OQS Assignment 1

Due: 31-10-2023

Consider the following equation $\frac{d\rho}{dt} = (\sigma_x \rho \sigma_x - \rho) + (\sigma_y \rho \sigma_y - \rho) - tanh(t)(\sigma_z \rho \sigma_z - \rho)$ where ρ is a qubit density matrix and σ_is are the Pauli operators.

1. Find the solution of this equation
2. Determine whether it represents a valid quantum evolution or not.
3. If it's a valid quantum evolution, then find it's Kraus operators.
4. If it's a valid quantum evolution, then determine whether it is CP-divisible or not.
5. Consider trace distance between two quantum states $\rho_1(t)$ and $\rho_2(t)$ be defined as T(t) = $\frac{1}{2}||\rho_1(t) - \rho_2(t)||_1$, where $||A||_1 = Tr\sqrt{A^\dagger A}$. Plot the evolution of T(t) with time and comment on it's nature.
   Note: The distance measure and the density matrices both are dependent on time t. Take the initial values(i.e., time t=0) of $\rho_1 =| 0\rangle \langle 0|$ and $\rho_2 =| 1\rangle \langle 1|$

1. Let $\rho = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Then $\frac{d\rho}{dt} = \begin{pmatrix} da_{11}/dt & da_{12}/dt \\ da_{21}/dt & da_{22}/dt \end{pmatrix} = (\sigma_x \rho \sigma_x - \rho) + (\sigma_y \rho \sigma_y - \rho) - \tan(t)(\sigma_z \rho \sigma_z - \rho)$

Here, $\sigma_x, \sigma_y, \sigma_z$ are pauli Matrices and

$$\tan(t) = \frac{e^t - 1}{e^t + 1} - 1 = \frac{-2}{e^t + 1}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x \rho \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}$$

$$\sigma_y \rho \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_{12} & -a_{11} \\ a_{22} & -a_{21} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$$

$$\sigma_z \rho \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$$

$$\therefore \text{RHS} = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} + \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} - 2\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \tan(t)\left[\begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2(a_{22}-a_{11}) & -2a_{12} \\ -2a_{21} & 2(a_{11}-a_{22}) \end{pmatrix} - \tan(t)\begin{pmatrix} 0 & -2a_{12} \\ -2a_{21} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2(a_{22}-a_{11}) & 2a_{12}(\tan(t)-1) \\ 2a_{21}(\tan(t)-1) & 2(a_{11}-a_{22}) \end{pmatrix} = \begin{pmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} \end{pmatrix}$$

We have 4 set of differential equations:

(1) $\frac{da_{11}}{dt} = 2(a_{22} - a_{11})$ (3) $\frac{da_{21}}{dt} = 2a_{12}(\tan(t) - 1)$

$|0\rangle + |1\rangle$

(2) $\frac{da_{12}}{dt} = 2a_{12}(\tan(t) - 1)$ (4) $\frac{da_{22}}{dt} = 2(a_{11} - a_{22})$

from (1) & (4) we get :-

$$\frac{d(a_{11}+a_{22})}{dt}=0 \Rightarrow \int_0^t d(a_{11}+a_{22})=\int_0^t 0\,dt$$

$$\Rightarrow a_{11}+a_{22}=1 \Rightarrow a_{22}=1-a_{11}$$

$$\therefore \frac{da_{11}}{dt}=2-4a_{11}$$

$$\therefore \int_{a_{11}(0)}^{a_{11}}\frac{da}{2-4a_{11}}=\int_0^t dt \Rightarrow -\frac{1}{4}\log\left(\frac{2-4a_{11}}{2-4a_{11}(0)}\right)=t-0$$

$$\therefore 4a_{11}-2=(4a_{11}(0)-2)e^{-4t} \Rightarrow a_{11}=\frac{1}{4}\left((4a_{11}(0)-2)e^{-4t}+2\right)$$

$$\frac{dy}{dt}=-4y$$

$$y=y_0 e^{-4t}$$

$$a_{11}(t)-a_{22}(t)=(a_{11}(0)-a_{22}(0))e^{-4t}$$

$$a_{11}(t)+a_{22}(t)=(a_{11}(0)+a_{22}(0))$$

$$a_{11}(t)=\frac{(1+e^{-4t})}{2}a_{11}(0)+\frac{(1-e^{-4t})}{2}a_{22}(0)$$

$$a_{22}(t)=\frac{(1-e^{-4t})}{2}a_{11}(0)+\frac{(1+e^{-4t})}{2}a_{22}(0)$$

Now, we solve (2)

let $a_{12}=y$, then we have

$$\frac{dy}{dt}=2y(\tan(t)-1)=2y\left(\frac{e^{2t}-1}{e^{2t}+1}-1\right)=2y\left(\frac{-2}{e^{2t}+1}\right)$$

$$\frac{dy}{y}=\frac{-4}{e^{2t}+1}dt=\frac{-4e^{-2t}}{e^{-2t}+1}dt=\frac{2\,d(e^{-2t})}{e^{-2t}+1}$$

$$y-y_0=2\log(e^{-t}+1)\Big|_0^t$$

$$y=y_0+2\log(e^{-2t}+$$

$2t$ 2 $-2t$ 2 $2t$ 2

$$\frac{dy}{dn}=\frac{-4y}{1+e^n}$$

$$\frac{dy}{y}\ \frac{1}{4}\left((4a_{11}(0)-2)e^{-4t}+2\right)\frac{dn}{1+e^n}$$

$$a_{12}(0)\left(\frac{e^{-2t}+1}{2}\right)^2$$

$$=\frac{-4e^{-n}}{e^{-n}+1}dn \qquad a_{21}(0)\left(\frac{e^{2t}+1}{2}\right)^2 \qquad \frac{1}{4}\left(-(4a_{11}(0)-2)e^{-4t}+2\right)$$

$$=\frac{4\,d(e^{-n})}{e^{-n}+1}=\frac{4\,dt}{t+1}=4\log(t+1)+c.$$

from (1) & (4) we get :-

$$\frac{d(a_{11}+a_{22})}{dt} = 0 \Rightarrow \int_0^t d(a_{11}+a_{22}) = \int_0^t 0\, dt$$

$$\Rightarrow a_{11}+a_{22} - (a_{11}(0)+a_{22}(0)) = 0 \Rightarrow a_{11}(t)+a_{22}(t) = a_{11}(0)+a_{22}(0)$$ [It is Trace preserving]

and $\frac{d}{dt}(a_{11}-a_{22}) = 4(a_{22}-a_{11})$

$\therefore$ we get,

$$a_{11}(t) = a_{11}(0)\left(\frac{1+e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1-e^{-4t}}{2}\right)$$

$$a_{22}(t) = a_{11}(0)\left(\frac{1-e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1+e^{-4t}}{2}\right)$$

Now, we solve (2)

let $a_{12} = y$, then we have

$$\frac{dy}{dt} = 2y(\tanh(t)-1) = 2y\left(\frac{e^{2t}-1}{e^{2t}+1}-1\right) = 2y\left(\frac{-2}{e^{2t}+1}\right)$$

$$\frac{dy}{y} = \frac{-4}{e^{2t}+1}dt = \frac{-4e^{-2t}}{e^{-2t}+1}dt = \frac{2\,d(e^{-2t})}{e^{-2t}+1}$$

$x = e^{-2t}+1$

$t=0,\ x=2$

$t=t,\ x=e^{-2t}+1$

$$\log\left(\frac{y}{y_0}\right) = 2\log(x)\Big|_2^{e^{-2t}+1}$$

$$\log\left(\frac{y}{y_0}\right) = 2\log(e^{-2t}+1) - 2\log 2$$

$$y = y_0\left(\frac{e^{-2t}+1}{2}\right)^2 \Rightarrow a_{12}(t) = a_{12}(0)\left(\frac{e^{-2t}+1}{2}\right)^2$$

similarly, the solution for (3) will be

$$a_{21}(t) = a_{21}(0)\left(\frac{e^{-2t}+1}{2}\right)^2$$

$$\therefore \rho = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} = \begin{pmatrix} a_{11}(0)\left(\frac{1+e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1-e^{-4t}}{2}\right) & a_{12}(0)\left(\frac{e^{-2t}+1}{2}\right)^2 \\ a_{21}(0)\left(\frac{e^{-2t}+1}{2}\right)^2 & a_{11}(0)\left(\frac{1-e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1+e^{-4t}}{2}\right) \end{pmatrix}$$

2. If $(\mathbb{I} \otimes \Lambda)\, \Phi_{AB} \geq 0$, then Λ is completely positive by Choi-Jamiolkowski Isomorphism.

Here, $\Phi_{AB} = |\phi_{AB}\rangle\langle\phi_{AB}|$ and $|\phi_{AB}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$\therefore\ \Phi_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right)$$

$$= \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\therefore\ (\mathbb{I} \otimes \Lambda)\, \Phi_{AB} = \begin{bmatrix} \Lambda\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \\ \Lambda\begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right)} & 0 & 0 & \boxed{\frac{1}{2}\cosh^2(t)\, e^{-2t}} \\ 0 & \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) & 0 \\ \boxed{\frac{1}{2}\cosh^2(t)\, e^{-2t}} & 0 & 0 & \boxed{\frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right)} \end{bmatrix}$$

$$= \rho_{Choi}$$

$$M_1 = \begin{bmatrix} \frac{1}{4}(1-e^{-4t}) & 0 \\ 0 & \frac{1}{4}(1-e^{-4t}) \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \frac{1}{4}(1+e^{-4t}) & \frac{1}{2}\cosh^2(t)\, e^{-2t} \\ \frac{1}{2}\cosh^2(t) e^{-2t} & \frac{1}{4}(1+e^{-4t}) \end{bmatrix}$$

Eigen values of M_1 = diagonals since it diagonal matrix

$\therefore\ \lambda_1 = \lambda_2 = \frac{1}{4}(1-e^{-4t})$

Eigen values of M_2:

$$\left(\frac{1}{4}(1+e^{-4t}) - \lambda\right)^2 - \left(\frac{1}{2}\cosh^2(t)\, e^{-2t}\right)^2 = 0$$

$$\Rightarrow \quad \frac{1}{4}(1+e^{-4t}) - \lambda = \pm \frac{1}{2}\cosh^2 t\, e^{-2t}$$

$$\Rightarrow \quad \lambda_3 = \frac{1}{4}(1+e^{-4t}) - \frac{1}{2}\cosh^2 t\, e^{-2t}$$

$$\lambda_4 = \frac{1}{4}(1+e^{-4t}) + \frac{1}{2}\cosh^2 t\, e^{-2t}$$

since $e^{-4t} \in [1, 0) \Rightarrow \frac{1}{4}(1-e^{-4t}) \in [0, \frac{1}{4})$

for $t \in [0, \infty)$

$\therefore\ \lambda_1 = \lambda_2 \geq 0$

Clearly, $\cosh^2 t \geq 0$ since $\cosh t \in \mathbb{R}$.

$$\lambda_3 = \frac{1}{4}(1+e^{-4t}) + \frac{1}{2}\cosh^2 t\, e^{-2t} \geq \frac{1}{4}(1+0) + \frac{1}{8} \geq 0.$$

$\therefore\ \lambda_3 \geq 0$

$$\lambda_4 = \frac{1}{4}(1+e^{-4t}) - \frac{1}{2}\left(\frac{e^{-2t}+1}{2}\right)^2$$

$$= \frac{1}{8}(1+2e^{-2t}+e^{-4t}) \geq \frac{1}{4} \geq 0.$$

$\therefore \lambda_4 \geq 0$

$\therefore$ The map is CP.

To prove positivity, we have to prove

$a_{11}, a_{22} \geq 0$ and $a_{11} a_{22} \geq a_{12}^2$

$$\rho = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} = \begin{pmatrix} a_{11}(0)\left(\frac{1+e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1-e^{-4t}}{2}\right) & a_{12}(0)\left(\frac{e^{-2t}+1}{2}\right)^2 \\ a_{21}(0)\left(\frac{e^{-2t}+1}{2}\right)^2 & a_{11}(0)\left(\frac{1-e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1+e^{-4t}}{2}\right) \end{pmatrix}$$

since $a_{11}(0), a_{22}(0) \geq 0$ for it to be a valid density matrix at $t=0$ (Assumption)

and since $e^{-4t} > 0$, $\therefore$ $a_{11}(t) \geq 0$ and $a_{22}(t) \geq 0$

and similarly,

$$a_{11}(t) \cdot a_{22}(t) = \left(a_{11}(0)\left(\frac{1+e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1-e^{-4t}}{2}\right)\right)\left(a_{11}(0)\left(\frac{1-e^{-4t}}{2}\right) + a_{22}(0)\left(\frac{1+e^{-4t}}{2}\right)\right)$$

$$\geq a_{12}(0)\left(\frac{e^{-2t}+1}{2}\right)^2$$

$\therefore$ The map is positive as well.

3. Assume $\vec{x}_1 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for $\lambda_1 = \frac{1}{4}(1-e^{-4t})$

$$\begin{bmatrix} \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right) & 0 & 0 & \frac{1}{2}\cosh^2(t)\,e^{-2t} \\ 0 & \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) & 0 \\ \frac{1}{2}\cosh^2(t)\,e^{-2t} & 0 & 0 & \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right) \end{bmatrix} \vec{x}_1$$

$$= \begin{bmatrix} \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right)a + \frac{1}{2}\cosh^2(t)\,e^{-2t}d \\ \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right)b \\ \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right)c \\ \frac{1}{2}\cosh^2(t)\,e^{-2t}a + \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right)d \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \end{bmatrix}$$

$\Rightarrow$ b & c are free variables

$$\Rightarrow \frac{1}{4}(1+e^{-4t})\,a + \frac{1}{2}\cosh^2(t)\,e^{-2t}d = \frac{1}{4}(1-e^{-4t})\,a$$

$$\Rightarrow \frac{1}{2}e^{-4t}a + \frac{1}{2}\cosh^2(t)\,e^{-2t}d = 0 \qquad [1]$$

$$\Rightarrow \frac{1}{2}\cosh^2(t)\,e^{-2t}a + \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right)d = \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right)d$$

$$\Rightarrow \frac{1}{2}\cosh^2(t)\,e^{-2t}a + \frac{1}{2}e^{-4t}d = 0 \qquad [2]$$

[1] - [2]

$$\frac{1}{2}e^{-4t}(a-d) - \frac{1}{2}\cosh^2(t)\, e^{-2t}(a-d) = 0$$

$$\Rightarrow \frac{1}{2}\left(e^{-2t} - \cosh^2(t)\right)(a-d) = 0$$

$$\Rightarrow \frac{1}{2}\left(e^{-2t} - \frac{(e^{-2t}+e^{2t}+2)}{4}\right)(a-d) = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{4e^{-2t} - e^{-2t} - e^{2t} - 2}{4}\right)(a-d) = 0$$

$\Rightarrow$ clearly, $4e^{-2t} - e^{-2t} - e^{2t} > 2$,

$$\frac{1}{2}\left(\frac{4e^{-2t} - e^{2t} - e^{2t} - 2}{4}\right) \neq 0.$$

$\therefore$ $a = d$, By plugging in value in the equation, we get $a = d = 0$.

$$\therefore (\lambda_1, \vec{x}_1) = \left(\frac{1}{4}(1-e^{-4t}), \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

let $\vec{x}_2$ be the eigenvector associated with λ_2.

since $\vec{x}_2$ is orthogonal, we get

$$(\lambda_2, \vec{x}_2) = \left(\frac{1}{4}(1-e^{-4t}), \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

Assume $\vec{x}_3 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for $\lambda_3 = \frac{1}{4}(1+e^{-4t}) - \frac{1}{2}\cosh^2 t\, e^{-2t}$

$$\begin{bmatrix} \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right) & 0 & 0 & \frac{1}{2}\cosh^2(t)\, e^{-2t} \\ 0 & \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) & 0 \\ \frac{1}{2}\cosh^2(t)\, e^{-2t} & 0 & 0 & \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right) \end{bmatrix} \vec{x}_2$$

$$= \begin{bmatrix} \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right) a + \frac{1}{2}\cosh^2(t)\, e^{-2t} d \\ \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) b \\ \frac{1}{2}\left(\frac{1-e^{-4t}}{2}\right) c \\ \frac{1}{2}\cosh^2(t)\, e^{-2t} a + \frac{1}{2}\left(\frac{1+e^{-4t}}{2}\right) d \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \end{bmatrix}$$

for $\lambda_3 = \frac{1}{4}(1+e^{-4t}) - \frac{1}{2}\cosh^2 t\, e^{-2t}$,

$$\frac{1}{4}(1+e^{-4t})\, a + \frac{1}{2}\cosh^2(t)\, e^{-2t} d = \frac{1}{4}(1+e^{-4t})\, a - \frac{1}{2}(\cosh^2(t)\, e^{-2t})\, d$$

$\Rightarrow$ d is a free variable

similarly, by symmetry, a is also a free variable

$$\frac{1}{4}(1-e^{-4t})\, b = \frac{1}{4}(1+e^{-4t})\, b - \frac{1}{2}\cosh^2(t)\, e^{-4t}\, b$$

$$\Rightarrow \left[\frac{1}{2}e^{-4t} - \frac{1}{2}\cosh^2(t)\right) b = 0 \quad \Rightarrow b = 0$$

Similarly, $e = 0$ by symmetry.

$\therefore$ we have,

$$(\lambda_3, \vec{x}_3) = \left(\frac{1}{4}(1+e^{-4t}) - \frac{1}{2}\cosh^2 t \, e^{-2t}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

and since $\vec{x}_4$ for λ_4 will be orthogonal to $\vec{x}_3$,

$$\therefore \quad (\lambda_4, \vec{x}_4) = \left(\frac{1}{4}(1+e^{-4t}) + \frac{1}{2}\left(\frac{e^{-2t}+1}{2}\right)^2, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$$

The Kraus operators will therefore be

$$K_i = \sqrt{\lambda_i} \text{ unstack}(\vec{x}_i)$$

$$\therefore \quad K_1 = \sqrt{\frac{1}{4}(1-e^{-4t})} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad K_2 = \sqrt{\frac{1}{4}(1-e^{-4t})} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$K_3 = \sqrt{\frac{1}{4}(1+e^{-4t}) - \frac{1}{2}\cosh^2 t \, e^{-2t}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_4 = \sqrt{\frac{1}{4}(1+e^{-4t}) + \frac{1}{2}\cosh^2 t \, e^{-2t}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nature of the dynamic map

At $t=0$, $T(t) = 1$ which is maximum, and from there the distance reduces exponentially and both ρ_1 & ρ_2 converge to the maximally entangled state $\mathbb{I}/2$.

This also indicates information loss as the previous state information has been loss by converging to $\mathbb{I}/2$

4. To show that Λ is CP-divisible, we need to check if

$l_\varepsilon = \mathbb{I} + \varepsilon\mathcal{L}$ is CP, where $\mathcal{L} = \dot{\rho}$

We have the 4 differential equations below describing the dynamic map:

(1) $\frac{da_{11}}{dt} = 2(a_{22} - a_{11})$ (3) $\frac{da_{21}}{dt} = 2a_{12}(\tan(t) - 1)$

(2) $\frac{da_{12}}{dt} = 2a_{12}(\tan(t) - 1)$ (4) $\frac{da_{22}}{dt} = 2(a_{11} - a_{22})$

$\therefore$ To check if l_ε is CP or not, we check if $(\mathbb{I} \otimes l_\varepsilon)\Phi_{AB}$ is ≥ 0 or not.

same as question 2.

This can be done as follows: $\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$

$$(\mathbb{I} \otimes l_\varepsilon)\Phi_{AB} = \begin{pmatrix} l_\varepsilon\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} & l_\varepsilon\begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \\ l_\varepsilon\begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} & l_\varepsilon\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \varepsilon(2\rho_{22} - 2\rho_{11}) & 0 & 0 & \frac{1}{2} + \varepsilon(\tanh(t) - 1) \\ 0 & \varepsilon(2\rho_{11} - 2\rho_{22}) & 0 & 0 \\ 0 & 0 & \varepsilon(2\rho_{22} - 2\rho_{11}) & 0 \\ \frac{1}{2} + \varepsilon(\tanh(t) - 1) & 0 & 0 & \frac{1}{2} + \varepsilon(2\rho_{11} - 2\rho_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} - \varepsilon & 0 & 0 & \frac{1}{2} + \varepsilon(\tanh(t) - 1) \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ \frac{1}{2} + \varepsilon(\tanh(t) - 1) & 0 & 0 & \frac{1}{2} - \varepsilon \end{pmatrix}$$

considering M_1 from the previous question we get

$\lambda_1 = \lambda_2 = \varepsilon$

$$M_2 = \begin{bmatrix} \frac{1}{2} - \varepsilon - \lambda & \frac{1}{2} + \varepsilon(\tanh|t| - 1) \\ \frac{1}{2} + \varepsilon(\tanh|t| - 1) & \frac{1}{2} - \varepsilon - \lambda \end{bmatrix}$$

$$\Rightarrow \left(\frac{1}{2} - \varepsilon - \lambda\right) = \pm\left(\frac{1}{2} + \varepsilon(\tanh|t|) - 1\right)$$

$$\frac{1}{2} - \varepsilon - \lambda_3 = \frac{1}{2} + \varepsilon(\tanh|t| - 1)$$

$$\Rightarrow \lambda_3 = -\varepsilon - \varepsilon(\tanh|t| - 1) = -\varepsilon\tanh|t| \leq 0$$

$\therefore$ we found an eigenvalue that is -ve

$\therefore$ ϱ_ε is not CPTP

and therefore our original map is not CP-divisible

5. $\rho_1 = |0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}[1\ 0] = \begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\rho_2 = |1\rangle\langle 1| = \begin{bmatrix}0\\1\end{bmatrix}[0\ 1] = \begin{bmatrix}0&0\\0&1\end{bmatrix}$

$$\Lambda(\rho_1) = \begin{bmatrix}a&b\\c&d\end{bmatrix}$$

$a = 1\left(\frac{1+e^{-4t}}{2}\right) + 0\left(\frac{1-e^{-4t}}{2}\right)$ $\qquad c = \cosh^2(t)\cdot e^{-2t}\,0 = 0$

$b = \cosh^2(t)\, e^{-2t}\cdot 0 = 0$ $\qquad d = \frac{(1-e^{-4t})}{2}\cdot 1$

$$\Lambda(\rho_1) = \begin{bmatrix}1+e^{-4t}/2 & 0\\ 0 & 1-e^{-4t}/2\end{bmatrix} = \rho_1(t)$$

$$\Lambda(\rho_2) = \begin{bmatrix}a&b\\c&d\end{bmatrix}, \quad \rho_2 = \begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$a = \left(\frac{1-e^{-4t}}{2}\right)$, $\qquad b = 0$

$c = 0$ $\qquad d = \left(\frac{1+e^{-4t}}{2}\right)$

$$\Lambda(\rho_2) = \begin{bmatrix}(1-e^{-4t})/2 & 0\\ 0 & (1+e^{-4t})/2\end{bmatrix} = \rho_2(t)$$

$$\frac{1}{2}\|\rho_1(t) - \rho_2(t)\|_1 = \frac{1}{2}\left\|\begin{bmatrix}e^{-4t} & 0\\ 0 & -e^{-4t}\end{bmatrix}\right\|_1$$

$$A = \begin{bmatrix}e^{-4t} & 0\\ 0 & -e^{-4t}\end{bmatrix} = e^{-4t}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$A^{+} = e^{-4t}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^{+}A = e^{-8t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt{A^{+}A} = e^{-4t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Tr(\sqrt{A^{+}A}) = 2e^{-4t}$$

$$T(t) = \frac{1}{2} Tr(\sqrt{A^{+}A}) = e^{-4t}$$

$t=0$, $T(0) = 1$ and $t>0$, e^{-4t} decreases exponentially,

$\therefore$ The graph $T(t)$ will be as follows: