# CS 3.307

# Performance Modeling for Computer Systems

**Tejas Bodas**

Assistant Professor, IIIT Hyderabad

# Logistics

- Feel free to contact me anytime at tejas.bodas@iiit.ac.in.
- Office @ A5304.
- Book– Performance modeling and design of computer systems (Cambridge press) by Mor Harchol-Balter (Professor, CMU)
- Other books: 1) Stochastic processes by Sheldon Ross 2) Probabilistic modeling by Isi Mitrani.
- Assignment 1 : 15%. Midsem exam: 30%. Assignment 2: 15% Endsem 40 %.

# Course Outline

- Module 1 (2 lectures)
  - Motivation, Probability refresher, Introduction to Stochastic Processes
- Module 2 (4 lectures)
  Poisson Process & Markov Chains
- Module 3 (2 lectures) Elementary Queues
- Module 4 Renewal theorems and Busy period analysis (3 lectures)
- Module 5 (3 lectures) Advanced Queues

# Performance modeling for Computer systems

- How do you measure the performance of your computer?
- Speed with which it runs programs. RAM, clock speed, GPU, Cores.
- Storage space ? SSD or not ?
- What is the key word here ? LATENCY!
- Performance metrics?
  - response time (run time, lag, delay, jitter)
  - blocking probability (screen freeze, no disk space, packet loss, buffer full)

# Modeling ?

- Design for performance: How many cores or GPU's? which core to use? how to schedule instructions in a core?
- Routing (which core) and scheduling (which program/ instruction to execute)
- How do you know which is a good design? via experimentation?(costly!)
- Performance analysis! via stochastic modeling

# Applications Beyond Computers

- Computer systems
  - server farms, cloud computing, distributed storage systems
  - Communication systems, Wifi, Sensor networks.
- Heathcare
  - How many OT? How many Specialists or nurses?
  - Scheduling operations, stocking of medicines, scheduling tests.
- Hospitality industry
  - Designing hotel lobbies for faster checkin
  - Restaurant seating! (How many tables of size 2,4,8?)
- Transportation systems
  - Airline or Railway scheduling
  - Priority scheduling, class differentiation
- Operation Research!
- Henceforth use the term Queueing system!

# A single server queue


- One server, one FIFO queue for jobs to wait.
- $\mu$ denotes service rate, $\lambda$ denotes the arrival rate.
- Service requirements $S_n$ and inter-arrival times $A_n$ are typically assumed to be i.i.d.
- In its simplest form, we will assume $S_n \sim Exp(\mu)$ and $A_n \sim Exp(\lambda)$.
- Jobs face queueing delay due to waiting for other jobs.
- This is the most basic $M/M/1$ queue. Modeling this as a Markov chain and solving its stationary distribution gives us mean response time (mean of service time + waiting time).
- $E[T] = \frac{1}{\mu - \lambda}$.

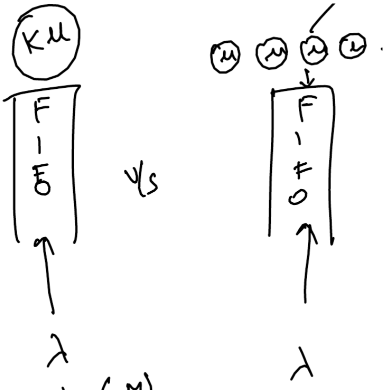
# A single server queue


- $E[T] = \frac{1}{\mu - \lambda}$.
- Let $N$ is the number of jobs in the system (Queue + server). Then what is $E[N]$?
- We will see Little's law that says that $E[N] = \lambda E[T]$.
- Mean number of jobs $E[N] = \frac{\lambda}{\mu - \lambda}$.
- This course is about Markov chain analysis to derive such formulas.

# Example 1: Doubling the arrival rate


- $E[T] = \frac{1}{\mu - \lambda}$.
- What would happen to $E[T]$ if $\lambda \to 2\lambda$?
- It could blow up if $\mu < 2\lambda$.
- If you want to maintain the same level of response time then do you need to double $\mu$?
- This course is about making such design choices!

# Example 2: A fast server versus many slow servers

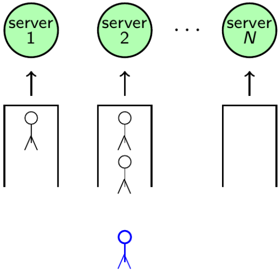



- Which system will have lower $E[T]$?
- Is a fast server $(K\mu)$ better that $K$ normal servers $(\mu)$?
- Does job variability impact this decision? Suppose job sizes were $XS, S, M, L, XL$.
- In the first model, an $S$, or $M$ job has to possibly wait behind $XL$. This is avoided in the second scenario.

# Example 3: Central queue or individual

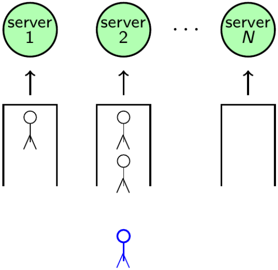

- At Airport immigration, Hotel check-ins you often see central queues.
- But at movie theatres, metro/train ticket counters, you see the second model.
- Which setting has a lower $E[T]$?
- This course will help you answer such performance modeling questions.

# Example 4: Supermarket queue and load balancing




- Load balancing concerns the questions which queue to join/assign?
- Popular policy is Join shortest Queue (JSQ).
- What should be ideally done is Join smallest work (JSW).
- $N$ is typically large and the overhead in obtaining queue length information is huge ($2N$).

# Example 4: Supermarket queue and load balancing




- In that case, sample $d$ servers randomly and join appropriate queue using $JSQ(d)$ or $JSW(d)$.
- Problem with $JSW$ or $JSW(d)$ is that the workload information is typically unknown. How to implement it then?
- How about replicating jobs on $d$ servers and cancelling copies when one copy starts service ?
- This is redundancy-d with cancel on start.
- We do this at super-markets all the time!

# Probability Refresher

# Random experiments and Sample space

- Random experiment : Experiment involving randomness
  - Coin toss
  - Roll a dice
  - Pick a number at random from $[0, 1]$.
- Sample space $\Omega$: set of all possible outcomes of the random experiment. It could be a finite or infinite set.
  - $\Omega_c = \{H, T\}$
  - $\Omega_d = \{1, 2, \ldots, 6\}$
  - $\Omega_u = [0, 1]$

# Events

- A subset $A \subseteq \Omega$ is called an **event**.
- Examples of events
  - Events in the coin experiment: $C_1 = \{T\}$.
  - Events in the dice experiment: $D_1 = 6, D_2 = \{1, 3, 5\}$
  - Events in $U[0, 1]$ experiment: $U_1 = \{0.5\}$, $U_2 = [.25, .75]$.
- Probability of event $A$ is denoted by $\mathbb{P}(A)$.

# Probability theory

$\{$Random experiment, Sample space, Events$\}$ are the key ingredients in probability theory.

In probability theory, we are interested in **measuring** the probability of subsets of $\Omega$ (events).

Probability measure $\mathbb{P}$ is a **set function**, i.e. it acts on sets and measures the probability of such sets.

## *sigma-algebra* as domain for $\mathbb{P}$

- Event space or *sigma-algebra* $\mathcal{F}$ is a collection of measurable sets that satisfy
  $\bullet\ \emptyset \in \mathcal{F} \quad \bullet\ A \in \mathcal{F} \implies A^c \in \mathcal{F}$
  $\bullet A_1, A_2, \ldots A_n, \ldots \in \Omega \implies \cup_{n=1}^{\infty} A_n \in \Omega$

- The $\sigma-$algebra is said to be closed under formation of compliments and countable unions.

- Is it also closed under the formation of countable intersections?

> When $\Omega$ is countable and finite, we will consider power-set $\mathcal{P}(\Omega)$ as the domain.

# Formal definition of Probability measure $\mathbb{P}$

## Definition

A probability measure $\mathbb{P}$ on the *measurable space* $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ s.t.

1. $\mathbb{P}(\emptyset) = 0, \quad \mathbb{P}(\Omega) = 1$
2. For a disjoint collection of event sets $A_1, A_2, \ldots$ from $\mathcal{F}$ we have
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$
(countable additivity)

- The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.

# Conditional probability

- Given/If dice rolls odd, what is the probability that the outcome is 1?
- Given/If $\bar{\omega} \in [0, 0.5]$ what is the probability that $\bar{\omega} \in [0, 0.25]$?
- The conditional probability of event $B$ given event $A$ is defined as $\mathbb{P}(B/A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ when $\mathbb{P}(A) > 0$.
- Bayes rule: $P(B/A) = \frac{P(A/B)P(B)}{P(A)}$.

# Independence and Mutually exclusive

- Two events $A, B$ are independent iff $P(A/B) = P(A)$ and $P(B/A) = P(B)$.
- Two events $A, B$ are independent iff $P(A \cap B) = P(A)P(B)$.

- If $A$ and $B$ are independent, then so are $A^c$ and $B^c$.
- What about $A$ and $B^c$? Are they independent?
- Two events $A$ and $B$ are mutually exclusive if occurrence of one implies that the other event cannot occur. Are they independent?
- If $A$ and $B$ are mutually exclusive, then they are not independent (and vice versa).

# Random variable

- Given a random experiment with associated $(\Omega, \mathcal{F}, \mathbb{P})$, it is sometimes difficult to deal directly with $\omega \in \Omega$. eg. rolling a dice ten times.
- Notice that each sample point $\omega \in \Omega$ is not a number but a sequence of numbers.
- Also, we may be interested in functions of these sample points rather than samples themselves. eg: Number of times 6 appears in the 10 rolls.
- In either case, it is often convenient to work in a new *simpler* probability space rather than the original space.
- Random variable is a device which precisely helps us make this mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to a 'simpler' $(\Omega', \mathcal{F}', P_X)$.
- $P_X$ is called as an induced probability measure on $\Omega'$.

# Random variable

- If $\Omega'$ is countable, then the random variable is called a discrete random variable.
- In this case it is convenient to use $\mathcal{F}'$ as power-set.
- If $\Omega' \subseteq \mathbb{R}$ or uncountable, then the random variable is a continuous random variable.
- In this case, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$.
- Notation: Random variables denoted by capital letters like $X, Y, Z$ etc. anf their realizations by small letters $x, y, z$..

# PMF and CDF of a Discrete r.v.

- Let $X : \Omega \to \Omega'$ be a discrete r.v.
- Let $p_X(x)$ for $x \in \Omega'$ denote the probability that $X$ takes the value $x$.
- $p_X(x)$ is called as a probability mass function.
- The cumulative distribution function (CDF) $F_X(\cdot)$ is defined as $F_X(x_1) := \sum_{x \leq x_1} p_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x_1\}$.

# Expectation, Moments, Variance

- The mean or expectation of a random variable $X$ is denoted by $E[X]$ and is given by $E[X] = \sum_{x \in \Omega'} x p_X(x)$.
- The $n^{th}$ moment of a random variable $X$ is denoted by $E[X^n]$ and is given by $E[X^n] = \sum_{x \in \Omega'} x^n p_X(x)$.
- Functions of random variables are random variables.
- For a function $g(\cdot)$ of a random variable $X$, its expectation is given by $E[g(X)] := \sum_{x \in \Omega'} g(x) p_X(x)$
- $Var(X) := E[(X - E[X])^2]$
- HW: Prove that $E[(X - E[X])^2] = E[X^2] - E[X]^2$
- For $Y = aX + b$, what is $E[Y]$? $E[Y] = aE[X] + b$. (Linearity of expectation)

# Bernoulli random variable

- Bernoulli random variable $X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{otherwise.} \end{cases}$
- Basic models of Multi-arm bandit problem assume Bernoulli Bandits.
- $E[X] = p, E[X^n] = p$.

# Binomial $B(n, p)$ random variable.

- Consider a biased coin (head with probability $p$) and toss it $n$ times.
- Denote head by 1 and tail by 0.
- Let random variable $N$ denote the number of heads in $n$ tosses.
- PMF of $N$?. $P_N(k) = \binom{n}{k} p^k (1-p)^{n-k}$.
- HW: What is $E[N], E[N^2], Var(X)$?

# Geometric random variable

- Consider a biased coin (head with probability $p$) and suppose you keep tossing it till head appears the first time.
- Let random variable $N$ denote the number of tosses needed for head to appear first time.
- What is the PMF of $N$? $p_N(k) = (1-p)^{k-1}p$.
- HW: What is $E[N], E[N^2], Var(N)$?

# Poisson random variable

- A Poisson random variable $X$ comes with a parameter $\lambda$ and has $\Omega' = \mathbb{Z}_{\geq 0}$
- PMF: $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$
- Intuitively its a limiting case of the Binomial distribution with $n$ increasing and $p$ decreasing such that $np$ converges to $\lambda$.
- Mean of binomial is $np$ so $p$ should decrease while $n$ increases.

# Continuous random variables

- A random variable $X$ is continuous if there exists a non-negative real valued probability density function (PDF) $f_X(\cdot)$ such that $F_X(x) = \int_{u=-\infty}^{x} f_X(u)du$.

- $P_X(a \leq X \leq b) = \int_a^b f_X(u)du$. (Area under the curve)

$$\boxed{\frac{dF_X(x)}{dx} = f_X(x) \text{ or } P_X(x < X \leq x + h) \simeq f_X(x)h.}$$

# Mean, Variance, Moments

- $E[X] = \int_{-\infty}^{\infty} u f_X(u) du$
- $E[X^n] = \int_{-\infty}^{\infty} u^n f_X(u) du$
- $E[g(X)] = \int_{-\infty}^{\infty} g(u) f_X(u) du$
- $Var[X] = E[g(X)]$ where $g(x) = (x - E[X])^2$.
- For $Y = aX + b$, $E[Y] = aE[X] + b$.

# Exponential random variable ($Exp(\lambda)$)

- This is a non-negative r.v. with parameter $\lambda$.
- Its pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
- Its CDF is given by $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
- $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$
- $E[X^n] = \frac{n!}{\lambda^n}$

# Summary: Multiple random variables

$p_{XY}(x,y) := \mathbb{P}\{\omega \in \Omega : X(w) = x \text{ and } Y(\omega) = y\}$.
$F_{XY}(x,y) := \mathbb{P}\{\omega \in \Omega : X(w) \leq x \text{ and } Y(\omega) \leq y\}$.

The marginal PMF's $p_X$ and $p_Y$ can be obtained from the joint PMF as follows:
$p_X(x) = \sum_y p_{XY}(x,y)$ and $p_Y(y) = \sum_x p_{XY}(x,y)$.

Two random variables, $X$ and $Y$ are independent if the following is true:
$p_{XY}(x,y) = p_X(x)p_Y(y), F_{XY}(x,y) = F_X(x)F_Y(y)$ and $E[XY] = E[X]E[Y]$.

$E[g(X,Y)] = \sum_{xy} g(xy)p_{XY}(xy)$

The rules for continuous random variables are similar.
Also revise conditioning of variables.

# Sums of independent random variable

- Consider $Z = X + Y$. What is the pdf of $Z$ when $X$ and $Y$?
- What is $p_Z(z)$ or $f_Z(z)$?
- $p_Z(z) = \sum_{(x,y):x+y=z} p_{X,Y}(x,y)$
- $f_Z(z) = \int_{(x,y):x+y=z} f_{X,Y}(x,y)dxdy$.
- Since $X$ and $Y$ are independent $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ and $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. This gives us

Convolution formula

$$p_Z(z) = \sum_x p_X(x)p_Y(z-x)$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

HW: What if $X$ and $Y$ are not independent?

# MGF of Sums of independent random variable

- Consider $Z = X + Y$. What is the pdf of $Z$ when $X$ and $Y$?
- Let $M_X(t)$ and $M_Y(t)$ be their MGF's. What is $M_Z(t)$ ?
- $M_Z(t) = E[e^{Zt}] = E[e^{(X+Y)t}]$.
- $M_Z(t) = E[e^{Xt}.e^{Yt}]$.
- If $X$ and $Y$ are independent, $E[XY] = E[X]E[Y]$ and $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.
- $M_Z(t) = E[e^{Xt}].E[e^{Yt}]$.

$$M_Z(t) = M_X(t)M_Y(t).$$

# MGF of Sums of independent random variable

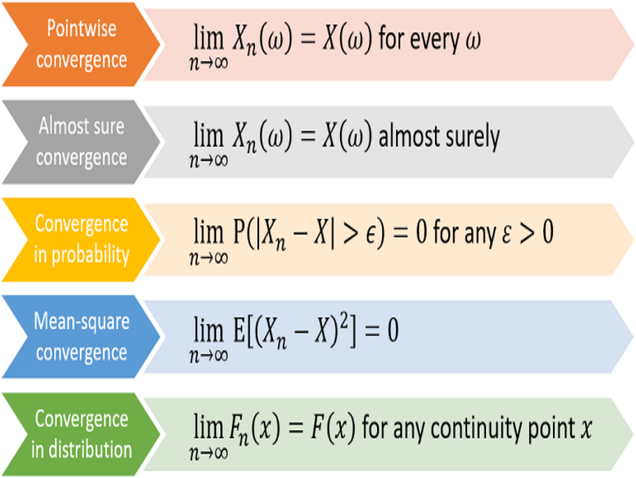
- Consider $Z = X + Y$. What is the MGF of $Z$ when $X$ and $Y$?

$$M_Z(t) = M_X(t)M_Y(t).$$

- What about $M_Z(t)$ when $Z = X_1 + X_2 + \ldots X_n$ and $X_i$ are iid.?
- $M_Z(t) = (M_X(t))^n$.
- What about $M_Z(t)$ when $Z = X_1 + X_2 + \ldots X_N$ where $N$ is a positive discrete random variable? section 4.5

# Convergence of Random Variables

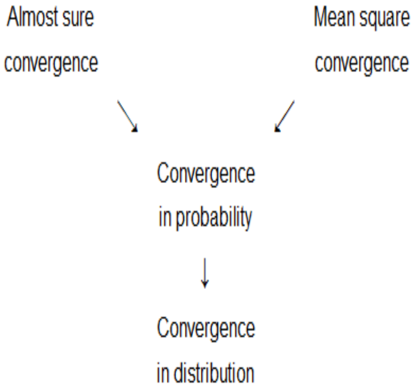
# Summary



[1]

[1] Image from probabilitycourse.com

# Relation between modes of convergence (no proofs)




https://en.wikipedia.org/wiki/Proofs_of_convergence_of_random_variables

# Introduction to Stochastic processes

# Introduction to Stochastic processes

- Stochastic process $\{X(t), t \in T\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables defined such that for every $t \in T$ we have $X(t) : \Omega \to \mathcal{S}$.
- $T$ is the parameter space (often resembles time) and $\mathcal{S}$ is the state space.
- Random variable $X(t)$ is often denoted by $X(\omega, t)$.
- When $t$ is fixed and $\omega$ is the only variable, we have a random variable $X(\cdot, t)$. When $\omega$ is fixed and $t$ is the variable, we have a $X(\omega, \cdot)$ as a function of time. This is also called as a realization or sampe path of a stochastic process.

# Introduction to Stochastic processes

- When $T$ is countable, we have a discrete time process.
- If $T$ is a subset of real line, we have a continuous time process.
- State space could be integers or real numbers
- State space could be $\mathbb{R}^n$ or $\mathbb{Z}^n$ valued

# Elementary Examples

- The process of rolling a dice 6 times.
- You bank balance over a week.
- Temperature fluctuations in a 1hr window.
- Number of customers in IKEA every day.

A c.t.s.p. is called an *independent increment process* if for any choice of parameters $t_0 < t_1 < \ldots < t_n$, the $n$ increment random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.

The c.t.m.p. is said to have *stationary and independent increments* if in addition $X(t_2 + s) - X(t_1 + s)$ has the same distribution as $X(t_2) - X(t_1)$ for all $t_1, t_2 \in T$ and any $s > 0$.

# Examples

- Sequence of i.i.d random variables.

- General random walk: If $X_1, X_2, \ldots$ is a sequence i.i.d of random variables, then $S_n = \sum_{i=1}^{n} X_i$ is a random walk.

- Weiner process: $\{X(t), t \geq 0\}$ is a Weiner process if
  1. $X(0) = 0$
  2. $\{X(t), t \geq 0\}$ has stationary and independent increments
  3. for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $t$.

- $\{X(t), t \geq 0\}$ is a Markov process if for $t_1 < t_2 < \ldots t_n < t$ we have
  $P(X(t) \leq x | X(t_1) = x_1, \ldots, X(t_n) = x_n) = P(X(t) \leq x | X(t_n) = x_n)$

- Random walk and Weiner process are examples of Markov processes.