with (24).

2.22) A two-dimensional Poisson process is a process of randomly occurring special points in the plane such that (i) for any region of area A the number of special points in that region has a Poisson distribution with mean λA , and (ii) the number of special points in nonoverlapping regions is independent. For such a process consider an arbitrary location in the plane and let X denote its distance from its nearest special point (where distance is measured in the usual Euclidean manner). Show that

- a) $P(X > t) = \exp(-\lambda \pi t^2)$
- b) $E[X] = 1/(2\sqrt{\lambda})$.

2.23) This problem is intended to show that one can analyze the long term behavior of queueing problems by using just notions of means and variances, but that such analysis is awkward, justifying understanding the strong law of large numbers. Consider an M/G/1 queue. The arrival process is Poisson with $\lambda = 1$. The expected service time, $E[Y]$, is $1/2$ and the variance of the service time is given to be 1 .

- a) Consider S_n , the time of the n th arrival, for $n = 10^{12}$. With high probability, S_n will lie within 3σ standard deviations of its mean. Find and compare this mean and the 3σ range.
- b) Let V_n be the total amount of time during which the server is busy with these n arrivals (i.e., the sum of 10^{12} service times). Find the mean and 3σ range of V_n .
- c) Find the mean and 3σ range of I_n , the total amount of time the server is idle up until S_n (take I_n as $S_n - V_n$, thus ignoring any service time after S_n).
- d) An idle period starts when the server completes a service and there are no waiting arrivals; it ends on the next arrival. Find the mean and variance of an idle period. Are successive idle periods IID?
- e) Combine (c) and (d) to estimate the total number of idle periods up to time S_n . Use this to estimate the total number of busy periods.
- f) Combine (e) and (b) to estimate the expected length of a busy period.

NOTES

1. With this density, $P(X_i=0) = 0$, so that we regard X_i as a positive random variable. Since events of probability zero can be ignored, the density $\lambda \exp(-\lambda x)$ for $x \geq 0$ and zero for $x < 0$ is effectively the same as the density $\lambda \exp(-\lambda x)$ for $x > 0$ and zero for $x \leq 0$.
2. Two processes $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are said to be independent if for all positive integers k and all sets of times t_1, \dots, t_k , the random variables $N_1(t_1), \dots, N_1(t_k)$ are independent of $N_2(t_1), \dots, N_2(t_k)$. Here it is enough to extend the independent increment property to independence between increments over the two processes; equivalently, one can require the inter-arrival intervals for one process to be independent of the inter-arrival intervals for the other process.
3. We assume that $\lambda(t)$ is right continuous, i.e., that for each t , $\lambda(t)$ is the limit of $\lambda(t+\epsilon)$ as ϵ approaches 0 from above. This allows $\lambda(t)$ to contain discontinuities, as shown in figure 2.6, but follows the convention that the value of the function at the discontinuity is the limiting value from the right. This convention is required in (15) to talk about the distribution of arrivals just to the right of time t .

Chapter 3

Renewal Processes

3.1 INTRODUCTION

Recall that a counting process, $\{N(t); t \geq 0\}$, is a stochastic process in which $N(t)$ models the number of arrivals to a system in the interval $(0, t]$. The corresponding arrival epochs are denoted S_1, S_2, \dots , and the inter-arrival intervals are denoted X_1, X_2, \dots . A renewal process is a counting process for which the inter-arrival intervals are positive independent identically distributed (IID) random variables.¹ These processes are called renewal processes because the process probabilistically starts over at each arrival epoch, $S_n = X_1 + \dots + X_n$. That is, if the n^{th} arrival occurs at $S_n = \tau$, then, counting from $S_n = \tau$, the j^{th} subsequent arrival epoch is at $S_{n+j} - S_n = X_{n+1} + \dots + X_{n+j}$. Thus $\{N(\tau+t) - N(\tau); t \geq 0\}$ is a renewal process with IID inter-arrival intervals of the same distribution as the original renewal process. Because of this renewal property, we shall usually refer to arrivals as renewals.

The major reason for studying renewal processes is that many complicated processes have randomly occurring instants at which the system returns to a state probabilistically equivalent to the starting state. These embedded renewal epochs allow us to separate the long term behavior of the process (which can be studied through renewal theory) from the behavior of the actual process within a renewal period.

EXAMPLE 1: Consider a G/G/m queue. The arrival process to a G/G/m queue is a renewal process, $\{A(t); t \geq 0\}$. Each arriving customer waits in the queue until one of m identical servers is free to serve it. The service time required by each customer is a random variable, IID over customers, and independent of arrival times and server. We define a new renewal process, $\{N(t); t \geq 0\}$, in which the renewal epochs are those arrival epochs in the original process $\{A(t); t \geq 0\}$ at which an arriving customer sees an empty system (i.e., no customer in queue and none in service). To make the time origin probabilistically identical to the renewal epochs in this new renewal process, we consider time 0 as the epoch of an arrival that starts a busy period. There are two renewal processes, $\{A(t); t \geq 0\}$ and $\{N(t); t \geq 0\}$ in this example, and the renewals in the second process are the subset of arrivals in the first process that arrive to an empty system.

Throughout our study of renewal processes, we use \bar{X} and $E[X]$ interchangeably to denote the mean inter-renewal interval and use σ^2 to denote the variance of the inter-

renewal interval. We will usually assume that \bar{X} is finite, but, except where explicitly stated, we need not assume that σ^2 is finite. This means, first, that this variance need not be calculated (which is often difficult if renewals are embedded into a more complex process), and second, that the results are relatively robust to modeling errors on the far tails of the inter-renewal distribution.

Much of this chapter will be devoted to understanding the behavior of the time average arrival rate, $N(t)/t$, and the aggregate number of arrivals, $N(t)$, as t becomes large. The first major result is the strong law of large numbers for renewal processes. This states that $N(t)/t$ approaches $1/\bar{X}$ with probability 1 as $t \rightarrow \infty$. This is an analog (and direct consequence) of the strong law of large numbers, theorem 3 of Chapter 1. The next major result is the elementary renewal theorem, which states that $E[N(t)]/t$ also approaches $1/\bar{X}$ as $t \rightarrow \infty$. It seems surprising that this does not follow from the strong law for renewal processes, but in fact it doesn't, and we shall develop several widely useful results in establishing this theorem. The final major result is Blackwell's theorem, which shows that, for appropriate values of δ , the expected number of renewals in an interval $(t, t+\delta]$ approaches δ/\bar{X} as $t \rightarrow \infty$. We shall thus interpret $1/\bar{X}$ as an ensemble average renewal rate. This rate is the same as the time average renewal rate which comes from the strong law of large numbers for renewal processes. We shall see the benefits of being able to work with both time averages and ensemble averages.

3.2 STRONG LAW OF LARGE NUMBERS FOR RENEWAL PROCESSES

To get an intuitive idea of why $N(t)/t$ should approach $1/\bar{X}$ for large t , note that S_n/n is the sample average of n inter-renewal intervals. From the strong law of large numbers, we know that S_n/n approaches \bar{X} with probability 1 as $t \rightarrow \infty$. From figure 3.1, observe that $N(S_n)$, the number of renewals at the epoch of the n^{th} renewal, is n , and thus $N(S_n)/S_n = n/S_n$. This is the reciprocal of the sample average of n inter-renewal intervals, and thus we should hypothesize that $N(S_n)/S_n$ approaches $1/\bar{X}$ as $n \rightarrow \infty$, and thus that $N(t)/t$ approaches $1/\bar{X}$ as $t \rightarrow \infty$. To make this precise, we need the following lemma. Essentially, it says that the first renewal occurs eventually, and after that, the second eventually occurs, and so forth.

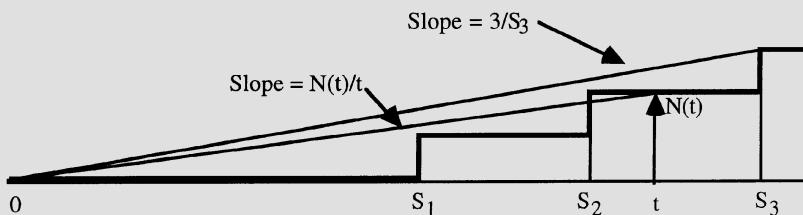


Figure 3.1. Comparison of S_n/n with $N(t)/t$.

LEMMA 1: Let $\{N(t); t \geq 0\}$ be a renewal process with inter-renewal intervals $\{X_n; n \geq 1\}$. Then (whether or not $E[X_n]$ is finite), $\lim_{t \rightarrow \infty} N(t) = \infty$ with probability 1 and $\lim_{t \rightarrow \infty} E[N(t)] = \infty$.

Proof: To show that $\lim_{t \rightarrow \infty} N(t) = \infty$ with probability 1, we must show that, for every $n \geq 1$, $\lim_{t \rightarrow \infty} P(N(t) \geq n) = 1$. From Eq. 2.2, $P(N(t) \geq n) = P(S_n \leq t)$, so we must show that $\lim_{t \rightarrow \infty} P(S_n \leq t) = 1$, i.e., that S_n is a random variable. For the sum of any two random variables X and Y to exceed t , at least one of them must exceed $t/2$, so $P(X+Y > t) \leq P(X > t/2) + P(Y > t/2)$. Since $\lim_{t \rightarrow \infty} P(X > t/2) = 0$ and $\lim_{t \rightarrow \infty} P(Y > t/2) = 0$, we have $\lim_{t \rightarrow \infty} P(X+Y > t) = 0$ and $X+Y$ is a random variable. Thus, $S_2 = X_1 + X_2$ is a random variable, and for any $n \geq 2$, if S_{n-1} is a random variable, then S_n is also. By induction, $\lim_{t \rightarrow \infty} P(S_n \leq t) = 1$ for all $n \geq 1$. Next, $E[N(t)] = \sum_{n \geq 1} P(N(t) \geq n) = \sum_{n \geq 1} P(S_n \leq t)$. Each term in this sum is 1 in the limit $t \rightarrow \infty$, so $\lim_{t \rightarrow \infty} E[N(t)] = \infty$, completing the proof.

For any given time $t > 0$, the random variable $N(t)$ is the number of renewal epochs in the interval $(0, t]$. The random variable $S_{N(t)}$ is then the epoch at which renewal $N(t)$ occurs, i.e., the latest renewal epoch before or at time t (see figure 3.2). Similarly $S_{N(t)+1}$ is the first arrival epoch after time t . Thus we have the inequalities

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \quad (1)$$

From lemma 1, $\lim_{t \rightarrow \infty} N(t) = \infty$ with probability 1. From the strong law of large numbers (theorem 3 of Chapter 1), we also have $\lim_{n \rightarrow \infty} S_n/n = \bar{X}$ with probability 1. For any sample function (i.e., sample point), $S_{N(t)}/N(t)$ runs through the same sequence of values with increasing t as S_n/n runs through with increasing n . Thus if Ω is the set of sample functions ω for which both $\lim_{n \rightarrow \infty} S_n/n = \bar{X}$ and $\lim_{t \rightarrow \infty} N(t) = \infty$, we have $\lim_{t \rightarrow \infty} S_{N(t)}/N(t) = \bar{X}$ for all sample functions in Ω . In the same way

$$\lim_{t \rightarrow \infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t \rightarrow \infty} \frac{S_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)} = \bar{X} \text{ for all } \omega \text{ in } \Omega \quad (2)$$

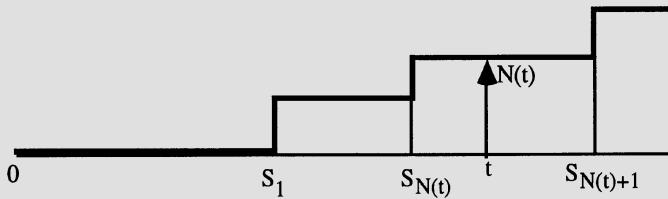


Figure 3.2. Relationship between t , $S_{N(t)}$, and $S_{N(t)+1}$.

Since $t/N(t)$ is between two random variables both converging to \bar{X} for all sample points in Ω , we see that $\lim_{t \rightarrow \infty} t/N(t) = \bar{X}$ for all sample functions in Ω , i.e., with

probability 1. Since \bar{X} must be greater than 0, it follows that $\lim_{t \rightarrow \infty} N(t)/t = 1/\bar{X}$ for all sample paths in Ω . Thus we have proved the strong law of large numbers for renewal processes.

THEOREM 1: STRONG LAW FOR RENEWAL PROCESSES: For a renewal process with mean inter-renewal interval \bar{X} , $\lim_{t \rightarrow \infty} N(t)/t = 1/\bar{X}$ with probability 1.

This theorem is true even if the mean inter-renewal interval is infinite; this can be seen by a truncation argument (see exercise 3.2). We could also prove a weak law for $N(t)$ (i.e., we could show that for any $\epsilon > 0$, $\lim_{t \rightarrow \infty} P(|N(t)/t - 1/\bar{X}| \geq \epsilon) = 0$). This could be done by using the weak law of large numbers for S_n (theorem 1 in Chapter 1) and the fact that the event $S_n \leq t$ is the same as $N(t) \geq n$. Such a derivation is very lengthy and tedious, however, and illustrates that the strong law of large numbers is often much easier to work with than the weak law. We shall not derive the weak law here, since the strong law for renewal processes implies the weak law and it is the strong law that is most often useful.

Figure 3.3 helps give some appreciation of what the strong law for $N(t)$ says and doesn't say. The strong law deals with time averages, $\lim_{t \rightarrow \infty} N(t, \omega)/t$, for individual sample points ω ; these are indicated in the figure as horizontal averages, one for each ω . It is also of interest to look at ensemble averages, $E[N(t)/t]$, shown in the figure as vertical averages. $N(t, \omega)/t$ is the time average number of renewals from 0 to t , and $E[N(t)/t]$ averages also over the ensemble. Finally, to focus on arrivals in the vicinity of a particular time t , it is of interest to look at the ensemble average $E[N(t+\delta)-N(t)]/\delta$.

Given the strong law for $N(t)$, one would hypothesize that $E[N(t)/t]$ approaches $1/\bar{X}$ as $t \rightarrow \infty$. One might also hypothesize that $\lim_{t \rightarrow \infty} E[N(t+\delta)-N(t)]/\delta = 1/\bar{X}$, subject to some minor restrictions on δ . These hypotheses are correct and are discussed in detail in what follows. This equality of time averages and limiting ensemble averages for renewal processes carries over to a large number of stochastic processes, and forms the basis of *ergodic theory*. These results are important for both theoretical and practical purposes. It is sometimes easy to find time averages (just like it was easy to find the time average $N(t, \omega)/t$ from the strong law of large numbers), and it is sometimes easy to find limiting ensemble averages. Being able to equate the two then allows one to alternate at will between time and ensemble averages.

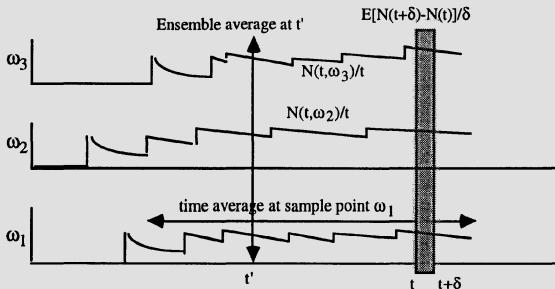


Figure 3.3. Time averages, time and ensemble averages, and ensemble averages in an interval $(t, t+\delta]$.

Note that in order to equate time averages and limiting ensemble averages, quite a few conditions are required. First, the time average must exist in the limit $t \rightarrow \infty$ with probability one and have a fixed value with probability one; second, the ensemble average must approach a limit as $t \rightarrow \infty$; and third, the limits must be the same. The following example, for a stochastic process very different from a renewal process, shows that equality between time and ensemble averages is not always satisfied for arbitrary processes.

EXAMPLE 2: Let $\{X_i; i \geq 1\}$ be a sequence of binary IID random variables, each taking the value 0 with probability 1/2 and 2 with probability 1/2. Let $\{M_n; n \geq 1\}$ be the product process in which $M_n = X_1 X_2 \dots X_n$. Since $M_n = 2^n$ if all the variables X_1 to X_n take the value 2 (an event of probability 2^{-n}) and $M_n = 0$ otherwise, we see that $\lim_{n \rightarrow \infty} M_n = 0$ with probability 1. Also $E[M_n] = 1$ for all $n \geq 1$. Thus the time average exists and equals 0 with probability 1 and the ensemble average exists and equals 1 for all n , but the two are different. The problem is that as n increases, the atypical event in which $M_n = 2^n$ has a probability approaching 0, but still has a significant effect on the ensemble average.

Before establishing the results about ensemble averages, we state and briefly discuss the central limit theorem for renewal processes.

THEOREM 2: CENTRAL LIMIT THEOREM FOR $N(t)$:² Assume that the inter-renewal intervals for a renewal process $\{N(T); t \geq 0\}$ have finite variance σ^2 . Then

$$\lim_{t \rightarrow \infty} P\left(\frac{N(t) - t\bar{X}}{\sigma \bar{X}^{-3/2} \sqrt{t}} < y\right) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx \quad (3)$$

This says that $N(t)$ tends to Gaussian with mean $t\bar{X}$ and standard deviation $\sigma t^{1/2}(\bar{X})^{-3/2}$.

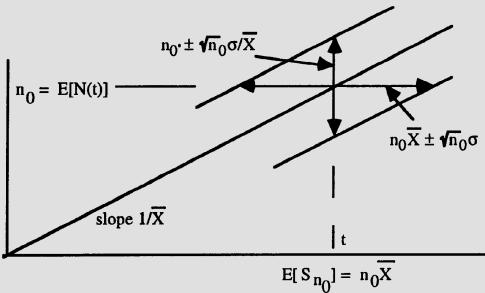


Figure 3.4. Illustration of the central limit theorem for renewal processes. A given integer n_0 is shown on the vertical axis, and the corresponding mean, $t = E[S_{n_0}]$ is shown on the horizontal axis. The horizontal line with arrows at height n_0 indicates the range of one standard deviation around $E[S_{n_0}]$, and the vertical line with arrows indicates a standard deviation around $E[N(t)]$ (see text).

The form of this standard deviation looks peculiar, but is somewhat demystified by figure 3.4. For large n , S_n is approximately Gaussian with mean $\bar{X}n$ and standard deviation $\sigma\sqrt{n}$. If we now vary n , $E[S_n]$ moves along a line of slope $1/\bar{X}$, and, assuming n large, the range of one standard deviation around $E[S_n]$ moves along almost parallel lines as shown in the figure. Now if S_n remains within one standard deviation of the mean as n varies, then $N(t)$ will stay between those parallel lines. For $t = E[S_{n_0}]$, the range of $N(t)$ corresponding to the one standard deviation limits on S_{n_0} is $\sigma\sqrt{n_0}/\bar{X}$. Replacing n_0 by t/\bar{X} , we get the standard deviation $\sigma t^{1/2}(\bar{X})^{-3/2}$ that appears in (3). The proof in [Ros83] uses this same analogy between S_n and $N(t)$.

3.3 EXPECTED NUMBER OF RENEWALS

Let $E[N(t)]$ be denoted by $m(t)$ in what follows. We first find an exact expression for $m(t)$. This is often quite messy for large t , so we then find the asymptotic behavior of $m(t)$. Since $N(t)/t$ approaches $1/\bar{X}$ with probability 1, we expect $m(t)$ to grow with a slope that asymptotically approaches $1/\bar{X}$, but we will find that this is not quite true in general. Two somewhat weaker results, however, are true. The first, called the elementary renewal theorem (theorem 4), states that $\lim_{t \rightarrow \infty} m(t)/t = 1/\bar{X}$. The second result, called Blackwell's theorem (theorem 5), states that, subject to some limitations on $\delta > 0$, $\lim_{t \rightarrow \infty} [m(t+\delta) - m(t)] = \delta/\bar{X}$. This says essentially that the expected renewal rate approaches steady state as $t \rightarrow \infty$. We will find a large number of applications of Blackwell's theorem throughout the remainder of the text.

The exact calculation of $m(t)$ makes use of the fact that the expectation of a non-negative random variable is the integral of the complementary distribution function,

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P(N(t) \geq n)$$

Since the event $N(t) \geq n$ is the same as $S_n \leq t$, $m(t)$ is expressed in terms of the distribution functions of S_n , $n \geq 1$ as follows.

$$m(t) = \sum_{n=1}^{\infty} P(S_n \leq t) \quad (4)$$

Although this expression looks fairly simple, it becomes increasingly complex with increasing t . As t increases, there is an increasing set of values of n for which $P(S_n \leq t)$ is significant, and $P(S_n \leq t)$ itself is not that easy to calculate if the interarrival distribution $F_X(x)$ is complicated. The main utility of (4) comes from the fact that it leads to an integral equation for $m(t)$. Since $S_n = S_{n-1} + X_n$ for each $n \geq 1$ (interpreting S_0 as 0), and since X_n and S_{n-1} are independent, we can use the convolution equation (1.11) to get

$$P(S_n \leq t) = \int_{x=0}^t P(S_{n-1} \leq t-x) dF_X(x) \quad \text{for } n \geq 2$$

For $n=1$, $X_1=S_1$ and $P(S_1 \leq t) = F_X(t)$. Substituting this in (4) and interchanging the order of integration and summation,

$$\begin{aligned} m(t) &= F_X(t) + \int_{x=0}^t \sum_{n=2}^{\infty} P(S_{n-1} \leq t-x) dF_X(x) = F_X(t) + \int_{x=0}^t \sum_{n=1}^{\infty} P(S_n \leq t-x) dF_X(x) \\ m(t) &= F_X(t) + \int_{x=0}^t m(t-x) dF_X(x) \quad ; t \geq 0 \end{aligned} \quad (5)$$

An alternative derivation is given in exercise 3.5. This integral equation is called the *renewal equation*.

LAPLACE TRANSFORM APPROACH: If we assume that X has a density $f_X(x)$, and that this density has a Laplace transform $L_X(r) = \int_0^\infty f_X(x)e^{-rx} dx$, we can take the Laplace transform of both sides of (5). Note that the final term in (5) is the convolution of m with f_X , so that the Laplace transform of $m(t)$ satisfies

$$\begin{aligned} L_m(r) &= L_X(r)/r + L_m(r) L_X(r) \\ L_m(r) &= \frac{L_X(r)}{r[1-L_X(r)]} \end{aligned} \quad (6)$$

EXAMPLE 3: As a simple example of how this can be used to calculate $m(t)$, suppose $f_X(x) = (1/2)e^{-x} + e^{-2x}$ for $x \geq 0$. The Laplace transform is given by

$$L_X(r) = \frac{1}{2(r+1)} + \frac{1}{r+2} = \frac{(3/2)r+2}{(r+1)(r+2)}$$

Substituting this into (6) yields

$$L_m(r) = \frac{(3/2)r+2}{r[(r+1)(r+2)-(3/2)r-2]} = \frac{(3/2)r+2}{r^2(r+3/2)} = \frac{4}{3r^2} + \frac{1}{9r} - \frac{1}{9(r+3/2)}$$

Taking the inverse Laplace transform, we then have

$$m(t) = \frac{4t}{3} + \frac{1 - \exp[-(3/2)t]}{9}$$

The procedure in this example can be used for any inter-renewal density $f_X(x)$ for which the Laplace transform is a rational function, i.e., a ratio of polynomials. In such cases, $L_m(r)$ will also be a rational function. The Heaviside inversion formula (i.e., factoring the denominator and expressing $L_m(r)$ as a sum of individual poles as done above) can then be used to calculate $m(t)$. In the example above, there was a second order pole at

$r=0$ leading to the linear term $4t/3$ in $m(t)$, there was a first order pole at $r=0$ leading to the constant $1/9$, and there was a pole at $r=-3/2$ leading to the exponentially decaying term.

We now show that a second order pole at $r=0$ always occurs when $L_X(r)$ is a rational function. To see this, note that $L_X(0)$ is just the integral of $f_X(x)$, which is 1; thus $1-L_X(r)$ has a zero at $r=0$ and $L_m(r)$ has a second order pole at $r=0$. To evaluate the residue for this second order pole, we recall that the first and second derivatives of $L_X(r)$ at $r=0$ are $-E[X]$ and $E[X^2]$ respectively. Expanding $L_X(r)$ in a power series around $r=0$ then yields $L_X(r) = 1 - rE[X] + (r^2/2)E[X^2]$ plus terms of order r^3 or higher. This gives us

$$L_m(r) = \frac{1 - r\bar{X} + (r^2/2)E[X^2] + \dots}{r^2[\bar{X} - (r/2)E[X^2] + \dots]} = \frac{1}{r^2\bar{X}} + \frac{1}{r} \left(\frac{E[X^2]}{2\bar{X}^2} - 1 \right) \dots \quad (7)$$

The remaining terms are the other poles of $L_m(r)$ with their residues. For values of r with $\text{Re}(r) \geq 0$, we have $|L_X(r)| = \int f_X(x) e^{-rx} dx \leq \int f_X(x) |e^{-rx}| dx \leq \int f_X(x) dx = 1$ with strict inequality except for $r=0$. Thus $L_X(r)$ cannot have any poles on the imaginary axis or the right half plane, and $1-L_X(r)$ cannot have any zeros there other than the one at $r=0$. It follows that all the remaining poles of $L_m(r)$ are strictly in the left half plane. This means that the inverse transforms for all these remaining poles die out as $t \rightarrow \infty$. Thus the inverse Laplace transform of $L_m(r)$ is

$$m(t) = \frac{t}{\bar{X}} + \frac{E[X^2]}{2\bar{X}^2} - 1 + R(t) \quad \text{for } t \geq 0 \quad (8)$$

where $\lim_{t \rightarrow \infty} R(t) = 0$.

We have derived (8) only for the special case in which $f_X(x)$ has a rational Laplace transform. For this case, (8) implies both the elementary renewal theorem ($\lim_{t \rightarrow \infty} m(t)/t = 1/\bar{X}$) and also Blackwell's theorem ($\lim_{t \rightarrow \infty} [m(t+\delta) - m(t)] = \delta/\bar{X}$). We will interpret the meaning of the constant term $E[X^2]/(2\bar{X}^2) - 1$ in section 3.4. In what follows, we derive the elementary renewal theorem for the general case. The machinery used is useful in its own right, and provides additional insight into renewal processes. We first develop a result called Wald's equality. This yields a relation between the expected epoch of the first arrival after some given time t and the expected number of arrivals up to that epoch. We then use this result to demonstrate that $m(t)/t \rightarrow 1/\bar{X}$.

WALD'S EQUALITY: Visualize a situation where one performs an experiment repeatedly, observing successive sample outputs of a given random variable (i.e., observing an outcome of X_1, X_2, \dots where the X_i are IID). The experiment is stopped when enough data has been accumulated for the purposes at hand.

This type of situation occurs frequently in applications. For example, we might be required to choose one of several hypotheses, and might repeat an experiment until the hypotheses are sufficiently discriminated. If the number of trials is allowed to depend

on the outcome, the mean number of trials required to achieve a given error probability is typically a small fraction of the number of trials required when the number is chosen in advance. Another example occurs in tree searches, where a path is explored until further extensions of the path appear to be unprofitable.

The first careful study of experimental situations where the number of trials depends on the data was made by the statistician Abraham Wald and led to the field of sequential analysis. The equality to follow, although simple, is crucial to the study of these situations. We will use it again, along with a generating function equality known as Wald's identity, when we study random walks.

An important part of experiments that stop after a random number of trials is the rule for stopping. Such a rule must specify, for each sample function, the trial at which the experiment stops. Thus the stopping rule is a positive, integer valued, random variable N , mapping sample functions into the trial at which the experiment stops. We view the sample space as the set of sample value sequences for the never ending sequence of random variables X_1, X_2, \dots . That is, even if the experiment is stopped after the second trial, we still visualize the 3rd, 4th, ... random variables as having sample values as part of the sample function. In other words, we visualize that the experiment continues forever, but that the observer stops watching at the stopping point. From the standpoint of applications, it doesn't make any difference whether the experiment continues or not after the observer stops watching. From a mathematical standpoint, however, it is far preferable to view the experiment as continuing so as to avoid confusion and ambiguity about what it means for the variables X_1, X_2, \dots to be IID when the very existence of later variables depends on earlier sample values.

The intuitive notion of stopping includes the notion that a decision to stop before trial n should depend only on the results before trial n . In other words, we want to exclude from stopping rules those rules that allow the experimenter to peek at subsequent values before making the decision to stop or not. Before formalizing this notion, we define a decision rule, I_n , for each $n \geq 1$. The n^{th} decision rule I_n is a random variable that takes the value 1 if the n^{th} observation is to be made, and takes the value 0 otherwise. That is, the observer makes a decision I_n before each trial n , with the value 1 if that trial is to be observed. Since we assume that the first observation is always made (i.e., N is a positive random variable), $I_1 = 1$ with probability 1.

Note that I_n is the indicator function of the event $\{N \geq n\}$, i.e., $I_n = 1$ if $N \geq n$ and $I_n = 0$ otherwise. Since $N \geq n+1$ implies that $N \geq n$, the indicator functions have the corresponding property that $I_{n+1} = 1$ implies that $I_n = 1$. Also, since N is a random variable, and thus finite with probability 1, $\lim_{n \rightarrow \infty} P(I_n = 1)$ must equal 0 with probability 1. We see that each decision rule I_n is a function of the stopping rule N , and the stopping rule N is also determined by all the decision rules (see exercise 3.3). The notion that a stopping rule should not allow peeking then means, for each $n > 1$, that I_n , the decision whether or not to observe X_n , should depend only on X_1, \dots, X_{n-1} .

STOPPING RULE—SIMPLE DEFINITION: A *stopping rule* for a set of random variables $\{X_n; n \geq 1\}$ is a positive, integer valued, random variable N such that, for each $n > 1$, the indicator function, I_n , of $\{N \geq n\}$ is a function of X_1, \dots, X_{n-1} . That is, the sample

values $X_1=x_1, \dots, X_{n-1}=x_{n-1}$ specify whether or not X_n is observed (i.e., whether $N \geq n$). For $n=1$, $I_n=1$ (i.e., X_1 is always observed).

Stopping rules are often called *optional stopping rules*. An implicit part of the definition above is that N , being a random variable, must take on a finite value with probability 1; that is, stopping must take place eventually (with probability 1). Occasionally, a set of random variables $\{X_n; n \geq 1\}$ are embedded into a more general probabilistic experiment, and it is desirable to have a more general stopping rule where the decision variable I_n can depend on more than X_1, \dots, X_{n-1} , but where peeking should still be forbidden. The following definition covers this.

STOPPING RULE—GENERAL DEFINITION: A *stopping rule* for a set of random variables $\{X_n; n \geq 1\}$ is a positive integer valued random variable N such that, for each $n > 1$, I_n , conditional on X_1, \dots, X_{n-1} , is independent of $\{X_i; i \geq n\}$.

THEOREM 3: WALD'S EQUALITY: Let $\{X_n; n \geq 1\}$ be IID random variables each of mean \bar{X} . Let N be a stopping rule for $\{X_n; n \geq 1\}$ and let $S_N = X_1 + X_2 + \dots + X_N$. Then

$$E[S_N] = \bar{X} E[N]. \quad (9)$$

Proof: We can express S_N as

$$S_N = \sum_{n=1}^{\infty} X_n I_n \text{ where } I_n = 1 \text{ if } N \geq n \text{ and } I_n = 0 \text{ if } N \leq n-1 \quad (10)$$

By the simple definition of a stopping rule, I_n is a function of X_1, \dots, X_{n-1} , and is thus independent of X_n since X_n is independent of X_1, \dots, X_{n-1} . The same conclusion follows from the general definition of a stopping rule. Thus, $E[X_n I_n] = E[X_n] E[I_n] = \bar{X} E[I_n]$. Using the fact that an expectation of a sum is equal to the sum of the expectations,

$$E[S_N] = \sum_{n=1}^{\infty} E[X_n I_n] = \bar{X} \sum_{n=1}^{\infty} E[I_n] = \bar{X} E[N] \quad (11)$$

The final step above comes from the observation that $E[I_n] = P(N \geq n)$ and $E[N] = \sum_{n=1}^{\infty} P(N \geq n)$. One can also obtain the last step by using $N = \sum_{n=1}^{\infty} I_n$ (see exercise 3.3).

To use this in evaluating $m(t)$, consider an experiment in which we observe successive inter-arrival intervals until the sum first exceeds t . From figure 3.5, note that $S_{N(t)+1}$ is the epoch of the first arrival after t (see figure 3.5), and thus $N(t)+1$ is the number of intervals observed until the sum exceeds t . We now show that $N(t)+1$ is a stopping rule for $\{X_n; n \geq 1\}$. Informally, the decision to stop when the sum exceeds t depends only on the intervals already observed. More formally, $N(t)+1$ is a random variable and the associated decision variable I_n has the value 1 for $N(t)+1 \geq n$, which is equivalent to $S_{n-1} \leq t$. This is a function of X_1, \dots, X_{n-1} , verifying that $N(t)+1$ is a stopping rule.

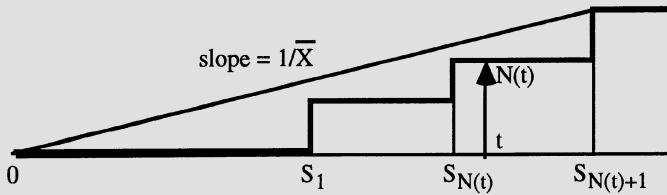


Figure 3.5. Illustration of Wald's equality applied to $N(t)+1$.

Note that $N(t)$ is not a stopping rule. If the random variable $N(t)$ happens to have a given value n , there is no way to tell this from the observation of X_1, X_2, \dots, X_n . One would have to peek ahead at X_{n+1} to verify that S_{n+1} exceeds t . Since $N(t)+1$ is a stopping rule, Wald's equality yields

$$\begin{aligned} E[S_{N(t)+1}] &= \bar{X} \quad E[N(t)+1] = \bar{X} (m(t)+1) \\ m(t) &= \frac{E[S_{N(t)+1}]}{\bar{X}} - 1 \end{aligned} \quad (12)$$

Since $E[S_{N(t)+1}] \geq t$, we have $m(t) \geq t / \bar{X} - 1$, and

$$\frac{m(t)}{t} \geq \frac{1}{\bar{X}} - \frac{1}{t} \quad (13)$$

If we had an upper bound on $E[S_{N(t)+1}] - t$, we could easily show that $m(t)/t$ approaches $1/\bar{X}$ in the limit $t \rightarrow \infty$, but unfortunately $E[S_{N(t)+1}]$ might be larger than t by a surprising amount. The difference $S_{N(t)+1} - t$ is known as the *residual life* of the renewal process at t , and we shall see subsequently that its expected value, in the limit as $t \rightarrow \infty$, is $E[X^2]/(2E[X])$. Substituting this into (12), we find the same limiting expression for $m(t)$ as in (8) for inter-renewal intervals with a rational Laplace transform. Since $E[X^2]$ can be arbitrarily large, and even infinite, this does not show that $m(t)/t \rightarrow 1/\bar{X}$. The reason that the expected residual life can be so large can be seen by an example. Suppose that X is 0 with probability $1-\epsilon$ and $1/\epsilon$ with probability ϵ , and that ϵ is very small. Then $\bar{X} = 1$, but arrivals occur bunched together with a large gap of $1/\epsilon$ between successive bunches. Most points t lie in these gaps, and the residual life is large over most of each gap (we discuss this example in more detail later). Fortunately, it turns out that the familiar truncation method allows us to circumvent these problems and prove the following theorem.

THEOREM 4: THE ELEMENTARY RENEWAL THEOREM: Let $\{N(t); t \geq 0\}$ be a renewal process with mean inter-renewal interval \bar{X} . Then $\lim_{t \rightarrow \infty} E[N(t)]/t = 1/\bar{X}$.

Proof: Let $\tilde{X}_i = X_i$ for $X_i \leq b$ and let $\tilde{X}_i = b$ for $X_i > b$. Since these truncated random variables are IID, they form a related counting process $\{\tilde{N}(t); t > 0\}$ with $\tilde{m}(t) = E[\tilde{N}(t)]$ and $\tilde{S}_{n^*} = \tilde{X}_1 + \dots + \tilde{X}_{n^*}$. Since the n^* th arrival in this truncated process arrives no later than the n^* th arrival in the original process, $\tilde{N}(t) \geq N(t)$, so $\tilde{m}(t) \geq m(t)$. Finally, in the truncated process, $E[\tilde{S}_{\tilde{N}(t)+1}] \leq t+b$. Thus, applying (12) to the truncated process, we have

$$E[\tilde{X}](m(t)+1) \leq E[\tilde{X}](\tilde{m}(t)+1) = E[\tilde{S}_{\tilde{N}(t)+1}] \leq t+b$$

Combining this equation with (13), we have upper and lower bounds on $m(t)$,

$$\frac{1}{E[X]} - \frac{1}{t} \leq \frac{m(t)}{t} \leq \frac{1}{E[\tilde{X}]} + \frac{b}{tE[\tilde{X}]} - \frac{1}{t} \quad (14)$$

Finally, choose $b = \sqrt{t}$. Then as $t \rightarrow \infty$, b also approaches infinity, so that $E[\tilde{X}] \rightarrow E[X]$. Also, both of the final terms on the right of (14) approach 0, completing the proof.

Note that this theorem (and its proof) have not assumed finite variance. The theorem also holds when $E[X]$ is infinite; $E[\tilde{X}]$ in (14) simply approaches ∞ in this case.

We have just shown that $m(t)$ has the property that $\lim_{t \rightarrow \infty} (m(t)/t) = 1/E[X]$. Note again that $N[t, \omega]/t$ is the average number of renewals from 0 to t for a sample function ω , and $m(t)/t$ is the average of this over the ensemble. Combining with theorem 1, the limiting time and ensemble average equals the time average renewal rate for each sample function except for a set of probability 0. Another interesting question is to determine the expected renewal rate in the limit of large t without averaging from 0 to t . That is, are there some values of t at which renewals are more likely than others for large t ? If the inter-renewal intervals have an integer distribution function (i.e., each inter-renewal interval must last for an integer number of time units), then each renewal epoch S_n must also be an integer. This means that $N(t)$ can increase only at integer times and the expected rate of renewals is zero at all non-integer times.

An obvious generalization of this behavior for integer valued inter-renewal intervals is that of inter-renewals that occur only at integer multiples of some real number $d > 0$. Such a distribution is called an *arithmetic distribution*. The *span* of an arithmetic distribution is the largest number d such that this property holds. Thus, for example if X takes on only the values 0, 2, and 6, its distribution is arithmetic with span 2. The remarkable thing, for our purposes, is that any inter-renewal distribution that is *not* an arithmetic distribution leads to a uniform expected rate of renewals in the limit of large t . This result is contained in Blackwell's theorem which we state without proof (see Section 11.1, theorem 1 of [Fel66]). Recall, however, that for the special case of an inter-renewal density that has a rational Laplace transform, Blackwell's theorem is a simple consequence of (8).

THEOREM 5 (BLACKWELL): If a renewal process has an inter-renewal distribution that is non-arithmetic, then for any $\delta > 0$,

$$\lim_{t \rightarrow \infty} [m(t+\delta) - m(t)] = \delta/E[X] \quad (15)$$

If the inter-renewal distribution is arithmetic with span d, then for any integer $n \geq 1$

$$\lim_{t \rightarrow \infty} [m(t+nd) - m(t)] = nd/E[X] \quad (16)$$

Eq. (15) says that for non-arithmetic distributions, the expected number of arrivals in the interval $(t, t+\delta]$ is equal to $\delta/E[X]$ in the limit $t \rightarrow \infty$. Since the theorem is true for arbitrarily small δ , the theorem almost, but not quite, says that $dm(t)/dt$ is equal to $1/E[X]$ in the limit $t \rightarrow \infty$. Unfortunately this latter statement is not true, and one can see the reason by looking at an example where X can take on only the values 1 and π . Then no matter how large t is, $N(t)$ can only increase at discrete points of time of the form $k + j\pi$ where k and j are integers; thus $dm(t)/dt$ is either 0 or ∞ for all t . As t gets larger, however, the jumps in $m(t)$ become smaller and closer together. Thus $[m(t+\delta) - m(t)]/\delta$ approaches $1/E[X]$ as $t \rightarrow \infty$ for any δ , no matter how small, but as δ gets smaller, the convergence gets slower. No matter how large one chooses t , $[m(t+\delta) - m(t)]/\delta$ does not approach $1/E[X]$ as $\delta \rightarrow 0$.

Since the inter-renewal intervals are positive random variables, multiple renewals cannot occur simultaneously, and thus (15) implies that the probability of a renewal in a small interval $(t, t+\delta]$ tends to $\delta/E[X] + o(\delta)$ as $t \rightarrow \infty$. Thus, for a non-arithmetic inter-renewal distribution, the limiting distribution of renewals in a small interval $(t, t+\delta]$ satisfies

$$\begin{aligned} \lim_{t \rightarrow \infty} P[N(t+\delta) - N(t) = 0] &= 1 - \delta/\bar{X} + o(\delta) \\ \lim_{t \rightarrow \infty} P[N(t+\delta) - N(t) = 1] &= \delta/\bar{X} + o(\delta) \\ \lim_{t \rightarrow \infty} P[N(t+\delta) - N(t) \geq 2] &= o(\delta) \end{aligned} \quad (17)$$

If we compare this with Eq. (2.10), associating the rate λ of a Poisson process with $1/\bar{X}$, we see that, asymptotically, a renewal process with a non-arithmetic inter-renewal distribution satisfies two of the three requirements in definition 3 of a Poisson process. That is, the increments are asymptotically stationary and the renewals do not occur simultaneously. If the renewal process is not Poisson, however, the increments are not independent.

For an arithmetic renewal process with span d , (16), with $n=1$, states that the probability of a renewal at time nd is given by

$$\lim_{n \rightarrow \infty} P[N(nd) - N(nd-d) = 1] = d/\bar{X} \quad (18)$$

3.4 RENEWAL REWARD PROCESSES; TIME AVERAGES

There are many situations in which, along with a renewal process $\{N(t); t \geq 0\}$, there is another randomly varying function of time, called a *reward function* $\{R(t); t \geq 0\}$. $R(t)$ models a rate at which the process is accumulating a reward. We shall illustrate many

examples of such processes and see that a “reward” could also be a cost or any randomly varying quantity of interest. The important restriction on these *reward functions* is that $R(t)$ at a given t depends only on the particular inter-renewal interval containing t . We start with several examples to illustrate the kinds of questions addressed by this type of process.

EXAMPLE 4—TIME AVERAGE RESIDUAL LIFE: For a renewal process $\{N(t), t \geq 0\}$, let $Y(t)$ be the *residual life* at time t . The residual life is defined as the interval from t until the next renewal epoch, i.e., as $S_{N(t)+1} - t$. For example, if we arrive at a bus stop at time t and buses arrive according to a renewal process, $Y(t)$ is the time we have to wait for a bus to arrive (see figure 3.6). We interpret $\{Y(t); t \geq 0\}$ as a reward function. The average of $Y(t)$, over the interval $(0, t]$, is given by³ $(1/t) \int_0^t Y(\tau) d\tau$. We are interested in the limit of this average as $t \rightarrow \infty$ (assuming that it exists in some sense). Figure 3.6 illustrates a sample function of a renewal process and the residual life for that sample function. Note that the integral $\int_0^t Y(\tau) d\tau$ is simply a sum of isosceles right triangles, with part of a final triangle at the end. Thus it can be expressed as

$$\int_0^t y(\tau) d\tau = \frac{\sum_{i=1}^{n(t)} x_i^2}{2} + \int_{\tau=s_{n(t)}}^t y(\tau) d\tau$$

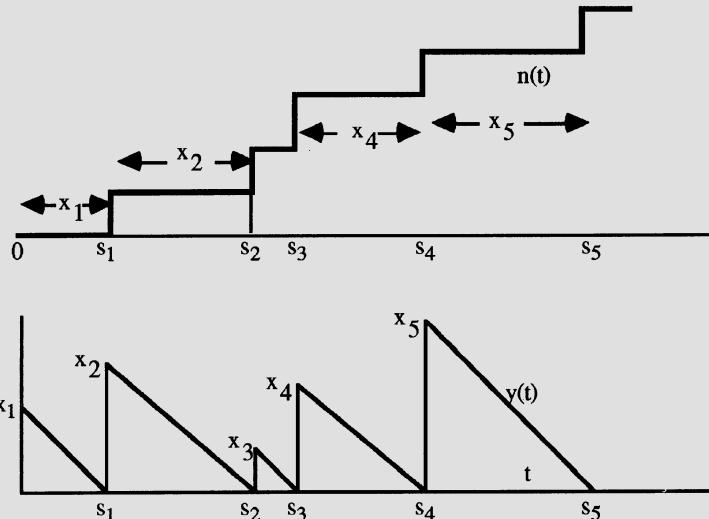


Figure 3.6. Residual life at time t . For a given sample function $n(t)$ of the renewal process, the sample function of residual life $y(t)$ decreases linearly with a slope of -1 from the beginning to the end of each inter-renewal interval.

Since this relationship holds for every sample function $n(t)$, we see that the random variable $\int_0^t Y(\tau) d\tau$ can be expressed in terms of the inter-renewal random variables X_n as

$$\int_{\tau=0}^t Y(\tau) d\tau = \frac{\sum_{n=1}^{N(t)} X_n^2}{2} + \int_{\tau=S_{N(t)}}^t Y(\tau) d\tau$$

Although the final term above can be easily evaluated, it is more convenient to use the following bound:

$$\frac{\sum_{n=1}^{N(t)} X_n^2}{2t} \leq \frac{\int_{\tau=0}^t Y(\tau) d\tau}{t} \leq \frac{\sum_{n=1}^{N(t)+1} X_n^2}{2t} \quad (19)$$

The term on the left can now be evaluated in the limit $t \rightarrow \infty$ (for all sample functions except a set of probability zero) as follows:

$$\lim_{t \rightarrow \infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{2t} = \lim_{t \rightarrow \infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{N(t)} \frac{N(t)}{2t} = \frac{E[X^2]}{2E[X]} \quad (20)$$

The second equality above follows by applying the strong law of large numbers (theorem 3, Chapter 1) to $\sum_{n \leq N(t)} X_n^2 / N(t)$ as $N(t)$ approaches infinity, and by applying the strong law for renewal processes (theorem 1, section 3.2) to $N(t)/t$ as $t \rightarrow \infty$. The right hand term of (19) is handled almost the same way:

$$\lim_{t \rightarrow \infty} \frac{\sum_{n=1}^{N(t)+1} X_n^2}{2t} = \lim_{t \rightarrow \infty} \frac{\sum_{n=1}^{N(t)+1} X_n^2}{N(t)+1} \frac{N(t)+1}{N(t)} \frac{N(t)}{2t} = \frac{E[X^2]}{2E[X]} \quad (21)$$

Combining these two results, we see that, with probability 1, the time average residual life is given by

$$\lim_{t \rightarrow \infty} \frac{\int_{\tau=0}^t Y(\tau) d\tau}{t} = \frac{E[X^2]}{2E[X]} \quad (22)$$

Note that this time average depends on the second moment of X ; this is always at least as large as the first moment squared, so the time average residual life is at least half the expected inter-renewal interval (which is not surprising). On the other hand, the second moment of X can be arbitrarily large (even infinite) for any given value of $E[X]$, so that

the time average residual life can be arbitrarily large relative to $E[X]$. This can be explained intuitively by observing that large inter-renewal intervals are weighted more heavily in this time average than small inter-renewal intervals. As an example, consider an inter-renewal random variable X that takes on value ϵ with probability $1-\epsilon$ and value $1/\epsilon$ with probability ϵ . Then, for small ϵ , $E[X] \approx 1$, $E[X^2] \approx 1/\epsilon$, and the time average residual life is approximately $1/(2\epsilon)$ (see figure 3.7).

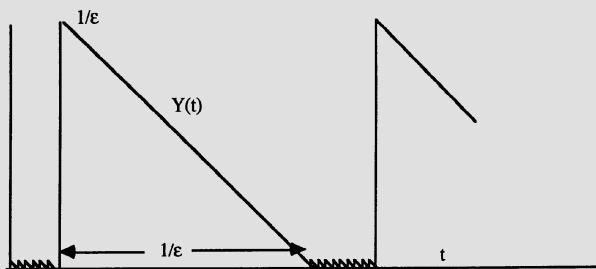


Figure 3.7. Average Residual life is dominated by large interarrival intervals. Note the large intervals of time (within the large triangles) over which $Y(t)$ is large, and small aggregate intervals over which it is small.

EXAMPLE 5—TIME AVERAGE AGE: Let $Z(t)$ be the *age* of a renewal process at time t . This is defined as the length of the interval back from t to the most recent renewal, i.e., $Z(t) = t - S_{N(t)}$ (see figure 3.8). We notice that the age process, for a given sample function of the renewal process, is almost the same as the residual life process—the isosceles right triangles are simply turned around. Thus the same analysis as before can be used to show that the time average of $Z(t)$ is the same as the time average of the residual life,

$$\lim_{t \rightarrow \infty} \frac{\int_{\tau=0}^t Z(\tau) d\tau}{t} = \frac{E[X^2]}{2E[X]} \quad (23)$$

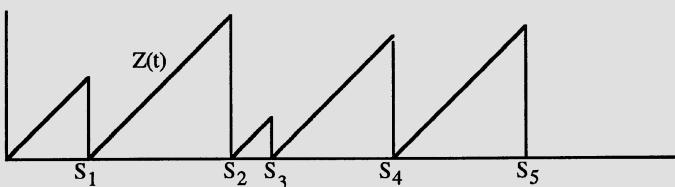


Figure 3.8. Age at time t : Here we show the renewal epochs and the age as random variables. This can be viewed as a shorthand for the approach in figure 3.6 which showed a particular sample function.

EXAMPLE 6—TIME AVERAGE DURATION: Let $X(t)$ be the *duration* of the inter-renewal interval containing time t , i.e., $X(t) = S_{N(t)+1} - S_{N(t)}$ (see figure 3.9). It is clear that $X(t) = Z(t) + Y(t)$, and thus the time average of the duration is given by

$$\lim_{t \rightarrow \infty} \frac{\int_{\tau=0}^t X(\tau) d\tau}{t} = \frac{E[X^2]}{E[X]} \quad (24)$$

Again, long intervals are heavily weighted in this average, so that the time average duration is at least as large as the mean inter-renewal interval and often much larger.

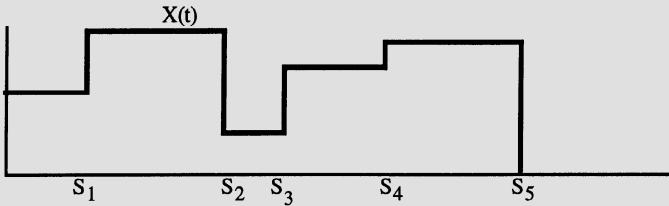


Figure 3.9. Duration of inter-renewal interval containing t .

In each of these examples, and in many other situations, we have a random function of time (i.e., $Y(t)$, $Z(t)$, or $X(t)$) whose value at time t depends only on where t is in the current inter-renewal interval (i.e., on the age $Z(t)$) and on the duration $X(t)$ of the current inter-renewal interval. We now investigate the general class of reward functions for which the reward function at time t depends only on the age and the duration at t , i.e., the reward $R(t)$ at time t is given explicitly as a function $R(Z(t), X(t))$ of the age and duration at t . For the three examples above, the function R is trivial. That is, the residual life, $Y(t)$, is given by $X(t) - Z(t)$ and the age and duration are given directly.

We now find the time average value of $R(t)$, namely, $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(\tau) d\tau$. As in examples 4 to 6 above, we first want to look at the accumulated reward over each inter-renewal period separately. Define R_n as the accumulated reward in the n^{th} renewal interval,

$$R_n = \int_{S_{n-1}}^{S_n} R(\tau) d\tau = \int_{S_{n-1}}^{S_n} R[Z(\tau), X(\tau)] d\tau \quad (25)$$

For example 4 (time average residual life), R_n is the area of the n^{th} isosceles right triangle in figure 3.6. In general, since $Z(\tau) = \tau - S_{n-1}$, we have

$$R_n = \int_{S_{n-1}}^{S_n} R(\tau - S_{n-1}, X_n) d\tau = \int_{z=0}^{X_n} R(z, X_n) dz \quad (26)$$

Note that R_n depends only on the value of X_n and the form of the function $R(Z, X)$. From this, it is clear that $\{R_n; n \geq 1\}$ is a set of IID random variables. For residual life, example 4, $R(z, X_n) = X_n - z$, so the integral in (26) is $X_n^2/2$, as calculated by inspection before.

The expected value of R_n is now given by

$$E[R_n] = \int_{x=0}^{\infty} R_n dF_X(x) = \int_{x=0}^{\infty} \int_{z=0}^x R(z, x) dz dF_X(x) \quad (27)$$

For residual life, example 4, $E[R_n] = E[X^2]/2$ as calculated by inspection before.

Breaking $\int_0^t R(\tau) d\tau$ into the reward over the successive renewal periods, we get

$$\begin{aligned} \int_0^t R(\tau) d\tau &= \int_0^{S_1} R(\tau) d\tau + \int_{S_1}^{S_2} R(\tau) d\tau + \dots + \int_{S_{N(t)-1}}^{S_{N(t)}} R(\tau) d\tau + \int_{S_{N(t)}}^t R(\tau) d\tau \\ &= \sum_{n=1}^{N(t)} R_n + \int_{S_{N(t)}}^t R(\tau) d\tau \end{aligned} \quad (28)$$

GENERAL RENEWAL REWARD FUNCTIONS: The following theorem now generalizes the results of examples 4 to 6 to general renewal reward functions.

THEOREM 6: Let $\{R(t); t > 0\}$ be a renewal reward function for a renewal process with expected inter-renewal time $E[X] = \bar{X}$. If $\bar{X} < \infty$ or $E[R_n] < \infty$, then with probability 1

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{\tau=0}^t R(\tau) d\tau = \frac{E[R_n]}{\bar{X}} \quad (29)$$

Proof: First assume that the reward function is non-negative. Then, using (28), the accumulated reward up to time t can be bounded between the accumulated reward up to the renewal before t and that to the next renewal after t ,

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{\int_{\tau=0}^t R(\tau) d\tau}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t} \quad (30)$$

The left hand side of (30) can now be broken into

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \frac{N(t)}{t} \quad (31)$$

As $t \rightarrow \infty$, $N(t) \rightarrow \infty$, and thus, by the strong law of large numbers, the first term on the right side of (31) approaches $E[R_n]$ with probability 1. Also the second term approaches $1/\bar{X}$ by the strong law for renewal processes. Thus (unless both $E[R_n]$ and \bar{X} are infinite), the product of the two terms approaches the limit $E[R_n]/\bar{X}$. The right hand term of (30) is handled almost the same way,

$$\frac{\sum_{n=1}^{N(t)+1} R_n}{t} = \frac{\sum_{n=1}^{N(t)+1} R_n}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \cdot \frac{N(t)}{t} \quad (32)$$

It is seen that the terms on the right side of (32) approach limits as before and thus the term on the left approaches $E[R_n]/E[X]$ with probability 1. Since the upper and lower bound in (30) approach the same limit, $(1/t) \int_0^t R(\tau) d\tau$ approaches the same limit and the theorem is proved for non-negative reward functions. By changing the directions of the inequalities in (30), the same result applies for non-positive reward functions, and the general result follows by splitting an arbitrary reward function into a positive and negative part.

EXAMPLE 7—DISTRIBUTION OF RESIDUAL LIFE: Example 4 treated the time average value of the residual life $Y(t)$. Suppose, however, that we would like to find the time average distribution function of $Y(t)$, i.e., the fraction of time that $Y(t) \leq y$ as a function of y . The approach, which applies to a wide variety of applications, is to use an indicator function (for a given value of y) as a reward function. That is, define $R(t)$ to have the value 1 for all t such that $Y(t) \leq y$ and to have the value 0 otherwise. Figure 3.10 illustrates this function for a given sample path. Expressing this reward function in terms of $Z(t)$ and $X(t)$, we have

$$R(t) = R(Z(t), X(t)) = \begin{cases} 1; & X(t)-Z(t) \leq y \\ 0; & \text{otherwise} \end{cases}$$

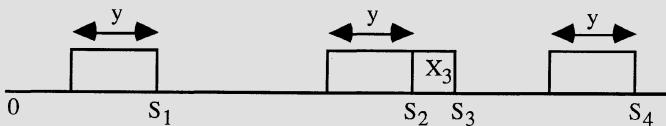


Figure 3.10. Reward function to find $P(Y(t) \leq y)$.

Note that if an inter-renewal interval is smaller than y (such as the third interval in figure 3.10), then $R(t)$ has the value one over the entire interval, whereas if the interval is greater than y , then $R(t)$ has the value one only over the final y units of the interval. Thus $R_n = \min[y, X_n]$. Note that the random variable $\min[y, X_n]$ is equal to X_n for $X_n \leq y$, and thus has the same distribution function as X_n in the range 0 to y . Figure 3.11 illus-

rates this in terms of the complementary distribution function. From the figure, we see that

$$E[R_n] = E[\min(X,y)] = \int_0^\infty P[\min(X,y) > x] dx = \int_0^y P(X > x) dx \quad (33)$$

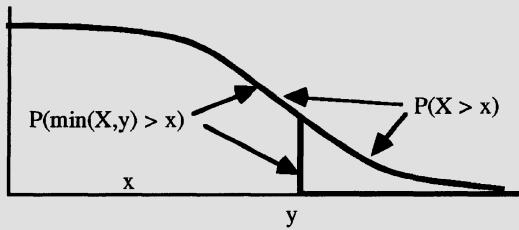


Figure 3.11. R_n for distribution of residual life.

Let $F_Y(y) = \lim_{t \rightarrow \infty} (1/t) \int_0^t R(\tau) d\tau$ denote the time average fraction of time that the residual life is less than or equal to y . From theorem 6 and Eq. (33), we then have

$$F_Y(y) = \frac{E[R_n]}{\bar{X}} = \frac{1}{\bar{X}} \int_{x=0}^y P(X > x) dx \quad (34)$$

As a check, note that this integral is increasing in y and approaches 1 as $y \rightarrow \infty$.

In the development so far, the reward function $R(t)$ has been a function solely of age and duration interval. In more general situations, where the renewal process is embedded in some more complex process, it is often desirable to define $R(t)$ to depend on other aspects of the process as well. The important thing here is for the reward function to depend only on the process within the given inter-renewal interval so that the accumulated rewards over successive inter-renewal intervals are IID random variables. Under this circumstance, theorem 6 clearly remains valid. The subsequent examples of Little's theorem and the M/G/1 expected queueing delay both use this more general type of renewal reward function.

The above time average can be visualized by the following type of experiment. For some given large time t , let T be a uniformly distributed random variable over $(0, t]$; T is independent of the renewal reward process under consideration. Then $(1/t) \int_0^t R(\tau) d\tau$ is the expected value (over T) of $R(T)$ for a given sample point $\{R(\tau); \tau > 0\}$. Theorem 6, however, states that in the limit $t \rightarrow \infty$, all sample points (except a set of probability 0) yield the same expected value over T . Viewing a renewal reward process at a randomly chosen time T is sometimes referred to as random incidence. Viewing the observation time as random, however, blurs the distinction between the time average of $R(t)$ and the ensemble average, $E[R(t)]$, at some particular t . In what follows, we use Blackwell's theorem to look at the ensemble average in the limit $t \rightarrow \infty$.

We could also find $\lim_{t \rightarrow \infty} (1/t)E[\int_0^t R(\tau)d\tau]$ (which not surprisingly is equal to $E[R_n]/E[X]$). This can be interpreted as the limiting expected value of the reward function over both the renewal reward process and over T. (See [Ros83], Theorem 3.6.1 for a proof that $\lim_{t \rightarrow \infty} (1/t)E[\int_0^t R(\tau)d\tau] = E[R_n]/E[X]$). We will not bother with that here, however, since what is more important is finding $E(R(t))$ as a function of t and finding the limiting behavior as $t \rightarrow \infty$. In concrete terms, if my expected delay in waiting for a bus depends strongly on when I start waiting, I would like to know about that time dependence. Thus we want to know if these expected reward functions depend critically, even after a long time, on when the renewal process started.

3.5 RENEWAL REWARD PROCESSES; ENSEMBLE AVERAGES

As in the last section, $\{N(t); t \geq 0\}$ is a renewal process, $Z(t)$ and $X(t)$, $t > 0$, are the age and duration random variables, $R(z,x)$ is a real valued function of the real variables z and x, and $\{R(t); t \geq 0\}$ is a reward process with $R(t)$ defined by $R[Z(t), X(t)]$. Our objective is to find (and to understand) $\lim_{t \rightarrow \infty} E[R(t)]$. We start out with an intuitive derivation which assumes that the inter-renewal intervals $\{X_n; n \geq 1\}$ have a probability density $f_X(x)$. Also, rather than finding $E[R(t)]$ for a finite t and then going to the limit, we simply assume that t is so large that $m(t+\delta)-m(t) = \delta/\bar{X}$, for all τ in the vicinity of t (i.e., we ignore the limit in (15)). After this intuitive derivation, we return to look at the limiting issues more carefully.

Since $R(t) = R[Z(t), X(t)]$, we start by finding the joint probability density, $f_{Z(t),X(t)}(z,x)$, of $Z(t), X(t)$. Since the duration at t is equal to the age plus residual life at t, we must have $X(t) \geq Z(t)$, and the joint probability density can be non-zero only in the triangular region shown in figure 3.12.

If $Z(t)$ has the value z, there must be a renewal at $t-z$. By assumption, $m(t-z+\delta) - m(t-z) = \delta/\bar{X}$ for any $\delta > 0$. Since X is positive, the probability of a renewal in a small interval $[t-z, t-z+\delta]$ is $\delta/\bar{X} - o(\delta)$ (i.e., the probability of multiple renewals in an interval of size δ is negligible for small δ). Note that, although $Z(t)=z$ implies a renewal at $t-z$, a renewal at $t-z$ does not imply that $Z(t)=z$, since there might be other renewals between $t-z$ and t. Given a renewal at $t-z$, however, the subsequent inter-renewal interval has probability density $f_X(x)$. Thus, the joint probability of a renewal in $[t-z, t-z+\delta]$ and a subsequent inter-renewal interval between x and $x+\delta$ is $\delta^2 f_X(x)/\bar{X} + o(\delta^2)$, i.e.,

$$P\{\text{renewal } \in [t-z, t-z+\delta], \text{next inter-renewal } \in (x, x+\delta)\} = \delta^2 f_X(x)/\bar{X} + o(\delta^2) \quad (35)$$

This is valid for arbitrary x. For $x > z$, however, the joint event inside (35) is the same as the joint event $\{Z(t) \in (z-\delta, z], X(t) \in (x, x+\delta]\}$. Thus, going to the limit $\delta \rightarrow 0$, we have

$$f_{Z(t),X(t)}(z,x) = \frac{f_X(x)}{\bar{X}} \quad , \quad x > z ; \quad f_{Z(t),X(t)}(z,x) = 0 \text{ elsewhere} \quad (36)$$

This joint density is illustrated in figure 3.12. Note that the argument z does not appear except in the condition $x > z \geq 0$, but this condition is very important. The marginal den-

sities for $Z(t)$ and $X(t)$ can be found by integrating (36) over the constraint region, and yield

$$f_{Z(t)}(z) = \int_{x=z}^{\infty} \frac{f_X(x) dx}{\bar{X}} = \frac{1-F_X(z)}{\bar{X}} \quad (37)$$

$$f_{X(t)}(x) = \int_{z=0}^x \frac{f_X(x) dz}{\bar{X}} = \frac{x f_X(x)}{\bar{X}} \quad (38)$$

The mean age can be calculated from (37) by integration by parts, yielding

$$E[Z(t)] = \frac{E[X^2]}{2\bar{X}} \quad (39)$$

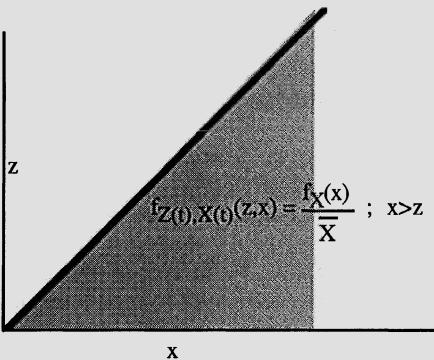


Figure 3.12. Joint density of age and duration.

This is the same as the time average age found in (23). The mean duration can also be found from (38). The result, which agrees with the time average in (24) is

$$E[X(t)] = \frac{E[X^2]}{\bar{X}} \quad (40)$$

In calculating time averages, the somewhat paradoxical result that the time average duration is greater than $E[X]$ was explained by the large inter-renewal intervals being weighted more heavily in the time average. Here the same effect occurs, but it can be given a different interpretation: the joint density, at z and x , for age and duration, is proportional to the inter-renewal density $f_X(x)$, but the marginal density for duration is weighted by x since the range for age is proportional to x .

Using the joint probability density of $Z(t)$ and $X(t)$ to evaluate the expectation of an arbitrary reward function $R(t) = R(Z(t), X(t))$ in the limit $t \rightarrow \infty$, we get

$$\lim_{t \rightarrow \infty} E[R(t)] = \int_{x=0}^{\infty} \int_{z=0}^x R(z, x) dz \frac{f_X(x) dx}{\bar{X}} = \frac{E[R_n]}{\bar{X}} \quad (41)$$

where $E[R_n]$ is defined in (27). Thus the limiting ensemble average is the same as the time average. This result should not be surprising. Since we are dealing with non-arithmetic renewals, the probability of a renewal in a small interval becomes independent of where the interval is, so long as the interval is far enough removed from 0 for the process to be in "steady state." Since the reward function depends on the process only through the current renewal interval, the reward function must also become independent of time.

In the intuitive derivations above, we assumed a density for the inter-renewal variables $\{X_n; n \geq 1\}$, and we ignored the mathematical issues of taking limits. Here we correct those defects, but the reader should bear in mind that the logical train of the argument is exactly the same as before. Assume throughout that X_n is non-arithmetic with distribution function $F_X(x)$. Since $m(\tau) = E[N(\tau)]$, the expected number of renewals in $(t-z, t-z+\delta]$ is $m(t-z+\delta) - m(t-z)$. Since the inter-renewal variables are strictly positive, the probability of more than one renewal in $(t-z, t-z+\delta]$ is of order $o(\delta)$, and thus⁴

$$P\{\text{renewal } \in (t-z, t-z+\delta]\} = m(t-z+\delta) - m(t-z) - o(\delta) \quad (42)$$

Conditional on a renewal in $(t-z, t-z+\delta]$, the probability that the next inter-renewal interval will exceed x is $1 - F_X(x)$. For $x > z$, the joint event

$$\{\text{renewal } \in (t-z, t-z+\delta], \text{ next inter-renewal interval } \in (x, x+\delta]\}$$

is the same as the joint event $\{Z(t) \in [z-\delta, z], X(t) \in (x, x+\delta]\}$, so, for $x > z$,

$$P\{Z(t) \in [z-\delta, z], X(t) \in (x, x+\delta]\} = [m(t-z+\delta) - m(t-z) - o(\delta)][F_X(x+\delta) - F_X(x)] \quad (43)$$

Now let $R(z, x)$ be a renewal reward function and assume $R(z, x) \geq 0$. Define

$$r(z) = \int_{x=z}^{\infty} R(z, x) dF_X(x) \quad (44)$$

In terms of figure 3.12, this is the contribution to $E[R(Z(t), X(t))]$ from a horizontal slice at $Z(t) = z$. We can then express $E[R(t)]$ as

$$E[R(t)] = E[R\{Z(t), X(t)\}] = \int_{z=0}^{\infty} r(z) dF_{Z(t)}(z)$$

We can upper bound this integral by quantizing z in disjoint intervals of fixed size δ and upper bounding $r(z)$ over an interval $(n\delta - \delta, n\delta]$ by $\bar{r}(n, \delta) = \sup_{n\delta - \delta \leq z \leq n\delta} r(z)$. Using (43), this yields

$$E[R(t)] \leq \sum_{n=1}^{t/\delta} [m(t-n\delta+\delta) - m(t-n\delta)]\bar{r}(n, \delta) \quad (45)$$

We can similarly lower bound $R(t)$ as

$$E[R(t)] \geq \sum_{n=1}^{t/\delta} [m(t-n\delta+\delta) - m(t-n\delta) - o(\delta)]\underline{r}(n, \delta) \quad (46)$$

where $\underline{r}(n, \delta) = \inf_{n\delta - \delta \leq z \leq n\delta} r(z)$. Aside from the term $o(\delta)$ in (46), we see from the definition of a Stieltjes integral in (1.7) that if the Stieltjes integral exists, we can express $E[R(t)]$ as

$$E[R(t)] = \int_{z=0}^t r(z) dm(t-z) \quad (47)$$

Finally, from Blackwell's theorem, the limit (if it exists) of the right hand side of (45) as $t \rightarrow \infty$ is

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{t/\delta} [m(t-n\delta+\delta) - m(t-n\delta)]\bar{r}(n, \delta) = \frac{\delta}{X} \sum_{n=1}^{\infty} \bar{r}(n, \delta) \quad (48)$$

Similarly, the limit of the right hand side of (46), if it exists, is

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{t/\delta} [m(t-n\delta+\delta) - m(t-n\delta) - o(\delta)]\underline{r}(n, \delta) = \left(\frac{\delta}{X} - o(\delta) \right) \sum_{n=1}^{\infty} \underline{r}(n, \delta) \quad (49)$$

A function $r(z)$ is called “directly Riemann integrable” if $\sum_{n \geq 1} \bar{r}(n, \delta)$ and $\sum_{n \geq 1} \underline{r}(n, \delta)$ are finite for all $\delta > 0$ and if $\lim_{\delta \rightarrow 0} \delta \sum_{n \geq 1} \bar{r}(n, \delta) = \lim_{\delta \rightarrow 0} \delta \sum_{n \geq 1} \underline{r}(n, \delta)$. If this latter equality holds, then each limit is equal to $\int_{z \geq 0} r(z) dz$. If $r(z)$ is directly Riemann integrable, then the right hand sides of (48) and (49) are equal in the limit $\delta \rightarrow 0$. Since one is an upper bound and the other a lower bound to $\lim_{t \rightarrow \infty} E[R(t)]$, we see that the limit exists and is equal to $[\int_{z \geq 0} r(z) dz] / X$. This can be summarized in the following theorem, known as the Key Renewal Theorem:

THEOREM 7 (KEY RENEWAL THEOREM): Let $r(z) \geq 0$ be a directly Riemann integrable function, and let $m(t) = E[N(t)]$ for a non-arithmetic renewal process. Then

$$\lim_{t \rightarrow \infty} \int_{z=0}^{\infty} r(z) dm(t-z) = \frac{\int_{z=0}^{\infty} r(z) dz}{\bar{X}} \quad (50)$$

Since $R(z,x) \geq 0$, $r(z) \geq 0$. Also, from (27), $E[R_n] = \int_{z \geq 0} r(z) dz$. Thus, combining (47) with (50), we have the corollary

COROLLARY: Let $\{N(t); t \geq 0\}$ be a non-arithmetic renewal process, let $R(z,x) \geq 0$, let $r(z)$ in (44) be directly Riemann integrable. Then

$$\lim_{t \rightarrow \infty} E[R(t)] = E[R_n]/\bar{X} \quad (51)$$

The major restrictions imposed by $r(z)$ being directly Riemann integrable are, first, that $E[R_n] = \int_{z \geq 0} r(z) dz$ is finite, second, that $r(z)$ contains no impulses, and third, that $r(z)$ is not too wildly varying (being continuous and bounded by a decreasing integrable function is enough). It is also not necessary to assume $R(z,x) \geq 0$, since one can break a more general R into positive and negative parts.

The above development assumed a non-arithmetic renewal process. For an arithmetic process, the situation is somewhat simpler mathematically, but in general $E[R(t)]$ depends on the remainder when t is divided by the span d . Usually with such processes, one is interested only in reward functions that remain constant over intervals of length d , or equivalently, in evaluating $E[R(t)]$ only for t equal to multiples of d , so we assume $t = nd$ here. Thus the function $R(z,x)$ is of interest only when z and x are multiples of d , and in particular, only for $x = d, 2d, \dots$ and for $z = d, 2d, \dots, x$. We follow the convention that an inter-renewal interval is open on the left and closed on the right, thus including the renewal that ends the interval.

$$E[R(nd)] = \sum_{i=1}^{\infty} \sum_{j=1}^i R(jd, id) P(\text{renewal at } (n-j)d, \text{ next renewal at } (n-j+i)d]$$

Let P_i be the probability that an inter-renewal interval has size id . Using (18) for the limiting probability of a renewal at $(n-j)d$, this becomes

$$\lim_{d \rightarrow \infty} E[R(nd)] = \sum_{i=1}^{\infty} \sum_{j=1}^i R(jd, id) \frac{d}{\bar{X}} P_i = \frac{E[R_n]}{\bar{X}} \quad (52)$$

where $E[R_n]$ is the expected reward over a renewal period.

3.6 APPLICATIONS OF RENEWAL REWARD THEORY

LITTLE'S THEOREM: Little's theorem is an important queueing result stating that the expected number of customers in a queueing system is equal to the expected time

each customer waits in the system times the rate of arrivals. This result is true under very general conditions; we use the G/G/1 queue as a specific example, but the reason for the greater generality will be clear as we proceed. Note that the theorem does not tell us how to find either the expected number or expected wait; it only says that if one can be found, the other can also be found.

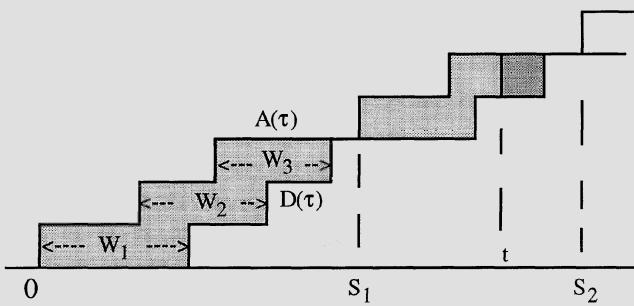


Figure 3.13. Arrival process, departure process, and waiting times for a queue. Renewals occur at S_1 and S_2 when an arrival sees an empty system. The lightly crosshatched region has area $\int_0^t L(\tau)d\tau$ where $L(\tau) = A(\tau)-D(\tau)$. The sum $W_1+\dots+W_{A(t)}$ also includes the heavily shaded region to the right of t .

Figure 3.13 illustrates the setting for Little's theorem. It is assumed that the first arrival occurs at time 0, and that the subsequent interarrival intervals are IID. $A(t)$ is the number of arrivals from time 0 to t , including the arrival at 0, so $\{A(t)-1; t \geq 0\}$ is a renewal process. The departure process $\{D(t); t \geq 0\}$ is the number of departures from 0 to t , and thus increases by one each time a customer leaves the system. The difference, $L(t) = A(t) - D(t)$, is the number in the system at time t . To be specific, we assume First Come First Serve (FCFS) service,⁵ which means that customers are served in their order of arrival to the system. This means that the system time of customer n , i.e., the time customer n spends in the system, is the interval from the n^{th} arrival to the n^{th} departure. Finally, the figure shows the renewal points S_1, S_2, \dots at which arriving customers find an empty system. As observed in example 1, the system probabilistically restarts at each of these renewal instants, and the behavior of the system in one inter-renewal interval is independent of that in each other inter-renewal interval.

It is important here to distinguish between two different renewal processes. The arrival process, or more precisely, $\{A(t)-1; t \geq 0\}$ is one renewal process, and the renewal epochs S_1, S_2, \dots in the figure generate another renewal process. In what follows, $\{A(t); t \geq 0\}$ is referred to as the *arrival process* and $\{N(t); t \geq 0\}$, with renewal epochs S_1, S_2, \dots , is referred to as the *renewal process*. The entire system can be viewed as starting anew at each renewal epoch, but not at each arrival epoch.

We now regard $L(t)$, the number of customers in the system at time t , as a reward function over the renewal process. This is slightly more general than the reward func-

tions of sections 3.4 and 3.5, since $L(t)$ depends on the arrivals and departures within a busy period (i.e., within an inter-renewal interval). Conditional on the age $Z(t)$ and duration $X(t)$ of the inter-renewal interval at time t , one could, in principle, calculate the expected value $R(Z(t), X(t))$ over the parameters other than $Z(t)$ and $X(t)$. Fortunately, this is not necessary and we can use the sample functions of the combined arrival and departure processes directly, which specify $L(t)$ as $A(t)-D(t)$. Assuming that the expected inter-renewal interval is finite, theorem 6 asserts that the time average number of customers in the system (with probability 1) is equal to $E[R_n]/E[X]$. $E[R_n]$ is the expected area between $A(t)$ and $D(t)$ (i.e., the expected integral of $L(t)$) over an inter-renewal interval. An inter-renewal interval is a busy period followed by an idle period, so $E[R_n]$ is also the expected area over a busy period. $E[X]$ is the mean inter-renewal interval.

From figure 3.13, we observe that $W_1 + W_2 + W_3$ is the area of the region between $A(t)$ and $D(t)$ in the first inter-renewal interval for the particular sample path in the figure. This is the aggregate reward over the first inter-renewal interval for the reward function $L(t)$. More generally, for any time t , $W_1 + W_2 + \dots + W_{A(t)}$ is the area between $A(t)$ and $D(t)$ up to a height of $A(t)$. It is equal to $\int_0^t L(\tau) d\tau$ plus the remaining waiting time of each of the customers in the system at time t (see figure 3.13). Since this remaining waiting time is at most the area between $A(t)$ and $D(t)$ from t until the next time when the system is empty, we have

$$\sum_{n=1}^{N(t)} R_n \leq \int_{\tau=0}^t L(\tau) d\tau \leq \sum_{i=1}^{A(t)} W_i \leq \sum_{n=1}^{N(t+1)} R_n \quad (53)$$

Assuming that the expected inter-renewal interval, $E[X]$, is finite, we can divide both sides of (53) by t and go to the limit $t \rightarrow \infty$. From the same argument as in theorem 6, we get

$$\lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{A(t)} W_i}{t} = \lim_{t \rightarrow \infty} \frac{\int_{\tau=0}^t L(\tau) d\tau}{t} = \frac{E[R_n]}{E[X]} \text{ with probability 1} \quad (54)$$

We denote $\lim_{t \rightarrow \infty} (1/t) \int_0^t L(\tau) d\tau$ as \bar{L} . The quantity on the left of (54) can now be broken up as waiting time per customer multiplied by number of customers per unit time, i.e.,

$$\lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{A(t)} W_i}{t} = \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{A(t)} W_i}{A(t)} \lim_{t \rightarrow \infty} \frac{A(t)}{t} \quad (55)$$

From (54), the limit on the left side of (55) exists (and equals \bar{L}) with probability 1. The second limit on the right also exists with probability 1 by the strong law for renewal processes, applied to $\{A(t)-1; t \geq 0\}$. This limit is called the arrival rate, λ , and is equal

to the reciprocal of the mean interarrival interval for $\{A(t)\}$. Since these two limits exist with probability 1, the first limit on the right, which is the time average waiting time per customer, denoted \bar{W} , also exists with probability 1. We have thus proved Little's theorem.

THEOREM 8 (LITTLE): For a FCFS G/G/1 queue in which the expected inter-renewal interval is finite, the time average number of customers in the system is equal, with probability 1, to the time average waiting time multiplied by the customer arrival rate, i.e., $\bar{L} = \lambda \bar{W}$.

The mathematics we have brought to bear here is quite formidable considering the simplicity of the idea. At any time t in an idle period, the sum of customer waiting periods up to time t is precisely equal to t times the time average number in the system up to t (see figure 3.13). Renewal theory informs us that the limits exist and that the edge effects (i.e., the customers in the system at an arbitrary time t) do not have any effect in the limit.

Recall that we assumed earlier that customers departed from the queue in the same order in which they arrived. From figure 3.14, however, it is clear that FCFS order is not required for the argument. Thus the theorem generalizes to systems with multiple servers and arbitrary service disciplines in which customers do not follow FCFS order. In fact, all that the argument requires is that the system has renewals (which are IID by definition of a renewal) and that the inter-renewal interval is finite with probability 1.

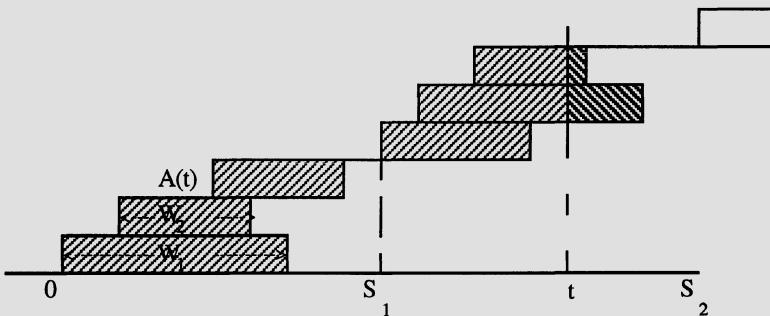


Figure 3.14. Arrivals and departures in non-FCFS systems. The aggregate reward (integral of number of customers in system) at time t is the first shaded area; the sum of waits of customers arriving by t includes the additional shaded wait to the right of t .

Finally, suppose the inter-renewal distribution is non-arithmetic; this occurs if the inter-arrival distribution is non-arithmetic. Then \bar{L} , the time average number of customers in the system, is also equal to⁶ $\lim_{n \rightarrow \infty} E[L(t)]$. It is also possible (see exercise 3.28) to replace the time average waiting time \bar{W} with $\lim_{n \rightarrow \infty} E[W_n]$. This gives us the following variant of Little's theorem:

$$\lim_{t \rightarrow \infty} E[L(t)] = \lambda W = \lim_{n \rightarrow \infty} \lambda E[W_n] \quad (56)$$

The same argument as in Little's theorem can be used to relate the average number of customers in the queue (not counting service) to the average wait in the queue (not counting service). Renewals still occur on arrivals to an empty system, but the integral of customers in queue over a busy period is still equal to the sum of the queue waiting times. Let $L_q(t)$ be the number in the queue at time t and let $\bar{L}_q = \lim_{t \rightarrow \infty} (1/t) \int_0^t L_q(\tau) d\tau$ be the time average queue wait. Letting \bar{W}_q be the time average waiting time in queue,

$$\bar{L}_q = \lambda \bar{W}_q \quad (57)$$

If the inter-renewal distribution is non-arithmetic, then

$$\lim_{t \rightarrow \infty} E[L_q(t)] = \lambda \bar{W}_q \quad (58)$$

The same argument can also be applied to the service facility. The time average of the number of customers in the server is just the fraction of time that the server is busy. Denoting this fraction by ρ and the expected service time by Z , we get

$$\rho = \lambda Z \quad (59)$$

EXPECTED QUEUEING TIME FOR AN M/G/1 QUEUE: For our last example of the use of renewal reward processes, we consider the expected queueing time in an M/G/1 queue. Renewals occur on arrivals to an empty system. At any given time t , let $L_q(t)$ be the number of customers in the queue (not counting the customer in service, if any) and let $R(t)$ be the residual life of the customer in service. If no customer is in service, $R(t) = 0$, and otherwise $R(t)$ is the remaining time until the current service will be completed. Let $U(t)$ be the waiting time in queue that a customer would experience if it arrived at time t . This is often called the unfinished work in the queueing literature and represents the delay until all the customers currently in the system complete service. Thus $U(t)$ is equal to $R(t)$, the residual life of the customer in service, plus the service times of each of the $L_q(t)$ customers currently waiting in the queue.

$$U(t) = \sum_{i=1}^{L_q(t)} Z_i + R(t) \quad (60)$$

where Z_i is the service time of the i^{th} customer in the queue at time t . Since the service times are independent of the arrival times and of the earlier service times, $L_q(t)$ is independent of $Z_1, Z_2, \dots, Z_{L_q(t)}$, so taking expected values,

$$E[U(t)] = E[L_q(t)] E[Z] + E[R(t)] \quad (61)$$

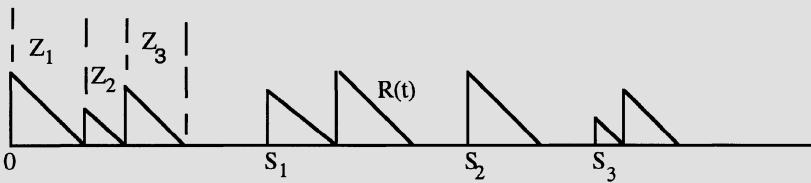


Figure 3.15. Sample value of the residual life function of customers in service.

Figure 3.15 illustrates how to find the time average of $R(t)$. Viewing $R(t)$ as a reward function, we can find the accumulated reward up to time t as the sum of triangular areas. First, consider $\int_{\tau=0}^{S_{N(t)}} R(\tau) d\tau$ from 0 to $S_{N(t)}$, i.e., the accumulated reward up to the last renewal epoch before t . $S_{N(t)}$ is not only a renewal epoch for the renewal process, but also an arrival epoch for the arrival process; in particular, it is the $A(S_{N(t)})^{\text{th}}$ arrival epoch, and the $A(S_{N(t)}) - 1$ earlier arrivals are the customers that have received service up to time $S_{N(t)}$. Thus,

$$\int_{\tau=0}^{S_{N(t)}} R(\tau) d\tau = \sum_{i=1}^{A(S_{N(t)})-1} \frac{Z_i^2}{2} \leq \sum_{i=1}^{A(t)} \frac{Z_i^2}{2}$$

We can similarly upper bound the term on the right above by $\int_{\tau=0}^{S_{N(t)+1}} R(\tau) d\tau$. We also know (from going through virtually the same argument many times) that $(1/t) \int_{\tau=0}^t R(\tau) d\tau$ will approach a limit with probability 1 as $t \rightarrow \infty$, and that the limit will be unchanged if t is replaced with $S_{N(t)}$ or $S_{N(t)+1}$. Thus, taking λ as the arrival rate,

$$\lim_{t \rightarrow \infty} \frac{\int_0^t R(\tau) d\tau}{t} = \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{A(t)} Z_i^2}{2A(t)} \cdot \frac{A(t)}{t} = \frac{\lambda E[Z^2]}{2} = \lim_{t \rightarrow \infty} E[R(t)] \quad (62)$$

In the last step, we have replaced the time average with the limiting ensemble average. Finally, we can use Little's theorem, in the limiting ensemble average form of (58), to assert that $\lim_{t \rightarrow \infty} E[L_q(t)] = \lambda \bar{W}_q$. Substituting this plus (62) into (61), we get

$$\lim_{t \rightarrow \infty} E[U(t)] = \lambda E[Z] \bar{W}_q + \frac{\lambda E[Z^2]}{2} \quad (63)$$

This shows that $\lim_{t \rightarrow \infty} E[U(t)]$ exists, so that $E[U(t)]$ is asymptotically independent of t . It is now important to distinguish between $E[U(t)]$ and \bar{W} . The first is the expected unfinished work at time t , which is the queue delay that a customer would incur by arriving at t ; the second is the time average expected queue delay. For Poisson arrivals, the probability of an arrival in $(t, t+\delta]$ is independent of $U(t)$.⁷ Thus, in the limit $t \rightarrow \infty$,

each arrival faces an expected delay $\lim_{t \rightarrow \infty} E[U(t)]$, so $\lim_{t \rightarrow \infty} E[U(t)]$ must be equal to \bar{W} . Substituting this into (63), we obtain the celebrated *Pollaczek–Khinchin* formula,

$$\bar{W} = \frac{\lambda E[Z^2]}{2(1 - \lambda E[Z])} \quad (64)$$

This queueing delay has some of the peculiar features of residual life, and in particular, if $E[Z^2] = \infty$, the limiting expected queueing delay is infinite even though the expected service time is less than the expected inter-arrival interval.

In trying to visualize why the queueing delay is so large when $E[Z^2]$ is large, note that while a particularly long service is taking place, numerous arrivals are coming in to the system, and all are being delayed by this single long service. In other words, the number of new customers held up by a long service is proportional to the length of the service, and the amount each of them are held up is also proportional to the length of the service. This visualization is rather crude, but does serve to explain the second moment of Z in (64). This phenomenon is sometimes called the “slow truck effect” because of the pile up of cars behind a slow truck on a single lane road.

For a G/G/1 queue, (63) is still valid, but arrival times are no longer independent of $U(t)$, so that typically $E[U(t)] \neq \bar{W}$. As an example, suppose that the service time is uniformly distributed between $1-\epsilon$ and $1+\epsilon$ and that the inter-arrival interval is uniformly distributed between $2-\epsilon$ and $2+\epsilon$. Assuming that $\epsilon < 1/2$, the system has no queueing and $\bar{W} = 0$. On the other hand, for small ϵ , $\lim_{t \rightarrow \infty} E[U(t)] \approx 1/4$ (i.e., the server is busy half the time with unfinished work ranging from 0 to 1).

3.7 DELAYED RENEWAL PROCESSES

We have seen a certain awkwardness in our discussion of Little’s theorem and the M/G/1 delay result because an arrival was assumed, but not counted, at time 0; this was necessary for the first interarrival interval to be statistically identical to the others. In this section, we correct that defect by allowing the epoch at which the first renewal occurs to be arbitrarily distributed. The resulting type of process is a generalization of the class of renewal processes known as *delayed renewal processes*. The word *delayed* does not necessarily imply that the first renewal epoch is in any sense larger than the other inter-renewal intervals. Rather, it means that the renewal process proper does not start until the epoch of that first renewal. What we shall discover is intuitive—both the time average behavior and the limiting ensemble behavior are not affected by the distribution of the first renewal epoch. It might be somewhat surprising, however, to find that this irrelevance of the distribution of the first renewal epoch holds even when the mean of the first renewal epoch is infinite.

To be more precise, we let $\{X_i; i \geq 1\}$ be a set of independent non-negative random variables. X_1 has some distribution function $G(x)$, whereas $\{X_i; i \geq 2\}$ are identically distributed with some distribution function $F(x)$. Typically, $G(x) \neq F(x)$, since if equality held, we would have an ordinary renewal process. Let $S_n = \sum_{i=1}^n X_i$ be the n^{th}

renewal epoch and let $N(t)$ be the number of renewal epochs up to and including time t (i.e., $N(t) \geq n$ if and only if $S_n \leq t$). $\{N(t); t \geq 0\}$ is then called a delayed renewal process. The following simple lemma follows from lemma 1.

LEMMA 2: Let $\{N(t); t \geq 0\}$ be a delayed renewal process. Then $\lim_{t \rightarrow \infty} N(t) = \infty$ with probability 1 and $\lim_{t \rightarrow \infty} E[N(t)] = \infty$.

Proof: Conditioning on $X_1 = x$, we can write $N(t) = 1 + N'(t-x)$ where $N'(\tau; \tau \geq 0)$ is the ordinary renewal process with inter-renewal intervals X_2, X_3, \dots . From lemma 1, $\lim_{t \rightarrow \infty} N'(t-x) = \infty$ with probability 1, and $\lim_{t \rightarrow \infty} E[N'(t-x)] = \infty$. Since this is true for every finite $x > 0$, and X_1 is finite with probability 1, the lemma is proven.

THEOREM 9 (STRONG LAW FOR DELAYED RENEWAL PROCESS): Let a delayed renewal process have mean inter-renewal interval $\bar{X}_2 = \int_{x=0}^{\infty} [1-F(x)]dx$. Then

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\bar{X}_2} \text{ with probability 1} \quad (65)$$

Proof: As in the proof of theorem 1, we have

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)} \quad (66)$$

$$\lim_{t \rightarrow \infty} \frac{S_{N(t)}}{N(t)} = \lim_{t \rightarrow \infty} \frac{X_1}{N(t)} + \lim_{t \rightarrow \infty} \frac{\sum_{n=2}^{N(t)} X_n}{N(t)-1} \frac{N(t)-1}{N(t)} \quad (67)$$

From lemma 2, $N(t)$ approaches ∞ as $t \rightarrow \infty$. Thus for any finite sample value of X_1 , the first limit on the right side of (67) approaches 0. Since X_1 is a random variable, it takes on a finite value with probability 1, so this first term is 0 with probability 1 (note that this does not require X_1 to have a finite mean). The second term in (67) approaches \bar{X}_2 with probability 1 by the strong law of large numbers. The same argument applies to the right side of (66), so that $\lim_{t \rightarrow \infty} \frac{t}{N(t)} = \bar{X}_2$ with probability 1. Eq. (65) then follows. A truncation argument, as in exercise 3.2, shows that the theorem is still valid if $\bar{X}_2 = \infty$.

Next we look at the elementary renewal theorem and Blackwell's theorem for delayed renewal processes. To do this, we view a delayed renewal process $\{N(t); t \geq 0\}$ as an ordinary renewal process that starts at a random non-negative epoch X_1 . Define $N_o(t-X_1)$ as the number of renewals that occur in the interval $(X_1, t]$. Conditional on any given sample value x for X_1 , $\{N_o(t-x); t-x \geq 0\}$ is an ordinary renewal process and thus, given $X_1=x$, $\lim_{t \rightarrow \infty} E[N_o(t-x)]/(t-x) = 1/\bar{X}_2$. Since $N(t) = 1 + N_o(t-X_1)$ for $t > X_1$, we see that, conditional on $X_1=x$,

$$\lim_{t \rightarrow \infty} \frac{E[N(t)|X_1=x]}{t} = \lim_{t \rightarrow \infty} \frac{E[N_0(t-x)]}{t-x} \frac{t-x}{t} = \frac{1}{\bar{X}_2} \quad (68)$$

Since this is true for every finite sample value x for X_1 , we can take the expected value over X_1 to get the following theorem:

THEOREM 10 (ELEMENTARY DELAYED RENEWAL THEOREM): For a delayed renewal process with $E[X_i] = \bar{X}_2$; $i \geq 2$,

$$\lim_{t \rightarrow \infty} \frac{E[N(t)]}{t} = \frac{1}{\bar{X}_2}. \quad (69)$$

The same approach gives us Blackwell's theorem. Specifically, if $\{X_i; i \geq 2\}$ are non-arithmetic, then, using Blackwell's theorem for ordinary renewal processes, for any $\delta > 0$

$$\lim_{t \rightarrow \infty} \frac{E[N_0(t+\delta)-N_0(t-x)]}{\delta} = \frac{1}{\bar{X}_2}. \quad (70)$$

Thus, conditional on any sample value $X_1=x$, $\lim_{t \rightarrow \infty} E[N(t+\delta)-N(t) | X_1=x] = \delta/\bar{X}_2$. Taking the expected value over X_1 gives us $\lim_{t \rightarrow \infty} E[N(t+\delta)-N(t)] = \delta/\bar{X}_2$. The case in which $\{X_i; i \geq 2\}$ are arithmetic with span d is somewhat more complicated. If X_1 is arithmetic with span d (or a multiple of d), then the first renewal epoch must be at some multiple of d and d/\bar{X}_2 gives the expected number of arrivals at time id in the limit as $i \rightarrow \infty$. If X_1 is non-arithmetic or arithmetic with some other span, then the effect of the first renewal epoch never dies out, since all subsequent renewals occur at multiples of d from this first epoch. This gives us the theorem:

THEOREM 11 (BLACKWELL FOR DELAYED RENEWAL): If $\{X_i; i \geq 2\}$ are non-arithmetic, then, for all $\delta > 0$,

$$\lim_{t \rightarrow \infty} \frac{E[N(t+\delta)-N(t)]}{\delta} = \frac{1}{\bar{X}_2} \quad (71)$$

If $\{X_i; i \geq 2\}$ are arithmetic with span d and X_1 is arithmetic with span md for some positive integer m , then

$$\lim_{t \rightarrow \infty} P(\text{renewal at } id) = \frac{d}{\bar{X}_2} \quad (72)$$

DELAYED RENEWAL REWARD PROCESSES: We have seen that the distribution of the first renewal epoch has no effect on the time or ensemble average behavior of a renewal process (other than the ensemble dependence on time for an arithmetic process). This carries over to reward functions with almost no change. In particular, the extended version of theorem 6 is as follows:

THEOREM 12: Let $\{N(t); t \geq 0\}$ be a delayed renewal process, let $Z(t) = t - S_{N(t)}$, let $X(t) = S_{N(t)+1} - S_{N(t)}$, and let $R(t) = R(Z(t), X(t))$ be a reward function. Assume that

$$E[R_n] = \int_{x=0}^{\infty} \int_{z=0}^x R(z, x) dz dF_{X_n}(x) < \infty \quad \text{for all } n.$$

Then, with probability one,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{\tau=0}^t R(\tau) d\tau = \frac{E[R_n]}{\bar{X}_2} \quad \text{for } n \geq 2 \quad (73)$$

We omit the proof of this since it is a minor variation of that of theorem 6. Finally, since Blackwell's theorem holds for delayed renewal processes, Eq. (51), giving the ensemble average reward for non-arithmetic processes, follows as before, yielding

$$\lim_{t \rightarrow \infty} E[R(t)] = \frac{E[R_n]}{\bar{X}_2} \quad (74)$$

TRANSIENT BEHAVIOR OF DELAYED RENEWAL PROCESSES: Let $m(t) = E[N(t)]$ for a delayed renewal process. As in (4), we have

$$m(t) = \sum_{n=1}^{\infty} P(N(t) \geq n) = \sum_{n=1}^{\infty} P(S_n \leq t) \quad (75)$$

For $n \geq 2$, $S_n = S_{n-1} + X_n$ where X_n and S_{n-1} are independent. From the convolution equation (1.12),

$$P(S_n \leq t) = \int_{x=0}^t P(S_{n-1} \leq t-x) dF(x) \quad \text{for } n \geq 2 \quad (76)$$

For $n=1$, $P(S_n \leq t) = G(t)$. Substituting this in (75) and interchanging the order of integration and summation,

$$\begin{aligned} m(t) &= G(t) + \int_{x=0}^t \sum_{n=2}^{\infty} P(S_{n-1} \leq t-x) dF(x) = G(t) + \int_{x=0}^t \sum_{n=1}^{\infty} P(S_n \leq t-x) dF(x) \\ m(t) &= G(t) + \int_{x=0}^t m(t-x) dF(x) \end{aligned} \quad (77)$$

[Fel66], section 11.1, theorem 1, shows that if a function $m(t)$ satisfies (77), then that function is equal to $E[N(t)]$.

There is another useful integral equation very similar to (77) that arises from breaking up S_n as the sum of X_1 and \widehat{S}_{n-1} where $\widehat{S}_{n-1} = X_2 + \dots + X_n$. Letting $\widehat{m}(t)$ be the expected number of renewals in time t for an ordinary renewal process with interarrival distribution F , a similar argument to that above, starting with $P(S_n \leq t) = \int_0^t P(\widehat{S}_{n-1} \leq t-x) dG(x)$ yields

$$m(t) = G(t) + \int_{x=0}^t \widehat{m}(t-x) dG(x) \quad (78)$$

This equation brings out the effect of the initial renewal interval clearly, and is useful in computation if one already knows $\widehat{m}(t)$.

Frequently, the most convenient way of dealing with $m(t)$ is through transforms. Following the same argument as that in (6), we get

$$L_m(r) = \frac{L_G(r)}{r} + L_m(r)L_F(r) = \frac{L_G(r)}{r[1 - L_F(r)]} \quad (79)$$

We can find $m(t)$ from (79) by finding the inverse Laplace transform, using the same procedure as in Example 3. There is a second order pole at $r=0$ again, and, evaluating the residue, it is $1/L'_F(0) = 1/\bar{X}_2$, which is not surprising in terms of Blackwell's theorem. We can also expand numerator and denominator of (79) in a power series, as in (7). The inverse transform, corresponding to (8), is

$$m(t) = \frac{t}{\bar{X}_2} + \frac{E[X_2^2]}{2\bar{X}_2} - \frac{\bar{X}_1}{\bar{X}_2} + R(t) \quad \text{for } t \geq 0 \quad (80)$$

where $\lim_{t \rightarrow \infty} R(t) = 0$.

THE EQUILIBRIUM PROCESS: In studying residual life for ordinary non-arithmetic renewal processes, we found that as $t \rightarrow \infty$, first, the expected number of renewals from t to $t+\delta$ approaches δ/\bar{X} , and, second, the interval from t to the next renewal approaches the distribution $F_Y(y) = (1/\bar{X}) \int_0^y [1-F(x)] dx$. This suggests that if we define a delayed renewal process in which X_i ; $i \geq 2$ each have the distribution function $F(x)$ and mean \bar{X}_2 , and if X_1 has the residual life distribution $F_Y(y) = (1/\bar{X}) \int_0^y [1-F(x)] dx$, then the process should start off in "steady state." To verify this, we show that $m(t) = t/\bar{X}_2$ is a solution to (77) if $G(t) = F_Y(y)$. Substituting $(t-x)/\bar{X}_2$ for $m(t-x)$, the right hand side is

$$\frac{1}{\bar{X}_2} \int_0^t [1-F(x)] dx + \frac{1}{\bar{X}_2} \int_0^t (t-x) dF(x) = \frac{1}{\bar{X}_2} \int_0^t [1-F(x)] dx + \frac{1}{\bar{X}_2} \int_0^t F(x) dx = \frac{t}{\bar{X}_2}$$

where we have used integration by parts for the first equality. This particular delayed renewal process is called the equilibrium process, since it starts off in steady state, and thus one need not worry about transients.

3.8 SUMMARY

Sections 1 to 3 give the central results about renewal processes that form the basis for many of the subsequent chapters. The chapter starts with the strong law for renewal processes, showing that the time average rate of renewals, $N(t)/t$, approaches $1/\bar{X}$ with probability 1 as $t \rightarrow \infty$. This, combined with the strong law of large numbers in Chapter 1, is the basis for most subsequent results about time averages. The next topic is the expected renewal rate, $E[N(t)]/t$. If the Laplace transform of the inter-renewal density is rational, $E[N(t)]/t$ can be easily calculated. In general, the Wald equality shows that $\lim_{t \rightarrow \infty} E[N(t)]/t = 1/\bar{X}$. Finally, Blackwell's theorem shows that the renewal epochs reach a steady state as $t \rightarrow \infty$. The form of this steady state depends on whether the inter-renewal distribution is arithmetic (see (16)) or non-arithmetic (see (15) and (17)).

Sections 4 and 5 add a reward function $R(t)$ to the underlying renewal process; $R(t)$ depends only on the inter-renewal interval containing t . The time average value of reward exists with probability 1 and is equal to the expected reward over a renewal interval divided by the expected length of an inter-renewal interval. Under some minor restrictions imposed by the key renewal theorem, we also found that, for non-arithmetic inter-renewal distributions, $\lim_{t \rightarrow \infty} E[R(t)]$ is the same as the time average value of reward. These general results were applied to residual life, age, and duration, and were also used to derive and understand Little's theorem and the Pollaczek–Khinchin expression for the expected delay in an M/G/1 queue. Finally, all the results above were shown to apply to delayed renewal processes.

For further reading on renewal processes, see [Fel66], [Ros83], or [Wol89]. Feller still appears to be the best source for deep understanding of renewal processes, but Ross and Wolff are somewhat more accessible.

EXERCISES

3.1) Let $\{X_i; i \geq 1\}$ be the inter-renewal intervals of a renewal process generalized to allow for inter-renewal intervals of size 0 and let $P(X_i = 0) = \alpha > 0$. Let $\{Y_i; i \geq 1\}$ be the sequence of *non-zero* interarrival intervals. That is, if $X_1 = x_1 > 0, X_2 = 0, X_3 = x_3 > 0, \dots$, then $Y_1 = x_1, Y_2 = x_3, \dots$.

- a) Find the distribution function of each Y_i in terms of that of each X_i .
- b) Find the PMF of the number of arrivals of the generalized renewal process at each epoch at which arrivals occur.
- c) Explain how to view the generalized renewal process as an ordinary renewal process with inter-renewal intervals $\{Y_i; i \geq 1\}$ and bulk arrivals at each renewal epoch.

3.2) Let $\{X_i; i \geq 1\}$ be the inter-renewal intervals of a renewal process and assume that $E[X_i] = \infty$. Let $b > 0$ be an arbitrary number and \tilde{X}_i be a truncated random variable defined by $\tilde{X}_i = X_i$ if $X_i \leq b$ and $\tilde{X}_i = b$ otherwise.

a) Show that for any constant $M > 0$, there is a b sufficiently large so that $E[\tilde{X}_i] \geq M$.

b) Let $\{\tilde{N}(t); t \geq 0\}$ be the renewal process with inter-renewal intervals $\{\tilde{X}_i; i \geq 1\}$ and show that for all $t > 0$, $\tilde{N}(t) \geq N(t)$.

c) Show that for all sample functions $n(t)$, except a set of probability 0, $n(t)/t < 2/M$ for all sufficiently large t . Note: Since M is arbitrary, this means that $\lim N(t)/t = 0$ with probability 1.

3.3 a) Let N be a stopping rule and I_n be the indicator random variable of the event $\{N \geq n\}$. Show that $N = \sum_{n \geq 1} I_n$.

b) Show that $I_1 \geq I_2 \geq I_3 \geq \dots$.

3.4) Is it true for a renewal process that:

a) $N(t) < n$ if and only if $S_n > t$?

b) $N(t) \leq n$ if and only if $S_n \geq t$?

c) $N(t) > n$ if and only if $S_n < t$?

3.5) Let $\{N(t); t \geq 0\}$ be a renewal process and let $m(t) = E[N(t)]$ be the expected number of arrivals up to and including time t . Let $\{X_i; i \geq 1\}$ be the inter-renewal times and assume that $F_X(0) = 0$.

a) Show that $E[N(t) | X_1 = x] = E[N(t-x)] + 1$ for $x < t$.

b) Use part (a) to show that $m(t) = F_X(t) + \int_0^t m(t-x)dF_X(x)$ for $t > 0$. This equation is known as the renewal equation and an alternative derivation is given in Eq. (5).

c) Suppose that X is an exponential random variable of parameter λ . Evaluate $L_m(s)$ from (6); verify that the inverse Laplace transform is λt ; $t \geq 0$.

3.6 a) Let the inter-renewal interval of a renewal process have a second order Erlang density, $f_X(x) = \lambda^2 x \exp(-\lambda x)$. Evaluate the Laplace transform of $m(t) = E[N(t)]$.

b) Use this to evaluate $m(t)$ for $t \geq 0$. Verify that your answer agrees with (8).

c) Evaluate the slope of $m(t)$ at $t=0$ and explain why that slope is not surprising.

d) View the renewals here as being the even numbered arrivals in a Poisson process of rate λ . Sketch $m(t)$ for the process here and show one half the expected number of arrivals for the Poisson process on the same sketch. Explain the difference between the two.

3.7 a) Let $N(t)$ be the arrivals in the interval $(0, t]$ for a Poisson process of rate λ . Show that the probability that $N(t)$ is even is $[1 + \exp(-2\lambda t)]/2$. Hint: Look at the power series expansion of $\exp(-\lambda t)$ and that of $\exp(\lambda t)$, and look at the sum of the two. Compare this with $\sum_{n \text{ even}} P(N(t) = n)$.

b) Let $\tilde{N}(t)$ be the number of even numbered arrivals in $(0, t]$. Show that $\tilde{N}(t) = N(t)/2 - I_{\text{odd}}(t)/2$ where $I_{\text{odd}}(t)$ is a random variable that is 1 if $N(t)$ is odd and 0 otherwise.

c) Use parts (a) and (b) to find $E[\tilde{N}(t)]$. Note that this is $m(t)$ for a renewal process with 2nd order Erlang inter-renewal intervals.

3.8) Use Wald's equality to compute the expected number of trials of a Bernoulli process up to and including the k^{th} success.

3.9) A gambler starts to play a dollar slot machine with an initial finite capital of $d > 0$ dollars. At each play, either his dollar is lost or is returned with some additional number of dollars. Let X_i be his change of capital on the i^{th} play. Assume that $\{X_i; i = 1, 2, \dots\}$ is a set of IID random variables taking on integer values $\{-1, 0, 1, \dots\}$. Assume that $E[X_i] < 0$. The gambler plays until losing all his money (i.e., the initial d dollars plus subsequent winnings).

a) Let N be the number of plays until the gambler loses all his money. Is the weak law of large numbers sufficient to argue that $\lim_{n \rightarrow \infty} P(N > n) = 0$ (i.e., that N is a random variable) or is the strong law necessary?

b) Find $E[N]$.

3.10) Let $\{X_i; i \geq 1\}$ be IID binary random variables with $P_X(0) = P_X(1) = 1/2$. Let N be a non negative integer valued random variable defined on the above sample space of binary sequences. Find the simplest example you can in which N is *not* a stopping rule for $\{X_i; i \geq 1\}$ and where $E[X]E[N] \neq E[S_N]$ where $S_N = \sum_{i=1}^N X_i$.

3.11) Let $N = \min\{n \mid S_n \leq B \text{ or } S_n \geq A\}$, where A is a positive integer, B is a negative integer, and $S_n = X_1 + X_2 + \dots + X_n$. Assume that $\{X_i; i \geq 1\}$ is a set of *zero mean* IID rv's that can take on only the set of values $\{-1, 0, \text{ or } +1\}$, each with positive probability.

a) Is N a stopping rule? Why or why not? Hint: Part of this is to argue that N is finite with probability 1; you do not need to construct a proof of this, but try to argue why it must be true.

b) What are the possible values of S_N ?

c) Find an expression for $E[S_N]$ in terms of p , A , and B , where $p = P\{S_N \geq A\}$.

d) Find an expression for $E[S_N]$ from Wald's equality. Use this to solve for p .

3.12) Let $\{N(t); t \geq 0\}$ be a renewal process generalized to allow for inter-renewal intervals $\{X_i\}$ of duration 0. Let each X_i have the PMF $P(X_i = 0) = 1 - \varepsilon$; $P(X_i = 1/\varepsilon) = \varepsilon$.

a) Sketch a typical sample function of $\{N(t); t \geq 0\}$. Note that $N(0)$ can be non-zero (i.e., $N(0)$ is the number of zero interarrival times that occur before the first non-zero interarrival time).

b) Evaluate $E[N(t)]$ as a function of t .

c) Sketch $E[N(t)]/t$ as a function of t .

d) Evaluate $E[S_{N(0)+1}]$ as a function of t (do this directly, and then use Wald's equality as a check on your work).

e) Sketch the lower bound $E[N(t)]/t \geq 1/E[X] - 1/t$ on the same graph with part (c).

f) Sketch $E[S_{N(0)+1} - t]$ as a function of t and find the time average of this quantity.

g) Evaluate $E[S_{N(0)}]$ as a function of t ; verify that $E[S_{N(0)}] \neq E[X]E[N(t)]$.

3.13) Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two-day's travel; door 2 returns him to his room after four-day's travel;

and door 3 returns him to his room after eight-day's travel. Suppose each door is equally likely to be chosen whenever he is in the room, and let T denote the time it takes the miner to become free.

- a) Define a sequence of independent and identically distributed random variables X_1, X_2, \dots and a stopping time N such that

$$T = \sum_{i=1}^N X_i$$

Note: You may imagine that the miner continues to randomly choose doors even after he reaches safety.

- b) Use Wald's equation to find $E[T]$.

- c) Compute $E\left[\sum_{i=1}^n X_i \mid N=n\right]$ and note that it is not equal to $E\left[\sum_{i=1}^N X_i\right]$.

- d) Use part (c) for a second derivation of $E[T]$.

3.14) Let $Y(t) = S_{N(t)+1} - t$ be the residual life at time t of a renewal process. First consider a renewal process in which the interarrival time has density $f_X(x) = e^{-x}$; $x \geq 0$, and next consider a renewal process with density

$$f_X(x) = \frac{3}{(x+1)^4}; \quad x \geq 0$$

For each of the above densities, use renewal reward theory to find:

- i) the time average of $Y(t)$
- ii) the second moment in time of $Y(t)$ (i.e., $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T Y^2(t) dt$)

For the exponential density, verify your answers by finding $E[Y(t)]$ and $E[Y^2(t)]$ directly.

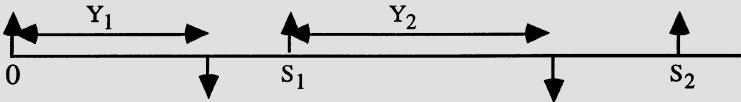
3.15) Consider a variation of an M/G/1 queueing system in which there is no facility to save waiting customers. Assume customers arrive according to a Poisson process of rate λ . If the server is busy, the customer departs and is lost forever; if the server is not busy, the customer enters service with a service time distribution function denoted by $F_Y(y)$.

Successive service times (for those customers that are served) are IID and independent of arrival times. Assume that the first customer arrives and enters service at time $t=0$.

- a) Show that the sequence of times S_1, S_2, \dots at which new customers enter service are the renewal times of a renewal process. Show that each inter-renewal interval $X_i = S_i - S_{i-1}$ (where $S_0 = 0$) is the sum of two independent random variables, $Y_i + U_i$ where Y_i is the i^{th} service time; find the probability density of U_i .

b) Assume that a reward (actually a cost in this case) of one unit is incurred for each customer turned away. Sketch the expected reward function as a function of time for the given sample below of inter-renewal intervals and service intervals; the expectation is to be taken over those (unshown) arrivals of customers that must be turned away.

c) Let $\int_0^t R(\tau)d\tau$ denote the accumulated reward (i.e., cost) from 0 to t and find the limit as $t \rightarrow \infty$ of $\int_0^t R(\tau)d\tau$. Explain (without any attempt to be rigorous or formal) why this limit exists with probability 1.



d) In the limit of large t , find the expected reward from time t until the next renewal. Hint: Sketch this expected reward as a function of t for a given sample of inter-renewal intervals and service intervals; then find the time average.

e) Now assume that the arrivals are deterministic, with the first arrival at time 0 and the n^{th} arrival at time $n-1$. Does the sequence of times S_1, S_2, \dots at which new customers start service still constitute the renewal times of a renewal process? Draw a sketch of arrivals, departures, and service time intervals. Again find $\lim_{t \rightarrow \infty} (\int_0^t R(\tau)d\tau)/t$

3.16) Let $Z(t) = t - S_{N(t)}$ be the age of a renewal process and $Y(t) = S_{N(t+1)} - t$ be the residual life. Let $F_X(x)$ be the distribution function of the inter-renewal interval and find the following as a function of $F_X(x)$:

- $P(Y(t) > x | Z(t) = s)$
- $P(Y(t) > x | Z(t + x/2) = s)$
- $P(Y(t) > x | Z(t+x) > s)$ for a Poisson process.

3.17) Let $Z(t)$, $Y(t)$, $X(t)$ denote the age, residual life, and duration at time t for a renewal process $\{N(t); t \geq 0\}$ in which the interarrival time has a density given by $f(x)$. Find the following probability densities; assume steady state.

- $f_{Y(t)}(y | Z(t+s/2)=s)$ for given $s>0$.
- $f_{Y(t),Z(t)}(y,z)$.
- $f_{Y(t)}(y | X(t)=x)$.
- $f_{Z(t)}(z | Y(t-s/2)=s)$ for given $s>0$.
- $f_{Y(t)}(y | Z(t+s/2)\geq s)$ for given $s>0$.

3.18 a) Find $\lim_{t \rightarrow \infty} \{E[N(t)] - t/E[X]\}$ for a renewal process $\{N(t); t \geq 0\}$ with inter-arrival times $\{X_i; i \geq 1\}$. Hint: Use Wald's equation.

b) Evaluate your result for the case in which X is an exponential random variable (you already know what the result should be in this case).

c) Evaluate your result for a case in which $E[X] < \infty$, $E[X^2] = \infty$. Explain (very briefly) why this does not contradict the elementary renewal theorem.

3.19) Customers arrive at a bus stop according to a Poisson process of rate λ . Buses arrive according to a renewal process with the inter-renewal interval distribution $F_X(x)$. At the epoch of a bus arrival, all waiting passengers enter the bus and the bus leaves immediately. Let $R(t)$ be the number of customers waiting at time t .

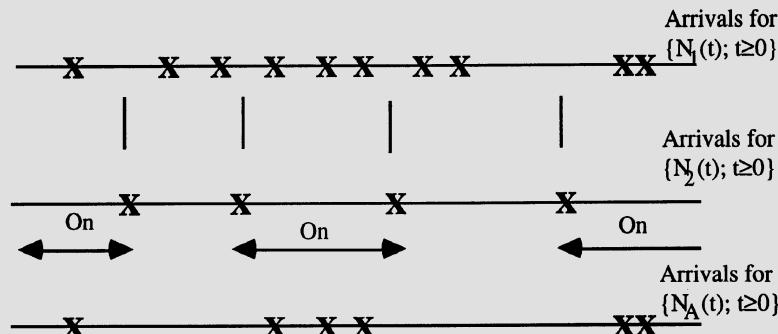
- a) Draw a sketch of a sample function $R(t)$.
- b) Given that the first bus arrives at time x (i.e. $X_1 = x$), find the expected number of customers picked up; then find $E\left[\int_0^x R(t)dt\right]$, again given the first bus arrival at x .
- c) Find $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(t)dt$ (with probability 1). Assuming that F_X is a non-arithmetic distribution, find $\lim_{t \rightarrow \infty} E[R(t)]$. Interpret what these quantities mean.
- d) Find the time average expected wait per customer.
- e) Find the fraction of time that there are no customers at the bus stop. (Hint: This part is independent of (a), (b), and (c); check your answer for $E[X] \ll 1/\lambda$).

3.20) Let $\{N_1(t); t \geq 0\}$ be a Poisson counting process of rate λ . Assume that the arrivals from this process are switched on and off by arrivals from a non-arithmetic renewal process $\{N_2(t); t \geq 0\}$. The two processes are independent.

Let $\{N_A(t); t \geq 0\}$ be the switched process; that is $N_A(t)$ includes arrivals from $\{N_1(t); t \geq 0\}$ while $N_2(t)$ is even and excludes arrivals from $\{N_1(t); t \geq 0\}$ while $N_2(t)$ is odd.

a) Is $N_A(t)$ a renewal process? Explain your answer and if you are not sure, look at several examples for $N_2(t)$.

b) Find $\lim_{t \rightarrow \infty} \frac{1}{t} N_A(t)$ and explain why the limit exists with probability 1. Hint: Use symmetry—that is, look at $N_1(t) - N_A(t)$. To show why the limit exists, use the renewal reward theorem. What is the appropriate renewal process to use here?



c) Now suppose that $\{N_1(t); t \geq 0\}$ is a non-arithmetic renewal process but not a Poisson process and let the expected inter-renewal interval be $1/\lambda$. For any given δ , find $\lim_{t \rightarrow \infty} E[N_A(t+\delta) - N_A(t)]$ and explain your reasoning. Why does your argument in (b) fail to demonstrate a time average?

3.21) An M/G/1 queue has arrivals at rate λ and a service time distribution given by $F_Y(y)$. Assume that $\lambda < \frac{1}{E[Y]}$. Epochs at which the system becomes empty define a renewal process. Let $F_Z(z)$ be the distribution of the inter-renewal intervals and let $E[Z]$ be the mean inter-renewal interval.

- a) Find the fraction of time that the system is empty as a function of λ and $E[Z]$. State carefully what you mean by such a fraction.
- b) Apply Little's theorem, not to the system as a whole, but to the number of customers in the server (i.e., 0 or 1). Use this to find the fraction of time that the server is busy.
- c) Combine your results in (a) and (b) to find $E[Z]$ in terms of λ and $E[Y]$; give the fraction of time that the system is idle in terms of λ and $E[Y]$.
- d) Find the expected duration of a busy period.

3.22) Consider a sequence X_1, X_2, \dots of IID binary random variables with $P(X_m = 1) = p$ and $P(X_m = 0) = 1-p$. A *renewal* is said to occur at time m if $X_{m-1} = 0$ and $X_m = 1$.

- a) Show that $\{N(m); m \geq 0\}$ is a renewal process where $N(m)$ is the number of renewals up to and including time m and $N(0)$ and $N(1)$ are taken to be 0.
- b) What is the probability that a renewal occurs at time m , $m \geq 2$?
- c) Find the expected inter-renewal interval; use Blackwell's theorem here.
- d) Now change the definition of renewal; a renewal now occurs at time m if $X_{m-1} = 1$ and $X_m = 1$. show that $\{N^*(m); m \geq 0\}$ is a delayed renewal process where N^*_m is the number of renewals up to and including m for this new definition of renewal ($N^*_0 = N^*_1 = 0$).
- e) Find the expected inter-renewal interval for the renewals of part (d).
- f) Given that a renewal (according to the definition in (d)) occurs at time m , find the expected time until the next renewal, conditional, first, on $X_{m+1} = 1$ and, next, on $X_{m+1} = 0$. Hint: use the result in (e) plus the result for $X_{m+1} = 1$ for the conditioning on $X_{m+1} = 0$.
- g) Use your result in (f) to find the expected time from time 0 to the first renewal according to the renewal definition in (d).
- h) Which pattern requires a larger expected time to occur: 0011 or 0101
- i) What is the expected time until the first occurrence of 011111?

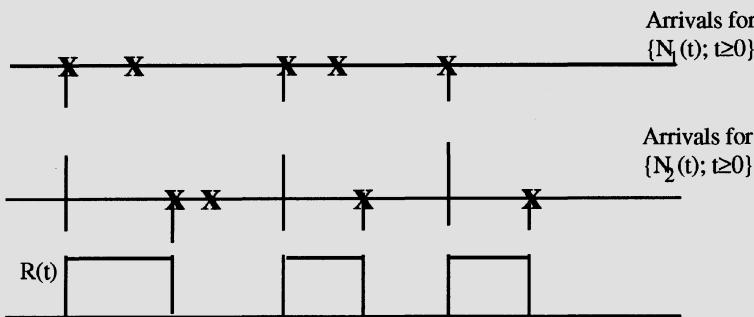
3.23) A large system is controlled by n identical computers. Each computer independently alternates between an operational state and a repair state. The duration of the operational state, from completion of one repair until the next need for repair, is a random variable X with finite expected duration $E[X]$. The time required to repair a computer is an exponentially distributed random variable with density $\lambda e^{-\lambda t}$. All oper-

ating durations and repair durations are independent. Assume that all computers are in the repair state at time 0.

- For a single computer, say the i^{th} , do the epochs at which the computer enters the repair state form a renewal process? If so, find the expected inter-renewal interval.
- Do the epochs at which it enters the operational state form a renewal process?
- Find the fraction of time over which the i^{th} computer is operational and explain what you mean by fraction of time.
- Let $Q_i(t)$ be the probability that the i^{th} computer is operational at time t and find $\lim_{t \rightarrow \infty} Q_i(t)$.
- The system is in failure mode at a given time if all computers are in the repair state at that time. Do the epochs at which system failure modes begin form a renewal process?
- Let $P(t)$ be the probability that the system is in failure mode at time t . Find $\lim_{t \rightarrow \infty} P(t)$. Hint: Look at part (d)
- For δ small, find the probability that the system enters failure mode in the interval $(t, t+\delta]$ in the limit as $t \rightarrow \infty$.
- Find the expected time between successive entries into failure mode.
- Next assume that the repair time of each computer has an arbitrary density rather than exponential, but has a mean repair time of $1/\lambda$. Do the epochs at which system failure modes begin now form a renewal process?
- Repeat part (f) for the assumption in (i).

3.24) Let $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ be independent renewal processes. Assume that each has the same distribution function $F(x)$ for interarrival intervals and assume that a density $f(x)$ exists for the interarrival intervals.

- Is the counting process $\{N_1(t) + N_2(t); t \geq 0\}$ a renewal process? Explain.
- Let $Y(t)$ be the interval from t until the first arrival (from either process) after t . Find an expression for the distribution function of $Y(t)$ in the limit $t \rightarrow \infty$ (you may assume that time averages and ensemble averages are the same).
- Assume that a reward of rate 1 unit per second starts to be earned whenever an arrival from process 1 occurs and ceases to be earned whenever an arrival from process 2 occurs.



Assume that $\lim_{t \rightarrow \infty} (1/t) \int_0^t R(\tau) d\tau$ exists with probability 1 and find its *numerical* value.

d) Let $Z(t)$ be the interval from t until the first time after t that $R(t)$ (as in part (c)) changes value. Find an expression for $E[Z(t)]$ in the limit $t \rightarrow \infty$. Hint: Make sure you understand why $Z(t)$ is not the same as $Y(t)$ in part (b). You might find it easiest to first find the expectation of $Z(t)$ conditional on both the duration of the $\{N_1(\tau); \tau \geq 0\}$ interarrival interval containing t and the duration of the $\{N_2(\tau); \tau \geq 0\}$ interarrival interval containing t ; draw pictures!

3.25) This problem provides another way of treating ensemble averages for renewal reward problems. Assume for notational simplicity that X is a continuous valued random variable.

a) Show that $P(\text{one or more arrivals in } (\tau, \tau+\delta)) = m(\tau+\delta) - m(\tau) + o(\delta)$ where $o(\delta) \geq 0$ and $\lim_{\delta \rightarrow 0} o(\delta)/\delta = 0$.

b) Show that $P\{Z(t) \in [z, z+\delta], X(t) \in (x, x+\delta)\} = [m(t-z) - m(t-z-\delta) + o(\delta)][F_X(x+\delta) - F_X(x)]$ for $x \geq z+\delta$.

c) Assuming that $m'(\tau) = dm(\tau)/d\tau$ exists for all τ , show that the joint density of $Z(t), X(t)$ is $f_{Z(t), X(t)}(z, x) = m'(t-z)f_X(x)$ for $x > z$.

d) Show that $E[R(t)] = \int_{z=0}^t \int_{x=z}^{\infty} R(z, x) f_X(x) dx m'(t-z) dz$. Note: In the absence of densities, this can be written $\int_{z=0}^t \int_{x=z}^{\infty} R(z, x) dF_X(x) dm(t-z)$. This is the same as (47), and if $r(z) = \int_{x \geq z} R(z, x) dF(x)$ is directly Riemann integrable, then, as shown in (48) to (50), this leads to (51).

3.26) This problem is designed to give you an alternate way of looking at ensemble averages for renewal reward problems. First we find an exact expression for $P(S_{N(t)} > s)$. We find this for arbitrary s and t , $0 < s < t$.

a) By breaking the event $\{S_{N(t)} > s\}$ into subevents $\{S_{N(t)} > s, N(t)=n\}$, explain each of the following steps:

$$\begin{aligned} P(S_{N(t)} > s) &= \sum_{n=1}^{\infty} P(t \geq S_n > s, S_{n+1} > t) \\ &= \sum_{n=1}^{\infty} \int_{y=s}^t P(S_{n+1} > t | S_n = y) dF_{S_n}(y) \\ &= \int_{y=s}^t [1 - F_X(t-y)] d \sum_{n=1}^{\infty} F_{S_n}(y) \\ &= \int_{y=s}^t [1 - F_X(t-y)] dm(y) \text{ where } m(y) = E[N(y)] \end{aligned}$$

b) Show that for $0 < s < t < u$,

$$P(S_{N(t)} > s, S_{N(t)+1} > u) = \int_{y=s}^t [1 - F_X(u-y)] dm(y)$$

c) Draw a two dimensional sketch, with age and duration as the axes, and show the region of (age, duration) values corresponding to the event $\{S_{N(t)} > s, S_{N(t)+1} > u\}$.

d) Assume that for large t , $dm(y)$ can be approximated (according to Blackwell) as $(1/\bar{X})dy$, where $\bar{X} = E[X]$. Assuming that X also has a density, use the result in parts (b) and (c) to find the joint density of age and duration.

3.27) In this problem, we show how to calculate the residual life distribution $Y(t)$ as a transient in t . Let $\mu(t) = dm(t)/dt$ where $m(t) = E[N(t)]$, and let the interarrival distribution have the density $f_X(x)$. Let $Y(t)$ have the density $f_{Y(t)}(y)$.

a) Show that these densities are related by the integral equation

$$\mu(t+y) = f_{Y(t)}(y) + \int_{u=0}^y \mu(t+u)f_X(y-u)du$$

b) Let $L_{\mu,t}(s) = \int_{y \geq 0} \mu(t+y)e^{-sy} dy$ and let $L_{Y(t)}(s)$ and $L_X(s)$ be the Laplace transforms of $f_{Y(t)}(y)$ and $f_X(x)$ respectively. Find $L_{Y(t)}(s)$ as a function of $L_{\mu,t}$ and L_X .

c) Consider the inter-renewal density $f_X(x) = (1/2)e^{-x} + e^{-2x}$ for $x \geq 0$ (as in example 3). Find $L_{\mu,t}(s)$ and $L_{Y(t)}(s)$ for this example.

d) Find $f_{Y(t)}(y)$. Show that your answer reduces to that of (34) in the limit as $t \rightarrow \infty$.

e) Explain how to go about finding $f_{Y(t)}(y)$ in general, assuming that f_X has a rational Laplace transform.

3.28) Show that for a G/G/1 queue, the time average wait in the system is the same as $\lim_{n \rightarrow \infty} E[W_n]$. Hint: Consider an integer renewal process $\{M(n); n \geq 0\}$ where $M(n)$ is the number of renewals in the G/G/1 process of section 3.6 that have occurred by the n^{th} arrival. Show that this renewal process has a span of 1. Then consider $\{W(n); n \geq 1\}$ as a reward within this renewal process.

3.29) If one extends the definition of renewal processes to include inter-renewal intervals of duration 0, with $P(X=0) = \alpha$, show that the expected number of simultaneous renewals at a renewal epoch is $1/(1-\alpha)$, and that, for a non-arithmetic process, the probability of 1 or more renewals in the interval $(t, t+\delta]$ tends to $(1-\alpha)\delta/E[X] + o(\delta)$ as $t \rightarrow \infty$.

NOTES

1. Renewal processes are often defined in a slightly more general way, allowing the inter-arrival intervals X_i to include the possibility $1 > P(X_i = 0) > 0$. All of the theorems in this chapter are valid under this more general assumption, as can be verified by checking the proofs. Allowing $P(X_i = 0) > 0$ allows multiple arrivals at the same instant, which makes it necessary to allow $N(0)$ to take on positive values, and appears to inhibit intuition about renewals. Exercise 3.1 shows how to view these more general renewal processes in terms of the definition here.

2. See page 62 of [Ros83].

3. $\int_0^t Y(t)dt$ is a random variable just like any other function of a set of random variables.

It has a sample value for each sample function of $\{N(t); t \geq 0\}$, and its distribution function could

be calculated in a straightforward but tedious way. For arbitrary stochastic processes, integration and differentiation can require great mathematical sophistication, but none of those subtleties occur here.

4. This is a slight abuse of notation. More precisely, $P\{\text{renewal} \in (t-z, t-z+\delta]\} \leq m(t-z+\delta) - m(t-z)$ and $P\{\text{renewal} \in (t-z, t-z+\delta]\} \geq m(t-z+\delta) - m(t-z) - o(\delta)$ where $o(\delta)$ is a function of δ such that $\lim_{\delta \rightarrow 0} o(\delta)/\delta = 0$.

5. FCFS is also frequently referred to as First In First Out (FIFO) service.

6. To show this mathematically requires a little care. One approach is to split the reward function into many individual terms. Let $R_n(t) = 1$ if the n^{th} arrival since the beginning of the busy period has arrived by time t , and let $R_n(t) = 0$ otherwise. Let $S_n(t) = 1$ if the n^{th} departure since the beginning of the busy period occurs by time t . It is easy to show the direct Riemann integrability condition holds for each of these reward functions, and $R(t)$ is $\sum_n R_n(t) - \sum_n S_n(t)$.

7. This is often called the *PASTA* property, standing for Poisson arrivals see time averages. This holds with great generality, requiring only that time averages exist and that the state of the system at a given time t is independent of future arrivals.

Chapter 4

Finite State Markov Chains

4.1 INTRODUCTION

The counting processes $\{N(t), t \geq 0\}$ of Chapters 2 and 3 have the property that $N(t)$ changes at discrete instants of time, but is *defined* for all real $t \geq 0$. Such stochastic processes are generally called continuous time processes. The Markov chains to be discussed in this and the next chapter are stochastic processes *defined* only at integer values of time, $n = 0, 1, \dots$. At each integer time $n \geq 0$, there is a random variable X_n called the *state* at time n , and the process is then the family of random variables $\{X_n, n \geq 0\}$. These processes are often called discrete time processes, but we prefer the more specific term *integer time processes*. An integer time process $\{X_n; n \geq 0\}$ can also be viewed as a continuous time process $\{X(t); t \geq 0\}$ by taking $X(t) = X_n$ for $n \leq t < n+1$, but since changes only occur at integer times, it is usually simpler to view the process only at integer times.

In general, for Markov chains, the set of possible values for each random variable X_n is a countable set usually taken to be $\{0, 1, 2, \dots\}$. In this chapter (except for theorems 1 and 2), we restrict attention to a finite set of possible values, say $\{1, \dots, J\}$. Thus we are looking at processes whose sample functions are sequences of integers, each between 1 and J . There is no special significance to using integer labels for states, and no compelling reason to include 0 as a state for the countably infinite case and not to include 0 for the finite case. For the countably infinite case, the most common applications come from queueing theory, and the state often represents the number of waiting customers, which can be zero. For the finite case, we often use vectors and matrices, and it is more conventional to use positive integer labels. In some examples, it will be more convenient to use more illustrative labels for states.

DEFINITION: A Markov chain is an integer time process, $\{X_n, n \geq 0\}$ for which each random variable X_n , $n \geq 1$, depends on the past random variables X_{n-1}, X_{n-2}, \dots only through the most recent variable X_{n-1} . That is, for all i, j, k, \dots, m ,

$$P(X_n=j | X_{n-1}=i, X_{n-2}=k, \dots, X_0=m) = P(X_n=j | X_{n-1}=i) = P_{ij} \quad (1)$$

The random variable X_n is called the *state* of the chain at time n . The possible values for the state at time n , namely $\{1, \dots, J\}$ or $\{0, 1, \dots\}$ are also generally called states, usually without too much confusion. Thus what (1) says is that P_{ij} is the probability of going to state j given that the previous state is i ; the new state, given the previous state, is independent of all earlier states. Thus, the use of the word *state* here conforms to the usual idea of the state of a system—that is, the state at a given time summarizes everything about the past that is relevant to the future. Note that the transition probabilities, $\{P_{ij}\}$, do not depend on n . Occasionally, a more general model is required where the transition probabilities do depend on n . In such situations, (1) is replaced by

$$P(X_n=j | X_{n-1}=i, X_{n-2}=k, \dots, X_0=m) = P(X_n=j | X_{n-1}=i) = P_{ij}(n) \quad (2)$$

A process that obeys (2), with a dependence on n , is called a non-homogeneous Markov chain. A process that obeys (1) is called a homogeneous Markov chain. We will discuss only the homogeneous case (since not much of general interest can be said about the non-homogeneous case) and thus omit the word homogeneous as a qualifier. An initial probability distribution for X_0 , combined with the transition probabilities $\{P_{ij}\}$ (or $\{P_{ij}(n)\}$ for the non-homogeneous case), define the probabilities for all events. The initial distribution is often not considered to be part of the definition of a Markov chain, and most of the results that we develop are conditioned on an initial state. One should keep in mind, however, that an initial distribution is essential to have a complete probabilistic description of the process.

Markov chains are often described by a directed graph (see figure 4.1). In the graphical representation, there is one node for each state and a directed arc for each non-zero transition probability. If $P_{ij} = 0$, then the arc from node i to node j is omitted; thus the difference between zero and non-zero transition probabilities stands out clearly in the graph. Several of the most important characteristics of a Markov chain depend only on which transition probabilities are zero, so the graphical representation is well suited for understanding these characteristics. Finite state Markov chains are also often described by a matrix $[P]$ (see figure 4.1). If the chain has J states, then $[P]$ is a J by J matrix with elements P_{ij} . The matrix representation is ideally suited for studying algebraic and computational issues.

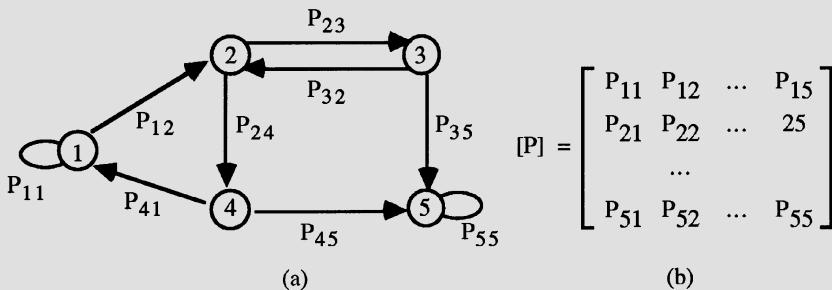


Figure 4.1. Graphical and Matrix Representation of a 5 state Markov Chain; the edge from i to j is included in the graph if and only if $P_{ij} > 0$.

4.2 CLASSIFICATION OF STATES

We say that state j is *accessible* from i (abbreviated as $i \rightarrow j$) if there is a directed path in the graph from i to j . For example, in figure 4.1a, there is a directed path from node 1 to node 3 (passing through node 2), so state 3 is accessible from 1. There is no directed path from node 5 to 3, so state 3 is not accessible from 5. What accessibility means probabilistically is that $i \rightarrow j$ if and only if (iff) $P(X_n=j | X_0=i) > 0$ for some $n \geq 1$; from now on, we denote $P(X_n=j | X_0=i)$ by P_{ij}^n . Thus, for $n \geq 1$, $P_{ij}^n > 0$ iff the graph has a directed walk¹ of n arcs (perhaps containing cycles) from i to j . For $n=0$, $P_{ij}^0 = 1$ for $j=i$ and $P_{ij}^0 = 0$ otherwise. For the example in figure 4.1a, $P_{13}^2 = P_{12} P_{23} > 0$. On the other hand, $P_{53}^n = 0$ for all $n \geq 1$. States i and j *communicate* (abbreviated $i \leftrightarrow j$) if i is accessible from j and j is accessible from i . By convention, we define i to communicate with itself for all i ($i \leftrightarrow i$).

An important fact about communicating states is that if $i \leftrightarrow j$ and $m \leftrightarrow j$ then $i \leftrightarrow m$. To see this, note that $i \leftrightarrow j$ and $m \leftrightarrow j$ implies that $i \rightarrow j$ and $j \rightarrow m$; extending a directed walk from i to j by a directed walk from j to m , we obtain a directed walk from i to m , so $i \rightarrow m$. Similarly, $m \rightarrow i$, so $i \leftrightarrow m$. A *class* of states is a non-empty set of states T such that all pairs of states in T communicate with each other and no state in T communicates with any state not in T . For the example of figure 4.1a, $\{1, 2, 3, 4\}$ is one class of states and $\{5\}$ is another class. The entire set of states in a given Markov chain is partitioned into one or more disjoint classes in this way.

For Markov chains with a finite number of states, a state i is called *recurrent* if i is accessible from all states that are accessible from i (i.e., if $i \rightarrow j$ implies that $j \rightarrow i$). This definition does not apply if the number of states is infinite. According to this definition, a state i is recurrent if there is no possibility of going to a state j from which there can be no return. As we shall see later, if a Markov chain ever enters a recurrent state, it returns to that state eventually with probability 1, and thus keeps returning infinitely often (in fact, this property serves as the definition of recurrence for the general case with a possibly infinite set of states). A state that is not recurrent is called *transient*. That is, i is transient if there is some j that is accessible from i but from which there is no possible return. Each time the system returns to i , there is a possibility of going to j ; eventually this possibility will occur, and then no more returns to i can occur (this can be thought of as a mathematical form of Murphy's law).

If i is transient, then all states in the same class as i are also transient. To see this, suppose that i is transient (i.e., for some j , $i \rightarrow j$ but there is no path from j to i) and suppose that i and m are in the same class (i.e., $i \leftrightarrow m$). Then $m \rightarrow i$ and $i \rightarrow j$, so $m \rightarrow j$. Now if $j \rightarrow m$, then the walk from j to m could be extended on to i ; this is a contradiction, and therefore there is no walk from j to m , and m is transient. Since we have just shown that all nodes in a class are transient if any are, it follows that a class is either all recurrent or all transient. For example in figure 4.1a, $\{1, 2, 3, 4\}$ is a transient class and $\{5\}$ is a recurrent class. In terms of the graph of a Markov chain, a class is transient if there are any arcs going from a node in the class to a node outside the class. Every finite state Markov chain must have at least one recurrent class of states (see exercise 4.1), and can have arbitrarily many additional classes of recurrent and transient states.

States can also be classified according to their periods (see figure 4.2). In figure 4.2a, given that $X_0 = 2$, we see that X_1 must be either 1 or 3, X_2 must then be either 2 or 4, and in general, X_n must be 2 or 4 for n even and 1 or 3 for n odd. On the other hand, if X_0 is 1 or 3, then X_n is 2 or 4 for n odd and 1 or 3 for n even. Thus the effect of the starting state never dies out. Figure 4.2b illustrates another example in which the state alternates from odd to even and the memory of the starting state never dies out. The states in both these Markov chains are said to be periodic with period 2.

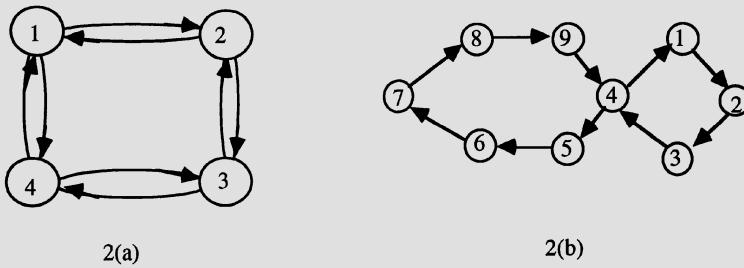


Figure 4.2. Periodic Markov Chains.

In general, the period² of a state i , denoted $d(i)$, is defined as the greatest common divisor (gcd) of those values of n for which $P_{ii}^n > 0$. If the period is 1, the state is said to be *aperiodic*, and if the period is 2 or more, the state is said to be *periodic*. For example, in figure 4.2a, $P_{11}^n > 0$ for $n = 2, 4, 6, \dots$. Thus $d(1)$, the period of state 1, is two. Similarly, $d(i)=2$ for all the states in figure 4.2a. For figure 4.2b, we have $P_{11}^n > 0$ for $n = 4, 8, 10, 12, \dots$; thus $d(1) = 2$, and it can be seen that $d(i)=2$ for all the states. These examples suggest the following theorems.

THEOREM 1: For any Markov chain (with either a finite or countably infinite number of states), all states in the same class have the same period.

Proof: Let i and j be any distinct pair of states in a class. Then $i \leftrightarrow j$ and there is some r such that $P_{ij}^r > 0$ and some s such that $P_{ji}^s > 0$. Since there is a walk of length $r+s$ going from i to j and back to i , $r+s$ must be divisible by $d(i)$. Let t be any integer such that $P_{jj}^t > 0$. Since there is a walk of length $r+t+s$ that goes first from i to j , then to j again, and then back to i , $r+t+s$ is divisible by $d(i)$, and thus t is divisible by $d(i)$. Since this is true for any t such that $P_{jj}^t > 0$, $d(j)$ is divisible by $d(i)$. Reversing the roles of i and j , $d(i)$ is also divisible by $d(j)$, so $d(i) = d(j)$.

THEOREM 2: If a recurrent class of states in a Markov chain (with either a finite or countably infinite number of states) has period $d > 1$, then the states in the class can be partitioned into d subclasses, T_1, T_2, \dots, T_d such that, for each m , all transitions out of

subclass T_m go to subclass T_{m+1} (or, for $m=d$, go to T_1). That is, if $j \in T_m$ and $P_{jk} > 0$, then $k \in T_{m+1}$ (or $k \in T_1$ for $m=d$).

Proof: See figure 4.3 for an illustration of the theorem. For a given state in the class, say state 1, define the sets T_1, \dots, T_d by

$$T_m = \{j: P_{1j}^{nd+m} > 0 \text{ for some } n \geq 0\}; 1 \leq m \leq d \quad (3)$$

For any state j in the class, since $1 \leftrightarrow j$, there is an r for which $P_{1j}^r > 0$ and an s for which $P_{j1}^s > 0$. Since there is a walk from 1 to 1 (through j) of length $r+s$, $r+s$ is divisible by d . Let m , $1 \leq m \leq d$, be defined by $r = m+nd$, where n is an integer. From (3), $j \in T_m$. Now let r' be any other integer such that $P_{1j}^{r'} > 0$. Then $r'+s$ is also divisible by d , and $r'-r$ is divisible by d . Thus $r' = m+n'd$ for some integer n' , and since r' is any integer such that $P_{1j}^{r'} > 0$, j is in T_m only for one value of m . Since j is arbitrary, this shows that the sets T_m are disjoint. Finally, suppose $j \in T_m$ and $P_{jk} > 0$. Given a walk of length $r = nd+m$ from state 1 to j , there is a walk of length $nd+m+1$ from state 1 to k , and thus $k \in T_{m+1}$ for $m < d$ and $k \in T_1$ for $m=d$, completing the proof.

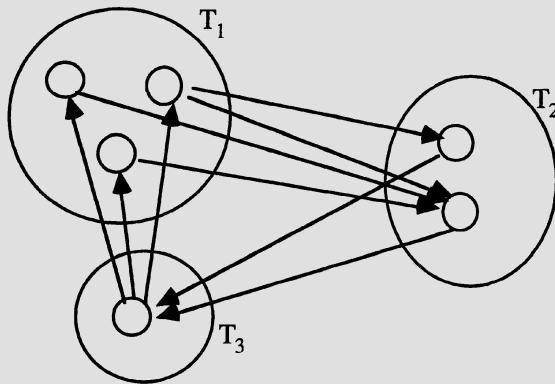


Figure 4.3. Structure of a Periodic Markov Chain with $d = 3$. Note that transitions only go from one subclass T_m to the next subclass T_{m+1} (or from T_d to T_1).

We have seen that classes of states (for finite state chains) can be classified both in terms of their period and in terms of whether or not they are recurrent. The most important case is that in which a class is both recurrent and aperiodic. Such classes are called *ergodic*³ and a chain consisting entirely of one ergodic class is called an ergodic chain. We shall see later that these chains have the desirable property that P_{ij}^n becomes independent of the starting state i as $n \rightarrow \infty$. The next theorem establishes the first part of this by showing that $P_{ij}^n > 0$ for all i and j when n is sufficiently large.

THEOREM 3: If a finite state Markov chain is ergodic and has J states, then $P_{ij}^m > 0$ for all i, j , and all $m \geq J-1$.

Proof*!⁴ Let i be any given state, fixed throughout the proof. Define $T(m)$, $m \geq 0$, as the set of states accessible from state i in m steps. Thus $T(0) = \{i\}$, $T(1) = \{j : P_{ij} > 0\}$, and for arbitrary $m \geq 0$,

$$T(m) = \{j : P_{ij}^m > 0\} \quad (4)$$

Let t be the smallest positive integer such that $P_{ii}^t > 0$; t must satisfy $t \leq J$ since no state is entered more than once in such a shortest walk from i back to i . If $j \in T(m)$ for $m \geq 1$, there is a walk from i to j in m steps, and also a walk from i to j in $t+m$ steps that passes through state i on the t^{th} step. Thus, for all $m \geq 1$

$$T(m) \subseteq T(t+m) \quad (5)$$

Eq(5) also holds for $m = 0$ since $i \in T(t)$. By starting with $m=0$ and iterating on (5),

$$T(0) \subseteq T(t) \subseteq T(2t) \subseteq T(3t) \subseteq \dots \quad (6)$$

Next assume, for some $m \geq 0$ and $s \geq 1$, that $T(m) = T(m+s)$. Note that $T(m+1)$ is the set of states that can be reached in one step from states in $T(m)$, and similarly $T(m+s+1)$ is the set reachable in one step from $T(m+s)$. Thus $T(m+1) = T(m+1+s)$. Iterating this result,

$$T(n) = T(n+s) \text{ for all } n \geq m \text{ if } T(m) = T(m+s) \quad (7)$$

Taking $s=t$, (7) shows us that if one of the inclusion relations in (6) is satisfied with equality, then all subsequent relations are satisfied with equality. For each strict inclusion, however, the set must be enlarged by at least one member. Since $T(0)$ has one member (state i), and the entire set has J members, there can be at most $J-1$ strict inclusions in (6), and $T((J-1)t) = T(Jt)$. Applying (7) iteratively to this,

$$T(nt) = T((J-1)t) \text{ for all integers } n \geq J-1 \quad (8)$$

We can also apply (7) to (8),

$$T(m+nt) = T(m+(J-1)t) \text{ for } m \geq 0, n \geq J-1 \quad (9)$$

Define T_∞ as $T((J-1)t)$. We see from (8) that $T(nt) = T_\infty$ for $n \geq J-1$. The rest of the proof consists of two parts: one is to show that

$$T(m) = T_\infty \text{ for all } m \geq (J-1)t. \quad (10)$$

The other is to show that (10) implies the statement of the theorem. We do the latter part first. Since $t \leq J$, (10) implies that $T(m) = T_\infty$ for all $m \geq (J-1)J$. Since each state j continues to be accessible after $(J-1)J$ steps, j must be in $T(m)$ for some $m \geq (J-1)t$, and thus $j \in T_\infty = T(m)$ for all $m \geq (J-1)t$.

To demonstrate (10), let μ be the smallest positive integer for which $T(\mu+(J-1)t) = T((J-1)t)$. From (8), $\mu \leq t$. Applying (7) iteratively to this, and then applying (7) to the result,

$$\begin{aligned} T((J-1)t+j\mu) &= T((J-1)t) \text{ for all integers } j \geq 0 \\ T(n+j\mu) &= T(n) \text{ for all integers } n \geq (J-1)t \text{ and all integers } j \geq 0 \end{aligned} \quad (11)$$

If $\mu=1$, then $T(m) = T_\infty$ for all $m \geq (J-1)t$, and we are done. Thus we assume $\mu > 1$ and show a contradiction. The chain is ergodic so i is aperiodic. Thus, $P_{ii}^{t'} > 0$ for some t' not divisible by μ . By the same argument used to derive (8), $T((J-1)t') = T(Jt')$, and from (7),

$$T(n) = T(n+t') \text{ for all integers } n \geq (J-1)t' \quad (12)$$

Now let m' be the remainder when t' is divided by μ , i.e., $t' = j\mu + m'$, $1 \leq m' < \mu$ and let $n' = (J-1)t'$. The desired contradiction is now:

$$\begin{aligned} T((J-1)t+m') &= T((J-1)t+j\mu+m') && \text{from (11)} \\ &= T((J-1)t+t'+j\mu+m') && \text{from (9)} \\ &= T(n't+t') && (t'=j\mu+m'; n'=(J-1)t') \\ &= T(n't) = T((J-1)t) && \text{from (12) and then (8)} \end{aligned} \quad (13)$$

Since $0 < m' < \mu$, this is a contradiction, and since i is arbitrary, this completes the proof.

4.3 THE MATRIX REPRESENTATION

The matrix $[P]$ of transition probabilities of a Markov chain is called a stochastic matrix; that is, a *stochastic matrix* is a square matrix of non-negative terms in which the elements in each row sum to 1. We first consider the n step transition probabilities P_{ij}^n in terms of $[P]$. The probability of going from state i to state j in two steps is the sum over h of all possible two step walks, from i to h and from h to j . Using the Markov condition in (1), this is

$$P_{ij}^2 = \sum_{h=1}^J P_{ih} P_{hj}$$

It can be seen that this is just the ij term of the product of matrix $[P]$ with itself; denoting $[P][P]$ as $[P]^2$, this means that P_{ij}^2 is the (i, j) element of the matrix $[P]^2$. Similarly, P_{ij}^n is the ij element of the n^{th} power of the matrix $[P]$. Since $[P]^{m+n} = [P]^m [P]^n$, this means that

$$P_{ij}^{m+n} = \sum_{h=1}^J P_{ih}^m P_{hj}^n \quad (14)$$

This is known as the *Chapman–Kolmogorov* equation. Note that if one wishes to compute $[P]^n$ for large n , an efficient approach is to multiply $[P]^2$ by $[P]^2$, then $[P]^4$ by $[P]^4$ and so forth and then multiply these binary powers together as needed.

The matrix $[P]^n$ (i.e., the matrix of transition probabilities raised to the n^{th} power) is very important for a number of reasons. The i, j element of this matrix is P_{ij}^n , which is the probability of being in state j at time n given state i at time 0. If memory of the past dies out with increasing n , then we would expect the dependence on both n and i to disappear in P_{ij}^n . This means, first, that $[P]^n$ should converge to a limit as $n \rightarrow \infty$, and, second, that each row of $[P]^n$ should tend to the same set of probabilities. If this convergence occurs (and we later determine the circumstances under which it occurs), $[P]^n$ and $[P]^{n+1}$ will be the same in the limit $n \rightarrow \infty$, which means $\lim [P]^n = (\lim [P]^n) P$. If all the rows of $\lim [P]^n$ are the same, equal to some row vector $\pi = (\pi_1, \pi_2, \dots, \pi_J)$, this simplifies to the equation

$$\pi = \pi [P]. \quad (15)$$

Since π is a probability vector (i.e., its components are the probabilities of being in the various states in the limit $n \rightarrow \infty$), its components must be non-negative and sum to 1. Any probability vector π which satisfies (15) is called a *steady state probability vector* or *steady state distribution* for the Markov chain with transition matrix $[P]$. The steady state distribution is also often called a *stationary distribution*. If a probability vector π satisfying (15) is taken as the initial probability assignment of the chain at time 0, then that assignment is maintained forever. That is, if $P(X_0=i) = \pi_i$ for all i , then $P(X_1=j) = \sum_i \pi_i P_{ij} = \pi_j$ for all j , and, by induction, $P(X_n=j) = \pi_j$ for all j and all $n > 0$.

If $[P]^n$ converges as above, then, the steady state distribution is reached asymptotically, and this is independent of the starting state. There are a number of questions that must be answered for a steady state distribution as defined above:

- 1) Does $\pi = \pi [P]$ always have a probability vector solution?
- 2) Does $\pi = \pi [P]$ have a unique probability vector solution?
- 3) Do the rows of $[P]^n$ converge to a probability vector solution of $\pi = \pi [P]$?

We first give the answers to these questions for finite state Markov chains and then derive them. First, $\pi = \pi [P]$ always has a probability vector solution (although this is not necessarily true for infinite state chains). Second, this probability vector solution is unique iff there is a single recurrent class in the Markov chain (there can be any number of transient classes). If there are r recurrent classes, then $\pi = \pi [P]$ has r linearly independent probability vector solutions. Third, if the Markov chain has a single recurrent class and is aperiodic (i.e., it is ergodic) then each row of $[P]^n$ converges to the unique probability vector solution of $\pi = \pi [P]$. If the Markov chain has multiple recurrent

classes, but all of them are aperiodic, then $[P]^n$ still converges, but to a matrix with non-identical rows. If the Markov chain has one or more periodic recurrent classes, then $[P]^n$ does not converge.

We first look at these answers from the standpoint of matrix theory and then proceed in Chapter 5 to look at the more general problem of Markov chains with a countably infinite number of states. There we use renewal theory to answer these same questions (and to discover the differences that occur for infinite state Markov chains). The matrix theory approach is useful computationally and also has the advantage of telling us something about rates of convergence. The approach using renewal theory is very simple (given an understanding of renewal processes), but is more abstract.

THE EIGENVALUES AND EIGENVECTORS OF P: A convenient way of dealing with the n^{th} power of a matrix is to find the eigenvalues and eigenvectors of the matrix. The row vector π is a *left eigenvector* of $[P]$ of eigenvalue λ if $\pi \neq 0$ and $\pi[P] = \lambda\pi$, and the column vector v is a *right eigenvector* of eigenvalue λ if $v \neq 0$ and $[P]v = \lambda v$. For the special case of a two state Markov chain, these equations can be written out as

$$\begin{aligned} \pi_1 P_{11} + \pi_2 P_{21} &= \lambda \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} &= \lambda \pi_2 \end{aligned} \quad \begin{aligned} P_{11}v_1 + P_{12}v_2 &= \lambda v_1 \\ P_{21}v_1 + P_{22}v_2 &= \lambda v_2 \end{aligned} \quad (16)$$

These equations have a non-zero solution iff the matrix $[P - \lambda I]$, where $[I]$ is the identity matrix, is singular (i.e., there must be a non-zero v for which $[P - \lambda I]v = 0$). Thus λ must be such that the determinant of $[P - \lambda I]$, namely $(P_{11} - \lambda)(P_{22} - \lambda) - P_{12}P_{21}$, is equal to 0. Solving this quadratic equation in λ , we find that λ has two solutions, $\lambda_1 = 1$ and $\lambda_2 = 1 - P_{12} - P_{21}$. Solving (16) for the eigenvectors $\pi^{(1)}$ and $v^{(1)}$ of λ_1 and $\pi^{(2)}$ and $v^{(2)}$ of λ_2 , (and assuming that P_{12} and P_{21} are not both zero), the result is (subject to an arbitrary scale factor in each eigenvector).

$$\begin{aligned} \pi_1^{(1)} &= \frac{P_{21}}{P_{12} + P_{21}} & \pi_2^{(1)} &= \frac{P_{12}}{P_{12} + P_{21}} & v_1^{(1)} &= 1 & v_2^{(1)} &= 1 \\ \pi_1^{(2)} &= 1 & \pi_2^{(2)} &= -1 & v_1^{(2)} &= \frac{P_{12}}{P_{12} + P_{21}} & v_2^{(2)} &= \frac{-P_{21}}{P_{12} + P_{21}} \end{aligned}$$

Now let $[\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and let $[U]$ be a matrix with columns $v^{(1)}$ and $v^{(2)}$. Then the two right eigenvector equations in (16) can be combined compactly as $[P][U] = [U][\Lambda]$.

It turns out (given the way we have normalized the eigenvectors) that the inverse of $[U]$ is just the matrix whose rows are the left eigenvectors of $[P]$ (this can be verified by direct calculation, and we show later that any right eigenvector of one eigenvalue must be orthogonal to any left eigenvector of another eigenvalue). We then see that $[P] = [U][\Lambda][U]^{-1}$ and consequently $[P]^n = [U][\Lambda]^n[U]^{-1}$. Multiplying this out, we get

$$[P]^n = \begin{bmatrix} \pi_1 + \pi_2 \lambda_2^n & \pi_2 - \pi_2 \lambda_2^n \\ \pi_1 - \pi_1 \lambda_2^n & \pi_2 + \pi_1 \lambda_2^n \end{bmatrix} \quad \text{where } \pi_1 = \frac{P_{21}}{P_{12}+P_{21}}, \pi_2 = 1 - \pi_1$$

Recalling that $\lambda_2 = 1 - P_{12} - P_{21}$, we see that $|\lambda_2| \leq 1$. In the special case where $P_{12}=P_{21}=0$, it turns out that $\lambda_2 = 1$ and $[P]$ and $[P]^n$ are simply identity matrices. In the special case $P_{12}=P_{21}=1$, it turns out that $\lambda_2 = -1$ and $[P]^n$ alternates between the identity matrix for n even and $[P]$ for n odd. In all other cases, $|\lambda_2| < 1$ and $[P]^n$ approaches the matrix whose rows are both equal to π .

Parts of this special case generalize to an arbitrary finite number of states. In particular, $\lambda=1$ is always an eigenvalue and the vector e whose components are all equal to 1 is always a right eigenvector of $\lambda=1$ (this follows immediately from the fact that each row of a stochastic matrix sums to 1). Unfortunately, not all stochastic matrices can be represented in the form $[P] = [U][\Lambda][U]^{-1}$ (since J independent right eigenvectors need not exist—see exercise 4.4). In general, the diagonal matrix of eigenvalues in $[P] = [U][\Lambda][U^{-1}]$ must be replaced by something called a Jordan form, which does not easily lead us to the desired results. In what follows, we develop the powerful Perron and Frobenius theorems, which are useful in their own right and also provide the necessary results about $[P]^n$ in general.

4.4 PERRON–FROBENIUS THEORY: A real vector x (i.e., a vector with real components) is defined to be *positive*, denoted $x > 0$ if $x_i > 0$ for each component i . Similarly a real matrix $[A]$ is *positive*, denoted $[A] > 0$, if $A_{ij} > 0$ for each i, j . Similarly, x is *non-negative*, denoted $x \geq 0$, if $x_i \geq 0$ for all i . $[A]$ is non-negative, denoted $[A] \geq 0$, if $A_{ij} \geq 0$ for all i, j . Note that it is possible to have $x \geq 0$ and $x \neq 0$ without having $x > 0$, since $x > 0$ means that all components of x are positive and $x \geq 0$, $x \neq 0$ means that at least one component of x is positive and all are non-negative. Also, $x > y$ and $y < x$ are both defined as $x - y > 0$; $x \geq y$ and $x \leq y$ and the corresponding matrix inequalities have corresponding definitions. We start by looking at the eigenvalues and eigenvectors of positive square matrices. In what follows, when we assert that a matrix, vector, or number is positive or non-negative, we implicitly mean that it is real also. We will prove Perron's theorem, which is the critical result for dealing with positive matrices. We then generalize Perron's theorem to the Frobenius theorem, which treats a class of non-negative matrices called irreducible matrices. We finally specialize the results to stochastic matrices.

THEOREM 4 (PERRON): Let $[A] > 0$ be a square matrix. Then $[A]$ has a positive eigenvalue λ that exceeds the magnitude of each other eigenvalue. There is a positive right eigenvector, $v > 0$, corresponding to λ , and the following properties hold for λ and v :

- 1) If $\lambda x \leq [A]x$ for $x \geq 0$, then $\lambda x = [A]x$.
- 2) If $\lambda x = [A]x$, then $x = \alpha v$ for some scalar α .

Discussion: Since λ exceeds the magnitude of any other eigenvalue, we refer to it in what follows as the *largest* eigenvalue of $[A]$. Part (2) asserts that the eigenvector v is unique (within a scale factor), not only among positive vectors but among all vectors. Finally, part (1) makes the surprising assertion that any non-zero $x \geq 0$ satisfying the inequality $\lambda x \leq [A]x$ is also an eigenvector of this special eigenvalue λ . It is this feature that is used to define λ and v for purposes of the proof. We start with a lemma about this definition. This is followed by the proof of the theorem and then the proof of the lemma.

LEMMA 1: Let $[A] > 0$ be a square matrix; for each non-zero column vector $x \geq 0$, define $L(x)$ as the largest real number L satisfying $Lx \leq [A]x$. Define $\lambda = \sup_{x \neq 0, x \geq 0} L(x)$. Then (a), $\lambda > 0$, (b), there is a vector $v \geq 0$ such that $\lambda = L(v)$, and (c), for any such v , $[A]v = \lambda v$.

Proof of theorem 4*: For the λ defined in the lemma, the lemma shows that $\lambda > 0$ is an eigenvalue of $[A]$ and that it has an eigenvector $v \geq 0$. Part (c) of the lemma shows also that if $x \geq 0$ and if $\lambda = L(x)$, i.e., if $\lambda x \leq [A]x$, then $\lambda x = [A]x$. This establishes property (1) of the theorem for the λ of the lemma. The lemma shows that that same λ has a non-negative eigenvector v . To show that $v > 0$, note that all elements of $[A]$ are positive and some elements of $v \geq 0$ are positive. Thus each component of $[A]v$, say $\sum_j A_{ij}v_j$, is positive, and $[A]v > 0$. Thus $\lambda v > 0$, and $v > 0$.

Next let $\lambda' \neq \lambda$ be any other eigenvalue of $[A]$. We show that $\lambda > |\lambda'|$. Let $x \neq 0$ be a right eigenvector for λ' (x might have complex components and/or negative components). Taking the magnitudes of both sides of $\lambda'x = [A]x$, we get the following for each component i

$$|\lambda'| |x_i| = |\sum_j A_{ij} x_j| \leq \sum_j A_{ij} |x_j| \quad (17)$$

Let $u = (|x_1|, |x_2|, \dots, |x_J|)$, so (17) becomes $|\lambda'| u \leq [A]u$. Since $u \geq 0$, $u \neq 0$, it follows from the lemma that $|\lambda'| \leq L(u)$ and $L(u) \leq \lambda$, so $|\lambda'| \leq \lambda$. Furthermore, if $|\lambda'| = \lambda$, then $\lambda u \leq [A]u$, and property 1 of the theorem asserts that $\lambda u = [A]u$. Thus u is an eigenvector of $\lambda = |\lambda'|$ and (17) is satisfied with equality. This means (see exercise 4.7) that $x = \beta u$ for some (perhaps complex) scalar β . It follows that x is an eigenvector of λ , so $\lambda' = \lambda$. Thus $\lambda > |\lambda'|$ for all eigenvalues $\lambda' \neq \lambda$.

To complete the proof of property (2), assume that x is any eigenvector of λ . The previous paragraph showed that $x = \beta u$ where β is a scalar and $u \neq 0$, $u \geq 0$ is also an eigenvector for λ . Since $v > 0$, we can choose $\alpha > 0$ so that $v - \alpha u \geq 0$ and $v_i - \alpha u_i = 0$ for some i . Since $\lambda(v - \alpha u) = [A](v - \alpha u)$, $v - \alpha u \geq 0$ is either 0 or is an eigenvector of λ , and thus strictly positive. Since $v_i - \alpha u_i = 0$ for some i , $v - \alpha u = 0$. Thus u and x are scalar multiples of v , completing the proof.

Proof of lemma 1*: (Part a): For any $x \neq 0$, $x \geq 0$, $L(x)$ is the largest L for which $Lx_i \leq ([A]x)_i$ for $1 \leq i \leq J$. Thus $L(x)$ is given by

$$L(\mathbf{x}) = \min_{i: x_i > 0} \frac{([A]\mathbf{x})_i}{x_i} \quad (18)$$

For $\mathbf{x} = (1, 1, \dots, 1)^T$, $L(\mathbf{x}) = \min_i \sum_j A_{ij} > 0$, so that $\lambda > 0$.

(Part b): For any $\alpha > 0$, $L(\mathbf{x}) = L(\alpha\mathbf{x})$. Let $S = \{\mathbf{x}: x_i \geq 0 \text{ for } 1 \leq i \leq J \text{ and } \max_i x_i = 1\}$. Then for any $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$, $\alpha\mathbf{x} \in S$ for some $\alpha > 0$, and $\lambda = \sup_{\mathbf{x} \in S} L(\mathbf{x})$. Now S is a closed and bounded region of vector space, and a continuous function achieves a maximum over a closed and bounded region. Thus, if $L(\mathbf{x})$ is continuous over S , there is a $\mathbf{v} \in S$ such that $\lambda = L(\mathbf{v})$. To show that $L(\mathbf{x})$ is continuous over S , note that for any $\mathbf{x} \in S$, $x_i = 1$ for some i , and for this i , $([A]\mathbf{x})_i / x_i \leq \sum_j A_{ij}$. Letting $L_{\max} = \max_i \sum_j A_{ij}$, we see that $L(\mathbf{x}) \leq L_{\max}$ for all $\mathbf{x} \in S$. Thus, $L(\mathbf{x})$ can be expressed as the minimum of L_{\max} and all the components in (18), and this can be expressed as

$$L(\mathbf{x}) = \min_i \left\{ \min \left[\frac{([A]\mathbf{x})_i}{x_i}, L_{\max} \right] \right\} \quad (19)$$

Since the expression in the inner minimum is continuous for each i , $L(\mathbf{x})$ is also continuous over $\mathbf{x} \in S$.

(Part c): For the \mathbf{v} of part b, $\lambda\mathbf{v} \leq [A]\mathbf{v}$. There are two possibilities: either $\lambda\mathbf{v} = [A]\mathbf{v}$ or $\lambda\mathbf{v} \neq [A]\mathbf{v}$. In the former case, the proof is complete, so we establish a contradiction for the latter case. In the latter case, there is some i for which $\lambda v_i < ([A]\mathbf{v})_i$. In this case, pre-multiplying both sides of $\lambda\mathbf{v} \leq [A]\mathbf{v}$ by $[A]$ yields $\lambda[A]\mathbf{v} < [A]^2\mathbf{v}$. To see that this inequality is indeed strict for all components, recall that $[A] > 0$, so that $\lambda v_i < ([A]\mathbf{v})_i$ causes $\lambda A_{ki} v_i < A_{ki}([A]\mathbf{v})_i$ for all k , and thus $\lambda[A]\mathbf{v} < [A]^2\mathbf{v}$. It follows that for small enough $\epsilon > 0$,

$$(\lambda + \epsilon)[A]\mathbf{v} \leq [A]^2\mathbf{v}. \quad (20)$$

Define $\mathbf{y} \geq \mathbf{0}$ by $\mathbf{y} = [A]\mathbf{v}$, so that (20) becomes $(\lambda + \epsilon)\mathbf{y} \leq [A]\mathbf{y}$, and $L(\mathbf{y})$ as defined in the lemma satisfies $L(\mathbf{y}) \geq \lambda + \epsilon$. This contradicts the definition of λ , completing the proof.

Next we apply the results above to a more general type of non-negative matrix called an *irreducible matrix*. Recall that we analyzed the classes of a finite Markov chain in terms of a directed graph where the nodes represent the states of the chain and a directed edge goes from i to j if $P_{ij} > 0$. We can draw the same type of directed graph for an arbitrary non-negative matrix $[A]$; i.e., a directed edge goes from i to j if $A_{ij} > 0$. A non-negative matrix is *irreducible* if for every pair of nodes i, j in this graph, there is a walk from i to j . For stochastic matrices, an irreducible matrix is thus the matrix for a Markov chain that consists of a single recurrent class of states. If we denote the i, j

element of $[A]^n$ by A_{ij}^n , then we see that $A_{ij}^n > 0$ iff there is a walk of length n from i to j in the graph. If $[A]$ is irreducible, a walk exists from any i to any $j \neq i$ with length at most $J-1$, since the walk need go through at most each of the other nodes. Thus $A_{ij}^n > 0$ for some n , $1 \leq n \leq J-1$, and $\sum_{n=1}^{J-1} A_{ij}^n > 0$. The key to analyzing irreducible matrices is the fact that the matrix $\sum_{n=0}^{J-1} [A]^n$ is positive. The $n=0$ term, $[A]^0$ is just the identity matrix, which covers the case $i=j$.

THEOREM 5 (FROBENIUS): Let $[A] \geq 0$ be a square J by J irreducible matrix. Then $[A]$ has a positive eigenvalue λ that is greater than or equal to the magnitude of each other eigenvalue. There is a positive right eigenvector, $v > 0$ corresponding to λ , and the following properties hold for λ and v :

- 1) For any non-zero $x \geq 0$, if $\lambda x \leq [A]x$, then $\lambda x = [A]x$.
- 2) If $\lambda x = [A]x$, then $x = \alpha v$ for some scalar α .

Proof*: Let $[B] = \sum_{n=0}^{J-1} [A]^n > 0$. Using theorem 4, we let λ_B be the largest eigenvalue of $[B]$ and let $v > 0$ be the corresponding right eigenvector. Then $[B]v = \lambda_B v$. Also, since $[B][A] = [A][B]$, we have $[B]\{[A]v\} = [A][B]v = \lambda_B [A]v$. Thus $[A]v$ is a right eigenvector of $[B]$ and thus equal to v times a scale factor. Let this scale factor be λ , so that $[A]v = \lambda v$. Since $[A]v$ and v are positive, $\lambda > 0$, establishing the existence of the positive eigenvalue eigenvector pair λ , v . Next, $[B]v = \sum_{n=0}^{J-1} [A]^n v = (1 + \lambda + \dots + \lambda^{J-1})v$. Thus $\lambda_B = 1 + \lambda + \dots + \lambda^{J-1}$. Let x be an arbitrary vector satisfying $[A]x = \lambda x$. Then x is also a right eigenvector of $[B]$ with eigenvalue λ_B , so from theorem 4, x must be a scalar multiple of v , establishing property (2) of the theorem. Now assume that $\lambda x \leq [A]x$ and $x \geq 0$. It follows, using induction on i , that $\lambda^i x \leq [A]^i x$ for all i , and thus $\lambda_B x \leq [B]x$. From theorem 4, x is a scalar multiple of v , and thus $\lambda x = [A]x$, establishing property (1). To complete the proof, we must show that $\lambda \geq |\lambda'|$ where λ' is an arbitrary eigenvalue of $[A]$ (not necessarily real). Let x be a right eigenvector for λ' , scaled so that $\max_i |x_i| = 1$. Taking the magnitudes of both sides of $\lambda x = [A]x$, we get the following for each component i

$$|\lambda| |x_i| = |\sum_j A_{ij} x_j| \leq \sum_j A_{ij} |x_j| \quad (21)$$

From property (1) of the theorem, applied to the vector $u = (|x_1|, |x_2|, \dots, |x_J|)$, we see that $|\lambda'| \leq \lambda$, completing the proof.

Note that the argument we used in theorem 4 to show that $|\lambda'|$ is strictly less than λ does not apply here since $[A]$ is not positive. $[A]$ can have other eigenvalues of magnitude equal to λ ; an example that we look at later is that where $[A]$ is the transition matrix of a periodic Markov chain. We refer to λ in what follows as the *largest real eigenvalue* of $[A]$. The theorem says that the largest real eigenvalue λ exists, and that $\lambda \geq |\lambda'|$ for each other eigenvalue λ' .

COROLLARY 1: The largest real eigenvalue λ of an irreducible matrix $[A] \geq 0$ has a positive left eigenvector π . π is unique (within a scale factor) and is the only non-negative non-zero vector (within a scale factor) that satisfies $\lambda\pi \leq \pi[A]$.

Proof: A left eigenvector of $[A]$ is a right eigenvector (transposed) of $[A]^T$. The graph corresponding to $[A]^T$ is the same as that for $[A]$ with all the arc directions reversed, so that all pairs of nodes still communicate and $[A]^T$ is irreducible. Since $[A]$ and $[A]^T$ have the same eigenvalues, the corollary is just a restatement of the theorem.

COROLLARY 2: Let λ be the largest real eigenvalue of an irreducible matrix and let the right and left eigenvectors of λ be $v > 0$ and $\pi > 0$. Then, within a scale factor, v is the only non-negative right eigenvector of $[A]$ (i.e., no other eigenvalues have non-negative eigenvectors). Similarly, π is the only non-negative left eigenvector of $[A]$.

Proof: Theorem 5 asserts that v is the unique right eigenvector of the largest real eigenvalue λ , so suppose that u is a right eigenvector of some other eigenvalue λ' . Letting π be the left eigenvector of λ , we have $\pi[A]u = \lambda'u$ and also $\pi[A]v = \lambda'v$. Thus $\pi u = 0$. Since $\pi > 0$, u cannot be non-negative and non-zero. The same argument shows the uniqueness of π .

COROLLARY 3: Let $[P]$ be a stochastic irreducible matrix (i.e., the matrix of a Markov chain consisting of a single recurrent class). Then $\lambda=1$ is the largest real eigenvalue of $[P]$, $e = (1, 1, \dots, 1)^T$ is the right eigenvector of $\lambda=1$, unique within a scale factor, and there is a unique probability vector $\pi > 0$ that is a left eigenvector of $\lambda=1$.

Proof: Since each row of $[P]$ adds up to 1, $[P]e = e$. Corollary 2 asserts the uniqueness of e and the fact that $\lambda=1$ is the largest real eigenvalue, and corollary 1 asserts the uniqueness of π .

The proof above shows that *any* stochastic matrix, whether irreducible or not, has an eigenvalue $\lambda=1$ with $e = (1, \dots, 1)^T$ as a right eigenvector. In general, a stochastic matrix with r recurrent classes has r independent non-negative right eigenvectors and r independent non-negative left eigenvectors; the left eigenvectors can be taken as the steady state probability vectors within the r recurrent classes (see exercise 4.10).

In many situations, we have a Markov chain consisting of a single recurrent class and one or more transient classes, and we call such chains *recurrent plus transient*. The following corollary, proved in exercise 4.9, modifies corollary 3 for this case. Note that π is non-negative rather than positive here.

COROLLARY 4: Let $[P]$ be the transition matrix of a recurrent plus transient Markov chain. Then $\lambda=1$ is the largest real eigenvalue of $[P]$, $e = (1, 1, \dots, 1)^T$ is the right eigenvector of $\lambda=1$, unique within a scale factor, and there is a unique probability vector $\pi \geq 0$ that is a left eigenvector of $\lambda=1$.

COROLLARY 5: The largest real eigenvalue λ of an irreducible matrix $[A] \geq 0$ is a strictly increasing function of each component of $[A]$.

Proof: For a given irreducible $[A]$, let $[A']$ satisfy $[A'] \geq [A]$, $[A'] \neq [A]$. Let λ be the largest real eigenvalue of $[A]$ and $v > 0$ be the corresponding right eigenvector. Then $\lambda v = [A]v \leq [A']v$, but $\lambda v \neq [A']v$. Let λ' be the largest real eigenvalue of $[A']$, which is also irreducible. If $\lambda' \leq \lambda$, then $\lambda'v \leq \lambda v \leq [A']v$, and $\lambda'v \neq [A']v$, which is a contradiction of property 1 of theorem 5. Thus, $\lambda' > \lambda$, completing the proof.

We are now ready to study the asymptotic behavior of $[A]^n$. The simplest and cleanest result holds for $[A] > 0$. We establish this in the following corollary and then look at the case of greatest importance, that of a stochastic matrix for an ergodic Markov chain. More general cases are treated in exercises 4.9 and 4.10.

COROLLARY 6: Let λ be the largest eigenvalue of $[A] > 0$ and let $\pi(v)$ be the positive left (right) eigenvector of λ normalized so that $\pi v = 1$. Then

$$\lim_{n \rightarrow \infty} \frac{[A]^n}{\lambda^n} = v\pi \quad (22)$$

Proof*: Since $v > 0$ is a column vector and $\pi > 0$ is a row vector, $v\pi$ is a positive matrix of the same dimension as $[A]$. Since $[A] > 0$, we can define a matrix $[B] = [A] - \alpha v\pi$ which is positive for small enough $\alpha > 0$. Note that $v(\pi)$ is a right (left) eigenvector of $[B]$ with eigenvalue $\mu = \lambda - \alpha$. We then have $\mu^n v = [B]^n v$, which when pre-multiplied by π yields

$$(\lambda - \alpha)^n = \pi[B]^n v = \sum_i \sum_j \pi_i B_{ij}^n v_j$$

where B_{ij}^n is the i, j element of $[B]^n$. Since each term in the above summation is positive, we have $(\lambda - \alpha)^n \geq \pi_i B_{ij}^n v$, and therefore $B_{ij}^n \leq (\lambda - \alpha)^n / (\pi_i v_j)$. Thus, for each i, j , $\lim_{n \rightarrow \infty} B_{ij}^n \lambda^{-n} = 0$, and therefore $\lim_{n \rightarrow \infty} [B]^n \lambda^{-n} = 0$. Next we use a convenient matrix identity: for any eigenvalue λ of a matrix A , and any right and left eigenvectors v and π , normalized so that $\pi v = 1$, we have $[(A) - \lambda v\pi]^n = [A]^n - \lambda^n v\pi$ (see exercise 4.8). Applying the same identity to B , we have $[(B) - \mu v\pi]^n = [B]^n - \mu^n v\pi$. Finally, $[B] - \mu v\pi = [A] - \alpha v\pi - (\lambda - \alpha)v\pi = [A] - \lambda v\pi$. Thus

$$[A]^n - \lambda^n v\pi = [B]^n - \mu^n v\pi \quad (23)$$

Dividing both sides of (23) by λ^n and taking the limit of both sides of (23) as $n \rightarrow \infty$, the right hand side goes to 0, completing the proof.

Note that for a stochastic matrix $[P] > 0$, this corollary simplifies to $\lim_{n \rightarrow \infty} [P]^n = \pi$. This means that $\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j$, which means that the probability of being in state j after a long time is π_j , independent of the starting state.

THEOREM 6: Let $[P]$ be the transition matrix of an ergodic finite state Markov chain. Then $\lambda=1$ is the largest real eigenvalue of $[P]$, and $\lambda > |\lambda'|$ for every other eigenvalue λ' .

Furthermore, $\lim_{m \rightarrow \infty} [P]^m = e\pi$, where $\pi \geq 0$ is the unique probability vector satisfying $\pi[P] = \pi$ and $e = (1, 1, \dots, 1)^T$ is the unique v (within a scale factor) satisfying $[P]e = e$.

Proof: From corollary 3, $\lambda=1$ is the largest real eigenvalue of $[P]$, e is the unique (within a scale factor) right eigenvector of $\lambda=1$, and there is a unique probability vector π such that $\pi[P]=\pi$. From theorem 3, $[P]^m$ is positive for sufficiently large m . Since $[P]^m$ is also stochastic, $\lambda=1$ is strictly larger than the magnitude of any other eigenvalue of $[P]^m$. Let λ' be any other eigenvalue of $[P]$ and let v' be a right eigenvector of λ' . Note that v' is also a right eigenvector of $[P]^m$ with eigenvalue $(\lambda')^m$. Since $\lambda=1$ is the only eigenvalue of $[P]^m$ of magnitude 1 or more, we either have $|\lambda'| < \lambda$ or $(\lambda')^m = \lambda$. If $(\lambda')^m = \lambda$, then v' must be a scalar times e . This is impossible, since v' cannot be an eigenvector of $[P]$ with both eigenvalue λ and λ' . Thus $|\lambda'| < \lambda$. Similarly, $\pi \geq 0$ is the unique left eigenvector of $[P]^m$ with eigenvalue $\lambda=1$, and $\pi e = 1$. Corollary 6 then asserts that $\lim_{n \rightarrow \infty} [P]^{mn} = e\pi$. Multiplying by $[P]^i$ for any i , $1 \leq i < m$, we get $\lim_{n \rightarrow \infty} [P]^{mn+i} = e\pi$, so $\lim_{n \rightarrow \infty} [P]^n = e\pi$.

Theorem 6 generalizes easily to the case of a single ergodic class of states with one or more transient classes (see exercise 4.11); in what follows, we call $[P]$ *ergodic plus transient* if it corresponds to a Markov chain with a single ergodic class and perhaps one or more transient classes. In this case, as one might suspect, $\pi_i = 0$ for each transient state i and $\pi_i > 0$ within the ergodic class. Theorem 6 becomes:

THEOREM 6A: Let $[P]$ be the transition matrix of an ergodic plus transient finite state Markov chain. Then $\lambda=1$ is the largest real eigenvalue of $[P]$, and $\lambda > |\lambda'|$ for every other eigenvalue λ' . Furthermore, $\lim_{m \rightarrow \infty} [P]^m = e\pi$, where $\pi \geq 0$ is the unique probability vector satisfying $\pi[P] = \pi$ and $e = (1, 1, \dots, 1)^T$ is the unique v (within a scale factor) satisfying $[P]v = v$.

If a chain has a periodic recurrent class, $[P]^m$ never converges. The existence of a unique probability vector solution to $\pi[P]=\pi$ for a periodic irreducible chain is somewhat mystifying at first. If the period is d , then the steady state vector π assigns probability $1/d$ to each of the d subclasses of theorem 2. If the initial probabilities for the chain are chosen as $P(X_0=i)=\pi_i$ for each i , then for each subsequent state n , $P(X_n=i)=\pi_i$. What is happening is that this initial probability assignment starts the chain in each of the d subclasses with probability $1/d$, and subsequent transitions maintain this randomness over subclass. On the other hand, $[P]^n$ cannot converge because P_{ii}^n , for each i , is zero except when n is a multiple of d . Thus the memory of starting state never dies out. An ergodic Markov chain does not have this peculiar property, and the memory of the starting state dies out (from theorem 6).

The intuition to be associated with the word ergodic is that of a process in which time averages are equal to ensemble averages. Using the general definition of ergodicity (which is beyond our scope here), a periodic irreducible Markov chain in steady state (i.e., with $P(X_n=i)=\pi_i$ for all n and i) is ergodic. Thus the notion of ergodicity for Markov chains is slightly different than that in the general theory.

4.5 MARKOV CHAINS WITH REWARDS

INTRODUCTION: Suppose that each state i in a Markov chain is associated with some reward, r_i . As the Markov chain proceeds from state to state, there is an associated sequence of rewards that are not independent, but are related by the statistics of the Markov chain. The situation is similar to, but simpler than, that of renewal reward processes. As with renewal reward processes, the reward r_i could equally well be a cost or an arbitrary real valued function of the state. We shall analyze the expected value of the aggregate reward over time.

The model of Markov chains with rewards is surprisingly broad. By making the number of states sufficiently large and the time increment sufficiently small, one can model almost any stochastic process by a Markov chain (although this is not always an insightful or desirable model). The concept of rewards is of course quite graphic in modeling such things as corporate profits or portfolio performance, but as we saw in the discussion of Little's theorem, the concept is applicable in many other types of problems as well.

In section 4.6, we shall study Markov decision theory, or dynamic programming. This can be viewed as a generalization of Markov chains with rewards in the sense that there is a “decision maker” or “policy maker” who in each state can choose between several different policies; each policy has associated with it a given set of transition probabilities to the next state and a given expected reward for the current state. Thus the decision maker must make a compromise between the expected reward of a given policy in the current state (i.e., the immediate reward) and the long term benefit from the next state to be entered. This is a much more challenging problem than our current study of Markov chains with rewards, but by thoroughly understanding the current problem, we shall develop the machinery to understand Markov decision theory also.

Frequently it is more natural to associate rewards with transitions rather than states. If r_{ij} denotes the reward associated with a transition from i to j and P_{ij} denotes the corresponding transition probability, then $r_i = \sum_j P_{ij}r_{ij}$ is the expected reward associated with a transition from state i . Since we analyze only expected rewards, and since the effect of transition rewards r_{ij} are summarized into state rewards r_i , we ignore transition rewards and consider only state rewards.

The steady state expected reward per unit time, given a single recurrent class of states, is easily seen to be $g = \sum_i \pi_i r_i$ where π_i is the steady state probability of being in state i . The following examples demonstrate that it is often important to understand the transient behavior of rewards. This transient behavior will turn out to be even more important when we study Markov decision theory and dynamic programming.

EXAMPLE 1—EXPECTED FIRST PASSAGE TIME: A common problem when dealing with Markov chains is that of finding the expected number of steps, starting in some initial state, before some final state is entered. Since the answer to this problem does not depend on what happens after the final state is entered, we can modify the chain to convert the final state, say state 1, into a trapping state. That is, we set $P_{11} = 1$, $P_{1j} = 0$ for all $j \neq 1$, and leave P_{ij} unchanged for all $i \neq 1$, all j (see figure 4.4).

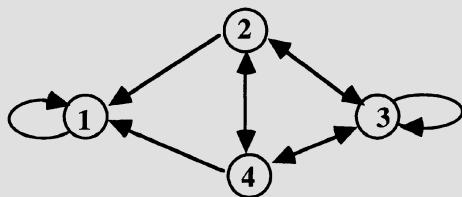


Figure 4.4. A four state Markov chain in which state 1 is a trapping state.

Let v_i be the expected number of steps to reach state 1 starting in state $i \neq 1$. This number of steps includes the first step plus the expected number of steps from whatever state is entered next (which is 0 if state 1 is entered next). Thus, for the chain in figure 4.4, we have the equations

$$\begin{aligned} v_2 &= 1 + P_{23}v_3 + P_{24}v_4 \\ v_3 &= 1 + P_{32}v_2 + P_{33}v_3 + P_{34}v_4 \\ v_4 &= 1 + P_{42}v_2 + P_{43}v_3 \end{aligned}$$

For an arbitrary chain of J states where 1 is a trapping state and all other states are transient, this set of equations becomes

$$v_i = 1 + \sum_{j \neq 1} P_{ij}v_j ; i \neq 1 \quad (24)$$

If we define $r_i = 1$ for $i \neq 1$ and $r_1 = 0$ for $i = 1$, then we can view r_i as being a unit reward for not yet entering the trapping state, and view v_i as the expected aggregate reward before entering the trapping state. Thus in this problem, if we take r_1 as zero, the reward ceases upon entering the trapping state, and v_i can be interpreted as the expected transient reward. Note that in this example, the steady state gain per unit time, $g = \sum_i \pi_i r_i$, is equal to 0.

If we define $v_1 = 0$, then we can write (24), along with $v_1 = 0$, in vector form as

$$\mathbf{v} = \mathbf{r} + [\mathbf{P}]\mathbf{v} ; v_1 = 0 \quad (25)$$

For a Markov chain with J states, (24) is a set of $J-1$ equations in the $J-1$ variables v_2 to v_J . The equation $\mathbf{v} = \mathbf{r} + [\mathbf{P}]\mathbf{v}$ is a set of J scalar equations, of which the first is the vacuous equation $v_1 = 0 + v_1$ and the last $J-1$ (with v_1 taken as 0) correspond to (24). It is not hard to show that (25) has a unique solution for \mathbf{v} under the condition that states 2 to J are all transient states and 1 is a trapping state, but we prove this later, in lemma 2, under more general circumstances.

EXAMPLE 2: Assume that a Markov chain has J states, $\{0, 1, \dots, J-1\}$ and that the state represents the number of customers in an integer time queueing system. Suppose we wish to find the expected sum of the times all customers spend in the system, start-

ing at an integer time where i customers are in the system and ending at the first instant when the system becomes idle. From our discussion of Little's theorem in section 3.6, we know that this sum of times is equal to the sum of the number of customers in the system, summed over each integer time from the initial time with i customers to the final time when the system becomes empty. As in the previous example, we modify the Markov chain to make state 0 a trapping state. We take $r_i = i$ as the “reward” in state i , and v_i as the expected aggregate reward until the trapping state is entered. Using the same reasoning as in the previous example, v_i is equal to the immediate “reward” $r_i = i$ plus the expected reward from whatever state is entered next. Thus $v_i = r_i + \sum_{j \geq 1} P_{ij} v_j$. With $v_0 = 0$, this is $\mathbf{v} = \mathbf{r} + [\mathbf{P}]\mathbf{v}$. This has a unique solution for \mathbf{v} as will be shown later in lemma 2. This same analysis is valid for any choice of reward r_i for each transient state i ; the reward in the trapping state must be 0 so as to keep the expected aggregate reward finite.

EXAMPLE 3: We now try to extend the analysis of the previous examples to a Markov chain without a trapping state. Suppose that $[\mathbf{P}]$ is the transition matrix for an ergodic plus transient Markov chain of J states, that $\pi = (\pi_1, \dots, \pi_J)$ is the steady state probability vector, and that $\mathbf{r} = (r_1, \dots, r_J)$ is the reward vector. Assume that $g = \sum_i \pi_i r_i = 0$ and let v_i be the expected aggregate reward starting in state i (assuming for the moment that v_i exists in some appropriate limiting sense). The same argument as before yields $\mathbf{v} = \mathbf{r} + [\mathbf{P}]\mathbf{v}$. We show later, in lemma 2, that this equation must have a family of solutions for \mathbf{v} . In fact, since $[\mathbf{P}]\mathbf{e} = \mathbf{e}$, it can be seen that if \mathbf{v} is a solution, then, for any scalar α , $\mathbf{v} + \alpha\mathbf{e}$ is also a solution. In the previous examples, this multiplicity of solutions was resolved by a gain of zero in the trapping state, but here there is no obvious way of resolving the ambiguity. Rather than pursuing this example further, we proceed to the more general case in which the steady state expected gain per unit time is arbitrary. In this more general analysis, we look at the gain over a finite number of time units, thus providing a clean way of going to the limit.

EXAMPLE 4: Some intuitive appreciation for the general problem can be obtained from figure 4.5. Note that the chain tends to persist in whatever state it is in for a relatively long time. Thus if the chain starts in state 2, not only is an immediate reward of 1 achieved, but there is a high probability of an additional gain of 1 on many successive transitions. Thus the aggregate value of starting in state 2 is considerably more than the immediate reward of 1.

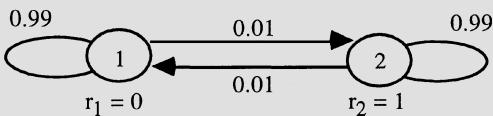


Figure 4.5. Markov chain with rewards.

Returning to the general case, it is convenient to analyze the problem by working backward from the final time rather than forward from the initial time. This will be

quite helpful later when we consider Markov decision theory. For any final time n , define *stage m* as m time units before the final time, i.e., as time $n-m$ in figure 4.6. Equivalently, we often view the final time as time 0, and then stage m corresponds to time $-m$.

0	1	2	3		$n-2$	$n-1$	n	Time
n	$n-1$	$n-2$	$n-3$		2	1	0	Stage
$-n$	$-n+1$	$-n+2$	$-n+3$		-2	-1	0	Time
n	$n-1$	$n-2$	$n-3$		2	1	0	Stage

Figure 4.6. Alternate views of Stages.

As a final generalization of the problem (which will be helpful in the solution), we allow the reward at the final time (i.e., in stage 0) to be different from that at other times. The final reward in state i is denoted $v_i(0)$, where the 0 denotes stage 0. Similarly, we denote the expected aggregate reward from stage n up to and including the final stage (stage zero), given state i at stage n , as $v_i(n)$. Note that the notation here is taking advantage of the Markov property. That is, given that the chain is in state i at time $-n$ (i.e., stage n), the expected aggregate reward up to and including time 0 is independent of the states before time $-n$ and is independent of when the Markov chain started prior to time $-n$.

The expected aggregate reward can be found by starting at stage 1. Given that the chain is in state i at time -1, the immediate reward is r_i . The chain then makes a transition (with probability P_{ij}) to some state j at time 0 with an additional reward of $v_j(0)$. Thus

$$v_i(1) = r_i + \sum_j P_{ij} v_j(0) \quad (26)$$

For the example of figure 4.5 (assuming the final reward is the same as that at the other stages, i.e., $v_i(0) = r_i$ for $i=1,2$), we have $v_2(1) = 1.99$ and $v_1(1) = 0.01$.

The expected reward for stage 2 can be calculated in the same way. Given state i at time -2 (i.e., stage 2), there is an immediate reward of r_i and with probability P_{ij} the chain goes to state j at time -1 (i.e., stage 1) with an expected additional gain of $v_j(1)$. Thus

$$v_i(2) = r_i + \sum_j P_{ij} v_j(1) \quad (27)$$

Iterating this argument to stage 3, 4, ..., n ,

$$v_i(n) = r_i + \sum_j P_{ij} v_j(n-1) \quad (28)$$

This can be written in vector form as

$$\mathbf{v}(n) = \mathbf{r} + [P]\mathbf{v}(n-1); n \geq 1 \quad (29)$$

where \mathbf{r} is a column vector with components r_1, r_2, \dots, r_J and $\mathbf{v}(n)$ is a column vector with components $v_1(n), \dots, v_J(n)$. By substituting (29), with n replaced by $n-1$, into the last term of (29),

$$\mathbf{v}(n) = \mathbf{r} + [P]\mathbf{r} + [P]^2\mathbf{v}(n-2); n \geq 2 \quad (30)$$

Applying the same substitution recursively, we eventually get an explicit expression for $\mathbf{v}(n)$,

$$\mathbf{v}(n) = \mathbf{r} + [P]\mathbf{r} + [P]^2\mathbf{r} + \dots + [P]^{n-1}\mathbf{r} + [P]^n\mathbf{v}(0) \quad (31)$$

Eq. (29), applied iteratively, is more convenient for calculating $\mathbf{v}(n)$ than (31), but neither give us much insight into the behavior of the expected aggregate reward, especially for large n . To obtain a more useful form for large n , we look at figure 4.5 again and make an educated guess about long term gain.

For stage $n \gg 1$, we would guess that the *relative* advantage of being in state 2 over state 1 would be almost independent of the final reward in stage 0, and thus the *relative* advantage between states should also be almost independent of n . Let w_2 denote this relative advantage of state 2 over state 1, let $w_1=0$, and let $\mathbf{w} = (w_1, w_2)^T$. Since there is a expected gain of $g = \sum_i \pi_i r_i$ per stage in steady state, we would hypothesize that for $n \gg 1$,

$$\mathbf{v}(n) \approx \mathbf{w} + n\mathbf{g}e + \beta\mathbf{e} \quad (32)$$

where β accounts for the effect of the final reward. We show in what follows that (32) is valid for figure 4.5, and more generally is valid for any ergodic Markov chain with rewards. Before showing this, however, we find how to calculate the *relative gain vector*, \mathbf{w} under the hypothesis in (32). From (29), $\mathbf{v}(n) = \mathbf{r} + [P]\mathbf{v}(n-1)$, and by hypothesis, $\mathbf{v}(n-1) = \mathbf{w} + (n-1)\mathbf{g}e + \beta\mathbf{e}$. Thus

$$\begin{aligned} \mathbf{v}(n) &= \mathbf{r} + [P]\mathbf{v}(n-1) = \mathbf{r} + [P]\{\mathbf{w} + (n-1)\mathbf{g}e + \beta\mathbf{e}\} = \\ &= \mathbf{r} + [P]\mathbf{w} + (n-1)\mathbf{g}e + \beta\mathbf{e} \end{aligned} \quad (33)$$

Combining this with (32), we see that $\mathbf{w} + n\mathbf{g}e = \mathbf{r} + [P]\mathbf{w}$. The following lemma shows that this equation has a solution.

LEMMA 2: Let $[P]$ be the transition matrix of a J state Markov chain containing a single recurrent class and perhaps some transient states. Let $\mathbf{r} = (r_1, \dots, r_J)$ be a reward vector, let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_J)$ be the steady state probabilities of the chain, and let $g = \sum_i \pi_i r_i$. Then the equation

$$\mathbf{w} + g\mathbf{e} = \mathbf{r} + [\mathbf{P}]\mathbf{w} \quad (34)$$

has a solution for \mathbf{w} . With the additional condition $w_1 = 0$, the solution for \mathbf{w} is unique.

Discussion: Note that (25) in example 1 is a special case of (34) for which $g = 0$, and therefore it has a unique solution. Similarly, $\mathbf{v} = \mathbf{r} + [\mathbf{P}]\mathbf{v}$ in example 2, along with $v_0 = 0$, has a unique solution.

Proof: The homogeneous equation $\mathbf{w} = [\mathbf{P}]\mathbf{w}$ is an eigenvector equation. Since $[\mathbf{P}]$ contains a single recurrent class plus transient states, theorem 6a shows that $\mathbf{w} = \mathbf{e}$ is a solution to $\mathbf{w} = [\mathbf{P}]\mathbf{w}$ and is unique within a scale factor. This means that J-1 of the equations in $\mathbf{w} = [\mathbf{P}]\mathbf{w}$ are linearly independent, and thus that J-1 of the equations in $\mathbf{w} = [\mathbf{P}]\mathbf{w} + \mathbf{r} - g\mathbf{e}$ are linearly independent. It follows that if the equations are consistent, they have a one dimensional family of solutions. Premultiplying both sides of the vector equation $\mathbf{w} = [\mathbf{P}]\mathbf{w} + \mathbf{r} - g\mathbf{e}$ by the row vector $\boldsymbol{\pi}$, (and using $\boldsymbol{\pi}[\mathbf{P}] = \boldsymbol{\pi}$ and $\boldsymbol{\pi} \mathbf{r} = g$) we get $\boldsymbol{\pi}\mathbf{w} = \boldsymbol{\pi}\mathbf{w}$, which demonstrates that the equations are consistent. Finally, since the family of solutions has the form $\mathbf{w} + \alpha\mathbf{e}$ for arbitrary choice of α , the additional constraint $w_1 = 0$ makes the solution unique.

As a matter of computational convenience, (34) can be solved without pre-computing g (which requires solving for $\boldsymbol{\pi}$). Instead, g can be taken as a variable, while taking w_1 to be 0. In this case (34) is a set of J linear equations in the J variables g, w_2, \dots, w_J , and it can be seen from the proof of the lemma that these equations have a unique solution.

We can now iterate the solution \mathbf{w} of (34) to get

$$\begin{aligned} \mathbf{w} &= \mathbf{r} - g\mathbf{e} + [\mathbf{P}]\mathbf{w} = \mathbf{r} - g\mathbf{e} + [\mathbf{P}](\mathbf{r} - g\mathbf{e} + [\mathbf{P}]\mathbf{w}) = \mathbf{r} + [\mathbf{P}]\mathbf{r} - 2g\mathbf{e} + [\mathbf{P}]^2\mathbf{w} = \\ &= \mathbf{r} + [\mathbf{P}]\mathbf{r} + [\mathbf{P}]^2\mathbf{r} - 3g\mathbf{e} + [\mathbf{P}]^3\mathbf{w} = \mathbf{r} + [\mathbf{P}]\mathbf{r} + \dots + [\mathbf{P}]^{n-1}\mathbf{r} - nge + [\mathbf{P}]^n\mathbf{w} \\ &\mathbf{w} + nge = \mathbf{r} + [\mathbf{P}]\mathbf{r} + \dots + [\mathbf{P}]^{n-1}\mathbf{r} + [\mathbf{P}]^n\mathbf{w} \end{aligned} \quad (35)$$

An explicit expression for expected aggregate gain after n stages was given in (31) as

$$\mathbf{v}(n) = \mathbf{r} + [\mathbf{P}]\mathbf{r} + [\mathbf{P}]^2\mathbf{r} + \dots + [\mathbf{P}]^{n-1}\mathbf{r} + [\mathbf{P}]^n\mathbf{v}(0)$$

Comparing this with (35), we see that if the final gain in stage 0 is taken as $\mathbf{v}(0) = \mathbf{w}$, then $\mathbf{v}(n) = \mathbf{w} + nge$. In general, (35) can be subtracted from (31) (with arbitrary $\mathbf{v}(0)$) to get

$$\mathbf{v}(n) = nge + \mathbf{w} + [\mathbf{P}]^n\{\mathbf{v}(0) - \mathbf{w}\} \quad (36)$$

This is summarized in the following theorem:

THEOREM 7: Let $[\mathbf{P}]$ be the transition matrix of a Markov chain with one recurrent class and perhaps some transient states. Let $\boldsymbol{\pi}$ be the steady state probability vector, \mathbf{r} a

reward vector, and \mathbf{w} a solution to (34) with $g = \sum_i \pi_i r_i$. Then the expected aggregate reward vector over n stages is given by (36). If the recurrent class is ergodic, then

$$\lim_{n \rightarrow \infty} \{ \mathbf{v}(n) - n\mathbf{e} \} = \mathbf{w} + \beta\mathbf{e}; \quad \beta = \sum_i \pi_i [v_i(0) - w_i] \quad (37)$$

Proof: The argument above established (36), and (37) follows from the fact that $[P]^n$ approaches a matrix each of whose rows equal π .

Note that the set of solutions to (34) has the form $\mathbf{w} + \alpha\mathbf{e}$ where \mathbf{w} is any given solution and α is any number. The undetermined factor α cancels out in (36) and (37), so any solution can be used. The *asymptotic relative gain* (or more briefly *relative gain*) vector is defined as the solution \mathbf{w} to (34) with $w_1=0$. The component w_i of \mathbf{w} is the *asymptotic relative gain* (or more briefly, *relative gain*) of starting in state i relative to state 1; because of the undetermined factor α , only the difference w_i-w_1 is relevant, which is why it is called relative gain. We include $w_1=0$ in the definition only to be specific; there is nothing special about state 1. From (37), $\lim_{n \rightarrow \infty} v_i(n) - ng = w_i + \beta$, so that $v_i(n)$ asymptotically depends on the initial state i only through w_i .

For the example in figure 4.5, $w_2-w_1 = 50$ (see exercise 4.16). The large relative gain for state 2 accounts for both the immediate reward and the high probability of multiple additional rewards through remaining in state 2. Note that w_2 is not the same as the expected reward up to the first transition from state 2 to 1. The reason for this is that the gain starting from state 1 cannot be ignored; this can be seen from figure 4.7, which modifies figure 4.5 by changing P_{12} to 1. In this case, (see exercise 4.16), $w_2-w_1 = 1/1.01 \approx 0.99$, reflecting the fact that state 1 is always left immediately, thus reducing the advantage of starting in state 2.

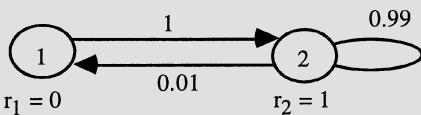


Figure 4.7. A variation of figure 4.5.

We can now interpret the general solution in (36) by viewing $g\mathbf{e}$ as the steady state gain per stage, \mathbf{w} as the dependence on the initial state, and $[P]^n \{\mathbf{v}(0) - \mathbf{w}\}$ as the final term, depending on the final reward vector $\mathbf{v}(0)$). Note that this final term is asymptotically independent of the starting state (which is why we call it a final term), but still contains the term $\beta\mathbf{e}$ of (37) in the limit. This interpretation is cleanest if \mathbf{w} is taken as the solution to (34) for which $\sum_i \pi_i w_i = 0$; in this case, the asymptotic final term is $\pi\mathbf{v}(0)$, which does not involve \mathbf{w} . We arbitrarily defined relative gain with $w_1=0$ rather than $\sum_i \pi_i w_i = 0$ for simplicity.

We now have the machinery to understand the limiting process in example 3. From (36), with $g=0$, the expected aggregate gain in stage n is $\mathbf{v}(n) = \mathbf{w} + [P]^n \{\mathbf{v}(0) - \mathbf{w}\}$. In the limit, $\lim_{n \rightarrow \infty} \mathbf{v}(n) = \mathbf{w} + \beta\mathbf{e}$ where $\beta = \sum_i \pi_i [v_i(0) - w_i]$. We see that the relative gain

of starting in each state is well defined by the solution of (34) with $w_1=0$. On the other hand, β depends on the abstraction of a final gain vector $v(0)$, which is undefined. A reasonable way to resolve the difficulty is to view $v_i(n)$ as the aggregate reward after n steps starting in state i (the original statement of the example was not precise enough to specify this). In this case, there is no final reward, i.e., $v(0) = \mathbf{0}$, so $\lim_{n \rightarrow \infty} v(n) = w + \beta e$, $\beta = -\sum_i \pi_i w_i$.

An interesting special case of example 3 is that where the reward is zero in all but the transient states and the final reward is zero. Since π_i is zero for the transient states, the steady state gain per stage g is 0 and (37) reduces to $\lim_{n \rightarrow \infty} v(n) = w - \{\sum_i \pi_i w_i\} e$. For the particular solution of (34) where $\sum_i \pi_i w_i = 0$, the solution is $\lim_{n \rightarrow \infty} v(n) = w$. Since the aggregate reward must be 0 whenever the initial state is in the ergodic class, we see that w_i must be 0 for each ergodic state i . If $r_i = 1$ for each transient state, then w becomes the expected time to leave the transient set of states. Note that w gives the expected time from each transient state to enter the ergodic set. It doesn't seem to be significantly easier to find just one component of this vector than to find the entire vector.

4.6 MARKOV DECISION THEORY AND DYNAMIC PROGRAMMING

INTRODUCTION: In the previous section, we analyzed the behavior of a Markov chain with rewards. In this section, we consider a much more elaborate structure in which a decision maker can select between various options for rewards and transition probabilities. In place of the reward r_i and the transition probabilities $\{P_{ij}; 1 \leq j \leq J\}$ associated with a given state i , there is a choice between some number K_i of different rewards, say $r_i^{(1)}, r_i^{(2)}, \dots, r_i^{(K_i)}$ and a corresponding choice between K_i different sets of transition probabilities, say $\{P_{ij}^{(1)}; 1 \leq j \leq J\}, \{P_{ij}^{(2)}; 1 \leq j \leq J\}, \dots, \{P_{ij}^{(K_i)}; 1 \leq j \leq J\}$. A decision maker then decides between these K_i selections each time the chain is in state i . Note that if the decision maker chooses selection k for state i , then the reward is $r_i^{(k)}$ and the transition probabilities from state i are $\{P_{ij}^{(k)}; 1 \leq j \leq J\}$; it is not possible to choose $r_i^{(k)}$ for one k and $\{P_{ij}^{(k)}; 1 \leq j \leq J\}$ for another k . We assume that, given $X_n = i$, and given decision k at time n , the probability of entering state j at time $n+1$ is $P_{ij}^{(k)}$, independent of earlier states and decisions.

Figure 4.8 shows an example of this situation in which the decision maker can choose between two alternatives in state 2 ($K_2=2$) and has no freedom of choice in state 1 ($K_1=1$). This figure illustrates the familiar tradeoff between instant gratification (alternative 2) and long term gratification (alternative 1).

It is also possible to consider the situation in which rewards are associated with transitions for each policy; that is, for policy k in state i , the reward $r_{ij}^{(k)}$ is associated with a transition from i to j . This means that the expected reward for a transition from i with policy k is given by $r_i^{(k)} = \sum_j P_{ij}^{(k)} r_{ij}^{(k)}$. Thus, as in the previous section, there is no essential loss in generality in restricting attention to the case in which rewards are associated with the states.

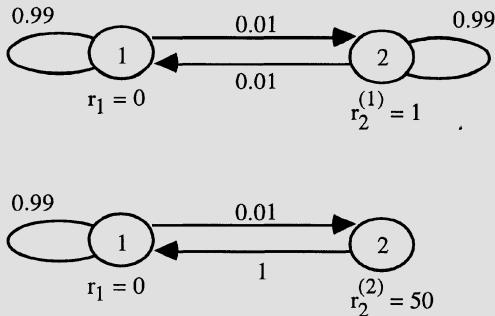


Figure 4.8. A Markov decision problem with two alternatives in state 2.

The set of rules used by the decision maker in selecting different alternatives at each stage of the chain is called a *policy*. We want to consider the expected aggregate reward over n trials of the “Markov chain,” as a function of the policy used by the decision maker. If the policy uses the same decision at each occurrence of state i , for each i , then that policy corresponds to a homogeneous Markov chain, and the aggregate reward is that found in the last section. Such a policy, i.e., making the decision for each state i independent of time, is called a *stationary policy*. Since both rewards and transition probabilities depend only on the state and the corresponding decision, and not on time, one feels intuitively that stationary policies make a certain amount of sense over a long period of time. On the other hand, assuming some final reward $v_i(0)$ for being in state i at the end of the n^{th} trial, one might expect the best policy to depend on time, at least close to the end of the n trials.

In what follows, we first derive the optimal policy for maximizing expected aggregate reward over an arbitrary number n of trials. We shall see that the decision at time m , $0 \leq m < n$, for the optimal policy does in fact depend both on m and on the final rewards $\{v_i(0); 1 \leq i \leq J\}$. We call this optimal policy the *optimal dynamic policy*. This policy is found from the dynamic programming algorithm, which, as we shall see, is really very simple. We then go on to find the relationship between the optimal dynamic policy and the optimal stationary policy and show that each has the same long term gain per trial.

DYNAMIC PROGRAMMING ALGORITHM: As in our development of Markov chains with rewards, we consider expected aggregate reward over n time periods and we use stages, counting backwards from the final trial. First consider the optimum decision with just one trial (i.e., with just one stage). We start in a given state i at stage 1, make a decision k , obtain the reward $r_i^{(k)}$, then go to state j with probability $P_{ij}^{(k)}$ and obtain the final reward $v_j(0)$. This expected aggregate reward is maximized over the choice of k , i.e.,

$$v_i^*(1) = \max_k \{ r_i^{(k)} + \sum_j P_{ij}^{(k)} v_j(0) \} \quad (38)$$

We use the notation $v_i^*(n)$ to represent the maximum expected aggregate reward for n stages starting in state i . Note that $v_i^*(1)$ depends on the final reward vector $v(0) = (v_1(0), v_2(0), \dots, v_J(0))$. Next consider the maximum expected aggregate reward starting in state i at stage 2. For each state j , $1 \leq j \leq J$, let $v_j(1)$ be the expected aggregate reward from stages 1 and 0 for some arbitrary policy, conditional on the chain being in state j at stage 1. Then if decision k is made in state i at stage 2, the expected aggregate reward for stage 2 is $r_i^{(k)} + \sum_j P_{ij}^{(k)} v_j(1)$. Note that no matter what policy is chosen at stage 2, this expression is maximized at stage 1 by choosing the stage 1 policy that maximizes $v_j(1)$. Thus, independent of what we choose at stage 2 (or at earlier times), we must use $v_j^*(1)$ for the aggregate gain from stage 1 onward in order to maximize the overall aggregate gain from stage 2. Thus, at stage 2, we achieve maximum expected aggregate gain, $v_i^*(2)$, by choosing the k that achieves the following maximum:

$$v_i^*(2) = \max_k \{ r_i^{(k)} + \sum_j P_{ij}^{(k)} v_j^*(1) \} \quad (39)$$

Repeating this argument for successively larger n , we obtain the general expression

$$v_i^*(n) = \max_k \{ r_i^{(k)} + \sum_j P_{ij}^{(k)} v_j^*(n-1) \} \quad (40)$$

Note that this is almost the same as (28), differing only by the maximization over k . We can also write this in vector form, for $n \geq 1$, as

$$v^*(n) = \max_A \{ r^A + [P^A] v^*(n-1) \} \quad (41)$$

where for $n=1$, we take $v^*(0) = v(0)$. Here A is a set (or vector) of decisions, $A = (k_1, k_2, \dots, k_J)$ where k_i is the decision to be used in state i . $[P^A]$ denotes a matrix whose (ij) element is $P_{ij}^{(k_i)}$, and r^A denotes a vector whose i^{th} element, $1 \leq i \leq J$, is $r_i^{(k_i)}$. The maximization over A in (41) is really J separate and independent maximizations, one for each state, i.e., (41) is simply a vector form of (40). Another frequently useful way to rewrite (40) or (41) is as follows:

$$\begin{aligned} v^*(n) &= r^B + [P]^B v^*(n-1) \text{ for } B \text{ such that} \\ r^B + [P]^B v^*(n-1) &\geq r^A + [P]^A v^*(n-1) \text{ for all policies } A \end{aligned} \quad (42)$$

If B satisfies (42), it is called an *optimal dynamic policy* for stage n . Note that (40), (41), and (42) are valid with no restrictions (such as recurrent or aperiodic states) on the transition probabilities $[P^A]$ for the various policies.

The *dynamic programming algorithm* is just the calculation of (40), (41), or (42), performed successively for $n=1, 2, 3, \dots$. The development of this algorithm, as a

systematic tool for solving this class of problems, is due to Bellman [Bel57]. This algorithm yields the optimal dynamic policy for any given final reward vector, $\mathbf{v}(0)$. Along with the calculation of $\mathbf{v}^*(n)$ for each n , the algorithm also yields the optimal set of decisions at each stage. The surprising simplicity of the algorithm is due to the Markov property. That is, $v_i^*(n)$ is the aggregate present and future reward conditional on the present state. Since it is conditioned on the present state, it is independent of the past (i.e., how the process arrived at state i from previous transitions and choices).

Although dynamic programming is computationally straightforward and convenient, the asymptotic behavior of $\mathbf{v}^*(n)$ as $n \rightarrow \infty$ is not evident from the algorithm. After working out a simple example, we look at the general question of asymptotic behavior.

Consider the example of figure 4.8 with the final rewards $v_2(0) = v_1(0) = 0$. The expected aggregate reward for stage 1, given state 1, is then 0 (i.e., using (38) with $r_1=0$ and $v_j(0)=0$ for $j = 1, 2$, we get $v_1^*(1)=0$). Alternatively, given state 2 at stage 1, there is a choice between decision 1 and decision 2. Decision 1 yields an immediate reward of 1, whereas decision 2 yields an immediate reward of 50. Since there is no final reward at stage 0, decision 2 maximizes the reward and $v_2^*(1)=50$. For stage 2, using (39), we have

$$\begin{aligned} v_1^*(2) &= r_1 + P_{11}v_1^*(1) + P_{12}v_2^*(1) = 0 + 0 + (0.01)(50) = 0.5 \\ v_2^*(2) &= \max \left\{ r_2^{(1)} + P_{22}^{(1)}v_2^*(1) + P_{21}^{(1)}v_1^*(1), r_2^{(2)} + P_{21}^{(2)}v_1^*(1) \right\} \\ &= \max \{ 1 + (0.99)(50) + 0, 50 + 0 \} = 50.5 \end{aligned}$$

Thus, in stage 2, decision 1 is best in state 2. What is happening is that the choice of decision 2 at stage 1 has made it very profitable to be in state 2 at stage 1. Thus if the chain is in state 2 at stage 2, it is preferable to choose the small gain (1) at stage 2 with the corresponding high probability of remaining in state 2 at stage 1. Continuing this computation for larger n , one finds that $v_1^*(n) = n/2$ and $v_2^*(n) = 50 + n/2$. The optimum dynamic policy is decision 2 for stage 1 and decision 1 for all stages $n>1$.

THE OPTIMAL STATIONARY POLICY: We assume throughout this section that for all stationary policies \mathbf{A} , the Markov chain with transitions $[P^{\mathbf{A}}]$ is recurrent. Although most of our previous development allows transient states along with a single recurrent class, it appears preferable here to treat situations with transient states on a case by case basis. The expected aggregate reward for a single recurrent class was determined in (36) to be

$$\mathbf{v}^{\mathbf{A}}(n) = ng^{\mathbf{A}}\mathbf{e} + \mathbf{w}^{\mathbf{A}} + [P^{\mathbf{A}}]^n\{\mathbf{v}(0)-\mathbf{w}^{\mathbf{A}}\} \quad (43)$$

where $g^{\mathbf{A}} = \sum_i \pi_i^{\mathbf{A}} r_i^{\mathbf{A}}$ is the steady state gain, $\pi^{\mathbf{A}}$ is the steady state probability vector for the Markov chain with transition probabilities $[P^{\mathbf{A}}]$, and $\mathbf{w}^{\mathbf{A}}$ satisfies

$$\mathbf{w}^A + g^A \mathbf{e} = \mathbf{r}^A + [P^A] \mathbf{w}^A \quad (44)$$

The relative gain vector for policy A is the solution \mathbf{w}^A to (44) for which $w_1^A = 0$. Recall that if \mathbf{w}^A satisfies (44), then $\mathbf{w}^A + \alpha \mathbf{e}$ also satisfies it for any α , but the term $\alpha \mathbf{e}$ cancels out in (43).

We define an *optimal stationary policy* to be a policy A with maximum g^A . If the A that maximizes g^A is unique, then for sufficiently large n , A will maximize the expected aggregate reward (over stationary policies) for each starting state i . If the A that maximizes g^A is not unique, then one maximizing policy might have a small advantage over another because of the final reward vector, but we still define any policy that maximizes g^A as an optimal stationary policy. The effect of multiple solutions to the maximization of g^A is considered in exercises 4.20 and 4.23.

The optimal stationary policy can be found simply by calculating g^A for each policy (there are $K_1 K_2 \dots K_J$ of them), but in what follows, we develop another way to calculate the optimal g^A that is often more attractive computationally. Our major objective in what follows, however, is to understand the asymptotic behavior of the optimal dynamic policy and see how it relates to the optimal stationary policy.

To motivate the following development, we hypothesize that the optimal dynamic policy becomes identical to the optimal stationary policy B for large n , while perhaps using other decisions for n less than some value m . If policy B is used for all stages $n \geq m$, and if the Markov chain corresponding to $[P^B]$ is ergodic, then, from (37), the difference between $v_i^*(n)$ and $v_i^*(m)$ for $n >> m$ becomes close to $w_i^B - w_1^B$. Thus, we would expect that one of the properties of the optimal stationary policy B is that the optimal dynamic policy at any stage $n+1$ should be B if $v^*(n) = w^B + \alpha \mathbf{e}$ for some α . In particular, as indicated by (42), if $v^*(0) = w^B$, the dynamic algorithm would choose B at stage 1 if

$$\mathbf{r}^B + [P^B] \mathbf{w}^B \geq \mathbf{r}^A + [P^A] \mathbf{w}^B \quad \text{for all policies } A \quad (45)$$

Note that this is a vector inequality and that \mathbf{w}^B appears on both sides. It must be satisfied for all components of (45). If we let (k_1, k_2, \dots, k_J) denote the policy B , then (45), written out by components, is

$$r_i^{(k_j)} + \sum_j P_{ij}^{(k_j)} w_j^B \geq r_i^{(k)} + \sum_j P_{ij}^{(k)} w_j^B \quad \text{for } 1 \leq k \leq K_i, 1 \leq i \leq J. \quad (46)$$

LEMMA 3: If $v^*(0) = w^B$ for some policy B and if (45) is satisfied, then policy B is an optimal dynamic policy at each stage and the optimal gain satisfies $v^*(n) = w^B + n g^B \mathbf{e}$.

Proof: By assumption, $v^*(0) = w^B$ and (45) is satisfied, so $\mathbf{r}^B + [P^B] v^*(0) \geq \mathbf{r}^A + [P^A] v^*(0)$ for all A . From (42), B is then an optimal dynamic policy for stage 1, and

$$v^*(1) = \mathbf{r}^B + [P^B] \mathbf{w}^B = \mathbf{w}^B + g^B \mathbf{e}$$

where we have used the definition of relative gain in (44). Substituting $\mathbf{v}^*(1) - g^B \mathbf{e}$ for \mathbf{w}^B in (45), we get

$$\mathbf{r}^B + [P^B]\mathbf{v}^*(1) - g^B[P^B]\mathbf{e} \geq \mathbf{r}^A + [P^A]\mathbf{v}^*(1) - g^B[P^A]\mathbf{e} \quad \text{for all } A$$

The last term on each side cancels since $[P^B]\mathbf{e} = [P^A]\mathbf{e} = \mathbf{e}$, so

$$\mathbf{r}^B + [P^B]\mathbf{v}^*(1) \geq \mathbf{r}^A + [P^A]\mathbf{v}^*(1) \quad \text{for all } A$$

Thus, from (42), \mathbf{B} is an optimal policy for stage 2 and

$$\begin{aligned} \mathbf{v}^*(2) &= \mathbf{r}^B + [P]^B\mathbf{v}^*(1) = \mathbf{r}^B + [P]^B[\mathbf{w}^B + g^B \mathbf{e}] \\ &= \mathbf{r}^B + [P]^B\mathbf{w}^B + g^B \mathbf{e} = \mathbf{w}^B + 2g^B \mathbf{e} \quad (\text{from (44)}) \end{aligned}$$

This same argument repeats for all n , yielding

$$\mathbf{v}^*(n) = \mathbf{w}^B + ng^B \mathbf{e} \tag{47}$$

completing the proof.

The following theorem carries this one step further by showing that (45) must be satisfied if \mathbf{B} is an optimal stationary policy.

THEOREM 8: Assume that for each policy A , the Markov chain $[P^A]$ is recurrent. Then the stationary policy \mathbf{B} is an optimal stationary policy iff (45) is satisfied. Furthermore, if \mathbf{w}^B is the final reward vector and policy \mathbf{B} is an optimal stationary policy, then \mathbf{B} is also an optimal dynamic policy at each stage.

Proof: *If:* First, assume that (45) is satisfied. From lemma 3, if the final reward vector is \mathbf{w}^B , then \mathbf{B} gives the optimal dynamic policy at each stage. If $g^A > g^B$ for some A , then the stationary policy A would yield a larger aggregate expected reward than policy \mathbf{B} for large enough n , contradicting the demonstration that \mathbf{B} is the optimal dynamic policy at each stage. Thus (45) implies that $g^A \leq g^B$ for all A .

Only if: Assume (45) is unsatisfied for some A . Letting $\mathbf{B} = (k_1, k_2, \dots, k_J)$, we see from (46) that there must be at least one i and one k for which

$$r_i^{(k_i)} + \sum_j P_{ij}^{(k_i)} w_j^B < r_i^{(k)} + \sum_j P_{ij}^{(k)} w_j^B \tag{48}$$

Choose a decision $A = (k'_1, k'_2, \dots, k'_J)$ in which for each i , $1 \leq i \leq J$, k'_i is the value of k that maximizes $r_i^{(k)} + \sum_j P_{ij}^{(k)} w_j^B$. In some cases, k'_i may be the same as k_i , but there is at least one value of i for which (48) is satisfied, and for all i ,

$$r_i^{(k_i)} + \sum_j P_{ij}^{(k_i)} w_j^{k_i} \leq r_i^{(k'_i)} + \sum_j P_{ij}^{(k'_i)} w_j^{k_i} \quad (49)$$

This means that, for the \mathbf{A} just chosen,

$$\mathbf{r}^{\mathbf{B}} + [\mathbf{P}^{\mathbf{B}}]\mathbf{w}^{\mathbf{B}} \leq, \neq \mathbf{r}^{\mathbf{A}} + [\mathbf{P}^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}} \quad (50)$$

where \leq, \neq means that the inequality is strict for at least one component of the vectors. Recall from (44) that $\mathbf{w}^{\mathbf{B}} + g^{\mathbf{B}}\mathbf{e} = \mathbf{r}^{\mathbf{B}} + [\mathbf{P}^{\mathbf{B}}]\mathbf{w}^{\mathbf{B}}$. Thus (50) becomes

$$\mathbf{w}^{\mathbf{B}} + g^{\mathbf{B}}\mathbf{e} \leq, \neq \mathbf{r}^{\mathbf{A}} + [\mathbf{P}^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}} \quad (51)$$

Since $\pi_i^{\mathbf{A}} > 0$ for each i , we can pre-multiply both sides of (51) by $\pi^{\mathbf{A}}$ to get

$$\pi^{\mathbf{A}}\mathbf{w}^{\mathbf{B}} + g^{\mathbf{B}}\pi^{\mathbf{A}}\mathbf{e} < \pi^{\mathbf{A}}\mathbf{r}^{\mathbf{A}} + \pi^{\mathbf{A}}[\mathbf{P}^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}} \quad \text{for the chosen } \mathbf{A}. \quad (52)$$

Cancelling the first term on the left with the second term on the right (since $\pi^{\mathbf{A}}[\mathbf{P}^{\mathbf{A}}] = \pi^{\mathbf{A}}$), we are left with $g^{\mathbf{B}} < g^{\mathbf{A}}$; thus \mathbf{B} is not an optimal stationary policy if (45) is unsatisfied, completing the proof. (For the algebraically inclined reader, the argument in (50) to (52), with inequalities reversed, can also be used to show that (45) implies that $g^{\mathbf{B}} \geq g^{\mathbf{A}}$; see exercise 4.23).

Theorem 8 suggests an algorithm for finding the optimal stationary policy: start with an arbitrary policy \mathbf{B} , calculate $\mathbf{w}^{\mathbf{B}}$, and then use (46) to find if there is a better policy. If so, repeat the above procedure with the newly found policy, and if not, the current policy is optimal. This algorithm is due to Howard [How60] and is called the policy improvement algorithm. More precisely, the algorithm is as follows.

HOWARD'S POLICY IMPROVEMENT ALGORITHM:

- 1) Choose an arbitrary policy \mathbf{B}
- 2) Calculate $\mathbf{w}^{\mathbf{B}}$ from Eq. (44).
- 3) If $\mathbf{r}^{\mathbf{B}} + [\mathbf{P}^{\mathbf{B}}]\mathbf{w}^{\mathbf{B}} \geq \mathbf{r}^{\mathbf{A}} + [\mathbf{P}^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}}$ for all \mathbf{A} , then stop— \mathbf{B} is optimal.
- 4) Otherwise, find \mathbf{A} such that $\mathbf{r}^{\mathbf{B}} + [\mathbf{P}^{\mathbf{B}}]\mathbf{w}^{\mathbf{B}} \leq, \neq \mathbf{r}^{\mathbf{A}} + [\mathbf{P}^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}}$.
- 5) Update policy \mathbf{B} to be the new policy \mathbf{A} and goto step 2.

There is some freedom in the algorithm about which policy \mathbf{A} to choose in step (4). The most reasonable approach is, for each state i , to choose the i^{th} component of \mathbf{A} to be some k that maximizes $r_i^{(k)} + \sum_j P_{ij}^{(k)} w_j^{k_B}$, but any \mathbf{A} that satisfies (50) is permissible. There is no assurance that any particular approach will achieve the maximum gain $g^{\mathbf{A}}$ in one step. If there were such an assurance, then there would be no need to iterate the algorithm more than once. Since there are a finite number of policies, however, and each iteration finds an improvement, the algorithm must terminate after a finite number of iterations.

In order to see the connection between the optimal dynamic policy and the optimal stationary policy, recall that if the final reward vector $\mathbf{v}(0)$ happens to be \mathbf{w}^B where B is an optimal stationary policy, then the optimal stationary and dynamic policies are equal and (from (47)) are given by $\mathbf{v}^*(n) = n\mathbf{g}^B \mathbf{e} + \mathbf{w}^B$. If \mathbf{w}^B is a solution to (44), then $\mathbf{w}^B + \gamma\mathbf{e}$ is also a solution, so we have

$$\mathbf{v}^*(n) = (n\mathbf{g}^B + \gamma)\mathbf{e} + \mathbf{w}^B \text{ for } \mathbf{v}(0) = \mathbf{w}^B + \gamma\mathbf{e} \quad (53)$$

We use this equation to find an upper bound on $\mathbf{v}^*(n)$ for an arbitrary final reward vector by using an important property of dynamic programming called the *monotonicity property*. This says that any increase in the final reward vector increases the expected aggregate reward for each stage. More precisely, the monotonicity property is given in the following lemma:

LEMMA 4: Let $\mathbf{v}(0)$ and $\mathbf{v}'(0)$ be final reward vectors satisfying $\mathbf{v}(0) \leq \mathbf{v}'(0)$. Then, for any stationary policy A , the corresponding expected aggregate reward vectors, $\mathbf{v}^A(n)$ and $\mathbf{v}'^A(n)$ at each stage n satisfy $\mathbf{v}^A(n) \leq \mathbf{v}'^A(n)$. Similarly, for the optimal dynamic policy, the corresponding optimal expected aggregate reward vectors $\mathbf{v}^*(n)$ and $\mathbf{v}'^*(n)$ at each stage n satisfy $\mathbf{v}^*(n) \leq \mathbf{v}'^*(n)$.

The proof of this follows by iteration (or induction) from (29) and (41) (see exercise 4.18). To use this lemma, let $\mathbf{v}(0)$ be an arbitrary final reward vector, let B be an optimal stationary policy, and let \mathbf{w}^B satisfy (44) with $[B]$ in place of $[A]$. We then choose γ to be the smallest number such that $\mathbf{v}(0) \leq \mathbf{w}^B + \gamma\mathbf{e}$. Applying the lemma with $\mathbf{v}'(0) = \mathbf{w}^B + \gamma\mathbf{e}$ and using (53) for $\mathbf{v}'(0)$ and $\mathbf{v}'^*(n)$, we have

$$\mathbf{v}^*(n) \leq (n\mathbf{g}^B + \gamma)\mathbf{e} + \mathbf{w}^B \text{ for } \mathbf{v}(0) \leq \mathbf{w}^B + \gamma\mathbf{e} \quad (54)$$

This tells us that the gain per stage of the optimal dynamic algorithm can be no more than the gain per stage \mathbf{g}^B of the optimal stationary algorithm; also, by comparing (54) with (43) (which gives the expected aggregate reward for a stationary policy), we get a lower bound on $\mathbf{v}^*(n)$ (since the optimal dynamic policy must be at least as good as the optimal stationary policy). We want to go a little further, however, and find an asymptotic expression for $\mathbf{v}^*(n)$ and find the conditions under which the dynamic policy becomes equal to B for sufficiently large n . As a prelude, we need the following lemma:

LEMMA 5: For the optimal stationary policy B , the function $f(n) = \pi^B \mathbf{v}^*(n) - n\mathbf{g}^B$ is monotonic non-decreasing in n and has some limit β' .

Proof: From (41),

$$\begin{aligned} \mathbf{v}^*(n) &= \max_A \{ \mathbf{r}^A + [P^A] \mathbf{v}^*(n-1) \} \geq \mathbf{r}^B + [P^B] \mathbf{v}^*(n-1) \\ \pi^B \mathbf{v}^*(n) &\geq \pi^B \mathbf{r}^B + \pi^B [P^B] \mathbf{v}^*(n-1) = \mathbf{g}^B + \pi^B \mathbf{v}^*(n-1) \end{aligned} \quad (55)$$

Subtracting $n\mathbf{g}^B$ from each side,

$$\pi^B v^*(n) - ng^B \geq \pi^B v^*(n-1) - (n-1)g^B \quad (56)$$

This establishes the monotonicity, and (54) implies that $f(n)$ has an upper bound. Thus $\lim_{n \rightarrow \infty} f(n)$ exists, completing the proof.

Unfortunately, there is no known way to calculate the limit β' in the lemma other than the brute force approach of calculating $v^*(n)$ from the Bellman algorithm for successively larger values of n . The lemma leads, however, to the following theorem, which is a special case of [ScF67]. This gives us the asymptotic form of $v^*(n)$ directly in terms of β' as defined in lemma 5.

THEOREM 9: Assume that the Markov chain for $[P]^A$ is recurrent for all policies A ; assume that B is an optimal stationary policy and that the Markov chain for $[P]^B$ is ergodic. Then

$$\lim_{n \rightarrow \infty} v^*(n) - ng^B e = w^B + (\beta' - \pi^B w^B) e \quad (57)$$

where β' is the constant in lemma 5.

Comments: This theorem says that the optimum dynamic policy has both the same gain per stage as the optimum stationary policy and the same relative gain vector, w^B . The optimum dynamic policy might have a larger asymptotic aggregate gain than the optimum stationary policy, this extra gain being independent of the starting state and equal to $\beta' - \pi^B v(0)$.

Proof*: Define $v^m(n)$ (for $m \leq n$) as the expected aggregate gain resulting from using the optimum dynamic policy for the first m stages and the fixed policy B for all subsequent stages up to n . It can be seen that $v^n(n) = v^*(n)$ and $v^0(n) = v^B(n)$. Eq. (55), in the notation here, asserts that $v^m(m) \geq v^{m-1}(m)$ for $1 \leq m \leq n$. Combining this with lemma 4 applied to the use of B on the last $n-m$ stages, we see that $v^m(n) \geq v^{m-1}(n)$. Since this is valid for $1 \leq m \leq n$, we have $v^n(n) \geq v^{n-1}(n) \geq \dots \geq v^1(n) \geq v^0(n)$. Note that $v^m(n)$ can be found in terms of $v^*(m)$ by applying the result in (43) on the stationary policy B for $n-m$ stages, using $v^*(m)$ as the final reward vector. Thus

$$v^m(n) = (n-m)g^B e + w^B + [P^B]^{n-m}\{v^*(m) - w^B\} \quad (58)$$

For any $\epsilon > 0$, lemma 5 asserts that for all m greater than some $m(\epsilon)$,

$$\pi^B v^*(m) \geq mg^B + \beta' - \epsilon \quad (59)$$

For any such m , the ergodicity of $[P^B]$ asserts that for all large enough $n-m$,

$$\begin{aligned} [P^B]^{n-m}\{v^*(m) - w^B\} &\geq e\pi^B\{v^*(m) - w^B\} - ee = [\pi^B\{v^*(m) - w^B\} - \epsilon]e \\ &\geq [mg^B + \beta' - \epsilon - \pi^B w^B - \epsilon]e \end{aligned} \quad (60)$$

where we have used (59). Substituting (60) into (58), with m and $n-m$ large enough,

$$v^m(n) \geq (ng^B + \beta' - 2\epsilon - \pi^B w^B) e + w^B$$

Since $v^*(n) \geq v^m(n)$,

$$v_i^*(n) \geq ng^B + \beta' - 2\epsilon - \pi^B w^B + w_i^B \quad (61)$$

for all i and large enough n . We next show, by contradiction, that for any n satisfying (61),

$$\frac{v_i^*(n)}{\pi_i} \leq ng^B + \beta' + \frac{2\epsilon}{\pi_i^B} - \pi^B w^B + w_i^B \quad \text{for all } i \quad (62)$$

If this is violated for some j , then

$$\begin{aligned} v_j^*(n) &> ng^B + \beta' + \frac{2\epsilon}{\pi_j^B} - \pi^B w^B + w_j^B \\ v_i^*(n) &\geq ng^B + \beta' - 2\epsilon - \pi^B w^B + w_i^B \quad \text{for } i \neq j \end{aligned}$$

Since $\pi_i > 0$ for all i , we can multiply by π_i^B and sum over i to get

$$\sum_i \pi_i^B v_i^*(n) > ng^B + \beta'$$

which contradicts lemma 5, thus establishing (62). Since $\epsilon > 0$ is arbitrary, (61) and (62) assert that $\lim_{n \rightarrow \infty} v_i^*(n) = ng^B + \beta' - \pi^B w^B + w_i^B$, completing the proof.

One of the curious consequences of this theorem is that it implies that if the optimum stationary policy is non-unique, then each of these optimum policies has the same relative gain vector w^B (this follows since the left side of (57) is independent of B , except for g^B which is the same for all optimum stationary policies). As shown in exercise 4.23, this can also be derived directly from (45). If the optimum stationary policy is non-unique, then it is possible for the optimum dynamic policy to alternate infinitely often between these different policies, but of course the benefit to be derived from this alternation approaches 0. If the optimum stationary policy B is unique, then there is some n_0 such that the optimal dynamic policy is B for all $n \geq n_0$ (see exercise 4.27).

We now return to the question of calculating the optimal gain per stage, g^B . Either the Howard algorithm or the Bellman algorithm can be used. If the Bellman algorithm is used, however, there is a question of estimating g^B from the gain at each stage. The following upper and lower bounds on g^B are due to Odoni, [Odo69], and are established in exercise 4.25.

$$\min_i [v_i^*(n) - v_i^*(n-1)] \leq g^B \leq \max_i [v_i^*(n) - v_i^*(n-1)] \quad (63)$$

These bound the error if g^B is estimated from the n^{th} stage of the Bellman algorithm. The lower bound is non-decreasing in n , the upper bound is non-increasing, and both converge to g^B with increasing n .

In comparing the use of the Bellman and Howard algorithms for calculating g^B , it should be noted that the iterations are fairly similar (at least if A is chosen to maximize each component of the inequality in step 4 of the Howard algorithm). The difference is that the Howard algorithm calculates a new relative gain vector w^B at each iteration, whereas the Bellman algorithm uses the previous value of expected aggregate gain in place of the relative gain. The Howard algorithm requires on the order of J^3 steps to find w^B (i.e., the computation required to solve J simultaneous linear equations) and requires on the order of $J(\sum_i K_i)$ steps for the check in step 2 and the maximization in step 3. The Bellman algorithm requires on the order of $J(\sum_i K_i)$ steps for the entire iteration. Thus if there are many decision alternatives and few states, the computation per iteration of the two algorithms is similar, whereas with many states and few alternatives, the Howard algorithm requires much more computation per iteration. Naturally, this does not help in seeing how many iterations are required with each algorithm, and there seems to be no easy way to answer this question.

Another related question is how the Howard algorithm compares with the brute force method of calculating g^A for every policy A . Finding g^A for a policy requires finding the steady state probability vector π^A for $[P^A]$, which is of comparable complexity to finding w^B . The number of different policies is $K_1 K_2 \dots K_J$, which is the number of times that a steady state probability must be calculated in the brute force method. In the Howard algorithm, on the other hand, each iteration yields a better policy than the one before. If one assumes (with little real justification) that the improved policy at each iteration is a random equiprobable choice among all possible improved algorithms, then it turns out (see [Ros83] section 4.6) that the expected number of required iterations is approximately equal to the natural log of the total number of policies.

4.7 SUMMARY

This chapter has developed the basic results about finite state Markov chains from a primarily algebraic standpoint. It was shown that the states of any finite state chain can be partitioned into classes, where each class is either transient or recurrent, and each class is periodic or aperiodic. If the entire chain is one recurrent class, then the Frobenius theorem, with all its corollaries, shows that $\lambda=1$ is an eigenvalue of largest magnitude and has positive right and left eigenvectors, unique within a scale factor. The left eigenvector (scaled to be a probability vector) is the steady state probability vector. If the chain is also aperiodic, then the eigenvalue $\lambda=1$ is the only eigenvalue of magnitude 1, and all rows of $[P]^n$ converge geometrically in n to the steady state vector. This same analysis can be applied to each aperiodic recurrent class of a general Markov chain, given that the chain ever enters that class.

For a periodic recurrent chain of period d , there are $d-1$ other eigenvalues of magnitude 1, with all d eigenvalues uniformly placed around the unit circle in the complex plane. Exercise 4.13 shows how to interpret these eigenvectors, and shows that $[P]^{\text{nd}}$ converges geometrically as $n \rightarrow \infty$.

For an arbitrary finite state Markov chain, if the initial state is transient, then the Markov chain will eventually enter a recurrent state, and the probability that this takes more than n steps approaches zero geometrically in n ; exercise 4.10 shows how to find the probability that each recurrent class is entered. Given an entry into a particular recurrent class, then the results above can be used to analyze the behavior within that class.

The results about Markov chains were extended to Markov chains with rewards. As with renewal processes, the use of reward functions provides a systematic way to approach a large class of problems ranging from first passage times to dynamic programming. The key result here is theorem 5, which provides both an exact expression and an asymptotic expression for the expected aggregate reward over n stages.

Finally, the results on Markov chains with rewards were used to approach Markov decision theory. We developed the Bellman dynamic programming algorithm, and also investigated the optimal stationary policy. Theorem 9 demonstrated the relationship between the optimal dynamic policy and the optimal stationary policy. This section provided only an introduction to dynamic programming. We omitted all discussion of the relation between optimal stationary and dynamic policies when the stationary chains contain transients and multiple recurrent classes; it appears that these situations are best treated on a case by case basis. Also we omitted discounting (in which future gain is considered worth less than present gain because of interest rates), and we omitted infinite state spaces.

For an introduction to vectors, matrices, and linear algebra, see any introductory text on linear algebra such as Strang [Str88]. Gantmacher [Gan59] has a particularly complete treatment of non-negative matrices and Perron-Frobenius theory. For further reading on Markov decision theory and dynamic programming, see Bertsekas, [Ber87]. Howard, [How60] and Bellman [Bel57] are of historic interest and provide very accessible introductory material.

EXERCISES

4.1 a) Prove that, for a finite state Markov chain, if $P_{ii} > 0$ for some i in a recurrent class A, then class A is aperiodic.

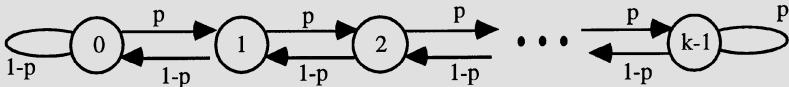
b) Show that every finite state Markov chain contains at least one recurrent set of states. Hint: Construct a directed graph in which the states are nodes and an edge goes from i to j if $i \rightarrow j$ but i is not accessible from j . Show that this graph contains no cycles, and thus contains one or more nodes with no outgoing edges. Show that each such node is in a recurrent class. Note: this result is not true for Markov chains with countably infinite state spaces.

4.2) A transition probability matrix P is said to be doubly stochastic if

$$\sum_j P_{ij} = 1 \text{ for all } i; \quad \sum_i P_{ij} = 1 \text{ for all } j$$

That is, both the row and the column sums each equal 1. If a doubly stochastic chain has J states and is ergodic (i.e., has a single class of states and is aperiodic), calculate its steady state probabilities.

- 4.3) a)** Find the steady state probabilities π_0, \dots, π_{k-1} for the Markov chain below. Express your answer in terms of the ratio $p = p/q$. Pay particular attention to the special case $p=1$.
- b) Sketch π_0, \dots, π_{k-1} . Give one sketch for $p=1/2$, one for $p=1$, and one for $p=2$.
- c) Find the limit of π_0 as k approaches ∞ ; give separate answers for $p < 1$, $p = 1$, and $p > 1$. Find limiting values of π_{k-1} for the same cases.



- 4.4) Answer each of the following questions for each of the following non-negative matrices [A]**

$$\text{i) } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{ii) } \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

- a) Find $[A]^n$ in closed form for arbitrary $n > 1$.
- b) Find all eigenvalues and all right eigenvectors of $[A]$.
- c) Use (b) to show that there is no diagonal matrix $[\Lambda]$ and no invertible matrix $[Q]$ for which $[A][Q] = [Q][\Lambda]$.
- d) Rederive the result of part (c) using the result of (a) rather than (b).

- 4.5) a)** Find the steady state probabilities for each of the Markov chains in figure 4.2 of section 4.1. Assume that all clockwise probabilities in the first graph are the same, say p , and assume that $P_{4,5} = P_{4,1}$ in the second graph.

- b) Find the matrices $[P]^2$ for the same chains. Draw the graphs for the Markov chains represented by $[P]^2$, i.e., the graph of two step transitions for the original chains. Find the steady state probabilities for these two step chains. Explain why your steady state probabilities are not unique.
- c) Find $\lim_{n \rightarrow \infty} [P]^{2n}$ for each of the chains.

- 4.6) Show that the graph for an ergodic Markov chain of J states must contain at least one cycle with $t \leq J-1$ nodes. Show that, for any $J \geq 3$, there is an ergodic Markov chain for which the graph consists of exactly one cycle of length J and one cycle of length $J-1$. Show that, for this chain, $P_{ij}^n = 0$ for some i, j , and for $n = (J-1)^2$. The point of this problem is to show that theorem 3 is relatively tight.**

4.7 a) Show that if x_1 and x_2 are real or complex numbers, then $|x_1+x_2| = |x_1|+|x_2|$ implies that for some β , βx_1 , and βx_2 are both real and non-negative.

b) Show from this that if the inequality in (17) is satisfied with equality, then there is some β for which $\beta x_i = |x_i|$ for all i .

4.8 a) Let λ be an eigenvalue of a matrix $[A]$, and let v and π be right and left eigenvectors respectively of λ , normalized so that $\pi v = 1$. Show that

$$[[A] - \lambda v \pi]^2 = [A]^2 - \lambda^2 v \pi.$$

b) Show that $[[A]^n - \lambda^n v \pi] [[A] - \lambda v \pi] = [A]^{n+1} - \lambda^{n+1} v \pi$.

c) Use induction to show that $[[A] - \lambda v \pi]^n = [A]^n - \lambda^n v \pi$.

4.9 Let $[P]$ be the transition matrix for a Markov chain with one recurrent class of states and one or more transient classes. Suppose there are J recurrent states, numbered 1 to J , and K transient states, $J+1$ to $J+K$. Thus $[P]$ can be partitioned as $[P] = \begin{bmatrix} P_r & 0 \\ P_{tr} & P_{tt} \end{bmatrix}$.

a) Show that $[P]^n$ can be partitioned as $[P]^n = \begin{bmatrix} [P_r]^n & [0] \\ [P_{ij}^n] & [P_{tt}]^n \end{bmatrix}$. That is, the blocks

on the diagonal are simply products of the corresponding blocks of $[P]$, and the lower left block is whatever it turns out to be.

b) Let Q_i be the probability that the chain will be in a recurrent state after K transitions, starting from state i , i.e., $Q_i = \sum_{j \in J} P_{ij}^K$. Show that $Q_i > 0$ for all transient i .

c) Let Q be the minimum Q_i over all transient i and show that $P_{ij}^{nK} \leq (1-Q)^n$ for all transient i, j (i.e., show that $[P_{tr}]^n$ approaches the all zero matrix $[0]$ with increasing n).

d) Let $\pi = (\pi_r, \pi_t)$ be a left eigenvector of $[P]$ of eigenvalue 1 (if one exists). Show that $\pi_t = 0$ and show that π_r must be positive and be a left eigenvector of $[P_r]$. Thus show that π exists and is unique (within a scale factor).

e) Show that e is the unique right eigenvector of $[P]$ of eigenvalue 1 (within a scale factor).

4.10 Generalize exercise 4.9 to the case of a Markov chain $[P]$ with r recurrent classes and one or more transient classes. In particular,

a) Show that $[P]$ has exactly r linearly independent left eigenvectors, $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(r)}$ of eigenvalue 1, and that the i^{th} can be taken as a probability vector that is positive on the i^{th} recurrent class and zero elsewhere.

b) Show that $[P]$ has exactly r linearly independent right eigenvectors, $v^{(1)}, v^{(2)}, \dots, v^{(r)}$ of eigenvalue 1, and that the i^{th} can be taken as a vector with $v_j^{(i)}$ equal to the probability that recurrent class i will ever be entered starting from state j .

4.11 Prove theorem 6A. Hint: Use theorem 6 and the results of exercise 4.9.

4.12) Generalize exercise 4.11 to the case of a Markov chain $[P]$ with r aperiodic recurrent classes and one or more transient classes. In particular, using the left and right eigenvectors $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(r)}$ and $v^{(1)}, \dots, v^{(r)}$ of exercise 4.10, show that

$$\lim_{n \rightarrow \infty} [P]^n = \sum_i v^{(i)} \pi^{(i)}.$$

4.13) Suppose a Markov chain with matrix $[P]$ is irreducible and periodic with period d and let T_i , $1 \leq i \leq d$, be the i^{th} subclass in the sense of theorem 2. Assume the states are numbered so that the first J_1 states are in T_1 , the next J_2 are in T_2 , and so forth. Thus $[P]$ has the block form given by

$$[P] = \begin{bmatrix} 0 & [P_1] & \ddots & 0 \\ 0 & 0 & [P_2] & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & [P_{d-1}] \\ [P_d] & 0 & \ddots & \ddots & 0 \end{bmatrix}$$

where $[P_i]$ has dimension J_i by $J_{(i+1)}$ for $i < d$ and J_d by J_1 for $i = d$.

a) Show that $[P]^d$ has the form

$$[P]^d = \begin{bmatrix} [Q_1] & 0 & \ddots & 0 \\ 0 & [Q_2] & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & [Q_d] \end{bmatrix}$$

where $[Q_i] = [P_1][P_{i+1}] \dots [P_d][P_1] \dots [P_{i-1}]$.

b) Show that $[Q_i]$ is the matrix of an ergodic Markov chain, so that with the eigenvectors defined in exercises 4.10 and 4.12, $\lim_{n \rightarrow \infty} [P]^nd = \sum_i v^{(i)} \pi^{(i)}$.

c) Show that $\hat{\pi}^{(i)}$, the left eigenvector of $[Q_i]$ of eigenvalue 1 satisfies $\hat{\pi}^{(i)}[P_i] = \hat{\pi}^{(i+1)}[P_{i+1}]$ for $i < d$ and $\hat{\pi}^{(d)}[P_d] = \hat{\pi}^{(1)}[P_1]$.

d) Let $\alpha = \frac{2\pi\sqrt{-1}}{d}$ and let $\pi^{(k)} = (\hat{\pi}^{(1)}, \hat{\pi}^{(2)}e^{i\alpha k}, \hat{\pi}^{(3)}e^{2i\alpha k}, \dots, \hat{\pi}^{(d)}e^{(d-1)i\alpha k})$. Show that $\pi^{(k)}$ is a left eigenvector of $[P]$ of eigenvalue $e^{-ik\alpha}$.

4.14) Assume a friend has developed an excellent program for finding the steady state probabilities for finite state Markov chains. More precisely, given the transition matrix $[P]$, the program returns $\lim_{n \rightarrow \infty} P_{ii}^n$ for each i . Assume all chains are aperiodic.

a) You want to find the expected time to first reach a given state k starting from a given state m for a Markov chain with transition matrix $[P]$. You modify the matrix to $[P']$ where $P'_{km} = 1$, $P'_{kj} = 0$ for $j \neq m$, and $P'_{ij} = P_{ij}$ otherwise. How do you find the

desired first passage time from the program output given $[P]$ as an input? Hint: The times at which a Markov chain enters any given state can be considered as renewals in a (perhaps delayed) renewal process.

b) Using the same $[P]$ as the program input, how can you find the expected number of returns to state m before the first passage to state k ?

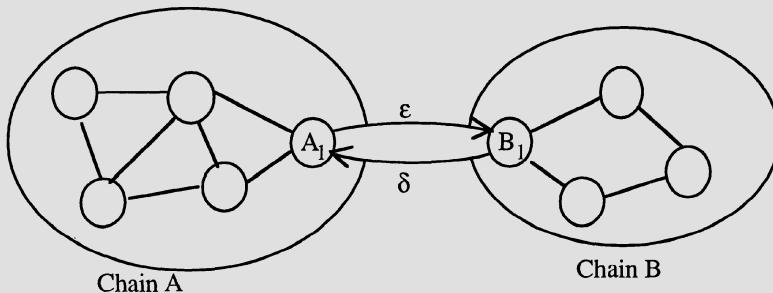
c) Suppose, for the same Markov chain $[P]$ and the same starting state m , you want to find the probability of reaching some given state n before the first passage to k . Modify $[P]$ to some $[P'']$ so that the above program with P'' as an input allows you to easily find the desired probability.

d) Let $P(X(0)=i) = Q_i$, $1 \leq i \leq J$ be an arbitrary set of initial probabilities for the same Markov chain $[P]$ as above. Show how to modify $[P]$ to some $[P'']$ for which the steady state probabilities allow you to easily find the expected time of the first passage to state k .

4.15) Suppose A and B are each ergodic Markov chains with transition probabilities $\{P_{A_i, A_j}\}$ and $\{P_{B_i, B_j}\}$ respectively. Denote the steady state probabilities of A and B by $\{\pi_{A_i}\}$ and $\{\pi_{B_i}\}$ respectively. The chains are now connected and modified as shown below. In particular, states A_1 and B_1 are now connected and the new transition probabilities P' for the combined chain are given by

$$\begin{aligned} P'_{A_1, B_1} &= \varepsilon, & P'_{A_1, A_j} &= (1-\varepsilon)P_{A_1, A_j} \text{ for all } A_j \\ P'_{B_1, A_1} &= \delta, & P'_{B_1, B_j} &= (1-\delta)P_{B_1, B_j} \text{ for all } B_j \end{aligned}$$

All other transition probabilities remain the same. Think intuitively of ε and δ as being small, but do not make any approximations in what follows. Give your answers to the following questions as functions of ε , δ , $\{\pi_{A_i}\}$ and $\{\pi_{B_i}\}$.



a) Assume that $\varepsilon>0$, $\delta=0$ (i.e., that A is a set of transient states in the combined chain). Starting in state A_1 , find the conditional expected time to return to A_1 given that the first transition is to some state in chain A.

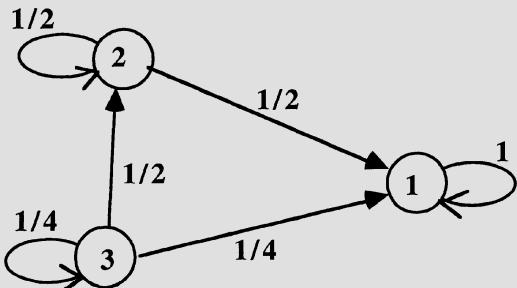
b) Assume that $\varepsilon>0$, $\delta=0$. Find $T_{A_1, B}$, the expected time to first reach state B_1 starting from state A_1 . Your answer should be a function of ε and the *original* steady state probabilities $\{\pi_{A_i}\}$ in chain A.

- c) Assume $\varepsilon > 0, \delta > 0$. find $T_{B,A}$, the expected time to first reach state A_1 , starting in state B_1 . Your answer should depend only on δ and $\{\pi_{B_i}\}$.
- d) Assume $\varepsilon > 0$ and $\delta > 0$. Find $P'(A)$, the steady state probability that the combined chain is in one of the states $\{A_j\}$ of the original chain A.
- e) Assume $\varepsilon > 0, \delta = 0$. For each state $A_j \neq A_1$ in A, find v_{A_j} , the expected number of visits to state A_j , starting in state A_1 , before reaching state B_1 . Your answer should depend only on ε and $\{\pi_{A_i}\}$.
- f) Assume $\varepsilon > 0, \delta > 0$. For each state A_j in A, find π'_{A_j} , the steady state probability of being in state A_j in the combined chain. Hint: Be careful in your treatment of state A_1 .

4.16) For the Markov chain with rewards in figure 4.5:

- a) Find the steady state reward per stage, g, using the steady state probability vector π .
- b) Let $w_1 = 0$ and use (34) to find w_2 .
- c) Assume g in (34) is an unknown. Again, let $w_1 = 0$ and solve (34) for w_2 and g.
- d) Now let w_1 be an arbitrary real number y. Again use (34) to solve for w_2 and g. How does your value of g compare to the values found in (a) and (c). Explain your answer.
- e) Let $w_1=0$, but take P_{12} as an arbitrary probability. Find g and w_2 again and give an intuitive explanation of why P_{12} effects the asymptotic relative gain of state 2.

4.17) Consider the Markov chain below:



- a) Suppose the chain is started in state i and goes through n transitions; let $v_i(n)$ be the expected number of transitions (out of the total of n) until the chain enters the trapping state, state 1. Find an expression for $v(n) = (v_1(n), v_2(n), v_3(n))$ in terms of $v(n-1)$ (take $v_1(n) = 0$ for all n). Hint: view the system as a Markov reward system; what is the value of r?
- b) Solve numerically for $\lim_{n \rightarrow \infty} v(n)$. Interpret the meaning of the elements v_i in the solution of Eq. (25).

c) Give a direct argument why (25) provides the solution directly to the expected time from each state to enter the trapping state.

4.18) Prove lemma 4; for the stationary policy result, use either induction on (29) or use (31) directly. For the dynamic policy, use induction on (41).

4.19) George drives his car to the theater, which is at the end of a one-way street. There are parking places along the side of the street and a parking garage that costs \$5 at the theater. Each parking place is independently occupied or unoccupied with probability 1/2. If George parks n parking places away from the theater, it costs him n cents (in time and shoe leather) to walk the rest of the way. George is myopic and can only see the parking place he is currently passing.

If George has not already parked by the time he reaches the n^{th} place, he first decides whether or not he will park if the place is unoccupied, and then observes the place and acts according to his decision. George can never go back and must park in the parking garage if he has not parked before.

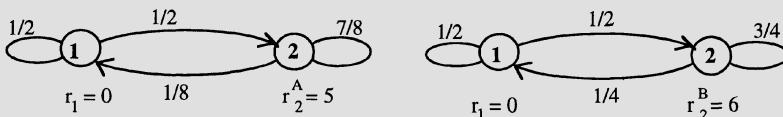
a) Model the above problem as a 2 state Markov decision problem. In the “driving” state, state 2, there are two possible decisions: park if the current place is unoccupied or drive on whether or not the current place is unoccupied.

b) Find $v^*_i(n)$, the *minimum* expected aggregate cost for n stages (i.e., immediately before observation of the n^{th} parking place) starting in state $i = 1$ or 2; it is sufficient to express $v^*_i(n)$ in terms of $v^*_i(n-1)$. The final costs, in cents, at stage 0 should be $v_2(0) = 500$, $v_1(0) = 0$.

c) For what values of n is the optimal decision the decision to drive on?

d) What is the probability that George will park in the garage, assuming that he follows the optimal policy?

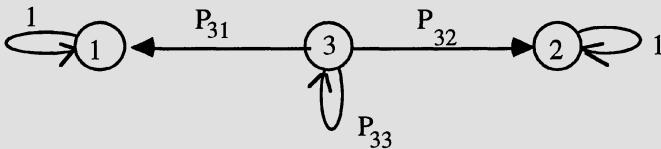
4.20) Consider a dynamic programming problem with two states and two possible policies, denoted **A** and **B**, in state 2; there is no choice of policies in state 1;



- a) Find the steady state gain per stage, g^A and g^B , for stationary policies **A** and **B**.
- b) Find the relative gain vectors, w^A and w^B , for stationary policies **A** and **B**.
- c) Suppose the final reward, at stage 0, is $v_1(0) = 0$, $v_2(0) = v$. For what range of v does the dynamic programming algorithm use decision A in state 2 at stage 1?
- d) For what range of v does the dynamic programming algorithm use decision A in state 2 at stage 2? at stage n?
- e) Find the optimal gain $v^*_2(n)$ and $v^*_1(n)$ as a function of stage n assuming $v = 10$.

4.21 a) Find $\lim_{n \rightarrow \infty} [P]^n$ for the Markov chain below. Hint: Think in terms of the long term transition probabilities. Recall that the edges in the graph for a Markov chain correspond to the positive transition probabilities.

b) Let $\pi^{(1)}$ and $\pi^{(2)}$ denote the first two rows of $\lim_{n \rightarrow \infty} [P]^n$ and let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ denote the first two columns of $\lim_{n \rightarrow \infty} [P]^n$. Show that $\pi^{(1)}$ and $\pi^{(2)}$ are independent left eigenvectors of $[P]$, and that $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are independent right eigenvectors of $[P]$. Find the eigenvalue for each eigenvector.



c) Let \mathbf{r} be an arbitrary reward vector and consider the equation

$$[P]\mathbf{w} + \mathbf{r} = \mathbf{g} + \mathbf{w}; \mathbf{g} = \alpha\mathbf{u}^{(1)} + \beta\mathbf{u}^{(2)}; \alpha, \beta \text{ scalars} \quad (\text{a})$$

Determine what values α and β (and thus \mathbf{g}) must have in order for (a) to have a solution. Argue that with the additional constraints $w_1=w_2=0$, (a) has a unique solution for \mathbf{w} and find that \mathbf{w} .

- d) Show that $\mathbf{w}' = \mathbf{w} + \alpha\mathbf{u}_1 + \beta\mathbf{u}_2$ satisfies (a) for all choices of scalars α and β .
 - e) Assume that the reward at stage 0 is $\mathbf{v}(0) = \mathbf{w}$. Show that $\mathbf{v}(n) = n\mathbf{g} + \mathbf{w}$.
 - f) For an arbitrary reward $\mathbf{v}(0)$ at stage 0, show that $\mathbf{v}(n) = n\mathbf{g} + \mathbf{w} + [P]^n(\mathbf{v}(0) - \mathbf{w})$.
- Why isn't theorem 7 applicable here?

4.22) Generalize exercise 4.21 to the general case of two ergodic Markov classes and one transient class.

4.23 a) Consider a Markov decision problem with J states and assume that for each policy $\mathbf{A} = (k_1, k_2, \dots, k_J)$, the Markov chain $[P^{\mathbf{A}}]$ is ergodic. Suppose that policy \mathbf{B} is a stationary optimal policy and \mathbf{A} is any other policy. Use equations (50) to (52), with the inequalities appropriately reversed, to show that $g^{\mathbf{B}} \geq g^{\mathbf{A}}$.

- b) Show that if $\mathbf{r}^{\mathbf{B}} + [P^{\mathbf{B}}]\mathbf{w}^{\mathbf{B}} \geq \mathbf{r}^{\mathbf{A}} + [P^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}}$ is not satisfied with equality, then $g^{\mathbf{B}} > g^{\mathbf{A}}$.
- c) Assume in parts (c) through (g) that \mathbf{A} and \mathbf{B} are both optimal stationary policies. Show that $\mathbf{r}^{\mathbf{B}} + [P^{\mathbf{B}}]\mathbf{w}^{\mathbf{B}} = \mathbf{r}^{\mathbf{A}} + [P^{\mathbf{A}}]\mathbf{w}^{\mathbf{B}}$. Hint: use part (b).
- d) Find the relationship between the relative gain vector $\mathbf{w}^{\mathbf{A}}$ for policy \mathbf{A} and the relative gain vector $\mathbf{w}^{\mathbf{B}}$ for policy \mathbf{B} . Hint: Show that $\mathbf{r}^{\mathbf{B}} + [P^{\mathbf{B}}]\mathbf{w}^{\mathbf{A}} = g^{\mathbf{B}}\mathbf{e} + \mathbf{w}^{\mathbf{A}}$; what does this say about $\mathbf{w}^{\mathbf{A}}$?
- e) Suppose that policy \mathbf{A} uses decision 1 in state 1 and policy \mathbf{B} uses decision 2 in state 1 (i.e., $k_1 = 1$ for policy \mathbf{A} and $k_1 = 2$ for policy \mathbf{B}). What is the relationship between $r_1^k, P_{11}^k, P_{12}^k, \dots, P_{1J}^k$ for k equal to 1 and 2?

f) Now suppose that policy **A** uses decision 1 in each state and policy **B** uses decision 2 in each state. Is it possible that $r_i^1 > r_i^2$ for all i ? Explain carefully.

g) Now assume that r_i^1 is the same for all i . Does this change your answer to part (f)? Explain.

4.24) Consider a Markov decision problem with three states. Assume that each stationary policy corresponds to an ergodic Markov chain. It is known that a particular policy $\mathbf{B} = (k_1, k_2, k_3) = (2, 4, 1)$ is the unique optimal stationary policy (i.e., the gain per stage in steady state is maximized by always using decision 2 in state 1, decision 4 in state 2, and decision 1 in state 3). As usual, r_i^k denotes the reward in state i under decision k , and P_{ij}^k denotes the probability of a transition to state j given state i and given the use of decision k in state i .

Consider the effect of changing the Markov decision problem in each of the following ways (the changes in each part are to be considered in the absence of the changes in the other parts):

- a) r_1^1 is replaced by $r_1^1 - 1$.
- b) r_1^2 is replaced by $r_1^2 + 1$.
- c) r_1^k is replaced by $r_1^k + 1$ for all state 1 decisions k .
- d) for all i , $r_i^{k_i}$ is replaced by $r_i^{k_i} + 1$ for the decision k_i of policy **B**.

For each of the above changes, answer the following questions; give explanations:

- i) Is the gain per stage, g^B , increased, decreased, or unchanged by the given change?
- ii) Is it possible that another policy, $\mathbf{A} \neq \mathbf{B}$, is optimal after the given change?

4.25) (The Odoni Bound) Let **B** be the optimal stationary policy for a Markov decision problem and let g^B and π^B be the corresponding gain and steady state probability respectively. Let $v_i^*(n)$ be the optimal dynamic expected reward for starting in state i at stage n .

a) Show that $\min_i [v_i^*(n) - v_i^*(n-1)] \leq g^B \leq \max_i [v_i^*(n) - v_i^*(n-1)]$; $n \geq 1$. Hint: Consider premultiplying $\mathbf{v}^*(n) - \mathbf{v}^*(n-1)$ by π^B or π^A where **A** is the optimal dynamic policy at stage n .

b) Show that the lower bound is non-decreasing in n and the upper bound is non-increasing in n and both converge to g^B with increasing n .

4.26) (Extension of theorem 8 to recurrent plus transient chains). Define a policy **B** that is recurrent plus transient to be a stationary optimal policy if, first, $g^B \geq g^A$ for any other policy (or any recurrent class of any other policy with multiple recurrent classes), and second, if $g^B = g^A$, $w^B \geq w^A$, where $w_i=0$ for both policy **A** and **B** and state 1 is in the recurrent class of policy **B**.

a) Show that if (46) is satisfied and if $\mathbf{v}(0) = \mathbf{w}^B$, then the optimal dynamic policy is **B** at every stage.

- b) Show that if (46) is satisfied, then $g^B \geq g^A$ for all recurrent classes of all A .
 c) Show that if (46) is satisfied, then B is a stationary optimal policy.
 d) Show that if (46) is not satisfied, then B is not a stationary optimal policy.

4.27) Assume that B is the stationary optimal policy and that B is unique.

- a) Let $B = (k_1, k_2, \dots, k_J)$. Show that for each i and each $k \neq k_i$, $k \in \{1, 2, \dots, K_i\}$,

$$r_i^{(k_i)} + \sum_j P_{ij}^{(k_i)} w_j^B > r_i^{(k)} + \sum_j P_{ij}^{(k)} w_j^B$$

Hint: Use the same line of argument as in theorem 8.

- b) Show (or observe) that there is some $\epsilon > 0$ such that for each i and $k \neq k_i$, $k \in \{1, \dots, K_i\}$,

$$r_i^{(k_i)} + \sum_j P_{ij}^{(k_i)} w_j^B \geq r_i^{(k)} + \sum_j P_{ij}^{(k)} w_j^B + \epsilon$$

- c) Show that there is some n_0 such that for all $n \geq n_0$, B is the optimal dynamic policy. Hint: Use (57) for $v_j^*(n-1)$ in (40), and perform the maximization with the help of part (b).

4.28) Consider an integer time queueing system with a finite buffer of size 2. At the beginning of the n^{th} time interval, the queue contains at most two customers. There is a cost of one unit for each customer in queue (i.e., the cost of delaying that customer). If there is one customer in queue, that customer is served. If there are two customers, an extra server is hired at a cost of 3 units and both customers are served. Thus the total immediate cost for two customers in queue is 5, the cost for one customer is 1, and the cost for 0 customers is 0. At the end of the n^{th} time interval, either 0, 1, or 2 new customers arrive (each with probability 1/3).

- a) Assume that the system starts with i ($0 \leq i \leq 2$) customers in queue at time -1 (i.e., in stage 1) and terminates at time 0 (stage 0) with a final cost of 5 units for each customer in queue (at the beginning of interval 0). Find the expected aggregate cost $v_i(1)$ for $0 \leq i \leq 2$.

- b) Assume now that the system starts with i customers in queue at time -2 with the same final cost at time 0. Find the expected aggregate cost $v_i(2)$ for $0 \leq i \leq 2$.

- c) For an arbitrary starting time $-n$, find the expected aggregate cost $v_i(n)$ for $0 \leq i \leq 2$.

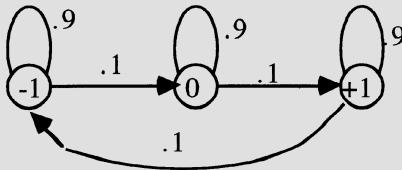
- d) Find the cost per stage and find the asymptotic relative cost (gain) vector.

- e) Now assume that there is a decision maker who can choose whether or not to hire the extra server when there are two customers in queue. If the extra server is not hired, the 3 unit fee is saved, but only one of the customers is served. If there are two arrivals in this case, assume that one is turned away at a cost of 5 units. Find the minimum dynamic aggregate expected cost $v_i^*(1)$, $0 \leq i \leq 2$, for stage 1 with the same final cost as before.

- f) Find the minimum dynamic aggregate expected cost $v_i^*(n)$ for stage n , $0 \leq i \leq 2$.

- g) Now assume a final cost in stage 0 of one unit per customer rather than 5, and find the new minimum dynamic aggregate expected cost $v^*_i(n)$, $0 \leq i \leq 2$.

- 4.29) Consider a finite state ergodic Markov chain $\{X_n; n \geq 0\}$ with an integer valued set of states $\{-K, -K+1, \dots, -1, 0, 1, \dots, +K\}$, a set of transition probabilities P_{ij} ; $-K \leq i, j \leq K$, and initial state $X_0=0$. One example of such a chain is given by:



Let $\{S_n; n \geq 0\}$ be a stochastic process with $S_n = \sum_{i=0}^n X_i$. Parts (a), (b), and (c) are independent of parts (d) and (e). Parts (a), (b), and (c) should be solved both for the special case in the above graph and for the general case.

a) Find $\lim_{n \rightarrow \infty} E[X_n]$ for the example and express $\lim_{n \rightarrow \infty} E[S_n]$ in terms of the steady state probabilities of $\{X_n; n \geq 0\}$ for the general case.

b) Show that $\lim_{n \rightarrow \infty} S_n/n$ exists with probability one and find the value of the limit. Hint: apply renewal reward theory to $\{X_n; n \geq 0\}$.

c) Assume that $\lim_{n \rightarrow \infty} E[X_n] = 0$. Find $\lim_{n \rightarrow \infty} E[S_n]$.

d) Show that

$$P(S_n=s_n | S_{n-1}=s_{n-1}, S_{n-2}=s_{n-2}, S_{n-3}=s_{n-3}, \dots, S_0=0) = P(S_n=s_n | S_{n-1}=s_{n-1}, S_{n-2}=s_{n-2}).$$

e) Let $\mathbf{Y}_n = (S_n, S_{n-1})$ (i.e., \mathbf{Y}_n is a random two dimensional integer valued vector). Show that $\{\mathbf{Y}_n; n \geq 0\}$ (where $\mathbf{Y}_0 = (0,0)$) is a Markov chain. Describe the transition probabilities of $\{\mathbf{Y}_n; n \geq 0\}$ in terms of $\{P_{ij}\}$.

NOTES

1. In graph theory, a *directed path of length n* is a sequence of $n+1 \geq 2$ *distinct* nodes x_0, x_1, \dots, x_n in which there is a directed arc from x_{i-1} to x_i for each i , $1 \leq i \leq n$. A *directed walk of length n* is a sequence of $n+1 \geq 2$ nodes x_0, x_1, \dots, x_n , not necessarily distinct, in which there is a directed arc from x_{i-1} to x_i for each i , $1 \leq i \leq n$. Thus a walk can contain cycles and a path cannot.
2. For completeness, we say that the period is infinite if $P_{ii}^n = 0$ for all $n \geq 1$.
3. For Markov chains with a countably infinite state space, ergodic means that the states are positive recurrent and aperiodic (see Chapter 5, section 1)
4. Proofs marked with an asterisk can be omitted without loss of continuity.

Chapter 5

Markov Chains with Countably Infinite State Spaces

5.1 INTRODUCTION AND CLASSIFICATION OF STATES

Markov chains with a countably infinite state space exhibit some types of behavior not possible for chains with a finite state space. Figure 5.1 helps explain how these new types of behavior arise. If $p > 1/2$, then transitions to the right occur with higher frequency than transitions to the left. Thus, reasoning heuristically, we expect X_n to be large for large n . This means that, given $X_0 = 0$, the probability P_{0j}^n should go to zero for any fixed j with increasing n . If one tried to define the steady state probability of state j as $\lim_{n \rightarrow \infty} P_{0j}^n$, then this limit would be 0 for all j . These probabilities would not sum to 1, and thus would not correspond to a limiting distribution. Thus we say that a steady state does not exist. In more heuristic terms, the state keeps increasing forever. The truncation of figure 5.1 to k states is analyzed in exercise 4.3. The solution there defines $p=p/q$ and shows that $\pi_i = (1-p)p^i/(1-p^k)$ for $p \neq 1$ and $\pi_i = 1/k$ for $p=1$. For $p < 1$, the limiting behavior as $k \rightarrow \infty$ is $\pi_i = (1-p)p^i$ for $p < 1$ and $\pi_i = 0$ otherwise. In section 5.3 we analyze birth death Markov chains, of which figure 5.1 is an example, without first truncating the chain.

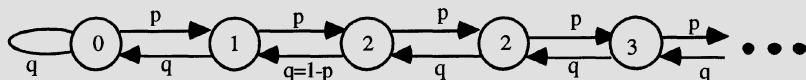


Figure 5.1. A Markov chain with countably infinite state space. If $p > 1/2$, then X_n is large with high probability for large n , i.e., $\lim_{n \rightarrow \infty} P(X_n \geq j) = 1$ for any integer j .

Fortunately, the strange behavior of figure 5.1 when $p > q$ is not typical of Markov chains in most applications. For typical chains with a countably infinite number of states, a steady state does exist, and the steady state probabilities of all but a finite number of states (the number depending on the chain) can almost be ignored for numerical calculations. In what follows we define the first passage time probabilities of

the Markov chain. These will help us to understand the difference between the strange cases and the more typical cases. The *first passage time probability*, $f_{ij}(n)$, is defined as the probability that the *first* entry to state j occurs at time $n \geq 1$, given that $X_0=i$; that is, $f_{ij}(1) = P_{ij}$, and for $n \geq 2$,

$$f_{ij}(n) = P(X_n=j, X_{n-1} \neq j, X_{n-2} \neq j, \dots, X_1 \neq j \mid X_0=i) \quad (1)$$

For $n \geq 2$, note the distinction between $f_{ij}(n)$ and $P_{ij}^n = P(X_n=j \mid X_0=i)$. The definition in (1) also applies for $j=i$; $f_{ii}(n)$ is thus the probability, given $X_0=i$, that the first occurrence of state i after time 0 occurs at time n . Since the transition probabilities are independent of time, $f_{ij}(n-1)$ is also the probability, given $X_1=i$, that the first subsequent occurrence of state j occurs at time n . Thus we can calculate $f_{ij}(n)$ from the iterative relations

$$f_{ij}(n) = \sum_{k \neq j} P_{ik} f_{kj}(n-1); \quad n > 1; \quad f_{ij}(1) = P_{ij} \quad (2)$$

With this iterative approach, the first passage time probabilities $f_{ij}(n)$ for a given n must be calculated for all i before proceeding to calculate them for the next larger value of n . This also gives us $f_{ij}(n)$, although $f_{ij}(n)$ is not used in the iteration.

Let $F_{ij}(n)$, for $n \geq 1$, be the probability, given $X_0=i$, that state j occurs at some time between 1 and n inclusive. Thus,

$$F_{ij}(n) = \sum_{m=1}^n f_{ij}(m) \quad (3)$$

We see that $F_{ij}(\infty)$, i.e., $\lim_{n \rightarrow \infty} F_{ij}(n)$, is the probability, given $X_0=i$, that state j will ever occur after time 0. If $F_{ij}(\infty) = 1$, then, given $X_0=i$, it is certain (with probability 1) that the chain will eventually enter state j . In this case, we can consider a random variable T_{ij} representing the first passage time from i to j . Then $f_{ij}(n)$ is the probability mass function of T_{ij} and $F_{ij}(n)$ is the distribution function of T_{ij} . If $F_{ij}(\infty) < 1$, then T_{ij} is a defective random variable, since, with some non-zero probability, there is no first passage to j . Defective random variables are not considered to be random variables (in the theorems here or elsewhere), but they do have many of the properties of random variables.

A state j is defined to be *recurrent* if $F_{jj}(\infty) = 1$ and is defined to be *transient* if $F_{jj}(\infty) < 1$. Thus a state is either recurrent or transient, and is recurrent if and only if (iff) it is certain that the state eventually returns to j , given that it starts in j . Equivalently, j is recurrent iff T_{jj} , the time to return to j , is a random variable. Note that for the special case of finite state Markov chains, this definition is consistent with the one in Chapter 4. For a countably infinite state space, however, the earlier definition is not adequate; for example, i and j communicate for all states i and j in figure 5.1, but for $p > 1/2$, each state is transient (this is shown in exercise 5.2, and further explained in section 5.3).

Since results about Markov chains depend very heavily on whether states are recurrent or transient, we will look carefully at the probabilities $F_{ij}(n)$. Substituting (2) into (3), we obtain

$$F_{ij}(n) = P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(n-1); \quad n > 1; \quad F_{ij}(1) = P_{ij} \quad (4)$$

In the expression $P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(n-1)$, P_{ij} is the probability that state j is entered on the first transition, and $\sum_{k \neq j} P_{ik} F_{kj}(n-1)$ is the sum, over every other state k , of the joint probability that k is entered on the first transition and that j is entered on one of the subsequent $n-1$ transitions.

For each i , $F_{ij}(n)$ is non-decreasing in n and upper bounded by 1 (this can be seen from (3), and can also be established directly from (4) by induction). Thus, the limit as $n \rightarrow \infty$ exists and satisfies

$$F_{ij}(\infty) = P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(\infty) \quad (5)$$

Eq. (5) does not always have a unique solution. That is, the set of equations

$$x_{ij} = P_{ij} + \sum_{k \neq j} P_{ik} x_{kj}; \quad \text{all } i \geq 0 \quad (6)$$

always has a solution in which $x_{ij}=1$ for all $i \geq 0$, but if state j is transient, there is another solution in which x_{ij} is the true value of $F_{ij}(\infty)$ and $F_{ij}(\infty) < 1$. Exercise 5.1 shows that if (6) is satisfied by a set of non-negative numbers $\{x_{ij}; 1 \leq i \leq J\}$, then $F_{ij}(\infty) \leq x_{ij}$ for each i .

The following lemma establishes the rather intuitive result that if state j is recurrent, then from any state i accessible from j , there must be an eventual return to j .

LEMMA 1: Let state i be accessible from j and let j be recurrent. Then $F_{ij}(\infty) = 1$.

Proof: First assume that $P_{ji} > 0$, and assume for the sake of establishing a contradiction that $F_{ij}(\infty) < 1$. Then (5), applied to $i=j$, is

$$F_{jj}(\infty) = P_{jj} + \sum_{k \neq j} P_{jk} F_{kj}(\infty) < P_{jj} + \sum_{k \neq j} P_{jk} = 1$$

where the strict inequality follows since $P_{ji}F_{ij}(\infty) < P_{ji}$ by assumption. This is a contradiction, so $F_{ij}(\infty) = 1$ for every i accessible from j in one step. Next assume that $P_{ji}^2 > 0$, say with $P_{jm} > 0$ and $P_{mi} > 0$. For a contradiction, assume that $F_{ij}(\infty) < 1$. From (5),

$$F_{mj}(\infty) = P_{mj} + \sum_{k \neq j} P_{mk} F_{kj}(\infty) < P_{mj} + \sum_{k \neq j} P_{mk} = 1$$

where the strict inequality follows since $P_{mi}F_{ij}(\infty) < 1$. This is a contradiction, since m is accessible from j in one step, and thus $F_{mj}(\infty) = 1$. It follows that every i accessible from j in two steps satisfies $F_{ij}(\infty) = 1$. Extending the same argument for successively larger numbers of steps, the conclusion of the lemma follows.

We next apply renewal theory to the successive occurrences of any given state. Given that $X_0 = i$, let $\{N_{ij}(t); t \geq 0\}$ be a counting process in which $N_{ij}(t)$ is the number of

transitions into state j by time t (this includes transitions from state j into state j). Initially, we assume that $X_0=j$ and look at the counting process $N_{jj}(t)$. If j is a recurrent state, then the inter-renewal time T_{jj} is a random variable, and the process probabilistically restarts itself on each transition to j . Thus $\{N_{jj}(t); t \geq 0\}$ is a renewal process if j is recurrent. Recall from lemma 1 of Chapter 3 that, whether or not the expected renewal time $E[T_{jj}]$ is finite, $\lim_{t \rightarrow \infty} N_{jj}(t) = \infty$ with probability 1 and $\lim_{t \rightarrow \infty} E[N_{jj}(t)] = \infty$.

Next assume that state j is transient. In this case, the inter-renewal time T_{jj} is not a random variable, and $\{N_{jj}(t); t \geq 0\}$ is not a renewal process. An eventual return to state j occurs only with probability $F_{jj}(\infty) < 1$, and, since subsequent returns are independent, the total number of returns to state j is a geometric random variable with mean $F_{jj}(\infty)/[1-F_{jj}(\infty)]$. Thus the total number of returns is finite with probability 1 and the expected total number of returns is finite.

Finally, note that P_{ij}^n , the probability of a transition to state j at time n , is equal to the expectation of a transition to j at the instant n (i.e., 1 transition occurs with probability P_{ij}^n and 0 occurs otherwise). Since $N_{jj}(t)$ is the sum of the number of transitions to j over integer times up to t , we have

$$E[N_{jj}(t)] = \sum_{1 \leq n \leq t} P_{jj}^n$$

Summarizing these results, we have the following lemma:

LEMMA 2: Let $\{N_{jj}(t); t \geq 0\}$ be the counting process for occurrences of state j up to time t in a Markov chain starting in state j . Then state j is recurrent iff $\lim_{t \rightarrow \infty} N_{jj}(t) = \infty$ with probability 1; also, j is recurrent iff $\lim_{t \rightarrow \infty} E[N_{jj}(t)] = \infty$; finally, j is recurrent iff $\lim_{t \rightarrow \infty} \sum_{1 \leq n \leq t} P_{jj}^n = \infty$.

LEMMA 3: If state j is recurrent and states i and j are in the same class, i.e. i and j communicate, then state i is recurrent.

Proof: From lemma 2, state j satisfies $\lim_{t \rightarrow \infty} \sum_{1 \leq n \leq t} P_{jj}^n = \infty$. Since j and i communicate, there are integers m and k such that $P_{ij}^m > 1$ and $P_{ji}^k > 1$. For every walk from state j to j in n steps, there is a corresponding walk from i to i in $m+n+k$ steps, going from i to j in m steps, j to j in n steps, and j back to i in k steps. Thus

$$\begin{aligned} P_{ii}^{m+n+k} &\geq P_{ij}^m P_{jj}^n P_{ji}^k \\ \sum_{n=1}^{\infty} P_{ii}^n &\geq \sum_{n=1}^{\infty} P_{ii}^{m+n+k} \geq P_{ij}^m P_{ji}^k \sum_{n=1}^{\infty} P_{jj}^n = \infty \end{aligned}$$

Thus, from lemma 2, i is recurrent, completing the proof.

LEMMA 4: Let $\{N_{ij}(t); t \geq 0\}$ be the counting process for transitions into state j up to time t for a Markov chain given $X_0=i$. Then if i and j are recurrent and in the same class, $\{N_{ij}(t); t \geq 0\}$ is a delayed renewal process (or an ordinary renewal process if $i=j$).

Proof: From lemma 1, T_{ij} , the time until the first transition into j , is a random variable. Also T_{jj} is a random variable by definition of recurrence, and subsequent intervals between occurrences of state j are IID, completing the proof.

If $F_{ij}(\infty) = 1$, the mean time \bar{T}_{ij} to first enter state j starting from state i is of interest. Since T_{ij} is a non-negative random variable, its expectation is the integral of its complementary distribution function,

$$\bar{T}_{ij} = 1 + \sum_{n=1}^{\infty} (1 - F_{ij}(n)) \quad (7)$$

It is possible to have $F_{ij}(\infty) = 1$ but $\bar{T}_{ij} = \infty$. A state j is defined to be *positive recurrent* if $F_{jj}(\infty) = 1$ and $\bar{T}_{jj} < \infty$. A state j is *null recurrent* if $F_{jj}(\infty) = 1$ and $\bar{T}_{jj} = \infty$. Each state of a Markov chain is thus classified as one of the following three types—positive recurrent, null recurrent, or transient. It will become apparent in section 5.3 that, for the chain in figure 5.1, all states are positive recurrent for $p < 1/2$, null recurrent for $p = 1/2$, and transient for $p > 1/2$. Null recurrence in this example lies on a boundary between positive recurrence and transience, and this is often a good way to look at null recurrence. Part (f) of exercise 6.1 illustrates another type of situation in which null recurrence can occur.

Assume that state j is recurrent and consider the renewal process $\{N_{jj}(t); t \geq 0\}$. The limiting theorems for renewal processes can be applied directly. From theorem 1 of Chapter 3,

$$\lim_{t \rightarrow \infty} N_{jj}(t)/t = 1/\bar{T}_{jj} \quad \text{with probability 1} \quad (8)$$

From theorem 4 of Chapter 3,

$$\lim_{t \rightarrow \infty} E[N_{jj}(t)/t] = 1/\bar{T}_{jj} \quad (9)$$

Eqs. (8) and (9) are valid whether j is positive recurrent or null recurrent.

Next we apply Blackwell's theorem to $\{N_{jj}(t); t \geq 0\}$. Recall that the period of a given state j in a Markov chain (whether the chain has a countable or finite number of states) is the greatest common divisor of the set of integers $n > 0$ such that $P_{jj}^n > 0$. If this period is d , then $\{N_{jj}(t); t \geq 0\}$ is arithmetic with span d (i.e., renewals occur only at times that are multiples of d). From Blackwell's theorem in the form of Eq. (3.18),

$$\lim_{n \rightarrow \infty} P(X_{nd} = j \mid X_0 = j) = d/\bar{T}_{jj} \quad (10)$$

If state j is aperiodic (i.e., $d=1$), this says that $\lim_{n \rightarrow \infty} P(X_n = j \mid X_0 = j) = 1/\bar{T}_{jj}$. Eqs. (8) and (9) suggest that $1/\bar{T}_{jj}$ has some of the properties associated with a steady state probability of state j , and (10) strengthens this if j is aperiodic. For a Markov chain consisting of a single class of states, all positive recurrent, we will strengthen this

association further in theorem 3 by showing that there is a unique *steady state distribution*, $\{\pi_j, j \geq 0\}$ such that $\pi_j = 1/\bar{T}_{jj}$ for all j and such that $\pi_j = \sum_i \pi_i P_{ij}$ for all $j \geq 0$ and $\sum_j \pi_j = 1$. The following theorem starts this development by showing that (8–10) are independent of the starting state.

THEOREM 1: Let j be a recurrent state in a Markov chain and let i be any state in the same class as j . Given $X_0=i$, let $N_{ij}(t)$ be the number of transitions into state j by time t and let \bar{T}_{jj} be the expected recurrence time of state j (either finite or infinite). Then

$$\lim_{t \rightarrow \infty} N_{ij}(t)/t = 1/\bar{T}_{jj} \text{ with probability 1} \quad (11)$$

$$\lim_{t \rightarrow \infty} E[N_{ij}(t)/t] = 1/\bar{T}_{jj} \quad (12)$$

If j is also aperiodic, then

$$\lim_{n \rightarrow \infty} P(X_n=j \mid X_0=i) = 1/\bar{T}_{jj} \quad (13)$$

Proof: Since i and j are recurrent and in the same class, lemma 4 asserts that $\{N_{ij}(t); t \geq 0\}$ is a delayed renewal process for $j \neq i$. Thus (11) and (12) follow from theorems 9 and 10 of Chapter 3. If j is aperiodic, then $\{N_{ij}(t); t \geq 0\}$ is a delayed renewal process for which the inter-renewal intervals T_{jj} have span 1 and T_{ij} has an integer span. Thus, (13) follows from Blackwell's theorem for delayed renewal processes, theorem 3.11. For $i=j$, Eqs. (11–13) follow from (8–10), completing the proof.

THEOREM 2: All states in the same class of a Markov chain are of the same type—either all positive recurrent, all null recurrent, or all transient.

Proof: Let j be a recurrent state. From lemma 3, any state in the same class as j is also recurrent. Thus all states in a class are recurrent or all are transient. Next suppose that j is positive recurrent, so that $1/\bar{T}_{jj} > 0$. Let i be in the same class as j , and consider the renewal reward process on $\{N_{ij}(t); t \geq 0\}$ for which $R(t) = 1$ whenever the process is in state i (i.e., if $X_n = i$, then $R(t) = 1$ for $n \leq t < n+1$). The reward is 0 whenever the process is in some state other than i . Let $E[R_n]$ be the expected reward in an inter-renewal interval; this must be positive since i is accessible from j . From theorem 3.6,

$$\lim_{n \rightarrow \infty} \frac{1}{t} \int_0^t R(\tau) d\tau = \frac{E[R_n]}{\bar{T}_{jj}} \text{ with probability 1.}$$

The term on the left is the time average number of transitions into state i , given $X_0=j$, and this is $1/\bar{T}_{ii}$ from (11). Since $E[R_n] > 0$ and $\bar{T}_{jj} < \infty$, we have $1/\bar{T}_{ii} > 0$, so i is positive recurrent. Thus if one state is positive recurrent, the entire class is, completing the proof.

If all of the states in a Markov chain are in a null recurrent class, then $1/\bar{T}_{jj} = 0$ for each state j , and one might think of $1/\bar{T}_{jj} = 0$ as a “steady state” probability for j in the

sense that 0 is both the time average rate of occurrence of j and the limiting probability of j . However, these “probabilities” do not add up to 1, so a steady state probability distribution does not exist. This appears rather paradoxical at first, but the example of figure 5.1, with $p = 1/2$ will help to clarify the situation. As time n increases (starting in state i , say), the random variable X_n spreads out over more and more states around i , and thus is less likely to be in each individual state. For each j , $\lim_{n \rightarrow \infty} P_{ij}^n = 0$. Thus, $\sum_j \{\lim_{n \rightarrow \infty} P_{ij}^n\} = 0$. On the other hand, for every n , $\sum_j P_{ij}^n = 1$. This is one of those unusual examples where a limit and a sum cannot be interchanged.

In Chapter 4, we defined the steady state distribution of a finite state Markov chain as a probability vector π that satisfies $\pi[P] = \pi$. Here, the *steady state distribution* $\{\pi_i; i \geq 0\}$ is defined in the same way, as a set of numbers that satisfy

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{for all } j; \quad \pi_j \geq 0 \text{ for all } j; \quad \sum_j \pi_j = 1 \quad (14)$$

Suppose that a set of numbers $\{\pi_i; i \geq 0\}$ satisfying (14) is chosen as the initial probability distribution for a Markov chain, i.e., if $P(X_0=i) = \pi_i$ for all i . Then $P(X_1=j) = \sum_i \pi_i P_{ij} = \pi_j$ for all j , and, by induction, $P(X_n=j) = \pi_j$ for all j and all $n \geq 0$. The fact that $P(X_n=j) = \pi_j$ for all j motivates the definition of steady state distribution above. Theorems 1 and 2 have also shown that $1/T_{jj}$ is a “steady state” probability for state j , both in a time average and a limiting sense. The following theorem brings these ideas together. An *irreducible Markov chain* is a Markov chain in which all pairs of states communicate. For finite state chains, irreducibility implied a single class of recurrent states, whereas for countably infinite chains, an irreducible chain is a single class that can be transient, null recurrent, or positive recurrent.

THEOREM 3: Assume an irreducible Markov chain with transition probabilities $\{P_{ij}\}$. If (14) has a solution, then the solution is unique, $\pi_i = 1/T_{ii} > 0$ for all $i \geq 0$, and the states are positive recurrent. Also, if the states are positive recurrent then (14) has a solution.

Proof*: Let $\{\pi_j; j \geq 0\}$ satisfy (14) and be the initial distribution of the Markov chain, i.e., $P(X_0=j) = \pi_j, j \geq 0$. Then, as shown above, $P(X_n=j) = \pi_j$ for all $n \geq 0, j \geq 0$. Let $\tilde{N}_j(t)$ be the number of occurrences of any given state j from time 1 to t . Equating $P(X_n=j)$ to the expectation of an occurrence of j at time n , we have,

$$(1/t)E[\tilde{N}_j(t)] = (1/t)\sum_{1 \leq n \leq t} P(X_n=j) = \pi_j \quad \text{for all integers } t \geq 1$$

Conditioning this on the possible starting states i , and using the counting processes $\{N_{ij}(t); t \geq 0\}$ defined earlier,

$$\pi_j = (1/t)E[\tilde{N}_j(t)] = \sum_i \pi_i E[N_{ij}(t)/t] \quad \text{for all integer } t \geq 1 \quad (15)$$

For any given state i , let T_{ij} be the time of the first occurrence of state j given $X_0=i$. Then if $T_{ij} < \infty$, we have $N_{ij}(t) \leq N_{ij}(T_{ij}+t)$. Thus, for all $t \geq 1$,

$$E[N_{ij}(t)] \leq E[N_{ij}(T_{ij}+t)] = 1 + E[N_{jj}(t)] \quad (16)$$

The last step follows since the process is in state j at time T_{ij} , and the expected number of occurrences of state j in the next t steps is $E[N_{jj}(t)]$.

Substituting (16) in (15) for each i , $\pi_j \leq 1/t + E[N_{jj}(t)/t]$. Taking the limit as $t \rightarrow \infty$ and using (12), $\pi_j \leq \lim_{t \rightarrow \infty} E[N_{jj}(t)/t]$. Since $\sum_i \pi_i = 1$, there is at least one value of j for which $\pi_j > 0$, and for this j , $\lim_{t \rightarrow \infty} E[N_{jj}(t)/t] > 0$, and consequently $\lim_{t \rightarrow \infty} E[N_{jj}(t)] = \infty$. Thus, from lemma 2, state j is recurrent, and from theorem 1, j is positive recurrent. From theorem 2, all states are then positive recurrent. For any j and any integer M , (15) implies that

$$\pi_j \geq \sum_{i \leq M} \pi_i E[N_{ij}(t)/t] \quad \text{for all } t. \quad (17)$$

From theorem 1, $\lim_{t \rightarrow \infty} E[N_{ij}(t)/t] = 1/\bar{T}_{jj}$ for all i . Substituting this into (17), we get $\pi_j \geq 1/\bar{T}_{jj} \sum_{i \leq M} \pi_i$. Since M is arbitrary, $\pi_j \geq 1/\bar{T}_{jj}$. Since we already showed that $\pi_j \leq \lim_{t \rightarrow \infty} E[N_{jj}(t)/t] = 1/\bar{T}_{jj}$, we have $\pi_j = 1/\bar{T}_{jj}$ for all j . This shows both that $\pi_j > 0$ for all j and that the solution to (14) is unique. Exercise 5.4 completes the proof by showing that if the states are positive recurrent, then choosing $\pi_j = 1/\bar{T}_{jj}$ for all j satisfies (14).

In practice, it is usually easy to see whether a chain is irreducible. We shall also see by a number of examples that the steady state distribution can often be calculated from (14). Theorem 3 then says that the calculated distribution is unique and that its existence guarantees that the chain is positive recurrent.

EXAMPLE 1: Consider a renewal process $\{N(t); t \geq 0\}$ in which the inter-renewal random variables $\{W_n; n \geq 1\}$ are arithmetic with span 1. We will use a Markov chain to model the age of this process (see figure 5.2). The probability that a renewal occurs at a particular integer time depends on the past only through the integer time back to the last renewal. The state of the Markov chain during a unit interval will be taken as the age of the renewal process at the beginning of the interval. Thus, each unit of time, the age either increases by one or a renewal occurs and the age decreases to 0 (i.e., if a renewal occurs at time t , the age at time t is 0).

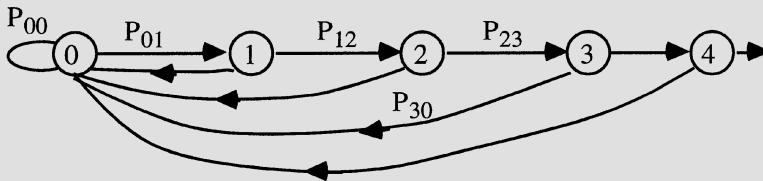


Figure 5.2. A Markov chain model of the age of a renewal process.

$P(W > n)$ is the probability that an inter-renewal interval lasts for more than n time units. We assume that $P(W > 0) = 1$, so that each renewal interval lasts at least one time

unit. The probability $P_{n,0}$ in the Markov chain is the probability that a renewal interval has duration $n+1$, given that the interval exceeds n . Thus, for example, P_{00} is the probability that the renewal interval is equal to 1. $P_{n,n+1}$ is $1 - P_{n,0}$, which is $P(W>n+1)/P(W>n)$. We can then solve for the steady state probabilities in the chain: for $n > 0$,

$$\pi_n = \pi_{n-1} P_{n-1,n} = \pi_{n-2} P_{n-2,n-1} P_{n-1,n} = \pi_0 P_{0,1} P_{1,2} \dots P_{n-1,n}$$

The first equality above results from the fact that state n ($n > 0$) can be entered only from state $n-1$. The subsequent equalities come from substituting in the same expression for π_{n-1} , then π_{n-2} , and so forth.

$$\pi_n = \pi_0 \frac{P(W>1)}{P(W>0)} \frac{P(W>2)}{P(W>1)} \cdots \frac{P(W>n)}{P(W>n-1)} = \pi_0 P(W>n) \quad (18)$$

We have cancelled out all the cross terms above and used the fact that $P(W>0) = 1$. Another way to see that $\pi_n = \pi_0 P(W>n)$ is to observe that state 0 occurs exactly once in each inter-renewal interval; state n occurs exactly once in those inter-renewal intervals of duration n or more.

Since the steady state probabilities must sum to 1, (18) can be solved for π_0 as

$$\pi_0 = \frac{1}{\sum_{n=0}^{\infty} P(W>n)} = \frac{1}{E[W]} \quad (19)$$

The second equality follows by expressing $E[W]$ as the integral of the complementary distribution function of W . Combining this with (18), the steady state probabilities for $n \geq 0$ are

$$\pi_n = \frac{P(W>n)}{E[W]} \quad (20)$$

In terms of the renewal process, π_n is the probability that, at some large integer time, the age of the process will be n . Note that if the age of the process at an integer time is n , then the age increases toward $n+1$ at the next integer time, at which point it either drops to 0 or continues to rise. Thus π_n can be interpreted as the fraction of time that the age of the process is between n and $n+1$. Recall from Eq. 3.44 (and the fact that residual life and age are equally distributed) that the distribution function of the time average age is given by $F_Z(n) = \int_0^n P(W>w)dw / E[W]$. Thus, the probability that the age is between n and $n+1$ is $F_Z(n+1) - F_Z(n)$. Since W is an integer random variable, this is $P(W>n)/E[W]$ in agreement with our result here. Thus, we have not learned anything new from this analysis, but it is reassuring to find the same answer with a different approach.

5.2 BRANCHING PROCESSES

Branching processes provide a simple model for studying the population of various types of individuals from one generation to the next. The individuals could be photons in a photo-multiplier, particles in a cloud chamber, micro-organisms, insects, or branches in a data structure.

Let X_n be the number of individuals in generation n of some population. Each of these X_n individuals, independently of each other, produces a random number of offspring, and these offspring collectively make up generation $n+1$. More precisely, a *branching process* is a Markov chain in which the state X_n at time n models the number of individuals in generation n . Denote the individuals of generation n as $\{1, 2, \dots, X_n\}$ and let $Y_{k,n}$ be the number of offspring of individual k . The random variables $Y_{k,n}$ are defined to be IID over k and n , with a PMF $p_j = P(Y_{k,n}=j)$. The state at time $n+1$, namely the number of individuals in generation $n+1$, is

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{k,n} \quad (21)$$

Assume a given distribution for the initial state X_0 . The transition probability, $P_{ij} = P(X_{n+1}=j | X_n=i)$, is just the probability that $Y_{1,n}+Y_{2,n}+\dots+Y_{i,n}=j$. The zero state (i.e., the state in which there are *no* individuals) is a trapping state (i.e., $P_{00} = 1$) since no future offspring can arise in this case.

One of the most important issues about a branching process is the probability that the population dies out eventually. Naturally, if p_0 (the probability that an individual has no offspring) is zero, then each generation must be at least as large as the generation before, and the population cannot die out unless $X_0 = 0$. We assume in what follows that $p_0 > 0$ and $X_0 > 0$.

Recall that $F_{ij}(n)$ was defined as the probability, given $X_0=i$, that state j is entered between times 1 and n . From (4), this satisfies the iterative relation

$$F_{ij}(n) = P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(n-1) , \quad n > 1; \quad F_{ij}(1) = P_{ij} \quad (22)$$

The probability that the process dies out by time n or before, given $X_0=i$, is thus $F_{i0}(n)$. For the n^{th} generation to die out, starting with an initial population of i individuals, the descendants of each of those i individuals must die out. Since each individual generates descendants independently, we have $F_{i0}(n) = [F_{1,0}(n)]^i$ for all i and n . Because of this relationship, it is sufficient to find $F_{1,0}(n)$, which can be determined from (22). Observe that P_{1k} is just p_k , the probability that an individual will have k offspring. Thus, (22) becomes

$$F_{1,0}(n) = p_0 + \sum_{k=1}^{\infty} p_k [F_{1,0}(n-1)]^k = \sum_{k=0}^{\infty} p_k [F_{1,0}(n-1)]^k \quad (23)$$

Let $g(z) = \sum_k p_k z^k$ be the z transform of the number of an individual's offspring. Then (23) can be written as $F_{1,0}(n) = g(F_{1,0}(n-1))$. This iteration starts with $F_{1,0}(1) = p_0$. Figure 5.3 shows a graphical construction for evaluating $F_{1,0}(n)$. Having found $F_{1,0}(n)$ as an ordinate on the graph for a given value of n , we find the same value as an abscissa by drawing a horizontal line over to the straight line of slope 1; we then draw a vertical line back to the curve $g(z)$ to find $g(F_{1,0}(n)) = F_{1,0}(n+1)$.

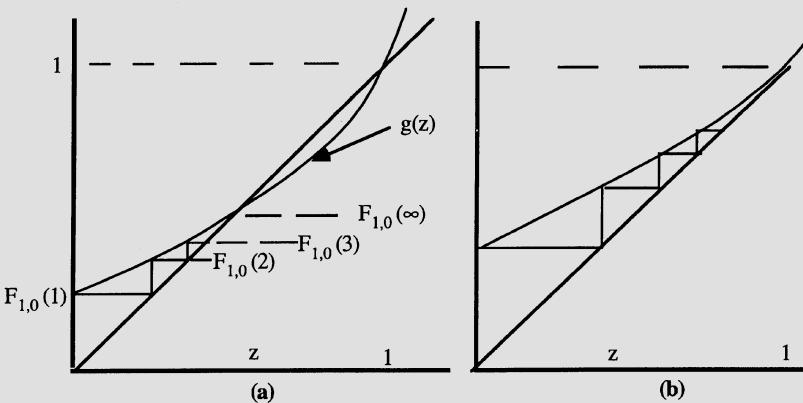


Figure 5.3. Graphical construction to find the probability that a population dies out.

For the two figures shown, it can be seen that $F_{1,0}(\infty)$ is equal to the smallest root of the equation $g(z)-z=0$. We next show that these two figures are representative of all possibilities. Since $g(z)$ is a z transform, we know that $g(1)=1$, so that $z=1$ is one root of $g(z)-z=0$. Also, $g'(1)=\bar{Y}$, where $\bar{Y}=\sum_k kp_k$ is the expected number of an individual's offspring. If $\bar{Y}>1$, as in figure 5.3a, then $g(z)-z$ is negative for z slightly smaller than 1. Also, for $z=0$, $g(z)-z=g(0)=p_0>0$. Since $g''(z)\geq 0$, there is exactly one root of $g(z)-z=0$ for $0<z<1$, and that root is equal to $F_{1,0}(\infty)$. By the same type of analysis, it can be seen that if $\bar{Y}\leq 1$, as in figure 5.3b, then there is no root of $g(z)-z=0$ for $z<1$, and $F_{1,0}(\infty)=1$.

As we saw earlier, $F_{1,0}(\infty)=[F_{1,0}(\infty)]^i$, so that for any initial population size, there is a probability strictly between 0 and 1 that successive generations eventually die out for $\bar{Y}>1$, and probability 1 that successive generations eventually die out for $\bar{Y}\leq 1$. Since state 0 is accessible from all i , but $F_{0i}(\infty)=0$, it follows from lemma 1 that all states other than state 0 are transient.

We next evaluate the expected number of individuals in a given generation. Conditional on $X_{n-1}=i$, (21) shows that the expected value of X_n is $i\bar{Y}$. Taking the expectation over X_{n-1} , we have

$$E[X_n] = \bar{Y} E[X_{n-1}] \quad (24)$$

Iterating this equation, we get

$$E[X_n] = \bar{Y}^n E[X_0] \quad (25)$$

Thus, if $\bar{Y} > 1$, the expected number of individuals in a generation increases exponentially with n , and \bar{Y} gives the rate of growth. Physical processes rarely grow exponentially forever, so branching processes are appropriate models of such physical processes only over some finite range of population.

It is important to realize that, for branching processes, the mean number of individuals is not a good measure of the actual number of individuals. For $\bar{Y} = 1$ and $X_0 = 1$, the expected number of individuals in each generation is 1, but the probability that $X_n = 0$ approaches 1 with increasing n ; this means that as n gets large, the n^{th} generation contains a large number of individuals with a very small probability and contains no individuals with a very large probability. For $\bar{Y} > 1$, we have just seen that there is a positive probability that the population dies out, but the expected number is growing exponentially.

A surprising result, which is derived from the theory of martingales in Chapter 7, is that if $X_0=1$ and $\bar{Y} > 1$, then the sequence of random variables X_n/\bar{Y}^n has a limit with probability 1. This limit is a random variable; it has the value 0 with probability $F_{1,0}(\infty)$, and has larger values with some given distribution. Intuitively, for large n , X_n is either 0 or very large. If it is very large, it tends to grow in an orderly way, increasing by a multiple of \bar{Y} in each subsequent generation.

5.3 BIRTH DEATH MARKOV CHAINS

A *birth death Markov chain* is a Markov chain in which the state space is the set of non-negative integers; for all $i \geq 0$, the transition probabilities satisfy $P_{i,i+1} > 0$ and $P_{i+1,i} > 0$, and for all $|i-j| > 1$, $P_{ij} = 0$ (see figure 5.4). A transition from state i to $i+1$ is regarded as a birth and one from $i+1$ to i as a death. Thus the restriction on the transition probabilities means that only one birth or death can occur in one unit of time. Many applications of birth death processes arise in queueing theory, where the state is the number of customers, births are customer arrivals, and deaths are customer departures. The restriction to only one arrival or departure at a time seems rather peculiar, but usually such a chain is a finely sampled approximation to a continuous time process, and the time increments are then small enough that multiple arrivals or departures in a time increment are unlikely and can be ignored in the limit.

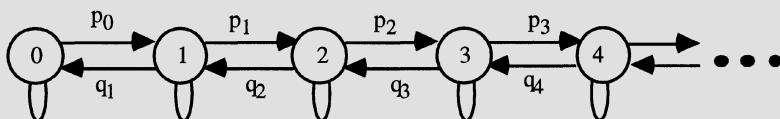


Figure 5.4. Birth Death Markov chain.

We denote $P_{i,i+1}$ by p_i and $P_{i,i-1}$ by q_i . Thus $P_{ii} = 1 - p_i - q_i$. There is an easy way to find the steady state probabilities of these birth death chains. In any sample function of the process, note that the number of transitions from state i to $i+1$ differs by at most 1 from the number of transitions from $i+1$ to i . If the process starts to the left of i and ends to the right, then one more $i \rightarrow i+1$ transition occurs than $i+1 \rightarrow i$, etc. Thus if we visualize a renewal reward process with renewals on occurrences of state i and unit reward on transitions from state i to $i+1$, the limiting time average number of transitions per unit time is $\pi_i p_i$. Similarly, the limiting time average number of transitions per unit time from $i+1$ to i is $\pi_{i+1} q_{i+1}$. Since these two must be equal in the limit,

$$\pi_i p_i = \pi_{i+1} q_{i+1} \quad (26)$$

The intuition in (26) is simply that the rate at which downward transitions occur from $i+1$ to i must equal the rate of upward transitions. Since this result is very important, both here and in our later study of continuous time birth death processes, we show that (26) also results from using the steady state equations in (14):

$$\pi_i = p_{i-1} \pi_{i-1} + (1-p_i-q_i)\pi_i + q_{i+1}\pi_{i+1} \quad ; \quad i > 0 \quad (27)$$

$$\pi_0 = (1-p_0)\pi_0 + q_1\pi_1 \quad (28)$$

From (28), $p_0 \pi_0 = q_1 \pi_1$. To see that (26) is satisfied for $i > 0$, we use induction on i , with $i=0$ as the base. Thus assume, for a given i , that $p_{i-1} \pi_{i-1} = q_i \pi_i$. Substituting this in (27), we get $p_i \pi_i = q_{i+1} \pi_{i+1}$, thus completing the inductive proof.

It is convenient to define ρ_i as p_i/q_{i+1} . Then we have $\pi_{i+1} = \rho_i \pi_i$, and iterating this,

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \rho_j \quad ; \quad \pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \rho_j} \quad (29)$$

If $\sum_{i \geq 1} \prod_{0 \leq j < i} \rho_j < \infty$, then π_0 is positive and all the states are positive recurrent. If this sum of products is infinite, none of the states are positive recurrent. If ρ_j is bounded below 1, say $\rho_j \leq 1-\epsilon$ for some fixed $\epsilon > 0$ and all sufficiently large j , then this sum of products will converge and the states will be positive recurrent.

For the simple birth death process of figure 5.1, if we define $\rho = q/p$, then $\rho_j = \rho$ for all j . For $\rho < 1$, (29) simplifies to $\pi_i = \pi_0 \rho^i$ for all $i \geq 0$, $\pi_0 = 1/\rho$, and thus $\pi_i = (1-\rho)\rho^i$ for $i \geq 0$. Exercise 5.2 shows how to find $F_{ij}(\infty)$ for all i, j in the case where $\rho \geq 1$.

5.4 REVERSIBLE MARKOV CHAINS

Many important Markov chains have the property that, in steady state, the sequence of states looked at backwards in time, i.e., $\dots, X_{n+1}, X_n, X_{n-1}, \dots$, has the same probabilistic structure as the sequence of states running forward in time. This equivalence between the forward chain and backward chain leads to a number of results that are intuitively quite surprising and that are quite difficult to derive without using this equivalence. We

shall study these results here and then extend them in Chapter 6 to Markov processes with a discrete state space. This set of ideas, and its use in queueing and queueing networks, has been an active area of queueing research over many years. It leads to many simple results for systems that initially look very complex. We only scratch the surface here and refer the interested reader to [Kel79] for a comprehensive treatment. Before going into reversibility, we describe the backward chain for an arbitrary Markov chain.

The defining characteristic of a Markov chain $\{X_n; n \geq 0\}$ is that for all $n \geq 0$,

$$P(X_{n+1} | X_n, X_{n-1}, \dots, X_0) = P(X_{n+1} | X_n) \quad (30)$$

For homogeneous chains, which we have been assuming throughout, $P(X_{n+1}=j | X_n=i) = P_{ij}$, independent of n . For any $k > 1$, we can extend (30) to get

$$\begin{aligned} P(X_{n+k}, X_{n+k-1}, \dots, X_{n+1} | X_n, X_{n-1}, \dots, X_0) \\ = P(X_{n+k} | X_{n+k-1}) P(X_{n+k-1} | X_{n+k-2}) \dots P(X_{n+1} | X_n) \\ = P(X_{n+k}, X_{n+k-1}, \dots, X_{n+1} | X_n) \end{aligned} \quad (31)$$

By letting E^+ be any event defined on the states X_{n+1} to X_{n+k} and letting E^- be any event defined on X_0 to X_{n-1} , this can be written more succinctly as

$$P(E^+ | X_n, E^-) = P(E^+ | X_n) \quad (32)$$

This says that, given state X_n , any future event E^+ is statistically independent of any past event E^- . This result, namely that past and future are independent given the present state, is equivalent to (30) for defining a Markov chain, but it has the advantage of showing the symmetry between past and future. This symmetry is best brought out by multiplying both sides of (32) by $P(E^- | X_n)$, obtaining

$$P(E^+, E^- | X_n) = P(E^+ | X_n) P(E^- | X_n) \quad (33)$$

This symmetric form says that, conditional on the current state, past and future are statistically independent. Dividing both sides by $P(E^+ | X_n)$ then yields

$$P(E^- | X_n, E^+) = P(E^- | X_n) \quad (34)$$

By letting E^- be X_{n-1} and E^+ be $X_{n+1}, X_{n+2}, \dots, X_{n+k}$, this becomes

$$P(X_{n-1} | X_n, X_{n+1}, \dots, X_{n+k}) = P(X_{n-1} | X_n)$$

This is the equivalent form to (30) for the backward chain, and says that the backward chain is also a Markov chain. By Bayes' law, $P(X_{n-1} | X_n)$ can be evaluated as

$$P(X_{n-1} | X_n) = \frac{P(X_n | X_{n-1})P(X_{n-1})}{P(X_n)} \quad (35)$$

Since the distribution of X_n can vary with n , $P(X_{n-1} | X_n)$ can also depend on n . Thus the backward Markov chain is not necessarily homogeneous. This should not be surprising, since the forward chain was defined with some arbitrary distribution for the initial state at time 0. This initial distribution was not relevant for equations (30) to (32), but as soon as $P(E | X_n)$ was introduced, the initial state implicitly became a part of each equation and destroyed the symmetry between past and future. For a chain in steady state, however, $P(X_n=j) = P(X_{n-1}=j) = \pi_j$ for all j , and we have

$$P(X_{n-1}=j | X_n=i) = P_{ji} \pi_j / \pi_i \quad (36)$$

Thus the backward chain is homogeneous if the forward chain is in steady state. For a chain with steady state probabilities $\{\pi_i; i \geq 0\}$, we define the backward transition probabilities P_{ij}^* as

$$\pi_i P_{ij}^* = \pi_j P_{ji} \quad (37)$$

The backward transition probability P_{ij}^* , for a Markov chain in steady state, is thus $P(X_{n-1}=j | X_n=i)$, the probability that the previous state is j given that the current state is i .

Now consider a new Markov chain with transition probabilities $\{P_{ij}^*\}$. Over some segment of time for which both this new chain and the old chain are in steady state, the set of states generated by the new chain is statistically indistinguishable from the backward running sequence of states from the original chain. It is somewhat simpler, in talking about forward and backward running chains, however, to visualize Markov chains running in steady state from $t=-\infty$ to $t=+\infty$. If one is uncomfortable with this, one can also visualize starting the Markov chain at some very negative time with the initial distribution equal to the steady state distribution.

A Markov chain is now defined to be *reversible* if $P_{ij}^* = P_{ji}$ for all states i and j . Thus the chain is reversible if, in steady state, the backward running sequence of states is statistically indistinguishable from the forward running sequence. Comparing (37) with the steady state equations (26) that we derived for birth death chains, we have the important theorem:

THEOREM 4: Every birth death chain with a steady state probability distribution is reversible.

We saw that for birth death chains, the equation $\pi_i P_{ij} = \pi_j P_{ji}$ (which only had to be considered for $|i-j| \leq 1$) provided a very simple way of calculating the steady state probabilities. Unfortunately, it appears that we must first calculate the steady state probabilities in order to show that a chain is reversible. The following simple theorem gives us a convenient escape from this dilemma.

THEOREM 5: Assume that an irreducible Markov chain has transition probabilities $\{P_{ij}\}$. Suppose $\{\pi_i\}$ is a set of positive numbers summing to 1 and satisfying

$$\pi_i P_{ij} = \pi_j P_{ji} ; \text{ all } i, j, \quad (38)$$

then, first, $\{\pi_i; i \geq 0\}$ is the steady state distribution for the chain, and, second, the chain is reversible.

Proof: Given a solution to (38) for all i and j , we can sum this equation over i for each j .

$$\sum_i \pi_i P_{ij} = \pi_j \sum_i P_{ji} = \pi_j \quad (39)$$

Thus the solution to (38), along with the constraints $\pi_i > 0$, $\sum_i \pi_i = 1$, satisfies the steady state equations, (14), and, from theorem 3, this is the unique steady state distribution. Since (38) is satisfied, the chain is also reversible.

It is often possible, sometimes by using an educated guess, to find a solution to (38). If this is successful, then we are assured both that the chain is reversible and that the actual steady state probabilities have been found.

Note that the theorem applies to periodic chains as well as to aperiodic chains. If the chain is periodic, then the steady state probabilities have to be interpreted as average values over the period, but, from theorem 3, (39) still has a unique solution (assuming an irreducible chain). On the other hand, for a chain with period d , there are d subclasses of states and the sequence $\{X_n\}$ must rotate between these classes in a fixed order. For this same order to be followed in the backward chain, the only possibility is $d = 2$. Thus periodic chains with periods other than 2 cannot be reversible.

There are several simple tests that can be used to show that some given irreducible chain is *not* reversible. First, the steady state probabilities must satisfy $\pi_i > 0$ for all i , and thus, if $P_{ij} > 0$ but $P_{ji} = 0$ for some i, j , then (38) cannot be satisfied and the chain is not reversible. Second, consider any set of three states, i, j, k . If $P_{ji} P_{ik} P_{kj}$ is unequal to $P_{jk} P_{ki} P_{ij}$ then the chain cannot be reversible. To see this, note that (38) requires that

$$\pi_i = \pi_j P_{ji} / P_{ij} = \pi_k P_{ki} / P_{ik}$$

Thus, $\pi_j P_{ji} P_{ik} = \pi_k P_{ki} P_{ij}$. Eq. (38) also requires that $\pi_j P_{jk} = \pi_k P_{kj}$. Taking the ratio of these equations, we see that $P_{ji} P_{ik} P_{kj} = P_{jk} P_{ki} P_{ij}$. Thus if this equation is not satisfied, the chain cannot be reversible. In retrospect, this result is not surprising. What it says is that for any cycle of three states, the probability of three transitions around the cycle in one direction must be the same as the probability of going around the cycle in the opposite (and therefore backwards) direction.

It is also true (see [Ros83] for a proof), that a necessary and sufficient condition for a chain to be reversible is that the product of transition probabilities around any cycle of arbitrary length must be the same as the product of transition probabilities going around the cycle in the opposite direction. This doesn't seem to be a widely useful way to demonstrate reversibility.

There is another result, similar to theorem 5, for finding the steady state probabilities of an arbitrary Markov chain and simultaneously finding the transition probabilities of the backward chain.

THEOREM 6: Assume that an irreducible Markov chain has transition probabilities $\{P_{ij}\}$. Suppose $\{\pi_i\}$ is a set of positive numbers summing to 1 and that $\{P^*_{ij}\}$ is a set of transition probabilities satisfying

$$\pi_i P_{ij} = \pi_j P^*_{ji} ; \text{ all } i, j \quad (40)$$

Then $\{\pi_i\}$ is the steady state distribution and $\{P^*_{ij}\}$ is the set of transition probabilities for the backward chain.

Proof: Summing (40) over i , we get the steady state equations for the Markov chain, so the fact that the given $\{\pi_i\}$ satisfy these equations asserts that they are the steady state probabilities. Eq. (40) then asserts that $\{P^*_{ij}\}$ is the set of transition probabilities for the backward chain.

As illustrated in section 5.6, one can sometimes guess the solution to (40) and thus find the steady state probabilities.

5.5 THE M/M/1 SAMPLED TIME MARKOV CHAIN

The M/M/1 Markov chain is a sampled time model of the M/M/1 queueing system. Recall that the M/M/1 queue has Poisson arrivals at some rate λ and IID exponentially distributed service times at some rate μ . We assume throughout this section that $\lambda < \mu$ (this is required to make the states positive recurrent). For some given small increment of time δ , we visualize observing the state of the system at the sample times $n\delta$. As indicated in figure 5.5, the probability of an arrival in the interval from $(n-1)\delta$ to $n\delta$ is modeled as $\lambda\delta$, independent of the state of the chain at time $(n-1)\delta$ and thus independent of all prior arrivals and departures. Thus the arrival process, viewed as arrivals in subsequent intervals of duration δ , is Bernoulli, thus approximating the Poisson arrivals. This is a sampled time approximation to the Poisson arrival process of rate λ for a continuous time M/M/1 queue

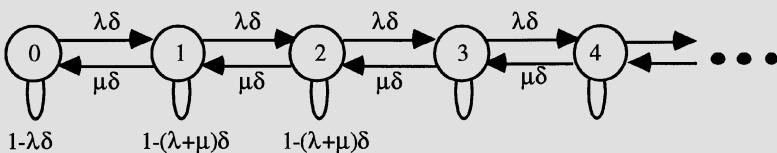


Figure 5.5. Sampled time approximation to M/M/1 queue for time increment δ .

When the system is non-empty (i.e., the state of the chain is one or more), the probability of a departure in the interval $(n-1)\delta$ to $n\delta$ is $\mu\delta$, thus modelling the exponential service times. When the system is empty, of course, departures cannot occur.

Note that in our sampled time model, there can be at most one arrival or departure in an interval $(n-1)\delta$ to $n\delta$. As in the Poisson process, the probability of more than one

arrival, more than one departure, or both an arrival and a departure in an increment δ is of order δ^2 for the actual continuous time M/M/1 system being modeled. Thus, for δ very small, we expect the sampled time model to be relatively good. At any rate, we can now analyze the model with no further approximations.

Since this chain is a birth death chain, we can use (29) to determine the steady state probabilities; they are

$$\pi_i = \pi_0 \rho^i ; \rho = \lambda/\mu < 1$$

Setting the sum of the π_i to 1, we find that $\pi_0 = 1 - \rho$, so

$$\pi_i = (1-\rho)\rho^i ; \text{ all } i \geq 0 \quad (41)$$

Thus the steady state probabilities exist and the chain is a birth death chain, so from theorem 4, it is reversible. We now exploit the consequences of reversibility to find some rather surprising results about the M/M/1 chain in steady state. Figure 5.6 illustrates a sample path of arrivals and departures for the chain. To avoid the confusion associated with the backward chain evolving backward in time, we refer to the original chain as the chain moving to the right and to the backward chain as the chain moving to the left.

There are two types of correspondence between the right moving and the left moving chain:

1. The left moving chain has the same Markov chain description as the right moving chain, and thus can be viewed as an M/M/1 chain in its own right. We still label the sampled time intervals from left to right, however, so that the left moving chain makes transitions from X_{n+1} to X_n to X_{n-1} . Thus, for example, if $X_n = i$ and $X_{n-1} = i+1$, the left moving chain has an arrival in the interval from $n\delta$ to $(n-1)\delta$.
2. Each sample function $\dots x_{n-1}, x_n, x_{n+1} \dots$ of the right moving chain corresponds to the same sample function $\dots x_{n+1}, x_n, x_{n-1} \dots$ of the left moving chain, where $X_{n-1} = x_{n-1}$ is to the left of $X_n = x_n$ for both chains. With this correspondence, an arrival to the right moving chain in the interval $(n-1)\delta$ to $n\delta$ is a departure from the left moving chain in the interval $n\delta$ to $(n-1)\delta$, and a departure from the right moving chain is an arrival to the left moving chain. Using this correspondence, each event in the left moving chain corresponds to some event in the right moving chain.

In each of the properties of the M/M/1 chain to be derived below, a property of the left moving chain is developed through correspondence 1 above, and then that property is translated into a property of the right moving chain by correspondence 2.

Property 1: Since the arrival process of the right moving chain is Bernoulli, the arrival process of the left moving chain is also Bernoulli (by correspondence 1). Looking

at a sample function x_{n+1}, x_n, x_{n-1} of the left moving chain (i.e., using correspondence 2), an arrival in the interval $n\delta$ to $(n-1)\delta$ of the left moving chain is a departure in the interval $(n-1)\delta$ to $n\delta$ of the right moving chain. Since the arrivals in successive increments of the left moving chain are independent and have probability $\lambda\delta$ in each increment δ , we conclude that departures in the right moving chain are similarly Bernoulli.

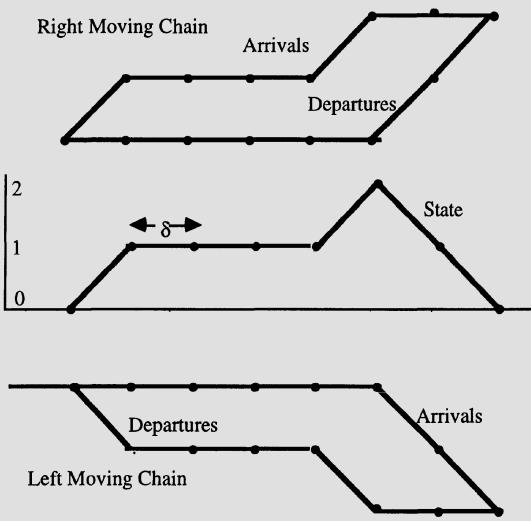


Figure 5.6. Sample function of $M/M/1$ chain over a busy period and corresponding arrivals and departures for right and left moving chains. Arrivals and departures are viewed as occurring between the sample times, and an arrival in the left moving chain between time $n\delta$ and $(n+1)\delta$ corresponds to a departure in the right moving chain between $(n+1)\delta$ and $n\delta$.

The fact that the departure process is Bernoulli with departure probability $\lambda\delta$ in each increment is surprising. Note that the probability of a departure in the interval $(n\delta - \delta, n\delta]$ is $\mu\delta$ conditional on $X_{n-1} \geq 1$ and is 0 conditional on $X_{n-1} = 0$. Since $P(X_{n-1} \geq 1) = 1 - P(X_{n-1} = 0) = \rho$, we see that the unconditional probability of a departure in the interval $(n\delta - \delta, n\delta]$ is $\rho\mu\delta = \lambda\delta$ as asserted above. The fact that successive departures are independent is much harder to derive without using reversibility (see exercise 5.12).

Property 2: In the original (right moving) chain, arrivals in the time increments after $n\delta$ are independent of X_n . Thus, for the left moving chain, arrivals in time increments to the left of $n\delta$ are independent of the state of the chain at $n\delta$. From the correspondence between sample paths, however, a left chain arrival is a right chain departure, so that for the right moving chain, departures in the time increments prior to $n\delta$ are indepen-

dent of X_n , which is equivalent to saying that the state X_n is independent of the prior departures. This means that if one observes the departures prior to time $n\delta$, one obtains no information about the state of the chain at $n\delta$. This is again a surprising result. To make it seem more plausible, note that an unusually large number of departures in an interval from $(n-m)\delta$ to $n\delta$ indicates that a large number of customers were probably in the system at time $(n-m)\delta$, but it doesn't appear to say much (and in fact it says exactly nothing) about the number remaining at $n\delta$. The following theorem summarizes these results.

THEOREM 7 (Burke's theorem for sampled time): Given an M/M/1 Markov chain in steady state with $\lambda < \mu$,

- a) the departure process is Bernoulli,
- b) the state X_n at any time $n\delta$ is independent of departures prior to $n\delta$.

The proof of Burke's theorem above did not use the fact that the departure probability is the same for all states except state 0. Thus these results remain valid for any birth death chain with Bernoulli arrivals that are independent of the current state (i.e., for which $P_{i,i+1} = \lambda\delta$ for all $i \geq 0$). One important example of such a chain is the sampled time approximation to an M/M/m queue. Here there are m servers, and the probability of departure from state i in an increment δ is $\mu_i\delta$ for $i \leq m$ and $\mu_m\delta$ for $i > m$. For the states to be recurrent, and thus for a steady state to exist, λ must be less than μ_m . Subject to this restriction, properties (a) and (b) above are valid for sampled time M/M/m queues.

5.6 ROUND-ROBIN AND PROCESSOR SHARING

Typical queueing systems have one or more servers who each serve customers in FCFS order, serving one customer completely while other customers wait. These typical systems have larger average delay than necessary. For example, if two customers with service requirements of 10 and 1 units respectively are waiting when a single server becomes empty, then serving the first before the second results in departures at times 10 and 11, for an average delay of 10.5. Serving the customers in the opposite order results in departures at times 1 and 11, for an average delay of 6. Supermarkets have recognized this for years and have special express checkout lines for customers with small service requirements.

Giving priority to customers with small service requirements, however, has some disadvantages; first, customers with high service requirements can feel discriminated against, and second, it is not always possible to determine the service requirements of customers before they are served. The following alternative to priorities is popular both in the computer and data network industries. When a processor in a computer system has many jobs to accomplish, it often serves these jobs on a time shared basis, spending a small increment of time on one, then the next, and so forth. In data networks, particularly high speed networks, messages are broken into small fixed length packets, and

then the packets from different messages can be transmitted on an alternating basis between messages.

A *round robin* service system is a system in which, if there are m customers in the system, say c_1, c_2, \dots, c_m , then c_1 is served for an incremental interval δ , followed by c_2 being served for an interval δ , and so forth up to c_m . After c_m is served for an interval δ , the server returns and starts serving c_1 for an interval δ again. Thus the customers are served in a cyclic, or “round robin” order, each getting a small increment of service on each visit from the server. When a customer’s service is completed, the customer leaves the system, m is reduced, and the server continues rotating through the now reduced cycle as before. When a new customer arrives, m is increased and the new customer must be inserted into the cycle of existing customers in a way to be discussed later.

Processor sharing is the limit of round robin service as the increment δ goes to zero. Thus, with processor sharing, if m customers are in the system, all are being served simultaneously, but each is being served at $1/m$ times the basic server rate. For the example of two customers with service requirement 1 and 10, each customer is initially served at rate $1/2$, so one customer departs at time 2. At that time, the remaining customer is served at rate 1 and departs at time 11. For round robin service with an increment of 1, the customer with unit service requirement departs at either time 1 or 2, depending on the initial order of service. With other increments of service, the results are slightly different.

We first analyze round robin service and then go to the processor sharing limit as $\delta \rightarrow 0$. As the above example suggests, the results are somewhat cleaner in the limiting case, but more realistic in the round robin case. Round robin provides a good example of the use of backward transition probabilities to find the steady state distribution of a Markov chain. The techniques used here are quite similar to those used in the next chapter to analyze queueing networks.

Assume a Bernoulli arrival process in which the probability of an arrival in an interval δ is $\lambda\delta$. Assume that the i^{th} arriving customer has a service requirement W_i . The random variables W_i , $i \geq 1$, are IID and independent of the arrival epochs. Thus, in terms of the arrival process and the service requirements, this is the same as an M/G/1 queue (see section 3.6), but with M/G/1 queues, the server serves each customer completely before going on to the next customer. We shall find that the round robin service here avoids the “slow truck effect” identified with the M/G/1 queue.

For simplicity, assume that W_i is arithmetic with span δ , taking on only values that are positive integer multiples of δ . Let $f(j) = P(W_i=j\delta)$, $j \geq 1$ and let $\bar{F}(j) = P(W_i > j\delta)$. Note that if a customer has already received j increments of service, then the probability that that customer will depart after 1 more increment is $f(j+1)/\bar{F}(j)$. This probability of departure on the next service increment after the j^{th} is denoted by

$$g(j) = f(j+1)/\bar{F}(j); j \geq 1 \quad (42)$$

The state s of a round robin system can be expressed as the number, m , of customers in the system, along with an ordered listing of how many service increments each of those m customers have received, i.e.,

$$\mathbf{s} = (m, z_1, z_2, \dots, z_m) \quad (43)$$

where $z_1\delta$ is the amount of service already received by the customer at the front of the queue, $z_2\delta$ is the service already received by the next customer in order, etc. In the special case of an idle queue, $\mathbf{s} = (0)$, which we denote as \emptyset .

Given that the state X_n at time $n\delta$ is $\mathbf{s} \neq \emptyset$, the state X_{n+1} at time $n\delta + \delta$ evolves as follows:

- i) A new arrival enters with probability $\lambda\delta$ and is placed at the front of the queue;
- ii) The customer at the front of the queue receives an increment δ of service;
- iii) The customer departs if service is complete;
- iv) Otherwise, the customer goes to the back of the queue.

It can be seen that the state transition depends, first, on whether a new arrival occurs (an event of probability $\lambda\delta$), and, second, on whether a departure occurs. If no arrival and no departure occurs, then the queue simply rotates. The new state is $\mathbf{s}' = r(\mathbf{s})$, where the rotation operator $r(\mathbf{s})$ is defined by $r(\mathbf{s}) = (m, z_2, \dots, z_m, z_1+1)$. If a departure but no arrival occurs, then the customer at the front of the queue receives its last unit of service and departs. The new state is $\mathbf{s}' = d(\mathbf{s})$, where the departure operator $d(\mathbf{s})$ is defined by $d(\mathbf{s}) = (m-1, z_2, \dots, z_m)$.

If an arrival occurs, the new customer receives one unit of service and goes to the back of the queue if more than one unit of service is required. In this case, the new state is $\mathbf{s}' = a(\mathbf{s})$ where the arrival operator $a(\mathbf{s})$ is defined by $a(\mathbf{s}) = (m+1, z_1, z_2, \dots, z_m, 1)$. If only one unit of service is required by a new arrival, the arrival departs and $\mathbf{s}' = \mathbf{s}$. In the special case of an empty queue, $\mathbf{s} = \emptyset$, the state is unchanged if either no arrival occurs or an arrival requiring one increment of service arrives. Otherwise, the new state is $\mathbf{s} = (1, 1)$, i.e., the one customer in the system has received one increment of service.

We next find the probability of each transition for $\mathbf{s} \neq \emptyset$. The probability of no arrival is $1 - \lambda\delta$. Given no arrival, and given a non-empty system, $\mathbf{s} \neq \emptyset$, the probability of a departure is $g(z_1) = f(z_1+1)/\bar{F}(z_1)$, i.e., the probability that one more increment of service allows the customer at the front of the queue to depart. Thus the probability of a departure is $(1 - \lambda\delta)g(z_1)$ and the probability of a rotation is $(1 - \lambda\delta)[1 - g(z_1)]$. Finally, the probability of an arrival is $\lambda\delta$, and given an arrival, the new arrival will leave the system after one unit of service with probability $g(0) = f(1)$. Thus the probability of an arrival and no departure is $\lambda\delta[1 - f(1)]$ and the probability of an unchanged system is $\lambda\delta f(1)$. To summarize, for $\mathbf{s} \neq \emptyset$,

$$\begin{aligned} P_{s,r(s)} &= (1 - \lambda\delta)[1 - g(z_1)] ; & r(\mathbf{s}) &= (m, z_2, \dots, z_m, z_1+1) \\ P_{s,d(s)} &= (1 - \lambda\delta)g(z_1) ; & d(\mathbf{s}) &= (m-1, z_2, \dots, z_m) \\ P_{s,a(s)} &= \lambda\delta[1 - f(1)] ; & a(\mathbf{s}) &= (m+1, z_1, z_2, \dots, z_m, 1) \\ P_{s,s} &= \lambda\delta f(1) \end{aligned} \quad (44)$$

For the special case of the idle state, $P_{\emptyset,\emptyset} = (1 - \lambda\delta) + \lambda\delta f(1)$ and $P_{\emptyset,(1,1)} = \lambda\delta(1 - f(1))$.

We now find the steady state distribution for this Markov chain by looking at the backward Markov chain. We will hypothesize backward transition probabilities, and then use theorem 6 to verify that the hypothesis is correct. Consider the backward transitions corresponding to each of the forward transitions in (44). A rotation in forward time causes the elements z_1, \dots, z_m in the state $s = (m, z_1, \dots, z_m)$ to rotate left, and the left most element (corresponding to the front of the queue) is incremented while rotating to the right end. The backward transition from $r(s)$ to s corresponds to the elements z_2, \dots, z_m, z_1+1 rotating to the right, with the right most element being decremented while rotating to the left end. If we view the transitions in backward time as a kind of round robin system, we see that the rotation is in the opposite direction from the forward time system.

In the backward time system, we view the numbers z_1, \dots, z_m in the state as the remaining service required before the corresponding customers can depart. Thus, these numbers decrease in the backward moving system. Also, since the customer rotation in the backward moving system is opposite to that in the forward moving system, z_m is the remaining service of the customer at the front of the queue, and z_1 is the remaining service of the customer at the back of the queue. We also view departures in forward time as arrivals in backward time. Thus the backward transition from $d(s) = (m-1, z_2, \dots, z_m)$ to $s = (m, z_1, \dots, z_m)$ corresponds to an arrival requiring z_1+1 units of service; the arrival goes to the front of the queue, receives one increment of service, and then goes to the back of the queue with z_1 increments of remaining service.

The nicest thing we could now hope for is that the arrivals in backward time are Bernoulli. This is a reasonable hypothesis to make, partly because it is plausible, and partly because it is easy to check via theorem 6. Fortunately, we shall find that it is valid. According to this hypothesis, the backward transition probability $P_{r(s),s}^*$ is given by $1-\lambda\delta$; that is, given that X_{n+1} is $r(s) = (m, z_2, \dots, z_m, z_1+1)$, and given that there is no arrival in the backward system at time $(n+1)\delta$, then the only possible state at time n is $s = (m, z_1, \dots, z_n)$. Next consider a backward transition from $d(s) = (m-1, z_2, \dots, z_n)$ to $s = (m, z_1, z_2, \dots, z_m)$. This corresponds to an arrival in the backward moving system; the arrival requires z_1+1 increments of service, one of which is provided immediately, leaving the arrival at the back of the queue with z_1 required increments of service remaining. The probability of this transition is $P_{d(s),s}^* = \lambda\delta f(z_1+1)$. Calculating the other backward transitions in the same way, the hypothesized backward transition probabilities are given by

$$\begin{aligned} P_{r(s),s}^* &= 1-\lambda\delta & P_{d(s),s}^* &= \lambda\delta f(z_1+1) \\ P_{a(s),s}^* &= 1-\lambda\delta & P_{s,s}^* &= \lambda\delta f(1) \end{aligned} \quad (45)$$

One should view (45) as an hypothesis for the backward transition probabilities. The arguments leading up to (45) are simply motivation for this hypothesis. If the hypothesis is correct, we can combine (44) and (45) to express the steady state equations of theorem 6 (for $s \neq \phi$) as

$$\pi_s P_{s,r(s)} = \pi_{r(s)} P_{r(s),s}^* ; \quad (1-\lambda\delta)[1-g(z_1)]\pi_s = (1-\lambda\delta)\pi_{r(s)} \quad (46)$$

$$\pi_s P_{s,d(s)} = \pi_{d(s)} P_{d(s),s}^* ; \quad (1-\lambda\delta)g(z_1)\pi_s = \lambda\delta f(z_1+1)\pi_{d(s)} \quad (47)$$

$$\pi_s P_{s,a(s)} = \pi_{a(s)} P_{a(s),s}^* ; \quad \lambda\delta[1-f(1)]\pi_s = (1-\lambda\delta)\pi_{a(s)} \quad (48)$$

$$\pi_s P_{s,s} = \pi_s P_{s,s}^* ; \quad \lambda\delta f(1)\pi_s = \lambda\delta f(1)\pi_s \quad (49)$$

We next show that (47), applied repeatedly, will allow us to solve for π_s (if λ is small enough for the states to be positive recurrent). Verifying that the solution also satisfies (46) and (48), will then verify the hypothesis. Since $f(z_1+1)/g(z_1)$ is $\bar{F}(z_1)$ from (42), we have

$$\pi_s = \frac{\lambda\delta}{1-\lambda\delta} \bar{F}(z_1) \pi_{d(s)} \quad (50)$$

For $m > 1$, $d(s) = (m-1, z_2, \dots, z_m)$, so we can apply (50) to $\pi_{d(s)}$, and substitute the result into (50), yielding

$$\pi_s = \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^2 \bar{F}(z_1) \bar{F}(z_2) \pi_{d(d(s))} \quad (51)$$

where $d(d(s)) = (m-2, z_3, \dots, z_m)$. Applying (50) repeatedly to $\pi_{d(d(s))}$, $\pi_{d(d(d(s)))}$, and so forth, we eventually get

$$\pi_s = \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^m \prod_{j=1}^m \bar{F}(z_j) \pi_\phi \quad (52)$$

Before this can be accepted as a steady state distribution, we must verify that it satisfies (46) and (48). The left hand side of (46) is $(1-\lambda\delta)[1-g(z_1)]\pi_s$, and, from (42), $1-g(z_1) = [\bar{F}(z_1)-f(z_1+1)]/\bar{F}(z_1) = \bar{F}(z_1+1)/\bar{F}(z_1)$. Thus using (52), the left side of (46) is

$$(1-\lambda\delta) \frac{\bar{F}(z_1+1)}{\bar{F}(z_1)} \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^m \prod_{j=1}^m \bar{F}(z_j) \pi_\phi = (1-\lambda\delta) \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^m \prod_{j=2}^m \bar{F}(z_j) \bar{F}(z_1+1) \pi_\phi$$

This is equal to $(1-\lambda\delta)\pi_{r(s)}$, verifying (46). Eq. (48) is verified in the same way. We now have to find whether there is a solution for π_ϕ such that these probabilities sum to 1. First define $P_m = \sum_{z_1, \dots, z_m} \pi_{(m, z_1, \dots, z_m)}$. This is the probability of m customers in the system. Whenever a new customer enters the system, it receives one increment of service immediately, so each $z_i \geq 1$. Using the hypothesized solution in (52),

$$P_m = \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^m \prod_{j=1}^m \left(\sum_{i=1}^{\infty} \bar{F}(i) \right) \pi_\phi \quad (53)$$

Since $\bar{F}(i) = P(W > i\delta)$, since W is arithmetic with span δ , and since the mean of a non-negative random variable is the integral of its complementary distribution function, we have

$$\delta \sum_{i=1}^{\infty} \bar{F}(i) = E[W] - \delta \quad (54)$$

$$P_m = \left(\frac{\lambda}{1-\lambda\delta} \right)^m (E[W]-\delta)^m \pi_\phi \quad (55)$$

Defining $\rho = [\lambda/(1-\lambda\delta)]\{E(W)-\delta\}$, we see that $P_m = \rho^m \pi_\phi$. If $\rho < 1$, then $\pi_\phi = 1-\rho$, and

$$P_m = (1-\rho)\rho^m ; m \geq 0 \quad (56)$$

The condition $\rho < 1$ is required for the states to be positive recurrent. The expected number of customers in the system for a round robin queue is $\sum_m m P_m = \rho/(1-\rho)$, and using Little's theorem, theorem 3.8, the expected delay is $\rho/[\lambda(1-\rho)]$. In using Little's theorem here, however, we are viewing the time a customer spends in the system as starting when the number m in the state increases; that is, if a customer arrives at time $n\delta$, it goes to the front of the queue and receives one increment of service, and then, assuming it needs more than one increment, the number m in the state increases at time $(n+1)\delta$. Thus the actual expected delay, including the original δ when the customer is being served but not counted in the state, is $\delta + \rho/[\lambda(1-\rho)]$.

The relation between ρ and $\lambda E[W]$ is shown in figure 5.7, and it is seen that $\rho < 1$ for $\lambda E[W] < 1$. The extreme case where $\lambda\delta = \lambda E[W]$ is the case for which each customer requires exactly one unit of service. Since at most one customer can arrive per time increment, the state always remains at $s=\phi$, and the delay is δ , i.e., the original increment of service received when a customer arrives.

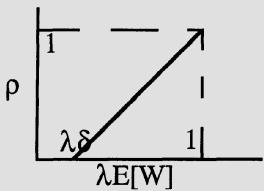


Figure 5.7. ρ as a function of $\lambda E[W]$ for given $\lambda\delta$.

Note that (56) is the same as the distribution of customers in the system for the M/M/1 Markov chain in (41), except for the anomaly in the definition of ρ here. We then have the surprising result that if round robin queueing is used rather than FCFS, then the distribution of the number of customers in the system is approximately the same as that for an M/M/1 queue. In other words, the slow truck effect associated with the M/G/1 queue has been eliminated.

Another remarkable feature of round robin systems is that one can also calculate the expected delay for a customer conditional on the required service of that customer. This is done in exercise 5.15, and it is found that the expected delay is linear in the required service.

Next we look at processor sharing by going to the limit as $\delta \rightarrow 0$. We first eliminate the assumption that the service requirement distribution is arithmetic with span δ . Assume that the server always spends an increment of time δ on the customer at the front of the queue, and if service is finished before the interval of length δ ends, the server is idle until the next sample time. The analysis of the steady state distribution above is still valid if we define $\bar{F}(j) = P(W > j\delta)$, and $f(j) = \bar{F}(j) - \bar{F}(j+1)$. In this case $\delta \sum_{i=1}^{\infty} F(i)$ lies between $E[W] - \delta$ and $E[W]$. As $\delta \rightarrow 0$, $\rho = \lambda E[W]$, and distribution of time in the system becomes identical to that of the M/M/1 system.

5.7 SEMI-MARKOV PROCESSES

Semi-Markov processes are generalizations of Markov chains in which the time intervals between transitions are random. To be specific, let $X(t)$ be the state of the process at time t and let $\{0, 1, 2, \dots\}$ denote the set of possible states (which can be finite or countably infinite). Let the random variables $S_1 < S_2 < S_3 < \dots$ denote the successive epochs at which state transitions occur. Let X_n be the new state entered at time S_n (i.e., $X_n = X(S_n)$, and $X(t) = X_n$ for $S_n \leq t < S_{n+1}$). Let $S_0 = 0$ and let X_0 denote the starting state at time 0 (i.e., $X_0 = X(0) = X(S_0)$). As part of the definition of a semi-Markov process, the sequence $\{X_n; n \geq 0\}$ is required to be a Markov chain, and the transition probabilities of that chain are denoted $\{P_{ij}; i \geq 0, j \geq 0\}$. This Markov chain is called the *embedded Markov chain* of the semi-Markov process. Thus, for $n \geq 1$,

$$P(X_n=j | X_{n-1}=i) = P[X(S_n)=j | X(S_{n-1})=i] = P_{ij} \quad (57)$$

Conditional on $X(S_{n-1})$, the state entered at S_n is independent of $X(t)$ for all $t < S_{n-1}$.

As the other part of the definition of a semi-Markov process, the intervals $U_n = S_n - S_{n-1}$ between successive transitions for $n \geq 1$ are random variables that depend only on the states $X(S_{n-1})$ and $X(S_n)$. More precisely, given X_{n-1} and X_n , the interval U_n is independent of the set of U_m for $m < n$ and independent of $X(t)$ for all $t < S_{n-1}$. The conditional distribution function for the intervals U_n is denoted by $G_{ij}(u)$, i.e.,

$$P(U_n \leq u | X_{n-1}=i, X_n=j) = G_{ij}(u) \quad (58)$$

The conditional mean of U_n , conditional on $X_{n-1}=i, X_n=j$, is denoted $\bar{U}(i, j)$, i.e.,

$$\bar{U}(i, j) = E[U_n | X_{n-1}=i, X_n=j] = \int_{u \geq 0} [1 - G_{ij}(u)] du \quad (59)$$

We can visualize a semi-Markov process evolving as follows: given an initial state, $X_0=i$ at time 0, a new state $X_1=j$ is selected according to the embedded chain with

probability P_{ij} . Then $U_1 = S_1$ is selected using the distribution $G_{ij}(u)$. Next a new state $X_2=k$ is chosen according to the probability P_{jk} ; then, given $X_1=j$ and $X_2=k$, the interval U_2 is selected with distribution function $G_{jk}(u)$. Successive state transitions and transition times are chosen in the same way. Because of this evolution from $X_0=i$, we see that $U_1 = S_1$ is a random variable, so S_1 is finite with probability 1. Also U_2 is a random variable, so that $S_2 = S_1 + U_2$ is a random variable and thus is finite with probability 1. By induction, S_n is a random variable and thus is finite with probability 1 for all $n \geq 1$. This proves the following simple lemma.

LEMMA 5: Let $M(t)$ be the number of transitions in a semi-Markov process in the interval $(0, t]$, (i.e., $S_{M(t)} \leq t < S_{M(t+1)}$) for some given initial state X_0 . Then $\lim_{t \rightarrow \infty} M(t) = \infty$ with probability 1.

Figure 5.8 shows an example of a semi-Markov process in which the transition times are deterministic but depend on the transitions. The important point that this example brings out is that the embedded Markov chain has steady state probabilities that are each 1/2. On the other hand, the semi-Markov process spends most of its time making long transitions from state 0 to state 1, and during these transitions the process is in state 0. This means that one of our first objectives must be to understand what steady state probabilities mean for a semi-Markov process.

In what follows, we assume that the embedded Markov chain is irreducible and positive recurrent. Define $\bar{U}(i)$ as the expected time in state i before a transition, i.e.,

$$\begin{aligned}\bar{U}(i) &= E[U_n | X_{n-1}=i] = \sum_j P_{ij} E[U_n | X_{n-1}=i, X_n=j] \\ &= \sum_j P_{ij} \bar{U}(i, j)\end{aligned}\quad (60)$$

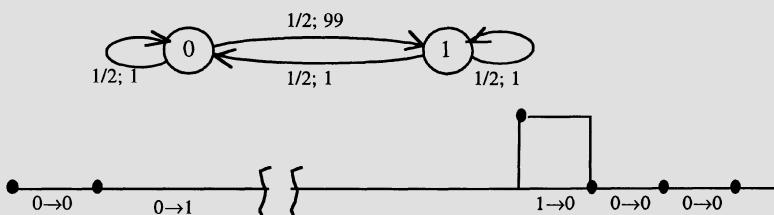


Figure 5.8. Example of a semi-Markov process with deterministic transition epochs. The label on each arc (i, j) in the graph gives P_{ij} followed by $\bar{U}(i, j)$. The solid dots on the sample function below the graph show the state transition epochs and show the new states entered. Note that the state at S_n is the new state entered, i.e., X_n , and the state remains X_n in the interval $[S_n, S_{n+1})$.

The steady state probabilities $\{\pi_i\}$ for the embedded chain tell us the fraction of transitions that enter any given state i . Since $\bar{U}(i)$ is the expected holding time in i per

transition into i , we would guess that the fraction of time spent in state i should be proportional to $\pi_i \bar{U}(i)$. Normalizing, we would guess that the time average probability of being in state i should be

$$p_i = \pi_i \bar{U}(i) / \sum_j \pi_j \bar{U}(j) \quad (61)$$

By now, it should be no surprise that renewal theory is the appropriate tool to make this precise. We continue to assume an irreducible positive recurrent chain; this ensures that the steady state probabilities for the embedded chain exist and are positive. It is possible for the denominator in (61) to be infinite. This can happen either because $\bar{U}(i)$ is infinite for some i , or because the sum does not converge (for example, there could be a countably infinite number of states and $\bar{U}(i)$ could be proportional to $1/\pi_i$). When the denominator is infinite, we say that the probabilities $\{p_i\}$ do not exist; this is a bizarre special case, and it is discussed further in section 1 of Chapter 6, but it doesn't have to be specifically excluded from the following analysis.

LEMMA 6: Consider a semi-Markov process with an irreducible recurrent embedded chain $\{X_n; n \geq 0\}$. Given $X_0=i$, let $\{M_{ij}(t); t \geq 0\}$ be the number of transitions into a given state j in the interval $(0, t]$. Then $\{M_{ij}(t); t \geq 0\}$ is a delayed renewal process (or, if $j=i$, is an ordinary renewal process).

Proof: Let $M(t)$ be the total number of state transitions over all states that occur in the interval $(0, t]$. From lemma 5, $\lim_{t \rightarrow \infty} M(t) = \infty$ with probability 1. Let $N_{ij}(n)$ be the number of transitions into state j that occur in the embedded Markov chain by the n^{th} transition of the embedded chain. From lemma 4, $\{N_{ij}(n); n \geq 0\}$ is a delayed renewal process. It follows from lemma 2 of Chapter 3 that $\lim_{n \rightarrow \infty} N_{ij}(n) = \infty$ with probability 1. Since $M_{ij}(t)$ is the number of transitions into j during the first $M(t)$ transitions of the embedded chain, we have $M_{ij}(t) = N_{ij}(M(t))$. Thus,

$$\lim_{t \rightarrow \infty} M_{ij}(t) = \lim_{t \rightarrow \infty} N_{ij}(M(t)) = \lim_{n \rightarrow \infty} N_{ij}(n) = \infty$$

It follows that the time W_1 at which the first transition into state j occurs, and the subsequent interval W_2 until the next transition, are both finite with probability 1. Subsequent intervals have the same distribution as W_2 , and all intervals are independent, so $\{M_{ij}(t); t \geq 0\}$ is a delayed renewal process with inter-renewal intervals $\{W_k; k \geq 1\}$. If $i=j$, then all W_k are identically distributed and we have an ordinary renewal process, completing the proof.

Let $\bar{W}(j)$ be the mean inter-renewal interval between successive transitions into state j (i.e., the mean of the inter-renewal intervals W_2, W_3, \dots in $\{M_{ij}(t); t \geq 0\}$). Consider a delayed renewal reward process defined on $\{M_{ij}(t); t \geq 0\}$ for which $R(t)=1$ whenever $X(t)=j$ (see figure 5.9). Define p_j as the time average fraction of time spent in state j . Then, if $\bar{U}(i) < \infty$, theorems 6 and 12 of Chapter 3 state that

$$p_j = \lim_{t \rightarrow \infty} \frac{\int_0^t R(\tau) d\tau}{t} = \frac{\bar{U}(j)}{\bar{W}(j)} \quad \text{with probability 1} \quad (62)$$

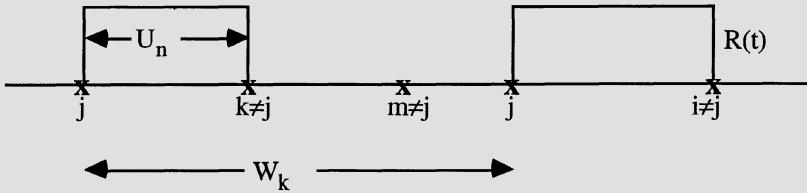


Figure 5.9. The delayed renewal reward process for time in state j . The reward is one from an entry into state j until the next transition, and the expected duration of such an interval U_n is $\bar{U}(j)$; the inter-renewal interval W_k lasts until the next entry into state j , with expected duration $\bar{W}(j)$.

We can also investigate the limit, as $t \rightarrow \infty$, of the probability that $X(t)=j$. This is equal to $E[R(t)]$ for the renewal reward process above. From Eq. 71 of Chapter 3, if the distribution of the inter-renewal time is non-arithmetic, then

$$p_j = \lim_{t \rightarrow \infty} E[R(t)] = \frac{\bar{U}(j)}{\bar{W}(j)} \quad (63)$$

Next we must express the mean inter-renewal time, $\bar{W}(j)$, in terms of more accessible quantities. From theorem 9 of Chapter 3,

$$\lim_{t \rightarrow \infty} M_{ij}(t)/t = 1/\bar{W}(j) \quad \text{with probability 1} \quad (64)$$

As before, $M_{ij}(t) = N_{ij}(M(t))$ where, given $X_0=i$, $M(t)$ is the total number of transitions in $(0,t]$ and $N_{ij}(n)$ is the number of transitions into state j in the embedded Markov chain by the n^{th} transition. Lemma 5 shows that $\lim_{t \rightarrow \infty} M(t) = \infty$ with probability 1, so

$$\lim_{t \rightarrow \infty} \frac{M_{ij}(t)}{M(t)} = \lim_{t \rightarrow \infty} \frac{N_{ij}(M(t))}{M(t)} = \lim_{n \rightarrow \infty} \frac{N_{ij}(n)}{n} = \pi_j \quad (65)$$

Combining (64) and (65), we have

$$\frac{1}{\bar{W}(j)} = \lim_{t \rightarrow \infty} \frac{M_{ij}(t)}{t} = \lim_{t \rightarrow \infty} \frac{M_{ij}(t)}{M(t)} \quad (66)$$

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = \pi_j \lim_{t \rightarrow \infty} \frac{M(t)}{t}$$

Substituting this in (62), we see that $p_j = \pi_j \bar{U}(j) \lim_{t \rightarrow \infty} \{M(t)/t\}$. Since $\lim_{t \rightarrow \infty} \{M(t)/t\}$ is independent of j , and since $\sum_j p_j = 1$, we see that $\lim_{t \rightarrow \infty} \{M(t)/t\}$ must be equal to $(\sum_j \pi_j \bar{U}(j))^{-1}$, thus yielding (61). Summarizing, we have the following theorem:

THEOREM 8: Assume that the embedded Markov chain of a semi-Markov process is irreducible and positive recurrent. If $\sum_i \pi_i \bar{U}(i) < \infty$, then, with probability 1, the limiting fraction of time spent in state j is $p_j = \pi_j \bar{U}(j) / \sum_i \pi_i \bar{U}(i)$.

From (63), p_j is also equal to $\lim_{t \rightarrow \infty} P(X(t)=j)$ if the distribution of the inter-renewal interval between transitions into j is non-arithmetic. A sufficient condition for this (assuming that $\sum_i \pi_i \bar{U}_i < \infty$) is that $G_{kj}(u)$ be a non-arithmetic distribution for at least one pair of states k, j such that $P_{kj} > 0$. It is not hard to see that if the distribution of inter-renewal intervals for one value of j is arithmetic with span d , then the distribution of inter-renewal intervals for each i is arithmetic with the same span (see exercise 5.19).

For the example of figure 5.8, we see by inspection that $\bar{U}(1) = 50$ and $\bar{U}(2) = 1$. Thus $p_1 = 50/51$, and $p_2 = 1/51$.

For a semi-Markov process, knowing the probability that $X(t) = j$ for large t does not completely specify the steady state behavior. Another important steady state question is to determine the fraction of time involved in i to j transitions. To make this notion precise, define $Y(t)$ as the residual time until the next transition after time t (i.e., $t+Y(t)$ is the epoch of the next transition after time t). We want to determine the fraction of time t over which $X(t)=i$ and $X(t+Y(t))=j$. Equivalently, for a non-arithmetic process, we want to determine $P(X(t)=i, X(t+Y(t))=j)$ in the limit as $t \rightarrow \infty$. Call this limit $Q(i, j)$.

Consider a renewal process, starting in state i and with renewals on transitions to state i . Define a reward $R(t) = 1$ for $X(t)=i, X(t+Y(t))=j$ and $R(t) = 0$ otherwise (see figure 5.10). That is, for each n such that $X(S_n)=i$ and $X(S_{n+1})=j$, $R(t)=1$ for $S_n \leq t < S_{n+1}$. The expected reward in an inter-renewal interval is then $P_{ij} \bar{U}(i, j)$. It follows that $Q(i, j)$ is given by

$$Q(i, j) = \lim_{t \rightarrow \infty} \frac{\int_0^t R(\tau) d\tau}{t} = \frac{P_{ij} \bar{U}(i, j)}{\bar{W}(i)} = \frac{P_i P_{ij} \bar{U}(i, j)}{\bar{U}(i)} \quad (67)$$

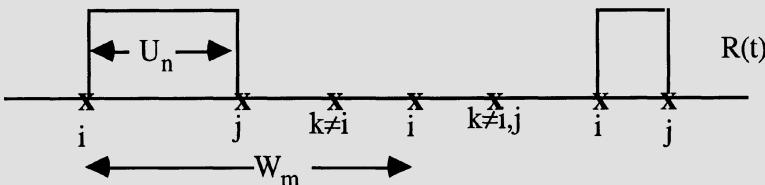


Figure 5.10. The renewal reward process for i to j transitions. The expected value of U_n for i into j is $\bar{U}(i, j)$ and the expected interval between entries to i is $\bar{W}(i)$.

5.8 EXAMPLE—M/G/1 QUEUE

As one example of a semi-Markov chain, consider an M/G/1 queue. Rather than the usual interpretation in which the state of the system is the number of customers in the system, we view the state of the system as changing only at departure times; the new state at a departure time is the number of customers left behind by the departure. This state then remains fixed until the next departure. New customers still enter the system according to the Poisson arrival process, but these new customers are not considered as part of the state until the next departure time. The number of customers in the system at arrival epochs does not in general constitute a “state” for the system, since the age of the current service is also necessary as part of the statistical characterization of the process.

One purpose of this example is to illustrate that it is often more convenient to visualize the transition interval $U_n = S_n - S_{n-1}$ as being chosen first and the new state X_n as being chosen second rather than choosing the state first and the transition time second. For the M/G/1 queue, first suppose that the state is some $i > 0$. In this case, service begins on the next customer immediately after the old customer departs. Thus, U_n , conditional on $X_n = i$ for $i > 0$, has the distribution of the service time, say $G(u)$. The mean interval until a state transition occurs is

$$\bar{U}(i) = \int_0^\infty [1-G(u)]du ; \quad i > 0 \quad (68)$$

Given the interval u for a transition from state $i > 0$, the number of arrivals in that period is a Poisson random variable with mean λu , where λ is the Poisson arrival rate. Since the next state j is the old state i , plus the number of new arrivals, minus the single departure,

$$P(X_{n+1} = j | X_n = i, U_n = u) = \frac{(\lambda u)^{j-i+1} \exp(-\lambda u)}{(j-i+1)!} \quad (69)$$

for $j \geq i-1$. For $j < i-1$, the probability above is 0. The unconditional probability P_{ij} of a transition from i to j can then be found by multiplying the right side of (69) by the probability density $g(u)$ of the service time and integrating over u .

$$P_{ij} = \int_0^\infty \frac{g(u)(\lambda u)^{j-i+1} \exp(-\lambda u)}{(j-i+1)!} du ; \quad j \geq i-1, i > 0 \quad (70)$$

For the case $i=0$, the server must wait until the next arrival before starting service. Thus the expected time from entering the empty state until a service completion is

$$\bar{U}(0) = (1/\lambda) + \int_0^\infty [1-G(u)] du \quad (71)$$

We can evaluate P_{0j} by observing that the departure of that first arrival leaves j customers in this system iff j customers arrive during the service time of that first customer; i.e., the new state doesn't depend on how long the server waits for a new customer to serve, but only on the arrivals while that customer is being served. Letting $g(u)$ be the density of the service time,

$$P_{0j} = \int_0^\infty \frac{g(u)(\lambda u)^j \exp(-\lambda u)}{j!} du ; \quad j \geq 0 \quad (72)$$

5.9 SUMMARY

This chapter extended the finite state Markov chain results of Chapter 4 to the case of countably infinite state spaces. It also provided an excellent example of how renewal processes can be used for understanding other kinds of processes. In section 5.1, the first passage time random variables were used to construct renewal processes with renewals on successive transitions to a given state. These renewal processes were used to rederive the basic properties of Markov chains using renewal theory as opposed to the algebraic Perron-Frobenius approach of Chapter 4. The central result of this was theorem 3, which showed that, for an irreducible chain, the states are positive recurrent iff the steady state equations, (14), have a solution. Also if (14) has a solution, it is positive and unique. We also showed that these steady state probabilities are, with probability 1, time averages for sample paths, and that, for an ergodic chain, they are limiting probabilities independent of the starting state.

We found that the major complications that result from countably infinite state spaces are, first, different kinds of transient behavior, and second, the possibility of null recurrent states. For finite state Markov chains, a state is transient only if it can reach some other state from which it can't return. For countably infinite chains, there is also the case, as in figure 5.1 for $p > 1/2$, where the state just wanders away, never to return. Null recurrence is a limiting situation where the state wanders away and returns with probability 1, but with an infinite expected time. There is not much engineering significance to null recurrence; it is highly sensitive to modeling details over the entire infinite set of states. One usually uses countably infinite chains to simplify models; for example, if a buffer is very large and we don't expect it to overflow, we assume it is infinite. Finding out, then, that the chain is transient or null recurrent simply means that the modeling assumption was not very good.

Branching processes were introduced in section 5.2 as a model to study the growth of various kinds of elements that reproduce. In general, for these models (assuming $p_0 > 0$), there is one trapping state and all other states are transient. Figure 5.3 showed how to find the probability that the trapping state is entered by the n^{th} generation, and also the probability that it is entered eventually. If the expected number of offspring of an element is at most 1, then the population dies out with probability 1, and otherwise, the population dies out with some given probability q , and grows without bound with probability $1-q$.

We next studied birth death Markov chains and reversibility. Birth death chains are widely used in queueing theory as sample time approximations for systems with Poisson arrivals and various generalizations of exponentially distributed service times. Eq. (29) gives their steady state probabilities if positive recurrent, and shows the condition under which they are positive recurrent. We showed that these chains are reversible if they are positive recurrent.

Theorems 5 and 6 provided a simple way to find the steady state distribution of reversible chains and also of chains where the backward chain behavior could be hypothesized or deduced. We used reversibility to show that M/M/1 and M/M/m Markov chains satisfy Burke's theorem for sampled time—namely that the departure process is Bernoulli, and that the state at any time is independent of departures before that time.

Round robin queueing was then used as a more complex example of how to use the backward process to deduce the steady state distribution of a rather complicated Markov chain; this also gave us added insight into the behavior of queueing systems and allowed us to show that, in the processor sharing limit, the distribution of number of customers is the same as that in an M/M/1 queue.

Finally, semi-Markov processes were introduced. Renewal theory again provided the key to analyzing these systems. Theorem 8 showed how to find the steady state probabilities of these processes, and it was shown that these probabilities could be interpreted both as time averages and, in the case of non-arithmetic transition times, as limiting probabilities in time.

For further reading on Markov chains with countably infinite state spaces, see [Fel66], [Ros83], or [Wol89]. Feller is particularly complete, but Ross and Wolff are somewhat more accessible. Harris, [Har63] is the standard reference on branching processes and Kelly, [Kel79] is the standard reference on reversibility. The material on round robin systems is from [Yat90] and is generalized there.

EXERCISES

5.1) Let $\{P_{ij}; i, j \geq 0\}$ be the set of transition probabilities for a countably infinite Markov chain. For each i, j , let $F_{ij}(n)$ be the probability that state j occurs sometime between time 1 and n inclusive, given $X_0=i$. For some given j , assume that $\{x_k; k \geq 0\}$ is a set of non-negative numbers satisfying $x_i = P_{ij} + \sum_{k \neq j} P_{ik}x_k$. Show that $x_i \geq F_{ij}(n)$ for all n and i , and hence that $x_i \geq F_{ij}(\infty)$ for all i . Hint: use induction.

5.2) a) For the Markov chain in figure 5.1, show that, for $p \geq 1/2$, $F_{00}(\infty) = 2(1-p)$ and show that $F_{i0}(\infty) = [(1-p)/p]^i$ for $i \geq 1$. Hint: First show that this solution satisfies (5) and then show that (5) has no smaller solution (see exercise 5.1). Note that you have shown that the chain is transient for $p > 1/2$ and that it is recurrent for $p = 1/2$.

b) Under the same conditions as part (a), show that $F_{ij}(\infty)$ equals $2(1-p)$ for $j=i$, equals $[(1-p)/p]^{i-j}$ for $i > j$, and equals 1 for $i < j$.

5.3) Let j be a transient state in a Markov chain and let j be accessible from i . Show that i is transient also. Interpret this as a form of Murphy's law (if something bad can

happen, it will, where the bad thing is the lack of an eventual return). Note: Give a direct demonstration rather than using lemma 1.

5.4) Consider an irreducible positive recurrent Markov chain. Consider the renewal process $\{N_{ij}(t); t \geq 0\}$ where, given $X_0=j$, $N_{ij}(t)$ is the number of times that state j is visited from time 1 to t . For each $i \geq 0$, consider a renewal reward function $R_i(t)$ equal to 1 whenever the chain is in state i and equal to 0 otherwise. Let π_i be the time average reward.

a) Show that $\pi_i = 1/\bar{T}_{ii}$ for each i with probability 1.

b) Show that $\sum_i \pi_i = 1$. Hint: Consider $\sum_{i \leq M} \pi_i$ for any integer M .

c) Consider a renewal reward function $R_{ij}(t)$ that is 1 whenever the chain is in state i and the next state is state j . $R_{ij}(t) = 0$ otherwise. Show that the time average reward is equal to $\pi_i P_{ij}$ with probability 1. Show that $\pi_k = \sum_i \pi_i P_{ik}$ for all k .

5.5) Let $\{X_n; n \geq 0\}$ be a branching process with $X_0=1$. Let \bar{Y}, σ^2 be the mean and variance of the number of offspring of an individual.

a) Argue that $\lim_{n \rightarrow \infty} X_n$ exists with probability 1 and either has the value 0 (with probability $P_{10}(\infty)$) or the value ∞ (with probability $1 - P_{10}(\infty)$).

b) Show that $\text{VAR}(X_n) = \sigma^2 \bar{Y}^{n-1} (\bar{Y} - 1)/(\bar{Y} - 1)$ for $\bar{Y} \neq 1$ and $\text{VAR}(X_n) = n\sigma^2$ for $\bar{Y} = 1$.

5.6) There are n states and for each pair of states i and j , a positive number $d_{ij} = d_{ji}$ is given. A particle moves from state to state in the following manner: Given that the particle is in any state i , it will next move to any $j \neq i$ with probability P_{ij} given by

$$P_{ij} = \frac{d_{ij}}{\sum_{j \neq i} d_{ij}}$$

Assume that $P_{ii} = 0$ for all i . Show that the sequence of positions is a reversible Markov chain and find the limiting probabilities.

5.7) Consider a reversible Markov chain with transition probabilities P_{ij} and limiting probabilities π_j . Also consider the same chain truncated to the states $0, 1, \dots, M$. That is, the transition probabilities $\{P'_{ij}\}$ of the truncated chain are

$$P'_{ij} = \begin{cases} \frac{P_{ij}}{\sum_{k=0}^M P_{ik}} & ; \quad 0 \leq i, j \leq M \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Show that the truncated chain is also reversible and has limiting probabilities given by

$$\bar{\pi}_i = \frac{\pi_i \sum_{j=0}^M P_{ij}}{\sum_{k=0}^M \pi_k \sum_{m=0}^M P_{km}}$$

5.8) A Markov chain (with states $\{0, 1, 2, \dots, J-1\}$ where J is either finite or infinite) has transition probabilities $\{P_{ij}; i, j \geq 0\}$. Assume that $P_{0j} > 0$ for all $j > 0$ and $P_{j0} > 0$ for all $j > 0$. Also assume that for all i, j, k , $P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji}$.

a) Assuming also that all states are positive recurrent, show that the chain is reversible and find the steady state probabilities $\{\pi_i\}$ in simplest form.

b) Find a condition on $\{P_{0j}; j \geq 0\}$ and $\{P_{j0}; j \geq 0\}$ that is sufficient to ensure that all states are positive recurrent.

5.9) a) Use the birth and death model described in figure 5.4 to find the steady state probability mass function for the number of customers in the system (queue plus service facility) for the following queues:

- i) M/M/1 with arrival probability $\lambda\delta$, service completion probability $\mu\delta$.
- ii) M/M/m with arrival probability $\lambda\delta$, service completion probability $i\mu\delta$ for i servers busy, $1 \leq i \leq m$.
- iii) M/M/ ∞ with arrival probability $\lambda\delta$, service probability $i\mu\delta$ for i servers. Assume δ so small that $i\mu\delta < 1$ for all i of interest.

Assume the system is positive recurrent.

b) For each of the queues above give necessary conditions (if any) for the states in the chain to be (i) transient, (ii) null recurrent, (iii) positive recurrent.

- c) For each of the queues find:

L = (steady state) mean number of customers in the system.

L_q = (steady state) mean number of customers in the queue.

W = (steady state) mean waiting time in the system.

W_q = (steady state) mean waiting time in the queue.

5.10) a) Given that an arrival occurs in the interval $(n\delta, (n+1)\delta)$ for the sampled time M/M/1 model in figure 5.5, find the conditional PMF of the state of the system at time $n\delta$ (assume n arbitrarily large and assume positive recurrence).

b) For the same model, find the expected number of customers seen in the system by the first arrival after time $n\delta$. Note: The purpose of this exercise is to make you cautious about the meaning of “the state seen by a random arrival.”

5.11) Find the backward transition probabilities for the Markov chain model of age in figure 5.2. Draw the graph for the backward Markov chain, and interpret it as a model for residual life.

5.12) Consider the sample time approximation to the M/M/1 queue in figure 5.5.

- a) Give the steady state probabilities for this chain (no explanations or calculations required—just the answer).

In parts (b) to (g) do not use reversibility and do not use Burke's theorem. Let X_n be the state of the system at time $n\delta$ and let D_n be a random variable taking on the value 1 if a departure occurs between $n\delta$ and $(n+1)\delta$, and the value 0 if no departure occurs. Assume that the system is in steady state at time $n\delta$.

- b) Find $P(X_n=i, D_n=j)$ for $i \geq 0, j = 0, 1$

- c) Find $P(D_n=1)$

- d) Find $P(X_n=i | D_n=1)$ for $i \geq 0$

- e) Find $P(X_{n+1}=i | D_n=1)$ and show that X_{n+1} is statistically independent of D_n .

Hint: Use part (d); also show that $P(X_{n+1}=i) = P(X_{n+1}=i | D_n=1)$ for all $i \geq 0$ is sufficient to show independence.

f) Find $P(X_{n+1}=i, D_{n+1}=j | D_n)$ and show that the pair of variables (X_{n+1}, D_{n+1}) is statistically independent of D_n .

g) For each $k > 1$, find $P(X_{n+k}=i, D_{n+k}=j | D_{n+k-1}, D_{n+k-2}, \dots, D_n)$ and show that the pair (X_{n+k}, D_{n+k}) is statistically independent of $(D_{n+k-1}, D_{n+k-2}, \dots, D_n)$. Hint: Use induction on k ; as a substep, find $P(X_{n+k}=i | D_{n+k-1}=1, D_{n+k-2}, \dots, D_n)$ and show that X_{n+k} is independent of $D_{n+k-1}, D_{n+k-2}, \dots, D_n$.

- h) What do your results mean relative to Burke's theorem?

5.13) Let $\{X_n, n \geq 1\}$ denote an irreducible Markov chain having a countable state space. Now consider a new stochastic process $\{Y_n, n \geq 0\}$ that only accepts values of the Markov chain that are between 0 and some integer m . For instance, if $m = 3$ and $X_1 = 1, X_2 = 3, X_3 = 5, X_4 = 6, X_5 = 2$, then $Y_1 = 1, Y_2 = 3, Y_3 = 2$.

- a) Is $\{Y_n, n \geq 0\}$ a Markov chain? Explain briefly.

b) Let π_j denote the proportion of time that $\{X_n, n \geq 1\}$ is in state j . If $\pi_j > 0$ for all j , what proportion of time is $\{Y_n, n \geq 0\}$ in each of the states $0, 1, \dots, m$?

c) Suppose $\{X_n\}$ is null recurrent and let $\pi_i(m)$, $i = 0, 1, \dots, m$ denote the long-run proportions for $\{Y_n, n \geq 0\}$. Show that

$$\pi_j(m) = \pi_i(m)E[\text{time the } X \text{ process spends in } j \text{ between returns to } i], \quad j \neq i.$$

5.14) Verify that (48) is satisfied by the hypothesized solution to π in (52). Also show that the equations involving the idle state ϕ are satisfied.

5.15) Replace the state $\mathbf{m} = (m, z_1, \dots, z_m)$ in section 5.6 with an expanded state $\mathbf{m} = (m, z_1, w_1, z_2, w_2, \dots, z_m, w_m)$ where m and $\{z_i; 1 \leq i \leq m\}$ are as before and w_1, w_2, \dots, w_m are the original service requirements of the m customers.

a) Hypothesizing the same backward round robin system as hypothesized in section 5.6, find the backward transition probabilities and give the corresponding equations to (46–49) for the expanded state description.

- b) Solve the resulting equations to show that

$$\pi_m = \pi_0 \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^m \prod_{j=1}^m f(w_j)$$

- c) Show that the probability that there are m customers in the system, and that those customers have original service requirements given by w_1, \dots, w_m , is

$$P(m, w_1, \dots, w_m) = \pi_0 \left(\frac{\lambda\delta}{1-\lambda\delta} \right)^m \prod_{j=1}^m (w_j - 1) f(w_j)$$

- d) Given that a customer has original service requirement w , find the expected time that customer spends in the system.

5.16) A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability 1/3 and to 3 with probability 2/3. From 3 it always goes to 1. The mean time between locations i and j are $t_{12} = 20$, $t_{13} = 30$, $t_{23} = 30$ ($t_{ij} = t_{ji}$).

- i) What is the (limiting) probability that the taxi's most recent stop was at location i , $i = 1, 2, 3$?
- ii) What is the (limiting) probability that the taxi is heading for location 2?
- iii) What fraction of time is the taxi traveling from 2 to 3. Note: Upon arrival at a location the taxi immediately departs.

5.17) Consider an M/G/1 queueing system with Poisson arrivals of rate λ and expected service time $E[X]$. Let $\rho = \lambda E[X]$ and assume $\rho < 1$. Consider a semi-Markov process model of the M/G/1 queueing system in which transitions occur on departures from the queueing system and the state is the number of customers immediately following a departure.

a) Suppose a colleague has calculated the steady state probabilities $\{p_i\}$ of being in state i for each $i \geq 0$. For each $i \geq 0$, find the steady state probability π_i of state i in the embedded Markov chain. Give your solution as a function of ρ , p_i , and p_0 .

- b) Calculate p_0 as a function of ρ .
- c) Find π_i as a function of ρ and p_i .
- d) Is p_i the same as the steady state probability that the queueing system contains i customers at a given time? Explain carefully.

5.18) Consider an M/G/1 queue in which the arrival rate is λ and the service time distribution is uniform $(0, 2W)$ with $\lambda W < 1$. Define a semi-Markov chain following the framework for the M/G/1 queue in section 5.8.

- a) Find $P_{0j}; j \geq 0$.
- b) Find P_{ij} for $i > 0; j \geq i-1$.

5.19) Consider a semi-Markov process for which the embedded Markov chain is irreducible and positive recurrent. Assume that the distribution of inter-renewal intervals for one state j is arithmetic with span d . Show that the distribution of inter-renewal intervals for all states is arithmetic with the same span.

Chapter 6

Markov Processes with Countable State Spaces

6.1 INTRODUCTION

A *Markov process* with a countable state space is a special case of a semi-Markov process in which, first, the interval between successive transitions has an exponential distribution, and second, that interval is independent of the next state. Thus, we can take the set of possible states as $\{0, 1, 2, \dots\}$ and the process as $\{X(t), t \geq 0\}$, where for each real $t \geq 0$, $X(t)$ is the state of the process at time t . The random variables S_1, S_2, \dots denote the successive epochs at which the process makes state transitions, and X_n denotes the state entered at time S_n , i.e., $X_n = X(S_n)$ and $X(t) = X_n$ for $S_n \leq t < S_{n+1}$. Let $S_0=0$, and let $X_0 = X(0) = X(S_0)$ denote the initial state. The *embedded Markov chain* $\{X_n, n \geq 0\}$ has transition probabilities $\{P_{ij}, i \geq 0, j \geq 0\}$, and we assume that $P_{ii}=0$ for all i (i.e., there are no self transitions). The assumption $P_{ii}=0$ will be removed later when we talk about uniformized Markov processes. The intervals $U_n = S_n - S_{n-1}$ between successive transition epochs satisfy

$$P(U_n \leq x \mid X_{n-1}=i, X_n=j) = 1 - \exp(-v_i x) \quad (1)$$

where, for each i , v_i is a positive number called the *transition rate* out of state i . Conditional on X_{n-1} , the interval U_n is independent of X_n and also independent of all earlier inter-transition intervals and states.

Let $Y(t)$ denote the residual time from t until the next transition after t . Given that $X(t)=i$, the memoryless property of the exponential distribution implies that $Y(t)$ has an exponential distribution, $1 - \exp(-v_i t)$, and that $Y(t)$ is independent of the next state and independent of $X(\tau)$ for all $\tau < t$. Thus, for all $j \neq i$,

$$P[Y(t) \leq x, X(t+Y(t))=j \mid X(t)=i, \{X(\tau); \tau < t\}] = P_{ij}[1 - \exp(-v_i x)] \quad (2)$$

For x sufficiently small, the probability of two transitions in $(0, x]$ is negligible, so the probability on the left is that of a transition to state j in $(0, x]$. Using δ in place of x , this becomes

$$P[X(t+\delta)=j \mid X(t)=i, \{X(\tau); \tau < t\}] = v_i P_{ij} \delta + o(\delta) ; j \neq i \quad (3)$$

By repeating the argument used in Chapter 2 to derive the stationary and independent increment properties of the Poisson process, we see that, conditional on $X(t)=i$, the state at any time $s>t$ is independent of the states at times before t ,

$$P[X(s)=j \mid X(t)=i, \{X(\tau); \tau < t\}] = P[X(s)=j \mid X(t)=i] \text{ for all } i,j \quad (4)$$

Eq. (4) is often used as the definition of a Markov process. It says that, conditional on the present state at time t , the state at any future time is independent of all past states. This can be extended to the statement that, conditional on the present state, the states at any set of future times are independent of all past states. Note how similar this is to the definition of a Markov chain. A semi-Markov process is not a Markov process, since, at an arbitrary time t , the residual time until the next transition can depend, not only on the current state, but also on how long the process has been in that state. In general, Markov processes can have arbitrary state spaces, but these are not discussed here. We implicitly assume in all that follows that the state space is countable.

From (3), we see that $v_i P_{ij}$ can be interpreted as the *rate* at which transitions occur to j , conditional on the process being in state i . Given that $X(t)=i$, such a transition occurs between t and $t+\delta$ with probability $v_i P_{ij} \delta + o(\delta)$, and this is independent of $X(\tau)$ for all $\tau < t$. This transition rate is denoted by

$$q_{ij} = v_i P_{ij} ; j \neq i ; v_i = \sum_{j \neq i} q_{ij} ; P_{ij} = q_{ij}/v_i \quad (5)$$

Eq. (5) shows that a Markov process is completely specified (aside from the initial state) by the set of transition rates $\{q_{ij}; j \neq i\}$. We can describe Markov processes by directed graphs as in figure 6.1. An edge from i to j indicates that $q_{ij} > 0$, and the label on the edge indicates the value of q_{ij} .

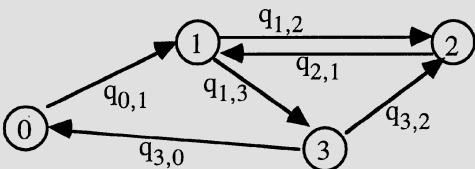


Figure 6.1. The graph of a Markov Process; the nodes indicate the states and the edges denote the non-zero transition rates.

The evolution of a Markov process can be visualized in several ways. In the approach above, the process starts in a known state $X_0=i$ at time 0. The next state X_1 is determined by the probabilities $\{P_{ij}; j \geq 0\}$, and the interval U_1 is independently determined by the exponential distribution of rate v_i . Given that $X_1=j$, the next state X_2 and next interval U_2 are independently determined by $\{P_{jk}; k \geq 0\}$ and v_j respectively. Subsequent transitions and intervals evolve in the same way.

For a second viewpoint, consider an independent Poisson process of rate v_i to be associated with each state i . When the Markov process enters a given state i , the next transition occurs at the next arrival epoch in the Poisson process for state i . At that epoch, a new state is chosen according to the transition probabilities P_{ij} . Since the choice of next state, given state i , is independent of the interval in state i , this view describes the same process as the first view.

For a third visualization, consider an independent Poisson process of rate q_{ij} to be associated with each possible transition i to j . When the Markov process enters a given state i , both the time of the next transition and the choice of the next state are determined by the set of i to j processes over all possible next states j . The transition occurs at the epoch of the first arrival to any of these i to j processes, and the next state is the j for which that first arrival occurred. Since such a collection of Poisson processes is equivalent to a single process of rate v_i followed by an independent selection according to the transition probabilities P_{ij} , this view again describes the same process as the other views. Note that the interarrival density for the Poisson process from i to a given j is $q_{ij} \exp(-q_{ij}x)$; on the other hand, conditional on a transition occurring from i to j , the interval has the density $v_i \exp(-v_i x)$.

Finally, visualize a *sampled time Markov chain* as an approximation to the Markov process (see figure 6.2). For a given increment size δ , the transition probability from i to j in the sampled time chain is defined to be $q_{ij}\delta$ for all $j \neq i$ and the self transition probability from i to i is $1 - \sum_j q_{ij}\delta = 1 - v_i\delta$ for each i . Note that this is an approximation to the Markov process in two ways. First, transitions occur only at integer multiples of the increment δ , and second, $q_{ij}\delta$ is an approximation to $P(X(\delta)=j | X(0)=i)$. From (3), $P(X(\delta)=j | X(0)=i) = q_{ij}\delta + o(\delta)$, so this second approximation is increasingly good as $\delta \rightarrow 0$, but we shall see later that the sampled time approximation has some interesting properties even when δ is not vanishingly small. Since the transition probability from i to itself in this approximation is $1 - v_i\delta$, we require that $v_i\delta \leq 1$ for all i . For a finite state space, this is satisfied for any $\delta \leq [\max_i v_i]^{-1}$. For a countably infinite set of states, however, the sampled time approximation requires the existence of some finite B such that $v_i \leq B$ for all i . Note that the M/M/1 Markov chain and the birth death generalizations in Chapter 5 are examples of sampled time approximations of Markov processes.

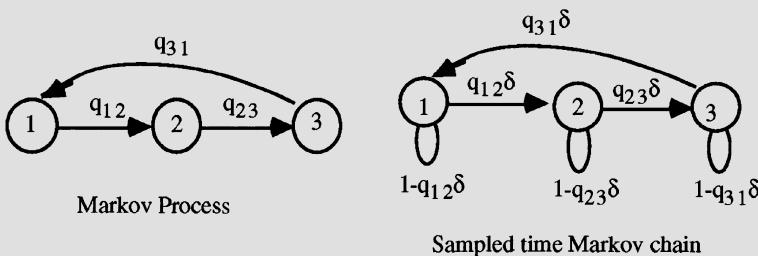


Figure 6.2. Approximating a Markov process by its sampled time Markov chain

Let us quickly review the results about semi-Markov processes in section 5.7 and apply them to the processes here. Assume that the embedded chain is irreducible. Then, from theorem 3 of Chapter 5, the chain is positive recurrent iff a solution exists to the steady state embedded chain equations,

$$\pi_j = \sum_i \pi_i P_{ij} \text{ for all } j; \pi_j \geq 0 \text{ for all } j; \sum_i \pi_i = 1 \quad (6)$$

Furthermore, if a solution exists, it is unique and $\pi_j > 0$ for all j . The mean time in state i until a transition is given by $\bar{U}(i) = 1/v_i$. Thus, from theorem 5.8, the steady state probabilities $\{p_i\}$ for the Markov process are given by

$$p_i = \frac{\pi_i/v_i}{\sum_k \pi_k/v_k} \quad (7)$$

These probabilities exist if the denominator in (7) is finite, and we assume this for now. Recall that p_i is the time average fraction of time spent in state i for all sample functions except a set of probability 0. Since there is a probability density for the transition intervals, the inter-renewal times are non-arithmetic, and therefore p_i is also the limiting probability of being in state i , i.e., $p_i = \lim_{t \rightarrow \infty} P(X(t)=i)$. Eq. (7) asserts that $p_i v_i = \alpha \pi_i$ where α is a normalization constant independent of i chosen to make $\sum_i p_i = 1$. Rewriting (7) to express $\{\pi_i\}$ in terms of $\{p_i\}$, and choosing the normalization to make $\sum_i \pi_i = 1$,

$$\pi_i = \frac{p_i v_i}{\sum_k p_k v_k} \quad (8)$$

We can substitute π_i as given by (8) into (6), obtaining, for each state j ,

$$p_j v_j = \sum_i p_i v_i P_{ij} = \sum_i p_i q_{ij} \quad (9)$$

Eq. (9) has the interpretation that the term on the left is the steady state rate at which transitions occur out of state j and the term on the right is the rate at which transitions occur into state j . Since the total number of entries to j must differ by at most 1 from the exits from j for each sample path, this equation is not surprising. The *steady state equations* for a Markov process are the set of equations in (9), along with $\sum_i p_i = 1$. In most of what follows, the Markov processes of interest have irreducible embedded Markov chains. Since this means that all states communicate with each other, we define an *irreducible Markov process* as a process where the embedded chain is irreducible.

THEOREM 1: Assume an irreducible Markov process and let $\{p_i; i \geq 0\}$ be a solution to (9) with $\sum_i p_i = 1$. If $\sum_i p_i v_i < \infty$, then, first, that solution is unique, second, for each i , $p_i = \lim_{t \rightarrow \infty} P(X(t)=i)$, third, for each i , p_i is the time average fraction of time spent in state

i with probability 1, and fourth, the embedded chain is positive recurrent. Conversely, if the embedded chain is positive recurrent with steady state probabilities $\{\pi_i; i \geq 0\}$, and if $\sum_i \pi_i/v_i < \infty$, then (9) and $\sum_i p_i = 1$ have a unique solution and $p_i > 0$ for each i .

Proof: If (9) has a solution with $\sum_i p_i = 1$, then $\{\pi_i; i \geq 0\}$ as given by (8) satisfies the steady state embedded chain equations in (6). From theorem 5.3, if (6) has a solution, it is unique and positive, so the solution to (9) is unique and $p_i > 0$ for all i . For $\{\pi_i; i \geq 0\}$ as defined by (8), the assumed solution to (9) satisfies (7), from which it follows that $p_i = \lim_{t \rightarrow \infty} P(X(t)=i)$ and p_i is the fraction of time in state i with probability 1. For the converse side, we have already verified that the solution to (7) also satisfies (9) and $\sum_i p_i = 1$. Thus the solution is unique and $p_i > 0$ for all i , completing the proof.

For an alternative view of the probabilities $\{p_i\}$, assume that the transition rates $\{v_i\}$ are bounded and consider the sampled time approximation to the process for a given increment size δ (see figure 6.2). Let $\{w_i; i \geq 0\}$ be the set of steady state probabilities for the sampled time chain, assuming that they exist. These steady state probabilities satisfy

$$w_j = \sum_{i \neq j} w_i q_{ij} \delta + w_j (1 - v_j \delta); w_j \geq 0; \sum_i w_i = 1 \quad (10)$$

The first equation simplifies to $w_j v_j = \sum_{i \neq j} w_i q_{ij} \delta$, which is the same as (9). It follows that the steady state probabilities $\{p_i; i \geq 0\}$ for the process are the same as the steady state probabilities $\{w_i; i \geq 0\}$ for the sampled time approximation. Note that this is not an approximation; w_i is exactly equal to p_i for all values of $\delta \leq 1/\sup_i v_i$. We shall see later that the dynamics of a Markov process are not quite so well modeled by the sampled time approximation except in the limit $\delta \rightarrow 0$.

PATHOLOGICAL CASES: Suppose the embedded Markov chain is positive recurrent but $\sum_i \pi_i/v_i = \infty$. This means that the embedded chain has steady state probabilities and, for each state, has a finite expected number of transitions between returns to that state. On the other hand, the mean *fraction of time* spent in each state is zero. This can be caused by increasingly long mean holding times for states of increasingly small values of π_i . This in turn can cause the mean *time interval* between returns to each state to be infinite. Exercise 6.1 gives some insight into this type of situation.

It is also possible for (9) to have a solution for $\{p_i\}$ with $\sum_i p_i = 1$, but for $\sum_i p_i v_i$ to be infinite. The interpretation here is quite different from that above, especially if the embedded Markov chain is transient rather than null recurrent. A transient chain means that there is a positive probability that the embedded chain will *never* return to a state after leaving it, and thus there can be no sensible kind of steady state behavior for the process. These processes are characterized by arbitrarily large transition rates from the various states, and these allow the process to transit through an infinite number of states in a finite time.

Processes for which there is a non-zero probability of passing through an infinite number of states in a finite time are called *irregular*. Exercises 6.2 and 6.3 give some

insight into irregular processes. Exercise 6.4 gives an example of a process that is not irregular, but for which (9) has a solution with $\sum_i p_i = 1$ and the embedded Markov chain is null recurrent. We restrict our attention in what follows to irreducible Markov chains for which (9) has a solution, $\sum_i p_i = 1$, and $\sum_i p_i v_i < \infty$. This is slightly more restrictive than necessary, but processes for which $\sum_i p_i v_i = \infty$ (see exercise 6.4) are not very robust. All of these pathological cases can also occur with semi-Markov chains.

6.2 THE KOLMOGOROV DIFFERENTIAL EQUATIONS

Let $P_{ij}(t)$ be the probability that a Markov process is in state j at time t given that $X(0)=i$,

$$P_{ij}(t) = P\{X(t)=j \mid X(0)=i\} \quad (11)$$

$P_{ij}(t)$ is analogous to the n^{th} order transition probabilities P_{ij}^n for Markov chains. We have already seen that $\lim_{t \rightarrow \infty} P_{ij}(t) = p_j$ for the case where the embedded chain is positive recurrent and $\sum_i p_i/v_i < \infty$. Here we want to find the transient behavior, and we start by deriving the Chapman–Kolmogorov equations. Let s and t be arbitrary times, $0 < s < t$. By including the state at time s , we can rewrite (11) as

$$\begin{aligned} P_{ij}(t) &= \sum_k P(X(t)=j, X(s)=k \mid X(0)=i) \\ &= \sum_k P(X(s)=k \mid X(0)=i) P(X(t)=j \mid X(s)=k); \text{ all } i, j \end{aligned} \quad (12)$$

where we have used (4), i.e., the Markov condition. Given that $X(s)=k$, the residual time until the next transition after s is exponential with rate v_k , and thus the process starting at s is statistically identical to that starting at 0. Thus, for any $s \geq 0$, we have

$$P\{X(t)=j \mid X(s)=k\} = P_{kj}(t-s)$$

Substituting this into (12), we have the *Chapman–Kolmogorov equations* for a Markov process,

$$P_{ij}(t) = \sum_k P_{ik}(s) P_{kj}(t-s) \quad (13)$$

These equations correspond to (4.14) for Markov chains. We now use these equations to derive two types of sets of differential equations for finding $P_{ij}(t)$. The first type are called the *Kolmogorov backward differential equations*, and the second are called the *Kolmogorov forward differential equations*. The backward equations are obtained by letting s approach 0 from above, and the forward equations are obtained by letting s approach t from below. First we derive the backward equations. For s small, (3) shows that $P_{ik}(s) = q_{ik}s + o(s)$ for $k \neq i$. Now, given that $X(0)=i$, the state remains at i throughout the interval $(0, s]$ with probability $\exp(-v_i s) = 1 - v_i s + o(s)$. The probability that the state changes in $(0, s]$ and returns to i by time s is also of order $o(s)$. Thus $P_{ii}(s) = 1 - v_i s + o(s)$ and (13) becomes

$$P_{ij}(t) = \sum_{k \neq i} [q_{ik}s P_{kj}(t-s)] + (1-v_i s) P_{ij}(t-s) + o(s) \quad (14)$$

Subtracting $P_{ij}(t-s)$ from both sides and dividing by s , we get

$$\frac{P_{ij}(t) - P_{ij}(t-s)}{s} = \sum_{k \neq i} [q_{ik}P_{kj}(t-s)] - v_i P_{ij}(t-s) + \frac{o(s)}{s} \quad (15)$$

Taking the limit as $s \rightarrow 0$, (and assuming that the summation and limit can be interchanged) we get the Kolmogorov backward equations,

$$\frac{dP_{ij}(t)}{dt} = \sum_{k \neq i} [q_{ik}P_{kj}(t)] - v_i P_{ij}(t) \quad (16)$$

For interpretation, the right hand side of (16) can be rewritten as $\sum_{k \neq i} q_{ik}[P_{kj}(t) - P_{ij}(t)]$. We see that q_{ik} is the rate of moving from i to k in an initial increment of time. If such a transition occurs in the initial increment, the change in probability of ending in state j after an additional interval of length t is given by $P_{kj}(t) - P_{ij}(t)$. Thus the rate of change of $P_{ij}(t)$ can be interpreted as arising from the initial rates of transition from i to other states.

Eq. (16) yields a set of linear differential first order equations which, for any given j , must be solved simultaneously for all i . For a finite state space, this set of equations, for all i and j , can be expressed most compactly in matrix notation. Letting $[P(t)]$ be a matrix for each t whose i, j element is $P_{ij}(t)$, and letting $[Q]$ be a matrix whose i, j term is q_{ij} for $i \neq j$ and $-v_i$ for $i=j$, we have

$$d[P(t)]/dt = [Q][P(t)] ; t \geq 0 \quad (17)$$

The initial conditions, at $t=0$, are $[P(t)] = I$, where I is the identity matrix. The solution to (17), subject to these initial conditions, is

$$[P(t)] = \sum_{m=0}^{\infty} \frac{t^m [Q]^m}{m!} \quad (18)$$

Eq. (18) can be verified by substitution into (17). We shall return to (17) and (18) after deriving the Kolmogorov forward equations.

For $t-s$ small and positive, (13) becomes

$$P_{ij}(t) = \sum_{k \neq j} [P_{ik}(s)(t-s) q_{kj}] + P_{ij}(s)[1 - (t-s)v_j] + o(t-s) \quad (19)$$

$$\frac{P_{ij}(t) - P_{ij}(s)}{t-s} = \sum_{k \neq j} [P_{ik}(s)q_{kj}] - P_{ij}(s)v_j \quad (20)$$

$$\frac{dP_{ij}(t)}{dt} = \sum_{k \neq j} [P_{ik}(t)q_{kj}] - P_{ij}(t)v_j \quad (21)$$

(where we have again assumed that the summation and limit can be interchanged). These are the Kolmogorov forward equations; their interpretation is similar to, but slightly simpler than, that of the backward equations. The incremental change in $P_{ij}(t)$ is equal to the difference of two terms. The first is the sum over k of the probability of being in state k at time t and then moving to j in the final incremental interval; the second is the probability of being in state j at time t and then moving away in the final incremental interval. For the finite state case, (21), in matrix notation is

$$d[P(t)]/dt = [P(t)][Q] \quad (22)$$

This equation has the same matrix solution as given in (18). Although (18) is a solution to these equations, it doesn't provide much insight into the form of the solution. In order to provide more of this insight, we go back to the sampled time approximation. With an increment of size δ between samples, the probability of a transition from i to j , $i \neq j$, is $q_{ij}\delta$, and the probability of remaining in state i is $1-v_i\delta$. Thus, in terms of the matrix $[Q]$ of transition rates, the transition probability matrix in the sampled time model is $I + \delta[Q]$, where I is the identity matrix. We denote this matrix by $[W_\delta] = I + \delta[Q]$. Note that λ is an eigenvalue of $[Q]$ iff $1+\lambda\delta$ is an eigenvalue of $[W_\delta]$, and the eigenvectors of these matrices are the same. That is, if $v(\pi)$ is a right (left) eigenvector of $[Q]$ with eigenvalue λ , then $v(\pi)$ is a right (left) eigenvector of $[W_\delta]$ with eigenvalue $1+\lambda\delta$, and conversely. We have already studied the eigenvalues and eigenvectors of transition matrices such as $[W_\delta]$ in section 4.4, and the following theorem translates some of these results into results about $[Q]$.

THEOREM 2: Consider an irreducible finite state Markov process with n states. Then the matrix $[Q]$ for that process has an eigenvalue $\lambda = 0$. That eigenvalue has a right eigenvector $e = (1, 1, \dots, 1)^T$ which is unique within a scale factor. It has a left eigenvector $p = (p_1, \dots, p_n)$ that is positive, sums to 1, satisfies (9), and is unique within a scale factor. All the other eigenvalues of $[Q]$ have strictly negative real parts.

Proof: Since all n states communicate, the sampled time chain is recurrent. From corollary 3 to theorem 4.5, $[W_\delta]$ has an eigenvalue $\lambda = 1$; the corresponding right eigenvector is e ; the left eigenvector is the steady state probability vector which is positive; and these eigenvectors are unique within a scale factor. From the equivalence of (9) and (10), p , as given by (9), is the steady state probability vector. Each eigenvalue λ_δ of $[W_\delta]$ corresponds to an eigenvalue λ of $[Q]$ with the correspondence $\lambda_\delta = 1 + \lambda\delta$, i.e., $\lambda = (\lambda_\delta - 1)/\delta$. Thus the eigenvalue 1 of $[W_\delta]$ corresponds to the eigenvalue 0 of $[Q]$. Since $|\lambda_\delta| \leq 1$ and $\lambda_\delta \neq 1$ for all other eigenvalues, the other eigenvalues of $[Q]$ all have strictly negative real parts, completing the proof.

We continue to look at an irreducible n state Markov process, and assume for simplicity that $[Q]$ has n distinct eigenvalues, $\lambda_1, \dots, \lambda_n$. Let v_1, v_2, \dots, v_n and π_1, \dots, π_n be the corresponding right and left eigenvectors respectively. Assume that these eigenvectors are scaled so that $\pi_i v_i = 1$ for each i , and recall that $\pi_i v_j = 0$ for $i \neq j$. Thus, if

$[V]$ is defined as the matrix with columns v_1, v_2, \dots, v_n , then its inverse, $[V]^{-1}$, is the matrix whose rows are π_1, \dots, π_n . Finally, let $[\Lambda]$ be the diagonal matrix with elements $\lambda_1, \dots, \lambda_n$. We then have the relationships

$$[\Lambda] = [V]^{-1}[Q][V]; [Q] = [V][\Lambda][V]^{-1} \quad (23)$$

We then also have $[Q]^m = [V]^{-1}[\Lambda]^m[V]$, and thus (18) can be written as

$$[P(t)] = \sum_{m=0}^{\infty} [V] \frac{t^m [\Lambda]^m}{m!} [V]^{-1} = [V][e^{t\Lambda}][V]^{-1} \quad (24)$$

where $[e^{t\Lambda}]$ is the diagonal matrix with elements $e^{t\lambda_1}, e^{t\lambda_2}, \dots, e^{t\lambda_n}$. Finally, if we break up the matrix $[e^{t\Lambda}]$ into a sum of n matrices, each with only a single non-zero element, then (24) becomes

$$[P(t)] = \sum_{i=1}^n v_i e^{t\lambda_i} \pi_i \quad (25)$$

Note that each term in (25) is a matrix formed by the product of a column vector v_i and a row vector π_i , scaled by $e^{t\lambda_i}$. From theorem 2, one of these eigenvalues, say λ_1 , is equal to 0, and thus $e^{t\lambda_1} = 1$ for all t . Each other eigenvalue λ_i has a negative real part, so $e^{t\lambda_i}$ goes to zero with increasing t for each of these terms. Thus the term for $i=1$ in (25) contains the steady state solution, \mathbf{ep} ; this is a matrix for which each row is the steady state probability vector \mathbf{p} .

Another way to express the solution to these equations (for finite n) is by the use of Laplace transforms. Let $L_{ij}(s)$ be the Laplace transform of $P_{ij}(t)$ and let $[L(s)]$ be the n by n matrix (for each s) of the elements $L_{ij}(s)$. Then the equation $d[P(t)]/dt = [Q][P(t)]$ for $t \geq 0$, along with the initial condition $[P(t)] = I$, becomes the Laplace transform equation

$$[L(s)] = [sI - [Q]]^{-1} \quad (26)$$

This appears to be simpler than (25), but it is really just more compact in notation. It can still be used, however, when $[Q]$ has fewer than n eigenvectors.

6.3 UNIFORMIZATION

Up until now, we have discussed Markov processes under the assumption that $q_{ii} = 0$ (i.e., no transitions from a state into itself are allowed). We now consider what happens if this restriction is removed. Suppose we start with some Markov process defined by a set of transition rates q_{ij} with $q_{ii} = 0$, and we modify this process by some arbitrary choice of $q_{ii} \geq 0$ for each state i . This modification changes the embedded Markov chain, since v_i is increased from $\sum_{k \neq i} q_{ik}$ to $\sum_{k \neq i} q_{ik} + q_{ii}$. From (5), P_{ij} is changed to q_{ij}/v_i for the new value of v_i for each i, j . Thus the steady state probabilities $\{\pi_i\}$ for the embedded chain are changed. The Markov process $\{X(t); t \geq 0\}$ is not changed, since a

transition from i into itself does not change $X(t)$ and does not change the distribution of the time until the next transition to a different state. The steady state probabilities for the process still satisfy

$$p_j v_j = \sum_k p_k q_{kj} ; \sum_i p_i = 1 \quad (27)$$

The addition of the new term q_{jj} increases v_j by q_{jj} , thus increasing the left hand side by $p_j q_{jj}$. The right hand side is similarly increased by $p_j q_{jj}$, so that the solution is unchanged (as we already determined it must be).

A particularly convenient way to add self transitions is to add them in such a way as to make the transition rate v_j the same for all states. Assuming that the transition rates $\{v_i; i \geq 0\}$ are bounded, we define v^* as $\sup_j v_j$ for the original transition rates. Then we set $q_{jj} = v^* - \sum_{k \neq j} q_{jk}$ for each j . With this addition of self transitions, all transition rates become v^* . From (8), we see that the new steady state probabilities, π^*_i , in the embedded Markov chain become equal to the steady state process probabilities, p_i . Naturally, we could also choose any v greater than v^* and increase each q_{jj} to make all transition rates equal to that value of v . When the transition rates are changed in this way, the resulting embedded chain is called a *uniformized chain* and the Markov process is called the *uniformized process*. The uniformized process is the same as the original process, except that quantities like the number of transitions over some interval are different because of the self transitions.

Assuming that all transition rates are made equal to v^* , the new transition probabilities in the embedded chain become $P^*_{ij} = q_{ij}/v^*$. Let $N(t)$ be the total number of transitions that occur from 0 to t in the uniformized process. Since the rate of transitions is the same from all states and the inter-transition intervals are independent and identically exponentially distributed, $N(t)$ is a Poisson counting process of rate v^* . Also, $N(t)$ is independent of the sequence of transitions in the embedded uniformized Markov chain. Thus, given that $N(t)=n$, the probability that $X(t)=j$ given that $X(0)=i$ is just the probability that the embedded chain goes from i to j in n steps, i.e., $P^*_{ij}^n$. This gives us another formula for calculating $P_{ij}(t)$, (i.e., the probability that $X(t)=j$ given that $X(0)=i$).

$$P_{ij}(t) = \sum_{n=0}^{\infty} P^*_{ij}^n \frac{e^{-v^*t}(v^*)^n}{n!} \quad (28)$$

Another situation where the uniformized process is useful is in extending Markov decision theory to Markov processes, but we do not pursue this.

6.4 BIRTH DEATH PROCESSES

Birth death processes are very similar to the birth death Markov chains that we studied earlier. Here transitions occur only between neighboring states, so it is convenient to define λ_i as $q_{i,i+1}$ and μ_i as $q_{i,i-1}$ (see figure 6.3). Since the number of transitions from i to $i+1$ is within 1 of the number of transitions from $i+1$ to i for every sample path, we conclude that

$$p_i \lambda_i = p_{i+1} \mu_{i+1} \quad (29)$$

This can also be obtained inductively from (9) using the same argument that we used earlier for birth death Markov chains.

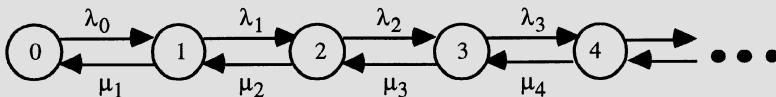


Figure 6.3. Birth death process.

Define p_i as λ_i/μ_{i+1} . Then applying (29) iteratively, we obtain the steady state equations

$$p_i = p_0 \prod_{j=0}^{i-1} p_j ; \quad i \geq 1 \quad (30)$$

We can solve for p_0 by substituting (30) into $\sum_i p_i$, yielding

$$p_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} p_j} \quad (31)$$

For the M/M/1 queue, the state of the Markov process is the number of customers in the system (i.e., waiting in queue or in service). The transitions from i to $i+1$ correspond to arrivals, and since the arrival process is Poisson of rate λ , we have $\lambda_i = \lambda$ for all $i \geq 0$. The transitions from i to $i-1$ correspond to departures, and since the service time distribution is exponential with parameter μ , say, we have $\mu_i = \mu$ for all $i \geq 1$. Thus, (31) simplifies to $p_0 = 1-\rho$, where $\rho = \lambda/\mu$ and thus

$$p_i = (1-\rho)\rho^i ; \quad i \geq 0. \quad (32)$$

We assume that $\rho < 1$, which is required for positive recurrence. The probability that there are i or more customers in the system in steady state is then given by $P\{X(t) \geq i\} = \rho^i$ and the expected number of customers in the system is given by

$$E[X(t)] = \sum_{i=1}^{\infty} P\{X(t) \geq i\} = \frac{\rho}{1-\rho} \quad (33)$$

The expected time that a customer spends in the system in steady state can now be determined by Little's formula (Theorem 3.8).

$$E[\text{System time}] = \frac{E[X(t)]}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu-\lambda} \quad (34)$$

The expected time that a customer spends in the queue (i.e., before entering service) is just the expected system time less the expected service time, so

$$E[\text{Queueing time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda} \quad (35)$$

Finally, the expected number of customers in the queue can be found by applying Little's formula to (35)

$$E[\text{Number in queue}] = \frac{\lambda\rho}{\mu-\lambda} = \frac{\rho^2}{1-\rho} \quad (36)$$

Note that the expected number in the system and in the queue depend only on ρ , so that if the arrival rate and service rate were both speeded up by the same factor, these expected values would remain the same. The expected system time and queueing time, however would decrease by the factor of the rate increases. Note also that as ρ approaches 1, all these quantities approach infinity as $1/(1-\rho)$. At the value $\rho = 1$, the embedded Markov chain becomes null-recurrent and the steady state probabilities (both $\{\pi_i\}$ and $\{p_i\}$) fail to exist.

There are many types of queueing systems that can be modeled as birth death processes. For example the arrival rate could vary with the number in the system and the service rate could vary with the number in the system. All of these systems can be analyzed in steady state in the same way, but (30) and (31) can become quite messy in these more complex systems. As an example, we analyze the M/M/m system. Here there are m servers, each with exponentially distributed service times with parameter μ . When i customers are in the system, there are i servers working for $i < m$ and all m servers are working for $i \geq m$. With i servers working, the probability of a departure in an incremental time δ is $i\mu\delta$, so that μ_i is $i\mu$ for $i < m$ and $m\mu$ for $i \geq m$ (see figure 6.4).

Define $\rho = \lambda/(m\mu)$. Then in terms of our general birth death process notation, $p_i = mp_i/(i+1)$ for $i < m$ and $p_i = \rho$ for $i \geq m$. From (30), we have

$$p_i = p_0 \frac{mp}{1} \frac{mp}{2} \cdots \frac{mp}{i} = \frac{p_0(m\mu)^i}{i!} ; \quad i \leq m \quad (37)$$

$$p_i = \frac{p_0 \rho^i m^m}{m!} ; \quad i \geq m \quad (38)$$

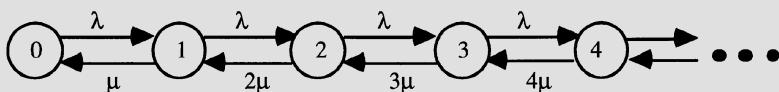


Figure 6.4. M/M/m queue for $m = 3$.

We can find p_0 by summing p_i and setting the result equal to 1; a solution exists if $\rho < 1$. Nothing simplifies much in this sum, except that $\sum_{i \geq m} p_i = p_0(\rho m)^m / [m!(1-\rho)]$, and the solution is

$$p_0 = \frac{1}{\frac{(mp)^m}{m!(1-\rho)} + \sum_{i=0}^{m-1} \frac{(mp)^i}{i!}} \quad (39)$$

6.5 REVERSIBILITY FOR MARKOV PROCESSES

In section 5.4 on reversibility for Markov chains, (5.37) showed that the backward transition probabilities P_{ij}^* in steady state satisfy

$$\pi_i P_{ij}^* = \pi_j P_{ji} \quad (40)$$

These equations are then valid for the embedded chain of a Markov process. Next, consider backward transitions in the process itself. Given that the process is in state i , the probability of a transition in an increment δ of time is $v_i \delta + o(\delta)$, and transitions in successive increments are independent. Thus, if we view the process running backward in time, the probability of a transition in each increment δ of time is also $v_i \delta + o(\delta)$ with independence between increments. Thus, going to the limit $\delta \rightarrow 0$, the distribution of the time backward to a transition is exponential with parameter v_i . This means that the process running backwards is again a Markov process with transition probabilities P_{ij}^* and transition rates v_i . Figure 6.5 helps to illustrate this.

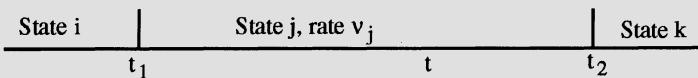


Figure 6.5. The forward process enters state j at time t_1 and departs at t_2 . The backward process enters state j at time t_2 and departs at t_1 . In any sample function, as illustrated, the interval in a given state is the same in the forward and backward process. Given $X(t) = j$, the time forward to the next transition and the time backward to the previous transition are each exponential with rate v_j .

Since the steady state probabilities $\{p_i\}$ for the Markov process are determined by

$$p_i = \frac{\pi_i/v_i}{\sum_k \pi_k/v_k}, \quad (41)$$

and since $\{\pi_i\}$ and $\{v_i\}$ are the same for the forward and backward processes, we see that the steady state probabilities in the backward Markov process are the same as the

steady state probabilities in the forward process. This result can also be seen by the correspondence between sample functions in the forward and backward processes.

The *transition rates* in the backward process are defined by $q^*_{ij} = v_i P^*_{ij}$. Using (40), we have

$$q^*_{ij} = v_i P^*_{ij} = \frac{v_i \pi_j P_{ji}}{\pi_i} = \frac{v_i \pi_j q_{ji}}{\pi_i v_j} \quad (42)$$

Substituting (41) in this equation for π_i/v_i and for π_j/v_j , we obtain

$$p_i q^*_{ij} = p_j q_{ji} \quad (43)$$

This equation can be used as an alternate definition of the backward transition rates. To interpret this, let δ be a vanishingly small increment of time and assume the process is in steady state at time t . Then $\delta p_j q_{ji} \approx P(X(t)=j) P(X(t+\delta)=i | X(t)=j)$ whereas $\delta p_i q^*_{ij} \approx P(X(t+\delta)=i) P(X(t)=j | X(t+\delta)=i)$.

A Markov process is defined to be *reversible* if $q^*_{ij} = q_{ji}$ for all i, j . If the embedded Markov chain is reversible, (i.e., $P^*_{ij} = P_{ij}$ for all i, j), then one can repeat the above steps using P_{ij} and q_{ij} in place of P^*_{ij} and q^*_{ij} to see that $p_i q_{ij} = p_j q_{ji}$ for all i, j . Thus, if the embedded chain is reversible, the process is also. Similarly, if the Markov process is reversible, the above argument can be reversed to see that the embedded chain is reversible. Thus, we have the following useful lemma.

LEMMA 1: Assume that steady state probabilities $\{p_i; i \geq 0\}$ exist in an irreducible Markov process (i.e., (9) has a solution with $\sum_i p_i = 1$) and that $\sum_i p_i v_i < \infty$. Then the Markov process is reversible if and only if the embedded chain is reversible.

One can find the steady state probabilities of a reversible Markov process and simultaneously show that it is reversible by the following useful theorem (which is directly analogous to theorem 5 of Chapter 5).

THEOREM 3: For an irreducible Markov process, assume that $\{p_i; i \geq 0\}$ is a set of non-negative numbers summing to 1, satisfying $\sum_i p_i v_i < \infty$, and satisfying

$$p_i q_{ij} = p_j q_{ji} \quad \text{for all } i, j \quad (44)$$

Then $\{p_i\}$ is the set of steady state probabilities for the process, $p_i > 0$ for all i , the process is reversible, and the embedded chain is positive recurrent.

Proof: Summing (44) over i , we obtain

$$\sum_i p_i q_{ij} = p_j v_j \quad \text{for all } j$$

These, along with $\sum_i p_i = 1$ are the steady state equations for the process. These equations thus have a solution, and by theorem 1, $p_i > 0$ for all i , the embedded chain is positive recurrent, and $p_i = \lim_{t \rightarrow \infty} P(X(t)=i)$. Comparing (44) with (43), we see that $q_{ij} = q_{ij}^*$, so the process is reversible.

There are many irreducible Markov processes that are not reversible but for which the backward process has interesting properties that can be deduced, at least intuitively, from the forward process. Jackson networks (to be studied shortly) and many more complex networks of queues fall into this category. The following simple theorem allows us to use whatever combination of intuitive reasoning and wishful thinking we desire to guess both the transition rates q_{ij}^* in the backward process and the steady state probabilities, and to then verify rigorously that the guess is correct. One might think that guessing is somehow unscientific, but in fact the art of educated guessing and intuitive reasoning is at the heart of all good scientific work.

THEOREM 4: For an irreducible Markov process, assume that a set of positive numbers $\{p_i; i \geq 0\}$ satisfy $\sum_i p_i = 1$ and $\sum_i p_i v_i < \infty$. Also assume that a set of non-negative numbers $\{q_{ij}^*\}$ satisfy the two sets of equations

$$\sum_j q_{ij} = \sum_j q_{ij}^* \text{ for all } i \quad (45)$$

$$p_i q_{ij} = p_j q_{ji}^* \text{ for all } i, j \quad (46)$$

Then $\{p_i\}$ is the set of steady state probabilities for the process, $p_i > 0$ for all i , the embedded chain is positive recurrent, and $\{q_{ij}^*\}$ is the set of transition rates in the backward process.

Proof: Sum (46) over i . Using the fact that $\sum_j q_{ij} = v_i$ and using (45), we obtain

$$\sum_i p_i q_{ij} = p_j v_i \text{ for all } j \quad (47)$$

These, along with $\sum_i p_i = 1$, are the steady state equations for the process. These equations thus have a solution, and by theorem 1, $p_i > 0$ for all i , the embedded chain is positive recurrent, and $p_i = \lim_{t \rightarrow \infty} P(X(t)=i)$. Finally, q_{ij}^* as given by (46) is the backward transition rate as given by (43) for all i, j .

We see that theorem 3 is just a special case of theorem 4 in which the guess about q_{ij}^* is that $q_{ij}^* = q_{ij}$.

Birth death processes are all reversible if the steady state probabilities exist. To see this, note that Eq. (29) (the equation to find the steady state probabilities) is just (44) applied to the special case of birth death processes. Due to the importance of this, we state it as a theorem.

THEOREM 5: For a birth death process, if there is a solution $\{p_i; i \geq 0\}$ to (29) with $\sum_i p_i = 1$ and $\sum_i p_i v_i < \infty$, then the process is reversible, and the embedded chain is positive recurrent and reversible.

Since the M/M/1 queueing process is a birth death process, it is also reversible. Burke's theorem, which was given as theorem 5.7 for sampled time M/M/1 queues, can now be established for continuous time M/M/1 queues.

THEOREM 6 (Burke's theorem): Given an M/M/1 queueing system in steady state with $\lambda < \mu$,

- a) the departure process is Poisson with rate λ ,
- b) the state $X(t)$ at any time t is independent of departures prior to t , and
- c) for FCFS service, given that a customer departs at time t , the arrival time of that customer is independent of the departures prior to t .

Proof: The proofs of parts (a) and (b) are the same as the proof of Burke's theorem for sampled time, theorem 5.7, and thus will not be repeated. For part (c), note that with FCFS service, the m^{th} customer to arrive at the system is also the m^{th} customer to depart. Figure 6.6 illustrates that the association between arrivals and departures is the same in the backward system as in the forward system (even though the customer ordering is reversed in the backward system). In the forward, right moving system, let τ be the epoch of some given arrival. The customers arriving after τ wait behind the given arrival in the queue, and have no effect on the given customer's service. Thus the interval from τ to the given customer's service completion is independent of arrivals after τ .

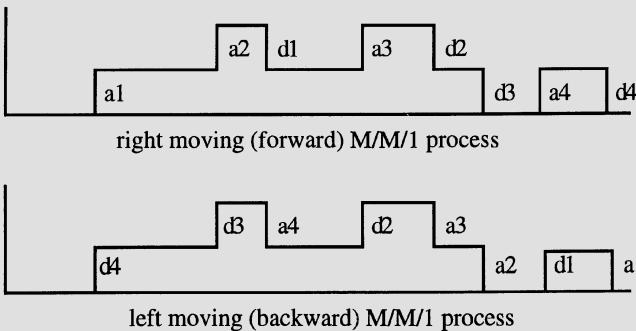


Figure 6.6. FCFS arrivals and departures in right and left moving M/M/1 processes.

Since the backward, left moving, system is also an M/M/1 queue, the interval from a given backward arrival, say at epoch t , moving left until the corresponding departure, is independent of arrivals to the left of t . From the correspondence between sample functions in the right moving and left moving systems, given a departure at epoch t in the right moving system, the departures before time t are independent of the arrival epoch of the given customer departing at t ; this completes the proof.

Part (c) of Burke's theorem does not apply to sampled time M/M/1 queues because the sampled time model does not allow for both an arrival and departure in the same increment of time.

Note that the proof of Burke's theorem (including parts (a) and (b) from section 5.5) does not make use of the fact that the transition rate $q_{i,i-1} = \mu$ for $i \geq 1$ in the M/M/1 queue. Thus Burke's theorem remains true for any birth death Markov process in steady state for which $q_{i,i+1} = \lambda$ for all $i \geq 0$. For example, parts (a) and (b) are valid for M/M/m queues; part (c) is also valid (see [Wol89]), but the argument here is not adequate since the first customer to enter the system might not be the first to depart.

We next show how Burke's theorem can be used to analyze a tandem set of queues. As shown in figure 6.7, we have an M/M/1 queueing system with Poisson arrivals at rate λ and service at rate μ_1 . The departures from this queueing system are the arrivals to a second queueing system, and we assume that a departure from queue 1 at time t instantaneously enters queueing system 2 at the same time t . The second queueing system has a single server and the service times are IID and exponentially distributed with rate μ_2 . The successive service times at system 2 are also independent of the arrivals to systems 1 and 2, and independent of the service times in system 1. Since we have already seen that the departures from the first system are Poisson with rate λ , the arrivals to the second system are Poisson with rate λ . Thus the second system is also M/M/1.

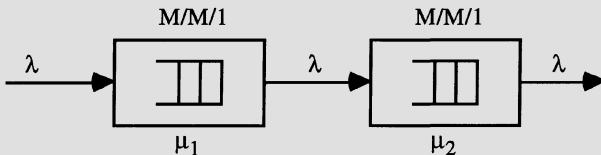


Figure 6.7. A tandem queueing system.

Let $X(t)$ be the state of queueing system 1 and $Y(t)$ be the state of queueing system 2. Since $X(t)$ at time t is independent of the departures from system 1 prior to t , $X(t)$ is independent of the arrivals to system 2 prior to time t . Since $Y(t)$ depends only on the arrivals to system 2 prior to t and on the service times that have been completed prior to t , we see that $X(t)$ is independent of $Y(t)$. This leaves a slight nit-picking question about what happens at the instant of a departure from system 1. We have considered the state $X(t)$ at the instant of a departure to be the number of customers remaining in system 1 not counting the departing customer. Also the state $Y(t)$ is the state in system 2 includ-

ing the new arrival at instant t . The state $X(t)$ then is independent of the departures up to and including t , so that $X(t)$ and $Y(t)$ are still independent.

Next assume that both systems use FCFS service. Consider a customer that leaves system 1 at time t . The time at which that customer arrived at system 1, and thus the waiting time in system 1 for that customer, is independent of the departures prior to t . This means that the state of system 2 immediately before the given customer arrives at time t is independent of the time the customer spent in system 1. It therefore follows that the time that the customer spends in system 2 is independent of the time spent in system 1. Thus the total system time that a customer spends in both system 1 and system 2 is the sum of two independent random variables.

This same argument can be applied to more than 2 queueing systems in tandem. It can also be applied to more general networks of queues, each with single servers with exponentially distributed service times. The restriction here is that there can not be any cycle of queueing systems where departures from each queue in the cycle can enter the next queue in the cycle. The problem posed by such cycles can be seen easily in the following example of a single queueing system with feedback (see figure 6.8).

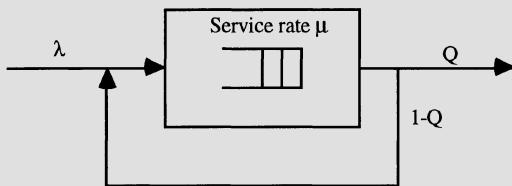


Figure 6.8. A queue with feedback.

We assume that the queueing system in figure 6.8 has a single server with IID exponentially distributed service times that are independent of arrival times. The exogenous arrivals from outside the system are Poisson with rate λ . With probability Q , the departures from the queue leave the entire system, and, alternatively, with probability $1-Q$, they return instantaneously to the input of the queue. Successive choices between leaving the system and returning to the input are IID and independent of exogenous arrivals and of service times. Figure 6.9 shows a sample function of the arrivals and departures in the case in which the service rate μ is very much greater than the exogenous arrival rate λ . Each exogenous arrival spawns a geometrically distributed set of departures and simultaneous re-entries. Thus the overall arrival process to the queue, counting both exogenous arrivals and feedback from the output, is not Poisson. Note, however, that if we look at the Markov process description, the departures that are fed back to the input correspond to self loops from one state to itself. Thus the Markov process is the same as one without the self loops with a service rate equal to μQ . Thus, from Burke's theorem, the exogenous departures are Poisson with rate λ . Also the steady state distribution of $X(t)$ is $P\{X(t)=i\} = (1-\rho)\rho^i$ where $\rho = \lambda/(\mu Q)$ (assuming, of course, that $\rho < 1$).

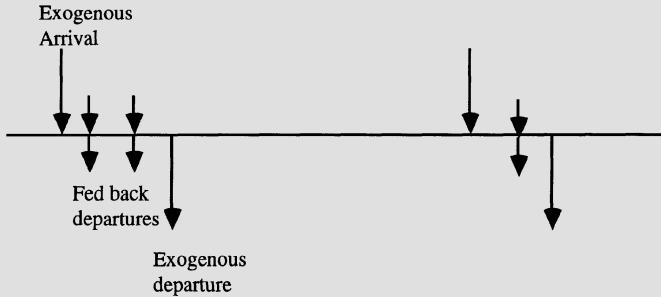


Figure 6.9. Sample path of arrivals and departures for queue with feedback.

The tandem queueing system of figure 6.7 can also be regarded as a combined Markov process in which the state at time t is the pair $(X(t), Y(t))$. The transitions in this process correspond to, first, exogenous arrivals in which $X(t)$ increases, second, exogenous departures in which $Y(t)$ decreases, and third, transfers from system 1 to system 2 in which $X(t)$ decreases and $Y(t)$ simultaneously increases. The combined process is *not* reversible since there is no transition in which $X(t)$ increases and $Y(t)$ simultaneously decreases. In the next section, we show how to analyze these combined Markov processes for more general networks of queues.

6.6 JACKSON NETWORKS

In many queueing situations, a customer has to wait in a number of different queues before completing the desired transaction and leaving the system. For example, when we go to the registry of motor vehicles to get a driver's license, we must wait in one queue to have the application processed, in another queue to pay for the license, and in yet a third queue to obtain a photograph for the license. In a multiprocessor computer facility, a job can be queued waiting for service at one processor, then go to wait for another processor, and so forth; frequently the same processor is visited several times before the job is completed. In a data network, packets traverse multiple intermediate nodes; at each node they enter a queue waiting for transmission to other nodes.

Such systems are modeled by a network of queues, and Jackson networks are perhaps the simplest models of such networks. In such a model, we have a network of k interconnected queueing systems which we call nodes. Each of the k nodes receives customers (i.e., tasks or jobs) both from outside the network (exogenous inputs) and from other nodes within the network (endogenous inputs). It is assumed that the exogenous inputs to each node i form a Poisson process of rate r_i and that these Poisson processes are independent of each other. For analytical convenience, we regard this as a single Poisson input process of rate λ_0 , with each input independently going to each node i with probability $Q_{0i} = r_i/\lambda_0$.

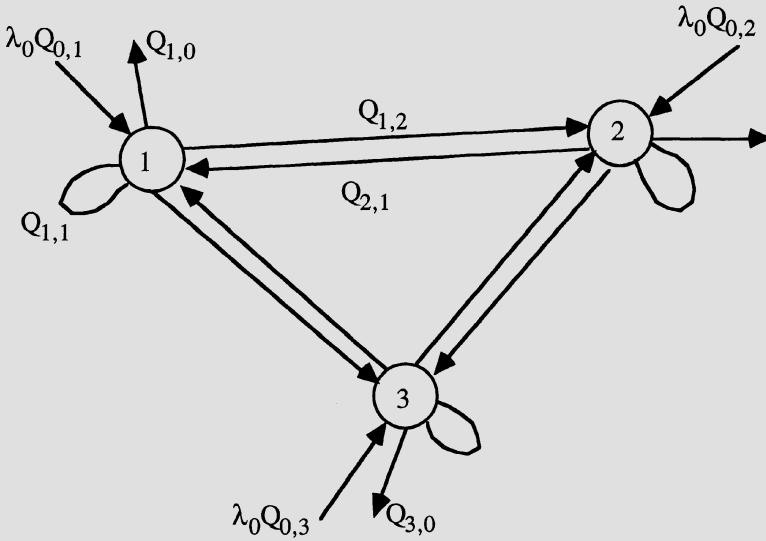


Figure 6.10. A Jackson network with 3 nodes; Q_{ij} is the probability that a departure from node i goes to node j (or, for $j=0$, departs the system). Note that a departure from node i can re-enter node i with probability Q_{ii} .

Each node i contains a single server, and the successive service times at node i are IID random variables with an exponentially distributed service time of rate μ_i . The service times at each node are also independent of the service times at all other nodes and independent of the exogenous arrival times at all nodes. When a customer completes service at a given node i , that customer is routed to node j with probability Q_{ij} (see figure 6.10). It is also possible for the customer to depart from the network entirely (called an exogenous departure), and this occurs with probability $Q_{i0} = 1 - \sum_{j \geq 1} Q_{ij}$. For a customer departing from node i , the next node j is a random variable with PMF $\{Q_{ij}, 0 \leq j \leq k\}$. Successive choices of the next node for customers at node i are IID, independent of the customer routing at other nodes, independent of all service times, and independent of the exogenous inputs. Notationally, we are regarding the outside world as a fictitious node 0 from which customers appear and to which they disappear.

When a customer is routed from node i to node j , it is assumed that the routing is instantaneous; thus at the epoch of a departure from node i , there is a simultaneous endogenous arrival at node j . Thus a node j receives Poisson exogenous arrivals from outside the system at rate $\lambda_0 Q_{0j}$ and receives endogenous arrivals from other nodes according to the probabilistic rules just described. We can visualize these combined

exogenous and endogenous arrivals as being served in FCFS fashion, but it really makes no difference in which order they are served, since the customers are statistically identical and simply give rise to service at node j at rate μ_j whenever there are customers to be served.

The Jackson queueing network, as just defined, is fully described by the exogenous input rate λ_0 , the service rates $\{\mu_i\}$, and the routing probabilities $\{Q_{ij}; 0 \leq i, j \leq k\}$. The network as a whole is a Markov process in which the state is a vector $\mathbf{m} = (m_1, m_2, \dots, m_k)$, where $m_i, 1 \leq i \leq k$, is the number of customers at node i . State changes occur upon exogenous arrivals to the various nodes, exogenous departures from the various nodes, and departures from one node that enter another node. In a vanishingly small interval δ of time, given that the state at the beginning of that interval is \mathbf{m} , an exogenous arrival at node j occurs in the interval with probability $\lambda_0 Q_{0j}\delta$ and changes the state to $\mathbf{m}' = \mathbf{m} + \mathbf{e}_j$ where \mathbf{e}_j is a unit vector with a one in position j . If $m_i > 0$, an exogenous departure from node i occurs in the interval with probability $\mu_i Q_{i0}\delta$ and changes the state to $\mathbf{m}' = \mathbf{m} - \mathbf{e}_i$. Finally, if $m_i > 0$, a departure from node i entering node j occurs in the interval with probability $\mu_i Q_{ij}\delta$ and changes the state to $\mathbf{m}' = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_j$. Thus, the transition rates are given by

$$q_{\mathbf{m}, \mathbf{m}'} = \lambda_0 Q_{0j} \quad \text{for } \mathbf{m}' = \mathbf{m} + \mathbf{e}_j, 1 \leq j \leq k \quad (48)$$

$$= \mu_i Q_{i0} \quad \text{for } \mathbf{m}' = \mathbf{m} - \mathbf{e}_i, m_i > 0, 1 \leq i \leq k \quad (49)$$

$$= \mu_i Q_{ij} \quad \text{for } \mathbf{m}' = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_j, m_i > 0, 1 \leq i, j \leq k \quad (50)$$

= 0 for all other choices of \mathbf{m}'

Note that a departure from node i that re-enters node i causes a transition from state \mathbf{m} back into state \mathbf{m} ; we disallowed such transitions in sections 6.1 and 6.2, but showed that they caused no problems in our discussion of uniformization. It is convenient to allow these self transitions here, partly for the added generality and partly to illustrate that the single node network with feedback of figure 6.8 is an example of a Jackson network.

Our objective is to find the steady state probabilities $p(\mathbf{m})$ for this type of process, and our plan of attack is in accordance with theorem 4; that is, we shall guess a set of transition rates for the backward Markov process, use these to guess $p(\mathbf{m})$, and then verify that the guesses are correct. Before making these guesses, however, we must find out a little more about how the system works, so as to guide the guesswork. Let us define λ_i for each $i, 1 \leq i \leq k$, as the time average overall rate of arrivals to node i , including both exogenous and endogenous arrivals. Since λ_0 is the rate of exogenous inputs, we can interpret λ_i/λ_0 as the expected number of visits to node i per exogenous input. The endogenous arrivals to node i are not necessarily Poisson, as the example of a single queue with feedback shows, and we are not even sure at this point that such a time average rate exists in any reasonable sense. However, let us assume for the time being that such rates exist and that the time average rate of departures from each node equals the time average rate of arrivals (i.e., the queue sizes do not grow linearly with time). Then these rates must satisfy the equation

$$\lambda_j = \sum_{i=0}^k \lambda_i Q_{ij} ; \quad 1 \leq j \leq k \quad (51)$$

To see this, note that $\lambda_0 Q_{0j}$ is the rate of exogenous arrivals to j . Also λ_i is the time average rate at which customers depart from queue i , and $\lambda_i Q_{ij}$ is the rate at which customers go from node i to node j . Thus, the right hand side of (51) is the sum of the exogenous and endogenous arrival rates to node j . Note the distinction between the time average rate of customers going from i to j in (51) and the rate $q_{m,m'} = \mu_i Q_{ij}$ for $m' = m - e_i + e_j$, $m_i > 0$ in (50). The rate in (50) is conditioned on a state m with $m_i > 0$, whereas that in (51) is the overall time average rate, averaged over all states.

Note that $\{Q_{ij}; 0 \leq i, j \leq k\}$ is a stochastic matrix and (51) is formally equivalent to the equations for steady state probabilities (except that steady state probabilities sum to 1). The usual equations for steady state probabilities include an equation for $j=0$, but that equation is redundant. Thus we know that, if there is a path between each pair of nodes (including the fictitious node 0), then (51) has a solution for $\{\lambda_i; 0 \leq i \leq k\}$, and that solution is unique within a scale factor. The known value of λ_0 determines this scale factor and makes the solution unique. Note that we don't have to be careful at this point about whether these rates are time averages in any nice sense, since this will be verified later; we do have to make sure that (51) has a solution, however, since it will appear in our solution for $p(m)$. Thus we assume in what follows that a path exists between each pair of nodes, and thus that (51) has a unique solution as a function of λ_0 .

We now make the final necessary assumption about the network, which is that $\mu_i > \lambda_i$ for each node i . This will turn out to be required in order to make the process positive recurrent. We also define ρ_i as λ_i / μ_i . We shall find that, even though the inputs to an individual node i are not Poisson in general, there is a steady state distribution for the number of customers at i , and that distribution is the same as that of an M/M/1 queue with the parameter ρ_i .

Now consider the backward time process. We have seen that only three kinds of transitions are possible in the forward process. First, there are transitions from m to $m' = m + e_j$ for any j , $1 \leq j \leq k$. Second, there are transitions from m to $m - e_i$ for any i , $1 \leq i \leq k$, such that $m_i > 0$. Third, there are transitions from m to $m' = m - e_i + e_j$ for $1 \leq i, j \leq k$ with $m_i > 0$. Thus in the backward process, transitions from m' to m are possible only for the m, m' pairs above. Corresponding to each arrival in the forward process, there is a departure in the backward process; for each forward departure, there is a backward arrival; and for each forward passage from i to j , there is a backward passage from j to i .

We now make the conjecture that the backward process is itself a Jackson network with Poisson exogenous arrivals at rates $\{\lambda_0 Q^*_{0j}\}$, service times that are exponential with rates $\{\mu_i\}$, and routing probabilities $\{Q^*_{ij}\}$. The backward routing probabilities $\{Q^*_{ij}\}$ must be chosen to be consistent with the transition rates in the forward process. Since each transition from i to j in the forward process must correspond to a transition from j to i in the backward process, we should have

$$\lambda_i Q_{ij} = \lambda_j Q^*_{ji} ; \quad 0 \leq i, j \leq k \quad (52)$$

Note that $\lambda_i Q_{ij}$ represents the rate at which forward transitions go from i to j , and λ_i represents the rate at which forward transitions *leave* node i . Eq.(52) takes advantage of the fact that λ_i is also the rate at which forward transitions *enter* node i , and thus the rate at which backward transitions *leave* node i . Using the conjecture that the backward time system is a Jackson network with routing probabilities $\{Q_{ij}^*; 0 \leq i, j \leq k\}$, we can write down the backward transition rates in the same way as (48)–(50),

$$q_{m,m'}^* = \lambda_0 Q_{0j}^* \quad \text{for } m' = m + e_j \quad (53)$$

$$= \mu_i Q_{i0}^* \quad \text{for } m' = m - e_i, m_i > 0, 1 \leq i \leq k \quad (54)$$

$$= \mu_i Q_{ij}^* \quad \text{for } m' = m - e_i + e_j, m_i > 0, 1 \leq i, j \leq k \quad (55)$$

If we substitute (52) into (53)–(55), we obtain

$$q_{m,m'}^* = \lambda_j Q_{j0} \quad \text{for } m' = m + e_j, 1 \leq j \leq k \quad (56)$$

$$= (\mu_i / \lambda_i) \lambda_0 Q_{0i} \quad \text{for } m' = m - e_i, m_i > 0, 1 \leq i \leq k \quad (57)$$

$$= (\mu_i / \lambda_i) \lambda_j Q_{ji} \quad \text{for } m' = m - e_i + e_j, m_i > 0, 1 \leq i, j \leq k \quad (58)$$

This gives us our hypothesized backward transition rates in terms of the parameters of the original Jackson network. To use theorem 4, we must verify that there is a set of positive numbers, $p(\mathbf{m})$, satisfying $\sum_m p(\mathbf{m}) = 1$ and $\sum_m v_m p_m < \infty$, and a set of non-negative numbers $q_{m',m}^*$ satisfying the following two sets of equations:

$$p(\mathbf{m}) q_{m,m'} = p(\mathbf{m}') q_{m',m}^* \quad \text{for all } \mathbf{m}, \mathbf{m}' \quad (59)$$

$$\sum_{m'} q_{m,m'} = \sum_{m'} q_{m',m}^* \quad \text{for all } \mathbf{m}. \quad (60)$$

We verify (59) by substituting (48)–(50) on the left side of (59) and (56)–(58) on the right side. Recalling that ρ_i is defined as λ_i / μ_i , and cancelling out common terms on each side, we have

$$p(\mathbf{m}) = p(\mathbf{m}') / \rho_j \quad \text{for } \mathbf{m}' = \mathbf{m} + e_j \quad (61)$$

$$p(\mathbf{m}) = p(\mathbf{m}') \rho_i \quad \text{for } \mathbf{m}' = \mathbf{m} - e_i, m_i > 0 \quad (62)$$

$$p(\mathbf{m}) = p(\mathbf{m}') \rho_i / \rho_j \quad \text{for } \mathbf{m}' = \mathbf{m} - e_i + e_j, m_i > 0 \quad (63)$$

Looking at the case $\mathbf{m}' = \mathbf{m} - e_i$, and using this equation repeatedly to get from state $(0, 0, \dots, 0)$ up to an arbitrary \mathbf{m} , we obtain

$$p(\mathbf{m}) = p(0, 0, \dots, 0) \prod_{i=1}^k \rho_i^{m_i} \quad (64)$$

It is easy to verify that (64) satisfies (61)–(63) for all possible transitions. Summing over all \mathbf{m} to solve for $p(0, 0, \dots, 0)$, we get

$$1 = \sum_{m_1, m_2, \dots, m_k} p(\mathbf{m}) = p(0, 0, \dots, 0) \sum_{m_1} \rho_1^{m_1} \sum_{m_2} \rho_2^{m_2} \dots \sum_{m_k} \rho_k^{m_k}$$

$$= p(0, 0, \dots, 0) (1-p_1)^{-1} (1-p_2)^{-1} \dots (1-p_k)^{-1}$$

Thus, $p(0, 0, \dots, 0) = (1-p_1)(1-p_2) \dots (1-p_k)$, and substituting this in (64), we get

$$p(\mathbf{m}) = \prod_{i=1}^k p_i(m_i) = \prod_{i=1}^k \left[(1-p_i)p_i^{m_i} \right] \quad (65)$$

where $p_i(m) = (1-p_i)p_i^m$ is the steady state distribution of a single M/M/1 queue. Now that we have found the steady state distribution implied by our assumption about the backward process being a Jackson network, our remaining task is to verify (60).

To verify (60), i.e., $\sum_{\mathbf{m}'} q_{\mathbf{m}, \mathbf{m}'} = \sum_{\mathbf{m}'} q^*_{\mathbf{m}, \mathbf{m}'}$, first consider the right side. Using (53) to sum over all $\mathbf{m}' = \mathbf{m} + \mathbf{e}_j$, then (54) to sum over $\mathbf{m}' = \mathbf{m} - \mathbf{e}_i$ (for i such that $m_i > 0$), and finally (55) to sum over $\mathbf{m}' = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_j$ (again for i such that $m_i > 0$), we get

$$\sum_{\mathbf{m}'} q^*_{\mathbf{m}, \mathbf{m}'} = \sum_{j=1}^k \lambda_0 Q^*_{0j} + \sum_{i: m_i > 0} \mu_i Q^*_{i0} + \sum_{i: m_i > 0} \mu_i \sum_{j=1}^k Q^*_{ij} \quad (66)$$

Using the fact Q^* is a stochastic matrix, then,

$$\sum_{\mathbf{m}'} q^*_{\mathbf{m}, \mathbf{m}'} = \lambda_0 + \sum_{i: m_i > 0} \mu_i \quad (67)$$

The left hand side of (60) can be summed in the same way to get the result on the right side of (67), but we can see that this must be the result by simply observing that λ_0 is the rate of exogenous arrivals and $\sum_{i: m_i > 0} \mu_i$ is the overall rate of service completions in state \mathbf{m} . Note that this also verifies that $v_{\mathbf{m}} = \sum_{\mathbf{m}'} q_{\mathbf{m}, \mathbf{m}'} \leq \lambda_0 + \sum_i \mu_i$, and since $v_{\mathbf{m}}$ is bounded, $\sum_{\mathbf{m}} v_{\mathbf{m}} p(\mathbf{m}) < \infty$. Since all the conditions of theorem 4 are satisfied, $p(\mathbf{m})$, as given in (65), gives the steady state probabilities for the Jackson network. This also verifies that the backward process is a Jackson network, and hence the exogenous departures are Poisson and independent.

Although the exogenous arrivals and departures in a Jackson network are Poisson, the endogenous processes of customers travelling from one node to another are typically not Poisson if there are feedback paths in the network. Also, although (65) shows that the numbers of customers at the different nodes are independent random variables at any given time in steady state, it is not generally true that the number of customers at one node at one time is independent of the number of customers at another node at another time.

There are many generalizations of the reversibility arguments used above, and many network situations in which the nodes have independent states at a common time. We discuss just two of them here and refer to Kelly, [Kel79], for a complete treatment.

For the first generalization, assume that the service time at each node depends on the number of customers at that node, i.e., μ_i is replaced by μ_{i,m_i} . Note that this includes the M/M/m type of situation in which each node has several independent exponential servers. With this modification, the transition rates in (49) and (50) are modified by replacing μ_i with μ_{i,m_i} . The hypothesized backward transition rates are modified in the same way, and the only effect of these changes is to replace ρ_i for each i (or j) in (61)–(63) with $\rho_{i,m_i} = \lambda_i / \mu_{i,m_i}$. With this change, (64) becomes

$$p(\mathbf{m}) = \prod_{i=1}^k p_i(m_i) = \prod_{i=1}^k p_i(0) \prod_{j=0}^{m_i} \rho_{i,j} \quad (68)$$

$$p_i(0) = \frac{1}{1 + \sum_{m=1}^{\infty} \prod_{j=0}^{m-1} \rho_{i,j}} \quad (69)$$

Thus, $p(\mathbf{m})$ is given by the product distribution of k individual birth death systems.

CLOSED JACKSON NETWORKS: The second generalization is to a network of queues with a fixed number M of customers in the system and with no exogenous inputs or outputs. Such networks are called *closed Jackson* networks, whereas the networks analyzed above are often called *open Jackson* networks. Suppose a k node closed network has routing probabilities Q_{ij} , $1 \leq i, j \leq k$, where $\sum_j Q_{ij} = 1$, and has exponential service times of rate μ_i (this can be generalized to μ_{i,m_i} as above). We make the same assumptions as before about independence of service variables and routing variables, and assume that there is a path between each pair of nodes. Since $\{Q_{ij}; 1 \leq i, j \leq k\}$ forms an irreducible stochastic matrix, there is a one dimensional set of solutions to the steady state equations

$$\lambda_j = \sum_i \lambda_i Q_{ij}; \quad 1 \leq j \leq k; \quad (70)$$

We interpret λ_i as the time average rate of transitions that go into node i . Since this set of equations can only be solved within an unknown multiplicative constant, and since this constant can only be determined at the end of the argument, we define $\{\pi_i; 1 \leq i \leq k\}$ as the particular solution of (70) satisfying

$$\pi_j = \sum_i \pi_i Q_{ij}; \quad 1 \leq j \leq k; \quad \sum_i \pi_i = 1 \quad (71)$$

Thus, for all i , $\lambda_i = \alpha \pi_i$, where α is some unknown constant. The state of the Markov process is again taken as $\mathbf{m} = (m_1, m_2, \dots, m_k)$ with the condition $\sum_i m_i = M$. The transition rates of the Markov process are the same as for open networks, except that there are no exogenous arrivals or departures; thus (48)–(50) are replaced by

$$q_{\mathbf{m}, \mathbf{m}'} = \mu_i Q_{ij} \quad \text{for } \mathbf{m}' = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_j, m_i > 0, 1 \leq i, j \leq k \quad (72)$$

We hypothesize that the backward time process is also a closed Jackson network, and as before, we conclude that if the hypothesis is true, the backward transition rates should be

$$q^*_{m,m'} = \mu_i Q^*_{ij} \quad \text{for } m' = m - e_i + e_j, m_i > 0, 1 \leq i, j \leq k \quad (73)$$

$$\text{where } \lambda_i Q_{ij} = \lambda_j Q^*_{ji} \quad \text{for } 1 \leq i, j \leq k \quad (74)$$

In order to use theorem 4 again, we must verify that a PMF $p(m)$ exists satisfying $p(m)q_{m,m'} = p(m')q^*_{m',m}$ for all possible states and transitions, and we must also verify that $\sum_{m'} q_{m,m'} = \sum_{m'} q^*_{m,m'}$ for all possible m . This latter verification is virtually the same as before and is left as an exercise. The former verification, with the use of (72), (73), and (74), becomes

$$p(m) (\mu_i / \lambda_i) = p(m') (\mu_j / \lambda_j) \quad \text{for } m' = m - e_i + e_j, m_i > 0 \quad (75)$$

Using the open network solution to guide our intuition, we see that the following choice of $p(m)$ satisfies (75) for all possible m (i.e., all m such that $\sum_i m_i = M$)

$$p(m) = A \prod_{i=1}^k (\lambda_i / \mu_i)^{m_i}; m \text{ such that } \sum_i m_i = M. \quad (76)$$

The constant A is a normalizing constant, chosen to make $p(m)$ sum to unity. The problem with (76) is that we do not know λ_i (except within a multiplicative constant independent of i). Fortunately, however, if we substitute π_i / α for λ_i , we see that α is raised to the power $-M$, independent of the state m . Thus, letting $A' = A \alpha^{-M}$, our solution becomes

$$p(m) = A' \prod_{i=1}^k (\pi_i / \mu_i)^{m_i}; m \text{ such that } \sum_i m_i = M. \quad (77)$$

$$\frac{1}{A'} = \sum_{m: \sum_i m_i = M}^{\infty} \prod_{i=1}^k \left(\frac{\pi_i}{\mu_i} \right)^{m_i} \quad (78)$$

Note that the steady state distribution of the closed Jackson network has been found without solving for the time average transition rates. Note also that the steady state distribution looks very similar to that for an open network; that is, it is a product distribution over the nodes with a geometric type distribution within each node. This is somewhat misleading, however, since the constant A' can be quite difficult to calculate. It is surprising at first that the parameter of the geometric distribution can be changed by a constant multiplier in (77) and (78) (i.e., π_i could be replaced with λ_i) and the solution does not change; the important quantity is the relative values of π_i / μ_i from one value of i to another rather than the absolute value.

In order to find λ_i (and this is important, since it says how quickly the system is doing its work), note that $\lambda_i = \mu_i p(m_i > 0)$. Solving for $p(m_i > 0)$ requires finding the constant A' in (72). In fact, the major difference between open and closed networks is that the relevant constants for closed networks are tedious to calculate (even by computer) for large networks and large M .

6.7 SUMMARY

We have seen that Markov processes with countable state spaces are remarkably similar to Markov chains with countable state spaces, and throughout the chapter, we frequently made use of both the embedded chain corresponding to the process and to the sampled time approximation to the process.

For irreducible processes, the steady state equations, (9) and $\sum_i v_i = 1$, were found to specify the steady state probabilities, p_i , which have significance both as time averages and as limiting probabilities. If the transition rates v_i are bounded, then the sampled time approximation exists and has the same steady state probabilities as the Markov process itself. If the transition rates v_i are unbounded but $\sum_i p_i v_i < \infty$, then the embedded chain is positive recurrent and has steady state probabilities, but the sampled time approximation does not exist. We assumed throughout the remainder of the chapter that $\sum_i p_i v_i < \infty$. This ruled out irregular processes in which there is no meaningful steady state, and also some peculiar processes such as that in exercise 6.4 where the embedded chain is null recurrent.

Section 6.2 developed the Kolmogoroff backward and forward differential equations for the transient probabilities $P_{ij}(t)$ of being in state j at time t given state i at time 0. We showed that for finite state processes, these equations can be solved by finding the eigenvalues and eigenvectors of the transition rate matrix Q . There are close analogies between this analysis and the algebraic treatment of finite state Markov chains in Chapter 4, and exercise 6.7 showed how the transients of the process are related to the transients of the sampled time approximation.

For irreducible processes with bounded transition rates, uniformization was introduced as a way to simplify the structure of the process. The addition of self transitions does not change the process itself, but can be used to adjust the transition rates v_i to be the same for all states. This changes the embedded Markov chain, and the steady state probabilities for the embedded chain become the same as those for the process. The epochs at which transitions occur then form a Poisson process which is independent of the set of states entered. This yields a separation between the transition epochs and the sequence of states.

The next two sections analyzed birth death processes and reversibility. The results about birth death Markov chains and reversibility for Markov chains carried over almost without change to Markov processes. These results are central in queueing theory, and Burke's theorem allowed us to look at simple queueing networks with no feedback and to understand how feedback complicates the problem.

Finally, Jackson networks were discussed. These are important in their own right and also provide a good example of how one can solve complex queueing problems by studying the reverse time process and making educated guesses about the steady state

behavior. The somewhat startling result here is that in steady state, and at a fixed time, the number of customers at each node is independent of the number at each other node and satisfies the same distribution as for an M/M/1 queue. Also the exogenous departures from the network are Poisson and independent from node to node. We emphasized that the number of customers at one node at one time is often dependent on the number at other nodes at other times. The independence holds only when all nodes are viewed at the same time.

For further reading on Markov processes, see [Kel79], [Ros83], [Wol89], and [Fel66].

EXERCISES

6.1) Consider a Markov process for which the embedded Markov chain is a birth death chain with transition probabilities $P_{i,i+1} = 2/5$ for all $i \geq 1$, $P_{i,i-1} = 3/5$ for all $i \geq 1$, $P_{01} = 1$, and $P_{ij} = 0$ otherwise.

- a) Find the steady state probabilities $\{\pi_i; i \geq 0\}$ for the embedded chain.
- b) Assume that the transition rate v_i out of state i , for $i \geq 0$, is given by $v_i = 2^i$. Find the transition rates $\{q_{ij}\}$ between states and find the steady state probabilities $\{p_i\}$ for the Markov process. Explain heuristically why $\pi_i \neq p_i$.
- c) Now assume in parts (c) to (f) that the transition rate out of state i , for $i \geq 0$, is given by $v_i = 2^{-i}$. Find the transition rates $\{q_{ij}\}$ between states and draw the directed graph with these transition rates.
- d) Show that there is no probability vector solution $\{p_i; i \geq 0\}$ to Eq. 9 in Chapter 6.
- e) Argue that the expected *time* between visits to any given state i is infinite. Find the expected *number of transitions* between visits to any given state i . Argue that, starting from any state i , an eventual return to state i occurs with probability 1.
- f) Consider the sampled time approximation of this process with $\delta=1$. Draw the graph of the resulting Markov chain and argue why it must be null-recurrent.

6.2) a) Consider a Markov process with the set of states $\{0, 1, \dots\}$ in which the transition rates $\{q_{ij}\}$ between states are given by $q_{i,i+1} = (3/5)2^i$ for $i \geq 0$, $q_{i,i-1} = (2/5)2^i$ for $i \geq 1$, and $q_{ij} = 0$ otherwise. Find the transition rate v_i out of state i for each $i \geq 0$ and find the transition probabilities $\{P_{ij}\}$ for the embedded Markov chain.

- b) Find a solution $\{p_i\}$ with $\sum_i p_i = 1$ to (9).
- c) Show that all states of the embedded Markov chain are transient.
- d) For each state i of the embedded Markov chain, associate a reward $r_i = 2^i$, i.e., the mean time until a transition is made from state i . Let $V_i(0)$ be a final reward in stage 0 and assume that $V_i(0) = W_i$ where W_i satisfies the equations

$$W_i = 2^{-i} + (3/5)W_{i+1} + (2/5)W_{i-1}; i \geq 1 \text{ and } W_0 = 1 + W_1 \quad (\text{a})$$

Show that for all $n > 0$, $V_i(n) = W_i$. Note that this is the expected time to make n transitions, plus the final reward after n transitions. Thus if (a) has a bounded solution, the expected time to make an infinite number of transitions is finite, and the process is irregular.

e) Define $\delta_i = W_i - W_{i+1}$, and show that (a) can be rewritten as $\delta_i = (2/3)\delta_{i-1} + (5/3)2^{-i}$ for $i \geq 1$. Note from (a) that $\delta_0 = 1$. Show that $\delta_i \leq 2i(2/3)^{i-1}$ for $i \geq 1$.

f) Show that $W_0 - \lim_{i \rightarrow \infty} W_i \leq \infty$. This shows that the process is irregular, and in particular shows that the solution $\{p_i; i \geq 0\}$ found in (a) is not a steady state solution, and in fact has no physical meaning.

6.3 a) Consider the process in the figure below. The process starts at $X(0) = 1$, and for all $i \geq 1$, $P_{i,i+1} = 1$ and $v_i = i^2$ for all i . Let T_n be the time that the n^{th} transition occurs. Show that

$$E[T_n] = \sum_{i=1}^n i^{-2} < \infty \quad \text{for all } n$$

Hint: Upper bound the sum from $i=2$ by integrating x^{-2} from $x=1$.



b) Use the Markov inequality to show that $P(T_n > 4) \leq 1/2$ for all n . Show that the probability of an infinite number of transitions by time 4 is at least $1/2$.

6.4 Let $q_{i,i+1} = 2^{i-1}$ for all $i \geq 0$ and let $q_{i,i-1} = 2^{i-1}$ for all $i \geq 1$. All other transition rates are 0.

- a) Solve the steady state equations and show that $p_i = 2^{-i-1}$ for all $i \geq 0$.
- b) Find the transition probabilities for the embedded Markov chain and show that the chain is null recurrent.

c) For any state i , consider the renewal process for which the Markov process starts in state i and renewals occur on each transition to state i . Show that, for each $i \geq 1$, the expected inter-renewal interval is equal to 2. Hint: Use renewal reward theory.

d) Show that the expected number of transitions between each entry into state i is infinite. Explain why this does *not* mean that an infinite number of transitions can occur in a finite time.

6.5 A two state Markov process has transition rates $q_{01} = 1$, $q_{10} = 2$. Find $P_{01}(t)$, the probability that $X(t) = 1$ given that $X(0) = 0$. Hint: You can do this by solving a single first order differential equation if you make the right choice between forward and backward equations.

6.6 a) Consider a two state Markov process with $q_{01} = \lambda$ and $q_{10} = \mu$. Find the eigenvalues and eigenvectors of the transition rate matrix $[Q]$.

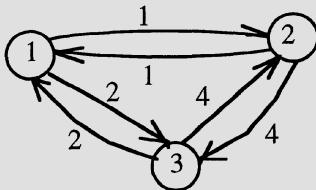
- b) Use (25) to solve for $[P(t)]$.
- c) Use the Kolmogorov forward equation for $P_{01}(t)$ directly to find $P_{01}(t)$ for $t \geq 0$. Hint: You don't have to use the equation for $P_{00}(t)$; why?

d) Check your answer in (b) with that in (c).

6.7) Consider an irreducible Markov process with n states and assume that the transition rate matrix $[Q] = [V][\Lambda][V]^{-1}$ where $[V]$ is the matrix of right eigenvectors of $[Q]$, $[\Lambda]$ is the diagonal matrix of eigenvalues of $\{Q\}$, and the inverse of $[Q]$ is the matrix of left eigenvectors.

- a) Consider the sampled time approximation to the process with an increment of size δ , and let $[W_\delta]$ be the transition matrix for the sampled time approximation. Express $[W_\delta]$ in terms of $[V]$ and $[\Lambda]$.
- b) Express $[W_\delta]^n$ in terms of $[V]$ and $[\Lambda]$.
- c) Expand $[W_\delta]^n$ in the same form as (25).
- d) Let t be an integer multiple of δ , and compare $[W_\delta]^{t/\delta}$ to $[P(t)]$. Note: What you see from this is that λ_i in (25) is replaced by $(1/\delta)\ln(1+\delta\lambda_i)$. For the steady state term, $\lambda_1 = 0$, this causes no change, but for the other eigenvalues, there is a change that vanishes as $\delta \rightarrow 0$.

6.8) Consider the three state Markov process below; the number given on edge (i, j) is the transition rate from i to j , q_{ij} . Assume that the process is in steady state.



- a) Is this process reversible?
- b) Find p_i , the time average fraction of time spent in state i for each i .
- c) Given that the process is in state i at time t , find the mean delay from t until the process leaves state i .
- d) Find π_i , the time average fraction of all transitions that go into state i for each i .
- e) Suppose the process is in steady state at time t . Find the steady state probability that the next state to be entered is state 1.
- f) Given that the process is in state 1 at time t , find the mean delay until the process first returns to state 1.
- g) Consider an arbitrary irreducible finite state Markov process in which $q_{ij} = q_{ji}$ for all i, j . Either show that such a process is reversible or find a counter example.

6.9) a) Consider an M/M/1 queueing system with arrival rate λ , service rate μ , $\mu > \lambda$. Assume that the queue is in steady state. Given that an arrival occurs at time t , find the probability that the system is in state i immediately *after* time t .

- b) Assuming FCFS service, and conditional on i customers in the system immediately after the above arrival, characterize the time until the above customer departs as a sum of random variables.

c) Find the unconditional probability density of the time until the above customer departs. Hint: You know (from splitting a Poisson process) that the sum of a geometrically distributed number of IID exponentially distributed random variables is exponentially distributed. Use the same idea here.

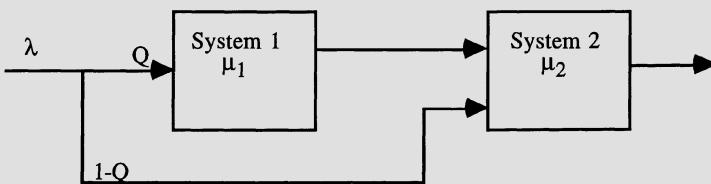
6.10) A small bookie shop has room for at most two customers. Potential customers arrive at a Poisson rate of 10 customers per hour; they enter if there is room and are turned away, never to return, otherwise. The bookie serves the admitted customers in order, requiring an exponentially distributed time of mean 4 minutes per customer.

a) Find the steady state distribution of number of customers in the shop.

b) Find the rate at which potential customers are turned away.

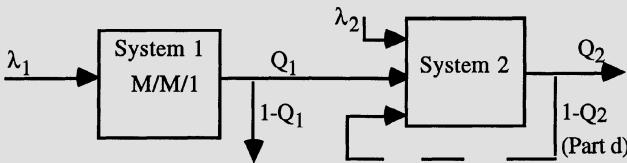
c) Suppose the bookie hires an assistant; the bookie and assistant, working together, now serve each customer in an exponentially distributed time of mean 2 minutes, but there is only room for one customer (i.e., the customer being served) in the shop. Find the new rate at which customers are turned away.

6.11) Consider the job sharing computer system illustrated below. Incoming jobs arrive from the left in a Poisson stream. Each job, independently of other jobs, requires pre-processing in system 1 with probability Q . Jobs in system 1 are served FCFS and the service times for successive jobs entering system 1 are IID with an exponential distribution of mean $1/\mu_1$. The jobs entering system 2 are also served FCFS and successive service times are IID with an exponential distribution of mean $1/\mu_2$. The service times in the two systems are independent of each other and of the arrival times. Assume that $\mu_1 > \lambda Q$ and that $\mu_2 > \lambda$. Assume that the combined system is in steady state.



- a) Is the input to system 1 Poisson? Explain.
- b) Are each of the two input processes coming into system 2 Poisson? Explain
- c) Considering the input process to system 1 and the two input processes to system 2, which are independent of each other? Explain carefully.
- d) Give the joint steady state PMF of the number of jobs in the two systems. Explain briefly.
- e) What is the probability that the first job to leave system 1 after time t is the same as the first job that entered the entire system after time t ?
- f) What is the probability that the first job to leave system 2 after time t both passed through system 1 and arrived at system 1 after time t ?

- 6.12)** Consider the following combined queueing system. The first queue system is M/M/1 with service rate μ_1 . The second queue system has IID exponentially distributed service times with rate μ_2 .

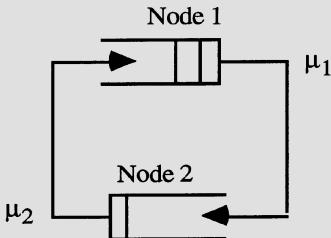


Each departure from system 1 independently goes to system 2 with probability Q_1 and leaves the system with probability $1-Q_1$. System 2 has an additional Poisson input of rate λ_2 , independent of inputs and outputs from the first system. Each departure from the second system independently leaves the combined system with probability Q_2 and re-enters system 2 with probability $1-Q_2$. *For parts (a), (b), (c) assume that $Q_2 = I$ (i.e., there is no feedback).*

- a) Characterize the process of departures from system 1 that enter system 2 and characterize the overall process of arrivals to system 2.
- b) Assuming FCFS service in each system, find the steady state distribution of time that a customer spends in each system.
- c) For a customer that goes through both systems, show why the time in each system is independent of that in the other; find the distribution of the combined system time for such a customer.
- d) Now assume that $Q_2 < 1$. Is the departure process from the combined system Poisson? Which of the three input processes to system 2 are Poisson? Which of the input processes are independent? Explain your reasoning, but do not attempt formal proofs.

- 6.13)** Suppose a Markov chain with transition probabilities $\{P_{ij}\}$ is reversible. For some given state, state 0 to be specific, suppose we change the transition probabilities out of state 0 from $\{P_{0j}\}$ to $\{P'_{0j}\}$. Assuming that $\{P_{ij}\}$ for all i, j with $i \neq 0$ are unchanged, what is the most general way in which we can choose $\{P'_{0j}\}$ so as to maintain reversibility? Your answer should be explicit about how the steady state probabilities $\{\pi_i\}$ are changed. Your answer should also indicate what this problem has to do with uniformization of reversible Markov processes, if anything. Hint: Given $\{P_{ij}\}$ a single additional parameter will suffice to specify $\{P'_{0j}\}$ for all j .

- 6.14)** Consider the closed queueing network in the figure below. There are three customers who are doomed forever to cycle between node 1 and node 2. Both nodes use FCFS service and have exponentially distributed IID service times. The service times at one node are also independent of those at the other node and are independent of the customer being served. The server at node i has mean service time $1/\mu_i$, $i = 1, 2$. Assume to be specific that $\mu_2 < \mu_1$.



- a) The system can be represented by a four state Markov process. Draw its graphical representation and label it with the individual states and the transition rates between them.
- b) Find the steady state probability of each state.
- c) Find the time average rate at which customers leave node 1.
- d) Find the time average rate at which a given customer cycles through the system.
- e) Is the Markov process reversible? Suppose that the backward Markov process is interpreted as a closed queueing network. What does a departure from node 1 in the forward process correspond to in the backward process? Can the transitions of a single customer in the forward process be associated with transitions of a single customer in the backward process?

6.15) Consider an M/G/1 queueing system with last come first serve (LCFS) service. That is, customers arrive according to a Poisson process of rate λ . A newly arriving customer interrupts the customer in service and enters service itself. When a customer is finished, it leaves the system and the customer that had been interrupted by the departing customer resumes service from where it had left off. For example, if customer 1 arrives at time 0 and requires 2 units of service, and customer 2 arrives at time 1 and requires 1 unit of service, then customer 1 is served from time 0 to 1; customer 2 is served from time 1 to 2 and leaves the system, and then customer 1 completes service from time 2 to 3. Let X_i be the service time required by the i^{th} customer; the X_i are IID random variables with expected value $E[X]$; they are independent of customer arrival times. Assume $\lambda E[X] < 1$.

- a) Find the mean time between busy periods (i.e., the time until a new arrival occurs after the system becomes empty).
- b) Find the time average fraction of time that the system is busy.
- c) Find the mean duration, $E[B]$, of a busy period. Hint: Use (a) and (b).
- d) Explain briefly why the customer that starts a busy period remains in the system for the entire busy period; use this to find the expected system time of a customer given that that customer arrives when the system is empty.
- e) Is there any statistical dependence between the system time of a given customer (i.e., the time from the customer's arrival until departure) and the number of customers in the system when the given customer arrives?

- f) Show that a customer's expected system time is equal to $E[B]$. Hint: Look carefully at your answers to (d) and (e).
- g) Let C be the expected system time of a customer conditional on the service time X of that customer being 1. Find (in terms of C) the expected system time of a customer conditional on $X=2$. Hint: Compare a customer with $X=2$ to two customers with $X=1$ each; repeat for arbitrary $X=x$.
- h) Find the constant C . Hint: Use (f) and (g); don't do any tedious calculations.

6.16) Consider a queueing system with two classes of customers, type A and type B. Type A customers arrive according to a Poisson process of rate λ_A and customers of type B arrive according to an independent Poisson process of rate λ_B .

- a) The queue has a single server with exponentially distributed IID service times of rate $\mu > \lambda_A + \lambda_B$. First come first serve service (FCFS) is used. Characterize the departure process of class A customers; explain carefully. Hint: Consider the combined arrival process and be judicious about how to select between A and B types of customers.
- b) Suppose now that last come first serve (LCFS) service is used (i.e., whenever a new customer arrives, the server drops what it is doing and starts work on the new customer; when a customer departs, the server resumes service on the most recently arrived remaining customer). Characterize the departure process of class A customers; explain carefully.
- c) Suppose now that LCFS service is used, but that now the single server requires independent exponentially distributed service times of rate μ_A for class A customers and rate μ_B for class B customers. Model this as a Markov process in which the state is the ordered set of customer classes in the queue and in service (see figure). What are the transition rates out of the state shown below? Is this process reversible? Assume $(\lambda_A/\mu_A) + (\lambda_B/\mu_B) < 1$.



Sample state of queueing system with type A customer most recently arrived, another type A next most recent, and customer B least recent.

- d) Characterize the departure process of class A customers for the system of part (c); explain carefully.

6.17) Consider a k node Jackson type network with the modification that each node i has s servers rather than one server. Each server at i has an exponentially distributed service time with rate μ_i . The exogenous input rate to node i is $r_i = \lambda_0 Q_{0i}$ and each output from i is switched to j with probability Q_{ij} and switched out of the system with probability Q_{i0} (as in the notes). Let λ_i , $1 \leq i \leq k$, be the solution, for given λ_0 , to

$$\lambda_j = \sum_{i=0}^k \lambda_i Q_{ij};$$

$1 \leq j \leq k$ and assume that $\lambda_i < s\mu_i$; $1 \leq i \leq k$. Show that the steady state probability of state \mathbf{m} is

$$P(\mathbf{m}) = \prod_{i=1}^k P_i(m_i)$$

where $P_i(m_i)$ is the probability of state m_i in an (M, M, s) queue. Hint: Simply extend the argument in the text to the multiple server case.

6.18) Suppose a Markov process with the set of states A is reversible and has steady state probabilities p_i ; $i \in A$. Let B be a subset of A and assume that the process is changed by setting $q_{ij} = 0$ for all $i \in B$, $j \notin B$. Assuming that the new process (starting in B and remaining in B) is irreducible, show that the new process is reversible and find its steady state probabilities.

6.19) Consider a sampled time $M/D/m/m$ queueing system. The arrival process is Bernoulli with probability $\lambda << 1$ of arrival in each time unit. There are m servers; each arrival enters a server if a server is not busy and otherwise the arrival is discarded. If an arrival enters a server, it keeps the server busy for d units of time and then departs; d is some integer constant and is the same for each server.

Let n , $0 \leq n \leq m$ be the number of customers in service at a given time and let x_i be the number of time units that the i^{th} of those n customers (in order of arrival) has been in service. Thus the state of the system can be taken as $(n, \mathbf{x}) = (n, x_1, x_2, \dots, x_n)$ where $0 \leq n \leq m$ and $1 \leq x_1 < x_2 < \dots < x_n \leq d$.

Let $A(n, \mathbf{x})$ denote the next state if the present state is (n, \mathbf{x}) and a new arrival enters service. That is,

$$A(n, \mathbf{x}) = (n+1, 1, x_1+1, x_2+1, \dots, x_n+1) \text{ for } n < m \text{ and } x_n < d \quad (1e)$$

$$A(n, \mathbf{x}) = (n, 1, x_1+1, x_2+1, \dots, x_{n-1}+1) \text{ for } n \leq m \text{ and } x_n = d \quad (2e)$$

That is, the new customer receives one unit of service by the next state time, and all the old customers receive one additional unit of service. If the oldest customer has received d units of service, then it leaves the system by the next state time. Note that it is possible for a customer with d units of service at the present time to leave the system and be replaced by an arrival at the present time (i.e., (2e) with $n=m$, $x_n=d$). Let $B(n, \mathbf{x})$ denote the next state if either no arrival occurs or if a new arrival is discarded.

$$B(n, \mathbf{x}) = (n, x_1+1, x_2+1, \dots, x_n+1) \text{ for } x_n < d \quad (3e)$$

$$B(n, \mathbf{x}) = (n-1, x_1+1, x_2+1, \dots, x_{n-1}+1) \text{ for } x_n = d \quad (4e)$$

a) Hypothesize that the *backward* chain for this system is also a sampled time $M/D/m/m$ queueing system, but that the state (n, x_1, \dots, x_n) ($0 \leq n \leq m$, $1 \leq x_1 < x_2 < \dots < x_n \leq d$) has a different interpretation: n is the number of customers as before, but x_i is now the remaining service required by customer i . Explain how this hypothesis leads to the following steady state equations:

$$\lambda \pi_{(n,x)} = (1-\lambda) \pi_{(A(n,x))} ; \quad n < m, x_n < d \quad (5e)$$

$$\lambda \pi_{(n,x)} = \lambda \pi_{(A(n,x))} ; \quad n \leq m, x_n = d \quad (6e)$$

$$(1-\lambda) \pi_{(n,x)} = \lambda \pi_{(B(n,x))} ; \quad n \leq m, x_n = d \quad (7e)$$

$$(1-\lambda) \pi_{(n,x)} = (1-\lambda) \pi_{(B(n,x))} ; \quad n \leq m, x_n < d \quad (8e)$$

- b) Using this hypothesis, find $\pi_{(n,x)}$ in terms of π_0 , where π_0 is the probability of an empty system. Hint: Use (7e) and (8e); your answer should depend on n , but not x .
- c) Verify that the above hypothesis is correct.
 - d) Find an expression for π_0 .
 - e) Find an expression for the steady state probability that an arriving customer is discarded.

Chapter 7

Random Walks and Martingales

7.1 INTRODUCTION

Let $\{X_i; i \geq 1\}$ be a sequence of IID random variables, and let $S_n = X_1 + X_2 + \dots + X_n$. The integer time stochastic process $\{S_n; n \geq 1\}$ is called a *random walk*. For any given n , S_n is just a sum of IID random variables, but here, we are more interested in the behavior of the random walk *process*, $\{S_n; n \geq 1\}$, and thus in such questions as finding the first n for which S_n exceeds some threshold α , or the probability that S_n exceeds α for any value of n . Since S_n drifts downward with increasing n for $E[X] = \bar{X} < 0$, and S_n drifts upward if $\bar{X} > 0$, the results to be obtained depend critically on whether $\bar{X} < 0$, $\bar{X} > 0$, or $\bar{X} = 0$. Since results for $\bar{X} < 0$ can be easily translated into results for $\bar{X} > 0$ by considering $\{-S_n; n \geq 0\}$, we will focus on the case $\bar{X} < 0$. As one might expect, both the results and the techniques have a very different flavor when $\bar{X} = 0$, since here the random walk does not drift but typically wanders around in a rather aimless fashion. We first give several representative examples of random walks and then treat the problem of threshold crossings. We then introduce a rather general type of stochastic process called a Martingale. The topic of Martingales is both a subject of interest in its own right and also a tool that provides additional insight into random walks, laws of large numbers, and other basic topics in probability and stochastic processes.

EXAMPLE 1—SIMPLE RANDOM WALK: Suppose X_1, X_2, \dots are IID binary random variables, taking on the value 1 with probability p and -1 with probability $q=1-p$. Then $\{S_n; n \geq 1\}$, $S_n = X_1 + \dots + X_n$, is called a *simple random walk*. S_n is the difference between positive and negative occurrences in the first n trials. Thus, if there are j positive occurrences for $0 \leq j \leq n$, then $S_n = 2j-n$, and

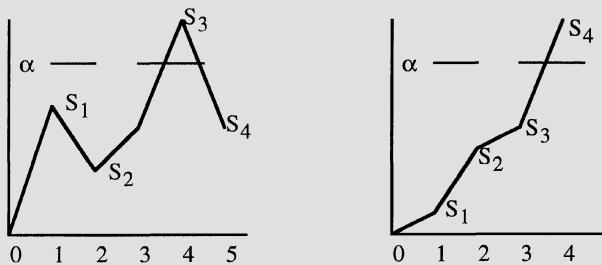
$$P(S_n = 2j-n) = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \quad (1)$$

This distribution allows us to answer questions about S_n for any given n , but it is not very helpful in answering such questions as the following: for any given integer $k > 0$,

what is the probability that the sequence S_1, S_2, \dots ever reaches or exceeds k ? This is the same as the probability that $\sup_{n \geq 1} S_n \geq k$ and is referred to as the probability that the random walk *crosses a threshold* at k . Exercise 7.1 illustrates an easy way to find $P(\sup_{n \geq 1} S_n \geq k)$ for the simple random walk, and we treat the same question for general random walks later. We shall also consider a number of other questions such as finding the distribution of the first time that $\{S_n; n \geq 1\}$ reaches a given value (conditional on reaching it at all), finding the probability that it reaches some positive value before reaching some negative value, etc.

EXAMPLE 2: Suppose next that X_1, X_2, \dots are arbitrary IID integer valued random variables. We can again ask for the probability that such an integer valued random walk crosses a threshold at k , i.e., that $\sup_n S_n \geq k$, but the question is considerably harder than for simple random walks. Since this random walk takes on only integer values, it can be represented as a Markov chain with the integers as the state space. In the Markov chain representation, threshold crossing problems are first passage time problems. These problems can be attacked by the Markov chain tools we already know, but the special structure of the random walk provides new approaches and simplifications.

EXAMPLE 3: If X_1, X_2, \dots are IID positive random variables, then $\{S_n; n \geq 1\}$ can be regarded as the sequence of arrival epochs of a renewal process, $\{N(t); t \geq 0\}$. The sequence $\{S_n; n \geq 1\}$ must eventually cross a threshold at any given positive value α . In the renewal process, $N(\alpha)$ is the largest n for which $S_n \leq \alpha$. It follows that $S_{N(\alpha)} \leq \alpha < S_{N(\alpha)+1}$ and $N(\alpha)+1$ is the smallest n for which $S_n > \alpha$, i.e., the smallest n for which a threshold at α is strictly exceeded. Thus studying threshold crossings at arbitrary α for positive random walks is the same as studying renewal processes (see figure 7.1).



Sample path of a random walk.
The barrier at α is crossed on trial 3, and the overshoot is $S_3 - \alpha$

Sample path of renewal process.
 $N(\alpha)=3$; note axes are reversed with t on vertical axis.

Figure 7.1. Comparison of a random walk with arbitrary $\{X_i; i \geq 1\}$ and a random walk with positive $\{X_i; i \geq 1\}$; i.e., a renewal process.

For random walks in general, the variables X_i can take on both positive and negative values, so that a threshold at a given value can be crossed several times (see figure

7.1). For threshold crossing problems with random walks, we are frequently interested in the *overshoot*, i.e., the extent by which a threshold crossing exceeds the threshold; the overshoot in a renewal process is the residual life.

7.2 THE G/G/1 QUEUE

The G/G/1 queue is another example of a system that can be modeled as a random walk. We shall show that, for FCFS service, the probability $P(W \geq w)$ of exceeding any given waiting time w in the queue is the same as the probability that a certain random walk ever crosses a threshold at w . Let X_1, X_2, \dots be the inter-arrival times of a G/G/1 queueing system; thus these variables are IID with a given distribution function $F(x) = P(X_i \leq x)$. Assume that arrival 0 enters an empty system at time 0, so that $S_n = X_1 + X_2 + \dots + X_n$ is the epoch of the n^{th} arrival after time 0. Let Y_i , $i \geq 0$, be the service time required by the i^{th} arrival. The sequence $\{Y_i; i \geq 0\}$ is IID, with some arbitrary distribution $G(y) = P(Y_i \leq y)$, and is independent of $\{X_i; i \geq 1\}$. Figure 7.2 shows a sample path of arrivals and departures and illustrates the waiting time in queue for each arrival.

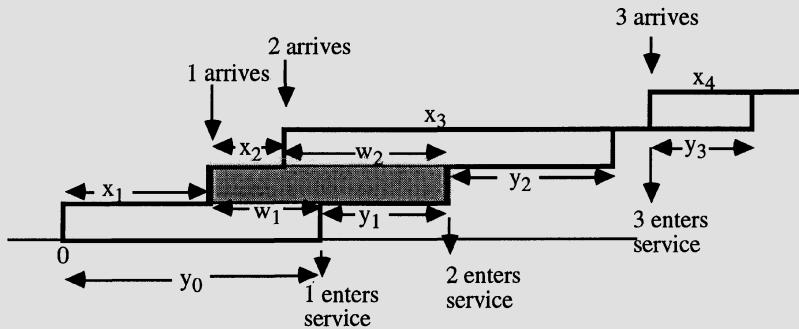


Figure 7.2. Customer 1 arrives at time x_1 and enters service after customer 0 completes service, i.e., at y_0 . Thus $w_1 = y_0 - x_1$. Customer 2 arrives at $x_2 + x_1$ and enters service after customer 1 finishes, i.e., at $y_1 + y_0$. Thus $w_2 = y_1 + y_0 - x_1 - x_2$. From the shaded area above, note that $x_2 + w_2 = w_1 + y_1$, so $w_2 = w_1 - y_1 - x_2$. Customer 3, however, enters when the queue is empty, so $w_3 = 0$.

To analyze the waiting time, note that the interval from arrival to departure of customer n is $W_n + Y_n$ (this is illustrated by the shaded area in figure 7.2 for customer 1). Customer $n+1$ arrives X_{n+1} time units after the beginning of this interval. If $X_{n+1} < W_n + Y_n$, then customer $n+1$ arrives while customer n is still in the system, and thus must wait in the queue until n finishes service (in the figure, for example, customer 2 arrives while customer 1 is still in the queue). Thus

$$W_{n+1} = W_n + Y_n - X_{n+1} \text{ if } X_{n+1} < W_n + Y_n \quad (2)$$

On the other hand, if $X_{n+1} > W_n + Y_n$, then customer $n+1$ starts service immediately and $W_{n+1}=0$. This is the case for customer 3 in the figure. These two cases can be combined in the single equation

$$W_{n+1} = \max [W_n + Y_n - X_{n+1}, 0]; n \geq 0; W_0 = 0 \quad (3)$$

Since Y_n and X_{n+1} are always coupled together in this equation for different n , it is convenient to define $U_{n+1} = Y_n - X_{n+1}$. Note that $\{U_i; i \geq 1\}$ is a sequence of IID random variables. We have $W_n = \max[W_{n-1} + U_n, 0]$, and iterating on this equation,

$$\begin{aligned} W_n &= \max[\{\max[W_{n-2} + U_{n-1}, 0] + U_n, 0\}] = \max[W_{n-2} + U_{n-1} + U_n, 0] \\ &= \max[W_{n-3} + U_{n-2} + U_{n-1} + U_n, U_{n-1} + U_n, 0] \\ &= \max[(U_1 + U_2 + \dots + U_n), (U_2 + U_3 + \dots + U_n), \dots, (U_{n-1} + U_n), U_n, 0] \end{aligned} \quad (4)$$

Although it is not necessary for the analysis, it can be verified that if (4) is maximized by $U_i + U_{i+1} + \dots + U_n$, then a busy period starts with the arrival of customer $i-1$ and continues through the service of customer n . Define $Z_1^n = U_n$, define $Z_2^n = U_n + U_{n-1}$, and in general, for $i \leq n$, define $Z_i^n = U_n + U_{n-1} + \dots + U_{n-i+1}$. Thus $Z_n^n = U_n + \dots + U_1$. With these definitions, (4) becomes $W_n = \max[0, Z_1^n, Z_2^n, \dots, Z_n^n]$. Note that the terms in $\{Z_i^n; 1 \leq i \leq n\}$ are the first n terms of a random walk, but it is not the random walk $\{S_n; n \geq 1\}$ where $S_n = U_1 + \dots + U_n$. Rather, it is a random walk going backward, starting with U_n . Note also that W_{n+1} , for example, is the maximum of a different set of variables, i.e., it is the walk going backward from U_{n+1} . Fortunately, this doesn't matter for our analysis. The probability that the wait is greater than or equal to a given value α is

$$P(W_n \geq \alpha) = P(\max[0, Z_1^n, Z_2^n, \dots, Z_n^n] \geq \alpha) \quad (5)$$

This says that, for the n^{th} customer, $P(W_n \geq \alpha)$ is equal to the probability that the random walk $\{Z_i^n; 1 \leq i \leq n\}$ crosses a threshold at α by the n^{th} try. Because of the initialization used in the analysis, we see that W_n is the waiting time in queue of the n^{th} arrival after the beginning of a busy period (although this n^{th} arrival might be in a later busy period). In the limit $n \rightarrow \infty$, (i.e., in steady state), $\lim_{n \rightarrow \infty} P(W_n \geq \alpha)$ equals the probability that the corresponding random walk ever crosses a threshold at α . Note that $P(W_n \geq \alpha)$ is nondecreasing in n . These results are summarized in the following theorem.

THEOREM 1: Let $\{X_i; i \geq 1\}$ be the interarrival intervals of a G/G/1 queue, let $\{Y_i; i \geq 0\}$ be the service times, and assume that the system is empty at time 0 when customer 0 arrives. Let W_n be the time that the n^{th} customer waits in the queue. Let $U_n = Y_{n-1} - X_n$ for $n \geq 1$. Then for any $\alpha > 0$, and $n \geq 1$, $P(W_n \geq \alpha)$ is equal to the probability that a random walk with the IID variables $\{U_n\}$ crosses a threshold at α by the n^{th} try. Let W be the time that a customer waits in the queue in steady state. Then $P(W \geq \alpha)$ is equal to the probability that the random walk ever crosses a threshold at α .

We shall study the probability that a random walk crosses a positive threshold later; from theorem 1, this problem is equivalent to finding the queueing delay for the G/G/1 queue.

7.3 DETECTION, DECISIONS, AND HYPOTHESIS TESTING

Consider a situation in which we make n noisy observations of a single sample of a binary random variable and then guess which of the binary values occurred. In communication, this is called a *detection* problem and models, for example, the problem of receiving binary data from a noisy communication link or the problem of knowing whether a target is present in a radar observation. In control theory, such situations are usually referred to as *decision* problems, whereas in statistics, they are referred to as *hypothesis testing*.

Specifically, let H_0 and H_1 (hypothesis 0 and 1) be the two possible values of a binary random variable and let $p_0 = P(H_0)$ and $p_1 = 1 - p_0 = P(H_1)$. One usually refers to p_0 and p_1 as *a priori* probabilities, and statisticians have argued since the beginning of statistics about the validity of assuming such an *a priori* probability. Our view is that we are simply creating a model with a parameter p_0 in order to obtain some insight about this class of problems. We shall find that there are some such problems whose answer is independent of p_0 , and these problems can be answered without assuming *a priori* probabilities. However, the answers to these problems are usually more transparent in a full probabilistic model in which all the relevant events have probabilities.

Let Y_1, Y_2, \dots, Y_n be the n observations. We assume that, conditional on H_0 , the observations Y_1, \dots, Y_n are IID random variables. Suppose, to be specific, that these variables have a density $f(y | H_0)$. Conditional on H_0 , the joint density of a sample sequence $y = (y_1, y_2, \dots, y_n)$ of observations is given by

$$f(y | H_0) = \prod_{i=1}^n f(y_i | H_0) \quad (6)$$

Similarly, conditional on H_1 , we assume that Y_1, \dots, Y_n are IID random variables with a conditional joint density given by (6) with H_1 in place of H_0 . Given a particular sequence of observations $y = y_1, y_2, \dots, y_n$, we can evaluate $P(H_1 | y)$ as

$$P(H_1 | y) = \frac{p_1 \prod_{i=1}^n f(y_i | H_1)}{p_1 \prod_{i=1}^n f(y_i | H_1) + p_0 \prod_{i=1}^n f(y_i | H_0)} \quad (7)$$

$P(H_0 | y)$ can be evaluated in the same way, and the ratio of these quantities is given by

$$\frac{P(H_1 | y)}{P(H_0 | y)} = \frac{p_1 \prod_{i=1}^n f(y_i | H_1)}{p_0 \prod_{i=1}^n f(y_i | H_0)} \quad (8)$$

If we observe y and choose H_0 , then $P(H_1 | y)$ is the resulting probability of error, and conversely if we choose H_1 , then $P(H_0 | y)$ is the resulting probability of error. Thus the probability of error is minimized, for a given y , by evaluating the above ratio and choosing H_1 if the ratio is greater than 1 and choosing H_0 otherwise. If the ratio is equal to 1, the error probability is the same whether H_0 or H_1 is chosen. For simplicity, we assume that $f(y | H_i)$ is finite and non-zero for all y , which means that the ratio in (8) is neither 0 nor ∞ . The above rule for choosing H_0 or H_1 is called the *Maximum A posteriori Probability* detection rule, usually abbreviated as the *MAP* rule. The rule has a more attractive form (and also brings us back to random walks) if we take the logarithm of each side of (8), getting

$$\ln \frac{P(H_1 | y)}{P(H_0 | y)} = \ln \frac{p_1}{p_0} + \sum_{i=1}^n z_i ; \text{ where } z_i = \ln \frac{f(y_i | H_1)}{f(y_i | H_0)} \quad (9)$$

The quantity z_i in (9) is called a *log likelihood ratio*. Each sample value y_i corresponds to some value of the log likelihood ratio as given above. The MAP rule is to choose H_1 or H_0 depending on whether the left hand equation above is positive or negative, i.e.,

$$\sum_{i=1}^n z_i \begin{cases} > \ln(p_0/p_1) ; \text{ choose } H_1 \\ < \ln(p_0/p_1) ; \text{ choose } H_0 \\ = \ln(p_0/p_1) ; \text{ don't care, choose either} \end{cases} \quad (10)$$

Conditional on H_0 , $\{Y_i; 1 \leq i \leq n\}$ is a sequence of IID random variables and thus, with $Z_i = \ln[f(Y_i | H_1)/f(Y_i | H_0)]$, $\{Z_i; 1 \leq i \leq n\}$ is also a sequence of IID random variables. It follows that $f(Y_i | H_1)$ is simply a numerical function of the random variable Y_i (conditional on H_0) and thus Z_i is another numerical function of Y_i (conditional on H_0). The sum on the left in (10) is thus a sample value of the n^{th} term in the random walk $S_n = Z_1 + \dots + Z_n$ (conditional on H_0). The MAP rule chooses H_1 , thus making an error conditional on H_0 , if S_n is greater than the threshold $\ln[p_0/p_1]$. Similarly, conditional on H_1 , $S_n = Z_1 + \dots + Z_n$ is the n^{th} term in a random walk with the conditional probabilities from H_1 , and an error is made, conditional on H_1 , if S_n is less than the threshold $\ln[p_0/p_1]$.

The decision rule in (10) is called a *threshold test* in the sense that S_n is compared with a threshold to make a decision. There are a number of other formulations of the problem that also lead to threshold tests. For example, *maximum likelihood detection* chooses the hypothesis i that maximizes $f(y | H_i)$, and thus corresponds to a threshold of 0. Also, costs might be associated with each hypothesis and error event. The minimum cost decision is then a threshold test (see exercise 7.4). Finally, we might impose the constraint that $P(\text{error} | H_1)$ must be less than some tolerable limit α , and then minimize $P(\text{error} | H_0)$ subject to this constraint. The solution to this is called a *Neyman–Pearson threshold test* (see exercise 7.5).

So far we have assumed that a decision is made after n observations. In many situations there is a cost associated with observations and one would prefer, after a given number of observations, to make a decision if the resulting probability of error is small enough, and to continue with more observations otherwise. This leads to the choice of H_1 if $S_n = Z_1 + \dots + Z_n$ exceeds some positive threshold α , to the choice of H_0 if S_n is less than some negative threshold β , and to further testing if S_n has not exceeded either threshold. This type of strategy is called *sequential analysis* and we see that it corresponds to a random walk with a positive and a negative threshold.

The previous examples have all involved random walks crossing thresholds, and we now turn to the systematic study of threshold crossing problems. First we look at single thresholds, so that one question of interest is to find $P(S_n \geq \alpha)$ for an arbitrary integer $n \geq 1$ and arbitrary real $\alpha > 0$. Another question is whether $S_n \geq \alpha$ for any $n \geq 1$. We then turn to random walks with both a positive and negative threshold. Here, some questions of interest are to find the probability that the positive threshold is crossed before the negative threshold, to find the distribution of the threshold crossing time given the particular threshold crossed, and to find the overshoot when a threshold is crossed.

7.4 THRESHOLD CROSSING PROBABILITIES

Let $\{X_i; i \geq 1\}$ be a sequence of IID random variables with the distribution function $F_X(x)$, and let $\{S_n; n \geq 1\}$ be a random walk with $S_n = X_1 + \dots + X_n$. We assume throughout that $E[X]$ exists and is finite. The reader should focus on the case $E[X] = \bar{X} < 0$ on a first reading, and consider $\bar{X} = 0$ and $\bar{X} > 0$ later. For $\bar{X} < 0$ and $\alpha > 0$, we shall develop upper bounds on $P(S_n \geq \alpha)$ that are exponentially decreasing in n and α . These bounds, and many similar results to follow, are examples of *large deviation theory*, i.e., probabilities of highly unlikely events

We assume throughout this section that X has a moment generating function $g(r) = E[e^{rX}] = \int e^{rx} dF_X(x)$, and that $g(r)$ is finite in some open interval around $r=0$. As pointed out in Chapter 1, X must then have moments of all orders and the tails of its distribution function $F_X(x)$ must decay at least exponentially in x as $x \rightarrow \infty$ and as $x \rightarrow -\infty$. Note that e^{rx} is increasing in r for $x > 0$, so that if $\int_{x \geq 0} e^{rx} dF_X(x)$ blows up for some $r_+ > 0$, it remains infinite for all $r > r_+$. Similarly, for $x < 0$, e^{rx} is increasing in $-r$, so that if $\int_{x \leq 0} e^{rx} dF_X(x)$ blows up at some $r_- < 0$, it is infinite for all $r < r_-$. Thus $g(r)$ is finite for $r_- < r < r_+$ and is infinite for $r > r_+$ and for $r < r_-$. The end points r_- and r_+ can each be finite or infinite, and the values $g(r_+)$ and $g(r_-)$ can each be finite or infinite. Exercise 7.6 provides some examples of these possibilities.

The moment generating function of $S_n = X_1 + \dots + X_n$ is given by

$$g_{S_n}(r) = E[\exp(rS_n)] = E[\exp(r(X_1 + \dots + X_n))] = \{E[\exp(rx)]\}^n = \{g(r)\}^n \quad (11)$$

and thus $g_{S_n}(r)$ is finite in the same interval (r_-, r_+) as $g(r)$.

First we look at the probability, $P(S_n \geq \alpha)$, that the n^{th} step of the random walk satisfies $S_n \geq \alpha$ for some threshold $\alpha > 0$. We could actually find the distribution of S_n

either by convolving the density of X with itself n times or by going through the transform domain. This would not give us much insight, however, and would be computationally tedious for large n . Instead, we explore the exponential, or Chernoff, bound, Eq. 22 in Chapter 1. For any $r \geq 0$, in the region where $g(r)$ is finite, i.e., for $0 \leq r < r_+$, we have

$$P(S_n \geq \alpha) \leq g_{S_n}(r) e^{-r\alpha} = \{g(r)\}^n e^{-r\alpha} \quad (12)$$

It is convenient to rewrite (12) in terms of the *semi-invariant moment generating function* $\gamma(r) = \ln[g(r)]$.

$$P(S_n \geq \alpha) \leq \exp[n\gamma(r) - r\alpha] ; \text{ any } r, 0 \leq r < r_+ \quad (13)$$

The first derivative of γ with respect to r is given by $\gamma'(r) = g'(r)/g(r)$ and similarly the second derivative is $\gamma''(r) = [g(r)g''(r) - (g'(r))^2]/[g(r)]^2$. At $r=0$, this simplifies to $\gamma'(0) = \bar{X} = E[X]$ and $\gamma''(0) = \text{VAR}(X)$. The fact that $\gamma'(0)$ is the second *central* moment of X is why γ is called a semi-invariant moment generating function. Unfortunately, the higher order derivatives of γ , evaluated at $r=0$, are not equal to the higher order central moments. In exercise 7.7, it is shown that $\gamma'(r) \geq 0$ over the range of r where $\gamma(r)$ is finite and that $\gamma'(r) > 0$ if $\gamma''(0) > 0$. Figure 7.3 sketches $\gamma(r)$ for the case $\bar{X} < 0$; in the next section, we explore the probability that S_n ever exceeds any given $\alpha > 0$ for any value of n . Exercise 7.10 explores $P(S_n \geq \alpha)$ for $\bar{X} \geq 0$.

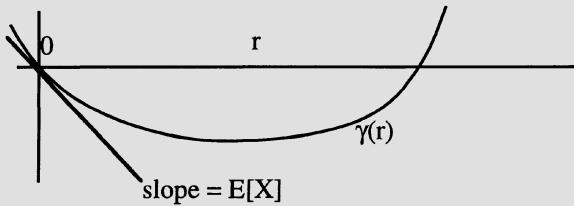


Figure 7.3. Semi-invariant moment generating function.

Since (13) is valid for all r , $0 \leq r < r_+$, we can minimize the right side of (13) over $0 \leq r < r_+$ to get the tightest bound. Assuming temporarily that $r_+ = \infty$, this minimum occurs at the r satisfying $\gamma'(r) = \alpha/n$, yielding

$$P(S_n \geq \alpha) \leq \exp\{-n[r\alpha/n - \gamma(r)]\} = \exp\{-\alpha[r - (n/\alpha)\gamma(r)]\}; \gamma'(r) = \alpha/n \quad (14)$$

This shows that for a fixed ratio α/n , $P(S_n \geq \alpha)$ decays exponentially in n , or equivalently, exponentially in α . Figure 7.4 gives a geometric interpretation of these exponents; consider a tangent of slope α/n to the curve $\gamma(r)$. The exponential decay rate in n , i.e., $r\alpha/n - \gamma(r)$, is the negative of the vertical axis intercept of this tangent line. Similarly, the exponential decay rate in α , i.e., $r - (n/\alpha)\gamma(r)$, is the horizontal axis intercept. Although we have only established this as an upper bound, exercise 7.9 shows that for

any fixed ratio $a=\alpha/n$, and any $\epsilon>0$, there is an $n_0(\epsilon)$ such that for all $n\geq n_0(\epsilon)$, $P(S_n \geq n(a-\epsilon)) > \exp\{-n[r\alpha - \gamma(r) + \epsilon]\}$ where r satisfies $\gamma'(r)=a$. This means that for fixed $a=\alpha/n$, (14) is exponentially tight, i.e., $P(S_n \geq na)$ decays exponentially with increasing n at the asymptotic rate $-ra+\gamma(r)$ where r satisfies $\gamma'(r)=a$.

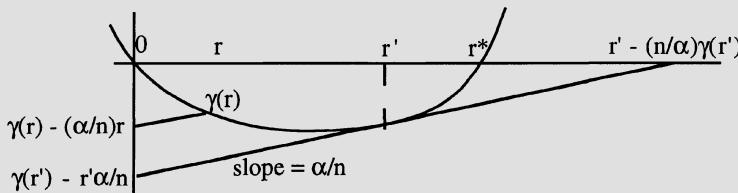


Figure 7.4. Graphical minimization of $\gamma(r) - (\alpha/n)r$. For any r , $\gamma(r) - (\alpha/n)r$ is found by drawing a line of slope (α/n) from the point $(r, \gamma(r))$ to the vertical axis. The minimum occurs when the line of slope α/n is tangent to the curve.

Note from figure 7.4 that, as α/n is increased, the horizontal axis intercept decreases until it reaches the point r^* defined as the positive root of $\gamma(r)$. The horizontal axis intercept then starts to increase again. This means that for each choice of n , we have the simpler (but usually less tight) bound

$$P(S_n \geq \alpha) \leq \exp(-r^*\alpha) \quad (15)$$

Figure 7.5 illustrates the minimization of (13) for the atypical case where $r_+<\infty$, $\gamma(r_+)<0$ and $\gamma'(r)=\infty$ for $r>r_+$. To include this case, (14) is generalized to

$$P(S_n \geq \alpha) \leq \exp\{-n[r\alpha/n - \gamma(r)]\} = \exp\{-\alpha[r - (n/\alpha)\gamma(r)]\} \quad (16)$$

where r satisfies $\gamma'(r) = \alpha/n$ if $\alpha/n < \lim_{r \rightarrow r_+} \gamma'(r)$ and $r=r_+$ otherwise.

If we extend the definition of r^* as the supremum of r such that $\gamma(r)\leq 0$, then both the cases of figure 7.4 and 7.5 are covered. For the case of figure 7.5, $r^* = r_+$ and $P(S_n \geq \alpha) \leq \exp(-r^*\alpha)$ still holds for all n .

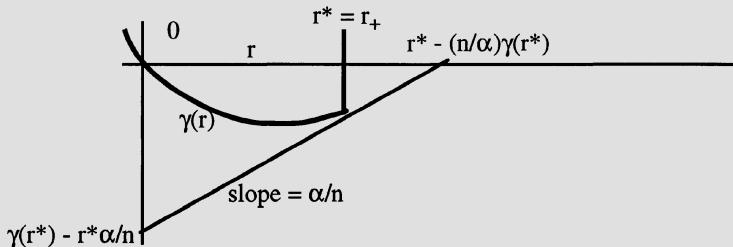


Figure 7.5. Example of a semi-invariant moment generating function that blows up at some point r_+ .

We next establish Wald's identity, which shows, among other things, that if $\bar{X} < 0$, then $\exp(-r^*\alpha)$ is an upper bound (and a reasonable approximation) to the probability that the walk ever crosses a threshold at $\alpha > 0$. This means that for such threshold crossing problems, the random variable X is characterized primarily by the parameter r^* rather than by the mean, variance, or other moments of X .

7.5 WALD'S IDENTITY AND WALKS WITH TWO THRESHOLDS

Let $\{X_i; i \geq 1\}$ be IID random variables that are not identically zero and that have a finite moment generating function $g(r)$ in an open region (r_-, r_+) including $r=0$. Let $\alpha > 0$ and $\beta < 0$ represent two thresholds, and let N denote the time at which the random walk first crosses either the threshold at α or the threshold at β (see figure 7.6). That is, N takes on the value n if either $S_n \geq \alpha$ or $S_n \leq \beta$, and, in addition, if $\beta < S_i < \alpha$ for $1 \leq i < n$. We first show that N is actually a random variable, i.e., that a threshold is eventually crossed with probability 1.

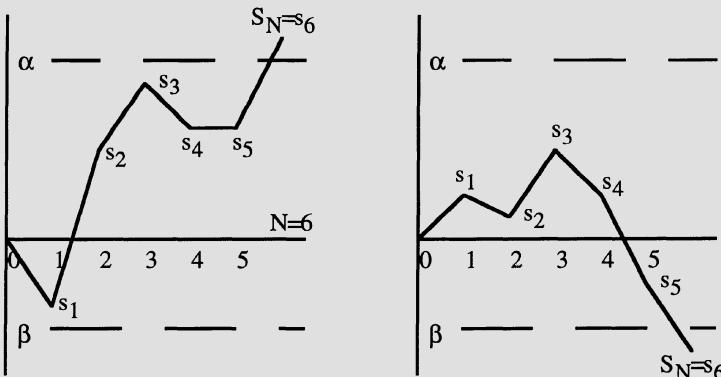


Figure 7.6. Two sample paths of a random walk with two thresholds. In each, the threshold is crossed at $N=6$. In the first $S_N > \alpha$, and in the second $S_N < \beta$.

LEMMA 1: N is a random variable (i.e., $\lim_{m \rightarrow \infty} P(N \geq m) = 0$) and N has finite moments of all orders.

Proof: Since X is not identically 0, there is some n for which either $P(S_n \leq -\alpha + \beta) > 0$ or for which $P(S_n \geq \alpha - \beta) > 0$. For any such n , let $\varepsilon = \max[P(S_n \leq -\alpha + \beta), P(S_n \geq \alpha - \beta)]$. For any integer $k \geq 1$, given that $N > n(k-1)$, and given any value of $S_{n(k-1)}$ in (β, α) , a threshold will be crossed by time nk with probability at least ε . Thus, $P[N > nk \mid N > n(k-1)] \leq 1 - \varepsilon$, and iterating on k , $P(N > nk) \leq (1 - \varepsilon)^k$. This shows that N is finite with probability 1 and that $P(N \geq j)$ goes to 0 at least geometrically in j . It follows that $g_N(r)$ is finite in a region around $r=0$, and thus that N has moments of all orders.

THEOREM 2 (WALD'S IDENTITY): Let $\{X_i; i \geq 1\}$ be IID and let $\gamma(r) = \ln\{E[e^{rX}]\}$ be the semi-invariant moment generating function of each X_i . Assume $\gamma(r)$ is finite in an open interval (r_-, r_+) , $r_- < 0 < r_+$. Let $S_n = X_1 + \dots + X_n$, let $\alpha > 0$ and $\beta < 0$, and let N be the smallest n for which either $S_n \geq \alpha$ or $S_n \leq \beta$. Then for all $r \in (r_-, r_+)$,

$$E[\exp(rS_N - N\gamma(r))] = 1 \quad (17)$$

We first show how to use and interpret this theorem, and then prove it through a sequence of lemmas. Wald's identity can be thought of as a generating function form of Wald's equality, $E[S_N] = \bar{X}E[N]$, as established in theorem 6 of chapter 3. In fact, if we take the derivative with respect to r of both sides of (17), we get

$$E[\{S_N - N\gamma'(r)\} \exp\{rS_N - N\gamma(r)\}] = 0,$$

which reduces to Wald's equality at $r=0$. Similarly, the second derivative, evaluated at $r=0$, yields

$$\text{Var}[S_N] = E[N] \text{ VAR}[X] \quad (18)$$

In the case where $\bar{X} = 0$, Wald's equality provides no information about $E[N]$, but (18) allows $E[N]$ to be calculated if $\text{VAR}[S_N]$ can be found. As an example, consider the simple random walk of example 1 with $P(X=1) = P(X=-1) = 1/2$, and assume that α and β are integers. Since S_n takes on only integer values and changes only by ± 1 , it takes on the value α or β before exceeding either of these values. Thus $S_N = \alpha$ or $S_N = \beta$. Let q_α denote $P(S_N = \alpha)$. From Wald's equality and from (18),

$$\begin{aligned} E[S_N] &= 0 = \alpha q_\alpha + \beta(1-q_\alpha) \\ E[S_N^2] &= E[N]E[X^2] = \alpha^2 q_\alpha + \beta^2(1-q_\alpha) \end{aligned}$$

Solving these equations yields

$$E[N] = -\beta\alpha/E[X^2] \quad (19)$$

Since $E[X^2] = 1$ for this example, $E[N] = -\alpha\beta$. For more general random walks, there is usually an overshoot when the threshold is crossed. If the magnitudes of α and β are large relative to the range of X , however, it is often reasonable to ignore the overshoots, and then (19) yields an approximation to $E[N]$. We now apply Wald's identity to upper bound $P(S_N \geq \alpha)$ for the case where $\bar{X} < 0$.

COROLLARY 1: Under the conditions of theorem 2, assume that $\gamma(r)$ has a root at $r^* > 0$. Then

$$P(S_N \geq \alpha) \leq \exp(-r^*\alpha) \quad (20)$$

Proof: Wald's identity, with $r=r^*$, reduces to $E[\exp(r^*S_N)] = 1$. We can express this as

$$P(S_N \geq \alpha)E[\exp(r^*S_N) | S_N \geq \alpha] + P(S_N \leq \beta)E[\exp(r^*S_N) | S_N \leq \beta] = 1 \quad (21)$$

Since the second term on the left is non-negative,

$$1 \geq P(S_N \geq \alpha)E[\exp(r^*S_N) | S_N \geq \alpha] \geq P(S_N \geq \alpha)\exp(r^*\alpha) \quad (22)$$

The second inequality follows because, conditional on $S_N \geq \alpha$, $\exp(r^*S_N) \geq \exp(r^*\alpha)$. It follows that $P(S_N \geq \alpha) \leq \exp(-r^*\alpha)$, concluding the proof.

This bound is valid for all $\beta < 0$, and thus is also valid in the limit $\beta \rightarrow -\infty$ (see exercise 7.12 for a more careful demonstration that (20) is valid without a lower threshold). Eq. (20) is also valid for the case of figure 7.5, where $\gamma(r^*) < 0$. The exponential bound in (15) shows that $P(S_n \geq \alpha) \leq \exp(-r^*\alpha)$ for each n ; (20) is stronger than this; in the limit $\beta \rightarrow -\infty$, it bounds the probability of the union of the events $\{S_n \geq \alpha\}$ over all $n \geq 1$. When applied to the G/G/1 queue, (20) is referred to as the *Kingman Bound*.

COROLLARY 2 (Kingman Bound): Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 0\}$ be the interarrival intervals and service times of a G/G/1 queue. Let $U_i = Y_{i-1} - X_i$, and let $\gamma(r) = \ln[E[e^{Ur}]]$ be the semi-invariant moment generating function of each U_i . Assume that $\gamma(r)$ has a root at $r^* > 0$. Then W , the steady state waiting time in queue, satisfies

$$P(W \geq \alpha) \leq \exp(-r^*\alpha); \alpha > 0 \quad (23)$$

Proof: From theorem 1, the probability $P(W \geq \alpha)$ that the steady state waiting time in queue is greater than or equal to any given $\alpha > 0$ is the same as the probability $P(S_N \geq \alpha)$, where $\{S_n; n \geq 1\}$ is the random walk with $S_n = U_1 + U_2 + \dots + U_n$, and N is the threshold crossing time in the limit as $\beta \rightarrow -\infty$. From corollary 1, $P(S_N \geq \alpha)$ is upper bounded by $\exp(-r^*\alpha)$, completing the proof.

In most applications, a positive threshold crossing for a random walk with a negative drift corresponds to some exceptional, and usually undesirable circumstance (for example an error in the hypothesis testing problem or an overflow in the G/G/1 queue). Thus an upper bound such as (20) provides an assurance of a certain level of performance and is often more useful than either an approximation or an exact expression that is very difficult to evaluate. For a random walk with $\bar{X} > 0$, the exceptional circumstance is $P(S_N \leq \beta)$, and one can upper bound this in the same way as before; r^* , the non-zero root of $\gamma(r)$, is then negative. These exponential bounds do not work for $\bar{X} = 0$. One can estimate $P(S_N \geq \alpha)$ from Wald's equality by ignoring the overshoots (see exercise 7.13), but either solving or bounding $P(S_N \geq \alpha)$ requires a more complex analysis involving the overshoots.

Note that (20) is an upper bound because, first, the effect of the second threshold in (21) was set to 0, and, second, the overshoot in the threshold crossing at α was set to 0

in the second inequality of (22). By recognizing that $P(S_N \leq \beta) = 1 - P(S_N \geq \alpha)$, we can solve (21), getting

$$P(S_N \geq \alpha) = \frac{1 - E[\exp(r^* S_N | S_N \leq \beta)]}{E[\exp(r^* S_N | S_N \geq \alpha)] - E[\exp(r^* S_N | S_N \leq \beta)]} \quad (24)$$

For the case of the simple random walk in example 1, overshoots never occur since the random walk always changes in unit steps. Thus, for α and β integers, we have $E[\exp(r^* S_N | S_N \leq \beta)] = \exp(r^* \beta)$ and $E[\exp(r^* S_N | S_N \geq \alpha)] = \exp(r^* \alpha)$. Substituting this in (24) yields the exact solution

$$P(S_N \geq \alpha) = \frac{\exp(-r^* \alpha) [1 - \exp(r^* \beta)]}{1 - \exp[-r^*(\alpha - \beta)]} \quad (25)$$

Solving the equation $\gamma(r^*)=0$ for the simple walk in example 1, $r^*=\ln(q/p)$. This is also valid if X takes on the three values -1, 0, and +1 with $p = P(X=1)$, $q = P(X=-1)$, and $1-p-q = P(X=0)$. It can be seen that if α and $-\beta$ are large positive integers, then the simple bound of (20) is almost exact for this example. Eq. (25) is sometimes taken as an approximation for (24). Unfortunately, for many applications, the overshoots are more significant than the effect of the opposite threshold so that (25) is only negligibly better than (20) as an approximation, and has the disadvantage of not being a bound.

If $P(S_N \geq \alpha)$ must actually be calculated, then the overshoots in (24) must be taken into account. See Chapter 12 of [Fel66] for a treatment of overshoots.

JOINT DISTRIBUTION OF N AND BARRIER: Next we look at $P(N \geq n, S_N \geq \alpha)$, where again we assume that $\bar{X} < 0$ and that $\gamma(r^*) = 0$ for some $r^* > 0$. For any r in the region where $\gamma(r) \leq 0$ (i.e., for $0 \leq r \leq r^*$), we have $-N\gamma(r) \geq -n\gamma(r)$ for $N \geq n$. Thus, from the Wald identity, we have

$$\begin{aligned} 1 &\geq E[\exp[rS_N - N\gamma(r)] | N \geq n, S_N \geq \alpha] P(N \geq n, S_N \geq \alpha) \\ &\geq \exp[r\alpha - n\gamma(r)] P(N \geq n, S_N \geq \alpha) \\ P(N \geq n, S_N \geq \alpha) &\leq \exp[-r\alpha + n\gamma(r)] ; \text{ all } r \text{ such that } 0 \leq r \leq r^* \end{aligned} \quad (26)$$

Under our assumption that $\bar{X} < 0$, we have $\gamma(r) \leq 0$ in the range $0 \leq r \leq r^*$, and (26) is valid for all r in this range. To obtain the tightest bound of this form, we should minimize the right hand side of (26). This is the same minimization (except for the constraint $r \leq r^*$) as in figure 7.4, and the result is

$$P(N \geq n, S_N \geq \alpha) \leq \exp[-r\alpha + n\gamma(r)] \quad (27)$$

where r satisfies $\gamma'(r) = \alpha/n$ if $\alpha/n < \gamma'(r^*)$ and $r=r^*$ otherwise. This is the same as the bound on $P(S_n \geq \alpha)$ in (16) except that $r \leq r^*$ in (27).

The bound in (27) is strictly tighter than the bound $P(S_N \geq \alpha) \leq \exp(-r^* \alpha)$ if $\alpha/n < \gamma'(r^*)$. For a given value of α , define $n^* = \alpha/\gamma'(r^*)$. We can then rewrite (27) as

$$P(N \geq n, S_N \geq \alpha) \leq \begin{cases} \exp[-r\alpha + \gamma(r)]; & \text{for } n > n^*, \gamma(r) = \alpha/n \\ \exp(-r^* \alpha); & \text{for } n \leq n^* \end{cases} \quad (28)$$

The interpretation of (28) is that n^* is an estimate of the typical value of N given that the threshold at α is crossed. For n greater than this typical value, (28) provides a tighter bound on $P(N \geq n, S_N \geq \alpha)$ than the bound on $P(S_N \geq \alpha)$ in (20), whereas (28) provides nothing new for $n \leq n^*$. In section 7.7, we shall derive the slightly stronger result that $P(\sup_{i \geq n} S_i \geq \alpha)$ is also upper bounded by the right hand side of (28).

We next develop an almost identical upper bound to $P(N \leq n, S_N \geq \alpha)$ by using the Wald identity for $r > r^*$. Here $\gamma(r) > 0$, so $-N\gamma(r) \geq -n\gamma(r)$ for $N \leq n$. It follows that

$$\begin{aligned} 1 &\geq E[\exp[rS_N - N\gamma(r)] \mid N \leq n, S_n \geq \alpha] P(N \leq n, S_N \geq \alpha) \\ &\geq \exp[r\alpha - n\gamma(r)] P(N \leq n, S_N \geq \alpha) \\ P(N \leq n, S_N \geq \alpha) &\leq \exp[-r\alpha + n\gamma(r)] \end{aligned} \quad (29)$$

Optimizing over r as before (except recognizing that $r \geq r^*$), we get

$$P(N \leq n, S_N \geq \alpha) \leq \begin{cases} \exp[-r\alpha + n\gamma(r)]; & \text{for } n < n^*, \gamma(r) = \alpha/n \\ \exp(-r^* \alpha); & \text{for } n \geq n^* \end{cases} \quad (30)$$

This strengthens the interpretation of n^* as the typical value of N conditional on crossing the threshold at α . That is, (30) provides information on the lower tail of the distribution of N (conditional on $S_N \geq \alpha$), whereas (28) provides information on the upper tail.

PROOF OF WALD'S IDENTITY: We prove Wald's identity with three lemmas, of which the first (lemma 2) is a truncated version of the final result.

LEMMA 2: Under the conditions of theorem 2, let m be a positive integer and let $N_m = \min[N, m]$. Then for $r \in (r_-, r_+)$, $E[\exp\{rS_{N_m} - N_m\gamma(r)\}] = 1$.

Proof: Let $I_n = 1$ for $N \geq n$ and $I_n = 0$ otherwise. Then $N_m = \sum_{n=1}^m I_n$ and $S_{N_m} = \sum_{n=1}^m X_n I_n$. N_m is a stopping rule for $\{X_i; i \geq 1\}$ (see section 3.4), and I_n is independent of X_n . Thus

$$\begin{aligned} E[\exp\{rS_{N_m} - N_m \gamma(r)\}] &= E\left[\exp\left(\sum_{n=1}^m (rX_n I_n) - \sum_{n=1}^m (I_n \gamma(r))\right)\right] \\ &= E\left[\prod_{n=1}^m \exp(I_n(rX_n - \gamma(r)))\right] \end{aligned}$$

The expected value of a product of random variables, $E[Y_1 Y_2]$ can be expressed as $E_{Y_1}[Y_1 E_{Y_2}[Y_2 | Y_1]]$, and similarly the product above can be expanded as

$$\begin{aligned} E_{X_1 I_1}[\exp\{I_1(rX_1 - \gamma(r))\} E_{X_2 I_2}[\exp\{I_2(rX_2 - \gamma(r))\} \dots \\ E_{X_m I_m}[\exp\{I_m(rX_m - \gamma(r))\} | X_1, \dots, X_{m-1}, I_1, \dots, I_{m-1}] \dots]] \end{aligned} \quad (31)$$

Consider the final term above. We look at this separately for the case $I_m = 0$ and $I_m = 1$.

$$\begin{aligned} E[\exp\{I_m(rX_m - \gamma(r))\} | X_1, \dots, I_{m-1}, I_m = 0] &= E[\exp(0)] = 1 \\ E[\exp\{I_m(rX_m - \gamma(r))\} | X_1, \dots, I_{m-1}, I_m = 1] &= E[\exp(rX_m - \gamma(r))] = 1 \end{aligned}$$

where the last step follows since X_m is independent of I_1, \dots, I_m and of X_1, \dots, X_{m-1} . Similarly, for each n , $1 \leq n \leq m$, we have

$$E[\exp\{I_n(rX_n - \gamma(r))\} | X_1, \dots, X_{n-1}, I_1, \dots, I_{n-1}] = 1 \quad (32)$$

Substituting (32) with $n=m$ into (31), and repeating this successively for $n=m-1, m-2, \dots, 1$, we get $E[\exp(rS_{N_m} - N_m \gamma(r))] = 1$, completing the proof.

LEMMA 3: Under the conditions of theorem 2, $E[\exp(rS_N - N \gamma(r))] \leq 1$, with equality if

$$\lim_{m \rightarrow \infty} E[\exp\{rS_m - m \gamma(r)\} | N \geq m] P(N > m) = 0 \quad (33)$$

Proof: Expanding $E[\exp(rS_{N_m} - N_m \gamma(r))]$ by conditioning on $N=n$ for $1 \leq n \leq m$, we get

$$\begin{aligned} 1 &= E\left[\exp\left(rS_{N_m} - N_m \gamma(r)\right)\right] \\ &= \sum_{n=1}^m E\left[\exp\left(rS_n - n \gamma(r)\right) | N=n\right] P(N=n) + E\left[\exp\left(rS_m - m \gamma(r)\right) | N > m\right] P(N > m) \end{aligned} \quad (34)$$

Since the final term in (34) is non-negative,

$$\sum_{n=1}^m E\left[\exp(rS_n - n\gamma(r)) \mid N=n\right] P(N=n) \leq 1 \quad (35)$$

Since $m \geq 1$ is arbitrary and all terms in the sum are non-negative, we can take the limit,

$$\lim_{m \rightarrow \infty} \sum_{n=1}^m E\left[\exp(rS_n - n\gamma(r)) \mid N=n\right] P(N=n) \leq 1 \quad (36)$$

From (34), we see that this is satisfied with equality if (33) is satisfied. We now expand $E[\exp\{rS_N - N\gamma(r)\}]$ in the same way, getting

$$E\left[\exp(rS_N - N\gamma(r))\right] = \lim_{m \rightarrow \infty} \sum_{n=1}^m E\left[\exp(rS_n - n\gamma(r)) \mid N=n\right] P(N=n) \quad (37)$$

From (36), we know that the limit exists and also that $E[\exp\{rS_N - N\gamma(r)\}] \leq 1$. Equality holds if (36) is satisfied with equality, i.e., if (33) is satisfied. This completes the proof.

To complete the proof of theorem 2, we must show that (33) is satisfied. The following lemma helps establish this result. Its proof is tedious, and exercise 7.16 provides a somewhat simpler but slightly less general proof.

LEMMA 4: For any given $r \in (r_-, r_+)$, there is a $\delta > 0$ such that

$$E[\exp(rS_n) \mid N=n] \geq \delta \text{ for all } n > 1 \text{ such that } P(N=n) > 0 \quad (38)$$

Proof*: Consider $r \geq 0$ initially. Given $N=n$, we have $S_n = S_{n-1} + X_n$, and $\beta < S_{n-1} < \alpha$. If X is bounded from below, i.e., if there is some B such that $F_X(B) = 0$, then $S_n > \beta + B$, and $E[\exp(rS_n \mid N=n)] > \exp(r(\beta+B)) > 0$. This satisfies (38) for $\delta = \exp(r(\beta+B))$. If X is not bounded from below, let $\varepsilon = F_X(\beta-\alpha)$ (which is positive), and choose $B > -\infty$ such that $F_X(B) \leq \varepsilon/2$. Thus

$$P(X \leq B) \leq (1/2)P(X < \beta-\alpha). \quad (39)$$

The event $\{N=n, S_{n-1}=s\}$ occurs only for $\beta < s < \alpha$ and can be written as $\{N \geq n, S_{n-1}=s, (X_n \leq \beta-s \cup X_n \geq \alpha-s)\}$. Thus

$$P\left[X_n \leq B \mid N=n, S_{n-1}=s\right] = P\left[X_n \leq B \mid (X_n \leq \beta-s \cup X_n \geq \alpha-s), N \geq n, S_{n-1}=s\right]$$

Since the event $N \geq n$ is determined by S_1, S_2, \dots, S_{n-1} , we see that X_n is independent of the event $\{N \geq n, S_{n-1}=s\}$.

$$P\left[X_n \leq B \mid N=n, S_{n-1}=s\right] = P\left[X_n \leq B \mid (X_n \leq \beta-s \cup X_n \geq \alpha-s)\right] \quad (40)$$

Since $B < \beta - \alpha < \beta - s$, the event $\{X_n \leq B\}$ is contained in $\{X_n \leq \beta - s \cup X_n \geq \alpha - s\}$, so

$$P\left[X_n \leq B \mid N=n, S_{n-1}=s\right] = \frac{P(X_n \leq B)}{P(X_n \leq \beta-s \cup X_n \geq \alpha-s)} \leq \frac{P(X_n \leq B)}{P(X_n \leq \beta-\alpha)} \leq \frac{1}{2}$$

where the final inequality comes from (39). This is valid for all s , $\beta < s < \alpha$, so it follows that $P(X_n \leq B \mid N=n) \leq 1/2$. Since $\beta < S_{n-1}$ for $N=n$, and since $S_n = S_{n-1} + X_n$,

$$P(S_n \leq B+\beta \mid N=n) \leq P(X_n \leq B \mid N=n) \leq 1/2 \quad (41)$$

This implies that $P(S_n > B+\beta \mid N=n) \geq 1/2$, so

$$E[\exp(rS_n) \mid N=n] > (1/2)\exp\{r(B+\beta)\} \quad (42)$$

Choosing δ equal to the right side of (42) completes the proof for $r \geq 0$. For $r < 0$, a similar argument applies to the upper tail of X_n , completing the proof for all $r \in (r_-, r_+)$.

Proof of theorem 2 (Wald's Identity)*: From lemma 3, the proof will be complete when we verify (33), i.e., $\lim_{m \rightarrow \infty} E[\exp\{rS_m - m\gamma(r)\} \mid N > m] P(N > m) = 0$. Since $\beta < S_m < \alpha$ for $N > m$, it suffices to show that $\lim_{m \rightarrow \infty} \exp(-m\gamma(r)) P(N > m) = 0$. From (36),

$$\lim_{m \rightarrow \infty} \sum_{n=1}^m E\left[\exp(rS_{n-m}\gamma(r)) \mid N=n\right] P(N=n) \leq 1 \quad (43)$$

From lemma 4, there is a $\delta > 0$, independent of n , such that $E[\exp(rS_n) \mid N=n] \geq \delta$. Substituting this in (43) yields

$$\lim_{m \rightarrow \infty} \delta \sum_{n=1}^m \exp(-n\gamma(r)) P(N=n) \leq 1 \quad (44)$$

This shows that $E[\exp(-N\gamma(r))]$ is finite. Now let Y be the non-negative random variable $\exp(-N\gamma(r))$. Since Y has a finite expectation, (1.20) shows that $\lim_{y \rightarrow \infty} y P(Y > y) = 0$. Taking y as $\exp(-N\gamma(r))$, we have $\lim_{n \rightarrow \infty} \exp(-n\gamma(r)) P(N > n) = 0$, completing the proof.

7.6 MARTINGALES AND SUBMARTINGALES

A *martingale* is defined as an integer time stochastic process $\{Z_n; n \geq 1\}$ with the properties that $E[|Z_n|] < \infty$ and

$$E[Z_n | Z_{n-1}, Z_{n-2}, \dots, Z_1] = Z_{n-1} \quad (45)$$

for all n . The name martingale comes from gambling terminology where martingales refer to gambling strategies in which the amount to be bet is determined by the past history of winning or losing. If one visualizes Z_n as representing the gambler's fortune at the end of the n^{th} play, the definition above means, first, that the game is fair (i.e., the expected increase in fortune from play $n-1$ to n is zero) and, second, that the expected fortune on the n^{th} play depends on the past only through the fortune on play $n-1$.

Eq. (45) can be interpreted in two different ways; the first and most straightforward is to view it as shorthand for $E[Z_n | Z_{n-1}=z_{n-1}, Z_{n-2}=z_{n-2}, \dots, Z_1=z_1] = z_{n-1}$ for all possible sample values z_1, z_2, \dots, z_{n-1} . The second is that $E[Z_n | Z_{n-1}=z_{n-1}, \dots, Z_1=z_1]$ is a function of the sample values z_1, \dots, z_{n-1} and thus $E[Z_n | Z_{n-1}, \dots, Z_1]$ is a random variable which is a function of the random variables Z_1, \dots, Z_{n-1} (and, for a martingale, a function only of Z_{n-1}). The student is encouraged to take the first viewpoint initially and to write out the expanded type of expression in cases of confusion.

EXAMPLE 4—ZERO MEAN RANDOM WALK: One example of a martingale is a zero mean random walk, since if $Z_n = X_1 + X_2 + \dots + X_n$, where the X_i are IID and zero mean, then

$$E[Z_n | Z_{n-1}, \dots, Z_1] = E[X_n + Z_{n-1} | Z_{n-1}, \dots, Z_1] = E[X_n] + Z_{n-1} = Z_{n-1} \quad (46)$$

Extending this example, suppose that $\{X_i; i \geq 1\}$ is an arbitrary set of IID random variables with mean \bar{X} and let $\tilde{X}_i = X_i - \bar{X}$. Then $\{S_n; n \geq 1\}$ is a random walk with $S_n = X_1 + \dots + X_n$ and $\{Z_n; n \geq 1\}$ is a martingale with $Z_n = \tilde{X}_1 + \dots + \tilde{X}_n$. The random walk and the martingale are simply related by $Z_n = S_n - n\bar{X}$, and thus general results about martingales can easily be converted into results about random walks.

EXAMPLE 5—SUMS OF DEPENDENT ZERO MEAN VARIABLES: Let $\{X_i; i \geq 1\}$ be a set of dependent random variables satisfying $E[X_i | X_{i-1}, \dots, X_1] = 0$. Then $\{Z_n; n \geq 1\}$, where $Z_n = X_1 + \dots + X_n$, is a zero mean martingale. To see this, note that

$$\begin{aligned} E[Z_n | Z_{n-1}, \dots, Z_1] &= E[X_n + X_{n-1} + \dots + X_1 | X_{n-1}, \dots, X_1] \\ &= E[X_n | X_{n-1}, \dots, X_1] + E[Z_{n-1} | Z_{n-1}, \dots, Z_1] = Z_{n-1} \end{aligned}$$

This is a more general example than it appears, since given any martingale $\{Z_n; n \geq 1\}$, we can define $X_n = Z_n - Z_{n-1}$ for $n \geq 2$ and define $X_1 = Z_1$. Then $E[X_n | X_{n-1}, \dots, X_1] = 0$ for $n \geq 2$, and if the martingale is zero mean (i.e., if $E[Z_1] = 0$), then $E[X_1] = 0$ also.

EXAMPLE 6—PRODUCT FORM MARTINGALES: Another example is a product of unit mean IID random variables. Thus if $Z_n = X_1 X_2 \dots X_n$, we have

$$E[Z_n | Z_{n-1}, \dots, Z_1] = E[X_n Z_{n-1} | Z_{n-1}, \dots, Z_1] = E[X_n] Z_{n-1} = Z_{n-1} \quad (47)$$

A particularly simple case of this product example is where $X_n=2$ with probability 1/2 and $X_n=0$ with probability 1/2. Then

$$P(Z_n=2^n) = 2^{-n}; P(Z_n=0) = 1-2^{-n}; E[Z_n] = 1 \quad (48)$$

Thus $\lim_{n \rightarrow \infty} Z_n = 0$ with probability 1, but $E[Z_n] = 1$ for all n and $\lim_{n \rightarrow \infty} E[Z_n] = 1$. This is an important example to keep in mind when trying to understand why proofs about martingales are necessary and non-trivial.

An important example of a product form martingale is as follows: let $\{X_i; i \geq 1\}$ be an IID sequence, and let $\{S_n = X_1 + \dots + X_n; n \geq 1\}$ be a random walk. Let $\gamma(r) = \ln\{E[\exp(rX)]\}$ be the semi-invariant moment generating function of the variables X_i (we assume $\gamma(r)$ to exist in some region of r around 0), and let Z_n be defined as

$$\begin{aligned} Z_n &= \exp(rS_n - n\gamma(r)) = \exp(rX_n - \gamma(r)) \exp(rS_{n-1} - (n-1)\gamma(r)) \\ &= \exp(rX_n - \gamma(r)) Z_{n-1} \end{aligned} \quad (49)$$

By taking expectations above, we see that $\{Z_n; n \geq 1\}$ is a martingale of the product form.

EXAMPLE 7—MARKOV MODULATED RANDOM WALKS: Frequently it is useful to generalize random walks to allow some dependence between the variables being summed. One can think of such processes equally well as Markov reward processes. Let $\{Y_m; m \geq 0\}$ be a sequence of (possibly dependent) random variables, and let

$$\{S_n; n \geq 1\} \text{ where } S_n = \sum_{m=0}^{n-1} Y_m \quad (50)$$

be the process of sums of these random variables. Let $\{X_n; n \geq 0\}$ be a Markov chain, and assume that each Y_n depends on X_n and X_{n+1} , but, conditional on X_n and X_{n+1} , is independent of Y_{n-1}, \dots, Y_1 , and independent of $\{X_i; i < n \text{ & } i > n+1\}$. Assume that Y_n , conditional on X_n and X_{n+1} has a distribution function $F_{ij}(y) = P(Y_n \leq y | X_n=i, X_{n+1}=j)$. Thus all variables Y_n , $n \geq 0$, associated with the same pair of states are identically distributed.

The process $\{S_n; n \geq 1\}$ is called a *Markov modulated random walk*. Y_m is associated with a transition from time m to $m+1$, and S_n is the aggregate reward up to time n ; thus S_n does not include Y_n . Let \bar{Y}_{ij} denote $E[Y_n | X_n=i, X_{n+1}=j]$ and \bar{Y}_i denote $E[Y_n | X_n=i]$. Let $\{P_{ij}\}$ be the set of transition probabilities for the Markov chain, so $\bar{Y}_i = \sum_j P_{ij} \bar{Y}_{ij}$. We may think of the process $\{Y_n; n \geq 0\}$ as evolving along with the Markov chain. The distributions of the variables Y_n are associated with the transitions from X_n to X_{n+1} , but the Y_n are otherwise independent random variables.

Note that in section 4.5, we were interested only in expected rewards, and thus characterized the reward in a state only by its expected value r_i (which corresponds to

\bar{Y}_i here). Here we focus on the random variables themselves, and thus must differentiate the rewards associated with different transitions. In order to define a martingale related to the process $\{S_n; n \geq 1\}$, we must subtract the mean reward from $\{S_n\}$ and must also compensate for the effect of the state of the Markov chain. The appropriate compensation factor turns out to be the relative gain vector defined in section 4.5.

For simplicity, consider only finite state irreducible Markov chains with J states. Let $\pi = (\pi_1, \dots, \pi_J)$ be the steady state probability vector for the chain, let $\bar{Y} = (\bar{Y}_1, \dots, \bar{Y}_J)^T$ be the vector of expected rewards, let $g = \pi \bar{Y}$ be the steady state gain per unit time, and let $w = (w_1, \dots, w_J)^T$ be the relative gain vector. From (4.34), w is the unique solution to

$$w + ge = \bar{Y} + [P]w ; w_1 = 0 \quad (51)$$

We assume a fixed starting state $X_0=k$. As we now show, the process $\{Z_n; n \geq 1\}$ given by

$$Z_n = S_n - ng + w_{X_n} - w_k ; n \geq 1 \quad (52)$$

is a martingale. In order to show this, we first condition on a given state at time $n-1$.

$$E[Z_n | Z_{n-1}, Z_{n-2}, \dots, Z_1, X_{n-1}=i] \quad (53)$$

Since $S_n = S_{n-1} + Y_{n-1}$, we can express Z_n as

$$Z_n = Z_{n-1} + Y_{n-1} - g + w_{X_n} - w_{X_{n-1}} \quad (54)$$

Since $E[Y_{n-1} | X_{n-1}=i] = \bar{Y}_i$ and $E[w_{X_n} | X_{n-1}=i] = \sum_j P_{ij}w_j$, we have

$$E[Z_n | Z_{n-1}, Z_{n-2}, \dots, Z_1, X_{n-1}=i] = Z_{n-1} + \bar{Y}_i - g + \sum_j P_{ij}w_j - w_i \quad (55)$$

From (51) the final four terms in (55) sum to 0, so

$$E[Z_n | Z_{n-1}, \dots, Z_1, X_{n-1}=i] = Z_{n-1}. \quad (56)$$

Since this is valid for all choices of X_{n-1} , we have $E[Z_n | Z_{n-1}, \dots, Z_1] = Z_{n-1}$. Since the expected values of all the reward variables \bar{Y}_i exist, we see that $E[Y_n] < \infty$, so that $E[Z_n] < \infty$ also. This verifies that $\{Z_n; n \geq 1\}$ is a martingale. It can be verified similarly that $E[Z_1] = 0$, so $E[Z_n] = 0$ for all $n \geq 1$.

In showing that $\{Z_n; n \geq 1\}$ is a martingale, we actually showed something a little stronger. That is, we demonstrated (56), and in the same way, it follows that for all $n > 1$,

$$E[Z_n | Z_{n-1}, X_{n-1}, Z_{n-2}, X_{n-2}, \dots, Z_1, X_1] = Z_{n-1} \quad (57)$$

In terms of the gambling analogy, this says that $\{Z_n; n \geq 1\}$ is fair for each possible past sequence of states. A martingale $\{Z_n; n \geq 1\}$ with this property (i.e., satisfying (57)) is

said to be a *martingale relative to the joint process* $\{Z_n, X_n; n \geq 1\}$. We will use this martingale later to discuss threshold crossing problems for Markov modulated random walks. We shall see that the added property of being a martingale relative to $\{Z_n, X_n\}$ gives us added flexibility in defining stopping rules.

As an added bonus to this example, note that if $\{X_n; n \geq 0\}$ is taken as the embedded chain of a Markov process (or semi-Markov process), and if Y_n is taken as the time interval from transition n to $n+1$, then S_n becomes the epoch of the n^{th} transition in the process.

EXAMPLE 8—GENERATING FUNCTION FOR MARKOV WALK: Consider the same Markov chain and reward variables as in the previous example, and assume that for each pair of states, i and j , the moment generating function

$$g_{ij}(r) = E[\exp(rY_n) | X_n=i, X_{n+1}=j] \quad (58)$$

exists over some open interval (r_-, r_+) containing 0. Let $[\Gamma(r)]$ be the matrix with terms $P_{ij}g_{ij}(r)$. Since $[\Gamma(r)]$ is an irreducible non-negative matrix, theorem 4.5 shows that $[\Gamma(r)]$ has a largest real eigenvalue, $\rho(r) > 0$, and an associated positive right eigenvector, $v(r) = (v_1(r), \dots, v_j(r))^T$ that is unique within a scale factor. We now show that the process $\{M_n(r); n \geq 1\}$ defined by

$$M_n(r) = \frac{\exp(rS_n)v_{X_n}(r)}{\rho(r)^n v_k(r)} \quad (59)$$

is a product type Martingale for each $r \in (r_-, r_+)$. Since $S_n = S_{n-1} + Y_{n-1}$, we can express $M_n(r)$ as

$$M_n(r) = M_{n-1}(r) \frac{\exp(rY_{n-1})v_{X_n}(r)}{\rho(r) v_{X_{n-1}}(r)} \quad (60)$$

The expected value of the ratio in (60), conditional on $X_{n-1} = i$, is

$$E\left[\frac{\exp(rY_{n-1})v_{X_n}(r)}{\rho(r) v_i(r)} \mid X_{n-1}=i \right] = \frac{\sum_j P_{ij} g_{ij}(r) v_j(r)}{\rho(r) v_i(r)} = 1 \quad (61)$$

Thus, $E[M_n(r) | M_{n-1}(r), \dots, M_1(r), X_{n-1}=i] = M_{n-1}(r)$. Since this is true for all choices of i , the condition on $X_{n-1}=i$ can be removed and $\{M_n(r); n \geq 1\}$ is a martingale. Also, for $n > 1$,

$$E[M_n(r) | M_{n-1}(r), X_{n-1}, \dots, M_1(r), X_1] = M_{n-1}(r) \quad (62)$$

so that $\{M_n(r); n \geq 1\}$ is also a martingale relative to the joint process $\{M_n(r), X_n; n \geq 1\}$. It can be verified by the same argument as in (61) that $E[M_1(r)] = 1$. It then follows that $E[M_n(r)] = 1$ for all $n \geq 1$.

One of the uses of this martingale is to provide exponential upper bounds, similar to (12), to the probabilities of threshold crossings for Markov modulated random walks. Define

$$\tilde{M}_n(r) = \frac{\exp(rS_n) \min_j(v_j(r))}{\rho(r)^n v_k(r)} \quad (63)$$

Then $\tilde{M}_n(r) \leq M_n(r)$, so $E[\tilde{M}_n(r)] \leq 1$. The Markov inequality applied to $\tilde{M}_n(r)$ is

$$P(\tilde{M}_n(r) \geq \mu) \leq E[\tilde{M}_n(r)]/\mu \leq 1/\mu \quad (64)$$

For any given α , we can choose $\mu = \exp(r\alpha) \rho(r)^{-n} \min_j(v_j(r))/v_k(r)$, and for $r > 0$, (64) becomes

$$P(S_n \geq \alpha) \leq \rho(r)^n \exp(-r\alpha) v_k(r)/ \min_j(v_j(r)) \quad (65)$$

This can be optimized over r to get the tightest bound in the same way as (12).

EXAMPLE 9—SCALED BRANCHING PROCESSES: A final example of a martingale is a “scaled down” version of a branching process $\{X_n; n \geq 0\}$. Let X_n be the aggregate number of elements in generation n . This is the number of offspring from the X_{n-1} elements in generation $n-1$, each of which has an IID number of offspring with some mean μ . Thus $E[X_n | X_{n-1}] = \mu X_{n-1}$. If we define $Z_n = X_n/\mu^n$, then it follows that

$$E[Z_n | Z_{n-1}, \dots, Z_1] = E\left[\frac{X_n}{\mu^n} | X_{n-1}, \dots, X_1\right] = \frac{\mu X_{n-1}}{\mu^n} = Z_{n-1} \quad (66)$$

Thus $\{Z_n; n \geq 1\}$ is a martingale.

A martingale, from one viewpoint, is a generalization of a Markov chain. That is, for a Markov chain, the future state, conditional on the present state, is independent of the past. For a martingale, the *expected value* of the martingale in the future, conditional on the value in the present, is independent of the past. The following lemma helps make this precise.

LEMMA 5: Let $\{Z_n; n \geq 1\}$ be a martingale. Then for any $n > i \geq 1$,

$$E[Z_n | Z_i, Z_{i-1}, \dots, Z_1] = Z_i \quad (67)$$

Proof: By definition of a martingale, $E[Z_{i+1} | Z_i, \dots, Z_1] = Z_i$. Next consider Z_{i+2} :

$$\begin{aligned} E[Z_{i+2} | Z_i, \dots, Z_1] &= \int_{Z_{i+1}} E[Z_{i+2} | Z_{i+1}=z_{i+1}, Z_i, \dots, Z_1] dP(Z_{i+1} \leq z_{i+1} | Z_i, \dots, Z_1) \\ &= \int_{Z_{i+1}} z_{i+1} dP(Z_{i+1} \leq z_{i+1} | Z_i, \dots, Z_1) = E[Z_{i+1} | Z_i, \dots, Z_1] = Z_i \end{aligned}$$

The same argument can be applied successively from Z_{i+3} to Z_n , completing the proof.

The above proof becomes more transparent if we view $E[Z_{i+2} | Z_{i+1}, Z_i, \dots, Z_1]$ as a random variable that is a function of Z_{i+1}, \dots, Z_1 . Since this random variable equals Z_{i+1} , we can take its expectation, conditional on (Z_i, \dots, Z_1) , to get $E[Z_{i+1} | Z_i, \dots, Z_1] = Z_i$. The same argument can be used (see exercise 7.18) to show that

$$E[Z_n] = E[Z_1] \text{ for all } n > 1 \quad (68)$$

SUBMARTINGALES AND SUPERMARTINGALES: Submartingales and supermartingales are simple generalizations of martingales that provide us with many useful results for very little additional work. We will subsequently derive the Kolmogorov submartingale inequality, which is a powerful generalization of the Markov inequality. We use this both to give a simple proof of the strong law of large numbers and also to better understand threshold crossing problems for random walks. A *submartingale* is an integer time stochastic process $\{Z_n; n \geq 1\}$ that satisfies the relations

$$E[|Z_n|] < \infty ; E[Z_n | Z_{n-1}, Z_{n-2}, \dots, Z_1] \geq Z_{n-1} ; n \geq 1 \quad (69)$$

In terms of our gambling analogy, a submartingale corresponds to a game that is at least fair, i.e., where the expected fortune of the gambler either increases or remains the same. A *supermartingale* is a process with the opposite type of inequality. That is, it is an integer time stochastic process $\{Z_n; n \geq 1\}$ that satisfies the relations

$$E[|Z_n|] < \infty ; E[Z_n | Z_{n-1}, Z_{n-2}, \dots, Z_1] \leq Z_{n-1} ; n \geq 1 \quad (70)$$

Since a martingale satisfies both (69) and (70) with equality, a martingale is both a submartingale and a supermartingale. Since a submartingale tends to grow with n , and a supermartingale tends to shrink with n , the notation is rather peculiar, but unfortunately it is quite standard. Note that if $\{Z_n; n \geq 1\}$ is a submartingale, then $\{-Z_n; n \geq 1\}$ is a supermartingale, and conversely. Thus, some of the results to follow are stated only for submartingales, with the understanding that they can be applied to supermartingales by changing signs as above.

Lemma 5, with the equality replaced by inequality, also applies to submartingales and supermartingales. That is, if $\{Z_n; n \geq 1\}$ is a submartingale, then

$$E[Z_n | Z_i, Z_{i-1}, \dots, Z_1] \geq Z_i ; 1 \leq i < n, \quad (71)$$

and if $\{Z_n; n \geq 1\}$ is a supermartingale, then

$$E[Z_n | Z_i, Z_{i-1}, \dots, Z_1] \leq Z_i; 1 \leq i < n. \quad (72)$$

Eqs. (71) and (72) are verified in the same way as lemma 5 (see exercise 7.20). Similarly, the generalization of (68) is that if $\{Z_n; n \geq 1\}$ is a submartingale, then

$$E[Z_n] \geq E[Z_i]; 1 \leq i < n, \quad (73)$$

and if $\{Z_n; n \geq 1\}$ is a supermartingale, then

$$E[Z_n] \leq E[Z_i] \text{ for all } i < n. \quad (74)$$

A random walk $\{S_n; n \geq 1\}$ with $S_n = X_1 + \dots + X_n$ is a submartingale, martingale, or supermartingale respectively for $\bar{X} \geq 0$, $\bar{X} = 0$, or $\bar{X} \leq 0$. Also, if $\{S_n; n \geq 1\}$ has a semi-invariant moment generating function $\gamma(r)$, the process $\{Z_n; n \geq 1\}$ where $Z_n = \exp(rS_n)$ is a submartingale, martingale, or supermartingale respectively for $\gamma(r) \geq 0$, $\gamma(r) = 0$, or $\gamma(r) \leq 0$. The next example helps show how martingales and submartingales are related.

EXAMPLE 10—CONVEX FUNCTIONS OF MARTINGALES: Figure 7.7 shows the graph of a convex function h of a real variable x . A function h of a real variable is defined to be *convex* if, for each point x_1 , there is a real number c with the property that $h(x_1) + c(x-x_1) \leq h(x)$ for all x . If $h(x)$ has a derivative at x_1 , then c is the value of that derivative and $h(x_1) + c(x-x_1)$ is the tangent line. If $h(x)$ has a discontinuous slope at x_1 , then there might be many choices for c ; for example, $h(x) = |x|$ is convex, and for $x_1 = 0$, one could choose any c in the range -1 to $+1$.

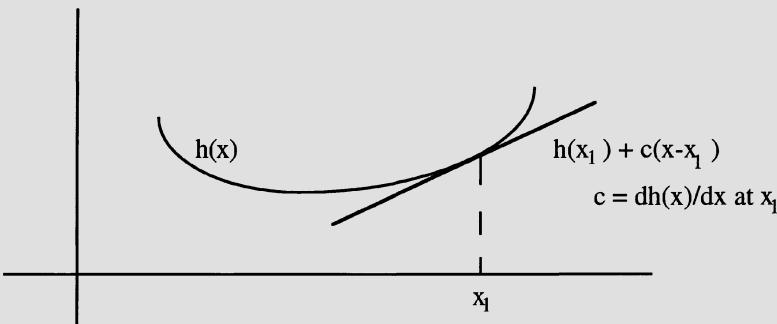


Figure 7.7. Convex functions. For each x_1 , there is a value of c such that, for all x , $h(x_1) + c(x-x_1) \leq h(x)$. If h is continuous at x_1 , then c is the derivative of h at x_1 .

Jensen's inequality states that if h is convex and X is a random variable with an expectation, then $h(E[X]) \leq E[h(X)]$. To prove this, let $x_1 = E[X]$ and choose c so that $h(x_1) + c(x - x_1) \leq h(x)$. Using the random variable X in place of x and taking expected values of both sides, we get Jensen's inequality. Note that for any particular event A , this same argument applies to X conditional on A , so that $h(E[X | A]) \leq E[h(X | A)]$. Jensen's inequality is very widely used; it is a minor miracle that we have not required it previously.

THEOREM 3: If h is a convex function of a real variable, $\{Z_n; n \geq 1\}$ is a martingale, and $E[|h(Z_n)|] < \infty$ for all n , then $\{h(Z_n); n \geq 1\}$ is a submartingale.

Proof: For any choice of z_1, \dots, z_{n-1} , we can use Jensen's inequality with the conditioning probabilities to get

$$E[h(Z_n) | Z_{n-1}=z_{n-1}, \dots, Z_1=z_1] \geq h(E[Z_n | Z_{n-1}=z_{n-1}, \dots, Z_1=z_1]) = h(z_{n-1}) \quad (75)$$

For any choice of numbers h_1, \dots, h_{n-1} in the range of the function h , let z_1, \dots, z_{n-1} be arbitrary numbers satisfying $h(z_1)=h_1, \dots, h(z_{n-1})=h_{n-1}$. For each such choice, (75) holds, so that

$$\begin{aligned} E[h(Z_n) | h(Z_{n-1})=h_{n-1}, \dots, h(Z_1)=h_1] &\geq h(E[Z_n | h(Z_{n-1})=h_{n-1}, \dots, h(Z_1)=h_1]) \\ &= h(z_{n-1}) = h_{n-1} \end{aligned} \quad (76)$$

completing the proof.

Some examples of this result, applied to a martingale $\{Z_n; n \geq 1\}$, are as follows:

$$\{|Z_n|; n \geq 1\} \text{ is a submartingale} \quad (77)$$

$$\{Z_n^2; n \geq 1\} \text{ is a submartingale if } E[Z_n^2] < \infty \quad (78)$$

$$\{\exp(rZ_n); n \geq 1\} \text{ is a submartingale for } r \text{ such that } E[\exp(rZ_n)] < \infty \quad (79)$$

A function of a real variable $h(x)$ is defined to be concave if $-h(x)$ is convex. It then follows from theorem 3 that if h is concave and $\{Z_n; n \geq 1\}$ is a martingale, then $\{h(Z_n); n \geq 1\}$ is a supermartingale (assuming that $E[|h(Z_n)|] < \infty$). For example, if $\{Z_n; n \geq 1\}$ is a positive martingale and $E[|\ln(Z_n)|] < \infty$, then $\{\ln(Z_n); n \geq 1\}$ is a supermartingale.

7.7 STOPPED PROCESSES AND STOPPING RULES

The discussion of stopping rules in section 3.3 applies to arbitrary integer time processes $\{Z_n; n \geq 1\}$ as well as to IID variables. Informally, recall that a positive, integer valued, random variable N is a *stopping rule* for $\{Z_n; n \geq 1\}$ if the decision whether or not to observe the process at time n depends only on Z_1, \dots, Z_{n-1} . More formally, the indicator function I_n of the event $\{N \geq n\}$ is required to be a function of Z_1, \dots, Z_{n-1} (or, more generally, I_n , conditional on Z_1, \dots, Z_{n-1} , must be independent of $\{Z_m; m \geq n\}$). For some of the results to follow, it is unimportant whether N is a random variable or a defective random variable (i.e., whether or not the process stops with probability 1). If

it is not specified whether N is a random variable or a defective random variable, we refer to the stopping rule as a *possibly defective stopping rule*; we consider N to take on the value ∞ if the process does not stop. The conditions on the indicator functions $\{I_n; n > 1\}$ for defective stopping rules are the same as those for ordinary stopping rules.

Given a possibly defective stopping rule N for a process $\{Z_n; n \geq 1\}$, the corresponding *stopped process* is defined as the process $\{Z_n^*; n \geq 1\}$ in which $Z_n^* = Z_n$ for $n \leq N$ and $Z_n^* = Z_N$ for $n > N$. As an example, suppose Z_n models the fortune of a gambler after the n^{th} trial of some game, and suppose the gambler then modifies the game by deciding to stop gambling under some given circumstances (i.e., the stopping rule). Thus, after stopping, the fortune remains constant, so the stopped process models the gambler's fortune in time, including the stopping rule. As another example, consider a random walk with a positive and negative threshold, and consider the process to stop after reaching or crossing a threshold. The stopped process then stays at that point beyond the threshold as an artifice to simplify analysis. The use of stopped processes is similar to the artifice that we employed in section 4.5 for first passage times in Markov chains; recall that we added an artificial trapping state after the desired passage to simplify analysis.

We next show that the possibly defective stopped process of a martingale is itself a martingale; the intuitive reason is that, before stopping, the stopped process is the same as the martingale, and, after stopping, $Z_n^* = Z_{n-1}^*$. The following theorem establishes this and the corresponding results for submartingales and supermartingales.

THEOREM 4: Given a stochastic process $\{Z_n; n \geq 1\}$ and a possibly defective stopping rule N for the process, the stopped process $\{Z_n^*; n \geq 1\}$ is a submartingale if $\{Z_n; n \geq 1\}$ is a submartingale, is a martingale if $\{Z_n; n \geq 1\}$ is a martingale, and is a supermartingale if $\{Z_n; n \geq 1\}$ is a supermartingale.

Proof: First we show that, for all three cases, the stopped process satisfies $E[|Z_n^*|] < \infty$ for any given $n \geq 1$. Conditional on $N=i$ for some $i < n$, we have $Z_n^* = Z_i$, so

$$E[|Z_n^*| | N=i] = E[|Z_i| | N=i] < \infty \text{ for each } i < n \text{ such that } P(N=i) > 0$$

The reason for this is that if $E[|Z_i| | N=i]$ were infinite, then $E[|Z_i|]$ would be infinite, which is impossible since $\{Z_n; n \geq 1\}$ is a martingale, submartingale, or supermartingale. Similarly, for $N \geq n$, we have $Z_n^* = Z_n$ so

$$E[|Z_n^*| | N \geq n] = E[|Z_n| | N \geq n] < \infty \text{ if } P(N \geq n) > 0$$

Averaging,

$$E[|Z_n^*|] = \sum_{i=1}^{n-1} E[|Z_n^*| | N=i] P(N=i) + E[|Z_n^*| | N \geq n] P(N \geq n) < \infty$$

Next assume that $\{Z_n; n \geq 1\}$ is a submartingale. For each n , the indicator function I_n of $\{N \geq n\}$ is independent of Z_n, Z_{n+1}, \dots given Z_1, \dots, Z_{n-1} . Consider some arbitrary initial sample sequence of the stopped process, $Z_1^* = z_1^*, Z_2^* = z_2^*, \dots, Z_{n-1}^* = z_{n-1}^*$ and con-

sider first the case in which $I_n=0$ and next the case $I_n=1$. For the case $I_n=0$, we have $N \leq n-1$ (i.e., stopping occurs by time $n-1$ or before) so that $Z_n^* = Z_{n-1}^* = z_{n-1}^*$. Thus, for any such sample sequence with $I_n=0$, we have

$$E[Z_n^* | Z_{n-1}^* = z_{n-1}^*, \dots, Z_1^* = z_1^*, I_n=0] = z_{n-1}^* \quad (80)$$

Next consider the condition $I_n=1$. Under this condition, $Z_i^* = Z_i$ for $1 \leq i \leq n$, so we have

$$\begin{aligned} E[Z_n^* | Z_{n-1}^* = z_{n-1}^*, \dots, Z_1^* = z_1^*, I_n=1] &= E[Z_n | Z_{n-1}^* = z_{n-1}^*, \dots, Z_1 = z_1^*, I_n=1] \\ &= E[Z_n | Z_{n-1} = z_{n-1}^*, \dots, Z_1 = z_1^*] \geq z_{n-1}^* \end{aligned} \quad (81)$$

The second equality above follows from the independence of Z_n and I_n conditional on Z_{n-1}, \dots, Z_1 . Since $E[Z_n^* | Z_{n-1}^* = z_{n-1}^*, \dots, Z_1^* = z_1^*, I_n=j] \geq z_{n-1}^*$ both for $j=0$ and 1, we have $E[Z_n^* | Z_{n-1}^* = z_{n-1}^*, \dots, Z_1^* = z_1^*] \geq z_{n-1}^*$, showing that the stopped process is a submartingale. The same argument works for martingales and supermartingales by replacing the inequality in (81) by equality for the martingale case and the opposite inequality for the supermartingale case. This completes the proof.

THEOREM 5: Given a stochastic process $\{Z_n; n \geq 1\}$ and a possibly defective stopping rule N for the process, the stopped process $\{Z_n^*; n \geq 1\}$ satisfies the following conditions for all $n \geq 1$ if $\{Z_n; n \geq 1\}$ is a submartingale, martingale, or supermartingale respectively:

$$E[Z_1] \leq E[Z_n^*] \leq E[Z_n] \quad (\text{submartingale}) \quad (82)$$

$$E[Z_1] = E[Z_n^*] = E[Z_n] \quad (\text{martingale}) \quad (83)$$

$$E[Z_1] \geq E[Z_n^*] \geq E[Z_n] \quad (\text{supermartingale}) \quad (84)$$

Proof: Since a process cannot stop before epoch 1, $Z_1 = Z_1^*$ in all cases. First consider the case in which $\{Z_n; n \geq 1\}$ is a submartingale. Theorem 4 shows that $\{Z_n^*; n \geq 1\}$ is a submartingale, and from (73), $E[Z_1] \leq E[Z_n^*]$ for all $n \geq 1$. This establishes the first half of (82) and we next prove the second half. Note that I_i is independent of Z_i, Z_{i+1}, \dots conditional on Z_1, \dots, Z_{i-1} , and thus independent of Z_{i+1}, Z_{i+2}, \dots conditional on Z_1, \dots, Z_i . I_{i+1} is also independent of Z_{i+1}, Z_{i+2}, \dots conditional on Z_1, \dots, Z_i . Thus $I_i - I_{i+1}$, which is the indicator function of the event $\{N=i\}$ is independent of Z_{i+1}, Z_{i+2}, \dots conditional on Z_1, \dots, Z_i . For any $n > i$, then, $\{N=i\}$ is independent of Z_n conditional on Z_1, \dots, Z_i , so

$$E[Z_n | Z_i, \dots, Z_1, N=i] = E[Z_n | Z_i, \dots, Z_1] \geq Z_i; n > i \quad (85)$$

where we have used (71). On the other hand, given that $N=i$, $Z_n^* = Z_i$ for $n > i$, so

$$E[Z_n^* | Z_i, \dots, Z_1, N=i] = Z_i; n > i \quad (86)$$

Thus, conditional on $N=i$, we have $E[Z_n - Z_n^* | N=i] \geq 0$ for each $n > i$. It follows from this that $E[Z_n - Z_n^* | N < n] \geq 0$. Also, conditional on $N \geq n$, $Z_n = Z_n^*$. Averaging over $N < n$ and $N \geq n$, we have $E[Z_n] \geq E[Z_n^*]$, verifying (82). If $\{Z_n; n \geq 1\}$ is a supermartingale, then

$\{-Z_n; n \geq 1\}$ is a submartingale, verifying (84). Since a martingale is both a submartingale and supermartingale, (83) follows and the proof is complete.

Consider a given stopping rule N for a martingale $\{Z_n; n \geq 1\}$. Since the stopped process is also a martingale, we have

$$E[Z_n^*] = E[Z_1^*] = E[Z_1]; n \geq 1 \quad (87)$$

Since $Z_n^* = Z_N$ for all $n \geq N$ and since N is finite with probability 1, we see that $\lim_{n \rightarrow \infty} Z_n^* = Z_N$ with probability 1. Unfortunately, in general, $E[Z_N]$ is unequal to $\lim_{n \rightarrow \infty} E[Z_n^*] = E[Z_1]$. An example in which this occurs is the binary product martingale in (48). Taking the stopping rule N to be the smallest n for which $Z_n=0$, we have $Z_N=0$ with probability 1, and thus $E[Z_N]=0$. But $Z_n^* = Z_n$ for all n , and $E[Z_n^*] = 1$ for all n . The problem here is that, given that the process has not stopped by time n , Z_n and Z_n^* each have the value 2^n . Fortunately, in most situations, this type of bizarre behavior does not occur and $E[Z_N] = E[Z_1]$. To get a better understanding of when $E[Z_N] = E[Z_1]$, note that for any n , we have

$$E[Z_n^*] = \sum_{i=1}^n E[Z_n^* | N=i] P(N=i) + E[Z_n^* | N>n] P(N>n) \quad (88)$$

$$= \sum_{i=1}^n E[Z_N | N=i] P(N=i) + E[Z_n | N>n] P(N>n) \quad (89)$$

The left side of this equation is $E[Z_1]$ for all n and the sum from 1 to n on the right approaches $E[Z_N]$ as $n \rightarrow \infty$. Thus we have established the following theorem.

THEOREM 6: Let N be a stopping rule for a martingale $\{Z_n; n \geq 1\}$. Then $E[Z_N] = E[Z_1]$ if and only if

$$\lim_{n \rightarrow \infty} E[Z_n | N>n] P(N>n) = 0 \quad (90)$$

EXAMPLE 11—RANDOM WALKS WITH THRESHOLDS: Recall the generating function product martingale of (49) in which $\{Z_n = \exp[rS_n - n\gamma(r)]; n \geq 1\}$ is a martingale defined in terms of the random walk $\{S_n = X_1 + \dots + X_n; n \geq 1\}$. From (83), we have $E[Z_n] = E[Z_1]$, and since $E[Z_1] = E[\exp\{rX_1 - \gamma(r)\}] = 1$, we have $E[Z_n] = 1$ for all n . Also, for any possibly defective stopping rule N , we have $E[Z_n^*] = E[Z_1] = 1$. If N is a non-defective stopping rule, and if (90) holds, then

$$E[Z_N] = E[\exp\{rS_N - N\gamma(r)\}] = 1 \quad (91)$$

If there are two thresholds, one at $\alpha > 0$, and the other at $\beta < 0$, and the stopping rule is to stop when either threshold is crossed, then (91) is just the Wald identity, (17). Note, however, that the difficult part of the derivation of (17) was lemma 4, which was used to show that (90) holds; the more general approach here does not avoid requiring that lemma.

The nice part about the approach here is that it also applies naturally to other stopping rules. For example, for some given integer n , let N_{n+} be the smallest integer $i \geq n$ for which $S_i \geq \alpha$ or $S_i \leq \beta$. Then, in the limit $\beta \rightarrow -\infty$, $P(S_{N_{n+}} \geq \alpha) = P\left(\bigcup_{i=n}^{\infty} (S_i \geq \alpha)\right)$. Given that $\bar{X} < 0$, we can find an upper bound to $P(S_{N_{n+}} \geq \alpha)$ for any $r > 0$ and $\gamma(r) \leq 0$ (i.e., for $0 \leq r \leq r^*$) by the following steps

$$\begin{aligned} 1 &= E[\exp\{rS_{N_{n+}} - N_{n+}\gamma(r)\}] \geq P(S_{N_{n+}} \geq \alpha) \exp[r\alpha - n\gamma(r)] \\ P(S_{N_{n+}} \geq \alpha) &\leq \exp[-r\alpha + n\gamma(r)]; \quad 0 \leq r \leq r^* \end{aligned} \quad (92)$$

This is almost the same result as (26), except that it is slightly stronger; (26) bounded the probability that the *first* threshold crossing crossed α at some epoch $i \geq n$, whereas this bounds the probability that α is ever exceeded at any epoch $i \geq n$.

STOPPING RULES FOR MARTINGALES RELATIVE TO A PROCESS: In example 7, we defined a martingale $\{Z_n; n \geq 1\}$ relative to a joint process $\{Z_n, X_n; n \geq 1\}$ as a martingale for which (57) is satisfied, i.e., $E[Z_n | Z_{n-1}, X_{n-1}, \dots, Z_1, X_1] = Z_{n-1}$. In the same way, we can define a *submartingale* or *supermartingale* $\{Z_n; n \geq 1\}$ relative to a joint process $\{Z_n, X_n; n \geq 1\}$ as a submartingale or supermartingale satisfying (57) with the $=$ sign replaced by \geq or \leq respectively. The purpose of this added complication is to make it easier to define useful stopping rules. In particular, N is defined¹ to be a *stopping rule for a martingale* $\{Z_n; n \geq 1\}$ relative to a joint process $\{Z_n, X_n; n \geq 1\}$ if the indicator function I_n of $\{N \geq n\}$ is a function of Z_i and X_i for $1 \leq i \leq n-1$. Similarly, N is a *stopping rule for a submartingale or supermartingale* $\{Z_n; n \geq 1\}$ relative to a joint process $\{Z_n, X_n; n \geq 1\}$ if the indicator function I_n is a function of Z_i and X_i for $1 \leq i \leq n-1$.

Theorems 4, 5, and 6 all carry over to martingales (submartingales or supermartingales) relative to a joint process. These theorems are stated more precisely in exercises 7.22 to 7.25. To summarize them here, assume that $\{Z_n; n \geq 1\}$ is a martingale (submartingale or supermartingale) relative to a joint process $\{Z_n, X_n; n \geq 1\}$ and assume that N is a stopping rule for $\{Z_n; n \geq 1\}$ relative to $\{Z_n, X_n; n \geq 1\}$. Then the stopped process is a martingale (submartingale or supermartingale) respectively, (82–84) are satisfied, and, for a martingale, $E[Z_N] = E[Z_1]$ is satisfied iff (90) is satisfied.

EXAMPLE 12—MARKOV MODULATED RANDOM WALKS WITH THRESHOLDS: In examples 7 and 8, we developed two martingales for Markov modulated random walks, both conditioned on a fixed initial state $X_0 = k$. The first, given in (52), is $\{Z_n = S_n - ng + w_{X_n} - w_k; n \geq 1\}$. Recall that $E[Z_n] = 0$ for all $n \geq 1$ for this martingale. Given two thresholds, $\alpha > 0$ and $\beta < 0$, define N as the smallest n for which $S_n \geq \alpha$ or $S_n \leq \beta$. I_n , the indicator function of $\{N \geq n\}$, is 1 iff $\beta < S_i < \alpha$ for $1 \leq i \leq n-1$, so I_n is a function of S_i for $1 \leq i \leq n-1$. Since $S_i = Z_i + ig - w_{X_i} + w_k$, S_i is a function of Z_i and X_i , so I_n is also a function of Z_i and X_i for $1 \leq i \leq n-1$. It follows that N is a stopping rule for $\{Z_n; n \geq 1\}$ relative to $\{Z_n, X_n; n \geq 1\}$. From theorem 6, we can assert that $E[Z_N] = E[Z_1] = 0$ if (90) is satisfied, i.e., if $\lim_{n \rightarrow \infty} E[Z_n | N > n] P(N > n) = 0$ is satisfied. Using the same argument

as in lemma 1, we can see that $P(N>n)$ goes to 0 at least geometrically in n . Conditional on $N>n$, $\beta < S_n < \alpha$, so S_n is bounded independent of n . Also w_{X_n} is bounded, since the chain is finite state, and ng is linear in n . Thus $E[Z_n | N>n]$ varies at most linearly with n , so (90) is satisfied, and

$$0 = E[Z_N] = E[S_N] - E[N]g + E[w_{X_n}] - w_k \quad (93)$$

Recall that Wald's equality for random walks is $E[S_N] = E[N]g$. For Markov modulated random walks, this is modified, as shown in (93), by the relative gain vector terms.

The same arguments can be applied to the generating function martingale of (59). Again, let N be the smallest n for which $S_n \geq \alpha$ or $S_n \leq \beta$. As before, S_i is a function of $M_i(r)$ and X_i , so I_n is a function of $M_i(r)$ and X_i for $1 \leq i \leq n-1$. It follows that N is a stopping rule for $\{M_n(r); n \geq 1\}$ relative to $\{M_n(r), X_n; n \geq 1\}$. Next we need the following lemma:

LEMMA 6: For the martingale $\{M_n(r); n \geq 1\}$ relative to $\{M_n(r), X_n; n \geq 1\}$ defined in (59), where $\{X_n; n \geq 0\}$ is a finite state Markov chain, and for the above stopping rule N ,

$$\lim_{n \rightarrow \infty} E[M_n(r) | N>n] P(N>n) = 0. \quad (94)$$

Proof: From lemma 4, slightly modified for the case here, there is a $\delta > 0$ such that for all states i, j , and all $n > 1$ such that $P(N=n, X_{n-1}=i, X_n=j) > 0$,

$$E[\exp(rS_n) | N=n, X_{n-1}=i, X_n=j] \geq \delta \quad (95)$$

Since the stopped process, $\{M_n(r)^*; n \geq 1\}$, is a martingale, we have for each m ,

$$1 = E[M_m(r)^*] \geq \sum_{n=1}^m \frac{E[\exp(rS_n)v_{X_n}(r) | N=n]}{\rho(r)^n v_k(r)} P(N=n) \quad (96)$$

From (95), we see that there is some $\delta > 0$ such that $E[\exp(rS_n)v_{X_n}(r) | N=n] \geq \delta$ for all n such that $P(N=n) > 0$, so that (96) is bounded by $1 \geq \delta' \sum_{n \leq m} \rho(r)^n P(N=n)$. Since this is valid for all m , it follows by the argument in the proof of theorem 2 that $\lim_{n \rightarrow \infty} \rho(r)^n P(N>n) = 0$. This, along with (95), establishes (94), completing the proof.

From theorem 6, we have the desired result:

$$E[M_N(r)] = \frac{E[\exp(rS_N)v_{X_N}(r)]}{[\rho(r)]^N v_k(r)} = 1 ; r_- < r < r_+ \quad (97)$$

This is the extension of the Wald identity to Markov modulated random walks, and is used in the same way as the Wald identity. As shown in exercise 7.27, the derivative of (97), evaluated at $r=0$, is the same as (93).

7.8 THE KOLMOGOROV INEQUALITIES

We now use the previous theorems to establish Kolmogorov's submartingale inequality, which is a major strengthening of the Markov inequality. Just as we used the Markov inequality in Section 1.7 to derive the Chebychev inequality and then the weak law of large numbers, we will use the Kolmogorov submartingale inequality to strengthen the Chebychev inequality, which we will use to prove the strong law of large numbers.

THEOREM 7—KOLMOGOROV'S SUBMARTINGALE INEQUALITY: Let $\{Z_n; n \geq 1\}$ be a non-negative submartingale. Then for any positive integer m and any $a > 0$,

$$P\left(\max_{1 \leq i \leq m} Z_i \geq a\right) \leq \frac{E[Z_m]}{a} \quad (98)$$

Proof: Given a non-negative submartingale $\{Z_n; n \geq 1\}$, given $a > 0$, and given a positive integer m , let N be the stopping rule defined as the smallest $n \leq m$ such that $Z_n \geq a$. If $Z_n < a$ for all $n \leq m$, then $N = m$. Thus the process must stop by time m , and $Z_N \geq a$ if and only if $Z_n \geq a$ for some $n \leq m$. Thus

$$P\left(\max_{1 \leq n \leq m} Z_n \geq a\right) = P(Z_N \geq a) \leq \frac{E[Z_N]}{a} \quad (99)$$

where we have used the Markov inequality. Finally, since the process must be stopped by time m , we have $Z_N = Z_m^*$. From (82), $E[Z_m^*] \leq E[Z_m]$, so the right hand side of (99) is less than or equal to $E[Z_m]/a$, completing the proof.

The following corollary bears the same relationship to the submartingale inequality as the Chebychev inequality does to the Markov inequality.

COROLLARY 1—KOLMOGOROV'S MARTINGALE INEQUALITY: Let $\{Z_n; n \geq 1\}$ be a martingale with $E[Z_n^2] < \infty$ for all $n \geq 1$. Then

$$P\left(\max_{1 \leq n \leq m} |Z_n| \geq b\right) \leq \frac{E[Z_m^2]}{b^2} ; \text{ for all integer } m \geq 1, \text{ all } b > 0 \quad (100)$$

Proof: Since $\{Z_n; n \geq 1\}$ is a martingale and Z_n^2 is a convex function of Z_n , it follows from theorem 3 that $\{Z_n^2; n \geq 1\}$ is a submartingale. Since Z_n^2 is non-negative, we can use the Kolmogorov submartingale inequality to see that

$$P(\max_{n \leq m} Z_n^2 \geq a) \leq E[Z_m^2]/a \text{ for any } a > 0$$

Substituting b^2 for a , we get (100).

COROLLARY 2—KOLMOGOROV'S RANDOM WALK INEQUALITY: Let $\{S_n; n \geq 1\}$ be a random walk with $S_n = X_1 + \dots + X_n$ where $\{X_i; i \geq 1\}$ is a set of IID random variables with mean \bar{X} and variance σ^2 . Then for any positive integer m and any $\epsilon > 0$,

$$P\left(\max_{1 \leq n \leq m} |S_n - n\bar{X}| \geq m\epsilon\right) \leq \frac{\sigma^2}{m\epsilon^2} \quad (101)$$

Proof: $\{Z_n = S_n - n\bar{X}; n \geq 1\}$ is a zero mean random walk, and thus a martingale. Since $E[Z_m^2] = m\sigma^2$, (101) follows by substituting $m\epsilon$ for b in (100).

Recall that the simplest form of the weak law of large numbers was given in (1.25) as $P(|S_m/m - \bar{X}| \geq \epsilon) \leq \sigma^2/(m\epsilon^2)$. This is strengthened in (101) to upper bound the probability that any of the first m terms deviate from the mean by more than $m\epsilon$. It is this strengthening that will allow us to prove the strong law of large numbers.

The following corollary yields essentially the same result as (29), but is included here as another example of the use of the Kolmogorov submartingale inequality.

COROLLARY 3: Let $\{S_n; n \geq 1\}$ be a random walk, $S_n = X_1 + \dots + X_n$ where X_i has mean $\bar{X} < 0$ and semi-invariant moment generating function $\gamma(r)$. For any $r > 0$ such that $0 < \gamma(r) < \infty$ (i.e., for $r > r^*$), and for any $\alpha > 0$.

$$P\left(\max_{1 \leq n \leq m} S_n \geq \alpha\right) \leq \exp\{-r\alpha + n\gamma(r)\} \quad (102)$$

Proof: For $r > r^*$, $\{\exp(rS_n); n \geq 1\}$ is a submartingale. Taking $a = \exp(r\alpha)$ in (98), we get (102).

The following theorem about supermartingales is, in a sense, the dual of the Kolmogorov submartingale inequality. Note, however, that it applies to the terms $n \geq m$ in the supermartingale rather than $n \leq m$.

THEOREM 8: Let $\{Z_n; n \geq 1\}$ be a non-negative supermartingale. Then for any positive integer m and any $a > 0$,

$$P\left(\sup_{i \geq m} Z_i \geq a\right) \leq \frac{E[Z_m]}{a} \quad (103)$$

Proof: For given $m \geq 1$ and $a > 0$, let N be a possibly defective stopping rule defined as the smallest $i \geq m$ for which $Z_i \geq a$. Let $\{Z_n^*; n \geq 1\}$ be the corresponding stopped process, which is also non-negative and is a supermartingale from theorem 4. For any $k > m$, note that $Z_k^* \geq a$ iff $\max_{m \leq i \leq k} Z_i \geq a$. Thus

$$P\left(\max_{m \leq i \leq k} Z_i \geq a\right) = P(Z_k^* \geq a) \leq \frac{E[Z_k^*]}{a}$$

Since $\{Z_n^*; n \geq 1\}$ is a supermartingale, (74) shows that $E[Z_k^*] \leq E[Z_m^*]$. On the other hand, $Z_m^* = Z_m$ since the process can not stop before epoch m . Thus $P(\max_{m \leq i \leq k} Z_i \geq a)$ is at most $E[Z_m]/a$. Since k is arbitrary, we can pass to the limit, getting (103) and completing the proof.

We now proceed to prove the strong law of large numbers. We assume a finite second moment. The theorem is true without this restriction, but the truncation argument we used for the weak law in theorem 1 of Chapter 1 does not carry over simply here.

THEOREM 9—STRONG LAW OF LARGE NUMBERS: Let $\{X_i; i \geq 1\}$ be a sequence of IID random variables with mean \bar{X} and standard deviation $\sigma < \infty$. Let $S_n = X_1 + \dots + X_n$. Then for any $\varepsilon > 0$,

$$\lim_{m \rightarrow \infty} P\left(\sup_{n \geq m} \left| \frac{S_n}{n} - \bar{X} \right| \geq \varepsilon\right) = 0 \quad (104)$$

Proof: The probability in (104) is non-increasing in m , so we can restrict attention to m of the form 2^k for integer k . For any given k , we can break up the supremum in (104) into a union of terms,

$$P\left(\sup_{n \geq m} \left| \frac{S_n}{n} - \bar{X} \right| \geq \varepsilon\right) \leq \sum_{j=k}^{\infty} P\left(\max_{2^j \leq n < 2^{j+1}} \left| \frac{S_n}{n} - \bar{X} \right| \geq \varepsilon\right) \quad (105)$$

Using (101), we can upper bound each of these terms by

$$\begin{aligned} P\left(\max_{2^j \leq n < 2^{j+1}} \left| \frac{S_n}{n} - \bar{X} \right| \geq \varepsilon\right) &\leq P\left(\max_{2^j \leq n < 2^{j+1}} \left| S_n - n\bar{X} \right| \geq \varepsilon 2^j\right) \\ &\leq P\left(\max_{1 \leq n \leq 2^{j+1}} \left| S_n - n\bar{X} \right| \geq \varepsilon 2^j\right) \leq \frac{2^{j+1} \sigma^2}{(\varepsilon 2^j)^2} \leq \frac{2^{1-j} \sigma^2}{\varepsilon^2} \end{aligned}$$

We can now substitute this into (105),

$$P\left(\sup_{n \geq 2^k} \left| \frac{S_n}{n} - \bar{X} \right| \geq \varepsilon\right) \leq \sum_{j=k}^{\infty} \frac{2^{1-j} \sigma^2}{\varepsilon^2} = \frac{2^{2-k} \sigma^2}{\varepsilon^2}$$

As k increases, the term on the right goes to zero, completing the proof.

Another famous result that follows from the Kolmogorov submartingale inequality is the martingale convergence theorem. This states that if a martingale $\{Z_n; n \geq 1\}$ has the property that there is some finite M such that $E[|Z_n|] \leq M$ for all n , then $\lim_{n \rightarrow \infty} Z_n$ exists (and is finite) with probability 1. This is a powerful theorem in more advanced work, but it is not quite as useful as it appears, since the restriction $E[|Z_n|] \leq M$ is more than a technical restriction; for example it is not satisfied by a zero mean random walk. We prove the theorem with the additional restriction that there is some finite M such that $E[Z_n^2] \leq M$ for all n .

THEOREM 10—MARTINGALE CONVERGENCE THEOREM: Let $\{Z_n; n \geq 1\}$ be a martingale and assume that there is some finite M such that $E[Z_n^2] \leq M$ for all n . Then there is a random variable Z such that, for all sample sequences except a set of probability 0, $\lim_{n \rightarrow \infty} Z_n = Z$.

Proof*: From theorem 3 and the assumption that $E[Z_n^2] \leq M$, $\{Z_n^2; n \geq 1\}$ is a submartingale. Thus, from (73), $E[Z_n^2]$ is nondecreasing in n , and since $E[Z_n^2]$ is bounded, $\lim_{n \rightarrow \infty} E[Z_n^2] = M'$ for some $M' \leq M$. For any integer k , the process $\{Y_n = Z_{k+n} - Z_k; n \geq 1\}$ is a zero mean martingale (see exercise 7.28). Thus from Kolmogorov's martingale inequality,

$$P(\max_{1 \leq n \leq m} |Z_{k+n} - Z_k| \geq b) \leq E[(Z_{k+m} - Z_k)^2]/b^2 \quad (106)$$

Next, observe that $E[Z_{k+m} Z_k | Z_k = z_k, Z_{k-1} = z_{k-1}, \dots, Z_1 = z_1] = z_k^2$, and therefore, $E[Z_{k+m} Z_k] = E[Z_k^2]$. Thus $E[(Z_{k+m} - Z_k)^2] = E(Z_{k+m}^2) - E(Z_k^2) \leq M' - E[Z_k^2]$. Since this is independent of m , we can pass to the limit, obtaining

$$P(\sup_{n \geq 1} |Z_{k+n} - Z_k| \geq b) \leq \{M' - E[Z_k^2]\}/b^2 \quad (107)$$

Since $\lim_{k \rightarrow \infty} E[Z_k^2] = M'$, we then have, for all $b > 0$,

$$\lim_{k \rightarrow \infty} P(\sup_{n \geq 1} |Z_{k+n} - Z_k| \geq b) = 0 \quad (108)$$

This means that with probability 1, a sample sequence of $\{Z_n; n \geq 1\}$ is a Cauchy sequence, and thus approaches a limit, concluding the proof.

This result can be relatively easily interpreted for branching processes. For a branching process $\{X_n; n \geq 1\}$ with an expected number μ of offspring, $\{X_n/\mu^n; n \geq 1\}$ is a martingale that satisfies the above conditions. If $\mu \leq 1$, the branching process dies out with probability 1, so X_n/μ^n approaches 0 with probability 1. For $\mu > 1$, however, the branching process dies out with some probability less than 1 and approaches ∞ otherwise. Thus, the limiting random variable Z is 0 with the probability that the process ultimately dies out, and is positive otherwise. In the latter case, for large n , the interpretation is that when the population is very large, a law of large numbers effect controls its growth in each successive generation, so that X_n/μ^n tends to grow in a random way for small n , and then changes increasingly little as n increases.

7.9 SUMMARY

Each term in a random walk $\{S_n; n \geq 1\}$ is a sum of IID random variables, and thus the study of random walks is closely related to that of sums of IID variables. The focus in random walks, however, as in most of the processes we have studied, is more in the relationship between the terms (such as which term first crosses a threshold) than in the individual terms. We started by showing that random walks are a generalization of renewal processes, are equivalent to queue waiting for G/G/1 queues, and equivalent to sequential analysis in hypothesis testing.

A major focus of the chapter was on estimating the probabilities of very unlikely events, a topic currently known as large deviation theory. We started by studying the exponential bound to $P(S_n \geq \alpha)$ for $\alpha > 0$ and $E[X] < 0$. We then developed the Wald identity, which can be used to find tight upper bounds to the probability that a threshold is ever crossed by a random walk. One of the insights gained here was that if a threshold at α is crossed, it is likely to be crossed at a time close to $n^* = \alpha/\gamma'(r^*)$, where r^* is the positive root of $\gamma(r)$. We also found that r^* plays a fundamental role in the probability of threshold crossings. For questions of typical behavior, the mean and variance of a random variable are the major quantities of interest, but for atypical large deviation behavior, r^* is the major parameter of interest.

We next introduced martingales, submartingales, and supermartingales. These are sometimes regarded as somewhat exotic topics in mathematics, but in fact they are very useful in a large variety of relatively simple processes. For example, we showed that all of the random walk issues of earlier sections can be treated as a special case of martingales, and that martingales can be used to model both sums and products of random variables. We also showed how Markov modulated random walks can be treated as martingales.

Stopping rules, as first introduced in Chapter 3, were then applied to martingales. We defined a stopped process $\{Z_n^*; n \geq 1\}$ to be the same as the original process $\{Z_n; n \geq 1\}$ up to the stopping point, and then constant thereafter. Theorems 4 and 5 showed that the stopped process has the same form (martingale, submartingale, or supermartingale) as the original process, and that the expected values $E[Z_n^*]$ are between $E[Z_1]$ and $E[Z_n]$. We also looked at $E[Z_N]$ and found that it is equal to $E[Z_1]$ iff (90) is satisfied. The Wald identity can be viewed as $E[Z_N] = E[Z_1] = 1$ for the Wald martingale, $Z_n = \exp\{rS_n - n\gamma(r)\}$. We then found a similar identity for Markov modulated random walks. In deriving results for Markov modulated random walks, it was necessary to define martingales relative to other processes in order to find suitable stopping rules, also defined on martingales relative to other processes. This added restriction on martingales is useful in other contexts.

The Kolmogorov inequalities were next developed. They are analogs of the Markov inequality and Chebyshev inequality, except sufficiently stronger to proof the strong law of large numbers and the martingale convergence theorem.

A standard reference on random walks, and particularly on the analysis of overshoots is [Fel66]. Dembo and Zeitouni, [DeZ93] develop large deviation theory in a much more general and detailed way than the introduction here. The classic reference on martingales is [Doo53], but [BhW90] and [Ros83] are more accessible.

EXERCISES

7.1) Consider the simple random walk $\{S_n; n \geq 1\}$ of example 1 with $S_n = X_1 + \dots + X_n$ and $P(X_i = 1) = p$; $P(X_i = -1) = 1-p$; assume that $p < 1/2$.

- a) Show that $P(\max_{i \geq 1} S_i \geq k) = [P(\max_{i \geq 1} S_i \geq 1)]^k$ for any positive integer k . Hint:

Given that the random walk ever reaches the value 1, consider a new random walk starting at that time and explore the probability that the new walk reaches a value 1 greater than its starting point.

- b) Write an equation for $P(\max_{i \geq 1} S_i \geq 1)$ by thinking of what has to happen after the first trial of the walk.

- c) Solve your equation and find $P(\max_{i \geq 1} S_i \geq k)$.

- d) Show that your answer is $\exp(-r^*k)$ where r^* is the positive root of $g(r) = 1$ where $g(r) = E[e^{rX}]$.

7.2) A G/G/1 queue has a deterministic service time of 2 and inter-arrival times that are 3 with probability p or 1 with probability $1-p$.

- a) Find the distribution of W_1 , the wait in queue of the first arrival after the beginning of a busy period.
 b) Find the distribution of W_∞ , the steady state wait in queue.
 c) Repeat parts (a) and (b) if the service times and inter-arrival times are exponentially distributed with rates μ and λ respectively.

7.3) A sales executive hears that one of his sales people is routing half of his incoming sales to a competitor. In particular, arriving sales are known to be Poisson at rate one per hour. According to the report (which we view as hypothesis 1), each second arrival is routed to the competition; thus under hypothesis 1 the inter-arrival density for successful sales is $f(y|H_1) = y e^{-y} y \geq 0$. The alternate hypothesis (H_0) is that the rumor is false and the inter-arrival density for successful sales is $f(y|H_0) = e^{-y}; y \geq 0$. Assume that, a priori, the hypotheses are equally likely. The executive, a recent student of stochastic processes, explores various alternatives for choosing between the hypotheses; he can only observe the times of successful sales however.

- a) Starting with a successful sale at time 0, let S_i be the arrival time of the i^{th} subsequent successful sale. The executive observes S_1, S_2, \dots, S_n ($n \geq 1$) and chooses the maximum a posteriori probability hypothesis given this data. Find the joint probability density $f(S_1, S_2, \dots, S_n | H_1)$ and $f(S_1, \dots, S_n | H_0)$ and give the decision rule.

- b) This is the same as part (a) except that the system is in steady state at time 0 (rather than starting with a successful sale). Find the density of S_1 (the time of the first arrival after time 0) conditional on H_0 and on H_1 . What is the decision rule now after observing S_1, \dots, S_n .

- c) This is the same as part (b), except rather than observing n successful sales, the successful sales up to some given time t are observed. Find the probability, under each hypothesis, that the first successful sale occurs in $(s_1, s_1 + \Delta]$, the second in $(s_2, s_2 + \Delta], \dots$, and the last in $(s_{N(t)}, s_{N(t)} + \Delta]$ (assume Δ very small). What is the decision rule now?

7.4) For the hypothesis testing problem of section 7.3, assume that there is a cost C_0 of choosing H_1 when H_0 is correct, and a cost C_1 of choosing H_0 when H_1 is correct. Find the test that minimizes the expected cost and express it in the form of equation (8).

7.5) a) For the hypothesis testing problem of section 7.3, assume that $Z_i = \ln[f(Y_i|H_1)/f(Y_i|H_0)]$, as a random variable conditional on H_i , $i = 1, 2$, has a continuous valued non-zero probability density. Show that, for any given α , a threshold test minimizes $P(\text{error} | H_0)$, under the constraint that $P(\text{error} | H_1) \leq \alpha$. Hint: Assume that some given test is better than any threshold test, and show, as a contradiction, that that test has a smaller probability of error than the MAP test for the appropriate $P(H_0)$. (This is called the Neyman–Pearson test.)

b) Repeat part (a) without the assumption that Z_i has a continuous density. Hint: Be careful about the don't care case in (10).

7.6) For each of the following random variables, find the interval (r_-, r_+) over which the moment generating function $g(r)$ exists, and determine at which end points $g(r)$ exists. For parts (a) and (b) you should also find and sketch $g(r)$. For part (c), $g(r)$ has no closed form.

a) Let a, b , be positive numbers and let X have the density

$$f_X(x) = (2a)^{-1}\exp(-ax); x \geq 0; f_X(x) = (2b)^{-1}\exp(bx); x < 0$$

b) Let Y be a Gaussian random variable with mean m and variance σ^2 .

c) Let Z be a non-negative random variable with density

$$f_Z(z) = k(1+z)^{-2} \exp(-az); z \geq 0$$

where $a > 0$ and $k = [\int_{z \geq 0} (1+z)^{-2} \exp(-az) dz]^{-1}$. Hint: You can find the Laplace transform for f_Z in Laplace transform tables, but the result is not in closed form and is not much help. Fortunately, there is no need to evaluate $g(r)$ to find r_+ or to find whether r_+ is finite.

7.7) a) Assume that the random variable X has a moment generating function $g_X(r)$ that is finite in the interval (r_-, r_+) , $r_- < 0 < r_+$, and assume $r_- < r < r_+$ throughout. For any finite constant c , express the moment generating function of $X - c$, i.e., $g_{(X-c)}(r)$ in terms of the moment generating function of X . Show that $g''_{(X-c)}(r) \geq 0$.

b) Show that $g''_{(X-c)}(r) = g''_X(r) - 2cg'_X(r) + c^2g_X(r)$.

c) Use (a) and (b) to show that $g''_X(r)g_X(r) - [g'_X(r)]^2 \geq 0$, and that $g''_X(r) \geq 0$.

d) Assume that X is non-atomic, i.e., that there is no value of c such that $P(X=c) = 1$. Show that the inequality sign “ \geq ” may be replaced by “ $>$ ” everywhere in (a), (b), and (c).

7.8) Define $\gamma(r)$ as $\ln g(r)$ where $g(r) = E[\exp(rX)]$. Assume that X is discrete with possible outcomes $\{a_i; i \geq 1\}$, let p_i denote $P(X=a_i)$, and assume that $g(r)$ exists in some

region (r_-, r_+) around $r=0$. For any given r , $r_- < r < r_+$, define a random variable X_r with the same set of possible outcomes $\{a_i; i \geq 1\}$ as X , but with a probability mass function $q_i = P(X_r = a_i) = p_i \exp[a_i r - \gamma(r)]$. X_r is not a function of X , and is not even to be viewed as in the same probability space as X ; it is of interest simply because of the behavior of its defined probability mass function. It is called a tilted random variable relative to X , and this exercise, along with exercise 7.9 will justify our interest in it.

- a) Verify that $\sum_i q_i = 1$.
- b) Verify that $E[X_r] = \sum_i a_i q_i$ is equal to $\gamma(r)$.
- c) Verify that $\text{Var}[X_r] = \sum_i a_i^2 q_i - (E[X_r])^2$ is equal to $\gamma''(r)$.
- d) Argue that $\gamma''(r) \geq 0$ for all r such that $g(r)$ exists, and that $\gamma''(r) > 0$ if $\gamma'(0) > 0$.
- e) Give a similar definition of X_r for a random variable X with a density, and modify parts (a) to (d) accordingly.

7.9) Assume that X is discrete, with possible values $\{a_i; i \geq 1\}$ and probabilities $P(X=a_i) = p_i$. Let X_r be the corresponding tilted random variable as defined in exercise 7.8. Let $S_n = X_1 + \dots + X_n$ be the sum of n IID rv's with the distribution of X , and let $S_{n,r} = X_{1,r} + \dots + X_{n,r}$ be the sum of n IID tilted rv's with the distribution of X_r . Assume that $\bar{X} < 0$.

- a) Show that $P(S_{n,r} = v) = P(S_n = v) \exp[vr - n\gamma(r)]$. Hint: First show that $P(X_{1,r} = v_1, \dots, X_{n,r} = v_n) = P(X_1 = v_1, \dots, X_n = v_n) \exp[vr - n\gamma(r)]$ where $v = v_1 + \dots + v_n$.
- b) Find the mean and variance of $S_{n,r}$ in terms of $\gamma(r)$.
- c) Define $a = \gamma(r)$ and $\sigma_r^2 = \gamma''(r)$. Show that $P(|S_{n,r} - na| \leq \sqrt{2n} \sigma_r) > 1/2$. Use this to show that

$$P(|S_n - na| \leq \sqrt{2n} \sigma_r) > (1/2) \exp[-r(an + \sqrt{2n} \sigma_r) + n\gamma(r)]$$

- d) Use this to show that for any ϵ and for all sufficiently large n ,

$$P(S_n \geq n(a-\epsilon)) > \exp[-rn + n\gamma(r) - n\epsilon]$$

7.10 a) Redraw figure 7.3 for the case $\bar{X} > 0$

b) Find the value of $r \geq 0$ that minimizes the right hand side of (13) and redraw figure 7.4 to illustrate your solution. Hint: The nature of your answer will change depending on the relationship between α/n and \bar{X} .

7.11) Evaluate the second derivative with respect to r of the Wald identity. By setting $r = 0$, find a relation between the first and second moments of N , S_N , and X . Verify Eq. (18) for the case $\mu=0$.

7.12 a) Consider a random walk with thresholds $\alpha > 0$, $\beta < 0$. We wish to find $P(S_N > \alpha)$ in the absence of a lower threshold. Use (24) to upper bound the probability that the random walk crosses α before β .

b) Given that the random walk crosses β first, find an upper bound to the probability that α is now crossed before a yet lower threshold at 2β is crossed.

c) Given that 2β is crossed before α , upperbound the probability that α is crossed before a threshold at 3β . Extending this argument to successively lower thresholds, find an upper bound to each successive term, and find an upper bound on the overall probability that α is crossed. By observing that β is arbitrary, show that (20) is valid with no lower threshold.

7.13) a) Use Wald's equality to show that if $\bar{X} = 0$, then $E[S_N] = 0$ where N is the time of threshold crossing with one threshold at $\alpha > 0$ and another at $\beta < 0$.

b) Obtain an expression for $P(S_N \geq \alpha)$. Your expression should involve the expected value of S_N conditional on crossing the individual thresholds (you need not try to calculate these expected values).

c) Evaluate your expression for the case of a simple random walk.

d) Evaluate your expression when X has an exponential density, $f_X(x) = a_1 e^{-\lambda x}$ for $x \geq 0$ and $f_X(x) = a_2 e^{\mu x}$ for $x < 0$ and where a_1 and a_2 are chosen so that $\bar{X} = 0$.

7.14) A random walk $\{S_n; n \geq 1\}$, with $S_n = \sum_{i=1}^n X_i$, has the following probability density for X_i

$$f_X(x) = \begin{cases} \frac{e^{-x}}{e - e^{-1}} & ; -1 \leq x \leq 1 \\ 0 & ; \text{ elsewhere} \end{cases}$$

a) Find the values of r for which $g(r) = E[\exp(rX)] = 1$.

b) Let P_A be the probability that the random walk ever crosses a threshold at A for some $A > 0$. Find an upper bound to P_A of the form $P_A \leq e^{-\alpha A}$ where α is independent of A ; evaluate α .

c) Find a lower bound to P_A of the form $P_A \geq \beta e^{-\alpha A}$ where α is the same as in part (b) and β is independent of A . Hint: Keep it simple—you are not expected to find an elaborate bound. Also recall that $E[e^{r*SN}] = 1$ where N is a stopping time for the random walk and $g(r^*) = 1$.

7.15) Let $\{X_n; n \geq 1\}$ be a sequence of IID integer valued random variables with the probability mass function $P_X(k) = Q_k$. Assume that $Q_k > 0$ for $|k| \leq 10$ and $Q_k = 0$ for $|k| > 10$. Let $\{S_n; n \geq 1\}$ be a random walk with $S_n = X_1 + \dots + X_n$. Let $\alpha > 0$ and $\beta < 0$ be integer valued thresholds, let N be the smallest value of n for which either $S_n \geq \alpha$ or $S_n \leq \beta$. Let $\{S_n^*; n \geq 1\}$ be the stopped random walk; i.e., $S_n^* = S_n$ for $n \leq N$ and $S_n^* = S_N$ for $n > N$. Let $\pi_i^* = P(S_N=i)$.

a) Consider a Markov chain in which this stopped random walk is run repeatedly until the point of stopping. That is, the Markov chain transition probabilities are given by $P_{ij} = Q_{j-i}$ for $\beta < i < \alpha$ and $P_{i0} = 1$ for $i \leq \beta$ and $i \geq \alpha$. All other transition probabilities are 0 and the set of states is the set of integers $[-9+\beta, 9+\alpha]$. Show that this Markov chain is ergodic.

- b) Let $\{\pi_i\}$ be the set of steady state probabilities for this Markov chain. Find the set of probabilities $\{\pi_i^*\}$ for the stopping states of the stopped random walk in terms of $\{\pi_i\}$.
- c) Find $E[S_N]$ and $E[N]$ in terms of $\{\pi_i\}$.

7.16) Assume that $\bar{X} < 0$ and show that the final term in (34) is upper bounded by $\exp[r\alpha - r_{\min}\beta + m\gamma(r_{\min}) - m\gamma(r)]$ where r_{\min} minimizes $g(r)$. Use this to prove lemma 4 for $r \neq r_{\min}$. Hint: Use $P(S_m > \beta)$ to upper bound $P(N \geq m)$ and recall that $\gamma(r)$ has a unique minimum.

7.17) a) Conditional on H_0 for the hypothesis testing problem, consider the random variables $Z_i = \ln[f(Y_i|H_1)/f(Y_i|H_0)]$. Show that r^* , the positive solution to $g(r)=1$, where $g(r) = E[\exp(rZ_i)]$, is given by $r^*=1$.

b) Assuming that Y is a discrete random variable (under each hypothesis), show that the tilted random variable with $r=1$ has the probability density $f(Y_i|H_1)$.

7.18) a) Suppose $\{Z_n; n \geq 1\}$ is a martingale. Verify (68); i.e., $E[Z_n] = E[Z_1]$ for $n > 1$.

b) If $\{Z_n; n \geq 1\}$ is a submartingale, verify (73), and if a supermartingale, verify (74).

7.19) Suppose $\{Z_n; n \geq 1\}$ is a martingale. Show that

$$E[Z_m | Z_{n_1}, Z_{n_1+1}, \dots, Z_{n_i}] = Z_{n_i} \text{ for all } 0 < n_1 < n_2 < \dots < n_i < m.$$

7.20) a) Assume that $\{Z_n; n \geq 1\}$ is a submartingale. Show that

$$E[Z_m | Z_n, Z_{n-1}, \dots, Z_1] \geq Z_n \text{ for all } n < m.$$

b) Show that $E[Z_m | Z_{n_1}, Z_{n_1+1}, \dots, Z_{n_i}] \geq Z_{n_i}$ for all $0 < n_1 < n_2 < \dots < n_i < m$.

c) Assume now that $\{Z_n; n \geq 1\}$ is a supermartingale. Show that parts (a) and (b) still hold with \geq replaced by \leq .

7.21) Let $\{Z_n = \exp[rS_n - n\gamma(r)]; n \geq 1\}$ be the generating function martingale of (49) where $S_n = X_1 + \dots + X_n$ and X_1, \dots, X_n are IID with mean $\bar{X} < 0$. Let N be the possibly defective stopping rule for which the process stops after crossing a threshold at $c > 0$ (there is no negative threshold). Show that $\exp[r^*\alpha]$ is an upper bound to the probability of threshold crossing by considering the stopped process $\{Z_n^*; n \geq 1\}$.

The purpose of this exercise is to illustrate that the stopped process can yield useful upper bounds even when the stopping rule is defective.

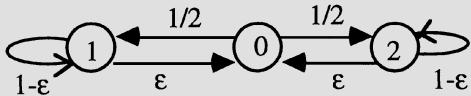
7.22) Show that theorem 3 is also valid for martingales relative to a joint process. That is, show that if h is a convex function of a real variable and if $\{Z_n; n \geq 1\}$ is a martingale relative to a joint process $\{Z_n, X_n; n \geq 1\}$, then $\{h(Z_n); n \geq 1\}$ is a submartingale relative to $\{h(Z_n, X_n); n \geq 1\}$.

7.23) Show that if $\{Z_n; n \geq 1\}$ is a martingale (submartingale or supermartingale) relative to a joint process $\{Z_n, X_n; n \geq 1\}$ and if N is a stopping rule for $\{Z_n; n \geq 1\}$ relative to $\{Z_n, X_n; n \geq 1\}$, then the stopped process is a martingale (submartingale or supermartingale) respectively.

7.24) Show that if $\{Z_n; n \geq 1\}$ is a martingale (submartingale or supermartingale) relative to a joint process $\{Z_n, X_n; n \geq 1\}$ and if N is a stopping rule for $\{Z_n; n \geq 1\}$ relative to $\{Z_n, X_n; n \geq 1\}$, then the stopped process satisfies (82), (83), or (84) respectively.

7.25) Show that if $\{Z_n; n \geq 1\}$ is a martingale relative to a joint process $\{Z_n, X_n; n \geq 1\}$ and if N is a stopping rule for $\{Z_n; n \geq 1\}$ relative to $\{Z_n, X_n; n \geq 1\}$, then $E[Z_N] = E[Z_1]$ iff (90) is satisfied.

7.26) Consider the Markov modulated random walk in the figure below. The random variables Y_n in this example take on only a single value for each transition, that value being 1 for all transitions from state 1, 10 for all transitions from state 2, and 0 otherwise. $\varepsilon > 0$ is a very small number, say $\varepsilon < 10^{-6}$.



a) Show that the steady state gain per transition is $5.5/(1+\varepsilon)$. Show that the relative gain vector is $w = (0, (\varepsilon-4.5)/[\varepsilon(1+\varepsilon)], (10\varepsilon+4.5)/[\varepsilon(1+\varepsilon)])$.

b) Let $S_n = Y_0 + Y_1 + \dots + Y_{n-1}$ and take the starting state X_0 to be 0. Let N be the smallest value of n for which $S_n \geq 100$. Find $P(N=11)$ and $P(N=101)$. Find an estimate of $E[N]$ that is exact in the limit $\varepsilon \rightarrow 0$.

c) Show that $P(X_N=1) = (1-45\varepsilon+o(\varepsilon))/2$ and that $P(X_N=2) = (1+45\varepsilon+o(\varepsilon))/2$. Verify, to first order in ε , that (93) is satisfied.

7.27) Show that (93) results from taking the derivative of (97) and evaluating it at $r=0$.

7.28) Let $\{Z_n; n \geq 1\}$ be a martingale, and for some integer m , let $Y_n = Z_{n+m} - Z_m$.

a) Show that $E[Y_n | Z_{n+m-1} = z_{n+m-1}, Z_{n+m-2} = z_{n+m-2}, \dots, Z_m = z_m, \dots, Z_1 = z_1] = z_{n+m-1} - z_m$

b) Show that $E[Y_n | Y_{n-1} = y_{n-1}, \dots, Y_1 = y_1] = y_{n-1}$

c) Show that $E[|Y_n|] < \infty$. Note that (b) and (c) show that $\{Y_n; n \geq 1\}$ is a martingale.

NOTES

1. It is not necessary here to use the more general form of stopping rule in which I_n is independent of Z_n, Z_{n+1}, \dots conditional on $Z_{n-1}, X_{n-1}, \dots, Z_1, X_1$, since that form was used to allow I_n to depend on more than Z_{n-1}, \dots, Z_1 . With a martingale relative to a joint process, that added flexibility is provided, but in a more controlled way.

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