MA 6.101
Probability and Statistics

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Generate samples using uniform distribution

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  import matplotlib.pyplot as plt
  uni_samples = np.random.uniform(0, 1, 5000)
  plt.hist(uni_samples, bins = 10, density = True)
  plt.show()
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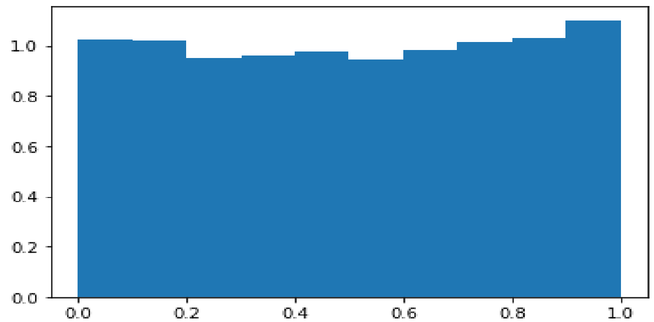
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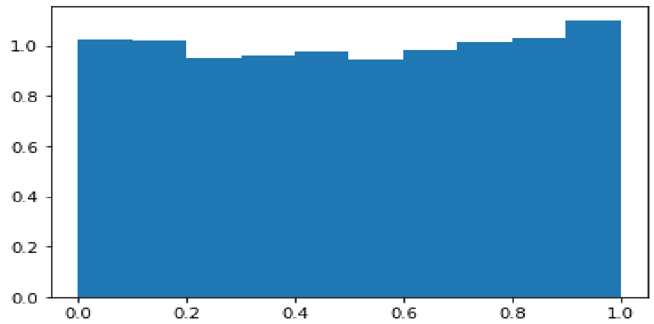

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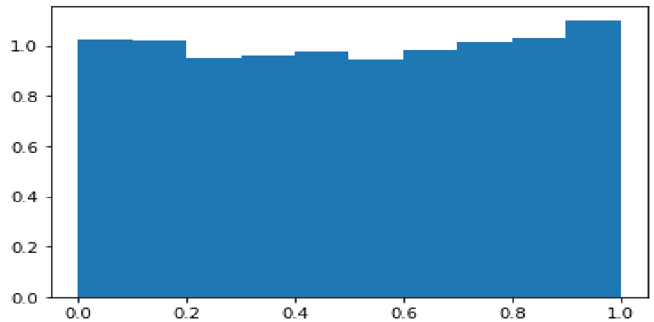

- *uni_samples* is a vector of 5000 realizations of uniform random variable U.

- You can also see it as a realization of $U_1, U_2, \dots U_{5000}$ i.i.d uniform variables.
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 t=0
dice_samples=np.zeros(5000)
for u in uni_samples:
  if u < 1/6:
    dice_sample = 1
  if 1/6 < u < 2/6:
    dice_sample = 2
  if 2/6 < u < 3/6:
    dice_sample = 3
  if 3/6 < u < 4/6:
    dice_sample = 4
  if 4/6 < u < 5/6:
    dice_sample = 5
  if 5/6 < u < 6/6:
    dice_sample = 6
  dice_samples[t] = dice_sample
  t = t+1
plt.hist(dice_samples, bins = 6, density = True)
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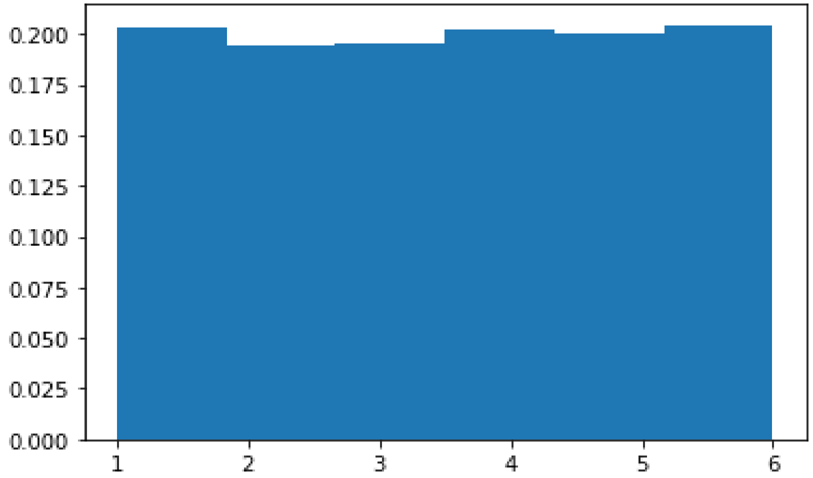
- [0.02, 0.8, 0.6, 0.03]
- [1, 5, 4, 1]



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- We shall now formally see the inverse transform method to do this.

The inverse transform method

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$$X = \begin{cases} x_0 \text{ if } u < p_0 \\ x_1 \text{ if } p_0 \leq u < p_0 + p_1 \\ x_2 \text{ if } p_0 + p_1 \leq u < p_0 + p_1 + p_2 \\ \vdots \\ x_j \text{ if } \sum_{i=0}^{j-1} p_i \leq u < \sum_{i=0}^{j} p_i \\ \vdots \end{cases}$$

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- Why is this method correct?

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  - **We are thus finding the inverse of $F_X(U)$!**

How to generate samples of a continuous random variable

(Using samples of a continuous uniform variable over $[0, 1]$)

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- Do you observe anything “special” about this lemma?

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- This property is known as "probability integral transform or universality of uniform".
- This property is used to test whether a set of observations can be modelled as arising from a specified distribution $G(.)$ or not.
  - Given set of data samples $s_1, s_2, \ldots, s_n$, plot $G(s_i)$ for different samples.
  - If these points are spread uniformly over the interval $[0, 1]$ then it indicates that the samples are indeed from $G(.)$.

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        - Hasting-Metropolis algorithm
        - Gibbs sampling
        - Slice sampling

Convergence of Random Variables

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If $N(\epsilon, x) = N(\epsilon)$ (i.e., independent of x) for every $x \in \mathbb{R}$, then such convergence of $F_n(\cdot)$ to $F(\cdot)$ is called as uniform convergence.