MA 6.101
Probability and Statistics

Tejas Bodas

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Logistics

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- Venue will be emailed later.

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- What is $p_Y(2)$?
- $p_Y(2) = \sum_{\{x:|x|=2\}} p_X(x) = p_X(-2) + p_x(2) = \frac{2}{9}$.

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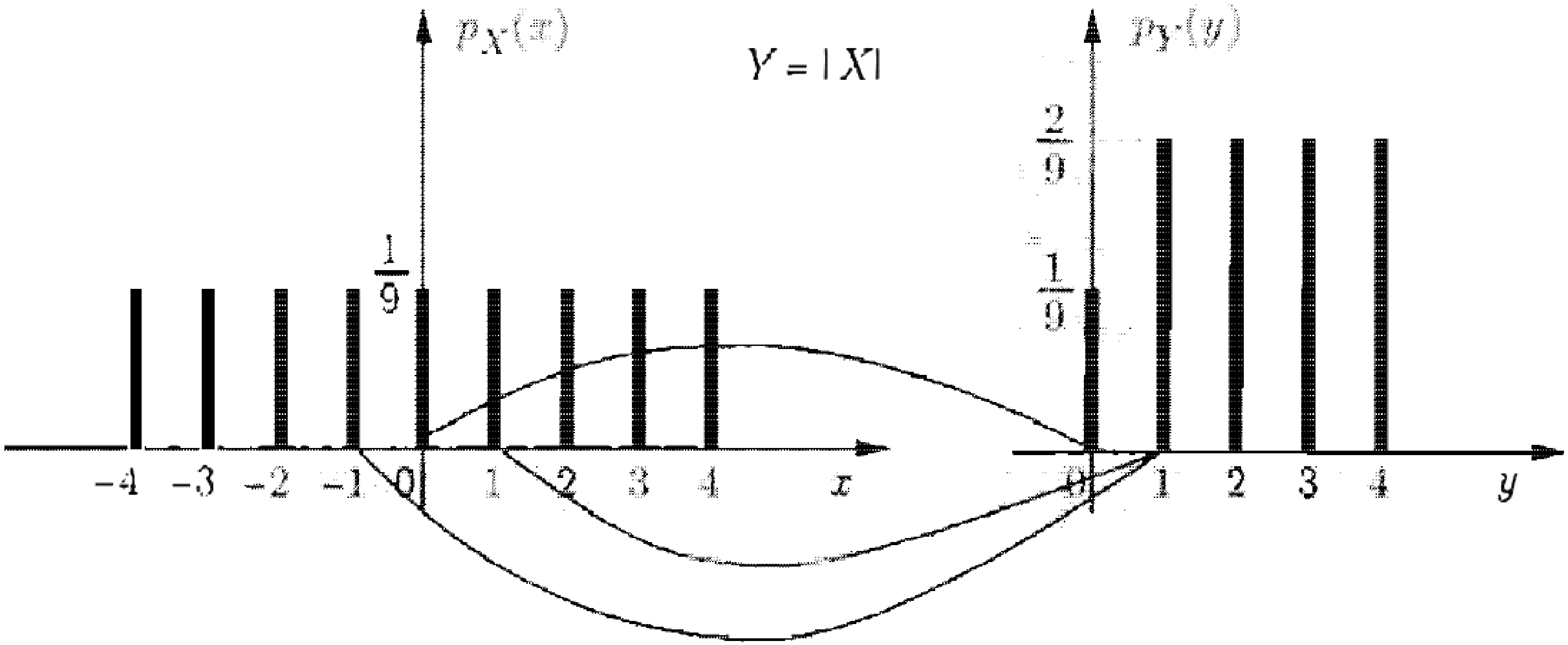

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- Is $g^{-1}(y)$ a value or a set?

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- Now $\mathbb{P}\{\omega \in \Omega : X(\omega) \in B\}$

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- Now $\mathbb{P}\{\omega \in \Omega : X(\omega) \in B\} = \sum_{\{x \in B\}} p_X(x)$ for $B \in \mathcal{F}'$.
- Proof follows after setting $B = \{x : g(x) = y\}$ $\square$

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- In general, $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$

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Consider $Y = g(X)$ where g is monotone, continuous, differentiable. Then $f_Y(y) = |\frac{dh}{dy}(y)| f_X(h(y))$ where h is the inverse function of g.

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Consider $Y = g(X)$ where g is monotone, continuous, differentiable. Then $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$ where h is the inverse function of g.

Proof:

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- $F_Y(y) = P(g(X) \leq y)$.

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- If they are same, then the two probabilities are equal.

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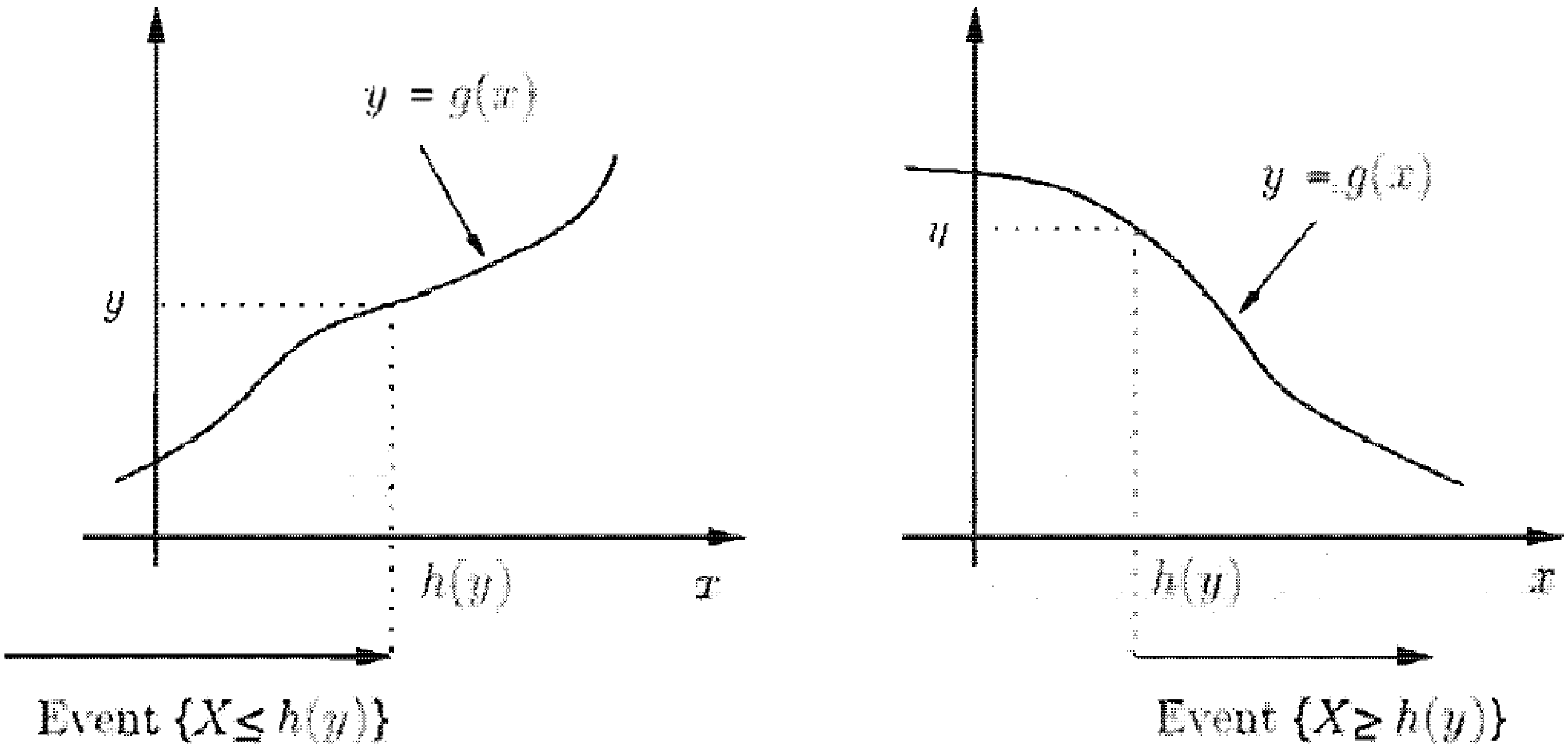

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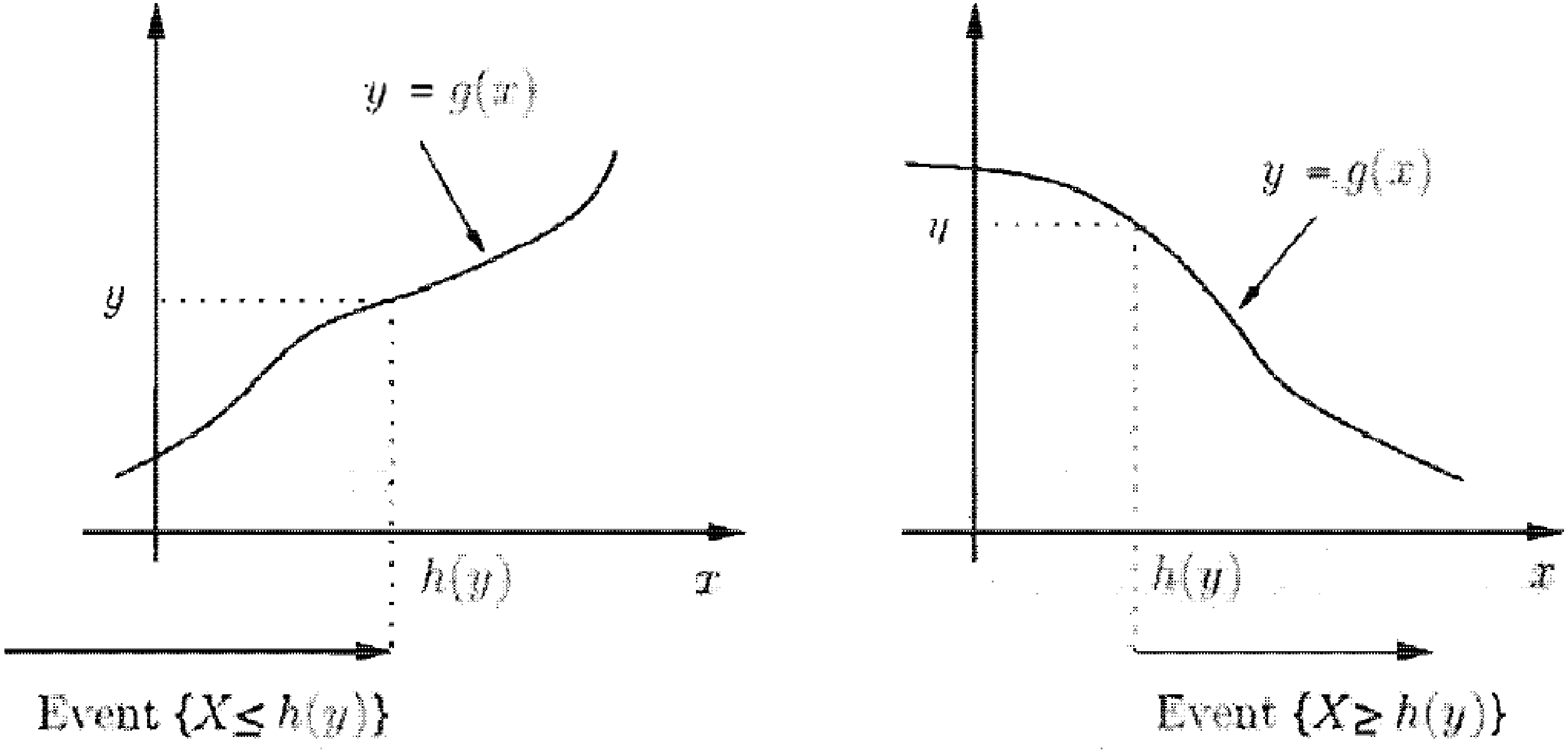

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HW: What about the case when g is not monotone ?

Multiple random variables

A running example

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- Suppose A is the event that you get a head and the roll is even. What is $P((X, Y) \in A)$?

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> The marginal PMF's p_X and p_Y can be obtained from the joint PMF as follows:
> $p_X(x) = \sum_y p_{XY}(x, y)$ and $p_Y(y) = \sum_x p_{XY}(x, y)$.

This is true in general, and requires a proof (later).