Thursday

09/02/2023

Lecture #9: Elementary Queues.

Geom(p) / Geom(q) / 1 / 1

- Geom(p) → Arrival process → Bernoulli process. ⟷ Geometric inter-arrival.
- Geom(q) → Service rate
- 1 → No. of server
- 1 → No. of positions in the system.

$$X(n) = \begin{cases} 1 & \text{if service is busy at } n \\ 0 & \text{otherwise.} \end{cases}$$

" Arrivals when server is busy are dropped "

$P(1 \mid 0) = p$

$P_{01} = p$

$P_{00} = 1-p$

$P_{10} = q$

$$\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

Memoryless prop. → Markov prop.

Assumption: ~~Ø~~ No ~~two~~ arrival & completion happen at the same time

(No simultaneous Transitions)

Arrivals can happen, but they will be dropped.

$$\pi = \pi P$$

$$[x \;\; y] \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = [x \;\; y]$$

$$x = \frac{q}{p+q}, \quad y = \frac{p}{p+q}$$

$$E(X_n) = \frac{p}{p+q}$$

(At stationary)

/ when the markov chain has become stationary.

$M/M/1/1$

$M \rightarrow PP(\lambda)$

$M \rightarrow$ service exp(μ)

$$X(t) = \begin{cases} 1 \\ 0 \end{cases}$$

$$Q = \begin{bmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{bmatrix}$$

$q_{00} = -\lambda \qquad q_{01} = \lambda$

$q_{10} = \mu \qquad q_{11} = -\mu.$

q_{ij} = rate at which the CTMC moves from state i to state j.

$$P(t) = e^{Qt} = \begin{bmatrix} e^{-\lambda t} & e^{\lambda t} \\ e^{\mu t} & e^{-\mu t} \end{bmatrix}$$

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$$\pi_0 = \frac{\mu}{\lambda+\mu}, \quad \pi_1 = \frac{\lambda}{\lambda+\mu}.$$

$$E(X) = \frac{\lambda}{\lambda+\mu}.$$

M/M/1/∞

N(t) = No. of jobs of the system at time t.

$$Q = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & \cdots & \\ 0 & -\lambda & \lambda & 0 & 0 & & \\ 1 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots & \\ 2 & 0 & \mu & -(\lambda+\mu) & \lambda & \cdots & \\ 3 & \vdots & & \vdots & & \ddots & \end{array}$$

or
$$q_{ij} = \begin{cases} \mu, & \text{if } i = j+1,\ i \neq 0 \\ -(\mu+\lambda), & \text{if } i = j,\ i \neq 0 \\ \lambda, & \text{if } i = j-1,\ i \neq 0 \\ -\lambda, & i = 0, j = 0 \\ \lambda, & i = 0, j = 1 \end{cases}$$

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$$\pi Q = 0.$$

$$\begin{bmatrix}\pi_0 & \pi_1 & \cdots\end{bmatrix}\begin{bmatrix}-\lambda & \lambda & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda \cdots\end{bmatrix} = 0.$$

$$-\lambda \pi_0 + \mu \pi_1 = 0.$$

Gen. eq:

$$\pi_1 = \frac{\lambda}{\mu}\pi_0.$$

$\lambda \pi_{n-1} + \mu \pi_{n+1} = (\lambda+\mu)\pi_n$

$$\lambda \pi_0 - (\lambda+\mu)\pi_1 + \mu \pi_2 = 0.$$

$$\pi_2 = \frac{\lambda}{\mu}\pi_1.$$

$\therefore$ generalizing, $\pi_i = \frac{\lambda}{\mu}\pi_{i-1}$.

$$\therefore \pi_i = \left(\frac{\lambda}{\mu}\right)^i \pi_0$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

$$\therefore \pi_0 = 1 - \frac{\lambda}{\mu}$$

$$\rho = \text{load} = \frac{\lambda}{\mu}.$$

$$\pi_i = \rho^i \cdot (1-\rho)$$

Homework : $E(N) = ?$

$$= \sum_0^\infty i\,\pi_i$$

$$= \lambda \quad i\,\rho^i (1-\rho)$$

$$= \frac{1}{\cancel{(1-\rho)}^2} \cdot \rho\,(\cancel{1-\rho})$$

$$= \frac{\rho}{1-\rho}.$$

$$= \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}$$

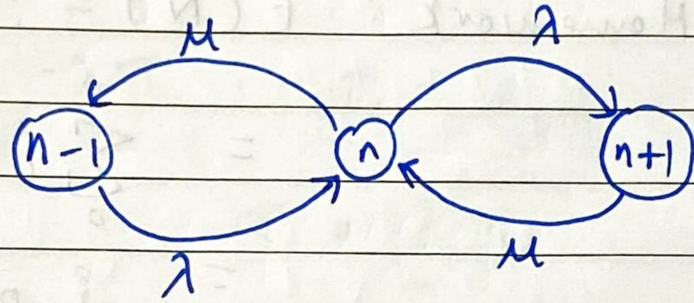
$$= \frac{\lambda}{\mu-\lambda}.$$

Lecture #10:

Global Balance equations.

$$\lambda \pi_{n-1} + \mu \pi_{n+1} = (\lambda + \mu)\pi_n$$

for any general equation.



$$\left\{\begin{matrix}\text{Rate of leaving } n \\ \pi_n(\lambda+\mu)\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate of entry } n \\ \pi_{n-1}\lambda + \pi_{n+1}\mu .\end{matrix}\right\}$$

Global balance equations used for solving solving ~~πQ~~ stationary distribution.

Little's Law:

$$E(N) = \lambda \cdot E(T)$$

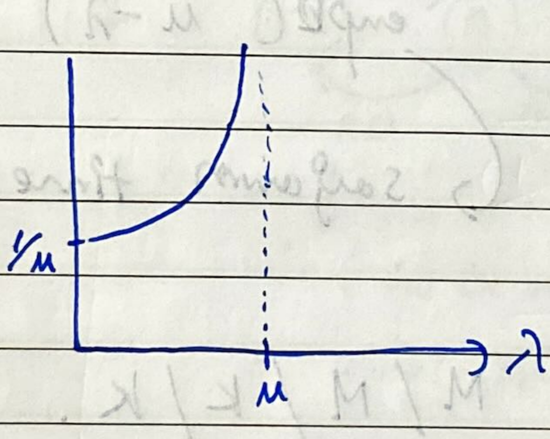
$N \to$ No. of people in the system

$T \to$ Response time

$T =$ service time + waiting time
$=$ time spent by each job in the system.

~~Rene~~ Renewal process → Generalization of poisson process.

$$E(T) = \frac{E(N)}{\lambda} = \frac{1}{\mu - \lambda}.$$



$$T = W + S$$

$$N = N_q + 1_s$$

$E[N_q] = \lambda E(W)$ → Try to derive (Homework).

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$P(T > t)$

$\Rightarrow \sum_n P(T > t / N = n)\, P(N = n)$

$\Rightarrow \sum_n P\left(\sum_{j=0}^{n} B_j > t\right)(1-\rho)\rho^n$ → Gamma / Erlang.

$\Rightarrow \downarrow$ ⟶ Homework

$Exp(\mu(1-\rho))$

$exp(\mu - \lambda)$

↳ sojourn time distribution.*

M / M / K / K.

① ② ③ . . . Ⓚ

One possible descriptor

(Hard) Desc 1 $= (I_1, I_2, \ldots, I_k)$

$$I_i = \begin{cases} 1 & \text{if busy} \\ 0 & \text{else} \end{cases}$$

This makes sense if μ_i's are different, but if they are same then

Desc. 2 = No. of busy servers

Easier to describe.

$$
\begin{tikzcd}
0 & 1 & 2 & \cdots & n-1 \arrow[r, bend left, "\lambda"] & n \arrow[l, bend left, "n\mu"] \arrow[r, bend left, "\lambda"] & n+1 \arrow[l, bend left, "(n+1)\mu"] & & K
\end{tikzcd}
$$

This will only go up till K.

$n\mu$

min (All the busy server)

$= \exp(\mu + \mu + \cdots + \mu)$

$= \exp(n\mu)$.

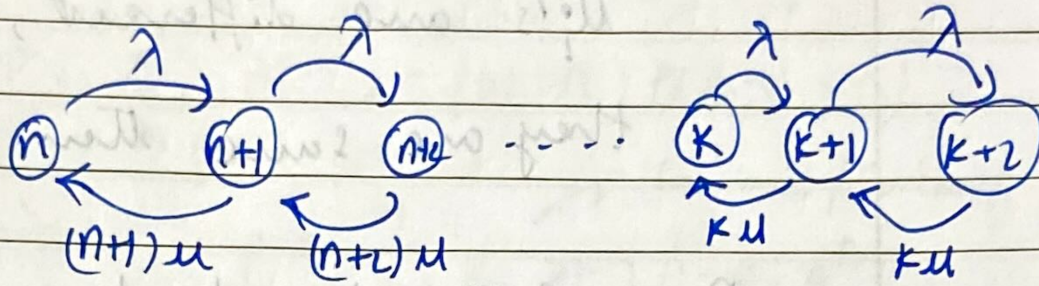
Home work: Calculate $E(N)$ & π_n.

The quantities we have to obtain

$E(W)$, $E(T)$ using Littles Law

π_n $E(W)$ $E(N_q)$

M / M / K / ∞



Check book by Mohr
Balter.

Renewal Process

→ A renewal process is a counting process for which the interarrival

→ Renewal process is
Generalization of poisson process.